SELF-OSCILLATIONS IN RELAY

CONTROL SYSTEMS WITH DELAY

A Thesis

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ABSTRACT

A general method for prediction and determination of the stability of self-oscillations in a system containing a discrete nonlinearity and delay is developed. This method is applied to a system for which Loeb's Rule fails.

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CHAPTER I

INTRODUCTION

Loeb's Rule,[1] for the prediction of symmetric limit cycles, fails in the case of a relay control system with delay (figure 1).

The describing function method is an approximate method of predicting limit cycles. The Nyquist Plot, $A(jw)$, and the negative inverse of the describing function, $-K_{eq}^{-1}(E)$, are plotted. Intersections of these two curves indicate possible limit cycles. The frequency, w, of the limit cycle is that value of w that maps, by means of the $A(jw)$ function, into the intersection point. Similarly the magnitude, E, is determined from the $-K_{eq}^{-1}(E)$ curve.

Loeb's Rule:

The limit cycle is stable, if the vector cross product

$$\overrightarrow{\frac{dA(jw)}{dw}} \times \frac{d}{dE}\overrightarrow{(-K_{eq}^{-1}(E))} \text{ is out of the page i.e. "positive"}$$

1. J. Loeb, "Phénomènes Héréditaires dans les Servomécanismes; un Critérium Général de Stabilité", Annales des Télécommunications, 6(12): 346 - 356 (1951).

where $\overrightarrow{\frac{dA(jw)}{dw}}$ is a vector at the intersection point tangent to the Nyquist Plot, in the direction of increasing frequency and

$\overrightarrow{\frac{d}{dE}(-K_{eq}^{-1}(E)}$ is a vector at the intersection point tangent to the negative inverse of the describing function, i.e. the critical locus, in the direction of increasing magnitude.

Loeb's Rule predicts an infinity of stable limit cycles, corresponding to each of the crossings of the critical locus in an upward direction. The inner limit cycles (those of higher frequency) are in fact unstable.

An intuitive reason for the failure can be given. Loeb's criterion presumes that the mapping from the s-plane, into the A(s) will be single valued. Because of the presence of the delay $e^{-\tau s}$ the inverse mapping is not single valued.

The Nyquist Plot is a mapping of the $s = jw$ line, from the s-plane into the A(s) plane. A(jw) represents the magnitude gain and phase shift given to an input of the form Esin wt.

Mapping of the $s = \sigma_i + jw$ lines in the s-plane (where σ_i is a real nonzero constant) can similarly be made into the A(s) plane. $A(\sigma_i + jw)$ represents the magnitude gain and phase shift, given to an input of $Ee^{\sigma_i t}\sin(wt)$.

Grensted's[2] definition of the describing function for an input of the form $Ee^{\sigma_i t}\sin wt$ gives $K_{eq}(E) = 4/(\pi Ee^{\sigma_i t})$. Thus the critical locus lies along the negative real axis for any σ_i.

2. P. E. W. Grensted, "Analysis of the Transient Response of Nonlinear Control Systems", A.S.M.E. Trans. 80, 1958, pp. 427 - 32.

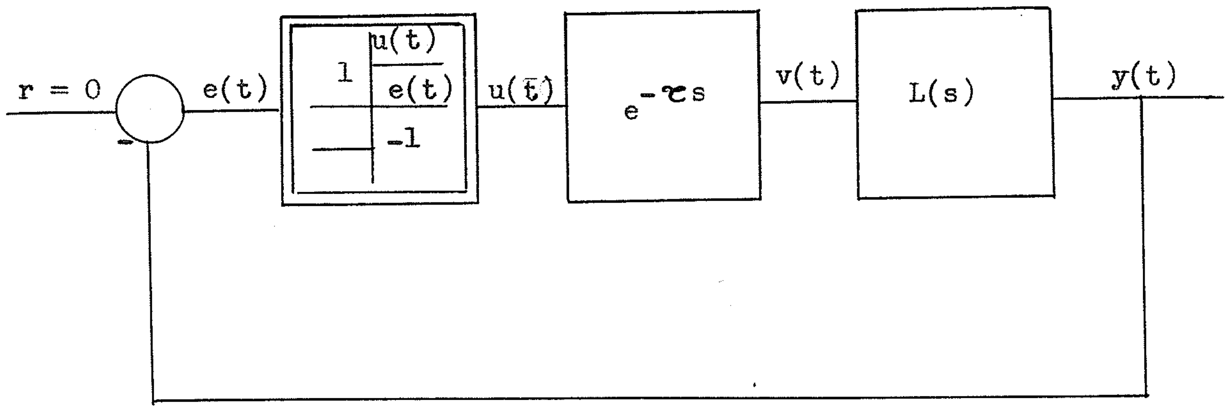


FIGURE I

A SYSTEM FOR WHICH LOEB'S RULE FAILS

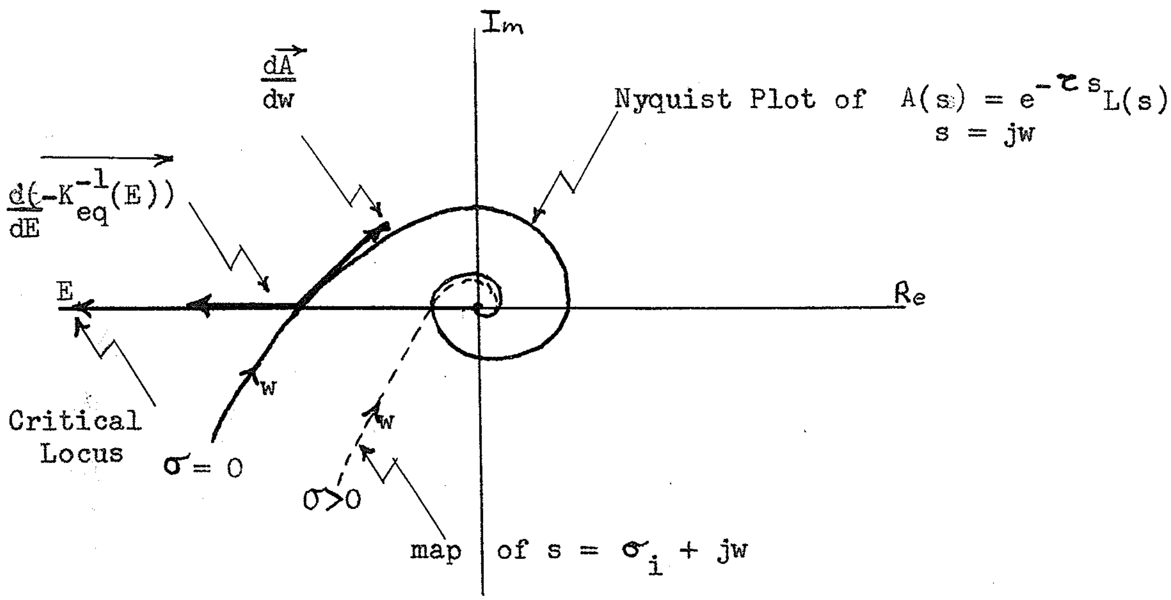


FIGURE 2

THE CRITICAL LOCUS AND PART OF THE NYQUIST PLOT FOR THE SYSTEM SHOWN IN FIGURE I

Because of the presence of the delay, $e^{-\tau s}$, an infinity of such $\sigma_i + jw$ curves cut the critical locus at any point (see figure 2). The closer the point is to the origin, the larger the number of the $\sigma_i > 0$.

Where the $s = jw$ curve cuts the critical locus closer to the origin, several $s = \sigma_i + jw$ curves with $\sigma_i > 0$ cut the critical locus.

Loeb's Rule informs us that the $E \sin wt$ mode would be stable. However other $Ee^{\sigma_i t} \sin wt$ modes are possible, with $\sigma_i > 0$. These modes are exponentially increasing and can not be stable.

It is not then reasonable to expect Loeb's Rule to work in a system containing a delay, unless one restricts its application to that portion of the plane where $\sigma \leqslant 0$ for all modes.

Brookes[3], in his M.Sc. thesis, obtained for the second predicted frequency of possible oscillation (m = 3), for a system in which $L(s) = 1/s(s+1)$ the following:

Frequency of Limit Cycle	(Hamel Locus)	$w = 6.41$ rad./sec.
Growing Exponential Term	(Tsypkin Method)	$w = 1.55$ rad./sec.
		$\sigma = 1.22$ nep./sec.

When $|A(s)| = 0.0243$ $\angle A(s) = 180^\circ$ the second crossing (second largest magnitude, E) of the critical locus by the Nyquist Plot occurs. Two modes of oscillation are possible with $\sigma \geqslant 0$.

[3] Barry Edward Brookes, "The Stability of Limit Cycles in Time-Lag Relay Control Systems" (M.Sc. Thesis, Dept. of Electrical Engineering, The University of Manitoba, May 1967). p.54

First mode of oscillation	$w = 6.40$ rad./sec.	
	$\sigma = 0$ nep./sec.	
Second mode of oscillation	$w = 1.74$ rad./sec.	
	$\sigma = 1.66$ nep./sec.	

This order of magnitude agreement gives some credence to the argument that it is the lower frequency modes with $\sigma > 0$ that cause the unstable behaviour.

However, in a relay control system it is not clear how two or more modes would interact if they existed simultaneously. An exact method is needed to determine what will happen.

In Chapter II, an exact method of predicting self-oscillations and determining their stability is developed for a system containing a delay and a discrete nonlinearity.

In Chapter III this method is applied to a system containing an ideal relay, a delay, and a linear part $L(s) = 1/s(s + 1)$.

It is shown that all the higher frequency self-oscillations are indeed unstable.

CHAPTER II

A PREDICTION OF SELF-OSCILLATIONS IN AN AUTONOMOUS SYSTEM CONTAINING A DISCRETE NONLINEARITY AND DELAY

In this chapter an exact method of determining the modes of oscillation is developed. A procedure for examining the stability of the predicted limit cycles is suggested.

1. DESCRIPTION OF THE SYSTEM:

The block diagram of the system is given in figure 3. It is a feedback system whose forward path contains:

(a) A discrete operator, $S(e,\mathrm{sgn}(\dot{e}))$, whose value may depend on both its input, e, and the sign of the time derivative of its input, $\mathrm{sgn}(\dot{e})$, and which assumes one of a set of constants so that

$$S(e,\mathrm{sgn}(\dot{e})) \equiv u(t) = c_j \qquad (2.1)$$

where c_j are constants and $j = 1, \ldots ,k$.

The current value of j depends on $e,\mathrm{sgn}(\dot{e})$, and the value j assumed, $j(t_i^+)$, at the previous switching instant of $u(t)$, $t_i^+ - \tau$.

(b) A fixed delay of magnitude, τ, so that

$$u(t - \tau) = v(t) \qquad (2.2)$$

(c) A linear operator whose input is $v(t)$ and output is $y(t)$. It is characterized by a rational transfer function $L(s)$. It may be described by the state equations:

$$\dot{\underline{x}} = A\underline{x} + Bv \tag{2.3}$$

$$y = C\underline{x} + Dv \tag{2.4}$$

where $\underline{x}$ is an n-dimensional state vector. A, B, C, and D are constant matrices of appropriate dimension.

The output $y(t)$ is fedback with a gain of -1 and the input is assumed to be zero.

Thus

$$e(t) = -y(t) \tag{2.5}$$

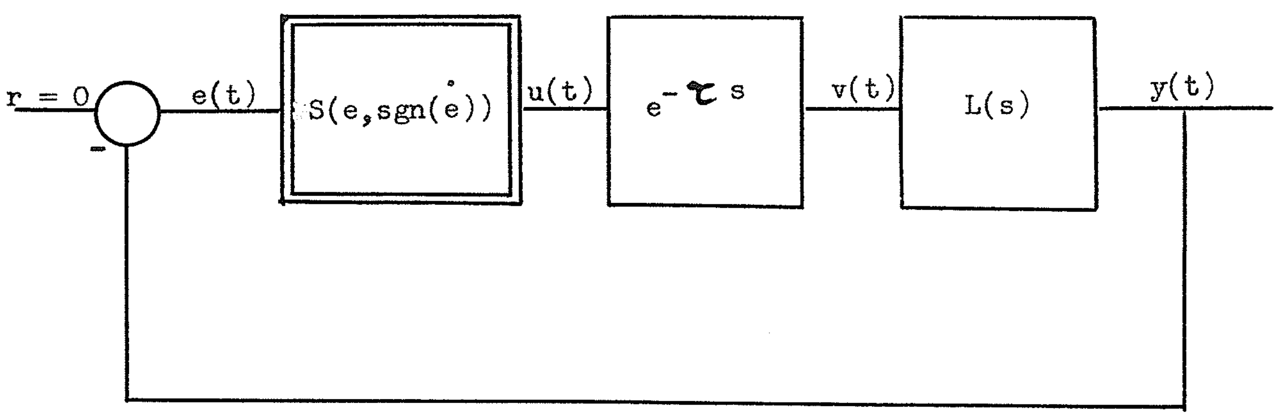


FIGURE 3

BLOCK DIAGRAM OF THE SYSTEM

2. THE PSEUDO-STATE SPACE APPROACH

As the system incorporates a delay, the conventional state space would be of infinite dimension. However, as $u(t)$ varies in discrete steps its value will be constant for finite intervals of time. The same information could be stored by recording the switching instants of $v(t)$, t_i, and the magnitude to which $v(t)$ will switch, $c_{j(t_i^+)}$. Thus the state of the system can be specified by a finite set of data.

3. PREDICTION OF SELF-OSCILLATIONS

In a system without delay, self-oscillation occurs if

$$\underline{x}(t+T) = \underline{x}(t).$$

The state variables of the linear part are the state variables of the system, and thus completely determine the future behavior of the system.

In a system with delay, the information stored in the delay (constituting an infinite number of state variables) also influences the future behavior of the system. In the case being considered, the waveform in the delay is a succession of constant levels of finite length.

$\underline{T}(t)$ is used to represent the intervals between switching instants stored in the delay and $\underline{C}(t)$ the magnitudes associated with these switching instants so that

$\underline{T}(t + T) = \underline{T}(t)$ and $\underline{C}(t + T) = \underline{C}(t)$ if self-oscillation occurs. The dimension as well as the values of $\underline{T}(t)$ and $\underline{C}(t)$ can vary with time.

(a) The condition $\underline{x}(t + T) = \underline{x}(t)$

The solution of the equation (2.3) yields

$$\underline{x}(t) = e^{A(t-t_o)}\underline{x}(t_o) + \int_{t_o}^{t} e^{A(t-t')}\, B\, v(t')\, dt' \tag{2.6}$$

$v(t)$ switches at $\{t_i\}_{-\infty}^{\infty}$. The t_i occur periodically with the period of i equal to r.

Let $t_o = 0$ then $t_r = T$ and $t_{sr} = sT$.

Let $\tau = sT + \theta$ where s is an integer and $0 \leqslant \theta < T$.

Thus $\tau = t_{sr} + \theta$.

Since $u(t - \tau) = v(t)$, so $u(t)$ switches at $t_i - \tau$.

As $$u(t) = c_{j(t_i^+)} \quad \text{when} \quad t_i - \tau < t \leqslant t_{i+1} - \tau \tag{2.7}$$

$$v(t) = c_{j(t_i^+)} \quad \text{when} \quad t_i < t \leqslant t_{i+1} \tag{2.8}$$

The solution for $\underline{x}(T)$ may be written as

$$\underline{x}(T) = e^{AT}\underline{x}(0) + \sum_{i=0}^{r-1} c_{j(t_i^+)} \int_{t_i}^{t_{i+1}} e^{A(T-t')}\, B\, dt' \tag{2.9}$$

As B is a constant matrix, a periodic solution requiring $\underline{x}(0) = \underline{x}(T)$ produces

$$\left[I - e^{AT}\right] \underline{x}(0) = \sum_{i=0}^{r-1} c_{j(t_i^+)}\, e^{AT} \int_{t_i}^{t_{i+1}} e^{-At'}\, dt'\, B \tag{2.10}$$

If A and $(I - e^{AT})$ are nonsingular this may be written as

$$\underline{x}(0) = \left[I - e^{AT}\right]^{-1} e^{AT} \sum_{i=0}^{r-1} c_{j(t_i^+)} \left[-e^{-At_{i+1}} + e^{-At_i}\right] A^{-1} B \qquad (2.11)$$

(b) The Conditions $\underline{T}(t + T) = \underline{T}(t)$ and $\underline{C}(t + T) = \underline{C}(t)$

Consider the signal $u(t)$ entering the delay in the time interval $(-\tau, T - \tau)$ as illustrated in figure 4(a). This determines the signal $v(t)$ in the time interval $(0,T)$ shown in figure 4(b).

At time $t = 0$ a waveform consisting of:

(i) s full periods as illustrated at (a) in figure 4 and

(ii) a periodic continuation for the interval $(-\theta, 0)$, is stored in the delay. This occupies the time interval $(-\tau, 0)$.

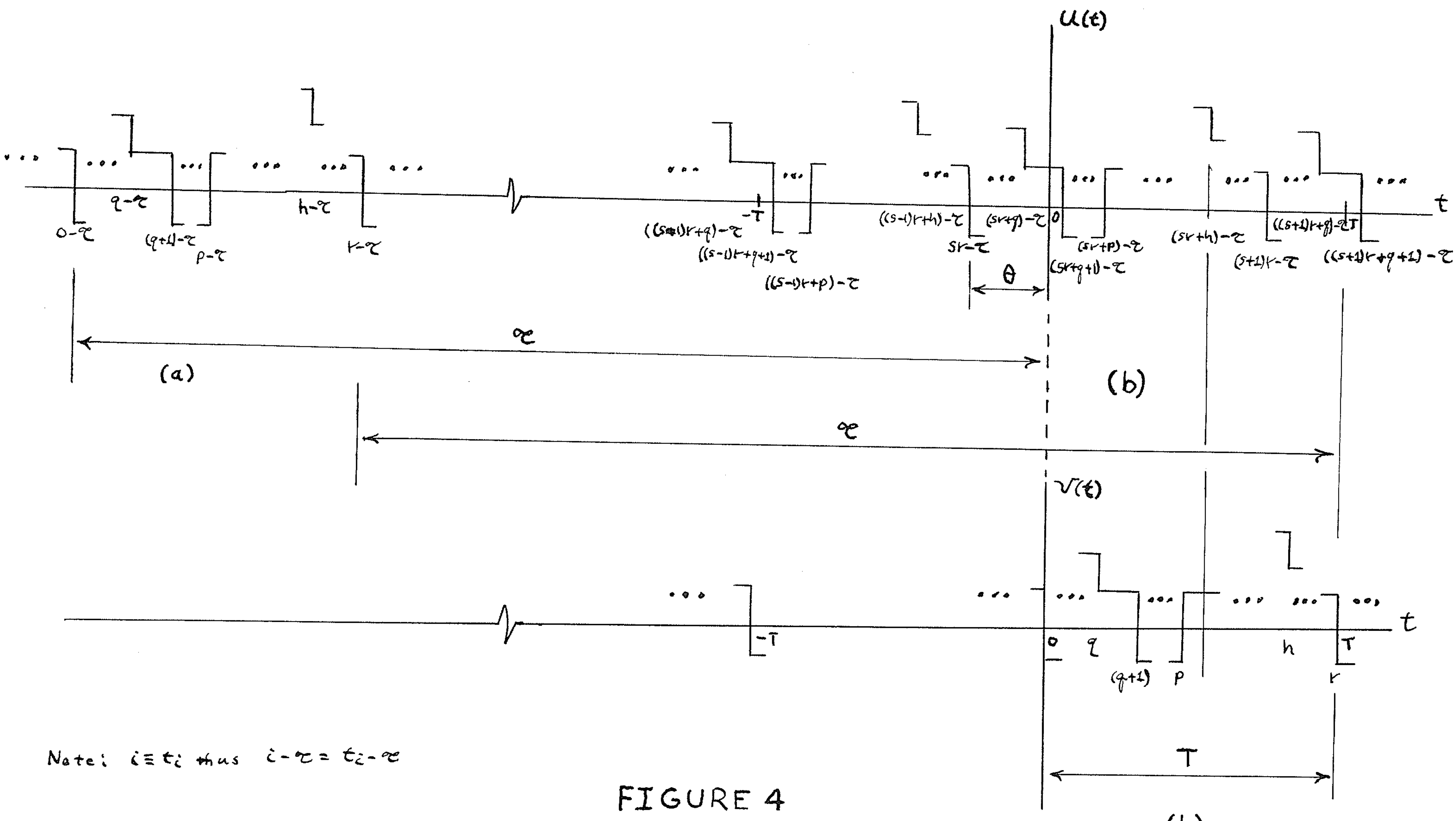
In order that the system have an oscillation of period T, the contents of the delay at time T must be identical to that at the time $t = 0$ described above. This occupies the time interval $(-\tau + T, T)$ illustrated in figure 4.

q is defined to be the maximum integer such that $t_q \leqslant \theta$. That is $t_{sr+q} - \tau$ is the last switching of $u(t)$ before $t = 0$.

Let h be an integer such that $q + 1 \leqslant h \leqslant q + r$. Then periodicity requires $t_{(s+1)r+h} - \tau = t_{sr+h} - \tau + T$, and

$$t_{sr+h} = t_h + sT \qquad (2.12)$$

Now let t_p be the largest switching time of $v(t)$ which satisfies $t_p \leqslant t_{sr+h} - \tau$.



Note: $i \equiv t_i$ thus $i - \tau = t_i - \tau$

FIGURE 4

SWITCHINGS OF $v(t)$ AND $u(t)$

Thus

$$t_p \leqslant t_h + sT - \tau < t_{p+1} \tag{2.13}$$

Now

$$\underline{x}(t_{sr+h} - \tau) = e^{A(t_h+sT-\tau)}\,\underline{x}(0) + \int_{t_0=0}^{t_h+sT-\tau} e^{A(t_h+sT-\tau-t')}\,B\,v(t')dt' \tag{2.14}$$

since $v(t) = c_{j(t_i^+)}$ where $t_i < t \leqslant t_{i+1}$ and B is a constant matrix.

$$\underline{x}(t_{sr+h} - \tau) = e^{A(t_h+sT-\tau)}\left\{\underline{x}(0) + \left[\sum_{i=0}^{p-1} c_{j(t_i^+)} \int_{t_i}^{t_{i+1}} e^{-At'}\,dt' + c_{j(t_p^+)} \int_{t_p}^{t_h+sT-\tau} e^{-At'}\,dt'\right] B\right\} \tag{2.15}$$

If A is nonsingular

$$\underline{x}(t^+_{sr+h} - \tau) = e^{A(t_h+sT-\tau)}\left\{\underline{x}(0) + \left[\sum_{i=0}^{p-1} c_{j(t_i^+)}(-e^{-At_{i+1}} + e^{-At_i}) + c_{j(t_p^+)}(-e^{-A(t_h+sT-\tau)} + e^{-At_p})\right] A^{-1} B\right\} \tag{2.16}$$

At this switching instant e and $\mathrm{sgn}(\dot{e})$ must have values required by the operator $S(e, \mathrm{sgn}(\dot{e}))$, in order that u(t) will switch from

$c_{j(t^+_{sr+h-1})}$ to $c_{j(t^+_{sr+h})}$.

Let $e_d(t^+_{sr+h} - \tau)$ be the specified value of $e(t^+_{sr+h} - \tau)$

Then

$$e_d(t^+_{sr+h} - \tau) = -y(t^+_{sr+h} - \tau) = -C\underline{x}(t^+_{sr+h} - \tau) - Dv(t^+_{sr+h} - \tau) \tag{2.17}$$

Since $v(t^+_{sr+h} - \tau) = c_{j(t^+_p)}$, this may be expressed as

$$e_d(t^+_{sr+h} - \tau) = -C\, e^{A(t_h + sT - \tau)} \left\{ \underline{x}(0) + \left[\sum_{i=0}^{p-1} c_{j(t^+_i)} \int_{t_i}^{t_{i+1}} e^{-At'}\, dt' + c_{j(t^+_p)} \int_{t_p}^{t_h+sT-\tau} e^{-At'}\, dt' \right] B \right\} - Dc_{j(t^+_p)} \tag{2.18}$$

and if A is nonsingular as

$$e_d(t^+_{sr+h} - \tau) = -C\, e^{A(t_h + sT - \tau)} \left\{ \underline{x}(0) + \left[\sum_{i=0}^{p-1} c_{j(t^+_i)} (-e^{-At_{i+1}} + e^{-At_i}) + c_{j(t^+_p)} (-e^{-A(t_h+sT-\tau)} + e^{-At_p}) \right] A^{-1} B \right\} - Dc_{j(t^+_p)} \tag{2.19}$$

When $\underline{x}(0)$ is substituted in terms of the unknown t_i $(i = 0, \ldots ,r)$ there result ~~are~~ r equations in r unknown t_i to solve. A wave shape for $u(t)$ must be chosen and the values of the t_i found numerically. One method of choosing the waveshape is to examine the trajectories in the state space in relation to the switching planes and to observe ~~observing~~ what limit cycles might be possible.

After a limit cycle is predicted, an examination of the trajectories in the state space will determine if any of the $\text{sgn}(\dot{e})$ conditions are unsatisfied or if any extraneous switchings occur. If neither of the preceding eventualities arise the limit cycle is possible.

4. A METHOD OF DETERMINING THE STABILITY OF THE LIMIT CYCLES

The method is as follows:

(a) Determine the relation between the incremental variations in one switching point, $\Delta \underline{x}_i$, the time between the i and the (i + 1) switchings of $v(t)$, ΔT_i, and the next switching point, $\Delta \underline{x}_{i+1}$, respectively.

Local linearization may be used to express this in the form:

$$\Delta \underline{x}_{i+1} = F_j \Delta \underline{x}_i + G_j \Delta T_i \qquad (2.20)$$

where $j = 0, \ldots ,r$, F_j is an nxn matrix and G_j is an nxl matrix.

In general there will be r equations, one for each switching point in the limit cycle; however, symmetry may make some of these equations identical.

(b) Determine the value of ΔT_i in terms of the $\Delta \underline{x}_b$'s where $b < i$. ΔT_i depends on the variation in time between the switchings of $u(t)$, that occurred τ seconds before. Local linearization may again be used to express this in the form:

$$\Delta T_i = \sum_{b=w}^{f} K_b \, \Delta \underline{x}_b \qquad (2.21)$$

where $f - w < r$

$f \leqslant i - sr$

K_b is a 1xn constant matrix.

If a switching point of $v(t)$ occurs on a switching plane of $u(t)$ there may be two equations. The one that is in fact applicable, depends on which side of the switching plane $\underline{x}_b$ places the switching point.

(c) Combine the relations into a set of difference equations. Then determine the stability of these equations.

$$\Delta \underline{x}_{i+1} = F_j \, \Delta \underline{x}_i + G_j \sum_{b=w}^{f} K_b \, \Delta \underline{x}_b \qquad (2.22)$$

The next chapter applies the procedures developed in this chapter to a specific problem and should clarify many of the details of the technique.

CHAPTER III

SELF-OSCILLATIONS IN A RELAY SYSTEM WITH DELAY

The methods of Chapter II are applied to a system in which the nonlinearity is an ideal relay, the delay is unity, and the linear part has a transfer function $L(s) = 1/s(s + 1)$.

1. DESCRIPTION OF THE SYSTEM

The block diagram of the system is given in figure 5. The elements of the forward path are:

(a) An ideal relay for which

$$u(t) = \operatorname{sgn}(e(t)) \tag{3.1}$$

(b) A unit relay for which

$$u(t - 1) = v(t) \tag{3.2}$$

(c) A linear operator, whose input is $v(t)$ and output is $y(t)$, characterized by the transfer function $1/s(s + 1)$.

Let the state variables for the linear subsystem be:

$$x_1 = y + \dot{y} \tag{3.3a}$$

$$x_2 = \dot{y} \tag{3.3b}$$

Then

$$\dot{\underline{x}} = A\,\underline{x} + B\,v \tag{3.4}$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and $\quad y = C\underline{x}$ $\qquad$ (3.5)

where $\quad C = \begin{bmatrix} 1 & -1 \end{bmatrix}$

The output $y(t)$ is fed back with a gain of -1 and the input is set to zero so that

$$e(t) = -y(t) \qquad (3.6)$$

2. STATE SPACE DESCRIPTION

(a) <u>The Trajectories</u>

Integration of the state equations produces for constant v

$$x_1 = vt + x_{10} \qquad (3.7)$$

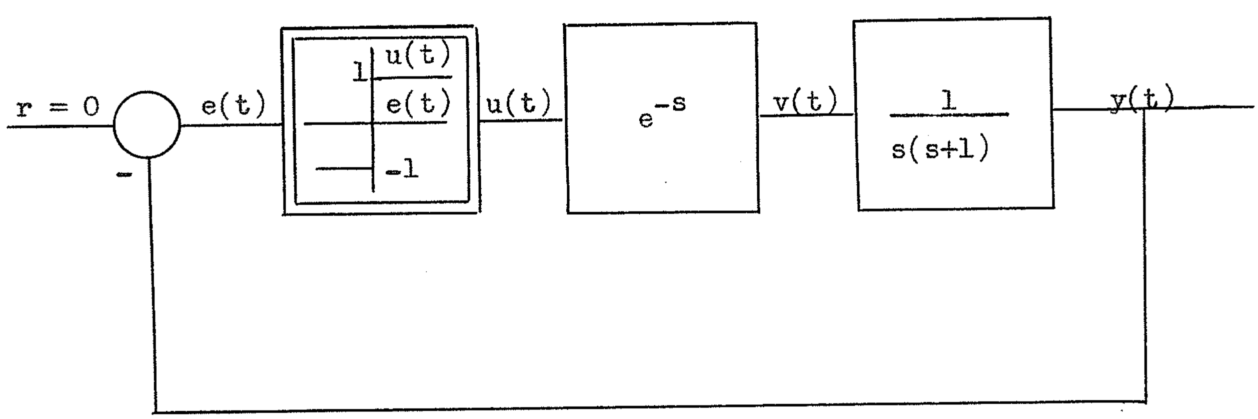


FIGURE 5

BLOCK DIAGRAM OF A RELAY SYSTEM WITH DELAY where x_{10} is the value of x_1 when $t = 0$, and

$$x_2 = v(1 - e^{-t}) + x_{20}\, e^{-t} \tag{3.8}$$

where x_{20} is the value of x_2 when $t = 0$

Elimination of t by combining (3.7) and (3.8) results in the family of curves

$$x_2 - v = (x_{20} - v)\, e^{-\frac{1}{v}(x_1 - x_{10})} \tag{3.9}$$

a sketch of which is given in figure 6.

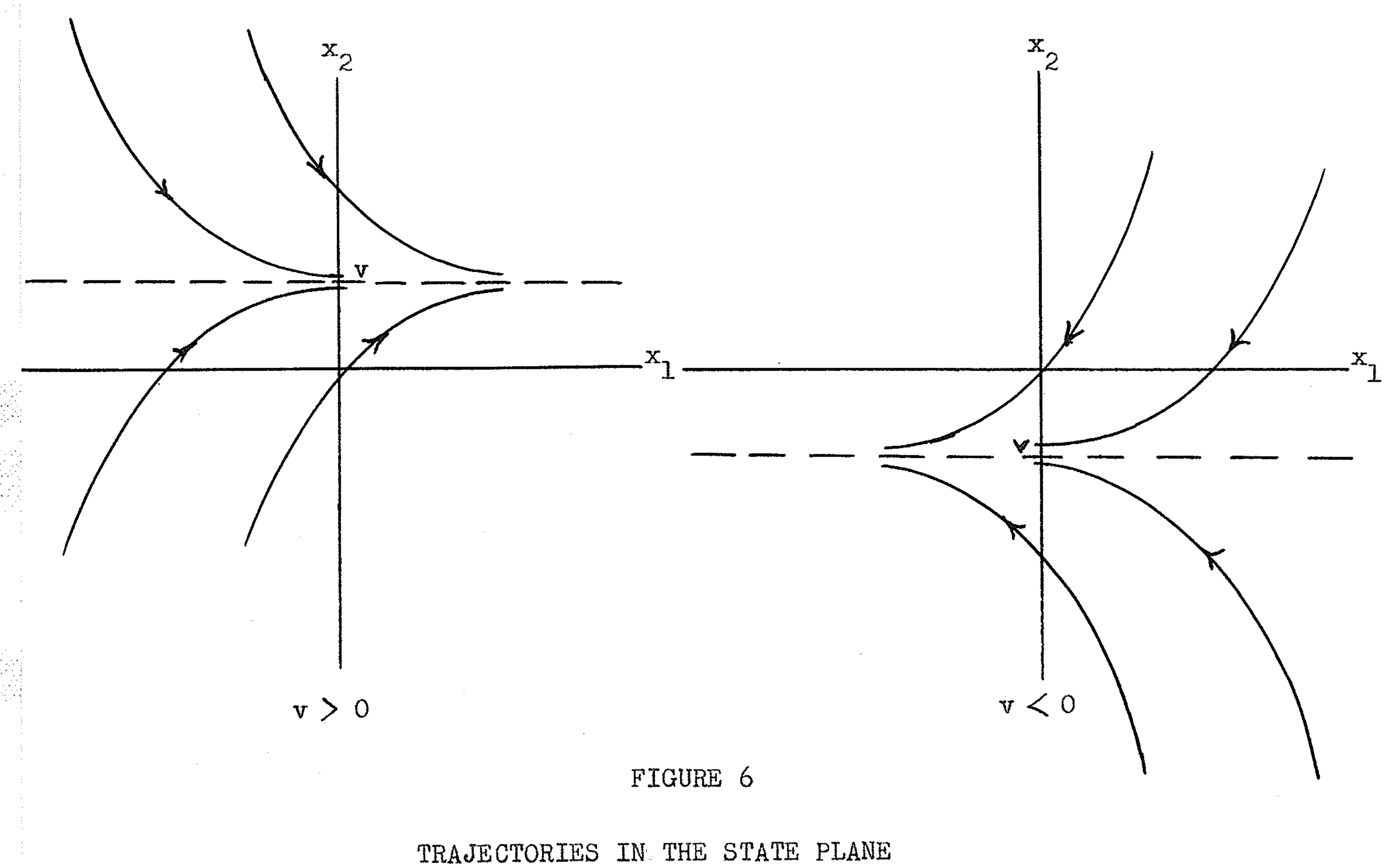


FIGURE 6

TRAJECTORIES IN THE STATE PLANE

(b) The Switching Line for u(t)

Equations (3.1), (3.5), and (3.6) yield

$$u(t) = -\text{sgn}(x_1 - x_2) \tag{3.10}$$

Therefore if $x_1 > x_2$ then $u = -1$

and if $x_1 < x_2$ then $u = 1$

(c) Possible Symmetric Self-Oscillations

Figure 7 shows possible limit cycles.

s is the number of full periods stored in the delay.

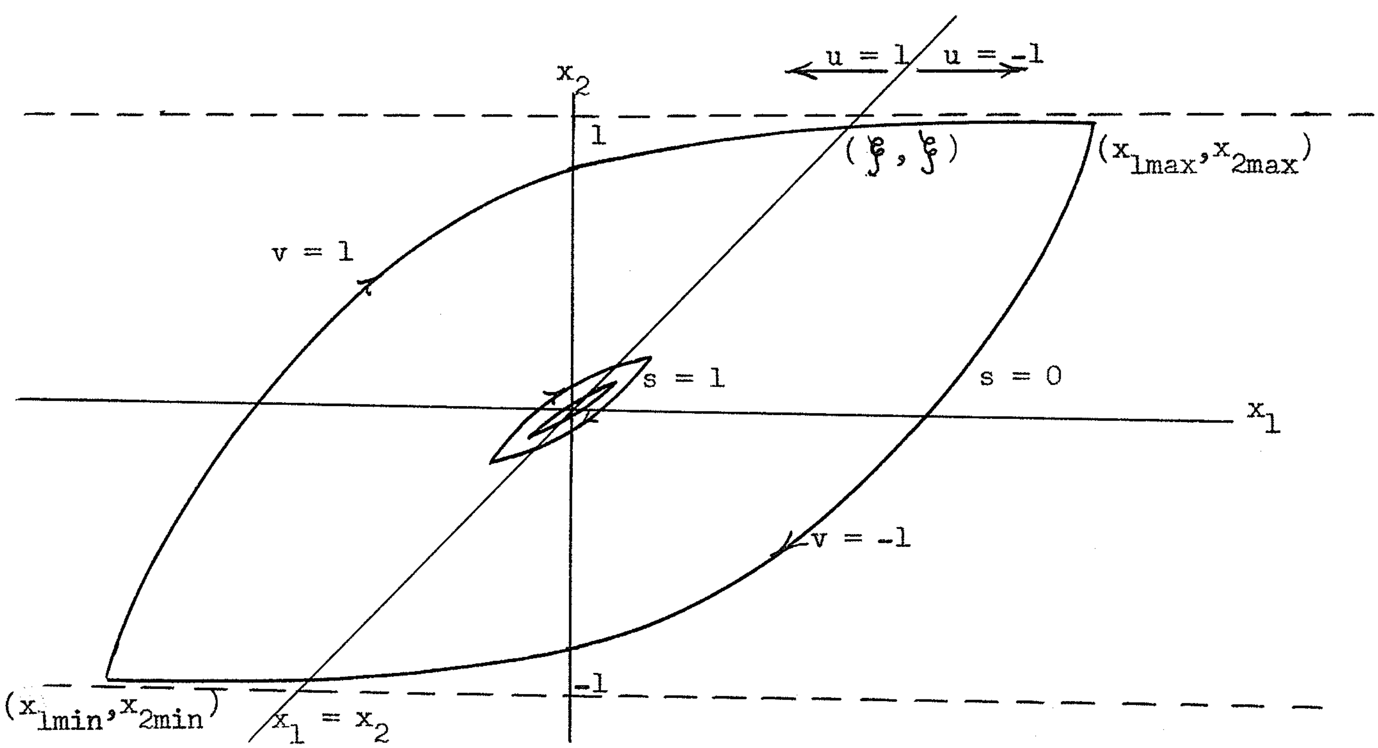


FIGURE 7

POSSIBLE SYMMETRIC LIMIT CYCLES

The Switching Lines for v(t)

From a consideration of the waveform stored in the delay, figure 7, and equation (3.7), and the fact* that $v = \pm 1$, it is apparent that $\tau = 1 = sT + (x_{1max} - \xi)$ where T is the period of the oscillation. The symmetry condition $x_{1min} = -x_{1max}$ and equation (3.7) produce

$$x_{1max} = T/4 \tag{3.11}$$

and so

$$\xi = (s + \tfrac{1}{4})T - 1 = (4s + 1)x_{1max} - 1 \tag{3.12}$$

Application of (3.9) to describe the trajectory between (ξ , ξ) and (x_{1max}, x_{2max}) and substitution from (3.12) yields

$$x_{2max} - 1 = ((4s + 1)x_{1max} - 2)e^{(4sx_{1max} - 1)} \tag{3.13a}$$

A similar procedure produces

$$x_{2min} + 1 = ((4s + 1)x_{1min} + 2)e^{-(4sx_{1min} + 1)} \tag{3.13b}$$

If $s = 0$ (3.13a) and (3.13b) represent straight lines. This case gives the switching lines for v(t) when no switchings are stored in the delay initially.

3. PREDICTION OF SELF-OSCILLATION

For the ideal relay $c_j = \pm 1$. If $t_o = 0$ and $c_{j(0+)} = 1$

then

$$c_{j(t_i^+)} = (-1)^i \tag{3.14}$$

* ξ is defined in figure 7.

Since

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-t} \end{bmatrix} \tag{3.15}$$

equation (2.10)* becomes

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 - e^{-T} \end{bmatrix} \underline{x}(0) = \sum_{i=0}^{r-1} (-1)^i \begin{bmatrix} \int_{t_i}^{t_{i+1}} dt' & 0 \\ 0 & \int_{t_i}^{t_{i+1}} e^{-(T-t')}\, dt' \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{3.16}$$

Thus

$$0 = \sum_{i=0}^{r-1} (-1)^i \,(t_{i+1} - t_i) \tag{3.17}$$

which specifies that the total time in a period for which $v = 1$ is the same as that for which $v = -1$, and

$$x_{20} = \frac{e^{-T}}{1 - e^{-T}} \sum_{i=0}^{r-1} (-1)^i \,(e^{t_{i+1}} - e^{t_i}) \tag{3.18}$$

For an ideal relay $e_d(t_i^+ - 1) = 0$ since $\tau = 1$.

Substitution of the values given in equations (3.4), (3.5), (3.14), (3.15), and (3.18) into equation (2.18) produces

*Since A is singular equation (2.11) cannot be used.

$$0 = x_{10} + \sum_{i=0}^{p-1} (-1)^i \, (t_{i+1} - t_i) + (-1)^p \, (t_h + sT - 1 - t_p)$$

$$- e^{-(t_h + sT - 1)} \left[\frac{e^{-T}}{1 - e^{-T}} \sum_{i=0}^{r-1} (-1)^i \, (e^{t_{i+1}} - e^{t_i}) \right.$$

$$\left. + \sum_{i=0}^{p-1} (-1)^i \, (e^{t_{i+1}} - e^{t_i}) + (-1)^p \, (e^{t_h + sT - 1} - e^{t_p}) \right] \tag{3.19}$$

where $t_p \leqslant t_h + sT - 1 < t_{p+1}$ (3.20)

This set of expressions for $h = q + 1, \ldots, q + r$ relates x_{10} to t_i where $i = 0, \ldots, r$ for the self oscillations that one assumes to exist.

Symmetric Oscillations

If $r = 2$ so that $t_2 = T$ then, as $t_o = 0$, equation (3.17) requires that $t_1 = T/2$.
Substitution into (3.18) gives

$$x_{20} = \frac{(1 - e^{T/2})}{(1 + e^{T/2})} = x_{2min} \tag{3.21}$$

for such an oscillation.

Now h was defined to be an integer that satisfies $q + 1 \leqslant h \leqslant q + r$. As $r = 2$ then $h = q + 1,\ q + 2$.

Also $\theta < T$ requires $q < r$. Thus q may only assume the two values 0 and 1.

The periodic conditions derived, in Section 3b of Chapter II, imply that switchings of $u(t)$ must occur at $(t_{q+1} + sT - 1)$, i.e. $h = q + 1$, and $(t_q + (s + 1)T - 1)$, i.e. $h = q + 2$.

(a) <u>$q = 0$</u>

Switchings of $u(t)$ occur at $(T/2 + sT - 1)$ and $((s + 1)T - 1)$. As these switchings must be $T/2$ apart, equation (2.13) yields

$$0 \leqslant (s + \tfrac{1}{2})T - 1 < T/2 \leqslant (s + 1)T - 1 < T$$

which upon rearrangement becomes (symmetry allows the use of either the condition for $h = 1$ or $h = 2$).

$$\frac{1}{s + 1/2} \leqslant T < \frac{1}{s} \tag{3.22}$$

Equation (3.14) states that odd switchings are switchings of $v(t)$ from +1 to -1. $T/2 + sT$ is an odd switching. With $v = 1$, a switching of u from +1 to -1 is required. Similarly, a switching of u from -1 to +1 is required when $v = -1$. An examination of figure 7 indicates that such a limit cycle is possible.

By evaluating equation (3.19) for the two switching points (i) $h = 1$, $p = 0$ and (ii) $h = 2$, $p = 1$, and eliminating x_{10} between them one obtains

$$0 = (4s + 1)T/4 - 2 + \frac{2e^{1-(2s+1)T/2}}{1 + e^{-T/2}} \tag{3.23}$$

As the oscillation is symmetric $x_{10} = -T/4$. This could have been substituted directly into equation (3.19) for (i) or (ii) and the above equation obtained. Alternatively x_{1max} given in (3.11) and x_{2max} given in (3.21) could have been substituted into the switching line (3.13a) again resulting in the above equation.

If the substitution $m = 2s + 1$ is made, this equation becomes identical with the real part of the Hamel Locus set to zero, one of the requisite conditions for oscillation in this system. The range of T/2 given by (3.22) is the range for which the imaginary part of the Hamel Locus satisfies the necessary switching condition.[4]

(b) $\underline{q = 1}$

Switching of u(t) occur at $(s + 1)T - 1$ and $(s + 3/2)T - 1$. As these switchings must be T/2 apart, equation (2.13) yields

$$0 \leqslant (s + 1)T - 1 < T/2 \leqslant (s + 3/2)T - 1 < T$$

which upon rearrangement gives

$$\frac{1}{s + 1} \leqslant T < \frac{1}{s + \frac{1}{2}} \tag{3.24}$$

Equation (3.14) states that even numbered switchings are switchings from -1 to +1. As $(s + 1)T$ is an even numbered switching, a switching of u from -1 to +1 while $v = +1$ is required. Similarily a switching of u from +1 to -1 while $v = -1$ is required. Examination of figure 7 indicates that such a limit cycle is not possible.

[4]Barry Edward Brookes, "The Stability of Limit Cycles in Time-Lag Relay Control Systems" (M.Sc. Thesis, Dept. of Electrical Engineering, 1967), p. 47.

By evaluating equation (3.19) for the two switching points (i) $h = 2$, $p = 0$ and (ii) $h = 3$, $p = 1$ and eliminating x_{10} between them one obtains

$$(4s + 3)T/4 - 2 + \frac{2e^{1-(s+1)T}}{(1 + e^{-T/2})} = 0 \tag{3.25}$$

If the substitution $m = 2s + 2$ is made, this again becomes the same as the real part of the Hamel Locus set to zero. The range of $T/2$ however, is the range for which the imaginary part of the Hamel Locus fails to satisfy the switching conditions.[5]

4. THE STABILITY OF THE LIMIT CYCLES

(a) Determination of Equation (2.20)

If $v = -1$ equation (3.7) specifies that for the trajectories shown on figure 8,

$$x_{1min} = -T/2 + x_{1max}$$

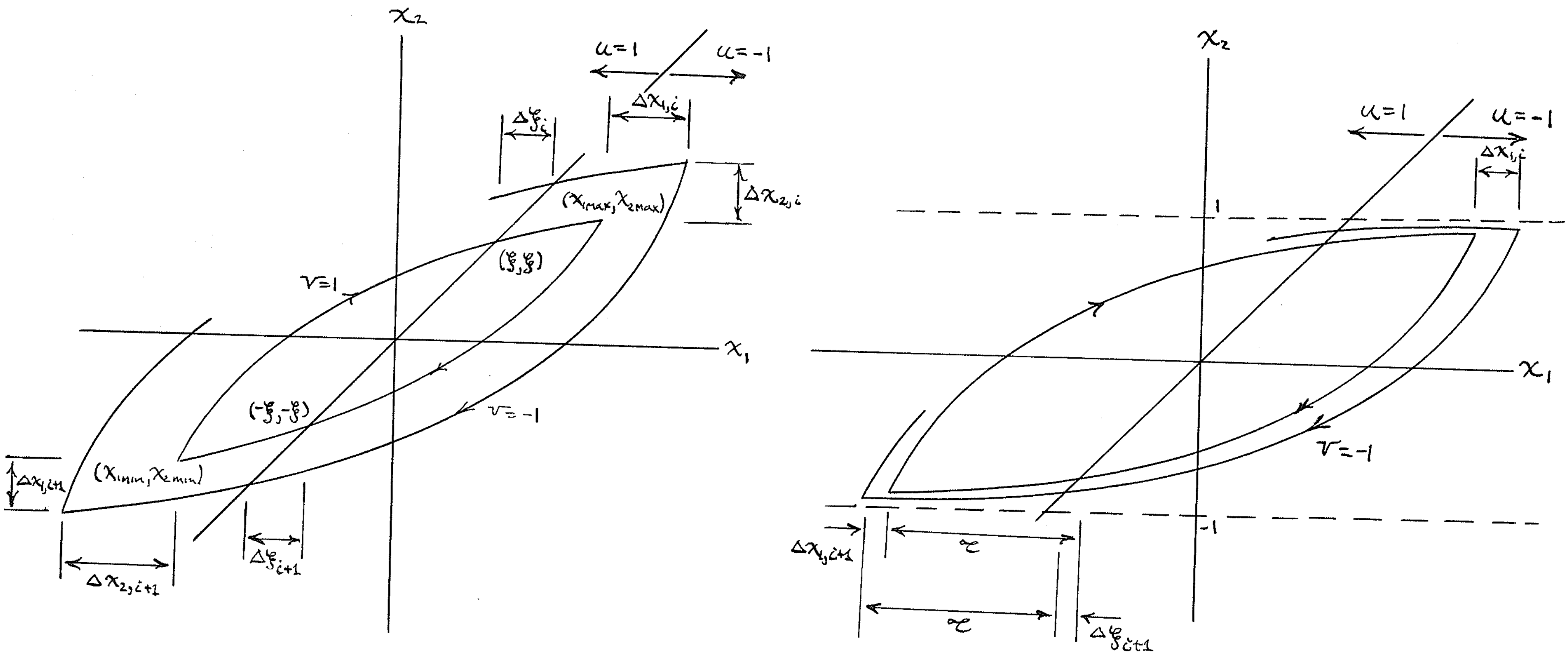
for the limit cycle and

$$x_{1min} + \Delta x_{1,i+1} = -(T/2 + \Delta T_i) + x_{1max} + \Delta x_{1,i}$$

for the perturbed trajectory. Thus

$$\Delta x_{1,i+1} = \Delta x_{1,i} - \Delta T_i \tag{3.26a}$$

[5] Ibid.



The Diagram illustrates the notation for the case $v=-1$.

FIGURE 8

INCREMENTAL VARIATIONS IN THE LIMIT CYCLE

Similarly if $v = 1$

$$\Delta x_{1,i+1} = \Delta x_{1,i} + \Delta T_i \qquad (3.26b)$$

Use of equations (3.9), (3.11), (3.21) and local linearization produces

$$\Delta x_{2,i+1} = \frac{2}{(1+e^{T/2})} (\Delta x_{1,i+1} - \Delta x_{1,i}) + e^{-T/2} \Delta x_{2,i} \qquad (3.27)$$

The same expression results if $v = +1$ or $v = -1$.

(b) <u>Determination of Equation (2.21)</u>

Let ξ be the value of x_1 and x_2, when u switches from +1 to -1 on the limit cycle and $\Delta \xi_i$ be the variation in x_1 and x_2 on the i-th crossing of the switching line.

(i) <u>$s = 0$</u>

From figure 8 and equation (3.7) it is apparent that

if $v = -1$ then

$$\Delta T_i = \Delta x_{1,i} - \Delta \xi_{i+1} \qquad (3.28a)$$

but if $v = 1$

$$\Delta T_i = - \Delta x_{1,i} + \Delta \xi_{i+1} \qquad (3.28b)$$

(ii) <u>$s \geq 1$</u>

The variation in time between the i and the (i + 1) switchings of $v(t)$, ΔT_i, is the variation in the time interval between the switchings of $u(t)$ that occurred $\tau = 1$ seconds previous to the switchings of $\tilde{x}(t)$, the $(i - 2s)$ and $(i + 1 - 2s)$ switchings of $u(t)$.

From equation (3.7) it is apparent that the change in the total distance travelled in the x_1 direction, between the $(i - 2s)$ and $(i + 1 - 2s)$ switching of $u(t)$, will be the variation in the time interval, between these switchings. Thus from figure 8 it follows when

$$v = -1$$

$$\Delta T_i = 2\,\Delta x_{1,i-2s} - \Delta\xi_{i-2s} - \Delta\xi_{i+1-2s} \tag{3.29a}$$

but if $v = 1$ then

$$\Delta T_i = -2\,\Delta x_{1,i-2s} + \Delta\xi_{i-2s} + \Delta\xi_{i+1-2s} \tag{3.29b}$$

Use of equations (3.9), (3.11), (3.12) and local linearization in a manner similar to that of section (a) above, the variations in ξ_{i+1} and ξ_i may be found to be

$$\Delta\xi_{i+1} = \frac{(s + 1/4)T - 2}{(s + 1/4)T - 1}\,\Delta x_{1,i} + \frac{e^{1-(s+1/2)T}}{(s + 1/4)T - 1}\,\Delta x_{2,i} \tag{3.30a}$$

and

$$\Delta\xi_i = \frac{(s + 1/4)T - 2}{(s + 1/4)T - 1}\,\Delta x_{1,i} + \frac{e^{1-st}}{(s + 1/4)T - 1}\,\Delta x_{2,i} \tag{3.30b}$$

(c) <u>Combination of Equations to Form (2.22)</u>

(i) <u>For $s = 0$</u>

From equations (3.26a) and (3.28a), or (3.26b) and (3.28b), combined with (3.30a) it follows that

$$\Delta \underline{x}_{i+1} = F_o \Delta \underline{x}_i \tag{3.31}$$

where

$$F_o = \frac{1}{(T/4 - 1)} \begin{bmatrix} T/4 - 2 & e^{1-T/2} \\ \dfrac{-2}{1 + e^{T/2}} & \left(\dfrac{2e^{1-T/2}}{1 + e^{T/2}} + e^{-T/2}\,(T/4 - 1)\right) \end{bmatrix}$$

Thus

$$\Delta \underline{x}_k = F_o^k \, \Delta \underline{x}_o$$

Therefore if λ_1 and λ_2 are the eigenvalues of F_o, the linear system is asymptotically stable if $|\lambda_j| < 1, \ j = 1, 2.$[6] Thus the limit cycle mode with $s = 0$ will be stable for arbitrarily small perturbations of the switching points and thus of the limit cycle.

For the system under consideration equation (3.23) for $s = 0$ produces $T/2 = 3.75$. For this value the eigenvalues are found to be $\lambda_{1,2} = 0.0580 \pm j\,0.0559$ so that the oscillation predicted for $s = 0$ is locally stable.

[6] Lotfi A. Zadeh and Charles A. Desoer, Linear System Theory (Toronto: McGraw-Hill Book Company, 1963), p. 482.

If
$$H(z) = z^{2s+1}\, I - z^{2s}\, F_s - G_s \tag{3.36}$$

then
$$\Delta \underline{X}(z) = H^{-1}(z) \left[I \sum_{i=0}^{2s} z^{2s+1-i}\, \Delta \underline{x}_i - F_s \sum_{i=0}^{2s-1} z^{2s-i}\, \Delta \underline{x}_i \right] \tag{3.37}$$

where
$$H^{-1}(z) = \frac{\text{adj } H(z)}{\det H(z)} = \frac{P(z)}{m(z)} \tag{3.38}$$

where $P(z)/m(z)$ is $H^{-1}(z)$ in the form where all the factors of $\det H(z)$ common to all the elements of $\text{adj } H(z)$ have been cancelled.

Theorem:[7]

Every z-matrix $H(z)$ of rank n can be reduced by elementary transformations to the Smith Normal form,

$$N(z) = \begin{bmatrix} f_i(z) & 0 & \cdots & 0 & \cdots & 0 \\ 0 & f_2(z) & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & f_n(z) & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \tag{3.39}$$

[7] Frank Ayres Jr., Matrices (New York: Schaum Publishing Co., 1962), p. 188.

where each $f_i(z)$ is monic (i.e. the coefficient of the highest power of z is unity) and $f_i(z)$ divides $f_{i+1}(z)$, $(i = 1, 2, \ldots, n-1)$.

When a z-matrix $H(z)$ of rank n has been reduced to (3.39) the greatest common divisor of all g-square minors of $H(z)$, $g \leq n$ is the greatest common divisor of all g-square minors of $N(z)$. This greatest common divisor is

$$d_g(z) = f_1(z)\, f_2(z) \ldots f_g(z) \qquad (g = 1, 2, \ldots, n)$$

Elementary transformations modify the determinant of the matrix by at most a multiplicative constant. Therefore if $H(z)$ can be inverted (i.e. is nonsingular)

$$\det H(z) = (f_1(z) f_2(z) \ldots f_n(z))k$$

in which k is a constant. The greatest common divisor of all $(n-1)$-square minors of $H(z)$ and thus of $\operatorname{adj} H(z)$ is therefore

$$d_{n-1}(z) = f_1(z)\, f_2(z) \ldots f_{n-1}(z)$$

and so

$$m(z) = f_n(z) \tag{3.40}$$

Since $f_i(z)$ divides into $f_{i+1}(z)$ evenly, $f_n(z)$ contains all the roots (perhaps to a lower power) that $\det H(z)$ contains.

Consider factors common to all elements of $P(z)\Delta(z)$ where

$$\Delta(z) = I \sum_{i=0}^{2s} z^{2s+1-i} \Delta \underline{x}_i - F_s \sum_{i=0}^{2s-1} z^{2s-i} \Delta \underline{x}_i$$

or

$$\Delta(z) = \begin{bmatrix} \sum_{i=0}^{2s} z^{2s+1-i} \Delta x_{1,i} - \sum_{i=0}^{2s-1} z^{2s-i} \Delta x_{1,i} \\ \sum_{i=0}^{2s} z^{2s+1-i} \Delta x_{2,i} - e^{-T/2} \sum_{i=0}^{2s-1} z^{2s-i} \Delta x_{2,i} \end{bmatrix} \qquad (3.41)$$

If the oscillation is to be stable, it must be stable for any small initial disturbance stored in the delay or placed on the system. In particular let the disturbance be such that

$$\underline{x}_i = \underline{0} \qquad i < 2s$$
$$\neq \underline{0} \qquad i = 2s$$

Then

$$\frac{P(z)\Delta(z)}{m(z)} = \frac{z}{m(z)} P(z) \Delta \underline{x}_{2s} \qquad (3.42)$$

Thus as $P(z)$ is already in reduced form, only the factor z may cancel in $m(z)$.

It is obvious from a partial fraction expansion of $\Delta \underline{X}(z)$, that if $\Delta \underline{x}(t)$ is to have stable behaviour the zeros of $m(z)$ and thus of $\det H(z)$ must all lie in the open unit circle: $|z| < 1$.

If $\Delta \underline{x}(t)$ is to have unstable behaviour $\det H(z)$ must have at least one zero outside the closed unit circle. If all the zeros of $\det H(z)$ lie in the closed unit circle with one or more on the unit circle higher order terms must be considered to determine stability.

Theorem:[8]

For a given polynomial $f(z) = a_o + a_1 z + \ldots + a_n z^n$, let all the a_k be real and the following sequence of polynomials $f_j(z)$ be constructed:

$$f_j(z) = \sum_{k=o}^{n-j} a_k^{(j)} z_k$$

where $f_o(z) = f(z)$ and

$$a_k^{(j+1)} = a_o^{(j)} a_k^{(j)} - a_{n-j}^{(j)} a_{n-j-k}^{(j)}, \; j = 0, 1, \ldots, n-1. \tag{3.40}$$

Let $P_k = a_o^{(1)} a_o^{(2)} \ldots a_o^{(k)}$, $k = 1, 2, \ldots n$

Then, if, for some $k < n$, $P_k \neq 0$ but $f_{k+1}(z) \equiv 0$, then $f(z)$ has $n - k$ zeros on the unit circle $|z| = 1$ at the zeros of $f_k(z)$; it has p zeros in this circle, where p is the number of negative $\cancel{\not{}}\,P_j$ for $j = 1, 2, \ldots, k$ and it has $q = k - p$ zeros outside this circle.

From equations (3.3~~2~~3), (3.34) and (3.36) and the definition of a determinant

$$\det H(z) = z^{2s}\Big\{ z^{2s+2} - (1 + e^{-T/2}) z^{2s+1} + e^{-T/2} z^{2s}$$

$$+ \frac{2(1 - e^{1-(s+1/2)T})}{(s + 1/4)T - 1} z + \frac{2(e^{1-(s+1/2)T} - e^{-T/2})}{(s + 1/4)T - 1} \Big\} \tag{3.41}$$

[8]Morris Marden, The Geometry of the Zeros of a Polynomial in a Complex Variable (New York: American Mathematical Society, 1949), p. 157.

$$\det H(z) = z^{2s}(z + 1)f(z)$$

where

$$f(z) = z^{2s+1} - (2 + e^{-T/2})z^{2s} - 2(1 + e^{-T/2}) \sum_{i=1}^{2s-1} (-1)^i z^i$$

$$+ \frac{2(e^{1-(s+1/2)T} - e^{-T/2})}{(s + 1/4)T - 1} \qquad (3.42)$$

Note: equation (3.23) may be used to show that $z = -1$ is a root of $\det H(z)$.

<u>Determination of the Location of the Zeros of $f(z)$</u>

$$a_o^{(1)} = a_o^2 - a_{2s+1}^2$$

$$= \left(\frac{2(e^{1-(s+1/2)T} - e^{-T/2})}{(s + 1/4)T - 1}\right)^2 - 1 \qquad (3.43)$$

a_o^2 is always less than 1 for $s \geqslant 1$ (for proof see appendix). Thus $-1 < a_o^{(1)} < 0$ for $s \geqslant 1$

Now

$$a_{2s}^{(1)} = a_o\, a_{2s} - a_{2s+1}\, a_1$$

$$= \frac{-2(e^{1-(s+1/2)T} - e^{-T/2})(2 + e^{-T/2})}{(s + 1/4)T - 1} - 2(1 + e^{-T/2}) \qquad (3.44)$$

From the geometry of figure 7 it is apparent that

$$(s + 1/4)T - 1 = \xi > 0 \quad \text{and}$$

$$(e^{1-(s+1/2)T} - e^{-T/2}) = (e^{-(\xi + T/4)} - e^{-T/2}) > 0 \quad \text{as}$$

$\xi < x_{1max} = T/4.$

Thus $a_{2s}^{(1)} < -2$ for $s \geqslant 1$

$$a_o^{(2)} = (a_o^{(1)})^2 - (a_{2s}^{(1)})^2 < 1 - (-2)^2 = -3 \tag{3.45}$$

Thus $P_1 = a_o^{(1)} < 0 \qquad P_2 = a_o^{(1)}\ a_o^{(2)} > 0$

and so there is at least one zero of $f(z)$ and thus of $\det H(z)$ that is outside the unit circle.

It is therefore proven that all the limit cycles for $s \geqslant 1$ are unstable. The predictions of Chapter I have been verified for this particular case.

CHAPTER IV

DISCUSSION OF RESULTS

In Chapter I, an intuitive discussion of the failure of Loeb's Rule in systems containing delay is given. It is predicted that all the higher frequency modes of oscillation in any relay control system with delay will be unstable.

In Chapter II an exact method of determining self-oscillations and their stability in a feedback control system containing a discrete nonlinearity, a delay, and a linear part is developed.

This method is stated in a very general form and can be simplified for special applications. For instance, if symmetric oscillations are expected the fact that $\underline{x}(t) = -\underline{x}(t + T/2)$ can be used instead of $\underline{x}(t) = \underline{x}(t + T)$ and the number of switching equations can be reduced by half and their forms simplified.

The chief weakness of the method is that one must first choose the form of the oscillation to be investigated. The method used in this thesis was to examine the trajectories in the state space in relation to the switching plane. This, however, requires judgement and it would be easy to miss a possible limit cycle in a more complex system. For very complex systems the method would be close to impossible.

A worthy project would be the development of a system for predicting possible forms of oscillations.

In Chapter III, the methods of Chapter II are applied to symmetric oscillations in a particular system. The prediction of Chapter I is confirmed for this system.

The methods of testing the stability of difference equations for variations in the switching points is applicable to other ideal relay systems, for which the state matrix is diagonal. They will have difference equations and F_s and G_s matrices of the same form.

When one considers the argument of Chapter I, one suspects there should be a general proof regarding instability of symmetric oscillations the frequencies of which correspond to the intersection of the Nyquist Plot and critical locus in a region where a mode of oscillations with $\sigma > 0$ exists regardless of the nonlinearity. ~~This~~ An investigation of this would also have merit.

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APPENDIX

Proof that $0 < a_o < 1$ for $s \geqslant 1$ in Equation (3.4~~2~~3)

$$a_o = \frac{2(e^{1-(s+1/2)T} - e^{-T/2})}{(s+1/4)T - 1} = \frac{2(e^{-(\xi + T/4)} - e^{-T/2})}{\xi}$$

Since $(e^{-(\xi + T/4)} - e^{-T/2}) > 0$ and $\xi > 0$ (proven on p. 36) then $a_o > 0$.

Use of (3.23) allows one to write

$$a_o = \frac{[2 - (4s+1)(T/4)](1+e^{-T/2})}{(s+1/4)T - 1}$$

It follows from the alternating series rule that

$$e^{-T/2} > 1 - T/2 \quad \text{if} \quad T > 6$$

Substitution of the above and use of (3.12) gives

$$a_o < -2 + T/2 + \frac{T/2}{\xi}$$

If $\xi > \dfrac{T}{6 - T}$ then $a_o < 1$

Therefore if $\xi > T/5$ and $T < 1$ then $a_o < 1$

When $s = 1$ equation (3.23) gives $T = 0.979422$. Since $1 = sT + \theta$ where $0 \leqslant \theta < T$, T decreases in a monotonic manner as s increases. Therefore $T < 1$ for $s \geqslant 1$.

A Lower Bound on ξ

Differentiation of (3.9) gives

$$\frac{dx_2}{dx_1} = -(x_2 - 1)$$

thus

$$\left.\frac{dx_2}{dx_1}\right|_{(\eta,\eta)} = 1 - \eta$$

where (η, η) is a point on the $x_1 = x_2$ switching line such that a tangent through this point of the trajectory with $v = 1$ passes through the (x_{1max}, x_{2max}) point.

As the slope of any trajectory decreases when x_1 and x_2 increase $(x_1 < 1)$ $\xi > \eta$.

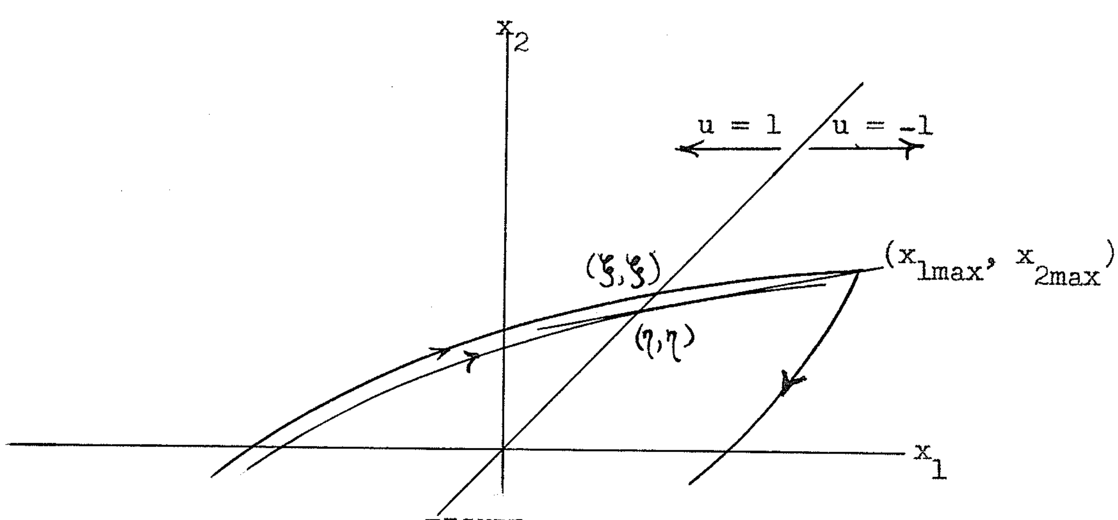

FIGURE 9

A LOWER BOUND ON ξ

Use of the point slope equation for a straight line gives

$$\frac{\eta - x_{2max}}{\eta - x_{1max}} = 1-\eta$$

Substitution from (3.11) and (3.~~12~~ 21) and use of the quadratic formula produces

$$\eta_{1,2} = T/8 \pm \sqrt{(T/8)^2 - T/4 + \frac{e^{T/2} - 1}{e^{T/2} + 1}}$$

Check That Figure 9 Correctly Represents the Geometry

As

$$x_{1max} > x_{2max} \quad \text{then} \quad T/4 > \frac{e^{T/2} - 1}{e^{T/2} + 1} = \tanh(T/4)$$

thus

$$\eta_1 = T/8 + \sqrt{(T/8)^2 - T/4 + \tanh(T/4)} < T/4$$

The geometry of figure 9 correctly represents this solution for η_1 as

$$\eta_1 < T/4 < 1 \quad \text{for} \quad s \geq 1.$$

Now $a_o < 1$ for $s \geq 1$ if

$$T/8 + \sqrt{(T/8)^2 - T/4 + \tanh(T/4)} > T/5$$

This is true if

$$L(T) \equiv \tanh T/4 > T/4 - T^2/100 \equiv R(T)$$

These are smooth curves. A calculation of values indicates that this inequality is valid for $0 < T < 2.12$ (approx.). The curves have a point of tangency at $T = 0$. However, the second derivatives

$$\left.\frac{d^2 L(T)}{dT^2}\right|_{T=0} = 0 \quad \text{and} \quad \frac{d^2 R(T)}{dT^2} = -1/50$$

indicate that for arbitrarily small $T > 0$ $L(T) > R(T)$.

The slopes of these functions are respectively

$$\frac{dL}{dT} = \frac{1}{4} \operatorname{sech}^2 T/4 \qquad \text{and} \qquad \frac{dR}{dT} = 1/4 - T/50$$

Since the shapes of these functions are well known it is easy to see that

$$(1/4) \operatorname{sech}^2 T/4 > 1/4 - T/50$$

on the range $0 < T < 1.39$ (approx.).

Thus the curves diverge properly at $T = 0$, and remain apart because of the slope condition over the range T assumes for $s \geqslant 1$. Thus

$$0 < a_o < 1 \quad \text{for} \quad s \geqslant 1.$$