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A NUMERICAL FORECAST MODEL USING BICUBIC SPLINES ON A TELESCOPING GRID

by

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SUMMARY

A two-level numerical forecast model is proposed in which bicubic polynomial splines are used to fit the spatial variations of the dependent variable fields on a variable area telescoping grid. The spline method generates spatial derivatives which inherently represent a form of smoothing of the slope estimates generated by finite difference methods; the telescoping grid is constructed to ensure computational stability at high latitudes without the need of high frequency filters, spatial staggering of the dependent variables and complex flux calculations.

Thirty-six hour numerical forecasts using the proposed model and using a 1969 version of the Mintz-Arakawa model are compared between themselves and the real weather. The spline method is shown to have advantages over the finite difference method in terms of decreased phase lag and lower root-mean-square forecast error. Computation time is decreased by a factor of 1/3 due to the telescoping nature of the grid, and there is no decrease in forecast accuracy in the fine grid region arising from the surrounding coarse grid region.

Extensions to the model are developed through the derivation of a generalized spline based on continuity of curvature and a numerical forecast technique using weighted residual methods.

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NOMENCLATURE

t	=	time
x	=	coordinate to the east
y	=	coordinate to the west
P	=	pressure
P_s	=	surface pressure
P_T	=	pressure at the base of the stratosphere, 200 mb
ζ	=	$P_s - P_T$
σ	=	$(P - P_T)/\zeta$, normalized pressure
u,v	=	horizontal wind components in the x and y directions, respectively
T	=	temperature
θ	=	potential temperature
q	=	mixing ratio (mass of water vapour/mass of dry air)
ϕ	=	geopotential
ϕ_s	=	surface geopotential
Θ	=	longitude
Φ	=	latitude
$\Delta\Theta$	=	longitude grid spacing in the Mintz-Arakawa model
$\Delta\Phi$	=	latitude grid spacing in the Mintz-Arakawa model
δ	=	extent of the polar cap, in radians of latitude
m,n	=	metrics
A	=	mean radius of the earth
u^*	=	$n\zeta u$, flux in the x direction
v^*	=	$m\zeta v$, flux in the y direction
Z	=	$mn\zeta$
a	=	specific volume

ρ	=	density
C_p	=	specific heat at constant pressure
R	=	gas constant for dry air
K	=	R/C_p
P^*	=	standard pressure, 1000 mb.
λ	=	level (altitude) sign parameter
F_x, F_y	=	x and y components of the horizontal friction force per unit mass
H	=	heating rate per unit mass
E	=	evaporation rate
C	=	precipitation rate
ψ, Ψ	=	represents any dependent variable
Δt	=	time step interval
B	=	the number of time steps between energy source calculations
i,j	=	general indexing pair specifying the grid point
N	=	the number of rows of grid points in the y direction
M_j	=	the number of grid points in the x direction on latitude circle j
d_j	=	grid point spacing on the j'th latitude circle
V	=	resultant velocity
S	=	S(x) or S(x,y), the equation for the cubic spline curve fit or the bicubic spline surface fit.
$u_i, u_{i,j}$	=	$S(x_i)$ or $S(x_i, y_j)$
$p_i, p_{i,j}$	=	$\frac{\partial S}{\partial x}(x_i)$ or $\frac{\partial S}{\partial x}(x_i, y_j)$, slope at the grid point in the x direction
$q_{i,j}$	=	$\frac{\partial S}{\partial x}(x_i, y_j)$, slope at the grid point in the y direction
$s_{i,j}$	=	$\frac{\partial^2 S}{\partial x \partial y}(x_i, y_j)$, cross derivative

$P_i, P_{i,j}$ = $\frac{\partial^2 S}{\partial x^2}(x_i)$ or $\frac{\partial^2 S}{\partial x^2}(x_i, y_j)$

$Q_{i,j}$ = $\frac{\partial^2 S}{\partial y^2}(x_i, y_j)$

$S_{i,j}$ = $\frac{\partial^4 S}{\partial x^2 \partial y^2}(x_i, y_j)$

h_i = $x_i - x_{i-1}$

k_j = $y_j - y_{j-1}$

λ_i = $h_{i+1} / (h_i + h_{i+1})$

μ_i = $h_i / (h_i + h_{i+1})$ $= 1 - \lambda_i$

u_p, v_p, V_p, Θ_p = expressions used in defining the polar boundary condition for the horizontal wind components

a, b, β_i, XS, US, $D_i, C_i^*, \lambda_i^*, \mu_i^*, \lambda_i^{**}, \mu_i^{**}$ = constants and expressions used in defining the generalized spline based on continuity of curvature

$a, b, c, \xi, \overline{h}, h_i$ = expressions used in interpolation method A

f_1, f_2, h_i = expressions used in interpolation method B

RMSE = 36 hour forecast root-mean-square error relative to the true weather at 36 hours

RMSE_i = average of RMSE along longitude line i

RMSE_j = average of RMSE around latitude circle j

RMSC = the root-mean-square change in the true weather over the 36 hour forecast period

RMSC_i = average of RMSC along longitude line i

RMSC_j = average of RMSC around latitude circle j

Glossary

balanced wind: Applying the assumption of non-divergent quasi-horizontal flow in which the horizontal velocity field is expressed in terms of a stream function, and other assumptions, the vector equation of motion may be reduced to the "balance equation". This equation involves the stream function and geopotential as dependent variables. Under certain conditions, with a known distribution of geopotential, the balance equation may be solved for the stream function. The velocity field obtained from the stream function is called a balanced wind.

barotropic: A barotropic atmosphere is one in which the surfaces of constant pressure are also surfaces of constant density and temperature.

computational mode: The computational mode is the portion of the solution of the difference equation which has no physical counterpart in the true solution of the differential equation.

filtered equation models: A numerical forecast model in which the governing equations are differentiated first and then numerically solved is termed a filtered equation model.

geopotential: The work done in moving a unit mass from mean sea level to some elevation above sea level is called the geopotential of that level.

grid points: These are points on the numerical forecast grid at which values of the dependent variables are estimated at discrete time intervals throughout the forecast.

latitudinal: The latitudinal direction is the south-north direction on the earth, perpendicular to the latitude circles.

longitudinal: The longitudinal direction is the west-east direction on the earth's surface, perpendicular to the longitude circles.

nine point difference operator: This denotes a finite difference expression involving values of the dependent variables at nine grid points.

node point: see grid point

numerical explosion: A numerical forecast is said to explode numerically when the magnitudes of the dependent variables exceed the allowable limits of computer storage.

mixing ratio: A measure of moisture content, the mixing ratio is the mass of water vapour per unit mass of dry air.

potential temperature: If a parcel of gas at temperature T and pressure P is brought adiabatically to standard pressure (1000 mb), the resulting temperature in the parcel is called the potential temperature.

phase lag: The position in longitude of the large scale meteoroligical waves in the numerical forecast is compared to the position of the waves in the true weather. The difference between the two positions is termed the phase lag of the numerical forecast.

physical mode: The physical mode refers to that portion of the solution of the difference equation which has a physical counterpart in the true solution of the differential equation.

primitive equation models: A numerical forecast model in which the governing equations are numerically solved in their usual form is termed a primitive equation model.

surface geopotential: This is the geopotential at the surface of the earth.

wave number n: The number of complete meteorological waves around a latitude circle is the wave number.

1. INTRODUCTION

Weather prediction by numerical methods deals with the numerical solution of the hydrodynamic and thermodynamic equations governing atmospheric flow. Such a solution involves an enormous number of arithmetic and logical operations for which reason electronic computers are now used. The basic principles underlying numerical weather prediction were discovered early in this century. At the time, it was recognized that the non-linear system of equations did not possess an analytic solution. Also, initial data defining the state of the atmosphere was inadequate. The first attempt to solve this system of equations using numerical methods was made by L.F. Richardson in 1921. His results were in considerable error and interest in numerical weather prediction declined. However, with the advent of the electronic computer in the 1940's, numerical forecasting was revived. The first successful numerical prediction was made by J. Charney in 1949. From this point on, the science developed rapidly and models with varying numbers of restrictions on the flow were studied by numerous research groups in many countries.

It is a common practice to classify numerical forecast models into one of two basic categories based on the form of the governing differential equations (Haltiner and Martin (1), Haltiner (2)). In primitive equation (PE) models*, the governing equations are numerically solved in their usual form, whereas in filtered equation (FE) models, the governing equations are differentiated first and then solved numerically. The

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* Refer to the Glossary for definitions of common meteorological terms.

PE models permit all types of wave motions (both long and short waves), while the FE models permit only the long meteorologically significant wave motions called Rossby waves. The earlier numerical forecast models are of the filtered type since they are the simplier model form (containing approximations which are only true on the average over long periods of time) and require considerably less computation time. This shorter computation time for FE models compared to PE models arises from the longer time steps and fewer dependent variables and equations in PE models. With the increase in computational speed of computers, it becomes feasible to investigate the more sophisticated PE models.

Numerical forecast models are primarily used for general circulation studies and short term weather prediction. In general circulation studies, the forecast usually begins with an atmosphere at rest and extends over a long period of time (several months). However, in short term weather prediction, the forecast begins with real weather data defining the initial state of the atmosphere and extends over a short period of time (up to a week). Three PE general circulation models which have been in operation for several years are a multi-level model developed by Smagorinsky at the Geophysical Fluid Dynamics Laboratory, U.S.A. (Smagorinsky (3,4), Smagorinsky, Manabe and Holloway (5), and others), a two level model developed by Mintz and Arakawa at UCLA (Mintz (6), Langlois and Kwok (7), and a multi-level model developed by Kasahara and Washington (8) at the National Center for Atmospheric Research, U.S.A. These models are being applied only in a limited fashion to short term forecasting. The Smagorinsky model shows promising results with real weather data for short term forecasts up to a week in duration (p.77 of ref 2), however the computation time is too long for operational purposes. Similarly, the Mintz-Arakawa model is being applied with some success to short term predictions (Kesel and Winninghoff (9), Price (10)). Two short term forecast models which have been in use for many years are an operational barotropic model developed by Shuman and Vanderman (11) and a six level PE short term forecast model developed by Shuman and Hovermale (12). In addition, many other short term forecast models are in use in various countries for both operational and experimental purposes.

Three major steps may be identified in the formulation of a numerical weather prediction model. The first step is to choose a system of hydrodynamic and thermodynamic equations, in terms of a suitable coordinate system, in order to explain mathematically the motion in the atmosphere. The relevant equations are Newton's second law of motion, the first law of thermodynamics, the equation of state for a perfect gas, and laws expressing conservation of dry air and water vapour. Next, it is necessary to approximate the continuous dependent variable fields by discrete values of the variables at specified nodes or grid points in the forecast region. This selection of the forecast grid is of major importance in determing the forecast resolution, accuracy, and computation time. The final step is to obtain an approximate numerical solution to the governing equations at the specified grid points, thereby advancing the dependent variable fields in time. In this thesis, emphasis will be placed on the second and third steps in the formulation of the forecast model: the selection of a forecast grid and the method used to solve the governing differential equations.

1.1 The Method of Solution of the Governing Differential Equations

Consider first the method used to solve the governing differential equations. In the majority of numerical forecast models, finite difference methods are used to obtain an approximate solution to the system of partial differential equations. The basic approximation in finite difference methods is to replace the continuous variables by discrete variables which vary stepwise by finite increments in space and time. Whereas the behavior of the continuous variables is governed by the system of differential equations, the behavior of the discrete variables is governed by a system of difference equations. Hence, a difference equation is simply the finite difference representation of a differential equation; and the solution of the difference equation yields an approximate solution to the differential equation at specified points in space and at discrete intervals in time. Associated with the numerical solution of the system of difference equations are a number of errors, primarily truncation error and discretization or computational error (Smith (13), Forsythe and Wasow (14)). The truncation error in the difference equation arises from representing the spatial derivatives in the differential equation by the first few terms in a Taylor series expansion of the derivative, in terms of specified values of the variable at adjacent nodes or grid points. This error depends on both the size of the finite space increment and the wavelength of the continuous field being estimated (Gates (15)). The most widely used procedure is the central space difference, which may be illustrated in the case of the first derivative of a continuous function f as (ref 15, 13)

$$\frac{\partial f}{\partial x} \simeq \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} \quad , \qquad (1.1)$$

where x denotes a typical spare variable and Δx is the grid point interval. The truncation error of this approximation is in the order of $(\Delta x)^2$. Other frequently used estimates for the first derivative are the forward difference and backward difference,

$$(f(x + \Delta x) - f(x))/\Delta x \text{ and } (f(x) - f(x - \Delta x))/\Delta x ,$$

respectively. The error in these approximations is in the order of Δx. The difference between the difference equation as a whole and the differential equation which it represents is called the truncation error of the difference equation.

The second error, discretization error, is the error in the exact numerical solution of the difference equation (Smith (13)). If ϕ represents the exact solution of the partial differential equation, and ϕ_D represents the exact solution of the difference equation, then the discretization error is $\phi - \phi_D$. The solution method is convergent if ϕ_D approaches ϕ as Δx, Δt become infinitisimally small. Here, Δx and Δt denote the finite space and time increments respectively. Closely associated with the discretization error is the computational stability of the difference scheme; that is, the time variation of the discretization error (Crandal (16), Kurihara (17)). Fundamentally, whenever $\Delta t/\Delta x$ becomes larger than some critical value, the computational mode in the numerical solution tends to grow in time and eventually destroys the physical mode. The physical mode refers to that portion of the solution of the difference equation which has a physical counterpart in the true solution of the differential equation; the computational mode is the remaining portion of the solution of the difference equation and has no physical counterpart in the true solution of the differential equation. Since there is no analytic solution to the governing partial differential equations for atmospheric flow, it is customary to examine the stability of this corresponding linearized version of the governing equations, with constant coefficients (the von Neumann stability condition, Kasahara (18)). To simplify the analysis further, a common approximation is to check the stability of the difference equations considering only one factor at a time (ref 18). For example, to examine the stability of the difference scheme for a typical advective term in the thermodynamic equation, one may examine the linear one dimensional advection equation for temperature, T,

$$\frac{\partial T}{\partial t} + c\,\frac{\partial T}{\partial x} = 0 \quad , \qquad (1.2)$$

where c is a constant (the zonal wind speed). Haltiner (pp. 18-25 of ref 2) shows that the difference scheme for equation 1.2 using central differences for both time and space, is computationally stable provided $c\Delta t/\Delta x \leq 1$. This means that the computational mode in the numerical solution approaches 0 as time increases, provided $c\Delta t/\Delta x \leq 1$. However, if forward time and central space differences are used, this difference scheme is computationally unstable for all values of $\Delta t/\Delta x$. It is interesting to note that the case of forward time and forward space differences is computationally unstable when $c>0$ and computationally stable when $c<0$. This corresponds to the so-called "upstream differencing" technique, in which a stable differencing scheme is obtained when the space differencing is in the opposite direction to the wave motion (Gosman, et.al. (19)). In addition to the computational stability of the difference scheme, it is necessary to consider the degree of phase lag and amplitude distortion of the physical mode of the numerical solution. For example, although the difference scheme using central finite differences in space and time for the one-dimensional advection equation 1.3 is computationally stable when $c\Delta t/\Delta x \leq 1$, the physical mode exhibits a phase lag and smaller amplitude when compared to the true solution.

The accuracy and stability characteristics of ten different finite difference schemes are discussed by Grammeltvedt (20) using the primitive equations in a barotropic fluid; with primary emphasis on the effects of the spatial differencing on the forecast. With an analytic wave for the initial condition, the analysis shows that the quadratic conservative difference schemes (or schemes which conserve both the first and second moments of the dependent variables) and total energy conservative difference schemes (or schemes which conserve the sum of available potential plus kinetic energy) are more stable than the other second order conservative schemes. However, the most stable schemes are those in which the advective terms are calculated using nine point spatial finite differences and therefore contain a form of smoothing, and the generalized Arakawa scheme which conserves mean vorticity, mean kinetic energy, and mean square vorticity in nondivergent flow. The most commonly used methods to suppress computational instabilities are to include artificial viscosity terms in the difference equations, or to write the finite difference equations in a form which conserves certain statistical moments (usually of quadratic form) of the dependent variables (ref 20). The Smagorinsky general circulation model (ref 3,4, 5) uses finite differences which conserve momentum and total energy. Therefore, the Smagorinsky model requires lateral eddy viscosity terms to suppress the nonlinear computational instabilities inherent in the difference scheme, but Mintz (6) feels that this may have the undesirable side effect of excessively damping the meteorologically significant wave motions. However, the Mintz-Arakawa general circulation model (ref 6,7,10) uses finite differences due to Arakawa which are both quadratic conservative and total energy conservative schemes. Therefore, the differencing in the Mintz-Arakawa model is inherently nonlinearly computationally stable without the use of explicit frictional dissipation. Of the short term prediction models, Shuman's scheme (ref 11) calculates the advective terms using a nine point difference operator which should yield the most stable forecast due to its smoothing effect (ref 20).

In addition to the space differencing scheme, the form of time differencing employed has a strong effect on stability. This was mentioned briefly in the discussion of computational stability, where, for example, it was noted that forward differencing in time is unstable whereas central differencing in time is conditionally stable (provided $c\Delta t/\Delta x \leq 1$). The stability characteristics of several implicit, explicit and iterative time differencing schemes were examined by Kurihara (17) using a linear system of equations. Of the methods investigated, the two stage leapfrog-trapezoidal method shows the most promise since it has little damping and little phase retardation effect on the physical mode, with strong damping of the spurious computational mode, for $c\Delta t/\Delta x < \sqrt{2}$. However, being a two stage scheme, it requires twice the computation time of the simple centered difference time differencing scheme (also called the centered leapfrog explicit scheme), which itself has no change in amplitude of both the physical and computational modes with only moderate acceleration of the physical mode. Therefore, the simple centered leapfrog explicit scheme is used in most models.

In the Mintz-Arakawa model, a modified Matsuno time integration (Matsuno (21), pp. 105-110 of ref 10) is employed. The original three stage Matsuno scheme gives strong damping of the high frequency waves (which are usually spurious). However, the modification used in the Mintz-Arakawa model essentially reduces the Matsuno method to a two stage Euler-backward scheme discussed by Kurihara (17). This scheme has no computational mode, with moderate selective damping and large phase acceleration of the physical mode.

In this thesis, a numerical forecast model is proposed in which double cubic polynomial spline functions are used to fit the spatial variation of the dependent variable fields, thereby eliminating the need for finite differencing in space to estimate the spatial derivatives. The cubic spline S(x) of interpolation to the ordinates u_i at mesh locations x_i, i,...M, is a piecewise continuous function defined as a cubic polynomial in each interval $x_{i-1} \leq x \leq x_i$ having continuous first and second derivatives (Ahlberg, Nilson and Walsh (22), Greville (23)). The generalization to two dimensions to obtain the double cubic (or bicubic) spline is straightforward. There are several reasons for proposing that the use of double cubic polynomial splines may be an improvement over finite difference methods in estimating spatial derivatives.

Firstly, cubic polynomial splines are an effective tool in the processes of numerical interpolation, differentiation, integration, and curve fitting (pp. 42-52 of ref 22). In particular, the slope estimates returned by a spline curve fit inherently represent a form of smoothing of the slope estimates returned by standard forward, backward or central finite differences. In the numerical forecast models developed up to this point, complex differencing schemes are necessary to obtain a smoothed slope estimate. For example, the Shuman model (ref 12) uses nine point difference estimates (a form of smoothing) to obtain estimates for the advective terms; and the Mintz-Arakawa model (Langlois and Kwok (7), Price (10)) was complex, multi-point difference estimates for the spatial slopes. These complex and somewhat arbitrary smoothing schemes used in finite difference methods are not required when the double cubic spline is used to estimate the first derivatives, due to the inherent "smoothed" nature of the spline curve fit. It should be mentioned that this does not hold true for the second derivative. Rather, the finite difference estimate of the second derivative given by Newton's second divided difference (a three point operator in one-dimension) represents a smoothing of the spline estimates for the second derivative (p.44 of ref 22). In order to employ the spline method to obtain good "smoothed" second derivative estimates, it is necessary to do a spline fit to the first derivatives, with the first derivative itself obtained from a previous spline fit. Ahlberg, Nilson and Walsh (p.44 or ref 22) discuss this "spline-on-spline" method of obtaining smoothed second derivatives.

A second reason for proposing the use of a spline function to obtain slope estimates, in place of finite difference methods, is the minimum norm property, or Holladay's theorem, for cubic splines (p.3 of ref 22). This theorem states that for any function $f(x) \varepsilon c^{2*}$ satisfying $f(x_i) = u_i$, $i = 1,\ldots M$, the integral of $|f''(x)|^2$ over the interval (x_1, x_M) is a minimum when $f(x) = S(x)$, provided $S''(x_1) = S''(x_M) = 0$.

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* f(x) and its first two derivatives are continuous.