WEAK AMENABILITY OF WEIGHTED GROUP ALGEBRAS AND OF THEIR CENTRES

BY

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ABSTRACT

Let G be a locally compact group, ω be a continuous weight function on G, and $L^1(G,\omega)$ be the corresponding Beurling algebra. In this thesis, we study weak amenability of $L^1(G,\omega)$ and of its centre $ZL^1(G,\omega)$ for non-commutative locally compact groups G.

We first give examples to show that the condition that characterizes weak amenability of $L^1(G,\omega)$ for commutative groups G is no longer sufficient for the non-commutative case. However, we prove that this condition remains necessary for all [IN] groups G. We also provide a necessary condition for weak amenability of $L^1(G,\omega)$ of a different nature, which, among other things, allows us to obtain a number of significant results on weak amenability of $\ell^1(\mathbb{F}_2,\omega)$ and $\ell^1((\boldsymbol{ax+b}),\omega)$.

We then study the relation between weak amenability of the algebra $L^1(G,\omega)$ on a locally compact group G and the algebra $L^1(G/H,\hat{\omega})$ on the quotient group G/H of G over a closed normal subgroup H with an appropriate weight $\hat{\omega}$ induced from ω. We give an example showing that $L^1(G,\omega)$ may not be weakly amenable even if both $L^1(G/H,\hat{\omega})$ and $L^1(H,\omega|_H)$ are weakly amenable. On the other hand, by means of constructing a generalized Bruhat function on G, we establish a sufficient condition under which weak amenability of $L^1(G,\omega)$ implies that of $L^1(G/H,\hat{\omega})$. In particular, with this approach, we prove that weak amenability of the tensor product $L^1(G_1,\omega_1)\hat{\otimes}L^1(G_2,\omega_2)$ implies weak amenability of both Beurling algebras $L^1(G_1,\omega_1)$ and $L^1(G_2,\omega_2)$, provided the weights ω_1, ω_2 are bounded away from zero. However, given a general weight on $G = G_1 \times G_2$, weak amenability of $L^1(G,\omega)$ usually does not imply that of $L^1(G_1,\omega|_{G_1})$, even if both G_1, G_2 are commutative. We provide an example to illustrate this.

While studying the centres $ZL^1(G,\omega)$ of $L^1(G,\omega)$, we characterize weak amenability of $ZL^1(G,\omega)$ for connected [SIN] groups G, establish a necessary condition for weak amenability of $ZL^1(G,\omega)$ in the case when G is an [FC] group, and give a sufficient condition for the case when G is an [FD] group. In particular, we obtain some positive results on weak amenability of $ZL^1(G,\omega)$ for a compactly generated [FC] group G with a polynomial weight ω.

Finally, we briefly discuss the derivation problem for weighted group algebras and present a partial solution to it.

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Contents

0.1 Notations

In this section, we list the notations used throughout the thesis. The notations introduced in the subsequent chapters are listed with a reference to the page they are defined on.

- $\mathbb{Z}$ denotes the set of all integer numbers;
- $\mathbb{N}$ denotes the set of all positive integer numbers;
- $\overline{1,n}$ denotes the set of all integer numbers in the interval $[1,n]$;
- $\mathbb{R}$ denotes the set of all real numbers;
- $\mathbb{R}^+$ denotes the set of all positive real numbers;
- $(\mathbb{R}^+,\cdot)$ denotes the multiplicative group of all positive real numbers;
- $\lfloor x \rfloor$ denotes the floor function of $x \in \mathbb{R}$, which is equal to the greatest integer number that does not exceed x;
- $\mathbb{C}$ denotes the set of all complex numbers;
- $\boldsymbol{i}$ denotes the complex unity of $\mathbb{C}$;
- $\mathbb{F}_2$ denotes the free group on two generators;
- $\boldsymbol{(ax+b)}$ denotes the group of all affine transformations $x \mapsto ax+b$ of $\mathbb{R}$ with $a > 0$ and $b \in \mathbb{R}$;
- $\#A$ denotes the cardinality of a set A;
- $\overline{A}$ denotes the closure of a set A;
- $\operatorname{lin} A$ denotes the linear span of a set A;
- μ denotes the left Haar measure of a locally compact group G;
- e denotes the identity of a locally compact group G;

- G' denotes the commutator subgroup of a group G generated by all elements of the form $xyx^{-1}y^{-1}$ $(x, y \in G)$;
- δ_x denotes the point mass at an element x of a locally compact group G;
- $[x]$ denotes the coset of an element x of a locally compact group G in the quotient group G/H of G over a normal subgroup H of G;
- $Aut(G)$ denotes the set of all continuous algebraic automorphisms of a topological group G;
- $I(G)$ denotes the set of all inner automorphisms of a topological group G;
- $C(K)$ denotes the set of all continuous functions on a compact set K;
- $C_c(G)$ denotes the set of all continuous functions with compact support on a locally compact group G;
- $C_0(G)$ denotes the set of all continuous functions vanishing at infinity on a locally compact group G;
- $L^1(G, \omega)$ denotes the weighted group algebra, or Beurling algebra, on a locally compact group G with respect to a weight ω (see page 6);
- $\ell^1(G, \omega)$ denotes the discrete weighted group algebra on a locally compact group G with respect to a weight ω;
- $L^\infty(G, 1/\omega)$ (see page 7);
- $\ell^\infty(G, 1/\omega)$ denotes the discrete analogue of $L^\infty(G, 1/\omega)$;
- $C_0(G, 1/\omega)$ (see page 14);
- $M(G, \omega)$ denotes the weighted measure algebra on a locally compact group G with respect to a weight ω (see page 14);
- $ZL^1(G, \omega)$ denotes the centre of the Banach algebra $L^1(G, \omega)$;

- $\operatorname{supp} f = \overline{\{x \in G : f(x) \neq 0\}}$ denotes the support of a function f on a locally compact group G;
- $f|_A$ denotes the restriction of a function f to a set A;
- $L_x f$ (resp. $R_x f$) denotes the left (resp. right) translation of the function f on a locally compact group G by an element $x \in G$ defined by $L_x f(t) = f(x^{-1}t)$ (resp. $R_x f(t) = f(tx)$), $t \in G$;
- $f * g$ denotes the convolution of functions f and g on a locally compact group G (see page 5);
- $a \cdot x$ (resp. $x \cdot a$) denotes the left (resp. right) module multiplication of $x \in X$ by $a \in A$, where A is a Banach algebra and X is a Banach left (resp. right) A-module (see page 7);
- $\mathcal{B}(X, Y)$ denotes the set of all bounded linear operators from a Banach space X to a Banach space Y;
- $\mathcal{L}(X)$ denotes the set of all bounded linear operators from a Banach space X to itself;
- X^* denotes the dual of a Banach space X;
- $\langle x, \varphi \rangle$ denotes the value of a continuous linear functional $\varphi \in X^*$ at an element $x \in X$;
- $\mathcal{M}(A)$ denotes the multiplier algebra of a Banach algebra A (see page 15);
- $A \hat{\otimes} B$ denotes the projective tensor product of Banach algebras A and B;
- $A \simeq B$ means that Banach algebras A and B are isomorphic;
- $A \cong B$ means that Banach algebras A and B are isometrically isomorphic.

Chapter 1

Introduction

We begin by introducing some basic concepts that are used throughout the thesis.

A topological group is a group G equipped with a topology with respect to which the group operations are continuous, i.e., $(x, y) \mapsto xy$ is continuous from $G \times G$ to G and $x \mapsto x^{-1}$ is continuous from G to G. The group G is called locally compact if there is a compact neighborhood of the identity element in G. The locally compact groups considered in this thesis are always assumed to be Hausdorff.

There is a special type of measure defined on locally compact groups. A left (respectively, right) Haar measure on a locally compact group G is a non-zero Radon measure μ on G (Radon measure is a locally finite inner regular Borel measure) that satisfies $\mu(xE) = \mu(E)$ (respectively, $\mu(Ex) = \mu(E)$) for every Borel set E and every $x \in G$. In other words, left (respectively, right) Haar measure is invariant under left (respectively, right) translations.

It is well-known (see, for example, [11, Theorem 2.10]) that every locally compact group possesses a unique up to a scalar multiple left (right) Haar measure. We always equip a locally compact group G with the left Haar measure. This allows us to consider the Banach space $L^1(G)$ of all absolutely Haar integrable Borel functions, where, as usual, we identify functions that are equal to each other locally almost everywhere on G. The integral of $f \in L^1(G)$ with respect to the left Haar measure is denoted by $\int_G f(x)\,dx$. There exists a continuous function Δ on G, defined by

$$\Delta(x) = \frac{\int_G f(tx^{-1})\,dt}{\int_G f(t)\,dt}, \quad x \in G,$$

when $\int_G f(t)\,dt \neq 0$, and is independent of the choice of $f \in L^1(G)$. If $\Delta \equiv 1$, then the group G is called unimodular.

The space $L^1(G)$ becomes a Banach algebra, called the group algebra of G, with the convolution product

$$(f * g)(x) = \int\limits_G f(y)g(y^{-1}x)\,dy \quad (\text{a.e. } x \in G,\ f, g \in L^1(G)).$$

Next we list several types of locally compact groups that are of special interest to us.

Definition 1.1. Let G be a locally compact group.

1. G is an [IN] group if there exists a compact neighborhood of identity in G invariant under all inner automorphisms of G (an inner automorphism is an automorphism of the form $x \mapsto gxg^{-1}$ for a fixed element $g \in G$);

2. G is an [SIN] group if there is a base of compact neighborhoods of identity invariant under inner automorphisms, i.e., for every neighborhood U of identity there exists a compact neighborhood $V \subset U$ of identity invariant under inner automorphisms.

It is obvious that the class of [IN] groups contains the class of [SIN] groups. We also remark that an [IN] group is always unimodular. A centre of a Banach algebra A is the set of all elements $a \in A$ that commute with any other element of A. It was proved in [30] that [IN] groups are precisely those groups G for which the centre $ZL^1(G)$ of the group algebra $L^1(G)$ is non-trivial, i.e., $ZL^1(G) \neq \{0\}$, and [SIN] groups G are those for which $ZL^1(G)$ has a bounded approximate identity. We recall that a bounded net $\{e_\gamma\} \subset A$ is called a bounded approximate identity for a Banach algebra A if $\lim e_\gamma x = \lim x e_\gamma = x$ for all $x \in A$. The group algebra $L^1(G)$

always has a bounded approximate identity. In fact, if $\{U_\gamma\}$ is a base of compact symmetric ($U_\gamma^{-1} = U_\gamma$) neighborhoods of identity e in G, then $\left\{\frac{1}{\mu(U_\gamma)}\chi_{U_\gamma}\right\}$ forms a bounded approximate identity for $L^1(G)$, where μ is the left Haar measure on G, and χ_U is the characteristic function of the set $U \subset G$.

The main objective of study in this thesis is the weighted group algebra $L^1(G,\omega)$.

Definition 1.2. Let G be a locally compact group. A weight on G is a measurable function $\omega : G \to (0,\infty)$ satisfying the weight inequality

$$\omega(xy) \leq \omega(x)\omega(y), \quad x, y \in G.$$

Two weights ω and $\tilde{\omega}$ are called equivalent if there exist constants $c_1, c_2 > 0$ such that $c_1\omega(x) \leq \tilde{\omega}(x) \leq c_2\omega(x)$, $x \in G$.

Given a weight ω on G, consider

$$L^1(G,\omega) = \left\{ \text{Borel measurable } f \;:\; \int_G |f(x)|\omega(x)\,dx < \infty \right\}.$$

Equipped with the norm

$$\|f\|_{L^1(G,\omega)} = \int_G |f(x)|\omega(x)\,dx$$

and the convolution product, $L^1(G,\omega)$ becomes a Banach algebra. This algebra is usually referred to as Beurling algebra after A. Beurling, who first studied $L^1(\mathbb{R},\omega)$ as a weighted convolution algebra in [4]. For the investigation of weighted convolution algebras on totally disconnected locally compact groups see [39]. It is easy to observe that if two weights ω_1 and ω_2 on G are equivalent, then the corresponding Beurling algebras $L^1(G,\omega_1)$ and $L^1(G,\omega_2)$ are isomorphic as Banach algebras. The Banach space dual of $L^1(G,\omega)$ is

$$L^\infty(G,1/\omega) = \left\{ \text{Borel measurable } \lambda \;:\; \|\lambda\|_{L^\infty(G,1/\omega)} = \operatorname*{ess\,sup}_{x\in G} \frac{|\lambda(x)|}{\omega(x)} < \infty \right\}.$$

Remark 1.3. *As follows from* [10, Proposition 2.1], *any weight ω on a locally compact group G must be locally bounded. Then, by* [35, Theorem 3.7.5] *any locally bounded weight ω on G is equivalent to a continuous weight on G. Combining these results with the observation above, we conclude that for any weight ω on G there is a continuous weight $\tilde{\omega}$ on G such that the Beurling algebra $L^1(G,\omega)$ is isomorphic to the Beurling algebra $L^1(G,\tilde{\omega})$. So, from now on we will always assume the weight ω to be continuous.*

We note that as Banach spaces $L^1(G,\omega)$ and $L^1(G)$ are isometric. In fact, the map $f \mapsto f\omega$ provides an isometry from $L^1(G,\omega)$ onto $L^1(G)$. However, as Banach algebras they are substantially different. For example, it is well-known that $L^1(G)$ is a quantum group algebra. But $L^1(G,\omega)$ is not, unless ω is equivalent to a multiplicative weight on G, which is regarded as the trivial case. Moreover, except for the trivial case, $L^1(G,\omega)$ is not even a member of the larger class of F-algebras. F-algebras were introduced in [22] as those algebras A whose dual A^* is a W^*-algebra such that the identity of A^* is a multiplicative linear functional on A. The class of F-algebras is rather wide. It contains the group algebra $L^1(G)$, the Fourier algebra $A(G)$, and the Fourier-Stieltjes algebra $B(G)$ of a locally compact group G. It also contains all semigroup algebras and all quantum group algebras (see [25] for details). To see that $L^1(G,\omega)$ is not an F-algebra, we note that $(L^1(G,\omega))^* = L^\infty(G,1/\omega)$ is a von Neumann algebra with the product $f \cdot g = fg/\omega$. So, the identity of $L^\infty(G,1/\omega)$ is ω, which is not a multiplicative linear functional on $L^1(G,\omega)$ unless ω is multiplicative, i.e., $\omega(xy) = \omega(x)\omega(y)$, $x, y \in G$. The notion of F-algebra has strong connections with amenability theory that we will discuss next.

We first give some definitions.

Definition 1.4. Let A be a Banach algebra. A Banach A-bimodule is a Banach space X together with bilinear maps $(a,x) \mapsto a \cdot x$ and $(a,x) \mapsto x \cdot a$ from $A \times X$ to X satisfying the following axioms:

$$a \cdot (b \cdot x) = ab \cdot x, \quad (x \cdot b) \cdot a = x \cdot ab, \quad a \cdot (x \cdot b) = (a \cdot x) \cdot b,$$

$$\max(\|a \cdot x\|, \|x \cdot a\|) \leq C\|a\|\|x\| \quad (a, b \in A,\, x \in X),$$

where $C > 0$ is a constant.

The Banach algebra A itself is naturally a Banach A-bimodule with the module actions implemented by the product of A. If X is a Banach A-bimodule then its dual X^* can also be equipped with the structure of a Banach A-bimodule in the following way:

$$\langle x, \varphi \cdot a \rangle = \langle a \cdot x, \varphi \rangle,\ \langle x, a \cdot \varphi \rangle = \langle x \cdot a, \varphi \rangle \quad (a \in A,\, x \in X,\, \varphi \in X^*).$$

In this case X^* is called a dual Banach A-bimodule.

Definition 1.5. Let A be a Banach algebra and X be a Banach A-bimodule. A linear map $D : A \to X$ is called a derivation if it satisfies the following relation:

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad a, b \in A.$$

A derivation is called bounded if it is a bounded linear operator, i.e., if it is continuous.

For every $x \in X$ the map $ad_x : A \to X$ defined by

$$ad_x(a) = a \cdot x - x \cdot a$$

is a bounded derivation, called an inner derivation.

Definition 1.6. A Banach algebra A is called amenable if every bounded derivation from A to any dual Banach A-bimodule is inner. A is called weakly amenable if every bounded derivation $D : A \to A^*$ is inner.

The notion of amenability was originally introduced for groups and semigroups in response to the Banach-Tarski paradox (see, for example, [36]) and has become an important concept in abstract harmonic analysis since the 1940-s. Amenability theory for Banach algebras started in 1972, when B.E. Johnson proved the remarkable result ([19]) that amenability of a locally compact group G is equivalent to amenability of the group algebra $L^1(G)$. Since then, many important investigations were conducted regarding amenability of various classes of Banach algebras. For example, it was shown that the unital uniform algebra is amenable if and only if it is isomorphic to $C(K)$ for some compact Hausdorff space K ([7]), and a C*-algebra is amenable if and only if it is nuclear ([17]). Extending the Johnson's celebrated result on $L^1(G)$, N. Gronbaek ([14]) characterized amenability for Beurling algebras $L^1(G,\omega)$.

Theorem 1.7. [14, Theorem 0] *The Beurling algebra $L^1(G,\omega)$ is amenable if and only if the group G is amenable and the weight ω is diagonally bounded, i.e,*

$$\sup\{\omega(g)\omega(g^{-1}) : g \in G\} < \infty.$$

M. White proved in [38] that for any weight ω on an amenable group G there is a continuous character function $\phi : G \to \mathbb{R}^+$ (i.e., $\phi(xy) = \phi(x)\phi(y)$, $x, y \in G$) such that $\phi \leq \omega$ on G. So, if ω is also diagonally bounded, then

$$\begin{aligned}\phi(x) \leq \omega(x) = \frac{\omega(x)\omega(x^{-1})}{\omega(x^{-1})} &\leq \frac{\omega(x)\omega(x^{-1})}{\phi(x^{-1})} = \phi(x)\omega(x)\omega(x^{-1}) \\ &\leq \phi(x)\sup_{x\in G}\omega(x)\omega(x^{-1}) \leq c\phi(x).\end{aligned}$$

This shows that except for the trivial case when the weight ω is equivalent to a multiplicative weight, $L^1(G,\omega)$ is intrinsically not amenable.

In 1986 W.G. Bade, P.C. Curtis and H.G. Dales introduced in [2] the concept of weak amenability for commutative Banach algebras as follows.

Definition 1.8. [2, Definition 1.1] Commutative Banach algebra A is called weakly amenable if every continuous derivation from A into a symmetric Banach A-bimodule is zero (here an A-bimodule X is symmetric if $a \cdot x = x \cdot a$, $a \in A$, $x \in X$).

Note that this is the same as to say that every continuous derivation from A into a symmetric Banach A-bimodule is inner, since an inner derivation into a symmetric bimodule, obviously, must be zero. In the same paper it was proved that a commutative Banach algebra is weakly amenable if and only if every bounded derivation

$D : A \to A^*$ is zero (inner). This motivated B.E. Johnson to introduce the notion of weak amenability for general Banach algebras ([19]) in the way we already given in Definition 1.6. The following theorem proved in [20] shows that we do not need to put any restrictions on the group G to guarantee weak amenability of $L^1(G)$.

Theorem 1.9. [20, Theorem, p.282] *The group algebra $L^1(G)$ is weakly amenable for every locally compact group G.*

This is quite different from the situation for amenability of $L^1(G)$, which depended on the properties of the group G. A shorter proof of the above result that uses the lattice structure of $L^\infty_{\mathbb{R}}(G)$ was given by M. Despic and F. Ghahramani in [9].

It turns out that the method of M. Despic and F. Ghahramani still works for the Beurling algebra $L^1(G, \omega)$ if we assume that the weight ω is diagonally bounded. Hence we have the following.

Theorem 1.10. [34, Theorem 3.14] *Let G be a locally compact group and ω be a diagonally bounded weight on G. Then the Beurling algebra $L^1(G, \omega)$ is weakly amenable.*

The first characterization of weights making $L^1(G, \omega)$ weakly amenable was given by N. Gronbaek for discrete Abelian groups.

Proposition 1.11. [13, Corollary 4.8] *Let G be an Abelian discrete group, and ω be a weight function on G. The Beurling algebra $\ell^1(G, \omega)$ is weakly amenable if and only if*

$$\sup_{g \in G} \frac{|\Phi(g)|}{\omega(g)\omega(-g)} = \infty$$

for every non-zero group homomorphism $\Phi : G \to \mathbb{C}$.

Recently, the N. Gronbaek's result was extended by Y. Zhang to all Abelian locally compact groups.

Theorem 1.12. [41, Theorem 3.1] *Let G be an Abelian locally compact group, and ω be a weight on G. The Beurling algebra $L^1(G, \omega)$ is weakly amenable if and only if*

$$\sup_{g \in G} \frac{|\Phi(g)|}{\omega(g)\omega(-g)} = \infty$$

for every continuous non-zero group homomorphism $\Phi : G \to \mathbb{C}$.

The situation for non-commutative groups is almost completely unknown, except for the diagonally bounded case of Theorem 1.10.

In this thesis, we study weak amenability of $L^1(G,\omega)$ for several types of locally compact non-commutative groups G. In Chapter 3 we consider polynomial weights on two basic groups: on $\mathbb{F}_2$ and on $(\mathbf{ax}+\mathbf{b})$ equipped with the discrete topology. We show that neither of the corresponding Beurling algebras is weakly amenable. In particular, $\ell^1(\mathbb{F}_2,\omega_\alpha)$ is not weakly amenable for any weight of the form $\omega_\alpha(x) = (1+|x|)^\alpha$, $\alpha > 0$, where $|x|$ denotes the length of the word x in $\mathbb{F}_2$. This contrasts with the following combination of the results of [2, Theorem 2.4 (iii), (iv)] and [13, Remark on p. 161].

Theorem 1.13. *Let* $\alpha \geq 0$ *and* $\omega_\alpha(n) = (1+|n|)^\alpha$, $n \in \mathbb{Z}$. *Then* $\ell^1(\mathbb{Z},\omega_\alpha)$ *is weakly amenable if and only if* $0 \leq \alpha < 1/2$.

The result of Theorem 1 is still true if one replaces $\mathbb{Z}$ with $\mathbb{R}$ ([41]). The situation for the free group $\mathbb{F}_2$ exposes how different it is for weak amenability of Beurling algebras on non-Abelian groups.

In Chapter 4 we present some general theory regarding weak amenability of non-Abelian Beurling algebras. In particular, we show that the necessary and sufficient condition on ω for weak amenability of $L^1(G,\omega)$ given in Theorem 1.12 for Abelian groups G remains necessary for all [IN] groups. Whereas, as follows from the discussion above, it is no longer sufficient even for $\ell^1(\mathbb{F}_2,\omega)$ to be weakly amenable. We give one more necessary condition for weak amenability of $L^1(G,\omega)$ for general locally compact group G, which is of a different nature. Using this condition, we are able to characterize weak amenability of $\ell^1(\mathbb{F}_2,\omega)$ for several important classes of weights. The free group $\mathbb{F}_2$ is of special interest since it is the simplest non-amenable group and is the source of many counterintuitive results. Some study concerning Beurling algebras on $\mathbb{F}_2$ was conducted by H.G. Dales and A.T.-M. Lau in [8]. The questions regarding weak amenability of $\ell^1(\mathbb{F}_2,\omega)$ remained open.

In Chapter 5 we consider the relation between weak amenability of the Beurling algebra $L^1(G,\omega)$ on a locally compact group G and the Beurling algebra $L^1(G/H,\hat{\omega})$

on the quotient group G/H of G over a closed normal subgroup H with an appropriate weight $\hat{\omega}$ induced from the original weight ω. More precisely, we follow [35] to define $\hat{\omega}$ by $\hat{\omega}([x]) = \inf_{z\in[x]} \omega(z)$, where $[x]$ stands for the coset of x in G/H. It is known that $L^1(G/H, \hat{\omega}) \cong L^1(G, \omega)/J_\omega(G, H)$ as Banach algebras, where $J_\omega(G, H)$ is a closed ideal in $L^1(G, \omega)$. We show that $J_\omega(G, H)$ is always complemented in $L^1(G, \omega)$ as a Banach subspace, which allows us to establish a sufficient condition under which weak amenability of $L^1(G, \omega)$ implies that of $L^1(G/H, \hat{\omega})$. In particular, with this approach we prove that weak amenability of the tensor product $L^1(G_1, \omega_1)\hat{\otimes}L^1(G_2, \omega_2)$ implies weak amenability of both Beurling algebras $L^1(G_1, \omega_1)$ and $L^1(G_2, \omega_2)$, provided the weights ω_1, ω_2 are bounded away from zero. In general, the question about relation between weak amenability of Banach algebras A and B and weak amenability of their tensor product $A\hat{\otimes}B$ is open. However, there are some partial results in [13] and [40], in particular, in the case when at least one of A, B is Abelian.

It was proved in [41] that for Abelian locally compact groups H and R, the Beurling algebra $L^1(H \times R, \omega)$ is weakly amenable whenever the algebras $L^1(H, \omega|_H)$ and $L^1(R, \omega|_R)$ are weakly amenable. In Chapter 3 we give an example showing that the converse is not true even for $H = R = \mathbb{Z}$. In Chapter 5 we prove that for any Abelian groups H and R admitting a non-zero continuous group homomorphism into $\mathbb{C}$ there exists a weight ω on $H \times R$ such that $L^1(H \times R, \omega)$ is weakly amenable, but $L^1(H, \omega|_H)$ is not weakly amenable. Finally, we conclude Chapter 5 by giving an example of a locally compact group G, a closed normal subgroup H of G, and a weight ω on G such that both Beurling algebras $L^1(H, \omega|_H)$ and $L^1(G/H, \hat{\omega})$ are weakly amenable, but $L^1(G, \omega)$ is not weakly amenable.

The centre $ZL^1(G, \omega)$ of a Beurling algebra $L^1(G, \omega)$ was extensively studied in [26]. However, weak amenability of this algebra is completely unknown except for the trivial cases. Note that when G is Abelian, $ZL^1(G, \omega) = L^1(G, \omega)$. So studying weak amenability of $ZL^1(G, \omega)$ is a natural extension of the study of that of $L^1(G, \omega)$ for Abelian groups G. Some results on the centre $ZL^1(G)$ of the group algebra $L^1(G)$ were obtained in [1]. More precisely, the following result was proved.

Theorem 1.14. [1, Theorem 0.2, Theorem 2.4] *If G is an [FC] group then $ZL^1(G)$ is weakly amenable.*

The same result was obtained independently by Y. Zhang in [41] for compact groups using the dual object and the representation theory.

In Chapter 6 we study weak amenability of $ZL^1(G,\omega)$. We prove that the necessary and sufficient condition on ω for weak amenability of $L^1(G,\omega)$ given in Theorem 1.12 for Abelian groups G is also necessary and sufficient for weak amenability of $ZL^1(G,\omega)$ if G is a connected [SIN] group. In the case when G has a compactness property of being an [FC] group, we show that this condition remains necessary. On the other hand, we provide a sufficient condition for weak amenability of $L^1(G,\omega)$ for [FD] groups G. In particular, we obtain some positive results on weak amenability of $ZL^1(G,\omega)$ for a compactly generated [FC] group G with a polynomial weight ω_α. Following [32], we define the length function $|\cdot|$ on a compactly generated group G by

$$|x| = \min\{n \in \mathbb{N} : x \in U^n\}, \quad x \in G,$$

where $U \subset G$ is an open symmetric neighborhood of identity with compact closure such that $G = \cup_{n=1}^{\infty} U^n$. We show that if G is a compactly generated [FC] group and $\omega_\alpha(x) = (1+|x|)^\alpha$, $\alpha \geq 0$, then $ZL^1(G,\omega_\alpha)$ is weakly amenable for $0 \leq \alpha < 1/2$. This perfectly agrees with the aforementioned Theorem 1 on weak amenability of $L^1(\mathbb{Z},\omega_\alpha)$ with the polynomial weight $\omega_\alpha = (1+|n|)^\alpha$, $n \in \mathbb{Z}$.

The derivation problem asks whether every continuous derivation D from a group algebra $L^1(G)$ to a measure algebra $M(G)$ must be inner. B.E. Johnson posed the question in 1970-s and pursued it over the years in developing his theory of cohomology in Banach algebras. The derivation problem for $L^1(G)$ has been attempted by many researchers and was completely solved affirmatively by V. Losert ([28]) in 2008. Recently, a shorter proof, that uses a special fixed point property for L-embedded Banach spaces, was given by U. Bader, T. Gelander and N. Monod in [3] in 2010. As for the weighted group algebra $L^1(G,\omega)$, the corresponding derivation problem is completely open. Using the method of U. Bader et. al., we were able to prove in Chapter 7 that continuous derivations from $L^1(G,\omega)$ into $M(G,\omega)$ are inner if the weight ω is diagonally bounded.

Chapter 2

Preliminaries

In this section, we define several more important objects and state some general results that we will use throughout the thesis.

We start from defining a measure algebra $M(G,\omega)$ for a locally compact group G and a weight ω on G:

$$M(G,\omega) = \left\{ \text{regular Borel measures}\, \mu \;:\; \|\mu\|_{M(G,\omega)} = \int_G \omega(x)\, d|\mu|(x) < \infty \right\},$$

where $|\mu|$ denotes the total variation of μ. With the norm $\|\cdot\|_{M(G,\omega)}$, $M(G,\omega)$ is a Banach space isometrically isomorphic to $M(G)$. Indeed, it is the dual space of

$$C_0(G,1/\omega) = \left\{ f \in C(G) : \frac{f}{\omega} \in C_0(G), \quad \|f\|_{C_0(G,1/\omega)} = \left\| \frac{f}{\omega} \right\|_{C_0(G)} < \infty \right\},$$

which becomes a Banach algebra with the convolution product

$$\int_G \phi(t)\, d(\nu * \sigma)(t) = \iint_{G\times G} \phi(xy)\, d\nu(x) d\sigma(y), \quad \nu, \sigma \in M(G,\omega),\; \phi \in C_c(G),$$

where $C_c(G)$ denotes the set of all continuous functions with compact support on G (Note that $C_c(G)$ is dense in $C_0(G,1/\omega)$). The map $f \mapsto f(x)\, dx$ embeds $L^1(G,\omega)$ isometrically into $M(G,\omega)$. Moreover, $L^1(G,\omega)$ is an ideal in $M(G,\omega)$.

There is also another relation between these two algebras. Namely, $M(G,\omega)$ can be identified with the so-called multiplier algebra of $L^1(G,\omega)$.

Definition 2.1. The multiplier algebra $\mathcal{M}(A)$ of a Banach algebra A is the set of pairs (L,R) of bounded linear operators on A satisfying the following properties:

$$L(ab) = L(a)b, \quad R(ab) = aR(b), \quad aL(b) = R(a)b, \quad a,b \in A.$$

Since $L^1(G,\omega)$ is an ideal in $M(G,\omega)$, each element $\mu \in M(G,\omega)$ gives rise to two bounded linear operators L_μ and R_μ on $L^1(G,\omega)$: $L_\mu(f) = \mu * f$ and $R_\mu(f) = f * \mu$, $f \in L^1(G,\omega)$. According to [12, Theorem 4], the map $\mu \mapsto (L_\mu, R_\mu)$ identifies $M(G,\omega)$ with the multiplier algebra of $L^1(G,\omega)$.

We will consider two more topologies on $M(G,\omega)$. The first is the strong operator (SO) topology. In general, if X is a Banach space and $\mathcal{L}(X)$ denotes the space of all bounded linear operators on X, the strong operator topology on $\mathcal{L}(X)\times\mathcal{L}(X)$ is induced by the family of seminorms $\{p_x\}_{x\in X}$ defined by

$$p_x(S,T) = \max\{\|S(x)\|, \|T(x)\|\}, \quad S,T \in \mathcal{L}(X).$$

Because $M(G,\omega)$ is identified with $\mathcal{M}(L^1(G,\omega)) \subset \mathcal{L}(X)\times\mathcal{L}(X)$, we can talk about SO topology on $M(G,\omega)$ with respect to $X = L^1(G,\omega)$. Then, by definition of SO topology, we have

$$\mu_\gamma \underset{SO}{\to} \mu \quad \Longleftrightarrow \quad f*\mu_\gamma \to f*\mu, \;\; \mu_\gamma * f \to \mu * f \;\text{ in }\; L^1(G,\omega), \quad f \in L^1(G,\omega).$$

Another topology on $M(G,\omega)$ we will deal with is the usual weak* topology generated by its Banach space predual $C_0(G,1/\omega)$.

In this thesis we will substantially use several times the following powerful tool from general amenability theory due to B.E. Johnson.

Proposition 2.2. [7, Theorem 2.9.53] *Let A be a Banach algebra with a bounded approximate identity, and let E be an essential Banach A-bimodule. Suppose that $D : A \to E^*$ is a derivation. Then there is a unique derivation $\widetilde{D} : \mathcal{M}(A) \to E^*$*

extending D, that is such that $\widetilde{D}|_A = D$. If D is continuous, then $\widetilde{D}$ is continuous in both norm and SO-weak topologies.*

Recall that a Banach A-bimodule E is called essential if $\overline{AE} = \overline{EA} = E$.

Another important technique for us is the following.

Lemma 2.3. *Let G be a locally compact group and ω be a weight on G. Then the linear space generated by the point masses δ_t, $t \in G$, is dense in $M(G, \omega)$ in strong operator topology.*

The non-weighted version of Lemma 2.3 is classical, for example, see [7, Proposition 3.3.41(i)]. The weighted case was proved in [41, Lemma 2.1].

We will sometimes need the weight ω to be bounded away from zero. The following result of M. White allows to assume without loss of generality that $\omega \geq 1$ if the group G is amenable.

Lemma 2.4. [38, Lemma 1] *Let G be an amenable group and ω be a weight on G. Then there is a continuous positive character (i.e., a multiplicative weight) $\phi : G \to (\mathbb{R}^+, \cdot)$ such that $\phi(x) \leq \omega(x)$ for all $x \in G$.*

It is evident that $\tilde{\omega} = \dfrac{\omega}{\phi} \geq 1$ is also a weight on G. Moreover, $L^1(G, \tilde{\omega})$ is Banach algebra isometrically isomorphic to $L^1(G, \omega)$. In fact, the map $\theta : L^1(G, \omega) \to L^1(G, \tilde{\omega})$ defined by $\theta(f) = f\phi$ is a Banach algebra isometry, where by $f\phi$ we mean the pointwise product of f and ϕ.

We can summarize the above observations as follows.

Remark 2.5. *Let G be an amenable group and ω be a weight on G. Then there exists a weight $\tilde{\omega} \geq 1$ on G such that $L^1(G, \omega)$ is isometrically isomorphic to $L^1(G, \tilde{\omega})$.*

In general, the pointwise supremum of a collection of continuous functions is not necessarily continuous. However, it must be Borel measurable.

Lemma 2.6. *Let E be a Hausdorff space and $\{f_\gamma\}_{\gamma \in \Gamma}$ be a collection of real-valued continuous functions on E. Suppose that the function f on E defined by*

$$f(x) = \sup_{\gamma \in \Gamma} f_\gamma(x), \quad x \in E$$

is finite on E. Then f is a Borel function. Analogous conclusion also holds for the point-wise infimum of continuous functions.

Although the proof is straightforward, we include it here for the sake of completeness.

Proof of Lemma 2.6. For every $a \in \mathbb{R}$, we have

$$\{x \in E : f(x) \leq a\} = \{x \in E : f_\gamma(x) \leq a,\ \gamma \in \Gamma\} = \bigcap_{\gamma \in \Gamma} \{x \in E : f_\gamma(x) \leq a\}.$$

Because all f_γ-s are continuous, each set $\{x \in G : f_\gamma(x) \leq a\}$ is closed. Therefore, as an intersection of closed sets, $\{x \in G : f(x) \leq a\}$ is also closed, and thus is a Borel set. This implies that f is a Borel function. $\square$

More generally, the pointwise supremum of a non-void collection of lower semicontinuous functions is still a lower semicontinuous function by [18, Theorem 11.10]. But we will only need Lemma 2.6.

Chapter 3

Testing examples

In this chapter we consider several specific weights on some basic non-commutative groups, and show that contrary to the expectations based on the theory of weak amenability for Abelian Beurling algebras, the corresponding weighted group algebras are not weakly amenable. Then we turn to the Abelian group $\mathbb{Z}^2$. We give a simple procedure of verifying whether $\ell^1(\mathbb{Z}^2, \omega)$ is weakly amenable. Finally, we present an example of the weight ω on $\mathbb{Z}^2$ making $\ell^1(\mathbb{Z}^2, \omega)$ weakly amenable, but whose restriction ω_1 to the first coordinate makes $\ell^1(\mathbb{Z}, \omega_1)$ not weakly amenable. This shows that the converse to the first part of [41, Theorem 3.8] does not hold.

3.1 Polynomial weights on $\mathbb{F}_2$

We start by a technical observation that will be used several times in this chapter.

Lemma 3.1. *Let G be a discrete group, and ω be a weight on G. Suppose a map D from $\{\delta_x\}_{x\in G}$ to $\ell^\infty(G, 1/\omega)$ has the following properties:*

$$D(\delta_{xy}) = D(\delta_x) \cdot \delta_y + \delta_x \cdot D(\delta_y), \quad x, y \in G, \quad \textit{and} \tag{3.1}$$

$$\|D(\delta_x)\|_{\ell^\infty(G,1/\omega)} \le c\,\omega(x), \quad x \in G, \tag{3.2}$$

where $c > 0$ is a constant. Then D can be extended to a bounded derivation from $\ell^1(G, \omega)$ to $\ell^\infty(G, 1/\omega)$.

Proof. We first extend D to the linear span of $\{\delta_x\}_{x\in G}$ by linearity. The bilinear mapping $D(f * g)$ satisfies the derivation relation

$$D(f * g) = D(f) \cdot g + f \cdot D(g)$$

for f, g from the generating set $\{\delta_x\}_{x\in G}$ by (3.1). So, the relation still holds for $f, g \in \operatorname{lin}\{\delta_x : x \in G\}$. Moreover,

$$\begin{aligned}\left\| D\left(\sum_{i=1}^{n} \alpha_i \delta_{x_i}\right)\right\| &\leq \sum_{i=1}^{n} |\alpha_i| \|D(\delta_{x_i})\| \underset{(3.2)}{\leq} c \sum_{i=1}^{n} |\alpha_i| \omega(x_i) \\ &= c \left\| \sum_{i=1}^{n} \alpha_i \delta_{x_i} \right\|_{\ell^1(G,\omega)}, \quad n \in \mathbb{N},\ x_i \in G,\ \alpha_i \in \mathbb{C}.\end{aligned}$$

Since $\operatorname{lin}\{\delta_x : x \in G\}$ is dense in $\ell^1(G, \omega)$, we can extend D to a bounded operator on $\ell^1(G, \omega)$, which is still a derivation by continuity. □

The necessity part of Proposition 1.11 also holds for a general discrete group.

Lemma 3.2. *Let G be a discrete group, and ω be a weight on G. If there exists a non-zero group homomorphism $\Phi : G \to \mathbb{R}$ such that*

$$\sup_{x\in G} \frac{|\Phi(x)|}{\omega(x)\omega(x^{-1})} < \infty,$$

then $\ell^1(G, \omega)$ is not weakly amenable.

Proof. It suffices to construct a non-inner bounded derivation $D : \ell^1(G, \omega) \to \ell^\infty(G, 1/\omega)$. We first define D on $\{\delta_x\}_{x\in G}$:

$$D(\delta_x) = \Phi(x)\delta_{x^{-1}}, \quad x \in G.$$

We claim that D satisfies the conditions of Lemma 3.1, and thus it can be extended to a bounded derivation from $\ell^1(G, \omega)$ to $\ell^\infty(G, 1/\omega)$. Indeed, since Φ is a group homomorphism, and

$$\delta_{(xy)^{-1}}(z) = \delta_{y^{-1}x^{-1}}(z) = \delta_{x^{-1}}(yz) = \delta_{y^{-1}}(zx),$$

we have

$$\begin{aligned} D(\delta_{xy})(z) &= \Phi(xy)\delta_{y^{-1}x^{-1}}(z) = \Phi(x)\delta_{x^{-1}}(yz) + \Phi(y)\delta_{y^{-1}}(zx) \\ &= D(\delta_x)(yz) + D(\delta_y)(zx) = (D(\delta_x)\cdot\delta_y)(z) + (\delta_x \cdot D(\delta_y))(z), \quad x, y, z \in G, \end{aligned}$$

and (3.1) is verified. If we denote $c = \sup_{x\in G} \dfrac{|\Phi(x)|}{\omega(x)\omega(x^{-1})}$, then for every $x \in G$ we have

$$\|D(\delta_x)\|_{\ell^\infty(G,1/\omega)} = \frac{|\Phi(x)|}{\omega(x^{-1})} = \omega(x)\frac{|\Phi(x)|}{\omega(x)\omega(x^{-1})} \le c\,\omega(x),$$

and (3.2) is also verified. Due to Lemma 3.1, D can be extended to a bounded derivation from $\ell^1(G,\omega)$ to $\ell^\infty(G,1/\omega)$. We now show that D is not inner. Assume, to the contrary, that there exists $\varphi \in \ell^\infty(G,1/\omega)$ such that $D(h) = h\cdot\varphi - \varphi\cdot h$, $h \in \ell^1(G,\omega)$. Then

$$D(\delta_x)(x^{-1}) = (\delta_x\cdot\varphi)(x^{-1}) - (\varphi\cdot\delta_x)(x^{-1}) = \varphi(e) - \varphi(e) = 0, \quad x \in G, \tag{3.3}$$

where e is the identity of G. On the other hand, according to our definition of D, $D(\delta_x)(x^{-1}) = \Phi(x)$. Combined with (3.3), this yields $\Phi \equiv 0$, which contradicts the assumption that Φ is non-zero. So, D is not inner, and hence, $\ell^1(G,\omega)$ is not weakly amenable. □

Later, in Section 4.2, we will see that Lemma 3.2 is true even for [IN] groups G.

We now examine the free group $\mathbb{F}_2$ with a polynomial weight. First, let us define several notions.

Definition 3.3. Let a and b denote the two generators of the free group $\mathbb{F}_2$. Then every $x \in \mathbb{F}_2$ can be written in a non-cancelable form $x = a^{k_1}b^{l_1}\dots a^{k_n}b^{l_n}$, where $k_i, l_i \in \mathbb{Z}$, and all k_i, l_i are non-zero except possibly k_1 and l_n, $1 \le i \le n$, $n \in \mathbb{N}$. We denote $|x| = \sum_{i=1}^{n}(|k_i| + |l_i|)$ and call it the *length* of x. The number $\sum_{i=1}^{n} k_i$ (resp. $\sum_{i=1}^{n} l_i$) will be called the *total power of* a (resp. the *total power of* b) in x and we denote it by $A(x)$ (resp. $B(x)$).

Example 3.4. *Let $\alpha > 0$ and ω_α be a function on $\mathbb{F}_2$ defined by $\omega_\alpha(x) = (1 + |x|)^\alpha$, $x \in \mathbb{F}_2$. Then ω_α is a weight on $\mathbb{F}_2$ (called a polynomial weight), and $\ell^1(\mathbb{F}_2, \omega_\alpha)$ is not weakly amenable.*

Proof. Since the length function $|\cdot|$ on $\mathbb{F}_2$ obviously satisfies the triangle inequality $|xy| \leq |x| + |y|$, $x, y \in \mathbb{F}_2$, it follows that ω_α is a weight on $\mathbb{F}_2$:

$$\omega_\alpha(xy) = (1 + |xy|)^\alpha \leq (1 + |x| + |y|)^\alpha \leq ((1 + |x|)(1 + |y|))^\alpha = \omega_\alpha(x)\omega_\alpha(y).$$

To prove that $\ell^1(\mathbb{F}_2, \omega_\alpha)$ is not weakly amenable, we first consider the case when $\alpha > 1/2$. Since the total power function $A : \mathbb{F}_2 \to \mathbb{Z}$ is, obviously, a group homomorphism and $|A(t)| \leq |t|$ for every $t \in \mathbb{F}_2$, we obtain:

$$\sup_{t\in\mathbb{F}_2} \frac{|A(t)|}{\omega_\alpha(t)\omega_\alpha(t^{-1})} = \sup_{t\in\mathbb{F}_2} \frac{|A(t)|}{(1 + |t|)^{2\alpha}} \leq \sup_{t\in\mathbb{F}_2} \frac{|t|}{(1 + |t|)^{2\alpha}} < \infty,$$

which, by Lemma 3.2, implies that $\ell^1(\mathbb{F}_2, \omega_\alpha)$ is not weakly amenable.

Now let $\alpha \leq 1/2$. In this case we will directly construct a non-inner derivation $D : \ell^1(\mathbb{F}_2, \omega) \to \ell^\infty(\mathbb{F}_2, 1/\omega)$. Take an arbitrary $\beta \in (\alpha, 2\alpha)$, and consider the function $\psi : \mathbb{F}_2 \to \mathbb{R}$ defined by

$$\psi(x) = \begin{cases} |t|^\beta, & \text{if } \ x = tat^{-1},\ t \in \mathbb{F}_2, \text{ and this representation is non-cancelable,} \\ 0, & \text{otherwise.} \end{cases}$$

We use ψ to define a map D from $\{\delta_x\}_{x\in\mathbb{F}_2}$ to $\ell^\infty(\mathbb{F}_2, 1/\omega)$:

$$D(\delta_x)(y) = \psi(xy) - \psi(yx), \quad x, y \in \mathbb{F}_2.$$

We claim that D satisfies the conditions of Lemma 3.1, and so it can be extended to a bounded derivation from $\ell^1(\mathbb{F}_2, \omega_\alpha)$ to $\ell^\infty(\mathbb{F}_2, 1/\omega_\alpha)$. The condition (3.1) holds since

$$D(\delta_{xy})(t) = \psi(xyt) - \psi(txy) = (\psi(xyt) - \psi(ytx)) + (\psi(ytx) - \psi(txy)) \tag{3.4}$$
$$= D(\delta_x)(yt) + D(\delta_y)(tx) = (D(\delta_x) \cdot \delta_y)(t) + (\delta_x \cdot D(\delta_y))(t), \quad x, y, t \in \mathbb{F}_2.$$

Now we prove that

$$|D(\delta_x)(y)| = |\psi(xy) - \psi(yx)| \le \omega_\alpha(x)\omega_\alpha(y), \quad x, y \in \mathbb{F}_2, \tag{3.5}$$

which will immediately imply (3.2) for $c = 1$. Indeed, in this case

$$\|D(\delta_x)\|_{\ell^\infty(\mathbb{F}_2, 1/\omega_\alpha)} = \sup_{y \in \mathbb{F}_2} \frac{|D(\delta_x)(y)|}{\omega_\alpha(y)} \le \sup_{y \in \mathbb{F}_2} \frac{\omega_\alpha(x)\omega_\alpha(y)}{\omega(y)} = \omega_\alpha(x).$$

By our definition of ψ, it vanishes off the conjugacy class $E = \{tat^{-1}\}_{t \in \mathbb{F}_2}$. Since $yx = y(xy)y^{-1}$, the elements xy and yx always belong to the same conjugacy class, and so we only need to prove (3.5) in the case when both xy and yx are in E. Let $xy = uau^{-1}$ and $yx = vav^{-1}$, both representations being non-cancelable. Assume without loss of generality that $|u| \le |v|$. Because

$$vav^{-1} = yx = y(xy)y^{-1} = yuau^{-1}y^{-1},$$

we have that $(u^{-1}y^{-1}v)a = a(u^{-1}y^{-1}v)$. So, the elements a and $u^{-1}y^{-1}v$ commute, which can happen in a free group only if both of them are powers of a third element (see, for example, [29, Proposition 2.17]). Since a is the generator of $\mathbb{F}_2$, it is only a power of itself, which implies that $u^{-1}y^{-1}v = a^k$ for some $k \in \mathbb{Z}$. In other words, $yu = va^{-k}$. We consider two cases: $k = 0$ and $k \ne 0$.

If $k = 0$, then $y = vu^{-1}$ and $x = (xy)y^{-1} = (uau^{-1})(uv^{-1}) = uav^{-1}$. In this case, the inequality (3.5) that we want to prove becomes the following:

$$\big||u|^\beta - |v|^\beta\big| \le (1 + |vu^{-1}|)^\alpha (1 + |uav^{-1}|)^\alpha.$$

Since $\alpha \le 1/2$, we have that $\beta < 2\alpha \le 1$, and so the real function $f(\tau) = \tau^\beta$ is concave for $\tau \geq 0$. It easily follows that $\big||u|^\beta - |v|^\beta\big| \leq \big||u| - |v|\big|^\beta$. We also have that $|vu^{-1}| \geq \big||u| - |v|\big|$, and $|uav^{-1}| \geq \big||u| - |v|\big| - 1$. Therefore,

$$\begin{aligned}(1 + |vu^{-1}|)^\alpha (1 + |uav^{-1}|)^\alpha &\geq (1 + \big||u| - |v|\big|)^\alpha \big||u| - |v|\big|^\alpha \geq \big||u| - |v|\big|^{2\alpha} \\ &\geq \big||u| - |v|\big|^\beta \geq \big||u|^\beta - |v|^\beta\big|,\end{aligned}$$

since $\beta \leq 2\alpha$ and $\big||u| - |v|\big| \in \mathbb{N} \cup \{0\}$. Hence, (3.5) is verified for the case $k = 0$.

Now let $k \neq 0$. Then $yu = va^{-k}$. Recall that both expressions uau^{-1} and vav^{-1} are non-cancelable. This means that both u and v end with a power of the second generator b of $\mathbb{F}_2$. Hence, the equality $yu = va^{-k}$ is only possible for $k \neq 0$ if $y = tu^{-1}$, and this expression is non-cancelable. In this case $t = va^{-k}$, and $|t| = |v| + |k|$, implying that $|v| = |t| - |k|$. We also have that $x = (xy)y^{-1} = (uau^{-1})(ut^{-1}) = uat^{-1}$. Thus, the inequality (3.5) that we want to prove becomes the following:

$$\big||u|^\beta - (|t| - |k|)^\beta\big| \leq (1 + |tu^{-1}|)^\alpha (1 + |uat^{-1}|)^\alpha.$$

Recall that we assumed from the very beginning that $|u| \leq |v| = |t| - |k|$, and so, using the same arguments as in the previous case, we obtain:

$$\begin{aligned}\big||u|^\beta - (|t| - |k|)^\beta\big| &= (|t| - |k|)^\beta - |u|^\beta \leq \big||t| - |k| - |u|\big|^\beta \leq \big||t| - |u|\big|^\beta \\ &\leq (1 + |tu^{-1}|)^\alpha (1 + |uat^{-1}|)^\alpha,\end{aligned}$$

and (3.5) is verified for $k \neq 0$ as well.

Therefore, we can use Lemma 3.1 to extend D to a bounded derivation from $\ell^1(\mathbb{F}_2, \omega_\alpha)$ to $\ell^\infty(\mathbb{F}_2, 1/\omega_\alpha)$. The only thing left to show is that D is not inner. Assume, to the contrary, that there exists $\varphi \in \ell^\infty(\mathbb{F}_2, 1/\omega_\alpha)$ such that $D(f) = \varphi \cdot f - f \cdot \varphi$ for every $f \in \ell^1(\mathbb{F}_2, \omega_\alpha)$. In particular,

$$D(\delta_x)(y) = (\varphi \cdot \delta_x)(y) - (\delta_x \cdot \varphi)(y) = \varphi(xy) - \varphi(yx), \quad x, y \in \mathbb{F}_2.$$

By the definition of D, we obtain that

$$\psi(xy) - \psi(yx) = \varphi(xy) - \varphi(yx), \quad x, y \in \mathbb{F}_2. \tag{3.6}$$

Taking $y = ax^{-1}$, we see that $\psi(xax^{-1}) - \psi(a) = \varphi(xax^{-1}) - \varphi(a)$ for all $x \in \mathbb{F}_2$. Therefore, the functions ψ and φ are different only by a constant $C = \psi(a) - \varphi(a)$ on the whole conjugacy class $E = \{tat^{-1}\}_{t \in \mathbb{F}_2}$. It follows that

$$\begin{aligned}\|\varphi\|_{\ell^\infty(\mathbb{F}_2, 1/\omega)} &= \sup_{t \in \mathbb{F}_2} \frac{|\varphi(t)|}{\omega(t)} \geq \sup_{n \in \mathbb{N}} \frac{|\varphi(b^n ab^{-n})|}{\omega(b^n ab^{-n})} \geq \sup_{n \in \mathbb{N}} \frac{\psi(b^n ab^{-n}) - C}{(2n+2)^\alpha} \\ &= \sup_{n \in \mathbb{N}} \frac{n^\beta - C}{(2n+2)^\alpha} = \infty,\end{aligned}$$

since $\beta > \alpha$. This is a contradiction to $\varphi \in \ell^\infty(\mathbb{F}_2, 1/\omega_\alpha)$ and proves that D is not inner. □

3.2 Polynomial weights on the group $(ax + b)$

In this section we consider the non-commutative amenable group $(\boldsymbol{ax + b})$ of all affine transformations $x \mapsto ax + b$ of $\mathbb{R}$ with $a > 0$ and $b \in \mathbb{R}$, where the map $x \mapsto ax + b$ is identified with the pair (a, b). Multiplication in this group is given by the composition of the corresponding transformations of $\mathbb{R}$, which can be expressed as

$$(a, b)(c, d) = (ac, ad + b), \quad a, c > 0,\ b, d \in \mathbb{R}.$$

The identity of $(\boldsymbol{ax + b})$ is a pair $(1, 0)$ corresponding to the identity map on $\mathbb{R}$. Therefore,

$$(a, b)^{-1} = \left(\frac{1}{a}, \frac{-b}{a}\right), \quad a > 0,\ b \in \mathbb{R}.$$

Throughout the remainder of this section, for the sake of notational convenience we denote the group $(\boldsymbol{ax + b})$ by G.

Example 3.5. *Let α be a positive number, and ω_α be the function on G defined by $\omega_\alpha(a, b) = (1 + |\ln a|)^\alpha$, $(a, b) \in G$. Then ω_α is a weight on G, and $\ell^1(G, \omega_\alpha)$ is not* *weakly amenable.*

Proof. To verify the weight inequality for ω_α, let (a,b), $(c,d) \in G$. Then

$$\begin{aligned}\omega_\alpha((a,b)(c,d)) = \omega_\alpha(ac, ad+b) = (1 + |\ln(ac)|)^\alpha &\leq (1 + |\ln a| + |\ln c|)^\alpha \\ &\leq (1 + |\ln a|)^\alpha (1 + |\ln c|)^\alpha = \omega_\alpha(a,b)\omega_\alpha(c,d).\end{aligned}$$

Again, as in the case of a polynomial weight on $\mathbb{F}_2$, we consider two possibilities: $\alpha \geq 1/2$ and $\alpha < 1/2$. Suppose first that $\alpha \geq 1/2$. Then

$$\begin{aligned}\sup_{(a,b)\in G} \frac{|\ln a|}{\omega_\alpha(a,b)\omega_\alpha\left((a,b)^{-1}\right)} &= \sup_{a>0} \frac{|\ln a|}{(1+|\ln a|)^\alpha (1 + |\ln(1/a)|)^\alpha} \\ &= \sup_{a>0} \frac{|\ln a|}{(1+|\ln a|)^{2\alpha}} < \infty,\end{aligned}$$

and since $(a,b) \mapsto \ln a$ is a group homomorphism from G to $\mathbb{R}$, we obtain that $\ell^1(G,\omega_\alpha)$ is not weakly amenable by Lemma 3.2.

Now suppose that $\alpha < 1/2$. In this case, to prove that $\ell^1(G,\omega_\alpha)$ is not weakly amenable, we construct a non-inner derivation D from $\ell^1(G,\omega_\alpha)$ to $\ell^\infty(G, 1/\omega_\alpha)$. We define the function $\psi : G \to \mathbb{R}$ as follows:

$$\psi(a,b) = \begin{cases} |\ln b|, & \text{if } \; a = 1, \, b > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Using ψ, we define D on $\{\delta_u\}_{u\in G}$:

$$D(\delta_u)(v) = \psi(uv) - \psi(vu), \quad u, v \in G.$$

We claim that D satisfies the conditions of Lemma 3.1, and so it can be extended to a bounded derivation from $\ell^1(G,\omega_\alpha)$ to $\ell^\infty(G, 1/\omega_\alpha)$. Note that (3.4) from the proof of Example 3.4 with $\mathbb{F}_2$ replaced by G still works to verify (3.1) in our case. So, we only need to show that there is a constant $C > 0$ such that

$$|D(\delta_u)(v)| = |\psi(uv) - \psi(vu)| \leq C\omega_\alpha(u)\omega_\alpha(v), \quad v, u \in G. \tag{3.7}$$

We will prove this inequality for $C = 1$. Let $u = (a, b)$ and $v = (c, d)$, where $a, c > 0$, $b, d \in \mathbb{R}$. Then $uv = (ac, ad{+}b)$, $vu = (ac, bc{+}d)$. If $ac \neq 1$, then $\psi(uv) = \psi(vu) = 0$, and (3.7) holds for any $C > 0$. Suppose now that $ac = 1$. Then

$$c = 1/a, \quad bc + d = b/a + d = (ad + b)/a.$$

Since $a > 0$, we have that either both $ad + b$ and $bc + d$ are negative, in which case (3.7) again holds for any $C > 0$, or both $ad + b$ and $bc + d$ are positive. In the latter case, we obtain

$$\begin{aligned} |\psi(uv) - \psi(vu)| &= \left|\psi(1, ad + b) - \psi\left(1, \frac{ad + b}{a}\right)\right| = \left||\ln(ad + b)| - \left|\ln \frac{ad + b}{a}\right|\right| \\ &= \big|\ln(ad + b)| - |\ln(ad + b) - \ln a|\big| \leq |\ln a| \leq (1 + |\ln a|)^{2\alpha} \\ &= \omega_\alpha(a, b)\omega_\alpha\left(\frac{1}{a}, d\right) = \omega_\alpha(a, b)\omega_\alpha(c, d) = \omega_\alpha(u)\omega_\alpha(v), \end{aligned}$$

and (3.7) is verified.

So, by Lemma 3.1, we can extend D to a bounded derivation from $\ell^1(G, \omega_\alpha)$ to $\ell^\infty(G, 1/\omega_\alpha)$. We now show that D is not inner. Assume, to the contrary, that there exists $\varphi \in \ell^\infty(G, 1/\omega_\alpha)$ such that $D(f) = \varphi \cdot f - f \cdot \varphi$ for every $f \in \ell^1(G, \omega_\alpha)$. Then, analogously to (3.6) from the proof of Example 3.4, we obtain:

$$\psi(uv) - \psi(vu) = \varphi(uv) - \varphi(vu), \quad u, v \in G.$$

For $u = (a, 1)$ and $v = \left(\frac{1}{a}, 0\right)$, $a > 0$, we have $uv = (1, 1)$, $vu = \left(1, \frac{1}{a}\right)$, and so

$$\varphi(uv) - \varphi(vu) = \varphi(1, 1) - \varphi\left(1, \frac{1}{a}\right) = \psi(1, 1) - \psi\left(1, \frac{1}{a}\right) = -|\ln a|.$$

Therefore, $\varphi\left(1, \dfrac{1}{a}\right) = |\ln a| + \varphi(1, 1)$, implying that

$$\sup_{t \in G} \frac{|\varphi(t)|}{\omega_\alpha(t)} \geq \sup_{a > 0} \frac{\left|\varphi\left(1, \frac{1}{a}\right)\right|}{\omega_\alpha\left(1, \frac{1}{a}\right)} = \sup_{a > 0}\big||\ln a| + \varphi(1, 1)\big| = \infty.$$

This is a contradiction to $\varphi \in \ell^\infty(G, 1/\omega_\alpha)$, proving that D is not inner. Hence, $\ell^1(G, \omega_\alpha)$ is not weakly amenable. $\square$

We also call the weight ω_α defined in Example 3.5 the polynomial weight on $(\boldsymbol{ax+b})$. Note that, unlike $\mathbb{F}_2$, the group $(\boldsymbol{ax+b})$ is amenable. Example 3.5 shows that even a "nice" weight on an amenable group may still make the corresponding weighted convolution algebra not weakly amenable.

Remark 3.6. *In fact, the proof of Example 3.5 can be adopted to produce an example of a finitely generated (and hence separable) non-commutative amenable group $\widetilde{G}$ such that Proposition 1.11 does not hold for $\widetilde{G}$. Indeed, all our arguments will work for the subgroup*

$$\widetilde{G} = \left\{ (2^n, b) : n \in \mathbb{Z},\, b \in \mathbb{Z}\left[\frac{1}{2}\right] \right\} = \big\langle (2,0), (1,1) \big\rangle$$

of $(\boldsymbol{ax+b})$-group and the weight $\omega_{1/3}$ restricted to $\widetilde{G}$. This shows that the pathology of the example is really the result of non-commutativity rather than of non-separability of the group.

3.3 Beurling algebras on $\mathbb{Z}^2$

We begin with noting that the complex-valued homomorphisms Φ in the characterization of weak amenability of $L^1(G, \omega)$ for Abelian groups G from Theorem 1.12 can, in fact, be replaced with real-valued homomorphisms, see [41, Theorem 3.5]. It follows that $\ell^1(\mathbb{Z}^2, \omega)$ is weakly amenable if and only if for every non-trivial group homomorphism $\Phi : \mathbb{Z}^2 \to \mathbb{R}$ we have that

$$\sup_{t \in \mathbb{Z}^2} \frac{|\Phi(t)|}{\omega(t)\omega(-t)} = \infty.$$

Because every such homomorphism has the form $\Phi(k,m) = ck+dm$ for some $c, d \in \mathbb{R}$ with $c^2 + d^2 \neq 0$, the group algebra $\ell^1(\mathbb{Z}^2, \omega)$ is weakly amenable if and only if

$$\sup_{k,m \in \mathbb{Z}} \frac{|ck + dm|}{\omega(k,m)\omega(-k,-m)} = \infty$$

for every pair $(c,d) \in \mathbb{R}^2$ with $c^2 + d^2 \neq 0$. We aim to find a procedure that allows us to determine weak amenability by checking the supremums for only two pairs (c,d), instead of all non-trivial pairs (c,d). This will significantly simplify the verification process in most cases. We start from proving the following simple technical lemma.

Lemma 3.7. *Suppose that ω is a weight on $\mathbb{Z}^2$. Let c_1, d_1, c_2, d_2 be real numbers satisfying the relation $c_1 d_2 - c_2 d_1 \neq 0$ and such that*

$$\sup_{k,m \in \mathbb{Z}} \frac{|c_i k + d_i m|}{\omega(k,m)\omega(-k,-m)} < \infty \quad (i = 1, 2).$$

Then for all $c, d \in \mathbb{R}$ we have that

$$\sup_{k,m \in \mathbb{Z}} \frac{|ck + dm|}{\omega(k,m)\omega(-k,-m)} < \infty.$$

Proof. Denote

$$M_i = \sup_{k,m \in \mathbb{Z}} \frac{|c_i k + d_i m|}{\omega(k,m)\omega(-k,-m)} \quad (i = 1, 2).$$

Then for every $k, m \in \mathbb{Z}$ we have

$$|c_i k + d_i m| \leq M_i \omega(k,m)\omega(-k,-m) \quad (i = 1, 2).$$

Since $c_1 d_2 - c_2 d_1 \neq 0$, the vectors (c_1, d_1) and (c_2, d_2) are linearly independent in $\mathbb{R}^2$. Fix an arbitrary $(c,d) \in \mathbb{R}^2$. Then, there exist real coefficients α, β such that $(c,d) = \alpha(c_1, d_1) + \beta(c_2, d_2)$, and we obtain

$$\begin{aligned} |ck + dm| &= |\alpha(c_1 k + d_1 m) + \beta(c_2 k + d_2 m)| \leq |\alpha| \cdot |c_1 k + d_1 m| + |\beta| \cdot |c_2 k + d_2 m| \\ &\leq (|\alpha| M_1 + |\beta| M_2)\omega(k,m)\omega(-k,-m) \quad (k, m \in \mathbb{Z}). \end{aligned}$$

This immediately implies that

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{\omega(k,m)\omega(-k,-m)} \leq |\alpha|M_1 + |\beta|M_2 < \infty.$$

The proof is complete. $\square$

It follows from Lemma 3.7 that for any weight ω on $\mathbb{Z}^2$ there are three possible situations:

S1. for every non-trivial pair $(c,d) \in \mathbb{R}^2$

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{\omega(k,m)\omega(-k,-m)} = \infty,$$

and $\ell^1(\mathbb{Z}^2,\omega)$ is weakly amenable;

S2. for every pair $(c,d) \in \mathbb{R}^2$

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{\omega(k,m)\omega(-k,-m)} < \infty,$$

and $\ell^1(\mathbb{Z}^2,\omega)$ is not weakly amenable;

S3. there is a unique, up to a non-zero multiple, non-trivial pair $(c,d) \in \mathbb{R}^2$ such that

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{\omega(k,m)\omega(-k,-m)} < \infty,$$

and, $\ell^1(\mathbb{Z}^2,\omega)$ is not weakly amenable.

Employing this observation, we can prove the following.

Proposition 3.8. *Let ω be a weight on $\mathbb{Z}^2$, which is symmetric and even with respect to the second variable, i.e., it satisfies the relation*

$$\omega(k,m) = \omega(k,-m) = \omega(m,k) \quad (k,m \in \mathbb{Z}). \tag{3.8}$$

Then $\ell^1(\mathbb{Z}^2,\omega)$ is weakly amenable if and only if there exist $c,d \in \mathbb{R}$ such that

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{\omega(k,m)\omega(-k,-m)} = \infty. \tag{3.9}$$

Remark 3.9. *The conclusion of Proposition 3.8 means that if (3.9) holds for one pair (c, d), then it holds for all pairs (c, d) of real numbers. So, in practice, if ω is symmetric and even, then one simply computes $\sup\limits_{k,m \in \mathbb{Z}} \dfrac{|ck + dm|}{\omega(k, m)\omega(-k, -m)}$ for any single non-trivial pair $(c, d) \in \mathbb{R}^2$ to determine whether $\ell^1(\mathbb{Z}^2, \omega)$ is weakly amenable. If the supremum is infinite, then $\ell^1(\mathbb{Z}^2, \omega)$ is weakly amenable; if the supremum is finite, then $\ell^1(\mathbb{Z}^2, \omega)$ is not weakly amenable.*

Proof of Proposition 3.8. We only need to prove that S3 is not possible for any weight ω satisfying (3.8). According to Lemma 3.7, it is enough to show that if for some non-trivial pair $(c_0, d_0) \in \mathbb{R}^2$ the corresponding supremum is finite, then there exists another pair $(c, d) \in \mathbb{R}^2$, not proportional to (c_0, d_0), for which the supremum is also finite. First we consider the case when $c_0 \neq \pm\, d_0$. Then the pair (d_0, c_0) is not proportional to (c_0, d_0), and for this pair we also have

$$\begin{aligned}
\sup_{k,m \in \mathbb{Z}} \frac{|d_0 k + c_0 m|}{\omega(k, m)\omega(-k, -m)} &\underset{\omega(k,m)=\omega(m,k)}{=} \sup_{k,m \in \mathbb{Z}} \frac{|d_0 k + c_0 m|}{\omega(m, k)\omega(-m, -k)} \\
&\underset{k \leftrightarrow m}{=} \sup_{k,m \in \mathbb{Z}} \frac{|c_0 k + d_0 m|}{\omega(k, m)\omega(-k, -m)} < \infty.
\end{aligned}$$

Now, if $c_0 = d_0$ or $c_0 = -d_0$, then $d_0 \neq 0$ (since the pair (c_0, d_0) is non-trivial), and so the pair $(c_0, -d_0)$ is not proportional to (c_0, d_0). For this pair we still have

$$\begin{aligned}
\sup_{k,m \in \mathbb{Z}} \frac{|c_0 k - d_0 m|}{\omega(k, m)\omega(-k, -m)} &\underset{\omega(k,m)=\omega(k,-m)}{=} \sup_{k,m \in \mathbb{Z}} \frac{|c_0 k - d_0 m|}{\omega(k, -m)\omega(-k, m)} \\
&\underset{m \leftrightarrow -m}{=} \sup_{k,m \in \mathbb{Z}} \frac{|c_0 k + d_0 m|}{\omega(k, m)\omega(-k, -m)} < \infty.
\end{aligned}$$

The proof is complete. $\square$

Example 3.10. *In particular, Proposition 3.8 holds for any weight of the form $\omega(k, m) = W(\|(k, m)\|)$, i.e., any weight depending only on the norm $\|(k, m)\| = \sqrt{k^2 + m^2}$, $k, m \in \mathbb{Z}$.*

Now let us consider the situation S3 in more detail. Let ω be a weight for which we have this situation. Without loss of generality, we can assume that the corresponding supremum is finite for a pair (c,d) with $c = 1$, i.e., that there exists a real d such that $\sup_{k,m\in\mathbb{Z}} \dfrac{|k+dm|}{\omega(k,m)\omega(-k,-m)} = M < \infty$. This implies the following:

$$\frac{1}{M}|k+dm| \le \omega(k,m)\omega(-k,-m) \quad (k,m\in\mathbb{Z}). \tag{3.10}$$

Since we are in the situation S3, the supremum is infinite for every pair (c',d') that is not proportional to $(1,d)$, in particular, for the pair $(0,1)$. So, we have that

$$\sup_{k,m\in\mathbb{Z}} \frac{|m|}{\omega(k,m)\omega(-k,-m)} = \infty,$$

which means that there exists a sequence $\{(k_n,m_n)\}_{n=1}^{\infty} \subset \mathbb{Z}^2$ such that

$$\frac{|m_n|}{\omega(k_n,m_n)\omega(-k_n,-m_n)} > n,$$

and so

$$\frac{|m_n|}{n} > \omega(k_n,m_n)\omega(-k_n,-m_n), \quad n\in\mathbb{N}.$$

Combining the last inequality with (3.10), we obtain

$$\frac{1}{M}|k_n+dm_n| \le \omega(k_n,m_n)\omega(-k_n,-m_n) < \frac{|m_n|}{n} \quad (n\in\mathbb{N}).$$

By dividing the whole inequality by (non-zero) $|m_n|$ and multiplying by M, we finally get that

$$\left|\frac{k_n}{m_n}+d\right| < \frac{M}{n} \quad (n\in\mathbb{N}).$$

It follows that $d = -\lim_{n\to\infty} \dfrac{k_n}{m_n}$.

Now we are ready to formulate the aforementioned procedure involving calculation of at most two supremums.

Procedure for verification of whether $\ell^1(\mathbb{Z}^2,\omega)$ is weakly amenable.

Step 1. We calculate $\sup_{k,m\in\mathbb{Z}} \dfrac{|m|}{\omega(k,m)\omega(-k,-m)}$. If it is finite, then $\ell^1(\mathbb{Z}^2,\omega)$ is not weakly amenable. If it is infinite, then we are either in situation S1 or in the situation S3, and we proceed to the second step.

Step 2. We choose $\{(k_n, m_n)\}_{n=1}^{\infty} \subset \mathbb{Z}^2$ such that $\dfrac{|m_n|}{\omega(k_n, m_n)\omega(-k_n, -m_n)} > n$, $n \in \mathbb{N}$, and consider $\lim\limits_{n\to\infty} \dfrac{k_n}{m_n}$. If the limit does not exist or is infinite, then, according to what we have discussed above, we cannot be in situation S3. This means that we are in the situation S1, and so $\ell^1(\mathbb{Z}^2, \omega)$ is weakly amenable. Now, if $\lim\limits_{n\to\infty} \dfrac{k_n}{m_n}$ exists and is finite, we denote it by $-d$ and proceed to the last step.

Step 3. We calculate $\sup\limits_{k,m\in\mathbb{Z}} \dfrac{|k+dm|}{\omega(k,m)\omega(-k,-m)}$. If it is finite, then $\ell^1(\mathbb{Z}^2, \omega)$ is not weakly amenable. On the other hand, if it is infinite, we cannot be in the situation S3, so we must be in the situation S1, which means that $\ell^1(\mathbb{Z}^2, \omega)$ is weakly amenable.

Remark 3.11. *The procedure above will also work if in the first step we start from any other* $\sup\limits_{k,m\in\mathbb{Z}} \dfrac{|ck+dm|}{\omega(k,m)\omega(-k,-m)}$, *instead of* $\sup\limits_{k,m\in\mathbb{Z}} \dfrac{|m|}{\omega(k,m)\omega(-k,-m)}$, *with minor adjustments in the next steps.*

It was proved in [41] that if ω is a weight on $\mathbb{Z}^2$ such that both $\ell^1(\mathbb{Z}, \omega_1)$ and $\ell^1(\mathbb{Z}, \omega_2)$ are weakly amenable, where $\omega_1(k) = \omega(k, 0)$, $\omega_2(k) = \omega(0, k)$, $k \in \mathbb{Z}$, then $\ell^1(\mathbb{Z}^2, \omega)$ is also weakly amenable. We finish this section by presenting an example showing that the converse is not true.

Consider the function ω on $\mathbb{Z}^2$ defined by

$$\omega(k,m) = (1+|k|)^{1/3}(1+|k+m|)^{1/3} \quad (k,m \in \mathbb{Z}). \tag{3.11}$$

It is easy to see that ω is a weight on $\mathbb{Z}^2$. This follows from the fact that both mappings $(k,m) \mapsto k$ and $(k,m) \mapsto k+m$ from $\mathbb{Z}^2$ to $\mathbb{Z}$ are linear, and from the obvious inequality

$$(1+|a+b|) \leq (1+|a|)(1+|b|) \quad (a,b \in \mathbb{Z}).$$

Example 3.12. *For the weight ω defined by (3.11), the algebra $\ell^1(\mathbb{Z}^2, \omega)$ is weakly amenable, but $\ell^1(\mathbb{Z}, \omega_1)$ is not weakly amenable, where $\omega_1(k) = \omega(k, 0)$, $k \in \mathbb{Z}$.*

Proof. The weight ω_1 is precisely given by $\omega_1(k) = \omega(k,0) = (1+|k|)^{2/3}$, and so $\ell^1(\mathbb{Z},\omega_1)$ is not weakly amenable by Theorem 1. We now prove that $\ell^1(\mathbb{Z}^2,\omega)$ is weakly amenable. According to Theorem 1.12, it is enough to show that

$$\sup_{t\in\mathbb{Z}^2} \frac{|\Phi(t)|}{\omega(t)\omega(-t)} = \infty$$

for every non-trivial group homomorphism $\Phi : \mathbb{Z}^2 \to \mathbb{C}$. Since every such homomorphism is of the form $\Phi(k,m) = ck + dm$, $k,m \in \mathbb{Z}$, for some complex numbers c,d with $|c|^2 + |d|^2 \neq 0$, we only need to show that

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{\omega(k,m)\omega(-k,-m)} = \sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{(1+|k|)^{2/3}(1+|k+m|)^{2/3}} = \infty$$

for all $c,d\in\mathbb{C}$ with $|c|^2+|d|^2\neq 0$. If $d\neq 0$, then

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{(1+|k|)^{2/3}(1+|k+m|)^{2/3}} \underset{\text{put } k=0}{\geq} \sup_{m\in\mathbb{Z}} \frac{|d|\cdot|m|}{(1+|m|)^{2/3}} = \infty.$$

Now, if $d=0$, then $c\neq 0$ since $|c|^2+|d|^2\neq 0$, and we have

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{(1+|k|)^{2/3}(1+|k+m|)^{2/3}} \underset{\text{put } m=-k}{\geq} \sup_{k\in\mathbb{Z}} \frac{|c|\cdot|k|}{(1+|k|)^{2/3}} = \infty.$$

So, we got that

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{(1+|k|)^{2/3}(1+|k+m|)^{2/3}} = \infty$$

for all non-trivial pairs $(c,d)\in\mathbb{C}^2$. Hence, $\ell^1(\mathbb{Z}^2,\omega)$ is, indeed, weakly amenable. $\square$

Chapter 4

Weak amenability of $L^1(G, \omega)$

In this section, we begin with proving Theorem 1.10 which provides us with a sufficient condition for weak amenability of the Beurling algebra $L^1(G, \omega)$. Then, in Section 4.2, we give two necessary conditions for weak amenability of $L^1(G, \omega)$. One of them is for [IN] groups; it generalizes [41, Remark 3.2]. The other one is for general locally compact groups. To prove these results, we need a characterization of bounded derivations from $L^1(G, \omega)$ to its dual $L^\infty(G, 1/\omega)$. This characterization generalizes the corresponding result of Johnson [20] for the weight $\omega \equiv 1$.

Section 4.3 is devoted to the study of weak amenability of $\ell^1(\mathbb{F}_2, \omega)$. We show that for two natural classes of weights ω, $\ell^1(\mathbb{F}_2, \omega)$ is weakly amenable if and only if ω is diagonally bounded. We also give an example of a weight on $\mathbb{F}_2$ which is diagonally bounded, but is not equivalent to a multiplicative weight. This contrasts with the case of amenable groups.

Finally, in Section 4.4 we consider the Beurling algebra $\ell^1(G, \omega)$ on a general discrete group G. We prove a result that can be considered a first step towards weakening the sufficient condition for weak amenability of $\ell^1(G, \omega)$ given in Theorem 1.10. Hence, this brings us closer to characterizing weak amenability of $\ell^1(G, \omega)$.

4.1 A sufficient condition for weak amenability of $L^1(G,\omega)$

As we have mentioned in the Introduction, this result was first proved in [34]. We have observed its validity independently applying the same method as was used by M. Despic and F. Ghahramani [9] to prove weak amenability of $L^1(G)$. Since the paper [34] is not easily accessible, we include a proof here for the sake of completeness.

Theorem 1.10. [34, Theorem 3.14] *Let G be a locally compact group and ω be a diagonally bounded weight on G. Then the Beurling algebra $L^1(G,\omega)$ is weakly amenable.*

Proof. Let D be a bounded derivation from $L^1(G,\omega)$ to $L^\infty(G,1/\omega)$. Since $L^\infty(G,1/\omega)$ is the dual of the essential $L^1(G,\omega)$-bimodule $L^1(G,\omega)$, and $M(G,\omega)=\mathcal{M}(L^1(G,\omega))$ (see Chapter 2), we can extend D to a bounded derivation $\widetilde{D}: M(G,\omega)\to L^\infty(G,1/\omega)$, which is continuous in SO-w* topology. If we show that $\widetilde{D}$ is inner, then it will automatically imply that D is inner. Consider the set

$$S=\{\mathrm{Re}(\delta_{t^{-1}}\cdot\widetilde{D}(\delta_t)) : t\in G\},$$

where δ_τ denotes the point mass at $\tau\in G$, and $\mathrm{Re}(\psi)$ stands for the real part of the function $\psi\in L^\infty(G,1/\omega)$. Then S is a bounded subset of the vector lattice $L^\infty_{\mathbb{R}}(G,1/\omega)$ of real-valued functions in $L^\infty(G,1/\omega)$. Indeed,

$$\|\mathrm{Re}(\delta_{t^{-1}}\cdot\widetilde{D}(\delta_t))\|_{L^\infty(G,1/\omega)}\le\|\widetilde{D}\|\|\delta_{t^{-1}}\|_{M(G,\omega)}\|\delta_t\|_{M(G,\omega)}=\|\widetilde{D}\|\omega(t)\omega(t^{-1})\le c\|\widetilde{D}\|,$$

because ω is diagonally bounded. Then, since $L^\infty_{\mathbb{R}}(G,1/\omega)$ is a complete vector lattice, $\psi_1=\sup(S)$ exists in $L^\infty_{\mathbb{R}}(G,1/\omega)$. Because D is a derivation, for every

$x \in G$ we have

$$\begin{aligned}
\delta_x \cdot \psi_1 &= \sup_{t\in G} \mathrm{Re}(\delta_x \cdot (\delta_{t^{-1}} \cdot \widetilde{D}(\delta_t))) = \sup_{t\in G} \mathrm{Re}(\delta_{xt^{-1}} \cdot \widetilde{D}(\delta_{tx^{-1}} * \delta_x)) \\
&= \sup_{t\in G} \mathrm{Re}(\delta_{xt^{-1}} \cdot \widetilde{D}(\delta_{(xt^{-1})^{-1}}) \cdot \delta_x + \delta_{xt^{-1}} \cdot (\delta_{tx^{-1}} \cdot \widetilde{D}(\delta_x))) \\
&= \sup_{xt^{-1}\in G} \mathrm{Re}(\delta_{xt^{-1}} \cdot \widetilde{D}(\delta_{(xt^{-1})^{-1}})) \cdot \delta_x + \mathrm{Re}(\widetilde{D}(\delta_x)) = \psi_1 \cdot \delta_x + \mathrm{Re}(\widetilde{D}(\delta_x)).
\end{aligned}$$

It follows that

$$\mathrm{Re}(\widetilde{D}(\delta_x)) = \delta_x \cdot \psi_1 - \psi_1 \cdot \delta_x, \quad x \in G.$$

Similarly, by considering imaginary parts, we obtain $\psi_2 \in L^\infty_{\mathbb{R}}(G, 1/\omega)$ such that

$$\mathrm{Im}(\widetilde{D}(\delta_x)) = \delta_x \cdot \psi_2 - \psi_2 \cdot \delta_x, \quad x \in G.$$

Therefore,

$$\widetilde{D}(\delta_x) = \delta_x \cdot \psi - \psi \cdot \delta_x, \quad x \in G,$$

where $\psi = \psi_1 + \boldsymbol{i}\psi_2$. Since by Lemma 2.3 every measure $\mu \in M(G,\omega)$ is the so-limit of a net of linear combinations of point masses and $\widetilde{D}$ is so-w* continuous, we obtain that

$$\widetilde{D}(\mu) = \mu \cdot \psi - \psi \cdot \mu, \quad \mu \in M(G,\omega).$$

This precisely means that $\widetilde{D}$ is inner, which completes the proof. □

4.2 Necessary conditions for weak amenability of $L^1(G,\omega)$

We first provide a characterization of bounded derivations from $L^1(G,\omega)$ to its dual $L^\infty(G, 1/\omega)$. It is particularly important to us for studying weak amenability of $L^1(G,\omega)$. We use the same approach as in [20], which dealt with the non-weighted case. For Abelian group G see [34].

Let G_1, G_2 be locally compact groups, and ω_i be a weight on G_i $(i = 1, 2)$. We denote by $\omega_1 \times \omega_2$ the weight on $G_1 \times G_2$ defined by

$$(\omega_1 \times \omega_2)(x_1, x_2) = \omega_1(x_1)\omega_2(x_2), \quad x_1 \in G_1,\ x_2 \in G_2.$$

Proposition 4.1. *Let G be a locally compact group, and ω be a weight on G. Then for every bounded derivation $D : L^1(G,\omega) \to L^\infty(G, 1/\omega)$ there exists a function $\alpha \in L^\infty(G \times G, 1/(\omega \times \omega))$ generating D in the following sense:*

$$\langle g, D(f)\rangle = \iint\limits_{G\times G} \alpha(x,y) f(x) g(y)\, dxdy \quad (f, g \in L^1(G,\omega)), \tag{4.1}$$

and satisfying the relation

$$\alpha(xy, z) = \alpha(x, yz) + \alpha(y, zx) \quad (\textit{for almost all } \ (x,y,z) \in G \times G \times G). \tag{4.2}$$

Conversely, every function $\alpha \in L^\infty(G \times G, 1/(\omega \times \omega))$ satisfying (4.2) defines a bounded derivation $D : L^1(G,\omega) \to L^\infty(G, 1/\omega)$ by the formula (4.1).

Proof. Let $D : L^1(G,\omega) \to L^\infty(G, 1/\omega)$ be a bounded derivation. Then, in particular, D belongs to $\mathcal{B}(L^1(G,\omega), (L^1(G,\omega))^*)$, the set of all bounded linear operators from $L^1(G,\omega)$ to its dual. It is well-known (see, for example, [33, Proposition 1.10.9]) that $\mathcal{B}(X, Y^*)$ is isometrically isomorphic to $(X\hat{\otimes}Y)^*$ by means of the map $F \mapsto T(F)$ defined by

$$\langle x \otimes y, T(F)\rangle = \langle y, F(x)\rangle, \quad F \in \mathcal{B}(X, Y^*)\ (x \in X,\ y \in Y).$$

So, there is an element $\alpha \in (L^1(G,\omega)\hat{\otimes}L^1(G,\omega))^*$ that corresponds to D and is related to D by

$$\langle f \otimes g, \alpha\rangle = \langle g, D(f)\rangle \quad (f, g \in L^1(G,\omega)). \tag{4.3}$$

Since

$$L^1(G,\omega)\hat{\otimes}L^1(G,\omega) \cong L^1(G \times G, \omega \times \omega), \quad \text{and}$$

$$(L^1(G \times G, \omega \times \omega))^* = L^\infty(G \times G, 1/(\omega \times \omega)),$$

we have that $\alpha \in L^\infty(G \times G, 1/(\omega \times \omega))$. From the action of a functional from $L^\infty(G \times G, 1/(\omega \times \omega))$ on $L^1(G \times G, \omega \times \omega)$, we immediately obtain (4.1) from (4.3). So, it is only left to prove that α satisfies (4.2). Since D is a derivation, we have that $D(f * g) = D(f) \cdot g + f \cdot D(g)$ for all $f, g \in L^1(G,\omega)$. Hence,

$$\begin{aligned}\langle h, D(f*g)\rangle &= \langle h, D(f)\cdot g\rangle + \langle h, f \cdot D(g)\rangle \\ &= \langle g * h, D(f)\rangle + \langle h * f, D(g)\rangle, \quad f, g, h \in L^1(G,\omega).\end{aligned}$$

Combining this with (4.1), we obtain

$$\begin{aligned}\iint\limits_{G\times G} \alpha(x,y)(f*g)(x)h(y)\,dxdy = &\iint\limits_{G\times G} \alpha(x,y)f(x)(g*h)(y)\,dxdy \\ &+ \iint\limits_{G\times G} \alpha(x,y)g(x)(h*f)(y)\,dxdy, \quad f,g,h \in L^1(G,\omega).\end{aligned} \tag{4.4}$$

Using the definition of convolution, we derive the following equalities for all $f, g, h \in L^1(G,\omega)$:

$$\begin{aligned}\iint\limits_{G\times G} \alpha(x,y)(f*g)(x)h(y)\,dxdy &= \iiint\limits_{G\times G\times G} \alpha(x,y)f(z)g(\underbrace{z^{-1}x}_{t})h(y)\,dxdydz \\ &= \iiint\limits_{G\times G\times G} \alpha(zt,y)f(z)g(t)h(y)\,dtdydz;\end{aligned}$$

$$\begin{aligned}\iint\limits_{G\times G} \alpha(z,x)f(z)(g*h)(x)\,dzdx &= \iiint\limits_{G\times G\times G} \alpha(z,x)f(z)g(t)h(\underbrace{t^{-1}x}_{y})\,dxdtdz \\ &= \iiint\limits_{G\times G\times G} \alpha(z,ty)f(z)g(t)h(y)\,dtdydz;\end{aligned}$$

$$\begin{aligned}\iint\limits_{G\times G} \alpha(t,x)g(t)(h*f)(x)h(y)\,dtdx &= \iiint\limits_{G\times G\times G} \alpha(t,x)g(t)h(y)f(\underbrace{y^{-1}x}_{z})\,dtdxdy \\ &= \iiint\limits_{G\times G\times G} \alpha(t,yz)f(z)g(t)h(y)\,dtdydz.\end{aligned}$$

Adding the last three equalities together and combining this with (4.4), we obtain

$$\begin{aligned}\iiint\limits_{G\times G\times G} \alpha(zt,y)f(z)g(t)h(y)\,dzdtdy &= \iiint\limits_{G\times G\times G} \alpha(t,yz)f(z)g(t)h(y)\,dtdydz \\ &+ \iiint\limits_{G\times G\times G} \alpha(z,ty)f(z)g(t)h(y)\,dzdtdy, \quad f,g,h\in L^1(G,\omega).\end{aligned} \tag{4.5}$$

Since $\alpha \in L^\infty(G\times G, 1/(\omega\times\omega))$, all maps $(x,y,z)\mapsto \alpha(xy,z)$, $(x,y,z)\mapsto\alpha(x,yz)$, and $(x,y,z)\mapsto\alpha(y,zx)$ belong to $L^\infty(G\times G\times G, 1/(\omega\times\omega\times\omega))$ which is the dual of $L^1(G\times G\times G,\omega\times\omega\times\omega)$. Then, because $L^1(G\times G\times G,\omega\times\omega\times\omega) = L^1(G,\omega)\hat{\otimes}L^1(G,\omega)\hat{\otimes}L^1(G,\omega)$, equality (4.5) is equivalent to

$$\alpha(zt,y) = \alpha(z,ty)+\alpha(t,yz) \quad \text{for almost all } (t,y,z)\in G\times G\times G,$$

which is the same as (4.2) up to a change of variables. So, we have shown that α satisfies all our requirements.

The proof of the converse statement follows the same lines in the reversed order. □

It is well-known that the left and right translation operators on $L^1(G)$ are continuous. We state this formally here for completeness (see, for example, [11, Proposition 2.41]).

Lemma 4.2. *Let G be a locally compact group with identity e and $f\in L^1(G)$. Then*

$$\lim_{y\to e} L_y f = \lim_{y\to e} R_y f = f,$$

where $(L_y f)(x) = f(y^{-1}x)$, $(R_y f)(x) = f(xy)$ stand for the left and right translations of f respectively, and the limits are taken with respect to the norm topology of $L^1(G)$.

Now we are ready to establish a necessary condition for weak amenability of $L^1(G,\omega)$ in the case when G is an [IN] group (see Definition 1.1). The construction of a non-inner derivation in our proof is the same as in [41, Remark 3.2]. However, some continuity arguments allow us to remove a restriction assumed there.

Theorem 4.3. *Let G be an [IN] group and ω be a weight on G. Suppose that there exists a non-trivial continuous group homomorphism $\Phi : G \to \mathbb{C}$ such that*

$$\sup_{t\in G} \frac{|\Phi(t)|}{\omega(t)\omega(t^{-1})} < \infty.$$

Then $L^1(G,\omega)$ is not weakly amenable.

Proof. To prove the theorem, it is enough to build a continuous derivation $D : L^1(G,\omega) \to L^\infty(G,1/\omega)$ that is not inner. Since G is an [IN] group, there exists a compact neighborhood B of identity that is invariant under all inner automorphisms of G. Then we define D as in [41, Theorem 3.1]:

$$D(h)(t) = \int\limits_B \Phi(t^{-1}\xi)h(t^{-1}\xi)\,d\xi, \quad h \in L^1(G,\omega),\ t \in G. \tag{4.6}$$

The fact that D is a derivation can be proved analogously to the corresponding part of [41, Theorem 3.1], but we will use a slightly different approach here. Note that we can use the duality of $L^1(G,\omega)$ and $L^\infty(G,1/\omega)$ to equivalently rewrite the formula for D in the following way:

$$\begin{aligned}\langle g, D(h)\rangle &= \int\limits_G \int\limits_{t^{-1}B} \Phi(\xi)h(\xi)\,d\xi\, g(t)\,dt = \int\limits_G\int\limits_G \chi_{t^{-1}B}(\xi)\Phi(\xi)h(\xi)g(t)\,d\xi dt \\ &= \int\limits_G\int\limits_G \underbrace{\chi_B(t\xi)\Phi(\xi)}_{\alpha(\xi,t)} h(\xi)g(t)\,d\xi dt, \quad g,h \in L^1(G,\omega).\end{aligned}$$

So, if we can show that α satisfies the conditions of Proposition 4.1, we then have shown that D is a bounded derivation from $L^1(G,\omega)$ to $L^\infty(G,1/\omega)$. We first verify that $\alpha \in L^\infty(G\times G, 1/(\omega\times\omega))$:

$$\begin{aligned}
\sup_{(\xi,t)\in G\times G} \frac{|\alpha(\xi,t)|}{\omega(\xi)\omega(t)} &= \sup_{\xi,t\in G} \frac{|\chi_B(t\xi)\Phi(\xi)|}{\omega(\xi)\omega(t)} = \sup_{\xi,t\in G,\, t\xi\in B} \frac{|\Phi(\xi)|}{\omega(\xi)\omega(t)} \\
&= \sup_{\xi,t\in G,\, t\xi\in B} \frac{|\Phi(\xi)|\omega(\xi^{-1})}{\omega(\xi)\omega(t)\omega(\xi^{-1})} \le \sup_{\xi\in G} \frac{|\Phi(\xi)|}{\omega(\xi)\omega(\xi^{-1})}\cdot \sup_{\xi,t\in G,\, t\xi\in B} \frac{\omega(\xi^{-1})}{\omega(t)} \\
&\underset{\omega(xy)\le\omega(x)\omega(y)}{\le} \sup_{\xi\in G} \frac{|\Phi(\xi)|}{\omega(\xi)\omega(\xi^{-1})}\cdot \sup_{\xi,t\in G,\, t\xi\in B} \omega((t\xi)^{-1}) < \infty,
\end{aligned}$$

since $\sup\limits_{\xi\in G} \dfrac{|\Phi(\xi)|}{\omega(\xi)\omega(\xi^{-1})} < \infty$, and the continuous function ω is bounded on the compact set B. Next we prove that

$$\alpha(xy,z) = \alpha(x,yz) + \alpha(y,zx), \quad x,y,z\in G.$$

Fix $x,y,z\in G$. Since $yzx = y(zxy)y^{-1}$ and B is invariant under inner automorphisms, we have that $\chi_B(zxy) = \chi_B(yzx)$. Then we can use the fact that Φ is a homomorphism to obtain

$$\begin{aligned}
\alpha(xy,z) &= \chi_B(zxy)\Phi(xy) = \chi_B(zxy)(\Phi(x)+\Phi(y)) = \chi_B(yzx)\Phi(x) + \chi_B(zxy)\Phi(y) \\
&= \alpha(x,yz) + \alpha(y,zx),
\end{aligned}$$

and we are done. So, we have shown that D is a bounded derivation from $L^1(G,\omega)$ to $L^\infty(G,1/\omega)$.

We now show that for every $h\in L^1(G,\omega)$ the function $D(h)\in L^\infty(G,1/\omega)$ is continuous. Fix any $t_0\in G$ and let C be a compact neighborhood of t_0. Then it is easy to see that the values of $D(h)$ on C depend only on the values of the functions Φ and h on $C^{-1}B$, and so $D(h)(t) = \int\limits_B \beta(t^{-1}\xi)\,d\xi$ for $t\in C$, where

$$\beta(x) = \begin{cases} \Phi(x)h(x), & x\in C^{-1}B, \\ 0, & x\notin C^{-1}B. \end{cases}$$

Because $C^{-1}B$ is compact, Φ is continuous, $h\in L^1(G,\omega)$, and ω is bounded on compact sets, we have that $\beta \in L^1(G)$. Then it follows from Lemma 4.2 that $L_t\beta \to L_{t_0}\beta$ in $L^1(G)$. Therefore, for $t \in C$ we have

$$|D(h)(t) - D(h)(t_0)| = \left| \int\limits_B \big(L_t\beta(\xi) - L_{t_0}\beta(\xi)\big)\, d\xi \right| \le \int\limits_G |L_t\beta(\xi) - L_{t_0}\beta(\xi)|\, d\xi$$
$$= \|L_t\beta - L_{t_0}\beta\|_{L^1(G)} \to 0, \quad t \to t_0,$$

which proves the continuity of $D(h)$ at t_0. Since t_0 was taken arbitrarily, we obtain the continuity of $D(h)$ on G for every $h \in L^1(G,\omega)$.

We are now ready to show that D is not an inner derivation, which will complete the proof of the theorem. Suppose, to the contrary, that there exists $f \in L^\infty(G, 1/\omega)$ such that

$$D(h) = f \cdot h - h \cdot f, \quad h \in L^1(G,\omega). \tag{4.7}$$

Fix any $t_0 \in G$ and take $h = \chi_{t_0^{-1}B}$. Then

$$D(h)(t_0) = (f \cdot h)(t_0) - (h \cdot f)(t_0) = \int\limits_G f(yt_0)h(y)\, dy - \int\limits_G f(t_0y)h(y)\, dy$$
$$\underset{h=\chi_{t_0^{-1}B}}{=} \int\limits_{t_0^{-1}B} f(yt_0)\, dy - \int\limits_{t_0^{-1}B} f(t_0y)\, dy = \int\limits_{t_0^{-1}Bt_0} f(y)\, dy - \int\limits_B f(y)\, dy = 0,$$

since G is unimodular as an [IN] group and B is invariant under inner automorphisms. As we have already shown, $D(h)$ is a continuous function. It is also standard that $f \cdot h - h \cdot f$ is a continuous function when $f \in L^\infty(G, 1/\omega)$ and $h \in L^1(G,\omega)$, since $L^\infty(G,1/\omega) \cdot L^1(G,\omega) = LUC(G,1/\omega)$ and $L^1(G,\omega) \cdot L^\infty(G,1/\omega) = RUC(G,1/\omega)$ (see, for example, [8, Proposition 7.17]). Therefore, the formulas (4.6) and (4.7) for the function $D(h)(t)$ must agree at every point $t \in G$, and, in particular, at $t = t_0$. Hence,

$$0 = D(h)(t_0) = \int\limits_B \Phi(t_0^{-1}\xi)h(t_0^{-1}\xi)\, d\xi = \int\limits_B \Phi(t_0^{-1}\xi)\chi_{t_0^{-1}B}(t_0^{-1}\xi)\, d\xi = \int\limits_B \Phi(t_0^{-1}\xi)\, d\xi.$$

Since Φ is a homomorphism, we obtain:

$$0 = \int\limits_B \Phi(t_0^{-1}\xi)\, d\xi = \int\limits_B (\Phi(\xi) - \Phi(t_0))\, d\xi = \int\limits_B \Phi(\xi)\, d\xi - \Phi(t_0)\mu(B),$$

which implies that

$$\Phi(t_0) = \frac{\int_B \Phi(\xi)\, d\xi}{\mu(B)},$$

where μ denotes the Haar measure on G ($\mu(B) > 0$ since B is a neighborhood of identity and thus contains an open subset). Because $t_0 \in G$ was chosen arbitrarily, it follows that $\Phi = const$, which can only happen if $\Phi \equiv 0$, since Φ is a homomorphism. The obtained contradiction shows that D cannot be an inner derivation, and the proof is complete. □

Remark 4.4. *Discrete groups are, obviously, [IN] groups. So, Lemma 3.2 in Chapter 3 is a special case of Theorem 4.3.*

Our next result provides a necessary condition for weak amenability of $L^1(G,\omega)$ for a general locally compact group G when ω is bounded below, away from zero.

Theorem 4.5. *Let G be a locally compact group, ω be a bounded away from zero weight on G, i.e, $\omega \geq \delta$ for some constant $\delta > 0$, and $B \neq \emptyset$ be an open set in G with compact closure. Define the set $C_B \subset G$ by $C_B = \{xyx^{-1} : x \in G,\, y \in B\}$. Suppose that there exists a measurable function $\psi : G \to \mathbb{C}$ bounded on B such that*

$$\operatorname*{ess\,sup}_{x,y\in G} \frac{|\psi(xy) - \psi(yx)|}{\omega(x)\omega(y)} < \infty, \quad \textit{and} \tag{4.8}$$

$$\operatorname*{ess\,sup}_{z\in C_B} \frac{|\psi(z)|}{\omega(z)} = \infty. \tag{4.9}$$

Then $L^1(G,\omega)$ is not weakly amenable.

Proof. To show that $L^1(G,\omega)$ is not weakly amenable, it is enough to build a non-inner bounded derivation $D : L^1(G,\omega) \to L^\infty(G, 1/\omega)$. We define D by

$$\langle g, D(f)\rangle = \int\limits_{G^2} (\psi(xy) - \psi(yx)) f(x) g(y)\, dx dy, \quad f, g \in L^1(G,\omega).$$

Note that the condition (4.8) implies that the function $\Psi(x,y) = \psi(xy) - \psi(yx)$ belongs to $L^\infty(G\times G, 1/(\omega\times\omega))$. Also, we can easily check that Ψ satisfies (4.2):

$$\begin{aligned}\Psi(xy,z) &= \psi(xyz) - \psi(zxy) = (\psi(xyz) - \psi(yzx)) + (\psi(yzx) - \psi(zxy)) \\ &= \Psi(x,yz) + \Psi(y,zx).\end{aligned}$$

Therefore, by Proposition 4.1, the operator D defined above is a bounded derivation from $L^1(G,\omega)$ to $L^\infty(G,1/\omega)$. Now we just need to show that D is not inner.

Suppose, to the contrary, that D is inner, which means that there exists a function $\varphi \in L^\infty(G,1/\omega)$ such that

$$D(f) = \varphi\cdot f - f\cdot\varphi, \quad f \in L^1(G,\omega).$$

Using the explicit formula for the module action of $L^1(G,\omega)$ on $L^\infty(G,\omega)$ and comparing the result to our definition of D, we obtain

$$\begin{aligned}\langle g, D(f)\rangle &= \int_{G^2} (\varphi(yx) - \varphi(xy)) f(y) g(x)\, dxdy \\ &= \int_{G^2} (\psi(yx) - \psi(xy)) f(y) g(x)\, dxdy, \quad f,g \in L^1(G,\omega).\end{aligned} \tag{4.10}$$

We already know that $\Psi(x,y) = \psi(yx) - \psi(xy) \in L^\infty(G\times G, 1/(\omega\times\omega))$. Also, since $\varphi \in L^\infty(G,1/\omega)$, we have that $\Phi(x,y) = \varphi(xy) - \varphi(yx) \in L^\infty(G\times G, 1/(\omega\times\omega))$. Recall that

$$L^\infty(G\times G, 1/(\omega\times\omega)) = \big(L^1(G\times G, \omega\times\omega)\big)^* \quad \text{and}$$

$$L^1(G\times G, \omega\times\omega) \cong L^1(G,\omega)\hat{\otimes} L^1(G,\omega),$$

which implies that the linear span of elementary tensor functions $F(x,y) = f(x)g(y)$, $f,g \in L^1(G,\omega)$ is dense in $L^1(G\times G, \omega\times\omega)$. Then from (4.10) it follows that $\Psi = \Phi$ as $L^\infty(G\times G, 1/(\omega\times\omega))$ functions, which means that there exists a locally null set

$A \subset G^2$ such that

$$\psi(yx) - \psi(xy) = \varphi(yx) - \varphi(xy), \quad (x,y) \in G^2 \setminus A. \tag{4.11}$$

By our assumption ψ is bounded on B, and ω is bounded on B since the closure of B is compact and the weight ω is assumed continuous. Therefore, we can use the condition (4.9) to find a subset K of C_B of positive Haar measure such that

$$\frac{|\psi(z)|}{\omega(z)} > \|\varphi\|_{L^\infty(G,1/\omega)} + \frac{1}{\delta}\left(\|\varphi\|_{L^\infty(G,1/\omega)} \sup_{t\in B}\omega(t) + \sup_{t\in B}|\psi(t)|\right) + 1 \quad (z \in K). \tag{4.12}$$

Moreover, because of the inner regularity of the Haar measure, we can assume without loss of generality that K is compact. Note that

$$K \subset C_B = \{xyx^{-1} : x \in G,\, y \in B\} = \bigcup_{x\in G} xBx^{-1},$$

and each of the sets $\{xBx^{-1}\}_{x\in G}$ is open, since so is the set B. Then, the compactness of K yields the existence of its finite subcover by the sets from $\{xBx^{-1}\}_{x\in G}$, i.e., there exist $\{x_i\}_{i=1}^n \subset G$ such that $K \subset \bigcup_{i=1}^n x_iBx_i^{-1}$. Because the measure of K is non-zero, we can assume without loss of generality that the set $K_1 = K \cap (x_1Bx_1^{-1})$ also has a non-zero measure. We claim that there exists a compact neighborhood U of the identity e such that $\mu((uK_1u^{-1}) \cap K_1) > 0$ for all $u \in U$. According to Lemma 4.2, for every function $f \in L^1(G)$ we have that $\|L_xf - f\|_{L^1(G)} \to 0$ and $\|R_xf - f\|_{L^1(G)} \to 0$ as $x \to e$. In particular, this is valid for $f = \chi_{K_1}$, the characteristic function of the set K_1. Hence, there exists a compact neighborhood U of e such that

$$\mu(K_1 \setminus u^{-1}K_1),\, \mu(K_1 \setminus K_1u^{-1}) < \frac{1}{3}\,\mu(K_1), \quad u \in U.$$

It follows that

$$\begin{aligned}\mu((uK_1u^{-1})\cap K_1) &= \mu((K_1u^{-1})\cap(u^{-1}K_1)) \geq \mu(K_1\cap K_1u^{-1}\cap u^{-1}K_1)\\ &= \mu(K_1\setminus((K_1\setminus K_1u^{-1})\cup(K_1\setminus u^{-1}K_1)))\\ &\geq \mu(K_1)-\frac{1}{3}\,\mu(K_1)-\frac{1}{3}\,\mu(K_1)=\frac{1}{3}\,\mu(K_1)>0, \quad u\in U,\end{aligned}$$

and the claim is proved. Consider the set $V=(Ux_1)\times(Bx_1U^{-1})\subset G^2$. Since U is compact and B has a compact closure, the set V has a finite measure. It then follows from (4.11) that

$$\psi(yx)-\psi(xy)=\varphi(yx)-\varphi(xy) \quad \text{for almost all } (x,y)\in V.$$

In particular, this implies the existence of $x_0=ux_1$, $u\in U$, such that

$$\psi(yx_0)-\psi(x_0y)=\varphi(yx_0)-\varphi(x_0y) \quad \text{for almost all} \quad y\in Bx_1^{-1}U^{-1},$$

and, in particular, for almost all $y\in Bx_0^{-1}$. If we let $t=yx$, then we obtain that

$$\varphi(t)-\varphi(x_0tx_0^{-1})=\psi(t)-\psi(x_0tx_0^{-1}) \quad \text{for almost all } t\in B. \tag{4.13}$$

Denote $K_2=(x_0Bx_0^{-1})\cap K_1$. Since $K_1\subset x_1Bx_1^{-1}$, we have that

$$K_2=(x_0Bx_0^{-1})\cap K=(ux_1Bx_1^{-1}u^{-1})\cap K_1\supset(uK_1u^{-1})\cap K_1,$$

and so from the choice of U it follows that $\mu(K_2)\geq\mu((uK_1u^{-1})\cap K_1)>0$. We then use (4.13) to obtain the following estimates:

$$\begin{aligned}
\operatorname*{ess\,sup}_{z\in K_2}\frac{|\psi(z)|}{\omega(z)} \underset{K_2\subset x_0Bx_0^{-1}}{\leq} \operatorname*{ess\,sup}_{t\in B}\frac{|\psi(x_0tx_0^{-1})|}{\omega(x_0tx_0^{-1})} &\underset{(4.13)}{=} \operatorname*{ess\,sup}_{t\in B}\left|\frac{\varphi(x_0tx_0^{-1})}{\omega(x_0tx_0^{-1})}+\frac{\psi(t)-\varphi(t)}{\omega(x_0tx_0^{-1})}\right| \\
&\leq \operatorname*{ess\,sup}_{t\in B}\left(\frac{|\varphi(x_0tx_0^{-1})|}{\omega(x_0tx_0^{-1})}+\frac{|\psi(t)|+|\varphi(t)|}{\omega(x_0tx_0^{-1})}\right) \\
&\underset{\omega\geq\delta}{\leq} \|\varphi\|_{L^\infty(G,1/\omega)}+\frac{1}{\delta}\operatorname*{ess\,sup}_{t\in B}\left(|\psi(t)|+\frac{|\varphi(t)|}{\omega(t)}\cdot\omega(t)\right) \\
&\leq \|\varphi\|_{L^\infty(G,1/\omega)}+\frac{1}{\delta}\left(\sup_{t\in B}|\psi(t)|+\|\varphi\|_{L^\infty(G,1/\omega)}\sup_{t\in B}\omega(t)\right).
\end{aligned}$$

Because $\mu(K_2)>0$, the inequality above contradicts (4.12), which should hold for every $z\in K_2$, since $K_2\subset K$. This completes the proof of the theorem. □

A direct consequence of Theorem 4.5 for discrete group G and $B=\{x_0\}$ is the following.

Corollary 4.6. *Let G be a discrete group, and ω be a weight on G. If there is a function $\psi: G\to\mathbb{R}$, $x_0\in G$, and a constant $c>0$ such that ω is bounded away from zero on $\{yx_0y^{-1}\}_{y\in G}$,*

$$|\psi(xy)-\psi(yx)|\leq c\,\omega(x)\omega(y),\quad x,y\in G,\quad \textit{and} \tag{4.14}$$

$$\sup_{y\in G}\frac{|\psi(yx_0y^{-1})|}{\omega(yx_0y^{-1})}=\infty, \tag{4.15}$$

then $L^1(G,\omega)=\ell^1(G,\omega)$ is not weakly amenable.

Corollary 4.6 was also obtained by C.R. Borwick in his PhD thesis [5]. Note, that, in fact, we have implicitly used it to prove that $\ell^1(\mathbb{F}_2,\omega)$ and $\ell^1((\boldsymbol{ax+b}),\omega)$ are both not weakly amenable for non-trivial polynomial weights.

4.3 Weak amenability of $\ell^1(\mathbb{F}_2,\omega)$

In this section we, as usual, denote the two generators of $\mathbb{F}_2$ by a and b. Then every $x\in\mathbb{F}_2$ can be uniquely written in the irreducible form $x=a^{k_1}b^{l_1}\ldots a^{k_n}b^{l_n}$

with $k_i, l_i \in \mathbb{Z}$, where all k_i, l_i are non-zero, possibly except k_1 and l_n. We start by characterizing weak amenability of $\ell^1(\mathbb{F}_2, \omega)$ for a special class of weights ω.

Theorem 4.7. *Let ω be a weight on $\mathbb{F}_2$ such that there exists an increasing function W from $\mathbb{N} \cup \{0\}$ to $\mathbb{R}^+$ and constants $c_1, c_2 > 0$ such that*

$$c_1 W(|x|) \leq \omega(x) \leq c_2 W(|x|), \quad x \in \mathbb{F}_2,$$

where $|x|$ is the length of the element $x \in \mathbb{F}_2$ as described in Definition 3.3. Then $\ell^1(\mathbb{F}_2, \omega)$ is weakly amenable if and only if ω is bounded.

We will need the following technical result.

Lemma 4.8. *Let W be an increasing function from $\mathbb{N} \cup \{0\}$ to $\mathbb{R}^+$. Then there exists a function $f : \mathbb{N} \cup \{0\} \to \mathbb{R}^+$ with the following properties:*

1. *f is increasing;*
2. *$f(m+n) - f(m-n) \leq W(m)W(n)$ for all $m \geq n$ in $\mathbb{N}$;*
3. *if W is not bounded and $\sup_{n \in \mathbb{N}} \dfrac{\sqrt{n}}{W(n)} = \infty$, then $\dfrac{f(k)}{W(k)}$ is not bounded.*

Proof. We define the function f inductively by the following formulas:

$$f(0) = 1, \quad f(1) = 1, \quad f(k) = \min_{1 \leq l \leq \left\lfloor \frac{k}{2} \right\rfloor} (W(l)W(k-l) + f(k-2l)), \quad k > 1.$$

We then prove that it satisfies conditions 1-3 (by $\lfloor x \rfloor$ we mean the standard floor function, which is equal to the greatest integer that does not exceed x).

We first show that the defined function f is increasing. We will do this by induction. The base is trivial: $f(0) = 1 \leq f(1)$. Suppose now that for all $0 \leq m < n \leq k$ we have that $f(m) \leq f(n)$. We then show that $f(k) \leq f(k+1)$. According to the definition of f, we have

$$f(k+1) = \min_{1 \leq l \leq \left\lfloor \frac{k+1}{2} \right\rfloor} (W(l)W(k+1-l) + f(k+1-2l)) \quad \text{and}$$

$$f(k) = \min_{1 \leq l \leq \left\lfloor \frac{k}{2} \right\rfloor} (W(l)W(k-l) + f(k-2l)).$$

For each l such that $1 \le l \le \left\lfloor \frac{k}{2} \right\rfloor$ we have that

$$W(l)W(k-l) + f(k-2l) \le W(l)W(k+1-l) + f(k+1-2l),$$

because $W > 0$ is increasing and $f(k-2l) \le f(k+1-2l)$ by our assumption. If k is even, then $\left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{k+1}{2} \right\rfloor$, and so the last inequality automatically implies $f(k) \le f(k+1)$. Now let $k = 2p+1$ be an odd integer. Then $\frac{k+1}{2} = p+1$, and so the minimum for $f(k+1)$ is taken over $(p+1)$ terms, whereas the minimum for $f(k)$ is taken over p terms. So, in this case we only need to show that the last term in the minimum for $f(k+1)$ (corresponding to $l = p+1$) is not smaller than $f(k)$. We have

$$\begin{aligned} W(l)W(k+1-l) + f(k+1-2l) &= W(p+1)W(p+1) + f(0) \\ &\ge W(p)W(p+1) + 1 = W(p)W(k-p) + f(k-2p) \\ &\ge \min_{1 \le l \le \left\lfloor \frac{k}{2} \right\rfloor} (W(l)W(k-l) + f(k-2l)) = f(k), \end{aligned}$$

so $f(k) \le f(k+1)$ also holds for any odd integer k. By induction, f is increasing.

We now verify the inequality:

$$f(m+n) - f(m-n) \le W(m)W(n), \quad m,n \in \mathbb{N}, \quad m \ge n.$$

By the definition of f, we have

$$f(m+n) = \min_{1 \le l \le \left\lfloor \frac{m+n}{2} \right\rfloor} (W(l)W(m+n-l) + f(m+n-2l)) \underset{\text{take } l=n}{\le} W(n)W(m) + f(m-n),$$

which gives the desired inequality.

Finally, we prove that if the function W is not bounded and $\sup_{n \in \mathbb{N}} \dfrac{\sqrt{n}}{W(n)} = \infty$, then $\sup_{k \in \mathbb{N}} \dfrac{f(k)}{W(k)} = \infty$. Suppose, to the contrary, that $\sup_{k \in \mathbb{N}} \dfrac{f(k)}{W(k)} \le N$ for some positive integer N. This means that $W(k) \ge \dfrac{f(k)}{N}$ for all $k \in \mathbb{N}$. By the definition of the function f, for each $k \in \mathbb{N}$ there exists an $l_1^{(k)}$ such that $1 \leq l_1^{(k)} \leq \left\lfloor \dfrac{k}{2} \right\rfloor$ and

$$f(k) = W\left(l_1^{(k)}\right) W\left(k - l_1^{(k)}\right) + f\left(k - 2l_1^{(k)}\right). \tag{4.16}$$

Then either $k-2l_1^{(k)}$ is equal to 0 or 1, or there exists an $l_2^{(k)}$ $\left(\text{in fact, } l_2^{(k)} = l_1^{\left(k-2l_1^{(k)}\right)}\right)$, such that $1 \leq l_2^{(k)} \leq \left\lfloor \dfrac{k}{2} \right\rfloor - l_1^{(k)}$ and

$$f\left(k - 2l_1^{(k)}\right) = W\left(l_2^{(k)}\right) W\left(k - 2l_1^{(k)} - l_2^{(k)}\right) + f\left(k - 2l_1^{(k)} - 2l_2^{(k)}\right).$$

Combining the last formula with (4.16), we get that

$$f(k) = W\left(l_1^{(k)}\right) W\left(k - l_1^{(k)}\right) + W\left(l_2^{(k)}\right) W\left(k - 2l_1^{(k)} - l_2^{(k)}\right) + f\left(k - 2l_1^{(k)} - 2l_2^{(k)}\right).$$

Continuing this process, we will eventually get the last term to be either $f(0) = 1$ or $f(1) = 1$:

$$\begin{aligned} f(k) = {} & W\left(l_1^{(k)}\right) W\left(k - l_1^{(k)}\right) + W\left(l_2^{(k)}\right) W\left(k - 2l_1^{(k)} - l_2^{(k)}\right) \\ & + W\left(l_3^{(k)}\right) W\left(k - 2l_1^{(k)} - 2l_2^{(k)} - l_3^{(k)}\right) + \ldots + 1. \end{aligned}$$

Because we assumed that $\sup\limits_{k\in\mathbb{N}} \dfrac{f(k)}{W(k)} \leq N < \infty$, we have that

$$W(k) \geq \frac{f(k)}{N} = \frac{W\left(l_1^{(k)}\right) W\left(k - l_1^{(k)}\right) + W\left(l_2^{(k)}\right) W\left(k - 2l_1^{(k)} - l_2^{(k)}\right) + \ldots + 1}{N} \tag{4.17}$$

for every $k \in \mathbb{N}$ and some $1 \leq l_1^{(k)} \leq \left\lfloor \dfrac{k}{2} \right\rfloor,\ 1 \leq l_2^{(k)} \leq \left\lfloor \dfrac{k}{2} \right\rfloor - l_1^{(k)}, \ldots$

Since W is unbounded, there is $n_0 \in \mathbb{N}$ such that $W(n_0) \geq 3N$. Our goal here is to prove by induction that

$$W\left(3^{p+1} N n_0\right) \geq 3^p N, \quad p \in \mathbb{N}. \tag{4.18}$$

For $p = 1$, we have that $W(3^2Nn_0) \geq W(n_0) \geq 3N$ by the monotonicity of W and the choice of n_0. So, the inequality (4.18) is true for $p = 1$. Now suppose that $W(3^{p+1}Nn_0) \geq 3^pN$ for some $p \in \mathbb{N}$. We aim to prove that $W(3^{p+2}Nn_0) \geq 3^{p+1}N$. The inequality (4.17) applied to $k = 3^{p+2}Nn_0$ gives us the following:

$$W\left(3^{p+2}Nn_0\right) \geq \frac{W(l_1)\,W(3^{p+2}Nn_0 - l_1) + W(l_2)\,W(3^{p+2}Nn_0 - 2l_1 - l_2) + ... + 1}{N} \tag{4.19}$$

for some $1 \leq l_1 \leq \left\lfloor \dfrac{3^{p+2}Nn_0}{2} \right\rfloor$, $1 \leq l_2 \leq \left\lfloor \dfrac{3^{p+2}Nn_0}{2} \right\rfloor - l_1, \ldots$

If $l_1 \geq n_0$, then $W(l_1) \geq W(n_0) \geq 3N$ by the monotonicity of W. Also, because $l_1 \leq \left\lfloor \frac{3^{p+2}Nn_0}{2} \right\rfloor$, we have that

$$3^{p+2}Nn_0 - l_1 \geq \frac{3^{p+2}Nn_0}{2} > 3^{p+1}Nn_0,$$

which implies that $W(3^{p+2}Nn_0 - l_1) \geq W(3^{p+1}Nn_0) \geq 3^pN$ by the assumption. So, in this case, using (4.19), we obtain:

$$W\left(3^{p+2}Nn_0\right) \geq \frac{W(l_1)\,W(3^{p+2}Nn_0 - l_1)}{N} \geq \frac{3N \cdot 3^pN}{N} = 3^{p+1}N,$$

and the desired inequality is verified in this case.

Now suppose that $l_1 < n_0$. Then $3^{p+2}Nn_0 - 2l_1 > 1$, and hence l_2 is present in (4.19). If $l_2 \geq n_0$, then $W(l_2) \geq W(n_0) \geq 3N$. Also, since $l_2 \leq \left\lfloor \frac{3^{p+2}Nn_0}{2} \right\rfloor - l_1$, we get

$$3^{p+2}Nn_0 - 2l_1 - l_2 \geq \frac{3^{p+2}Nn_0}{2} - l_1 > \frac{3^{p+2}Nn_0 - 2n_0}{2} > 3^{p+1}Nn_0$$

(the last inequality holds since $3^{p+1}Nn_0 - 2n_0 > 0$). From this it follows that $W(3^{p+2}Nn_0 - 2l_1 - l_2) \geq W(3^{p+1}Nn_0) \geq 3^pN$ by our assumption, and so from (4.19) we obtain

$$W\left(3^{p+2}Nn_0\right) \geq \frac{W(l_2)\,W(3^{p+2}Nn_0 - 2l_1 - l_2)}{N} \geq \frac{3N \cdot 3^pN}{N} = 3^{p+1}N.$$

So, we verified the inequality for the case when $l_1 < n_0$ and $l_2 \geq n_0$.

Suppose now that both $l_1 < n_0$ and $l_2 < n_0$. Continuing in the same manner, we will either find some l_q ($q < 3N$) such that $l_1, l_2, \ldots, l_{q-1} < n_0$, $l_q \geq n_0$, or we will have that $l_1, l_2, \ldots, l_{3N} < n_0$. In the first case, similar argument shows that $W(l_q) \geq 3N$ and

$$\begin{aligned} W\left(3^{p+2}Nn_0 - 2l_1 - \ldots - 2l_{q-1} - l_q\right) &\geq W\left(\frac{3^{p+2}Nn_0}{2} - l_1 - l_2 - \ldots - l_{q-1}\right) \\ \geq W\left(\frac{3^{p+2}Nn_0}{2} - qn_0\right) &\geq W\left(\frac{3^{p+2}Nn_0}{2} - 3Nn_0\right) \geq W\left(3^{p+1}Nn_0\right) \geq 3^pN \end{aligned} \tag{4.20}$$

by the assumption and the fact that $3^{p+1}Nn_0 - 6Nn_0 > 0$. In view of (4.19), this implies

$$W\left(3^{p+2}Nn_0\right) \geq \frac{W(l_q)\, W(3^{p+2}Nn_0 - 2l_1 - \ldots - 2l_{q-1} - l_q)}{N} \geq \frac{3N \cdot 3^pN}{N} = 3^{p+1}N,$$

which is exactly what we need.

So, the only remaining case is when $l_1, l_2, \ldots, l_{3N} < n_0$. Similarly to (4.20), we have that

$$\begin{aligned} W\left(3^{p+2}Nn_0 - l_1\right) &\geq W\left(3^{p+2}Nn_0 - 2l_1 - l_2\right) \geq \ldots \\ &\geq W\left(3^{p+2}Nn_0 - 2l_1 - \ldots - 2l_{3N-1} - l_{3N}\right) \geq W(3^{p+1}Nn_0) \geq 3^pN. \end{aligned}$$

Then, since all $l_i \geq 1$, and hence $W(l_i) \geq W(1) = 1$, the inequality (4.19) gives us the following:

$$\begin{aligned} &W\left(3^{p+2}Nn_0\right) \\ &\quad \geq \frac{W(l_1)\, W(3^{p+2}Nn_0 - l_1) + \ldots + W(l_{3N})\, W(3^{p+2}Nn_0 - 2l_1 - \ldots - l_{3N})}{N} \\ &\quad \geq \frac{\overbrace{1 \cdot 3^pN + \ldots + 1 \cdot 3^pN}^{3N}}{N} = 3^{p+1}N. \end{aligned}$$

Therefore, we have proved that $W(3^{p+1}Nn_0) \geq 3^pN$ for every positive integer p. We now use this to get a contradiction to the condition that $\sup_{n\in\mathbb{N}} \dfrac{\sqrt{n}}{W(n)} = \infty$.

For each $n \in \mathbb{N}$ with $n \geq 9Nn_0$ there exists a unique number $p \in \mathbb{N}$ such that $3^{p+1}Nn_0 \leq n < 3^{p+2}Nn_0$. Then, the monotonicity of W implies that $W(n) \geq W\left(3^{p+1}Nn_0\right) \geq 3^pN$ by what we have just proved. Also, $\sqrt{n} < \sqrt{3^{p+2}Nn_0}$, and so

$$\frac{\sqrt{n}}{W(n)} < \frac{\sqrt{3^{p+2}Nn_0}}{3^pN} \underset{p\to\infty}{\longrightarrow} 0.$$

From this it follows immediately that $\sup_{n\in\mathbb{N}} \dfrac{\sqrt{n}}{W(n)} < \infty$, which gives us the desired contradiction. Thus, $\sup_{k\in\mathbb{N}} \dfrac{f(k)}{W(k)} = \infty$, which completes the proof. □

Proof of Theorem 4.7. If ω is a bounded weight, then $\ell^1(\mathbb{F}_2,\omega)$ is isomorphic to $\ell^1(\mathbb{F}_2)$, and it is weakly amenable by Theorem 1.9. So, the non-trivial part is to prove that if ω is not bounded and satisfies the conditions of Theorem 4.7, then $\ell^1(\mathbb{F}_2,\omega)$ is not weakly amenable.

Recall that the total power $A(x)$ of a in x (see Definition 3.3) is a group homomorphism from $\mathbb{F}_2$ to $\mathbb{Z}$. So, according to Remark 4.4, $\ell^1(\mathbb{F}_2,\omega)$ is not weakly amenable if

$$\sup_{x\in\mathbb{F}_2} \frac{|A(x)|}{\omega(x)\omega(x^{-1})} < \infty.$$

Assume now that

$$\sup_{x\in\mathbb{F}_2} \frac{|A(x)|}{\omega(x)\omega(x^{-1})} = \infty.$$

Since, obviously, $|A(x)| \leq |x|$, $|x^{-1}| = |x|$, and $\omega(x) \geq c_1W(|x|)$, it follows that

$$\sup_{x\in\mathbb{F}_2} \frac{|x|}{(W(|x|))^2} = \infty,$$

and hence

$$\sup_{n\in\mathbb{N}} \frac{\sqrt{n}}{W(n)} = \infty.$$

Therefore, we can apply Lemma 4.8 to the function W to get an increasing function

$f : \mathbb{N} \cup \{0\} \to \mathbb{R}$ such that

$$f(m+n) - f(m-n) \leq W(m)W(n), \quad m, n \in \mathbb{N},\ m \geq n, \quad \text{and} \tag{4.21}$$

$$\sup_{n \in \mathbb{N}} \frac{f(n)}{W(n)} = \infty. \tag{4.22}$$

We show that $\psi(x) = f(|x|)$ satisfies the conditions of Corollary 4.6 either for $x_0 = a$, or for $x_0 = a^2$, implying that $\ell^1(\mathbb{F}_2, \omega)$ is not weakly amenable. Note that because $\omega(x) \geq c_1 W(|x|)$ and $W : \mathbb{N} \cup \{0\} \to \mathbb{R}^+$ is an increasing function, we have that the weight ω is bounded away from zero on the whole group $\mathbb{F}_2$ and, in particular, on any conjugacy class $\{y x_0 y^{-1}\}_{y \in \mathbb{F}_2}$.

We now aim to find a constant $c > 0$ such that

$$|\psi(xy) - \psi(yx)| \leq c\omega(x)\omega(y), \quad x, y \in \mathbb{F}_2. \tag{4.23}$$

Let $x, y \in \mathbb{F}_2$ be given. According to the definition of ψ, we have that

$$|\psi(xy) - \psi(yx)| = |f(|xy|) - f(|yx|)|.$$

Let $|x| = m$, $|y| = n$, and assume without loss of generality that $m \geq n$. By the triangle inequality,

$$m - n = \big||x| - |y|\big| \leq |xy|, |yx| \leq |x| + |y| = m + n.$$

Since f is an increasing function, it follows that

$$|f(|xy|) - f(|yx|)| \leq f(m+n) - f(m-n).$$

Together with (4.21) and the inequality $\omega(x) \geq c_1 W(|x|)$, this implies the desired inequality (4.23) with $c = 1/c_1^2$:

$$|\psi(xy) - \psi(yx)| = \big|f(|xy|) - f(|yx|)\big| \leq f(m+n) - f(m-n)$$
$$\leq W(m)W(n) = W(|x|)W(|y|) \leq \frac{1}{c_1^2}\,\omega(x)\omega(y).$$

We now check the second condition of Corollary 4.6 for the function ψ. We take x_0 to be either a or a^2 and consider conjugacy classes $\{xax^{-1} : x \in \mathbb{F}_2\}$ and $\{xa^2x^{-1} : x \in \mathbb{F}_2\}$:

$$\sup_{y\in G} \frac{\psi(yay^{-1})}{\omega(yay^{-1})} \geq \sup_{n\in\mathbb{N}} \frac{\psi(b^nab^{-n})}{\omega(b^nab^{-n})} \geq \sup_{n\in\mathbb{N}} \frac{f(2n+1)}{c_2W(2n+1)} = \frac{1}{c_2} \sup_{n\in\mathbb{N}} \frac{f(2n+1)}{W(2n+1)},$$

$$\sup_{y\in G} \frac{\psi(ya^2y^{-1})}{\omega(ya^2y^{-1})} \geq \sup_{n\in\mathbb{N}} \frac{\psi(b^na^2b^{-n})}{\omega(b^na^2b^{-n})} \geq \frac{1}{c_2} \sup_{n\in\mathbb{N}} \frac{f(2n+2)}{W(2n+2)}.$$

Therefore, it is enough to show that either

$$\sup_{n\in\mathbb{N}} \frac{f(2n+1)}{W(2n+1)} \quad \text{or} \quad \sup_{n\in\mathbb{N}} \frac{f(2n+2)}{W(2n+2)}$$

is infinite. But this is a direct consequence of (4.22), and the proof is complete. □

In fact, we can extend Theorem 4.7 to a more general class of groups. Using the approach of [32], we define a length function on all finitely generated discrete groups as follows.

Definition 4.9. Let G be a finitely generated discrete group with identity e and minimal set of generators $\{a_1, a_2, \ldots, a_n\}$, $n \in \mathbb{N}$. Denote $U = \{a_k\}_{k=1}^n \cup \{a_k^{-1}\}_{k=1}^n \cup \{e\}$. The length function $|\cdot| : G \to \mathbb{N}$ is defined as follows:

$$|x| = \min\{m \in \mathbb{N} : x \in U^m\}, \quad x \in G.$$

It is easy to see that the length function $|\cdot|$ satisfies the triangle inequality $|xy| \leq |x| + |y|$, $x, y \in G$. It also follows from the definition of the set U that $|x| = |x^{-1}|$ for all $x \in G$. There are two places in the proof of Theorem 4.7 where we used some structural properties of the free group $\mathbb{F}_2$. The first one is for the existence of a group homomorphism $A : \mathbb{F}_2 \to \mathbb{R}$ satisfying the inequality $|A(x)| \le |x|$, $x \in \mathbb{F}_2$. The second one is for the existence of two conjugacy classes $C_1 = \{xax^{-1} : x \in \mathbb{F}_2\}$ and $C_2 = \{xa^2x^{-1} : x \in \mathbb{F}_2\}$, such that $\{|y| : y \in C_1 \cup C_2\}$ covers the whole set of positive integers $\mathbb{N}$ except for finitely many numbers. We will show that any discrete group G that can be written as a free product $G = G_1 * G_2$ of an infinite finitely generated Abelian group G_1 and a non-trivial finitely generated group G_2 also has these properties, if the set of generators of G is taken to be the union of the sets of generators of G_1 and G_2. Here, by the free product of two groups we mean the most general group generated by the elements of these groups. In particular, any finitely generated free group $\mathbb{F}_n$, $n \ge 2$, can be written as such a product: $\mathbb{F}_n = \mathbb{F}_1 * \mathbb{F}_{n-1}$, where $\mathbb{F}_1$ is, obviously, infinite and Abelian and $\mathbb{F}_{n-1}$ is non-trivial.

Lemma 4.10. *Let $G = G_1 * G_2$, where G_1 is an infinite finitely generated Abelian group and G_2 is a non-trivial finitely generated group. Further, let $|\cdot|$ be the length function on G defined by the set of generators described above. Then the following hold:*

1. there exists a group homomorphism $\Phi : G \to \mathbb{R}$ such that $|\Phi(x)| \le |x|$, $x \in G$;

2. there exist conjugacy classes C_1, C_2 in G such that $\{|x| : x \in C_1 \cup C_2\} \supset \mathbb{N} \setminus \{1\}$.

Proof. By the fundamental theorem of finitely generated Abelian groups (see, for example, [6, Theorem 19.2.2]), G_1 admits a decomposition $G_1 = \mathbb{Z}^k \oplus \mathbb{Z}_{p_1} \oplus \ldots \mathbb{Z}_{p_m}$, where $\mathbb{Z}_{p_i}$ is a cyclic group of prime order p_i, $1 \le i \le m$, $m \ge 0$, and $k > 0$ since G_1 is infinite. Without loss of generality we may assume that a_1 is a generator of G_1 of infinite order. We define Φ by setting $\Phi(a_1) = 1$, $\Phi(a_i) = 0$ for any other generator a_i of G, $i \in \overline{2,n}$, and then extend it to a group homomorphism on the whole G. It is easy to see that Φ satisfies the inequality $|\Phi(x)| \le |x|$, $x \in G$, and the first property is verified.

Now we show that the conjugacy classes $C_1 = \{xa_1x^{-1} : x \in G\}$ and $C_2 = \{xa_1^2x^{-1} : x \in G\}$ possess the second property. Let $a_k \in G_2$ be one of the generators of G, $2 \leq k \leq n$. Then

$$\begin{aligned}
&\{|x| : x \in C_1 \cup C_2\} \\
&\qquad \supset \{|a_1^{m-1} a_k a_1 a_k^{-1} a_1^{-m+1}| : m \in \mathbb{N}\} \cup \{|a_1^{m-1} a_k a_1^2 a_k^{-1} a_1^{-m+1}| : m \in \mathbb{N}\} \\
&\qquad = \{2m+1 : m \in \mathbb{N}\} \cup \{2m+2 : m \in \mathbb{N}\} = \mathbb{N} \setminus \{1\},
\end{aligned}$$

and the lemma is proved. $\square$

Following the proof of Theorem 4.7 and applying Lemma 4.10 when needed, we obtain the following result.

Proposition 4.11. *Let $G = G_1 * G_2$, where G_1 is an infinite finitely generated Abelian group and G_2 is a non-trivial finitely generated group. Let ω be a weight on G and suppose that there exists an increasing function $W : \mathbb{N} \to \mathbb{R}^+$ together with constants $c_1, c_2 > 0$ such that*

$$c_1 W(|x|) \leq \omega(x) \leq c_2 W(|x|), \quad x \in G.$$

Then $\ell^1(G,\omega)$ is weakly amenable if and only if ω is bounded.

Now we consider the class of weights ω on $\mathbb{F}_2$ that can be written as a function of a group homomorphism, i.e., $\omega(x) = W(\varphi(x))$, $x \in \mathbb{F}_2$, where $\varphi : \mathbb{F}_2 \to \mathbb{C}$ is a group homomorphism. We characterize the weights of this type that make $\ell^1(\mathbb{F}_2,\omega)$ weakly amenable.

Proposition 4.12. *Let $\varphi : \mathbb{F}_2 \to \mathbb{C}$ be a group homomorphism and ω be a weight on $\mathbb{F}_2$ of the form $\omega(x) = W(\varphi(x))$, $x \in \mathbb{F}_2$, for some function $W : \mathbb{C} \to \mathbb{R}^+$. Then the Beurling algebra $\ell^1(\mathbb{F}_2,\omega)$ is weakly amenable if and only if ω is diagonally bounded.*

Proof. The sufficiency part of this proposition is a direct consequence of Proposition 1.10. So, we only need to show that if ω is not diagonally bounded, then $\ell^1(\mathbb{F}_2,\omega)$ is not weakly amenable. Let $x = aba^{-1}b^{-1} \in \mathbb{F}_2$. Then $\varphi(x) = 0$ since φ is a group homomorphism. We show that the function

$$\psi(t) = \begin{cases} \ln(\omega(y)\omega(y^{-1})), & \text{if } t = yxy^{-1}, \\ 0, & \text{otherwise}, \end{cases}$$

satisfies the conditions of Corollary 4.6 for the conjugacy class $\{yxy^{-1} : y \in \mathbb{F}_2\}$. First, we note that since $\omega(x) = W(A(x), B(x))$ and both A and B are group homomorphisms, we have that ω is constant, and hence, bounded away from zero, on each conjugacy class, in particular on $\{yxy^{-1}\}_{y\in\mathbb{F}_2}$. Next we check that ψ is well-defined, i.e., if $t \in \mathbb{F}_2$ has two different representations $t = y_1xy_1^{-1} = y_2xy_2^{-1}$, then $\psi(y_1xy_1^{-1}) = \psi(y_2xy_2^{-1})$. To this end, it is enough to show that $\varphi(y_1) = \varphi(y_2)$. Note that

$$(y_1xy_2^{-1})(y_2y_1^{-1}) = y_1xy_1^{-1} = y_2xy_2^{-1} = (y_2y_1^{-1})(y_1xy_2^{-1}), \tag{4.24}$$

which means that the elements $y_1xy_2^{-1}$ and $y_2y_1^{-1}$ commute. In a free group two elements commute if and only if both of them are powers of a third element (see, for example, [29, Proposition 2.17]). So, since $\mathbb{F}_2$ is a free group, (4.24) implies the existence of $u \in \mathbb{F}_2$ and integers k, l such that $y_1xy_2^{-1} = u^k$ and $y_2y_1^{-1} = u^l$. Because φ is a homomorphism, we have that

$$\varphi(y_2) - \varphi(y_1) = \varphi(y_2y_1^{-1}) = l\,\varphi(u).$$

Hence, to prove that $\varphi(y_1) = \varphi(y_2)$, it suffices to show that $\varphi(u) = 0$. Recalling that $\varphi(x) = 0$, we obtain:

$$0 = \varphi(x) = \varphi(y_1xy_1^{-1}) = \varphi((y_1xy_2^{-1})(y_2y_1^{-1})) = \varphi(u^{k+l}) = (k+l)\varphi(u).$$

In the case when $k + l \neq 0$, it immediately follows that $\varphi(u) = 0$, and our claim is proved. If $k+l = 0$, then $y_1xy_1^{-1} = u^{k+l} = e$, which implies that $x = e$, contradicting the choice of x. This proves that the function ψ is well-defined.

Our next goal is to show that ψ satisfies the conditions of Corollary 4.6. First, we prove that there exists a constant $c > 0$ such that

$$|\psi(uv) - \psi(vu)| \leq c\,\omega(u)\omega(v), \quad u, v \in \mathbb{F}_2.$$

Since ψ is non-zero only on the conjugacy class $\{yxy^{-1} : y \in \mathbb{F}_2\}$, the inequality above will obviously hold if both uv and vu do not belong to this class. We also note that $vu = v(uv)v^{-1}$, and so uv and vu always belong to the same conjugacy class, which implies that we only need to consider the case when $uv, vu \in \{yxy^{-1} : y \in \mathbb{F}_2\}$. Let $uv = yxy^{-1}$. Then $vu = (vy)x(vy)^{-1}$, and we have

$$\begin{aligned}|\psi(uv) - \psi(vu)| &= |\psi(yxy^{-1}) - \psi((vy)x(vy)^{-1})| \\ &= |\ln(\omega(y)\omega(y^{-1})) - \ln(\omega(vy)\omega((vy)^{-1}))| = \left|\ln \frac{\omega(y)\omega(y^{-1})}{\omega(vy)\omega(y^{-1}v^{-1})}\right|.\end{aligned}$$

Using the weight inequality for ω, we obtain

$$\omega(y) \leq \omega(v^{-1})\omega(vy),\ \omega(y^{-1}) \leq \omega(y^{-1}v^{-1})\omega(v),$$

which implies

$$\frac{\omega(y)\omega(y^{-1})}{\omega(vy)\omega(y^{-1}v^{-1})} \leq \omega(v^{-1})\omega(v), \tag{4.25}$$

and

$$\omega(vy) \leq \omega(v)\omega(y),\ \omega(y^{-1}v^{-1}) \leq \omega(y^{-1})\omega(v^{-1}),$$

which yields that

$$\frac{\omega(vy)\omega(y^{-1}v^{-1})}{\omega(y)\omega(y^{-1})} \leq \omega(v)\omega(v^{-1}). \tag{4.26}$$

From the inequalities (4.25) and (4.26) it follows that

$$\left|\ln \frac{\omega(y)\omega(y^{-1})}{\omega(vy)\omega(y^{-1}v^{-1})}\right| \leq \ln(\omega(v)\omega(v^{-1})).$$

Since $\omega(v)\omega(v^{-1}) \geq \omega(e) = const > 0$, by elementary calculus there exists a constant $c > 0$ such that

$$\ln(\omega(v)\omega(v^{-1})) \leq c\,\omega(v)\omega(v^{-1}). \tag{4.27}$$

Combining all of the above, we get

$$|\psi(uv) - \psi(vu)| \leq c\,\omega(v)\omega(v^{-1}).$$

Recalling that $uv = yxy^{-1}$, we obtain that $v^{-1} = yx^{-1}y^{-1}u$, and so $\varphi(v^{-1}) = \varphi(u)$, since $\varphi(x) = 0$ and φ is a group homomorphism. Therefore, $\omega(v^{-1}) = W(\varphi(v^{-1})) = W(\varphi(u)) = \omega(u)$, which implies the desired inequality

$$|\psi(uv) - \psi(vu)| \leq c\,\omega(u)\omega(v).$$

Finally, we show that $\sup\limits_{y\in\mathbb{F}_2} \dfrac{\psi(yxy^{-1})}{\omega(yxy^{-1})} = \infty$. Employing the fact that φ is a group homomorphism, we obtain

$$\begin{aligned}\sup_{y\in\mathbb{F}_2} \frac{\psi(yxy^{-1})}{\omega(yxy^{-1})} &= \sup_{y\in\mathbb{F}_2} \frac{\ln(\omega(y)\omega(y^{-1}))}{W(\varphi(yxy^{-1}))} = \sup_{y\in\mathbb{F}_2} \frac{\ln(\omega(y)\omega(y^{-1}))}{W(\varphi(x))} \\ &= \sup_{y\in\mathbb{F}_2} \frac{\ln(\omega(y)\omega(y^{-1}))}{\omega(x)} = \frac{1}{\omega(x)} \ln\left(\sup_{y\in\mathbb{F}_2} \omega(y)\omega(y^{-1})\right) = \infty,\end{aligned}$$

since ω is not diagonally bounded. Applying Corollary 4.6, we conclude that $\ell^1(\mathbb{F}_2,\omega)$ is not weakly amenable, and the proposition is proved. □

The results of Propositions 1.10 and 4.12, and Theorem 4.7 lead us to the following.

Conjecture 4.13. *Let ω be a weight on $\mathbb{F}_2$. Then $\ell^1(\mathbb{F}_2,\omega)$ is weakly amenable if and only if ω is diagonally bounded.*

As we noted in the Introduction (see p. 9), every diagonally bounded weight on an amenable group is equivalent to a multiplicative weight. For the non-amenable group $\mathbb{F}_2$ the situation is different, as shows [8, Example 10.1] which is based on the function of B.E. Johnson from [19, Proposition 2.8] (notice that if $\ell^1(G,\omega)$ is not isomorphic to $\ell^1(G)$ then, obviously, ω is not equivalent to a multiplicative weight). We finish this section by providing another example of a diagonally bounded weight on $\mathbb{F}_2$ that is not equivalent to a multiplicative weight.

Consider the function $g : \mathbb{F}_2 \to \mathbb{Z}$ defined as follows:

$$g(a^{k_1}b^{m_1}a^{k_2}b^{m_2}\dots a^{k_n}b^{m_n}) = \#\{i:\ k_i = m_i = 1\} - \#\{i:\ m_i = k_{i+1} = -1\},$$

where # stands for the number of elements in a finite set, and the representation $a^{k_1}b^{m_1}a^{k_2}b^{m_2}\dots a^{k_n}b^{m_n}$ is non-cancelable. It is easy to see that

$$g(x^{-1}) = -g(x), \quad x \in \mathbb{F}_2. \tag{4.28}$$

We claim that g also satisfies the following inequality:

$$g(x) + g(y) - 3 \le g(xy) \le g(x) + g(y) + 3, \quad x, y \in \mathbb{F}_2. \tag{4.29}$$

To prove this, take arbitrary $x, y \in \mathbb{F}_2$ and write them in the form $x = x_1z$, $y = z^{-1}y_1$, where z, a factor of x, is chosen in such a way that there is no further cancelation in x_1y_1, and, of course, no cancelation in x_1z or $z^{-1}y_1$. Then it is easy to see that

$$g(x_1) + g(z) - 1 \le g(x_1z) = g(x) \le g(x_1) + g(z) + 1, \tag{4.30}$$

$$g(z^{-1}) + g(y_1) - 1 \le g(z^{-1}y_1) = g(y) \le g(z^{-1}) + g(y_1) + 1, \quad \text{and} \tag{4.31}$$

$$g(x_1) + g(y_1) - 1 \le g(x_1y_1) = g(xy) \le g(x_1) + g(y_1) + 1. \tag{4.32}$$

Adding inequalities (4.30) and (4.31) and using (4.28), we obtain

$$g(x_1) + g(y_1) - 2 \le g(x) + g(y) \le g(x_1) + g(y_1) + 2.$$

Combining this inequality with (4.32), we get

$$g(x) + g(y) - 3 \le g(xy) \le g(x) + g(y) + 3,$$

and (4.29) is verified. We define $\omega : \mathbb{F}_2 \to \mathbb{R}^+$ by

$$\omega(x) = \begin{cases} 2^{g(x)+3}, & \text{if } x \neq e,\ x \in \mathbb{F}_2, \\ 1, & \text{if } x = e. \end{cases} \tag{4.33}$$

Example 4.14. *The function ω on $\mathbb{F}_2$ defined in (4.33) is a diagonally bounded weight, but it is not equivalent to a multiplicative weight.*

Proof. First we show that ω is a weight. In view of (4.29), we have that

$$2^{-6}\omega(x)\omega(y) \leq \omega(xy) \leq \omega(x)\omega(y), \tag{4.34}$$

when $x, y, xy \neq e$. If $x = e$ or $y = e$, then, obviously, $\omega(xy) = \omega(x)\omega(y)$, and the inequality (4.34) is still satisfied. Finally, if $xy = e$, then $y = x^{-1}$, implying that $g(y) = -g(x)$, and, hence, $1 = \omega(xy) \leq 2^6 = \omega(x)\omega(y)$. So, the inequality (4.34) holds for all $x, y \in \mathbb{F}_2$. It follows that ω is, indeed, a weight on $\mathbb{F}_2$. Moreover, ω is diagonally bounded since $\omega(x)\omega(x^{-1}) \leq 2^6\omega(e)$, $x \in \mathbb{F}_2$.

It only remains to show that ω is not equivalent to a multiplicative weight. Let $\tilde{\omega}$ be any multiplicative weight. Then for the generators a and b of $\mathbb{F}_2$ we have $\tilde{\omega}(a^n) = (\tilde{\omega}(a))^n$ and $\tilde{\omega}(b^n) = (\tilde{\omega}(b))^n$ for every $n \in \mathbb{Z}$. From our definition of ω it follows that $\omega(a^n) = \omega(b^n) = 2^3$, $n \in \mathbb{Z} \setminus \{0\}$, and so ω can only be equivalent to $\tilde{\omega}$ if $\tilde{\omega}(a) = \tilde{\omega}(b) = 1$. This, in turn, implies that $\tilde{\omega} \equiv 1$ on $\mathbb{F}_2$, and hence, ω must be bounded in order to be equivalent to $\tilde{\omega}$. However, ω is, obviously, unbounded, and the proof is complete. □

4.4 The form of derivations for discrete groups

In this section we deal with discrete groups G, in which case $L^1(G,\omega) = \ell^1(G,\omega)$. According to Proposition 1.10, if ω is diagonally bounded, then $L^1(G,\omega)$ is always weakly amenable. On the other hand, [41, Theorem 3.1] implies that the condition of ω being diagonally bounded is not necessary for weak amenability of $L^1(G,\omega)$ for most Abelian groups G. For non-Abelian groups, we still have no example of a weakly amenable group algebra with a weight that is not diagonally bounded. In this section we study the form of the derivations from $L^1(G,\omega)$ to $L^\infty(G,1/\omega)$ for a special class of weights ω on discrete groups G.

Theorem 4.15. *Let G be a discrete group and ω be a weight on G such that*

$$\sup_{n\in\mathbb{N}} \frac{n}{\omega(x^n)\omega(x^{-n})} = \infty, \quad x \in G.$$

Then for every bounded derivation $D : \ell^1(G,\omega) \to \ell^\infty(G,1/\omega)$ there exists a function f on G such that

$$D(\delta_x)(y) = f(xy) - f(yx) \quad (x, y \in G). \tag{4.35}$$

Remark 4.16. *If we could guarantee that $f \in \ell^\infty(G,1/\omega)$, we would get that $D(\delta_x) = \delta_x \cdot f - f \cdot \delta_x$ for every $x \in G$. In turn, this would imply that D is an inner derivation implemented by f, since $\{\delta_x\}_{x\in G}$ is a basis for $\ell^1(G,\omega)$.*

To prove Theorem 4.15, we need the following technical result.

Lemma 4.17. *Let G be a discrete group, ω be a weight on G, and $D : \ell^1(G,\omega) \to \ell^\infty(G,1/\omega)$ be a bounded derivation. If for all commuting elements $x, y \in G$ we have that $D(\delta_x)(y) = 0$, then there exists a function f such that (4.35) holds.*

Proof. Replacing x with xy^{-1}, we obtain the following condition equivalent to (4.35):

$$D(\delta_{xy^{-1}})(y) = f(x) - f(yxy^{-1}) \quad (x, y \in G). \tag{4.36}$$

It is easy to see that the right hand side of (4.36) only depends on the values of the function f on the same conjugacy class. Since different conjugacy classes have empty intersection, we construct the function f separately on each class. Fix $x_0 \in G$ and define f on $\{yx_0y^{-1} : y \in G\}$ as follows:

$$f(yx_0y^{-1}) = -D(\delta_{x_0y^{-1}})(y), \quad y \in G.$$

First, we check that f is well-defined, i.e., if an element u has two representations

$u = yx_0y^{-1} = zx_0z^{-1}$, then

$$-D(\delta_{x_0y^{-1}})(y) = -D(\delta_{x_0z^{-1}})(z).$$

Since $yx_0y^{-1} = zx_0z^{-1}$, we have that $x_0y^{-1} = y^{-1}zx_0z^{-1}$, and using the derivation identity, we obtain

$$\begin{aligned} D(\delta_{x_0y^{-1}})(y) &= D(\delta_{(y^{-1}z)(x_0z^{-1})})(y) = (D(\delta_{y^{-1}z}) \cdot \delta_{x_0z^{-1}})(y) + (\delta_{y^{-1}z} \cdot D(\delta_{x_0z^{-1}}))(y) \\ &= D(\delta_{y^{-1}z})(x_0z^{-1}y) + D(\delta_{x_0z^{-1}})(z). \end{aligned}$$

Therefore, we need to show that $D(\delta_{y^{-1}z})(x_0z^{-1}y) = 0$. Because of our condition on D, it is enough to prove that $y^{-1}z$ and $x_0z^{-1}y$ commute. Indeed, since $zx_0z^{-1} = yx_0y^{-1}$, we have

$$(y^{-1}z)(x_0z^{-1}y) = y^{-1}(zx_0z^{-1})y = y^{-1}(yx_0y^{-1})y = x_0 = (x_0z^{-1}y)(y^{-1}z).$$

Thus, the function f is well-defined. The next step is to show that f satisfies (4.36) for any $x = zx_0z^{-1}$ and any $y \in G$. If we denote $u = yz$, then

$$yxy^{-1} = y(zx_0z^{-1})y^{-1} = (yz)x_0(yz^{-1}) = ux_0u^{-1}, \quad xy^{-1} = (zx_0z^{-1})(zu^{-1}) = zx_0u^{-1},$$

and (4.36) becomes

$$D(\delta_{zx_0u^{-1}})(uz^{-1}) = -D(\delta_{x_0z^{-1}})(z) + D(\delta_{x_0u^{-1}})(u), \quad z, u \in G.$$

Using the derivation identity for D again, we obtain

$$\begin{aligned} &D(\delta_{zx_0u^{-1}})(uz^{-1}) = (D(\delta_z) \cdot \delta_{x_0u^{-1}})(uz^{-1}) + (\delta_z \cdot D(\delta_{x_0u^{-1}}))(uz^{-1}) \\ &= D(\delta_z)(x_0z^{-1}) + D(\delta_{x_0u^{-1}})(u) = (\delta_{x_0z^{-1}} \cdot D(\delta_z))(e) + D(\delta_{x_0u^{-1}})(u) \\ &= D(\delta_{(x_0z^{-1})z})(e) - (D(\delta_{x_0z^{-1}}) \cdot \delta_z)(e) + D(\delta_{x_0u^{-1}})(u) \\ &= D(\delta_{x_0})(e) - D(\delta_{x_0z^{-1}})(z) + D(\delta_{x_0u^{-1}})(u) = -D(\delta_{x_0z^{-1}})(z) + D(\delta_{x_0u^{-1}})(u), \end{aligned}$$

since $D(\delta_{x_0})(e) = 0$, as x_0 and e commute. So, (4.36) holds for all $x = zx_0z^{-1}$ and $y \in G$. The proof is complete. □

Proof of Theorem 4.15. Given a continuous derivation $D : \ell^1(G,\omega) \to \ell^\infty(G, 1/\omega)$, our goal is to find a function f on G such that

$$D(\delta_x)(y) = f(xy) - f(yx), \quad x, y \in G.$$

Due to Lemma 4.17, it is enough to show that the conditions listed in the theorem imply that $D(\delta_x)(y) = 0$ for all commuting elements $x, y \in G$. Suppose to the contrary that $xy = yx$ and $D(\delta_x)(y) = c \neq 0$. Then, by induction,

$$D(\delta_{x^n})(yx^{1-n}) = cn, \quad n \in \mathbb{N}. \tag{4.37}$$

Indeed, the base for $n = 1$ is just the definition of c. Now assume that (4.37) holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned} &D(\delta_{x^{n+1}})(yx^{-n}) = D(\delta_x * \delta_{x^n})(yx^{-n}) = (D(\delta_x)\cdot\delta_{x^n})(yx^{-n}) + (\delta_x \cdot D(\delta_{x^n}))(yx^{-n}) \\ &= D(\delta_x)(x^n yx^{-n}) + D(\delta_{x^n})(yx^{1-n}) \underset{yx=yx}{=} D(\delta_x)(y) + cn = c + cn = c(n+1). \end{aligned}$$

It follows that

$$\begin{aligned} \|D\| &= \sup_{f\in\ell^1(G,\omega)} \frac{\|D(f)\|_{\ell^\infty(G,1/\omega)}}{\|f\|_{\ell^1(G,\omega)}} \geq \sup_{n\in\mathbb{N}} \frac{\|D(\delta_{x^n})\|_{\ell^\infty(G,1/\omega)}}{\|\delta_{x^n}\|_{\ell^1(G,\omega)}} \\ &\geq \sup_{n\in\mathbb{N}} \frac{\frac{|D(\delta_{x^n})(yx^{1-n})|}{\omega(yx^{1-n})}}{\omega(x^n)} = \sup_{n\in\mathbb{N}} \frac{|c|n}{\omega((yx)x^{-n})\omega(x^n)} \geq \sup_{n\in\mathbb{N}} \frac{|c|n}{\omega(yx)\omega(x^{-n})\omega(x^n)} \\ &= \frac{|c|}{\omega(yx)} \sup_{n\in\mathbb{N}} \frac{n}{\omega(x^{-n})\omega(x^n)} = \infty \end{aligned}$$

by our assumption on ω, and because $|c| \neq 0$. This contradicts the fact that D is a bounded derivation, and thus completes the proof of the theorem. □

Chapter 5

Weak amenability of Beurling algebras on quotient groups and subgroups

We start this chapter by relating weak amenability of $L^1(G,\omega)$ to weak amenability of $L^1(G/H,\hat{\omega})$, where H is a closed normal subgroup of G, and the weight $\hat{\omega}$ on G/H is defined by

$$\hat{\omega}([x]) = \inf_{z\in[x]} \omega(z) \quad ([x] \in G/H),$$

where $[x]$ stands for the coset of x in G/H. According to the theory established in [35],

$$L^1(G/H,\hat{\omega}) \cong L^1(G,\omega)/J_\omega(G,H),$$

where $J_\omega(G,H)$ is a closed ideal in $L^1(G,\omega)$. We show that $J_\omega(G,H)$ is complemented in $L^1(G,\omega)$ as a Banach subspace, which allows us to obtain a sufficient condition under which weak amenability of $L^1(G,\omega)$ implies that of $L^1(G/H,\hat{\omega})$. We also consider a special case when $G = G_1 \times G_2$, $H = G_2$, and $\omega = \omega_1 \times \omega_2$, where ω_i is a bounded away from zero weight on G_i, $i = 1,2$. We then prove that weak amenability of $L^1(G,\omega) = L^1(G_1,\omega_1)\hat{\otimes}L^1(G_2,\omega_2)$ implies weak amenability of both $L^1(G_1,\omega_1)$ and $L^1(G_2,\omega_2)$.

In Section 5.2 we consider Beurling algebras on subgroups of Abelian groups. We show that if a group G is a direct product of locally compact Abelian groups G_1 and G_2 both admitting a non-trivial continuous group homomorphism to $\mathbb{R}$, then there exists a weight ω on G such that $L^1(G, \omega)$ is weakly amenable, but $L^1(G_1, \omega|_{G_1})$ is not weakly amenable. This result generalizes Example 3.12. On the other hand, we prove that if H is an open normal subgroup of an Abelian group G such that G/H is compact, then weak amenability of $L^1(G, \omega)$ always implies weak amenability of $L^1(H, \omega|_H)$.

Finally, in Section 5.3 we present a locally compact group G, a closed normal subgroup H of G, and a weight ω on G such that both $L^1(G/H, \hat{\omega})$ and $L^1(H, \omega|_H)$ are weakly amenable, but $L^1(G, \omega)$ is not weakly amenable.

5.1 Weak amenability of Beurling algebras on quotient groups

Let G be a locally compact group, H be its closed normal subgroup, and ω be a weight on G. As we mentioned earlier, our goal in this section is to relate weak amenability of $L^1(G, \omega)$ and weak amenability of $L^1(G/H, \hat{\omega})$.

We will always assume in this chapter that the weight ω is bounded away from zero. As we noted in Remark 2.5, this assumption holds automatically if the group G is amenable. First, we formally prove that $\hat{\omega}$ defined by $\hat{\omega}([x]) = \inf\{\omega(z) : z \in [x]\}$, $[x] \in G/H$, is a weight on G/H. Because ω is bounded away from zero, we have that $\hat{\omega} > 0$. We also have that $\hat{\omega}$ is measurable, since the function

$$\tilde{\omega}(x) = \hat{\omega}([x]) = \inf_{h \in H} \omega(hx), \quad x \in G,$$

is a pointwise infimum of continuous functions $\omega_h(x) = \omega(hx)$, $x \in G$, $h \in H$, and, hence, is measurable by Lemma 2.6. Finally, for every $x, y \in G$ and arbitrary $x_0 \in [x]$

we have

$$\begin{aligned}\hat{\omega}([x]\cdot[y]) = \hat{\omega}([xy]) &= \inf_{z\in[xy]} \omega(z) \le \inf_{z\in[xy]} \omega(x_0)\omega(x_0^{-1}z) = \omega(x_0)\cdot \inf_{z\in[x_0y]} \omega(x_0^{-1}z) \\ &= \omega(x_0)\cdot \inf_{x_0^{-1}z\in[y]} \omega(x_0^{-1}z) = \omega(x_0)\hat{\omega}([y]).\end{aligned}$$

Because $x_0 \in [x]$ is arbitrary, it follows that

$$\hat{\omega}([x]\cdot[y]) \le \inf_{x_0\in[x]} \omega(x_0)\hat{\omega}([y]) = \hat{\omega}([x])\hat{\omega}([y]),$$

which proves that $\hat{\omega}$ is a weight on G/H.

Note that the following result is an immediate corollary of Theorem 4.3.

Proposition 5.1. *Let G be an [IN] group, H be a closed normal subgroup of G such that G/H is Abelian, and ω be a bounded away from zero weight on G. Then weak amenability of $L^1(G,\omega)$ implies that of $L^1(G/H,\hat{\omega})$.*

Proof. Suppose, to the contrary, that $L^1(G/H,\hat{\omega})$ is not weakly amenable. In this case, according to Theorem 1.12, there exists a continuous non-trivial group homomorphism $\Phi : G/H \to \mathbb{C}$ such that

$$\sup_{[x]\in G/H} \frac{|\Phi([x])|}{\hat{\omega}([x])\hat{\omega}([x]^{-1})} < \infty.$$

Then the natural extension $\tilde{\Phi}$ of Φ to G defined by $\tilde{\Phi}(x) = \Phi([x])$ $(x \in G)$ is a non-trivial continuous group homomorphism from G to $\mathbb{C}$ and

$$\sup_{x\in G} \frac{|\tilde{\Phi}(x)|}{\omega(x)\omega(x^{-1})} \le \sup_{[x]\in G/H} \frac{|\Phi([x])|}{\hat{\omega}([x])\hat{\omega}([x]^{-1})} < \infty,$$

since $\hat{\omega}([x]) = \inf_{h\in H} \omega(xh) \le \omega(x)$ $(x \in G)$. By Theorem 4.3 this implies that $L^1(G,\omega)$ is not weakly amenable contradicting our assumptions. □

We now obtain a sufficient condition under which weak amenability of $L^1(G,\omega)$ implies that of $L^1(G/H,\hat{\omega})$ for a general group G.

According to [35], there is a standard construction of a continuous algebra-homomorphism $T : L^1(G, \omega) \to L^1(G/H, \hat{\omega})$:

$$(Tf)([x]) = \int_H f(xh)\, dh, \qquad f \in L^1(G, \omega),\ x \in G.$$

The kernel of T is denoted by $J_\omega(G, H)$. It is proved in [35, Theorem 3.7.13] that, as Banach algebras,

$$L^1(G/H, \hat{\omega}) \cong L^1(G, \omega)/J_\omega(G, H),$$

and the homomorphism T is an isometric isomorphism from $L^1(G, \omega)/J_\omega(G, H)$ to $L^1(G/H, \hat{\omega})$.

For the sake of completeness, we verify that T is an algebra-homomorphism, which also automatically implies that $J_\omega(G, H)$ is a two-sided ideal in $L^1(G, \omega)$. For this we need the following Weil's formula (see, for example, [35, Theorem 3.4.6(iii)]).

Theorem 5.2 (Weil's formula)**.** *Let H be a closed normal subgroup of a locally compact group G and $f \in L^1(G)$. Then*

$$\int_G f(x)\, dx = \int_{G/H} \left(\int_H f(xh)\, dh \right) d[x], \tag{5.1}$$

provided that the Haar measures dx, dh, and $d[x]$ are canonically related, i.e., $dh\, d[x] = dx$ meaning that the relation (5.1) is satisfied for every continuous function f on G that has a compact support.

For arbitrary functions $f, g \in L^1(G, \omega)$ and $x \in G$ we have

$$\begin{aligned}
T(f * g)([x]) &= \int\limits_H (f * g)(xh)\, dh = \int\limits_H \int\limits_G f(y) g(y^{-1}xh)\, dy\, dh \\
&= \int\limits_G f(y) \int\limits_H g(y^{-1}xh)\, dh\, dy = \int\limits_G f(y)\, T(g)([y^{-1}x])\, dy \\
&\underset{\text{Weil's formula}}{=} \int\limits_{G/H} \left(\int\limits_H f(yt)\, T(g)[(yt)^{-1}x]\, dt \right) d[y] \\
&\underset{t \in H \Rightarrow [(yt)^{-1}x] = [y^{-1}x]}{=} \int\limits_{G/H} T(g)([y^{-1}x]) \left(\int\limits_H f(yt)\, dt \right) d[y] \\
&= \int\limits_{G/H} T(g)([y^{-1}][x])\, T(f)([y])\, d[y] = (T(f) * T(g))\, ([x]).
\end{aligned}$$

This shows that T is an algebra homomorphism.

Since $L^1(G/H, \hat{\omega}) \cong L^1(G, \omega)/J_\omega(G, H)$ and $J_\omega(G, H)$ is a closed two-sided ideal in $L^1(G, \omega)$, we are in the situation considered in the following well-known result.

Proposition 5.3. [7, Proposition 2.8.66(iv)] *Let A be a Banach algebra and I be a closed ideal in A. Suppose that A is weakly amenable and I has the trace extension property. Then A/I is weakly amenable.*

The trace extension property is defined as follows.

Definition 5.4. Let I be a closed ideal in a Banach algebra A. Then I has the trace extension property if for every functional $\lambda \in I^*$ satisfying $a \cdot \lambda = \lambda \cdot a$, $a \in A$, there is a continuous functional $\tau \in A^*$ such that $\tau|_I = \lambda$ and $\tau(ab) = \tau(ba)$, $a, b \in A$ (linear functional τ satisfying the second condition is called a trace).

So, to relate weak amenability of $L^1(G, \omega)$ to weak amenability of $L^1(G/H, \hat{\omega})$, it is natural to investigate when $J_\omega(G, H)$ has the trace extension property in $L^1(G, \omega)$. We start from proving that $J_\omega(G, H)$ is always complemented in $L^1(G, \omega)$ as a Banach subspace.

Theorem 5.5. *Let G be a locally compact group, H be a closed normal subgroup of G, and ω be a bounded away from zero weight on G. Then there exists a continuous projection $P : L^1(G,\omega) \to L^1(G,\omega)$ whose kernel is $J_\omega(G,H)$.*

To prove Theorem 5.5, we need two lemmas. The first one is a result from [35].

Lemma 5.6. [35, Proposition 8.1.16] *Let H be a closed subgroup of a locally compact group G, and U be a non-empty open set in G with compact closure. Then there is a subset Y of G such that the family $\{UyH\}_{y\in Y}$ covers G and is locally finite, i.e., every point of G has a neighborhood intersecting at most finitely many members of the family.*

The second lemma we need guarantees the existence of the function g that generalizes the Bruhat function studied in Section 8.1 of [35], which is used in the theory of quasi-invariant measures on quotient groups.

Lemma 5.7. *Let G, H, and ω be as in Theorem 5.5. Then there exists a continuous function $g \geq 0$ on G and a constant $c > 0$ such that the following two conditions are satisfied:*

$$\int_H g(xh)\,dh = 1, \quad x \in G, \quad \text{and} \tag{5.2}$$

$$\int_H g(xh)\omega(xh)\,dh \leq c\,\hat{\omega}([x]), \quad [x] \in G/H. \tag{5.3}$$

Proof. We start from constructing a continuous function g_1 on G such that

$$0 < \int_H g_1(xh)\,dh < \infty, \quad x \in G, \qquad \text{and} \qquad \operatorname{supp} g_1 \subset \{x \in G : \omega(x) \leq c\,\hat{\omega}([x])\} \tag{5.4}$$

for some constant $c > 0$, which we will determine later. Let $f \geq 0$ be a non-trivial continuous function on G with compact support. Denote $U = \{x \in G : f(x) > 0\}$. Then U is an open set with a compact closure, and so by Lemma 5.6 there exists a set $Y \subset G$ such that the family $\{UyH\}_{y\in Y}$ covers G, and every point of G has a neighborhood intersecting at most finitely many sets from the family. For every $y \in Y$ we build a continuous function $g_{1,y}$ on G vanishing outside UyH and satisfying the conditions in (5.4) for $x \in UyH$. Since ω is continuous and the set U has compact closure, there exists a constant $\tilde{c}$ such that $\omega(u), \omega(u^{-1}) \leq \tilde{c}$ for every $u \in U$. According to the definition of $\hat{\omega}$, there exists $y_0 \in [y]$ such that $\omega(y_0) \leq 2\hat{\omega}([y])$. Consider the set Uy_0. Using the choice of y_0 and the weight inequality for ω, we obtain for every $u \in U$

$$\omega(uy_0) \leq \omega(u)\omega(y_0) \leq \tilde{c} \cdot 2\hat{\omega}([y]) = 2\tilde{c}\,\hat{\omega}([y]), \quad \text{and}$$

$$\hat{\omega}([uy_0]) = \inf_{h \in H} \omega(uy_0 h) \geq \inf_{h \in H} \frac{\omega(y_0 h)}{\omega(u^{-1})} = \frac{\hat{\omega}([y])}{\omega(u^{-1})} \geq \frac{\hat{\omega}([y])}{\tilde{c}}.$$

Therefore,

$$\omega(uy_0) \leq 2\tilde{c}\,\hat{\omega}([y]) \leq 2\tilde{c} \cdot \tilde{c}\,\hat{\omega}([uy_0]) = 2\tilde{c}^2\,\hat{\omega}([uy_0]).$$

So, if we put $c = 2\tilde{c}^2$ (which does not depend on y), we will have that

$$Uy_0 \subset \{x \in G : \omega(x) \leq c\,\hat{\omega}([x])\}. \tag{5.5}$$

We claim that then the function $g_{1,y}(x) = f(xy_0^{-1})$, $x \in G$, satisfies all our requirements. It is easy to see that $g_{1,y}$ is a continuous function,

$$\{x : g_{1,y}(x) \neq 0\} = \{x : f(x) > 0\} \cdot y_0 = Uy_0 \subset UyH,$$

and the second condition in (5.4) is satisfied because of (5.5). It remains to verify that

$$0 < \int_H g_{1,y}(xh)\,dh < \infty, \quad x \in UyH. \tag{5.6}$$

Because $g_{1,y}(x) = f(xy_0^{-1})$, $x \in G$, and f is a continuous function with compact support, it is obvious that the integral above is finite. Now we show that it is strictly positive. Let $x = uyt$, $u \in U$, $t \in H$. From the definition of U it follows that $f(u) > 0$, and since f is continuous, there exists $\varepsilon > 0$ and an open neighborhood V of u such that $f(x) > \varepsilon$ for every $x \in V$. Because f is non-negative, it follows that $f \geq \varepsilon\chi_V$, where χ_V denotes the characteristic function of the set V. Therefore,

$g_{1,y} \geq \varepsilon \chi_{Vy_0}$, and we obtain

$$\int\limits_H g_{1,y}(xh)\,dh \geq \varepsilon \int\limits_H \chi_{Vy_0}(uy(th))\,dh = \varepsilon \int\limits_H \chi_{Vy_0}(uyh)\,dh = \varepsilon \mu_H(H \cap (uy)^{-1}Vy_0),$$

where μ_H stands for the Haar measure on H. Since V is open in G, so is the set $(uy)^{-1}Vy_0$, which implies that $H \cap (uy)^{-1}Vy_0$ is open in H. Hence, to prove that $H \cap (uy)^{-1}Vy_0$ has non-zero measure, it is enough to show that it is non-empty. Indeed, $y^{-1}y_0 \in H$, because $y_0 \in [y]$, and

$$y^{-1}y_0 = y^{-1}u^{-1}uy_0 = (uy)^{-1}uy_0 \in (uy)^{-1}Vy_0.$$

Thus, $y^{-1}y_0 \in H \cap (uy)^{-1}Vy_0$, and (5.6) is verified.

We now show that the function

$$g_1 = \sum_{y \in Y} g_{1,y}$$

satisfies the conditions (5.4). First, we note that since $\{x : g_{1,y}(x) \neq 0\} \subset UyH$, $y \in Y$, and the family $\{UyH\}_{y \in Y}$ is locally finite, the sum in the definition of g_1 is finite in a neighborhood of every point. This implies that g_1 is well-defined and continuous. Also, because the family $\{UyH\}_{y \in Y}$ is locally finite and covers the whole G, it follows from (5.6) that

$$0 < \int\limits_H g_1(xh)\,dh < \infty, \quad x \in G.$$

Finally, the second condition in (5.4) is satisfied for g_1 because it is satisfied for every $g_{1,y}$, $y \in Y$.

We then define the function g by

$$g(x) = \frac{g_1(x)}{\int\limits_H g_1(xh)\,dh}, \quad x \in G.$$

It is easy to see that g is a continuous non-negative function satisfying

$$\int_H g(xh)\,dh = \int_H \frac{g_1(xh)}{\int_H g_1(xh)\,dh}\,dh = \frac{\int_H g_1(xh)\,dh}{\int_H g_1(xh)\,dh} = 1, \quad x \in G.$$

It remains to prove that

$$\int_H g(xh)\omega(xh)\,dh \leq c \cdot \hat{\omega}([x]), \quad [x] \in G/H.$$

But this follows directly from the second condition in (5.4) and (5.2):

$$\int_H g(xh)\omega(xh)\,dh \leq c\,\hat{\omega}([x]) \int_H g(xh)\,dh = c\,\hat{\omega}([x]).$$

The proof is complete. □

Proof of Theorem 5.5. We construct a continuous projection $P : L^1(G,\omega) \to L^1(G,\omega)$ with $\ker P = J_\omega(G,H)$. Let g be a function constructed in Lemma 5.7. We claim that the operator P defined by

$$(Pf)(x) = (Tf)([x])\,g(x), \quad x \in G,\ f \in L^1(G,\omega),$$

is a projection satisfying our requirement. Obviously, $\ker P = \ker T = J_\omega(G,H)$. Hence, we only need to prove that P is a continuous projection. We first show that P ranges in $L^1(G,\omega)$. In fact, for every $f \in L^1(G,\omega)$ the function $P(f)$ is measurable and

$$\begin{aligned}
\int_G |(Pf)(x)|\,\omega(x)\,dx &\underset{\text{Weil's formula}}{=} \int_{G/H}\int_H |(Pf)(xh)|\,\omega(xh)\,dhd[x] \\
&= \int_{G/H}\int_H |(Tf)([x])| g(xh)\omega(xh)\,dhd[x] = \int_{G/H} |(Tf)([x])| \int_H g(xh)\omega(xh)\,dhd[x] \\
&\underset{(5.3)}{\leq} \int_{G/H} |(Tf)([x])| \cdot c\,\hat{\omega}([x])\,d[x] = c\,\|Tf\|_{1,\hat{\omega}} \leq c\,\|f\|_{1,\omega} < \infty.
\end{aligned}$$

So, $P(f) \in L^1(G,\omega)$. Moreover, from the inequality above it also follows that $P : L^1(G,\omega) \to L^1(G,\omega)$ is a bounded operator with $\|P\| \leq c$. Finally, we verify that P is a projection, i.e., $P^2 = P$:

$$\begin{aligned}
(P^2 f)(x) = (P(Pf))(x) = (T(Pf))([x])\,g(x) &= \int_H (Pf)(xh)\,dh \cdot g(x) \\
&= g(x) \int_H (Tf)([xh]) g(xh)\,dh \underset{[xh]=[x]}{=} g(x)(Tf)([x]) \int_H g(xh)\,dh \\
&\underset{(5.2)}{=} (Tf)([x])\,g(x) = (Pf)(x), \quad x \in G.
\end{aligned}$$

□

The next lemma provides a sufficient condition for a complemented ideal to have the trace extension property.

Lemma 5.8. *Let A be a Banach algebra and I be a closed complemented ideal in A, i.e., there exists a Banach subspace X of A such that $A = I \oplus X$ as a Banach space. Suppose also that*

$$xy - yx \in X, \quad \text{whenever} \quad x, y \in X.$$

Then I has the trace extension property.

Remark 5.9. *The lemma was proved in* [23, Lemma 2.3] *in the case when X is a subalgebra of A.*

Proof of Lemma 5.8. Let $\lambda \in I^*$ satisfy $\lambda \cdot f = f \cdot \lambda$, $f \in A$. We need to show that there exists an extension $\tau \in A^*$ of λ such that $\tau(fg) = \tau(gf)$ for every $f, g \in A$. Since $A = I \oplus X$, we have that $A^* = I^* \oplus X^*$. We claim that $\tau = \lambda \oplus 0$ will satisfy our requirements. Obviously, τ is a continuous linear functional on A and $\tau|_I = \lambda$, so τ, indeed, extends λ. Now let $f, g \in A$. There exist $f_1, g_1 \in I$, $f_2, g_2 \in X$ such that $f = f_1 + f_2$ and $g = g_1 + g_2$. Since I is an ideal and the condition for λ can be rewritten as $\lambda(kh) = \lambda(hk)$ for all $k \in I$, $h \in A$, we have

$$\begin{aligned}
\tau(fg) &= \tau((f_1+f_2)(g_1+g_2)) = \tau(\underbrace{f_1g_1 + f_1g_2 + f_2g_1}_{\text{belongs to } I}) + \tau(f_2g_2) \qquad (5.7)\\
&= \lambda(f_1g_1 + f_1g_2 + f_2g_1) + \tau(f_2g_2) = \lambda(f_1g_1) + \lambda(f_1g_2) + \lambda(f_2g_1) + \tau(f_2g_2)\\
&= \lambda(g_1f_1) + \lambda(g_2f_1) + \lambda(g_1f_2) + \tau(f_2g_2) = \lambda(\underbrace{g_1f_1 + g_2f_1 + g_1f_2}_{\text{belongs to } I}) + \tau(f_2g_2)\\
&= \tau(g_1f_1 + g_2f_1 + g_1f_2) + \tau(f_2g_2) + (\tau(g_2f_2) - \tau(g_2f_2))\\
&= \tau((g_1+g_2)(f_1+f_2)) + \tau(f_2g_2) - \tau(g_2f_2) = \tau(gf) + (\tau(f_2g_2) - \tau(g_2f_2)).
\end{aligned}$$

Hence, to prove that $\tau(fg) = \tau(gf)$, it suffices to show that $\tau(f_2g_2) = \tau(g_2f_2)$ for all $f_2, g_2 \in X$. But by our assumption, $f_2g_2 - g_2f_2 \in X$, and so $\tau(f_2g_2 - g_2f_2) = 0$ by the definition of τ. This completes the proof. □

Combining Theorem 5.5 with Proposition 5.3 and Lemma 5.8, we obtain the following result.

Proposition 5.10. *Let G be a locally compact group, H be a closed normal subgroup of G, and ω be a bounded away from zero weight on G. Suppose that X is a Banach space complement of $J_\omega(G,H)$ in $L^1(G,\omega)$, and*

$$xy - yx \in X, \quad \textit{whenever} \quad x, y \in X.$$

Then weak amenability of $L^1(G,\omega)$ implies weak amenability of $L^1(G/H,\hat{\omega})$.

We consider the special case when $G = G_1 \times G_2$, $H = G_2$, and $\omega = \omega_1 \times \omega_2$, where ω_i is a bounded away from zero weight on the locally compact group G_i,

$i = 1, 2$. In this case $G/H = G_1$,

$$\hat{\omega}(x_1) = \omega_1(x_1) \inf_{x_2 \in G_2} \omega_2(x_2) = const \cdot \omega_1(x_1),$$

and the operator $T : L^1(G, \omega) \to L^1(G/H, \hat{\omega})$ is precisely given by

$$T(f)(x_1) = \int_{G_2} f(x_1, x_2)\, dx_2, \quad x_1 \in G_1.$$

It is easy to see that if $h \geq 0$ is a continuous function on G_2 with compact support and

$$\int\limits_{G_2} h(x_2)\, dx_2 = 1,$$

then the function $g(x_1, x_2) = h(x_2)$ will satisfy the conditions of Lemma 5.7. Indeed,

$$\int\limits_{G_2} g(x_1, x_2)\, dx_2 = \int\limits_{G_2} h(x_2)\, dx_2 = 1, \quad x_1 \in G_1, \quad \text{and}$$

$$\begin{aligned} \int\limits_{G_2} g(x_1, x_2)\, \omega(x_1, x_2)\, dx_2 = \int\limits_{G_2} h(x_2)\, \omega_1(x_1)\, \omega_2(x_2)\, dx_2 = \omega_1(x_1) \int\limits_{G_2} h(x_2) \omega_2(x_2)\, dy \\ = const \cdot \hat{\omega}(x_1), \end{aligned}$$

since h has a compact support and ω_2 is continuous. Because $J_\omega(G, H) = \ker(T)$ and $L^1(G, \omega) = L^1(G_1, \omega_1) \hat{\otimes} L^1(G_2, \omega_2)$, we have that

$$J_\omega(G, H) = L^1(G_1, \omega_1) \hat{\otimes} I_2 \quad \text{and} \quad X = L^1(G_1, \omega_1) \hat{\otimes} (\mathbb{C} h),$$

where $I_2 = \left\{ f \in L^1(G_2, \omega_2) : \int_{G_2} f(x_2)\, dx_2 = 0 \right\}$ is the augmentation ideal of the Beurling algebra $L^1(G_2, \omega_2)$.

We claim that in this case $J_\omega(G, H)$ always has the trace extension property implying the following.

Proposition 5.11. *Let G_1, G_2 be locally compact groups and ω_i be a bounded away* *from zero weight on G_i, $i = 1, 2$. Suppose that $L^1(G_1 \times G_2, \omega_1 \times \omega_2)$ is weakly amenable. Then both $L^1(G_1, \omega_1)$ and $L^1(G_2, \omega_2)$ are also weakly amenable.*

Proof. Because of the symmetry, it is enough to show that $L^1(G_1, \omega_1)$ is weakly amenable. Taking into account the discussion preceding the statement of Proposition 5.11, by Proposition 5.3, we only need to prove that $J_\omega(G_1 \times G_2, G_2) = L^1(G_1, \omega_1)\hat{\otimes} I_2$ has the trace extension property in $L^1(G_1 \times G_2, \omega_1 \times \omega_2)$. So, let $\lambda \in (J_\omega(G_1 \times G_2, G_2))^*$ satisfy $\lambda \cdot f = f \cdot \lambda$, $f \in L^1(G \times G_2, \omega_1 \times \omega_2)$. It follows that for every $f = f_1 \otimes f_2$, $f_i \in L^1(G_i, \omega_i)$, $i = 1, 2$, and every $g = g_1 \otimes j_2$, $g_1 \in L^1(G_1, \omega_1)$, $j_2 \in I_2$, we have

$$\begin{aligned} 0 &= \lambda(g * f - f * g) = \lambda\big((g_1 \otimes j_2) * (f_1 \otimes f_2) - (f_1 \otimes f_2) * (g_1 \otimes j_2)\big) \\ &= \lambda\big((g_1 * f_1) \otimes (j_2 * f_2) - (f_1 * g_1) \otimes (f_2 * j_2)\big). \end{aligned}$$

Recall that $L^1(G_2, \omega_2)$ has a bounded approximate identity, say $\{f_{2_\gamma}\}$. We replace f_2 with f_{2_γ} and then take the limit with respect to γ. Using the continuity of λ, we obtain

$$\lambda\big((g_1 * f_1 - f_1 * g_1) \otimes j_2\big) = 0, \quad f_1, g_1 \in L^1(G_1, \omega_1),\ j_2 \in I_2. \tag{5.8}$$

Our goal is to find $\tau \in (L^1(G \times G_2, \omega_1 \times \omega_2))^*$ such that $\tau|_{J_\omega(G_1 \times G_2, G_2)} = \lambda$ and $\tau \cdot f = f \cdot \tau$ for every $f \in L^1(G \times G_2, \omega_1 \times \omega_2)$. According to Theorem 5.5, $J_\omega(G_1 \times G_2, G_2)$ is complemented in $L^1(G \times G_2, \omega_1 \times \omega_2)$. As we have already observed,

$$L^1(G \times G_2, \omega_1 \times \omega_2) = J_\omega(G_1 \times G_2, G_2) \oplus \big(L^1(G_1, \omega_1)\hat{\otimes}(\mathbb{C}h)\big),$$

where $h \geq 0$ is a continuous function on G_2 with compact support satisfying

$$\int_{G_2} h(x_2)\, dx_2 = 1.$$

We claim that $\tau = \lambda \oplus 0$ will meet our requirements. Obviously, τ is a continuous linear functional. Hence, we only need to show that $\tau \cdot f = f \cdot \tau$ for every $f \in L^1(G_1 \times G_2, \omega_1 \times \omega_2)$, i.e., $\tau(g * f - f * g) = 0$ for all $f, g \in L^1(G_1 \times G_2, \omega_1 \times \omega_2)$.

As we have seen in (5.7), it is, in fact, enough to prove that $\tau(g * f - f * g) = 0$ for all $f, g \in X$. Since the elementary tensors span a dense subset of X, and τ is a continuous linear functional, we only need to show that for all $f_1, g_1 \in L^1(G_1, \omega_1)$ we have

$$0 = \tau\big((g_1 \otimes h) * (f_1 \otimes h) - (f_1 \otimes h) * (g_1 \otimes h)\big) = \tau\big((g_1 * f_1 - f_1 * g_1) \otimes (h * h)\big). \quad (5.9)$$

For every $f_1, g_1 \in L^1(G_1, \omega_1)$ we can write the element $a = (g_1 * f_1 - f_1 * g_1) \otimes (h * h)$ of $L^1(G \times G_2, \omega_1 \times \omega_2)$ uniquely as $a = a_J + a_h$, where

$$a_J = (g_1 * f_1 - f_1 * g_1) \otimes j_2 \in J_\omega(G_1 \times G_2, G_2), \quad j_2 \in I_2, \quad \text{and}$$

$$a_h = \big((g_1 * f_1 - f_1 * g_1) \otimes ch\big) \in X, \quad c \in \mathbb{C}.$$

Hence, by definition of τ and (5.8), $\tau(a) = \lambda(a_J) = 0$. This means that (5.9) is verified, and the proof is complete. □

5.2 Weak amenability of Beurling algebras on subgroups of Abelian groups

Let G_1, G_2 be Abelian locally compact groups and $G = G_1 \times G_2$. Suppose that there exist continuous non-zero group homomorphisms $\Phi_i : G_i \to \mathbb{R}$, $i = 1, 2$. Given α, $\beta > 0$ we define the function ω on G as follows:

$$\omega(g_1, g_2) = (1 + |\Phi_1(g_1)|)^\alpha \, (1 + |\Phi_1(g_1) + \Phi_2(g_2)|)^\beta, \quad g_i \in G_i, \; i = 1, 2. \qquad (5.10)$$

It is readily seen that ω is a weight on $G_1 \times G_2$, and

$$\omega_1(g_1) = \omega(g_1, e_2) = (1 + |\Phi_1(g_1)|)^{\alpha+\beta}, \quad g_1 \in G_1,$$

where e_2 denotes the identity of G_2.

Proposition 5.12. *Let G_1 and G_2 be Abelian locally compact groups and $G =$*

$G_1 \times G_2$. Suppose that $\Phi_i : G_i \to \mathbb{R}$, $i = 1, 2$, is a non-trivial continuous group homomorphism, and ω is a weight on G defined by (5.10). If $0 < \alpha, \beta < 1/2$ and $\alpha + \beta \geq 1/2$, then $L^1(G, \omega)$ is weakly amenable, but $L^1(G_1, \omega_1)$ is not weakly amenable.

Proof. Using Theorem 1.12, it is easy to see that $L^1(G_1, \omega_1)$ is not weakly amenable if $\alpha + \beta \geq 1/2$. Indeed, Φ_1 is a non-trivial continuous group homomorphism from G_1 to $\mathbb{R}$, and

$$\sup_{g_1 \in G_1} \frac{|\Phi_1(g_1)|}{\omega_1(g_1)\omega_1(g_1^{-1})} = \sup_{g_1 \in G_1} \frac{|\Phi_1(g_1)|}{(1 + |\Phi_1(g_1)|)^{2(\alpha+\beta)}} < \infty.$$

We now show that $L^1(G, \omega)$ is weakly amenable. According to Theorem 1.12, it suffices to prove that

$$\sup_{g \in G} \frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} = \infty \tag{5.11}$$

for every non-trivial continuous group homomorphism $\Phi : G \to \mathbb{R}$. Fix such a homomorphism Φ. Since $(g_1, g_2) = (g_1, e_2)(e_1, g_2)$, where e_i is the identity of G_i, $g_i \in G_i$, $i = 1, 2$, we have

$$\sup_{g \in G} \frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} = \sup_{g_1 \in G_1,\ g_2 \in G_2} \frac{|\Phi(g_1, e_2) + \Phi(e_1, g_2)|}{(1 + |\Phi_1(g_1)|)^{2\alpha} (1 + |\Phi_1(g_1) + \Phi_2(g_2)|)^{2\beta}}.$$

We first consider the case when there is no constant $c \in \mathbb{R}$ such that $\Phi(g_1, e_2) = c\Phi_1(g_1)$ for all $g_1 \in G_1$. Since Φ_2 is non-trivial, we can choose $g_{2_0} \in G_2$ such that $\Phi_2(g_{2_0}) \neq 0$. Then for every $g_1 \in G_1$ there exists an $n(g_1) \in \mathbb{Z}$ such that

$$\left|\Phi_1(g_1) + \Phi_2\left(g_{2_0}^{n(g_1)}\right)\right| = |\Phi_1(g_1) + n(g_1)\Phi_2(g_{2_0})| < |\Phi_2(g_{2_0})|, \tag{5.12}$$

and so

$$\left|n(g_1) + \frac{\Phi_1(g_1)}{\Phi_2(g_{2_0})}\right| \leq \frac{|\Phi_2(g_{2_0})|}{|\Phi_2(g_{2_0})|} = 1. \tag{5.13}$$

From this we obtain the following:

$$
\begin{aligned}
\sup_{g\in G}\frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} &= \sup_{g_1\in G_1,\ g_2\in G_2}\frac{|\Phi(g_1,e_2)+\Phi(e_1,g_2)|}{\left(1+|\Phi_1(g_1)|\right)^{2\alpha}\left(1+|\Phi_1(g_1)+\Phi_2(g_2)|\right)^{2\beta}}\\
&\underset{g_2=g_{2_0}^{n(g_1)}}{\geq}\ \sup_{g_1\in G_1}\frac{|\Phi(g_1,e_2)+\Phi(e_1,g_{2_0}^{n(g_1)})|}{\left(1+|\Phi_1(g_1)|\right)^{2\alpha}\left(1+\left|\Phi_1(g_1)+\Phi_2\left(g_{2_0}^{n(g_1)}\right)\right|\right)^{2\beta}}\\
&\underset{(5.12)}{\geq}\ \sup_{g_1\in G_1}\frac{|\Phi(g_1,e_2)+n(g_1)\Phi(e_1,g_{2_0})|}{\left(1+|\Phi_1(g_1)|\right)^{2\alpha}\left(1+|\Phi_2(g_{2_0}|)\right)^{2\beta}}\\
&= \sup_{g_1\in G_1}\frac{\left|\left(\Phi(g_1,e_2)-\frac{\Phi_1(g_1)}{\Phi_2(g_{2_0})}\Phi(e_1,g_{2_0})\right)+\left(\frac{\Phi_1(g_1)}{\Phi_2(g_{2_0})}+n(g_1)\right)\Phi(e_1,g_{2_0})\right|}{\left(1+|\Phi_1(g_1)|\right)^{2\alpha}\left(1+|\Phi_2(g_{2_0}|)\right)^{2\beta}}\\
&\underset{(5.13)}{\geq}\ \sup_{g_1\in G_1}\frac{\left|\Phi(g_1,e_2)-\frac{\Phi(e_1,g_{2_0})}{\Phi_2(g_{2_0})}\Phi_1(g_1)\right|-|\Phi(e_1,g_{2_0})|}{\left(1+|\Phi_1(g_1)|\right)^{2\alpha}\left(1+|\Phi_2(g_{2_0}|)\right)^{2\beta}}.
\end{aligned}
$$

Because $\dfrac{\Phi(e_1,g_{2_0})}{\Phi_2(g_{2_0})}$ is a constant, and we assumed that there is no constant c such that $\Phi(g_1,e_2)=c\Phi_1(g_1)$ for all $g_1\in G_1$, there exists $g_{1_0}\in G_1$ for which $\Phi(g_{1_0},e_2)-\dfrac{\Phi(e_1,g_{2_0})}{\Phi_2(g_{2_0})}\Phi_1(g_{1_0})\neq 0$. Then

$$
\begin{aligned}
&\sup_{g_1\in G_1}\frac{\left|\Phi(g_1,e_2)-\frac{\Phi(e_1,g_{2_0})}{\Phi_2(g_{2_0})}\Phi_1(g_1)\right|-|\Phi(e_1,g_{2_0})|}{\left(1+|\Phi_1(g_1)|\right)^{2\alpha}\left(1+|\Phi_2(g_{2_0}|)\right)^{2\beta}}\\
&\qquad\underset{g_1=g_{1_0}^m}{\geq}\ \sup_{m\in\mathbb{N}}\frac{\left|\Phi(g_{1_0}^m,e_2)-\frac{\Phi(e_1,g_{2_0})}{\Phi_2(g_{2_0})}\Phi_1(g_{1_0}^m)\right|-|\Phi(e_1,g_{2_0})|}{\left(1+|\Phi_1(g_{1_0}^m)|\right)^{2\alpha}\left(1+|\Phi_2(g_{2_0}|)\right)^{2\beta}}\\
&\qquad=\sup_{m\in\mathbb{N}}\frac{m\left|\Phi(g_{1_0},e_2)-\frac{\Phi(e_1,g_{2_0})}{\Phi_2(g_{2_0})}\Phi_1(g_{1_0})\right|-|\Phi(e_1,g_{2_0})|}{\left(1+m|\Phi_1(g_{1_0})|\right)^{2\alpha}\left(1+|\Phi_2(g_{2_0}|)\right)^{2\beta}}=\infty.
\end{aligned}
$$

So, in this case (5.11) holds.

Now let $\Phi(g_1,e_2)=c\,\Phi_1(g_1)$ for some constant c and all $g_1\in G_1$. Assume, in addition, that $\Phi(e_1,g_2)$ is non-trivial as a function of g_2, i.e., there exists $g_{2_0}\in G_2$

such that $\Phi(e_1, g_{2_0}) \neq 0$. Then

$$\begin{aligned}\sup_{g\in G}\frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} &= \sup_{g_1\in G_1,\ g_2\in G_2}\frac{|c\Phi_1(g_1)+\Phi(e_1,g_2)|}{(1+|\Phi_1(g_1)|)^{2\alpha}\,(1+|\Phi_1(g_1)+\Phi_2(g_2)|)^{2\beta}}\\ &\underset{g_1=e_1,\,g_2=g_{2_0}^n}{\geq}\sup_{n\in\mathbb{N}}\frac{|\Phi(e_1,g_{2_0}^n)|}{(1+|\Phi_2(g_{2_0}^n)|)^{2\beta}}=\sup_{n\in\mathbb{N}}\frac{n|\Phi(e_1,g_{2_0})|}{(1+n|\Phi_2(g_{2_0})|)^{2\beta}}=\infty,\end{aligned}$$

since $\beta < 1/2$. So (5.11) holds in this case too.

Finally, let $\Phi(g_1, e_2) = c\Phi_1(g_1)$ for all $g_1 \in G_1$ and some constant c, and $\Phi(e_1, g_2) \equiv 0$ as a function of g_2. Then, since Φ and Φ_2 are non-trivial, there exist $g_{1_0} \in G_1$ and $g_{2_0} \in G_2$ such that $\Phi(g_{1_0}, e_2) \neq 0$ and $\Phi_2(g_{2_0}) \neq 0$. Using the same arguments as in the very first case, for every $g_1 \in G_1$ we can find $n(g_1) \in \mathbb{Z}$ such that $|\Phi_1(g_1) + \Phi_2(g_{2_0}^{n(g_1)})| < |\Phi_2(g_{2_0})|$. Thus we obtain

$$\begin{aligned}\sup_{g\in G}\frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} &= \sup_{g_1\in G_1,\ g_2\in G_2}\frac{|c\Phi_1(g_1)|}{(1+|\Phi_1(g_1)|)^{2\alpha}\,(1+|\Phi_1(g_1)+\Phi_2(g_2)|)^{2\beta}}\\ &\underset{g_2=g_{2_0}^{n(g_1)}}{\geq}\sup_{g_1\in G_1}\frac{|c|\,|\Phi_1(g_1)|}{(1+|\Phi_1(g_1)|)^{2\alpha}(1+|\Phi_2(g_{2_0})|)^{2\beta}}\\ &\underset{g_1=g_{1_0}^m}{\geq}\sup_{m\in\mathbb{N}}\frac{m|c|\,|\Phi_1(g_{1_0})|}{(1+m|\Phi_1(g_{1_0})|)^{2\alpha}(1+|\Phi_2(g_{2_0})|)^{2\beta}}=\infty,\end{aligned}$$

since $\alpha < 1/2$. So, we have shown that (5.11) holds for each non-trivial continuous group homomorphism $\Phi : G \to \mathbb{R}$, and the proof is complete. □

Remark 5.13. *In particular, the result of Proposition 5.12 holds for $G_1 = G_2 = \mathbb{Z}$ or $G_1 = G_2 = \mathbb{R}$ and $\Phi_1 = \Phi_2 = \mathrm{id}$.*

Remark 5.14. *Proposition 5.12 implies that, in general, it is not true, even for Abelian groups G, that weak amenability of the Beurling algebra $L^1(G, \omega)$ implies weak amenability of $L^1(H, \omega|_H)$, where H is a subgroup of G and ω_H is the restriction of ω to H.*

However, the implication is true for certain open subgroups.

Proposition 5.15. *Let G be a locally compact Abelian group and H be its open* *subgroup such that G/H is compact. Then for any weight ω on G weak amenability of $L^1(G,\omega)$ implies weak amenability of $L^1(H,\omega|_H)$.*

To prove this proposition we need a technical lemma:

Lemma 5.16. *Let G be a locally compact Abelian group, and H be an open subgroup of G. Then any continuous group homomorphism $\Phi : H \to \mathbb{C}$ can be extended to a continuous group homomorphism $\tilde{\Phi} : G \to \mathbb{C}$.*

Proof. By Zorn's Lemma, it is enough to show that for every $g \in G$ we can extend Φ to the open subgroup $H_g = \bigcup_{n\in\mathbb{Z}} g^n H = \{g^n h : h \in H,\, n \in \mathbb{Z}\}$ of G. We first consider the case when there exists $m \in \mathbb{N}$ such that $g^m \in H$. Let m_0 be the smallest such number. Then we denote $\alpha = \frac{1}{m_0}\Phi(g^{m_0})$ and define $\tilde{\Phi}(g^n h) = n\alpha + \Phi(h)$ for $h \in H$, $n \in \mathbb{Z}$. It is easy to see that $\tilde{\Phi}$ is a group homomorphism on H_g. In fact, the only non-obvious thing to check is that the extension is well-defined, that is if $g^{n_1}h_1 = g^{n_2}h_2$ then $n_1\alpha + \Phi(h_1) = n_2\alpha + \Phi(h_2)$. In this case $g^{n_1-n_2} = h_2h_1^{-1} \in H$, and so $n_1 - n_2 = km_0$ for some $k \in \mathbb{Z}$. Because Φ is a group homomorphism on H, we have that

$$\Phi(h_2) - \Phi(h_1) = \Phi(h_2h_1^{-1}) = \Phi(g^{n_1-n_2}) = k\Phi(g^{m_0}) = km_0\alpha = (n_1 - n_2)\alpha,$$

which implies the desired equality $n_1\alpha + \Phi(h_1) = n_2\alpha + \Phi(h_2)$. We will now show that $\tilde{\Phi}$ is continuous on H_g. Consider a net $\{t_\gamma = g^{n_\gamma}h_\gamma\}_{\gamma\in\Gamma} \subset H_g$ that converges to some $t = g^n h \in H_g$. Our goal is to prove that $\tilde{\Phi}(t_\gamma)$ converges to $\tilde{\Phi}(t)$. Since $g^{n_\gamma}h_\gamma \to g^n h$, we have that $g^{n_\gamma - n}h_\gamma \to h$, and because H is open, this implies that $g^{n_\gamma-n} \in H$ for $\gamma \geq \gamma_0$, $\gamma_0 \in \Gamma$. Then from the continuity of Φ on H it follows that

$$\tilde{\Phi}(g^{n_\gamma-n}h_\gamma) = \Phi(g^{n_\gamma-n}h_\gamma) \to \Phi(h) = \tilde{\Phi}(h),$$

and using the fact that $\tilde{\Phi}$ is a group homomorphism, we finally obtain that

$$\tilde{\Phi}(t_\gamma) = \tilde{\Phi}(g^{n_\gamma}h_\gamma) = \tilde{\Phi}(g^{n_\gamma-n}h_\gamma) + \tilde{\Phi}(g^n) \to \tilde{\Phi}(h) + \tilde{\Phi}(g^n) = \tilde{\Phi}(g^n h) = \tilde{\Phi}(t).$$

Now assume that $g^n \notin H$ for any $n \in \mathbb{N}$. Then we put $\tilde{\Phi}(g^n h) = \Phi(h)$ for

$h \in H$, $n \in \mathbb{Z}$. Obviously, $\tilde{\Phi}$ is a group homomorphism on H_g, and we just need to prove that it is continuous. Let $g^{n_\gamma} h_\gamma \to g^n h$, $n_\gamma, n \in \mathbb{Z}$, $h_\gamma, h \in H$. Then, as above, $g^{n_\gamma - n} h_\gamma \to h$ and because H is open, this implies that $g^{n_\gamma - n} \in H$ for $\gamma \geq \gamma_0$. But our condition on g implies that $n_\gamma = n$ for $\gamma \geq \gamma_0$, and so $h_\gamma \to h$. Then we have that $\tilde{\Phi}(g^{n_\gamma} h_\gamma) = \Phi(h_\gamma) \to \Phi(h) = \tilde{\Phi}(g^n h)$ since Φ is continuous on H. So $\tilde{\Phi}$ is continuous, and the proof is complete. $\square$

Proof of Proposition 5.15. Suppose to the contrary that $L^1(G, \omega)$ is weakly amenable, but $L^1(H, \omega|_H)$ is not. Since $H \subset G$ is Abelian, we can apply Theorem 1.12 to find a non-trivial continuous group homomorphism $\Phi : H \to \mathbb{R}$ such that

$$\sup_{h \in H} \frac{|\Phi(h)|}{\omega(h)\omega(h^{-1})} < \infty.$$

Since $H \subset G$ is open, we can apply Lemma 5.16 to extend Φ to a continuous group homomorphism $\tilde{\Phi} : G \to \mathbb{R}$. Another consequence of the openness of H in G is that the quotient group G/H is discrete, and so its compactness implies that G/H is finite. Therefore, we can choose $\{x_i\}_{i=1}^n \subset G$ so that each coset from G/H has its representative among $\{x_i\}_{i=1}^n$. It follows that for every $g \in G$ there is an $i \in \overline{1, n}$ such that $x_i^{-1} g \in H$. Also, since ω is a weight, we have that $\omega(x_i h) \geq \omega(h)/\omega(x_i^{-1})$ and $\omega((x_i h)^{-1}) = \omega(h^{-1} x_i^{-1}) \geq \omega(h^{-1})/\omega(x_i)$. Using all of the above, we obtain

$$\begin{aligned}
\sup_{g \in G} \frac{|\tilde{\Phi}(g)|}{\omega(g)\omega(g^{-1})} &= \sup_{h \in H,\, 1 \leq i \leq n} \frac{|\tilde{\Phi}(x_i h)|}{\omega(x_i h)\omega((x_i h)^{-1})} \\
&\leq \sup_{h \in H,\, 1 \leq i \leq n} \frac{|\tilde{\Phi}(h)| + |\tilde{\Phi}(x_i)|}{(\omega(h)/\omega(x_i^{-1})) \cdot (\omega(h^{-1})/\omega(x_i))} \\
&\leq \sup_{h \in H} \frac{|\Phi(h)| + \sup_{1 \leq i \leq n} |\tilde{\Phi}(x_i)|}{\omega(h)\omega(h^{-1})} \cdot \sup_{1 \leq i \leq n} \omega(x_i)\omega(x_i^{-1}) < \infty,
\end{aligned}$$

since $\sup_{h \in H} \dfrac{|\Phi(h)|}{\omega(h)\omega(h^{-1})} < \infty$, and both supremums in i are also finite. Since G is Abelian, last inequality contradicts weak amenability of $L^1(G, \omega)$ by Theorem 1.12. Therefore, $L^1(H, \omega|_H)$ must be weakly amenable if so is $L^1(G, \omega)$. $\square$

5.3 Weak amenability of $L^1(H, \omega|_H)$ and $L^1(G/H, \hat{\omega})$ does not imply that of $L^1(G, \omega)$

Given a closed normal subgroup H of a locally compact group G, it is well-known that $L^1(G)$ is amenable if and only if both $L^1(H)$ and $L^1(G/H)$ are amenable. Returning to our weak amenability problem for weighted group algebras, Proposition 5.10 provides conditions under which weak amenability of $L^1(G, \omega)$ implies weak amenability of $L^1(G/H, \hat{\omega})$. We also note that Example 3.12 shows that $L^1(H, \omega|_H)$ may not be weakly amenable even when $L^1(G, \omega)$ is. In view of the above, it is natural to consider the following problem.

Question 5.17. *Let H be a closed normal subgroup of a locally compact group G and ω be a weight on G. Assume that both Beurling algebras $L^1(H, \omega|_H)$ and $L^1(G/H, \hat{\omega})$ are weakly amenable. Does this imply weak amenability of $L^1(G, \omega)$?*

It turns out that the answer to this question is negative in general. Below we will construct a counterexample using the $(\boldsymbol{ax+b})$ group equipped with the discrete topology. Recall, that each element of $(\boldsymbol{ax+b})$ is identified with a pair $(a, b) \in \mathbb{R}^+ \times \mathbb{R}$, and the group operations are defined as follows:

$$(a,b)(c,d) = (ac, ad+b), \quad (a,b)^{-1} = \left(\frac{1}{a}, \frac{-b}{a}\right), \quad a, c > 0,\ b, d \in \mathbb{R}.$$

It is easy to see that

$$H = \{(1,b) : b \in \mathbb{R}\} \tag{5.14}$$

is a closed normal subgroup of $(\boldsymbol{ax+b})$, and $(\boldsymbol{ax+b})/H \cong (\mathbb{R}^+, \cdot)$ through the map $[(a,b)] \mapsto a$. Before giving the desired example, we prove the following general result.

Proposition 5.18. *Let ω be a weight on $(\boldsymbol{ax+b})$ that is bounded on H. Then $\ell^1((\boldsymbol{ax+b}), \omega)$ is weakly amenable if and only if ω is diagonally bounded.*

Proof. The sufficiency is trivial due to Proposition 1.10.

For the necessity, we assume that ω is not diagonally bounded. We show that in this case $\ell^1((\boldsymbol{ax+b}),\omega)$ is not weakly amenable. First observe that if $\hat{\omega}$ is the weight on $(\boldsymbol{ax+b})/H$ defined by the formula $\hat{\omega}[z] = \inf_{h\in H}\omega(zh)$, then $\tilde{\omega}$ defined by

$$\tilde{\omega}(a,b) = \hat{\omega}([(a,b)])(= \hat{\omega}(a)), \quad a>0, b\in\mathbb{R},$$

is a weight on $(\boldsymbol{ax+b})$. Moreover, because ω is bounded on H, $\tilde{\omega}$ is equivalent to ω. Indeed,

$$\begin{aligned}\omega(a,b) &= \omega((a,b)(1,0)) \geq \inf_{t\in\mathbb{R}}\omega((a,b)(1,t)) = \tilde{\omega}(a,b) \geq \inf_{t\in\mathbb{R}}\frac{\omega(a,b)}{\omega((1,t)^{-1})}\\ &= \frac{\omega(a,b)}{\sup\limits_{t\in\mathbb{R}}\omega(1,-t)} = \omega(a,b)\cdot\frac{1}{\sup\limits_{h\in H}\omega(h)} = \tilde{c}\,\omega(a,b),\end{aligned}$$

where $\tilde{c} = \dfrac{1}{\sup\limits_{h\in H}\omega(h)} > 0$ is a constant. Hence, $\tilde{c}\omega \leq \tilde{\omega} \leq \omega$, which precisely means that ω and $\tilde{\omega}$ are equivalent.

According to Corollary 4.6, to show that $\ell^1((\boldsymbol{ax+b}),\omega)$ is not weakly amenable it is enough to find a function $\psi : (\boldsymbol{ax+b}) \to \mathbb{R}$, a conjugacy class $\{yx_0y^{-1} : y\in (\boldsymbol{ax+b})\}$, $x_0 \in (\boldsymbol{ax+b})$, and a constant $c>0$ such that ω is bounded away from zero on $\{yx_0y^{-1} : y \in (\boldsymbol{ax+b})\}$,

$$|\psi(zy) - \psi(yz)| \leq c\,\omega(y)\omega(z), \quad y,z\in(\boldsymbol{ax+b}), \quad \text{and} \tag{5.15}$$

$$\sup_{y\in(\boldsymbol{ax+b})}\frac{|\psi(yx_0y^{-1})|}{\omega(yx_0y^{-1})} = \infty. \tag{5.16}$$

We take $x_0 = (1,1)$ and claim that its conjugacy class in $(\boldsymbol{ax+b})$ coincides with the set $B = \{(1,b) : b>0\}$. Indeed, if $y = (a,b)\in(\boldsymbol{ax+b})$, then

$$yx_0y^{-1} = (a,b)(1,1)\left(\frac{1}{a},\frac{-b}{a}\right) = (a,a+b)\left(\frac{1}{a},\frac{-b}{a}\right) = (1,a),$$

and since $a>0$ was arbitrary, the claim is proved. Since $B\subset H$, H is a subgroup of $(\boldsymbol{ax+b})$, and the weight ω is bounded on H, we have that ω is bounded away from zero on B:

$$\inf_{h\in B}\omega(h) \geq \inf_{h\in H}\frac{\omega(e)}{\omega(h^{-1})} = \frac{\omega(e)}{\sup\limits_{h\in H}\omega(h^{-1})} > 0.$$

We now aim to show that the function $\psi : (\boldsymbol{ax+b}) \to \mathbb{R}$ defined by

$$\psi(a,b) = \begin{cases} \ln(\hat{\omega}(b)\hat{\omega}(b^{-1})), & \text{if } \ a=1,\, b>0, \\ 0, & \text{otherwise.} \end{cases}$$

satisfies the conditions (5.15) and (5.16). By definition, ψ vanishes outside the conjugacy class B. So, since zy and yz always belong to the same conjugacy class, in order to show that $|\psi(zy)-\psi(yz)| \leq c\,\omega(y)\omega(z)$ for all $y,z \in (\boldsymbol{ax+b})$, we only need to ensure that this inequality is valid in the case $zy, yz \in B$. Let $yz = (1,b)$, and $z=(k,l)$, $b,k>0$, $l\in\mathbb{R}$. Then

$$y = (yz)z^{-1} = (1,b)\left(\frac{1}{k},\frac{-l}{k}\right) = \left(\frac{1}{k},\frac{-l+bk}{k}\right), \quad \text{and so}$$

$$zy = (k,l)\left(\frac{1}{k},\frac{-l+bk}{k}\right) = (1,bk)\,.$$

Using the same arguments as for obtaining (4.25) and (4.27) in the proof of Proposition 4.12, we get

$$\begin{aligned} |\psi(zy)-\psi(yz)| &= |\psi(1,bk)-\psi(1,b)| = |\ln(\hat{\omega}(bk)\hat{\omega}((bk)^{-1})) - \ln(\hat{\omega}(b)\hat{\omega}((b)^{-1}))| \\ &= \left|\ln\frac{\hat{\omega}(bk)\hat{\omega}((bk)^{-1})}{\hat{\omega}(b)\hat{\omega}((b)^{-1})}\right| \leq \ln\bigl(\hat{\omega}(k)\hat{\omega}(k^{-1})\bigr) \leq c\,\hat{\omega}(k)\hat{\omega}\left(\frac{1}{k}\right) \end{aligned} \tag{5.17}$$

for some constant $c>0$. On the other hand, since $\omega(t) \geq \hat{\omega}([t])$ for every $t \in (\boldsymbol{ax+b})$, we have

$$\omega(y)\omega(z) = \omega\left(\frac{1}{k},\frac{-l+bk}{k}\right)\omega(k,l) \geq \hat{\omega}\left(\frac{1}{k}\right)\hat{\omega}(k) \tag{5.18}$$

Combining (5.17) and (5.18), we obtain that $|\psi(zy)-\psi(yz)| \leq c\,\omega(y)\omega(z)$, and so the first condition for ψ is verified.

We now check the second condition for ψ. Recall that $\hat{\omega}(a) = \tilde{\omega}(a,b) \geq \tilde{c}\,\omega(a,b)$ for all $(a,b) \in \boldsymbol{(ax+b)}$. Then

$$\sup_{y\in\boldsymbol{(ax+b)}} \frac{|\psi(yx_0y^{-1})|}{\omega(yx_0y^{-1})} = \sup_{b>0} \frac{|\psi(1,b)|}{\omega(1,b)} = \sup_{b>0} \frac{|\ln(\hat{\omega}(b)\hat{\omega}(b^{-1}))|}{\omega(1,b)}$$
$$\geq \frac{\sup\limits_{z\in\boldsymbol{(ax+b)}} |\ln(\omega(z)\omega(z^{-1}))| - |\ln\tilde{c}^2|}{\sup\limits_{h\in H}\omega(h)} = \infty,$$

since ω is not diagonally bounded on G, but is bounded on H. So, the second condition for ψ is also true. Due to Corollary 4.6, $\ell^1(\boldsymbol{(ax+b)},\omega)$ is not weakly amenable. $\square$

We are now ready to give a counter-example to answer Question 5.17.

Example 5.19. *Let H be the normal subgroup of $\boldsymbol{(ax+b)}$ defined by (5.14). Suppose W is a weight on $(\mathbb{R}^+,\cdot)$ that is not diagonally bounded, but is such that $\ell^1(\mathbb{R}^+,W)$ is weakly amenable. We define a weight ω on $\boldsymbol{(ax+b)}$ by $\omega(a,b) = W(a)$, $a>0$. With this weight both $\ell^1(H,\omega|_H)$ and $\ell^1(\boldsymbol{(ax+b)}/H,\hat{\omega})$ are weakly amenable, but $\ell^1(\boldsymbol{(ax+b)},\omega)$ is not weakly amenable.*

Proof. From the definition of ω it follows that $\omega|_H = W(1) = const$. So $\ell^1(H,\omega|_H)$ is isomorphic to $\ell^1(H)$ and, hence, is weakly amenable. It is easy to see that $\hat{\omega}([(a,b)]) = \hat{\omega}(a) = W(a)$. Since ω is, obviously, bounded on H, Proposition 5.18 asserts that $\ell^1(\boldsymbol{(ax+b)},\omega)$ is weakly amenable only if ω is diagonally bounded. This is equivalent to W being diagonally bounded as a weight on $(\mathbb{R}^+,\cdot)$. Since W was chosen not to be diagonally bounded, $\ell^1(\boldsymbol{(ax+b)},\omega)$ is not weakly amenable. However, $\ell^1(\boldsymbol{(ax+b)}/H,\hat{\omega}) \simeq \ell^1(\mathbb{R}^+,\omega)$, which is weakly amenable as assumed. The proof is complete. $\square$

Remark 5.20. *A natural choice of the function W in Example 5.19 is $W(a) = (1+|\ln a|)^\alpha$, $0<\alpha<1/2$.*

Remark 5.21. *The arguments from Remark 3.6 can also be used in the proof of Example 5.19 to produce a separable counter-example to Question 5.17.*

Chapter 6

Weak amenability of centres of Beurling algebras

In this chapter we deal with the algebras $ZL^1(G,\omega)$, where G is a locally compact group and ω is a weight on G. Recall, that $ZL^1(G,\omega)$ is non-empty if and only if G is an [IN] group, and $ZL^1(G,\omega)$ has a bounded approximate identity if and only if G is an [SIN] group ([30]). We will also consider several other classes of locally compact groups, such as [FC], [FD], and [FIA] groups. Before defining these classes of groups, let us recall the definition of the topology on the group $Aut(G)$ of topological automorphisms of G.

Definition 6.1. [18, Definition 26.3] Let G be a topological group and $Aut(G)$ be the set of all continuous algebraic automorphisms of G. For a compact subset F of G and a neighborhood U of identity e in G, let $\mathfrak{B}(F,U)$ be the set of all $\tau \in Aut(G)$ such that $\tau(x) \in Ux$ and $\tau^{-1}(x) \in Ux$ for all $x \in F$.

Proposition 6.2. [18, Theorem 26.5] *Let G be a locally compact group. The family of sets $\{B\tau\}$, where B runs through $\{\mathfrak{B}(F,U) : F$ is compact, U is a neighborhood of $e\}$ and τ runs through $Aut(G)$, is an open subbasis for a topology on $Aut(G)$ under which it is a topological group.*

In the sequel, for a locally compact group G, we always equip $Aut(G)$ with the topology ensured in Proposition 6.2. Now we can define [FC], [FD], and [FIA] groups.

Definition 6.3. Let G be a locally compact group.

1. G is an [FC] group if all conjugacy classes $\{gxg^{-1}\}_{g\in G}$, $x \in G$, have compact closure in G;

2. G is an [FD] group if the closure of the commutator subgroup G' of G is compact in G (the commutator subgroup of G is the group generated by all elements of the form $xyx^{-1}y^{-1}$, $x, y \in G$);

3. G is an [FIA] group if the closure of the set of all inner automorphisms $I(G)$ of G is compact in $Aut(G)$.

One can find a brief overview of the history of these notions in [15]. We just mention almost obvious inclusion of the class of [FD] groups in the class of [FC] groups, and a deeper result that the class of [FIA] groups coincides with the intersection of the classes of [SIN] groups and [FC] groups. Also, every [FC] group is an [IN] group.

6.1 Weak amenability of $ZL^1(G, \omega)$ on a connected [SIN] group G

It is well-known (see, for example, [33, Theorem 1.10.11]) that the group algebra $L^1(G_1 \times G_2)$ on a direct product of locally compact groups G_1, G_2 is isometrically isomorphic to a projective tensor product of group algebras $L^1(G_1)$ and $L^1(G_2)$. We start this section by proving the corresponding result for centres $ZL^1(G_1 \times G_2)$, $ZL^1(G_1)$, and $ZL^1(G_2)$ for [FIA] groups G_1, G_2.

Proposition 6.4. *Let G_1 and G_2 be locally compact $[FIA]$ groups. Then*

$$ZL^1(G_1 \times G_2) \cong ZL^1(G_1)\hat{\otimes} ZL^1(G_2).$$

Proof. It is known that for every $[FIA]$ group G the map $P : L^1(G) \to ZL^1(G)$ defined by

$$(Pf)(x) = \int_{\overline{I(G)}} f(\beta^{-1}x)\, d\beta \tag{6.1}$$

is a norm one projection onto $ZL^1(G)$ (see, for example, [31, Proposition 1.5]). Here $\overline{I(G)}$ is the closure of $I(G)$ in $Aut(G)$, which is compact since G is an $[FIA]$ group. We consider the following diagram:

$$\begin{array}{ccc} L^1(G_1 \times G_2) & \xleftarrow{T} & L^1(G_1)\hat{\otimes}L^1(G_2) \\ \downarrow P & & \downarrow P_1 \otimes P_2 \\ ZL^1(G_1 \times G_2) & \xleftarrow{\widetilde{T}} & ZL^1(G_1)\hat{\otimes}ZL^1(G_2) \end{array},$$

where by P, P_1, and P_2 we mean the projections defined by (6.1) for G being $G_1 \times G_2$, G_1, and G_2 respectively, T is the standard isomorphism that sends $f_1 \otimes f_2$ to $f_1 f_2$ (that is $T(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1)f_2(x_2)$, $f_i \in L^1(G_i)$, $x_i \in G_i$, $i = 1, 2$), and $\widetilde{T}$ is the restriction of T to $ZL^1(G_1)\hat{\otimes}ZL^1(G_2)$. It is easy to see that $\widetilde{T}$ ranges in $ZL^1(G_1 \times G_2)$. So if its range is dense in $ZL^1(G_1 \times G_2)$, the desired isomorphism of $ZL^1(G_1)\hat{\otimes}ZL^1(G_2)$ and $ZL^1(G_1 \times G_2)$ will be established by means of $\widetilde{T}$. Since T is an isomorphism, the set $\{T(f_1 \otimes f_2)\}_{f_i \in L^1(G_i)}$ is dense in $L^1(G_1 \times G_2)$. Moreover, because P is a norm one projection, the set $\{(P \circ T)(f_1 \otimes f_2)\}_{F_i \in L^1(G_i)}$ is dense in $ZL^1(G_1 \times G_2)$. Therefore, it suffices to prove that the diagram above is commutative on the elementary tensors $f_1 \otimes f_2$, i.e., that $(P \circ T)(f_1 \otimes f_2) = (\widetilde{T} \circ (P_1 \otimes P_2))(f_1 \otimes f_2)$ for all $f_i \in L^1(G_i)$, $i = 1, 2$. Recalling the definitions of the projections P, P_1, and P_2, we see that it only remains to prove the following:

$$\int\limits_{\overline{I(G_1 \times G_2)}} T(f_1 \otimes f_2)(\beta^{-1}(x_1, x_2))\, d\beta = \int\limits_{\overline{I(G_1)}} f_1(\beta_1^{-1} x_1)\, d\beta_1 \cdot \int\limits_{\overline{I(G_2)}} f_2(\beta_2^{-1} x_2)\, d\beta_2.$$

This is obvious if we show that $\overline{I(G_1 \times G_2)} = \overline{I(G_1)} \times \overline{I(G_2)}$. First, $I(G_1 \times G_2) \simeq I(G_1) \times I(G_2)$ holds in a natural way. So it will be enough for us to prove that $\overline{I(G_1 \times G_2)} \subset Aut(G_1) \times Aut(G_2)$, and that the restriction of the topology of $Aut(G_1 \times G_2)$ to $Aut(G_1) \times Aut(G_2)$ coincides with the original topology of $Aut(G_1) \times Aut(G_2)$.

First, we show that $\overline{I(G_1 \times G_2)} \subset Aut(G_1) \times Aut(G_2)$, i.e., for every $\beta \in \overline{I(G_1 \times G_2)}$ there exist $\beta_i \in Aut(G_i)$ such that $\beta(x_1, x_2) = (\beta_1(x_1), \beta_2(x_2))$, $x_i \in G_i$, $i = 1, 2$. We start by proving that for every fixed $x_1 \in G_1$ there exists a $y_1 \in G_1$ such that if the first coordinate of $x \in G_1 \times G_2$ is equal to x_1, then the first coordinate of

$\beta(x)$ equals y_1. Assume, to the contrary, that there exist $x_1 \in G_1$ and $x_2, \tilde{x}_2 \in G_2$ such that $\beta(x_1, x_2) = (y_1, y_2)$ and $\beta(x_1, \tilde{x}_2) = (\tilde{y}_1, \tilde{y}_2)$, where $y_1 \neq \tilde{y}_1$. Then there exists an open neighborhood U_1 of the identity in G_1 that does not contain $y_1\tilde{y}_1^{-1}$. Since G_1 is an $[FIA]$ group, it must be also a $[SIN]$ group, and so we can choose an open symmetric neighborhood V_1 of the identity in G_1 such that V_1 is invariant under $I(G_1)$, $\overline{V}_1$ is compact, and $\overline{V}_1^2 \subset U_1$. We can also choose a compact set $K_2 \subset G_2$ so that both x_2 and $\tilde{x}_2$ belong to K_2. Since $\beta \in \overline{I(G_1 \times G_2)}$, there exists an $\alpha \in I(G_1 \times G_2)$ such that $(\alpha^{-1}\beta)x \in (V_1 \times G_2)x$ for every $x \in (\overline{V}_1 x_1) \times K_2$. Note that, in particular, both (x_1, x_2) and $(x_1, \tilde{x}_2)$ belong to $(\overline{V}_1 x_1) \times K_2$. Let α be represented by $(g_1, g_2) \in G_1 \times G_2$. Then the first coordinate of $(\alpha^{-1}\beta)(x_1, x_2)$ is equal to $g_1 y_1 g_1^{-1}$, and the first coordinate of $(\alpha^{-1}\beta)(x_1, \tilde{x}_2)$ equals $g_1\tilde{y}_1 g_1^{-1}$. Both these first coordinates must belong to $V_1 x_1$, implying that

$$g_1 y_1 \tilde{y}_1^{-1} g_1^{-1} = (g_1 y_1 g_1^{-1})(g_1 \tilde{y}_1 g_1^{-1})^{-1} \in V_1 x_1 (V_1 x_1)^{-1} = V_1 V_1^{-1} = V_1^2,$$

and so $y_1\tilde{y}_1^{-1} \in g_1^{-1} V_1^2 g_1 = V_1^2$ since V_1 is invariant under inner automorphisms. Hence, $y_1\tilde{y}_1^{-1} \in V_1^2 \subset U_1$ by our choice of V_1, and we obtain a contradiction. So, we have proved that for every $x_1 \in G_1$ and every $x_2, \tilde{x}_2 \in G_2$ the first coordinates of $\beta(x_1, x_2)$ and $\beta(x_1, \tilde{x}_2)$ coincide. This allows us to define $\beta_1 \in Aut(G_1)$ by $\beta_1(x_1)$ to be the first coordinate of $\beta(x_1, x_2)$. In the same manner, one can prove that $\beta_2 \in Aut(G_2)$ can be defined by $\beta_2(x_2)$ to be the second coordinate of $\beta(x_1, x_2)$. We then get that $\beta(x_1, x_2) = (\beta_1(x_1), \beta_2(x_2))$.

Finally, since every open neighborhood U of the identity in $G_1 \times G_2$ contains a neighborhood of the form $U_1 \times U_2$, where U_i is a neighborhood of the identity in G_i, and also every compact set K in $G_1 \times G_2$ is included in $K_1 \times K_2$, where $K_i \subset G_i$ is compact, $i = 1, 2$, one can easily see that the restriction of the topology of $Aut(G_1 \times G_2)$ to $Aut(G_1) \times Aut(G_2)$ coincides with the original topology of $Aut(G_1) \times Aut(G_2)$. This completes the proof. $\square$

To extend this proposition to the weighted case, we will need the following technical result.

Lemma 6.5. *Let G be an $[FIA]$ group and $\omega \geq 1$ be a weight on G such that $\omega(gxg^{-1}) \leq c\,\omega(x)$ for some constant $c > 0$ and all $x, g \in G$. Then there exists a continuous projection from $L^1(G,\omega)$ onto $ZL^1(G,\omega)$.*

Proof. We already know that the map $P : L^1(G) \to ZL^1(G)$ defined by

$$(Pf)(x) = \int\limits_{\overline{I(G)}} f(\beta^{-1}x)\, d\beta$$

is a norm one projection from $L^1(G)$ onto $ZL^1(G)$. Since $\omega \geq 1$, we have that $L^1(G,\omega) \subset L^1(G)$, and so P can be restricted to $L^1(G,\omega)$. Therefore, it will be enough to prove that the restricted projection P ranges in $ZL^1(G,\omega)$ and that it is bounded as an operator from $L^1(G,\omega)$ to $ZL^1(G,\omega)$. Since $ZL^1(G,\omega) = L^1(G,\omega) \cap ZL^1(G)$, we only need to show that the restricted P ranges in $L^1(G,\omega)$ and is bounded as an operator from $L^1(G,\omega)$ to $L^1(G,\omega)$. So, let $f \in L^1(G,\omega)$. Then

$$\begin{aligned}\|Pf\|_{L^1(G,\omega)} = \int\limits_G \left| \int\limits_{\overline{I(G)}} f(\beta^{-1}x)\, d\beta \right| \omega(x)\, dx \leq \int\limits_G \int\limits_{\overline{I(G)}} |f(\beta^{-1}x)|\, d\beta\, \omega(x)\, dx = \\ = \int\limits_{\overline{I(G)}} \int\limits_G |f(\beta^{-1}x)| \omega(x)\, dx\, d\beta = \int\limits_{\overline{I(G)}} \int\limits_G |f(x)| \omega(\beta x)\, dx\, d\beta,\end{aligned}$$

where the last equality holds due to the fact that G is an $[FIA]$ group. Indeed, an [FIA] group G is, in particular, an [IN] group, hence, it is unimodular, which implies that the Haar measure is invariant under each $\beta \in \overline{I(G)}$. Using that $\omega(gxg^{-1}) \leq c\,\omega(x)$, $x, g \in G$, we obtain

$$\begin{aligned}\|Pf\|_{L^1(G,\omega)} \leq \int\limits_{\overline{I(G)}} \int\limits_G |f(x)| \omega(\beta x)\, dx\, d\beta = c \int\limits_{\overline{I(G)}} \int\limits_G |f(x)| \omega(x)\, dx\, d\beta = \\ = c \int\limits_{\overline{I(G)}} \|f\|_{L^1(G,\omega)}\, d\beta \leq c\, \mu(\overline{I(G)}) \|f\|_{L^1(G,\omega)},\end{aligned}$$

where μ is a Haar measure on the compact group $\overline{I(G)}$. The last inequality ensures both that $Pf \in L^1(G,\omega)$ and that P is bounded as an operator from $L^1(G,\omega)$ to $L^1(G,\omega)$. $\square$

Corollary 6.6. *Let G_1 and G_2 be $[FIA]$ groups, and $\omega_i \geq 1$ be a weight on G_i invariant under $\overline{I(G_i)}$, $i = 1, 2$. Then $ZL^1(G_1 \times G_2, \omega_1 \times \omega_2) \cong ZL^1(G_1, \omega_1)\hat{\otimes}ZL^1(G_2, \omega_2)$.*

The proof of this corollary follows exactly the same way as the one of Proposition 6.4. Note that the projection is asserted by Lemma 6.5.

Our next goal is to characterize weak amenability of $ZL^1(G,\omega)$ in the case when G is a connected [SIN] group. For this we need one more lemma.

Lemma 6.7. *Let $G = V \times K$ be a direct product of an Abelian group V and a compact group K. Further let $\omega \geq 1$ be a weight on G. Then $ZL^1(G,\omega)$ is topologically isomorphic to $L^1(V,\hat{\omega})\hat{\otimes}ZL^1(K)$, where $\hat{\omega}(v) = \omega(v, e_K)$, and e_K is the identity of K.*

Proof. First, we note that $\hat{\omega}$ is, obviously, a weight on V. Secondly, we show that the weight ω on $G = V \times K$ is equivalent to the weight $\tilde{\omega}$ on G defined by $\tilde{\omega}(v,k) = \hat{\omega}(v)$, $v \in V$, $k \in K$. Since K is compact, so is the subset $\{e_V\} \times K$ of G, where e_V denotes the identity of V. Because ω is assumed continuous, this implies the existence of a constant $M > 0$ such that $\sup_{k\in K} \omega(e_V, k) = M < \infty$. Moreover, we can make the following estimates for all $(v,k) \in G$:

$$\tilde{\omega}(v,k) = \hat{\omega}(v) = \omega(v, e_K) = \omega(v \cdot e_V, k \cdot k^{-1}) \leq \omega(v,k)\omega(e_V, k^{-1}) \leq M\omega(v,k) \quad \text{and}$$

$$\omega(v,k) = \omega(v \cdot e_V, e_K \cdot k) \leq \omega(v, e_K)\omega(e_v, k) = \tilde{\omega}(v,k)\omega(e_V, k) \leq M\tilde{\omega}(v,k).$$

Therefore,

$$\frac{1}{M} \leq \frac{\tilde{\omega}(v,k)}{\omega(v,k)} \leq M, \quad (v,k) \in G.$$

This proves the equivalence of ω and $\tilde{\omega}$. Hence, we have that $ZL^1(G,\omega) \cong ZL^1(G,\tilde{\omega})$. Note that $\tilde{\omega}(v,k) = \hat{\omega}(v)$ can be regarded as the product of the weight $\hat{\omega}$ on V and the constant weight on K. Since V is Abelian, all inner automorphisms of V are trivial which makes $\hat{\omega}$ invariant under $\overline{I(V)}$. The constant weight on K is also invariant under $\overline{I(K)}$. So, by Corollary 6.6, $ZL^1(V \times K, \tilde{\omega}) \cong ZL^1(V, \hat{\omega})\hat{\otimes}ZL^1(K)$. Since $G = V \times K$, $ZL^1(G, \omega)$ is topologically isomorphic to $ZL^1(G, \tilde{\omega})$, and V is Abelian, we finally obtain that $ZL^1(G, \omega)$ is topologically isomorphic to $ZL^1(V, \hat{\omega})\hat{\otimes}ZL^1(K) = L^1(V, \hat{\omega})\hat{\otimes}ZL^1(K)$. The lemma is proved. □

According to [16, Theorem 4.3], if G is a connected $[SIN]$ group, then G can be written as a direct product of a vector group V and a compact group K. So, we can use Lemma 6.7 to prove the following result.

Theorem 6.8. *Let G be a connected $[SIN]$ group, and ω be a weight on G. Then $ZL^1(G, \omega)$ is weakly amenable if and only if there is no non-trivial continuous group homomorphism $\Phi : G \to \mathbb{C}$ such that*

$$\sup_{g \in G} \frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} < \infty. \tag{6.2}$$

Proof. As noted above, we can write G in the form $G = V \times K$ for some vector group V and compact group K. Then, in particular, G is amenable, and we can apply Remark 2.5 to assume without loss of generality that $\omega \geq 1$, because replacing the weight ω with the quotient ω/ϕ for any continuous positive character $\phi : G \to (\mathbb{R}^+, \times)$ does not change the product $\omega(g)\omega(g^{-1})$, $g \in G$. Since any vector group is, obviously, Abelian, we can apply Lemma 6.7 to get that

$$ZL^1(G, \omega) \simeq L^1(V, \hat{\omega})\hat{\otimes}ZL^1(K),$$

where $\hat{\omega}(v) = \omega(v, e_K)$, and e_K is the identity of K. Because all algebras $ZL^1(G, \omega)$, $L^1(V, \hat{\omega})$, and $ZL^1(K)$ are Abelian, $ZL^1(G, \omega)$ is weakly amenable if both $L^1(V, \hat{\omega})$ and $ZL^1(K)$ are weakly amenable due to [13, Proposition 2.6]. As was proved in [1] and [41], $ZL^1(K)$ is weakly amenable for any compact group K. Hence, $ZL^1(G, \omega)$ is weakly amenable if $L^1(V, \hat{\omega})$ is weakly amenable. Conversely, the operator T :

$L^1(V,\hat{\omega})\hat{\otimes}ZL^1(K) \to L^1(V,\hat{\omega})$ defined by

$$T(f \otimes g) = \int_K g(x)\,dx\,f, \quad f \in L^1(V,\hat{\omega}), g \in ZL^1(K),$$

is a Banach algebra epimorphism between Abelian algebras $L^1(V,\hat{\omega})\hat{\otimes}ZL^1(K) \simeq ZL^1(G,\omega)$ and $L^1(V,\hat{\omega})$. Therefore, by [7, Proposition 2.8.64(iii)], $L^1(V,\hat{\omega})$ is weakly amenable if so is $ZL^1(G,\omega)$. So we have that $ZL^1(G,\omega)$ is weakly amenable if and only if $L^1(V,\hat{\omega})$ is weakly amenable. Since V is Abelian, we know from Theorem 1.12 that $L^1(V,\hat{\omega})$ is weakly amenable if and only if there is no non-trivial continuous group homomorphism $\hat{\Phi} : V \to \mathbb{C}$ such that

$$\sup_{v\in V} \frac{|\hat{\Phi}(v)|}{\hat{\omega}(v)\hat{\omega}(v^{-1})} < \infty. \tag{6.3}$$

So, the only thing left is to prove the equivalence of the existence of a non-trivial continuous group homomorphism $\hat{\Phi} : V \to \mathbb{C}$ such that (6.3) holds and the existence of a non-trivial continuous group homomorphism $\Phi : G \to \mathbb{C}$ such that

$$\sup_{g\in G} \frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} < \infty.$$

For this, we note that every group homomorphism $\hat{\Phi} : V \to \mathbb{C}$ can be extended to $\Phi : G \to \mathbb{C}$ by $\Phi(v,k) = \hat{\Phi}(v)$, and vice versa — every group homomorphism $\Phi : G \to \mathbb{C}$ has the form $\Phi(v,k) = \hat{\Phi}(v)$ for some group homomorphism $\hat{\Phi} : V \to \mathbb{C}$ since K is compact. Also, as we have proved in Lemma 6.7, the weight $\tilde{\omega}(v,k) = \hat{\omega}(v)$ on G is equivalent to ω. These facts immediately lead to the desired equivalence. □

6.2 Weak amenability of $ZL^1(G,\omega)$ for [FD] group G

We start by proving the following almost obvious characterization of [FD] groups.

Lemma 6.9. *Let G be a locally compact group. Then G is an [FD] group if and only if there exists a compact normal subgroup K of G such that the quotient G/K is Abelian.*

Proof. Suppose first that G is an [FD] group. Then, by definition, its commutator subgroup G' has a compact closure $\overline{G'}$ in G. Since G' is a normal subgroup of G, and the quotient group G/G' is Abelian, we can take $K = \overline{G'}$. Conversely, let K be a compact normal subgroup of G such that the quotient G/K is Abelian. Then by the fundamental property of a commutator subgroup, G' must be contained in K. Since K is compact in G, we automatically obtain that $\overline{G'} \subset K$ is also compact, and so G is an [FD] group. □

We will also make use of the following structural result.

Lemma 6.10. [32, Lemma 1 (applied for $B = I(G)$)] *Suppose a locally compact group G contains a compact normal subgroup K such that the quotient group G/K is Abelian. Let $\omega \geq 1$ be a weight function on G satisfying $\lim_{n\to\infty}(\omega(x^n))^{1/n} = 1$ for all $x \in G$, and $\hat{\omega}$ be the induced weight on G/K defined by $\hat{\omega}([x]) = \inf_{t\in[x]} \omega(t)$, $x \in G$. Then $ZL^1(G,\omega)$ may be written as the closure of the linear span of a family of complemented ideals, each of which is isomorphic to a Beurling algebra $L^1(S/K,\hat{\omega})$, where $S \supset K$ is an open normal subgroup of G.*

Remark 6.11. *As follows from Lemma 6.9, Lemma 6.10 holds precisely for [FD] groups G.*

It is well-known that if an Abelian Banach algebra can be written as a closed span of closed subalgebras each of which is weakly amenable, then the algebra itself must be weakly amenable. We include a proof of this fact here for the sake of completeness.

Lemma 6.12. *Let A be a commutative Banach algebra, and $\{A_\gamma\}_{\gamma\in\Gamma}$ be a family of closed subalgebras of A such that $A = \overline{\mathrm{lin}}\{A_\gamma\}_{\gamma\in\Gamma}$ and each A_γ is weakly amenable. Then A is also weakly amenable.*

Proof. Let $D : A \to A^*$ be a bounded derivation. We prove that D must be trivial. Since the span of $\{A_\gamma\}_{\gamma\in\Gamma}$ is dense in A, and D is a continuous operator, it is enough to show that D equals zero on each A_γ. Consider the restriction $D|_{A_\gamma}$. Obviously, it is a bounded derivation from A_γ to A^*. Because A_γ is Abelian and weakly amenable, and A^* is a symmetric Banach A_γ-bimodule, we obtain that $D|_{A_\gamma}$ must be zero according to Definition 1.8. This completes the proof. □

Since $ZL^1(G,\omega)$ and all algebras $L^1(S/K,\hat{\omega})$ in Lemma 6.10 are Abelian, we can use the last two lemmas together with Theorem 1.12 to study weak amenability of $ZL^1(G,\omega)$ in the case when G is an [FD] group. We need one more simple technical lemma.

Lemma 6.13. *Let $\omega \geq 1$ be a weight on a locally compact group G, K be a compact normal subgroup of G, and $\hat{\omega}([x]) = \inf_{k\in K} \omega(xk)$ be the induced weight on G/K. Then there exist constants $c_1, c_2 > 0$ such that $c_1\omega(x) \leq \hat{\omega}([x]) \leq c_2\omega(x)$, $x \in G$.*

Proof. In fact,

$$\hat{\omega}([x]) = \inf_{k\in K} \omega(xk) \leq \omega(x) \inf_{k\in K} \omega(k) \quad \text{and} \quad \hat{\omega}([x]) = \inf_{k\in K} \omega(xk) \geq \omega(x) \inf_{k\in K} \frac{1}{\omega(k^{-1})}.$$

So we can take

$$c_1 = \inf_{k\in K} \frac{1}{\omega(k^{-1})} = \frac{1}{\sup_{k\in K^{-1}} \omega(k)} \quad \text{and} \quad c_2 = \inf_{k\in K} \omega(k).$$

□

Now we are ready to prove the following.

Theorem 6.14. *Let G be an [FD] group, and $\omega \geq 1$ be a weight on G satisfying*

$$\sup_{n\in\mathbb{N}} \frac{n}{\omega(x^n)\omega(x^{-n})} = \infty, \qquad x \in G. \tag{6.4}$$

Then $ZL^1(G,\omega)$ is weakly amenable.

Proof. First we show that (6.4) implies that $\lim_{n\to\infty}(\omega(x^n))^{1/n} = 1$ for every $x \in G$. Since $\omega \geq 1$, it is enough to prove that

$$\limsup_{n\to\infty}(\omega(x^n))^{1/n} \leq 1, \quad x \in G.$$

Fix $x \in G$ and let $\varepsilon > 0$ be arbitrary. Because $\lim_{n\to\infty} n^{1/n} = 1$, there exists $N_\varepsilon \in \mathbb{N}$ such that $n^{1/n} \leq (1+\varepsilon)$ for every $n \geq N_\varepsilon$. Using the assumption (6.4) and the inequality $\omega \geq 1$, we can find $n_\varepsilon > N_\varepsilon$ such that

$$\omega(x^{n_\varepsilon}) \leq \omega(x^{n_\varepsilon})\omega(x^{-n_\varepsilon}) \leq n_\varepsilon = (n_\varepsilon^{1/n_\varepsilon})^{n_\varepsilon} \leq (1+\varepsilon)^{n_\varepsilon}.$$

Consider any $m \in \mathbb{N}$. There exist $k \in \mathbb{N} \cup \{0\}$ and $0 \leq l < n_\varepsilon$ such that $m = kn_\varepsilon + l$. Using the weight inequality for ω, we can make the following estimates:

$$\omega(x^m) = \omega(x^{kn_\varepsilon + l}) \leq \left(\omega\left(x^{n_\varepsilon}\right)\right)^k \omega(x^l) \leq (1+\varepsilon)^{kn_\varepsilon}\omega(x^l) \leq \frac{(1+\varepsilon)^m \omega(x^l)}{(1+\varepsilon)^l} \leq c_\varepsilon (1+\varepsilon)^m,$$

where

$$c_\varepsilon = \sup_{0 \leq l < n_\varepsilon} \frac{\omega(x^l)}{(1+\varepsilon)^l}$$

is a constant that does not depend on m. The last inequality implies that

$$\limsup_{n\to\infty} (\omega(x^n))^{1/n} \leq \limsup_{n\to\infty} (c_\varepsilon (1+\varepsilon)^n)^{1/n} = 1+\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we obtain that $\limsup_{n\to\infty} (\omega(x^n))^{1/n} \leq 1$, $x \in G$, as desired.

According to Lemma 6.9, there exists a compact normal subgroup K of G such that the quotient G/K is Abelian. So, the conditions of Lemma 6.10 are satisfied, which means that there exists a family of complemented ideals $\{J_\gamma\}$ of $ZL^1(G, \omega)$ such that the span of $\{J_\gamma\}$ is dense in $ZL^1(G, \omega)$ and for each γ there exists an open subgroup $S_\gamma \supset K$ of G for which $J_\gamma \cong L^1(S_\gamma/K, \hat{\omega})$. Fix any γ. Since G/K is Abelian, S_γ/K is also Abelian. For any non-trivial continuous group homomorphism $\Phi : S_\gamma/K \to \mathbb{C}$ let $t_\gamma \in S_\gamma/K$ be such that $\Phi(t_\gamma) \neq 0$. Then

$$\sup_{t \in S_\gamma/K} \frac{|\Phi(t)|}{\hat{\omega}(t)\hat{\omega}(t^{-1})} \geq \sup_{n\in\mathbb{N}} \frac{|\Phi(t_\gamma^n)|}{\hat{\omega}(t_\gamma^n)\hat{\omega}(t_\gamma^{-n})} = \sup_{n\in\mathbb{N}} \frac{n\,|\Phi(t_\gamma)|}{\hat{\omega}(t_\gamma^n)\hat{\omega}(t_\gamma^{-n})}.$$

Let $x_\gamma \in S_\gamma$ be a representative of $t_\gamma \in S_\gamma/K$. By Lemma 6.13, we have that

$$\hat{\omega}(t_\gamma^n) \leq c_2 \omega(x_\gamma^n) \quad \text{and} \quad \hat{\omega}(t_\gamma^{-n}) \leq c_2 \omega(x_\gamma^{-n}),$$

thus

$$\sup_{n\in\mathbb{N}} \frac{n}{\hat{\omega}(t_\gamma^n)\hat{\omega}(t_\gamma^{-n})} \geq \frac{1}{c^2} \sup_{n\in\mathbb{N}} \frac{n}{\omega(x_\gamma^n)\omega(x_\gamma^{-n})} = \infty.$$

This shows that

$$\sup_{t\in S_\gamma/K} \frac{|\Phi(t)|}{\hat{\omega}(t)\hat{\omega}(t^{-1})} = \infty$$

for each non-trivial continuous group homomorphism $\Phi : S_\gamma/K \to \mathbb{C}$. Then $J_\gamma \cong L^1(S_\gamma/K, \hat{\omega})$ is weakly amenable by Theorem 1.12, and so $ZL^1(G,\omega)$ is weakly amenable by Lemma 6.12. □

Let G be a compactly generated group. Then there is an open symmetric neighborhood of identity U in G with compact closure such that $G = \bigcup_{n=1}^\infty U^n$. Following [32], we consider the length function $|\cdot| : G \to \mathbb{N}$ defined by

$$|x| = \min\{n \in \mathbb{N} : x \in U^n\}, \quad x \in G.$$

It is readily checked that the corresponding polynomial weight $\omega_\alpha(x) = (1+|x|)^\alpha$, $\alpha > 0$, $x \in G$, is indeed a continuous weight on G. If, in addition, G is an [FC] group, we can obtain the following corollary from Theorem 6.14.

Corollary 6.15. *Let G be a compactly generated [FC] group and ω_α be the weight on G defined as above. Then $ZL^1(G,\omega_\alpha)$ is weakly amenable if $0 < \alpha < 1/2$.*

Proof. According to [16, Theorem 3.20], a compactly generated [FC] group is an [FD] group. So, if we verify (6.4) for ω_α in the case $0 < \alpha < 1/2$, the result will follow immediately from Theorem 6.14. By the definition of $|\cdot|$, it is obvious that $|x^{-1}| = |x|$ and $|x^n| \leq n|x|$ for every $x \in G$, $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}\sup_{n\in\mathbb{N}} \frac{n}{\omega_\alpha(x^n)\omega_\alpha(x^{-n})} &= \sup_{n\in\mathbb{N}} \frac{n}{\omega(x^n)\omega(x^{-n})} = \sup_{n\in\mathbb{N}} \frac{n}{(1+|x^n|)^\alpha(1+|x^{-n}|)^\alpha} \\ &\geq \sup_{n\in\mathbb{N}} \frac{n}{(1+n|x|)^{2\alpha}} = \infty, \quad x \in G,\end{aligned}$$

since $\alpha < 1/2$. The proof is completed. □

It is easy to see that the condition (6.4) in Theorem 6.14 and in the corollaries above is, in general, stronger than the condition (6.2) in Theorem 6.8 (see, for example, [41, Corollary 3.6]). But as is shown in [41], these conditions are equivalent for the group $\mathbb{R}$ and, similarly, for $\mathbb{Z}$.

Proposition 6.16. [41, Corollary 3.7] *Let ω be a weight on $\mathbb{R}$. Then the following statements are equivalent:*

(1) *the Beurling algebra $L^1(\mathbb{R}, \omega)$ is weakly amenable.*

(2) $\sup_{t\in\mathbb{R}} \dfrac{|\phi(t)|}{\omega(t)\omega(t^{-1})} = \infty$ *for every continuous nonzero group homomorphism $\phi:\ \mathbb{R} \to \mathbb{C}$.*

(3) $\sup_{n\in\mathbb{N}} \dfrac{n}{\omega(t^n)\omega(t^{-n})} = \infty$ *for all $t \in \mathbb{R}$.*

This leads to the following corollary from Theorem 6.14.

Corollary 6.17. *Let G be a locally compact group and K be a compact normal subgroup of G such that $G/K \cong \mathbb{R}$ or $G/K \cong \mathbb{Z}$. Suppose that $\omega \geq 1$ is a weight on G such that there is no non-trivial continuous group homomorphism $\Phi :\ G \to \mathbb{C}$ for which*

$$\sup_{t\in G} \frac{|\Phi(t)|}{\omega(t)\omega(t^{-1})} < \infty. \tag{6.5}$$

Then $ZL^1(G, \omega)$ is weakly amenable.

Proof. We first show that the condition that there is no non-trivial continuous group homomorphism $\Phi : G \to \mathbb{C}$ for which (6.5) holds is equivalent to weak amenability of $L^1(G/K, \hat{\omega})$. From Proposition 6.16 we know that $L^1(G/K, \hat{\omega})$ is weakly amenable if and only if there is no non-trivial continuous group homomorphism $\phi : G/K \to \mathbb{C}$ for which

$$\sup_{t\in G/K} \frac{|\phi(t)|}{\hat{\omega}(t)\hat{\omega}(t^{-1})} < \infty. \tag{6.6}$$

We note the following relation between continuous group homomorphisms $\Phi : G \to \mathbb{C}$ and $\phi : G/K \to \mathbb{C}$: every Φ can be written in the form $\Phi = \phi \circ q$, where $q : G \to G/K$ is the quotient map, and, conversely, for every ϕ the map $\phi \circ q$ is a continuous group homomorphism from G to $\mathbb{C}$. This follows easily from the compactness of K and the continuity of the quotient map q. We also note that the weights ω and $\hat{\omega}$ are equivalent due to Lemma 6.13. All these immediately lead to the equivalence of (6.5) and (6.6).

Since $G/K \cong \mathbb{R}$ or $G/K \cong \mathbb{Z}$, it follows from Proposition 6.16 that (6.6) implies

$$\sup_{n\in\mathbb{N}} \frac{n}{\hat{\omega}(t^n)\hat{\omega}(t^{-n})} = \infty, \quad t \in G/K.$$

By Lemma 6.13, the last condition leads to

$$\sup_{n\in\mathbb{N}} \frac{n}{\omega(x^n)\omega(x^{-n})} = \infty, \quad x \in G.$$

Then, applying Theorem 6.14, we conclude that $ZL^1(G,\omega)$ is weakly amenable. $\square$

The proof of Theorem 6.14 leads to the following.

Proposition 6.18. *Let G be an [FD] group and $\omega \geq 1$ be a weight on G satisfying $\lim_{n\to\infty} (\omega(x^n))^{1/n} = 1$ for every $x \in G$. Then $ZL^1(G,\omega)$ is weakly amenable if and only if each ideal J_γ ensured in Lemma 6.10 is weakly amenable.*

Proof. Let K be a compact normal subgroup of G such that G/K is Abelian. The sufficiency follows directly from Lemma 6.12. Suppose, conversely, that $ZL^1(G,\omega)$ is weakly amenable. Since each J_γ is a closed ideal in $ZL^1(G,\omega)$, according to [7, Theorem 2.8.69], J_γ is weakly amenable if and only if it is essential, i.e., $\overline{J_\gamma^2} = J_\gamma$. But this immediately follows from the facts that $J_\gamma \cong L^1(S_\gamma/K,\hat{\omega})$ and that $L^1(S_\gamma/K,\hat{\omega})$ has a bounded approximate identity. $\square$

6.3 A necessary condition for weak amenability of $ZL^1(G,\omega)$ on [FC] groups

According to [27, Proposition 3.1], if G is an [FC] group, then it is also an [IN] group. So the centre algebra $ZL^1(G,\omega)$ is non-trivial. We reveal a property of [FC] groups that we will use later.

Lemma 6.19. *Let $G \in$ [FC]. Then for every $x \in G$ there exists a compact set $K_x \subset G$ invariant under inner automorphisms of G, and such that x belongs to the interior of K_x.*

Proof. Let $x \in G$ be fixed. Since G is an [FC] group, the set $C_x = \overline{\{gxg^{-1} : g \in G\}}$ is compact as the closure of a conjugacy class. Also, C_x is, obviously, invariant under inner automorphisms. Because G is an [IN] group, there exists a compact invariant neighborhood U of identity. We claim that the set $K_x = C_x \cdot U$ satisfies all our requirements. Indeed, K_x is compact as a product of two compact sets, and $xU \subset K_x$, which means that x belongs to the interior of K_x. Finally, K_x is invariant under inner automorphisms. To see this, let $g \in G$ and $y \in K_x$. Then $y = cu$, where $c \in C_x$ and $u \in U$, and so

$$gyg^{-1} = g(cu)g^{-1} = (gcg^{-1})(gug^{-1}) \in C_xU = K_x,$$

since both C_x and U are invariant under inner automorphisms. This proves that K_x is invariant under inner automorphisms. The proof is complete. □

We now can give a necessary condition for weak amenability of the centre algebra $ZL^1(G,\omega)$ in the case when G is an [FC] group.

Proposition 6.20. *Let G be a locally compact [FC] group and ω be a weight on G. Suppose that there exists a non-trivial continuous group homomorphism $\Phi : G \to \mathbb{C}$ such that*

$$\sup_{t\in G} \frac{|\Phi(t)|}{\omega(t)\omega(t^{-1})} < \infty.$$

Then $ZL^1(G,\omega)$ is not weakly amenable.

Proof. We note that the Banach algebra $ZL^1(G,\omega)$ is commutative and

$$ZL^\infty(G,1/\omega) = \{f \in L^\infty(G,1/\omega) : f \circ \beta = f,\ \beta \in I(G)\}$$

is a symmetric $ZL^1(G,\omega)$-bimodule. So, according to the Definition 1.8, to prove that $ZL^1(G,\omega)$ is not weakly amenable, it suffices to construct a non-trivial continuous derivation $D : ZL^1(G,\omega) \to ZL^\infty(G,1/\omega)$. Since Φ is non-trivial, there exists $x \in G$ such that $\Phi(x) \neq 0$. We now apply Lemma 6.19 to get a compact set K_x invariant under inner automorphisms and whose interior contains the point x. Because of the continuity of Φ, there exists a neighborhood U_x of x contained in K_x, and such that Φ is bounded away from zero on U_x. We then define D as follows:

$$D(h)(t) = \int_{K_x} \Phi(t^{-1}\xi)h(t^{-1}\xi)\, d\xi, \quad t \in G, \quad h \in ZL^1(G,\omega).$$

First we note that D is non-trivial. Indeed, we can use the argument from [41, Remark 3.2] and take $h_\Phi = \overline{\Phi} \cdot \chi_{K_x}$, where χ_{K_x} is a characteristic function of K_x. Then h_Φ belongs to $ZL^1(G,\omega)$ since Φ is a homomorphism, and K_x is invariant under inner automorphisms. Moreover,

$$D(h_\Phi)(t) = \int\limits_{K_x} \Phi(t^{-1}\xi)h_\Phi(t^{-1}\xi)\, d\xi = \int\limits_{K_x \cap tK_x} |\Phi(t^{-1}\xi)|^2\, d\xi = \int\limits_{t^{-1}K_x \cap K_x} |\Phi(\xi)|^2\, d\xi.$$

From the formula above we see that $D(h_\Phi)(t) > 0$ for t in a neighborhood of identity, since Φ is bounded away from zero on $U_x \subset K_x$. Hence, D is non-trivial. The same argument as in the proof of Theorem 4.3 shows that D is a bounded derivation from $ZL^1(G,\omega)$ to $L^\infty(G,1/\omega)$. So to complete the proof we only need to show that D ranges in $ZL^\infty(G,1/\omega)$, or, equivalently, that $D(h)$ is invariant under inner automorphisms of G for each $h \in ZL^1(G,\omega)$. Fix any $g \in G$. Using the facts that K_x is invariant under inner automorphisms and that G is unimodular (since it is an [IN] group), we obtain

$$\begin{aligned} D(h)(gtg^{-1}) &= \int\limits_{K_x} \Phi(gt^{-1}g^{-1}\xi)h(gt^{-1}g^{-1}\xi)\, d\xi \\ &= \int\limits_{K_x} \Phi(gt^{-1}(\underbrace{g^{-1}\xi g}_{\zeta})g^{-1})h(gt^{-1}(g^{-1}\xi g)g^{-1})\, d\xi \\ &= \int\limits_{K_x} \Phi(gt^{-1}\zeta g^{-1})h(gt^{-1}\zeta g^{-1})\, d\zeta, \quad t \in G. \end{aligned} \tag{6.7}$$

Because Φ is a homomorphism, $\Phi(gt^{-1}\zeta g^{-1}) = \Phi(t^{-1}\zeta)$. Also, since $h \in ZL^1(G,\omega)$, we have that $h(gzg^{-1}) = h(z)$ for almost all $z \in G$. Taking these observations into account, we finally get the following from (6.7):

$$D(h)(gtg^{-1}) = \int\limits_{K_x} \Phi(gt^{-1}\zeta g^{-1})h(g(t^{-1}\zeta)g^{-1})\, d\zeta = \int\limits_{K_x} \Phi(t^{-1}\zeta)h_\Phi(t^{-1}\zeta)\, d\zeta = D(h)(t).$$

Therefore, $D(h) \in ZL^\infty(G, 1/\omega)$, and the proposition is proved. $\square$

Remark 6.21. We only used the condition that G is an [FC] group to obtain the invariant compact set K_x on which Φ is non-trivial. The same idea still works to prove the following (cf. [41, Remark 3.2]).

Proposition 6.22. *Let G be an [IN] group, ω be a weight on G, and U be a compact neighborhood of identity invariant under inner automorphisms of G. Suppose that there exists a continuous group homomorphism $\Phi : G \to \mathbb{C}$ nontrivial on U and such that*

$$\sup_{t\in G} \frac{|\Phi(t)|}{\omega(t)\omega(t^{-1})} < \infty.$$

Then $ZL^1(G,\omega)$ is not weakly amenable.

Chapter 7

Derivation problem on Beurling algebras

In this chapter we consider the following analogue of the derivation problem on Beurling algebras.

Question 7.1. *Let G be a locally compact group and ω be a weight on G. Does every bounded derivation $D : L^1(G, \omega) \to M(G, \omega)$ have to be inner?*

Note that in comparison with the original derivation problem, we have an extra restriction on D, i.e., it is bounded. We remark that in [21] it was proved that every derivation on $L^1(G)$ must be continuous, so the original derivation problem is actually concerned with continuous derivations.

As we have already mentioned in the Introduction, our goal will be to give an affirmative answer to Question 7.1 in the case when the weight ω is diagonally bounded. Following the idea from [3] that gives a simple and elegant solution to the original derivation problem, we will use the following fixed point theorem.

Theorem 7.2. [3, Theorem A] *Let A be a non-empty bounded subset of an L-embedded Banach space V. Then there is a point in V fixed for every isometry of V preserving A.*

Here an L-embedded Banach space V is a space such that its bidual V^{**} admits a decomposition $V^{**} = V \oplus_1 V_0$ for some $V_0 \subset V^{**}$, where $\oplus_1$ indicates that the norm on V^{**} is the sum of the norms on V and V_0. According to [37, III.2.14], any von Neumann algebra and, in particular, a dual of any C^*-algebra is L-embedded. We will use Theorem 7.2 for a subset of $M(G,\omega)$, which is the dual of the C^*-algebra $C_0(G,1/\omega)$, and hence, is L-embedded. The isometric self maps will be actions of the group G on $M(G,\omega)$. We also remark here that Theorem 7.2 has been extended to non-expansive semigroup mappings on L-embedded sets of Banach spaces in [24].

We will need the following technical result.

Lemma 7.3. *Let ω be a diagonally bounded weight on a locally compact group G. Then the function ω' defined by*

$$\omega'(x) = \sup_{g\in G} \omega(gxg^{-1}), \quad g \in G,$$

is a weight on G equivalent to ω, and it satisfies the relation

$$\omega'(gxg^{-1}) = \omega'(x), \quad x, g \in G. \tag{7.1}$$

Proof. We first note that since ω is diagonally bounded, there is a constant $M > 0$, such that $\omega(g)\omega(g^{-1}) \leq M$ for every $g \in G$. Therefore,

$$\omega(gxg^{-1}) \leq \omega(x)(\omega(g)\omega(g^{-1})) \leq M\omega(x), \quad x, g \in G,$$

which means that the supremum in the definition of ω' is finite for every $x \in G$, and so ω' is well-defined. Since, by definition, ω' is a pointwise supremum of the set of continuous functions $\{\omega_g\}_{g\in G}$ defined by $\omega_g(x) = \omega(gxg^{-1})$, $x \in G$, we can apply Lemma 2.6 to conclude that ω' is a measurable function. Now we check the weight inequality for ω':

$$\begin{aligned}\omega'(xy) &= \sup_{g\in G} \omega(g(xy)g^{-1}) = \sup_{g\in G} \omega((gxg^{-1})(gyg^{-1})) \leq \sup_{g\in G} \omega(gxg^{-1})\omega(gyg^{-1}) \\ &\leq \sup_{g\in G} \omega(gxg^{-1}) \cdot \sup_{g\in G} \omega(gyg^{-1}) = \omega'(x)\omega'(y).\end{aligned}$$

So, ω' is, indeed, a weight on G. We also have that

$$\omega(x) = \omega(exe^{-1}) \leq \sup_{g\in G} \omega(gxg^{-1}) = \omega'(x),$$

and since we have already shown that $\omega'(x) \leq M\omega(x)$, we obtain that ω' is equivalent to ω. Finally,

$$\omega'(gxg^{-1}) = \sup_{k\in G} \omega((kg)x(kg)^{-1}) = \sup_{\tilde{g}\in G} \omega(\tilde{g}x\tilde{g}^{-1}) = \omega'(x), \quad x, g \in G,$$

and the relation (7.1) is proved. □

Now we can prove the following.

Proposition 7.4. *Let G be a locally compact group and $\omega \geq 1$ be a diagonally bounded weight on G. Then every continuous derivation $D : L^1(G,\omega) \to M(G,\omega)$ is inner.*

Proof. We start by noting that as in the case $\omega \equiv 1$, D must map into $L^1(G,\omega)$. Indeed, since $L^1(G,\omega)$ has a bounded approximate identity, by Cohen's Factorization Theorem (see, for example, [8, Theorem 2.3]) every $f \in L^1(G,\omega)$ can be written as $f = f_1 * f_2$, for some $f_1, f_2 \in L^1(G,\omega)$. So,

$$D(f) = D(f_1 * f_2) = f_1 * D(f_2) + D(f_1) * f_2,$$

and because $L^1(G,\omega)$ is an ideal in $M(G,\omega)$, we obtain that $D(f) \in L^1(G,\omega)$.

Denote $M = \sup_{g\in G} \omega(g)\omega(g^{-1})$, and let ω' be the weight from Lemma 7.3. Since ω' is equivalent to ω, we have that $L^1(G,\omega') \simeq L^1(G,\omega)$ and $M(G,\omega') \simeq M(G,\omega)$, which means that we can view D as a continuous derivation from $L^1(G,\omega')$ to $M(G,\omega')$. Because on the one hand $M(G,\omega)$ is a dual of an essential $L^1(G,\omega)$-bimodule $C_0(G,1/\omega)$, and on the other hand $M(G,\omega) = \mathcal{M}(L^1(G,\omega))$, we can apply Proposition 2.2 to extend D to a bounded derivation $\widetilde{D} : M(G,\omega') \to M(G,\omega')$, which is continuous in SO-w* topology.

Consider the function $b : G \to M(G,\omega')$ defined as $b(g) = \widetilde{D}(\delta_g) * \delta_{g^{-1}}$, where $\delta_g \in M(G,\omega')$ is a point mass at g. We claim that b satisfies the relation

$$b(xy) = b(x) + x \cdot b(y), \quad x, y \in G,$$

with respect to the action $x \cdot \mu = \delta_x * \mu * \delta_{x^{-1}}$ of G on $M(G,\omega')$, and the set $b(G) = \{b(g) : g \in G\}$ is bounded (such functions are called *crossed homomorphisms*). Indeed,

$$\begin{aligned} b(xy) &= \widetilde{D}(\delta_{xy}) * \delta_{(xy)^{-1}} = \widetilde{D}(\delta_x * \delta_y) * \delta_{y^{-1}x^{-1}} = (\widetilde{D}(\delta_x) * \delta_y + \delta_x * \widetilde{D}(\delta_y)) * \delta_{y^{-1}} * \delta_{x^{-1}} \\ &= \widetilde{D}(\delta_x) * \delta_{x^{-1}} + \delta_x * (\widetilde{D}(\delta_y) * \delta_{y^{-1}}) * \delta_{x^{-1}} = b(x) + x \cdot b(y) \end{aligned}$$

and

$$\|b(g)\| = \|\widetilde{D}(\delta_g) * \delta_{g^{-1}}\| \le \|\widetilde{D}\| \cdot \|\delta_g\|_{\omega'} \cdot \|\delta_{g^{-1}}\|_{\omega'} = \|\widetilde{D}\| \cdot \omega'(g)\omega'(g^{-1}),$$

which is bounded for all $g \in G$ since ω' is equivalent to ω, and ω is diagonally bounded.

Using b we can define an action of G on $M(G,\omega')$ in the following way:

$$g(\mu) = g \cdot \mu + b(g) = \delta_g * \mu * \delta_{g^{-1}} + b(g), \quad g \in G,\ \mu \in M(G,\omega').$$

We claim that this action is isometric:

$$\begin{aligned} \|g(\mu_1) - g(\mu_2)\|_{M(G,\omega')} &= \|g \cdot (\mu_1 - \mu_2)\|_{M(G,\omega')} = \int_G \omega'(x)\, d|\delta_g * (\mu_1 - \mu_2) * \delta_{g^{-1}}|(x) \\ &= \int_G \omega'(gxg^{-1})\, d|\mu_1 - \mu_2|(x) = \int_G \omega'(x)\, d|\mu_1 - \mu_2|(x) = \|\mu_1 - \mu_2\|_{M(G,\omega')}, \end{aligned}$$

because, by (7.1), $\omega'(gxg^{-1}) = \omega'(x)$ for all $x, g \in G$.

We will now apply Theorem 7.2 to a bounded set $A = b(G)$ in a Banach space $M(G,\omega)$. To this end, we need to check that all the conditions of the theorem are satisfied. We already know that $M(G,\omega')$ is an L-embedded Banach space. Also, by the properties of b, we have that $b(G)$ is a non-empty bounded subset of $M(G,\omega')$ invariant under the isometric action of G on $M(G,\omega')$ defined above:

$$\begin{aligned} g(b(G)) &= \{g(b(x)) : x \in G\} = \{g \cdot b(x) + b(g) : x \in G\} = \{b(gx) : x \in G\} \\ &= b(G), \quad g \in G. \end{aligned}$$

So, all conditions are verified, and hence Theorem 7.2 provides us with a measure $\mu \in M(G,\omega')$ such that $g(\mu) = \mu$ for all $g \in G$. It follows that $b(g) = g(\mu) - g \cdot \mu = \mu - g \cdot \mu$, $g \in G$. Recalling the definition of b, we see that

$$b(g) = \widetilde{D}(\delta_g) * \delta_{g^{-1}} = \mu - g \cdot \mu = \mu - \delta_g * \mu * \delta_{g^{-1}},$$

and so, convoluting this equality with δ_g on the right, we obtain

$$\widetilde{D}(\delta_g) = \mu * \delta_g - \delta_g * \mu, \quad g \in G.$$

Our next goal is to prove that

$$D(f) = \widetilde{D}(f) = \mu * f - f * \mu, \quad f \in L^1(G,\omega'),$$

which will automatically mean that D is inner.

Let $f \in L^1(G,\omega')$. Using Lemma 2.3, we can find a net $\{f_\alpha\}_{\alpha\in A}$ from $\mathrm{lin}\{\delta_g : g \in G\}$, such that $f_\alpha \underset{SO}{\longrightarrow} f$. Then for each f_α we know that

$$\widetilde{D}(f_\alpha) = \mu * f_\alpha - f_\alpha * \mu,$$

because the formula is true for all δ_g. So, if we show that $\widetilde{D}(f_\alpha) \underset{SO}{\longrightarrow} \widetilde{D}(f)$ and $\mu * f_\alpha - f_\alpha * \mu \underset{SO}{\longrightarrow} \mu * f - f * \mu$, then we are done.

Take arbitrary $h \in L^1(G,\omega')$. Since $\widetilde{D}$ is a derivation, we have that

$$h * \widetilde{D}(f_\alpha) = \widetilde{D}(h * f_\alpha) - \widetilde{D}(h) * f_\alpha.$$

Because $f_\alpha \underset{SO}{\longrightarrow} f$, we get that $h * f_\alpha \to h * f$ in the norm topology. Hence, $\widetilde{D}(h * f_\alpha) \to \widetilde{D}(h * f)$ as $\widetilde{D}$ is continuous. Also, $\widetilde{D}(h) = D(h) \in L^1(G, \omega')$, and so $\widetilde{D}(h) * f_\alpha \to \widetilde{D}(h) * f$. Because $\widetilde{D}$ is a derivation, the above conclusions imply that

$$h * \widetilde{D}(f_\alpha) = \widetilde{D}(h * f_\alpha) - \widetilde{D}(h) * f_\alpha \to \widetilde{D}(h * f) - \widetilde{D}(h) * f = h * \widetilde{D}(f) = h * D(f).$$

So, we have just proved that $h * \widetilde{D}(f_\alpha) \to h * \widetilde{D}(f)$ for every $h \in L^1(G, \omega')$. Similarly, $\widetilde{D}(f_\alpha) * h \to \widetilde{D}(f) * h$ for every $h \in L^1(G, \omega')$, which means that $\widetilde{D}(f_\alpha) \underset{SO}{\longrightarrow} D(f)$.

Now, let us investigate the behavior of $\mu * f_\alpha - f_\alpha * \mu$. We again take arbitrary $h \in L^1(G, \omega')$ and convolute our expression on the left with it:

$$h * (\mu * f_\alpha - f_\alpha * \mu) = h * \mu * f_\alpha - h * f_\alpha * \mu.$$

Since $f_\alpha \underset{SO}{\longrightarrow} f$, we have that $h * f_\alpha \to h * f$ in $L^1(G, \omega')$. Now, because convolution with μ on the right is a continuous operator, we obtain that $h * f_\alpha * \mu \to h * f * \mu$. Since $L^1(G, \omega')$ is an ideal in $M(G, \omega')$, we get that $h * \mu \in L^1(G, \omega')$, and using again the fact that $f_\alpha \underset{SO}{\longrightarrow} f$, we obtain that $h * \mu * f_\alpha \to h * \mu * f$. It now follows that

$$h * \mu * f_\alpha - h * f_\alpha * \mu \to h * \mu * f - h * f * \mu = h * (\mu * f - f * \mu), \quad h \in L^1(G, \omega').$$

Similarly, $(\mu * f_\alpha - f_\alpha * \mu) * h \to (\mu * f - f * \mu) * h$ for every $h \in L^1(G, \omega')$. This precisely means that $\mu * f_\alpha - f_\alpha * \mu \underset{SO}{\longrightarrow} \mu * f - f * \mu$, and the proposition is proved. □

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