

Functional Calculus with Applications to Tadmor-Ritt Operators

by

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**A Thesis submitted to the Faculty of Graduate Studies of
The University of Manitoba
in partial fulfillment of the requirements of the degree of**

MASTER OF SCIENCE

**Department of Mathematics
University of Manitoba, Winnipeg**

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UNIVERSITY OF MANITOBA
DEPARTMENT OF MATHEMATICS

The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance the thesis entitled "*Functional Calculus with Applications to Tadmor-Ritt Operators*" by Stefan Juncu in partial fulfillment of the requirements for the degree of *Master of Science*.

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Abstract.

One can give various rigorous definitions to the notion of "functional calculus", but a functional calculus is ultimately just a mathematically meaningful way of talking about an operator $f(T)$, where, T is an operator and f is a function. This thesis is concerned with this concept and with one of its applications, the finding of bounds for powers of operators. It is actually this very application that has prompted the entire investigation presented here. This application is relevant to various fields, such as the numerical analysis of PDE and Markov chains. Chapter I presents various abstract approaches to the notion of "functional calculus" that are given content by three major examples: the Riesz-Dunford functional calculus, the Weyl functional calculus and the functional calculus for sectorial operators. Chapter II investigates various conditions that ensure power boundedness for operators, putting the Tadmor-Ritt condition at its center. The Riesz-Dunford calculus is instrumental for the proofs in this chapter. Chapter III investigates Pascale Vitse's use of Cauchy-Stieltjes integrals and their multipliers for obtaining bounds on powers of operators; the chapter closes with an investigation of partially power bounded operators.

My gratitude goes, first and foremost, to Professor S. H. Lui, for his generous support during this program.

I would like to thank Dr. Stephanie Portet and Dr. Stephen Kirkland, whose help contributed decisively to my undertaking.

I would also like to thank the members of my examination commission, Dr. Stephanie Portet again, and Dr. Puyan Mojabi, for their patience.

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Notation:

$\pi\sigma(T)$	the peripheral spectrum of the operator T
$\varrho(T)$	the resolvent set associated with the operator T
$\sigma(T)$	the spectrum of the operator T
$\sigma_p(T)$	the point spectrum (the set of eigenvalues) of the operator T
\mathbf{C}	the field of complex numbers
$\mathcal{C}_A(\mathbf{C}\setminus\text{cl}(\mathbf{D}))$	the algebra of functions holomorphic on $\mathbf{C}\setminus\mathbf{D}$ and continuous on $\mathbf{C}\setminus\text{cl}(\mathbf{D})$
$\mathcal{C}_A(\mathbf{C}\setminus\text{cl}(\mathbf{D}))^\perp$	the space of measures on the unit circle orthogonal to $\mathcal{C}_A(\mathbf{C}\setminus\text{cl}(\mathbf{D}))$
$\text{CSI}(\mathbf{D})$	the space of Cauchy-Stieltjes integrals on \mathbf{D}
\mathbf{D}	the open unit disk
\mathcal{E}	the extended Riesz-Dunford class
H_0^∞	the class of functions regularly decaying at zero and infinity (the Riesz-Dunford class)

$H_{(0)}^{\infty}$	the class of functions holomorphic at zero and regularly decaying at infinity
H^p	the Hardy space of order p
Hol	the space of holomorphic functions
$\mathcal{L}(X)$	the algebra of bounded linear operators on the Banach space X
$\mathcal{M}(\mathbf{T})$	the space of measures on the unit circle
Mult (D)	the algebra of multipliers of Cauchy-Stieltjes integrals on D
$R(\cdot, T)$	the resolvent function associated with the operator T
$\mathcal{R}(T)$	the range of operator T
S_{φ}	the sector of angle φ
T	the unit circle

Introduction

Finding bounds on powers of matrices has important consequences for the numerical analysis of partial differential equations; as is well known, in this field, partial differential operators are approximated by matrices, therefore successive applications of such an operator correspond to powers of matrices. Naturally, knowing that all powers of the relevant matrices are bounded from above proves crucial for the task of the numerical analyst.

Take, for example, the following discretization scheme for the heat equation $\partial_t u = \Delta u$:

$$\delta^\tau u^{(k)} - \Delta_h u^{(k)} = 0.$$

This is the forward Euler scheme in time and the central difference scheme in space (see Lui [17], p. 292).

We obtain

$$u^{(k+1)} = (I + \tau \Delta_h) u^{(k)} = A u^{(k)}.$$

So, $u^{(k)} = A^k u^{(0)}$.

For a fixed $h > 0$, suppose we wish to solve for the solution to final time $t = 1$. The number of iterations required is $1/\tau$. In the limit $\tau \rightarrow 0$, we need $|A^k|$ bounded as $k \rightarrow \infty$.

Another important application that comes immediately to mind lies in the field of Markov chains. A Markov chain is a sequence $(X_n)_n$ of random variables that obey the following Markov property:

For all integers $n \geq 0$ and all states $i_0, i_1, \dots, i_{n-1}, i, j \in E$,

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = P(X_{n+1} = j | X_n = i).$$

In the study of such processes, an important role is played by the transition matrix $\mathbf{P}=\{p_{i,j}\}_{i,j \in E}$, where

$$p_{i,j} = P(X_{n+1} = j | X_n = i).$$

An element $p_{i,j}$ gives the probability of transition from one state to another, so we can see that the matrix \mathbf{P} gives the evolution of the system. It is of course imperative that the powers of this matrix do not become increasingly large after many iterations.

Obviously, keeping a bound on powers of matrices is a crucial factor in any field that uses matrices and recursion.

But the problem cannot so be isolated as to restrict it to matrices only; indeed, it is naturally embedded in the problem of finding bounds on powers of bounded linear operators on general Banach spaces. One can therefore attempt to obtain results for such general operators; or, one can study the problem under additional constraints, such as those of a Hilbert space, or any other such conditions related to the geometry of the space. This results in mathematics of extraordinary richness and beauty, comprising results not only from operator theory, but also from measure theory, complex analysis and the theory of functions. This last part is particularly relevant, since the construction of a functional calculus for an operator may prove an essential tool for solving the problem.

Now, of course, not all operators are power-bounded and the same is true for matrices. So one has to select a certain class of matrices (operators) by imposing certain conditions that would ensure their power - boundedness. One such condition is the Kreiss condition, encountered in the celebrated Kreiss matrix theorem. This condition states:

$$\|(T - \lambda I)^{-1}\| \leq \frac{C}{|\lambda| - 1} \quad ((\forall)\lambda, |\lambda| > 1, C \text{ a positive constant}). \quad (0.1)$$

In this thesis I will be more interested, however, in another condition, namely the Tadmor-Ritt condition, which imposes the following stronger requirement on an operator T :

$$\|(T - \lambda I)^{-1}\| \leq \frac{C}{|\lambda - 1|} \quad ((\forall)\lambda, |\lambda| > 1, C \text{ a positive constant}). \quad (0.2)$$

Other conditions as well will form part of the subject matter of the thesis, such as the more general condition studied by El-Fallah and Ransford, of which the Kreiss and the Tadmor-Ritt conditions are particular instances.

This thesis deals primarily with linear operators, but this is simply not the place to give a thorough account of linear functional analysis. Therefore, I will restrict the following introduction to three main concepts: spectrum, resolvent set and resolvent function.

The *spectrum* $\sigma(T)$ of a linear operator T over a complex Banach space X is the set of complex numbers λ for which the operator $\lambda I - T$ is not invertible.

Although the notion of spectrum applies to operators in general, I will restrict the following discussion to bounded linear operators over a Banach space X , since this is the case which interests us the most. Now, there are two main ways of dividing the spectrum into parts. First, notice that $\lambda I - T$ is invertible if and only if it is bijective. Since, of course, bijective means injective plus surjective, the operator $\lambda I - T$ may fail to be bijective in two ways: namely, it may not be injective, in which case λ is an *eigenvalue* and the set of eigenvalues is called the *point spectrum* of T , or it may be injective, and therefore not surjective. The set of all λ such that $\lambda I - T$ is not surjective, but its range is dense in X , is called the *continuous spectrum* of T ; the set of all λ for which $\mathcal{R}(\lambda I - T)$ is a proper subspace of X is called the *residual spectrum*.

Another way of dividing the spectrum belongs to the theory of bounded linear operators on Hilbert spaces; for such an operator, we have a theorem that says that it is invertible if and only if it is bounded below and has a dense range. If the first condition fails, we have the *approximate point spectrum* of T ; if the second does not occur, we have the *compression spectrum* of T .

For a bounded linear operator, the spectrum is a compact set included in the disk centered at zero with radius $\|T\|$. We have the important value: $\max\{|\lambda|: \lambda \in \sigma(T)\} = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}$ (this is called the spectral radius).

The *resolvent set* $\rho(T)$ is the complement of the spectrum, and it is an open set. Naturally, on this set, the *resolvent function* $R(\lambda, T) = (\lambda I - T)^{-1}$ makes sense; notice that this is an operator-valued function defined on a set of complex numbers. We have the following important properties of the resolvent function:

- $R(\lambda, T)$ is analytic on $\rho(T)$. For $|\lambda| > \sup|\sigma(T)|$, we have the representation

$$R(\lambda, T) = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}.$$

- The resolvent identity:

$$R(\lambda, T) - R(\mu, T) = (\mu - \lambda)R(\lambda, T)R(\mu, T).$$

- $|R(\lambda, T)| \geq \frac{1}{d(\lambda, \sigma(T))}$.
- If $\lambda \in \rho(T)$, then

$$\{\mu: |\mu - \lambda| < \|R(\lambda, T)\|^{-1}\} \subset \rho(T)$$

and the series

$$R(\mu, T) = \sum_{k=0}^{\infty} R(\lambda, T)^{k+1}(\mu - \lambda)^k$$

converges uniformly on this disk.

References.

For numerical analysis of PDE, see Lui [17]; for Markov chains, see Bremaud [6]. For the Kreiss condition and the famous Kreiss matrix theorem, see Strikwerda, Wade [30]. The discussion around the spectrum is from MacCluer [19], pp. 96-97, 115; for the properties of the resolvent function, see Dunford and Schwartz [8], ch. 7.

Outline.

In the following, \mathbf{D} denotes the open unit disk, \mathbf{T} the unit circle, $\mathcal{L}(X)$ the space of all bounded linear operators on the Banach space X .

The thesis is structured into three chapters. The first chapter investigates the notion of *functional calculus*; the applications of the concept go well beyond finding bounds for powers of operators. Nevertheless, this latter enterprise is such an application. This chapter skips many of the details involved in such a construction, because the interest will be mainly in identifying the structure on which it is based. The examples given in this first chapter will exhibit a common pattern: one starts with a relatively small class of functions for which a calculus is defined via an integral representation, and then extends that calculus to a larger class of functions through the properties of the representation. These examples are: the Riesz-Dunford functional calculus, the Weyl functional calculus, the functional calculus for sectorial operators.

The second chapter deals with the topic that actually prompted the writing of this thesis, namely, condition that ensure the power boundedness of an operator (matrix). The condition highlighted here is the Tadmor-Ritt condition, although other constraints are described as well. The core of the chapter consists in giving a very thorough characterization of Tadmor-Ritt operators.

The third chapter represents a synthesis of the previous two, in that it follows very closely the considerations involved in choosing suitable classes of functions relevant for the purpose of finding

bounds on powers of Tadmor-Ritt operators. The construction is due to Vitse, and the classes of functions are those of Cauchy-Stieltjes integrals and their multipliers. This chapter concludes by describing some relevant connections between measures on the unit circle and the growth of powers of operators.

Chapter I

1. Functional calculus

In this section we introduce the notion of "functional calculus". Usually, when one speaks of a "functional calculus" one has in mind a type of calculus that would allow one to build functions of an operator; so, basically, what we have is a tool through which we can build $f(A)$, where A is an operator and f is a function with real or complex values. So, in essence, we are applying every function from a certain class to the operator A and obtain $f(A)$.

Of course, in building such a functional calculus, we have to choose the class to which the operator belongs, and also the class of functions. The two have to obey certain properties which allow one to establish a correspondence between them, and this correspondence *is* the functional calculus.

We begin with the idea of an *abstract functional calculus*. Let F be an algebra of functions and K an algebra of continuous linear operators on a topological vector space X . We define the abstract functional calculus on K relative to F as the map $\theta: F \times K \rightarrow K$, that obeys the following properties:

$$(1) \theta(1, A) = I, \text{ for every } A \in K;$$

$$(2) \theta(t, A) = A, \text{ for every } A \in K;$$

$$(3) \theta(\cdot, A) \text{ is an algebra homomorphism for every } A \in K: \theta(fg, A) = \theta(f, A)\theta(g, A), \forall f, g \in F, \forall A \in K.$$

We say that θ is *complete* if F is closed under composition and

$$\theta(f \circ g, A) = \theta(f, \theta(g, A))$$

for every $f, g \in F, A \in K$.

In the following, P denotes the algebra of polynomials (either real or complex).

Theorem 1.1 (Bosch, Swartz [4]). Assume that F and K are Frechet algebras with P dense in F .

Let θ be an abstract functional calculus on K relative to P .

(1) If $\theta(\cdot, A)$ is continuous for every $A \in K$, then θ has a unique extension, θ' , to an abstract functional calculus on K relative to F such that $\theta'(\cdot, A)$ is continuous for every $A \in K$.

(2) If the bilinear map $(f, g) \rightarrow f \circ g$ from $F \times F$ into F is separately continuous and $\theta(p, \cdot)$ is continuous from K into K for every $p \in P$, then the extension θ' is complete.

(3) If the map $\theta(\cdot, A): P \rightarrow K$ ($\theta(p, A) = p(A)$) is uniformly continuous when A runs over bounded subsets of K , then $\theta'(f, \cdot): K \rightarrow K$ is continuous for every $f \in F$.

Proof. Since K is complete, every $\theta(\cdot, A)$ has a unique continuous linear extension to F , so we have (1).

(2) Let $f, g \in F$ and pick $p_j, q_j \in P$ such that $p_j \rightarrow f, q_j \rightarrow g$. Then $p_j \circ q_k(A) = p_j(q_k(A))$ implies

$$\theta(p_j \circ q_k, A) = \theta(p_j, \theta(q_k(A))).$$

Fix j . Then $p_j \circ q_k \rightarrow p_j \circ g$ as $k \rightarrow \infty$ by hypothesis. By continuity of $\theta'(\cdot, A)$,

$$\theta(p_j \circ q_k, A) \rightarrow \theta'(p_j \circ g, A).$$

Also $\theta(q_k, A) \rightarrow \theta'(g, A)$ so

$$\theta(p_j, \theta(q_k, A)) \rightarrow \theta(p_j, \theta'(g, A))$$

by hypothesis ($\theta(p, \cdot)$ is continuous). Therefore

$$\theta'(p_j \circ g, A) = \theta(p_j, \theta'(g, A)).$$

Since $p_j \circ g \rightarrow f \circ g$ by hypothesis,

$$\theta'(p_j \circ g, A) \rightarrow \theta'(f \circ g, A).$$

Also, $\theta(p_j, \theta'(g, A)) \rightarrow \theta(f, \theta'(g, A))$ so

$$\theta'(f, \theta'(g, A)) = \theta'(f \circ g, A).$$

(3) First observe that $\theta'(f, \cdot)$ is uniformly continuous when f runs over bounded subsets of F . Let $f \in F$.

Pick $p_j \in P$ such that $p_j \rightarrow f$ in F . Let $A \in K$ and $A_j \in K, A_j \rightarrow A$. Let $\|\cdot\|$ be a continuous semi-norm in

K . Then

$$\begin{aligned} \|\theta'(f, A_j) - \theta'(f, A)\| &\leq \|\theta'(f, A_j) - \theta'(p_k, A_j)\| + \|\theta'(p_k, A_j) - \theta'(p_k, A)\| + \\ &+ \|\theta'(p_k, A) - \theta'(f, A)\|. \end{aligned}$$

Let $\varepsilon > 0$. By uniform continuity there exists k_1 , such that

$$\|\theta'(f, A_j) - \theta'(p_k, A_j)\| < \varepsilon$$

for $k \geq k_1, j \in N$. Since $\theta'(\cdot, A)$ is continuous, there exists $k_2 > k_1$, such that

$$\|\theta'(p_k, A) - \theta'(f, A)\| < \varepsilon$$

for $k \geq k_2$. Now, $\theta(p_{k_2}, \cdot)$ is continuous, so there exists j_0 such that

$$\|\theta'(p_{k_2}, A_j) - \theta'(p_{k_2}, A)\| < \varepsilon.$$

for $j > j_0$. Adding the previous three inequalities, we obtain

$$\|\theta'(f, A_j) - \theta'(f, A)\| < 3\varepsilon$$

for $j > j_0$. Therefore, $\theta'(f, A_j) \rightarrow \theta'(f, A)$.

■

Theorem 1.1 gives us a method for extending a functional calculus from the class P of polynomials to an algebra F of functions which contains P as a dense subset. So, basically, if we already have an abstract functional calculus for polynomials which obeys certain properties, then we can extend this calculus to an algebra F of functions provided this algebra in turn contains P as a dense subset and satisfies the properties mentioned in the theorem.

Extending by continuity the functional calculus from a smaller, relatively controllable class of functions, to a larger class, is one of the two main procedures involved in building functional calculi. The smaller class (sub-algebra) can be the algebra of polynomials, or the algebra of infinitely differentiable functions with compact support defined on a certain set. The idea is to have an algebra of functions for which the calculus is easily definable, and then extend this to a larger algebra by continuity.

The second procedure is to use a reproducing formula for functions from F ,

$$f(x) = \int k(x, y)f(y)dy$$

(e.g., a transform, or a Cauchy-type formula). If the kernel $k(x, y)$ can be used to define an operator

$k(A, y)$, then the functional calculus can be defined through the formula

$$\theta(f, A) = f(A) = \int k(A, y)f(y)dy.$$

In his seminal paper on "Condition Numbers of Large Matrices and Analytic Capacities", N. K. Nikolski works with a more applied notion of functional calculus. Namely, he introduces first the notion of *function algebra* on \mathbf{D} as a unital Banach algebra \mathbf{A} such that

(i) \mathbf{A} is continuously embedded in the algebra $\mathbf{Hol}(\mathbf{D})$;

(ii) \mathbf{A} contains all polynomials and $\lim_n \|z^n\|_A^{1/n} = 1$;

(iii) $(a \in \mathbf{A}, \lambda \in \mathbf{D}, a(\lambda) = 0) \Rightarrow \frac{a}{z-\lambda} \in \mathbf{A}$.

Given such a function algebra, he defines an *\mathbf{A} -functional calculus* for an operator T on Banach space X to be a bounded homomorphism $\mathbf{A} \rightarrow \mathcal{L}(X)$ such that $z \rightarrow T, 1 \rightarrow I$. Also,

$$\|a(T)\| \leq C_T \|a\|_A$$

for every $a \in \mathbf{A}$, where C_T stands for the norm of the homomorphism.

We must also take into consideration the following remark of Nikolski, to which I will come back later; I will give here the full quote:

"In order to define and to use a calculus, we do not really need to require that A be an algebra. Below, we often work with merely a *function space* on \mathbf{D} , that is, a Banach space A satisfying (i)-(iii) and such that $pf \in A$ for every $f \in A$ and every polynomial p . The *calculus over such a space* A means a bounded mapping $f \rightarrow f(T)$ such that $(pf)(T) = p(T)f(T)$ for every $f \in A$ and every polynomial p ." (Nikolski, [24], p. 652).

2. Examples

(a) The Riesz-Dunford functional calculus

In the following, we will give content to the abstract structure from the first part by describing two functional calculi. The first is the Riesz-Dunford functional calculus which constitutes the framework for the proofs used in Chapter II. The two main ideas behind this calculus are the following:

- the Cauchy integral formula, which gives the reproducing integral formula for functions analytic on a domain;
- the Bochner integral, which allows us to integrate operator-valued functions.

Given that Cauchy integrals and Cauchy's theorem in complex analysis (in its various forms) are well known, we will just define the second concept. Also, this concept is relevant for the other functional calculi described in this thesis.

Let (S, Σ, μ) be a measure space and $f: S \rightarrow X$ a function, where X is a Banach space. We say that such a function is *strongly measurable* if there exists a sequence of Σ -simple functions $\{g_j\}$ such that $g_j \rightarrow f$ in norm μ -a.e. A simple function

$$g = \sum_{j=1}^n x_j \chi_{A_j} : S \rightarrow X, \quad \chi_{A_j}(x) = \begin{cases} 1, & x \in A_j \\ 0, & x \notin A_j \end{cases}$$

is Bochner integrable over A in Σ if and only if $\mu(A \cap A_j) < \infty$ for every j , and, in such a case, the Bochner integral of f over A is

$$\int_A g d\mu = \sum_{j=1}^n x_j \mu(A \cap A_j).$$

A strongly measurable function $f: S \rightarrow X$ is *Bochner integrable* if

(a) there exists a sequence of simple Bochner integrable functions $\{g_j\}$ that converges to f μ -a.e., and

$$(b) \lim_{j \rightarrow \infty} \int_S \|f(\cdot) - g_j(\cdot)\| d\mu = 0.$$

We have in this case, the Bochner integral of f

$$\int_S f d\mu = \lim_{j \rightarrow \infty} \int_S g_j d\mu.$$

We have the following important fact about the Bochner integral:

- f is Bochner-integrable if and only if $\|f\|$ is Lebesgue integrable and

$$\left\| \int_S f d\mu \right\| \leq \int_S \|f\| d\mu.$$

Let A be an operator in $\mathcal{L}(X)$.

Definition. Let $F(A)$ be the set of all functions f which are analytic on an open neighborhood U of $\sigma(A)$.

By a Jordan curve C we mean a finite number C_1, \dots, C_k of disjoint, simple, closed, positively oriented rectifiable curves. We say that V is an *admissible* neighborhood if $\sigma(A) \subset V \subset cl(V) \subset U$ and ∂V is a Jordan curve.

Definition. Let f be an analytic function on an open neighborhood U of $\sigma(A)$. We define

$$f(A) = \frac{1}{2\pi i} \int_{\partial V} f(z) R(z, A) dz$$

where V is any admissible domain with ∂V positively oriented.

Theorem 1.2 (Bosch, Swartz [4]). Let $f, g \in F(A)$ and $s, t \in \mathbf{C}$. Then

$$(1) sf + tg \in F(A) \text{ and } (sf + tg)(A) = sf(A) + tg(A);$$

(2) $fg \in F(A)$ and $(fg)(A) = f(A)g(A)$;

(3) if $f \equiv 1$, then $f(A) = I$;

(4) if $f(z) = z$ for all $z \in \mathbf{C}$, then $f(A) = A$;

(5) if $f(z) = z^k$ for $k = 0, 1, 2, \dots$, then $f(A) = A^k$;

(6) if f has a Maclaurin expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in a neighborhood of $\sigma(A)$, then $f(A) = \sum_{k=0}^{\infty} a_k A^k$ (norm convergence);

(7) $f \in F(A')$ and $f(A') = f(A)'$, where A' denotes the transpose or adjoint of A .

Proof. (3) If $f \equiv 1$,

$$f(A) = \frac{1}{2\pi i} \int_C R(z, A) dz$$

where C is a circle with center at the origin and radius $L > \|A\|$. Then

$$f(A) = \frac{1}{2\pi i} \int_C R(z, A) dz = \frac{1}{2\pi i} \int_C (z - A)^{-1} dz = \frac{1}{2\pi i} \int_C \frac{1}{z} \left(I - \frac{A}{z}\right)^{-1} dz = \frac{1}{2\pi i} \int_C \frac{1}{z} \sum_{k=0}^{\infty} \frac{A^k}{z^k} dz.$$

Since $\| \frac{A}{z} \| < 1$ for $|z| = L$, the series converges uniformly over C , and we can integrate term by term,

$$f(A) = \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_C \frac{A^k}{z^{k+1}} dz = I$$

since $\int_C \frac{1}{z^{k+1}} dz = 0$ if $k > 1$ and $\int_C \frac{1}{z^{k+1}} dz = 2\pi i$ if $k=0$.

(4) If $f(z) = z$ for all z , then

$$f(A) = \frac{1}{2\pi i} \int_C zR(z, A)dz = \frac{1}{2\pi i} \int_C \left(I - \frac{A}{z}\right)^{-1} dz = \frac{1}{2\pi i} \int_C \sum_{k=0}^{\infty} \frac{A^k}{z^k} dz,$$

where C is a circle as in (3). By the same considerations as the ones from (3), we can integrate the series term by term and, after performing some simple contour integrations, we arrive at the result $f(A) = A$.

(6) The power series for f converges uniformly on a circle C with center 0 and radius $L > r(A)$ (the spectral radius). From (5), we get

$$f(A) = \frac{1}{2\pi i} \int_C f(z)R(z, A)dz = \frac{1}{2\pi i} \int_C \left(\sum_{k=0}^{\infty} a_k z^k\right) R(z, A)dz = \frac{1}{2\pi i} \sum_{k=0}^{\infty} a_k \int_C z^k R(z, A)dz = \sum_{k=0}^{\infty} a_k A^k$$

We have, of course, that if p is a complex polynomial, $p(z) = \sum_{k=0}^n a_k z^k$, then

$$p(A) = \sum_{k=0}^n a_k A^k.$$

(a complete proof can be found in the reference, pp. 90-93).

■

Corollary 1.3 (Bosch, Swartz [4]). Let $f \in F(A)$. Then $f(A)$ is invertible iff $f(z) \neq 0$ for $z \in \sigma(A)$. The inverse is $g(A)$, where g is any member of $F(A)$ such that $g = \frac{1}{f}$ on a neighborhood of $\sigma(A)$.

(a complete proof can be found in the reference, p. 93).

We give the following results without proof.

Theorem 1.4 (Bosch, Swartz [4]) (Spectral Mapping Theorem). If $f \in F(A)$, then $f(\sigma(A)) = \sigma(f(A))$.

(a complete proof can be found in the reference, pp. 94-95).

Theorem 1.5 (Bosch, Swartz [4]). Let $f \in F(A), g \in F(f(A))$ and $h = g \circ f$. Then $h \in F(A)$ and $h(A) = g(f(A))$.

(a complete proof can be found in the reference, p. 95).

Theorem 1.6 (Bosch, Swartz [4]). Let $f, f_k \in F(A)$ and suppose each f_k is analytic on an open neighborhood U of $\sigma(A)$. If $\{f_k\}$ converges to f uniformly on compact subsets of U , then $\|f_k(A) - f(A)\| \rightarrow 0$.

Proof. Let V be an open neighborhood of $\sigma(A)$ such that $V \subset U$ is admissible. By the properties of the resolvent function (see Introduction), $R(z, A)$ is bounded for $z \in \partial V$ so

$$\|f_k(z)R(z, A) - f(z)R(z, A)\| \rightarrow 0$$

uniformly for $z \in \partial V$ and

$$\|f_k(A) - f(A)\| = \frac{1}{2\pi} \left\| \int_{\partial V} (f_k(z) - f(z))R(z, A)dz \right\| \leq \frac{1}{2\pi} \int_{\partial V} \|(f_k(z) - f(z))R(z, A)\| |dz| \rightarrow 0.$$

■

Theorem 1.7 (Bosch, Swartz [4]) (Perturbation). Let $\varepsilon > 0$. There exists $\delta > 0$ such that if $A_1 \in L(X)$ and $\|A - A_1\| < \delta$, then

$$\sigma(A_1) \subset \{z: \text{dist}(z, \sigma(A)) < \varepsilon\} := S(\sigma(A), \varepsilon)$$

and

$$\|R(z, A_1) - R(z, A)\| < \varepsilon$$

for $z \notin S(\sigma(A), \varepsilon)$.

(a complete proof can be found in the reference, pp. 98-99).

Theorem 1.8 (Bosch, Swartz [4]). If $f \in F(A)$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $A_1 \in \mathcal{L}(X)$ with $\|A - A_1\| < \delta$, then $f \in F(A_1)$ and $\|f(A) - f(A_1)\| < \varepsilon$.

Proof. Let U be a neighborhood of $\sigma(A)$, where f is analytic and let $V \subset U$ be admissible. The **Perturbation Theorem** asserts that the location of the spectrum depends continuously on the operator, so there exists $\delta_1 > 0$ such that $\sigma(A_1) \subset V$ for $\|A - A_1\| < \delta_1$ when $A_1 \in \mathcal{L}(X)$. Since f is holomorphic on V , $f \in F(A_1)$ when $\|A - A_1\| < \delta_1$. Also, by the **Perturbation Theorem**, $R(z, A)$ is near $R(z, A_1)$ for $\|A - A_1\|$ small and z not too close to $\sigma(A)$ so we may choose $0 < \delta < \delta_1$, such that

$$\begin{aligned} \|f(A) - f(A_1)\| &\leq \frac{1}{2\pi} \left\| \int_{\partial V} f(z)[R(z, A) - R(z, A_1)]|dz| \right\| \\ &\leq \frac{1}{2\pi} \int_{\partial V} |f(z)| \|R(z, A) - R(z, A_1)\| |dz| < \varepsilon \end{aligned}$$

for all $\|A - A_1\| < \delta$.

■

Now, the construction given in the preceding pages is not yet an abstract functional calculus, nor even a functional calculus for an operator. The reason for the latter is that $F(A)$ is not an algebra. However, we can make it one if we introduce an equivalence relation on it, identifying two functions if they agree on a neighborhood of $\sigma(A)$ (working, thus, with *germs* of functions instead of the functions themselves). We then have the following result:

Theorem 1.9 (Bosch, Swartz [4]). Suppose $\psi: F(A) \rightarrow \mathcal{L}(X)$ is an algebra homomorphism satisfying

(1) $\psi(1) = I,$

(2) $\psi(z) = A,$

(3) If $\{f_n\}$ is a sequence of functions analytic on an open neighborhood D of $\sigma(A)$ such that $f_k \rightarrow f$ uniformly on compact subsets of D , then $\psi(f_j) \rightarrow \psi(f).$

Then $\psi(f) = f(A)$ for every $f \in F(A).$

Proof. First, we show $\psi(f) = f(A),$ for f a rational function in $F(A).$ Now, by the fact that ψ is a homomorphism, for $m \geq 1,$

$$\psi(z^m) = \psi(z)^m = A^m$$

so for any polynomial

$$\psi(p) = p(A).$$

Let q be any polynomial such that $\frac{1}{q} \in F(A).$ Then

$$I = \psi(1) = \psi\left(q \frac{1}{q}\right) = \psi(q)\psi\left(\frac{1}{q}\right) = q(A)\psi\left(\frac{1}{q}\right)$$

so

$$\psi\left(\frac{1}{q}\right) = (q(A))^{-1}$$

and $q(A)$ is invertible. If p, q are polynomials such that $\frac{1}{q} \in F(A),$ then

$$\psi\left(\frac{p}{q}\right) = \psi(p)\psi\left(\frac{1}{q}\right) = p(A)(q(A))^{-1} = \frac{p}{q}(A)$$

by **Theorem 1.2**.

Let $f \in F(A)$. By Runge's theorem, f can be approximated uniformly on compact sets by rational functions $\{r_n\} \in F(A)$. By property (3), $\psi(r_n) \rightarrow \psi(f)$. But $\psi(r_n) = r_n(A)$ so $\psi(f) = f(A)$ by **Theorem 1.6**.

■

Still, the functional calculus built thus far is not yet an abstract functional calculus, in that it is defined for an operator only and so it depends on the spectrum of that operator. Starting from this, however, we can abstract more and build our desired functional calculus.

We take F to be the algebra of all entire functions with the topology of uniform convergence on compact subsets of \mathbf{C} . F is a Frechet algebra which contains the space of polynomials. By the above construction, we already know that, for a function f of F , $f(A) \in L(X)$ is well defined. So our abstract functional calculus will take the form

$$\theta: F \times \mathcal{L}(X) \rightarrow \mathcal{L}(X)$$

$$\theta(f, A) = f(A).$$

By **Theorem 1.2** above, θ is an algebra homomorphism, and we have $\theta(p, A) = p(A)$ for every polynomial p and $A \in \mathcal{L}(X)$. Of course, F will be closed under composition so, by **Theorem 1.5** we have

$$\theta(f \circ g, A) = \theta(f, \theta(g, A))$$

and thus the calculus is complete.

Also, by **Theorems 1.6, 1.8**, the functional calculus is continuous in each variable. We can prove more:

Theorem 1.10 (Bosch, Swartz [4]). The map $\theta: F \times \mathcal{L}(X) \rightarrow \mathcal{L}(X)$ is jointly continuous.

Proof. Let $f_n \rightarrow f$ in F and $A_n \rightarrow A$ in $\mathcal{L}(X)$. We have

$$\|f_n(A_n) - f(A)\| \leq \|f_n(A_n) - f_n(A)\| + \|f_n(A) - f(A)\|,$$

and so by continuity in the first variable, the second term in the right goes to 0 as $n \rightarrow \infty$. We take the first term. Let $\varepsilon > 0$ and $\delta > 0$ as in **Theorem 1.7**. Pick N such that for $n \geq N$ we have $\|A_n - A\| < \delta$. Let $L > \|A\| + \varepsilon$ and C be the positively oriented circle with center at the origin and radius L . We know that, for $n \geq N$, the spectrum of A_n , $\sigma(A_n)$, lies inside C and (given that the points on C lie outside the ε -neighborhood of $\sigma(A)$),

$$\|R(z, A_n) - R(z, A)\| < \varepsilon$$

for $z \in C$ by **Theorem 1.7**. Take

$$M = \sup\{|f_n(z)|; z \in C, n \geq N\}.$$

(Note that $M < \infty$, since $f_n \rightarrow f$ uniformly on compact sets of \mathbf{C})

For $n \geq N$,

$$\|f_n(A_n) - f_n(A)\| = \left\| \frac{1}{2\pi i} \int_C f_n(z) [R(z, A_n) - R(z, A)] dz \right\| \leq M L \varepsilon$$

so $f_n(A_n) - f_n(A) \rightarrow 0$, which gives our result. ■

(b) The Weyl functional calculus

The next functional calculus that I am about to describe was created by Hermann Weyl in his *Theory of groups and quantum mechanics*, New York: Dover, 1931, in order to deal with the unbounded

self-adjoint operators of differentiation and multiplication by a position coordinate in quantum mechanics.

We will, however, define the Weyl calculus for bounded self-adjoint operators, since quantum mechanics is not the concern of this thesis.

Given that, in one of the following proofs we will use the spectral integral, we will sketch briefly its construction. One starts with the notion of *spectral measure*.

Definition. Let (S, Σ) be a measurable space. A *spectral measure* on S is a function $E: S \rightarrow \mathcal{L}(X)$, where X is a Hilbert space, such that:

(a) For each A in Σ , $E(A)$ is an orthogonal projection.

(b) $E(\emptyset) = 0$ and $E(S) = I$.

(c) For A_1, A_2 in Σ such that $A_1 \cap A_2 = \emptyset$, we have

$$E(A_1)X \perp E(A_2)X.$$

(d) If $\{A_k\}$ is a sequence of pairwise disjoint sets from Σ , then

$$\sum_{k=1}^n E(A_k)h \rightarrow E\left(\bigcup_{k=1}^{\infty} A_k\right)h$$

as $n \rightarrow \infty$, for each $h \in X$.

Now, given a spectral measure E , we can define a complex measure on S

$$\mu_{h,g}(A) = \langle E(A)h, g \rangle,$$

with $\|\mu_{h,g}\| \leq \|h\|\|g\|$.

For a bounded measurable function f on S , we consider

$$b(h, g) \equiv \int_S f d\mu_{h,g}$$

which defines a bounded sesquilinear form on X^2 . We have a theorem (see Theorem 2.11 in MacCluer [19], p. 35) that guarantees the existence of a unique bounded operator $\pi(f)$ on X such that

$$b(h, g) = \langle \pi(f)h, g \rangle \text{ with}$$

$$\|\pi(f)\| \leq \|f\|_S$$

We define

$$M = \int_S f dE = \pi(f).$$

So we will have

$$\langle Mh, g \rangle = \langle \pi(f)h, g \rangle = \int_S f d\mu_{h,g}.$$

Finally, the spectral theorem (or, rather, a version of it) says that for every normal bounded operator T on a separable Hilbert space X , there exists a unique spectral measure E on $\sigma(T)$ such that

$$T = \int_S z dE.$$

Moreover, for every continuous function f on $\sigma(T)$,

$$f(T) = \int_S f dE.$$

As MacCleur puts it, the intuition behind this theorem is that "bounded normal operators can be approximated by linear combinations of projections with pairwise orthogonal ranges" (MacCleur [19], p. 181), just like a bounded Borel function can be approximated by simple functions.

We start by acknowledging the following result, which we will take without proof. We have that for a bounded self-adjoint operators A in $\mathcal{L}(X)$ (X being this time a Hilbert space), $e^{2\pi itA}$ is a unitary operator, for every $t \in \mathbf{R}$ (Theorem 4.12 in Bosch, Swartz [4], pp. 47-48). Taking a function f in the Schwartz space $\mathbf{S}(\mathbf{R})$, we define

$$W(f, A) = \int_{-\infty}^{\infty} e^{2\pi i t A} \hat{f}(t) dt,$$

where $\hat{f}(t)$ is the Fourier transform of f . The formula is well defined as a Bochner integral, and we have

$$\|W(f, A)\| \leq \int_{-\infty}^{\infty} \|e^{2\pi i t A} \hat{f}(t)\| dt \leq \int_{-\infty}^{\infty} |\hat{f}(t)| dt = \|\hat{f}\|_1.$$

We are in need of more if we want to call $W(f, A)$ the function $f(A)$. For this, we can use the spectral theorem for bounded normal operators, mentioned above; thus, if E denotes the spectral measure for the operator A , we can write

$$e^{2\pi i \xi A} = \int_{\mathbf{R}} e^{2\pi i \xi t} dE(t).$$

Substituting this into the formula for $W(f, A)$, we get

$$W(f, A) = \int_{\mathbf{R}} \int_{\mathbf{R}} \hat{f}(\xi) e^{2\pi i \xi t} dE(t) d\xi.$$

We can reverse the order of integration (again a result that we take as given, see Theorem 5.24 in Bosch, Swartz [4], pp. 68-69) and obtain

$$W(f, A) = \int_{\mathbf{R}} \int_{\mathbf{R}} \hat{f}(\xi) e^{2\pi i \xi t} d\xi dE(t) = \int_{\mathbf{R}} f(t) dE(t) = f(A)$$

since $\int_{\mathbf{R}} f(t) dE(t)$ is defined for $f \in \mathcal{S}(\mathbf{R})$. Thus, we arrive at $W(f, A) = f(A)$.

This allows us to extend the calculus to the space F ,

$$F = \{f \in L^1(\mathbf{R}) : \hat{f} \in L^1(\mathbf{R})\}$$

with the norm $\|f\| = \|\hat{f}\|_1$.

The next step would be to bring in polynomials, and even $C^\infty(\mathbf{R})$ functions, but this poses a conceptual problem caused by the convergence conditions of improper integrals of such functions. Basically, one cannot interpret \hat{p} in $W(p, A)$ as a function; \hat{p} has to be seen as a distribution and, consequently one must consider $e^{2\pi itA}$ as an operator-valued distribution. Two lemmas whose full contents I will not give here help us in our endeavour (see Bosch and Swartz [4], Lemmas 8.4, 8.5); the second lemma, basically, says that $W(\psi, A)$, for $\psi \in C_0^\infty(\mathbf{R})$ vanishes if $\text{supp}(\psi)$ lies outside an interval centered at the origin and with radius $\|A\|$. Naturally, in this context, $W(\psi, A)$ can be interpreted as a distribution.

Given these facts, one can extend the Weyl calculus to C^∞ functions by taking a function $\phi \in C_0^\infty(\mathbf{R})$ such that $\phi \equiv 1$ for $\{t: |t| \leq \|A\|\}$ and define

$$W(\psi, A) = W(\phi\psi, A)$$

for $\psi \in C^\infty(\mathbf{R})$.

The definition does not depend on the choice of infinitely differentiable function with compact support, as long as it obeys, of course the above mentioned condition, since any difference between two such functions occurs outside the interval with radius $\|A\|$, and so, considering the second lemma, it is a difference that makes no difference.

We will have:

Theorem 1.11 (Bosch, Swartz [4]). If p is a polynomial, then $W(p, A) = p(A)$.

(a complete proof can be found in the reference, p. 133).

And also the more general theorem:

Theorem 1.12 (Bosch, Swartz [4]). $W: C^\infty(\mathbf{R}) \times A \rightarrow A$ is a complete functional calculus, continuous in both variables simultaneously and is an extension of the Riesz functional calculus.

Proof. I will just prove here that we are dealing with a functional calculus so, basically, just the existence of the homomorphism, leaving continuity aside.

For $\alpha, \beta \in C^\infty, \psi \in C_0^\infty$, with $\psi \equiv 1$ on a neighborhood of $\{t \in \mathbf{R}: |t| \leq \|A\|\}$, we have

$$\begin{aligned} W(\alpha\beta, A) &= \int_{-\infty}^{\infty} e^{2\pi itA} \widehat{\psi\alpha\psi\beta}(t) dt = \int_{-\infty}^{\infty} e^{2\pi itA} (\widehat{\psi\alpha} * \widehat{\psi\beta})(t) dt = \int_{-\infty}^{\infty} e^{2\pi itA} \left(\int_{-\infty}^{\infty} \widehat{\psi\alpha}(s) \widehat{\psi\beta}(t-s) ds \right) dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi itA} \widehat{\psi\alpha}(s) \cdot \widehat{\psi\beta}(t-s) ds dt. \end{aligned}$$

In the above we have used a well known fact relating convolution and the Fourier transform; now we make the change of variables $t \rightarrow r = t - s$ and, given that $e^{2\pi isA}, e^{2\pi irA}$ commute, we have

$$\begin{aligned} W(\alpha\beta, A) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i(r+s)A} \widehat{\psi\alpha}(s) \cdot \widehat{\psi\beta}(r) ds dr = \int_{-\infty}^{\infty} e^{2\pi irA} \widehat{\psi\beta}(r) dr \int_{-\infty}^{\infty} e^{2\pi isA} \widehat{\psi\alpha}(s) ds \\ &= W(\alpha, A)W(\beta, A). \end{aligned}$$

Linearity of $W(\psi, A)$ follows from that of the integral; also, we will have $W(1, A) = I, W(x, A) = A$, for every bounded self-adjoint A , from **Theorem 1.11**.

Thus, $W(\psi, A)$ is a functional calculus.

(a complete proof can be found in the reference, pp. 134-136). ■

3. Functional calculus for sectorial operators

We can look at the building process of a functional calculus also from another angle. This time, we will not limit the calculus to bounded linear operators; instead, we will deal with sectorial operators, namely operators whose spectra lie in a sector around the positive axis. Differential operators are an important class of sectorial operators. The spectrum of such an operator can be an unbounded set, the

only requirement being that it be included in such a sector. The procedure described here will exhibit the relevant structure needed for building such a calculus. The source of the following is Markus Haase, *The Functional Calculus for Sectorial Operators* [13]. Tadmor-Ritt operators are sectorial operators, but they are also bounded, so the Riesz-Dunford calculus is sufficient for dealing with them. However, describing the procedure for building a functional calculus for sectorial operators gives more content to the idea of "functional calculus" presented here.

(a) Once again, we start in an abstract setting. We call the triple $(\mathcal{E}, \mathcal{M}, \Phi)$ an *abstract functional calculus* (afc) over a Banach space X , where $\mathcal{E} \subset \mathcal{M}$ is a subalgebra of a commutative unital algebra \mathcal{M} ($1 \notin \mathcal{E}$ in general) and Φ a homomorphism: $\mathcal{E} \rightarrow \mathcal{L}(X)$. We call the afc *non-degenerate* or *proper* if the set:

$$Reg(\mathcal{E}) := \{e \in \mathcal{E} : \Phi(e) \text{ is injective}\}$$

is not empty. A member of $Reg(\mathcal{E})$ is called a *regulariser*. We call a function $f \in \mathcal{M}$ *regularisable* if there exists $e \in Reg(\mathcal{E})$ such that $ef \in \mathcal{E}$; in this case, e is called a *regulariser* for f . We observe that $\mathbf{1}$ is regularisable if and only if the afc is proper. In the case of such a proper afc, the set

$$\mathcal{M}_r = \{f \in \mathcal{M} : f \text{ is regularisable}\}$$

is a subalgebra of \mathcal{M} that contains \mathcal{E} .

Now, given $(\mathcal{E}, \mathcal{M}, \phi)$ a proper afc, for $f \in \mathcal{M}_r$ we define

$$\Phi(f) = \Phi(e)^{-1}\Phi(ef) \quad (1)$$

where e is such a regulariser for f .

We have the following lemma:

Lemma 1.13 (Haase [13]). Let $(\mathcal{E}, \mathcal{M}, \Phi)$ be a proper afc. Then, by (1) a closed operator on X is well defined and the so-defined mapping

$$\Phi = (f \mapsto \Phi(f)): \mathcal{M}_r \rightarrow \{\text{closed operators on } X\}$$

extends the original mapping $\Phi: \mathcal{E} \rightarrow \mathcal{L}(X)$.

(a complete proof can be found in the reference, p. 5)

We call the original mapping $\Phi: \mathcal{E} \rightarrow \mathcal{L}(X)$ the *primary functional calculus*, and the extension described in the lemma the *extended functional calculus*. The algebra \mathcal{M}_r is called the *domain* of the afc $(\mathcal{E}, \mathcal{M}, \Phi)$.

We define

$$\mathcal{M}_b := \{f \in \mathcal{M}_r: \Phi(f) \in \mathcal{L}(X)\}.$$

We will have the following:

Proposition 1.14 (Haase [13]). Let $(\mathcal{E}, \mathcal{M}, \Phi)$ be a proper afc over the Banach space X .

(i) If $T \in \mathcal{L}(X)$ commutes with each $\Phi(e)$, then it commutes with each $\Phi(f)$, $f \in \mathcal{M}_r$.

(ii) $\mathbf{1} \in \mathcal{M}_r$, $\Phi(\mathbf{1}) = I$.

(iii) For $f, g \in \mathcal{M}_r$, we have

$$\Phi(f) + \Phi(g) \subset \Phi(f + g)$$

$$\Phi(f)\Phi(g) \subset \Phi(fg)$$

with $\mathcal{D}(\Phi(f)\Phi(g)) = \mathcal{D}(\Phi(fg)) \cap \mathcal{D}(\Phi(g))$.

(iv) If $f, g \in \mathcal{M}$ such that $fg = \mathbf{1}$, then $\Phi(f)$ is injective with $(\Phi(f))^{-1} = \Phi(g)$.

(v) Let $f \in \mathcal{M}_r$ and let F be a subspace of $\mathcal{D}(\Phi(f))$. Suppose that there is a sequence $\{e_n\}_n \subset \mathcal{E}$ such that $\Phi(e_n) \rightarrow I$ in the strong topology and $\mathcal{R}(\Phi(e_n)) \subset F, \forall n \in \mathbb{N}$. Then F is a core for $\Phi(f)$ (that is a set dense in $\mathcal{D}(\Phi(f))$ in the norm $\|x\|_{\Phi(f)} = \|x\| + \|\Phi(f)x\|$).

(a complete proof can be found in the reference, pp. 5-6).

Corollary 1.15 (Haase [13]). With the same structures as before, we have:

(i) $f \in \mathcal{M}_r, g \in \mathcal{M}_b \Rightarrow \Phi(f) + \Phi(g) = \Phi(f + g)$ and $\Phi(f)\Phi(g) = \Phi(fg)$.

(ii) \mathcal{M}_b is a subalgebra with $\mathbf{1}$ of \mathcal{M} , and the map

$$f \mapsto \Phi(f): \mathcal{M} \rightarrow \mathcal{L}(X)$$

is a homomorphism of unital algebras.

(iii) If $f \in \mathcal{M}_b$ is such that $\Phi(f)$ is injective, then

$$\Phi(f)^{-1}\Phi(g)\Phi(f) = \Phi(g)$$

holds for all $g \in \mathcal{M}_r$.

(a complete proof can be found in the reference, pp. 6-7).

Corollary 1.16 (Haase [13]). With the same structures as before, suppose that $f \in \mathcal{M}_r, g \in \mathcal{M}$ such that $fg = \mathbf{1}$. Then

$$g \in \mathcal{M}_r \Leftrightarrow \Phi(f) \text{ is injective.}$$

In this case, we have $\Phi(g) = \Phi(f)^{-1}$.

(a complete proof can be found in the reference, p. 7).

We define some more structure. Given a proper afc $(\mathcal{E}, \mathcal{M}, \Phi)$ over the Banach space X , we call a subalgebra $\mathcal{D} \subset \mathcal{M}_b$ *admissible* if the set $\{f \in \mathcal{D}: \Phi(f) \text{ is injective}\}$ is not empty. In such a case, $(\mathcal{D}, \mathcal{M}, \Phi)$ is a proper afc over X . We define by

$$\langle D \rangle = \{f \in \mathcal{M}: \text{there is a } d \in \mathcal{D} \text{ such that } df \in \mathcal{D} \text{ and } \Phi(d) \text{ is injective}\}$$

the regularisable elements of this new afc.

We call a *generator* of the afc an admissible subalgebra \mathcal{D} such that $\langle D \rangle = \mathcal{M}_r$.

We will have the following facts.

For $(\mathcal{E}, \mathcal{M}, \Phi)$ a proper afc over the Banach space X , and \mathcal{D} an admissible subalgebra of \mathcal{M}_b ; if $f \in \mathcal{M}$ and $g \in \langle D \rangle$ so that $\Phi(g)$ is injective and $gf \in \langle D \rangle$, then $f \in \langle D \rangle$.

Of course, for two admissible subalgebras $\mathcal{D}, \mathcal{D}'$ of \mathcal{M}_b , if $\mathcal{D}' \subset \langle D \rangle$, then $\langle \mathcal{D}' \rangle \subset \langle D \rangle$. A subalgebra \mathcal{D} of \mathcal{M}_b is a generator of the afc iff $\mathcal{E} \subset \langle D \rangle$.

(b) In this subsection we will describe the mathematical tools used in building what in subsection **(a)** is called a primary functional calculus; the operators of this calculus are sectorial (possibly unbounded) operators.

Before proceeding, a technical note is in order, given the mathematical apparatus that is about to be used. One will encounter in the pages that follow sentences that deal with the behaviour of functions at ∞ or with curves that pass through the point at infinity. Now, in the literature, some authors (e.g., Michele Audin, see below) interpret these sentences as sentences regarding the behaviour of the function $f\left(\frac{1}{z}\right)$ at zero and, in the case of complex integrals, one obtains the corresponding improper Riemann integral. One example is encountered in the following definition:

- It is said that a function f defined on the complement of a disk of \mathbf{C} (respectively, the complement of a disk minus a discrete set) is *holomorphic at ∞* (respectively *meromorphic at ∞*), if the application

$$z \mapsto f\left(\frac{1}{z}\right)$$

is holomorphic (respectively meromorphic) at 0. (Michele Audin [1], p. 74)

Now, whether one wants to actually add the point at infinity and work with $\mathbf{C} \cup \{\infty\}$, depends on the objectives that one pursues. Freitag and Busan [11], for example, define the following topology on $\mathbf{C} \cup \{\infty\}$:

- A subset $D \subseteq \hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ is open if:
 - (a) $D \cap \mathbf{C}$ is open in \mathbf{C} ;
 - (b) If $\infty \in D$, then there exists an $R > 0$ so that

$$D \supseteq \{z \in \mathbf{C} : |z| > R\}.$$

Now, using the conventions

$$\frac{1}{0} = \infty, \quad \frac{1}{\infty} = 0,$$

they define a function

$$f: D \rightarrow \mathbf{C}, D \subset \hat{\mathbf{C}} \text{ open}$$

to be *meromorphic* if (a) it is meromorphic in $D \cap \mathbf{C}$;

$$(b) \hat{f}(z) := f\left(\frac{1}{z}\right) \text{ is meromorphic in the open set } \hat{D} = \{z \in \mathbf{C} : 1/z \in D\} \text{ (pp. 156-}$$

157).

Again, the transformation $z \rightarrow \frac{1}{z}$ is essential; one will obtain, for example, the Laurent expansion of a function f at ∞ , by the expansion of \hat{f} at 0. Other concepts, including the various types of singularities carry over in the same fashion.

As we have mentioned, one can deal with integration on curves that pass through infinity (a line, for example) by writing them as improper Riemann integrals. A more elegant and thorough treatment (although, in terms of actually calculating the integral, it will come down to the same thing) would be to consider them as curvilinear integrals on the Riemann sphere, considered as a Riemann surface. Things become a little more complicated, in that one cannot integrate directly functions on an abstract surface, only differential forms.

The Riemann sphere, viewed as a Riemann surface, is not difficult to deal with; one has two charts,

$$z \rightarrow z, \text{ for the whole of the sphere without } \{\infty\}, \text{ and}$$

$$z \rightarrow \frac{1}{z}, \text{ for a neighborhood of infinity,}$$

the values of the charts lying, of course, in \mathbf{C} .

The mathematics used is very elegant, as it often happens in differential geometry; one integrates a differential form on the sphere, and calculates the integral by pulling it back to \mathbf{C} through the parametrizations that are the inverses of the charts. It is interesting that in the second parametrization, a circle around zero becomes a circle around infinity. So, for example, if one has a function analytic on \mathbf{C} , except for a finite number of poles, if one integrates the function on a circle around zero whose radius is greater than the modulus of every pole, one can then, by using the second chart, consider the curve as a curve around infinity (reversing also its orientation, of course and taking into account the change of parametrization), so that, in calculating its integral one is concerned with the behaviour of the function at infinity. In the case at hand, one is concerned over whether the function has a residue at infinity or not. If the function is holomorphic at ∞ , then we will have a version of Cauchy's theorem for the integral and its value will thus be 0.

When dealing strictly with \mathbf{C} , the residue at infinity can be defined as

$$Res_{\infty} f = Res_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right]$$

The discussion above can then be summed up in the following theorem:

Theorem 1.17. If f is analytic everywhere on \mathbf{C} except for a finite number of singular points interior to a counterclockwise contour C , then

$$\int_C f(z)dz = 2\pi i \operatorname{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right].$$

The proof is very simple (it basically amounts to a change of variables, without forgetting that this change also reverses the orientation of the circle) and will not be given here. Notice, again, that, in terms of the Riemann sphere, we are dealing with a change of parametrization (or of chart; we are using $w = \frac{1}{z}$). Obviously, when it comes to integrating over a curve that passes through ∞ , one again will be interested in the behaviour of the function at ∞ . If the function is analytic at infinity, one may very well not only integrate over such a curve, but also use various Cauchy-like theorems.

For $\omega \in [0, \pi)$, denote by S_ω the sector of angle 2ω around the positive axis (if $\omega = 0$, then we are dealing with the positive real axis); we have that an operator A on X is *sectorial* of angle ω , written $A \in \operatorname{Sect}(\omega)$, if:

- (1) $\sigma(A) \subset \overline{S_\omega}$;
- (2) $\sup\{\|\lambda R(\lambda; A)\|: \lambda \in \mathbf{C} \setminus \overline{S_{\omega'}}\} < \infty, \forall \omega' \in (\omega, \pi)$.

Notice that in the terminology used here (borrowed from Hasse's book [13], p. 19), a Tadmor-Ritt operator does not qualify as a sectorial operator. However, it is easily seen, that all the mathematics developed here with regard to what Hasse calls sectorial operators can be transferred to the case of Tadmor-Ritt operators.

Let $\varphi \in (0, \pi]$ and let $f \in \mathcal{M}(S_\varphi)$. We say that f has *polynomial limit* $c \in \mathbf{C}$ at 0 if there is $\alpha > 0$ such that $f(z) - c = O(|z|^\alpha)$ as $z \rightarrow 0$. We say that f has *polynomial limit* ∞ at 0 if $1/f$ has polynomial limit 0 at 0. Similarly, we say that f has *polynomial limit* $d \in \hat{\mathbf{C}}$ at ∞ if $f(z^{-1})$ has polynomial limit d at 0.

If the respective limit belongs to \mathbf{C} (in various cases), then we say that f has *finite polynomial limit*. Also, if f has polynomial limit 0 at 0 (at ∞), we call f *regularly decaying* at 0 (at ∞).

Next, we define the Riesz-Dunford class on S_φ :

$$H_0^\infty(S_\varphi) := \{f \in H^\infty(S_\varphi) : f \text{ is regularly decaying at 0 and at } \infty\}$$

endowed with the supremum norm $\|f\|_\infty$.

Given the behavior at zero and infinity of the functions from the defined class, we will have

$$f(z) \in H_0^\infty(S_\varphi) \Rightarrow f\left(\frac{1}{z}\right) \in H_0^\infty(S_\varphi).$$

The Riesz-Dunford class is not quite enough for our purposes; we must, therefore, define the extended Riesz-Dunford class:

$$\mathcal{E}(S_\varphi) = H_0^\infty(S_\varphi) \oplus \langle (1+z)^{-1} \rangle \oplus \langle \mathbf{1} \rangle.$$

We have the following important lemma:

Lemma 1.18 (Haase [13]). Let $\varphi \in (0, \pi]$, $f: S_\varphi \rightarrow \mathbf{C}$ holomorphic. Then

(i) $f \in \mathcal{E}(S_\varphi)$

iff

(ii) f is bounded and has finite polynomial limits at 0 and ∞ .

(a complete proof can be found in the reference, p. 28).

Let $\varphi \in (0, \pi)$. We denote by $\Gamma_\varphi := \partial S_\varphi$ the boundary of the sector S_φ , oriented in the positive sense, i.e.,

$$\Gamma_\varphi := -\mathbf{R}_+ e^{i\varphi} \oplus \mathbf{R}_+ e^{-i\varphi}.$$

For $\varphi \in (\omega, \pi)$, $f \in H_0^\infty(S_\varphi)$ we define

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma_{\omega'}} f(z) R(z, A) dz$$

where $\omega' \in (\omega, \varphi)$ is arbitrary (see Figure 1).

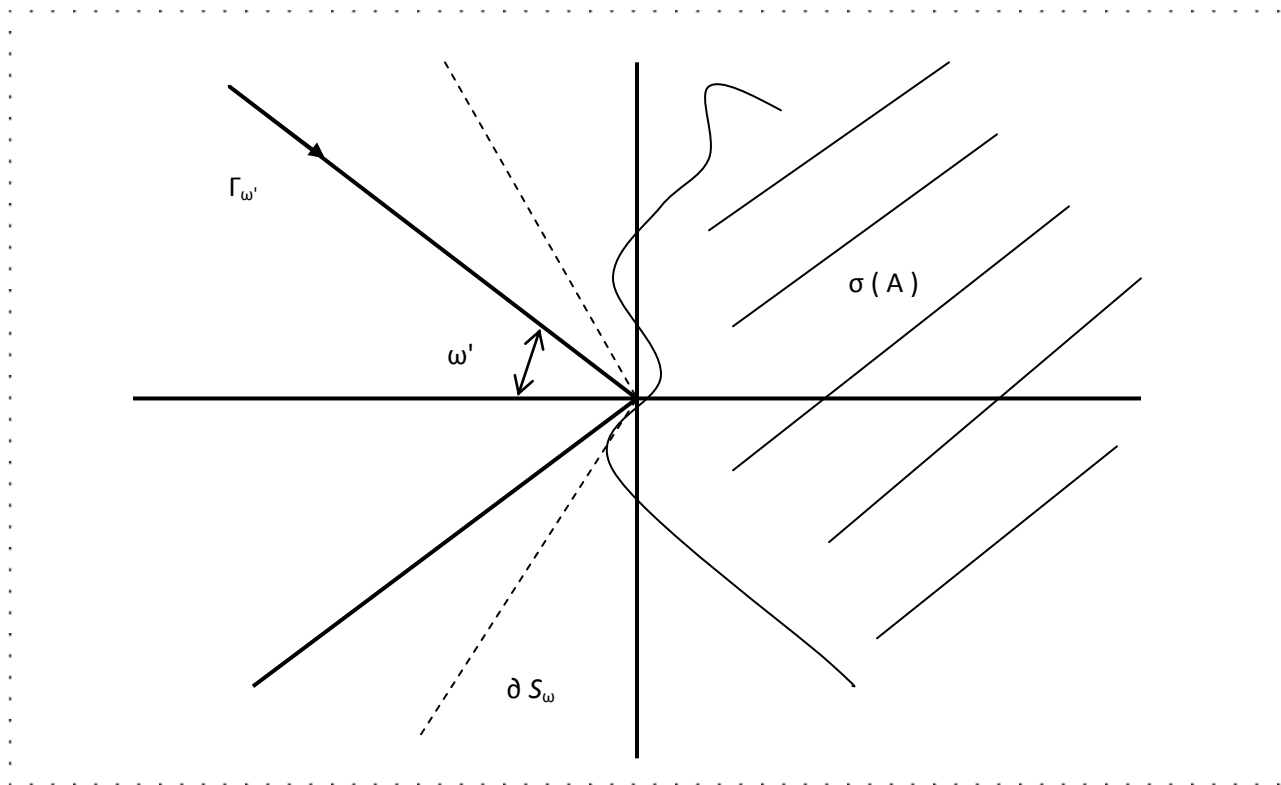


Figure 1. $\Gamma_{\omega'}$.

Lemma 1.19 (Haase [13]). Let $A \in \text{Sect}(\omega)$, and let $\varphi \in (\omega, \pi)$. Then we have the following:

(a) The mapping $(h \mapsto h(A)): H_0^\infty(S_\varphi) \rightarrow \mathcal{L}(X)$ is a homomorphism of algebras.

(b) If B is a closed operator commuting with the resolvent of A , then B commutes with $f(A)$. In particular, $f(A)$ commutes with A and with $R(\lambda, A)$ for all $\lambda \in \rho(A)$.

(c) We have $R(\lambda, A)f(A) = ((\lambda - z)^{-1}f)(A)$ for each $\lambda \notin cl(S_\varphi)$.

Proof. (a) This part is quite straightforward and it involves essentially Fubini's theorem and the resolvent identity. Although it is a beautiful proof, it is not very complicated and it is too long, so it will be omitted here; for details regarding the calculation, see Bosch and Swartz, *Functional Calculi*, pp. 91-92.

For (b), we have

$$Bf(A)x = \frac{1}{2\pi i} \int_{\Gamma} f(z)BR(z, A)dz x = \frac{1}{2\pi i} \int_{\Gamma} f(z)R(z, A)Bdz x = \frac{1}{2\pi i} \int_{\Gamma} f(z)R(z, A)dz Bx = f(A)Bx$$

(c) Define $g := (\lambda - z)^{-1}f$ and $\Gamma := \Gamma_\omega$. Then

$$\begin{aligned} (\lambda - A)g(A) &= \frac{1}{2\pi i} \int_{\Gamma} g(z)(\lambda - A)R(z, A)dz \\ &= \frac{1}{2\pi i} \int_{\Gamma} g(z)[(\lambda - z)R(z, A) + I]dz \\ &= \frac{1}{2\pi i} \int_{\Gamma} f(z)R(z, A)dz + \frac{1}{2\pi i} \int_{\Gamma} g(z)dz \\ &= f(A). \end{aligned}$$

since the latter summand equals 0 by Cauchy's theorem. ■

We now extend the definition of $f(A)$ from $H_0^\infty(S_\varphi)$ to all $f \in \mathcal{E}(S_\varphi)$. This can be done by defining

$$g(A) := f(A) + c(1 + A)^{-1} + d$$

for $g = f + (1 + z)^{-1} + d$ with $f \in H_0^\infty(S_\varphi), c, d \in \mathbf{C}$. We will show that this yields an algebra homomorphism

$$\Phi(A) := (g \mapsto g(A)): \mathcal{E}(S_\varphi) \rightarrow \mathcal{L}(X)$$

which gives rise to a meromorphic functional calculus $(\mathcal{E}(S_\varphi), \mathcal{M}(S_\varphi), \Phi)$.

In order to do this, we first define

$$H_{(0)}^\infty(S_\varphi) := \{f \in H^\infty(S_\varphi) : f \text{ is regularly decaying at } \infty \text{ and holomorphic at } 0\}.$$

Lemma 1.20 (Haase [13]). Let $f \in H_{(0)}^\infty(S_\varphi), \omega' \in (\omega, \varphi)$, and let $\delta > 0$ be small enough so that f is holomorphic in a neighborhood of $cl(B_\delta(0))$. Then

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z)R(z, A)dz$$

where $\Gamma = \Gamma_{\omega', \delta}$ is the positively oriented boundary of $S_{\omega'} \cup B_\delta(0)$ (see Figure 2).

Proof. First, note that holomorphy at 0 ensures that f belongs to the extended Riesz-Dunford class $\mathcal{E}(S_\varphi)$. If $f \in H_{(0)}^\infty(S_\varphi) \cap H_0^\infty(S_\varphi)$ then the statement is true since one can shrink the path around the point 0 to 0 without changing the value of the integral (it is a consequence of Cauchy's theorem that one can deform the path within the integrand's domain of analyticity, without changing the value of the integral). As a general $f \in H_{(0)}^\infty$ may be written as $f = g + c(1 + z)^{-1}$ with $g \in H_{(0)}^\infty \cap H_0^\infty$, we are left to show the claim for $f(z) = (1 + z)^{-1}$. The idea is to introduce the contour $\Gamma' := -\Gamma_{\omega', R}$ with $R > 1$. Cauchy's theorem will imply that

$$\int_{\Gamma'} (1+z)^{-1} R(z, A) dz = 0.$$

The reason for this is given by the following calculation:

Take $h(z) = \frac{1}{1+z} \frac{1}{z-A}$. Then

$$\frac{1}{z^2} h\left(\frac{1}{z}\right) = \frac{1}{z^2} \frac{1}{1 + \frac{1}{z}} \frac{1}{\frac{1}{z} - A} = \frac{1}{z^2} \frac{z}{z+1} \frac{z}{1-zA} = \frac{1}{(z+1)(1-zA)}$$

which is analytic at 0 (so h has no residue at ∞ ; see also the comments at the beginning of this subsection).

However, if we add the integral over Γ' and the initial integral over Γ , some parts cancel and there remains only a simple closed curve around the singularity -1 . To find out the residue at -1 , given that the resolvent function is analytic at this point, we expand

$$\begin{aligned} \frac{1}{1+z} &= \frac{1}{1-(z+1)} \frac{-z}{z+1} = \frac{-z}{z+1} (1 + (z+1) + (z+1)^2 + \dots) \\ &= \left(-1 - \frac{1}{z+1}\right) (1 + (z+1) + (z+1)^2 + \dots) \end{aligned}$$

so the residue at -1 is

$$-1 \cdot \frac{1}{-1-A} = \frac{1}{1+A}.$$

■

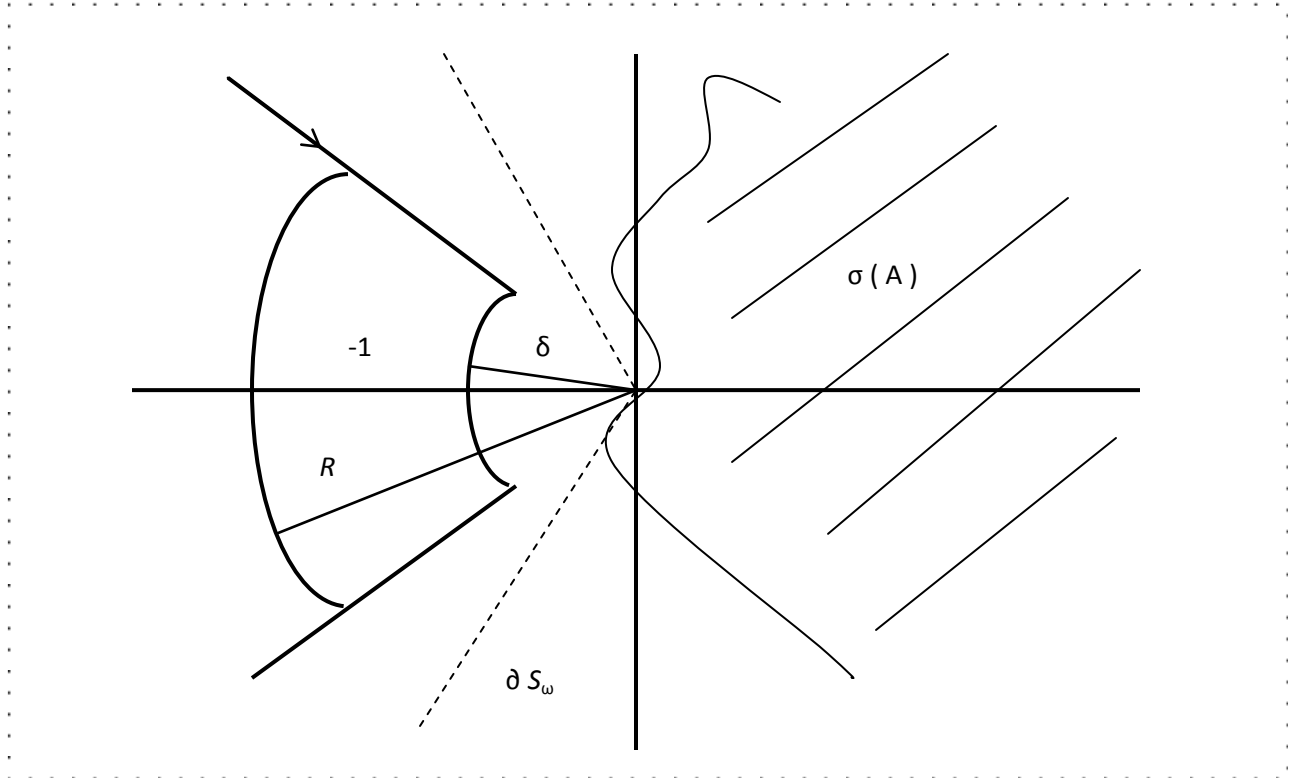


Figure 2. $\Gamma_{\omega',\delta}$ and $\Gamma_{\omega',R}$.

We have the following theorem:

Theorem 1.21 (Haase [13]). Let $A \in Sect(\omega)$ on X , and let $\varphi \in (\omega, \pi)$. The mapping

$$\Phi_A := (g \mapsto g(A)): \mathcal{E}(S_\varphi) \rightarrow \mathcal{L}(X)$$

defined above is a homomorphism of algebras. Moreover, it has the following properties:

(a) $(z(1+z)^{-1})(A) = A(1+A)^{-1}$.

(b) If B is a closed operator commuting with the resolvent of A , the B also commutes with $f(A)$ for each $f \in \mathcal{E}(S_\varphi)$. In particular, each $f(A)$ commutes with A .

(c) If $x \in \mathcal{N}(A)$ and $f \in \mathcal{E}(S_\varphi)$, then $f(A)x = f(0)x$.

(d) Let B be the injective part of A . Then $Y := \overline{\mathcal{R}(A)}$ is invariant under the action of each $f(A)$, and one has $f(B) = f(A)|_Y$.

Proof. We see that for two functions $g_i = f_i + c_i(1+z)^{-1} + d_i$, ($i = 1,2$) in $\mathcal{E}(S_\varphi)$, with $f_i \in H_0^\infty$, $c_i, d_i \in \mathbf{C}$, one has to deal separately with all the mixed products (linearity is obvious, given that we are working with an integral). When one of the members of the product is a constant, the results are immediate; when we have products of the form $f_1 f_2$ or $f_i(1+z)^{-1}$, one uses **lemma 1.19**, while for products of the form $(1+z)^{-1}(1+z)^{-1} = (1+z)^{-2}$ one uses **lemma 1.20** plus Fubini's theorem and the resolvent identity.

For (c), we have: $x \in \mathcal{N}(A), z \in \rho(A)$ imply that: denote $y = R(z, A)x$, therefore $y = \frac{1}{z-A}x, x = (z-A)y = zy - A(y) = zy$ (because the resolvent commutes with A), so $R(z, A)x = \left(\frac{1}{z}\right)x$. Therefore

$$g(A)x = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(z)}{z} dz x = 0 \cdot x = 0,$$

for $g \in H_0^\infty$ (by Cauchy's theorem; so we have, in part what we expected, since functions in this class vanish at zero). So, for $f \in \mathcal{E}$, we will obtain immediately the desired result (one notices that $\frac{1}{1+A}x = x = 1 \cdot x = \frac{1}{1+z}(1) \cdot x$).

(a complete proof can be found in the reference, pp. 33-34)

■

We call the algebra homomorphism

$$\Phi_A := (f \mapsto f(A)): \mathcal{E}(S_\varphi) \rightarrow \mathcal{L}(X)$$

the *primary functional calculus* on S_φ for A as a sectorial operator.

Corollary 1.22 (Haase [13]). Let $A \in Sect(\omega)$, $\varphi \in (\omega, \pi)$, and let $f \in \mathcal{O}(S_\varphi)$ be holomorphic at both points 0 and ∞ . Then $f \in E(S_\varphi)$ and $f(A)$ is given by

$$f(A) = f(\infty) + \frac{1}{2\pi i} \int_{\Gamma} f(z)R(z, A)dz$$

with the contour $\Gamma = \Gamma_{\omega', \delta, R}$ from Figure 3.

Proof. We set $g(z) = f(z) - f(\infty) \in H_{(0)}^\infty$. Then, we will have $f(A) = f(\infty) + g(A)$; for $g(A)$ we can use the formula from lemma 7

$$g(A) = \frac{1}{2\pi i} \int_{\Gamma_{\omega', \delta}} g(z)R(z, A)dz$$

and then, using the fact that the integrand is holomorphic in a neighborhood of ∞ , we deform the path of integration to a finite path $\Gamma_{\omega', \delta, R}$ (basically, we take R so big that no pole lies outside the circle of center 0 and radius R , and we take the path of integration so that the interior is the part that contains ∞ , and so, by Cauchy's theorem the value of the integral remains unchanged), so

$$g(A) = \frac{1}{2\pi i} \int_{\Gamma_{\omega', \delta, R}} (f(z) - f(\infty))R(z, A)dz = \frac{1}{2\pi i} \int_{\Gamma_{\omega', \delta, R}} f(z)R(z, A)dz - \frac{1}{2\pi i} \int_{\Gamma_{\omega', \delta, R}} f(\infty)R(z, A)dz$$

(it makes sense to break the integral into two, because now we are dealing with a domain in which both integrands are holomorphic functions). The second term vanishes (again by Cauchy's theorem), so we are left with the desired formula. ■

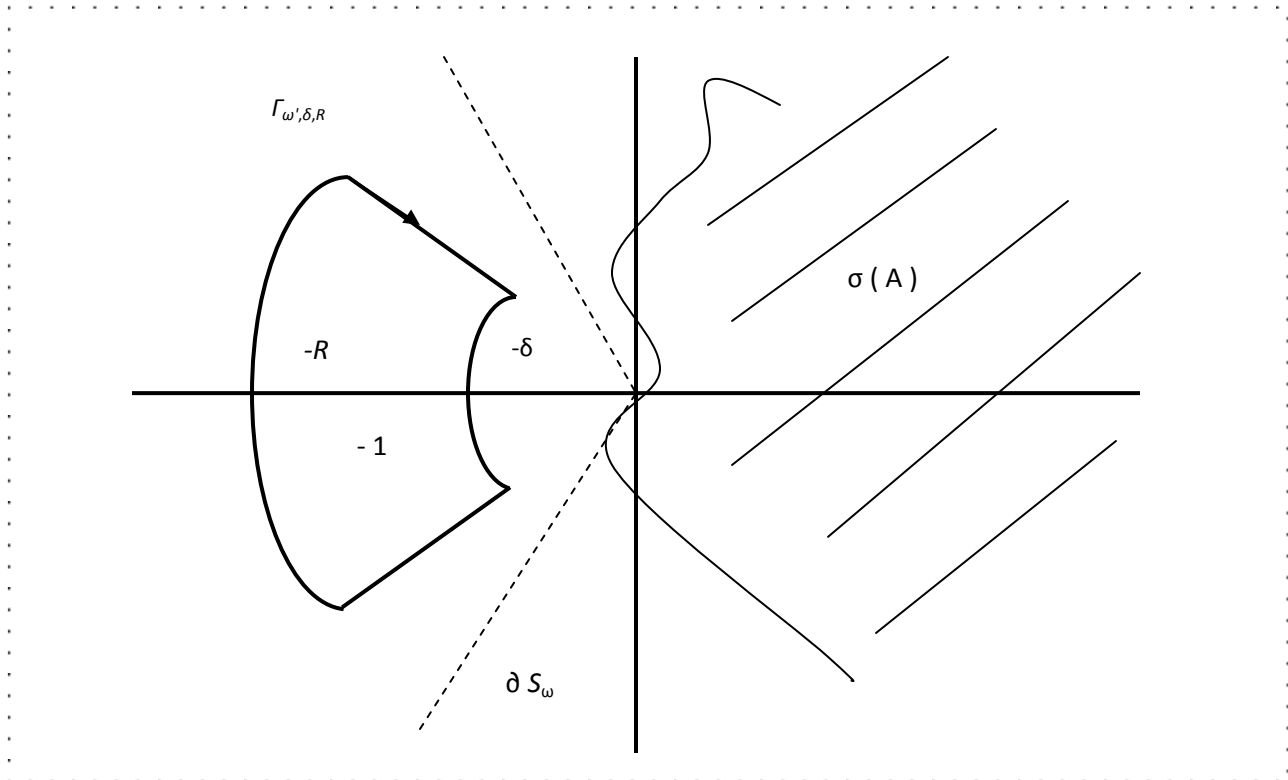


Figure 3. $\Gamma_{\omega', \delta, R}$.

As mentioned, the mathematical structure built so far is only a primary functional calculus; one will then extend it, as described in the first part, to what Hasse calls the *natural functional calculus* for sectorial operators. I will not, however, prolong the discussion farther than this point; the interested reader may consult Hasse's book. Extending the functional calculus opens the way for exciting mathematics, but I am here interested more in describing a procedure than in filling in the details. Notice that although the primary functional calculus operates with bounded operators, it is not created for a bounded operator: A is sectorial and may be unbounded. Another way to see this is by noticing that the function $f(z) = z$ does not belong to the extended Riesz-Dunford class, so that the operator A will not exist as a value of a function $f(A)$. Also, one must notice the similarity of the procedure encountered here with what we have seen in the case of the Weyl calculus; namely, the starting point is given by a class of decaying functions for which a calculus is given by an integral formula, and which is then extended through the properties

of this formula. These facts, along with integrating along curves that pass through the infinity point, make the mathematical facts presented in this section very exciting.

References.

The source of much of the chapter is Bosch and Swartz [4]; thus, section 1 has as its source ch. 6, while section 2, ch. 7 and 8. See also the paper by Nikolski [24]. The part regarding the Bochner integral is also from Bosch and Swartz [4], ch. 5; the construction of the spectral integral is taken from MacCluer [19], ch. 6. See also the more general concept of *operator-valued measure* in Bosch and Swartz [4], ch. 1. The material in section 3 is largely drawn from Haase [13]; I have supplanted it with facts related to complex analysis (see Bibliography); for integration on a Riemann surface, see Miranda [22], ch. IV.

Chapter II

This chapter deals with conditions on operators that ensure power boundedness. It comes as a natural continuation of the previous chapter, in that the proofs that these conditions imply power boundedness use the Riesz-Dunford functional calculus. The Tadmor-Ritt operators are at the center of the chapter.

1. Characterization of Tadmor-Ritt operators

We will start this part with the following theorem that provides a characterization of Tadmor-Ritt operators.

First, we define the following sets:

$$\Sigma_C = \mathbf{D} \setminus \bigcup_{|z|=1} \bar{D}\left(z, \frac{|z-1|}{C}\right), C \text{ a constant,}$$

$$S_\theta = \{z \in \mathbf{C} : |\arg(z-1)| \leq \pi - \theta\},$$

$$\mathcal{B}_C = \mathcal{B}(\delta, \sin \delta) = \left\{z \in \mathbf{D} \setminus S_\delta : |\arg z| < \frac{\pi}{2} - \delta\right\} \cup D(0, \sin \delta), \text{ where } \delta = \arccos \frac{1}{C}.$$

The third set \mathcal{B}_C is an example of a Stolz domain; these sets are used in complex analysis on the disk in order to investigate how functions on the unit disk approach their values on the unit circle (see Figure 4).

In the following, we will also assume as known the **Katznelson-Tzafriri Theorem [Katznelson, Tzafriri [14]:**

- Let T be a linear contraction on a Banach space X . Then $\lim_{n \rightarrow \infty} \|T^n - T^{n+1}\| = 0$ if and only if the peripheral spectrum $\pi\sigma(T)$ of T (the intersection of the spectrum with the unit circle) consists of at most the point $z = 1$.

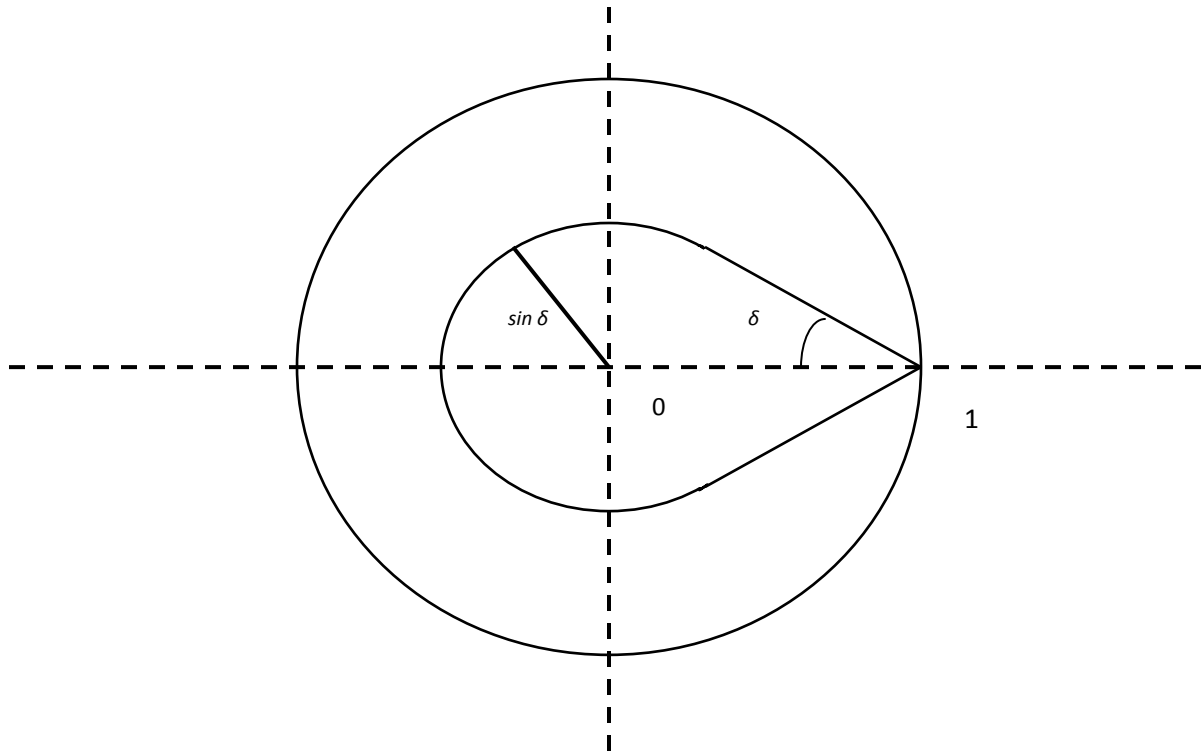


Figure 4. The set \mathcal{B}_C .

Theorem 2.1 (Malinen, Nevanlinna, Yuan [20]). The following are equivalent:

- (i) T satisfies (0.2) for all $|\lambda| > 1$;
- (ii) $\sigma(T) \subseteq \mathbf{D} \cup \{1\}$ (it is actually included in a Stolz cone) and there exists $\delta > 0$ and $C = C(\delta)$ such that (1.2) holds for all $\lambda \in S_\delta$, and
- (iii) T is power bounded, and it satisfies the Tauberian condition

$$\sup_{n \geq 1} (n + 1) \| (I - T)T^n \| \leq M \tag{2.1}$$

for some $M < \infty$.

Proof: (i) \rightarrow (ii). We start by proving that if T is a Tadmor-Ritt operator with constant C , then $\sigma(T) \subseteq \overline{\mathcal{B}_C}$.

This is done in two steps:

(a) We prove that $\sigma(T) \subseteq \overline{\Sigma_C}$. This is achieved by adding to the Tadmor-Ritt inequality the following inequality for the resolvent: $\|R(z, T)\| \geq \frac{1}{\text{dist}(z, \sigma(T))}$. These two inequalities will give us $\text{dist}(z, \sigma(T)) \geq \frac{|z-1|}{C}$.

(b) We show that $\Sigma_C \subseteq \mathcal{B}_C$. In order to do this, we will show that if $z \notin \mathcal{B}_C$, then $z \notin \Sigma_C$. We have two cases:

(b1) $z \in S_\delta$. Let $\alpha_0 = |\arg(z-1)| - \frac{\pi}{2}$; then $0 < \alpha_0 \leq \frac{\pi}{2} - \delta$. We denote by z_0 the point such that $\text{Re}(z_0) = 1$ and $\left| \arg\left(\frac{z-z_0}{1-z}\right) \right| = \frac{\pi}{2}$. We will have

$$\sin \alpha_0 = \left| \frac{z-z_0}{z_0-1} \right| \leq \sin\left(\frac{\pi}{2} - \delta\right) = \frac{1}{C}$$

so $|z-z_0| \leq \frac{|z_0-1|}{C}$ with $|z_0| \geq 1$ and thus $z \notin \Sigma_C$.

(b2) $z \in \mathbf{D} \setminus S_\delta$. Let $z_1 = \frac{z}{|z|}$. To show that $z \notin \Sigma_C$, it is sufficient to show that $|z-z_1| \leq \frac{|z_1-1|}{C}$. This follows from the fact that $|z-z_1| \leq 1 - \sin \delta \leq (\cos \delta)^2 = \frac{1}{C^2}$ and $|z_1-1| \geq \cos \delta = \frac{1}{C}$.

See Figure 5.

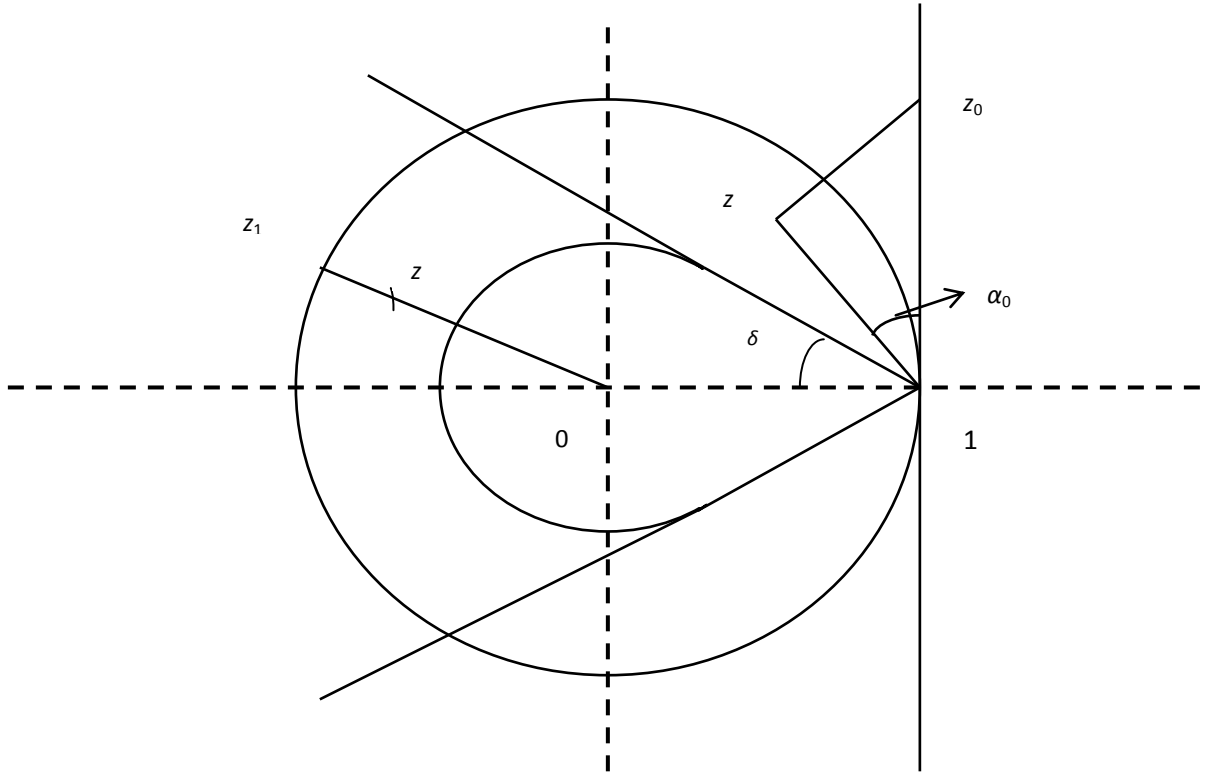


Figure 5.

So, we will have $S_\delta / \{1\} \subseteq \rho(T)$. We still need to prove that the Tadmor-Ritt condition applies to these new points of the resolvent set. We will prove that for any point μ in a sector $S_{\delta'}$, with δ' slightly greater than δ , we will have

$$R(\mu, T) < \frac{C(\delta')}{|\mu - 1|}.$$

Take a point $\mu \in S_{\delta'}$, with $|\mu| < 1$. The vector $\mu - 1$ determines a chord in the unit disk. Consider the corresponding arc of the unit circle lying in the sector and let λ be the point in the arc whose orthogonal projection onto the chord is μ . Let θ be the acute angle between the vectors $\mu - 1$ and $\lambda - 1$. Then

$\theta < \frac{\pi}{2} - \delta$, hence $\sin\theta < \cos\delta' < \frac{1}{C}$. Looking at the triangle with vertices $1, \lambda, \mu$ we see that $|\mu - \lambda| = |\lambda - 1|\sin\theta$ (See Figure 6.) The Tadmor-Ritt condition yields

$$\|R(\lambda, T)\| |\mu - \lambda| \leq \frac{C|\mu - \lambda|}{|\lambda - 1|} = C\sin\theta < C\cos\delta' < 1.$$

Now we can apply the following formula

$$R(\mu, T) = \sum_{k=0}^{\infty} R(\lambda, T)^{k+1} (\mu - \lambda)^k$$

to get

$$\|R(\mu, T)\| \leq \frac{\|R(\lambda, T)\|}{1 - \|R(\lambda, T)\| |\mu - \lambda|} < \frac{\|R(\lambda, T)\|}{1 - C\cos\delta'}$$

Using the Tadmor-Ritt condition again we obtain the estimate:

$$\|R(\mu, T)\| < \frac{C}{1 - C\cos\delta'} \frac{1}{|\lambda - 1|} < \frac{C}{1 - C\cos\delta'} \frac{1}{|\mu - 1|}$$

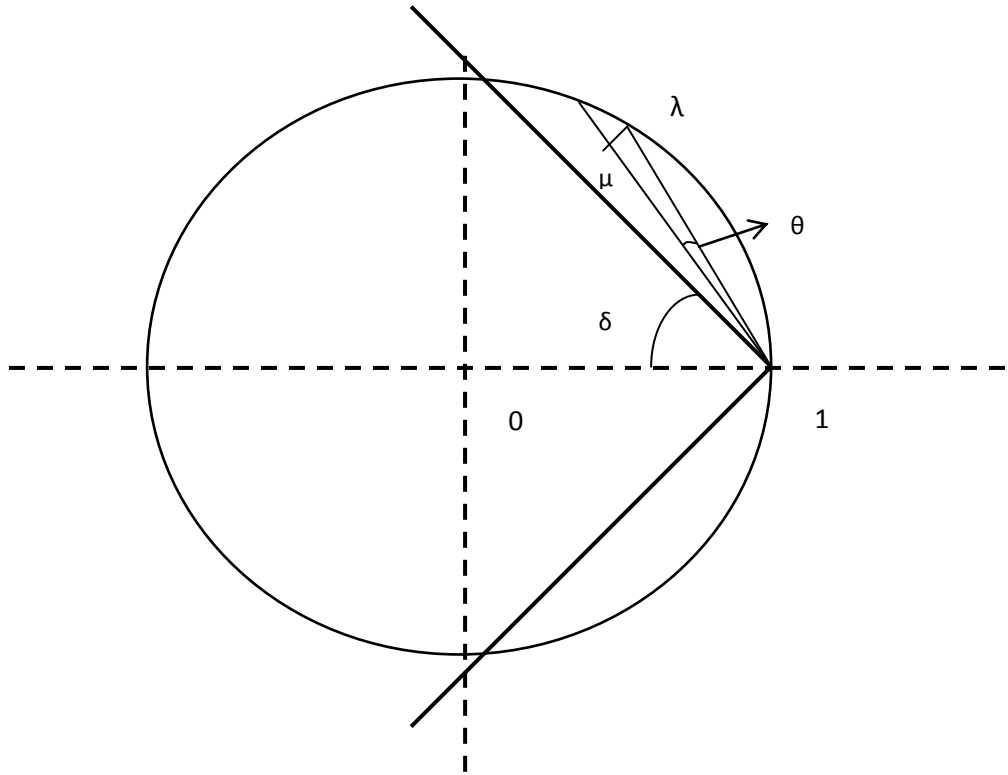


Figure 6.

(ii)→(iii). Consider the path in the integral

$$T^k(T-1) = \frac{1}{2\pi i} \int_{\Gamma} \lambda^k(\lambda-1)R(\lambda, T)d\lambda.$$

We decompose Γ into three parts $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ of which Γ_2 denotes a circular arc of the form $\{\lambda = \rho e^{i\theta} \mid \rho < 1 \text{ is fixed and } \theta \text{ varies}\}$ while Γ_1 is a line segment of the form $\{\lambda = 1 + \frac{1}{k} +$

$te^{i(\frac{\pi}{2}+\delta)}$: where $t \geq 0$ } (δ is the appropriately chosen sector angle; see the previous part); and Γ_3 is symmetric with Γ_1 (see Figure 7). The spectrum of the operator is included in a Stolz cone, so we may choose ρ in such a way that $\|(\lambda - 1)R(\lambda, T)\|$ is uniformly bounded over Γ_2 . For Γ_1 (and for Γ_3) this is also true; one has only to look again at the remarks on the geometry of the spectrum in the previous part. Notice that our definition of the path does depend on k and we need large enough values of k , otherwise the lines and the arcs need not intersect. However, for large values of k we have a connected path and the resolvent estimates then hold independently of k .

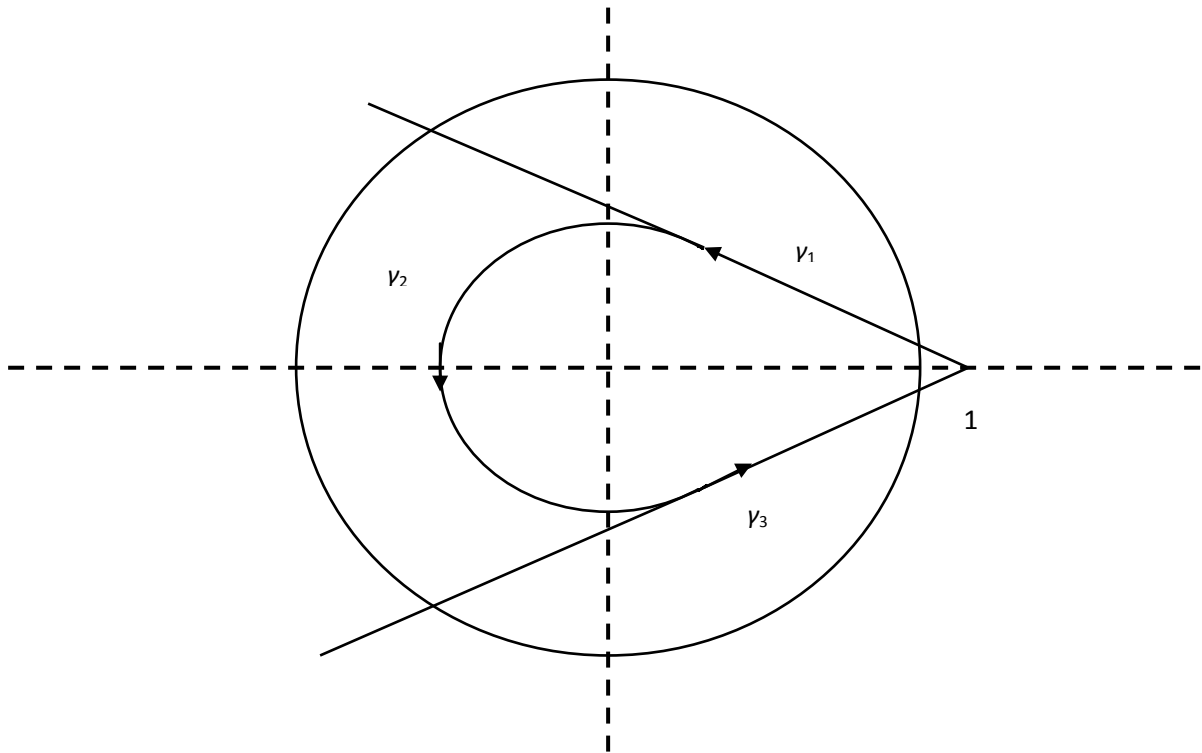


Figure 7.

We estimate the integral over Γ_2 first. We obtain:

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_2} \lambda^k (\lambda - 1) R(\lambda, T) d\lambda \right\| \leq C_1 \rho^k.$$

In order to estimate the integral over Γ_1 we need to estimate $|\lambda(t)|$. There exist positive constants, c_1 and c_2 , such that (for large k and all relevant t) $|\lambda(t)| \leq [1 + \frac{c_1}{k}] e^{-c_2 t}$. Thus

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_1} \lambda^k (\lambda - 1) R(\lambda, T) d\lambda \right\| \leq \frac{C}{2\pi} \int_0^\infty [1 + \frac{c_1}{k}]^k e^{-c_2 k t} dt \leq \frac{C e^{c_1}}{2\pi c_2 k}.$$

The integral over Γ_3 is analogous, so we have $\|T^k(T - 1)\| \leq C_2[\rho^k + \frac{1}{k}]$ and the desired result follows. We still have to show that T is power bounded. We use the same path for integration:

$$T^k = \frac{1}{2\pi i} \int_{\Gamma} \lambda^k R(\lambda, T) d\lambda$$

We start again with Γ_2 . We have

$$\frac{1}{2\pi} \int_{\Gamma_2} |\lambda^k| \|R(\lambda, T)\| |d\lambda| \leq C_3 \rho^k.$$

Further, over Γ_1 we have

$$\begin{aligned}
& \frac{1}{2\pi} \int_{\Gamma_1} \frac{|\lambda^k|}{|\lambda - 1|} |\lambda - 1| \|R(\lambda, T)\| |d\lambda| \\
& \leq \frac{C}{2\pi} \int_{\Gamma_1} \frac{|\lambda^k|}{|\lambda - 1|} |d\lambda| \\
& \leq \frac{C}{2\pi} \int_0^\infty e^{c_1} \frac{e^{-c_2 kt}}{\left| \frac{1}{k} + te^{i\left(\frac{\pi}{2} + \delta\right)} \right|} dt \\
& = \frac{Ce^{c_1}}{2\pi} \int_0^\infty \frac{e^{-c_2 \tau}}{\left| 1 + \tau e^{i\left(\frac{\pi}{2} + \delta\right)} \right|} d\tau =: C_4.
\end{aligned}$$

As the integral over Γ_3 is analogous we obtain $\|T^k\| \leq 2C_4 + C_3\rho^k$ and the proof is complete.

(iii)→(i). The proof of the third part is mainly a semigroup proof. Namely, it is the application of a general proof for a uniformly bounded C_0 semigroup $T(t)$ generated by the operator A to the special case of the semigroup $e^{t(T-1)}$ generated by $T - 1$, plus the fact that we are dealing with the discrete, and not the continuous, case. As such, it is divided into five parts:

a) We start the proof by showing the following:

$$\|(T - 1)e^{t(T-1)}\| \leq C \frac{1 - e^{-t}}{t}, \quad (t > 0).$$

We have

$$\|(T - 1)e^{tT}\| \leq \sum_{n=0}^{\infty} \|T^n(T - 1)\| \frac{t^n}{n!} \leq \frac{M}{t} \sum_{n=1}^{\infty} \frac{t^n}{n!} = M \frac{e^t - 1}{t},$$

and so, we obtain the desired result.

b) Now, we will prove that the C_0 semigroup $e^{t(T-1)}$ can be extended to an analytic semigroup $e^{z(T-1)}$ in a sector $\Delta_\delta = \{z: |\arg z| < \delta\}$ with $\|T(z)\|$ uniformly bounded in every closed subsector $\bar{\Delta}_{\delta'}, \delta' < \delta$ of Δ_δ .

Thus, we start from the fact that $e^{t(T-1)}$ is differentiable for $t > 0$, which implies that

$$\|(e^{t(T-1)})^{(n)}\| = \left\| \left(\left(e^{\frac{t}{n}(T-1)} \right)' \right)^n \right\| \leq \left\| \left(e^{\frac{t}{n}(T-1)} \right)' \right\|^n.$$

(notice that, in the last two derivatives, the variable is t/n .)

From this, together with the inequality $n! e^n \geq n^n$ and

$$\|(T-1)e^{t(T-1)}\| = \|(e^{t(T-1)})'\| \leq C/t \quad (t > 0)$$

we have that

$$\frac{1}{n!} \|(e^{t(T-1)})^{(n)}\| \leq \left(\frac{Ce}{t} \right)^n.$$

Next, consider the power series

$$e^{z(T-1)} = e^{t(T-1)} + \sum_{n=1}^{\infty} \frac{(e^{t(T-1)})^{(n)}}{n!} (z-t)^n.$$

This series converges uniformly in $\mathcal{L}(X)$ for $|z-t| \leq k(t/eC)$ for every $k < 1$. Therefore, $e^{z(T-1)}$ is analytic in $\Delta = \{z: |\arg z| < \arctan 1/Ce\}$. Since they have identical values for real numbers, $e^{z(T-1)}$ extends $e^{t(T-1)}$ to the sector Δ . Also, given that $e^{z(T-1)}$ is analytic, it follows that it is a semigroup, and from the above series expansion one can see that $e^{z(T-1)}x \rightarrow x$ as $z \rightarrow 0$ in Δ . This also shows that $\|e^{z(T-1)}\|$ is uniformly bounded in every closed subsector $\bar{\Delta}_\varepsilon = \{z: |\arg z| \leq \arctan(1/Ce) - \varepsilon\}$.

(c) The core of the proof resides in the following integral representation for the resolvent function:

$$(\lambda - T)^{-1} = \int_0^{\infty} e^{-(\lambda-1)t} e^{t(T-1)} dt, \quad \lambda > 1.$$

This part will consist in obtaining a first bound for the resolvent function.

As such, having extended the representation to complex numbers in a sector around the positive real axis, we let $0 < \delta' < \delta$ such that $\|e^{z(T-1)}\| \leq C_1$ for $\overline{\Delta_{\delta'}}$. Taking $x \in X$ and $\sigma > 1$, we have

$$(\sigma + i\tau - T)^{-1}x = \int_0^{\infty} e^{-(\sigma+i\tau-1)t} e^{t(T-1)} x dt$$

The analyticity and the uniform boundedness of the semigroup in Δ_{δ} , allows us to shift the path of integration in the above integral representation from \mathbf{R}^+ to any ray $\rho e^{i\nu}$ within the subsector. For $\tau > 0$, shifting the path of integration to the ray $\rho e^{-i\delta'}$ and estimating the resulting integral, we obtain that

$$\begin{aligned} \|(\sigma + i\tau - T)^{-1}x\| &\leq \int_0^{\infty} e^{-\rho((\sigma-1)\cos\delta' + \tau\sin\delta')} C_1 \|x\| d\rho \\ &\leq \frac{C_1 \|x\|}{(\sigma-1)\cos\delta' + \tau\sin\delta'} \leq \frac{C \|x\|}{\tau}. \end{aligned}$$

Notice that we must ensure that the real exponent in the above evaluation stays positive, otherwise the integral will diverge; but, of course, the proper angle δ can be found; for example, if the imaginary part is very large in absolute value, we can take the angle to be very small, thus its cosine will be large and its sine small. The same considerations play a role in the next inequalities above.

For negative imaginary part, we shift the path of integration to the ray $\rho e^{i\delta'}$ and we obtain the estimate

$$\|(\sigma + i\tau - T)^{-1}\| \leq C/\tau, \text{ and so we have}$$

$$\|(\sigma + i\tau - T)^{-1}\| \leq \frac{C}{|\tau|}.$$

(d) We approach now the final evaluation.

We know that, since $T - 1$ is the generator of a C_0 semigroup, we have $\|(\lambda - T)^{-1}\| \leq M_1/Re \lambda - 1$ for $Re \lambda > 1$. From (c) we have that $\|(\lambda - T)^{-1}\| \leq C/|Im \lambda|$ and so $\|(\lambda - T)^{-1}\| \leq C_1/|\lambda - 1|$ right of the imaginary axis. For $\sigma > 1$, we write the Taylor expansion for the resolvent around $\sigma + i\tau$:

$$(\lambda - T)^{-1} = \sum_{n=0}^{\infty} ((\sigma + i\tau - T)^{-1})^{n+1} (\sigma + i\tau - \lambda)^n.$$

The series converges in $\mathcal{L}(X)$ for $\|R(\sigma + i\tau; T)\| |\sigma + i\tau - \lambda| \leq k < 1$. In the above series expansion, we fix the imaginary part of λ to be τ , and using the estimate from point (b), we get that the series converges uniformly in $\mathcal{L}(X)$ for $|\sigma - Re \lambda| \leq k|\tau|/C$; but, as both $\sigma > 1$ and $k < 1$ are arbitrary, this will imply that we can extend the series representation to all λ with $Re \lambda - 1 \leq 0$ satisfying $\frac{|Re \lambda - 1|}{|Im \lambda|} < 1/C$ and so

$$\varrho(T) \supset \{\lambda: |\arg(\lambda - 1)| \leq \frac{\pi}{2} + \delta\}$$

with $\delta = k \arctan 1/C$, $0 < k < 1$. Moreover, in this region

$$\|(\lambda - T)^{-1}\| \leq \frac{C}{1 - k} \frac{1}{|\tau|} \leq \frac{\sqrt{C^2 + 1}}{(1 - k)} \frac{1}{|\lambda - 1|} = \frac{M}{|\lambda - 1|}.$$

(the latter estimates come from considering the inequalities that provide the conditions of membership in the sector, conditions who are given in terms of relations between the imaginary parts and the real parts of the complex numbers involved).

(e) We extend the inequality to every μ such that $|\mu| > 1$.

By the Katznelson-Tzafriri theorem, a power bounded operator satisfying $\sup\|T^n - T^{n+1}\| < \infty$ has at most one point in the peripheral spectrum $(\sigma(T) \cap \mathbf{T})$, namely the point $\{1\}$. Now, this means that, for the part of the complex plane that we are interested in (the part outside the sector), the points on the unit circle are part of the resolvent set. Take one such point μ , with imaginary part $Im\mu \geq 0$, such that we can take a point λ close to the boundary of the sector for which the Tadmor-Ritt inequality has already been proven, such that $\lambda - 1 \perp \mu - \lambda$ and the series

$$R(\mu, T) = \sum_{n=0}^{\infty} R(\lambda, T)^{n+1} (\mu - \lambda)^n$$

converges (see Figure 8). This will give us the estimate

$$\|R(\mu, T)\| \leq \frac{M}{|\lambda - 1|} \frac{1}{1 - k} = \frac{M}{|\mu - 1| \cos \theta} \frac{1}{1 - k}$$

where θ is the angle between $\mu - 1$ and $\lambda - 1$ and k is the factor that allows the convergence of the series. Notice that for each point on the ray that goes from 1 through μ the estimate remains valid, because the modulus of μ increases. We continue this procedure with this new ray and eventually evaluate the resolvent at every point outside the sector with modulus greater than 1. We can do this because we know that this area is part of the resolvent set. Also, for any point μ with modulus greater than 1, the resolvent is bounded above, so $\|R(\mu, T)\|^{-1}$ will be bounded below, which gives us a minimum radius of the balls through which we can extend the estimate.

■

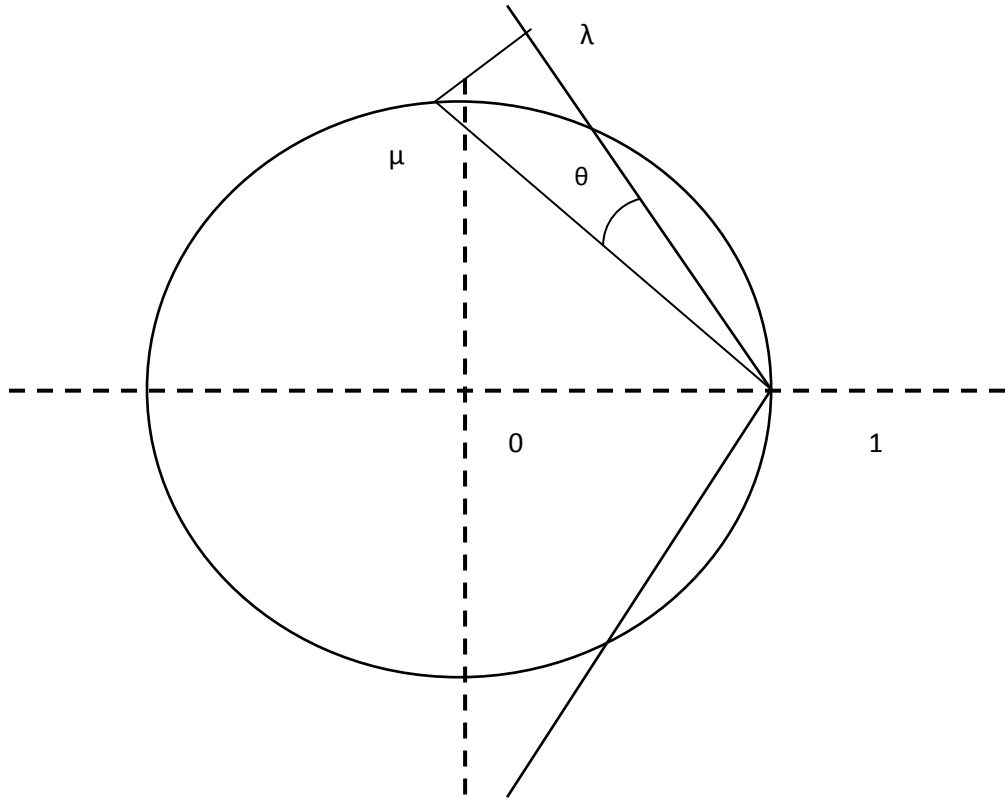


Figure 8.

We add the following theorem, which refines the previous result. We need first to introduce a new term: Let $a = (a_k)_{k \geq 0}$ be a sequence in a vector space; we write $\Delta a = (a_k - a_{k+1})_{k \geq 0}$, $\Delta^2 a = \Delta(\Delta a) = (a_k - 2a_{k+1} + a_{k+2})_{k \geq 0}$ and $\Delta^n a = \Delta(\Delta^{n-1} a)$. For $a = (T^k)_{k \geq 0}$, we write $(\Delta^n T^k)_{k \geq 0}$ for $\Delta^n a$, so $(\Delta^n T^k)_{k \geq 0} = (T^k (I - T)^n)_{k \geq 0}$.

Theorem 2.2 (Vitse [33]). Let $T: X \rightarrow X$ be a power bounded operator, where X is a Banach space, $c_0(T) := \sup_{k \geq 0} \|T^k\| < \infty$ and let $n > 0$ be an integer. The following are equivalent:

- (1) $c_n(T) := \sup_{k \geq 0} (k + 1)^n \|\Delta^n T^k\| < \infty$;
- (2) $c_1(T) := \sup_{k \geq 0} (k + 1) \|\Delta T^k\| < \infty$;
- (3) T is a Tadmor-Ritt operator.

Proof. (1) \Rightarrow (2). Since $c_0(T) < \infty$, the spectrum of T is contained in the closed unit disk: $\sigma(T) \subset cl(\mathbf{D})$. Moreover, the spectral mapping theorem implies that if $\lambda \in \sigma(T)$, $|\lambda| = 1$, then

$$|1 - \lambda|^n = |\lambda^k(1 - \lambda)^n| \leq \| \Delta^n T^k \|,$$

and hence $\lambda = 1$ because of condition (1). So, the only complex number of modulus 1 in the spectrum can be the point $\{1\}$. This implies that $\sigma(T) \subset \mathbf{D} \cup \{1\}$. By the Katznelson-Tzafriri theorem, we will have $\lim_k \| \Delta T^k \| = 0$, and hence $\lim_k \| \Delta^{n-1} T^k \| = 0$. Therefore,

$$\Delta^{n-1} T^k = \lim_m (\Delta^{n-1} T^k - \Delta^{n-1} T^m) = \lim_m \sum_{j=k}^{m-1} \Delta^n T^j,$$

and for $n \geq 2$,

$$\| \Delta^{n-1} T^k \| \leq \sum_{j \geq k} \| \Delta^n T^j \| \leq c_n(T) \sum_{j \geq k} \frac{1}{(j+1)^n} \leq \frac{2c_n(T)}{n-1} \frac{1}{(k+1)^{n-1}}$$

Therefore, T satisfies (1) also with $n - 1$. By induction, T satisfies (2). Obviously, $c_1(T) \leq 2c_2(T)$.

(2) \Rightarrow (3). Already shown, see **Theorem 2.1** in this chapter.

(3) \Rightarrow (1). For $n \geq 1$, this is shown (see **Theorem 2.1** in this chapter). For the general case, we have

$$\Delta^n T^k = \frac{1}{2\pi i} \int_{\gamma} z^k (1-z)^n R(z, T) dz.$$

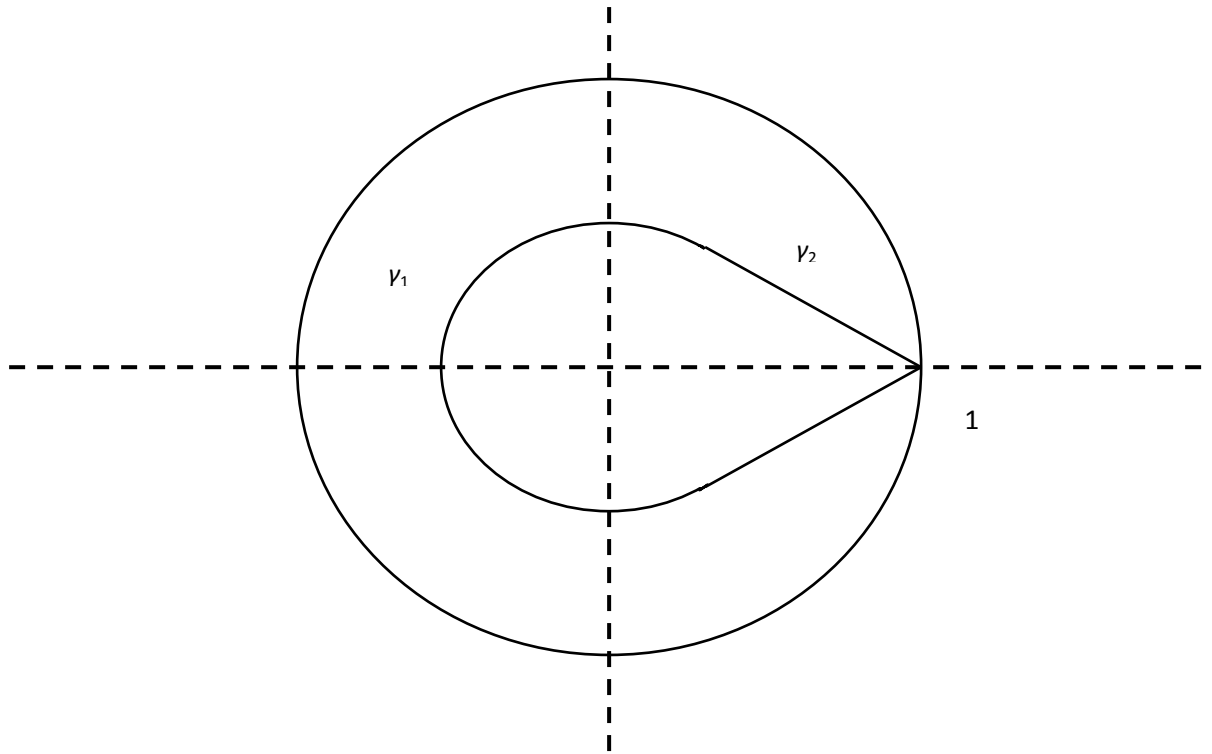


Figure 9. (Orientation: positive).

We know that $\sigma(T) \subset B_\theta \cup \{1\}$, where B_θ is a Stolz domain with $\theta = \arccos \frac{1}{2C(T)}$, and that $\|R(z, T)\| \leq \frac{2C(T)}{|z-1|}$ for $z \in \mathcal{C} \setminus cl(B_\theta)$. We integrate directly on the boundary of the spectrum in order to simplify the calculations that follow; to be mathematically correct, however, one would have to use a perturbation rT of T ($0 < r < 1$) if she\he wants to use integration on the boundary of the spectrum of T , and then pass to limit as $r \rightarrow 1$. This approach is used in chapter III (see Theorem 3.3, but the proof of this theorem is considerably shorter). However, this shortcut does not affect the final result, given that rT is also Tadmor-Ritt with constant $C(rT) \leq \frac{2}{1+r}C(T)$, and so passing to the limit would occasion no surprises. We will have:

$$\Delta^n T^k = \frac{1}{2\pi i} \int_{\partial B_\theta} z^k (1-z)^n R(z, T) dz.$$

Now,

$$\|\Delta^n T^k\| \leq \frac{1}{2\pi} 2C(T) \int_{\partial B_\theta} |z|^k |1-z|^{n-1} |dz| = \frac{C(T)}{\pi} \left(\int_{\gamma_1} + \int_{\gamma_2} \right),$$

where $\gamma_1 = \partial B_\theta \cap \{z \in \mathbf{C}: |z| = \sin\theta\}$, γ_2 is the union of the two segments $\{z = 1 - te^{\mp i\theta}, 0 \leq t \leq \cos\theta\}$ (see Figure 9). For γ_1 , we have $|z| = \sin\theta$, $|z-1| \leq 1 + \sin\theta$, and hence

$$\begin{aligned} \frac{C(T)}{\pi} \int_{\gamma_1} |z|^k |1-z|^{n-1} |dz| &\leq \frac{C(T)2\pi}{\pi} \sin\theta (1 + \sin\theta)^{n-1} (\sin\theta)^k \\ &\leq 2C(T)(1 + \sin\theta)^{n-1} \left(\frac{n}{e \log \frac{1}{\sin\theta}} \right)^n \frac{1}{(k+1)^n} \\ &\leq \left(\frac{16n}{e} \right)^n \frac{C(T)^{2n+1}}{(k+1)^n} \end{aligned}$$

since $\sup (t^n (\sin\theta)^t) = \left(\frac{n}{e \log \frac{1}{\sin\theta}} \right)^n$ and $\log \frac{1}{\sin\theta} = -\log(1 - \frac{1}{4C(T)^2})^{1/2} \geq \frac{1}{8C(T)^2}$.

For γ_2 , we have

$$\begin{aligned} \frac{C(T)}{\pi} \int_{\gamma_2} |z|^k |z-1|^{n-1} |dz| &\leq \frac{2C(T)}{\pi} \int_0^{\cos\theta} |1 - te^{i\theta}|^k t^{n-1} dt \\ &\leq \frac{2C(T)}{\pi} \int_0^{\cos\theta} (1 - t\cos\theta)^{k/2} t^{n-1} dt \end{aligned}$$

since $|1 - te^{i\theta}|^k = (1 + t^2 - 2t\cos\theta)^{k/2} \leq (1 - t\cos\theta)^{k/2}$ for $0 \leq t \leq \cos\theta$. Hence, taking $t\cos\theta = x$ and using $1 - x \leq e^{-x}$, we get

$$\begin{aligned}
\frac{C(T)}{\pi} \int_{\gamma_2} |z|^k |z - 1|^{n-1} |dz| &\leq \frac{2C(T)}{\pi(\cos\theta)^n} \int_0^{(\cos\theta)^2} (1-x)^{k/2} x^{n-1} dx \\
&\leq \frac{2^{n+1}C(T)^{n+1}}{\pi} \int_0^{(\cos\theta)^2} e^{-x\frac{k}{2}} x^{n-1} dx \\
&\leq \frac{2^{n+1}C(T)^{n+1}}{\pi} \int_0^\infty e^{-x\frac{k}{2}} x^{n-1} dx \\
&= \frac{2^{n+1}C(T)^{n+1}}{\pi} \left(\frac{2}{k}\right)^n \int_0^\infty e^{-y} y^{n-1} dy \\
&= \frac{2^{2n+1}C(T)^{n+1}(n-1)!}{\pi} \frac{1}{k^n} \\
&\leq \frac{2^{3n+1}C(T)^{n+1}(n-1)!}{\pi} \frac{1}{(k+1)^n}.
\end{aligned}$$

Finally, $\|\Delta^n T^k\| \leq c_n(T)(k+1)^{-n}$, where

$$c_n(T) \leq C(T)^{2n+1} \left(\left(\frac{16n}{e}\right)^n + \frac{2^{3n+1}(n-1)!}{\pi} \right).$$

■

As one can see from the previous theorems, there is a relation between the constant appearing in the resolvent condition and the constant that bounds the powers of the operators. The sharpest estimate known in the literature is given in a paper by Th. Ransford and O. El-Fallah [10], where the

bound for the powers of the operators is C^2 (C being the constant from the resolvent condition). Next, we present the proof which obtains this result.

The theorem, of which the desired result is an immediate corollary is the following:

Proposition 2.3 (El-Fallah, Ransford [10]). Let X be a complex Banach space and let T be a bounded linear operator on X . Suppose that T satisfies:

$$\| (I - zT)^{-1} \| \leq |f(z)| \quad (|z| < 1) \quad (2.2)$$

where $f(z) = \sum_0^\infty a_k z^k$ is a function holomorphic on the open unit disk. If p is a polynomial of degree n , then

$$\| p'(T) \| \leq \max_{\mathbf{T}} |p| \sum_0^{n-1} |a_k|^2.$$

Proof. First, notice that in the condition (2.2), we have $R(\frac{1}{2}, T)$. Given that the condition is valid for complex numbers inside the open unit disk, we must assume that the resolvent function is defined outside the closed unit disk. This is a method for moving inside the unit disk an investigation that usually takes place on its complement.

Let $0 < r < 1$. Using the Riesz-Dunford functional calculus and performing an integration by parts, we have

$$p'(T) = \frac{1}{2\pi i} \int_{|\lambda|=1/r} p'(\lambda)(\lambda I - T)^{-1} d\lambda = \frac{1}{2\pi i} \int_{|\lambda|=1/r} p(\lambda)(\lambda I - T)^{-2} d\lambda.$$

We make the substitution $z = \frac{1}{\lambda}$ and obtain

$$p'(T) = -\frac{1}{2\pi i} \int_{|z|=r} p\left(\frac{1}{z}\right) (I - zT)^{-2} dz.$$

Let g be a holomorphic function, in the open unit disk, to be chosen later. By Cauchy's theorem,

$$\int_{|z|=r} g(z)(I - zT)^{-2} dz = 0$$

and hence

$$p'(T) = -\frac{1}{2\pi i} \int_{|z|=r} \left(p\left(\frac{1}{z}\right) - g(z) \right) (I - zT)^{-2} dz.$$

Using (2.2), we obtain

$$\|p'(T)\| \leq \frac{1}{2\pi} \int_{|z|=r} |p(1/z) - g(z)| |f(z)|^2 |dz|.$$

We write $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $f_1(z) = \sum_{k=0}^{n-1} a_k z^k$. Then

$$f(z)^2 - f_1(z)^2 = z^n f_2(z)$$

for some f_2 holomorphic on the unit disk. We choose now $g(z)$:

$$g(z) = \frac{p(1/z)z^n f_2(z)}{f(z)^2}.$$

By the way it is defined, g is holomorphic on the unit disk, and

$$\left(p(1/z) - g(z)\right) f(z)^2 = p(1/z) f_1(z)^2.$$

Substituting this into the estimate for $\|p'(T)\|$ and letting $r \rightarrow 1$, we deduce

$$\|p'(T)\| \leq \frac{1}{2\pi} \int_{|z|=1} |p(1/z)| |f_1(z)|^2 |dz| \leq \max_{\mathbf{T}} |p| \sum_{k=0}^{n-1} |a_k|^2.$$

This completes the proof. ■

Theorem 2.4 (El-Fallah, Ransford [10]). We have the following sharpest estimate for a Tadmor-Ritt operator with constant C :

$$\|T^n\| \leq C^2.$$

Proof. The desired result is obtained from putting $f(z) = \frac{c}{1-z}$ and $p(z) = \frac{z^n}{n}$ in the result from the previous theorem. ■

Before we proceed to the following section, we would like to make a historical note. The Tadmor-Ritt condition was initially two conditions. The first one, the Ritt condition stated that

$$\sigma(T) \subseteq \mathbf{D} \cup \{1\} \text{ and}$$

$$\| (z - 1)(zI - T)^{-1} \| \leq M \text{ for all } z \in \mathbf{C} \text{ with } |z| \geq 1, 0 < |z - 1| \leq \eta. \quad (2.3)$$

Later, Tadmor proposed the following condition:

$$\sigma(T) \subseteq cl(\mathbf{D}) \text{ and}$$

$$\| (zI - T)^{-1} \| \leq \frac{L}{|z-1|} \text{ for all } z \in \mathbf{C} \text{ with } |z| > 1. \quad (2.4)$$

The next lemma shows that the two conditions are equivalent, which guarantees that the phrase "Tadmor-Ritt condition" is correct.

Lemma 2.5 (Borovykh, Drissi, Spijker [3]). There exist constants M and $\eta > 0$ such that the Ritt condition holds, if and only if, there is an L such that the Tadmor condition is valid.

Proof. We will use the following statement, which is based on the properties of the resolvent set and resolvent function (see the Introduction):

The function $R(\zeta, T)$ is analytic on $\rho(T)$. Furthermore, if $\xi \in \rho(T)$ and $\alpha = |\zeta - \xi| \|R(\xi, T)\| < 1$, then also $\zeta \in \rho(T)$ and $\|R(\zeta, T)\| \leq \frac{\|R(\xi, T)\|}{1-\alpha}$.

1. Assume (2.3). Since the function $F(\zeta) = |\zeta - 1| \|R(\zeta, T)\|$ is continuous on $\rho(T)$, and $F(\zeta) \rightarrow 1$ for $|\zeta| \rightarrow \infty$ (from the fact that the resolvent function is analytic plus a limit calculation) there is a finite constant K such that

$$F(\zeta) \leq K \text{ for all } \zeta \in \mathbf{C} \text{ with } |\zeta| \geq 1, |\zeta - 1| \geq \eta.$$

Condition (2.3) gives us the bound M for $F(\zeta)$ with $|\zeta| \geq 1, |\zeta - 1| \leq \eta$, so we obtain (2.4) with $L = \max\{K, M\}$.

2. For the second part, we use the statement at the beginning of the proof, in which one immediately recognizes the standard procedure for extending the resolvent set and resolvent function inequalities. So, assume that (2.4) is true, and let ζ be given with $|\zeta| = 1, \zeta \neq 1$. Choosing ξ , with $|\xi| > 1$, sufficiently close to ζ , we have, according to the statement at the beginning of the proof, that

$$\alpha = |\zeta - \xi| \|R(\xi)\| \leq |\zeta - \xi| \frac{L}{|\xi - 1|} < 1,$$

It follows that $\zeta \in \rho(T)$ with $|\xi - 1|(1 - \alpha) \|R(\zeta, T)\| \leq L$. We take the limit $\xi \rightarrow \zeta$, and obtain that $|\zeta - 1| \|R(\zeta, T)\| \leq L$.

Thus, we have (2.3) with $M = L$ and any $\eta > 0$.

■

2. Similar constraints

As mentioned before, in their seminal paper [10], El-Fallah and Ransford work with a condition that subsumes the Kreiss and Tadmor-Ritt conditions. Their condition takes the following form:

$$\| (T - \lambda I)^{-1} \| \leq \frac{C}{\text{dist}(\lambda, E)} \quad (|\lambda| > 1) \quad (2.5)$$

where E is a compact subset of the unit circle \mathbf{T} . To obtain an upper bound for powers of operators on complex Banach spaces, they use the following function:

$$\Phi_E(n) = \frac{n}{\pi} |E_{\pi/2n}|.$$

Now, a short presentation of notation is in order. First, E_δ stands for the δ - neighborhood of the set E on \mathbf{T} :

$$E_\delta = \{e^{i\theta} : \exists e^{i\varphi} \in E, |\theta - \varphi| \leq \delta\}.$$

$|E|$ is the Lebesgue measure of the set E ; we will also use $\#E$, which stands for the cardinal of E . We have the following immediate relations:

$$\lim_n \Phi_E(n) = \#E \quad \text{and} \quad \lim_n \frac{\Phi_E(n)}{n} = \frac{|E|}{\pi}.$$

El-Fallah and Ransford also introduce the following auxiliary functions. First, it must be mentioned that, given a compact E , its complement $\mathbf{T} \setminus E$ is composed of a at most countable set of disjoint open arcs I_j .

Now, we define

$$N_E(\delta) = \#\{I_j: |I_j| > \delta\},$$

$$L_E(\delta) = |\mathbf{T} \setminus \cup \{I_j: |I_j| > \delta\}|.$$

We have the following result:

$$|E_{\delta/2}| = L_E(\delta) + \delta N_E(\delta) \quad (\delta > 0)$$

and so

$$\Phi_E(n) = \frac{n}{\pi} L_E\left(\frac{\pi}{n}\right) + N_E\left(\frac{\pi}{n}\right), \quad n \geq 1. \quad (2.6)$$

We have the following result:

Theorem 2.6 (El-Fallah, Ransford [10]). Let X be a complex Banach space, and let T be a bounded linear operator on X . Suppose that T satisfies (2.5) for some compact set $E \subseteq \mathbf{T}$ and some constant C . Then

$$\|T^{n-1}\| \leq \frac{e}{2} C^2 \Phi_E(n) \quad (n \geq 1).$$

Proof. The outline of the proof is the following: we use the Riesz-Dunford functional calculus, we integrate by parts and then we use the resolvent condition to estimate the resulting integral by a functional expression whose minimum value we then try to determine.

We need the following lemma:

Lemma 2.7 (El-Fallah, Ransford [10]). Let E be a compact subset of \mathbf{T} , let I be an arc of \mathbf{T} such that $E \cap I = \emptyset$ and let $R > 1$. Then,

$$\int_I \frac{|dz|}{\text{dist}(Rz, E)^2} \leq \frac{2\pi}{R^2 - 1}.$$

(a complete proof can be found in the reference, p. 140).

If $n \geq 1$, the result is obvious. Let $n \geq 2$, and $R > 1$. By the Riesz-Dunford functional calculus and integration by parts, we have

$$T^{n-1} = \frac{1}{2\pi i} \int_{|\lambda|=R} \lambda^{n-1} (\lambda I - T)^{-1} d\lambda = \frac{1}{2\pi i} \int_{|\lambda|=R} \frac{\lambda^n}{n} (\lambda I - T)^{-2} d\lambda.$$

As T satisfies (2.5), it follows that

$$\|T^{n-1}\| \leq \frac{1}{2\pi} \int_{|\lambda|=R} \frac{|\lambda|^n}{n} \frac{C^2}{\text{dist}(\lambda, E)^2} |d\lambda|.$$

We make the substitution $z = \lambda/R$ and obtain

$$\|T^{n-1}\| \leq \frac{C^2 R^{n+1}}{2\pi n} \int_{|z|=1} \frac{|dz|}{\text{dist}(Rz, E)^2}.$$

We estimate the integral by breaking it into two parts; to this result, take $\delta > 0$, and let I_1, \dots, I_N be the components of $\mathbf{T} \setminus E$ such that $|I_j| > \delta$. Then

$$\begin{aligned} \int_{\mathbf{T}} \frac{|dz|}{\text{dist}(Rz, E)^2} &= \sum_{j=1}^N \int_{I_j} \frac{|dz|}{\text{dist}(Rz, E)^2} + \int_{\mathbf{T} \setminus \cup_1^N I_j} \frac{|dz|}{\text{dist}(Rz, E)^2} \\ &\leq \frac{2\pi N}{R^2 - 1} + \frac{|\mathbf{T} \setminus \cup_1^N I_j|}{(R - 1)^2}. \end{aligned}$$

For the integrals over the I_j we have used lemma 2.7 and for the integral over the rest of \mathbf{T} the estimate $\text{dist}(Rz, E)^2 \geq (R - 1)^2$. In terms of the functions N_E and L_E , this inequality becomes

$$\int_{\mathbf{T}} \frac{|dz|}{\text{dist}(Rz, E)^2} \leq \frac{2\pi N_E(\delta)}{R^2 - 1} + \frac{L_E(\delta)}{(R - 1)^2}.$$

As a consequence, we arrive at the following estimate for $\|T^{n-1}\|$:

$$\|T^{n-1}\| \leq C^2 \frac{R^{n+1}}{n(R^2 - 1)} \left(N_E(\delta) + \frac{1}{2\pi} \frac{R + 1}{R - 1} L_E(\delta) \right).$$

This inequality is valid for all $R > 1$ and all $\delta > 0$. Since we are looking for a best bound, we choose R so as to minimize $R^{n+1}/(R^2 - 1)$,

namely $R = (n + 1)^{1/2}/(n - 1)^{1/2}$. With this choice, $(R + 1)/(R - 1) \leq 2n$. Hence,

$$\|T^{n-1}\| \leq \frac{C^2}{2} \frac{(n+1)^{\frac{n+1}{2}}}{n(n-1)^{\frac{n-1}{2}}} \left(N_E(\delta) + \frac{n}{\pi} L_E(\delta) \right).$$

Taking $\delta = \frac{\pi}{n}$ and using (2.6), we obtain

$$\|T^{n-1}\| \leq \frac{C^2}{2} \frac{(n+1)^{\frac{n+1}{2}}}{n(n-1)^{\frac{n-1}{2}}} \Phi_E(n).$$

The function

$$x \rightarrow \frac{(x+1)^{\frac{x+1}{2}}}{x(x-1)^{\frac{x-1}{2}}}$$

is increasing for $x > 1$ and tends to e as $x \rightarrow \infty$, so it is bounded above by e . Thus,

$$\|T^{n-1}\| \leq \frac{C^2}{2} e \Phi_E(n).$$

This completes the proof. ■

In closing this first part, we will mention a few other conditions that are present in the literature. Borovykh-Drissi-Spijker [3] mention the following two conditions

$$\begin{aligned} \sigma(T) &\subseteq W_\delta, \text{ and} \\ \|\zeta I - T\|^{-1} &\leq \frac{K}{|\zeta - 1|} \text{ for all } \zeta \in \mathbf{C} \setminus W_\delta. \end{aligned} \quad (2.7)$$

$$\begin{aligned} \sigma(T) &\subseteq W_\delta, \text{ and} \\ \|\zeta I - T\|^{-1} &\leq \frac{K}{d(\zeta, W_\delta)} \text{ for all } \zeta \in \mathbf{C} \setminus W_\delta. \end{aligned} \quad (2.8)$$

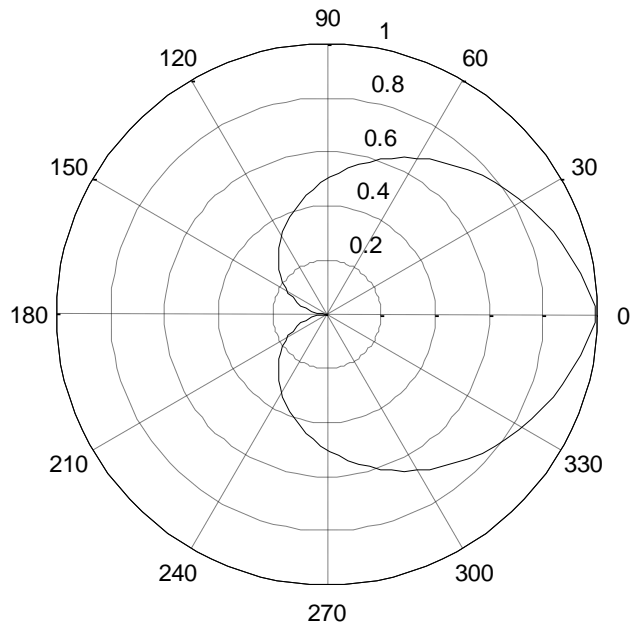
For these conditions, W_δ is the set

$$W_\delta = \{\xi: \xi \in \mathbf{C} \text{ with } |\xi| \leq 1 - \delta |\text{Arg}(\xi)|\}$$

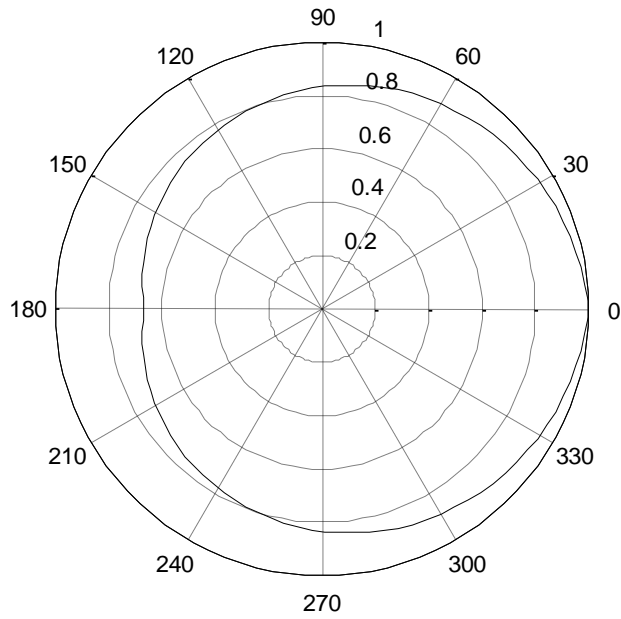
where δ is a constant satisfying

$$0 \leq \delta \leq 1/\pi$$

and $\text{Arg}(\xi) \in [-\pi, \pi]$ (see Figure 10).



(a)



(b)

Figure 10. The set W_δ . (a) $\delta = \frac{1}{\pi}$. (b) $\delta = \frac{1}{3\pi}$.

Borovykh-Drissi-Spijker [3] also prove a theorem that connects these two conditions with the Kreiss and Tadmor-Ritt conditions.

Theorem 2.8 (Borovykh, Drissi, Spijker [3]). (a) For $0 < \delta \leq \frac{1}{\pi}$ condition (2.7) implies (2.8).

(b) For $0 < \delta \leq \frac{1}{\pi}$ condition (2.8) implies (2.4) with $L = cK$, where $c = \frac{1+\epsilon}{(1-\epsilon\pi)\epsilon}$ and $\epsilon = \min \left\{ \delta, \sqrt{1 + \frac{1}{\pi}} - 1 \right\}$.

(c) condition (2.4) implies (2.7) with $\delta = \frac{1}{\pi L}$, $K = 3L$.

We will sketch here the proof of (b).

Taking δ in the given interval and a complex ζ with $|\zeta| > 1$, $\theta = \arg(\zeta) \in (-\pi, \pi]$. The proof starts from the following fact:

$$d(\zeta, W_\epsilon) \geq \frac{1-\epsilon\pi}{1+\epsilon} (|\zeta| - 1 + \epsilon|\theta|), \text{ for } 0 \leq \epsilon \leq 1/\pi.$$

For $f(\epsilon) = \frac{(1-\epsilon\pi)\epsilon}{1+\epsilon}$, we have

$$d(\zeta, W_\epsilon) \geq f(\epsilon)|\zeta - 1|. \quad (2.9)$$

The function f attains its maximum value at

$$\epsilon = \sqrt{1 + \frac{1}{\pi}} - 1$$

We choose $\epsilon = \min \{\delta, \sqrt{1 + 1/\pi} - 1\}$, and with this value we will have in (2.9)

$$d(\zeta, W_\delta) \geq \frac{|\zeta-1|}{c}, \text{ where } c = \frac{1+\epsilon}{(1-\epsilon\pi)\epsilon}.$$

This completes the proof of point (b).

(a complete proof can be found in the reference, pp. 428-430).

■

Finally, one has to take into account the Yosida approximation of a bounded operator and the related theorem regarding power boundedness.

The Yosida approximation of a bounded operator T is defined in the following way:

$$Y(\lambda, T) = \lambda TR(\lambda, T).$$

The Yosida approximation is analytic in $\rho(T)$, and we have

$$Y(\lambda, T) = \lambda^2 R(\lambda, T) - \lambda,$$

$$Y(\lambda, T) = \sum_{j=0}^{\infty} \frac{T^{j+1}}{\lambda^j},$$

$$\|Y(\lambda, T) - T\| \leq \frac{\|T\|}{|\lambda| - \|T\|} \text{ for } |\lambda| > \|T\|.$$

Theorem 2.9 (Nevanlinna [25]). Let T be a bounded operator and $Y(\lambda, T)$ its Yosida approximation. Then the following are equivalent:

$$(2.10) \quad \|T^k\| \leq C, \text{ for } k=1,2,\dots$$

$$(2.11) \quad \|Y(\lambda, T)^k\| \leq \frac{C}{\left(1 - \frac{1}{|\lambda|}\right)^k} \text{ for } k=1,2,\dots \text{ and } |\lambda| > 1.$$

Proof. Assuming (2.10), the spectral radius is ≤ 1 and the series representation for the Yosida approximation holds in the complement of the closed unit disk. By the binomial formula we have again, in the same set

$$\begin{aligned} Y(\lambda, T)^k &= (\lambda TR(\lambda, T))^k = \lambda^k T^k (R(\lambda, T))^k = \lambda^k T^k \frac{1}{(\lambda - T)^k} = \lambda^k T^k \frac{1}{\lambda^k \left(1 - \frac{T}{\lambda}\right)^k} \\ &= T^k \sum_{j=0}^{\infty} \binom{j+k-1}{j} \frac{T^j}{\lambda^j} = \sum_{j=0}^{\infty} (-1)^j \binom{-k}{j} T^{k+j} / \lambda^j. \end{aligned}$$

Since $(-1)^j \binom{-k}{j} > 0$, we obtain using (2.10)

$$\|Y(\lambda, T)^k\| \leq C \sum_{j=0}^{\infty} (-1)^j \binom{-k}{j} |\lambda|^{-j} = \frac{C}{\left(1 - \frac{1}{|\lambda|}\right)^k},$$

so (2.11) holds.

For the reverse direction, we observe that, by the third formula related to the Yosida approximation (just before the theorem), for any fixed k ,

$$T^k = \lim_{\lambda \rightarrow \infty} Y(\lambda, T)^k.$$

This, together with (2.11) implies (2.10).

■

References.

For section 1, the statement of Theorem 2.1 is taken from Malinen- Nevanlinna-Yuan [20], although its proof is not given in this paper; for (i) \Rightarrow (ii), see Vitse [32], Lemma 1.1 (the definitions of the various sets at the beginning of section 1 are from the same paper) and Lyubich [18], p. 158. (ii) \Rightarrow (iii) is taken from Nevanlinna [25] (Th. 4.5.4, pp. 101-102). As mentioned in the text, the bulk of (iii) \Rightarrow (i) is adapted from a semigroup proof in Pazy [27] (Th. 5.2, pp. 61-63); I have also used Nevanlinna [26], Th. 2.1. Part (e) is mine. Theorem 2.2 is in Vitse [33]; this theorem is instrumental in Vitse's construction of a Besov functional calculus for Tadmor-Ritt operators. For the Katznelson-Tzafriri Theorem, see [14]. For another paper related to the characterization of Tadmor-Ritt operators, see Nagy, Zemanek [23].

The proof of Proposition 2.3 is from El-Fallah-Ransford [10] , Th. 1.5. Theorem 2.4 follows immediately from it. Lemma 2.5, which states the equivalence of the Tadmor and the Ritt conditions, is taken from Borovikh-Drissi-Spijker [3], Th. 2.1.

For section 2, the sources are El-Fallah-Ransford [10] (Th. 1.1), Borovikh-Drissi-Spijker [3] (Th. 2.3); th. 2.9 and the Yosida-approximation material are taken from Nevanlinna [25] (pp. 29-31, Th. 2.7.1)

Chapter III

1. Cauchy-Stieltjes integrals and their multipliers

As a way of concluding the topic discussed in this thesis, we will present a functional calculus for the Tadmor-Ritt operators. This will constitute a good synthesis of the material discussed so far - namely, the classical results for Tadmor-Ritt operators and the idea of "functional calculus".

In this, we will follow the construction of P. Vitse, as presented in two papers [31] and [32]. The starting point is the space $\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))$ of all functions holomorphic on $\mathbf{C} \setminus cl(\mathbf{D})$, continuous on $\mathbf{C} \setminus \mathbf{D}$ and vanishing at ∞ , endowed with the supremum norm. The two following remarks need to be mentioned:

- $\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))$ is a closed subspace of $\mathcal{C}(\mathbf{T})$, the space of continuous functions on the unit circle;
- $\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D})) = span_{\mathcal{C}(\mathbf{T})} \left\{ \frac{1}{z-\zeta} : \zeta \in \mathbf{D} \right\}$. In other words, $\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))$ is spanned by the rational functions on the unit circle with poles in the open unit disk.

The next structure used is the dual space of $\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))$, which is

$$\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))^* = \mathcal{M}(\mathbf{T}) / \mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))^\perp,$$

where $\mathcal{M}(\mathbf{T})$ is the space of complex Borel measures on \mathbf{T} , and $\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))^\perp$ is the set of measures orthogonal to $\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))$:

$$\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))^\perp = \{ \mu \in \mathcal{M}(\mathbf{T}) : (f, \mu) = 0 \text{ for every } f \in \mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D})) \}.$$

The duality used is integration on the unit circle:

$$(f, \mu) = \int_{\mathbf{T}} f d\mu.$$

This fact, combined with the previous remark that rational functions on the circle with poles in the unit disk generate $\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))$, gives us the following description of $\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))^\perp$:

$$\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))^\perp = \left\{ \mu \in \mathcal{M}(\mathbf{T}) : \int_{\mathbf{T}} \frac{d\mu(\zeta)}{z - \zeta} = 0 \text{ for every } z \in \mathbf{D} \right\}.$$

We define, for every $\mu \in \mathcal{M}(\mathbf{T})$, a *Cauchy-Stieltjes integral* C^μ (CSI) by

$$C^\mu(z) = \int_{\mathbf{T}} \frac{d\mu(\zeta)}{z - \zeta}, z \in \mathbf{C} \setminus \mathbf{T}.$$

We have the following fact: For every function in $\mathbf{C} \setminus \mathbf{T}$ representable in this form, there exists a unique measure which allows this representation. The reason for this is that a measure μ satisfying

$$\int_{\mathbf{T}} \frac{d\mu(\zeta)}{z - \zeta} = 0, z \in \mathbf{C} \setminus \mathbf{T}.$$

must annihilate the trigonometric polynomials, so it can only be the null measure. Therefore, the mapping $\mu \rightarrow C^\mu$ is injective.

Example. $f(z) = \frac{1}{1-z} \in C^\mu$ ($z \in \mathbf{D}$), with $\mu = \delta_1$, because

$$\frac{1}{1-z} = \int_{\mathbf{T}} \frac{d\delta_1(\zeta)}{\zeta - z}.$$

Next, we must bring in a new concept, namely that of Hardy class. Hardy spaces in the complex plane are usually defined on the unit disk or in the upper half plane. There is, however, a workable definition for arbitrary complex domains as well. We have:

Definition. Let $0 < p < \infty$; a holomorphic function on the domain Ω is in $H^p(\Omega)$ if the subharmonic function $|f(z)|^p$ has a harmonic majorant in Ω ; that is, there is a harmonic function $v(z)$ with

$$|f(z)|^p \leq v(z), z \in \Omega.$$

The function f is in $H^\infty(\Omega)$ if it is both holomorphic in Ω and bounded there.

One usually works on such spaces with *regular exhaustions* $\{\Omega_n\}$ of the domain Ω . We define this concept in the following way:

Definition. Let Ω be a domain. A regular exhaustion of Ω is a sequence $\{\Omega_n\}$ of subdomains of Ω satisfying

$$cl(\Omega_n) \subset \Omega_{n+1}, n = 1, 2, \dots \quad (1)$$

$$\bigcup_{n=1}^{\infty} \Omega_n = \Omega \quad (2)$$

$$\text{each component of } \partial\Omega_n \text{ is nontrivial} \quad (3)$$

For $\mathbf{C} \setminus cl(\mathbf{D})$, one such regular exhaustion is given by the annuli

$$\left\{ z \in \mathbf{C}: 1 + \frac{1}{n} < |z| < n, n \text{ a positive integer} \right\}.$$

Working with general domains Ω proves more difficult, and yet many of the facts proven in the disk carry over to the general case. One such fact that is of interest to us in this context is the following:

- $H^1(\Omega)$ is the set of functions f holomorphic in Ω and representable by a Cauchy-Stieltjes integral which vanishes in the complement of Ω .

This fact allows us to identify $\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))^\perp$ with $H^1(\mathbf{C} \setminus cl(\mathbf{D}))$.

Next, we bring attention to the following fact:

- Every Cauchy-Stieltjes integral defines two functions: one holomorphic in \mathbf{D} , the other holomorphic in $\mathbf{C} \setminus cl(\mathbf{D})$ (see Duren [9], p. 39; the same is true for a Cauchy integral, with φ integrable on the unit circle).

Given the facts mentioned so far, what this all adds up to is that we can identify the dual of $\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))$ with the space of holomorphic functions on the unit disk that are representable as Cauchy-Stieltjes integrals. That is,

$$\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))^* = CSI(\mathbf{D}) = \{f \in Hol(\mathbf{D}): f = C^\mu \text{ for some } \mu \in \mathcal{M}(\mathbf{T})\}.$$

On this space, we have the quotient norm, obtained from the total variation norm for measures:

$$\|f\|_{CSI(D)} = \inf\{\|\mu\| : f = C^\mu\} = \|\mu\|_{\mathcal{M}(\mathbb{T})/H^1}.$$

As Vitse does, we can take into consideration two special cases:

- $f \in H^1(\mathbf{D})$. In this case, f can be represented as a Cauchy integral on the unit circle, and so, we will have

$$\|f\|_{CSI(D)} = \left\| \frac{1}{2\pi i} \int_{\mathbb{T}} f d\zeta \right\|_{\mathcal{M}(\mathbb{T})/H^1}.$$

- $f = C^\mu$, with μ a singular measure with respect to the arclength.

The reasoning that will guide us in this case is based on the following: We remark that, for a measure μ to belong to $\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))^\perp$, it has to annihilate all functions of the form $\frac{1}{z^n}$, $z \in \mathbf{C} \setminus \mathbf{D}$ (these are holomorphic on $\mathbf{C} \setminus cl(\mathbf{D})$, continuous on $\mathbf{C} \setminus \mathbf{D}$, and vanish at ∞). Now, what this means is that such a measure has the property that

$$\hat{\mu}(1) = \hat{\mu}(2) = \dots = 0$$

where

$$\hat{\mu}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} d\mu(t).$$

At this point comes into play the F. and M. Riesz theorem, which says that such a measure (one can call such a measure *analytic*, although usually one defines an analytic measure to be one for which the negative Fourier coefficients $\hat{\mu}(-n)$, n positive, vanish; see Masreghi [21], pp. 116-119, also Duren [9]; this is because one usually works just on the disk, not on its complement; but the Riesz Brothers theorem is valid for our types of measures as well; what happens when one works on the complement is that one gets the "conjugate" results of those on the disk, so to speak) is absolutely continuous with respect to Lebesgue measure.

So, given all this, we can say that for such a measure there exists a function $\phi \in H^1(\mathbf{C} \setminus cl(\mathbf{D}))$ which vanishes at ∞ and whose values on the unit circle make it an $L^1(\mathbf{T})$ function, such that

$$d\mu = \phi dm \quad (m \text{ is Lebesgue measure})$$

So we can, in fact, characterize $\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))^\perp$ by means of such a function ϕ . We have the following Proposition:

Proposition 3.1. Let $K\mu$ denote the Cauchy-Stieltjes integral associated with $\mu \in \mathcal{M}(\mathbf{T})$, and

$$R_f = \{\mu \in \mathcal{M}(\mathbf{T}) : f = K\mu\}$$

the set of all measures which represent the Cauchy-Stieltjes integral f .

The following hold:

- (1) $K\mu \equiv 0$ iff $d\mu = \phi dm$, for a ϕ as above.
- (2) For $\mu, \nu \in R_f$, $d\mu - d\nu = \phi$, for ϕ as above.
- (3) If $\mu, \nu \in R_f$, then $\mu_s = \nu_s$.

(see Cima, Matheson and Ross *The Cauchy Transform*, [7] p. 42 for something similar).

So, we will have, in fact, the following (see Bourdon and Cima [5], p. 468):

$$\mathcal{C}^\mu = \mathcal{C}^{\mu_a} \oplus \mathcal{C}^{\mu_s},$$

where \mathcal{C}^{μ_a} is the set of all Cauchy-Stieltjes integrals with measure absolutely continuous with respect to the Lebesgue measure, and \mathcal{C}^{μ_s} is the set of all Cauchy-Stieltjes integrals with measure singular with respect to the Lebesgue measure. This will yield, of course, that

$$\|f\|_{\mathcal{C}SI(\mathbf{D})} = \|\mu\|_{\mathcal{M}(\mathbf{T})}$$

in our case (μ is singular). We will also have, as another consequence of the above decomposition, that

$$\|C^\mu\|_{\mathcal{C}SI(\mathbf{D})} = \|C^{\mu_a}\|_{\mathcal{C}SI(\mathbf{D})} + \|C^{\mu_s}\|_{\mathcal{C}SI(\mathbf{D})},$$

where $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of the measure μ .

The duality between $CSI(\mathbf{D})$ and $\mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))$ will be given, of course, by integration with respect to the measure of C^μ :

$$\langle g, C^\mu \rangle = (g, \mu) = \int_{\mathbf{T}} g d\mu.$$

For $f = \sum_{k \geq 0} a_k z^k \in CSI(D)$ and $g = \sum_{k \geq 0} \frac{b_k}{z^{k+1}} \in \mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))$, the duality will take the form (if one of the two series converges uniformly on \mathbf{T})

$$(g, f) = \sum_{k \geq 0} a_k b_k.$$

Indeed, if g is uniformly convergent on \mathbf{T} ,

$$(g, f) = \frac{1}{2\pi i} \int_{\mathbf{T}} g(z) f(z) dz = \sum_{k \geq 0} b_k \frac{1}{2\pi i} \int_{\mathbf{T}} f(z) \frac{1}{z^{k+1}} dz = \sum_{k \geq 0} a_k b_k.$$

There are other equivalent forms of obtaining the above duality. One starts with the duality $\mathcal{C}(\mathbf{T})^* = \mathcal{M}(\mathbf{T})$ via $(f, \mu) = \int_{\mathbf{T}} f d\bar{\mu}$, where $\bar{\mu}$ is the conjugate measure of μ ($\bar{\mu}(\sigma) = \overline{\mu(\sigma)}$). We will have

$$\mathcal{C}_A(\mathbf{D})^* = \mathcal{M}(\mathbf{T}) / \mathcal{C}_A^\perp = \mathcal{M}(\mathbf{T}) / H_-^1(\mathbf{D}) = CSI(\mathbf{D})$$

with the duality

$$(f, g) = \sum_{n \geq 0} \hat{f}(n) \overline{\hat{g}(n)}, \quad f \in \mathcal{C}_A(D), g \in CSI(\mathbf{D}).$$

Now, the crucial step in this line of reasoning is the identification of \mathcal{C}_A^\perp with $H_-^1(\mathbf{D})$ (this is the space conjugate to $H^1(\mathbf{D})$). This is equivalent to the following remark (see Cima, Matheson, Ross [7], p. 42):

- If $\phi \in H_0^1 = \{\phi \in H^1: f(0) = 0\}$, then

$$\hat{\phi}(n) = \int_{\mathbf{T}} \bar{\zeta}^n \overline{\phi(\zeta)} dm(\zeta) = 0, \quad \forall n \text{ positive integer}$$

So, basically, this says that the space of Cauchy-Stieltjes integrals can be factorized to the space of functions $\bar{\phi}$ such that $\phi \in H^1(\mathbf{D})$ and ϕ vanishes at zero. This will give that

$$K\mu \equiv 0 \text{ iff } d\mu = \bar{\phi}dm, \text{ for some } \phi \in H_0^1(\mathbf{D})$$

with the notations as in **Proposition 3.1**.

The reasoning is parallel to what I have said about factorizing $CSI(\mathbf{D})$ with respect to $H^1(\mathbf{C} \setminus \mathbf{D})$, so I will not repeat it here.

The next result is used in the proof of the following lemma:

- If $f \in \mathbf{Hol}(cl(\mathbf{D}))$, then

$$\frac{f}{1-z} = \frac{f(1)}{1-z} + \frac{f-f(1)}{1-z} \in CSI(\mathbf{D}).$$

Also,

$$\langle g, \frac{f}{1-z} \rangle = \frac{1}{2\pi i} \int_{\mathbf{T}} \frac{f(z)}{1-z} g(z) dz.$$

for every $g \in \mathcal{C}_A(\mathbf{C} \setminus cl(\mathbf{D}))$ with $g(1) = 0$.

This follows because

$$\begin{aligned} \langle g, \frac{f}{1-z} \rangle &= f(1) \langle g, \frac{1}{1-z} \rangle + \frac{1}{2\pi i} \int_{\mathbf{T}} \frac{f(z) - f(1)}{1-z} g(z) dz \\ &= f(1)g(1) + \frac{1}{2\pi i} \int_{\mathbf{T}} \frac{f(z) - f(1)}{1-z} g(z) dz = \frac{1}{2\pi i} \int_{\mathbf{T}} \frac{f(z)}{1-z} g(z) dz \end{aligned}$$

given that $\int_{\mathbf{T}} \frac{g(z)}{1-z} dz = 0$ by Cauchy's theorem.

We have the lemma:

Lemma 3.2 (Vitse [31]). Let T be Tadmor-Ritt with constant C , $1 \notin \sigma(T)$, $f \in \mathbf{Hol}(cl(D))$, $x \in X$, $y \in X^*$.

Then

$$(f(T)x, y) = \left\langle \frac{zf}{1-z}, g \right\rangle = f(1)(x, y) + \left\langle \frac{f-f(1)}{1-z}, zg + (x, y) \right\rangle,$$

where $g = g_{x,y}(z) = \frac{(1-z)(R(z,T)x,y)}{z}$, $z \in \mathbf{C} \setminus \mathbf{D}$. Therefore,

$$\|f(T)\| \leq (C+1) \left\| \frac{f}{1-z} \right\|_{CSI(\mathbf{D})}.$$

Proof. The resolvent function $R(z,T)$ is holomorphic on $\mathbf{C} \setminus \mathbf{D}$, so we will have

$$(f(T)x, y) = \frac{1}{2\pi i} \int_{\mathbf{T}} f(z)(R(z,T)x, y) dz = \frac{1}{2\pi i} \int_{\mathbf{T}} \frac{zf(z)}{1-z} g(z) dz = \left\langle \frac{zf}{1-z}, g \right\rangle$$

(from the remark made just before the lemma).

We now write $f(z) = f(1) + (f(z) - f(1))$ and we obtain the equality from the statement. Next, we use the Tadmor-Ritt condition for $z \in \mathbf{T}$ (we know that $1 \notin \sigma(T)$, therefore the spectrum of the operator lies entirely in the open disk \mathbf{D})

$$|(f(T)x, y)| \leq \left(|f(1)| + (C+1) \left\| \frac{f-f(1)}{1-z} \right\|_{CSI(\mathbf{D})} \right) \cdot \|x\| \cdot \|y\|$$

so

$$\begin{aligned} \|f(T)\| &\leq (C+1) \left(|f(1)| + \left\| \frac{f-f(1)}{1-z} \right\|_{CSI(\mathbf{D})} \right) \\ &= (C+1) \left(\left\| \frac{f(1)}{1-z} \right\|_{CSI(\mathbf{D})} + \left\| \frac{f-f(1)}{1-z} \right\|_{CSI(\mathbf{D})} \right) \\ &= (C+1) \left\| \frac{f}{1-z} \right\|_{CSI(\mathbf{D})}. \end{aligned}$$

The last equality comes from the previously mentioned decomposition of C^μ , which has as a consequence the fact that

$$\|C^\mu\|_{CSI(\mathbf{D})} = \|C^{\mu_a}\|_{CSI(\mathbf{D})} + \|C^{\mu_s}\|_{CSI(\mathbf{D})}$$

and the remark that the measure corresponding to $\frac{f(1)}{1-z}$ is singular with respect to Lebesgue measure (it is in fact the δ_1 measure), and the measure corresponding to $\frac{f(z)-f(1)}{1-z}$ is absolutely continuous with respect to Lebesgue measure. ■

Theorem 3.3 (Vitse [31]). Let T be a Tadmor-Ritt operator with constant C , $f \in \text{Hol}(cl(\mathbf{D}))$. Then

$$\|f(T)\| \leq (C + 1) \left\| \frac{f}{1-z} \right\|_{CSI(\mathbf{D})}.$$

Proof. The proof consists in applying the preceding lemma to the operator rT , $0 < r < 1$. We have

$$\|R(z, rT)\| = \frac{1}{r} \left\| R\left(\frac{z}{r}, T\right) \right\| \leq \frac{C}{|z-r|} \leq \frac{C}{|z-1|} \frac{2}{1+r}$$

for every z with $|z| > 1$. As a consequence, the operator (rT) satisfies the Tadmor-Ritt condition with constant $C \frac{2}{1+r}$. Also, by the continuity of the Riesz-Dunford functional calculus, we have

$$\lim_{r \rightarrow 1} \|f(rT) - f(T)\| = 0.$$

Therefore,

$$\|f(rT)\| \leq \left(C \frac{2}{1+r} + 1 \right) \left\| \frac{f}{1-z} \right\|_{CSI(\mathbf{D})}$$

and passing to the limit as $r \rightarrow 1$ gives us the result.

See also the comments made with regard to this procedure in the proof of Theorem 2.2. ■

Next, we use the space of multipliers of $CSI(\mathbf{D})$,

$$\text{Mult}(\mathbf{D}) = \{f \in \mathbf{Hol}(\mathbf{D}): g \in CSI(\mathbf{D}) \Rightarrow fg \in CSI(\mathbf{D})\}$$

with the multiplier norm

$$\|f\|_{\text{Mult}(\mathbf{D})} = \sup\{\|fg\|_{CSI(\mathbf{D})} : \|g\|_{CSI(\mathbf{D})} \leq 1\}$$

which makes it a Banach algebra. We have therefore:

Corollary 3.4 (Vitse [31]). If T is a Tadmor-Ritt operator with constant C , then

$$\|f(T)\| \leq (C + 1)\|f\|_{Mult(\mathbf{D})}, \quad \forall f \in \mathbf{Hol}(cl(\mathbf{D})).$$

Given all this, we have the following result related to the growth of polynomials of Tadmor-Ritt operators:

Theorem 3.5 (Vitse [31]). For T a Tadmor-Ritt Banach space operator with constant C , we have

$$C_T(n) \leq (C + 1) \log(e^2 n), \quad n \geq 1$$

where

$$C_T(n) = \sup\{\|p(T)\| : \deg(p) \leq n, \|p\|_{\infty, \mathbf{D}} \leq 1\}.$$

The proof is quite straightforward (see the reference, pp. 52-53; the proof uses the Riesz projection).

Notice that $CSI(D)/H^1(\mathbb{C} \setminus \bar{\mathbf{D}})$ is not an algebra, it is just a Banach space on \mathbf{D} . So, we do not talk in this

case of a functional calculus in the sense of a homomorphism between two algebras. Now is the moment to recall the remark made by Nikolski that we have quoted in Chapter I. According to that quote, what we sometimes need is just a coherent way to talk about $f(T)$ for some functions f , so that we may evaluate the norms of these operators relative to the norms of the functions f . So, one can see that, after all, the notion of a functional calculus is not a rigid one, rather it is relative to the objectives that the mathematician has.

The space of multipliers, on the other hand, is, as we have seen, a Banach algebra, and the correspondence between it and the algebra of bounded operators is already given in the formulation of the Riesz-Dunford calculus. In fact, as one immediately notices, both calculi use the Cauchy-integral formulation from the Riesz-Dunford calculus, given that in both cases we deal with holomorphic functions on the spectrum of the operator. So, basically, in both cases, we start with the holomorphic functions and the Riesz-Dunford calculus and then restricts the class of functions so that one may use better norms in estimating the operators.

2. Measures as tools for determining the growth of powers of operators

The facts presented so far involve measures on the unit circle; one can, however, work on a different set: we have seen that the spectrum of a Tadmor-Ritt operator lies in a Stolz domain, and so one can work with measures on the boundary of such a domain. The main lesson to be learned here is that the set relevant to the mathematical facts presented is the spectrum, and especially its boundary. Vitse does that (that is, work on the boundary of the Stolz domain) in a different paper. The unit disk is, however, more generally relevant for the growth of powers of operators, in that if the spectrum of an operator goes beyond the unit circle (that is, outside the unit disk), its powers cannot be bounded. So, of course, the question is, what happens when part of the spectrum lies on the unit circle? That is, what properties the intersection of the spectrum with the unit circle must have so that the powers of the operators remain bounded? One interesting result that we have already mentioned (without proof) is the Katznelson-Tzafriri theorem which says that a contraction satisfies $\lim_{n \rightarrow \infty} \|T^n - T^{n+1}\| = 0$ if and only if the *peripheral spectrum* $\mu\sigma(T)$ of T consists of at most the point $z = 1$ (remember, this is precisely $\sigma(T) \cap \mathbf{T}$). We will close this section with two theorems related to the same question, namely how is the growth of the powers of an operator is influenced by the intersection of its spectrum with the unit circle.

For this investigation, we have chosen operators which exhibit what we may call an intermediary behaviour when it comes to the growth of their powers namely, partially power bounded operators. In the following, $\sigma_p(T)$ denotes the point spectrum of T (see Introduction).

We start with the following lemma:

Lemma 3.6 (Ransford [28]). Let T be an operator on a Banach space X , with $\alpha, \beta \in \sigma_p(T)$, and let $x, y \in X$ be unit eigenvectors corresponding to α, β respectively. Then

$$|\alpha^n - \beta^n| \leq 2 \|T^n\| \|x - y\| \quad (n \geq 1).$$

Proof. We fix $n \geq 1$ and we consider $\|T^n x - T^n y\|$; first, we have

$$\begin{aligned}\|T^n x - T^n y\| &= \|\alpha^n x - \beta^n y\| \\ &\geq \|\alpha^n x - \beta^n x\| - \|\beta^n x - \beta^n y\| \\ &= |\alpha^n - \beta^n| - |\beta^n| \|x - y\|.\end{aligned}$$

Second, we have

$$\|T^n x - T^n y\| \leq \|T^n\| \|x - y\|.$$

By the spectral theorem, we have that

$$|\beta^n| \leq \|T^n\|$$

and so we obtain the desired result. ■

This lemma will set the stage for the investigation of the intersection of the unit circle with the point spectrum. The tools used in this investigation are two measures, namely the *Rajchman measure* and the *Hausdorff measure* (through the related notion of *Hausdorff dimension*). So we must first define these notions.

Definition. A measure $\mu \in \mathcal{M}(\mathbf{T})$ which is a pseudofunction (i.e. $\hat{\mu}(n) \rightarrow 0$ as $|n| \rightarrow \infty$) is called a *Rajchman measure*.

Definition. A set $P \subseteq \mathbf{T}$ is called a set of *extended uniqueness* if for every positive Rajchman measure μ we have $\mu(P) = 0$. Otherwise, P is called a set of *restricted multiplicity*.

Definition. Let $h: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ be a *defining function* (an increasing function such that $\lim_{t \rightarrow 0} h(t) = 0$). Given $\delta > 0$, we define

$$m_{h,\delta}(E) = \inf \left\{ \sum_j h(|I_j|) : E \subseteq \bigcup_j I_j, |I_j| \leq \delta \right\}$$

where the infimum is taken over all countable covers of E by arcs I_j of length at most δ . We define the *h-Hausdorff measure* of E by

$$m_h(E) = \lim_{\delta \rightarrow 0} m_{h,\delta}(E).$$

If $h(t) = t^\alpha$, we write m_α for m_h . Given this, the *Hausdorff dimension* of E is defined as

$$\dim E = \inf\{\alpha > 0: m_\alpha(E) = 0\}.$$

Definition. A subset E of \mathbf{T} is *analytic* if there exists a continuous surjection from a complete separable metric space (a *Polish space*) onto E . It is well known that every Borel subset of \mathbf{T} is analytic, and every analytic subset of \mathbf{T} is *universally measurable*, that is, it is measurable for every probability measure on \mathbf{T} .

Lemma 3.7 (Ransford [28]). Let X be a separable Banach space and T an operator on X . Then $\sigma_p(T) \cap \mathbf{T}$ is an analytic subset of \mathbf{T} , and there exists a universally measurable function $g: \mathbf{T} \rightarrow X$ such that

$$(3.1) \quad Tg(\lambda) = \lambda g(\lambda) \quad (\lambda \in \mathbf{T}) \quad \text{and} \quad \|g(\lambda)\| = \begin{cases} 1, & \lambda \in \sigma_p(T) \cap \mathbf{T} \\ 0, & \lambda \notin \sigma_p(T) \cap \mathbf{T} \end{cases}$$

(a complete proof can be found in the reference, p. 98).

We have the lemma:

Lemma 3.8 (Ransford [28]). Let X be a separable Banach space and let T be an operator on X , and μ a probability measure on \mathbf{T} such that $\mu(\sigma_p(T) \cap \mathbf{T}) > 0$. Then, given $\varepsilon > 0$, there exists a Borel probability measure ν on \mathbf{T} such that $\nu \ll \mu$ and

$$|\hat{\nu}(n)|^2 \geq 1 - \varepsilon \|T^n\| \quad (n \geq 1).$$

Proof. By the preceding lemma 3.6, there exists a universally measurable function $g: \mathbf{T} \rightarrow X$ satisfying

(3.1). Combining this result with Lemma , we have that

$$(3.2) \quad |\alpha^n - \beta^n| \leq 2 \|T^n\| \|g(\alpha) - g(\beta)\| \quad (\alpha, \beta \in \sigma_p(T) \cap \mathbf{T}, n \geq 1).$$

X is separable, and so its unit sphere can be covered by a countable set of balls B_j of diameter $\varepsilon/2$ (we assume that none of these balls contains 0, reducing ε otherwise, if necessary). The sets $g^{-1}(B_j)$ cover $\sigma_p(T) \cap \mathbf{T}$ so at least one of them satisfies $\mu(g^{-1}(B_j)) > 0$. With this j , we define a new Borel measure ν on \mathbf{T} by

$$\nu(E) = \frac{\mu(E \cap g^{-1}(B_j))}{\mu(g^{-1}(B_j))}.$$

Then ν is a probability measure concentrated on $g^{-1}(B_j)$; it is obvious that ν is absolutely continuous with respect to μ . Also, by (3.2), we have

$$\int \int |e^{in\theta} - e^{in\phi}| d\nu(\theta) d\nu(\phi) \leq 2\|T^n\| 2\frac{\varepsilon}{2} = \varepsilon \|T^n\| \quad (n \geq 1).$$

On the other hand, for each integer n , we have

$$\begin{aligned} \int \int |e^{in\theta} - e^{in\phi}| d\nu(\theta) d\nu(\phi) &\geq \left| \int \int (1 - e^{in(\theta-\phi)}) d\nu(\theta) d\nu(\phi) \right| = \left| 1 - \iint e^{in(\theta-\phi)} d\nu(\theta) d\nu(\phi) \right| \\ &= \left| 1 - \int e^{in\theta} d\nu(\theta) \int e^{-in\phi} d\nu(\phi) \right| = 1 - |\hat{\nu}(n)|^2. \end{aligned}$$

Combining these two inequalities, we get the desired result. ■

Lemma 3.9 (Kechris, Louveau [15]). Let σ be a Rajchman measure and $\rho \ll \sigma$. Then ρ is also a Rajchman measure. In particular, if σ is a Rajchman measure, so is $|\sigma|$.

Proof. We have $d\rho = f d\sigma$, $f \in L^1(|\sigma|)$. The trigonometric polynomials are dense in $L^1(|\sigma|)$, so for each $\varepsilon > 0$ we can find $P(x) = \sum_{n=-N}^N c_n e^{inx}$ such that $\|f - P\|_{L^1(|\sigma|)} < \varepsilon$. Consider the measure $d\rho_1 = P d\sigma$. Then

$$\widehat{\rho}_1(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} d\rho_1(\theta) = \int_0^{2\pi} e^{-in\theta} \sum_{k=-N}^N c_k e^{ik\theta} d\sigma(\theta) = \sum_{k=-N}^N c_k \widehat{\sigma}(n-k) \rightarrow 0 \quad (|n| \rightarrow \infty)$$

so ρ_1 is a Rajchman measure. We have

$$|\widehat{\rho}(n) - \widehat{\rho}_1(n)| \leq \|f - P\|_{L^1(|\sigma|)} < \varepsilon$$

therefore $\limsup_n |\widehat{\rho}(n)| \leq \varepsilon$ and do $\widehat{\rho}(n) \rightarrow 0$ as $|n| \rightarrow \infty$. ■

Theorem 3.10 (Ransford [28]). Let X be a separable Banach space, and let T be an operator on X such that $\|T^n\| \rightarrow \infty$ as $n \rightarrow \infty$. Then the set $\sigma_p(T) \cap \mathbf{T}$ is a set of extended uniqueness.

Proof. By assumption, there exists a constant M and a strictly increasing sequence of positive integers $(n_j)_{j \geq 1}$ such that $\|T^{n_j}\| \leq M, \forall j \geq 1$.

Assume, for a contradiction, that $\sigma_p(T) \cap \mathbf{T}$ is not a set of extended uniqueness. This means that there exists a positive Rajchman measure μ on \mathbf{T} such that $\mu(\sigma_p(T) \cap \mathbf{T}) > 0$. We may suppose that μ is a probability measure. By **Lemma 3.8**, with $\varepsilon = 1/(2M)$, there exists a probability measure $\nu \ll \mu$ such that

$$|\hat{\nu}(n_j)|^2 \geq \frac{1}{2} \quad (j \geq 1).$$

By Lemma 3.9, ν is also a Rajchman measure, that is $\hat{\nu}(n) \rightarrow 0$ as $|n| \rightarrow \infty$. We have reached the desired contradiction, which shows that our initial assumption is false. Therefore, $\sigma_p(T) \cap \mathbf{T}$ is a set of extended uniqueness. ■

We have the following immediate corollary, given by the fact that Lebesgue measure is a Rajchman measure:

Corollary 3.11 (Ransford [28]). Assume T is an operator on a separable Banach space X and $\sigma_p(T) \cap \mathbf{T}$ has positive Lebesgue measure. Then $\|T^n\| \rightarrow \infty$ as $n \rightarrow \infty$.

Now we will focus on the other concept, that of *Hausdorff dimension (measure)*. We start with two lemmas.

Lemma 3.12 (Ransford, Roginskaya [29]). Let X be a separable Banach space and T an operator on X . Suppose that $\sup_{k \geq 1} \|T^{n_k}\| < \infty$, where n_k is an increasing sequence of positive integers. Then, for $\varepsilon > 0$, there exists a countable subset $\{\mu_l\}_{l \geq 1}$ of \mathbf{T} such that

$$\sigma_p(T) \cap \mathbf{T} \subseteq \bigcup_{l \geq 1} \mu_l E$$

where

$$E := \bigcap_{k \geq 1} \{\lambda \in \mathbf{T} : |\lambda^{n_k} - 1| \leq \varepsilon\}.$$

(a complete proof can be found in the reference, pp. 434-435).

Lemma 3.13 (Ransford, Roginskaya [29]). Let $(n_k)_{k \geq 1}$ be an increasing sequence of positive integers, and let $\varepsilon \in (0,1)$. For each $k \geq 1$, define

$$E_k := \bigcap_{j=1}^k \{\lambda \in \mathbf{T} : |\lambda^{n_j} - 1| \leq \varepsilon\}.$$

Then E_k is composed of N_k disjoint arcs, where each arc is of length at most $\pi\varepsilon/n_k$ and

$$N_k \leq n_1 \prod_{j=2}^k \left[1 + \varepsilon \frac{n_j}{n_j - 1} \right].$$

(a complete proof can be found in the reference, pp. 435-436).

These results are interesting in themselves (as it usually happens when one works on the unit circle), but their proofs are not very difficult, so they will not be given here. They serve to prove the following theorem:

Theorem 3.14 (Ransford, Roginskaya [29]). Let T be an operator on a separable Banach space X . Suppose that $\sup_{k \geq 1} \|T^{n_k}\| < \infty$, where $(n_k)_{k \geq 1}$ is an increasing sequence of positive integers such that

$$P := \liminf_{k \rightarrow \infty} \frac{n_{k+1}}{n_k} > 1, \quad Q := \liminf_{k \rightarrow \infty} n_k^{1/k} < \infty.$$

Then $\sigma_p(T) \cap \mathbf{T}$ is of Hausdorff dimension at most $1 - \left(\log P / \log Q \right)$.

Proof. Given the definition of Hausdorff dimension, it suffices to prove that $m_\alpha(\sigma_p(T) \cap \mathbf{T}) = 0$ for all $\alpha > 1 - \log P / \log Q$.

We fix such an $\alpha > 1 - \log P / \log Q$ and we choose $P' \in (1, P)$ such that $\alpha > 1 - \frac{\log P'}{\log Q}$ and $\varepsilon \in (0, 1)$

such that $(1/P' + \varepsilon)Q^{1-\alpha} < 1$. We will have that $\liminf_{k \rightarrow \infty} n_{k+1}/n_k > P'$ and so there exists k_0 such that $n_{k+1}/n_k \geq P'$ for all $k \geq k_0$. For $k > k_0$, E_k (defined as in **Lemma 3.13**) is a union of N_k disjoint arcs of length at most $\pi\varepsilon/n_k$, where

$$N_k \leq n_1 \prod_{j=2}^k \left[1 + \varepsilon \frac{n_j}{n_{j-1}} \right] \leq n_1 \prod_{j=2}^{k_0} \left[1 + \varepsilon \frac{n_j}{n_{j-1}} \right] \prod_{j=k_0+1}^k \left((1/P' + \varepsilon) \frac{n_j}{n_{j-1}} \right) \leq C n_k (1/P' + \varepsilon)^k,$$

where C is a constant independent of k (given that $\frac{n_j}{n_{j-1}} \geq 1$ for all $k \geq k_0$). Writing $E := \bigcap_{k \geq 1} E_k$, we

have

$$m_\alpha(E) \leq \liminf_{k \rightarrow \infty} C n_k (1/P' + \varepsilon)^k \left(\frac{\pi\varepsilon}{n_k} \right)^\alpha \leq C (\pi\varepsilon)^\alpha \liminf_{k \rightarrow \infty} \left(\left(\frac{1}{P'} + \varepsilon \right) (n_k^{1/k})^{1-\alpha} \right)^k.$$

With our notation

$$Q = \liminf_{k \rightarrow \infty} n_k^{1/k}$$

we will have

$$\liminf_{k \rightarrow \infty} \left(\frac{1}{P'} + \varepsilon \right) (n_k^{1/k})^{1-\alpha} = \left(\frac{1}{P'} + \varepsilon \right) Q^{1-\alpha} < 1.$$

As a consequence, we have that $m_\alpha(E) = 0$. From **Lemma 3.12**, we deduce that $m_\alpha(\sigma_p(T) \cap \mathbf{T}) = 0$. ■

It is important to notice that the conditions mentioned in the two main theorems of this section (**Theorems 3.10 and 3.14**) are necessary conditions; so given that they hold for partially power bounded operators, they will automatically hold for power bounded operators as well. The obvious drawback is that they are not sufficient conditions; still, they are mathematically relevant. One has to see this section as essentially an investigation into the relation between the growth of operators and the "size" of the set $\sigma_p(T) \cap \mathbf{T}$.

References.

The two main sources of section 1 in this chapter are Vitse [31], [32]; I have tried, however, to highlight every step in her argument (she does not give too many explanations). For Cauchy-Stieltjes integrals, see Bourdon and Cima [5], Duren [9], pp. 39-41. For Hardy spaces on general domains, see Fisher [12]. A Cauchy-Stieltjes integral is also known as the Cauchy transform of a measure; see the book by Cima, Matheson and Ross [7].

Maybe the most important mathematical theorem present in this chapter is the Theorem of F. and M. Riesz; for this, see Duren [9], ch. 3, Masreghi [21], ch. 5. For some detailed proof, see Koosis [16], ch. II; see also ch. 7 for some interesting results related to duality to H^p spaces. For generalizations, see Barbey and Konig [2].

For section 2, see mainly Ransford [28] and Ransford and Roginskaya [29]; for the proofs of lemma 3.12, lemma 3.13, see [29]. The definition of Hausdorff dimension is taken from El-Fallah, Ransford [10]. Lemma 3.9 is taken from Kechriss and Louveau [15]; one should consult this source for many interesting facts related to Rajchman measures.

Conclusion

Functional calculus is a relatively new area of mathematical interest and discoveries. Its applications span a vast number of fields, from quantum mechanics to numerical analysis. But we must not forget that we are not dealing with a rigid notion, but with one that is relative to the various interests that drive the mathematicians. This is a point that I have made in the thesis and it is one that must be emphasized. One has definite intuitions, or expectations that the calculus must fulfill, and these drive the building process of the calculus. In Markus Haase's apt choice of words, a functional calculus must be "reasonable". "Here 'reasonable' means *at least* that $f(A)$ should have the expected meaning *if* one expects something." (Haase [13], p. 2). He lists two such expectations:

(1) If $\lambda \in \rho(A)$, then $(\lambda - z)^{-1}(A) = R(\lambda, A)$.

(2) If A generates a semigroup T , then $e^{tz}(A) = T(t)$.

A similar idea was found in Nikolski's statement quoted in chapter I.

The first chapter is a review of literature regarding the notion of "functional calculus". The focus has been not so much on the proofs and theorems (as these can be found in the books listed in the Bibliography), but on making clear the notions and the procedures involved in building a functional calculus. Even though we are not dealing with a fixed and rigid notion, a pattern must be discerned beyond the variety of definitions and examples. One must notice the central importance of the spectrum, as the set on which the functions used must be defined. As the spectrum of a linear operator lies in the complex plane, naturally mathematical facts related to complex analysis will prove crucial in building the calculus. The connection between the class of functions and the class of operators is achieved via a homomorphism, that must establish a significant correspondence between the two

classes. This homomorphism, in turn, can be defined in various ways; we have seen that integral representations, like Cauchy formulas or Fourier transforms can be used for this purpose.

Chapter II sees the Riesz-Dunford functional calculus at work. The main goal of the chapter is to provide a thorough analysis of Tadmor-Ritt operators. This is achieved mainly in Theorem 2.1, whose statement consists in the equivalence of three mathematical facts. Although the statement of the Theorem is taken from Malinen, Nevanlinna, Yuan [20], its proof is not given there. As a consequence, I have pieced together the various parts of the proof and in doing so, I adapted proofs from various sources listed in the Bibliography. Making explicit every step in the proof is a contribution of this thesis.

The main aspect that needs to be noticed with regard to the Tadmor-Ritt condition is that it imposes a condition on the growth of the resolvent function. The same holds for all the other resolvent conditions encountered in the chapter. As we know from the Introduction, the resolvent function is greater than or equal to the inverse of the distance to the spectrum. As a consequence, as this distance tends to zero, the resolvent will grow to infinity. In order to ensure power boundedness through resolvent conditions, we need to control the growth. And the way to do this is by requiring that the product between the resolvent function and the distance to the spectrum (or a relevant part of the spectrum) is bounded in norm.

I have mentioned above that, given the fact that the spectrum of a linear operator lies in the complex plane, one will have to use complex analysis in working with this set and the functions defined on it. This aspect came to the fore in the third chapter; this chapter investigated the construction by Pascale Vitse of a functional calculus for Tadmor-Ritt operators. As such, Chapter III provides a synthesis of the two previous chapters that introduced the notion of "functional calculus" (Chapter I) and gave a mathematical characterization of Tadmor-Ritt operators (Chapter II). We used here notions related to advanced complex analysis, such as Hardy spaces, measures on the unit circle, or Cauchy-Stieltjes

integrals. These latter functions and their multipliers are used by Vitse in building a functional calculus for Tadmor-Ritt operators. This calculus, however, does not involve a new type of procedure; it is based on the Riesz-Dunford calculus which uses integration on a Jordan curve that surrounds the spectrum. What is significant is the selection of a type of functions with norms adequate for finding bounds for polynomials of Tadmor-Ritt operators. Another way to put this is to draw attention to the fact that one still uses holomorphic functions, but holomorphic functions that can be characterized in a certain way (as a Cauchy-Stieltjes integral or a multiplier of such an integral), and this characterization provides adequate norms for the purposes that are pursued. The contribution of the thesis is that it clearly defines the notions and makes explicit the steps in Vitse's construction.

Chapter III ended with an investigation of partially power boundedness; this intermediary phenomenon was presented as a way of shedding light on the connection between power boundedness and the (peripheral) spectrum. This goal was achieved by presenting two necessary conditions (in terms of Rajchman measure and Hausdorff dimension on the unit circle) for partially power boundedness.

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