Residual Finite-dimensionality and Realizations of Approximate Rigidity Phenomena for Operator Algebras

by

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Abstract

Representations on a Hilbert space are a common tool for understanding both C*-algebras and (non self-adjoint) operator algebras. For operator algebras, a major theme has been to capitalize on the structural theory of an enveloping C*-algebra (or C*-cover) to better understand the underlying subalgebra. This philosophy has spanned several advancements, yet there is a subtlety involved in the process: the C*-algebra can vary greatly depending on the choice of representation of the operator algebra. Thus, it becomes advantageous to leverage information from several C*-covers to better understand the underlying operator algebra. Here, we focus on interactions between an operator algebra and its C*-covers, with an emphasis on sufficiency and maximality conditions for two fundamental classes of representations: the finite-dimensional representations and the non-commutative Choquet boundary, respectively. Both families offer separate advantages because the former class is more tractable, yet the latter induces a minimal representation due to work of Arveson.

In Chapter 3, we analyze the residual finite-dimensionality of the maximal C*-cover of an operator algebra. To this end, we consider several C*-covers that are formed through families of finite-dimensional representations, and compare these C*-covers with the maximal C*-cover. Along the way, we generalize Hadwin's characterization of separable residually finite-dimensional C*-algebras.

In Chapter 4, we study maximality conditions on the non-commutative Choquet boundary. A conjecture of Arveson asserts that maximality conditions imply a rigidity property that is, a priori, much stronger. In this chapter, we uncover a significant localization procedure that generalizes several past attempts at Arveson's conjecture. This localization procedure is also applied to another conjecture of Arveson that concerns quotient modules of the Drury-Arveson space.

Arveson's rigidity conjecture is originally inspired by a development in approximation theory due to Šaškin. In Chapter 5, we achieve one non-commutative analogue to Šaškin's theorem. In the setting of classical function theory, this encodes a maximality condition for the Choquet boundary with a rigidity property. We find that a similar phenomenon is still true for a large class of C*-algebras.

Contributions of Authors

Chapters 3 and 4 are based off peer-reviewed publications for which I am the solo author (see [85] and [86], respectively). Chapter 5 is an early draft of a manuscript that, since writing the content of this thesis, has been made public and written jointly with Raphaël Clouâtre [28].

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Chapter 1

Introduction

Over the course of the twentieth century, rings of operators have been critical to the development of quantum physics and functional analysis, as well as showcasing deep connections to complex function theory, group theory, and algebraic topology (among others). Within this thesis, we study spaces of Hilbert space operators. In particular, we will predominantly be concerned with unital operator spaces, (non self-adjoint) operator algebras, as well as the C*-algebras that they generate. We start, in Chapter 2, by describing preliminary details that will be necessary for later chapters.

In Chapter 3, we are concerned with operator algebras and C*-algebras that are well-understood through their finite-dimensional $*$-representations. This philosophy underlies nuclearity and quasidiagonality for C*-algebras, which has generated a flurry of recent progress [48],[87]. However, we will study a different choice of finite-dimensional approximation property. Namely, we study residual finite-dimensionality for operator algebras.

Residually finite-dimensional C*-algebras constitute a significant class of C*-algebras and possess numerous equivalent characterizations [3],[26],[32],[46],[50]. Motivated by these successes in the C*-algebra theory, we focus on residual finite-dimensionality of non self-adjoint operator algebras. We say that an operator algebra is *residually finite-dimensional* (or RFD) if it may be embedded into a product of matrix algebras. This concept was introduced in [24] and there are still only a few characterizations of RFD operator algebras [21],[24],[55].

We consider the collection of all C*-covers of an operator algebra $\mathcal{A}$ to study residual finite-dimensionality. A C*-*cover* of $\mathcal{A}$ is a pair $(\mathfrak{A}, \iota)$ where ι is a completely isometric representation of $\mathcal{A}$ and $\mathfrak{A} = \mathrm{C}^*(\iota(\mathcal{A}))$. In the literature on C*-covers, there are two well-studied covers which dominate the theory: the minimal C*-cover and the maximal C*-cover. These two C*-covers are known to always exist for any operator algebra [13, Proposition 2.4.2],[53]. In Chapter 3, the primary focus is to analyze residual finite-dimensionality of the maximal C*-cover by constructing suitable replacements to the maximal C*-cover for when it may fail to be RFD.

However, for the majority of this thesis, we will be concerned with the structure of the minimal C*-cover, or C*-envelope, for a unital operator space $\mathcal{M}$. Existence of the minimal C*-cover was the basis of a research program spanning nearly half a century [4],[7],[36],[44],

[53],[71]. The existence of the C*-envelope was proposed by Arveson [4], but has only started to reach its full potential in recent years (for example, see [7],[15],[19],[26],[27],[29],[37],[38], [39],[41],[62]). Within Arveson's original work [4], they conjectured that a special family of irreducible $*$-representations of $C^*(\mathcal{M})$, called boundary representations, completely determine the norm of $\mathcal{M}$. As a consequence, the boundary representations can then be seen to determine the C*-envelope as well. For spaces of continuous functions, the C*-envelope is the space of continuous functions over the Shilov boundary (or, alternatively, the closure of the Choquet boundary). In this case, the family of boundary representations coincide with the Choquet boundary. Thus, in general, the completion of Arveson's program suggests that the class of boundary representations can be seen to be a suitable non-commutative analogue to the Choquet boundary.

Chapters 4 and 5 are primarily motivated by Arveson's so-called hyperrigidity conjecture, which concerns the consequences of a maximality constraint for the non-commutative Choquet boundary [7]. For spaces of continuous functions, Šaškin proved that a maximality constraint on the Choquet boundary may be connected with approximation theory [83]. Fifteen years ago, Arveson conjectured that a maximality constraint on the non-commutative Choquet boundary is connected with a type of non-commutative approximation theory. Moreover, Arveson's conjecture appears to be, a priori, much stronger than Šaškin's conclusion. Many authors have attempted Arveson's conjecture and verified it in isolated scenarios [8],[19],[20],[22],[25],[38],[60],[62],[64],[67],[82]. Although, quite recently, Bilich and Dor-On found a counterexample to Arveson's conjecture [10]. At the beginning of Chapters 4 and 5, we will be more explicit but, for now, we will refrain from many of the technical details.

In Chapter 4, we study approximate unitary equivalence classes of $*$-representations that are of interest for the purposes of Arveson's conjecture. Among the known results that point positively toward Arveson's conjecture, we find that a handful of these conclusions may be achieved merely locally, rather than as a consequence to the (global) maximality constraint on the non-commutative Choquet boundary. In turn, we find a couple examples of operator spaces that possess many of the known properties of those in Arveson's conjecture, yet their non-commutative Choquet boundary is not maximal.

Additionally, due to the recent counterexample for Arveson's hyperrigidity conjecture [10], an open question that has remained is to prove alternative formulations of a non-commutative Šaškin theorem. In Chapter 5, we investigate one new interpretation for a non-commutative Šaškin theorem. The centrepiece of this chapter is an analogue to Šaškin's theorem for a notable class of non-commutative spaces.

Finally, in Chapter 6, we recall a few of the central developments in this thesis, as well as proposing some new directions for exploration.

Chapter 2

Preliminaries

To start, we will make some of our notation explicit. Given a (complex) Hilbert space $\mathcal{H}$, let $B(\mathcal{H})$ denote the space of all bounded linear maps from $\mathcal{H}$ to itself and let $\mathfrak{K}(\mathcal{H}) \subset B(\mathcal{H})$ denote the corresponding subspace of compact operators. For $r \in \mathbb{N}$, we let $\mathbb{M}_r$ denote the space of all $r \times r$ complex-valued matrices. Given a compact Hausdorff space X, we also let $\mathrm{C}(X)$ denote the space of continuous complex-valued functions defined on X.

2.1 C^*-algebras and their Representation Theory

In this section, we will provide a whirlwind tour of a few pertinent examples and some of the not so foundational aspects of C*-algebra theory.

2.1.1 The spectral topology of a C^*-algebra

For some of our analyses, we work with topologies on spaces of representations. Here, we recount basic facts on one such instance: the spectrum of a C*-algebra. For a detailed treatment, the reader can also consult [40, Chapter 3].

Let $\mathfrak{A}$ be a C*-algebra. An ideal $\mathfrak{J}$ is *primitive* whenever $\mathfrak{J}$ is the kernel of an irreducible $*$-representation of $\mathfrak{A}$. We define the *primitive ideal space*, denoted $\mathrm{Prim}(\mathfrak{A})$, to be the collection of all primitive ideals. The primitive ideal space is equipped with a natural topology: whenever $\mathcal{J}$ is a collection of primitive ideals, the closure of $\mathcal{J}$ is the collection of all primitive ideals containing $\bigcap_{\mathfrak{J}\in\mathcal{J}} \mathfrak{J}$. The *spectrum* of $\mathfrak{A}$, denoted $\widehat{\mathfrak{A}}$, is the collection of unitary equivalence classes of irreducible $*$-representations of $\mathfrak{A}$. For an irreducible $*$-representation $\pi : \mathfrak{A} \to B(\mathcal{H})$, we let $[\pi]$ denote the unitary equivalence class of π. The topology on $\widehat{\mathfrak{A}}$ is defined to be the weakest topology such that the natural mapping

$$\widehat{\mathfrak{A}} \to \mathrm{Prim}(\mathfrak{A}), \qquad [\pi] \mapsto \ker \pi,$$

is continuous. If π and σ are $*$-representations of $\mathfrak{A}$, then π is *weakly contained* in σ, denoted $\pi \prec \sigma$, if $\ker \sigma \subset \ker \pi$. Equivalently, $\pi \prec \sigma$ if and only if there is a $*$-homomorphism

$\Lambda : \sigma(\mathfrak{A}) \to \pi(\mathfrak{A})$ such that $\Lambda \circ \sigma = \pi$. Further, this is equivalent to stipulating that $\|\pi(t)\| \leq \|\sigma(t)\|$ for each $t \in \mathfrak{A}$. When π and σ are weakly contained in one another, we say that the $*$-representations are *weakly equivalent*. The closure of a singleton $[\pi] \in \widehat{\mathfrak{A}}$ is $\{[\sigma] \in \widehat{\mathfrak{A}} : \sigma \prec \pi\}$. Let $\mathcal{D} \subset \widehat{\mathfrak{A}}$ and $\mathcal{D}_0$ be a set of irreducible $*$-representations such that $\mathcal{D} = \{[\sigma] \in \widehat{\mathfrak{A}} : \sigma \in \mathcal{D}_0\}$. Note that if the $*$-representation $\bigoplus_{\sigma \in \mathcal{D}_0} \sigma$ is injective, then the subset $\mathcal{D}$ is dense within $\widehat{\mathfrak{A}}$.

Recall that every irreducible $*$-representation of a closed two-sided ideal $\mathfrak{J} \subset \mathfrak{A}$ extends uniquely to $\mathfrak{A}$. Conversely, if σ is an irreducible $*$-representation of $\mathfrak{A}$, then $\sigma |_{\mathfrak{J}}$ is irreducible if and only if $\sigma |_{\mathfrak{J}} \neq 0$ [34, Lemmata 1.9.14-15]. These facts allow for a description of the spectrum that is central to Chapter 3.

Theorem 2.1.1. *[40, Propositions 3.2.1, 3.2.2] Let $\mathfrak{J}$ be a closed two-sided ideal of a* C*-*algebra* $\mathfrak{A}$. *Let* $\mathcal{G}_{\mathfrak{J}} = \{[\sigma] \in \widehat{\mathfrak{A}} : \sigma |_{\mathfrak{J}} = 0\}$ *and* $\mathcal{U}_{\mathfrak{J}} = \{[\sigma] \in \widehat{\mathfrak{A}} : \sigma |_{\mathfrak{J}} \neq 0\}$. *Then, the following statements hold.*

(i) *We have that $\widehat{\mathfrak{A}}$ is the disjoint union of $\mathcal{U}_{\mathfrak{J}}$ and $\mathcal{G}_{\mathfrak{J}}$.*

(ii) *The subsets $\mathcal{U}_{\mathfrak{J}}$ and $\mathcal{G}_{\mathfrak{J}}$ are open and closed, respectively, in $\widehat{\mathfrak{A}}$.*

(iii) *The natural mapping $\mathcal{G}_{\mathfrak{J}} \to \widehat{\mathfrak{A}/\mathfrak{J}}$ is a homeomorphism.*

(iv) *The natural mapping $\mathcal{U}_{\mathfrak{J}} \to \widehat{\mathfrak{J}}$ is a homeomorphism.*

Furthermore, the mapping $\mathfrak{J} \mapsto \mathcal{U}_{\mathfrak{J}}$ is an inclusion-preserving bijection between the closed two-sided ideals of $\mathfrak{A}$ and the open subsets of $\widehat{\mathfrak{A}}$.

2.1.2 Residual finite-dimensionality

A C*-algebra $\mathfrak{A}$ is *residually finite-dimensional* (or RFD) if there is a collection $\{r_\lambda : \lambda \in \Lambda\}$ of positive integers, and an isometric $*$-representation

$$\iota : \mathfrak{A} \to \prod_{\lambda \in \Lambda} \mathbb{M}_{r_\lambda}.$$

There are many alternative formulations of residual finite-dimensionality and we will recall some of the most relevant characterizations for our work. Let $\mathfrak{A}$ be a C*-algebra and $\pi : \mathfrak{A} \to B(\mathcal{H})$ be a $*$-representation. Then, π is *residually finite-dimensional* (or RFD) if there is a net of (possibly degenerate) $*$-representations $\pi_\lambda : \mathfrak{A} \to B(\mathcal{H})$ such that $\pi_\lambda(\mathfrak{A})\mathcal{H}$ is finite-dimensional and

$$\text{SOT} \lim_{\lambda} \pi_\lambda(t) = \pi(t), \qquad t \in \mathfrak{A}.$$

Due to [40, 3.5.2], this is the same as the topology of pointwise-WOT convergence. The following may be found in [3].

Theorem 2.1.2. *Let $\mathfrak{A}$ be a* C*-*algebra. Then, the following statements are equivalent.*

(i) *The* C*-*algebra* $\mathfrak{A}$ *is RFD.*

(ii) *The collection of unitary equivalence classes of irreducible finite-dimensional* $*$*-representations in* $\widehat{\mathfrak{A}}$ *is dense.*

(iii) *Every* $*$*-representation for* $\mathfrak{A}$ *is residually finite-dimensional.*

We analyze residual finite-dimensionality for more general algebras of operators in Chapter 3. In particular, statements (ii) and (iii) of Theorem 2.1.2 are the impetus behind Sections 3.2 and 3.3, respectively.

2.1.3 Cuntz Algebras and the Toeplitz Algebra

In [33], Cuntz studied C*-algebras generated by isometries with pairwise orthogonal ranges. We will frequently consider these algebras for sources of examples.

To this end, fix an integer $n \geq 2$. We let $\mathcal{O}_n$ denote the C*-algebra generated by a collection of isometries $V_1, \ldots, V_n \in B(\mathcal{H})$ such that

$$\sum_{i=1}^{n} V_i V_i^* = I$$

and satisfying the following universal property: whenever $T_1, \ldots, T_n \in B(\mathcal{K})$ are a collection of isometries such that $\sum_{i=1}^{n} T_i T_i^* = I$, then there is a $*$-homomorphism $\pi : \mathcal{O}_n \to \mathrm{C}^*(T_1, \ldots, T_n)$ such that $\pi(V_i) = T_i$ for $i = 1, \ldots, n$. Note that the stipulation that $\sum_{i=1}^{n} V_i V_i^* = I$ forces the isometries $V_1, \ldots, V_n$ to have pairwise orthogonal ranges.

Due to [34, Corollary V.4.7], $\mathcal{O}_n$ is a simple C*-algebra. Therefore, for any collection of isometries $W_1, \ldots, W_n \in B(\mathcal{L})$ satisfying $\sum_{i=1}^{n} W_i W_i^* = I$, we have that $\mathrm{C}^*(I, W_1, \ldots, W_n) \cong \mathcal{O}_n$.

In the case of a single isometry, a C*-algebra with a different type of representation theory makes an appearance. To this end, we let $\mathcal{T}$ denote the C*-algebra generated by a single isometry $V \in B(\mathcal{H})$ and satisfying the following universal property: whenever $T \in B(\mathcal{K})$ is an isometry, then there is a $*$-homomorphism $\pi : \mathcal{T} \to \mathrm{C}^*(T)$ such that $\pi(V) = T$. The C*-algebra $\mathcal{T}$ is called the *Toeplitz algebra*. Due to a classical theorem of Coburn [30], the Toeplitz algebra is $*$-isomorphic to the C*-algebra generated by the unilateral shift $S \in B(\ell^2)$. Moreover, there is a short exact sequence of C*-algebras

$$0 \longrightarrow \mathfrak{K}(\mathcal{H}) \longrightarrow \mathcal{T} \longrightarrow \mathrm{C}(\mathbb{T}) \longrightarrow 0$$

where $\mathbb{T} \subset \mathbb{C}$ denotes the unit circle.

For technical reasons, we will also consider an infinite version of the Cuntz algebras. For this, we let $\mathcal{O}_\infty$ denote the C*-algebra generated by a sequence of isometries $(V_n)_{n\geq 1} \subset B(\mathcal{H})$ such that

$$\sum_{i=1}^{\infty} V_i V_i^* \leq I$$

and satisfying the following universal property: whenever $(T_n)_{n\geq 1} \subset B(\mathcal{K})$ is a sequence of isometries such that $\sum_{i=1}^{\infty} T_i T_i^* \leq I$, then there is a $*$-homomorphism $\pi : \mathcal{O}_\infty \to \mathrm{C}^*(T_1, T_2, \ldots)$ such that $\pi(V_i) = T_i$ for each $i \in \mathbb{N}$.

2.1.4 Liminal and Postliminal C*-algebras

Throughout this thesis, we will frequently consider special classes of C*-algebras whose representation is suitably well-behaved.

First, a C*-algebra $\mathfrak{A}$ is *liminal* if, for every irreducible $*$-representation $\pi : \mathfrak{A} \to B(\mathcal{H}_\pi)$, we have that $\pi(\mathfrak{A}) = \mathfrak{K}(\mathcal{H}_\pi)$. When $\mathfrak{A}$ is unital, a C*-algebra is liminal precisely when all irreducible $*$-representations are finite-dimensional.

Second, a C*-algebra $\mathfrak{A}$ is *postliminal* if, for every irreducible $*$-representation $\pi : \mathfrak{A} \to B(\mathcal{H}_\pi)$, we have that $\pi(\mathfrak{A}) \supset \mathfrak{K}(\mathcal{H}_\pi)$. In particular, the Toeplitz algebra $\mathcal{T}$ is one example of a postliminal C*-algebra that is not liminal. For us, the major benefit of a postliminal C*-algebra will be that, whenever two irreducible $*$-representations share the same kernel, then they are unitarily equivalent [40, Theorem 4.3.7]. However, not all C*-algebras are postliminal. For example, both $B(\mathcal{H})$, for $\mathcal{H}$ infinite-dimensional, and the Cuntz algebras $\mathcal{O}_n, n \in \mathbb{N}$, fail to be postliminal.

2.1.5 Voiculescu's theorem

Here, we will record a few facts that are central to Chapter 4. Let $\mathfrak{A}$ be a C*-algebra and suppose that $\pi : \mathfrak{A} \to B(\mathcal{H}_\pi)$ and $\sigma : \mathfrak{A} \to B(\mathcal{H}_\sigma)$ are $*$-representations. We say that π and σ are *approximately unitarily equivalent* if there is a net of unitary operators $u_\beta : \mathcal{H}_\sigma \to \mathcal{H}_\pi$ satisfying

$$\lim_\beta \|u_\beta^* \pi(t) u_\beta - \sigma(t)\| = 0, \qquad t \in \mathfrak{A}.$$

When $\mathfrak{A}$ and $\mathcal{H}_\pi$ are separable, then a standard argument reveals that the net of unitaries $(u_\beta)_\beta$ may be taken to be a sequence.

In [90], there are a few central results on approximate unitary equivalence that are commonly referred to as Voiculescu's theorem. We focus on highlighting two of Voiculescu's major insights. For this, the *rank* of a Hilbert space operator $T \in B(\mathcal{H})$ is defined to be the cardinality of the dimension of $\overline{T\mathcal{H}}$.

Theorem 2.1.3. *Let $\mathfrak{A}$ be a C*-algebra and π, σ be $*$-representations for $\mathfrak{A}$. Then, the following statements are equivalent.*

(i) *We have that π and σ are approximately unitarily equivalent.*

(ii) *We have that* $\mathrm{rank}(\pi(t)) = \mathrm{rank}(\sigma(t))$ *for every* $t \in \mathfrak{A}$.

We note that Voiculescu originally proved Theorem 2.1.3 for $*$-representations of separable C*-algebras that act on separable Hilbert spaces [34, Theorem II.5.8]. Since their original work, Hadwin had uncovered the non-separable adaptation [51, Theorem 3.14], which is the statement we chose to record. Nevertheless, the second form of Voiculescu's theorem we require is specific to the separable setting [90, Corollary 1.6].

Theorem 2.1.4. *Let $\mathfrak{A}$ be a separable* C*-*algebra and* $\pi : \mathfrak{A} \to B(\mathcal{H}_\pi)$ *be a* $*$*-representation on a separable Hilbert space. Then,* π *is approximately unitarily equivalent to a direct sum of irreducible* $*$*-representations.*

We remark that Hadwin obtained a partial extension of Theorem 2.1.4 to the non-separable setting as well [51, Corollary 4.3].

To conclude this subsection, we record one well-known observation of the rank function.

Lemma 2.1.5. *Let $\mathcal{H}$ be a separable Hilbert space. Then, the function*

$$\mathrm{rank} : B(\mathcal{H}) \to \mathbb{R} \cup \{\infty\}, \qquad T \mapsto \mathrm{rank}(T)$$

is lower semicontinuous in the weak operator topology.

Proof. Fix $r \in \mathbb{R}$. We show that

$$\mathcal{L}_r = \{T \in B(\mathcal{H}) \ : \ \mathrm{rank}(T) \leq r\}$$

is closed in the weak operator topology. We may clearly assume without loss of generality that r is a positive integer. Suppose that $(T_n)_{n\geq 1} \subset \mathcal{L}_r$ and that $T_n \to T$ in the weak operator topology. We show that for any $h_1, \dots, h_{r+1} \in \mathcal{H}$, we have that $Th_1, \dots, Th_{r+1}$ are linearly dependent. Note that $T_n h_1, \dots, T_n h_{r+1}$ are linearly dependent for each $n \geq 1$ since $T_n \in \mathcal{L}_r$. So for each $n \in \mathbb{N}$, we find complex scalars $c_{n,1}, \dots, c_{n,r+1} \in \mathbb{C}$, not all zero, such that

$$\sum_{i=1}^{r+1} c_{n,i} T_n h_i = 0. \tag{2.1}$$

By possibly rescaling, we may assume that

$$\sum_{i=1}^{r+1} |c_{n,i}|^2 = 1.$$

Since the closed unit ball of $\mathbb{C}^{r+1}$ is compact, we may extract a subsequence $((c_{n_k,1}, \dots, c_{n_k,r+1}))_{k\geq 1}$ of $((c_{n,1}, \dots, c_{n,r+1}))_{n\geq 1}$ that converges to some $(c_1, \dots, c_{r+1}) \in \mathbb{C}^{r+1}$ satisfying $\sum_{i=1} |c_i|^2 = 1$.

Note that $\sup_{n\geq 1} \|T_n\| < \infty$ since (T_n) converges in the weak operator topology. Thus, for each $k \in \mathcal{H}$, we have that

$$\begin{aligned}
\left| \left\langle \sum_{i=1}^{r+1} c_{n_k,i} T_{n_k} h_i, k \right\rangle - \left\langle \sum_{i=1}^{r+1} c_i T h_i, k \right\rangle \right| &\leq \sum_{i=1}^{r+1} |\langle (c_{n_k,i} T_{n_k} - c_i T) h_i, k \rangle| \\
&\leq \sum_{i=1}^{r+1} \left(|(c_{n_k,i} - c_i) \langle T_{n_k} h_i, k \rangle| + |c_i \langle T_{n_k} h_i, k \rangle - c_i \langle T h_i, k \rangle| \right) \\
&\leq \sum_{i=1}^{r+1} \left(\left(\sup_{n\geq 1} \|T_n\| \right) |c_{n_k,i} - c_i| + c_i |\langle (T_{n_k} - T) h_i, k \rangle| \right).
\end{aligned}$$

So, we have that

$$\lim_{k\to\infty} \left| \left\langle \sum_{i=1}^{r+1} c_{n_k,i} T_{n_k} h_i, k \right\rangle - \left\langle \sum_{i=1}^{r+1} c_i T h_i, k \right\rangle \right| = 0.$$

However, by equation (2.1), we may then conclude that

$$\left\langle \sum_{i=1}^{r+1} c_i T h_i, k \right\rangle = 0, \qquad k \in \mathcal{H}.$$

Consequently, we obtain that

$$\sum_{i=1}^{r+1} c_i T h_i = 0.$$

In other words, we have that the vectors $Th_1, Th_2, \dots, Th_{r+1}$ are linearly dependent for any choice of $h_1, \dots, h_{r+1} \in \mathcal{H}$. So, we have that $\mathrm{rank}(T) \le r$ as desired. □

2.2 Completely positive maps and operator algebras

An *operator space* is a subspace $\mathcal{M} \subset B(\mathcal{H})$. If, in addition, $\mathcal{M}$ is unital and self-adjoint, then $\mathcal{M}$ is said to be an *operator system*. Moreover, an operator space that is closed under multiplication is called an *operator algebra*.

The definitions for these spaces of operators is dependent on an underlying Hilbert space. However, similar to C*-algebras, there is a purely axiomatic (or, alternatively, abstract) definition for each such structure that is not dependent on a specific choice of Hilbert space (see [73, Chapters 13 and 16] for details).

Given an operator space $\mathcal{M} \subset B(\mathcal{H})$ and $r \in \mathbb{N}$, we let $\mathbb{M}_r(\mathcal{M})$ denote the space of $r \times r$ matrices with entries in $\mathcal{M}$. We may regard $M = [m_{ij}] \in \mathbb{M}_r(\mathcal{M})$ as a bounded linear operator on

$$\mathcal{H}^{(r)} = \mathcal{H} \oplus \ldots \oplus \mathcal{H}$$

by setting

$$[m_{ij}] \begin{bmatrix} h_1 \\ \vdots \\ h_r \end{bmatrix} = \begin{bmatrix} \sum_j m_{1j} h_j \\ \vdots \\ \sum_j m_{rj} h_j \end{bmatrix}.$$

This correspondence yields an isometric $*$-isomorphism $\mathbb{M}_r(B(\mathcal{H})) \cong B(\mathcal{H}^{(r)})$ of C*-algebras.

Whenever X is a compact Hausdorff space and $\mathcal{M} \subset \mathrm{C}(X)$ is a subspace, then we say that $\mathcal{M}$ is a *function space*. Similarly, if $\mathcal{M}$ is unital and self-adjoint, then $\mathcal{M}$ is said to be a *function system*.

2.2.1 Completely positive maps

A linear mapping $\varphi : \mathcal{M} \to B(\mathcal{H}_\varphi)$ is said to be *completely contractive* if the mapping

$$\varphi^{(r)} : \mathbb{M}_r(\mathcal{M}) \to \mathbb{M}_r(B(\mathcal{H}_\varphi)), \qquad [m_{ij}] \mapsto [\varphi(m_{ij})]$$

is contractive for each $r \in \mathbb{N}$. Similarly, φ is *completely positive* if $\varphi^{(r)}$ is positive for each $r \in \mathbb{N}$ and *completely isometric* if $\varphi^{(r)}$ is isometric for each $r \in \mathbb{N}$. Additionally, if $\mathcal{M}$ happens to be an operator algebra, then a completely contractive algebra homomorphism is called a *representation* of $\mathcal{M}$. A standard computation reveals that, if $\mathcal{S} \subset B(\mathcal{H})$ is an operator system and $\varphi : \mathcal{S} \to B(\mathcal{H}_\varphi)$ is a completely positive map, then $\varphi(s^*) = \varphi(s)^*$ for each $s \in \mathcal{S}$.

Throughout, we will typically work with unital operator spaces as well as unital mappings. However, in Chapter 3, we will also consider non-unital operator algebras. For this, we recall a couple relevant details from [69] on unitizations of operator algebras. Given a non-unital operator algebra $\mathcal{A}$, we let $\widetilde{\mathcal{A}}$ denote the unitization. If $\mathcal{A}$ is unital, we will simply refer to $\widetilde{\mathcal{A}}$ as $\mathcal{A}$ itself. Given a non-unital operator algebra $\mathcal{A}$, there is a completely isometric representation $\mathcal{A} \to \widetilde{\mathcal{A}}$. Moreover, any representation $\rho : \mathcal{A} \to B(\mathcal{H}_\rho)$ extends uniquely to a unital representation $\rho^+ : \widetilde{\mathcal{A}} \to B(\mathcal{H}_\rho)$ and ρ is completely isometric precisely when ρ^+ is completely isometric.

For the remainder of this section, we will study unital operator spaces and unital mappings. We recall a few standard facts for these classes of mappings, starting with [73, Proposition 2.12].

Theorem 2.2.1. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space. If $\varphi : \mathcal{M} \to B(\mathcal{H}_\varphi)$ is a unital completely contractive map, then*

$$\widetilde{\varphi} : \mathcal{M} + \mathcal{M}^* \to B(\mathcal{H}_\varphi), \qquad m + n^* \mapsto \varphi(m) + \varphi(n)^*$$

is a well-defined unital completely positive map. Moreover, $\widetilde{\varphi}$ is the unique completely positive map $\psi : \mathcal{M} + \mathcal{M}^* \to B(\mathcal{H}_\varphi)$ satisfying $\psi \mid_{\mathcal{M}} = \varphi$.

Theorem 2.2.1 guarantees that, to study completely contractive maps on unital operator spaces, it is often sufficient to consider completely positive maps on operator systems. Additionally, a unital linear mapping defined on an operator system is completely contractive precisely when it is completely positive. For completely positive maps, there are a few critical facts that allow for a well-understood theory. First, we record the Schwarz inequality for completely positive maps [73, Proposition 3.3].

Theorem 2.2.2. *Let $\mathfrak{A}$ be a unital C^*-algebra and suppose that $\varphi : \mathfrak{A} \to B(\mathcal{H}_\varphi)$ is a unital completely positive map. Then, we have that $\varphi(t)^*\varphi(t) \leq \varphi(t^*t)$ for each $t \in \mathfrak{A}$.*

When equality is achieved in the Schwarz inequality, this allows for an explicit identification of when φ is multiplicative [73, Theorem 3.18].

Theorem 2.2.3. *Let $\mathfrak{A}$ be a unital C^*-algebra and suppose that $\varphi : \mathfrak{A} \to B(\mathcal{H}_\varphi)$ is a unital completely positive map. Then, the following statements hold.*

(i) *The family of operators*

$$\mathit{Mult}(\varphi) := \{t \in \mathfrak{A} \ : \ \varphi(t^*t) = \varphi(t)^*\varphi(t), \ \varphi(tt^*) = \varphi(t)\varphi(t)^*\}$$

coincides with

$$\{t \in \mathfrak{A} \ : \ \varphi(bt) = \varphi(b)\varphi(t), \ \varphi(tb) = \varphi(t)\varphi(b), \ \text{for every } b \in \mathfrak{A}\}.$$

(ii) *We have that $\mathit{Mult}(\varphi)$ is a C^*-subalgebra of $\mathfrak{A}$ and $\varphi \mid_{\mathit{Mult}(\varphi)}$ is a $*$-representation.*

The subset $\mathrm{Mult}(\varphi)$ will be referred to as the *multiplicative domain* of φ. Next, we record Arveson's extension theorem [73, Theorem 7.5], which may be regarded as the completely positive counterpart to the Hahn–Banach extension theorem.

Theorem 2.2.4. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space and $\varphi : \mathcal{M} \to B(\mathcal{H}_\varphi)$ be a unital completely contractive map. Then, there exists a completely contractive map $\Phi : B(\mathcal{H}) \to B(\mathcal{H}_\varphi)$ satisfying $\Phi \mid_{\mathcal{M}} = \varphi$.*

As a consequence of Arveson's extension theorem, there may be instances where we are not completely descriptive for the domain of a completely contractive map. In addition to this extension property, the space of completely contractive maps enjoy a compactness property [73, Theorem 7.4].

Theorem 2.2.5. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space. Then,*

$$\mathrm{UCC}(\mathcal{M}, \mathcal{K}) = \{\varphi : \mathcal{M} \to B(\mathcal{K}) \mid \varphi \text{ unital completely contractive}\}$$

and

$$\mathrm{UCP}(\mathcal{M}, \mathcal{K}) = \{\varphi : \mathcal{M} \to B(\mathcal{K}) \mid \varphi \text{ unital completely positive}\}$$

are both compact in the topology of pointwise-WOT convergence.

Finally, we record Stinespring's dilation theorem [73, Theorem 4.1].

Theorem 2.2.6. *Let $\mathfrak{A}$ be a unital* C*-*algebra and let $\varphi : \mathfrak{A} \to B(\mathcal{H}_\varphi)$ be a unital completely positive map. Then, there exists a triple $(\sigma, \mathcal{K}, V)$ consisting of a Hilbert space $\mathcal{K}$, a unital $*$-representation $\sigma : \mathfrak{A} \to B(\mathcal{K})$, and an isometry $V : \mathcal{H}_\varphi \to \mathcal{K}$ with the property that*

$$V^*\sigma(t)V = \varphi(t), \qquad t \in \mathfrak{A}.$$

The converse to Theorem 2.2.6 is also true. That is, a map $\varphi : \mathfrak{A} \to B(\mathcal{H}_\varphi)$ is unital and completely positive precisely when φ is the compression of a $*$-representation for $\mathfrak{A}$.

A triple $(\sigma, \mathcal{K}, V)$ as in Theorem 2.2.6 is said to be a *representation* for the mapping φ. If, in addition, $\mathcal{K} = \overline{\mathrm{span}}\{\sigma(\mathfrak{A})V\mathcal{H}_\varphi\}$, then the representation is said to be *minimal*. Due to [73, Proposition 4.2], a minimal representation always exists and is unique up to a unitary equivalence that simultaneously intertwines $*$-representations and isometries. Accordingly, we let $(\sigma_\varphi, \mathcal{K}_\varphi, V_\varphi)$ denote the minimal representation of φ and refer to this triple as the *minimal Stinespring dilation* of φ.

2.2.2 C*-covers of operator algebras

The C*-algebra that an operator algebra $\mathcal{A} \subset B(\mathcal{H})$ generates often carries strikingly different information based off the choice of representation for the algebra. We will make this explicit. Within $B(\mathcal{H})$, there is a smallest C*-algebra containing $\mathcal{A}$, denoted $\mathrm{C}^*(\mathcal{A})$. However, given a completely isometric representation $\iota : \mathcal{A} \to B(\mathcal{K})$, the C*-algebra $\mathrm{C}^*(\iota(\mathcal{A}))$ could be quite unlike $\mathrm{C}^*(\mathcal{A})$ (Example 1).

For an operator algebra $\mathcal{A} \subset B(\mathcal{H})$, we call the pair $(\mathfrak{A}, \iota)$ a C*-*cover* of $\mathcal{A}$ if $\iota : \mathcal{A} \to B(\mathcal{K})$ is a completely isometric representation and $\mathfrak{A} = \mathrm{C}^*(\iota(\mathcal{A}))$. Throughout this dissertation, we will be concerned with a few notable C*-covers and, occasionally, we will simultaneously work with several C*-covers at the same time. For this reason, in Chapter 3, we will commonly think of $\mathcal{A}$ as an abstract operator algebra, rather than being concretely represented on a specific Hilbert space (see [73, Chapter 16] for details). This is purely for simplicity of notation. For now, we will address a couple of the most natural C*-covers to consider.

The *maximal* C*-*cover* of an operator algebra $\mathcal{A}$ is the C*-cover, denoted $(\mathrm{C}^*_{max}(\mathcal{A}), \mu)$, satisfying the following universal property: whenever $\rho : \mathcal{A} \to B(\mathcal{H}_\rho)$ is a representation of $\mathcal{A}$, there is a unique $*$-representation $\theta : \mathrm{C}^*_{max}(\mathcal{A}) \to B(\mathcal{H}_\rho)$ such that $\theta \circ \mu = \rho$. That is, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A} & \xhookrightarrow{\mu} & \mathrm{C}^*_{max}(\mathcal{A}) \\ & \overset{\rho}{\searrow} & \downarrow{\scriptstyle\theta} \\ & & \mathrm{C}^*(\rho(\mathcal{A})) \end{array}$$

In particular, whenever $(\mathfrak{A}, \iota)$ is a C*-cover for $\mathcal{A}$, there is a unique surjective $*$-representation $q_{\mathfrak{A}} : \mathrm{C}^*_{max}(\mathcal{A}) \to \mathfrak{A}$ such that $q_{\mathfrak{A}} \circ \mu = \iota$. We will utilize this notation in Chapter 3. The maximal C*-cover is known to exist for any operator algebra and can be constructed in a concrete way [13, Proposition 2.4.2]. That is, one constructs the completely isometric representation $\mu : \mathcal{A} \to \mathrm{C}^*_{max}(\mathcal{A})$ by taking an appropriately large direct sum of representations of $\mathcal{A}$.

The second main C*-cover we are interested in is the C*-*envelope* of an operator algebra, denoted $(\mathrm{C}^*_e(\mathcal{A}), \varepsilon)$. The C*-envelope satisfies the following universal property: whenever $(\mathfrak{A}, \iota)$ is a C*-cover of $\mathcal{A}$, there is a surjective $*$-representation $\pi : \mathfrak{A} \to \mathrm{C}^*_e(\mathcal{A})$ such that $\pi \circ \iota = \varepsilon$. In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A} & \xhookrightarrow{\iota} & \mathfrak{A} \\ & \overset{\varepsilon}{\searrow} & \downarrow{\scriptstyle\pi} \\ & & \mathrm{C}^*_e(\mathcal{A}) \end{array}$$

It is not obvious, but the C*-envelope will always exist. Existence of the C*-envelope was originally shown by Hamana [53] through so-called injective envelopes. An alternative, dilation theoretic, proof was more recently completed due to several magnificent contributions [7],[36],[44],[71]. We will elaborate on the structure of the C*-envelope, as well as the dilation theoretic proof, in Section 2.4.

Finally, we introduce some terminology. Given a C*-cover $(\mathfrak{A}, \iota)$ of $\mathcal{A}$, there is a surjective $*$-representation $\pi : \mathfrak{A} \to \mathrm{C}^*_e(\mathcal{A})$ such that $\pi \circ \iota = \varepsilon$. We refer to the kernel of the $*$-representation π as the *Shilov ideal* of $\mathcal{A}$ relative to $(\mathfrak{A}, \iota)$, denoted $\mathrm{Sh}(\mathcal{A}, \mathfrak{A}, \iota)$. The Shilov ideal is known to be the largest closed two-sided ideal of $\mathfrak{A}$ such that the quotient map $\mathfrak{A} \to \mathfrak{A}/\mathrm{Sh}(\mathcal{A}, \mathfrak{A}, \iota)$ is completely isometric on $\iota(\mathcal{A})$ [7].

2.2.3 Residual finite-dimensionality of operator algebras

An operator algebra $\mathcal{A}$ is said to be *residually finite-dimensional* (or RFD) if there exists a completely isometric representation $\iota : \mathcal{A} \to \prod_{\lambda \in \Lambda} \mathbb{M}_{r_\lambda}$ where $\{r_\lambda : \lambda \in \Lambda\}$ is some collection of positive integers. Furthermore, we say a C*-cover $(\mathfrak{A}, \iota)$ is *residually finite-dimensional* if $\mathfrak{A}$ is an RFD C*-algebra.

Residual finite-dimensionality for operator algebras was only recently introduced [24],[70]. Within [24], an analysis discussing residual finite-dimensionality of C*-covers was provided and, in Chapter 3, our main interest is regarding the residual finite-dimensionality of the maximal C*-cover. For now, we remark that there is no transparent understanding of how to determine the RFD C*-covers of an operator algebra. The following, Example 1, demonstrates the subtlety to this phenomenon. In particular, there may be several RFD C*-covers of an operator algebra, while there are intermediate C*-covers that fail to be RFD.

Example 1. Let $\mathrm{A}(\mathbb{D})$ denote the algebra of complex functions which are holomorphic on the open unit disk $\mathbb{D}$ and continuous up to the boundary $\mathbb{T}$. The algebra $\mathrm{A}(\mathbb{D})$ is called the disc algebra. There are several well-studied C*-covers for the disc algebra. Indeed, there are obvious completely isometric representations $\iota : \mathrm{A}(\mathbb{D}) \to \mathrm{C}(\mathbb{T})$ and $j : \mathrm{A}(\mathbb{D}) \to \mathrm{C}(\overline{\mathbb{D}})$ that determine RFD C*-covers of $\mathrm{A}(\mathbb{D})$. Indeed, the restriction mapping $\iota : \mathrm{A}(\mathbb{D}) \to \mathrm{C}(\mathbb{T})$ is completely isometric due to the maximum modulus principle. Moreover, $\mathrm{C}(\mathbb{T})$ is an RFD C*-algebra since

$$\mathrm{C}(\mathbb{T}) \to \prod_{\xi \in \mathbb{T}} \mathbb{C}, \qquad f \mapsto (f(\xi))_{\xi \in \mathbb{T}}$$

is an isometric $*$-representation. For similar reasons, $\mathrm{C}(\overline{\mathbb{D}})$ is also an RFD C*-algebra.

Moreover, the matrical von-Neumann inequality [73, Corollary 3.12] implies that the maximal C*-cover of $\mathrm{A}(\mathbb{D})$ is the universal C*-algebra generated by a contraction (alternatively, see [12, Example 2.3]). By [31, Proposition 2.2], it follows that the maximal C*-cover of $\mathrm{A}(\mathbb{D})$ is RFD.

Despite the fact that $\mathrm{C}^*_{max}(\mathrm{A}(\mathbb{D}))$ is RFD, there are other C*-covers of $\mathrm{A}(\mathbb{D})$ that are not RFD. To show this, we utilize standard facts on the Toeplitz algebra that may be found within [1],[6],[43]. Let H^2 denote the classical Hardy space on the disc. Each $f \in \mathrm{A}(\mathbb{D})$ determines a multiplication operator $M_f : H^2 \to H^2$ such that $\|M_f\| = \|f\|$. Furthermore, the map

$$\omega : \mathrm{A}(\mathbb{D}) \to B(H^2), \qquad f \mapsto M_f$$

is a completely isometric representation. The C*-algebra $\mathfrak{T} = \mathrm{C}^*(\omega(\mathrm{A}(\mathbb{D})))$ is the Toeplitz algebra and there is a short exact sequence

$$0 \longrightarrow \mathfrak{K}(H^2) \longrightarrow \mathfrak{T} \longrightarrow \mathrm{C}(\mathbb{T}) \longrightarrow 0.$$

Note that residual finite-dimensionality passes to subalgebras and $\mathfrak{T}$ contains the ideal of compact operators $\mathfrak{K}(H^2)$. The C*-algebra $\mathfrak{K}(H^2)$ is not RFD as it does not possess any finite-dimensional $*$-representations. Hence, $\mathfrak{T}$ cannot be RFD either.

2.3 Classical Choquet Theory

In this short section, we recall a few facts from classical Choquet theory. For additional resources, one can consult [2],[47],[76].

Let X be a compact Hausdorff space and suppose that $\mathcal{M} \subset \mathrm{C}(X)$ is a unital subspace that separates points in X. A subset $\Delta \subset X$ is said to be a *boundary* for $\mathcal{M}$ if

$$\|f\| = \max_{\xi \in \Delta} |f(\xi)|, \qquad f \in \mathcal{M}.$$

Certainly, the entire space X is always a boundary for $\mathcal{M}$. In fact, Urysohn's Lemma guarantees that X is the only boundary for $\mathrm{C}(X)$. For arbitrary subspaces, boundaries require a more careful analysis. To this end, we will require terminology.

Let $\xi \in X$. A *representing measure* at ξ will refer to a Borel probability measure μ on X with the property that

$$\int_X f d\mu = f(\xi), \qquad f \in \mathcal{M}.$$

The *Choquet boundary* for $\mathcal{M}$ is the collection of all points $\xi \in X$ for which Dirac measure is the unique representing measure at ξ. As a consequence to the Choquet–Bishop–de-Leeuw integral representation theorem [11], the Choquet boundary is always a boundary for $\mathcal{M}$.

Unfortunately, in practice, it can be cumbersome to identify the Choquet boundary. For subalgebras, this is made easier due to a theorem of Bishop [76, Section 8].

Theorem 2.3.1. *Let X be a compact Hausdorff space and $\mathcal{M} \subset \mathrm{C}(X)$ be a unital norm-closed subspace that separates points in X. Let $\xi \in X$ and consider the following statements.*

(i) *There is $f \in \mathcal{M}$ such that*

$$f(\xi) = 1 > |f(\zeta)|, \qquad \zeta \in X, \zeta \neq \xi.$$

(ii) *The point ξ lies in the Choquet boundary for $\mathcal{M}$.*

Then, we have that (i)$\Rightarrow$(ii) *and, if $\mathcal{M}$ is a subalgebra, then* (ii)$\Rightarrow$(i).

Accordingly, the Choquet boundary of a subalgebra may be identified with the set of *peak points* for $\mathcal{M}$, which are those points $\xi \in X$ that satisfy condition (i). For our purposes, this allows for some concrete examples.

Example 2. Let $\mathrm{A}(\mathbb{D}) \subset \mathrm{C}(\overline{\mathbb{D}})$ denote the disc algebra. For each $\xi \in \mathbb{T}$, it is easy to check that

$$f_\xi(z) = \frac{1}{2}(1 + \overline{\xi} z)$$

lies in $\mathrm{A}(\mathbb{D})$ and peaks at ξ. On the other hand, a peak point necessarily lies on $\mathbb{T}$ by the maximum modulus principle. Thus, $\mathbb{T}$ is the Choquet boundary for $\mathrm{A}(\mathbb{D})$.

A consequence to Theorem 2.3.1 and Choquet's Theorem [76, Chapter 3] is that the Choquet boundary is a minimal boundary for a subalgebra. For arbitrary subspaces, the closure of the Choquet boundary is the *Shilov boundary* for $\mathcal{M}$, which is the smallest closed boundary for $\mathcal{M}$. Moreover, a near-immediate consequence to the universal property of the C^*-envelope implies the following.

Theorem 2.3.2. *Let X be a compact Hausdorff space and $\mathcal{M} \subset \mathrm{C}(X)$ be a unital norm-closed subspace that separates points in X. Then, we have that $\mathrm{C}^*_e(\mathcal{M}) = \mathrm{C}(Y)$ where $Y \subset X$ is the Shilov boundary for $\mathcal{M}$.*

Therefore, we have one concrete class of examples for which the structure of the C^*-envelope is well-understood.

To close this section, we discuss maximality constraints on the Choquet boundary. The most notable advancement, due to Šaškin [83], is that a maximality constraint on the Choquet boundary is linked to approximation theory. Before we state Šaškin's theorem, we record the following classical result of Korovkin [68], which is one example where Šaškin's theorem was first actively witnessed.

Theorem 2.3.3. *Suppose that $\varphi_n : \mathrm{C}[0,1] \to \mathrm{C}[0,1]$ is a sequence of positive linear maps such that*

$$\lim_{n\to\infty} \|\varphi_n(x^k) - x^k\| = 0, \qquad k = 0, 1, 2.$$

Then, we necessarily have that

$$\lim_{n\to\infty} \|\varphi_n(f) - f\| = 0, \qquad f \in \mathrm{C}[0,1].$$

Šaškin's theorem was originally proven for metric spaces [76, Section 9], but we record the non-metrizable analogue [38, Theorem 5.3].

Theorem 2.3.4. *Let X be a compact Hausdorff space and $\mathcal{M} \subset \mathrm{C}(X)$ be a unital norm-closed subspace that separates points in X. Then, the following statements are equivalent.*

(i) *We have that X is equal to the Choquet boundary for $\mathcal{M}$.*

(ii) *For every net of unital positive maps $\varphi_\lambda : \mathrm{C}(X) \to \mathrm{C}(X)$ satisfying*

$$\lim_{\lambda} \|\varphi_\lambda(m) - m\| = 0, \qquad m \in \mathcal{M},$$

we have that

$$\lim_{\lambda} \|\varphi_\lambda(f) - f\| = 0, \qquad f \in \mathrm{C}(X).$$

(iii) *For every compact Hasudorff space Y, every $*$-homomorphism $\pi : \mathrm{C}(X) \to \mathrm{C}(Y)$, and every net of unital positive maps $\varphi_\lambda : \mathrm{C}(X) \to \mathrm{C}(Y)$ satisfying*

$$\lim_{\lambda} \|\varphi_\lambda(m) - \pi(m)\| = 0, \qquad m \in \mathcal{M},$$

we have that

$$\lim_{\lambda} \|\varphi_\lambda(f) - \pi(f)\| = 0, \qquad f \in \mathrm{C}(X).$$

Theorem 2.3.4 will be central to the main developments in Chapter 5.

2.4 The Non-commutative Choquet Boundary

While we briefly introduced the C*-envelope in Subsection 2.2.2, an explicit representation of the C*-envelope remains mysterious in general. Here, we will discuss some of the known structure for this object. In particular, motivated by Theorem 2.3.2, we will discuss a non-commutative variation of the Choquet boundary.

First, we address a minor point. While the C*-envelope of an operator algebra has been introduced, the C*-envelope of a unital operator space $\mathcal{M}$ may be defined in an analogous way. Indeed, the C*-*envelope* of $\mathcal{M}$ is a pair $(\mathrm{C}_e^*(\mathcal{M}), \varepsilon)$ consisting of a C*-algebra $\mathrm{C}_e^*(\mathcal{M})$ and a completely isometric linear map $\varepsilon : \mathcal{M} \to \mathrm{C}_e^*(\mathcal{M})$ such that $\mathrm{C}^*(\varepsilon(\mathcal{M})) = \mathrm{C}_e^*(\mathcal{M})$ and satisfying the following universal property: whenever $\iota : \mathcal{M} \to B(\mathcal{H}_\iota)$ is a completely isometric linear map, then there is a surjective $*$-representation $\pi : \mathrm{C}^*(\iota(\mathcal{M})) \to \mathrm{C}_e^*(\mathcal{M})$ such that $\pi \circ \iota = \varepsilon$. Recall that, when $\mathcal{M}$ is a subalgebra, then we only considered *multiplicative* completely isometric mappings for $\mathcal{M}$. However, the universal property of the C*-envelope ensures that these two definitions will coincide. Indeed, if $\mathcal{M}$ is a subalgebra, then ε will be multiplicative [13, Proposition 4.3.5].

Over 50 years ago, Arveson had conjectured the existence of a non-commutative Choquet boundary [4]. For this, let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space and $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a $*$-representation. We say that π has the *unique extension property* if, whenever $\psi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a completely positive map satisfying $\psi \mid_{\mathcal{M}} = \pi \mid_{\mathcal{M}}$, then we have that $\psi = \pi$. If, in addition, π is irreducible, then π is said to be a boundary representation for $\mathcal{M}$. For function spaces, it is easy to see that a $*$-representation π is a boundary representation for $\mathcal{M}$ precisely when π is evaluation against a point in the Choquet boundary. Consequently, we will refer to the class of all boundary representations for $\mathcal{M}$ as the *non-commutative Choquet boundary* for $\mathcal{M}$. It is natural to wonder whether the non-commutative Choquet boundary is as fruitful of an invariant for an operator space. Indeed, it is not immediately clear whether a $*$-representation with the unique extension property is guaranteed to exist. Nevertheless, since Arveson's original work, it has been verified that there are always sufficiently many boundary representations to completely determine the norm of $\mathcal{M}$.

Theorem 2.4.1. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space. Then, for each $n \in \mathbb{N}$ and $M \in \mathbb{M}_n(\mathcal{M})$, there is a boundary representation $\beta : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\beta)$ and a finite-dimensional subspace $F \subset \mathcal{H}_\beta^{(n)}$ such that $\|P_F \beta^{(n)}(M) \mid_F \| = \|M\|$.*

The proof of Theorem 2.4.1 is the culmination of work by many authors [7],[36],[44],[71]. For the purposes of this dissertation, we simply remark that the proof hinges on a few notable steps. First, equate the unique extension property with those unital completely contractive maps on $\mathcal{M}$ that are maximal under non-trivial dilations (see [7, Proposition 2.4]). Second, in the presence of a suitable extremality condition, show that every unital completely contractive map on $\mathcal{M}$ can be dilated to a unital completely contractive map that extends (uniquely) to a boundary representation [36, Lemma 2.3 and Theorem 2.4].

For the purposes of this section, we record structural facts about the unique extension property and the C*-envelope. We start with a fundamental fact.

Lemma 2.4.2. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space and $\{\pi_i : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_i) : i \in I\}$ be a collection of $*$-representations. Then, the following statements are equivalent.*

(i) *For each $i \in I$, π_i has the unique extension property with respect to $\mathcal{M}$.*

(ii) *We have that $\bigoplus_{i\in I} \pi_i$ has the unique extension property with respect to $\mathcal{M}$.*

Proof. See [8, Proposition 4.4] and [26, Lemma 2.8]. □

Finally, we record a few additional facts that will be integral to Chapters 4 and 5.

Theorem 2.4.3. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space and $(\mathrm{C}^*_e(\mathcal{M}), \varepsilon)$ denote the C^*-envelope of $\mathcal{M}$. Let $\theta : \mathrm{C}^*(\mathcal{M}) \to \mathrm{C}^*_e(\mathcal{M})$ denote the $*$-representation that satisfies $\theta\mid_{\mathcal{M}} = \varepsilon$. Then, the following statements hold.*

(i) *There is an isometric $*$-representation of $\mathrm{C}^*_e(\mathcal{M})$ that possesses the unique extension property with respect to $\varepsilon(\mathcal{M})$.*

(ii) *If $\beta : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\beta)$ is a boundary representation for $\mathcal{M}$, then there is a $*$-representation $\widetilde{\beta} : \mathrm{C}^*_e(\mathcal{M}) \to B(\mathcal{H}_\beta)$ such that $\beta = \widetilde{\beta} \circ \theta$.*

(iii) *If every irreducible $*$-representation of $\mathrm{C}^*(\mathcal{M})$ is a boundary representation for $\mathcal{M}$, then $\mathrm{C}^*(\mathcal{M}) = \mathrm{C}^*_e(\mathcal{M})$.*

Proof. (i): By Theorem 2.4.1, there is a collection Δ consisting of boundary representations for $\varepsilon(\mathcal{M})$, that are defined on $\mathrm{C}^*_e(\mathcal{M})$, such that $\bigoplus_{\delta\in\Delta} \delta$ is isometric. Thus, by Lemma 2.4.2, $\bigoplus_{\delta\in\Delta} \delta$ is an isometric $*$-representation of $\mathrm{C}^*_e(\mathcal{M})$ that has the unique extension property with respect to $\varepsilon(\mathcal{M})$.

(ii): By Theorem 2.4.1, there is a collection Σ consisting of boundary representations for $\mathcal{M}$, that are defined on $\mathrm{C}^*(\mathcal{M})$, such that $\bigoplus_{\sigma\in\Sigma} \sigma$ is completely isometric on $\mathcal{M}$. Define a closed two-sided ideal of $\mathrm{C}^*(\mathcal{M})$ by

$$\mathfrak{J}_\Sigma = \bigcap_{\sigma\in\Sigma} \ker \sigma.$$

By [4, Theorem 2.2.3], the ideal $\mathfrak{J}_\Sigma$ coincides with the intersection of the kernels of all boundary representations for $\mathcal{M}$ that are defined on $\mathrm{C}^*(\mathcal{M})$. Then, by [4, Proposition 2.2.4 and Theorem 2.2.5], we have that $\mathrm{C}^*(\mathcal{M})/\mathfrak{J}_\Sigma$ is the C^*-envelope of $\mathcal{M}$ and that $\ker \theta = \mathfrak{J}_\Sigma$. Since $\mathfrak{J}_\Sigma \subset \ker \beta$, statement (ii) is immediate.

(iii): Again, let $\mathfrak{J}_\Sigma$ denote the intersection of the kernels of all boundary representations for $\mathcal{M}$ that are defined on $\mathrm{C}^*(\mathcal{M})$. By [4, Proposition 2.2.4 and Theorem 2.2.5], $\mathrm{C}^*(\mathcal{M})/\mathfrak{J}_\Sigma$ is the C^*-envelope of $\mathcal{M}$. So (iii) is immediate since $\mathfrak{J}_\Sigma = \{0\}$. □

2.5 Complete Pick Spaces on the Unit Ball

In this section, we introduce a class of operator algebras for which the non-commutative Choquet boundary is well-understood. While there are several technical definitions that we introduce, only the conclusion of Theorem 2.5.1 will be necessary for understanding the remainder of this dissertation.

For this, let $d \in \mathbb{N}$ and $\mathbb{B}_d$ denote the open unit ball in $\mathbb{C}^d$. Suppose that $\mathcal{H}$ is a *reproducing kernel Hilbert space* on $\mathbb{B}_d$. In other words, $\mathcal{H}$ is a Hilbert space of functions on $\mathbb{B}_d$ for which the evaluation mappings are continuous. Thus, for each $z \in \mathbb{B}_d$, the Riesz representation theorem implies that there is $k_z \in \mathcal{H}$ such that $f(z) = \langle f, k_z \rangle$ for every $f \in \mathcal{H}$. Then, the *reproducing kernel* of $\mathcal{H}$ is the function $k : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ defined by $k(z, w) = \langle k_z, k_w \rangle$. It is unnecessary outside of this subsection but, for background information on reproducing kernel Hilbert spaces, one may refer to [1]. A frequently studied example of a reproducing kernel is the Hardy space on the unit disc H^2, which has reproducing kernel

$$k(z, w) = \frac{1}{1 - z\overline{w}}, \qquad z, w \in \mathbb{D}.$$

We will be concerned with reproducing kernel Hilbert spaces that possess several regularity conditions. First, we say that $\mathcal{H}$ is a *regular unitarily invariant* space if there is a sequence $(a_n)_{n \geq 0}$ of strictly positive real numbers such that

$$k(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n, \qquad \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = 1,$$

and $a_0 = 1$. We will also require that $\mathcal{H}$ satisfies the complete Pick property [1, Section 2.8], which is a technical constraint that will not be explicitly needed. There are several well-studied examples of regular, unitarily invariant complete Pick spaces on the unit ball. In particular, the Hardy space on the unit disc H^2, the Drury-Arveson space H^2_d, and the Dirichlet space are all examples of regular, unitarily invariant, complete Pick spaces on the unit ball. In particular, the Drury-Arveson space has reproducing kernel given by

$$k(z, w) = \frac{1}{1 - \langle z, w \rangle_{\mathbb{C}^d}}, \qquad z, w \in \mathbb{B}_d.$$

When $d = 1$, the Drury-Arveson space is the Hardy space on the unit disc H^2. If $\sum_{n=0}^{\infty} a_n < \infty$, then we say that k is *bounded*. Otherwise, we say that k is *unbounded*.

A function $\theta : \mathbb{B}_d \to \mathbb{C}$ is said to be a *multiplier* for $\mathcal{H}$ if $\theta f \in \mathcal{H}$ for every $f \in \mathcal{H}$. If θ is a multiplier for $\mathcal{H}$, then the multiplication operator

$$M_\theta : \mathcal{H} \to \mathcal{H}, \qquad f \mapsto \theta f$$

defines a bounded linear operator due to the closed graph theorem. Now, we let $\mathrm{A}(\mathcal{H})$ denote the norm-closed unital subalgebra of $B(\mathcal{H})$ generated by the polynomial multipliers, and let $\mathfrak{T}(\mathcal{H}) = \mathrm{C}^*(\mathrm{A}(\mathcal{H}))$.

By [49, Theorem 4.6], we have a short exact sequence of C*-algebras

$$0 \longrightarrow \mathfrak{K}(\mathcal{H}) \longrightarrow \mathfrak{T}(\mathcal{H}) \longrightarrow \mathrm{C}(\mathbb{S}_d) \longrightarrow 0$$

where $\mathbb{S}_d \subset \mathbb{C}^d$ denotes the unit sphere. Let $q : \mathfrak{T}(\mathcal{H}) \to \mathrm{C}(\mathbb{S}_d)$ denote the corresponding quotient map and, for each $z \in \mathbb{S}_d$, let $\chi_z : \mathrm{C}(\mathbb{S}_d) \to \mathbb{C}$ denote the evaluation mapping at z. Then, we have the following description of the boundary representations for $\mathrm{A}(\mathcal{H})$.

Theorem 2.5.1. *Let $\mathcal{H}$ be a regular, unitarily invariant, complete Pick space on the unit ball. Then, the following statements hold.*

(i) *An irreducible $*$-representation of $\mathfrak{T}(\mathcal{H})$ is either unitarily equivalent to the identity representation or to $\chi_z \circ q$ for some $z \in \mathbb{S}_d$.*

(ii) *If the reproducing kernel k is unbounded and $\mathcal{H}$ is not the Hardy space on the unit disc, then every irreducible $*$-representation for $\mathfrak{T}(\mathcal{H})$ is a boundary representation for $\mathrm{A}(\mathcal{H})$.*

(iii) *If the reproducing kernel k is bounded, then the boundary representations for $\mathrm{A}(\mathcal{H})$ are precisely those $*$-representations for $\mathfrak{T}(\mathcal{H})$ that are unitarily equivalent to the identity representation.*

Proof. (i): This follows from [22, Lemma 3.3].

(ii), (iii): These statements follow from [22, Theorem 6.2]. □

Suppose that $\mathcal{H}$ is the Hardy space on the unit disc. Then, M_z is unitarily equivalent to the unilateral shift on ℓ^2. In particular, $\mathfrak{T}(\mathcal{H})$ is unitarily equivalent to the Toeplitz algebra.

Chapter 3

Maximal C^*-covers and Residual Finite-Dimensionality

Recall that the maximal C^*-cover of an operator algebra $\mathcal{A}$, denoted $(\mathrm{C}^*_{max}(\mathcal{A}), \mu)$, satisfies the universal property that every representation of $\mathcal{A}$ lifts to a unique $*$-representation of $\mathrm{C}^*_{max}(\mathcal{A})$. Hence, the representations of $\mathcal{A}$ are in one-to-one correspondence with the $*$-representations of $\mathrm{C}^*_{max}(\mathcal{A})$. As residual finite-dimensionality is encoded in the structural properties of spaces of representations, one could ask whether residual finite-dimensionality of operator algebras is closely linked to that of their maximal C^*-covers. The main inquiry for this chapter is regarding a conjecture of Clouâtre and Ramsey, which was originally proposed in [24]. They had asked if $\mathrm{C}^*_{max}(\mathcal{A})$ is an RFD C^*-algebra whenever $\mathcal{A}$ is an RFD operator algebra.

There have been many classes of examples for which Clouâtre and Ramsey's conjecture has been answered in the affirmative [21],[24]. In particular, it is known that $\mathrm{C}^*_{max}(\mathcal{A})$ is RFD whenever $\mathcal{A}$ is finite-dimensional. The maximal C^*-cover is also RFD for Popescu's non-commutative disk algebra, the Schur-Agler class of functions, as well as some semi-group algebras and spaces of analytic functions. We remark that Clouâtre and Ramsey's conjecture is also related to a result of Pestov [75, Theorem 4.1], which states that the free C^*-algebra of any operator space is residually finite-dimensional. However, since publication of the manuscript that this chapter is based on, a counterexample to Clouâtre and Ramsey's conjecture has been found [55]. Throughout this chapter, we will record the various ways that this counterexample impacts our work.

In Section 3.1, we study a natural partial ordering on the collection of all C^*-covers of a fixed operator algebra. This was previously observed in [13],[54]. Under a natural interpretation, we show that this partial ordering is equivalent to the partial ordering given by inclusion of the spectra of the C^*-covers (Proposition 3.1.2). Moreover, this produces an isomorphism of complete lattices (Theorem 3.1.3) and allows us to equate topological statements on the spectrum of the maximal C^*-cover with order-theoretic statements on the collection of C^*-covers. This identification supplies us with a natural framework to study residual finite-dimensionality in Section 3.2.

Fix an RFD operator algebra $\mathcal{A}$. In Subsection 3.2.1, we derive the existence of a largest RFD C*-cover of $\mathcal{A}$ relative to the partial ordering of C*-covers (Theorem 3.2.2). This is referred to as the *RFD-maximal* C*-*cover*, denoted $(\mathfrak{R}(\mathcal{A}), \mu_r)$. As a consequence, an operator algebra agrees with Clouâtre and Ramsey's conjecture precisely when the RFD-maximal C*-cover is the maximal C*-cover (Corollary 3.2.3).

In Subsection 3.2.2, we provide an explicit representation of the RFD-maximal C*-cover. Typically, the embedding $\mu : \mathcal{A} \to C^*_{max}(\mathcal{A})$ is taken to be an appropriately large direct sum of completely contractive representations of $\mathcal{A}$. We show that the RFD-maximal C*-cover can be constructed in an analogous way and enjoys a universal property (Theorem 3.2.6).

Theorem A. *Let $\mathcal{A}$ be an RFD operator algebra and let $(\mathfrak{R}(\mathcal{A}), \mu_r)$ denote the RFD-maximal* C*-*cover of $\mathcal{A}$. For any representation $\rho : \mathcal{A} \to B(\mathcal{H}_\rho)$ on a finite-dimensional Hilbert space, there is a unique $*$-representation $\theta : \mathfrak{R}(\mathcal{A}) \to B(\mathcal{H}_\rho)$ such that $\theta \circ \mu_r = \rho$. Moreover, $(\mathfrak{R}(\mathcal{A}), \mu_r)$ is minimal among all* C*-*covers of $\mathcal{A}$ which satisfy this property.*

As previously noted, to verify that an operator algebra satisfy Clouâtre and Ramsey's conjecture, we must show that the RFD-maximal C*-cover and the maximal C*-cover coincide. In Subsection 3.2.3, we exhibit several aspects of the RFD-maximal C*-cover which are reminiscent of known properties of the maximal C*-cover. In particular, for unital operator algebras, the maximal C*-cover is known to respect countable direct sum and free products of finitely many operator algebras [12, Proposition 2.2],[24, Theorem 5.2]. We show that the RFD-maximal C*-cover also preserves these constructions (Theorems 3.2.10 and 3.2.11).

Theorem B. *The following statements hold.*

(i) *If $\mathcal{A}_n, n \in \mathbb{N}$, are unital RFD operator algebras, $\mathfrak{R}(\bigoplus_{n=1}^\infty \mathcal{A}_n) \cong \bigoplus_{n=1}^\infty \mathfrak{R}(\mathcal{A}_n)$.*

(ii) *If $\mathcal{A}, \mathcal{B}$ are unital RFD operator algebras, then $\mathfrak{R}(\mathcal{A} * \mathcal{B}) \cong \mathfrak{R}(\mathcal{A}) * \mathfrak{R}(\mathcal{B})$.*

In Section 3.3, we study the representation theory of RFD operator algebras and relate this to properties of the RFD-maximal C*-cover. Due to work of Exel and Loring, a C*-algebra is RFD precisely when all $*$-representations are point-strong limits of (possibly degenerate) finite-dimensional representations [46, Theorem 2.4]. In this setting, the point-strong and point-strong$*$ topologies coincide. This subtlety regarding the adjoint is responsible for some of the phenomena we witness.

Specifically, the possible discrepancy of the point-strong and point-strong$*$ topologies results in some uncertainty over how to interpret residually finite-dimensional representations for operator algebras. Here we consider two possible candidates: those representations which are point-strong limits of finite-dimensional representations are referred to as *residually finite-dimensional* representations (or RFD representations). Alongside RFD representations, we consider representations which are point-strong$*$ limits of finite-dimensional representations, called $*$-*residually finite-dimensional* representations (or $*$-RFD representations).

Recently, Clouâtre and Dor-On showed that $C^*_{max}(\mathcal{A})$ is RFD precisely when every representation of $\mathcal{A}$ is $*$-RFD [21, Theorem 3.3]. Utilizing this result, we are presented with an intermediate question. Indeed, whenever an operator algebra satisfies the conjecture of Clouâtre and Ramsey, it must be the case that RFD and $*$-RFD representations of the operator algebra coincide. This provides us with a suitable motivation for analyzing both classes of representations.

Finally, in Section 3.4, we study Hadwin's characterization of separable RFD C*-algebras [50] and provide a non self-adjoint version of their result. We recount the details.

Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for ℓ^2. For each $n \in \mathbb{N}$, let P_n be the orthogonal projection onto the linear span of $\{e_1, \dots, e_n\}$ and let $\mathcal{M}_n = P_n B(\ell^2) P_n$. Further, let $\mathfrak{B}$ be the C*-subalgebra of $\prod_{n=1}^{\infty} \mathcal{M}_n$ consisting of all sequences $(T_n)_{n\geq 1}$ which converge $*$-strongly in $B(\ell^2)$. Let $\pi : \mathfrak{B} \to B(\ell^2)$ denote the $*$-representation defined by $\pi((T_n)) = *\text{SOT} \lim_n T_n$. A representation $\rho : \mathcal{A} \to B(\ell^2)$ is *$*$-liftable in the sense of Hadwin* if there is a representation $\tau : \mathcal{A} \to \mathfrak{B}$ such that $\pi \circ \tau = \rho$.

Hadwin showed that a separable C*-algebra $\mathfrak{A}$ is RFD if and only if every unital $*$-representation $\sigma : \widetilde{\mathfrak{A}} \to B(\ell^2)$ is $*$-liftable [50, Theorem 11]. In Theorem 3.4.2, we prove a suitable analogue of Hadwin's theorem.

Theorem C. *Let $\mathcal{A}$ be a separable operator algebra. Then, $\mathrm{C}^*_{max}(\mathcal{A})$ is an RFD C*-algebra if and only if every unital representation $\rho : \widetilde{\mathcal{A}} \to B(\ell^2)$ is $*$-liftable in the sense of Hadwin.*

3.1 Topology and Ordering for C^*-covers

In this section, we equate order theoretic statements on the collection of all C^*-covers for a fixed operator algebra with topological data from the spectrum of the maximal C^*-cover. Since publication of [85], David Blecher has informed the author that this was also mentioned in [13, Section 2.8]. However, as the notation introduced here is important to future sections, we opt to include the finer details.

To begin, fix an operator algebra $\mathcal{A}$. Then, as in [54, Proposition 2.1.1], one may define an ordering on the collection of all C^*-covers of $\mathcal{A}$. Explicitly, if $(\mathfrak{A},\iota)$ and $(\mathfrak{B},j)$ are C^*-covers for $\mathcal{A}$, then $(\mathfrak{A},\iota)\preceq(\mathfrak{B},j)$ precisely when there is a surjective $*$-representation $\pi:\mathfrak{B}\to\mathfrak{A}$ such that $\pi\circ j=\iota$. Whenever we have that $(\mathfrak{A},\iota)\preceq(\mathfrak{B},j)$ and $(\mathfrak{B},j)\preceq(\mathfrak{A},\iota)$, we say the C^*-covers are *equivalent*, denoted $(\mathfrak{A},\iota)\sim(\mathfrak{B},j)$. It is easy to verify that two C^*-covers are equivalent if and only if there is a $*$-isomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ such that $\pi\circ j=\iota$.

Up to equivalence of C^*-covers, the maximal C^*-cover is the unique maximal element under this ordering whereas the C^*-envelope is the unique minimal element [54, Examples 2.1.8 and 2.1.13]. Moreover, we remark that the equivalence classes associated with this order structure form a complete lattice (see [54, Section 2.1]). For more details on the ordering for C^*-covers, the reader can consult the thesis of Hamidi [54, Chapters 1-2]. The approach we take in this section will complement Hamidi's work.

We start by identifying the space of all C^*-covers for $\mathcal{A}$ with a family of closed subsets of the spectrum for the maximal C^*-cover. To this end, recall that $\widehat{\mathrm{C}^*_{max}(\mathcal{A})}$ denotes the collection of unitary equivalence classes of irreducible $*$-representations for $\mathrm{C}^*_{max}(\mathcal{A})$ and that $\widehat{\mathrm{C}^*_{max}(\mathcal{A})}$ can be equipped with a natural topology (Subsection 2.1.1). The spectrum of any C^*-cover of $\mathcal{A}$ may be identified as a closed subset of $\widehat{\mathrm{C}^*_{max}(\mathcal{A})}$. To this end, recall that, given a C^*-cover $(\mathfrak{A},\iota)$ of $\mathcal{A}$, we let $q_{\mathfrak{A}}:\mathrm{C}^*_{max}(\mathcal{A})\to\mathfrak{A}$ denote the unique surjective $*$-representation satisfying $q_{\mathfrak{A}}\circ\mu=\iota$. Now, let

$$\mathcal{S}(\mathfrak{A},\iota):=\left\{[\pi]\in\widehat{\mathrm{C}^*_{max}(\mathcal{A})}:\pi=\sigma\circ q_{\mathfrak{A}},\ \sigma\text{ irreducible }*\text{-representation of }\mathfrak{A}\right\}.$$

Theorem 2.1.1 reveals that $\mathcal{S}(\mathfrak{A},\iota)$ is naturally homeomorphic to $\widehat{\mathfrak{A}}$. In fact, every closed subset which contains the spectrum of the C^*-envelope is of this form.

Theorem 3.1.1. *Let $\mathcal{A}$ be an operator algebra and $\mathcal{C}\subset\widehat{\mathrm{C}^*_{max}(\mathcal{A})}$ be some subset. Then, $\mathcal{C}=\mathcal{S}(\mathfrak{A},\iota)$ for some C^*-cover $(\mathfrak{A},\iota)$ of $\mathcal{A}$ if and only if $\mathcal{C}$ is closed and contains $\mathcal{S}(\mathrm{C}^*_e(\mathcal{A}),\varepsilon)$.*

Proof. ($\Rightarrow$) By Theorem 2.1.1 (ii), the subset $\mathcal{S}(\mathfrak{A},\iota)$ is closed. Since $q_{\mathfrak{A}}:\mathrm{C}^*_{max}(\mathcal{A})\to\mathfrak{A}$ is completely isometric on $\mu(\mathcal{A})$, we have that $\ker q_{\mathfrak{A}}\subset\mathrm{Sh}(\mathcal{A},\mathrm{C}^*_{max}(\mathcal{A}),\mu)$. Then, Theorem 2.1.1 implies that $\mathcal{S}(\mathrm{C}^*_e(\mathcal{A}),\varepsilon)\subset\mathcal{S}(\mathfrak{A},\iota)$.

($\Leftarrow$) As $\mathcal{C}$ is closed, by Theorem 2.1.1, $\mathcal{C}$ is the collection of all unitary equivalence classes consisting of irreducible $*$-representations of $\mathrm{C}^*_{max}(\mathcal{A})$ which vanish on some ideal $\mathfrak{J}$. Similarly, the subset $\mathcal{S}(\mathrm{C}^*_e(\mathcal{A}),\varepsilon)$ consists of all equivalence classes of irreducible $*$-representations of $\mathrm{C}^*_{max}(\mathcal{A})$ which vanish on the Shilov ideal $\mathrm{Sh}(\mathcal{A},\mathrm{C}^*_{max}(\mathcal{A}),\mu)$. As $\mathcal{S}(\mathrm{C}^*_e(\mathcal{A}),\varepsilon)\subset\mathcal{C}$, we have that $\mathfrak{J} \subset \mathrm{Sh}(\mathcal{A}, \mathrm{C}^*_{max}(\mathcal{A}), \mu)$ by Theorem 2.1.1. Let $\mathfrak{A} = \mathrm{C}^*_{max}(\mathcal{A})/\mathfrak{J}$ and $q : \mathrm{C}^*_{max}(\mathcal{A}) \to \mathfrak{A}$ be the corresponding quotient map.

Let $q_e : \mathrm{C}^*_{max}(\mathcal{A}) \to \mathrm{C}^*_e(\mathcal{A})$ denote the corresponding quotient map that is completely isometric on $\mu(\mathcal{A})$ and, in particular, with the property that $\ker q_e = \mathrm{Sh}(\mathcal{A}, \mathrm{C}^*_{max}(\mathcal{A}), \mu)$. As $\mathfrak{J} \subset \mathrm{Sh}(\mathcal{A}, \mathrm{C}^*_{max}(\mathcal{A}), \mu)$, it follows that $q_e = Q \circ q$ where $Q : \mathfrak{A} \to \mathrm{C}^*_e(\mathcal{A})$ is a surjective $*$-representation.

Define a representation of $\mathcal{A}$ by $\iota = q \circ \mu$. Note that $Q \circ \iota = q_e \circ \mu$ and so the map ι must be completely isometric. Furthermore,

$$\mathrm{C}^*(\iota(\mathcal{A})) = \mathrm{C}^*(q \circ \mu(\mathcal{A})) = q(\mathrm{C}^*(\mu(\mathcal{A}))) = \mathfrak{A}.$$

So $(\mathfrak{A}, \iota)$ is a C^*-cover of $\mathcal{A}$ and $q_{\mathfrak{A}} = q$ by uniqueness of the representation $q_{\mathfrak{A}}$. Therefore $\mathcal{C}$ is the collection of all unitary equivalence classes consisting of irreducible $*$-representations which vanish on $\mathfrak{J} = \ker q_{\mathfrak{A}}$ and moreover, $\mathcal{C} = \mathcal{S}(\mathfrak{A}, \iota)$. □

It will be a consequence of the subsequent Proposition 3.1.2 that the C^*-cover obtained in Theorem 3.1.1 is unique up to equivalence. Also, despite the conclusion of Theorem 3.1.1, the relative topology of $\mathcal{S}(\mathrm{C}^*_e(\mathcal{A}), \varepsilon)$ does not impose an obvious restriction regarding residual finite-dimensionality of C^*-covers. In fact, there are operator algebras whose C^*-envelope does not possess any finite-dimensional $*$-representations, while there exist other C^*-covers which are RFD.

Example 3. Take a pair of isometries $V, W \in B(\mathcal{H})$ satisfying $VV^* + WW^* = I$ and let $\mathcal{M}$ be the unital subspace of $B(\mathcal{H})$ generated by V and W. Form the unital operator algebra $\mathcal{A}$ consisting of elements of the form

$$\begin{bmatrix} \lambda I & T \\ 0 & \mu I \end{bmatrix} \in B(\mathcal{H}^{(2)})$$

where $\lambda, \mu \in \mathbb{C}$ and $T \in \mathcal{M}$. We have that $\mathrm{C}^*_e(\mathcal{A}) \cong \mathbb{M}_2(\mathcal{O}_2)$ where $\mathcal{O}_2$ denotes the Cuntz algebra [26, Lemma 4.2]. In particular, $\mathrm{C}^*_e(\mathcal{A})$ is a simple, infinite-dimensional C^*-algebra. Consequently, the subset $\mathcal{S}(\mathrm{C}^*_e(\mathcal{A}), \iota_{env})$ does not possess any equivalence classes of finite-dimensional irreducible $*$-representations. On the other hand, $\mathrm{C}^*_{max}(\mathcal{A})$ is RFD by [24, Theorem 5.1].

As there can easily be many inequivalent C^*-covers of a fixed operator algebra, Theorem 3.1.1 indicates that the topology of $\widehat{\mathrm{C}^*_{max}(\mathcal{A})}$ is non-trivial. This is the case for even the simplest choice of non self-adjoint algebra.

Example 4. Let $\mathcal{T}_2$ denote the algebra of upper-triangular 2×2 matrices. We show that $\widehat{\mathrm{C}^*_{max}(\mathcal{T}_2)}$ is not even Hausdorff. Let

$$\mathfrak{M} = \{f \in C([0,1], \mathbb{M}_2) : f(0) \text{ is a diagonal matrix}\}$$

and let $\mu : \mathcal{T}_2 \to \mathfrak{M}$ be defined by

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mapsto \begin{bmatrix} ax & b\sqrt{x} \\ 0 & cx \end{bmatrix}$$

where $x \in [0,1]$. In [12, Example 2.4], it was shown that μ is completely isometric and that $(\mathfrak{M}, \mu)$ is equivalent to the maximal C*-cover of $\mathcal{T}_2$.

Let $\iota : \mathcal{T}_2 \to \mathbb{M}_2$ be the identity representation. As $\mathbb{M}_2$ is simple, $(\mathbb{M}_2, \iota)$ is the C*-envelope for $\mathcal{T}_2$. Let $\gamma_\xi : \mathfrak{M} \to \mathbb{M}_2$ denote evaluation at $\xi \in (0,1]$. Observe that Theorem 3.1.1 implies that any closed subset of $\widehat{\mathfrak{M}}$ containing $[\gamma_1]$ determines the spectrum of a C*-cover for $\mathcal{T}_2$. Indeed $\gamma_1 \circ \mu = \iota$ and then it is easy to see that $\mathcal{S}(\mathbb{M}_2, \iota) = \{[\gamma_1]\}$ as there is a unique irreducible $*$-representation of $\mathbb{M}_2$.

Note that $\mathrm{C}([0,1], \mathbb{M}_2) \cong \mathrm{C}([0,1]) \otimes \mathbb{M}_2$ is a liminal C*-algebra [88, Theorem 2 (c)]. As a result, $\mathfrak{M}$ is also a liminal C*-algebra [40, Proposition 4.2.4]. As $\mathfrak{M}$ is unital, it follows that all irreducible $*$-representations of $\mathfrak{M}$ are finite-dimensional. Whence, $\widehat{\mathfrak{M}}$ is homeomorphic to $\mathrm{Prim}(\mathfrak{M})$ via the natural mapping [40, Proposition 3.1.6].

For $f \in \mathfrak{M}$ and $j = 1, 2$, define $\eta_j(f)$ to be the (j,j)-entry of $f(0)$. Then $\eta_j : \mathfrak{M} \to \mathbb{C}$ is a character. Take a sequence of points $(\xi_n)_{n\geq 1} \subset (0,1]$ converging to 0. Let $\mathfrak{J}_n = \ker \gamma_{\xi_n}$ and for $j = 1, 2$, let $\mathfrak{L}_j = \ker \eta_j$. Then, we see that $\bigcap_{n=1}^\infty \mathfrak{J}_n \subset \mathfrak{L}_1, \mathfrak{L}_2$. Therefore, $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are distinct accumulation points for the sequence $(\mathfrak{J}_n)_{n\geq 1}$. In particular, $\widehat{\mathfrak{M}}$ is not Hausdorff.

The reader should compare the conclusion of Example 4 with the following classical result of Kaplansky [59, Theorem 4.2]. Therein, it was shown that if all irreducible $*$-representations of a C*-algebra are finite-dimensional and of the same dimension, then the spectrum is Hausdorff.

Now we showcase how the partial ordering on the C*-covers of an operator algebra can be completely understood by topological statements.

Proposition 3.1.2. *Let $\mathcal{A}$ be an operator algebra and $(\mathfrak{A}, \iota), (\mathfrak{B}, j)$ be C*-covers of $\mathcal{A}$. Then, the following statements are equivalent:*

(i) *$\mathcal{S}(\mathfrak{A}, \iota)$ is a subset of $\mathcal{S}(\mathfrak{B}, j)$;*

(ii) *$q_{\mathfrak{A}}$ is weakly contained in $q_{\mathfrak{B}}$;*

(iii) *$(\mathfrak{A}, \iota) \preceq (\mathfrak{B}, j)$.*

In particular, $\mathcal{S}(\mathfrak{A}, \iota) = \mathcal{S}(\mathfrak{B}, j)$ if and only if $(\mathfrak{A}, \iota) \sim (\mathfrak{B}, j)$.

Proof. (i)⇒(ii): The assumption immediately implies that $\ker q_{\mathfrak{B}} \subset \ker q_{\mathfrak{A}}$ by Theorem 2.1.1.

(ii)⇒(iii): Since $q_{\mathfrak{A}}$ is weakly contained in $q_{\mathfrak{B}}$, there is a surjective $*$-representation $q : \mathfrak{B} \to \mathfrak{A}$ such that $q \circ q_{\mathfrak{B}} = q_{\mathfrak{A}}$. We see that

$$q \circ j = q \circ q_{\mathfrak{B}} \circ \mu = q_{\mathfrak{A}} \circ \mu = \iota.$$

Therefore $(\mathfrak{A}, \iota) \preceq (\mathfrak{B}, j)$.

(iii)⇒(i): Let $q : \mathfrak{B} \to \mathfrak{A}$ be a surjective $*$-representation such that $q \circ j = \iota$. As $q \circ q_{\mathfrak{B}} \circ \mu = \iota$, we have that $q \circ q_{\mathfrak{B}} = q_{\mathfrak{A}}$. Take $[\pi] \in \mathcal{S}(\mathfrak{A}, \iota)$ and let π be a representative for $[\pi]$. Express $\pi = \sigma \circ q_{\mathfrak{A}}$ where σ is an irreducible $*$-representation of $\mathfrak{A}$. Then $\pi = (\sigma \circ q) \circ q_{\mathfrak{B}}$ where $\sigma \circ q$ is an irreducible $*$-representation of $\mathfrak{B}$. So $[\pi] \in \mathcal{S}(\mathfrak{B}, j)$. □

Next, we provide a refinement of Proposition 3.1.2. For this, we recall the complete lattice structure of C*-covers in [54, Section 2.1]. Let $\mathrm{Cov}(\mathcal{A})$ denote the collection of all equivalence classes of C*-covers for a fixed operator algebra $\mathcal{A}$. Define a partial ordering $\preceq_q$ on $\mathrm{Cov}(\mathcal{A})$ by $[(\mathfrak{A}, \iota)] \preceq_q [(\mathfrak{B}, j)]$ if and only if $(\mathfrak{A}, \iota) \preceq (\mathfrak{B}, j)$. For brevity, we will refer solely to the ordering $\preceq$. A complete lattice structure on $\mathrm{Cov}(\mathcal{A})$ is defined as follows. Let $\mathcal{C} = \{[(\mathfrak{A}_\lambda, \iota_\lambda)]\} \subset \mathrm{Cov}(\mathcal{A})$ be some collection. Then, $\sup \mathcal{C}$ is the equivalence class $[(\mathrm{C}^*(\iota(\mathcal{A})), \iota)]$ where $\iota = \bigoplus_\lambda \iota_\lambda$. For each λ, let $\mathfrak{J}_\lambda = \ker q_{\mathfrak{A}_\lambda}$ and let $\mathfrak{J} = \overline{\sum_\lambda \mathfrak{J}_\lambda}$ be the norm closure of the ideal generated by $\bigcup_\lambda \mathfrak{J}_\lambda$. Then, $\inf \mathcal{C}$ is the equivalence class $[(\mathrm{C}^*_{max}(\mathcal{A})/\mathfrak{J}, q \circ \mu)]$ where $q : \mathrm{C}^*_{max}(\mathcal{A}) \to \mathrm{C}^*_{max}(\mathcal{A})/\mathfrak{J}$ is the quotient map. To see that $(\mathrm{C}^*_{max}(\mathcal{A})/\mathfrak{J}, q \circ \mu)$ defines a C*-cover of $\mathcal{A}$, one can refer to [54, Theorem 2.1.11].

We show that $\mathrm{Cov}(\mathcal{A})$ is isomorphic to a complete lattice arising naturally from $\widehat{\mathrm{C}^*_{max}(\mathcal{A})}$. To this end, consider

$$\mathfrak{Q}(\mathcal{A}) := \left\{ F \subset \widehat{\mathrm{C}^*_{max}(\mathcal{A})} \ : \ F \text{ closed and containing } \mathcal{S}(\mathrm{C}^*_e(\mathcal{A}), \iota_{env}) \right\},$$

ordered by inclusion. Then, $\mathfrak{Q}(\mathcal{A})$ is a complete lattice since, whenever $\mathcal{D} = \{F_\lambda : \lambda \in \Lambda\}$ is a subset of $\mathfrak{Q}(\mathcal{A})$, we may define $\inf \mathcal{D} = \bigcap_\lambda F_\lambda$ and $\sup \mathcal{D} = \overline{\bigcup_\lambda F_\lambda}$.

Theorem 3.1.3. *Let $\mathcal{A}$ be an operator algebra. Then, the mapping*

$$\Omega : Cov(\mathcal{A}) \to \mathfrak{Q}(\mathcal{A}), \qquad [(\mathfrak{A}, \iota)] \mapsto \mathcal{S}(\mathfrak{A}, \iota),$$

is an isomorphism of complete lattices.

Proof. The map Ω is a well-defined bijection by Theorem 3.1.1 and Proposition 3.1.2. We will show that Ω respects (i) infima and (ii) suprema. Let $\mathcal{C} = \{[(\mathfrak{A}_\lambda, \iota_\lambda)]\} \subset \mathrm{Cov}(\mathcal{A})$ be some collection. Let π be an irreducible $*$-representation of $\mathrm{C}^*_{max}(\mathcal{A})$ and, for each λ, let $\mathfrak{J}_\lambda = \ker q_{\mathfrak{A}_\lambda}$.

(i) Set $\mathfrak{J} = \overline{\sum_\lambda \mathfrak{J}_\lambda}$ and let $q : \mathrm{C}^*_{max}(\mathcal{A}) \to \mathrm{C}^*_{max}(\mathcal{A})/\mathfrak{J}$ be the corresponding quotient map. Observe that $\mathcal{S}(\mathrm{C}^*_{max}(\mathcal{A})/\mathfrak{J}, q \circ \mu)$ consists of the equivalence classes of irreducible $*$-representations of $\mathrm{C}^*_{max}(\mathcal{A})$ which vanish on $\mathfrak{J}$. As π vanishes on $\mathfrak{J}$ if and only if π vanishes on each $\mathfrak{J}_\lambda$, we conclude that

$$\mathcal{S}(\mathrm{C}^*_{max}(\mathcal{A})/\mathfrak{J}, q \circ \mu) = \bigcap_\lambda \mathcal{S}(\mathfrak{A}_\lambda, \iota_\lambda).$$

(ii) Let $\iota = \bigoplus_\lambda \iota_\lambda$ and $\mathfrak{A} = \mathrm{C}^*(\iota(\mathcal{A}))$. We show that

$$\mathcal{S}(\mathfrak{A}, \iota) = \overline{\bigcup_\lambda \mathcal{S}(\mathfrak{A}_\lambda, \iota_\lambda)}. \tag{3.1}$$

Suppose that $\pi = \sigma \circ q_{\mathfrak{A}_\lambda}$ for some λ and some irreducible $*$-representation σ of $\mathfrak{A}_\lambda$. As $\mathfrak{A} \subset \prod_\mu \mathfrak{A}_\mu$, we have a $*$-representation $\gamma_\lambda : \mathfrak{A} \to \mathfrak{A}_\lambda$ given by the projection mapping. Moreover,

$$\gamma_\lambda(\mathfrak{A}) = \mathrm{C}^*((\gamma_\lambda \circ \iota)(\mathcal{A})) = \mathrm{C}^*(\iota_\lambda(\mathcal{A})) = \mathfrak{A}_\lambda$$

and so, γ_λ is surjective. So, $\sigma \circ \gamma_\lambda$ is an irreducible $*$-representation of $\mathfrak{A}$ with the property that

$$\sigma \circ \gamma_\lambda \circ q_{\mathfrak{A}} \circ \mu = \sigma \circ \gamma_\lambda \circ \iota = \sigma \circ \iota_\lambda = \sigma \circ q_{\mathfrak{A}_\lambda} \circ \mu = \pi \circ \mu.$$

So $\pi = \sigma \circ \gamma_\lambda \circ q_{\mathfrak{A}}$. Hence, $\mathcal{S}(\mathfrak{A}_\lambda, \iota_\lambda) \subset \mathcal{S}(\mathfrak{A}, \iota)$. By Theorem 3.1.1, $\mathcal{S}(\mathfrak{A}, \iota)$ is closed and thus, we have that $\overline{\bigcup_\lambda \mathcal{S}(\mathfrak{A}_\lambda, \iota_\lambda)} \subset \mathcal{S}(\mathfrak{A}, \iota)$.

Conversely, by Theorem 3.1.1, there is a C*-cover $(\mathfrak{B}, j)$ of $\mathcal{A}$ such that $\mathcal{S}(\mathfrak{B}, j) = \overline{\bigcup_\lambda \mathcal{S}(\mathfrak{A}_\lambda, \iota_\lambda)}$. For each λ, Proposition 3.1.2 implies that there is a surjective $*$-representation $\beta_\lambda : \mathfrak{B} \to \mathfrak{A}_\lambda$ satisfying $\beta_\lambda \circ j = \iota_\lambda$. Let $\beta = \bigoplus_\lambda \beta_\lambda$. Then β is a $*$-representation satisfying $\beta \circ j = \iota$. So

$$\beta(\mathfrak{B}) = \beta(\mathrm{C}^*(j(\mathcal{A}))) = \mathrm{C}^*(\beta \circ j(\mathcal{A})) = \mathfrak{A}$$

and we have that $(\mathfrak{A}, \iota) \preceq (\mathfrak{B}, j)$. By Proposition 3.1.2, we have that $\mathcal{S}(\mathfrak{A}, \iota) \subset \mathcal{S}(\mathfrak{B}, j)$. So equation (3.1) holds. □

We remark that Humeniuk-Ramsey [58] have further developed the structure of the complete lattice Cov$(\mathcal{A})$. In particular, they extrapolated various ways that the lattice structure for Cov$(\mathcal{A})$ fails to classify $\mathcal{A}$ up to complete isometric isomorphism.

As the representations of $\mathcal{A}$ are in one-to-one correspondence with the $*$-representations of $\mathrm{C}^*_{max}(\mathcal{A})$, it is feasible to identify the spectrum of $\mathcal{A}$ as the spectrum of the maximal C*-cover. However, it is unclear to what degree one can identify the topology at the level of the operator algebra. To close this section, we provide a partial answer to this question.

Let $\mathcal{A}$ be an operator algebra. If $\pi : \mathcal{A} \to B(\mathcal{H}_\pi)$ and $\rho : \mathcal{A} \to B(\mathcal{H}_\rho)$ are representations, then π and ρ are *approximately unitarily equivalent* if

$$\lim_\lambda \|U_\lambda^* \rho(a) U_\lambda - \pi(a)\| = 0, \qquad a \in \mathcal{A},$$

where $U_\lambda : \mathcal{H}_\pi \to \mathcal{H}_\rho, \lambda \in \Lambda$, is a net of unitary operators.

By [40, Corollary 4.1.10] and [34, Corollary 2.5.6], it is easily seen that irreducible $*$-representations of a separable C*-algebra are weakly equivalent if and only if they are approximately unitarily equivalent. Hence, two equivalence classes of irreducible $*$-representations possess the same closure precisely when they are approximately unitarily equivalent. Therefore, due to the subsequent proposition, the spectrum of a separable operator algebra $\mathcal{A}$ can be identified with $\widehat{\mathrm{C}^*_{max}(\mathcal{A})}$ pointwise.

Proposition 3.1.4. *Let $\mathcal{A}$ be an operator algebra and let $(\mathrm{C}^*_{max}(\mathcal{A}), \mu)$ denote the maximal C*-cover of $\mathcal{A}$. Let π and ρ be representations of $\mathcal{A}$ and let θ_π, θ_ρ be $*$-representations of $\mathrm{C}^*_{max}(\mathcal{A})$ satisfying $\theta_\pi \circ \mu = \pi$ and $\theta_\rho \circ \mu = \rho$. Then, π and ρ are approximately unitarily equivalent if and only if θ_π and θ_ρ are approximately unitarily equivalent.*

Proof. (⇒) Suppose that

$$\pi(a) = \lim_{\lambda} U_\lambda^* \rho(a) U_\lambda, \qquad a \in \mathcal{A},$$

where $U_\lambda : \mathcal{H}_\pi \to \mathcal{H}_\rho, \lambda \in \Lambda$, is a net of unitaries. So

$$\theta_\pi(\mu(a)) = \lim_{\lambda} U_\lambda^* \theta_\rho(\mu(a)) U_\lambda, \qquad a \in \mathcal{A}.$$

As addition, multiplication and the adjoint are continuous in the norm topology, we infer that

$$\theta_\pi(w) = \lim_{\lambda} U_\lambda^* \theta_\rho(w) U_\lambda$$

for each $w \in \mathrm{C}^*_{max}(\mathcal{A})$ which lies in the linear span of words in $\mu(\mathcal{A}) \cup \mu(\mathcal{A})^*$. This set is dense in $\mathrm{C}^*_{max}(\mathcal{A})$ and so θ_π is approximately unitarily equivalent to θ_ρ.

(⇐) Suppose that

$$\theta_\pi(t) = \lim_{\lambda} U_\lambda^* \theta_\rho(t) U_\lambda, \qquad t \in \mathrm{C}^*_{max}(\mathcal{A}),$$

where $U_\lambda : \mathcal{H}_\pi \to \mathcal{H}_\rho$ is a net of unitaries. In particular, for each $a \in \mathcal{A}$,

$$\pi(a) = \theta_\pi(\mu(a)) = \lim_{\lambda} U_\lambda^* \theta_\rho(\mu(a)) U_\lambda = \lim_{\lambda} U_\lambda^* \rho(a) U_\lambda.$$

So π and ρ are approximately unitarily equivalent. □

In spite of Proposition 3.1.4, it is non-trivial whether one can make a similar statement involving weak *containment*, which determines the topology on $\widehat{\mathrm{C}^*_{max}(\mathcal{A})}$. Indeed, if one were to define weak containment of representations of $\mathcal{A}$ (in any of the equivalent forms presented in Subsection 2.1.1), then it is not true that this is equivalent to weak containment of the respective lifts to $\mathrm{C}^*_{max}(\mathcal{A})$. This can be seen by taking two completely isometric representations of an operator algebra which induce inequivalent C^*-covers.

3.2 The RFD-Maximal C^*-cover

Fix an RFD operator algebra $\mathcal{A}$. Recall that a C^*-cover $(\mathfrak{A}, \iota)$ is RFD if $\mathfrak{A}$ is an RFD C^*-algebra. Note that residual finite-dimensionality is preserved under equivalence of C^*-covers. Our goal in this section is to analyze the largest RFD C^*-cover of $\mathcal{A}$ in the ordering among all C^*-covers of $\mathcal{A}$. We refer to this C^*-cover as the *RFD-maximal* C^**-cover* of $\mathcal{A}$. It is unclear what likeness the RFD-maximal C^*-cover might have to the maximal C^*-cover. Whenever the Clouâtre–Ramsey conjecture has an affirmative answer, then the RFD-maximal C^*-cover inherits all possible characteristics of the maximal C^*-cover. This will be the motivation behind the material found within Subsection 3.2.3.

We outline our procedure in this section. First we show that the RFD-maximal C^*-cover of an RFD operator algebra always exists (Theorem 3.2.2). Following this, in Theorem 3.2.4, we identify the spectrum of the RFD-maximal C^*-cover through spectral data of the maximal C^*-cover. In Theorem 3.2.6, we provide a concrete representation of the RFD-maximal C^*-cover which is akin to the classical construction of the maximal C^*-cover. This naturally endows the RFD-maximal C^*-cover with a certain universal property among RFD C^*-covers. Ending this section, we provide two instances in which the RFD-maximal C^*-cover shares commonality with the maximal C^*-cover (Theorems 3.2.10 and 3.2.11).

3.2.1 Abstract Characterization

To determine the RFD-maximal C^*-cover, we require an observation on how residual finite-dimensionality is affected in $\mathrm{Cov}(\mathcal{A})$.

Lemma 3.2.1. *Let $\mathcal{A}$ be an operator algebra and $\mathcal{R} = \{[(\mathfrak{A}_\lambda, \iota_\lambda)]\} \subset Cov(\mathcal{A})$ be some collection of RFD C^*-covers. Then, $\sup \mathcal{R}$ is an RFD C^*-cover.*

Proof. Letting $\iota = \bigoplus_\lambda \iota_\lambda$, we have that $\sup \mathcal{R} = [(\mathrm{C}^*(\iota(\mathcal{A})), \iota)]$ and

$$\mathrm{C}^*(\iota(\mathcal{A})) \subset \prod_\lambda \mathfrak{A}_\lambda.$$

As each $\mathfrak{A}_\lambda$ is an RFD C^*-algebra, $\mathrm{C}^*(\iota(\mathcal{A}))$ is RFD as well. So, $\sup \mathcal{R}$ is RFD as desired. □

As an immediate consequence, we may deduce the existence of the RFD-maximal C^*-cover:

Theorem 3.2.2. *Let $\mathcal{A}$ be an RFD operator algebra and*

$$\mathcal{R} = \{[(\mathfrak{A}, \iota)] \in Cov(\mathcal{A}) : \mathfrak{A} \text{ is RFD}\}.$$

Then, there is a unique equivalence class of C^*-covers $[(\mathfrak{R}(\mathcal{A}), \mu_r)] \in \mathcal{R}$ which is maximal for $\mathcal{R}$.

Proof. As $\mathcal{A}$ is RFD, there is an RFD C^*-cover and so $\mathcal{R}$ is non-empty. Then, by Lemma 3.2.1, $\sup \mathcal{R}$ both lies in $\mathcal{R}$ and defines a (necessarily unique) equivalence class of C^*-covers for $\mathcal{A}$. □

Throughout, we will let $(\mathfrak{R}(\mathcal{A}), \mu_r)$ denote the *RFD-maximal* C**-cover* of an RFD operator algebra $\mathcal{A}$. Since the RFD-maximal C*-cover is unique up to equivalence of C*-covers, we will frequently fix a representative of $[(\mathfrak{R}(\mathcal{A}), \mu_r)]$, as with the maximal and minimal C*-covers. Later, we will derive some explicit representations of this C*-cover. Additionally, we remark that, since publication of [85], Humeniuk-Katsoulis-Ramsey [57, Section 3] have constructed a similar C*-cover that is maximal among another well-studied family of C*-covers.

Now, we start by highlighting the connection that $(\mathfrak{R}(\mathcal{A}), \mu_r)$ has to the conjecture of Clouâtre and Ramsey:

Corollary 3.2.3. *Let $\mathcal{A}$ be an operator algebra. Then,* $\mathrm{C}^*_{max}(\mathcal{A})$ *is an RFD* C**-algebra if and only if* $(\mathfrak{R}(\mathcal{A}), \mu_r) \sim (\mathrm{C}^*_{max}(\mathcal{A}), \mu)$.

Proof. ($\Rightarrow$) When $\mathrm{C}^*_{max}(\mathcal{A})$ is RFD, $\mathcal{A}$ is RFD itself. By Theorem 3.2.2, we have that the RFD-maximal C*-cover for $\mathcal{A}$ exists. Also, we have that $(\mathrm{C}^*_{max}(\mathcal{A}), \mu) \preceq (\mathfrak{R}(\mathcal{A}), \mu_r)$ as $\mathrm{C}^*_{max}(\mathcal{A})$ is RFD. However, as the maximal C*-cover is maximal in the ordering, we have $(\mathfrak{R}(\mathcal{A}), \mu_r) \preceq (\mathrm{C}^*_{max}(\mathcal{A}), \mu)$.

($\Leftarrow$) If $(\mathfrak{R}(\mathcal{A}), \mu_r) \sim (\mathrm{C}^*_{max}(\mathcal{A}), \mu)$, then there is a $*$-isomorphism $\pi : \mathfrak{R}(\mathcal{A}) \to \mathrm{C}^*_{max}(\mathcal{A})$. Since $\mathfrak{R}(\mathcal{A})$ is an RFD C*-algebra, so is $\mathrm{C}^*_{max}(\mathcal{A})$. □

We also record an example of Hartz [55] which demonstrates that $(\mathfrak{R}(\mathcal{A}), \mu_r)$ is not always equal to the maximal C*-cover. We will present some of the inner-workings of this example in future sections.

Example 5. Let $\mathrm{A}(\mathbb{D}) \subset \mathrm{C}(\overline{\mathbb{D}})$ denote the disc algebra. Since $\mathrm{C}^*_e(\mathrm{A}(\mathbb{D})) = \mathrm{C}(\mathbb{T})$, we may view the disc algebra (completely isometrically) as a subalgebra of $\mathrm{C}(\mathbb{T})$. Form an operator algebra

$$\mathcal{B} = \left\{ \begin{bmatrix} f & 0 \\ h & g \end{bmatrix} \in \mathbb{M}_2(\mathrm{C}(\mathbb{T})) \ : \ f, g \in \mathrm{A}(\mathbb{D}), h \in \mathrm{C}(\mathbb{T}) \right\}.$$

Then, $\mathrm{C}^*_{max}(\mathcal{B})$ is not RFD by [55, Corollary 1.4] and hence, $(\mathfrak{R}(\mathcal{B}), \mu_r)$ is not the maximal C*-cover by Corollary 3.2.3.

Next, we provide some defining properties for the RFD-maximal C*-cover of an operator algebra. This is achieved by analyzing the spectrum of the maximal C*-cover. Here, we identify a dense subset of the spectrum of the RFD-maximal C*-cover. This is very natural when utilizing Proposition 3.1.2: the RFD-maximal C*-cover is the largest RFD C*-cover of $\mathcal{A}$, both under the ordering of C*-covers and under inclusion of the closed subsets $\mathcal{S}(\mathfrak{A}, \iota)$ where $(\mathfrak{A}, \iota)$ is an RFD C*-cover of $\mathcal{A}$.

Theorem 3.2.4. *Let $\mathcal{A}$ be an RFD operator algebra and let* $(\mathrm{C}^*_{max}(\mathcal{A}), \mu), (\mathfrak{R}(\mathcal{A}), \mu_r)$ *denote the maximal and RFD-maximal* C**-covers of $\mathcal{A}$, respectively. Let* $q : \mathrm{C}^*_{max}(\mathcal{A}) \to \mathfrak{R}(\mathcal{A})$ *denote the unique surjective $*$-representation satisfying $q \circ \mu = \mu_r$ and let $\mathfrak{J} = \ker q$. Then, the following statements hold:*

(i) $\mathcal{S}(\mathfrak{R}(\mathcal{A}),\mu_r)$ *is the closure of all equivalence classes which consist of finite-dimensional irreducible* $*$*-representations of* $\mathrm{C}^*_{max}(\mathcal{A})$*;*

(ii) $\mathfrak{J}$ *does not possess any finite-dimensional* $*$*-representations;*

(iii) $\mathcal{S}(\mathfrak{R}(\mathcal{A}),\mu_r)$ *is the unique maximal closed subset of* $\widehat{\mathrm{C}^*_{max}(\mathcal{A})}$ *possessing a dense subset of equivalence classes of finite-dimensional irreducible* $*$*-representations.*

Proof. (i): In Theorem 2.1.1 (iii), it is straightforward to see that the homeomorphism produces a bijection between equivalence classes of finite-dimensional $*$-representations. Hence, by Theorem 2.1.2, a C*-cover $(\mathfrak{A},\iota)$ of $\mathcal{A}$ is RFD if and only if there is a dense subset of equivalence classes in $\mathcal{S}(\mathfrak{A},\iota)$ consisting of finite-dimensional $*$-representations.

Let $\mathfrak{F}\subset\widehat{\mathrm{C}^*_{max}(\mathcal{A})}$ be the closure of all equivalence classes of finite-dimensional irreducible $*$-representations. As $\mathcal{A}$ is RFD, we have that $\mathfrak{F}$ contains $\mathcal{S}(\mathrm{C}^*_e(\mathcal{A}),\iota_{env})$. Indeed, if $(\mathfrak{A},\iota)$ is an RFD C*-cover, then $\mathcal{S}(\mathfrak{A},\iota)$ contains $\mathcal{S}(\mathrm{C}^*_e(\mathcal{A}),\iota_{env})$ and possesses a dense subset of equivalence classes consisting of finite-dimensional $*$-representations. By Theorem 3.1.1, there is a C*-cover $(\mathfrak{A},\iota)$ such that $\mathcal{S}(\mathfrak{A},\iota)=\mathfrak{F}$. Note that $\mathfrak{F}$ is the largest closed subset of $\widehat{\mathrm{C}^*_{max}(\mathcal{A})}$ containing a dense subset of equivalence classes of finite-dimensional irreducible $*$-representations. By Theorem 3.1.3, $(\mathfrak{A},\iota)$ is the largest RFD C*-cover up to equivalence. By Theorem 3.2.2, we have that $(\mathfrak{A},\iota)\sim(\mathfrak{R}(\mathcal{A}),\mu_r)$. Whence, $\mathcal{S}(\mathfrak{R}(\mathcal{A}),\mu_r)=\mathfrak{F}$ by Proposition 3.1.2.

(ii): Let π be a finite-dimensional irreducible $*$-representation of $\mathfrak{J}$. As $\mathfrak{J}$ is an ideal of $\mathrm{C}^*_{max}(\mathcal{A})$, we may extend π to a finite-dimensional irreducible $*$-representation of $\mathrm{C}^*_{max}(\mathcal{A})$, still denoted π. As $\pi\mid_{\mathfrak{J}}$ is non-zero, we see that $[\pi]\notin\mathcal{S}(\mathfrak{R}(\mathcal{A}),\mu_r)$, a contradiction by (i).

(iii): This is obvious by (i). □

Using the language of Section 3.1, we may view Theorem 3.2.4 (iii) as the topological counterpart to the order-theoretic statement of Theorem 3.2.2. We highlight an additional consequence that allows for a clear identification of the RFD-maximal C*-cover:

Corollary 3.2.5. *Let* $\mathcal{A}$ *be an RFD operator algebra and* $(\mathfrak{A},\iota)$ *be a* C**-cover for* $\mathcal{A}$*. Then, the following statements are equivalent.*

(i) *We have that* $(\mathfrak{A},\iota)\sim(\mathfrak{R}(\mathcal{A}),\mu_r)$*.*

(ii) *We have that* $(\mathfrak{A},\iota)$ *is RFD and* $[\sigma]\in\mathcal{S}(\mathfrak{A},\iota)$ *whenever* $[\sigma]\in\widehat{\mathrm{C}^*_{max}(\mathcal{A})}$ *is an equivalence class consisting of finite-dimensional* $*$*-representations.*

Proof. (i)⇒(ii): This follows by Theorem 3.2.4 (i).

(ii)⇒(i): Since $\mathfrak{A}$ is an RFD C*-algebra, we have that $(\mathfrak{A},\iota)\preceq(\mathfrak{R}(\mathcal{A}),\mu_r)$. By Proposition 3.1.2, we have that $\mathcal{S}(\mathfrak{A},\iota)\subset\mathcal{S}(\mathfrak{R}(\mathcal{A}),\mu_r)$. By Theorem 3.1.1 and Theorem 3.2.4 (i), we see that $\mathcal{S}(\mathfrak{R}(\mathcal{A}),\mu_r)\subset\mathcal{S}(\mathfrak{A},\iota)$. Hence, $(\mathfrak{A},\iota)\sim(\mathfrak{R}(\mathcal{A}),\mu_r)$ by Proposition 3.1.2. □

3.2.2 Concrete Characterization

In this subsection, we exhibit an explicit identification of the RFD-maximal C*-cover. The maximal C*-cover can be constructed in a concrete way [13, Proposition 2.4.2] and this is a common method of proving existence. We show that the RFD-maximal C*-cover can be constructed through analogous methods.

Theorem 3.2.6. *Let $\mathcal{A}$ be an RFD operator algebra. Then, there is an RFD* C^**-cover* $(\mathfrak{Q}, \iota)$ *of $\mathcal{A}$ with the following properties.*

(i) *For any representation $\rho : \mathcal{A} \to B(\mathcal{H}_\rho)$ on a finite-dimensional Hilbert space, there is a unique $*$-representation $\theta : \mathfrak{Q} \to B(\mathcal{H}_\rho)$ such that $\theta \circ \iota = \rho$.*

(ii) *The* C^**-cover* $(\mathfrak{Q}, \iota)$ *is minimal among all* C^**-covers of $\mathcal{A}$ which satisfy* (i)*.*

(iii) *We have that* $(\mathfrak{Q}, \iota) \sim (\mathfrak{R}(\mathcal{A}), \mu_r)$*.*

Proof. (i): Let $\mathcal{F}$ be the set of representations $\{\rho : \mathcal{A} \to \mathbb{M}_r \mid r \in \mathbb{N}\}$. As $\mathcal{A}$ is RFD, the map $\iota = \bigoplus_{\rho \in \mathcal{F}} \rho$ defines a completely isometric representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H} = \bigoplus_{\rho\in\mathcal{F}} \mathbb{C}^{r_\rho}$. Letting $\mathfrak{Q} = \mathrm{C}^*(\iota(\mathcal{A}))$, we see that $\mathfrak{Q}$ is an RFD C*-algebra and that $(\mathfrak{Q}, \iota)$ is a C*-cover of $\mathcal{A}$.

Let $\rho : \mathcal{A} \to B(\mathcal{H}_\rho)$ denote a representation of $\mathcal{A}$ on a finite-dimensional Hilbert space. Then there is some $r_\chi \in \mathbb{N}$ and a unitary operator $U : \mathbb{C}^{r_\chi} \to \mathcal{H}_\rho$ such that $\chi = U^*\rho(\cdot)U \in \mathcal{F}$. By construction of $\mathfrak{Q}$, we obtain a $*$-representation $\pi : \mathfrak{Q} \to \mathbb{M}_{r_\chi}$ defined by

$$\pi(T) = P_{\mathbb{C}^{r_\chi}} T \mid_{\mathbb{C}^{r_\chi}} .$$

Then, this dictates that

$$(\pi \circ \iota)(a) = \chi(a) = U^*\rho(a)U, \qquad a \in \mathcal{A}.$$

Taking $\theta = U\pi(\cdot)U^* : \mathfrak{Q} \to B(\mathcal{H}_\rho)$, we see θ is a finite-dimensional $*$-representation such that $\theta \circ \iota = \rho$.

For uniqueness of the map θ, suppose that θ and τ are $*$-representations of $\mathfrak{Q}$ such that $\theta \circ \iota = \rho = \tau \circ \iota$. As τ and θ are $*$-representations which agree on $\iota(\mathcal{A})$, we infer that τ and θ agree on the linear span of words in $\iota(\mathcal{A}) \cup \iota(\mathcal{A})^*$. This latter set is dense within $\mathfrak{Q}$ and so $\theta = \tau$.

(ii): Suppose that $(\mathfrak{A}, j)$ is a C*-cover of $\mathcal{A}$ such that every representation of $\mathcal{A}$ on a finite-dimensional Hilbert space lifts to $\mathfrak{A}$. For each $\rho \in \mathcal{F}$, we obtain a $*$-representation $\theta_\rho : \mathfrak{A} \to B(\mathcal{H}_\rho)$ such that $\theta_\rho \circ j = \rho$. In particular, we have that

$$\iota = \bigoplus_{\rho\in\mathcal{F}} \rho = \bigoplus_{\rho\in\mathcal{F}} \theta_\rho \circ j.$$

Then, $\Theta = \bigoplus_{\rho\in\mathcal{F}} \theta_\rho$ is a $*$-representation of $\mathfrak{A}$ such that $\Theta \circ j = \iota$. We see that

$$\Theta(\mathfrak{A}) = \Theta(\mathrm{C}^*(j(\mathcal{A}))) = \mathrm{C}^*((\Theta \circ j)(\mathcal{A})) = \mathrm{C}^*(\iota(\mathcal{A})) = \mathfrak{Q}$$

and so $(\mathfrak{Q}, \iota) \preceq (\mathfrak{A}, j)$ as desired.

(iii): Since $\mathfrak{Q}$ is an RFD C*-algebra, we show that the conditions in Corollary 3.2.5 hold. To this end, let $\pi : \mathrm{C}^*_{max}(\mathcal{A}) \to B(\mathcal{H}_\pi)$ be a finite-dimensional irreducible $*$-representation. Then $\pi \circ \mu$ is a representation of $\mathcal{A}$ on a finite-dimensional Hilbert space. Hence, there exists a $*$-representation $\theta : \mathfrak{Q} \to B(\mathcal{H}_\pi)$ such that $\theta \circ \iota = \pi \circ \mu$. So, $\theta \circ q_{\mathfrak{Q}}$ is a $*$-representation of $\mathrm{C}^*_{max}(\mathcal{A})$ satisfying

$$\theta \circ q_{\mathfrak{Q}} \circ \mu = \theta \circ \iota = \pi \circ \mu.$$

We may conclude that $\theta \circ q_{\mathfrak{Q}} = \pi$ and so $[\pi] \in \mathcal{S}(\mathfrak{Q}, \iota)$ as desired. □

Remark 3.2.7. If $\mathcal{A}$ is a unital RFD operator algebra, then a slight modification to the proof of Theorem 3.2.6 shows that the embedding ι can taken to be unital. In turn, the universal property of Theorem 3.2.6 is equivalent to the following: for any unital completely contractive representation $\rho : \mathcal{A} \to B(\mathcal{H}_\rho)$ on a finite-dimensional Hilbert space, there is a unique (unital) $*$-representation $\theta : \mathfrak{Q} \to B(\mathcal{H}_\rho)$ such that $\theta \circ \iota = \rho$.

3.2.3 Preservation of Algebraic Properties

We conclude this section by supplying a few instances in which the RFD-maximal C*-cover is similar to the traditional maximal C*-cover.

The maximal C*-cover of an operator algebra is known to preserve certain algebraic constructions. For example, see [12, Proposition 2.2], [13, 2.4.3], [24, Theorem 5.2], [45, Theorem 4.1]. Here, we show that the RFD-maximal C*-cover preserves a few of the same algebraic constructions (Theorems 3.2.10 and 3.2.11). Due to Example 5, this allows us to construct additional counterexamples to the conjecture of Clouâtre and Ramsey. First we address a minor, albeit worthwhile, point.

Proposition 3.2.8. *Let $\mathcal{A}$ be a non-unital RFD operator algebra and let $\widetilde{\mathcal{A}}$ denote the unitization of $\mathcal{A}$. Let $(\mathfrak{R}(\widetilde{\mathcal{A}}), \mu_{r_1})$ be the RFD-maximal* C*-*cover of $\widetilde{\mathcal{A}}$ and let $(\mathfrak{R}(\mathcal{A}), \mu_r)$ be the RFD-maximal* C*-*cover of $\mathcal{A}$. If $\iota = \mu_{r_1}|_{\mathcal{A}}$ and $\mathfrak{A} = \mathrm{C}^*(\iota(\mathcal{A}))$, then we have that $(\mathfrak{A}, \iota) \sim (\mathfrak{R}(\mathcal{A}), \mu_r)$.*

Proof. First note that $\mathfrak{A}$ is an RFD C*-algebra. Let $\rho : \mathcal{A} \to B(\mathcal{H})$ be a representation of $\mathcal{A}$ on a finite-dimensional Hilbert space. Let $\rho^+ : \widetilde{\mathcal{A}} \to B(\mathcal{H})$ be the unital representation which extends ρ. By Theorem 3.2.6, there is a $*$-representation $\theta : \mathfrak{R}(\widetilde{\mathcal{A}}) \to B(\mathcal{H})$ such that $\theta \circ \mu_{r_1} = \rho^+$. Then $\sigma := \theta|_{\mathfrak{A}}$ is a $*$-representation of $\mathfrak{A}$ such that $\sigma \circ \iota = \rho$. So $(\mathfrak{A}, \iota) \sim (\mathfrak{R}(\mathcal{A}), \mu_r)$ by Theorem 3.2.6. □

We proceed with a technical fact which will aid in our derivations for the remainder of this subsection. The following roughly states that if all finite-dimensional irreducible $*$-representations of an *isomorphic* copy of the maximal C*-cover factor through a fixed RFD C*-cover, then this RFD C*-cover is isomorphic to the RFD-maximal C*-cover.

Lemma 3.2.9. *Let $\mathcal{A}$ be an RFD operator algebra. Suppose $(\mathfrak{R}, \hat{\mu_r})$ and $(\mathfrak{M}, \hat{\mu})$ are* C*-*covers of $\mathcal{A}$ where $(\mathfrak{R}, \hat{\mu_r}) \preceq (\mathfrak{M}, \hat{\mu})$ and $\mathfrak{M} \cong \mathrm{C}^*_{max}(\mathcal{A})$. Let $q : \mathfrak{M} \to \mathfrak{R}$ be the surjective $*$-representation satisfying $q \circ \hat{\mu} = \hat{\mu_r}$. If every finite-dimensional $*$-representation π of $\mathfrak{M}$ is of the form $\pi = \sigma \circ q$ for some $*$-representation σ of $\mathfrak{R}$, then $\mathfrak{R} \cong \mathfrak{R}(\mathcal{A})$.*

Proof. Let $\mathcal{F}_{max} \subset \widehat{\mathrm{C}^*_{max}(\mathcal{A})}$ and $\mathcal{F}_{\mathfrak{M}} \subset \widehat{\mathfrak{M}}$ denote the unitary equivalence classes of finite-dimensional $*$-representations in the respective spectra. Let $\chi : \mathrm{C}^*_{max}(\mathcal{A}) \to \mathfrak{M}$ be a $*$-isomorphism. Then the mapping

$$\widehat{\mathfrak{M}} \to \widehat{\mathrm{C}^*_{max}(\mathcal{A})}, \qquad [\beta] \mapsto [\beta \circ \chi],$$

is a homeomorphism which maps $\mathcal{F}_{\mathfrak{M}}$ bijectively onto $\mathcal{F}_{max}$. Let $\mathcal{F}'_{\mathfrak{M}}$ be a set of irreducible $*$-representations of $\mathfrak{M}$ such that $\mathcal{F}_{\mathfrak{M}} = \{[\beta] : \beta \in \mathcal{F}'_{\mathfrak{M}}\}$. Then $\mathcal{F}'_{max} = \{\beta \circ \chi : \beta \in \mathcal{F}'_{\mathfrak{M}}\}$ is a complete system of representatives for $\mathcal{F}_{max}$.

For each $\beta \in \mathcal{F}'_{\mathfrak{M}}$, there is an irreducible $*$-representation $\beta_{\mathfrak{R}}$ of $\mathfrak{R}$ satisfying $\beta = \beta_{\mathfrak{R}} \circ q$. Conversely, if σ is a finite-dimensional irreducible $*$-representation of $\mathfrak{R}$, then $\sigma \circ q$ is unitarily equivalent to an element of $\mathcal{F}'_{\mathfrak{M}}$. Whence, σ is unitarily equivalent to $\beta_{\mathfrak{R}}$ for some $\beta \in \mathcal{F}'_{\mathfrak{M}}$. So $\{[\beta_{\mathfrak{R}}] \in \widehat{\mathfrak{R}} : \beta \in \mathcal{F}'_{\mathfrak{M}}\}$ is the collection of all unitary equivalence classes of finite-dimensional irreducible $*$-representations of $\mathfrak{R}$. As $\mathfrak{R}$ is an RFD C*-algebra,

$$\mathfrak{R} \cong \left(\bigoplus_{\beta \in \mathcal{F}'_{\mathfrak{M}}} \beta_{\mathfrak{R}} \right) (\mathfrak{R}) = \left(\bigoplus_{\beta \in \mathcal{F}'_{\mathfrak{M}}} \beta \right) (\mathfrak{M}).$$

For each $\beta \in \mathcal{F}'_{\mathfrak{M}}$, we have that $[\beta \circ \chi] \in \mathcal{F}_{max}$. Conversely, if σ is a finite-dimensional irreducible $*$-representation of $\mathrm{C}^*_{max}(\mathcal{A})$, then σ is unitarily equivalent to $\beta \circ \chi$ for some $\beta \in \mathcal{F}'_{\mathfrak{M}}$. By Theorem 2.1.2 (ii), the collection of unitary equivalence classes of irreducible finite-dimensional $*$-representations in $\widehat{\mathfrak{R}(\mathcal{A})}$ is dense. By Theorem 2.1.1 (iii), we see that the homeomorphism $\mathcal{S}(\mathfrak{R}(\mathcal{A}), \mu_r) \to \widehat{\mathfrak{R}(\mathcal{A})}$ maps equivalence classes of finite-dimensional $*$-representations bijectively onto themselves. In turn, we may infer that

$$\mathfrak{R}(\mathcal{A}) \cong \left(\bigoplus_{\theta \in \mathcal{F}'_{max}} \theta \right) (\mathrm{C}^*_{max}(\mathcal{A})).$$

Then,

$$\left(\bigoplus_{\theta \in \mathcal{F}'_{max}} \theta \right) (\mathrm{C}^*_{max}(\mathcal{A})) = \left(\bigoplus_{\beta \in \mathcal{F}'_{\mathfrak{M}}} \beta \circ \chi \right) (\mathrm{C}^*_{max}(\mathcal{A})) = \left(\bigoplus_{\beta \in \mathcal{F}'_{\mathfrak{M}}} \beta \right) (\mathfrak{M}) \cong \mathfrak{R}.$$

□

Our first consequence will pertain to countable direct sums of operator algebras. For unital operator algebras $\mathcal{A}_n, n \in \mathbb{N}$, define

$$\mathcal{A} = \bigoplus_{n=1}^{\infty} \mathcal{A}_n = \left\{ (a_n) \in \prod_{n=1}^{\infty} \mathcal{A}_n : \lim_{n\to\infty} \|a_n\| = 0 \right\}.$$

In [24, Theorem 5.2], it was shown that

$$\mathrm{C}^*_{max}(\mathcal{A}) \cong \bigoplus_{n=1}^{\infty} \mathrm{C}^*_{max}(\mathcal{A}_n).$$

We show the RFD-maximal C*-cover shares this property.

Theorem 3.2.10. *For each $n \in \mathbb{N}$, let $\mathcal{A}_n$ be a unital RFD operator algebra and let $\mathcal{A} = \bigoplus_{n=1}^{\infty} \mathcal{A}_n$. Then, $\mathfrak{R}(\mathcal{A}) \cong \bigoplus_{n=1}^{\infty} \mathfrak{R}(\mathcal{A}_n)$.*

Proof. For each $n \in \mathbb{N}$, let μ_n and μ_{r_n} denote the completely isometric representations of $\mathcal{A}_n$ into $\mathrm{C}^*_{max}(\mathcal{A}_n)$ and $\mathfrak{R}(\mathcal{A}_n)$, respectively. Moreover, let $q_n : \mathrm{C}^*_{max}(\mathcal{A}_n) \to \mathfrak{R}(\mathcal{A}_n)$ denote the surjective $*$-representation satisfying $q_n \circ \mu_n = \mu_{r_n}$ and

$$\mathfrak{R} = \bigoplus_{n=1}^{\infty} \mathfrak{R}(\mathcal{A}_n), \qquad \mathfrak{M} = \bigoplus_{n=1}^{\infty} \mathrm{C}^*_{max}(\mathcal{A}_n).$$

For $n \in \mathbb{N}$, let $\iota_n : \mathrm{C}^*_{max}(\mathcal{A}_n) \to \mathfrak{M}$ denote the obvious embedding. We verify the conditions of Lemma 3.2.9.

First note that $\mathfrak{R}$ is an RFD C*-algebra. Define completely isometric representations of $\mathcal{A}$ by

$$\hat{\mu_r} : (a_n)_{n\geq 1} \mapsto (\mu_{r_n}(a_n))_{n\geq 1}, \qquad \hat{\mu} : (a_n)_{n\geq 1} \mapsto (\mu_n(a_n))_{n\geq 1}.$$

We show that $\mathrm{C}^*(\hat{\mu}(\mathcal{A})) = \mathfrak{M}$. It is clear that $\mathrm{C}^*(\hat{\mu}(\mathcal{A})) \subset \mathfrak{M}$. For the reverse inclusion, note that

$$\iota_n(\mathrm{C}^*_{max}(\mathcal{A}_n)) \subset \mathrm{C}^*(\hat{\mu}(\mathcal{A})), \qquad n \in \mathbb{N}.$$

Upon taking finite sums,

$$\bigoplus_{n=1}^{m} \mathrm{C}^*_{max}(\mathcal{A}_n) \oplus 0 \subset \mathrm{C}^*(\hat{\mu}(\mathcal{A})), \qquad m \in \mathbb{N}.$$

For $t = (t_n)_{n\geq 1} \in \mathfrak{M}$, we let

$$s^{(m)} = (t_1, t_2, \ldots, t_m, 0, 0, \ldots) \in \mathrm{C}^*(\hat{\mu}(\mathcal{A})), \qquad m \in \mathbb{N}.$$

As $\|t_n\|$ tends to 0, it follows that $s^{(m)}$ converges to t in the norm topology of $\mathfrak{M}$. Whence, we have that $t \in \mathrm{C}^*(\hat{\mu}(\mathcal{A}))$. Therefore $\mathfrak{M} = \mathrm{C}^*(\hat{\mu}(\mathcal{A}))$. Similarly, we may establish that $\mathfrak{R} = \mathrm{C}^*(\hat{\mu_r}(\mathcal{A}))$.

Define a $*$-representation $Q : \mathfrak{M} \to \mathfrak{R}$ by $Q((t_n)) = (q_n(t_n))$. Note that $Q \circ \hat{\mu} = \hat{\mu_r}$. In particular, Q is surjective as

$$Q(\mathfrak{M}) = Q(\mathrm{C}^*(\hat{\mu}(\mathcal{A}))) = \mathrm{C}^*(Q \circ \hat{\mu}(\mathcal{A})) = \mathrm{C}^*(\hat{\mu_r}(\mathcal{A})) = \mathfrak{R}.$$

Therefore $(\mathfrak{R}, \hat{\mu_r})$ and $(\mathfrak{M}, \hat{\mu})$ are C*-covers for $\mathcal{A}$ such that $(\mathfrak{R}, \hat{\mu_r}) \preceq (\mathfrak{M}, \hat{\mu})$.

Let $\pi : \mathfrak{M} \to B(\mathcal{H}_\pi)$ be an irreducible finite-dimensional $*$-representation. Let $n \in \mathbb{N}$ and note that the C*-algebra $\iota_n(\mathrm{C}^*_{max}(\mathcal{A}_n))$ is an ideal of $\mathfrak{M}$. Hence, the representation $\pi \mid_{\iota_n(\mathrm{C}^*_{max}(\mathcal{A}_n))}$ is either identically zero or irreducible (Subsection 2.1.1). In particular, $\pi \circ \iota_n$ is either identically zero or an irreducible $*$-representation of $\mathrm{C}^*_{max}(\mathcal{A}_n)$. In the latter case, Theorem 3.2.4 yields that $\pi \circ \iota_n = \sigma_n \circ q_n$ where σ_n is a finite-dimensional irreducible $*$-representation of $\mathfrak{R}(\mathcal{A}_n)$. In the former case, one can take σ_n to be the zero representation in order to establish that $\pi \circ \iota_n = \sigma_n \circ q_n$.

For each $n \in \mathbb{N}$, let $\mathfrak{J}_n = \ker q_n \subset \mathrm{C}^*_{max}(\mathcal{A}_n)$. Then, $\mathfrak{J} = \bigoplus_{n=1}^{\infty} \mathfrak{J}_n = \ker Q$ is a closed two-sided ideal of $\mathfrak{M}$. Let $t = (t_n)_{n \geq 1} \in \mathfrak{J}$ and

$$s^{(m)} = (t_1, t_2, \ldots, t_m, 0, 0, \ldots), \qquad m \in \mathbb{N}.$$

Then

$$\begin{aligned} \pi(t) &= \lim_{m\to\infty} \pi(s^{(m)}) \\ &= \lim_{m\to\infty} \sum_{n=1}^{m} (\pi \circ \iota_n)(t_n) \\ &= \lim_{m\to\infty} \sum_{n=1}^{m} (\sigma_n \circ q_n)(t_n) = 0 \end{aligned}$$

as $t_n \in \mathfrak{J}_n = \ker q_n$ for each $n \in \mathbb{N}$. So π vanishes on $\mathfrak{J}$. Whence, $\pi = \sigma \circ Q$ where σ is a finite-dimensional $*$-representation of $\mathfrak{R}$. By [24, Theorem 5.2], we have that $\mathrm{C}^*_{max}(\mathcal{A}) \cong \mathfrak{M}$. Hence, Lemma 3.2.9 yields that $\mathfrak{R} \cong \mathfrak{R}(\mathcal{A})$. $\square$

Next, we provide a similar statement for free products of operator algebras. Given unital operator algebras $\mathcal{A}$ and $\mathcal{B}$, the *free product* of $\mathcal{A}$ and $\mathcal{B}$, denoted $\mathcal{A} * \mathcal{B}$, is the unique operator algebra satisfying the following universal property:

(i) there exist unital completely isometric representations $\iota : \mathcal{A} \to \mathcal{A} * \mathcal{B}$ and $j : \mathcal{B} \to \mathcal{A} * \mathcal{B}$ such that $\iota(\mathcal{A}), j(\mathcal{B})$ generate $\mathcal{A} * \mathcal{B}$ and if $\pi : \mathcal{A} \to \mathcal{C}$ and $\theta : \mathcal{B} \to \mathcal{C}$ are representations into a common operator algebra $\mathcal{C}$, then there exists a unique representation $\pi * \theta : \mathcal{A} * \mathcal{B} \to \mathcal{C}$ such that $(\pi * \theta) \circ \iota = \pi$ and $(\pi * \theta) \circ j = \theta$.

In the self-adjoint setting, this coincides with the usual notion of free products for C*-algebras. In [14, Theorem 4.1], it was shown that such an object exists for any pair of unital operator algebras. Moreover, the norm of $X \in \mathbb{M}_n(\mathcal{A} * \mathcal{B})$ may be defined by

$$\|X\| = \inf\{\|X_1\|\|X_2\| \ldots \|X_k\|\} \tag{3.2}$$

where the infimum is over all expressions $X = X_1 X_2 \ldots X_k$ where X_1 is an $n \times m_1$ matrix in $\iota(\mathcal{A})$, X_2 is an $m_1 \times m_2$ matrix in $j(\mathcal{B})$, X_3 is an $m_2 \times m_3$ matrix in $\iota(\mathcal{A})$, etc.

Theorem 3.2.11. *Let $\mathcal{A}, \mathcal{B}$ be unital RFD operator algebras. Then,* $\mathfrak{R}(\mathcal{A}*\mathcal{B}) \cong \mathfrak{R}(\mathcal{A})*\mathfrak{R}(\mathcal{B})$.

Proof. Let $(\mathrm{C}^*_{max}(\mathcal{A}), \mu_{\mathcal{A}})$ and $(\mathfrak{R}(\mathcal{A}), \mu_{r_{\mathcal{A}}})$ denote the maximal and RFD-maximal C*-covers of $\mathcal{A}$, respectively. Let $q_{\mathcal{A}} : \mathrm{C}^*_{max}(\mathcal{A}) \to \mathfrak{R}(\mathcal{A})$ be the surjective unital $*$-representation satisfying $q_{\mathcal{A}} \circ \mu_{\mathcal{A}} = \mu_{r_{\mathcal{A}}}$. Let $r_{\mathcal{A}} : \mathfrak{R}(\mathcal{A}) \to \mathfrak{R}(\mathcal{A}) * \mathfrak{R}(\mathcal{B})$ and $m_{\mathcal{A}} : \mathrm{C}^*_{max}(\mathcal{A}) \to \mathrm{C}^*_{max}(\mathcal{A}) * \mathrm{C}^*_{max}(\mathcal{B})$ be the usual unital isometric $*$-representations. Let $\iota_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} * \mathcal{B}$ be the unital completely isometric map given in (i). Define representations of $\mathcal{B}$ and $*$-representations of $\mathrm{C}^*_{max}(\mathcal{B})$ and $\mathfrak{R}(\mathcal{B})$, denoted $\mu_{\mathcal{B}}, \mu_{r_{\mathcal{B}}}, q_{\mathcal{B}}, r_{\mathcal{B}}, m_{\mathcal{B}}$ and $\iota_{\mathcal{B}}$, in an analogous way.

We check the conditions of Lemma 3.2.9. First note that $\mathfrak{R}(\mathcal{A}) * \mathfrak{R}(\mathcal{B})$ is an RFD C*-algebra by [46, Theorem 3.2] and that $\mathrm{C}^*_{max}(\mathcal{A}*\mathcal{B}) \cong \mathrm{C}^*_{max}(\mathcal{A}) * \mathrm{C}^*_{max}(\mathcal{B})$ by [12, Proposition 2.2]. Take

$$q = (r_{\mathcal{A}} \circ q_{\mathcal{A}}) * (r_{\mathcal{B}} \circ q_{\mathcal{B}}) : \mathrm{C}^*_{max}(\mathcal{A}) * \mathrm{C}^*_{max}(\mathcal{B}) \to \mathfrak{R}(\mathcal{A}) * \mathfrak{R}(\mathcal{B}).$$

Then q is a surjective unital $*$-representation. Indeed, as $q_{\mathcal{A}}$ and $q_{\mathcal{B}}$ are surjective, we have that

$$r_{\mathcal{A}}(\mathfrak{R}(\mathcal{A})) \cup r_{\mathcal{B}}(\mathfrak{R}(\mathcal{B})) \subset q(\mathrm{C}^*_{max}(\mathcal{A}) * \mathrm{C}^*_{max}(\mathcal{B})).$$

The former set generates $\mathfrak{R}(\mathcal{A}) * \mathfrak{R}(\mathcal{B})$ and so q is surjective.

Define a representation

$$\mu_* = (m_{\mathcal{A}} \circ \mu_{\mathcal{A}}) * (m_{\mathcal{B}} \circ \mu_{\mathcal{B}}) : \mathcal{A} * \mathcal{B} \to \mathrm{C}^*_{max}(\mathcal{A}) * \mathrm{C}^*_{max}(\mathcal{B}).$$

Note that $\mu_* \circ \iota_{\mathcal{A}} = m_{\mathcal{A}} \circ \mu_{\mathcal{A}}$ and $\mu_* \circ \iota_{\mathcal{B}} = m_{\mathcal{B}} \circ \mu_{\mathcal{B}}$ are completely isometric. By [42, Proposition 4.3], we have that $\mu_*(\mathcal{A} * \mathcal{B}) \cong (m_{\mathcal{A}} \circ \mu_{\mathcal{A}})(\mathcal{A}) * (m_{\mathcal{B}} \circ \mu_{\mathcal{B}})(\mathcal{B})$. Thus, for $X \in \mathbb{M}_r(\mathcal{A} * \mathcal{B})$, we have that

$$\begin{aligned} \|\mu_*(X)\| &= \inf\{\|Y_1\| \dots \|Y_k\| \ : \ \mu_*(X) = Y_1 \dots Y_k\} \\ &= \inf\{\|X_1\| \dots \|X_m\| \ : \ X = X_1 \dots X_m\} = \|X\| \end{aligned}$$

where $Y_1 \in \mathbb{M}_{r,r_1}((m_{\mathcal{A}} \circ \mu_{\mathcal{A}})(\mathcal{A})), Y_2 \in \mathbb{M}_{r_1,r_2}((m_{\mathcal{B}} \circ \mu_{\mathcal{B}})(\mathcal{B})), \dots Y_k \in \mathbb{M}_{r_{k-1},r}((m_{\mathcal{A}} \circ \mu_{\mathcal{A}})(\mathcal{A}))$ and, likewise, we have that $X_1 \in \mathbb{M}_{r,r_1}(\mathcal{A}), X_2 \in \mathbb{M}_{r_1,r_2}(\mathcal{B}) \dots, X_m \in \mathbb{M}_{r_{m-1},r}(\mathcal{A})$. So, μ_* is completely isometric.

Similarly, the map

$$\mu_{r_*} = (r_{\mathcal{A}} \circ \mu_{r_{\mathcal{A}}}) * (r_{\mathcal{B}} \circ \mu_{r_{\mathcal{B}}}) : \mathcal{A} * \mathcal{B} \to \mathfrak{R}(\mathcal{A}) * \mathfrak{R}(\mathcal{B})$$

is also a completely isometric representation. Next, note that $\mathrm{C}^*(\mu_*(\mathcal{A} * \mathcal{B})) = \mathrm{C}^*_{max}(\mathcal{A}) * \mathrm{C}^*_{max}(\mathcal{B})$: Indeed, by definition of μ_*, we see that

$$\mathrm{C}^*(\mu_* \circ \iota_{\mathcal{A}}(\mathcal{A})) = \mathrm{C}^*(m_{\mathcal{A}} \circ \mu_{\mathcal{A}}(\mathcal{A})) = m_{\mathcal{A}}(\mathrm{C}^*_{max}(\mathcal{A})) \subset \mathrm{C}^*(\mu_*(\mathcal{A} * \mathcal{B})).$$

Similarly, we obtain that $m_{\mathcal{B}}(\mathrm{C}^*_{max}(\mathcal{B})) \subset \mathrm{C}^*(\mu_*(\mathcal{A} * \mathcal{B}))$. Therefore

$$\mathrm{C}^*(\mu_*(\mathcal{A} * \mathcal{B})) = \mathrm{C}^*_{max}(\mathcal{A}) * \mathrm{C}^*_{max}(\mathcal{B}).$$

Observe that

$$q \circ \mu_* \circ \iota_{\mathcal{A}} = q \circ m_{\mathcal{A}} \circ \mu_{\mathcal{A}} = r_{\mathcal{A}} \circ q_{\mathcal{A}} \circ \mu_{\mathcal{A}} = r_{\mathcal{A}} \circ \mu_{r_{\mathcal{A}}}.$$

Similarly, we obtain $q \circ \mu_* \circ \iota_{\mathcal{B}} = r_{\mathcal{B}} \circ \mu_{r_{\mathcal{B}}}$. By the universal property of the free product, we have that $q \circ \mu_* = \mu_{r_*}$. Whence,

$$\mathrm{C}^*(\mu_{r_*}(\mathcal{A} * \mathcal{B})) = q(\mathrm{C}^*(\mu_*(\mathcal{A} * \mathcal{B}))) = q(\mathrm{C}^*_{max}(\mathcal{A}) * \mathrm{C}^*_{max}(\mathcal{B})) = \mathfrak{R}(\mathcal{A}) * \mathfrak{R}(\mathcal{B}).$$

So $(\mathfrak{R}(\mathcal{A}) * \mathfrak{R}(\mathcal{B}), \mu_{r_*}) \preceq (\mathrm{C}^*_{max}(\mathcal{A}) * \mathrm{C}^*_{max}(\mathcal{B}), \mu_*)$.

Let $\pi : \mathrm{C}^*_{max}(\mathcal{A}) * \mathrm{C}^*_{max}(\mathcal{B}) \to B(\mathcal{H}_\pi)$ be a finite-dimensional irreducible $*$-representation. By considering $\sigma_{\mathcal{A}} := \pi \circ m_{\mathcal{A}}$ and $\sigma_{\mathcal{B}} := \pi \circ m_{\mathcal{B}}$, uniqueness in the universal property of the free product dictates that $\pi = \sigma_{\mathcal{A}} * \sigma_{\mathcal{B}}$. By Theorem 3.2.4 (i), there are $*$-representations $\hat{\sigma}_{\mathcal{A}} : \mathfrak{R}(\mathcal{A}) \to B(\mathcal{H}_\pi)$ and $\hat{\sigma}_{\mathcal{B}} : \mathfrak{R}(\mathcal{B}) \to B(\mathcal{H}_\pi)$ such that $\hat{\sigma}_{\mathcal{A}} \circ q_{\mathcal{A}} = \sigma_{\mathcal{A}}$ and $\hat{\sigma}_{\mathcal{B}} \circ q_{\mathcal{B}} = \sigma_{\mathcal{B}}$. Define a $*$-representation of $\mathfrak{R}(\mathcal{A}) * \mathfrak{R}(\mathcal{B})$ by $\Pi = \hat{\sigma}_{\mathcal{A}} * \hat{\sigma}_{\mathcal{B}}$. Observe that

$$\Pi \circ q \circ m_{\mathcal{A}} = \Pi \circ r_{\mathcal{A}} \circ q_{\mathcal{A}} = \hat{\sigma}_{\mathcal{A}} \circ q_{\mathcal{A}} = \sigma_{\mathcal{A}}$$

and similarly, $\Pi \circ q \circ m_{\mathcal{B}} = \sigma_{\mathcal{B}}$. So $\Pi \circ q = \pi$. Therefore, by Lemma 3.2.9, we obtain that $\mathfrak{R}(\mathcal{A} * \mathcal{B}) \cong \mathfrak{R}(\mathcal{A}) * \mathfrak{R}(\mathcal{B})$. □

3.3 Residually Finite-Dimensional Representations

In this section, we pursue a finer characterization of the RFD-maximal C*-cover. Indeed, Theorem 3.2.4 only characterized the RFD-maximal C*-cover up to a dense subset of representations. Now we uncover a wider class of representations by taking appropriate pointwise limits of representations. We can identify this wider class of $*$-representations with the so-called $*$-RFD representations of the operator algebra (Theorem 3.3.4). Alongside our analysis for $*$-RFD representations, we also consider a seemingly more general class of representations for $\mathcal{A}$, called RFD representations, and ask whether these two classes coincide.

Accordingly, the following two definitions are central to this section. A representation $\rho : \mathcal{A} \to B(\mathcal{H})$ is *residually finite-dimensional* (or RFD) if there is a net of representations $\rho_\lambda : \mathcal{A} \to B(\mathcal{H})$ such that $\mathrm{C}^*(\rho_\lambda(\mathcal{A}))\mathcal{H}$ is finite-dimensional for each λ and

$$\text{SOT}\lim_{\lambda} \rho_\lambda(a) = \rho(a), \qquad a \in \mathcal{A}.$$

We call the representation $*$-*residually finite-dimensional* (or $*$-RFD) if we further impose that

$$\text{SOT}\lim_{\lambda} \rho_\lambda(a)^* = \rho(a)^*, \qquad a \in \mathcal{A}.$$

In other words, ρ is RFD (respectively, $*$-RFD) if it is the point-strong limit (or point-strong$*$ limit) of certain finite-dimensional representations of the operator algebra. These two concepts were introduced in [21] using different terminology.

In the self-adjoint setting, RFD and $*$-RFD representations coincide and agree with the notion of a residually finite-dimensional $*$-representation (as introduced in Subsection 2.1.2). We warn the reader that these refer to three different notions. An RFD or $*$-RFD representation refers to a possibly non self-adjoint setting. On the other hand, an RFD $*$-representation refers to the self-adjoint setting.

First, we remark that residual finite-dimensionality of an operator algebra is equivalent to the algebra possessing either an RFD or $*$-RFD embedding:

Proposition 3.3.1. *Let $\mathcal{A}$ be an operator algebra. Then, the following statements are equivalent:*

(i) *$\mathcal{A}$ is RFD;*

(ii) *there exists a completely isometric $*$-RFD representation of $\mathcal{A}$;*

(iii) *there exists a completely isometric RFD representation of $\mathcal{A}$.*

Proof. (i)⇒(ii): Let $\iota : \mathcal{A} \to \prod_\lambda \mathbb{M}_{r_\lambda}$ be a completely isometric representation. For each λ, let $\gamma_\lambda : \mathrm{C}^*(\iota(\mathcal{A})) \to \mathbb{M}_{r_\lambda}$ denote the projection mapping. Then the representation $j := \bigoplus_\lambda \gamma_\lambda \circ \iota$ is completely isometric. Moreover, since $\gamma_\lambda \circ \iota$ is $*$-RFD for every λ, it follows that j is $*$-RFD by [21, Lemma 2.2].

(ii)⇒(iii): Obvious.

(iii)⇒(i): Let $\iota : \mathcal{A} \to B(\mathcal{H})$ be a completely isometric RFD representation. Then there is a net of representations $\pi_\lambda : \mathcal{A} \to B(\mathcal{H}), \lambda \in \Lambda$, such that $\mathrm{C}^*(\pi_\lambda(\mathcal{A}))\mathcal{H}$ is finite-dimensional and $\pi_\lambda(a)$ converges strongly to $\iota(a)$ for each $a \in \mathcal{A}$. For each λ, we may find a positive integer r_λ and a unitary operator $U_\lambda : \mathrm{C}^*(\pi_\lambda(\mathcal{A}))\mathcal{H} \to \mathbb{C}^{r_\lambda}$. Then $\bigoplus_\lambda U_\lambda \pi_\lambda(\cdot) U_\lambda^*$ is a completely isometric representation of $\mathcal{A}$. □

For the remainder of this section, we will be concerned with two C*-covers of an RFD operator algebra. This is similar to the concrete representation of the RFD-maximal C*-cover but we require extra machinery to account for set-theoretic technicalities. This process can be seen in the classical construction of the maximal C*-cover [13, Proposition 2.4.2]. Our notation will be consistent throughout.

Let $\mathcal{A}$ be an RFD operator algebra. Suppose that the cardinality of $\mathcal{A}$ is less than or equal to a cardinal κ satisfying $\kappa^{\aleph_0} = \kappa$. For each cardinal $\alpha \leq \kappa$, fix a Hilbert space $\mathcal{H}_\alpha$ of dimension α. In particular, $\mathcal{H}_n$ will denote an n-dimensional Hilbert space for $n \in \mathbb{N}$. Let $\mathcal{F}$ denote the set of all representations $\{\rho : \mathcal{A} \to B(\mathcal{H}_\alpha) \mid \alpha \leq \kappa, \ \rho \text{ RFD}\}$. Define a representation $\mu_{r_s} = \bigoplus_{\rho \in \mathcal{F}} \rho$. Necessarily, κ is infinite. In particular, $\mathcal{F}$ contains a representative from each unitary equivalence class consisting of representations of $\mathcal{A}$ which act on a finite-dimensional Hilbert space. Thus, as $\mathcal{A}$ is RFD, μ_{r_s} is completely isometric. Letting $\mathfrak{R}_s(\mathcal{A}) = \mathrm{C}^*(\mu_{r_s}(\mathcal{A}))$, we see that $(\mathfrak{R}_s(\mathcal{A}), \mu_{r_s})$ is a C*-cover of $\mathcal{A}$.

Similarly, let $\mathcal{G}$ denote the set of all representations $\{\rho : \mathcal{A} \to B(\mathcal{H}_\alpha) \mid \alpha \leq \kappa, \ \rho \ *\text{-RFD}\}$. Define a representation $\mu_{r_{*s}} = \bigoplus_{\rho \in \mathcal{G}} \rho$. Similarly, the map $\mu_{r_{*s}}$ is completely isometric. Letting $\mathfrak{R}_{*s}(\mathcal{A}) = \mathrm{C}^*(\mu_{r_{*s}}(\mathcal{A}))$, we have that $(\mathfrak{R}_{*s}(\mathcal{A}), \mu_{r_{*s}})$ is a C*-cover of $\mathcal{A}$.

It is not immediately clear whether either of the C*-covers, $(\mathfrak{R}_s(\mathcal{A}), \mu_{r_s})$ or $(\mathfrak{R}_{*s}(\mathcal{A}), \mu_{r_{*s}})$, are RFD. We will work towards showing that the latter is in fact RFD. Due to the succeeding Lemma 3.3.2, this will imply that the C*-cover $(\mathfrak{R}_{*s}(\mathcal{A}), \mu_{r_{*s}})$ is equivalent to the RFD-maximal C*-cover.

Lemma 3.3.2. *Let $\mathcal{A}$ be an RFD operator algebra and let $(\mathfrak{R}(\mathcal{A}), \mu_r)$ denote the RFD-maximal* C**-cover of $\mathcal{A}$. Then, we have that $(\mathfrak{R}(\mathcal{A}), \mu_r) \preceq (\mathfrak{R}_{*s}(\mathcal{A}), \mu_{r_{*s}}) \preceq (\mathfrak{R}_s(\mathcal{A}), \mu_{r_s})$.*

Proof. Note that the subsets $\mathcal{F}$ and $\mathcal{G}$ defined above satisfy $\mathcal{G} \subset \mathcal{F}$. Whence, there is a natural projection mapping $\Theta : \mathfrak{R}_s(\mathcal{A}) \to \mathfrak{R}_{*s}(\mathcal{A})$ such that $\Theta \circ \mu_{r_s} = \mu_{r_{*s}}$. So $(\mathfrak{R}_{*s}(\mathcal{A}), \mu_{r_{*s}}) \preceq (\mathfrak{R}_s(\mathcal{A}), \mu_{r_s})$.

To show that $(\mathfrak{R}(\mathcal{A}), \mu_r) \preceq (\mathfrak{R}_{*s}(\mathcal{A}), \mu_{r_{*s}})$, we use minimality of Theorem 3.2.6. Let $\rho : \mathcal{A} \to B(\mathcal{H}_\rho)$ be a representation of $\mathcal{A}$ on a finite-dimensional Hilbert space. Then, there is $n \in \mathbb{N}$ and a unitary operator $U_\rho : \mathcal{H}_n \to \mathcal{H}_\rho$ such that $\chi = U_\rho^* \rho(\cdot) U_\rho : \mathcal{A} \to B(\mathcal{H}_n)$. The construction of $\mathfrak{R}_{*s}(\mathcal{A})$ allows us to obtain a $*$-representation $\pi : \mathfrak{R}_{*s}(\mathcal{A}) \to B(\mathcal{H}_n)$ defined by $\pi(T) = P_{\mathcal{H}_n} T \mid_{\mathcal{H}_n}$. Define a $*$-representation $\theta : \mathfrak{R}_{*s}(\mathcal{A}) \to B(\mathcal{H}_\rho)$ by $\theta = U\pi(\cdot)U^*$. Then $\theta \circ \mu_{r_{*s}} = \rho$. Therefore, $(\mathfrak{R}(\mathcal{A}), \mu_r) \preceq (\mathfrak{R}_{*s}(\mathcal{A}), \mu_{r_{*s}})$. □

We now illustrate how the two new C*-covers are equipped with lifting properties similar to that exposed in Theorem 3.2.6 with the RFD-maximal C*-cover.

Proposition 3.3.3. *Let $\mathcal{A}$ be an RFD operator algebra and $\rho : \mathcal{A} \to B(\mathcal{H})$ be a representation. Then, the following statements hold:*

(i) *If ρ is RFD, then there exists a unique $*$-representation $\theta : \mathfrak{R}_s(\mathcal{A}) \to B(\mathcal{H})$ such that $\theta \circ \mu_{r_s} = \rho$;*

(ii) *If ρ is $*$-RFD, then there exists a unique $*$-representation $\theta : \mathfrak{R}_{*s}(\mathcal{A}) \to B(\mathcal{H})$ such that $\theta \circ \mu_{r_{*s}} = \rho$.*

Proof. We show (ii) with the proof of (i) being similar. First, assume that $\dim \mathcal{H} \leq \kappa$. It is easy to verify that $*$-RFD representations are stable under unitary equivalence. So we may find a unitary operator U such that $\chi = U^*\rho(\cdot)U \in \mathcal{G}$. Define $\pi : \mathfrak{R}_{*s}(\mathcal{A}) \to B(\mathcal{H}_\chi)$ by $\pi(T) = P_{\mathcal{H}_\chi} T \mid_{\mathcal{H}_\chi}$. By construction of $\mathfrak{R}_{*s}(\mathcal{A})$, we see that π is a $*$-representation. If we let $\theta = U\pi(\cdot)U^*$, then $\theta \circ \mu_{r_{*s}} = \rho$ as desired.

Now, suppose that $\dim \mathcal{H} > \kappa$. By [13, Proposition 2.4.2], we may express $\rho = \bigoplus_i \pi_i$ where each $\pi_i : \mathcal{A} \to B(\mathcal{K}_i)$ is a representation of $\mathcal{A}$ on a Hilbert space with dimension at most κ. Due to [21, Lemma 2.4], each π_i is also a $*$-RFD representation of $\mathcal{A}$ (or RFD for statement (i)). By the previous paragraph, for each i, we obtain a $*$-representation $\theta_i : \mathfrak{R}_{*s}(\mathcal{A}) \to B(\mathcal{K}_i)$ such that $\theta_i \circ \mu_{r_{*s}} = \pi_i$. Taking $\theta = \bigoplus_i \theta_i$, we see that $\theta \circ \mu_{r_{*s}} = \rho$ as desired. Uniqueness holds for the same reason as in Theorem 3.2.6. □

We remark that Proposition 3.3.3 makes no statement on the residually finite-dimensionality of the resulting $*$-representations. This will be clarified by showing that the lift of a $*$-RFD representation as in Proposition 3.3.3 is necessarily an RFD $*$-representation. However, we do not know of a corresponding statement about RFD representations. The following reconceptualizes [21, Theorem 3.3].

Theorem 3.3.4. *Let $\mathcal{A}$ be an operator algebra and let $(\mathrm{C}^*_{max}(\mathcal{A}), \mu)$ denote the maximal C^*-cover of $\mathcal{A}$. Let $\rho : \mathcal{A} \to B(\mathcal{H})$ be a representation and let $\theta : \mathrm{C}^*_{max}(\mathcal{A}) \to B(\mathcal{H})$ denote the $*$-representation satisfying $\theta \circ \mu = \rho$. Then, ρ is a $*$-RFD representation if and only if θ is an RFD $*$-representation.*

Proof. ($\Rightarrow$) Let $\rho_\lambda : \mathcal{A} \to B(\mathcal{H}), \lambda \in \Lambda$, be a net of representations such that $\mathrm{C}^*(\rho_\lambda(\mathcal{A}))\mathcal{H}$ is finite-dimensional and

$$*\text{SOT} \lim_\lambda \rho_\lambda(a) = \rho(a), \qquad a \in \mathcal{A}.$$

For each ρ_λ, we obtain a corresponding $*$-representation $\theta_\lambda : \mathrm{C}^*_{max}(\mathcal{A}) \to B(\mathcal{H})$ satisfying $\theta_\lambda \circ \mu = \rho_\lambda$. Note that

$$\theta_\lambda(\mathrm{C}^*_{max}(\mathcal{A}))\mathcal{H} = \mathrm{C}^*((\theta_\lambda \circ \mu)(\mathcal{A}))\mathcal{H} = \mathrm{C}^*(\rho_\lambda(\mathcal{A}))\mathcal{H}$$

is finite-dimensional for each λ. For any $w \in \mathrm{C}^*_{max}(\mathcal{A})$ which is a finite sum of words in $\mu(\mathcal{A}) \cup \mu(\mathcal{A})^*$, we obtain that

$$*\text{SOT} \lim_\lambda \theta_\lambda(w) = \theta(w)$$

as the adjoint, addition and multiplication are jointly $*$SOT-continuous over bounded sets. This set is dense in $\mathrm{C}^*_{max}(\mathcal{A})$ and so θ is an RFD $*$-representation.

($\Leftarrow$) Suppose that $\theta_\lambda, \lambda \in \Lambda$, is a net of $*$-representations such that $\theta_\lambda(\mathrm{C}^*_{max}(\mathcal{A}))\mathcal{H}$ is finite-dimensional and

$$\text{SOT}\lim_\lambda \theta_\lambda(t) = \theta(t), \qquad t \in \mathrm{C}^*_{max}(\mathcal{A}).$$

Then the net $(\theta_\lambda \circ \mu)_\lambda$ demonstrates that ρ is a $*$-RFD representation of $\mathcal{A}$. $\square$

As a result, we have that $(\mathfrak{R}_{*s}(\mathcal{A}), \mu_{r_s})$ is equivalent to the RFD-maximal C*-cover.

Corollary 3.3.5. *If $\mathcal{A}$ is an RFD operator algebra, then $(\mathfrak{R}(\mathcal{A}), \mu_r) \sim (\mathfrak{R}_{*s}(\mathcal{A}), \mu_{r_{*s}})$. In particular, for any $*$-RFD representation $\rho : \mathcal{A} \to B(\mathcal{H}_\rho)$, there is a unique $*$-representation $\theta : \mathfrak{R}(\mathcal{A}) \to B(\mathcal{H}_\rho)$ such that $\theta \circ \mu_r = \rho$. Furthermore, θ is an RFD $*$-representation.*

Proof. As $\mu_{r_{*s}} : \mathcal{A} \to B(\mathcal{H})$ is a direct sum of $*$-RFD representations, it follows that $\mu_{r_{*s}}$ is also a $*$-RFD representation [21, Lemma 2.2]. Let $q : \mathrm{C}^*_{max}(\mathcal{A}) \to \mathfrak{R}_{*s}(\mathcal{A})$ be the surjective $*$-representation satisfying $q \circ \mu = \mu_{r_{*s}}$. By Theorem 3.3.4, we have that q is an RFD $*$-representation.

Let $\theta_\lambda : \mathrm{C}^*_{max}(\mathcal{A}) \to B(\mathcal{H}), \lambda \in \Lambda$, be a net of $*$-representations such that $\theta_\lambda(\mathrm{C}^*_{max}(\mathcal{A}))\mathcal{H}$ is finite-dimensional for each λ and

$$\text{SOT}\lim_\lambda \theta_\lambda(t) = q(t), \qquad t \in \mathrm{C}^*_{max}(\mathcal{A}).$$

For each $\lambda \in \Lambda$, define a representation of $\mathcal{A}$ by $\rho_\lambda = \theta_\lambda \circ \mu$. By Theorem 3.3.4, ρ_λ is a $*$-RFD representation of $\mathcal{A}$. Hence, by Proposition 3.3.3 (ii), there is a $*$-representation $\widehat{\theta}_\lambda : \mathfrak{R}_{*s}(\mathcal{A}) \to B(\mathcal{H})$ such that $\widehat{\theta}_\lambda \circ \mu_{r_{*s}} = \rho_\lambda$. Observe that

$$\widehat{\theta}_\lambda \circ q \circ \mu = \widehat{\theta}_\lambda \circ \mu_{r_{*s}} = \rho_\lambda = \theta_\lambda \circ \mu.$$

Hence, $\widehat{\theta}_\lambda \circ q = \theta_\lambda$ and so

$$\text{SOT}\lim_\lambda (\widehat{\theta}_\lambda \circ q)(t) = q(t), \qquad t \in \mathrm{C}^*_{max}(\mathcal{A}).$$

Since $\widehat{\theta}_\lambda(\mathfrak{R}_{*s}(\mathcal{A}))\mathcal{H} = \theta_\lambda(\mathrm{C}^*_{max}(\mathcal{A}))\mathcal{H}$ is finite-dimensional for each $\lambda \in \Lambda$, we obtain that the identity representation of $\mathfrak{R}_{*s}(\mathcal{A})$ is an RFD $*$-representation. By Proposition 3.3.1 (or [46, Theorem 2.4]), we obtain that $\mathfrak{R}_{*s}(\mathcal{A})$ is an RFD C*-algebra. So $(\mathfrak{R}_{*s}(\mathcal{A}), \mu_{r_{*s}}) \preceq (\mathfrak{R}(\mathcal{A}), \mu_r)$ and Lemma 3.3.2 implies that the C*-covers are equivalent. The last two statements are direct consequences of Proposition 3.3.3 (ii) and Theorem 3.3.4. $\square$

The reader should exercise some care in utilizing Corollary 3.3.5. Indeed, let $\mathcal{A}$ be an RFD operator algebra and $(\mathfrak{A}, \iota)$ be a C*-cover such that $(\mathfrak{A}, \iota) \preceq (\mathfrak{R}(\mathcal{A}), \mu_r)$. Suppose ρ is a representation of $\mathcal{A}$ which lifts to a $*$-representation θ of $\mathfrak{A}$. Then, as $(\mathfrak{A}, \iota) \preceq (\mathfrak{R}(\mathcal{A}), \mu_r)$, we have that ρ also lifts to a $*$-representation of $\mathfrak{R}(\mathcal{A})$. As ρ lifts to a $*$-representation of the RFD C*-algebra $\mathfrak{R}(\mathcal{A})$, we have that ρ is a $*$-RFD representation. Moreover, the lift of ρ to $\mathfrak{R}(\mathcal{A})$ is an RFD $*$-representation. Nevertheless, it is not necessarily true that θ itself will be RFD.

Example 6. Let $\mathcal{H}$ be a separable, infinite-dimensional Hilbert space with orthonormal basis $\{e_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let P_n be the orthogonal projection onto the subspace spanned by $\{e_1, \ldots, e_n\}$. Let $\mathcal{A}$ denote the operator algebra consisting of triangular operators. That is, $T \in \mathcal{A}$ precisely when $P_n T P_n = T P_n$ for every $n \in \mathbb{N}$. It is known that $\mathcal{A}$ is an RFD operator algebra [23, Example 3.3] and, moreover, it is clear that $\mathfrak{K}(\mathcal{H}) \subset \mathrm{C}^*(\mathcal{A})$.

Define a representation $\pi_n : \mathcal{A} \to B(\mathcal{H})$ by $\pi_n(T) = TP_n$. We see that $\mathrm{C}^*(\pi_n(\mathcal{A}))\mathcal{H}$ is finite-dimensional and that

$$*\text{SOT} \lim_{n\to\infty} \pi_n(T) = T, \qquad T \in \mathcal{A},$$

as $(P_n)_{n\geq 1}$ converges strongly to the identity operator on $\mathcal{H}$. Therefore the identity representation of $\mathcal{A}$ is $*$-RFD.

On the other hand, the identity representation of $\mathcal{A}$ lifts to the identity representation of $\mathrm{C}^*(\mathcal{A})$, which is not an RFD $*$-representation. Indeed, otherwise $\mathrm{C}^*(\mathcal{A})$ is an RFD C*-algebra by Proposition 3.3.1. This is a contradiction because $\mathrm{C}^*(\mathcal{A})$ contains $\mathfrak{K}(\mathcal{H})$, which is not RFD.

For the remainder of this section, we would like to quantify the possible discrepancy between RFD and $*$-RFD representations. To this end, it will be worth elaborating on the example of Hartz [55].

Proposition 3.3.6. *Let $\mathcal{A}$ be an operator algebra. Then, $(\mathfrak{R}_s(\mathcal{A}), \mu_{r_s}) \sim (\mathrm{C}^*_{max}(\mathcal{A}), \mu)$ if and only if every representation of $\mathcal{A}$ is RFD.*

Proof. If all representations of $\mathcal{A}$ are RFD, then $(\mathfrak{R}_s(\mathcal{A}), \mu_{r_s})$ has the universal property of the maximal C*-cover by Proposition 3.3.3. So, $(\mathfrak{R}_s(\mathcal{A}), \mu_{r_s}) \sim (\mathrm{C}^*_{max}(\mathcal{A}), \mu)$.

Conversely, assume that $(\mathfrak{R}_s(\mathcal{A}), \mu_{r_s}) \sim (\mathrm{C}^*_{max}(\mathcal{A}), \mu)$ and that $\mathfrak{R}_s(\mathcal{A})$ is concretely represented on a Hilbert space $\mathcal{H}_s$. If $\rho : \mathcal{A} \to B(\mathcal{H})$ is a representation, then, by assumption, there is a $*$-representation $\theta : \mathfrak{R}_s(\mathcal{A}) \to B(\mathcal{H})$ such that $\theta \circ \mu_{r_s} = \rho$. Define two $*$-representations for $\mathfrak{R}_s(\mathcal{A})$ by $\Sigma = id^{(\kappa)}$ where κ is some cardinal and $\Theta = \Sigma \oplus \theta$.

We may assume that κ is suitably large so that Σ and Θ are approximately unitarily equivalent (Theorem 2.1.3). Thus, there is a net $U_\lambda : \mathcal{H}_s^{(\kappa)} \oplus \mathcal{H} \to \mathcal{H}_s^{(\kappa)}$ of unitary operators such that

$$\lim_\lambda \|U_\lambda^* \Sigma(t) U_\lambda - \Theta(t)\| = 0, \qquad t \in \mathfrak{R}_s(\mathcal{A}).$$

We will show that $\Theta \circ \mu_{r_s}$ is an RFD representation. To this end, note that $\Sigma \circ \mu_{r_s} = \mu_{r_s}^{(\kappa)}$ is an RFD representation by [21, Lemma 2.2]. So, there is a net of representations $\rho_\mu : \mathcal{A} \to B(\mathcal{H}_s^{(\kappa)})$ such that

$$\text{SOT} \lim_\mu \rho_\mu(a) = \mu_{r_s}^{(\kappa)}(a), \qquad a \in \mathcal{A}$$

and $\mathrm{C}^*(\rho_\mu(\mathcal{A}))\mathcal{H}_s^{(\kappa)}$ is finite-dimensional for each μ. Therefore, we have that

$$\text{SOT} \lim_\lambda \lim_\mu U_\lambda^* \rho_\mu(a) U_\lambda = (\Theta \circ \mu_{r_s})(a) = (\mu_{r_s}^{(\kappa)} \oplus \rho)(a), \qquad a \in \mathcal{A}.$$

Let $(\tau_\alpha)_\alpha$ be the subnet of $(U_\lambda^* \rho_\mu(\cdot) U_\lambda)_{\lambda,\mu}$ that corresponds to the above iterated limit (for example – see [61, pg. 69]). Then, we may conclude that $\mu_{r_s}^{(\kappa)} \oplus \rho$ is an RFD representation. By [21, Lemma 2.4], we have that ρ is an RFD representation. $\square$

Let μ denote Lebesgue measure on the unit circle and let $H^2(\mathbb{T}) \subset \mathrm{L}^2(\mathbb{T}, \mu)$ denote those functions whose negative Fourier coefficients vanish, i.e. the Hardy space on the unit circle. Accordingly, we let $P : \mathrm{L}^2(\mathbb{T}, \mu) \to H^2(\mathbb{T})$ denote the corresponding orthogonal projection. Moreover, given $h \in \mathrm{C}(\mathbb{T})$, the operator $T_h \in B(H^2(\mathbb{T}))$ defined by $T_h f = P(hf)$ is the *Toeplitz operator* with symbol h.

Example 7. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} f & 0 \\ h & \overline{g} \end{bmatrix} \in \mathbb{M}_2(\mathrm{C}(\mathbb{T})) \ : \ f, g \in \mathrm{A}(\mathbb{D}), h \in \mathrm{C}(\mathbb{T}) \right\}.$$

Then, $\mathcal{B}$ is an RFD operator algebra and the representation

$$\rho : \mathcal{B} \to B(H^2(\mathbb{T}) \oplus H^2(\mathbb{T})), \qquad \begin{bmatrix} f & 0 \\ h & \overline{g} \end{bmatrix} \mapsto \begin{bmatrix} T_f & 0 \\ T_h & T_{\overline{g}} \end{bmatrix}$$

fails to be RFD [55, Theorem 1.3]. The main technical reason why ρ fails to be RFD is that

$$\rho\left(\begin{bmatrix} z & 0 \\ 0 & \overline{z} \end{bmatrix} \right) = \begin{bmatrix} T_z & 0 \\ 0 & T_{\overline{z}} \end{bmatrix}$$

cannot be approximated by unitaries in the SOT (see [55, Discussion after Theorem 4.3] for more information).

Consequently, by Proposition 3.3.6, we have that $(\mathfrak{R}_s(\mathcal{A}), \mu_{r_s})$ is not equivalent to the maximal C*-cover. In particular, the maximal C*-cover for $\mathcal{B}$ is not RFD.

Remark 3.3.7. Let $\mathcal{A}$ be an RFD operator algebra. Suppose we construct a C*-cover $(\mathfrak{R}_w(\mathcal{A}), \mu_{r_w})$ that is built from point-WOT limits of finite-dimensional representations (in an analogous way to the construction of $(\mathfrak{R}_s(\mathcal{A}), \mu_{r_s})$ and $(\mathfrak{R}_{*s}(\mathcal{A}), \mu_{r_{*s}})$). Then, by [55, Theorem 2.1] and an argument analogous to Proposition 3.3.3, we can conclude that $(\mathfrak{R}_w(\mathcal{A}), \mu_{r_w})$ is always equivalent to the maximal C*-cover.

In spite of Example 7, we remark that Hartz's example does not demonstrate whether RFD and $*$-RFD representations are actually distinct. We summarize when this does hold.

Theorem 3.3.8. *Let $\mathcal{A}$ be an RFD operator algebra and let $(\mathfrak{R}(\mathcal{A}), \mu_r)$ denote the RFD-maximal C*-cover of $\mathcal{A}$. Then, the following statements are equivalent.*

(i) *For any completely isometric RFD representation $\iota : \mathcal{A} \to B(\mathcal{H})$, we have that $(\mathrm{C}^*(\iota(\mathcal{A})), \iota) \preceq (\mathfrak{R}(\mathcal{A}), \mu_r)$.*

(ii) *We have that $(\mathfrak{R}_s(\mathcal{A}), \mu_{r_s}) \sim (\mathfrak{R}(\mathcal{A}), \mu_r)$.*

(iii) *Every RFD representation of $\mathcal{A}$ is $*$-RFD.*

(iv) *For any RFD representation $\rho : \mathcal{A} \to B(\mathcal{H})$, there is an RFD $*$-representation $\theta : \mathrm{C}^*_{max}(\mathcal{A}) \to B(\mathcal{H})$ satisfying $\theta \circ \mu = \rho$.*

Proof. (i)⇒(ii): The representation μ_{r_s} is RFD by [21, Lemma 2.2]. So $(\mathfrak{R}_s(\mathcal{A}), \mu_{r_s}) \preceq (\mathfrak{R}(\mathcal{A}), \mu_r)$. By Lemma 3.3.2, we infer that (ii) holds.

(ii)⇒(iii): Let $\rho : \mathcal{A} \to B(\mathcal{H})$ be an RFD representation. By Proposition 3.3.3, there exists a $*$-representation $\theta : \mathfrak{R}_s(\mathcal{A}) \to B(\mathcal{H})$ such that $\theta \circ \mu_{r_s} = \rho$. The assumption implies that $\mathfrak{R}_s(\mathcal{A})$ is an RFD C*-algebra. By [46, Theorem 2.4], θ is a $*$-RFD representation. Let $q : \mathrm{C}^*_{max}(\mathcal{A}) \to \mathfrak{R}_s(\mathcal{A})$ be the surjective $*$-representation satisfying $q \circ \mu = \mu_{r_s}$. Then $\theta \circ q$ is also a $*$-RFD representation. Hence, by Theorem 3.3.4, we obtain that

$$\theta \circ q \circ \mu = \theta \circ \mu_{r_s} = \rho$$

is a $*$-RFD representation of $\mathcal{A}$.

(iii)⇒(iv): This is Theorem 3.3.4.

(iv)⇒(i): Let $\iota : \mathcal{A} \to B(\mathcal{H})$ be a completely isometric RFD representation. By assumption, there is an RFD $*$-representation $\theta : \mathrm{C}^*_{max}(\mathcal{A}) \to B(\mathcal{H})$ satisfying $\theta \circ \mu = \iota$. By Theorem 3.3.4, ι is in fact a $*$-RFD representation. Whence, by Corollary 3.3.5, there is a $*$-representation $\beta : \mathfrak{R}(\mathcal{A}) \to B(\mathcal{H})$ such that $\beta \circ \mu_r = \iota$. We have that

$$\beta(\mathfrak{R}(\mathcal{A})) = \beta(\mathrm{C}^*(\mu_r(\mathcal{A}))) = \mathrm{C}^*(\iota(\mathcal{A}))$$

and so $(\mathrm{C}^*(\iota(\mathcal{A})), \iota) \preceq (\mathfrak{R}(\mathcal{A}), \mu_r)$. □

3.4 Hadwin Liftings for Operator Algebras

To close this chapter, we obtain a non self-adjoint version of Hadwin's characterization of separable RFD C*-algebras [50]. We will recount the details of Hadwin's work here.

Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for ℓ^2. For each $n \in \mathbb{N}$, let P_n be the orthogonal projection onto the linear span of $\{e_1, \dots, e_n\}$ and let $\mathcal{M}_n = P_n B(\ell^2) P_n$. Let $\mathfrak{B}$ be the C*-subalgebra of $\prod_{n=1}^{\infty} \mathcal{M}_n$ consisting of those operators $(T_n)_{n\geq 1}$ for which there is $T \in B(\ell^2)$ such that

$$*\text{SOT} \lim_{n\to\infty} T_n = T.$$

Define a $*$-representation

$$\pi : \mathfrak{B} \to B(\ell^2), \qquad (T_n)_{n\geq 1} \mapsto *\text{SOT} \lim_{n\to\infty} T_n.$$

It is routine to check that π is a surjective unital $*$-representation [50, Lemma 1].

Now, take an operator algebra $\mathcal{A}$. We say that a representation $\rho : \mathcal{A} \to B(\ell^2)$ is *$*$-liftable in the sense of Hadwin* if there is a representation $\tau : \mathcal{A} \to \mathfrak{B}$ such that $\pi \circ \tau = \rho$. Recall that $\widetilde{\mathfrak{A}}$ denotes the unitization of a C*-algebra $\mathfrak{A}$. Hadwin's result is then stated as follows [50, Theorem 11].

Theorem 3.4.1. *Let $\mathfrak{A}$ be a separable C*-algebra. Then, $\mathfrak{A}$ is an RFD C*-algebra if and only if every unital $*$-representation $\sigma : \widetilde{\mathfrak{A}} \to B(\ell^2)$ is $*$-liftable in the sense of Hadwin.*

We present a non-self adjoint version to Theorem 3.4.1.

Theorem 3.4.2. *Let $\mathcal{A}$ be a separable operator algebra. Then, $C^*_{max}(\mathcal{A})$ is an RFD C*-algebra if and only if every unital representation $\rho : \widetilde{\mathcal{A}} \to B(\ell^2)$ is $*$-liftable in the sense of Hadwin.*

Proof. ($\Rightarrow$) If $C^*_{max}(\mathcal{A})$ is an RFD C*-algebra, then $C^*_{max}(\widetilde{\mathcal{A}})$ is also RFD by [13, 2.4.3]. Hence, we have that $(C^*_{max}(\widetilde{\mathcal{A}}), \mu_1)$ is the RFD-maximal C*-cover of $\widetilde{\mathcal{A}}$.

Let $\rho : \widetilde{\mathcal{A}} \to B(\ell^2)$ be a unital representation. Then, there is a $*$-representation $\theta : C^*_{max}(\widetilde{\mathcal{A}}) \to B(\ell^2)$ such that $\theta \circ \mu_1 = \rho$. As in Remark 3.2.7, μ_1 may taken to be unital. Hence, θ is also unital. As $C^*_{max}(\widetilde{\mathcal{A}})$ is a separable RFD C*-algebra, Theorem 3.4.1 yields that there is a unital $*$-representation $\sigma : C^*_{max}(\widetilde{\mathcal{A}}) \to \mathfrak{B}$ such that $\pi \circ \sigma = \theta$. Then we see that

$$\pi \circ \sigma \circ \mu_1 = \theta \circ \mu_1 = \rho.$$

Taking $\tau = \sigma \circ \mu_1$ shows that ρ is $*$-liftable in the sense of Hadwin.

($\Leftarrow$) By [13, 2.4.3] and Theorem 3.3.4 (or, alternatively, [21, Theorem 3.3]), it suffices to show that every representation of $\widetilde{\mathcal{A}}$ is $*$-RFD. We first show that every unital representation $\rho : \widetilde{\mathcal{A}} \to B(\ell^2)$ is $*$-RFD. As ρ is $*$-liftable in the sense of Hadwin, there is a representation $\tau : \widetilde{\mathcal{A}} \to \mathfrak{B}$ such that $\pi \circ \tau = \rho$. By definition of $\mathfrak{B}$, we see that for each $n \in \mathbb{N}$, there is a representation $\tau_n : \widetilde{\mathcal{A}} \to \mathcal{M}_n$ and $\tau(a) = (\tau_n(a))_{n\geq 1}$. Note that $\mathrm{C}^*(\tau_n(\widetilde{\mathcal{A}}))\ell^2 \subset \mathcal{M}_n$ is finite-dimensional for each $n \in \mathbb{N}$. Also,

$$\rho(a) = (\pi \circ \tau)(a) = *\text{SOT} \lim_{n\to\infty} \tau_n(a), \qquad a \in \widetilde{\mathcal{A}},$$

and so ρ is a $*$-RFD representation.

Now let $\rho : \widetilde{\mathcal{A}} \to B(\mathcal{H})$ be an arbitrary representation. If ρ is non-unital, then we may express $\rho = \widehat{\rho} \oplus 0$ where $\widehat{\rho}$ is a unital representation of $\widetilde{\mathcal{A}}$. We have that ρ is $*$-RFD if and only if $\widehat{\rho}$ is $*$-RFD. Whence, we may assume that ρ is unital. Further, we may assume the non-trivial case of when $\mathcal{H}$ is infinite-dimensional. For each $h \in \mathcal{H}$, let $\mathcal{H}_h = \overline{\text{span}}\{\mathrm{C}^*(\rho(\widetilde{\mathcal{A}}))h\}$. Note that $\mathcal{H}_h$ is a separable Hilbert space which is reducing for $\rho(\widetilde{\mathcal{A}})$. Then, for each $h \in \mathcal{H}$, $\rho\,|_{\mathcal{H}_h}$ is a representation of $\widetilde{\mathcal{A}}$ on a separable Hilbert space, and ρ is unitarily equivalent to $\bigoplus_{h\in\mathcal{H}} \rho\,|_{\mathcal{H}_h}$. Hence, $\rho\,|_{\mathcal{H}_h}$ is unitarily equivalent to a unital representation of $\widetilde{\mathcal{A}}$ on ℓ^2 and so $\rho\,|_{\mathcal{H}_h}$ is $*$-RFD. Therefore ρ is a $*$-RFD representation by [21, Lemma 2.2]. □

Chapter 4

An Approximate Unique Extension Property for Completely Positive Maps

Motivated by classical results of Korovkin and Šaškin, Arveson proposed a non-commutative approximate rigidity property that is encoded in terms of boundary representations [8]. Initially, Korovkin had proven a rigidity result concerning sequences of positive maps with domain and codomain equal to $C[0,1]$ [68]. Then, Šaškin had demonstrated that Korovkin's result is a consequence to a particular unital operator space $\mathcal{M} \subset C[0,1]$ satisfying the property that all irreducible $*$-representations of $C[0,1]$ are boundary representations for $\mathcal{M}$ (Section 2.3). Explicitly, Šaškin had showed that, given a compact metric space X, every irreducible $*$-representation for $C(X)$ is a boundary representation for a unital operator space $\mathcal{M} \subset C(X)$ if and only if, whenever $\psi_n : C(X) \to C(X)$ is a sequence of unital positive maps such that $\|\psi_n(g) - g\| \to 0$ for every $g \in \mathcal{M}$, then we have that $\|\psi_n(f) - f\| \to 0$ for every $f \in C(X)$. Decades later, Arveson provided a non-commutative adaptation of this phenomenon [8, Theorem 2.1]. Given a separable unital operator space $\mathcal{M} \subset B(\mathcal{H})$, Arveson showed that every $*$-representation of $C^*(\mathcal{M})$ has the unique extension property with respect to $\mathcal{M}$ if and only if, whenever $\omega : C^*(\mathcal{M}) \to B(\mathcal{K})$ is a faithful $*$-representation and $\psi_n : B(\mathcal{K}) \to B(\mathcal{K})$ is a sequence of unital completely positive maps satisfying

$$\lim_{n\to\infty} \|\psi_n(\omega(m)) - \omega(m)\| = 0, \qquad m \in \mathcal{M},$$

then we necessarily have that

$$\lim_{n\to\infty} \|\psi_n(\omega(t)) - \omega(t)\| = 0, \qquad t \in C^*(\mathcal{M}).$$

A unital operator space $\mathcal{M}$ with the property that all $*$-representations of $C^*(\mathcal{M})$ have the unique extension property is said to be *hyperrigid*. Unfortunately, Arveson was unable to frame hyperrigidity in terms of irreducible $*$-representations. Accordingly, Arveson conjectured the following.

Arveson's hyperrigidity conjecture: Let $\mathcal{M} \subset B(\mathcal{H})$ be a separable unital operator space. If every irreducible $*$-representation for $\mathrm{C}^*(\mathcal{M})$ is a boundary representation, then $\mathcal{M}$ is hyperrigid.

Significant effort has gone towards attempting to provide a positive answer to Arveson's hyperrigidity conjecture [8],[19],[20],[22],[38],[60],[62],[64],[67],[82], although Bilich and Dor-On have just recently announced a counterexample in the case when $\mathrm{C}^*(\mathcal{M})$ is postliminal [10]. Nevertheless, Arveson's conjecture is true in a wide variety of examples and is a natural conclusion to expect. Moreover, the conjecture still remains open in the commutative case. Indeed, the reader should note that it is not clear whether Šaškin's result coincides with the commutative case of Arveson's characterization of hyperrigidity due to the difference in ranges for the completely positive maps. We will return to this aspect of Arveson's conjecture in Chapter 5.

For the non-trivial direction of Arveson's conjecture, suppose that every irreducible $*$-representation of $\mathrm{C}^*(\mathcal{M})$ is a boundary representation for $\mathcal{M}$. It is well-known that the unique extension property is preserved under direct sums (Lemma 2.4.2). So, Voiculescu's Theorem (Theorem 2.1.4) implies that every $*$-representation of $\mathrm{C}^*(\mathcal{M})$ that acts on a separable Hilbert space is approximately unitarily equivalent to a $*$-representation with the unique extension property. If Arveson's conjecture were true for $\mathcal{M}$, then these $*$-representations would necessarily have the unique extension property. Accordingly, the main purpose of our work is to study those $*$-representations that are approximately unitarily equivalent to a $*$-representation with the unique extension property.

In Section 4.1, we record preliminary information regarding those $*$-representations of $\mathrm{C}^*(\mathcal{M})$ that are approximately unitarily equivalent to a $*$-representation with the unique extension property. In particular, we address whether *irreducible* $*$-representations that are approximately unitarily equivalent to a boundary representation are necessarily boundary representations. In the literature, there are a couple of examples demonstrating that this question has a negative answer (Example 9 and surrounding discussion) and we record the details of one such example.

In Section 4.2, we study a variation on the unique extension property. Our first main development (Theorem 4.2.6) shows that this variation is inherited among those $*$-representations that are approximately unitarily equivalent to a $*$-representation with the unique extension property.

Theorem D. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space and $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a $*$-representation that is approximately unitarily equivalent to a $*$-representation with the unique extension property. If $\psi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a unital completely positive map satisfying $\psi|_{\mathcal{M}} = \pi|_{\mathcal{M}}$, then there is a net of unitary operators $u_\beta \in B(\mathcal{H}_\pi)$ such that*

$$\text{WOT}\lim_{\beta} u_\beta^* \psi(t) u_\beta = \pi(t), \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

We say that a $*$-representation of $\mathrm{C}^*(\mathcal{M})$ that satisfies the conclusion of Theorem D has the *approximate unique extension property* with respect to $\mathcal{M}$. Throughout Section

4.2, we study the class of $*$-representations with the approximate unique extension property and develop a collection of structural properties that are satisfied by these $*$-representations (Propositions 4.2.5, 4.2.7, and Corollary 4.2.8). Consequently, this allows us to relate our work to Arveson's hyperrigidity conjecture (Theorems 4.2.11 and 4.3.8).

Theorem E. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a separable unital operator space. Consider the following statements:*

(i) *The operator space $\mathcal{M}$ is hyperrigid.*

(ii) *Every irreducible $*$-representation of $\mathrm{C}^*(\mathcal{M})$ is a boundary representation.*

(iii) *Every irreducible $*$-representation of $\mathrm{C}^*(\mathcal{M})$ is approximately unitarily equivalent to a boundary representation.*

(iv) *Every $*$-representation of $\mathrm{C}^*(\mathcal{M})$ has the approximate unique extension property.*

Then, we have that (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) *and* (iii)$\not\Rightarrow$(ii). *Moreover, if $\mathrm{C}^*(\mathcal{M})$ is postliminal, then we have that* (iv)$\Rightarrow$(ii).

Finally, we unveil the main result of this chapter, which imposes a structural constraint on the space of completely positive extensions of $*$-representations with the approximate unique extension property (Theorems 4.3.1 and 4.3.4).

Theorem F. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space and $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a $*$-representation that possesses the approximate unique extension property. Suppose that $\psi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a unital completely positive map satisfying $\psi|_{\mathcal{M}} = \pi|_{\mathcal{M}}$. Then, the following statements hold.*

(i) *There is a homomorphic conditional expectation $\Gamma_\psi : \mathrm{C}^*(\mathrm{Im}\psi) \to \mathrm{Im}\pi$ satisfying $\Gamma_\psi \circ \psi = \pi$.*

(ii) *The conditional expectation Γ_ψ has the unique extension property with respect to $\mathrm{Im}\psi$.*

(iii) *We have that $\ker \Gamma_\psi$ is the smallest closed two-sided ideal of $\mathrm{C}^*(\mathrm{Im}\psi)$ that contains $\mathrm{Im}(\psi - \pi)$. In particular, $\psi = \pi$ if and only if Γ_ψ is isometric.*

As a consequence, Theorem F demonstrates that there is a split exact sequence of C^*-algebras given by

$$0 \longrightarrow \langle\langle \mathrm{Im}(\psi - \pi) \rangle\rangle \longrightarrow \mathrm{C}^*(\mathrm{Im}\psi) \longrightarrow \mathrm{Im}\pi \longrightarrow 0.$$

where the second term is the smallest closed two-sided ideal of $\mathrm{C}^*(\mathrm{Im}\psi)$ that contains $\mathrm{Im}(\psi - \pi)$. The subsequent results in Section 4.3 are consequences to the above developments. In particular, this allows us to apply our machinery to obtain new results on the structure of completely positive extensions of arbitrary isometric $*$-representations of the C^*-envelope (Corollary 4.3.2).

In Section 4.4, we consider two classes of concrete examples where our results may apply. The first is a reformulation of Šaškin's Theorem (Proposition 4.4.1) that addresses commutativity and the approximate unique extension property. For the second application, note that Voiculescu's Theorem and Arveson's program guarantee that the collection of $*$-representations that possess the approximate unique extension property is always quite large. Therefore, one would expect that requiring a single $*$-representation to have the approximate unique extension property to be a weak assumption. Nevertheless, we apply our machinery to operator systems arising from quotient modules of multipliers on the Drury-Arveson space and find that hyperrigidity is encoded in a single $*$-representation possessing the approximate unique extension property (Theorem 4.4.2). Due to work of Kennedy and Shalit [62], our development allows us to reframe Arveson's essential normality conjecture.

4.1 Approximate Unitary Equivalence and the Unique Extension Property

We start this section by showing that the collection of $*$-representations with the unique extension property is rarely closed under approximate unitary equivalence.

Proposition 4.1.1. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space and assume that $(\mathrm{C}^*(\mathcal{M}), id_{\mathcal{M}})$ is equivalent to the C^*-envelope of $\mathcal{M}$. Then, the following statements are equivalent.*

(i) *The operator space $\mathcal{M}$ is hyperrigid.*

(ii) *Whenever $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a $*$-representation that is approximately unitarily equivalent to a $*$-representation with the unique extension property, then π has the unique extension property.*

Proof. It is trivial that (i)⇒(ii). For the converse, let $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a $*$-representation. We show that π has the unique extension property. To this end, note that π may be assumed to be injective. Indeed, as $\mathcal{M}$ generates its C^*-envelope, by Theorem 2.4.3 (i), there is an isometric $*$-representation $\theta : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\theta)$ that has the unique extension property with respect to $\mathcal{M}$. Thus, by Lemma 2.4.2 and Theorem 2.4.3 (i), we have that π has the unique extension property if and only if $\theta \oplus \pi$ has the unique extension property. So, we may assume that π is injective. In which case, define an infinite cardinal $\kappa = \max\{\aleph_0, \dim \mathcal{H}_\pi\}$. By considering $\pi^{(\kappa)}$ if necessary, we may additionally assume that $\mathrm{rank}(\pi(t)) = \kappa$ for each non-zero $t \in \mathrm{C}^*(\mathcal{M})$.

Now, by Theorem 2.4.3 (i), there is an injective $*$-representation σ of $\mathrm{C}^*(\mathcal{M})$ that possesses the unique extension property with respect to $\mathcal{M}$. Again, we may assume that $\mathrm{rank}(\sigma(t)) = \kappa$ for every non-zero $t \in \mathrm{C}^*(\mathcal{M})$. By Theorem 2.1.3, we have that π and σ are approximately unitarily equivalent. Whence, π has the unique extension property with respect to $\mathcal{M}$ by assumption. □

Remark 4.1.2. Assume that $\mathrm{C}_e^*(\mathcal{M})$ is separable and postliminal. To study hyperrigidity, the reader may find it desirable to consider how the unique extension property behaves with respect to direct integrals as this machinery can be used to classify multiplicity-free $*$-representations up to unitary equivalence [9, Theorem 4.3.4]. However, due to [51, Corollary 6.3] and Lemma 2.4.2, the unique extension property is stable under direct integrals whenever it is stable under approximate unitary equivalence. We opt to study the latter, but remark that there has been some progress using direct integrals [8],[67].

Now, we discuss whether those *irreducible* $*$-representations with the unique extension property (i.e. boundary representations) are closed under approximate unitary equivalence. As the next two examples demonstrate, the answer to this question is more subtle than the previous one.

We first note that a variation of Proposition 4.1.1 fails even in the classical setting of function spaces. For this, recall that postliminal C^*-algebras satisfy the property that irreducible $*$-representations that possess the same kernel are necessarily unitarily equivalent [40, Corollary 4.1.10]. Thus, whenever the C^*-envelope is postliminal, the non-commutative Choquet boundary is automatically stable under approximate unitary equivalence.

Example 8. There is a compact metric space X and a unital subalgbera $\mathcal{A} \subset \mathrm{C}(X)$ that separates points in X with the property that the Choquet boundary is a proper subset of the Shilov boundary [76, pg. 42]. In other words, not every irreducible $*$-representation of $\mathrm{C}_e^*(\mathcal{A})$ is a boundary representation for $\mathcal{A}$ [47, Theorem II.11.3].

On the other hand, suppose that π is a boundary representation for $\mathcal{A}$. If σ is an irreducible $*$-representation of $\mathrm{C}_e^*(\mathcal{A})$ that is approximately unitarily equivalent to π, then π and σ are unitarily equivalent by the previous discussion. Whence, σ is also a boundary representation for $\mathcal{A}$. So, the boundary representations for $\mathcal{A}$ are stable under approximate unitary equivalence, yet not every irreducible $*$-representation of $\mathrm{C}_e^*(\mathcal{A})$ is a boundary representation for $\mathcal{A}$.

The above conclusion can be achieved for any other unital operator space $\mathcal{M}$ such that $\mathrm{C}_e^*(\mathcal{M})$ is postliminal and possesses an irreducible $*$-representation that is not a boundary representation. However, when we remove the postliminal assumption, the non-commutative Choquet boundary is not necessarily closed under approximate unitary equivalence. For this, we recall the details of an example due to Muhly-Solel [72, Continuation of Example 2.7].

Example 9. Let $\mathcal{O}_\infty$ denote the infinite Cuntz algebra with generators $(V_n)_{n\in\mathbb{N}}$, and consider the algebra $\mathcal{A}$ that is generated by $\{V_n : n \in \mathbb{N}\}$. As $\mathrm{C}^*(\mathcal{A}) \cong \mathcal{O}_\infty$ is simple [33, Theorem 1.12], by Voiculescu's Theorem (Theorem 2.1.3), we have that all irreducible $*$-representations of $\mathcal{O}_\infty$ are approximately unitarily equivalent.

However, up to unitary equivalence, there is a unique irreducible $*$-representation of $\mathcal{O}_\infty$ that is not a boundary representation for $\mathcal{A}$. Indeed, the $*$-representations of $\mathcal{O}_\infty$ are in one-to-one correspondence with sequences of isometries $(W_n)_{n\geq 0}$ satisfying $\sum_{n=1}^\infty W_n W_n^* \leq I$. An irreducible $*$-representation is a boundary representation for $\mathcal{A}$ if and only if the associated sequence of isometries satisfies $\sum_{n=1}^\infty W_n W_n^* = I$ (see [72] for details). By work of Popescu [78, Theorem 1.3], up to unitary equivalence, there is a unique such irreducible $*$-representation with $\sum_{n=1}^{\infty} W_n W_n^* \neq I$.

Within [37, Example 6.6.3], there is an example of another operator space where the class of boundary representations are not closed under approximate unitary equivalence. Therein, an uncountable class of irreducible $*$-representations were constructed where all but a finite number were boundary representations. Simultaneously, one can obtain that all irreducible $*$-representations of the C*-envelope are approximately unitarily equivalent for the same reasoning as in Example 9.

4.1.1 Asymptotics of the Unique Extension Property

The unique extension property enjoys a well-known characterization in the framework of pointwise convergent sequences. The separable version can be found within [67, Proposition 2.2]. We record the non-separable version, which is obtained by replacing sequences with nets.

Proposition 4.1.3. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space and let $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a $*$-representation. Then, the following statements are equivalent.*

(i) *The $*$-representation π has the unique extension property with respect to $\mathcal{M}$.*

(ii) *If $\psi_\alpha : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a net of unital completely positive maps such that $\psi_\alpha(m) \to \pi(m)$ in the weak operator topology for every $m \in \mathcal{M}$, then $\psi_\alpha(t) \to \pi(t)$ in the weak operator topology for every $t \in \mathrm{C}^*(\mathcal{M})$.*

(iii) *If $\psi_\alpha : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a net of unital completely positive maps such that $\psi_\alpha(m) \to \pi(m)$ in the strong operator topology for every $m \in \mathcal{M}$, then $\psi_\alpha(t) \to \pi(t)$ in the strong operator topology for every $t \in \mathrm{C}^*(\mathcal{M})$.*

We present a reinterpretation of Proposition 4.1.3 that allows us to connect the unique extension property with a notion of approximate unitary equivalence for unital completely positive extensions. We will return to this philosophy in the next section.

Proposition 4.1.4. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space and $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a $*$-representation. Then, the following statements are equivalent.*

(i) *The $*$-representation π has the unique extension property with respect to $\mathcal{M}$.*

(ii) *If $\psi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a unital completely positive map satisfying $\psi|_{\mathcal{M}} = \pi|_{\mathcal{M}}$, then there exists a net of unitary operators $w_\beta \in B(\mathcal{H}_\pi)$ such that*

$$\textsc{sot}\lim_{\beta} w_\beta^* \pi(t) w_\beta = \psi(t), \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

Proof. (i) $\Rightarrow$ (ii) : In this case, $\psi = \pi$ and so we may take w_β to be the identity operator on $\mathcal{H}_\pi$.

(ii) $\Rightarrow$ (i) : Let $\psi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a unital completely positive map satisfying $\psi \mid_{\mathcal{M}} = \pi \mid_{\mathcal{M}}$ and let $w_\beta \in B(\mathcal{H}_\pi)$ be a net of unitary operators such that $w_\beta^* \pi(t) w_\beta \to \psi(t)$ in the strong operator topology for each $t \in \mathrm{C}^*(\mathcal{M})$. As multiplication over bounded subsets is continuous in the strong operator topology, and as the net $(w_\beta \pi(\cdot) w_\beta)_\beta$ consists of $*$-representations, we may conclude that $\psi(st) = \psi(s)\psi(t)$ for all $s, t \in \mathrm{C}^*(\mathcal{M})$. Since $\psi \mid_{\mathcal{M}} = \pi \mid_{\mathcal{M}}$, we then obtain that $\psi = \pi$. □

As we shall see in Section 4.2, the equivalence of Proposition 4.1.4 is no longer true if one were to interchange the roles of the maps ψ and π in statement (ii).

4.2 Approximate Unique Extension Property

In accordance with studying the approximate unitary equivalence class of a $*$-representation possessing the unique extension property, we focus our attention on a variation of the unique extension property.

Definition 4.2.1. Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space and let $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a $*$-representation. We say that π has the *approximate unique extension property* with respect to $\mathcal{M}$ if, whenever $\psi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a unital completely positive map satisfying $\psi\,|_{\mathcal{M}} = \pi\,|_{\mathcal{M}}$, then there is a net of unitaries $u_\beta \in B(\mathcal{H}_\pi)$ such that

$$\text{WOT}\lim_{\beta} u_\beta^* \psi(t) u_\beta = \pi(t), \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

When there is no confusion, we will refrain from referencing the operator space and simply state that a $*$-representation possesses the approximate unique extension property. Moreover, when $\mathcal{M}$ and $\mathcal{H}_\pi$ are separable, then the net of unitaries may be replaced by a sequence.

We remind the reader that the approximate unique extension property differs from the characterization of the unique extension property that was seen within Proposition 4.1.4 (ii). For example, we will see that this follows from Theorem 4.2.6. Additionally, we remark that Definition 4.2.1 is somewhat reminiscent of different conditions seen in the literature but remains largely independent [35, Theorem 5.2],[52, Theorem 2.4].

Recall that boundary representations factor through the C*-envelope by Theorem 2.4.3 (ii). Since the approximate unique extension property appears to be a significant constraint on the space of completely positive extensions, one may wonder whether representation with the approximate unique extension property also factor through the C*-envelope. While the author is unaware of such a restriction, we start with a fact that is sufficient for our purposes.

Lemma 4.2.2. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space and $\mathfrak{A} \subset B(\mathcal{K})$ be a* C*-*algebra. Let $q : \mathrm{C}^*(\mathcal{M}) \to \mathfrak{A}$ be a surjective $*$-homomorphism and $\pi : \mathfrak{A} \to B(\mathcal{H}_\pi)$ be a $*$-representation. If $\pi \circ q$ possesses the approximate unique extension property with respect to $\mathcal{M}$, then π possesses the approximate unique extension property with respect to $q(\mathcal{M})$.*

Proof. Suppose that $\psi : \mathfrak{A} \to B(\mathcal{H}_\pi)$ is a unital completely positive map such that $\psi\,|_{q(\mathcal{M})} = \pi\,|_{q(\mathcal{M})}$. Then $\varphi := \psi \circ q$ is a unital completely positive map satisfying $\varphi\,|_{\mathcal{M}} = (\pi \circ q)\,|_{\mathcal{M}}$. By the assumption, there is a net of unitary operators $u_\beta \in B(\mathcal{H}_\pi)$ such that

$$\text{WOT}\lim_{\beta} u_\beta^* \varphi(t) u_\beta = (\pi \circ q)(t), \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

Thus,

$$\text{WOT}\lim_{\beta} u_\beta^* \psi(s) u_\beta = \pi(s), \qquad s \in \mathfrak{A}.$$

So π has the approximate unique extension property. $\square$

Given a unital completely isometric map $\iota : \mathcal{M} \to \mathcal{N}$ between unital operator spaces, there is a $*$-isomorphism $\theta : \mathrm{C}^*_e(\mathcal{M}) \to \mathrm{C}^*_e(\mathcal{N})$ satisfying $\theta |_{\mathcal{M}} = \iota$ [4, Theorem 2.2.5]. Thus, Lemma 4.2.2 implies that the $*$-representations of $\mathrm{C}^*_e(\mathcal{M})$ that possess the approximate unique extension property are invariant under completely isometric isomorphism.

Proposition 4.2.3. *Let $\mathcal{M} \subset B(\mathcal{H})$ and $\mathcal{N} \subset B(\mathcal{K})$ be unital operator spaces such that $(\mathrm{C}^*(\mathcal{M}), id_{\mathcal{M}})$ and $(\mathrm{C}^*(\mathcal{N}), id_{\mathcal{N}})$ are the C^*-envelope of $\mathcal{M}$ and $\mathcal{N}$, respectively. Suppose that $\iota : \mathcal{M} \to \mathcal{N}$ is a unital completely isometric linear map and let $\theta : \mathrm{C}^*(\mathcal{M}) \to \mathrm{C}^*(\mathcal{N})$ be the corresponding $*$-isomorphism that satisfies $\theta |_{\mathcal{M}} = \iota$. Then, $\pi : \mathrm{C}^*(\mathcal{N}) \to B(\mathcal{H}_\pi)$ possesses the approximate unique extension property with respect to $\mathcal{N}$ if and only if $\pi \circ \theta$ possesses the approximate unique extension property with respect to $\mathcal{M}$.*

Proof. If $\pi \circ \theta$ has the approximate unique extension property with respect to $\mathcal{M}$, then π has the approximate unique extension property with respect to $\mathcal{N}$ by Lemma 4.2.2.

Conversely, if $\pi = (\pi \circ \theta) \circ \theta^{-1}$ has the approximate unique extension property with respect to $\mathcal{N}$, then again $\pi \circ \theta$ has the approximate unique extension property with respect to $\mathcal{M}$ by Lemma 4.2.2. □

Additionally, we remark that the approximate unique extension property has a characterization that is reminiscent of Proposition 4.1.3.

Proposition 4.2.4. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space and let $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a $*$-representation. Then, the following statements are equivalent.*

(i) *The $*$-representation π possesses the approximate unique extension property.*

(ii) *If $\psi_\alpha : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a net of unital completely positive maps such that $\psi_\alpha(m) \to \pi(m)$ in the weak operator topology for every $m \in \mathcal{M}$, then there is a subnet $(\psi_{\alpha_\eta})_\eta$ of unital completely positive maps and a net of unitary operators $u_\beta \in B(\mathcal{H}_\pi)$ such that*

$$\mathrm{WOT}\lim_\beta \lim_\eta u_\beta^* \psi_{\alpha_\eta}(t) u_\beta = \pi(t), \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

Proof. (i)⇒(ii): Let $\psi_\alpha : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a net of unital completely positive maps such that $\psi_\alpha(m) \to \pi(m)$ in the weak operator topology for every $m \in \mathcal{M}$. By compactness (Theorem 2.2.5), we may obtain a subnet $(\psi_{\alpha_\eta})_\eta$ of unital completely positive maps that converges to a unital completely positive map $\varphi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ in the pointwise weak operator topology. Consequently, we have that $\varphi |_{\mathcal{M}} = \pi |_{\mathcal{M}}$.

Since π has the approximate unique extension property, there is a net of unitary operators $u_\beta \in B(\mathcal{H}_\pi)$ such that $u_\beta^* \varphi(t) u_\beta \to \pi(t)$ in the weak operator topology for every $t \in \mathrm{C}^*(\mathcal{M})$. So we conclude that

$$\mathrm{WOT}\lim_\beta \lim_\eta u_\beta^* \psi_{\alpha_\eta}(t) u_\beta = \mathrm{WOT}\lim_\beta u_\beta^* \varphi(t) u_\beta = \pi(t), \qquad t \in \mathrm{C}^*(\mathcal{M}),$$

as desired.

(ii)⇒(i): Trivial. □

We now work towards identifying a large class of $*$-representations that possess the approximate unique extension property. For this, we require that the approximate unique extension property be preserved under approximate unitary equivalence.

Proposition 4.2.5. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space and let $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a $*$-representation. Suppose that π possesses the approximate unique extension property. If $\sigma : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\sigma)$ is a $*$-representation that is approximately unitarily equivalent to π, then σ also possesses the approximate unique extension property.*

Proof. The approximate unique extension property is easily seen to be stable under unitary equivalence. Thus, since there is a unitary operator $U : \mathcal{H}_\sigma \to \mathcal{H}_\pi$, we may assume for simplicity that $\mathcal{H}_\sigma = \mathcal{H}_\pi$. In this case, there is a net of unitary operators $w_\gamma \in B(\mathcal{H}_\pi)$ such that

$$\lim_\gamma \|w_\gamma^* \sigma(t) w_\gamma - \pi(t)\| = 0, \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

In particular, we also have that

$$\lim_\gamma \|\sigma(t) - w_\gamma \pi(t) w_\gamma^*\| = 0, \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

For the remainder of the proof, all limits will be in the weak operator topology. Let $\psi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a unital completely positive map such that $\psi\mid_{\mathcal{M}} = \sigma\mid_{\mathcal{M}}$. By compactness, we may obtain a subnet $(w_{\gamma_\eta}^* \psi(\cdot) w_{\gamma_\eta})_\eta$ of unital completely positive maps and a unital completely positive map $\varphi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ such that

$$\lim_\eta w_{\gamma_\eta}^* \psi(t) w_{\gamma_\eta} = \varphi(t), \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

In particular,

$$\varphi(m) = \lim_\eta w_{\gamma_\eta}^* \psi(m) w_{\gamma_\eta} = \lim_\eta w_{\gamma_\eta}^* \sigma(m) w_{\gamma_\eta} = \pi(m), \qquad m \in \mathcal{M}.$$

As π has the approximate unique extension property, there is a net of unitaries $u_\beta \in B(\mathcal{H}_\pi)$ such that

$$\pi(t) = \lim_\beta u_\beta^* \varphi(t) u_\beta = \lim_\beta \lim_\eta u_\beta^* w_{\gamma_\eta}^* \psi(t) w_{\gamma_\eta} u_\beta, \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

Then we see that

$$\lim_\gamma \lim_\beta \lim_\eta w_\gamma u_\beta^* w_{\gamma_\eta}^* \psi(t) w_{\gamma_\eta} u_\beta w_\gamma^* = \lim_\gamma w_\gamma \pi(t) w_\gamma^* = \sigma(t), \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

Thus, by [61, pg. 69], there is a net of unitaries $v_\alpha := w_{\gamma_\alpha} u_{\beta_\alpha}^* w_{\gamma_{\eta_\alpha}}^*$ such that

$$\lim_\alpha v_\alpha^* \psi(t) v_\alpha = \sigma(t), \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

Therefore σ has the approximate unique extension property. □

This allows us to obtain a notable class of $*$-representations that possess the approximate unique extension property.

Theorem 4.2.6. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space. Suppose that $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a $*$-representation that is approximately unitarily equivalent to a $*$-representation with the unique extension property. Then, we have that π has the approximate unique extension property.*

Proof. By Proposition 4.2.5, it suffices to show that $*$-representations with the unique extension property have the approximate unique extension property. Thus, we assume that π has the unique extension property. In which case, if $\psi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a unital completely positive map satisfying $\psi\mid_{\mathcal{M}} = \pi\mid_{\mathcal{M}}$, then $\psi = \pi$ and so we may take u_β to be the identity operator. $\square$

Since the usual unique extension property is not preserved under approximate unitary equivalence (Example 9), by Theorem 4.2.6 we obtain that the approximate unique extension property is truly different from the usual unique extension property. Nevertheless, the approximate unique extension property still behaves well enough to form a well-behaved class of $*$-representations.

Proposition 4.2.7. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space. If $\{\pi_i : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_i) : i \in I\}$ is a collection of $*$-representations that possess the approximate unique extension property, then $\pi = \bigoplus_{i\in I} \pi_i$ also possesses the approximate unique extension property.*

Proof. Let $\psi : \mathrm{C}^*(\mathcal{M}) \to B(\bigoplus_{i\in I} \mathcal{H}_i)$ be a unital completely positive map satisfying $\psi\mid_{\mathcal{M}} = \pi\mid_{\mathcal{M}}$. For each i, define a unital completely positive map

$$\psi_i : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_i), \qquad t \mapsto P_{\mathcal{H}_i}\psi(t)\mid_{\mathcal{H}_i},$$

and note that $\psi_i\mid_{\mathcal{M}} = \pi_i\mid_{\mathcal{M}}$. In turn, as each π_i possesses the approximate unique extension property, we may obtain a directed set Λ_i and a corresponding net of unitary operators $u_{\lambda_i} \in B(\mathcal{H}_i)$, $\lambda_i \in \Lambda_i$, such that

$$\text{WOT} \lim_{\lambda_i \in \Lambda_i} u_{\lambda_i}^* \psi_i(t) u_{\lambda_i} = \pi_i(t), \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

Now, let $\mathcal{F}$ denote the directed set consisting of finite subsets of I. Given $F = \{i_1, \dots, i_n\} \in \mathcal{F}$ and $\lambda_{i_k} \in \Lambda_{i_k}$, define a unitary operator on $\bigoplus_{i\in I} \mathcal{H}_i$ by

$$u_{F,\lambda_{i_1},\dots,\lambda_{i_n}} = \bigoplus_{i\in\mathcal{F}} u_{\lambda_i} \oplus \bigoplus_{i\notin\mathcal{F}} I_{\mathcal{H}_i}$$

where $I_{\mathcal{H}_i}$ denotes the identity operator on $\mathcal{H}_i$. For $t \in \mathrm{C}^*(\mathcal{M})$, we have that

$$P_{\mathcal{H}_i} u^*_{F,\lambda_{i_1},\dots,\lambda_{i_n}} \psi(t) u_{F,\lambda_{i_1},\dots,\lambda_{i_n}} \mid_{\mathcal{H}_i} = \begin{cases} u^*_{\lambda_i}\psi_i(t)u_{\lambda_i}, & i \in F \\ \psi_i(t), & i \notin F. \end{cases}$$

Thus, we must have that the iterated limit

$$\lim_{\lambda_{i_1}\in\Lambda_{i_1}} \lim_{\lambda_{i_2}\in\Lambda_{i_2}} \cdots \lim_{\lambda_{i_n}\in\Lambda_{i_n}} P_{\mathcal{H}_i} u^*_{F,\lambda_{i_1},\ldots,\lambda_{i_n}} \psi(t) u_{F,\lambda_{i_1},\ldots,\lambda_{i_n}} |_{\mathcal{H}_i}$$

exists in the weak operator topology for any $i \in I$ and any $F = \{i_1, \ldots, i_n\} \in \mathcal{F}$. Indeed, if $i \in \mathcal{F}$, then the limit equals $\pi_i(t)$ and, when $i \notin \mathcal{F}$, we have that the limit is equal to $\psi_i(t)$. So

$$\lim_{\{i_1,\ldots,i_n\}\in\mathcal{F}} \lim_{\lambda_{i_1}\in\Lambda_{i_1}} \lim_{\lambda_{i_2}\in\Lambda_{i_2}} \cdots \lim_{\lambda_{i_n}\in\Lambda_{i_n}} P_{\mathcal{H}_i} u^*_{F,\lambda_{i_1},\ldots,\lambda_{i_n}} \psi(t) u_{F,\lambda_{i_1},\ldots,\lambda_{i_n}} |_{\mathcal{H}_i} = \pi_i(t)$$

in the weak operator topology for every $i \in I$. Thus, there is a net of unitary operators $u_\beta \in B(\bigoplus_{i\in I} \mathcal{H}_i)$ such that

$$\text{WOT} \lim_{\beta} P_{\mathcal{H}_i} u^*_\beta \psi(t) u_\beta |_{\mathcal{H}_i} = \pi_i(t), \qquad t \in \mathrm{C}^*(\mathcal{M}),$$

for every $i \in I$.

By compactness, we may extract a subnet $(u^*_{\beta_\eta}\psi(\cdot)u_{\beta_\eta})_\eta$ and a unital completely positive map $\varphi : \mathrm{C}^*(\mathcal{M}) \to B(\bigoplus_{i\in I}\mathcal{H}_i)$ such that

$$\text{WOT} \lim_{\eta} u^*_{\beta_\eta} \psi(t) u_{\beta_\eta} = \varphi(t), \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

From here, the argument is a direct adaptation of [8, Proposition 4.4]. Indeed, observe that $P_{\mathcal{H}_i}\varphi(t)P_{\mathcal{H}_i} = \pi(t)P_{\mathcal{H}_i}$ for each $i \in I$ and so, by the Schwarz inequality,

$$\begin{aligned} P_{\mathcal{H}_i}\varphi(t)^*(I - P_{\mathcal{H}_i})\varphi(t)P_{\mathcal{H}_i} &= P_{\mathcal{H}_i}\varphi(t)^*\varphi(t)P_{\mathcal{H}_i} - P_{\mathcal{H}_i}\varphi(t)^*P_{\mathcal{H}_i}\varphi(t)P_{\mathcal{H}_i} \\ &\leq P_{\mathcal{H}_i}\varphi(t^*t)P_{\mathcal{H}_i} - P_{\mathcal{H}_i}\varphi(t)^*P_{\mathcal{H}_i}\varphi(t)P_{\mathcal{H}_i} \\ &= \pi(t^*t)P_{\mathcal{H}_i} - \pi(t)^*\pi(t)P_{\mathcal{H}_i} = 0. \end{aligned}$$

Therefore $(I - P_{\mathcal{H}_i})\varphi(t)P_{\mathcal{H}_i} = 0$ and so

$$\text{WOT} \lim_{\eta} u^*_{\beta_\eta}\psi(t)u_{\beta_\eta} = \varphi(t) = \sum_{i\in I}\varphi(t)P_{\mathcal{H}_i} = \sum_{i\in I}P_{\mathcal{H}_i}\varphi(t)P_{\mathcal{H}_i} = \sum_{i\in I}\pi(t)P_{\mathcal{H}_i} = \pi(t).$$

□

Recall that Lemma 2.4.2 states that the implication in Proposition 4.2.7 is known to be an equivalence when the approximate unique extension property is replaced by the usual unique extension property. On the contrary, the converse will be shown to be far from true for $*$-representations that possess the approximate unique extension property (Proposition 4.2.9). First, we require a corollary to Proposition 4.2.7 that allows us to construct many more $*$-representations that possess the approximate unique extension property.

Corollary 4.2.8. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space. Suppose that $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a $*$-representation possessing the approximate unique extension property. Then, the following statements hold.*

(i) *If* $\sigma : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\sigma)$ *is a* $*$*-representation such that* $\ker\sigma = \ker\pi$*, then there is some cardinal* κ *such that* $\sigma^{(\kappa)}$ *possesses the approximate unique extension property.*

(ii) *If* $\rho : \mathrm{C}^*_e(\mathcal{M}) \to B(\mathcal{H}_\rho)$ *is an isometric* $*$*-representation, then there is a cardinal* μ *such that* $\rho^{(\mu)}$ *possesses the approximate unique extension property.*

Proof. (i): Define an infinite cardinal $\kappa = \max\{\dim\mathcal{H}_\pi, \dim\mathcal{H}_\sigma, \aleph_0\}$. For each $t \in \mathrm{C}^*(\mathcal{M}) \setminus \ker\sigma$, we have that

$$\mathrm{rank}(\sigma^{(\kappa)}(t)) = \kappa \cdot \mathrm{rank}(\sigma(t)) = \kappa,$$

and likewise, $\mathrm{rank}(\pi^{(\kappa)}(t)) = \kappa$. Since $\ker\pi = \ker\sigma$, Theorem 2.1.3 implies that $\sigma^{(\kappa)}$ is approximately unitarily equivalent to $\pi^{(\kappa)}$. By Propositions 4.2.5 and 4.2.7, we find that $\sigma^{(\kappa)}$ possesses the approximate unique extension property.

(ii): By Theorem 2.4.3 (i), there is an isometric $*$-representation $\beta : \mathrm{C}^*_e(\mathcal{M}) \to B(\mathcal{H}_\beta)$ that possesses the unique extension property. Thus, it follows by (i) that there is a cardinal μ such that $\rho^{(\mu)}$ possesses the approximate unique extension property. □

Due to Corollary 4.2.8 (ii), we may confirm that the approximate unique extension property is badly behaved with respect to sub-representations.

Proposition 4.2.9. *Let* $\mathcal{M} \subset B(\mathcal{H})$ *be a unital operator space. Assume that whenever* $\pi : \mathrm{C}^*_e(\mathcal{M}) \to B(\mathcal{H}_\pi)$ *is a* $*$*-representation possessing the approximate unique extension property and* $\mathcal{F} \subset \mathcal{H}_\pi$ *is a closed subspace that is reducing for* $\mathrm{Im}\pi$*, then we have that the* $*$*-representation*

$$t \mapsto \pi(t)\,|_{\mathcal{F}}, \qquad t \in \mathrm{C}^*_e(\mathcal{M}),$$

also possesses the approximate unique extension property. Then, we have that every $*$*-representation of* $\mathrm{C}^*_e(\mathcal{M})$ *has the approximate unique extension property.*

Proof. Let $\beta : \mathrm{C}^*_e(\mathcal{M}) \to B(\mathcal{H}_\beta)$ be an isometric $*$-representation that possesses the approximate unique extension property (which exists by either Theorem 2.4.3 or Corollary 4.2.8) and let $\pi : \mathrm{C}^*_e(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a $*$-representation. Then, there is a cardinal κ such that $\beta^{(\kappa)}$ has the approximate unique extension property by Proposition 4.2.7. By Theorem 2.1.3, we have that $\beta^{(\kappa)} \oplus \pi$ is approximately unitarily equivalent to $\beta^{(\kappa)}$. So $\beta^{(\kappa)} \oplus \pi$ possesses the approximate unique extension property by Propostion 4.2.5. As π is a subrepresentation of $\beta^{(\kappa)} \oplus \pi$, the assumption allows us to conclude that π has the approximate unique extension property. □

We conclude this section by connecting our work with Arveson's hyperrigidity conjecture. To start, we note the following.

Proposition 4.2.10. *Let* $\mathcal{M} \subset B(\mathcal{H})$ *be a separable unital operator space. Then, the following statements are equivalent.*

(i) *Every irreducible* $*$*-representation of* $\mathrm{C}^*(\mathcal{M})$ *possesses the approximate unique extension property.*

(ii) *Every $*$-representation of* $\mathrm{C}^*(\mathcal{M})$ *possesses the approximate unique extension property.*

Proof. For the non-trivial direction, assume that every irreducible $*$-representation of $\mathrm{C}^*(\mathcal{M})$ possesses the approximate unique extension property and let $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a $*$-representation. As $\mathrm{C}^*(\mathcal{M})$ is separable, we may express $\pi = \bigoplus_{i\in I} \pi_i$ where $\pi_i : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_i)$ is a $*$-representation on a separable Hilbert space $\mathcal{H}_i$. Since $\mathcal{H}_i$ is separable, we have that π_i is approximately unitarily equivalent to a direct sum of irreducible $*$-representations [90, Corollary 1.6]. By the assumption, Proposition 4.2.5, and Proposition 4.2.7, we may conclude that each π_i has the approximate unique extension property. Thus, π has the approximate unique extension property by Proposition 4.2.7. □

Proposition 4.2.10 may be viewed as a positive answer to an approximate version of Arveson's hyperrigidity conjecture. Indeed, as reflected in Proposition 4.2.10, to determine the class of $*$-representations that possess the approximate unique extension property, it is sufficient to study those *irreducible* $*$-representations that possess the approximate unique extension property.

We may now state the main result of this section.

Theorem 4.2.11. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a separable unital operator space. Consider the following statements:*

(i) *The operator space $\mathcal{M}$ is hyperrigid.*

(ii) *Every irreducible $*$-representation of* $\mathrm{C}^*(\mathcal{M})$ *is a boundary representation.*

(iii) *Every irreducible $*$-representation of* $\mathrm{C}^*(\mathcal{M})$ *is approximately unitarily equivalent to a boundary representation.*

(iv) *Every $*$-representation of* $\mathrm{C}^*(\mathcal{M})$ *has the approximate unique extension property.*

Then, we have that (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) *and* (iii)$\not\Rightarrow$(ii).

Proof. The directions (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are trivial and (iii)$\Rightarrow$(iv) is a consequence to Theorem 4.2.6 and Proposition 4.2.10. Finally, (iii)$\not\Rightarrow$(ii) by considering Example 9. □

Later, it will be revealed that the approximate unique extension property coincides with the usual unique extension property for those irreducible $*$-representations whose image contains a compact operator (Theorem 4.3.8). Consequently, statements (ii), (iii), and (iv) will be equivalent whenever $\mathrm{C}^*(\mathcal{M})$ is postliminal.

4.3 Split Sequences for Completely Positive Extensions

Suppose that π has the approximate unique extension property with respect to a unital operator space $\mathcal{M}$. We will now present a restriction on the structure of the unital completely positive extensions of $\pi\,|_{\mathcal{M}}$. This will be the driving force of our arguments throughout this section and is the main result of this chapter.

For this recall that, given C*-algebras $\mathfrak{A} \subset \mathfrak{B}$, a *conditional expectation* is a contractive linear projection $E : \mathfrak{B} \to \mathfrak{A}$. When E is multiplicative, we say that the map is *homomorphic*. By Tomiyama's Theorem [16, Theorem 1.5.10], a conditional expectation is automatically completely positive and satisfies $E(aba') = aE(b)a'$ for every $a, a' \in \mathfrak{A}$ and $b \in \mathfrak{B}$. Consequently, a homomorphic conditional expectation is a $*$-homomorphism.

Theorem 4.3.1. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space. Suppose that $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a $*$-representation possessing the approximate unique extension property and let $\psi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a unital completely positive map satisfying $\psi\,|_{\mathcal{M}} = \pi\,|_{\mathcal{M}}$. Then, the following statements hold.*

- (i) *There is a homomorphic conditional expectation $\Gamma_\psi : \mathrm{C}^*(\mathrm{Im}\psi) \to \mathrm{Im}\pi$ satisfying $\Gamma_\psi \circ \psi = \pi$.*
- (ii) *If $\mathcal{H}_\pi$ is separable, then $\mathrm{rank}(\pi(t)) \leq \mathrm{rank}(\psi(t))$ for each $t \in \mathrm{C}^*(\mathcal{M})$.*
- (iii) *We have that $\ker \Gamma_\psi$ is the smallest closed two-sided ideal of $\mathrm{C}^*(\mathrm{Im}\psi)$ that contains $\mathrm{Im}(\psi - \pi)$.*
- (iv) *We have that $\mathrm{Im}(id - \Gamma_\psi) = \ker \Gamma_\psi$ and $\mathrm{C}^*(\mathrm{Im}\psi) = \ker \Gamma_\psi + \mathrm{Im}\pi$.*
- (v) *We have that $\psi = \pi$ if and only if Γ_ψ is isometric.*

Proof. (i): As π possesses the approximate unique extension property, there is a net of unitaries $u_\beta \in B(\mathcal{H}_\pi)$ such that $u_\beta^* \psi(t) u_\beta \to \pi(t)$ in the weak operator topology for each $t \in \mathrm{C}^*(\mathcal{M})$. It follows by the Schwarz inequality that convergence is in the strong operator topology (see [67, Lemma 2.1]). Consequently, given a polynomial p in non-commuting variables and finitely many elements $t_1, \ldots, t_n \in \mathrm{C}^*(\mathcal{M})$, we have that

$$\mathrm{SOT}\lim_\beta u_\beta^* p(\psi(t_1), \ldots, \psi(t_n)) u_\beta = p(\pi(t_1), \ldots, \pi(t_n)).$$

Since the linear span of words in $\mathrm{Im}\psi$ is dense within $\mathrm{C}^*(\mathrm{Im}\psi)$, we obtain a $*$-homomorphism

$$\Gamma_\psi : \mathrm{C}^*(\mathrm{Im}\psi) \to \mathrm{Im}\pi, \qquad x \mapsto \mathrm{SOT}\lim_\beta u_\beta^* x u_\beta.$$

Evidently, $\Gamma_\psi \circ \psi = \pi$. Note that

$$\mathrm{Im}\pi = \mathrm{C}^*(\pi(\mathcal{M})) = \mathrm{C}^*(\psi(\mathcal{M})) \subset \mathrm{C}^*(\mathrm{Im}\psi).$$

Since

$$(\Gamma_\psi \circ \pi) \mid_{\mathcal{M}} = (\Gamma_\psi \circ \psi) \mid_{\mathcal{M}} = \pi \mid_{\mathcal{M}}$$

and Γ_ψ is a $*$-representation, it follows that $\Gamma_\psi \circ \pi = \pi$. Therefore, Γ_ψ is a conditional expectation as desired.

(ii): Recall, by Lemma 2.1.5, the rank function is lower semicontinous in the weak operator topology provided that $\mathcal{H}_\pi$ is separable. Then (ii) follows instantly since there is a net of unitaries $u_\beta \in B(\mathcal{H}_\pi)$ such that $u_\beta^* \psi(t) u_\beta \to \pi(t)$ in the weak operator topology.

(iii): Since $\Gamma_\psi \circ \psi = \pi = \Gamma_\psi \circ \pi$, it follows that

$$\mathrm{Im}(\psi - \pi) \subset \ker \Gamma_\psi.$$

Let $\mathfrak{J}$ be the smallest closed two-sided ideal of $\mathrm{C}^*(\mathrm{Im}\psi)$ containing $\mathrm{Im}(\psi - \pi)$.

We will show that each operator $\zeta \in \mathrm{C}^*(\mathrm{Im}\psi)$ may be expressed as $J + \pi(t)$ where $J \in \mathfrak{J}$ and $t \in \mathrm{C}^*(\mathcal{M})$. Observe that

$$\mathfrak{D} = \{J + \pi(t) \ : \ J \in \mathfrak{J}, \ t \in \mathrm{C}^*(\mathcal{M})\}$$

is a closed subset of $\mathrm{C}^*(\mathrm{Im}\psi)$ as it is the pre-image of $\mathrm{Im}\pi/\mathfrak{J}$ under the quotient map $q : \mathrm{C}^*(\mathrm{Im}\psi) \to \mathrm{C}^*(\mathrm{Im}\psi)/\mathfrak{J}$. Hence, by density, it suffices to consider when $\zeta \in \mathrm{C}^*(\mathrm{Im}\psi)$ lies within the linear span of words in $\mathrm{Im}\psi$.

In this case, since ψ and $\mathrm{C}^*(\mathcal{M})$ are unital, we may obtain a finite sum of words

$$\zeta = \sum_i \psi(t_{i_1}) \dots \psi(t_{i_n})$$

where n is a fixed integer and $t_{k_\ell} \in \mathrm{C}^*(\mathcal{M})$ for each k, ℓ. For a fixed i, we will show that $\psi(t_{i_1}) \dots \psi(t_{i_n})$ may be expressed as $J + \pi(t)$ by an easy inductive argument. For the base step, observe that

$$\psi(t_{i_1}) = \psi(t_{i_1}) - \pi(t_{i_1}) + \pi(t_{i_1})$$

and $(\psi(t_{i_1}) - \pi(t_{i_1})) \in \mathfrak{J}$. Now, for the inductive step, we may obtain $K \in \mathfrak{J}$ and $s \in \mathrm{C}^*(\mathcal{M})$ such that $\psi(t_{i_1}) \dots \psi(t_{i_{n-1}}) = K + \pi(s)$. Then we see that

$$\begin{aligned}
\psi(t_{i_1}) \dots \psi(t_{i_n}) &= (K + \pi(s))\psi(t_{i_n}) \\
&= (K + \pi(s))(\psi(t_{i_n}) - \pi(t_{i_n}) + \pi(t_{i_n})) \\
&= K(\psi(t_{i_n}) - \pi(t_{i_n}) + \pi(t_{i_n})) + \pi(s)(\psi(t_{i_n}) - \pi(t_{i_n})) + \pi(st_{i_n}).
\end{aligned}$$

Therefore, we have that the claim also holds for ζ.

Now, let $K \in \ker \Gamma_\psi$ and express $K = J + \pi(t)$ where $J \in \mathfrak{J}$ and $t \in \mathrm{C}^*(\mathcal{M})$. Then we obtain that

$$K - J = \pi(t) = \Gamma_\psi(\pi(t)) = \Gamma_\psi(K - J) = 0.$$

So $K = J$ and we have that $\mathfrak{J} = \ker \Gamma_\psi$.

(iv): This is a standard fact about linear idempotents, but we record it for later arguments. Since Γ_ψ is a linear idempotent, it is easy to see that $\mathrm{Im}(id - \Gamma_\psi) = \ker \Gamma_\psi$. Then,

$$\mathrm{C}^*(\mathrm{Im}\psi) = \mathrm{Im}(id - \Gamma_\psi) + \mathrm{Im}\Gamma_\psi = \ker \Gamma_\psi + \mathrm{Im}\pi.$$

(v): This is an immediate consequence of (iii). □

A conclusion similar to Theorem 4.3.1 (i) has already been proven for isometric $*$-representations of separable nuclear C*-algebras with the property that all factor representations possess the unique extension property [67, Theorem 4.2]. The construction therein required non-trivial machinery on direct integrals and disintegration of measures. In contrast, our methods in Theorem 4.3.1 provide a non-trivial implication for all unital operator spaces that fail to be hyperrigid. Indeed, whenever $\mathcal{M} \subset \mathrm{C}^*_e(\mathcal{M})$ fails to be hyperrigid, there is a $*$-representation of $\mathrm{C}^*_e(\mathcal{M})$ that has the approximate unique extension property but not the usual unique extension property (this follows by Proposition 4.1.1 and Theorem 4.2.6).

Additionally, as a consequence to Theorem 4.3.1, we may derive a statement of independent interest.

Corollary 4.3.2. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space. Suppose that $\pi : \mathrm{C}^*_e(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is an isometric $*$-representation and let $\psi : \mathrm{C}^*_e(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a unital completely positive map satisfying $\psi |_{\mathcal{M}} = \pi |_{\mathcal{M}}$. Then, the following statements hold.*

(i) *The map ψ is completely isometric.*

(ii) *We have that $\mathrm{C}^*_e(\mathrm{Im}\psi) \cong \mathrm{C}^*_e(\mathcal{M})$.*

Proof. (i): By Corollary 4.2.8, there is a cardinal κ such that $\Pi := \pi^{(\kappa)}$ has the approximate unique extension property. Note that the map $\Psi := \psi^{(\kappa)}$ agrees with Π over $\mathcal{M}$. Thus, by Theorem 4.3.1, there is a homomorphic conditional expectation $\Gamma_\Psi : \mathrm{C}^*(\mathrm{Im}\psi) \to \mathrm{Im}\Pi$ satisfying $\Gamma_\Psi \circ \Psi = \Pi$. Since Π is isometric, it immediately follows that Ψ is completely isometric. Finally, this implies that ψ itself must be completely isometric.

(ii): By statement (i), there is a completely isometric isomorphism $\iota : \mathrm{Im}\psi \to \mathrm{Im}\pi$ such that $\iota \circ \psi = \pi$. Since $\mathrm{Im}\pi$ is a C*-algebra, it then follows that $\mathrm{Im}\pi \cong \mathrm{C}^*_e(\mathcal{M})$ is the C*-envelope for $\mathrm{Im}\psi$.

Indeed, suppose that $(\mathfrak{A}, j)$ is a C*-cover for $\mathrm{Im}\psi$ such that $(\mathfrak{A}, j) \preceq (\mathrm{Im}\pi, \iota)$. So, there is a surjective $*$-homomorphism $q : \mathrm{Im}\pi \to \mathfrak{A}$ such that $q \circ \iota = j$. We will verify that q is injective. To this end, take $t \in \mathrm{C}^*(\mathcal{M})$ such that $(q \circ \pi)(t) = 0$. Then, we have that

$$0 = (q \circ \pi)(t) = (q \circ \iota \circ \psi)(t) = (j \circ \psi)(t).$$

As j and ψ are completely isometric, it follows that $t = 0$. So, indeed, q is injective and we must have that $(\mathfrak{A}, j) \sim (\mathrm{Im}\pi, \iota)$. Therefore, $(\mathrm{Im}\pi, \iota)$ is the C*-envelope for $\mathrm{Im}\psi$. □

The argument in Corollary 4.3.2 may be repurposed. If π has the approximate unique extension property and ψ is a completely positive extension such that $\ker \pi \subset \ker \psi$, then in fact we have equality of kernels and hence, there is a completely isometric map $\iota : \mathrm{Im}\psi \to \mathrm{Im}\pi$ given by $\Gamma_\psi |_{\mathrm{Im}\psi}$.

We briefly remark that the existence of the conditional expectation Γ_ψ provides a new tool for determining when a completely positive extension ψ of $\pi |_{\mathcal{M}}$ will coincide with the $*$-representation π.

Corollary 4.3.3. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space. Suppose that $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a $*$-representation possessing the approximate unique extension property and let $\psi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a unital completely positive map satisfying $\psi |_{\mathcal{M}} = \pi |_{\mathcal{M}}$. Then, the following statements are equivalent.*

(i) *The equality $\psi = \pi$ holds.*

(ii) *The linear map $(id - \Gamma_\psi)$ is contractive.*

Proof. It is clear from Theorem 4.3.1 that Γ_π is the identity representation and so, (i)$\Rightarrow$ (ii) is immediate.

(ii)$\Rightarrow$(i): The map $(id - \Gamma_\psi)$ is a linear projection onto $\ker \Gamma_\psi$ by Theorem 4.3.1 (iv). By Tomiyama's Theorem [16, Theorem 1.5.10], contractive idempotents are completely positive and conditional expectations. So, the assumption allows us to conclude that $(id - \Gamma_\psi)$ is a conditional expectation. Thus, given $k \in \ker \Gamma_\psi$ and $x \in \mathrm{C}^*(\mathrm{Im}\psi)$, we have that

$$xk = (id - \Gamma_\psi)(xk) = (id - \Gamma_\psi)(x)k = xk - \Gamma_\psi(x)k$$

and so $\Gamma_\psi(x)k = 0$. As $x \in \mathrm{C}^*(\mathrm{Im}\psi)$ is arbitrary, we may take $x = I$ and thus, $0 = \Gamma_\psi(I)k = k$. So Γ_ψ is injective and thus, $\psi = \pi$ by Theorem 4.3.1 (v). $\square$

For our next development, we will require some terminology. Let $\mathcal{S} \subset \mathcal{T}$ be (closed) operator systems and let $\mathrm{UCP}(\mathcal{S}, \mathcal{T})$ denote the collection of unital completely positive maps from $\mathcal{S}$ to $\mathcal{T}$. Given a subspace $\mathcal{F}$ of $\mathcal{S}$, there is an ordering on the idempotent elements of $\mathrm{UCP}(\mathcal{S}, \mathcal{T})$ that fix $\mathcal{F}$. Indeed, if $\varphi, \psi \in \mathrm{UCP}(\mathcal{S}, \mathcal{T})$ are idempotent and fix a subspace $\mathcal{F} \subset \mathcal{S}$, then $\varphi \prec \varphi'$ if and only if $\varphi \circ \varphi' = \varphi' \circ \varphi = \varphi$. An idempotent φ is said to be a *minimal $\mathcal{F}$-projection* in $\mathrm{UCP}(\mathcal{S}, \mathcal{T})$ if φ is minimal among the idempotent elements of $\mathrm{UCP}(\mathcal{S}, \mathcal{T})$ that fix $\mathcal{F}$. Minimal $\mathcal{F}$-projections have found previous success in helping to prove existence of the injective envelope of an operator system [73, Chapter 15]. We find that the map Γ_ψ enjoys this minimality condition and is uniquely determined by its behaviour over $\mathrm{Im}\psi$.

Theorem 4.3.4. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space. Suppose that $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a $*$-representation possessing the approximate unique extension property and let $\psi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a unital completely positive map satisfying $\psi |_{\mathcal{M}} = \pi |_{\mathcal{M}}$. Then, the following statements hold.*

(i) *Every $*$-representation $\sigma : \mathrm{C}^*(\mathrm{Im}\psi) \to B(\mathcal{H}_\sigma)$ such that $\sigma |_{\ker \Gamma_\psi} = 0$ has the unique extension property with respect to* $\mathrm{Im}\psi$.

(ii) *The map Γ_ψ is a minimal* $\mathrm{Im}\pi$*-projection in* $\mathrm{UCP}(\mathrm{C}^*(\mathrm{Im}\psi), B(\mathcal{H}_\pi))$.

Proof. (i): Let $\varphi : \mathrm{C}^*(\mathrm{Im}\psi) \to B(\mathcal{H}_\sigma)$ be a unital completely positive map satisfying $\varphi \mid_{\mathrm{Im}\psi} = \sigma \mid_{\mathrm{Im}\psi}$. As σ vanishes over $\ker \Gamma_\psi$, we may express $\sigma = \widetilde{\sigma} \circ \Gamma_\psi$ where $\widetilde{\sigma} : \mathrm{Im}\pi \to B(\mathcal{H}_\sigma)$ is a $*$-representation. Observe that, by Theorem 4.3.1,

$$(\varphi \circ \psi)(t) = (\sigma \circ \psi)(t) = (\widetilde{\sigma} \circ \Gamma_\psi \circ \psi)(t) = (\widetilde{\sigma} \circ \pi)(t), \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

By the Schwarz inequality, we have that for each $t \in \mathrm{C}^*(\mathcal{M})$,

$$\varphi(\psi(t)^*\psi(t)) \leq (\varphi \circ \psi)(t^*t) = (\widetilde{\sigma} \circ \pi)(t^*t) = \varphi(\psi(t))^*\varphi(\psi(t)) \leq \varphi(\psi(t)^*\psi(t))$$

and so $\varphi(\psi(t)^*\psi(t)) = \varphi(\psi(t))^*\varphi(\psi(t))$. Consequently, we have that $\mathrm{Im}\psi$ is a subset of the multiplicative domain for φ [73, Theorem 3.18]. Therefore, since $\varphi \mid_{\mathrm{Im}\psi} = \sigma \mid_{\mathrm{Im}\psi}$, we must have that $\varphi = \sigma$.

(ii): Let $\Omega : \mathrm{C}^*(\mathrm{Im}\psi) \to B(\mathcal{H}_\pi)$ be a unital completely positive idempotent such that $\Omega \circ \pi = \pi$ and $\Omega \prec \Gamma_\psi$. In this case, by Theorem 4.3.1,

$$\Omega \circ \psi = \Omega \circ \Gamma_\psi \circ \psi = \Omega \circ \pi = \pi.$$

As $\Omega \mid_{\mathrm{Im}\psi} = \Gamma_\psi \mid_{\mathrm{Im}\psi}$, it follows by (i) that $\Omega = \Gamma_\psi$. □

As a result of Theorem 4.3.4, we show that there is a one-to-one correspondence between the idempotents Γ_ψ and the domain of such maps. In particular, a completely positive extension ψ will be equal to a $*$-representation π with the approximate unique extension property provided that the image of the completely positive extension is sufficiently small. This generalizes [20, Corollary 3.3].

Corollary 4.3.5. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space. Suppose that $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a $*$-representation possessing the approximate unique extension property and let $\psi, \psi' : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be unital completely positive maps satisfying $\psi \mid_{\mathcal{M}} = \psi' \mid_{\mathcal{M}} = \pi \mid_{\mathcal{M}}$. Then, the following statements are equivalent.*

(i) *We have that* $\ker \Gamma_\psi \subset \ker \Gamma_{\psi'}$.

(ii) *We have that* $\mathrm{C}^*(\mathrm{Im}\psi) \subset \mathrm{C}^*(\mathrm{Im}\psi')$.

(iii) *We have that* $\Gamma_{\psi'} \mid_{\mathrm{C}^*(\mathrm{Im}\psi)} = \Gamma_\psi$.

In particular, we have that $\Gamma_\psi = \Gamma_{\psi'}$ *if and only if* $\mathrm{C}^*(\mathrm{Im}\psi) = \mathrm{C}^*(\mathrm{Im}\psi')$. *Further,* $\psi = \pi$ *whenever* $\mathrm{Im}\psi \subset \mathrm{Im}\pi$.

Proof. The implication (iii)⇒(i) is trivial and (i)⇒(ii) follows directly from Theorem 4.3.1 because

$$\mathrm{C}^*(\mathrm{Im}\psi) = \ker \Gamma_\psi + \mathrm{Im}\pi \subset \ker \Gamma_{\psi'} + \mathrm{Im}\pi = \mathrm{C}^*(\mathrm{Im}\psi').$$

(ii)⇒(iii): In this case, we obtain a well-defined unital completely positive map $\varphi := \Gamma_{\psi'} \circ \psi : \mathrm{C}^*(\mathcal{M}) \to \mathrm{Im}\pi$ satisfying

$$\varphi \mid_{\mathcal{M}} = (\Gamma_{\psi'} \circ \psi) \mid_{\mathcal{M}} = (\Gamma_{\psi'} \circ \pi) \mid_{\mathcal{M}} = \pi \mid_{\mathcal{M}} .$$

Thus, by Theorem 4.3.1 (i),

$$\pi(t) = (\Gamma_\varphi \circ \varphi)(t) = \varphi(t), \qquad t \in \mathrm{C}^*(\mathcal{M}),$$

on account of $\varphi(t)$ lying in the image of the idempotent $\Gamma_\varphi : \mathrm{C}^*(\mathrm{Im}\varphi) \to \mathrm{Im}\pi$. It follows that $\Gamma_{\psi'} \circ \psi = \pi$. However, by Theorem 4.3.4 (i), the map Γ_ψ has the unique extension property with respect to $\mathrm{Im}\psi$. So, since $\Gamma_{\psi'} \mid_{\mathrm{Im}\psi} = \Gamma_\psi \mid_{\mathrm{Im}\psi}$, it follows that $\Gamma_{\psi'} \mid_{\mathrm{C}^*(\mathrm{Im}\psi)} = \Gamma_\psi$.

For the final statement note that, whenever $\mathrm{Im}\psi \subset \mathrm{Im}\pi$, it follows from condition (iii) that

$$\Gamma_\psi = \Gamma_\pi \mid_{\mathrm{C}^*(\mathrm{Im}\psi)} = id,$$

where id denotes the identity representation of $\mathrm{C}^*(\mathrm{Im}\psi)$. By Theorem 4.3.1 (v), we conclude that $\psi = \pi$. □

Recall that, by Proposition 4.2.10, to determine the class of $*$-representations that possess the approximate unique extension property it suffices to determine the class of *irreducible* $*$-representations that possess the approximate unique extension property (provided that $\mathcal{M}$ is separable). Any other $*$-representation is the result of a direct sum or an approximate unitary equivalence of $*$-representations of this form. Here, we will mention how the conditional expectation Γ_ψ behaves with respect to direct sums and approximate unitary equivalences of completely positive extensions. We start with direct sums.

Corollary 4.3.6. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space and $\{\pi_i : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_i) : i \in I\}$ be a collection of $*$-representations that possess the approximate unique extension property. For each $i \in I$, let $\psi_i : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_i)$ be a unital completely positive map satisfying $\psi_i \mid_{\mathcal{M}} = \pi_i \mid_{\mathcal{M}}$. Then, defining $\pi = \bigoplus_{i\in I} \pi_i$ and $\psi = \bigoplus_{i \in I} \psi_i$, we have that the map $\Gamma_\psi : \mathrm{C}^*(\mathrm{Im}\psi) \to \mathrm{Im}\pi$ satisfies*

$$\Gamma_\psi((x_i)_{i\in I}) = (\Gamma_{\psi_i}(x_i))_{i\in I}, \qquad (x_i)_{i\in I} \in \mathrm{C}^*(\mathrm{Im}\psi).$$

Proof. Note that ψ is a unital completely positive map satisfying $\psi \mid_{\mathcal{M}} = \pi \mid_{\mathcal{M}}$ and recall that π has the approximate unique extension property by Proposition 4.2.7. Define a $*$-homomorphism

$$\Lambda : \mathrm{C}^*(\mathrm{Im}\psi) \to \prod_{i\in I} \mathrm{Im}\pi_i, \qquad (x_i)_{i\in I} \mapsto (\Gamma_{\psi_i}(x_i))_{i\in I}.$$

Note that $\Lambda \circ \psi = \pi$ and so,

$$\Lambda(\mathrm{C}^*(\mathrm{Im}\psi)) = \mathrm{C}^*(\Lambda(\mathrm{Im}\psi)) = \mathrm{Im}\pi.$$

Therefore, we may conclude that $\Lambda = \Gamma_\psi$ by Theorem 4.3.4 (i). □

Next, we address how the conditional expectation Γ_ψ is affected by approximately unitarily equivalent completely positive extensions. We say that unital completely positive maps

$\psi, \psi' : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ are *approximately unitarily equivalent* whenever there is a net of unitary operators $w_\gamma \in B(\mathcal{H}_\pi)$ satisfying

$$\lim_\gamma \|w_\gamma^* \psi'(t) w_\gamma - \psi(t)\| = 0, \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

In particular, when ψ and ψ' are approximately unitarily equivalent, there is a $*$-isomorphism $\Omega : \mathrm{C}^*(\mathrm{Im}\psi) \to \mathrm{C}^*(\mathrm{Im}\psi')$ satisfying $\Omega \circ \psi = \psi'$.

Corollary 4.3.7. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space. Suppose that $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a $*$-representation possessing the approximate unique extension property and let $\psi, \psi' : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be unital completely positive maps satisfying $\psi|_\mathcal{M} = \psi'|_\mathcal{M} = \pi|_\mathcal{M}$. Assume that there is a $*$-isomorphism $\Omega : \mathrm{C}^*(\mathrm{Im}\psi) \to \mathrm{C}^*(\mathrm{Im}\psi')$ satisfying $\Omega \circ \psi = \psi'$. Then, we have that $\mathrm{C}^*(\mathrm{Im}\psi) = \mathrm{C}^*(\mathrm{Im}\psi')$ and $\Gamma_\psi = \Gamma_{\psi'}$.*

Proof. Observe that, by Theorem 4.3.1 (i), we have that

$$\pi = \Gamma_{\psi'} \circ \psi' = \Gamma_{\psi'} \circ \Omega \circ \psi.$$

By Theorem 4.3.4 (i), it follows that $\Gamma_{\psi'} \circ \Omega = \Gamma_\psi$. Whence, we obtain that $\ker \Gamma_{\psi'} = \ker \Gamma_\psi$ on account of Ω being a $*$-isomorphism. By Corollary 4.3.5, we conclude that $\mathrm{C}^*(\mathrm{Im}\psi) = \mathrm{C}^*(\mathrm{Im}\psi')$ and $\Gamma_\psi = \Gamma_{\psi'}$. □

We close this section by providing an example of when the approximate unique extension property is equivalent to the usual unique extension property. This allows us to improve Theorem 4.2.11 in a special case.

Theorem 4.3.8. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space and let $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a $*$-representation such that $\mathfrak{K}(\mathcal{H}) \subset \mathrm{Im}\pi$. Then, the following statements are equivalent.*

(i) *π possesses the unique extension property.*

(ii) *π is approximately unitarily equivalent to a boundary representation.*

(iii) *π possesses the approximate unique extension property.*

In particular, if $\mathcal{M}$ is separable and $\mathrm{C}^*(\mathcal{M})$ is postliminal, then every irreducible $*$-representation of $\mathrm{C}^*(\mathcal{M})$ is a boundary representation if and only if every $*$-representation of $\mathrm{C}^*(\mathcal{M})$ possesses the approximate unique extension property.

Proof. The direction (i) ⇒(ii) is trivial and (ii)⇒(iii) is Theorem 4.2.6. For (iii)⇒(i), let $\psi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a completely positive extension of $\pi|_\mathcal{M}$. Then, observe that Γ_ψ is a $*$-representation that fixes $\mathfrak{K}(\mathcal{H})$ and consequently, must be the identity representation (for example, see [74, Proposition 3.5]). In which case, by Theorem 4.3.1 (v), we have that $\psi = \pi$ and so π has the unique extension property.

For the last statement, if every irreducible $*$-representation of $\mathrm{C}^*(\mathcal{M})$ is a boundary representation, then every $*$-representation of $\mathrm{C}^*(\mathcal{M})$ possesses the approximate unique extension property by Theorem 4.2.11. For the converse, note that every irreducible $*$-representation $\sigma : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\sigma)$ possesses the approximate unique extension property. Since $\mathrm{C}^*(\mathcal{M})$ is postliminal, we have that $\mathfrak{K}(\mathcal{H}_\sigma) \subset \mathrm{Im}\sigma$. Therefore, the previous paragraph guarantees that σ has the unique extension property. $\square$

Finally, note that Theorem 4.3.8 allows us to readily identify many instances where $*$-representations fail to have the approximate unique extension property.

Example 10. Let H^2 denote the classical Hardy space and $\mathrm{A}(\mathbb{D}) \subset \mathrm{C}(\overline{\mathbb{D}})$ denote the disc algebra. Consider the subalgebra $\mathcal{A} \subset B(H^2)$ of multiplication operators M_φ where $\varphi \in \mathrm{A}(\mathbb{D})$. Recall that there is a well-known split exact sequence of C*-algebras

$$0 \longrightarrow \mathfrak{K}(H^2) \longrightarrow \mathrm{C}^*(\mathcal{A}) \longrightarrow \mathrm{C}(\mathbb{T}) \longrightarrow 0$$

and that $\mathrm{C}^*(\mathcal{A})$ is the Toeplitz algebra (one can consult [1],[43],[56] for details). Moreover, we have that $\mathrm{C}^*_e(\mathcal{A}) \cong \mathrm{C}(\mathbb{T})$ (for example, this follows by [22, Corollary 6.4]).

Note that the image of the identity representation $\pi : \mathrm{C}^*(\mathcal{A}) \to B(H^2)$ contains $\mathfrak{K}(H^2)$. Since π does not factor through the C*-envelope of $\mathcal{A}$, we have that π cannot be a boundary representation for $\mathcal{A}$ by Theorem 2.4.3 (ii). So π does not possess the approximate unique extension property either by Theorem 4.3.8.

4.4 Concrete Applications

In this section, we apply our work to a few classes of operator spaces. This allows us to analyze how the approximate unique extension property is behaved for spaces of continuous functions (Proposition 4.4.1), as well as obtain new formulations of Arveson's essential normality conjecture (Theorem 4.4.2).

4.4.1 Function Spaces

Let X be a compact metric space and $\mathcal{M} \subset \mathrm{C}(X)$ be a unital function space. Recall that Šaškin proved that every irreducible $*$-representation of $\mathrm{C}(X)$ is a boundary representation for $\mathcal{M}$ if and only if, whenever $\psi_n : \mathrm{C}(X) \to \mathrm{C}(X)$ is a sequence of unital completely positive maps satisfying $\|\psi_n(m) - m\| \to 0$ for every $m \in \mathcal{M}$, then we have that $\|\psi_n(t) - t\| \to 0$ for every $t \in \mathrm{C}(X)$. We note that a slight variation on Šaškin's theorem is possible.

Proposition 4.4.1. *Let X be a compact Hausdorff space and $\mathcal{M} \subset \mathrm{C}(X)$ be a unital function space. Assume that $(\mathrm{C}(X), id_{\mathcal{M}})$ is equivalent to the C^*-envelope of $\mathcal{M}$. Then, the following statements are equivalent.*

(i) *Whenever $\pi : \mathrm{C}(X) \to B(\mathcal{H}_\pi)$ is a $*$-representation possessing the approximate unique extension property and $\psi : \mathrm{C}(X) \to B(\mathcal{H}_\pi)$ is a unital completely positive map such that $\mathrm{C}^*(\mathrm{Im}\psi)$ is commutative and $\psi\mid_{\mathcal{M}} = \pi\mid_{\mathcal{M}}$, we have that $\psi = \pi$.*

(ii) *Whenever $\psi_\alpha : \mathrm{C}(X) \to \mathrm{C}(X)$ is a net of unital completely positive maps satisfying $\|\psi_\alpha(g) - g\| \to 0$ for every $g \in \mathcal{M}$, then we have that $\|\psi_\alpha(f) - f\| \to 0$ for every $f \in \mathrm{C}(X)$.*

(iii) *Every irreducible $*$-representation for $\mathrm{C}(X)$ is a boundary representation.*

Proof. (ii)⇔(iii): This is the non-separable version of Šaškin's Theorem (Theorem 2.3.4).

(iii)⇒(i): Let $\pi : \mathrm{C}(X) \to B(\mathcal{H}_\pi)$ be a $*$-representation possessing the approximate unique extension property and let $\psi : \mathrm{C}(X) \to B(\mathcal{H}_\pi)$ be a unital completely positive map such that $\mathrm{C}^*(\mathrm{Im}\psi)$ is commutative and $\psi\mid_{\mathcal{M}} = \pi\mid_{\mathcal{M}}$. If Y is a compact Hausdorff space and $\Omega : \mathrm{C}^*(\mathrm{Im}\psi) \to \mathrm{C}(Y)$ is $*$-isomorphism, then $\Omega \circ \psi : \mathrm{C}(X) \to \mathrm{C}(Y)$ is a unital completely positive map satisfying $(\Omega \circ \psi)\mid_{\mathcal{M}} = (\Omega \circ \pi)\mid_{\mathcal{M}}$. Thus, $\Omega \circ \psi = \Omega \circ \pi$ by [38, Theorem 5.1] and so, $\psi = \pi$ as desired.

(i)⇒(ii): Let $\pi : \mathrm{C}(X) \to B(\mathcal{H}_\pi)$ be an isometric $*$-representation and let $\psi_\alpha : \mathrm{C}(X) \to \mathrm{C}(X)$ be a net of unital completely positive maps satisfying $\|\psi_\alpha(g) - g\| \to 0$ for every $g \in \mathcal{M}$. Define a $*$-homomorphism

$$\pi_{\mathfrak{u}} : \mathrm{C}(X) \to \left(\prod_\alpha \mathrm{Im}\pi\right)/\mathfrak{J}, \qquad f \mapsto (\pi(f))_\alpha + \mathfrak{J},$$

where $\mathfrak{J}$ is the closed two-sided ideal of $\prod_\alpha \mathrm{Im}\pi$ consisting of those nets $(\pi(f_\alpha))_\alpha$ such that $\|\pi(f_\alpha)\| = \|f_\alpha\| \to 0$. By definition of $\mathfrak{J}$, we have that $\ker \pi_{\mathfrak{u}} = \ker \pi$ and so $\pi_{\mathfrak{u}}$ is isometric.

Upon representing $(\prod_\alpha \mathrm{Im}\pi)/\mathfrak{J}$ on some Hilbert space via an isometric $*$-homomorphism ι, we may apply Corollary 4.2.8 (ii) to obtain a cardinal κ such that $(\iota \circ \pi_\mathfrak{u})^{(\kappa)}$ has the approximate unique extension property.

Now, define a unital completely positive map

$$\psi : \mathrm{C}(X) \to \left(\prod_\alpha \mathrm{Im}\pi\right)/\mathfrak{J}, \qquad f \mapsto ((\pi \circ \psi_\alpha)(f))_\alpha + \mathfrak{J}.$$

By assumption, we have that ψ and $\pi_\mathfrak{u}$ agree on $\mathcal{M}$. Therefore, we have that $(\iota \circ \psi)^{(\kappa)}$ and $(\iota \circ \pi_\mathfrak{u})^{(\kappa)}$ agree on $\mathcal{M}$ as well. However, since $\mathrm{C}^*(\mathrm{Im}(\iota \circ \psi)^{(\kappa)})$ is a commutative C^*-algebra, the assumption allows us to conclude that $(\iota \circ \psi)^{(\kappa)} = (\iota \circ \pi_\mathfrak{u})^{(\kappa)}$. As ι and $\pi_\mathfrak{u}$ are isometric, and by definition of $\pi_\mathfrak{u}$, it then follows that $\|\psi_\alpha(f) - f\| \to 0$ for every $f \in \mathrm{C}(X)$. □

We also remark that it appears notable that there are many examples of unital operator spaces where condition (i) fails (Example 8).

4.4.2 Essentially Normal Operator Tuples

Next, we apply our machinery to a different class of operator systems and relate the approximate unique extension property to another well-studied conjecture of Arveson.

For these purposes, let $d \geq 2$ be a fixed positive integer, $\mathbb{B}_d \subset \mathbb{C}^d$ denote the open unit ball and $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_d]$ denote the algebra of complex d-variate polynomials. We require terminology from Section 2.5. Recall that the Drury-Arveson space H^2_d is the reproducing kernel Hilbert space on $\mathbb{B}_d$ that is associated with the kernel

$$k(z, w) = \frac{1}{1 - \langle z, w \rangle}, \qquad z, w \in \mathbb{B}_d.$$

Let $M_p \in B(H^2_d)$ denote the polynomial multiplier associated with $p \in \mathbb{C}[z]$. The row operator

$$M_z = (M_{z_1}, \ldots, M_{z_d}) : (H^2_d)^{(d)} \longrightarrow H^2_d$$

is called the d-shift of the Drury-Arveson space. Next, fix a homogeneous ideal $I \lhd \mathbb{C}[z]$, i.e. I is an ideal that is generated by a family of homogeneous polynomials. It is clear that IH^2_d is invariant for M_z. Then, we define a d-tuple $S = (S_1, \ldots, S_d)$ of operators on $\mathcal{F}_I = H^2_d \ominus IH^2_d$ given by

$$S_i = P_{\mathcal{F}_I} M_{z_i} |_{\mathcal{F}_I}, \qquad 1 \leq i \leq d.$$

Let $\mathcal{S}_I \subset B(\mathcal{F}_I)$ denote the operator system generated by $S_1, \ldots, S_d$. If we let $\mathfrak{T}_I = \mathrm{C}^*(\mathcal{S}_I)$, then it is known that $\mathfrak{K}(\mathcal{F}_I) \subset \mathfrak{T}_I$ [79, Theorem 1.3]. In particular, we may define $\mathcal{O}_I = \mathfrak{T}_I/\mathfrak{K}(\mathcal{F}_I)$. Throughout, we let $q : \mathfrak{T}_I \to \mathcal{O}_I$ denote the quotient map. We now present the main conjecture on the operator tuple $(S_1, \ldots, S_d)$.

Arveson's Essential Normality Conjecture: If $I \lhd \mathbb{C}[z]$ is a homogeneous ideal, then

$S_iS_j^* - S_j^*S_i$ is a compact operator for every $1 \leq i, j \leq d$.

Tuples of operators that satisfy the above criterion are said to be *essentially normal*. Among the main interests in Arveson's essential normality conjecture is that S is the universal operator d-tuple for commuting row contractions which satisfy $p(S) = 0$ for each $p \in I$ [84, Theorem 8.4]. Accordingly, the essential normality conjecture is connected with a theory of varieties on the ball.

Kennedy and Shalit revealed a connection between the essential normality conjecture and Arveson's hyperrigidity conjecture. Assume that $I \lhd \mathbb{C}[z]$ is a non-zero homogeneous ideal that contains no linear polynomials and is not of finite codimension in $\mathbb{C}[z]$. Then, essential normality of the tuple S is equivalent to hyperrigidity of the operator system $\mathcal{S}_I$ [62, Theorem 4.12],[63]. We will further reframe essential normality of the tuple in terms of the approximate unique extension property.

Theorem 4.4.2. *Let $I \lhd \mathbb{C}[z]$ be a non-zero homogeneous ideal that contains no linear polynomials and such that I is not of finite codimension in $\mathbb{C}[z]$. Then, the following statements are equivalent.*

(i) *There is an isometric $*$-representation of $\mathcal{O}_I$ that has the approximate unique extension property with respect to $q(\mathcal{S}_I)$.*

(ii) *The tuple $S = (S_1, \ldots, S_d)$ is essentially normal.*

(iii) *The operator system $\mathcal{S}_I$ is hyperrigid.*

(iv) *Every irreducible $*$-representation of $\mathfrak{T}_I$ is a boundary representation for $\mathcal{S}_I$.*

Proof. The direction (iii)$\Rightarrow$ (iv) is trivial, (iv)$\Rightarrow$(i) is a consequence of Lemma 4.2.2 and Theorem 4.2.11, and (ii)$\Leftrightarrow$(iii) is [62, Theorem 4.12],[63].

(i)$\Rightarrow$ (ii): Let $\pi : \mathcal{O}_I \to B(\mathcal{H}_\pi)$ be an isometric $*$-representation that has the approximate unique extension property with respect to $q(\mathcal{S}_I)$ and consider $\sigma := \pi \circ q$. Observe that σ vanishes over the compact operators on $\mathcal{F}_I$ and so, by [62, Proposition 4.13], there exists a unital completely positive map $\psi : \mathfrak{T}_I \to B(\mathcal{H}_\pi)$ such that $\psi \mid_{\mathcal{S}_I} = \sigma \mid_{\mathcal{S}_I}$ and

$$\psi(S_i^* S_j) = \sigma(S_j S_i^*), \qquad 1 \leq i, j \leq d.$$

Moreover, by [62, Lemma 4.6], we may express $\psi = \varphi \circ q$ where $\varphi : \mathcal{O}_I \to B(\mathcal{H}_\pi)$ is a unital completely positive map. Since $\varphi \mid_{q(\mathcal{S}_I)} = \pi \mid_{q(\mathcal{S}_I)}$, we obtain a net of unitaries $u_\beta \in B(\mathcal{H}_\pi)$ such that $u_\beta^* \varphi(x) u_\beta \to \pi(x)$ in the weak operator topology for every $x \in \mathcal{O}_I$. Therefore, $u_\beta^* \psi(t) u_\beta \to \sigma(t)$ in the weak operator topology for every $t \in \mathfrak{T}_I$. As in the proof of Theorem 4.3.1, we obtain a homomorphic conditional expectation $\Gamma_\psi : \mathrm{C}^*(\mathrm{Im}\psi) \to \mathrm{Im}\sigma$ satisfying $\Gamma_\psi \circ \psi = \sigma$. In particular, we see that

$$\sigma(S_i^* S_j) = (\Gamma_\psi \circ \psi)(S_i^* S_j) = \sigma(S_j S_i^*), \qquad 1 \leq i, j \leq d.$$

Since $\ker \sigma = \mathfrak{K}(\mathcal{F}_I)$, we obtain that $S_i^* S_j - S_j S_i^*$ is compact for every $1 \leq i, j \leq d$. In other words, the tuple $S = (S_1, \ldots, S_d)$ is essentially normal. $\square$

Remark 4.4.3. Let $I \lhd \mathbb{C}[z]$ be a non-zero homogeneous ideal. For Theorem 4.4.2, we assumed that I contains no linear polynomials and is not of finite codimenision in $\mathbb{C}[z]$. However, there are other stipulations that work in place of our assumptions on I (see [63]).

A priori, condition (i) appears to be a weak stipulation and thus, one may expect that there are many choices of homogeneous ideals that will result in essentially normal tuples. However we remind the reader that essential normality of the tuple S is equivalent to $\mathcal{O}_I$ being the C*-envelope of $q(\mathcal{S}_I)$. As such, it is unclear whether $\mathcal{O}_I$ will possess an isometric $*$-representation with the approximate unique extension property.

Chapter 5

Non-commutative Korovkin Approximations for Operator Spaces

In this chapter, our primary purpose is to further address Arveson's hyperrigidity conjecture. For this, we will first restate Arveson's conjecture.

Arveson's hyperrigidity conjecture: Let $\mathcal{M} \subset B(\mathcal{H})$ be a separable unital operator space. If every irreducible $*$-representation for $\mathrm{C}^*(\mathcal{M})$ is a boundary representation, then $\mathcal{M}$ is hyperrigid.

Arveson's conjecture was partially motivated by statements in classical approximation theory due to Korovkin and Šaškin [68],[83]. Initially, Korovkin had proved that, whenever $\varphi_n : \mathrm{C}[0,1] \to \mathrm{C}[0,1]$ is a sequence of positive linear maps such that

$$\lim_{n\to\infty} \|\varphi_n(x^k) - x^k\| = 0, \qquad k = 0,1,2,$$

then we necessarily have that

$$\lim_{n\to\infty} \|\varphi_n(f) - f\| = 0, \qquad f \in \mathrm{C}[0,1].$$

Afterwards, Šaškin demonstrated that Korovkin's theorem is an instance of all irreducible $*$-representations of $\mathrm{C}[0,1]$ being boundary representations for the unital span of $\{x, x^2\}$. In particular, recall that Šaškin proved the following statements to be equivalent for a separable unital subspace $\mathcal{M} \subset \mathrm{C}(X)$ that separates points in X.

(i) Whenever $\varphi_n : \mathrm{C}(X) \to \mathrm{C}(X)$ is a sequence of positive linear maps such that $\|\varphi_n(g) - g\| \to 0$ for each $g \in \mathcal{M}$, then we have that $\|\varphi_n(f) - f\| \to 0$ for every $f \in \mathrm{C}(X)$.

(ii) For every compact metric space Y, every $*$-representation $\pi : \mathrm{C}(X) \to \mathrm{C}(Y)$, and every sequence of positive linear maps $\varphi_n : \mathrm{C}(X) \to \mathrm{C}(Y)$ such that $\|\varphi_n(g) - \pi(g)\| \to 0$ for every $g \in \mathcal{M}$, we have that $\|\varphi_n(f) - \pi(f)\| \to 0$ for every $f \in \mathrm{C}(X)$.

(iii) Every irreducible $*$-representation for $\mathrm{C}(X)$ is a boundary representation for $\mathcal{M}$.

Upon replacing sequences by nets, a non-separable version of Šaškin's theorem is also known (Theorem 2.3.4).

The primary focus of our work in this chapter is, instead of addressing Arveson's conjecture directly, we analyze non-commutative approximations and extensions of Šaškin's theorem. Non-commutative Šaškin and Korovkin-type theorems have been addressed several times [17],[67],[80],[81],[89], although our primary contributions appear to be distinct. As we progress, we will highlight the underlying connections our work has with Arveson's non-commutative Choquet boundary, as well as the counterexample to Arveson's hyperrigidity conjecture [10].

In Section 5.1, we analyze an approximate rigidity property on the identity representation. Namely, we say that a unital operator space $\mathcal{M} \subset B(\mathcal{H})$ is *Korovkin* relative to $\mathrm{C}^*(\mathcal{M})$ if, whenever $\varphi_\lambda : \mathrm{C}^*(\mathcal{M}) \to \mathrm{C}^*(\mathcal{M})$ is a net of unital completely positive maps such that

$$\|\varphi_\lambda(m) - m\| \to 0, \qquad m \in \mathcal{M},$$

then we necessarily have that

$$\|\varphi_\lambda(t) - t\| \to 0, \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

The major result in this chapter is that, in the presence of the lifting property, a faithful analogue of Šaškin's theorem may be achieved (Theorem 5.1.6).

Theorem G. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space such that $\mathrm{C}^*(\mathcal{M})$ has the lifting property. Then, the following statements are equivalent.*

(i) *We have that $\mathcal{M}$ is Korovkin relative to $\mathrm{C}^*(\mathcal{M})$.*

(ii) *For every $*$-representation $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ and every net of unital completely positive maps $\varphi_\lambda : \mathrm{C}^*(\mathcal{M}) \to \pi(\mathrm{C}^*(\mathcal{M}))$ satisfying*

$$\lim_\lambda \|\varphi_\lambda(m) - \pi(m)\| = 0, \qquad m \in \mathcal{M},$$

we have that

$$\lim_\lambda \|\varphi_\lambda(t) - \pi(t)\| = 0, \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

5.1 A non-commutative Šaškin theorem

In this section, we will analyze Korovkin operator spaces. To start, we remark that an operator space being Korovkin can be reinterpreted as a particular choice of $*$-representation possessing a relative unique extension property. This is inspired by an argument of Arveson [7, Theorem 2.1]. For simplicity of notation, we restrict ourselves to the separable setting.

Proposition 5.1.1. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital separable operator space. Consider the closed two-sided ideal*

$$\mathfrak{J} = \left\{ (t_n)_{n\geq 1} \in \prod_{n=1}^{\infty} \mathrm{C}^*(\mathcal{M}) \ : \ \|t_n\| \to 0 \right\}$$

and define a $*$-homomorphism

$$\Pi_{\mathfrak{u}} : \mathrm{C}^*(\mathcal{M}) \to \left(\prod_{n=1}^{\infty} \mathrm{C}^*(\mathcal{M}) \right) / \mathfrak{J}, \qquad t \mapsto (t)_{n\geq 1} + \mathfrak{J}.$$

Then, the following statements are equivalent.

(i) *We have that $\mathcal{M}$ is Korovkin relative to $\mathrm{C}^*(\mathcal{M})$.*

(ii) *Whenever $\Psi : \mathrm{C}^*(\mathcal{M}) \to \left(\prod_\lambda \mathrm{C}^*(\mathcal{M})\right)/\mathfrak{J}$ is a unital completely positive map satisfying $\Psi\,|_{\mathcal{M}} = \Pi_{\mathfrak{u}}\,|_{\mathcal{M}}$, then we have that $\Psi = \Pi_{\mathfrak{u}}$.*

Proof. Let $\varphi_n : \mathrm{C}^*(\mathcal{M}) \to \mathrm{C}^*(\mathcal{M})$ be a sequence of unital completely positive maps satisfying $\|\varphi_n(m) - m\| \to 0$ for each $m \in \mathcal{M}$. Define

$$\Phi : \mathrm{C}^*(\mathcal{M}) \to \left(\prod_{n=1}^{\infty} \mathrm{C}^*(\mathcal{M}) \right) / \mathfrak{J}, \qquad t \mapsto (\varphi_n(t))_{n\geq 1} + \mathfrak{J}.$$

It is easily verified that Φ is a unital completely positive map satisfying $\Phi\,|_{\mathcal{M}} = \Pi_{\mathfrak{u}}\,|_{\mathcal{M}}$. Moreover, $\Phi = \Pi_{\mathfrak{u}}$ if and only if $\|\varphi_n(t) - t\| \to 0$ for every $t \in \mathrm{C}^*(\mathcal{M})$. So, the conclusion is immediate. □

The most fruitful source of non-commutative spaces that are Korovkin are the hyperrigid operator spaces. Indeed, recall that an operator space $\mathcal{M} \subset B(\mathcal{H})$ is *hyperrigid* if, whenever $\varepsilon : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{K})$ is an isometric $*$-representation and $\varphi_\lambda : B(\mathcal{K}) \to B(\mathcal{K})$ is a net of unital completely positive maps satisfying

$$\lim_\lambda \|\varphi_\lambda(\varepsilon(m)) - \varepsilon(m)\| = 0, \qquad m \in \mathcal{M},$$

then we necessarily have that

$$\lim_\lambda \|\varphi_\lambda(\varepsilon(t)) - \varepsilon(t)\| = 0, \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

When *there exists* an isometric $*$-representation ε that has the above properties, then $\mathcal{M}$ is automatically Korovkin relative to $\mathrm{C}^*(\mathcal{M})$. Indeed, this follows as a consequence to the following lemma.

Lemma 5.1.2. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space. Suppose that $\theta : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\theta)$ is an isometric $*$-representation. Then, $\mathcal{M}$ is Korovkin relative to $\mathrm{C}^*(\mathcal{M})$ if and only if $\theta(\mathcal{M})$ is Korovkin relative to $\theta(\mathrm{C}^*(\mathcal{M}))$.*

Proof. (⇒) Assume that $\mathcal{M}$ is Korovkin relative to $\mathrm{C}^*(\mathcal{M})$ and let $\psi_\lambda : \theta(\mathrm{C}^*(\mathcal{M})) \to \theta(\mathrm{C}^*(\mathcal{M}))$ be a net of unital completely positive maps such that

$$\lim_\lambda \|\psi_\lambda(\theta(m)) - \theta(m)\| = 0, \qquad m \in \mathcal{M}.$$

Then, define a net of unital completely positive maps $\varphi_\lambda : \mathrm{C}^*(\mathcal{M}) \to \mathrm{C}^*(\mathcal{M})$ by $\varphi_\lambda = \theta^{-1} \circ \psi_\lambda \circ \theta$. It is clear that

$$\lim_\lambda \|\varphi_\lambda(m) - m\| = 0, \qquad m \in \mathcal{M}.$$

As $\mathcal{M}$ is Korovkin relative to $\mathrm{C}^*(\mathcal{M})$, we conclude that

$$\lim_\lambda \|\varphi_\lambda(t) - t\| = 0, \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

Thus, as θ^{-1} is isometric, we have that

$$\lim_\lambda \|\psi_\lambda(\theta(t)) - \theta(t)\| = \lim_\lambda \|\theta^{-1}(\psi_\lambda(\theta(t)) - \theta(t))\| = \lim_\lambda \|\varphi_\lambda(t) - t\| = 0, \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

So, $\theta(\mathcal{M})$ is Korovkin relative to $\theta(\mathrm{C}^*(\mathcal{M}))$.

(⇐) This follows from the previous direction upon considering θ^{-1}. □

Outside of the function theoretic setting, one might expect that $\mathcal{M}$ being Korovkin does not necessarily imply that all irreducible $*$-representations for $\mathrm{C}^*(\mathcal{M})$ are boundary representations for $\mathcal{M}$. Moreover, it appears even less likely that being Korovkin is equivalent to hyperrigidity. Nevertheless, we will quantify one instance of this, which the author assumes is known amongst experts. For this, we require a minor simplification.

Lemma 5.1.3. *Let $\mathcal{M} \subset \mathbb{M}_r$ be a unital operator space. Then, the following statements are equivalent.*

(i) *We have that $\mathcal{M}$ is Korovkin relative to $\mathrm{C}^*(\mathcal{M})$.*

(ii) *Whenever $\varphi : \mathrm{C}^*(\mathcal{M}) \to \mathrm{C}^*(\mathcal{M})$ is a unital completely positive map satisfying $\varphi\,|_{\mathcal{M}} = id_{\mathcal{M}}$, we have that $\varphi = id_{\mathrm{C}^*(\mathcal{M})}$.*

Proof. (i)⇒(ii): Obvious.

(ii)⇒(i): Let $\varphi_n : \mathrm{C}^*(\mathcal{M}) \to \mathrm{C}^*(\mathcal{M})$ be a sequence of unital completely positive maps satisfying

$$\lim_{n\to\infty} \|\varphi_n(m) - m\| = 0, \qquad m \in \mathcal{M}.$$

By compactness, there exists a point-WOT limit point φ for the sequence $(\varphi_n)_{n\geq 1}$. It is evident that $\varphi\,|_{\mathcal{M}} = id_{\mathcal{M}}$. Moreover, we have that $\varphi(\mathrm{C}^*(\mathcal{M})) \subset \mathrm{C}^*(\mathcal{M})$ because $\mathrm{C}^*(\mathcal{M})$ is closed in the weak operator topology. Therefore, it follows that φ is the identity representation for $\mathrm{C}^*(\mathcal{M})$.

In other words, the identity representation for $\mathrm{C}^*(\mathcal{M})$ is the unique point-WOT limit point for the sequence $(\varphi_n)_{n\geq 1}$. Due to finite-dimensionality, we may conclude that $(\varphi_n)_{n\geq 1}$ necessarily converges to the identity representation in the point-norm topology. So, $\mathcal{M}$ is Korovkin relative to $\mathrm{C}^*(\mathcal{M})$. □

Therefore, in this particular scenario, Korovkin operator spaces are automatically hyperrigid.

Corollary 5.1.4. *Let $\mathcal{M} \subset \mathbb{M}_r$ be a unital operator space. Then, the following statements are equivalent.*

(i) *We have that $\mathcal{M}$ is Korovkin relative to $\mathrm{C}^*(\mathcal{M})$.*

(ii) *Every irreducible $*$-representation for $\mathrm{C}^*(\mathcal{M})$ is a boundary representation for $\mathcal{M}$.*

(iii) *The operator space $\mathcal{M}$ is hyperrigid.*

Proof. Statements (ii) and (iii) are known to be equivalent (see [7, Theorem 5.1]). We show that statements (i) and (ii) are equivalent.

To this end, due to Lemma 5.1.2, we may assume that $\mathrm{C}^*(\mathcal{M}) = \bigoplus_{i=1}^n \mathbb{M}_{r_i}$ for some collection of positive integers $\{r_1, \dots, r_n\}$. By Lemma 5.1.3, $\mathcal{M}$ is Korovkin relative to $\mathrm{C}^*(\mathcal{M})$ precisely when any unital completely positive map $\varphi : \mathrm{C}^*(\mathcal{M}) \to \mathrm{C}^*(\mathcal{M})$ fixing $\mathcal{M}$ is the identity representation. Note that any unital completely positive map $\varphi : \mathrm{C}^*(\mathcal{M}) \to \mathrm{C}^*(\mathcal{M})$ may be expressed as $\varphi = \bigoplus_{i=1}^n \varphi_i$ where

$$\varphi_i : \mathrm{C}^*(\mathcal{M}) \to \mathbb{M}_{r_i}, \qquad t \mapsto P_{\mathbb{C}^{r_i}} \varphi(t) \mid_{\mathbb{C}^{r_i}},$$

because the projections $P_{\mathbb{C}^{r_1}}, \dots, P_{\mathbb{C}^{r_n}}$ are reducing for $\varphi(\mathrm{C}^*(\mathcal{M})) \subset \bigoplus_{i=1}^n \mathbb{M}_{r_i}$. Thus, φ fixes $\mathcal{M}$ precisely when $\varphi_i \mid_{\mathcal{M}} = \pi_i \mid_{\mathcal{M}}$ for each i, where $\pi_i : \mathrm{C}^*(\mathcal{M}) \to \mathbb{M}_{r_i}$ is the natural projection mapping. In particular, we have that φ is necessarily the identity representation if and only if $\varphi_i = \pi_i$ for each i, which occurs precisely when each projection mapping is a boundary representation for $\mathcal{M}$. □

Before proceeding, we record an infinite-dimensional example of an operator space that fails to be Korovkin.

Example 11. Let $S \in B(\ell^2)$ be the unilateral shift and let $\mathcal{T} = \mathrm{C}^*(S)$ denote the Toeplitz algebra (Subsection 2.1.3). Let $\mathcal{M}$ be the unital operator space generated by S. Then,

$$\varphi : \mathcal{T} \to \mathcal{T}, \qquad X \mapsto S^* X S$$

is a unital completely positive map with the property that $\varphi(m) = m$ for each $m \in \mathcal{M}$. On the other hand, we have that $\varphi(SS^*) = I$ and so φ is not the identity representation for $\mathcal{T}$. Thus, $\mathcal{M}$ is not Korovkin relative to $\mathcal{T}$.

For simplicity, we will require some terminology in our next developments.

Definition 5.1.5. Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space. We say that a $*$-representation $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is *Korovkin* with respect to $\mathcal{M}$ if, whenever $\varphi_\lambda : \mathrm{C}^*(\mathcal{M}) \to \pi(\mathrm{C}^*(\mathcal{M}))$ is a net of unital completely maps satisfying

$$\lim_\lambda \|\varphi_\lambda(m) - \pi(m)\| = 0, \qquad m \in \mathcal{M},$$

we have that

$$\lim_\lambda \|\varphi_\lambda(t) - \pi(t)\| = 0, \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

Next, we require an additional technical constraint. A unital C*-algebra $\mathfrak{A}$ is said to have the *lifting property* if, whenever $\mathfrak{J} \subset \mathfrak{B}$ is a closed two-sided ideal of a unital C*-algebra $\mathfrak{B}$, $\varphi : \mathfrak{A} \to \mathfrak{B}/\mathfrak{J}$ is a unital completely positive map, there exists a unital completely positive map $\Phi : \mathfrak{A} \to \mathfrak{B}$ with the property that $q \circ \Phi = \varphi$ where $q : \mathfrak{B} \to \mathfrak{B}/\mathfrak{J}$ is the quotient mapping. Those C*-algebras with the lifting property form a large class that includes many examples of interest. For example, the Choi–Effros lifting theorem [18] asserts that all nuclear C*-algebras have the lifting property.

We now present the main result of this chapter.

Theorem 5.1.6. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space and assume that* $\mathrm{C}^*(\mathcal{M})$ *has the lifting property. Then, the following statements are equivalent.*

(i) *We have that $\mathcal{M}$ is Korovkin relative to* $\mathrm{C}^*(\mathcal{M})$.

(ii) *Every $*$-representation of* $\mathrm{C}^*(\mathcal{M})$ *is Korovkin with respect to $\mathcal{M}$.*

The proof is inspired by Arveson's lifting theorem [5, Section 3] and the original proof of Šaškin's theorem for function spaces [76, Section 9]. Additionally, we require the fact that every closed two-sided ideal $\mathfrak{J}$ of a C*-algebra $\mathfrak{A}$ admits a *quasicentral approximate unit* [34, Theorem 1.9.16]. That is, there is an increasing net $(s_\mu)_\mu$ of positive contractions in $\mathfrak{J}$ such that

$$\lim_\mu \|s_\mu k - k\| = \lim_\mu \|k s_\mu - k\| = 0, \qquad k \in \mathfrak{J},$$

and

$$\lim_\mu \|s_\mu t - t s_\mu\| = 0, \qquad t \in \mathfrak{A}.$$

Proof. (ii)⇒(i): Obvious.

(i)⇒(ii): As $\mathrm{C}^*(\mathcal{M})$ has the lifting property, we may find a net of unital completely positive maps $\Psi_\lambda : \mathrm{C}^*(\mathcal{M}) \to \mathrm{C}^*(\mathcal{M})$ such that $\pi \circ \Psi_\lambda = \psi_\lambda$. Now, let $(s_\mu)_\mu$ denote a quasicentral approximate unit for $\ker \pi$ relative to $\mathrm{C}^*(\mathcal{M})$ and define a net of unital completely positive maps $\Psi_{\lambda,\mu} : \mathrm{C}^*(\mathcal{M}) \to \mathrm{C}^*(\mathcal{M})$ by

$$\Psi_{\lambda,\mu}(t) = s_\mu^{1/2} t s_\mu^{1/2} + (I - s_\mu)^{1/2} \Psi_\lambda(t) (I - s_\mu)^{1/2}.$$

Since $(s_\mu)_\mu$ is quasicentral, a standard approximation guarantees that

$$\lim_\mu \|t f(s_\mu) - f(s_\mu) t\| = 0, \qquad t \in \mathrm{C}^*(\mathcal{M}),$$

whenever f is a uniform limit of polynomials on the unit interval. In particular,

$$\limsup_\mu \|s_\mu^{1/2} t s_\mu^{1/2} - s_\mu t\| \le \limsup_\mu \|t s_\mu^{1/2} - s_\mu^{1/2} t\| = 0, \qquad t \in \mathrm{C}^*(\mathcal{M}),$$

and so, $\|s_\mu^{1/2} t s_\mu^{1/2} - s_\mu t\| \to 0$ whenever $t \in \mathrm{C}^*(\mathcal{M})$.

Fix $m \in \mathcal{M}$ and $\varepsilon > 0$. Then, note that

$$\lim_\lambda \|\pi(m - \Psi_\lambda(m))\| = \lim_\lambda \|\pi(m) - \psi_\lambda(m)\| = 0$$

and so, we may find some λ_0 such that

$$\|\pi(m - \Psi_\lambda(m))\| < 2\varepsilon, \qquad \lambda \geq \lambda_0.$$

By the first isomorphism theorem for C*-algebras and by definition of the quotient norm, we may then extract a net $(K_\lambda)_{\lambda \geq \lambda_0}$ of operators in $\ker \pi$ such that

$$\|(m - \Psi_\lambda(m)) - K_\lambda\| < \varepsilon, \qquad \lambda \geq \lambda_0.$$

For each $\lambda \geq \lambda_0$, we may find some μ_λ such that

$$\|(I - s_\mu)K_\lambda\| < \varepsilon, \qquad \mu \geq \mu_\lambda,$$

as $(s_\mu)_\mu$ is an approximate unit for $\ker \pi$. Additionally, as $(s_\mu)_\mu$ is quasicentral, we may assume that μ_λ is chosen so that

$$\|s_\mu m - s_\mu^{1/2} m s_\mu^{1/2}\|, \|(I - s_\mu)\Psi_\lambda(m) - (I - s_\mu)^{1/2}\Psi_\lambda(m)(I - s_\mu)^{1/2}\| < \varepsilon$$

whenever $\mu \geq \mu_\lambda$ and $\lambda \geq \lambda_0$. Therefore, whenever $\lambda \geq \lambda_0$ and $\mu \geq \mu_\lambda$,

$$\begin{aligned}
\|m - \Psi_{\lambda,\mu}(m)\| &= \|(m - s_\mu^{1/2} m s_\mu^{1/2}) - (I - s_\mu)^{1/2}\Psi_\lambda(m)(I - s_\mu)^{1/2}\| \\
&\leq \|(I - s_\mu)(m - \Psi_\lambda(m))\| + \|s_\mu m - s_\mu^{1/2} m s_\mu^{1/2}\| \\
&\qquad + \|(I - s_\mu)\Psi_\lambda(m) - (I - s_\mu)^{1/2}\Psi_\lambda(m)(I - s_\mu)^{1/2}\| \\
&< \|(m - \Psi_\lambda(m)) - K_\lambda\| + \|(I - s_\mu)K_\lambda\| + 2\varepsilon \\
&< 4\varepsilon.
\end{aligned}$$

So, we have that

$$\lim_\lambda \|m - \Psi_{\lambda,\mu_\lambda}(m)\| = 0, \qquad m \in \mathcal{M}.$$

As $\mathcal{M}$ is Korovkin relative to $\mathrm{C}^*(\mathcal{M})$, we can conclude that

$$\lim_\lambda \|t - \Psi_{\lambda,\mu_\lambda}(t)\| = 0, \qquad t \in \mathrm{C}^*(\mathcal{M}).$$

Finally, observe that

$$(\pi \circ \Psi_{\lambda,\mu})(t) = (\pi \circ \Psi_\lambda)(t) = \psi_\lambda(t), \qquad t \in \mathrm{C}^*(\mathcal{M}),$$

for each μ and λ. In particular,

$$\limsup_\lambda \|\pi(t) - \psi_\lambda(t)\| \leq \limsup_\lambda \|t - \Psi_{\lambda,\mu_\lambda}(t)\| = 0, \qquad t \in \mathrm{C}^*(\mathcal{M}),$$

and so,

$$\lim_\lambda \|\psi_\lambda(t) - \pi(t)\| = 0, \qquad t \in \mathrm{C}^*(\mathcal{M}),$$

as desired.

□

It is worth discussing some partial refinements and consequences to Theorem 5.1.6. First, the lifting property is not completely necessary to achieve some of the conclusions in Theorem 5.1.6. Indeed, for (i)⇒(ii), we can still retain the same approximation property for nets of *nuclear* completely positive maps. For this, recall that a unital completely positive map $\theta : \mathfrak{A} \to \mathfrak{B}$ between unital C*-algebras is *nuclear* if there exists a pair of nets $\eta_\lambda : \mathfrak{A} \to \mathbb{M}_{r_\lambda}$ and $\tau_\lambda : \mathbb{M}_{r_\lambda} \to \mathfrak{B}$ of contractive completely positive maps such that

$$\lim_{\lambda} \|(\tau_\lambda \circ \eta_\lambda)(t) - \theta(t)\| = 0, \qquad t \in \mathfrak{A}.$$

Corollary 5.1.7. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a separable unital operator space and assume that $\mathcal{M}$ is Korovkin relative to $\mathrm{C}^*(\mathcal{M})$. Then, whenever $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ is a $*$-representation and $\psi_\lambda : \mathrm{C}^*(\mathcal{M}) \to \pi(\mathrm{C}^*(\mathcal{M}))$ is a net of nuclear unital completely positive maps satisfying $\|\psi_\lambda(m) - \pi(m)\| \to 0$ for each $m \in \mathcal{M}$, then we necessarily have that $\|\psi_\lambda(t) - \pi(t)\| \to 0$ for each $t \in \mathrm{C}^*(\mathcal{M})$.*

Proof. By [5, Theorem 7], the proof becomes identical to Theorem 5.1.6 (i)⇒(ii). □

Therefore, we find that Korovkin operator spaces admit some connections to Arveson's non-commutative Choquet boundary. Moreover, due to Corollary 5.1.7, our consequence does not require the local lifting property.

Corollary 5.1.8. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a separable unital operator space. If $\mathcal{M}$ is Korovkin relative to $\mathrm{C}^*(\mathcal{M})$, then all finite-dimensional $*$-representations for $\mathrm{C}^*(\mathcal{M})$ have the unique extension property with respect to $\mathcal{M}$.*

Proof. As the unique extension property passes to subrepresentations (Lemma 2.4.2), it suffices to prove the statement for irreducible finite-dimensional $*$-representations. The latter maps are surjective and so this becomes a consequence to Corollary 5.1.7. □

We are unsure whether the converse to Corollary 5.1.8 holds for the class of liminal C*-algebras. Nevertheless, the class of liminal C*-algebras are not the only possible examples that showcase a connection between the non-commutative Choquet boundary and being Korovkin.

Example 12. Let $\mathcal{H}$ be a regular unitarily invariant complete Pick space on $\mathbb{B}_d$ (Section 2.5) and assume that $\mathcal{H}$ is not the Hardy space on the unit disc. Let $\mathrm{A}(\mathcal{H})$ denote the norm-closure of the polynomial multipliers and $\mathfrak{T}(\mathcal{H}) := \mathrm{C}^*(\mathrm{A}(\mathcal{H}))$. By Theorem 2.5.1, the irreducible $*$-representations for $\mathfrak{T}(\mathcal{H})$ consist of the identity representation and some family of characters.

Moreover, by Theorem 2.5.1, the identity representation for $\mathfrak{T}(\mathcal{H})$ is always a boundary representation for $\mathrm{A}(\mathcal{H})$. Moreover, every character for $\mathfrak{T}(\mathcal{H})$ will be a boundary representation for $\mathrm{A}(\mathcal{H})$ precisely when the corresponding reproducing kernel is unbounded. As $\mathfrak{T}(\mathcal{H})$ is postliminal, it is nuclear [16, Proposition 2.7.4],[40, Theorem 9.1]. So, $\mathfrak{T}(\mathcal{H})$ has the local lifting property by the Choi–Effros lifting theorem [18]. Thus, by Theorem 5.1.6, $\mathrm{A}(\mathcal{H})$ is Korovkin relative to $\mathfrak{T}(\mathcal{H})$ if and only if the kernel is unbounded. By [22, Theorem 6.3], we conclude that $\mathrm{A}(\mathcal{H})$ is Korovkin precisely when it is hyperrigid.

Note that, when $\mathcal{H}$ is the Hardy space, we have the same conclusion by Example 11. Moreover, there are examples where the reproducing kernel for $\mathcal{H}$ can be bounded (see [22, Subsection 2.4]). Thus, Example 12 provides a family of operator spaces where the identity representation for $\mathrm{C}^*(\mathcal{M})$ is a boundary representation, yet $\mathcal{M}$ fails to be Korovkin relative to $\mathrm{C}^*(\mathcal{M})$.

Additionally, the family of irreducible $*$-representations for $\mathrm{C}^*(\mathcal{M})$ that are Korovkin are not always abundant enough to determine a completely isometrically isomorphic copy of $\mathcal{M}$. Indeed, assume that the kernel for $\mathcal{H}$ is bounded and let $\mathscr{F}$ be the collection of unitary equivalence classes $[\pi]$ of irreducible $*$-representations for $\mathfrak{T}(\mathcal{H})$ with the property that, whenever $\psi_\lambda : \mathfrak{T}(\mathcal{H}) \to \pi(\mathfrak{T}(\mathcal{H}))$ is a net of unital completely positive maps satisfying $\|\psi_\lambda(a) - \pi(a)\| \to 0$ for each $a \in \mathrm{A}(\mathcal{H})$, then $\|\psi_\lambda(t) - \pi(t)\| \to 0$ for each $t \in \mathfrak{T}(\mathcal{H})$. As the kernel is bounded, by Example 12, the family $\mathscr{F}$ does not contain the unitary equivalence class for the identity representation of $\mathfrak{T}(\mathcal{H})$. On the other hand, the identity representation is a boundary representation for $\mathrm{A}(\mathcal{H})$ and so, the C^*-envelope of $\mathrm{A}(\mathcal{H})$ is $\mathfrak{T}(\mathcal{H})$. By the universal property of the C^*-envelope, the only possible way for the family $\mathscr{F}$ to completely norm $\mathrm{A}(\mathcal{H})$ is if $\mathscr{F}$ includes the identity representation. Indeed, otherwise $\left(\bigoplus_{[\pi]\in\mathscr{F}} \pi\right)|_{\mathrm{A}(\mathcal{H})}$ is completely isometric and thus, $\mathrm{C}^*_e(\mathrm{A}(\mathcal{H})) \cong \mathfrak{T}(\mathcal{H})$ is a quotient of $\left(\bigoplus_{[\pi]\in\mathscr{F}} \pi\right)(\mathfrak{T}(\mathcal{H}))$. This is a contradiction as $\left(\bigoplus_{[\pi]\in\mathscr{F}} \pi\right)(\mathfrak{T}(\mathcal{H}))$ is commutative if $\mathscr{F}$ does not include the identity representation. Therefore, we may conclude that $\bigoplus_{[\pi]\in\mathscr{F}} \pi$ is not completely isometric for $\mathrm{A}(\mathcal{H})$. In particular, the irreducible $*$-representations for $\mathrm{C}^*(\mathcal{M})$ that are Korovkin do not always appear to be as faithful of an extension of the classical Choquet boundary when compared to Arveson's boundary representations.

Chapter 6

Conclusion and future directions

In this chapter, we provide a brief summary of some of the major components of this thesis, as well as some open questions.

In Chapter 3, we studied residually finite-dimensional (RFD) operator algebras, which are those operator algebras that may be embedded completely isometrically into a product of matrix algebras. A conjecture of Clouâtre-Ramsey [24] asserted that the maximal C*-cover of an RFD operator algebra is RFD. Hartz has recently found a counterexample to the conjecture [55], but we focus on describing suitable replacements to the maximal C*-cover. In particular, we show that every RFD operator algebra $\mathcal{A}$ admits an RFD C*-cover $(\mathfrak{R}(\mathcal{A}), \mu_r)$ that is universal among all RFD C*-covers of $\mathcal{A}$ (Theorem 3.2.2). We describe this C*-cover topologically (Theorem 3.2.4), concretely (Theorem 3.2.6), and describe some of the properties that it satisfies (Subsection 3.2.3). Afterwards, we provide an alternative analysis in terms of representations of the operator algebra that can be approximated in different choices of topologies (Section 3.3). In particular, we construct one such C*-cover, $(\mathfrak{R}_s(\mathcal{A}), \mu_{r_s})$, that is built from representations of $\mathcal{A}$ that may be approximated by finite-dimensional representations of $\mathcal{A}$ in the pointwise-SOT topology. By Lemma 3.3.2, we know that

$$(\mathfrak{R}(\mathcal{A}), \mu_r) \preceq (\mathfrak{R}_s(\mathcal{A}), \mu_{r_s}) \preceq (\mathrm{C}^*_{max}(\mathcal{A}), \mu)$$

and, from Example 7, we may infer that $(\mathfrak{R}_s(\mathcal{A}), \mu_{r_s})$ is not always the maximal C*-cover of $\mathcal{A}$. However, we have been unable to determine whether $(\mathfrak{R}_s(\mathcal{A}), \mu_{r_s})$ can differ from the RFD-maximal C*-cover.

Question 1. Is there an example of an RFD operator algebra $\mathcal{A}$ such that $(\mathfrak{R}_s(\mathcal{A}), \mu_{r_s})$ is not the RFD-maximal C*-cover? Equivalently, find an example of a $*$-RFD representation of $\mathcal{A}$ that is not RFD.

It appears feasible that a suitable alteration of Hartz's proof [55] would yield a counterexample to Question 1. Indeed, recall that a central aspect to Hartz's argument is related to a particular operator T being able to be approximated by unitaries in the WOT, yet not in the SOT [55, Discussion after Theorem 4.3]. Here, the main obstruction from T being able to approximated in the SOT by unitaries is that T fails to be an isometry. It is feasible that one can construct a similar counterexample that addresses Question 1 because any (non-unitary) isometry can be approximated in the SOT by unitaries, yet cannot be approximated by unitaries in the $*$SOT [52, Theorem 4.4].

The remainder of this thesis is primarily concerned with Arveson's hyperrigidity conjecture, which has recently been disproven in the postliminal setting [10]. Arveson conjectured that, if every irreducible $*$-representation for $C^*(\mathcal{M})$ is a boundary representation for a separable unital operator space $\mathcal{M}$, then every $*$-representation for $C^*(\mathcal{M})$ has the unique extension property with respect to $\mathcal{M}$ [7]. In Chapter 4, we leveraged Voiculescu's theorem [90] to identify approximate forms of the unique extension property. Most notably, we found that any completely positive extension ψ of a $*$-representation π for $C^*(\mathcal{M})$ lies in the pointwise-WOT closed unitary orbit of π (Theorem 4.2.11). We refer to this constraint on the completely positive extensions of π as the approximate unique extension property. We find that the approximate unique extension property implies the existence of a $*$-homomorphism $\Gamma_\psi : C^*(\text{Im}\psi) \to \text{Im}\pi$ that is also a projection (Theorem 4.3.1). When all irreducible $*$-representations for $C^*(\mathcal{M})$ are boundary representations, then the mapping Γ_ψ unveils quite notable structural constraints on the completely positive extensions of all $*$-representations for $C^*(\mathcal{M})$. However, our approach is purely local in the sense that Theorem 4.3.1 may be applied solely to the approximate unitary equivalence class of a single $*$-representation rather than with the (global) assumption that every irreducible $*$-representation be a boundary representation. For Arveson's conjecture, this local approach is notable due to some troubling examples (Example 9) as well as in light of the recent counterexample [10].

In the future, it would be interesting to further unravel the structure of the completely positive extensions of a $*$-representation with the approximate unique extension property. Here, we propose one such possibility.

Question 2. Let $\mathcal{M} \subset B(\mathcal{H})$ be a unital operator space. Let $\pi : C^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a $*$-representation with the approximate unique extension property and let $\psi : C^*(\mathcal{M}) \to B(\mathcal{H}_\pi)$ be a unital completely positive map such that $\psi |_{\mathcal{M}} = \pi |_{\mathcal{M}}$. If ω is a pure state for $\text{Im}\pi$, then is ω the weak-$*$ limit of pure states for $\text{Im}\pi$ that uniquely extend to $C^*(\text{Im}\psi)$?

Clouâtre proved that, when all irreducible $*$-representations for $C^*(\mathcal{M})$ are boundary representations, then Arveson's conjecture reduces to proving that *every* pure state for $\text{Im}\pi$ uniquely extends to $C^*(\text{Im}\psi)$ [20, Theorem 3.10]. Thus, a positive answer to Question 2 would be one realization of an approximate answer to Arveson's conjecture.

In Chapter 5, we recast Arveson's hyperrigidity conjecture and focus on non-commutative extensions of Šaškin's theorem. Recall that Šaškin's theorem asserts that, for function spaces, an approximate rigidity condition for the identity representation is equivalent to all representations sharing an approximate rigidity property, and that this is further equivalent to all irreducible $*$-representations being boundary representations. For a large class of C^*-algebras (including nuclear C^*-algebras), we find an analogue to Šaškin's theorem is attainable (Theorem 5.1.6).

We call an operator space Korovkin if it satisfies an appropriate rigidity condition for the identity representation. At the present time, verifying that an operator space is Korovkin remains somewhat difficult. Thus, it would be rewarding to find more examples of Korovkin operator spaces. We suggest characterizing when a specific class of operator systems is Korovkin.

As in Subsection 4.4.2, let $I \subset \mathbb{C}[z_1, \dots, z_d]$ be a homogeneous ideal and let $\mathcal{S}_I \subset B(\mathcal{F}_I)$ denote the operator system generated by the operator d-tuple $S = (S_1, \dots, S_d)$. Recall that Arveson conjectured S is essentially normal and, for sufficiently non-trivial homogeneous ideals, Arveson's conjecture is equivalent to $\mathcal{S}_I$ being hyperrigid [62]. In proving this equivalence, Arveson's boundary theorem [4] was used to verify that the identity representation for $C^*(\mathcal{S}_I)$ is a boundary representation for $\mathcal{S}_I$ [62],[63]. However, at this time, the author is unaware of whether $\mathcal{S}_I$ is Korovkin relative to $C^*(\mathcal{S}_I)$. In fact, we suggest that being Korovkin is a highly restrictive condition on the operator system $\mathcal{S}_I$.

Question 3. Let $I \subset \mathbb{C}[z_1, \dots, z_d]$ be a non-zero homogeneous ideal that contains no linear polynomials and that is not of finite codimension in $\mathbb{C}[z_1, \dots, z_d]$. Determine whether the following statements are equivalent.

(i) We have that $\mathcal{S}_I$ is Korovkin relative to $C^*(\mathcal{S}_I)$.

(ii) The operator d-tuple $S = (S_1, \dots, S_d)$ is essentially normal.

It appears plausible that Question 3 has a positive answer. As demonstrated in Theorem 5.1.6, Korovkin operator spaces tend to impose a significant rigidity property on the space of all irreducible $*$-representations. In Theorem 4.4.2, it was shown that essential normality of $S = (S_1, \dots, S_d)$ is equivalent to all irreducible $*$-representations for $C^*(\mathcal{S}_I)$ being boundary representations for $\mathcal{S}_I$, which is one realization of a rigidity property on the space of all irreducible $*$-representations.

In addition, we remark that the proof for why the identity representation for $C^*(\mathcal{S}_I)$ has the unique extension property is dependent on Arveson's boundary theorem [62],[63]. In other words, as $C^*(\mathcal{S}_I)$ contains the ideal of compact operators $\mathfrak{K}(\mathcal{F}_I)$, to show that the identity representation is a boundary representation, it suffices to prove that the quotient mapping $q : C^*(\mathcal{S}_I) \to C^*(\mathcal{S}_I)/\mathfrak{K}(\mathcal{F}_I)$ is not completely isometric on $\mathcal{S}_I$. At present, the author is unaware of whether an asymptotic formulation of Arveson's boundary theorem can hold. Indeed, the simplest proof for Arveson's boundary theorem is dependent on the existence of sufficiently many boundary representations (Theorem 2.4.1). However, as demonstrated in Example 12, there do not always exist sufficiently many irreducible $*$-representations that are Korovkin. In particular, we are not completely certain whether $\mathcal{S}_I$ can be Korovkin apart from the circumstance where S is essentially normal.

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