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SOME COMBINATORIAL PROPERTIES OF COMPLEMENTARY SEQUENCES

by

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CHAPTER 1: INTRODUCTION

1.1 Definitions

Two sequences of 1's and -1's are said to be complementary if the number of like pairs having a separation of k elements in the two sequences equals the number of unlike pairs so separated, for k= 1,2, ... Complementary sequences have combinatorial properties of considerable interest, partially due to their applications to physical problems in more than one field. As a consequence, the theory of these sequences has passed through two distinct generations , as different researchers have bent their studies toward different ends. In the process, complementary sequences have been generalized in several ways. In this work, these developments are reviewed, and generalized in the form of self-complementary sequences of orthogonal vectors (SCoSOV's).

Complementary sequences in their original form can be defined more explicitly with the aid of correlation functions. Given a pair of finite sequences, A = (a_i) and B = (b_i), for i=1(1)L, the correlation function of A with B for a spacing k is defined to be

$$R_{AB}(k) = a_1 b_{1+k} + a_2 b_{2+k} + \ldots + a_{L-k} b_L .$$

This expression can be more concisely written as

$$R_{AB}(k) = \sum_{i=-\infty}^{\infty} a_i b_{i+k} ,$$

if the assertion is made that $a_i = b_i = 0$ for i<1 and for i>L. Note that

$$R_{AB}(k) = R_{BA}(-k)$$

using this form of the summation. The autocorrelation of A with itself for a spacing k is, by analogy,

$$R_{AA}(k) = R_{AA}(-k) = \sum_{i=-\infty}^{\infty} a_i \; a_{i+k} .$$

With the aid of these functions, a pair of sequences, $A = (a_i)$ and $B = (b_i)$ will be said to be complementary if

$$R_{AA}(k) + R_{BB}(k) = 0,$$

for k not equal to 0. That is,

$$\sum_{i=-\infty}^{\infty} (a_i \; a_{i+k} + b_i \; b_{i+k}) = 0,$$

for k not equal to 0, if the domains of the sequences are again suitably extended. Henceforth, any subscripted variable bearing a subscript outside its domain of definition, whether a vector or a scalar, will be assumed to have the appropriate zero value.

1.2 History and Applications

The known properties of complementary sequences were all given by Marcel Golay in his landmark paper of 1961 [5]. This paper was , in fact, his third paper on the subject, the first two [3, 4] being in the field of optics. Golay used complementary sequences in the late 1940's in the design of multi-slit spectrometers. Whereas a single-slit spectrometer has a long narrow aperture, through which a light source shines , the multi-slit device has a series of such slits, parallel and carefully spaced. Light shining through such apertures is diffracted, by an amount determined by its frequency and by the width of the slits, to produce a beam-splitting effect, similar in some respects to the refractive dispersion of light in a prism. However the image of each slit on a screen or recording device is complicated by the appearance of light and dark bands, caused by reinforcement and cancellation of light waves. For very narrow slits, these fringes will be wide and dim. For wider slits, the bands will be narrow, but brighter. In the many-slit spectrometer, several slits are used to enhance efficiency, but narrow slits can be used to permit greater dispersion.

Complementary series were first used in the design of the slit arrangement of such a spectrometer, as shown in figure 1, which was taken from Golay's 1961 paper. The slits were equally spaced, with some open or uncovered, and some closed, or sealed, in a pattern specified by two complementary sequences, A and B. The detector was placed behind a similarly-slitted screen, so that the autocorrelation function was automatically calculated by summing the radiation incident on the screen apertures. This design permitted efficient use of the light from the source, and the selection of just one frequency range for examination through effective cancellation of all others.

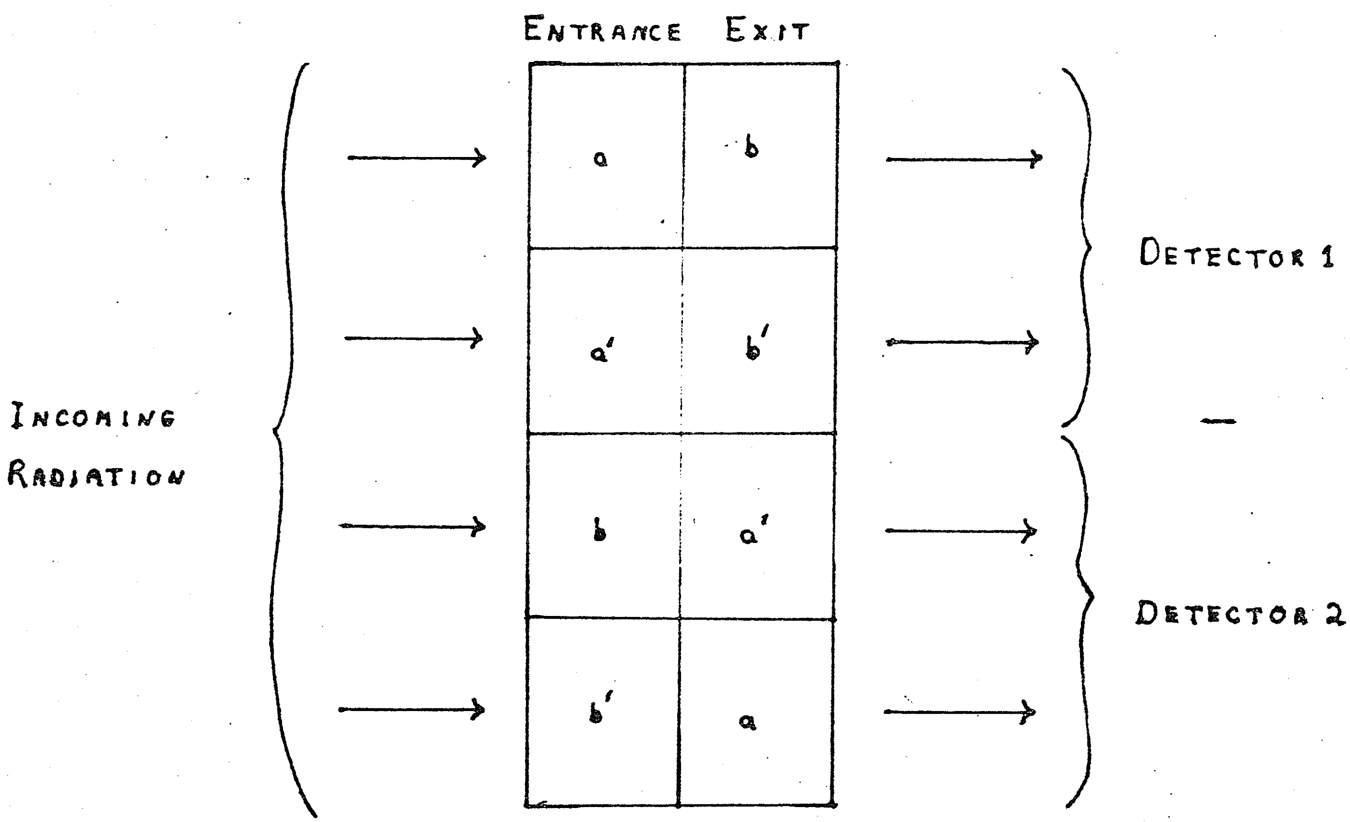

FIG. 1 - MULTI-SLIT SPECTROMETER

The interests of Golay in the combinatorial aspects of complementary series apparently extended beyond their usage in spectroscopy, for his 1961 paper refers only briefly to optics. Primarily, he discusses combinatorial properties of complementary series , as well as methods of synthesis of longer sequences from certain short basis series of lengths 2 and 10 . Having given a necessary condition (which will be stated later as Theorem 2.7) for the existence of complementary series, he also used a counter-example of length 18 to show that the condition was not sufficient. In a later note [6], Golay was also the first to publish an example of complementary series of length 26. It is interesting to note that although extensive computer searches have been made by myself and others [8], the last new complementary pair discovered was found by Golay in a "by hand" search.

In other fields besides optics, physical phenomena can often be represented by correlation functions, and the utility of complementary series rests on this fact. Correlations appear in communications theory as an intrinsic aspect of the separation of signals from noise [24]. It is not surprising then to find that the theory of radar, born in the 1940's, should have found applications for complementary sequences in the 1950's and 1960's. The use of special codings for transmitted radar signals was explored to some extent in the frequently referenced paper by Siebert [14]. Then, at the same time that Golay was publishing his paper , George Welti [23] produced a paper showing how certain codings could be successfully used in pulsed radar for range detection. Richard Turyn later established the isomorphism between the sequences of Golay and the codes of Welti [20].

Communications theory, to which radar studies belong, covers many smaller disciplines, but stripped to the essentials, the problem of extracting signals from noise has the same characteristics, whether the medium be radar or surface acoustic waves on crystals, or something even more exotic.

The basic objectives are commonly twofold. First, a signal which has been transmitted must be recognized, usually after power dissipation during transmission, and in the presence of noise. Secondly, the time of arrival of the signal must be determined. Meeting the first objective is hampered by a limitation on the maximum power output of the transmitter. (Otherwise, the signal strength could be stepped up until it blotted out all interference.) The second aim is limited by the frequency bandwidth of the transmitted signal. (This is the principle that makes laser ranging much more accurate than radar ranging, because light waves have a higher frequency than radar waves.) Conflict arises in attempts to satisfy both criteria, because the signal which packs the most energy will have a continuous power output, while a spike output will be easiest to accurately measure in the time domain. A compromise can be achieved by using a receiver which continuously correlates the incoming signal with the form of the transmitted signal. In the discrete-coding case with which we are concerned, the signal consists of a long pulse containing many shorter pulses, separated by possible phase reversals. In effect, the result is a time-varying sequence of 1's and -1's. If the receiver is matched to the transmitter, the general form of the received correlation function will appear as in figure 2, which shows a pseudo-random code. The key features are the central peak and the smaller side-peaks(side-lobes). If clever coding can keep the ratio of side-lobe height to central peak height as low as possible, then the energy transmitted will be concentrated into the main peak. Of course, to transmit information, the whole pattern must be repeated for each bit of the message. Even this feature turns out not to be a disadvantage in the world of surface acoustic wave (S.A.W.) devices.

Many codes have been constructed for the purpose of making the side-lobe to centre peak ratio as low as possible. These include the Barker codes [1], illustrated in figure 3, which feature side-lobes of constant minimal amplitude, and of constant sign, the Welti codes [23], which as we have said, are isomorphic to complementary series, but

a) 1 - 1 1 - - 1 - 1 1 1 - 1 - - -

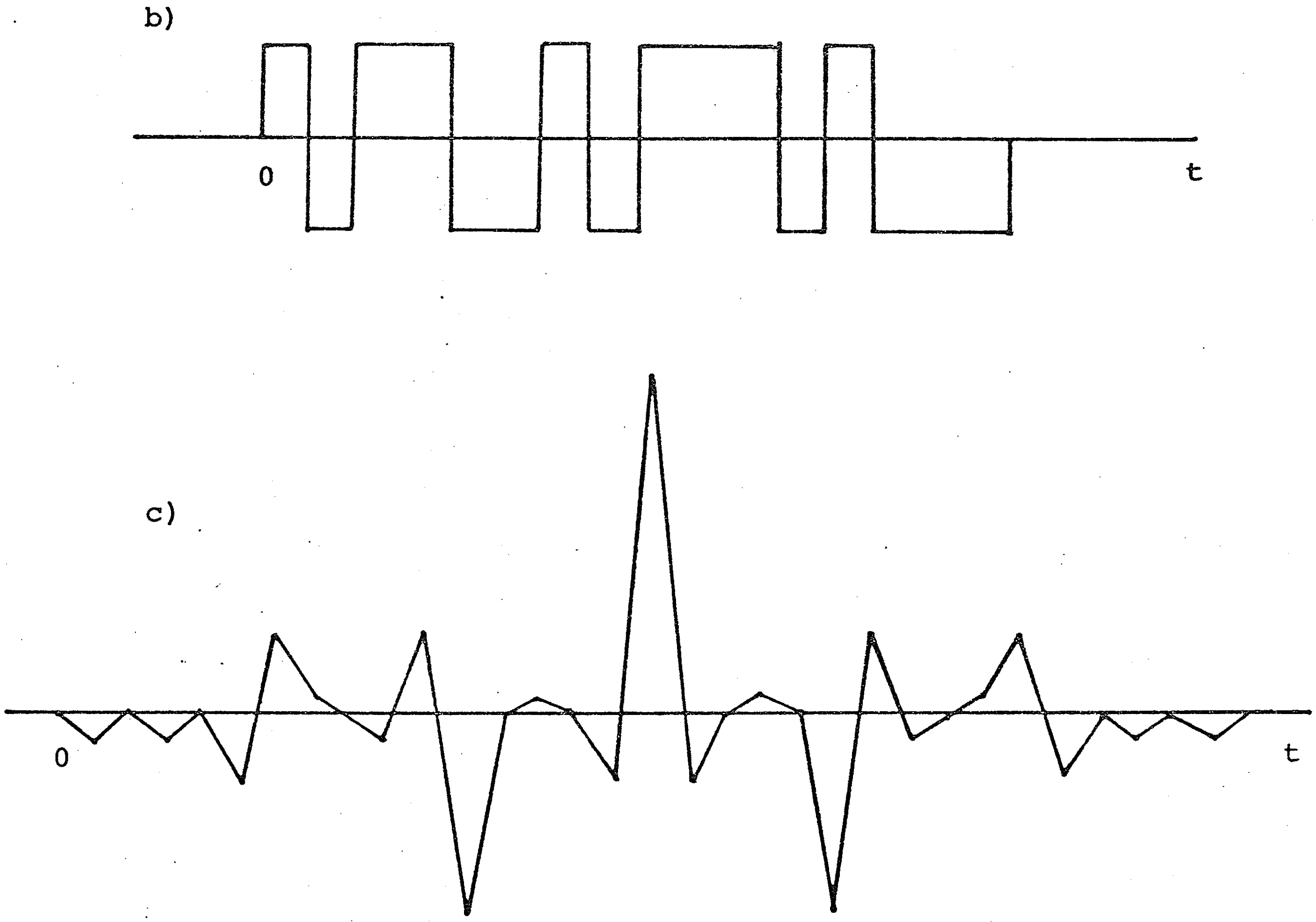

FIG. 2 - A PSEUDO-RANDOM CODE: a) A CODE OF LENGTH 16; b) THE CODE AS A FUNCTION OF TIME; c) THE AUTO-CORRELATION FUNCTION.

a) 1 1 1 - - 1 -

b)

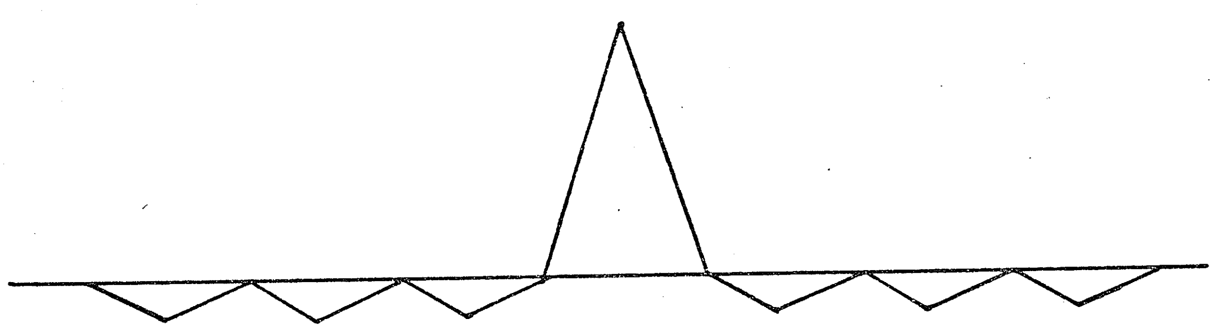

FIG. 3 - A BARKER CODE: a) THE CODE, OF LENGTH 7

b) THE AUTO-CORRELATION FUNCTION

which are not binary; the Frank codes [2], which are also not binary, etc. Any such codes must necessarily have non-zero side-lobes, caused by the arrival of the first sub-pulse, which is correlated in the receiver with the last sub-pulse of the transmitted pattern.

Using complementary series, however, an ideal situation can be (theoretically) achieved where no side-lobes, but only the central peak, exist. In exchange the use of complementary series extracts the penalty of using two channels. Each signal received must be independently correlated with the corresponding transmitter signal, and the two channel outputs can be summed to produce the net output. This procedure is illustrated in figure 4. The central peak will correspond to a zero-shift correlation, and as such , it will have a magnitude of 2L.

In practice, only if the channels are matched extremely well will the theoretical predictions be reached. In range-finding radar applications, for instance, signals could be transmitted at two different frequencies (requiring expensive duplication of equipment). The transmitters , receivers, and amplifiers would then have to be similar in response, and stable over long periods of time. Even then, spurious results might be obtained if the target responded differently at the two frequencies. Alternatively, the two signals could alternate on the same

a) A = 1 1 1 - - 1 - -
B = 1 1 1 - 1 - 1 1

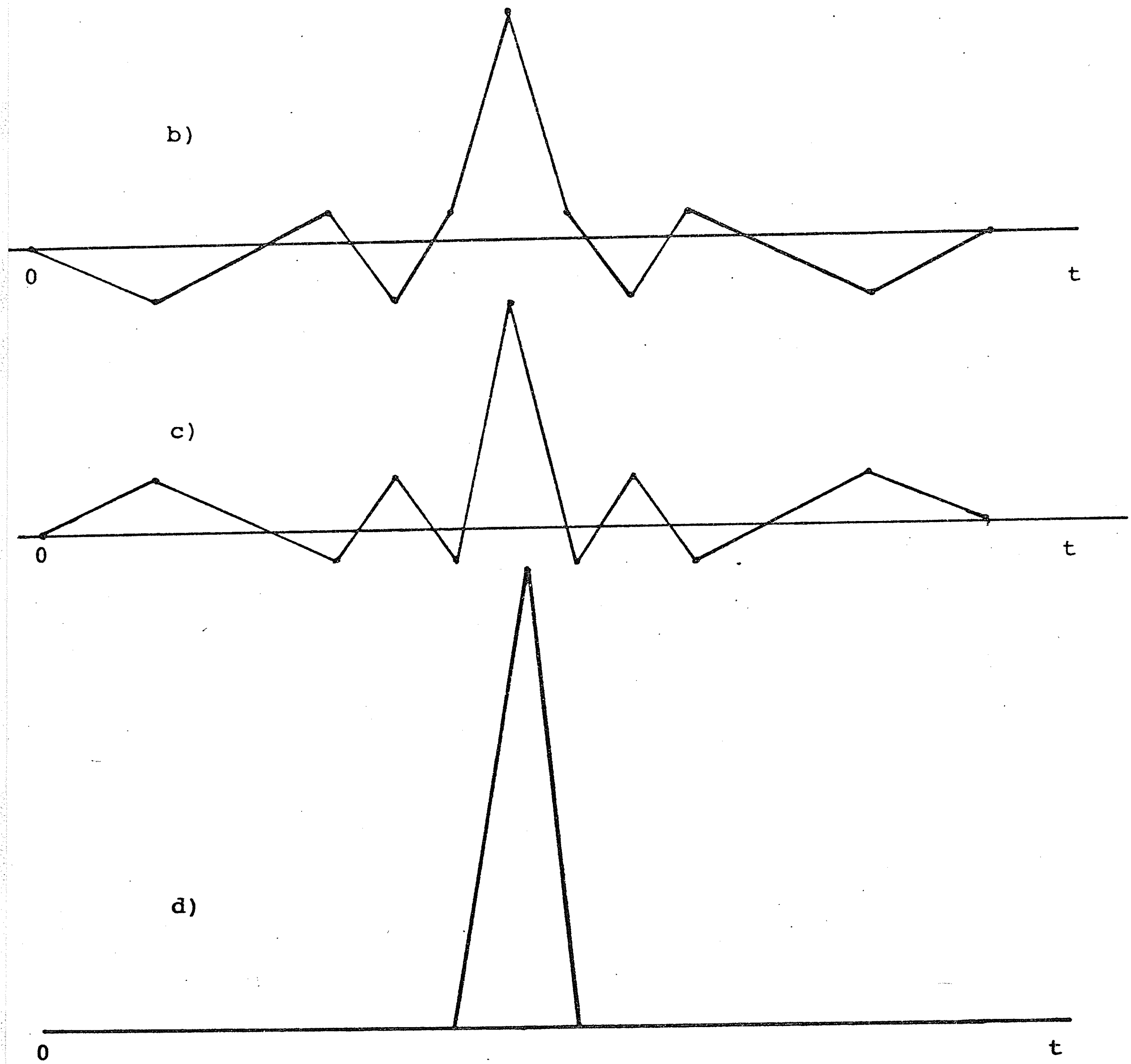

FIG. 4 - A PAIR OF COMPLEMENTARY SEQUENCES: a) SEQUENCES A AND B; b) AUTO-CORRELATION FUNCTION OF A; c) AUTO-CORRELATION FUNCTION OF B; AND d) SUM OF b) AND c)

equipment, if one of the signals could be delayed to permit the necessary summing when the second signal arrived. That technique would prove unsatisfactory should the echo change appreciably in the time required for one pulse. The difficulties involved in creating two channels in radar usage might very well outweigh the advantages of using complementary series, except in very high signal density situations.

Surface-acoustic wave (S.A.W.) devices [9] have proved much more amenable to the application of complementary series. These devices utilize the piezo-electric properties of certain crystals to convert electronic signals to and from Rayleigh waves, which propagate across the crystal at a speed much less than the speed of light. This slow velocity makes S.A.W. delay lines very compact. Furthermore, various signal-modifying and filtering functions can be performed simply through the design of the transducers on the crystals. It is however in delay lines that Golay sequences have been widely used [15, 16, 18, 21].

The first favourable circumstance involves the ease with which two identical channels can be created in a S.A.W. device. The transmitters consist of two transducers lying side-by-side on the surface of the crystal. The transducers appear as in figure 5. Each one consists of a sequence of

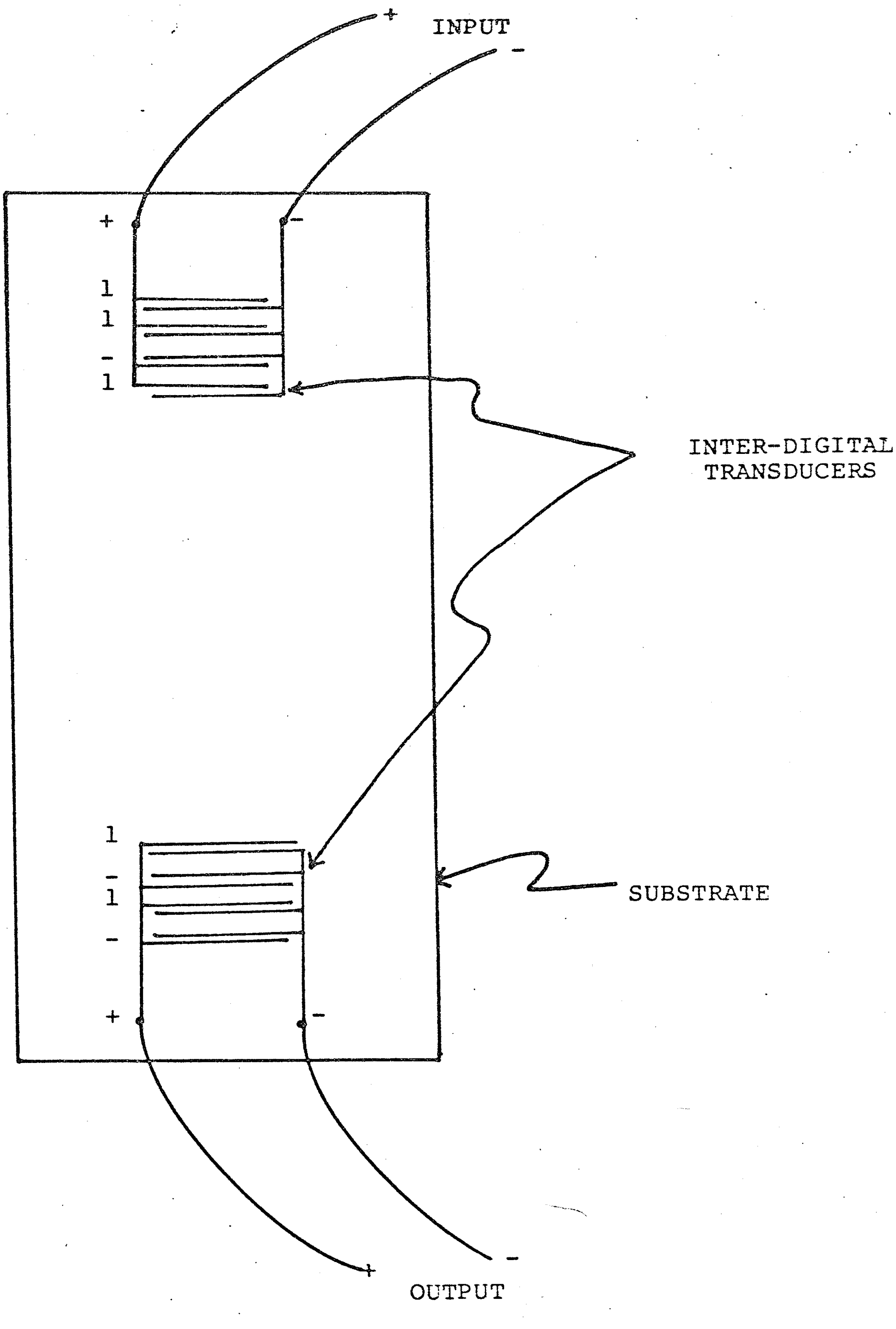

FIG. A S.A.W. DEVICE

metallic digit pairs deposited on the crystal material. In each case, one of the digits is grounded, and the other is given the input signal. A piezo-electric crystal will always change shape in the presence of an electric field, and create an electric field when subjected to physical stress. Therefore, when a signal is applied to one of the inter-digital pairs, it will create a surface wave which will propagate at right angles to the digits. The pairs themselves are separated by gaps sufficiently large to prevent interference. An identical set of inter-digital transducers acts as a receiver some distance along in the path of the acoustic waves. The correlation functions are automatically and continuously formed, and summed by connecting the output leads of the two sets of receiving electrodes. The revolution in micro-circuit technology since the mid 1960's has ensured that to all intents and purposes the two sets of transducers can be made identical.

Of course, as in radar applications, the duplication of anything involves additional expense. In this case, the area on the crystal used for acoustic pathways costs the most. The nature of this cost has been considerably reduced since C. C. Tseng published a paper [18] showing that two acoustic pathways could simultaneously carry two independent signals in a non-interfering manner, provided that these signals were orthogonal. This development started the search for orthogonal sets of complementary series, known as orthogonal mates. Figure 6 shows how two acoustic pathways could be efficiently shared by two independent, but non-interacting, signal lines.

Mathematically, this non-interfering quality can be expressed in the following way. Two pairs of complementary series (A,B) and (C,D) are orthogonal if

$$R_{AC}(k) + R_{BD}(k) = 0,$$

for k = ... -2, -1, 0, 1, 2, ...

In his doctoral dissertation, Bernard Schweitzer has shown how n sequences can form a complementary system, which he calls a code, and how n such codes can form a mutually orthogonal set, which he calls a complementary code set. He proceeds to show how such complementary code sets (CCS) can be synthesized from certain primitive code sets, using various transformations devised by him. For binary sequences, the primitive elements in his chains of synthesis are Hadamard matrices [22], the properties of which are well known.

Using other orthogonal matrices, Schweitzer was able to construct real-valued CCS. These were found to have certain properties not shared by the binary CCS. In fact, the nature of this generalization is so broad that the real codes share very little in common with the binary codes. SCOSoV's were designed to fill this large gap, by allowing

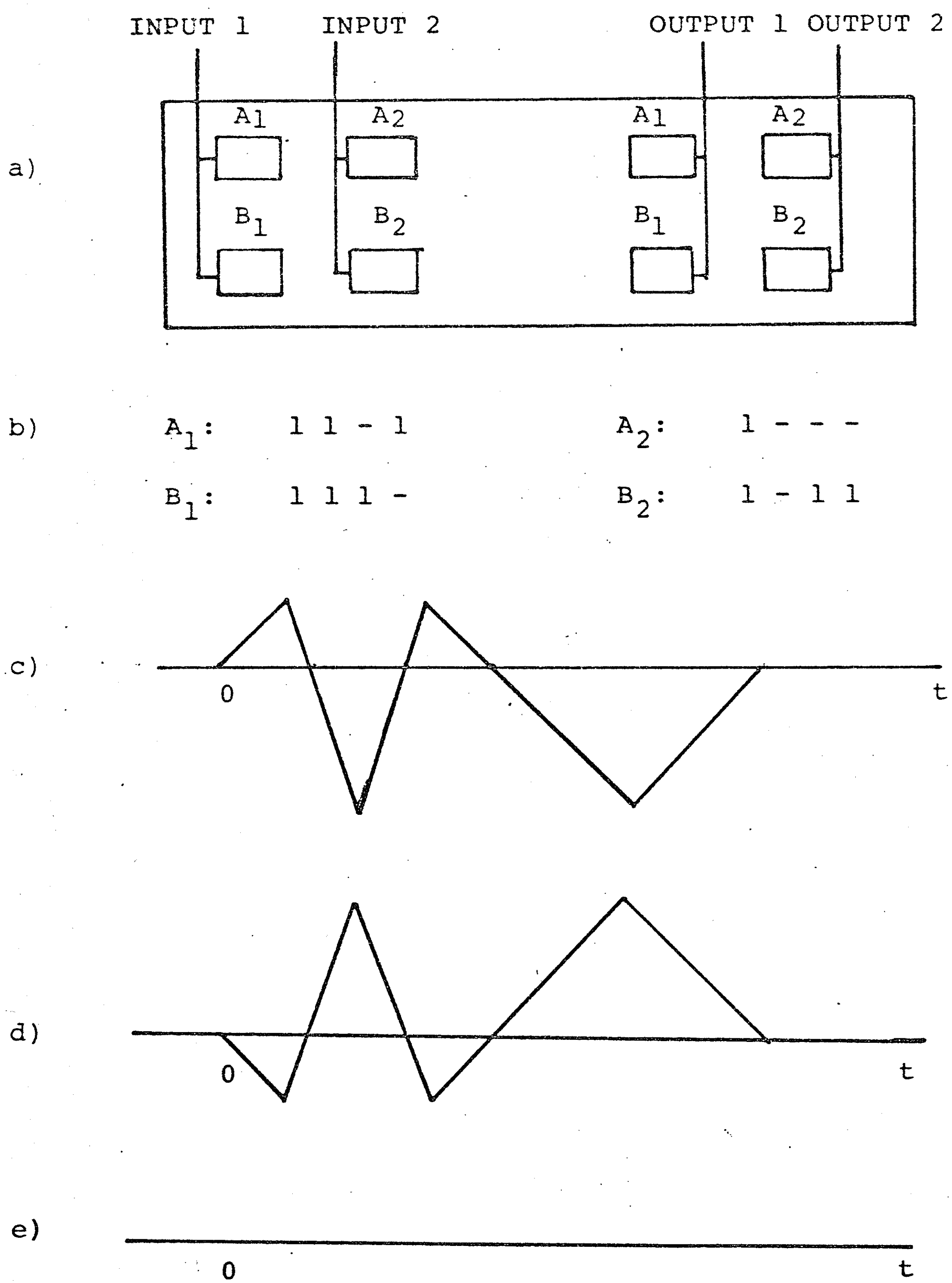

FIGURE 6 - ORTHOGONAL COMPLEMENTARY SEQUENCES

a) MULTIPLYING S.A.W. DEVICE; b) ORTHOGONAL COMPLEMENTARY PAIRS, (A_1, B_1), (A_2, B_2); c) $R_{A_1A_2}(t)$; d) $R_{B_1B_2}(t)$; e) $R_{A_1A_2}(t)+R_{B_1B_2}($