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TOPICS IN THE FOUNDATIONS OF GEOMETRY AND LOGIC

BY

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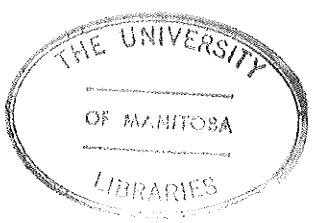
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Contents

Preface

Chapter 1: Preliminary Results and Definitions

Chapter 2: The Theory of Affine Constructible Sets

Chapter 3: The Configurations of Linear Projective Geometry

Chapter 4: The Model Theory of Universal Classes

Footnotes

Bibliography

Preface

This thesis consists of four chapters. In the first chapter we introduce some necessary preliminaries. The second chapter is concerned with the theory of constructible sets in algebraic geometry. The third chapter deals with problems in projective geometry and the fourth with problems in logic. In the fourth chapter we generalize some results of algebraic geometry to model theory.

The second chapter of the dissertation develops a general theory of constructible sets over fields of any characteristic. To this end we introduce the systematic use of logic and the techniques of model theory to the foundations of algebraic geometry. In the course of the study most of the known results of constructible sets are reproven and new technical theorems are obtained. We describe the main technical results of each subsection.

(i) We first introduce the Stone topology [2.1] in Ω_n over F and obtain a characterization of this topology [2.1]. We prove next [2.2] an intersection theorem for constructible sets. This result is a definability property of fields which we call the trace property. In fact fields have the equational trace property. For later reference we abbreviate these properties by TP and ETP. In the final portion of the first part of this section we prove the compactness of the Stone topology [2.3] and generalize this to projective space [2.4].

The compactness theorem provides us with a tool for proving the existence of solutions of systems of equations and inequations. An

immediate application [2.5] is a proof that an almost homogeneous subset of an algebraically closed field is almost homogeneous in a universal domain. Following this result we provide a geometric proof of the Representation Theorem [2.6] for almost homogeneous constructible sets. This latter result allows us to extend the trace property to projective space [2.7] and to prove two results [2.8][2.4] concerning rational maps in projective space. The first section is concluded with the fundamental result: There is a smallest field of definition for constructible sets [2.10].

(ii) The main results of this section are first [2.12] that an infinite compact field extension has infinite transcendence degree over the base field and second [2.13] that an infinite compact homogeneous extension is a universal domain. There are some other results and examples.

(iii) In this section we obtain new proofs for the known results concerning images of varieties under regular maps and prove some additional results.

(iv) The main theorem [2.21] of this section asserts that general finite correspondences preserve the extensional dimension [2.12] of constructible sets.

(v) This section of the chapter applies amalgamation and other methods to the study of product and splitting properties.

(vi) This subsection contains some further technical results. The first main result [3.1] is a definability theorem whose proof uses the trace property. The second main result [3.10] asserts that the set of homogeneous types constitute a closed subspace. This allows us to define the projective Stone space as the closed and compact subspace of all homogeneous types.

(vii) The final subsection of the second chapter of this dissertation linear structure defined on the Stone space of an algebraically closed field. The existence of this linear algebraic structure [3.1] depends on the existence of linearly disjoint coordinates of types Γ_1 and Γ_2 in a class of coordinates chosen in a universal domain. In [2.31] and [2.32] we study the basic dimensionality and continuity properties of the linear operations and prove that regular maps lift to open continuous maps of the Stone spaces [2.35]. Next we examine linear subspaces and linear transformations. The section ends with an analysis of some detailed questions concerning the monoid structure.

The third and fourth chapters each consist of several separate sections. We describe the main results of the third chapter section by section as follows.

(i) The first section begins with an axiomatic proof [3.7] of the Perspective Tetrad Theorem. We next give a new proof [3.3] of the equivalence of the Desargue Theorem and the Dual Desargue Theorem. The final two results [3.4] and (3.5) are applications of the Desargue Theorem.

(ii) The second section begins with the introduction of a triangle relation [3.1] that is then applied to obtain a new self dual equivalent of the Pappus Theorem [3.6]. This equivalent form of the Pappus Theorem yields a new and simple proof [3.7] of the equivalence of the Pappus Theorem and the Dual Pappus Theorem. The last portion of the section is devoted to the study of a configurational plane polarity related to the Pappus figure.

(iii) The third section of the second chapter is concerned with invariant line and pencil constructions. The three trace constructions are first described and shown to be equivalent to the dual constructions. We next prove that the assumption of the invariance of the three trace constructions implies the Desargue Theorem.

(iv) The fourth section is devoted to the proof of the invariance of four new line and pencil constructions related to the trace constructions.

The main results of the fourth chapter are described section by section as follows.

(i) We first prove three diagram theorems for universal classes. The results are followed by the presentation of analogues for universal classes with the amalgamation property of the Prime Model Theorem [4.4] and the Categoricity Theorem [4.5]. These results are concerned with existentially closed structures. The section concludes with three Lowenheim-Skolem theorems.

(ii) This section is initiated by a study of the relationship of sub-model completeness and the amalgamation property. We go on to prove a quantifier reduction theorem [4.13] and give alternative proofs for this result in two special cases. The next result [4.15] is a generalization to strongly minimal sets of the Chevalley Theorem of algebraic geometry. For a discussion of this theorem see for example [30]. The section concludes with a discussion of the model companion problem for universal classes with the amalgamation property.

(iii) The final section of the chapter is devoted to the generalization to strongly minimal sets of the Principle of Counting Constants [4.20]. This correspondence result of algebraic geometry can be found for example in [42].

CHAPTER 1: Preliminary Results and Definitions

1.1 Results and Definitions from Algebraic Geometry

In this section we review some fundamental definitions and results of algebraic geometry. We first present some standard definitions. Let F be a field. A universal domain Ω extending F is an algebraically closed extension of F having infinite transcendence degree over F. A variety defined over F in Ω^n is the solution set of a set of polynomial equations in n variables defined over F. If V is a variety in Ω^n then F is a field of definition for V if there is a set of equations defined over F having V as solution set. A field F is the smallest field of definition for a variety if it is contained in all fields of definition. A variety V defined over F is irreducible over F if it cannot be decomposed into the union of two distinct varieties defined over F. A variety is absolutely irreducible if it is irreducible over all fields of definition. A complete discussion of absolutely irreducible varieties can be found in [1].

Suppose that F is a field and (x) is a point of Ω^n. A point (y) is a specialization of (x) if the map σ (defined in the natural way) from F[x] to F[y] sending (x) to (y) is a ring homomorphism. The set of all specializations of a point (x) over F is the locus of (x) over F. The locus of a point over a field is an irreducible variety defined over the field. On the other hand given an irreducible variety V defined over the field F there is always a point (x) such that V is the locus over F of (x). Such a point is a generic point of V. The locus of (x) over the field F is the solution set of the ideal of polynominal equations

satisfied by (x) over F. The dimension of an irreducible variety V defined over F is the transcendence degree of a generic point over F.

Let F be a field. A Zariski open subset of the affine space F^n is the set theoretic complement of a variety. The family of all Zariski open subsets of the affine space F^n constitute the Zariski topology on F^n. See [19] for a detailed discussion of this topology. Let $p_1,\ldots,p_m$ be polynomials in n variables defined over F. If (x) is a point of F^n then the image of (x) under the mapping $\mathfrak{F}$ defined by the above polynomials in F^m is the point $(p_1(x),\ldots,p_m(x))$. Such a function of affine space is called a regular map. A hypersurface in F^m is a variety defined by a single polynomial. A constructible subset of F^m is a Boolean combination of varieties. A homogeneous equation over F is a polynomial equation $p = o$ in which all terms of the polynomial p have the same degree. A homogenous variety in F^n is one defined by homogeneous polynomials. See [1] for the basic properties of homogeneous varieties. A linear variety in F^n is a variety defined by linear equations. A regular map is linear if the polynomials defining it are linear. If (x) is a point of F^{n+1} then the ray through (x) is the set of points of the form (λx) where $\lambda \in F$. The partial ray through (x) is the ray through (x) minus the origin of F^{n+1}. The set of all partial rays in F^{n+1} constitutues the projective space $\mathbb{P}^n$ over F. A homogeneous variety V in F^{n+1} contains the rays through each of its points. The points of $\mathbb{P}^n$ corresponding to the rays in V constitute a projective variety in $\mathbb{P}^n$. A rational map $\mathfrak{F}$ of the projective space $\mathbb{P}^n$ is defined by polynomials $p_1,\ldots,p_m$ in n+1 variables of the same degree d as follows. Given a point P of $\mathbb{P}^n$ let (x) be a repre-

sentative point in F^{n+1}. Then $\mathfrak{G}(P)$ is the point with representative $(p_1(x),\ldots,p_m(x))$. See [1] and [42] for a detailed discussion of projective varieties and rational maps. We define finally the following operation on varieties. Let V be a variety in the affine space F^n. A projection of V is defined as follows. If $(X_1,\ldots,X_i,X_{i+1},\ldots,X_n)$ is a representative point then $(X_{i+1},\ldots,X_n)$ is the projection. The set of all such points is the projection of V.

The following basic theorems are required in the development of this thesis. These results are proven in [1].

Theorem 1.1 (Basis Theorem)

Suppose that F is a field. Let I be an ideal in the polynomial ring $F[X_i]$. Then I has a finite basis.

Theorem 1.2 (Nullstellensatz)

Suppose that F is algebraically closed. If a set of equations defined over F has a solution in an extension of F then it has a solution in F.

Theorem 1.3 (Smallest Field of Definition)

If V is a variety in Ω^n there is a smallest field of definition for V in Ω.

We note finally a convention of the second chapter of this thesis. That is that the term equation means always a polynomial equation defined over a field or integral domain.

1.3 Results and Definitions from Projective Geometry

We will assume familiarity with the definition of n-dimensional projective space and the axioms of incidence for this space. References for this background are [3] and [41]. If L_1 and L_2 are two linear subspace we denote the intersection of L_1 and L_2 by $L_1 \cdot L_2$. The subspace generated by these subspaces is denoted by $L_1 L_2$. For the most part we will be concerned with the projective plane although three dimensional geometry is involved with this study. The following two theorems and their duals are fundamental.

Theorem 1.4 (Desargue Theorem)

Suppose that ABC and $A^1B^1C^1$ are perspective triangles. Then corresponding sides meet in line.

Theorem 1.5 (Pappus Theorem)

Suppose that L_1 and L_2 are lines and A,B,C and A^1,B^1,C^1 are triads of points on L_1 and L_2 respectively. The points $AB^1 \cdot A^1B$, $AC^1 \cdot A^1C$ and $BC^1 \cdot B^1C$ are in line.

2.3 Results and Definitions from Logic

In this section we present some results and definitions from logic and illustrate these with some examples. General references to mathematical logic are [6] and [38]. We first consider the logic of field theory and next general model theory. The polynomial language of fields is

first extended to the open language of fields by the adjunction of the symbols $\wedge$, $\vee$ and $\neg$. These symbols are interpreted respectively as conjunction, disjunction and negation. Formulas built up from polynomial equations by the application of the connectives are called open formulas. The first order language of fields is obtained from the open language of fields by the adjunction of the universal quantifier $\forall$. Formulas built up from open formulas by quantification and further application of the connectives are called first order formulas.

Let F be a field. The set of open formulas obtained from polynomials is n variables with coefficients in F is denoted by $L_A^n(F)$. Suppose that F_1 is an extension field of F. A sequence (x) from F_1^n is a solution of a formula $\psi \in L_A^n(F)$ if $\psi(x)$ is valid over F_1. A subset Γ of $L_A^n(F)$ is an open type if Γ is the set of all formulas from $L_A^n(F)$ having a sequence (x) in an extension F_1 as a solution. If Γ is an open type and (y) is a solution of each formula from Γ then (y) is said to realize Γ. Two sequences (x) and (y) have the same open type over F iff F(x) and F(y) and F-isomorphic. The sequences (x) and (y) are then said to be conjugate over F. The set of all open types in n variables defined over F is denoted by $S_A^n(F)$. This space is topologized by taking the family of sets $\Gamma_\psi = \{\Gamma \in S_A^n(F): \psi \in \Gamma\}$ as a base for a topology. $S_A^n(F)$ with this topology is the Stone Space with the open (or Stone) topology.

Suppose that F is a field and $\mathfrak{M}$ an algebraically closed field having infinite transcendence degree over F. A subset X of $\mathfrak{M}^n$ is constructible over F if there is an open formula defined over F such that X is the solution set of ψ in the space $\mathfrak{M}^n$. If $\psi \in L_A^n(F)$

then ψ has dimension m if m is the maximal dimension of a solution of ψ in Ω^n. A formula $\psi \in L_A^n(F)$ is minimal over F if it cannot be decomposed into two disjoint constructible sets of equal dimension over F. A formula $\psi \in L_A^n(F)$ of dimension m has degree r if there is an extension F_1 of F such that ψ decomposes into r disjoint constructible sets of dimension m defined over F_1 and no decomposition into more than r disjoint constructible sets of dimension m is possible over any extension. An independent subset of a constructible set in Ω^n defined over F is a set of algebraically free points over F having maximal dimension. If $\psi \in L_A^n(F)$ is minimal over F the variety associated with ψ is the variety having as generic point any independent point of ψ. If ψ is not minimal the associated variety is the union of the varieties that are associated with the minimal components of ψ.

We now present a list of logical results that are of key importance in the next chapter.

Theorem 1.6 (Compactness Theorem)

A set of first order sentences has a model if every finite subset has a model.

A proof of the above result can be found in [6]. The Compactness Theorem can be translated to obtain the following result.

Theorem 1.7 (Compactness of the Stone Space)

If F is a field the Stone space $S_A^n(F)$ is compact.

This result follows from the more general formulation given by Lemma

4.1 of [13]. We say that a theory admits elimination of quantifiers if every formula is equivalent in T to an open formula. For a general discussion of quantifier elimination and references one can refer to [6].

Theorem 1.8 (Quantifier Elimination Theorem)
The theory of algebraically closed fields admits elimination of quantifiers.

A theory is model complete if whenever R and L are models of T and R is a substructure of L then given a formula if (a_i) is valid in R it is valid in L.

Theorem 1.9 (Model Completeness Theorem)
The theory of algebraically closed fields is model complete.

This fact follows from well known properties of fields by an application of Lindstroms Theorem. A proof of the latter can be found in [6]. The following is Theorem 4.55 of [13].

Theorem 1.10 (Splitting Theorem)
Suppose that F is a field and $L_A^n(F)$ has dimension m. There is an unique minimal finite normal extension F_1 of F such that decomposes over F_1 into disjoint components of degree 1 and dimension m.

We now present some examples illustrating the notions introduced in the first part of this section.

Example 1.1

Consider the field Q of rational numbers. The formula $X^6 + 1 = 0 \wedge X^2 + 1 = 0$ is open over Q. The formula $\neg \exists x\,(X^2+1 = 0)$ is a first order formula valid in Q but not in the complex numbers. The open formula $X^2+1 = 0$ is satisfied by i in the complex numbers.

Example 1.2

Let the groundfield be the rational numbers and consider the set $\Gamma = \{\neg p = 0 : p \text{ is a polynomial in one variable over } Q\}$. Then Γ is the type of a transcendental element over Q.

Example 1.3

Consider the formula $x = x$ over the rational numbers. It has dimension equal to 1 and also degree 1. If we extend Q to a field F_1 of transcendence degree m then $x = x$ has independent subsets of cardinality m. In particular a single transcendental element is independent.

Example 1.4

A polynomial p of degree d over the rational numbers if irreducible yields a formula $p = 0$ which is minimal over Q and of degree d.

Example 1.5

The Stone space in one variable over the rational numbers consists of all types defined by irreducible algebraic polynomials and the type of a transcendental element.

Example 1.6

Consider the formula $x = x \wedge \neg x = 0$. Over the rational numbers this formula defines a constructible set. The variety associated with this constructible set is Q.

We next consider results and notions from abstract model theory. Detailed references for this subject are [6] and [13]. A universal class is the class of all models of a universal theory. If T is a first order theory then Σ_T is the class of all submodels of models of T. For any theory T the class Σ_T is a universal class. If L is a first order language then L_A denotes the open or quantifier free part of L. If B is an L structure then $L_A^n(B)$ is the open part of $L^n(B)$. An open type is a maximal consistent set of open formulas in some fixed number of variables. If A is a structure in a universal class Σ the Stone Space of A in n-variables is the set of all open types in $L_A^n(A)$. For a detailed discussion of Stone spaces and the Stone topology we refer to [13]. Suppose that A is a structure in a universal class. The finite open diagram $D_A^F(A)$ of A is the set of all open types realized in A. A finite type diagram D is the finite open diagram of some structure. Given such a diagram D we denote by Σ_D the class of all structures in the given universal class whose finite open diagrams are contained in D. Given a diagram or structure we denote the cardinality of that diagram or structure X by $|X|$.

Suppose that Σ is a universal class. A structure A in Σ is weakly L_A - saturated if for substructures B of A on less than $|A|$ generators all open types over B are realized in A. A structure A in Σ is existentially closed if for any open formula over A if that formula has a solu-

tion in an extension of A it has a solution already in A. A structure A in is a prime existentially closed structure if it embeds in all infinite existentially closed structures in .

Suppose that A is a structure in a universal class and is an open A formula that has an infinite set of solutions in some extension of A. Then is strongly minimal if there is no extension B of A and an open B formula θ such θ and θ are both infinite. For the notions of rank and independent points of open formulas we refer to [13]. A set X over the structure A is definable by an open A formula if for some open formula over A the set X is exactly the solution set of . Given an open formula (v,w) having the blocks of variables v and w the projection of on the first or second coordinate is the corresponding set theoretic projection of the solution set of in A. An open type is a generalized principal type if there is an atom (v,w) such that is the type of the second block of variables w.

If is a universal class then has a model companion if there is a model complete theory such that is the class of a substructures of models of T. We refer to [20] for the theory of model companions.

CHAPTER 2: The Theory of Affine Constructible Sets

2.0 Introduction

Algebraic geometry begins with the study of affine varieties which are the fundamental building blocks from which other geometric objects are constructed. This secondary construction procedure extends algebraic geometry in several directions; the construction of abstract varieties, projective space and the varieties embedded in it and algebraic spaces are typical examples. Each such extension is motivated by geometric considerations and either provides a solution to some problem or provides a means of modeling algebraically geometric phenomenon for which affine space alone is not adequate.

All of the examples cited above are global geometric objects constructed from varieties in affine space by patching together the latter in some smooth fashion. Thus local problems in global geometry are often reduced to the case of affine varieties. The question naturally arises: Are the affine varieties adequate for the solution of all affine geometric problems or must the class of local objects be extended? Now a well-known theorem of projective algebraic geometry states that the image under a regular map of a projective variety is a projective variety. This result is no longer true for affine varieties. It was to resolve this latter problem that Chevalley introduced the class of constructible sets which are formed from the varieties by finite applications of the Boolean operations: The image of an affine variety under a regular map is a constructible set.

Classical algebraic geometry is not just the study of varieties as

point sets embedded in some space; inherent to the subject is the consideration of these point sets as solution sets of polynomial equations. The purpose of this linkage with polynomials is threefold. First it provides a linkage with the algebraic mechanisms associated with polynomial rings and thus allows translation of geometric problems into algebra. In this respect we consider polynomials in their algebraic aspect. But polynomials also provide a primitive logic which describes structural features. The relation between the syntactical form of polynomials and the geometry of the solution set is one of the oldest topics of algebraic geometry. A good example is provided by the 27 lines on the cubic surface, a proof of which can be found in [31]. Closely related to this second use of the basic linkage of language and point sets is the third use of polynomials to define certain geometric objects. In fact definability proofs of one sort or another abound in algebraic geometry: A good example is provided by the proof of completeness of projective space in [37]. Another typical example is the proof of Theorem 2.2 in this chapter. In these cases a definability argument is used as an intermediate mechanism. But algebraic geometers are also concerned with representing geometric objects in terms of polynomials. Thus for example moduli spaces can be represented as quasiprojective varieties. See [32] for a discussion of the latter problem. A more classical example is provided by elimination theory: The sets of coefficients for which n forms in n variables have a nontrivial zero is definable by a single polynomial. Another example of a similar sort is given in the Proposition on page 47 of [31]. This latter result tells us that if polynomials are represented as points in a projective space then the reducible polynomials

constitute a projective variety.

From the point of view of logic one cannot expect the polynomial language to provide a strong language for the solution of definability problems. In fact it is necessary to extend the class of point sets to the class of constructible sets in order to define the images of varieties under regular maps. If we expand the corresponding logic by considering all Boolean combinations of polynomial equations we obtain the class of all open formulas. If the ground field is algebraically closed the Elimination of Quantifiers Theorem of Tarski tells us that this extension is sufficient to define all first order sets.

In order to make systematic and full use of constructible sets one has to introduce some of the mechanisms of logic. On the primary algebraic level the logical machinery is the open language and the Stone space of the ground field, the latter consisting of all open types over the field. These are linked to the geometric structure in the following way: open formulas define constructible sets and a type defines an isomorphism type of a generic point of some irreducible variety over the ground field in the affine space of a universal domain extending the ground field.

Important to this theory is the use of quantifiers to extend the algebraic open language to full first order logic. There are three fundamental tools here: (1) Compactness of the first order language (2) Elimination of Quantifiers Theorem (3) Model Completeness of the theory of algebraically closed fields. The first property of the language has the geometric interpretation that the Stone topology on the affine space $\mathfrak{Q}^n$ over the ground field is compact ($\mathfrak{Q}$ a universal domain extending

F). The second basic result allows reduction of first order formulas to the algebraic level and thus yields strong definability properties for the class of constructible sets. The third property is a transfer principle which we use for example in the proof of Theorem 2.26.

The general conclusions we reach on the basis of our analysis of constructible sets and the foundations of algebraic geometry as developed in this chapter and in [13] is that the use of these sets and the associated mechanisms of language provide a simplifying and unifying set of devices in algebraic geometry. To explain this conclusion we list the classical results we have reproven utilizing these methods.

(1) The Nullstellensatz

(2) Existence of the unique normal finite splitting extension of a variety into degree 1 subvarieties.

(3) The product of two degree 1 varieties has degree 1.

(4) An irreducible variety over an algebraically closed field is absolutely irreducible.

(5) Chevalley's Theorem.

The proofs of (1) and (5) use the open language and Quantifier Elimination. The proofs of (2) and (3) use the open language and algebraic methods (the amalgamation property is useful in many such arguments and is central to the proofs we obtain of (2) and (3)). The proof of (4) utilizes Model Completeness.

The new proofs we obtain for the above fundamental results cast light on the basic linkages of geometry and algebra. For consider the following hierarchy of algebraic results:

(A) The Basis Theorem

(B) The Nullstellensatz

(C) The Extension Theorem for Places.

We demonstrate that the proofs of (1), (2), (3), (5) depend only on the Basis Theorem and elementary field theory. The proof of (4) is a reduction to the Nullstellensatz. Thus we eliminate the use of valuations or of the Extension Theorem from the proof of the elementary theorems.

The first point where we require the Extension Theorem is to prove that components of a variety which splits over an extension of the ground field all have the same dimension. But the latter is equivalent to a fundamental property of specializations: If (z) is a point on the variety V and F_1 is an extension of the ground field F then there is a generic specialization (y) of the generic point (x) of V over F such that dim(y) = dim V and (y)--->(z) is a specialization over F_1. Thus the use of a key result in the theory of places of fields is natural at this point in the development of the foundations of algebraic geometry.

1. Elements of the Theory of Constructible Sets

1.0

This section begins with the proof of a set of fundamental results concerning constructible sets some of which are applied later. We define the Stone topology on an affine space over an integral domain and give a simple description of the open subsets of a variety. We next obtain a characterization of the Stone topology in terms of certain key properties of regular maps and algebraic points. An important definability property

of constructible sets and proofs of the compactness of the Stone topology in affine and projective space complete the first subsection.

The compactness of the Stone topology is a strategic result and in consequence there are a number of illustrations given of its use in this chapter. One expects such applications since this geometrical property of affine space is the direct translation of the Compactness Theorem of logic (Theorem 1.3.22 of [5]). Our first proof of this property uses geometric links between affine space and the Stone space to translate compactness from the Stone space to affine space. Hence it is a translation from logic to geometry. The second proof is direct and uses the connection between open types and prime ideals. In this latter proof the maximality of open types is the key property. Our next result is an application of compactness that also uses the maximality of types.

Another strategic result in the theory of constructible sets is the representation theorem for almost homogeneous constructible sets first proven by W. Whiteley. We obtain a number of fundamental properties of almost homogeneous constructible sets and the associated varieties and then apply these properties to give a geometric proof of the representation theorem. To illustrate the strategic character of this result we prove three projective results as direct applications.

The proof of the representation theorem for almost homogeneous constructible sets uses a tree argument. As a further application of this method we prove next that constructible sets have a smallest field of

definition. The proof reduces the problem to that of varieties by means of the associated variety of a constructible set. We prove a number of preliminary results concerning fields of definition of associated varieties before presenting the main result. The section concludes with a definability result whose proof is an application of compactness.

1.1 Definition 2.1

Let R be an integral domain and F some integral domain extending R. By the Stone topology on F^n over R we mean the topology whose base is the family of all open n-formulas over R. We also alternatively call this topology the open R-topology.

Lemma 2.1

Suppose that F is a field and V is a variety defined over F in the space F_1^n. Then the basic open sets on V in the relative Stone topology are the subvarieties of V and the Zariski open subsets of these subvarieties.

Proof

By definition the subvarieties of V are basic open sets in the open topology. Suppose that ψ is an open formula over F and $\psi \cap V \neq \emptyset$. Say $\psi = \psi_1 \vee, \ldots, \vee \psi_m$ in disjunctive normal form where the ψ_i are all the disjuncts. Then each $\psi_i = (f_1^i = 0 \wedge, \ldots, \wedge f_j^i = 0 \wedge \neg f1_{j+1} = 0 \wedge, \ldots, \wedge \neg f_k^i = 0)$ for appropriate polynomials. There is a subset $\psi r_1, \ldots, \psi r_s$ of the disjuncts with nonempty inter-

section with V. Given Ψ_i in this set the conjunction of polynomial equations in the above formula defines a variety V_1 so $V_1 \cap V = V_2$ is a subvariety of V. Now the conjunction of inequations in Ψ_i is equivalent to the expression $\neg$ ($f^1_{k+1} = 0$ V,...,V $f_k{}^i = 0$). The expression within the brackets defines a union of hypersurfaces so is a variety V_3 and V_3 is a basic Zariski open set. By assumption $\sim V_3 \cap V_2 \neq \emptyset$ and this latter set is a basic Zariski open subset of the subvariety V_2. ⊠

Theorem 2.1 (1)

Suppose that F is a field and Ω is a universal domain extending F. Let $\{\Gamma_i\}_{i<\omega}$ be a family of topologies on the affine spaces $\{\Omega^i\}_{i<\omega}$. Suppose this family has the following properties: (a) Regular maps are open and continuous (b) Algebraic points are isolated. Then each of the topologies Γ_i is a refinement of the open F-topology on Ω^i.

Proof

Let V be a variety in Ω^n. Say $f_1,\ldots,f_m$ is a basis for the ideal of V. The origin $(0,\ldots,0)$ in Ω^m is an algebraic point hence by (b) it is open in Γ_m. By condition (a) the regular map $\Pi = (f,\ldots,f_m)$ is continuous so the set $\Pi^{-1}(0)$ is open in the topology Γ_n. Thus all varieties are open in this topology for $V = \Pi^{-1}(0)$. Now let X be a constructible set of the form $V_1 - H_g$ for a variety V_1 and a hypersurface H_g (defined by the polynomial g). Let $F_1,\ldots,F_r$ be a basis for the ideal of V_1. Consider the variety V_0 defined by the

following equations: $F_1 = 0,\ldots,F_r = 0$, $gx_{n+1} = 1$ where x_{n+1} is a new variable. Then V_0 is open in $\mathfrak{A}^{n+1}$ in the topology Γ_{n+1}. Furthermore the projection of V_0 onto the first n-coordinates is the constructible set X. By condition (a) the projection is an open map hence X is open in the topology Γ_n. Suppose the Y is an arbitrary constructible set. Then by Lemma 1 Y is the union of sets of the form $Y_0 = V_1 - V_2$ for varieties V_1 and V_2. Let $g_1,\ldots,g_s$ be a basis of the ideal of V_2. Then $V_2 = \bigcap H_{g_i}$ so $Y_0 = V_1 - \bigcap H_{g_i}$. But then $Y_0 = \bigcup V_1 - H_{g_i}$. Hence by what was shown above Y_0 is a union of a finite number of open sets hence is an open set. So Y is an open set. Therefore all constructible sets are open sets and it follows that Γ_n is a refinement of the open topology. ☒

The above result characterizes the family of open topologies on the affine spaces $\{\mathfrak{A}^i\}_{i<\omega}$ as the smallest topologies which satisfy conditions (a) and (b) in the statement of the theorem.

Corollary 2.1 (2)

Every constructible set is the image of a variety under a regular map.

Proof

As in the proof of Theorem 2.1 a constructible set Y is the union of sets of the form $V - H_g$ for a variety V and hypersurface H_g. Again as above each such set is the projection of a variety in $\mathfrak{A}^{n+1}$ if $Y \subseteq \mathfrak{A}^n$ so Y is the projection of the union of these varieties. The latter is a variety so Y is the projection of a variety. ☒

Theorem 2.2

Suppose that Ψ is a constructible set defined over F_1 and F is a subfield of F_1. Then $F^n \cap \Psi(F_1)$ is a constructible set defined over F.

If F_1 is an extension of a field F and Ψ is an open F_1-formula then the trace of Ψ on F^n is the set of all solutions of Ψ with coordinates in F. In order to prove the above theorem we need the following lemma.

Lemma 2.2

Suppose that F is a subfield of a field F_1 and H_g is a hypersurface defined by a polynomial g over F_1. Then the trace of H_g on F^n is definable by a set of equations with parameters from F.

Proof

Suppose $g = \sum c_\alpha M$ where $c_\alpha \in F_1$ and M_α is a basic monomial. We can consider F_1 as a vector space over F say with a basis X. Then each $c_\alpha = \sum a_{i_\alpha} x_i$ where the $x_1,\ldots,x_r$ which appear in the right hand expression are a subset of the basis set X. We thus obtain by substitution an algebraic expression $\sum_\alpha \sum_i a_{i_\alpha} x_i M_\alpha = 0$. We can rewrite this expression in terms of the x_i to obtain an expression of the form $g_1(b_i)x_1+\ldots+g_r(b_i)x_r = 0$ where each g_i is a polynomial defined over F in n variables. Explicitly $g_i = \sum_\alpha a_{i_\alpha} M_\alpha$. Now if the sequence (b_i) is a solution of the defining equation for H_g and (b_i) is in F then our derived expression yeilds $g_1(b_i)x_1+\ldots, +g_r(b_i)x_r = 0$ so since $g_j(b_i)$ is an element of F and $[x_1,\ldots, ,x_r]$ is linearly independent over F we must have $g_j(b_i) = 0$ for

each j. But on the other hand if (b_i) is a point of F^n and $g_j(b_i) = 0$ for each i then also (b_i) is a point of H_g. Hence the polynomials $g_1,\ldots,g_r$ define the trace of H_g on F^n. ⊠

Suppose now that $f_1,\ldots,f_k$ is a basis for the ideal of the algebraic set V defined over F. Then each polynomial f_r defines a hypersurface. By the lemma for each r there is an algebraic set V_r defined over F whose solutions in F^n are the trace of the hypersurface of f_r. Let S be the intersection of the family V_r. Then if (b_i) is a point of S in F^n, (b_i) is a point of each V_r in F^n so (b_i) is a point of H_{f_r} and so (b_i) is in the trace of V on F^n. The other inclusion is also immediate so $S \cap F^n$ is the trace of V on F^n.

The above argument proves Theorem 2.2 in the case of the algebraic sets. We can continue the argument to extend to arbitrary open formulas as follows. The proof proceeds by induction on the level of formulas. We have shown that it holds for varieties. So suppose the trace of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ $L_A(F_1)$ on F^n is definable by open formulas θ_1 and θ_2 respectively, where θ_1, θ_2 $L_A(F)$. Then clearly the trace of $\mathcal{Y}_1 \wedge \mathcal{Y}_2$ and $\mathcal{Y}_1 \vee \mathcal{Y}_2$ is defined by $\theta_1 \wedge \theta_2$ and $\theta_1 \vee \theta_2$ respectively. Also the trace $\neg\mathcal{Y}_1$ on F^n is defined by $\neg\theta_1$. Hence the trace of F^n of an arbitrary open formula is the solution set in F of an open F-formula. This completes the proof of Theorem 2.2.

Consider an open type $\Gamma \subseteq S_A^n(F)$ over a field F. Let $\mathfrak{U}$ be a universal domain extending F. A solution (x) of Γ in $\mathfrak{U}$ is a generic point of a variety V in $\mathfrak{U}^n$ over F. Let I be the prime ideal of $F[X_i]$ whose variety is V. The ideal I is a point of the spectral space over the field F. The reader can consult [13] or [16] for the definition and

and properties of the spectral space. On the other hand given a point I of the spectral space let V be the variety defined by I. A generic point (x) of V in $\mathfrak{U}$ defines a type Γ in the Stone space $S_A^n(F)$. Thus there is a one-one correspondence between points of the Stone space and spectral space over F. Consequently the Stone space with the open topology is the spectral space of modern algebraic geometry but endowed with a refinement of the Zariski topology. One advantage of the representation of points of the spectral space as open types is the fact that types are maximal while prime ideals are so only if they define algebraic points. The points of the spectral space have another representation which we will utilize later. Call two points of $\mathfrak{U}^n$ conjugate (over F) if there is an F-automorphism of $\mathfrak{U}$ which maps one point onto the other. Clearly the relation so defined is an equivalence relation on $\mathfrak{U}^n$ and the equivalence classes are in one-one correspondence with types. The equivalence class of points associated with a type provides a set of coordinates for points of the spectral space.

Lemma 2.3

Suppose the F is a field and $\mathfrak{U}$ is a universal domain extending F. Let R be the relation on $\mathfrak{U}^n$ defined as follows: $x_i \equiv_R y_i$ iff x_i and y_i are conjugate over F. Let $\mathfrak{U}^n/_R$ have the quotient topology relative to the open topology on $\mathfrak{U}^n$. Then the quotient map $\delta : \mathfrak{U}^n \dashrightarrow \mathfrak{U}^n/_R$ is an open map.

Proof

Let U be an open set in $\mathfrak{U}^n$. We want to show that R(U) is also

open. Let $\bar{x}_i \in R(U)$. Then by definition there is $\bar{y}_i$ in U such that $\bar{x}_i$ and $\bar{y}_i$ are conjugate over F. Let U_1 be a basic open set such that $\bar{y}_i \in U_1 \subseteq U$. By definition U_1 is defined by some open formula $\psi(v_i)$ in $L_A(F)$. Now the fact that $\bar{x}_i$ is conjugate to $\bar{y}_i$ over F implies that $\mathfrak{U} \models \psi(x_i)$. Hence $\bar{x}_i \in U_1$. Hence R(U) contains a neighborhood of $\hat{x}_i$ so R(U) is open. ⊠

Lemma 2.4

The space $\mathfrak{U}^n/_R$ is homeomorphic to the space $S_A^n(F)$.

Proof

We define the map $f: \mathfrak{U}^n/_R \dashrightarrow S_A^n(F)$ as follows: Let Π be a conjugacy class in $\mathfrak{U}^n$ over F. Then if $\bar{x}_i \in \Pi$ let $\Gamma(\bar{v}_i)$ be the $L_A(F)$-type of $\bar{x}_i$. By definition each $\bar{y}_i$ in Π has type Γ. Hence defining $f(\Pi) = \Gamma$ our map is indeed well defined. Clearly f is one-one. Since $\mathfrak{U}$ is a universal domain extending F each type Γ is realized in $\mathfrak{U}$ so that the map f is also onto. Let U be a basic open set in $S_A^n(F)$. Then $U = U_\psi$ for some open formula $\psi(\bar{v}_i) \in L_A(F)$. The inverse of U under f is the set $\{\Pi : \exists\, \bar{x}_i \in \Pi \ni \models \psi(\bar{x}_i)\}$. Then $b^{-1}f^{-1}(U)$ is the set of all solutions in $\mathfrak{U}$ of $\psi(v_i)$. Hence this set is open in $\mathfrak{U}^n$. By lemma 2.3 we have that $f^{-1}(U)$ is open in $\mathfrak{U}^n/_R$. It follows that f is continuous. Now let U be an open set in $\mathfrak{U}^n/_R$. Let Γ be some point in f(U). Let $\bar{x}_i$ be a representative of $f^{-1}(\Gamma)$. Since U is open in $\mathfrak{U}^n/_R$ we must have $b^{-1}(U)$ open in $\mathfrak{U}^n$ so let $\psi(v_i)$ define a basic open neighborhood of $\bar{x}_i$ in $\mathfrak{U}^n$ contained in $b^{-1}(U)$. Such a set exists since the open formulas are a base for the topology. Then if

U_1 is the open set defined by ψ the set $f(U_1) \subseteq U$ and by lemma 2.3 the set $f(U_1)$ is open. Furthermore f $f(U_1)$ contains Γ and equals U . To see that suppose that $\Gamma_1 \in U_\psi$. Then there is $\bar{z}_i$ in Ω^n with $\Omega \models \psi(\bar{z}_i)$ and $\bar{z}_i$ has type Γ_1. So $\Gamma_1 \in f f(U_1)$. The other direction is clear. Since f(U) contains a neighborhood of each of its points it is an open set. Hence f is a homeomorphism. ⊠

Theorem 2.3

Suppose that F is a field and Ω is a universal domain extending F. Then Ω^n is compact in the open F-topology.

Proof

Let $\{U_\alpha\}_{\alpha<\lambda}$ be an open cover of Ω^n. Then by Lemma 2.3 the family of sets $\{f U_\alpha\}_{\alpha<\lambda}$ is an open cover of Ω^n/R. By Lemma 2.4 Ω^n/R is homeomorphic to the compact space $S_A^n(F)$ so that $\{f U_\alpha\}_{\alpha<\lambda}$ admits a finite subcover. Now in the proof of Lemma 2.3 we actually prove that R(U) = U for any open set U in Ω^n so that $f^{-1} f(U) = U_\alpha$ for each α. Hence we can pull back our finite subcover of Ω^n/R to obtain a finite subcover of Ω^n. Hence Ω^n is compact. ⊠

Note that utilizing the patch topology M. Hochster in [22] proves a generalization to rings of the compactness of the Stone topology over fields.

We give a second proof of the compactness of the Stone topology on Ω^n over F by a direct argument as follows. Let Π be a consistent set of open F-formulas in Ω^n. It follows by Zorn's Lemma that Π admits extension to a maximal consistent set Γ. As is well known (or can easily

be proven by the reader) Γ has the property that given a formula ψ either ψ or $\neg\psi$ is in Γ. Let I be the set of polynomials defined by the condition: $p \in I$ if and only if $p = 0 \in \Gamma$. Then I is a prime ideal. For suppose p_1 and p_2 to be in I. If $p_1 + p_2$ is not in I then $\neg(p_1 + p_2 = 0) \in \Gamma$ since Γ is maximal. It follows that the set $p_1 = 0$, $p_2 = 0, \neg(p_1 + p_2 = 0)$ is contained in Γ. But this set is inconsistent. Hence $p_1 + p_2 = 0$ is in and so $p_1 + p_2 \in I$. The proof that $hp \in I$ when $p \in I$ and h is an arbitrary polynomial is similar to the above. Suppose then that $p_1p_2 \in I$ but p_1 and p_2 are not in I. Then $\neg p_1 = 0$ and $\neg p_2 = 0$ are in Γ. But $p_1p_2 = 0 \in \Gamma$ so that $\{\neg p_1 = 0, \neg p_2 = 0, p_1p_2 = 0\}$ is inconsistent. This contradiction proves that I is a prime ideal. Let $F_1 = F[X]/I$. Then F_1 is a finite extension of F with $(z) = (x)/I$ having ideal I. Map F_1 in Ω over F. Claim: (z) is a solution of Γ. For the ideal I determines the open formulas of the form

$$f_1 = 0 \wedge \ldots \wedge f_m = 0 \wedge \neg f_{m+1} = 0 \wedge \ldots \wedge \neg f_k = 0$$

satisfied by (z). If (z) is not a solution of Γ let $\psi \in \Gamma$ be such that $\neg\psi(z)$. Now consider whether each of the disjuncts of ψ (in normal form) is consistent with I. Choose for each disjunct an equation or inequation inconsistent with I. By definition of I the negation of this equation or inequation is in Γ. The set of all such formulas with the formula ψ constitute an inconsistent subset of Γ. Contradiction. Hence some disjunct of ψ is of the correct form and so (z) is a solution of ψ.

We can apply the above affine compactness theorem to prove an analogous result for projective space. We first state some necessary definitions.

Definition 2.2

A subset X of the projective space $\mathbb{P}^n(\mathfrak{U})$ over the ground field F $\leq \mathfrak{U}$, where $\mathfrak{U}$ is a universal domain over F, is a projective constructible set if it is a Boolean combination of projective varieties.

If X is a projective constructible set then X can be represented as a Boolean combination of the solution sets of homogeneous equations. A formula representing this combination we call a projective open formula. We often call a set defined by such a formula projective even with solutions considered in affine space.

Definition 2.3

A constructible set γ defined over the field F with solutions in a universal domain $\mathfrak{U}$ is almost homogeneous if for each solution (x) of γ and $t \neq 0$ in $\mathfrak{U}$ the sequence (tx) is also a solution of γ.

Defintion 2.4

The Stone toplogy on the space $\mathbb{P}^n(\mathfrak{U})$ over F $\leq \mathfrak{U}$ is the topology with base the family of projective F-constructible sets.

Lemma 2.5

A projective open formula γ is almost homogeneous.

Proof

We can assume that γ is in disjunctive normal form and each polynomial in the formula is a form. Suppose that

(*) $F_1 = 0 \wedge, \ldots, \wedge F_m = 0 \wedge \neg G_1 = 0 \wedge, \ldots, \wedge \neg G_r = 0$

is one of the disjuncts of Υ say (x) is a solution of (*). As the F_i are forms we have then $F_i(tx) = 0$ for any $t \in \Omega$ so in particular for $t \neq 0$. Now if for some $t \neq 0$ in Ω we have $G_j(tx) = 0$ then $G_j(tx)(\frac{1}{t})^d = 0$ where d is the degree of G_j. As G_j is a form the latter equation implies $G_j(X) = 0$. Contradiction. ⊠

Theorem 2.4

Suppose that F is a field and Ω is a universal domain extending F. The Stone toplogy over F on $\mathbb{P}^n(\Omega)$ is compact.

We give two proofs. The first was communicated by H. Lakser. Let $\{\Psi_i : i \in I\}$ be a family of projective open formulas having the finite intersection property in $\mathbb{P}^n(\Omega)$. Then the family of formulas $\Sigma = \{\Psi i \wedge (\neg X_0 = 0 \vee, \ldots, \vee \neg X_n = 0) : i \in I\}$ has the finite intersection property in Ω^{n+1}. By the compactness of the Stone topology in Ω^{n+1} over F there is a solution (x) of Σ. Since $(x) \neq 0$ this sequence yields a solution of $\{\Psi_i : i \in I\}$ in $\mathbb{P}^n(\Omega)$.

For the second proof note that the projective formulas are closed under conjunction and disjunction and so the solution sets form a basis for a topology in Ω^{n+1}. This homogeneous topology is clearly compact. Furthermore the origin is defined by the projective formula $X_0 = 0 \wedge, \ldots, \wedge X_n = 0$ so that the space $X = \Omega^{n+1} \sim \{(0)\}$ is a closed hence compact subspace of Ω^{n+1}. The Stone toplogy on $\mathbb{P}^n(\Omega)$ is the quotient topology of the subspace topology on X and so it is compact.

1.2 We next investigate the precise relationship between the properties of an open formula being projective and almost homogeneous. Before

proving the main result we obtain a theorem which is an application of the compactness of the Stone topology.

Theorem 2.5

Suppose that F is an algebraically closed field and Ψ is an almost homogeneous constructible subset of F^n. If Ω is a universal domain extending F then Ψ is an almost homogeneous subset of Ω^n.

Proof

Suppose that $(x) \in \Psi(\Omega)^n$. Say $t \neq 0$ is in Ω. Let Γ_1 be the type of (t,x) over F. Given any finite subset s of Γ_1 there is a solution (t',x') of $\wedge s \wedge \Psi$ in F since F is existentially closed. Now as $t \neq 0$ we can stipulate that s also contains the formula $v_0 \neq 0$. So then $t' \neq 0$. As Ψ is almost homogeneous in F^n it follows that $\Psi(t'\ x')$. Hence $\wedge s$ and the set $\Psi(v_0v_1,\ldots,v_0v_n)$ have non-trivial intersection. The family of closed sets on $\Psi(v_0v_1,\ldots,v_0v_n)$ determined by the formulas $\wedge s$ thus has the finite intersection property in Ω^{n+1} over F. By compactness of the Stone topology there is a solution in Ω^{n+1} of $\Gamma_1 \cup \{\Psi(v_0v_1,\ldots,v_0v_n)\}$. But Γ_1 is maximal so that the formula $\Psi(v_0v_1,\ldots,v_0v_n) \in \Gamma_1$. This tells us that $\Omega \models (tx)$ and so Ψ is almost homogeneous. ⊠

A second proof communicated by H. Lakser avoids the use of the compactness of the Stone toplogy. Let Ψ_1 be the following sentence:

$\forall t \forall x_1,\ldots,\forall x_n((\neg t = 0 \wedge \Psi(X_1,\ldots,X_n)) \rightarrow \Psi(tx_1,\ldots,tx_n))$.

Suppose that $\Omega \models \neg\Psi_1$. The sentence $\neg\Psi_1$ is an existential sentence and since F is algebraically closed it is existentially closed. Hence

Hence $F \models \neg\psi_1$ which contradicts the fact that ψ is an almost homogeneous constructible subset of F^n.

The following result is the key to the respresentation theorem for almost homogeneous constructible sets.

Lemma 2.6

An almost homogeneous constructible set of dimension 1 and degree 1 is either a ray or a ray minus (0).

Proof

Let Ω be a universal domain over the groundfield F and choose an independent point (x) of $\psi(\Omega^n)$ over F. Then ψ contains the ray L_ψ generated by (x) except possibly for (0). If $L \subseteq \psi$ it follows by the fact that ψ has degree 1 we have $L = \psi$. So assume that $(0) \in \psi$ and let $L_1 = L \sim (0)$. Then $\psi \sim L_1$ can be at most a finite set of points. But if $(z) \in \psi \sim L_1$ then since ψ is almost homogeneous the partial ray L_2 (minus the origin) generated by (z) is contained in ψ. But then $L_1 \cap L$ is infinite and as both curves L and $L' = L_2 \cup \{(0)\}$ are irreducible we have $L = L'$. Hence $L_1 = L_2$. Contradiction. So $\psi = L_1$. ☒

Corollary 2.2

An almost homogeneous constructible set of dimension and degree 1 can be represented as a Boolean combination of homogeneous equations.

We state the next lemma for purposes of reference.

Lemma 2.7

Suppose that F is a field and (x) is a point whose coordinates are chosen in a universal domain Ω extending F. If $t \in \Omega$ is algebraically independent over F(x) then $\dim_F(x) \leq \dim_F(tx)$.

Lemma 2.8

Suppose that $\mathcal{V}$ is an almost homogeneous constructible set defined and minimal over F with points in the universal domain Ω. The associated variety V is homogeneous.

Proof

Let (x) be an independent point of $\mathcal{V}$ over F in Ω^n. Then $\dim_F(x) = \dim \mathcal{V}$ so if $t \in \Omega$ is free with respect to (x) over F it follows by lemma 2.7 that $\dim_F(tx) = \dim \mathcal{V}$. By assumption $(tx) \in \mathcal{V}$ and so $(tx) \in V$, since (tx) is an independent point of $\mathcal{V}$ and thus a generic point of V. Let G be a polynomial in the ideal of V over F. Then $G(tx) = 0$. We can rewrite this relation in the form $\sum t^i p_i(x) = 0$ and we deduce $p_i(x) = 0$. So the homogeneous parts of G are in the ideal of V and therefore this ideal is homogenous. So V is homogeneous. ☒

Lemma 2.9

Suppose that $\mathcal{V}$ is an almost homogeneous minimal constructible set and V is the associated variety. Then the sets $\mathcal{V} \sim V$ and $V \sim \mathcal{V}$ are almost homogenous.

Proof

Given $(x) \in \mathcal{P} \sim V$ and $t \neq 0$ in Ω the sequence $(tx) \in \mathcal{P}$ so must be in $V_{\mathcal{P}}$ if it is not in $\mathcal{P} \sim V$. By the previous lemma we have then $(t^{-1}tx) = (x) \in V_{\mathcal{P}}$. Contradiction. So $(tx) \in \mathcal{P} \sim V$. The second case follows by the same argument. ☒

Lemma 2.10

Suppose that $\mathcal{P}$ is an almost homogeneous constructible set. Then $\mathcal{P}$ decomposes into a finite union of almost homogeneous constructible sets all minimal and of dimension equal to $\dim \mathcal{P}$.

Proof

Let V be the algebraic set associated with $\mathcal{P}$. Say $(x^1), \ldots, (x^r)$ are realizations of all the types through $\mathcal{P}$ that have dimensions equal to $\dim \mathcal{P}$. Given a point $t \in \Omega$ free over the groundfield F with respect to say (x^i) the point $(tx^i) \in \mathcal{P}$ and by lemma 2.7 we deduce that $\dim_F(tx^i) = \dim \mathcal{P}$. So (tx^1) is a generic point of a component V_r of V over F. Let G be a polynomial in the ideal of V_r. Then as $G(tx) = 0$ we deduce a relation $\sum t^i p_i(x) = 0$ which is identical in the free variable t over F(x). So given say $t_0 \in \Omega$ we have $\sum t_0^i p_i(x) = 0$ and so $G(t_0 x) = 0$. This tells us that V_r contains the ray generated by (tx) and so in particular the point (x). Applying the argument of lemma 2.8 it follows that V_r is homogeneous. Now V has exactly r components (as follows from Theorem 4.61, page 354 of [13]) so all components of V are homogeneous. Let $\theta_1 = V_1 \cap \mathcal{P}, \ldots, \theta_{r-1} = V_{r-1} \cap \mathcal{P}$, $\theta_r = (V_r \cap \mathcal{P}) \cup (\mathcal{P} \sim V)$. Now as the

V_i are homogeneous so is V hence by the argument of lemma 2.9 the set $\Psi \sim V$ is almost homogeneous. Also it is clear that the intersection of almost homogeneous sets is almost homogeneous. As each set $V_i \cap \Psi$ has dimension equal to dimΨ and is F-minimal (again by the basic result of [31]) and $\Psi = \theta_1 V, .., V\theta_r$ we have obtained the desired decomposition. ☒

Theorem 2.6 (3)

Suppose that Ψ is an almost homogeneous constructible set. Then Ψ is equivalent to a Boolean combination of homogeneous equations.

Proof

We describe a geometric construction that will allow us to construct the desired formula. First decompose Ψ into almost homogeneous minimal components. Say these components are $\Psi_1, \ldots, \Psi_m$. With each components Ψ_i we have by lemma 2.8 that the associated variety V_i is homogenous. By lemma 2.9 the sets $\Psi_{i,1} = \Psi_i \sim V_i$ and $\Psi_{i,2} = V_i \sim \Psi_i$ are almost homogeneous. Each of the sets $\Psi_{i,1}$, $\Psi_{i,2}$ split into components which are almost homogeneous. Denote these components by $\Psi_{i,1,j}$ and $\Psi_{i,2,j}$ for some set of indices j. Now proceed with each component as above. Note that at even stages of the construction the two sets formed have dimension less than the sets of the previous stage. So the construction of this tree of formulas will terminate at a finite stage; namely each branch can have length at most 2n if n is the dimension of Ψ. Each branch then terminates in a homogeneous variety or in an almost homogeneous constructible set which is minimal and of dimension 1. A

slight adjustment in the proof of lemma 2.6 shows that such a set if not a variety is an irreducible homogeneous curve minus the origin. Again the origin is expressible as a Boolean combination of homogeneous equations so our set is expressible also in this form. Now we construct a formula by a reverse procedure beginning with the end points of the tree and representing the next stage as Boolean combinations of previous stages and homogeneous associated varieties. We finally obtain the presentation of $\mathcal{Y}$ as a Boolean combination of homogeneous equations. ⊠

Corollary 2.3

The points in $\mathbb{P}^n$ corresponding to the partial rays of an almost homogeneous constructible subset of Ω^{n+1} constitute a projective constructible subset of $\mathbb{P}^n$.

1.3 The representation theorem proved above allows us to deduce immediately a number of results on projective space. The study of constructible subsets of projective space is motivated by geometric representation problems. For example, the birational equivalence classes of curves of given genus are represented by the points of a constructible subset of a projective space. For a discussion of moduli of curves and for results in the theory of quasiprojective varieties the reader can consult [19].

Definition 2.5

Suppose that F is a field and F_1 extends F. Let $\mathcal{Y}$ be a constructible set defined over F_1 in $\mathbb{P}^n(F_1)$. Then the trace of $\mathcal{Y}$ on

$\mathbb{P}^n(F)$ is the set of all points in $\mathbb{P}^n(F)$ having a set of coordinates which are solution of the open formula ψ in F_1.

Note that this definition is justified by the fact that is $F_1 \models \psi(x)$ for $(x) \in F^{n+1}$ and if $\lambda \neq 0$ is in F then also (λx) lies on ψ. Intuitively the trace is the intersection of ψ and $\mathbb{P}^n(F)$.

Lemma 2.11

Suppose that F is a field and F_1 is an extension. If ψ is an almost homogeneous constructible set defined over F_1 then the trace of ψ on F^n is an almost homogeneous constructible set.

Proof

Immediate.

Theorem 2.7

Suppose that Ω is a universal domain and F is an extension. If ψ is a projective constructible set in $\mathbb{P}^n(F)$ then the trace of ψ on $\mathbb{P}^n(\Omega)$ is a projective constructible set.

Proof

By definition ψ is the solution set of a Boolean combination of homogeneous equations so $\psi(F^{n+1})$ is almost homogeneous. By lemma 2.11 it follows that the trace ψ_1 of ψ on Ω^{n+1} is an almost homogeneous constructible set. Hence by Theorem 6 ψ_1 can be represented as a Boolean combination of homogeneous equations. So ψ_1 defines a constructible subset of $\mathbb{P}^n(\Omega)$. By definition the set ψ_1 in $\mathbb{P}^n(\Omega)$ is the trace of ψ. ⊠

We next prove some results concerning rational projective maps. As defined in [42] a map $\mathfrak{G} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ is a rational map if $\mathfrak{G} = (F_1, \ldots, F_{m+1})$ where each F_i is a form of fixed degree d in n + 1 variables.

Lemma 2.12

Suppose that $\mathfrak{Y} \in \Omega^{n+1}$ is an almost homogeneous constructible set defined over the subfield F of the universal domain Ω. Let $\mathfrak{G} = (F_1, \ldots, F_{m+1})$ be a regular map such that each F_i is a form in n+1 variables having degree d. Then $\mathfrak{G}(\mathfrak{Y})$ is an almost homogeneous constructible set.

Proof

That $\mathfrak{G}(\mathfrak{Y})$ is a constructible set follows by Chevalley's Theorem (which we prove in a later section). Assume then that $\mathfrak{Y}_1$ defines $\mathfrak{G}(\mathfrak{Y})$. Say $(x) \in \mathfrak{Y}_1$ and $t \neq 0$ is in Ω. Let t_0 be a d-th root of t. Then $t_0 \neq 0$ so if $(y) \in \mathfrak{Y}$ is such that $(y) = (x)$ we have $(t_0 y) \in \mathfrak{Y}$. Thus $\mathfrak{G}(t_0 y) = (F_1(t_0 y), \ldots, F_{m+1}(t_0 y)) = (t_0^d F_1(y), \ldots, t_0^d F_{m+1}(y)) = (t_0^d x) = (tx)$ is in $\mathfrak{Y}_1$. So $\mathfrak{Y}_1$ is almost homogeneous. ⊠

Theorem 2.8

Suppose that Ω is a universal domain and $\mathfrak{Y}$ is a constructible subset of $\mathbb{P}^n(\Omega)$. Let $\mathfrak{G}$ be a projective rational map defined on $\mathbb{P}^n(\Omega)$ and such that $\mathfrak{G}$ is defined at some point P of $\mathfrak{Y}$. Then $\mathfrak{G}(\mathfrak{Y})$ is a projective constructible set.

Proof

The formula Ψ defines an almost homogeneous constructible subset of Ω^{n+1}. As σ induces a regular map on Ω^{n+1} it follows by lemma 2.12 that $\sigma(\Psi) = \Psi_1$ is an almost homogeneous constructible subset of Ω^{m+1} which by assumption does not collapse to (0). By Theorem 2.6 Ψ_1 can be represented by a Boolean combination of homogeneous equations and as $\Psi_1 \neq (0)$ the set Ψ_1 is a projective constructible set in $\mathbb{P}^m(\Omega)$. It is also the image under σ of Ψ and this proves the theorem. ☒

The projective form of Chevalley's Theorem is proved in [31]. An ancestor of this result appears in [42].

Theorem 2.9

Suppose that F is a field and Ω is a universal domain extending F. Let $\Pi : \mathbb{P}^m(\Omega) \dashrightarrow \mathbb{P}^n(\Omega)$ be a rational map defined over F. If Π is regular at a point P then Π is continuous at P in the Stone topology.

Proof

Let $\Pi = (F_1, \ldots, F_{n+1})$ where the F_i are forms of degree d. Let $Q = \Pi(P)$ and suppose that Ψ_0 is a basic open subset of $\mathbb{P}^n(\Omega)$ with $Q \in \Psi_0$. There are coordinates $(x, \ldots, x_{m+1})$ and $(y_1, \ldots, y_{n+1})$ of P and Q respectively such that $(F_1(x), \ldots, F_{n+1}(x)) = (y_1, \ldots, y_{n+1})$. Let $\hat{\Psi}_0$ be the almost homogeneous and constructible subset of Ω^{n+1} corresponding to Ψ_0. Then $\hat{\Psi}_0$ is a neighborhood of (y). Consider the formula:

(*) $\exists (z) \in \Psi_0 ((z) = \Pi(v))$.

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There is a quantifier free formula $\hat{\psi}_1$ defined over F which is equivalent in Ω^{n+1} to (*). Claim: The set $\hat{\psi}_1$ is almost homogeneous. For suppose $(z) \in \hat{\psi}_1$ and $t \in \Omega$ is nonzero. By definition of $\hat{\psi}_1$ there is some $(w) \in \hat{\psi}_0$ such that:

$$(F_1(z),\ldots,F_{n+1}(z)) = (w).$$

Then

$$(F_1(tz),\ldots,F_{n+1}(tz) = (t^dF_1(z),\ldots,t^dF_{n+1}(z)) = t^d w)$$

and the latter point is in $\hat{\psi}_0$. It follows from * that $(tz) \in \hat{\psi}_1$ and so $\hat{\psi}_1$ is almost homogeneous. By Theorem 2.6 the set ψ_1 defines a projective constructible set Ψ_1 in $\mathbb{P}^m(\Omega)$. As $P \in \Psi_1$ and $\pi^{-1}(\Psi_0) = \Psi_1$ the map π is continuous at P. ⊠

An alternate proof avoiding the use of Theorem 2.6 was communicated by H. Lakser. Let $\pi = (F_1,\ldots,F_{n+1})$ where the F_i are forms of degree d and let define a basic open neighborhood of Q= (P) in $\mathbb{P}^n(\Omega)$. Then $\pi^{-1}(\psi)$ is defined by the formula:

$$\psi(F_1(V_i),\ldots,F_{n+1}(V_i)) \wedge (\neg V_1=0 \vee \ldots \vee \neg V_{m+1}=0).$$

This formula is projective since substituting forms of a fixed degree in a projective formula yields a projective formula. Thus $\pi^{-1}(\Psi)$ is a neighborhood of P.

1.4 The main result of this subsection is a further application of the method of proof of Theorem 2.6. We begin with a definition.

Definition 2.6

Suppose that Ω is a universal domain and $X \subseteq \Omega^n$ is a constructible

set. A field of definition $F \leq \mathfrak{M}$ for X is a field F such that some open formula over F defines X.

Lemma 2.13

Let $\mathfrak{M}$ be a universal domain and $\mathcal{Y}$ a degree 1 constructible subset of $\mathfrak{M}^n$. If F_1 and F_2 are fields of definition of $\mathcal{Y}$ such that $\mathfrak{M}$ is a universal extension of F_1F_2 then the variety associated with $\mathcal{Y}$ over F_1 and F_2 is the same.

Proof

Let (x) be an independent point of $\mathcal{Y}$ over F_1F_2; such exists by hypothesis. Then $\dim_{F_1}(x) \geq \dim_{F_1F_2}(x) = \dim \mathcal{Y}$ but $\mathcal{Y}$ is defined over F_1. So $\dim_{F_1}(x) = \dim_{F_1F_2} = \dim \mathcal{Y}$ and thus (x) is an independent point of $\mathcal{Y}$ over F_1. By the same argument it is an independent point of $\mathcal{Y}$ over F_2. Let V_1 be the locus of (x) over F_1. If V_1 has two distinct components W_1 and W_2 over F_1F_2, which must then have dimension equal to dim V_1, let (x) be generic for say W_1, and (y) generic for W_2. Then as $(y) \in V_1$ we have $\dim_{F_1}(y) = \dim_{F_1F_2}(y)$ so (y) is a generic point of V_1. Hence (y) is an independent point of $\mathcal{Y}$ over F_1. It follows that as $\dim_{F_1F_2}(y) = \dim V_1 = \dim \mathcal{Y}$ the point (y) is an independent point of $\mathcal{Y}$ over F_1F_2. But then (x) and (y) have the same locus over F_1F_2. Contradiction. Therefore V_1 does not split over F_1F_2 and

is thus the locus of (x) over F_1F_2. Hence V_1 is the variety associated to $\mathfrak{P}$ over both F_1 and F_2. ⊠

The above lemma allows us to define the associated variety of a degree 1 constructible set $\mathfrak{P}$ as the associated variety of any open formula $\mathfrak{P}_0$ defining $\mathfrak{P}$ over its parameter field.

Corollary 2.4

Suppose that Ω is a universal domain and $\mathfrak{P}$ is a degree 1 constructible subset of Ω^n. Let V be the variety associated with $\mathfrak{P}$ and F_0 the smallest field of definition of V. If F is any field of definition of $\mathfrak{P}$ then $F_0 \leq F$.

The next lemma extends the above corollary to sets not necessarily having degree 1.

Lemma 2.14

Suppose that Ω is a universal domain and $\mathfrak{P}$ is a constructible subset of Ω^n. Let V be the algebraic set which is the union of the associated varieties of the degree 1 components of $\mathfrak{P}$. If F_0 is the smallest field of definition of V then F_0 is a subfield of any field of defintion of $\mathfrak{P}$.

Proof

By lemma 2.13 the algebraic set V is well defined. Suppose then that F is a field of definition of $\mathfrak{P}$. Let $\mathfrak{P}_1 V, \ldots, V \mathfrak{P}_m$ be a decomposition over F of $\mathfrak{P}$ into distinct F-minimal sets of dimension

equal to dim $\mathfrak{P}$. Then the independent points of each $\mathfrak{P}_i$ in $\mathfrak{U}^n$ over F define algebraic irreducible sets V_i. Let $W = \cup V_i$. Claim: $W = V$. For consider a splitting $W_1 V,\ldots,V W_r$ of W into irreducible varieties over $\bar{F}$. The set $\mathfrak{P}$ also splits over $\bar{F}$ into degree 1 components (by 4.55 of [13])(4). Now a standard result (by 2.1 of [1]) tells us that each W_i is of dimension equal to dim W = dim $\mathfrak{P}$. Thus each W_i corresponds to a component of $\mathfrak{P}$. For let (x) be a generic point of W_i over F. Then $\dim_{\bar{F}}(x) = \dim_F (x) = \dim W$ so as (x) W there is V_j such that (x) is a generic point of V_j over F. But then (x) $\in \mathfrak{P}_j$ so (x)$\in \mathfrak{P}$. Thus (x) is an element of a degree 1 component. But then W = W_i is equal to V by definition. This proves the claim. Now W is defined over F so F_0 F. ☒

Theorem 2.10

Suppose $\mathfrak{U}$ is a universal domain and $\mathfrak{P}$ is a constructible subset of $\mathfrak{U}^n$. There is a smallest field of definition F_0 of $\mathfrak{P}$ in $\mathfrak{U}$.

Proof

Let F_1 be the smallest field of definition of the algebraic set W associated with $\mathfrak{P}$. Such exists by our argument above. Let $\mathfrak{P}_0 = W \sim \mathfrak{P}$ and $\mathfrak{P}_1 = \mathfrak{P} \sim W$. Then again we have smallest fields of definition for the algebraic sets V_0 and V_1 associated with $\mathfrak{P}_0$ and $\mathfrak{P}_1$. Let F_2 be the amalgam of F_1 and the latter fields. Define now $\mathfrak{P}_{0,0}$, $\mathfrak{P}_{0,1}$ and $\mathfrak{P}_{1,0}$ and $\mathfrak{P}_{1,1}$ in the same fashion as above. Proceeding inductively in this way we terminate the construc-

tion at say the r-th stage, forming the field F_r. Claim: The field F_r is the smallest field of definition of $\mathcal{Y}$. To prove this we first note that via an argument similar to that of Theorem 6 it follows that some open formula defined over F_r defines $\mathcal{Y}$. For the sets at each end of every branch of the tree are varieties of some dimension ≥ 0, the next sets are formed as Boolean combinations of these algebraic sets and associated varieties defined over F_r and so on. As all the algebraic sets are defined over F_r so is $\mathcal{Y}$. Now suppose that F is a field of definition of $\mathcal{Y}$. We can assume that Ω is a universal domain over F as F can be taken to be finitely generated. Proceed now in reverse of the above argument utilizing lemma 15 as follows. As $\mathcal{Y}$ is defined over F so is the associated algebraic set W hence by the lemma the field of definition F_1 of W is a subfield of F. We continue down the tree of construction of F_r applying the lemma at each stage. Finally we conclude that $F_r \leq F$. ☒

The tree utilized in the above proof (and in the proof of Theorem 2.6) we call the tree associated with the constructible set $\mathcal{Y}$. One can consider a constructible set to be, in terms of this tree, a configuration of varieties. As the tree is entirely canonical, not depending on any representation of $\mathcal{Y}$ as a solution set of an open formula, so too constructible sets must be viewed as canonical geometric objects. The above theorem is a formal reflection of this geometric principle.

We conclude this subsection with one further basic result on constructible affine sets. We need first the following well known lemma (see for example [19]) for which we present a very simple proof.

Lemma 2.15

A constructible F-set $\mathfrak{P}$ is a variety iff for each point (x) in the locus of (x) in affine space $\mathfrak{M}^n$ over F is contained in $\mathfrak{P}$.

Proof

Let V(x) be the locus of $(x) \in \mathfrak{P}$ over F. Then V(x) is open in the Stone topology. As the family of all such varieties covers $\mathfrak{P}$ and $\mathfrak{P}$ is compact there is a finite subfamily $V(x^i),\ldots,V(x^m)$ which covers $\mathfrak{P}$. But $\bigcup V(x^i) \subseteq \mathfrak{P}$ and $\bigcup V(x^i)$ is a variety. Thus $\mathfrak{P} = \bigcup V(x^i)$ and so $\mathfrak{P}$ is a variety. ⊠

Theorem 2.11

Suppose that F is a field and $\mathfrak{P}$ is a constructible set defined over F. If there is $F_1 \geq F$ and equations $P_1 = 0,\ldots, P_m = 0$ defined over F_1 such that the variety so defined is equal to $\mathfrak{P}$ then $\mathfrak{P} = V$ for a variety defined over F.

Proof

Since $\mathfrak{P}$ is a variety V_1 over F_1 if $(x) \in \mathfrak{P}$ then the locus V(x) of (x) over F is contained in V_1. For if (x) ---> (y) is a specialization of (x) there is by Proposition 7 of Chapter II of [43] a generic specialization $(\bar{x})$ of (x) over $\bar{F}$ such that $(\bar{x})$ ---> (y) is a specialization over F. Since the latter is algebraically closed $(\bar{x})$ is regular over $\bar{F}$. Thus its locus is defined by a generic point (z) over $\bar{F}_1 \geq \bar{F}$.

We have $(z) \in \mathfrak{P}$ and V_1 is defined over F_1 so the locus of (z) is contained in V_1. As $(z) \dashrightarrow (y)$ over F_1 we have then $(y) \in V_1 = \mathfrak{P}$. As the locus of each point of $\mathfrak{P}$ over F is contained in $\mathfrak{P}$ it follows by Lemma 2.15 that $\mathfrak{P}$ is already a variety over F. ☒

2. A discussion of Some Associated Field Extensions

2.0

The fact that the affine space of a universal domain extending a field is compact in the Stone topology defined over the field is not surprising since the universal domain is saturated over the groundfield. That is a universal domain over a field F realizes all open types over F. The first result of this section shows that if an infinite extension of a field F is such that all affine spaces are compact in the topology over F then the extension has infinite transcendence degree over F. The proof of this result illustrates the use of logical compactness arguments in field theory.

The saturated models of logic are precisely the homogeneous universal models. The classical theory can be found in [6] and a related theory for universal classes in [13]. We introduce the notion of a homogeneous extension of a field and prove that an infinite compact and homogeneous extension of an algebraically closed field is a universal domain. The proof uses compactness to prove that the extension is algebraically closed. We show that normal algebraic extensions are homogeneous but a simple transcendental extension is not homogeneous.

The next two results are concerned with properties of homogeneous extensions. The first proves a universal property of an infinitely generated homogeneous extension which is analogous to a property of homogeneous models. See for example Proposition 3.2.9 of [6]. The second proves a uniformity property of maximal independent subsets of degree 1 constructible sets defined over the groundfield. The section concludes with some examples.

2.1

It is an open question to completely characterize the extensions F_1 of a given field F with the property that F_1^n is compact in the open F-topology for each $n < w$. We call such an extension compact. The following result is a partial converse of Theorem 2.3.

Theorem 2.12

Suppose that F is a field and F_1 an infinite extension of F such that for each $n > 1$ the space F_1^n is compact in the open topology over F. Then F_1 has infinite transcendence degree over F.

Proof

Let $n > 1$ and $< \omega$ be given. For each $1 \leq m \leq n$ let $\{f_\alpha^m: \alpha < \lambda\}$ be the set of all irreducible polynomials over F in exactly m variables. Let X be the direct product of the above defined sets so that the m-th set is on the m-th coordinate. Let X_1 be the set of all finite subsets of X. For each $s \in X_1$ we define a set ψ_s as follows. Say the

elements of S are $\langle f^1_{\alpha 1,1},\ldots,f^n_{\alpha 1,n}\rangle,\ldots,\langle f^1_{\alpha r,1},\ldots,f^n_{\alpha r,n}\rangle$. For each $1 \leq j \leq r$ let $\theta_j(v_1,..,v_n)$ be the formula $\neg f^1_{\alpha j,1}(v_1) = 0 \wedge,\ldots,\wedge \neg f^n_{\alpha j,n}(v_1,\ldots,v_n) = 0$. We then define ψ_s to be $\bigwedge_{1\leq j\leq r}\theta_j$. Let $Y = \{\psi_s : s \in X_1\}$. Then each element of Y defines a closed set over F. Call this family of sets Y_1. We claim that Y_1 has the finite intersection property. Since the union of a finite number of elements of X_1 is an element of X_1 to prove our claim we need only show that each set ψ_s in Y has a solution in $F_1{}^n$. As above let $\psi_s = \bigwedge \theta_j$. Consider first the set of polynomials which are in first conjuncts of θ_j for $1 \leq j \leq r$. Each has at most a finite solution set in F_1 so that, since F_1 is an infinite extension of F there is $X_1 \in F_1$ which is not the solution of any of these polynomials. Now consider the polynomials which appear in the second conjuncts of θ_j. Substitute x_1 for the variable v_1 in each. The resulting polynomials are not identically zero since by our construction each such polynomial is irreducible and is a polynomial in exactly v_1 and v_2. Again there are a finite number of polynomials resulting from our construction and since each is $\neq 0$ there is X_2 in F_1 which is not a solution of any of these polynomials. We proceed inductively in this fashion to construct a solution $\langle x_1,\ldots,x_n\rangle$ of ψ_s. This proves our claim. It now follows by the compactness of $F_1{}^n$ that the entire set Y has a common solution, say $\langle y_1,\ldots,y_n\rangle$. We claim that the set $(y_1,\ldots,y_n)$ is algebraically independent over F. If not, let r be the least number $\leq n$ such that $\langle y_1,\ldots,y_r\rangle$ satisfies some nontrivial algebraic relation $f(v_1,\ldots,v_r) = 0$ where f is a poly-

nomial over f. We can choose f to be irreducible. It follows that f = f^r_α for some $\alpha < \lambda$. Hence by definition of X there is some element $< f^1_{\alpha 1}, \ldots, f^r_{\alpha r}, \ldots, f^n_{\alpha n} >$ with $\alpha_r = \alpha$ in X. Then the set $\{< f^1_{\alpha 1}, \ldots, f^r_{\alpha r}, \ldots, f^n_{\alpha n} >\} = S$ is in X_1 so that $<y_1, \ldots, y_n>$ is a solution of $\psi_S \in Y$. But by definition $\psi_S = \neg f^1_\alpha (v_1) = 0 \wedge, \ldots, \neg f^r_\alpha (v_1, \ldots, v_r) = 0 \wedge, \ldots, \wedge \neg f^n_{\alpha n} (v_1, \ldots, v_n) = 0$ so that $<y_1, \ldots, y_r>$ satisfies the inequation $\neg f^r_\alpha (v_1, \ldots, v_r) = 0$. Contradiction. So our claim is proven. Since F_1 contains arbitrarily large finite algebraically independent sets it follows that F_1 has infinite transcendence degree over F. ☒

Problem 1

Is there a field F and a compact extension F_1 of F which is infinite over F but contains no universal domain extending F?

Lemma 2.16

Suppose that F is a field and F_1 is an extension such that for some $m > 1$ the space F_1^m is compact in the open F-topology. Then if $n < m$ the space F_1^n is also compact in the open F-topology.

Proof

Let $\{\psi_\alpha(v_i): \alpha < \lambda\}$ be a family of basic closed sets with the finite intersection property. We define a corresponding family in F_1^m by letting $\psi_\alpha{}^1(v_1) = \psi_\alpha(v_i) \wedge v_{n+1} = v_{n+1} \wedge, \ldots, \wedge v_m = v_m$. We claim that the corresponding family of solution sets has the finite intersection property. This is immediate from the corres-

ponding property for $\{\psi_\alpha\}_{\alpha<\lambda}$ since the last m-n coordinates are arbitrary. So by compactness of F_1^m we have a common solution to all the sets $\{\psi_\alpha^1\}_{\alpha<\lambda}$, say $\langle x_1,\ldots,x_m\rangle$. Then $\langle x_1,\ldots,x_n\rangle$ is a common solution of $\{\psi_\alpha\}_{\alpha<\lambda}$. ⊠

Note that by an argument similar to the proof of Theorem 2.12 it follows that if F_1^n is compact in the open topology over F then F_1 has transcendence degree $\geq n$ over F. We now introduce a class of extensions of a field F for which we can obtain a resolution of Problem 1.

Definition 2.7

Suppose that F is a field and F_1 is an extension of F. Then F_1 is homogeneous over F (or is a homogeneous extension of F) if whenever $\bar{x}_1$ and $\bar{y}_i$ are sequences of the same type over F (or what is the same are conjugate in a universal domain Ω extending F_1) and $x \in F_1$ there is $y \in F_1$ such that the sequences $\bar{x}_i \frown \{x\}$ and $\bar{y}_i \frown \{y\}$ have the same type $p \in S_A^{n+1}(F)$.

If F is a field an extension field F_1 of F is a compact extension if for all $n < \omega$ the space F_1^n is compact in the open F-topology.

Theorem 2.13

A compact homogeneous infinite extension F_1 of an algebraically closed field F is a universal domain.

Proof

Let $f(c_1,\ldots,c_n,v_0)$ be a polynomial over F_1 with $c_1,\ldots,c_n$ all the parameters of the polynomial from the set $F_1 - F$. Then there is an extension F_2 of F_1 which contains a solution x of the equation $f(c_i,v_0) = 0$. Let $\Gamma(v_i,v_0)$ be the open type of $\langle c_i, x\rangle$ over F. Then since F is algebraically closed it is existentially closed hence Γ is a set of open formulas with the finite intersection property in F_1^{n+1}. Since F_1 is a compact extension of F it follows that there is a solution $\langle d_1,\ldots,d_n,x_0\rangle$ of Γ in F_1. Not it is immediate that $\bar{c}_i$ and $\bar{d}_i$ have the same type over F so since x_0 extends $\bar{d}_i$ in F_1 to a sequence of type Γ it follows from homogeneity that there is $y \in F_1$ such that $\langle c_1,\ldots,c_n,y\rangle$ has type Γ. But the formula $f(v_iv_0) = 0$ is in Γ so that $f(c_iy) = 0$. Hence $f(c_iv_0)$ has a root in F_1 and thus F_1 is algebraically closed. Since F_1 is an infinite extension of F it follows from Theorem 12 that F_1 has infinite transcendence degree over F and so F_1 is a universal domain. ⊠

Example 2.1

Suppose that F_1 is a simple transcendental extension of a field F. Then F_1 is not a homogeneous extension of F. For let x be a transcendental point of F_1 over F. Let $y = x^2$. Then by Luroth's Theorem y is transcendental over F so that y and x have the same open type over F. But x is a square root of y but x itself has no square root in F_1. ⊠

Example 2.2

If F is any field and F_1 is a normal extension of F then F_1 is a homogeneous extension of F. For suppose that $\bar{x}_i$ and $\bar{y}_i$ are two sequences in F_1 of type Γ. Let x be a point of F_1 that extends $\bar{x}_i$ to a sequence of type Γ_1. Then x is algebraic over F and say f(v) is the minimal polynomial of x over the field. Let g(v) be the minimal polynomial of x over the field $F(\bar{x}_i)$. Then $F(\bar{x}_i)$ and $F(\bar{y}_i)$ are F-isomorphic via a map σ such that $\sigma(x_i) = y_i$. Let $g_1 = \sigma(g)$. Now g divides f and $\sigma(f) = f$ so that g_1 divides f and is irreducible. Furthermore F_1 is normal so that f(v) splits in F_1 since it is the minimal polynomial of $x \in F_1$. Since g_1 divides f it follows that g_1 splits in F_1. Let y be a solution. It is immediate that we can extend σ by an isomorphism σ_1 of $F(x_1,x)$ to $F(y_i,y)$ such that $\sigma_1(x) = y$. Hence $\bar{y}_i \frown \{(y)\}$ has type Γ_1 so F_1 is homogeneous. ⊠

The second example above shows that not all homogeneous extensions are universal domains. Nevertheless a homogeneous extension is universal in a very precise sense.

Definition 2.8

Suppose F is a field and F_1 is an extension of F. The extension diagram $D_A^F(F_1)$ of F_1 over F is the set of all open types over F which are realized in F_1.

Theorem 2.14

Suppose that F is a field. Then two homogeneous extensions of F which are countably generated over F are F-isomorphic iff they have the same extension diagrams. If F_1 is a homogeneous extension with no less than ω-generators over F and if F_2 is an extension of F with the same extension diagram but $\leq \omega$ generators over F then F_2 embeds in F_1 over F.

Proof

Let F_1 and F_2 be two homogeneous extensions of F with the same extension diagram and assume that $\langle x_i \rangle i < \omega$ and $\langle y_i \rangle i < \omega$ are sets of generators for F_1 and F_2 respectively over F. We define an isomorphism via a back and forth argument which proceeds by induction. Let $\Gamma(v_0)$ be the type of x_1 over F. Then Γ is in the extension diagram of F_1 so Γ is realized by say z_1 in F_2. Let f_1 be an isomorphism over F such that $f_1(x_1) = z_1$. Let y_r be the first generator of F_2 not in $F(z_1)$. Then the type Γ_1 of $\langle z_1, y_r \rangle$ is also realized in F_1 say be a sequence $\langle w_1, w_2 \rangle$. Then w_1 and x_1 have the same open type over F and so by definition of homogeneous extension there is a point s_1 in F_1 such that $\langle x_1, s_1 \rangle$ realizes Γ_1. Then clearly f_2 can be defined so that f_2 extends f_1 and $f_2(s_1) = y_r$. Now choose the least generator x_m not in the field $F(x_1, s_1)$ and proceed as above to extend f_2. In this way we define an F-isomorphism $f = \bigcup f_i$ of F_1 onto F_2. This proves the first assertion. The universal property of homogeneous extensions can be

proven in a similar fashion constructing an embedding into the homogeneous extension by induction. ☒

Theorem 2.15

Suppose that F is a field and F_1 is a homogeneous extension of F. Let $\psi(v_i)$ be a degree one open formula over F of dimension ≥ 1. Then all maximal independent subsets of ψ in F_1^n have the same power.

Proof

If there are maximal independent subsets of ψ in F_1^n of infinite power the result follows from Theorem 4.12 of [13]. So assume that all such subsets are finite. Then if F_1 has a set of $\leq \omega$ generators over F then it is clear from the proof of Theorem 2.15 that if $\bar{x}_i$ and $\bar{y}_i$ are sequences in F_1 of the same type Γ of $S_A^n(F)$ then there is an automorphism σ of F_1 over F such that $\sigma(x_i) = y_i$. Now if X is an independent subset of ψ and Y is another such set of power $\leq |X|$ from the definition of independent subset there is a subset $(x^1),\ldots,(x^r)$ of X such that $Y = \{(y^i),\ldots,(y^r)\}$ and the sequence $(x^1)^\frown,\ldots,^\frown(x^r)$ and $(y^1)^\frown,\ldots,^\frown(y^r)$ have the same type over F. Applying the automorphism σ of F_1 defined as above for these sequences we obtain an extension Y_1 of Y to an independent subset of ψ of power $|X|$. In the case where F_1 has $>\omega$ generators over F we extend the canonical map obtained from the above sequences to an embedding of some finite subfield of F_1 which contains all the coordinates of the

points of X into F. This latter map has an image in which Y extends to a set of power equal to $|X|$. ⊠

Example 2.3

Let F be an arbitrary field and x,y elements which are algebraically independent over F. Then $x^{1/2}$ is also algebraically independent of y. Let $z = y^{1/2}$. Then x,z are algebraically independent over F. Now let $F_1 = F(z)$, $F_2 = F(x)$ and $F_3 = F(y,x^{1/2})$. Then F_1 and F_2 are free extensions of F but the pairs F_3, F_2 and F_1, F_3 are not free over F. ⊠

The above example shows that the relation "F_1 is not free over F_2 (relative to F)" is not a transitive relation. This gives some insight into the difficulties involved in the question of obtaining a full exchange theorem for independent sets. In [13] we prove a partial exchange theorem for these sets.

We call an extension F_1 of a field F uniform for a degree one constructible set if all maximal independent subsets of ψ in F_1^n have the same power.

Example 2.4. Let F be a field and consider the variety V which is the entire affine plane over F (say with points from a universal domain Ω extending F). Let $x_1,\ldots,x_6$ be algebraically independent points of Ω over F. Let $F_1 = F(x_1,x_2,x_3)$ and $F_2 = F(x_4,x_5,x_6)$. Then a maximal free subset of generic points of V in F_1 has cardinality 1 and similarly such a set has cardinality 1 in F_2. Furthermore

F_1, F_2 are linearly disjoint over F. But there is a set of free generic points of V in F_1F_2 over F of cardinality 3. ⊠

3. Regular Maps and Constructible Sets

3.0

The section begins with a proof of Chevalley's Theorem. Using the Elimination of Quantifiers Theorem for algebraically closed fields we first prove the result for projections and then reduce the general problem to this special case. Our next result shows precisely the relationship between the varietal and set theoretic projection of a variety.

To completely resolve the question of how close to a variety is the image of a variety under a regular map one has to show that the image contains a nonempty Zariski open subset of the closure of the image in the Zariski topology. We give a new proof of this fact utilizing a key lemma: The closure of the image of an irreducible variety is irreducible. As a corollary of our proof we show that if σ is a regular isomorphism of a constructible set and a variety then σ maps the Zariski closure of the constructible set onto the variety.

We next deduce two properties of regular isomorphisms of constructible sets. The first result tells us an isomorphism of degree 1 sets maps independent subsets to independent subsets. The second result says that isomorphic minimal constructible sets have the same splitting extension. The final result of the section is a proposition concerning the

fibers of a regular map. It is given as an illustration of the use of compactness.

3.1

A regular map σ on an affine F^n to a space F^m is defined by the condition that for any point (x) of F^n we have $\sigma(x) = \langle p_1,(x), \ldots,p_m(x)\rangle$ where $p_1,\ldots,P_m$ are polynominals over the field F. A projection for example is a regular map.

Theorem 2.16 (6)

Suppose that R is an integral domain and $\Psi(v_iv_j)$ is an open formula over R in n + m variables. There is an open R-formula $\Psi 1(v_j)$ such that if F is any algebraically closed extension of R then Ψ_1 defines the projection of Ψ onto the last m variables in the space F^m.

Proof

Let F_1 be the algebraic closure of the quotient field of R. Say $= \Psi_2(c_1,\ldots,c_sv_i,v_j)$ where Ψ_2 is an open formula in the language of fields and $c_1,\ldots,c_s$ are parameters from R. Now by Tarski's Theorem the theory of algebraically closed fields admits elimination of quantifiers, so there is an open formula Ψ_3 such that $F_1 \models \Psi_3(v_r,v_j) \equiv \exists v_i \Psi_2(v_r,v_i,v_j)$ and so $F_1 \models \Psi_3(c_1,\ldots,c_sv_j) \equiv \exists \tilde{v}_i \Psi_2$ $(c_1,\ldots c_s\tilde{v}_i,\tilde{v}_j)$. Now $\Psi_1 = \Psi_3 (c_1,\ldots,c_sv_j)$ is an open R-formula and we claim that Ψ_1 satisfies the conditions of our theorem. For let F

be any algebraically closed extension of R. Then since $F \equiv F_1$ we have that $F \models \Psi_3(\bar{v}_r \bar{v}_j) \equiv \exists \bar{v}_i \Psi_2 (\bar{v}_r \bar{v}_i \bar{v}_j)$ so since $F \geq R$ we have also that $F \models \Psi_3(c_1, \ldots, c_s, \bar{v}_j) \equiv \exists \bar{v}_i \Psi_2 (c_1, \ldots, c_s \bar{v}_i \bar{v}_j)$. But the latter formula defines the projection on the second factor of Ψ in F^m so that Ψ_1 also defines this projection. ⊠

We note that another purely algebraic proof, but one which is not so elementary, can be obtained by a reduction to Theorem 2.6 page 42 of [28]. The following result is an immediate corollary.

Corollary 2.5 (Chevalley)

Suppose that R is an integral domain and $\mathfrak{b}$ is a regular map defined over R in the space F^n to F^m where F is an algebraically closed extension of R. Let Ψ be a constructible R-formula. Then the image of Ψ under $\mathfrak{b}$ is a constructible R-formula.

Proof

Suppose $\mathfrak{b} = (f_1, \ldots, f_m)$ for polynomials f_i defined over R. Let Ψ_2 be the following formula: $\Psi(v_i) \wedge v_{n+1} = f_1(\bar{v}_i) \wedge \ldots \wedge f_m(\bar{v}_i) = v_{n+m}$. By the previous theorem the projection of Ψ_2 on the second factor in F_m is definable by an open R-formula Ψ_1. But this projection is clearly the image of Ψ under $\mathfrak{b}$. ⊠

The reader is referred to [30] page 97 for a proof of the above result over an algebraically closed field. Other versions of the theorem appear in [19] and [28].

Note that Corollary 2.5 implies that if $\mathfrak{U}^n$ and $\mathfrak{U}^m$ have the open F-topology then a regular map $\pi : \mathfrak{U}^n \dashrightarrow \mathfrak{U}^m$ is an open map. For if O is an open set in $\mathfrak{U}^n$ then O is the union of a family $\{U_\alpha\}_{\alpha<\lambda}$ of constructible sets and we have $\pi(\bigcup_\alpha U_\alpha) = \bigcup \pi(U_\alpha)$. Each of the sets $\pi(U_\alpha)$ is a constructible set so that it is open in $\mathfrak{U}^m$. Hence $\bigcup \pi(U_\alpha)$ is open in $\mathfrak{U}^m$.

Lemma 2.17

Suppose that F is a field and $\mathfrak{U}$ a universal domain extending F. Let V be an irreducible variety defined over F in $\mathfrak{U}^{n+m}$. If (v_j) defines the projection of V onto the second factor then the variety associated with the open formula ψ in $\mathfrak{U}^m$ is the varietal projection V_1 of V in $\mathfrak{U}^m$.

Proof

Let $\langle x_1^1,\ldots,x_n^1,x_1^2,\ldots,x_m^2\rangle$ be a generic point of V over F. Then V_1 is the locus of $\langle x_1^2,\ldots,x_m^2\rangle$. Clearly, $\langle x_1^2,\ldots,x_{m2}\rangle$ is a solution of and we must have $\dim\psi = \dim x_j^2$. For if $\bar{y}_j$ is any point in ψ there is $\bar{z}_i$ such that $\bar{z}_i \wedge \bar{y}_j$ is in V so $(x_j^1,x_j^2)\dashrightarrow(z_i,y_j)$ is a specialization over F. Hence $(x_j^2)\dashrightarrow(y_j^1)$ is a specialization over F so that $\dim \bar{x}_j^2 \geq \dim \bar{y}_j$. On the other hand $\dim V_1 = \dim \bar{x}_j^2$ and $\bar{x}_j^2 \in V_1 \cap \psi$. Hence $\dim\psi = \dim V_1 = \dim V_1$. We also know that $\psi \subseteq V_1$ as is immediate from our above argument. It therefore suffices to show that $V_1 \sim \psi$ has dimension $< \dim V_1$. If not then there is a point $\bar{w}_j$

in this set with dim w_j = dim V_1. But V_1 is irreducible so that (x_j^2)-->(w_j) is a generic specialization. But then (x_j^2) and (w_j) are conjugate points of Ω over F so that (w_j) is an element of $\mathcal{Y}$. This contradiction proves the lemma. ∎

In projective space the image of a variety under a regular map is a variety. The Theorem of Chevalley partially answers the question in affine space: How close to an irreducible variety is the image under a regular map of an irreducible variety? The following two results give a complete resolution of this problem.

Theorem 2.17

Let V be a variety defined over the field F in the space Ω^n where Ω is a universal domain over F. If $\pi:\Omega^n \to \Omega^m$ is a regular map and V is irreducible then $\pi(V)$ contains a nonempty Zariski open subset of the closure V_1 of $\pi(V)$ in the Zariski topology.

This is a classical theorem for which we obtain an elementary proof via the following lemma.

Lemma 2.18

Suppose that V is an irreducible variety over the field $F \leq \Omega$ where Ω is a universal domain extending F. Let π be a regular map from V into Ω^r. Then the closure of the constructible set $\pi(V)$ in the Zariski topology is an irreducible variety.

Proof

Let (x) be a generic point of V. We claim that every point of (V) is in the locus of the point $\Pi(x)$. For let (y) be a point of V and (z) = $\Pi(y)$. Since (x) is a generic point of V we have a specialization (x)--->(y). Let Π = $(f_1,\ldots,f_r)$. Then the points $f_1(x),\ldots, f_r(x)$ are elements of $F[\bar{x}]$. Consider the subring $F[f_1(x),\ldots, f_r(x)]$. The specialization Φ induces a homomorphism Φ_1:$F[f_1(x), \ldots,f_r(x)]$ ---> $F[f_1(y),\ldots,f_r(y)]$ since $\Phi(f_i(x)) = f_i(y)$ as f_i is a polynomial over F and Φ leaves F fixed. Thus Φ induces a specialization $(f_1(x)_1,\ldots,f_r(x)\text{--->}(f_1(y),\ldots,f_r(y))$ = (z). Hence (z) is in the locus of $(f_1(x), \ldots,f_r(x))$. Hence $\Pi(V)$ = X is contained in the locus of $(f_1(x), \ldots,f_r(x))$ and since the latter point is in X we have dimX = dim $(F_1(x),\ldots,F_r(x))$ = $\dim V_1$ where V_1 is the locus of $(f_1(x), \ldots,f_r(x))$. Hence V_1 is the closure of X. Since clearly V_1 is irreducible this proves the lemma. ☒

Now suppose Π is a regular map defined over the field F and V is an irreducible variety with points in a universal domain Ω extending F. The image $\Pi(V)$ is a constructible set and by the above lemma the closure of (V) is irreducible. The theorem now follows from the following fact: If Ψ is a constructible subset of an irreducible variety V then Ψ contains a nonempty Zariski open subset of V if V is the closure of Ψ in the Zariski topology. For Ψ is the union of sets of the form $V_1 - V_2$ for subvarieties V_1 of V and one such set must have dimension equal to dim V since the closure of is just the algebraic set V. But V is irreducible so any subvariety of the same dimension is equal to V. Thus Ψ contains a set $V - V_2$ and this is a Zariski open subset of V.

To summarize our above discussion: If V is an irreducible variety then the image of V under a regular map is a constructible set X whose closure V_1 is irreducible and X contains a nonempty Zariski open subset of V_1. Thus although the image of V is not in general a variety, as in the projective case, it is almost a variety in the above precise sense.

Corollary 2.6

Suppose that F is a field and $\mathfrak{U}$ is a universal domain extending F. Suppose $\mathcal{Y}$ is a constructible set defined over F in $\mathfrak{U}^n$ and $\varphi : \mathcal{Y}$ ---> V is a regular isomorphism of $\mathcal{Y}$ onto an irreducible variety V. Then φ maps the closure V_1 of $\mathcal{Y}$ onto V.

Proof

Since V is irreducible it follows by lemma 2.18 that the closure V_1 of $\mathcal{Y}$ in the Zariski topology is irreducible, for φ^{-1} is a regular map and $\varphi^{-1}(V) = \mathcal{Y}$. Then $\overline{\varphi(V_1)}$ is an irreducible variety again by lemma 2.18. Now $V_1 = \bar{\mathcal{Y}}$ so that $\dim V_1 = \dim \mathcal{Y}$ and so a generic point (x) of V_1 is in $\mathcal{Y}$. Now it was shown in the proof of lemma 2.18 that $\overline{\varphi(V_1)}$ is the locus of the point (x). But $(x) \in \mathcal{Y}$ so $\varphi(x) \in V$. Hence $V \supseteq \overline{\varphi(V_1)}$. As $\mathcal{Y} \subseteq V_1$ we have $\varphi(\mathcal{Y}) \subseteq \varphi(V_1)$ so $V \subseteq \varphi(V_1)$. From these inclusions we conclude that $V = \varphi(V_1)$. ⊠

Problem 2 (R. Venkataraman) are regular maps characterized by the properties of continuity in the Stone topology and mapping constructible sets to constructible sets?

3.2

Suppose F is a field and ψ_1 and ψ_2 are open formulas over F. Then ψ_1 and ψ_2 are <u>isomorphic in an extension</u> F_1 of F if there are regular maps $\varphi: \psi_1(F_1^n) \dashrightarrow \psi_2(F_1^m)$ and $\theta: \psi_2(F_1^m) \dashrightarrow \psi_1(F_1^n)$ which are inverse on points with coordinates in F_1.

<u>Lemma 2.19</u> (7)

Suppose that F is a field and ψ_1 and ψ_2 are constructible sets defined over F. If F_1 is an extension of F and $\varphi: \psi_1 \dashrightarrow \psi_2$ is an isomorphism of ψ_1 and ψ_2 in F_1 then φ maps points (x) of dimension r in $\psi_1(F_1^n)$ to points of dimension r in $\psi_2(F_1^m)$. In particular if F_1 is a universal domain extending F then $\dim \psi_1 = \dim \psi_2$.

<u>Proof</u>

Since $\dim \bar{x}_i = r$ the field $F(x_1,\ldots,x_n)$ has transcendence degree r over F. Let $(x_i) = (y_1,\ldots,y_m)$. Then since φ is a regular map we have $F(y_1,\ldots,y_m) \leq F(x_1,\ldots,y_n)$. Applying the same argument to the inverse we see that $F(x_1,\ldots, x_n) \leq F(y_1,\ldots,$

$y_m)$. Hence the field $F(y_1,\ldots,y_m)$ has transcendence degree r and so dim $\bar{x}_i$ = dim $\bar{y}_j$ = r. ⊠

Theorem 2.18

Suppose that F is a field and $\mathcal{V}_1$ and $\mathcal{V}_2$ are constructible sets defined and isomorphic over F in an extension F_1 or F. Say $\mathcal{V}_1$ and $\mathcal{V}_2$ have degree 1 and dim$\mathcal{V}_1$ = dim$\mathcal{V}_2$. If $\varphi : \mathcal{V}_1 \dashrightarrow \mathcal{V}_2$ is our isomorphism then φ carries independent subsets of $\mathcal{V}_1$ to independent subsets of $\mathcal{V}_2$.

Proof

Let X be an independent subset of $\mathcal{V}_1(F_1^n)$. Say dim $\mathcal{V}_1$ = r. Now if Y = φ(X) is not independent over F then some point $\bar{y}_j \in Y$ is dependent on $Y_1 = Y - \{\bar{y}_j\}$. Let $F_2 = F(Y_1)$ be the extension of F by all elements of F_1 which are coordinates of a point in Y_1 and say $\varphi_1 : \mathcal{V}_2 \dashrightarrow \mathcal{V}_1$ is the inverse of φ in F_1. Then φ and φ_1 still define an isomorphism of $\mathcal{V}_1$ and $\mathcal{V}_2$ in F_1 over F_2. Hence by the previous lemma $\varphi_1(\bar{y}_j)$ has dimension < r over F_2. Say $\varphi_1(\bar{y}_j) = \bar{x}_i$. Note that since φ is an isomorphism with inverse φ_1 the set $\varphi_1(Y_1) = X_1$ has the property that $F(X_1) = F_2$. Hence x_i has dimension < r over F(X). But this contradicts the independence of X over F. ⊠

Theorem 2.19

Suppose that F is a field and $\mathcal{V}_1$ and $\mathcal{V}_2$ are isomorphic

F-minimal sets. Then the splitting extension F_1 of ψ_1 is the splitting extension of ψ_2.

Proof

Say that ψ_1 splits into components $\theta_1 V, \ldots, V\theta_r$ over the splitting extension F_1 of ψ_1. By Chevalley's Theorem the image of each set θ_i (with points in a universal domain extending F_1) is an open F-formula Π_i. The restriction of our isomorphism to the set θ_i and Π_i is clearly an isomorphism. By lemma 2.19 the sets θ_i and Π_i have the same dimension. Since the isomorphism is 1-1 and onto the isomorphism is 1-1 and onto ψ_2 we must have that $\psi_2 \equiv \Pi_1 V, \ldots V \Pi_r$, and each pair of sets Π_i, Π_j are disjoint if $i \neq j$. Hence $\deg \psi_2 \geq r$. But if F_1 is not the splitting extension of ψ_2 then some set Π_i splits over a proper extension F_2 of F_1. But the same argument as above can be applied to Π_i and so θ_i would split over F_2. Contradiction. Hence each Π_i has degree 1 and so F_1 is the splitting extension of ψ_2. ⊠

The following result illustrates how definability and compactness argument can be utilized to prove purely geometrical theorems.

Proposition 2.1

Suppose that F is a field and Ω is a universal domain extending F. Let Π be a regular map of the constructible set ψ_1 onto the con-

structible set φ_2. If for all $n<\omega$ there is (x) on φ_2 such that $\pi^{-1}(x)$ has power $\geq$ n then there is (y) on φ_2 such that $\pi^{-1}(y)$ is infinite.

Proof

Let θ_i be the following formula: $\exists \geq i(x)[\varphi_1(\bar{x}) \wedge \pi(\bar{x}) = (\bar{y}_i) \wedge \varphi_2(y_i)]$ (written with the obvious abbreviations). There is a quantifier free formula σ_i such that $\Omega \models \theta \equiv \sigma_i$. We claim that the sequence of formulas $\{\sigma_i\}_{i<\omega}$ has the finite intersection property. For note that $\sigma_i \neq 0$ and $\sigma_{i+1} \subseteq \sigma_i$ by our assumptions and the definition of σ_i. Hence $\cap \sigma_{i\leq n} \neq 0$. Thus by the compactness of the open topology on Ω^m the sequence has nonempty intersection so say (y) is in this intersection. Again by definition of σ_i this implies that $\pi^{-1}(y)$ has power $\geq i$ for all $i < \omega$. Hence $\pi^{-1}(y)$ is infinite. ☒

4. Correspondences Defined by Constructible Sets

4.0

The section begins with the definition of a finite correspondence between constructible sets. This concept is shown to be an extension of the classical notion of (n,m) - correspondence. Three properties of finite correspondences are derived in this section. First it is shown that if all points of maximal dimension in the domain set have disjoint images in the range set then the correspondence is one-many expect on a

constructible subset of the domain set of dimension less than that of the domain. We prove next that finite correspondences define an equivalence relation on the constructible sets. Finally we prove that a complete finite correspondence preserves the cardinality of maximal independent subsets.

In classical algebraic and analytic geometry the theory of correspondences extends the theory of regular and projective mappings. Fundamental correspondence theorems in analytic projective geometry can be found in [41]. For classical results in algebraic correspondence theory the reader can consult [42]. Algebraic geometry extends the correspondence theory of a single equation to the consideration of arbitrary varieties. This is suggestive of the following general definition.

Definition 2.9

Suppose R is an integral domain and Ω is a universal domain extending R. A correspondence Π of $\mathcal{V}_1$ and $\mathcal{V}_2 \in L_A(R)$ with points in Ω is a constructible set Π defined over R such that if $(x,y) \in \Pi$ then (x) is in $\mathcal{V}_1$ and (y) is in $\mathcal{V}_2$.

Note Later we will define the product $\mathcal{V}_1 \times \mathcal{V}_2$ of two constructible sets. Then a correspondence is a constructible subset $\mathcal{V}_1 \times \mathcal{V}_2$. As with mappings we will also consider correspondences with points restricted to an extension R_1 or R in the universal domain Ω.

The following example answers a question of R. Venkataraman.

Example 2.5

Let A be an algebraically closed field. Consider the polynomial $x^2 - y^2$. The equation $x^2 - y^2 = 0$ defines a finite proper correspondence between A^1 and A^1. Each $x \in A^1$ has an image under this correspondence namely the set $\{x, -x\}$. Since the correspondence is one - many it is not equivalent to any polynomial map. ⊠

A correspondence Π with finite nonempty image and preimage we call a finite correspondence. Thus the correspondence given in the above example is a finite correspondence. The notion of finite correspondence is equivalent to the classical concept of an (n,m)-correspondence (in the case where Π is an algebraic set). See [42] for an exposition of some of the main results of the classical theory. To prove that the two definitions are equivalent note that we can apply the argument used in the proof of the Proposition first on one factor and then on the other factor. The correspondence described in Example 5 is a 2-2 correspondence. This correspondence has another property we call disjointness. Namely if (x) and (y) are free sequences (so in this special case x and y are free variables) then the image or preimage sets are disjoint. The following lemma implies as an immediate corollary that finite correspondences are disjoint.

Lemma 2.20

Suppose that R is an integral domain and Ω is a universal domain which extends R. Let Ψ_1 and Ψ_2 be constructible sets defined

over R and suppose π is a finite R-correspondence between $\mathcal{V}_1$ and $\mathcal{V}_2$. If $\bar{x}_i$ and $\bar{z}_i$ are points in $\mathcal{V}_1$ such that $\pi(\bar{x}_i, \mathfrak{M}^m) \cap \pi(\bar{z}_i, \mathfrak{M}^m) \neq \emptyset$ then $\bar{z}_i$ is algebraic over $R[x_1, \ldots, x_n]$.

Proof

Consider the formula $\pi(\bar{v}_i 1 \quad \bar{v}_j) \wedge \pi(\bar{v}_i 2 v_j)$. Since $\pi(\bar{x}_i, \mathfrak{M}^m) \wedge \pi(\bar{z}_i, \mathfrak{M}^m) \neq \emptyset$ there is y_j (in $\mathcal{V}_2$) such that $\mathfrak{M} \models \pi(\bar{x}_i, \bar{y}_j) \wedge \pi(\bar{z}_i, \bar{y}_j)$. Let $\theta(\bar{v}_i 1 \bar{v}_i 2)$ define the projection of the above formula on the coordinates $\bar{v}_i 1$ and $\bar{v}_i 2$. Then $\theta(\bar{x}_i \bar{v}^2)$ must be algebraic. For $\pi(\bar{x}_i \ \bar{v}_j)$ has a finite solution set, including $\bar{y}_j$ and each solution $\bar{z}_j$ has a finite preimage in $\mathcal{V}_1$. Thus by definition $\theta(\bar{x}_i \ \bar{v}_i 2)$ is the union of all the preimages of solutions of $\pi(\bar{x}_i \ \bar{v}_j 1)$ and so must be finite by our above remark. Now in [13] we show that if $\bar{z}_1$ is a solution of an algebraic formula it is in fact algebraic in the field theoretic sense. This proves the lemma. ☒

Theorem 2.20

Suppose that R is an integral domain and π is a finite correspondence of the constructible R-sets $\mathcal{V}_1$ and $\mathcal{V}_2$. If all points $\bar{x}_1 1 \neq \bar{z}_1 1$ of maximal dimension in $\mathcal{V}_1$ have disjoint images under the correspondence π then π is one-many except on a constructible subset of $\mathcal{V}_1$ of dimension < $\dim \mathcal{V}_1$.

Proof

Suppose that $\bar{x}_i$ is a point of $\mathcal{V}_1$ of maximal dimension. If

$\bar{y}_i$ is a point of $\mathcal{Y}_1$ such that for some $\bar{z}_j$ we have $\Pi(\bar{x}_i, \bar{z}_j)$ and $\Pi(\bar{y}_i, \bar{z}_j)$ then the above lemma implies that $\bar{x}_i$ is algebraic over $R[y_i,\ldots,y_n]$. Hence the transcendence degree of $R[y_1,\ldots, y_n]$ over R is $\geq$ dim $\bar{x}_i$ = dim $\mathcal{Y}_1$. Since $\bar{y}_i \in \mathcal{Y}_1$ this implies that dim y_i = dim $\mathcal{Y}_1$. But then by assumption the images of $\bar{x}_i$ and $\bar{y}_i$ are disjoint. Contradiction unless $\bar{y}_i = \bar{x}_i$. Thus Π is one-many on points of $\mathcal{Y}_1$ of maximal dimension. Let X be the set of all points of $\mathcal{Y}_1(\Omega^n)$ on which Π is not on-many. Consider the following formula:

$$(*)\quad \Pi(\bar{v}_i\bar{v}_j) \wedge \Pi(\bar{v}_i{}^1\bar{v}_j) \wedge (v_1 \neq v_1{}^1 \vee,\ldots,\vee\, v_n \neq v_n{}^1).$$

Clearly points $\bar{x}_i$ and $\bar{z}_i$ on $\mathcal{Y}_1$ appear in a solution $\bar{x}_i \frown \bar{z}_i \frown \bar{y}_j$ iff both $\bar{x}_i$ and $\bar{z}_i$ are in X. Hence if we let $\theta(v_i)$ be an open formula which defines the projection of (*) on the coordinate $\bar{v}_i$ then $\theta \wedge \mathcal{Y}_1$ defines the set X. Hence X is constructible. Now by our above remarks no point of dimension = dim $\mathcal{Y}_1$ is in X hence $\theta \wedge \mathcal{Y}_1$ is of dimension < dim $\mathcal{Y}_1$. ⊠

Suppose R is an integral domain and is a universal domain extending R. Then constructible sets $\mathcal{Y}_1$ and $\mathcal{Y}_2$ defined over R are said to <u>correspond</u> if there exists a finite correspondence between them which is defined over R. We write in that case $\mathcal{Y}_1 \equiv_R \mathcal{Y}_2$. The following lemma shows that the relation $_R$ is an equivalence relation.

Lemma 2.21

Suppose that R is an integral domain and $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ are constructible R-sets. Then $\mathcal{V}_1 \Xi_R \mathcal{V}_2$ and $\mathcal{V}_2 \Xi_R \mathcal{V}_3$ implies that $\mathcal{V}_1 \Xi_R \mathcal{V}_3$.

Proof

Let $\Pi_1(\bar{v}_i, \bar{v}_j)$ be a finite correspondence between $\mathcal{V}_1$ and $\mathcal{V}_2$ and let $\Pi_2(v_j, v_s)$ be a finite correspondence between $\mathcal{V}_2$ and $\mathcal{V}_3$. Consider the formula: $(*) = \Pi_1(\bar{v}_i \bar{v}_j) \wedge \mathcal{V}_2(\bar{v}_j, \bar{v}_s)$. Let $\bar{x}_i$ be a point of $\mathcal{V}_1$. Then $\bar{x}_i$ has a finite nonempty solution set Y in $\mathcal{V}_2$ under Π_1 and Π_2 maps Y to a finite nonempty set Y_1 in $\mathcal{V}_3$. Hence $\Pi_1(\bar{x}_i \bar{v}_j) \wedge \Pi_2(\bar{v}_j, \bar{v}_s)$ has a finite nonempty solution set. Let $\theta(v_i, v_s)$ define the projection of the formula (*) on the coordinates $\bar{v}_i$ and $\bar{v}_s$. Then our above remarks show that $\theta(v_i v_s)$ has a finite nonempty image set in $\mathcal{V}_3$ for each point of $\mathcal{V}_1$. A similar argument can be applied to preimages of points on $\mathcal{V}_3$ and so $\theta(v_i, v_s)$ defines a finite correspondence of $\mathcal{V}_1$ and $\mathcal{V}_3$. This proves the lemma. ☒

It is easy to see now that Ξ_R is an equivalence relation for the formula $\mathcal{V}_1(v_i) \wedge \mathcal{V}_1(\bar{v}_{n+i}) \wedge v_1 = v_{n+1}, \wedge \ldots, \wedge v_n = v_{n+m}$ defines the identity correspondence of $\mathcal{V}_1$ with itself. We note that this equivalence relation is considerably weaker than isomorphism of constructible sets. But it is not clear how the three equivalence relations defined on varieties or constructible sets by isomorphism, bi-rational isomorphism and finite

correspondence are related with regard to geometric properties. And so it is worthwhile to state the following problem.

Problem 3

What geometric properties of constructible sets are invariant under the relation $\equiv_R$? In particular what properties of varieties over fields are invariant under this relation?

Definition 2.10

Let R be an integral domain and R_1 a domain extending R in the universal domain Ω. Suppose that π is a finite correspondence of constructible R-sets Ψ_1 and Ψ_2. Then is complete over R_1 if for each $\bar{x}_i$ in $\Psi_1(R_1^n)$ we have $\pi(x_i\Omega^n) \subseteq \Psi_2(R_1^m)$ and similarly for preimages of points in $\Psi_2(R_1^m)$.

Definition 2.11

Suppose that R is an integral domain and R_1 is a domain extending R. Let Ψ be a degree 1 constructible set defined over R. Then $\lambda(\Psi, R_1)$ is the least upper bound of the cardinalities of maximal independent subsets of $\Psi(R_1^m)$.

Theorem 2.21

Suppose R is an integral domain and R_1 is a domain extending R in a universal domain Ω. Let π be a finite disjoint correspondence of the

constructible R-sets Ψ_1 and Ψ_2 such that Π is defined over R and complete over R_1. If both Ψ_1 and Ψ_2 have degree 1 then $\lambda(\Psi_1, R_1) = \lambda(\Psi_2, R_1)$.

Proof. First note that lemma 2.20 implies that Ψ_1 and Ψ_2 have the same dimension. Let X be an independent subset of $\Psi_1(R_1^n)$. For each $\bar{x}_i^{\alpha}$ (in some well ordering of X) choose a point $\bar{y}_j^{\alpha}$ in the image set $\Pi(\bar{x}_i^{\alpha}\bar{v}_j)$ in $\Psi_2(R_1^m)$. Such a point exists since Π is finite and complete over R_1. Note that disjointness implies our correspondence $\bar{x}_i^{\alpha} \dashrightarrow \bar{y}_i^{\alpha}$ is one-one. Suppose that the set Y of all such elements $\bar{y}_j^{\alpha}$ is dependent. Then there are elements $\bar{y}_j^{\alpha}{}^1, \ldots, \bar{y}_j^{\alpha}{}^r$ such that $\bar{y}_j^{\alpha}{}^1, \ldots, \bar{y}_j^{\alpha}{}^{r-1}$ is independent but $\bar{y}_j^{\alpha}{}^r$ is dependent on this latter set. Say $\bar{y}_j^{\alpha}{}^r = \langle y_1, \ldots, y_s, y_{s+1}, \ldots, y_{s+k}, \ldots, y_m \rangle$ where $y_1, \ldots, y_s$ are algebraic over $\bar{y}_j^{\alpha}{}^1, \ldots, \bar{y}_j^{\alpha}{}^{r-1}$, the coordinates $y_{s+1}, \ldots, y_{s+k}$ are algebraically independent over this set and the final coordinates are algebraic over the first s+k coordinate. Now we similarly write $\bar{x}_i^{\alpha}{}^r = \langle x_1, \ldots, x_t, x_{t+1}, \ldots, x_{t+p}, \ldots, x_n \rangle$ where $p = \dim \Psi_1 = \dim \Psi_2$ and the elements $x_{t+1}, \ldots, x_{t+p}$ are algebraically independent over $y_j^{\alpha}{}^1, \ldots, y^{\alpha}{}^{r-1}$. There clearly must be p such coordinates of $x_i^{\alpha}{}^r$ free over $y_j^{\alpha}{}^1, \ldots, y_j^{\alpha}{}^{r-1}$ since the latter set is algebraic over $x_i^{\alpha}{}^1, \ldots, x_1^{\alpha}{}^{r-1}$ and $x_i^{\alpha}{}^1, \ldots, x_i^{\alpha}{}^r$ is independent (noting that algebraic relations are transitive so that if there were fewer than p such coordinates we would have a contradiction of independence). Now note that $k<p$. This is because $\dim \Psi_2 = \dim \Psi_1$ so $k \leq p$ in any case and if $k = p$ then $y_j^{\alpha}{}^r$ would not be

dependent on $y_j^{\alpha_1},\ldots,y_j^{\alpha_{r-1}}$. The sequence $x_i^{\alpha_r}$ is independent over the latter set but algebraic over $y_i^{\alpha_r}$ so we can obtain an immediate contradiction by considering the extension of quotient field of R by the $y_j^{\alpha_i}$. ⊠

5. The Product and Splitting Properties of Constructible Sets

5.0

This section begins with the definition of the product of two constructible sets and the proof of two basic and related properties of the product. We show first that if ψ_1 has degree 1 and ψ_2 is minimal then the product is minimal. This generalizes the corresponding product theorem for varieites. The proof makes use of the amalgamation property. The second product theorem tells us that the variety associated with the product of two constructible sets is the product variety. This allows us to deduce the first product theorem from the theorem for varieties. Finally we give a new and very simple proof using techniques of logic that an irreducible variety over an algebraically closed field is absolutely irreducible.

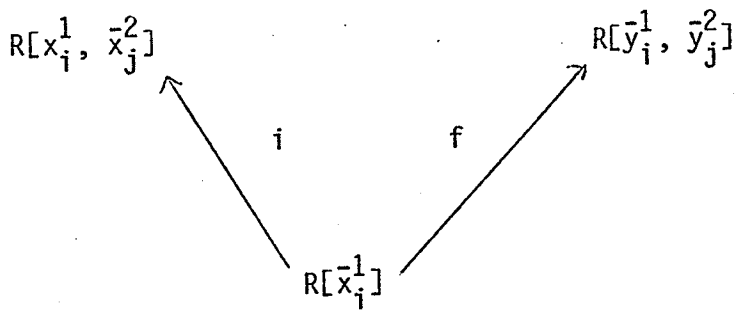
5.1 We begin this section of the chapter with some basic results on the product of constructible sets. If ψ_1 and ψ_2 are open R-formulas over an integral domain R define $\psi_1 \times \psi_2$ to be the set $\psi_1(\vec{v}_i) \wedge \psi_2(\vec{v}_j)$ where the free variables in the two formulas in the latter conjunction form disjoint sets.

Lemma 2.21. Suppose R is an integral domain and $\psi_1(\vec{v}_i)$ and $\psi_2(\vec{v}_j)$ are open formulas over R. Then $\dim \psi_1 \times \psi_2 = \dim \psi_1 + \dim \psi_2$.

Proof. Let (x) and (y) be free points of maximal dimension of Ψ_1 and Ψ_2 respectively. Then dim(x)$^\frown$(y) = dim(x) + dim(y) and so dim Ψ_1 x $\Psi_2 \geq$ dim Ψ_1 + dim Ψ_2. As it cannot be greater the lemma is proved. ☒

Theorem 2.22. Suppose that R is an integral domain and $\Psi_1(v_i)$ and $\Psi_2(v_j)$ are open formulas over R such that Ψ_2 has degree 1 and Ψ_1 is R-minimal. Then Ψ_1 x Ψ_2 is R-minimal.

Proof. Suppose that $\bar{x}_i$ and $\bar{y}_i$ are solutions of $\Psi_1 x \Psi_2$ having dimension equal to dim Ψ_1 x Ψ_2 in a universal domain Ω extending F. Say $\bar{x}_i = \langle x_1^1,\ldots,x_r^1,x_1^2,\ldots,x_m^2\rangle$ and $\bar{y}_i = \langle y_1^1,\ldots,y_r^1, y_1^2,\ldots,y_m^2\rangle$ where Ψ_1 has r variables and Ψ_2 has m variables. We must have dim $\langle x_1^1,\ldots,x_r^1\rangle$ = dim Ψ_1 for dim (x_1^1, x_j^2) = dim Ψ_1 x Ψ_2 = dim Ψ_1 + dim Ψ_2 by the above lemma. Hence if dim $\bar{x}_i^1$ < dim Ψ_1 then dim $\bar{x}_j^2$ > dim Ψ_2 by the above lemma. Hence if dim $\bar{x}_i^1$ < dim Ψ_1 then dim x_j^2 > dim Ψ_2. Contradiction. Similarly we have dim $\bar{y}_i^1$ = dim Ψ_1 so it follows that $\bar{x}_i^1$ and $\bar{y}_i^1$ are both independent points of Ψ_1. Next note that $\bar{x}_j^2$ and $\bar{x}_i^1$ are free over R, for if not then $\bar{x}_j^2$ has transcendence degree greater over R than over $R[x_1^1,\ldots,x_r^1]$ = R_1 and so we have $\dim_{R_1} \bar{x}_j^2 < \dim \Psi_2$. But then dim Ψ_1 x Ψ_2 = dim Ψ_1 + dim Ψ_2 = dim $\bar{x}_1^1$ + dim Ψ_2 > dim$(\bar{x}_i^1, \bar{x}_j^2)$. Contradiction. Similarly $\bar{y}_i^1$ and $\bar{y}_j^2$ are free over R. Now consider the following diagram:



where i is the identity map and f is an embedding mapping R onto R identically and $f(x_i{}^1) = y_i{}^1$ for each $i \leq r$. Such a map exists by our remarks (noting that independent points of an R-minimal set have the same open type over R). Let $\mathcal{J}$ with maps $g_2: R[\tilde{x}_i{}^1, \tilde{x}_j{}^2] \dashrightarrow \mathcal{J}$ (with g_2 equal and identity) and $g_1: R[\bar{y}_i{}^1, \bar{y}_j{}^2] \dashrightarrow \mathcal{J}$ amalgamate this diagram over $R[\tilde{x}_i{}^1]$. Then $(g_1(\bar{y}_i{}^1), g_1(\bar{y}_j{}^2))$ is a solution of $\Psi_1 \times \Psi_2$ with $g_1(\bar{y}_j{}^2)$ an independent point of Ψ_2 over $R[g_1(\bar{y}_i{}^1)] = R[\tilde{x}_i{}^1]$. This follows from the commutativity of the diagram and the fact that Ψ_2 has degree 1 hence does not split over $R[\tilde{x}_i{}^1]$, for as noted above $\bar{x}_j{}^2$ is free over R_1 and $\dim \bar{x}_j{}^2 = \dim \Psi_2$. Since $x_j{}^2$ is an independent point of Ψ_2 over R_1 there is an isomorphism g over R_1 with $g(x_j{}^2) = g_1(y_j)$ for $j \leq m$. Then we obtain the following sequence of maps:

$$R[\tilde{x}_i{}^1, \tilde{x}_j{}^2] \overset{g}{\dashrightarrow} R[\tilde{x}_i{}^1, g_1(\bar{y}_j{}^2)] \overset{g^{-1}}{\dashrightarrow} R[\bar{y}_i{}^1, \bar{y}_j{}^2]$$

so that $g_1^{-1}g$ is an R-isomorphism of $R[\vec{x}_i^1, \vec{x}_j^2]$ onto $R[\vec{y}_i^1, \vec{y}_j^2]$ mapping x_i^1 to y_i^1 for $i \leq r$ and x_j^2 to y_j^2 for $j \leq m$. Hence (x) and (y) have the same open type over R and so $\psi_1 \times \psi_2$ does not split over R. ☒

Theorem 2.23. Suppose that F is a field and $\psi_1(v_i)$ and $\psi_2(v_j)$ are open F-minimal formulas. Let V_1 and V_2 be the associated varieties. Then $V_1 \times V_2$ is the variety associated with $\psi_1 \times \psi_2$.

Proof. Let $W_1,\ldots,W_n$ be the components of $V_1 \times V_2$ over F. Then each W_i has dimension r+s where $\dim \psi_1 = r$ and $\dim \psi_2 = s$. Let $S_i = W_i \cap \psi_1 \times \psi_2$. We claim that S_i has dimension equal to r+s. For let $x_i = \langle x_i^1,\ldots,x_m^1, x_1^2,\ldots,s_x^2\rangle$, where ψ_1 has m variables and ψ_2 had k variables, be a generic point of a given component W_p. Then we can argue as in the proof of the previous theorem to show that $\dim \langle x_1^1,\ldots,x_m^1\rangle = r$ and $\dim \langle x_1^2,\ldots, x_k^2\rangle = s$ so that $\vec{x}_i^1$ is a generic point of V_1 and $\vec{x}_j^2$ is a generic point of V_2. Hence $\vec{x}_i^1$ is a solution of ψ_1 and $\vec{x}_j^2$ is a solution of ψ_2 (by definition: see [4]) so that (x_i^1, x_j^2) is a solution of $\psi_1 \times \psi_2$. This proves our claim. Now let θ_i be an open formula which defines S_i and let $\psi = \theta_1 \vee,\ldots, \vee\theta_n$. Then ψ is an open formula over F with $\psi \equiv \psi_1 \times \psi_2$. We claim that $\psi_1 \times \psi_2 \wedge \neg\psi$ has dimension $< \dim \psi_1 \times \psi_2$. For if not then some point $\vec{x}_i$ of this set has dimension equal r+s. Say $\vec{x}_i = \langle x_1^1,\ldots, x_m^1, x_1^2,\ldots, x_k^2\rangle$. Then (as in the proof of the previous theorem) we can show that $\dim \vec{x}_i^1 = r$ and $\dim \vec{x}_i^2 = s$ so $\vec{x}_i^1$ is a generic point of V_1 and $\vec{x}_j^2$ is a generic point of V_2 and

$\vec{x}_1$ is a point of $V_1 \times V_2$ of dimension $r + s$. Hence $\vec{x}_1$ is a generic point of a component of $V_1 \times V_2$. Contradiction. Arguing in the other direction we can show that the open formula which defines $W_i - S_i$ has dimension $< r+s$ so that $V_1 \times V_2$ differs from be a set of smaller dimension than $r+s$. Also $\mathcal{Y} \subseteq V_1 \times V_2$ so $V_1 \times V_2$ is the variety associated with $\mathcal{Y}$. As we showed above that $\mathcal{Y}$ differs from $\mathcal{Y}_1 \times \mathcal{Y}_2$ by a set of smaller dimension and $\mathcal{Y} \subseteq \mathcal{Y}_1 \times \mathcal{Y}_2$ it follows that $V_1 \times V_2$ is the variety associated with $\mathcal{Y}_1 \times \mathcal{Y}_2$. ☒

Corollary 2.7. Suppose F is a field and $\mathcal{Y}_1(v_i^1)$ and $\mathcal{Y}_2(v_j^1)$ are open F-minimal formulas. Then degree $\mathcal{Y}_1 \times \mathcal{Y}_2$ = degree $\mathcal{Y}_1$ · degree $\mathcal{Y}_2$.

Proof. This follows from the above theorem and the corresponding fact for varieties. ☒

Theorem 2.24. (9) Suppose that V is an irreducible variety over an algebraically closed field F. Then V is absolutely irreducible.

Proof. Let $f_1(x_i,v_0),\ldots,f_r(x_i,v_j)$ be a defining set of polynomials for the variety V where $\vec{x}_i$ is the sequence of all parameters from F which appear in any of the polynomials. If V is not absolutely irreducible then there is some algebraically closed extension F_1 of F such that V splits over F_1 into varieties V_1 and V_2. Say $p_1(z,v_j),\ldots,P_m(z,v_j)$ is a defining set polynomials for V_1 and $F_1(z,v_j),\ldots,F_k(z,v_j)$ is a defining set of polynomials for V_2.

Then the following sentence is true in F_1:

$$(*)\ \exists z(\forall v_j\ (p(\vec{z},\vec{v}_j) = 0 \wedge,\ldots,\wedge p_m(\vec{z}\vec{v}_j) = 0 \vee$$

$$F_1(\vec{z},\vec{v}_j) = 0 \wedge,\ldots,\wedge F_k(\vec{z},\vec{v}_j) = 0 \leftrightarrow f_1(\vec{x}_i,\vec{v}_j) = 0 \wedge,\ldots,f_r(\vec{x}_i,\vec{v}_j) = 0)$$

$$\wedge \exists v_j(p_1(\vec{z},\vec{v}_j) = 0 \wedge,\ldots,\wedge p_m(\vec{z},\vec{v}_j) = 0 \wedge \neg(F_1(\vec{z},\vec{v}_j) = 0 \wedge,\ldots,\wedge$$

$$F_k(\vec{z},\vec{v}_j) = 0)) \wedge \exists\ v_j(F_1(\vec{z},\vec{v}_j) = 0 \wedge,\ldots,\wedge F_k(\vec{z}\vec{v}_j) = 0 \wedge \neg(p_1(\vec{z},\vec{v}_j) = 0$$

$$,\ldots,\wedge p_m(\vec{z},\vec{v}_j) = 0))).$$

Now $F_1 \geq F$ so $F_1 > F$ as F_1 and F are algebraically closed. Thus (*) holds in F. This tells us that there are varieties $\vec{V}_1$ and $\vec{V}_2$ both nonempty such that $\vec{V}_1 \vee \vec{V}_2 = V$ in the space F^n. But as F is algebraically closed the Nullstellensatz tells us that therefore this relation holds in the affine space $\mathfrak{A}^n$ over F for the universal domain $\mathfrak{U} \geq F$. But then V splits over F. Contradiction. ⊠

6. Some Definability Properties of the Stone Space

6.0

In this section we prove two definability results concerning the stone space. The first gives a topological and dimension criteria for a type over a field to be defined over a subfield. The proof is an application of the trace property (Theorem 2.2). The second and final

result of the section tells us that the homogeneous types in the stone space over a field constitute a closed subspace. The proof is an application of the compactness of the Stone topology.

The results of this section and the next continue the study of the spectral space with the Stone topology that was begun in the first section of this chapter. The results of this section cast some light on the question of the linkage of the Stone topology to the geometry of the spectral space of a field.

6.1 The Zariski topology is defined in terms of polynomials and so is linked to ideal theoretic properties of the space. In contrast the Stone topology is defined through formulas and can thus be expected to be more closely related to field theoretic definability properties of the underlying space. The following theorems provide some evidence in this direction.

If F_1 is a field a point $\Gamma \in S_A^n(F)$ is said to be <u>defined</u> over $F \leq F_1$ if the variety of Γ has F as a field of definition.

<u>Theorem 2.25</u>

Suppose that F is an algebraically closed field and F_1 is an extension of F. Let Γ be a point of dimension m+1 in $S_A^n(F_1)$ and suppose that Γ is an accumulation point of a set S of points of dimension m in $S_A^n(F)$ defined over F. If V is the variety of Γ over F_1 and if $V \cap F^n$ has dimension m+1 then Γ is defined over F.

Proof

We can assume that each point of S is contained in V. If $\mathfrak{Y} = V \cap F^n$ then $\mathfrak{Y}$ is definable by an open F-formula as follows by Theorem 2.2. For each point Γ_1 in S the variety V_1 of Γ_1 is defined over F and so the restriction $\bar{\Gamma}_1$ of Γ_1 to F has dimension m. Let S_1 be the set of all restrictions of points of S to F. Then the set $S_1 \cap \mathfrak{Y}$ is infinite so it has an accumulation point Γ_0 in $S_A^n(F)$. Since $\mathfrak{Y}$ is closed it follows that $\Gamma_0 \in \mathfrak{Y}$ and since $\dim \mathfrak{Y} = m+1$ the type Γ_0 must also have dimension m+1. The field F is algebraically closed and so the locus V_0 of Γ_0 over F is absolutely irreducible. It follows that Γ_0 has a unique extension $\bar{\Gamma}_0$ of dimension m+1 in $S_A^n(F_1)$. The variety V is closed and clearly the point $\bar{\Gamma}_0$ is an accumulation point of S. For the field F is algebraically closed and thus it follows by the Nullstellensatz that the varieties of an infinite subset of S_1 are subvarieties of V_0. Therefore V_0 contains an infinite subset of S and as each point of S is of dimension m and V_0 is irreducible the points of S must have $\bar{\Gamma}_0$ as a limit point in V_0. It follows that $\bar{\Gamma}_0$ is a point of V and since V is irreducible we must have $\bar{\Gamma}_0 = \Gamma$. ⊠

If F is a field then $\Gamma \in S_A^n(F)$ is a homogeneous type if a solution (x) of Γ in a universal domain extending F is a homogeneous point.

Theorem 2.26

Suppose that F is a field. Then the set of all homogeneous types in $S_A^n(F)$ is a closed subspace.

Proof

Suppose that Γ is a point in the closure of the set of all homogeneous types. Let S be the set of all finite subsets of irreducible polynomials in $F[x_0,\ldots,x_n]$. If $\psi \in \Gamma$ then given $s \in S$ we define $\psi_s = \psi(v_0v_1,\ldots,v_0v_n) \wedge \psi(v_1,\ldots,v_n) \wedge \bigwedge_{s_i} \neg f_{s_i}(v_jv_0) = 0$ where s_i varies over all the elements of S. Define Γ_ψ to be the set of all formulas ψ_s for $s \in S$. Let $X = \cup \Gamma_{\psi_s}$. Then X has the finite intersection property in the space Ω^{n+1} defined by a universal domain Ω over F. For let $Y \subseteq X$ be a finite subset. There is then a set $\{\psi_1,\ldots,\psi_m\} \subseteq \Gamma$ such that all formulas which occur in Y are in some Γ_{ψ_i}. Let $\psi = \wedge\psi_i$; then $\psi \in \Gamma$. Let s be the union of all the finite subsets s' of S such that $\psi_{is'} \in Y$ for some $1 \leq i \leq m$. Now as Γ is the limit of homogeneous points there is $\Gamma_1 \in \psi$ with Γ_1 homogeneous. Say (x) is a solution of Γ_1 in Ω^n. Let t be algebraically independent of (x) over F in Ω. Then by homogeneity the point (tx) is in the variety V of (x) over F. Hence (tx) is a generic point of V so has the same type Γ_1 over F as the point (x). So (tx) is a solution of ψ. Thus (t,x) will be a solution of ψ_s. So then (t,x) satisfies all the formulas in Y. This proves that X has the finite intersection property. By compactness it follows that X has a solution (t_1,z) in Ω. Then by definition of X we have (t_1z) a solution of Γ. Also t_1 is algebraically independent of (z) over F. This implies that the ideal I of (z) over F is homogeneous. For let $f \in I$. Then $f(z) = 0$ so $f = 0$ since by construction (z) is a solution of Γ and Γ is maximal. As $f = 0 \in \Gamma$ we have $f(t_1z) = 0$. Writing

$$f(t_1z) = \sum t_1^i g_i(z) = 0$$

we deduce $g_i(z) = 0$ for each i. But the g_i are the homogeneous terms of f and $g_i(z) = 0$ implies that $g_i \in I$. Hence I is homogeneous. So Γ is a homogeneous point of $S_A^n(F)$. ☒

Theorems 2.25 and 2.26 of this subsection were inspired by a geometrical problem which we call the stability problem for algebraic manifolds. In complex projective space the stability question first arose as an intuitive principle known as the Poncelet continuity principle. It would appear from examples of physical phenomenon and from analogous work in differential equations that this principle is certainly false. A more conservative local version was utilized by the Italian school. See for example the proof of Bertini's Theorem in [36]. We can formulate the following two specific problems.

Problem 4

For what properties P of a projective variety V is it true that there is a neighborhood of the parameters of V in a suitable projective complex parameter space such that P is valid for all varieties defined by points of the neighborhood?

Problem 5

Given a complex projective parameter space and a net of points in that space converging to a point P what properties of the corresponding varieties are stable at P?

We take the opportunity now to mention another problem in algebraic geometry related to the Stone space. If R is a commutative ring the Stone topology on Spec(R) is the topology generated by the least Boolean

algebra containing all the Zariski open sets.

Problem 6

Suppose that R is a commutative ring and Spec(R) is endowed with the Stone topology. Is Spec(R) compact if R is Noetherian?

7. The Linear Topological Structure of the Stone Space

7.0

The section begins with the definition of the monoid operation on the Stone space of an algebraically closed field. Two proofs are given that this operation is well defined and abelian and that it is an extension of the additive group structure of the n-dimensional vector space over the groundfield. The first proof uses properties of the tensor product while the second is direct and uses types and logic.

We next give proofs of two fundamental properties of the monoid operation. The first tells us that the dimension of the sum of two types of the same dimension is greater than or equal this dimension. The second tells us that relative to the topology induced on $S^n(F)$ x $S^n(F)$ by an embedding into $S_A{}^{2n}(F)$ the monoid operation is continuous.

After introducing the monoid operation and describing its basic properties we show that scalar multiplication also extends from the vector space over the groundfield to the stone space over that field. Thus the linear structure of the vector space lifts to the Stone space. We prove the basic algebraic properties of scalar multiplication in a

lemma and then prove the dimension and continuity theorems in the following two results.

The third subsection is devoted to the presentation of two results concerning maps between Stone spaces over fields. The first says that regular maps between affine spaces lift to unique open and continuous maps of the respective Stone spaces. In particular projections lift to continuous linear monoid homomorphisms. The second theorem shows that the map induced by a field extension is a continuous linear monoid homorphism of the stone space of the extension onto the stone space of the groundfield.

Having shown that regular maps in general lift to the Stone space it is natural to study the particular case of linear maps. But first we give two results concerning linear subspaces. The first shows that linear polynomials define linear subspaces. The second result is a partial converse of this linear subspace theorem.

We prove next that linear maps on affine spaces lift to linear maps on the Stone spaces. To obtain the converse we must add the conditions that the map be open and continuous. As a corollary to this linear map theorem we prove that the image of a linear variety under an open continuous linear map of the Stone space is a linear variety.

The final results of the section are concerned with some detailed properties of the monoid operation. We prove first that the sum of a type with the type of the generic point of affine space is the type of the generic point. After giving two examples we show that there are subgroups of the Stone space not contained in the affine space. We next define the annihilator of a point of the Stone space and prove that the

annihilator is always a closed submonoid. The proof uses continuity of the monoid operation. The final result of the section shows if X is a set of types defined over an algebraically closed subfield then the monoid generated by X is defined over the subfield.

7.1 In this subsection we begin to apply the language and methods developed in previous sections to the study of the intrinsic geometry of the Stone space. The Stone space has a natural linear structure extending the vector space over an algebraically closed ground field and this structure is linked to the Stone topology. For other examples in geometry of algebras growing out of geometrical structures the reader can consult [32] and [19].

Definition 12

Let F be an algebraically closed field. Suppose Γ_1 and Γ_2 are open n-types over F. We define $\Gamma_1 + \Gamma_2$ as follows: Let (x) and (y) be algebraically free sequences of type Γ_1 and Γ_2 respectively in the universal domain $\mathfrak{U}$ extending F. Then $\Gamma_1 + \Gamma_2$ is defined to be the type Γ_3 of the sequence (x) + (y) in the space $\mathfrak{U}^n$.

Theorem 2.27

Suppose that F is an algebraically closed field. Then the operation + is well defined on $S_A^n(F)$ and the latter is an abelian monoid extending the group F^n.

Proof. If (x) and (y) are free over F then (x) and (y) are linearly disjoint sequences (that is Fx and Fy are linearly disjoint over F) since both (x) and (y) are regular points (since F is algebraically closed). Thus the compositum F(x)F(y) in Ω contains the tensor product F(x) $\otimes$ F(y) (or an isomorphic copy of it under the canonical embedding which maps (x) to (x) and (y) to (y)). Now the latter depends only on the type of (x) and (y) and thus the type of the coordinates sum of (x) and (y) is invariant of our choice of free realizations of the types Γ_1 and Γ_2 of (x) and (y). Hence $\Gamma_1 + \Gamma_2$ is well defined. This operation clearly coincides with the group operation of F^n. Furthermore if Γ is the type of the origin then by definition if Γ_1 $S_A^n(F)$ we have $\Gamma+\Gamma_1 = \Gamma_1$. From the associative law of tensor products (as proved for example in [47]) our operation is associative and it is commutative from both the definition and the corresponding property of the tensor product. Hence with this operation $S_A^n(F)$ is an abelian monoid extending the group F^n. ☒

We give a second proof which avoids the use of tensor products. Since F is algebraically closed a sequence (x) in Ω^n is regular over F. Let (x) and (y) be free realizations of the types Γ_1 and Γ_2 respectively. Then (x) and (y) are linearly disjoint over F. Hence (x,y) is a generic point of the product variety $V_1 \times V_2$ where V_1 is the locus of (x) over F and (y) is a generic point of V_2. This product is irreducible over F and so the type Γ of (x,y) is independent of the choice of free realizations of Γ_1 and Γ_2, for all realizations of Γ_1 are generic points of V_1 and similarly all realizations of Γ_2 are generic for V_2. The type Γ determines the type of (x) + (y). For

given any open formula ψ over F in n variables we have $\psi(V_1^1 + V_1^2, \ldots, V_n^1 + V_n^2)$ or its negation in Γ. For each such formula call $\hat{\psi}$ the formula or its negation according to the above prescription. We form a new formula θ from by substituting the variable V_i for the term $V_i^1 + V_i^2$. Let Γ_0 be the set of all formulas θ obtained in this way. Then clearly Γ_0 is the type of (x) + (y). Be definition Γ_0 is equal to the sum $\Gamma_1 + \Gamma_2$ which is thus well defined. The other assertions of the theorem can be obtained in a similar fashion.

Theorem 2.28

Assume that F is albegraically closed. If $\Gamma_1, \Gamma_2 \in S_A^n(F)$ and $\dim \Gamma_1 = \dim \Gamma_2 = r$ then $\dim(\Gamma_1 + \Gamma_2) \geq r$.

Proof

Let (x) and (y) be algebraically free realizations of Γ_1 and Γ_2 respectively. Say $x_{\alpha_1}, \ldots, x_{\alpha_r}$ is a set of algebraically independent coordinates of (x) and $y_{\beta_1}, \ldots, y_{\beta_r}$ is a similar set of coordinates of (y). Let $\lambda_1, \ldots, \lambda_s$ be the coordinates which agree in the above two sets. We claim first that $x_{\lambda_1} + y_{\lambda_1}, \ldots, x_{\lambda_s} + y_{\lambda_s}$ are algebraically independent over F. We prove this by induction. Suppose $x_{\lambda_1} + y_{\lambda_1}$ is algebraic over F. Then $X_{\lambda_1} + Y_{\lambda_1} \in F$. Hence $X_{\lambda_1} \in F(Y_{\lambda_1})$ and so x_{λ_1} is algebraic over this field. But (x) and (y) are algebraically free over F and x_1 is transcendental over F. Contradiction. The remainder of the argument is similar. Now if α_j is a coordinate not among the set $\lambda_1, \ldots, \lambda_s$ then by our definition of the latter set the point y_{α_j} is algebraic over $y_{\beta_1}, \ldots, y_{\beta_r}$. We claim that $x_{\alpha_j} + y_{\alpha_j}$ is transcendental over $F(x_{\lambda_1} + y_{\lambda_1}, \ldots, x_{\lambda_s} +$

y_{λ_S}). If not then it is algebraic over $F(x_{\lambda_1}+ y_{\lambda_1},\ldots,x_{\lambda_S} + y_{\lambda_S}$ $y_{\beta_1},\ldots,y_{\beta_r})$. But y_{α_j} is algebraic over the latter field and this implies x_{α_j} is also. But again (x) and (y) are algebraically free over F so that the set $x_{\lambda_1},\ldots,x_{\lambda_S}$, $y_{\beta_1},\ldots,y_{\beta_r}$, x_{α_j} is algebraically free over F. Contradiction. We can proceed in this way by choosing α_j to be the least coordinte not in the set $\lambda_1,\ldots,\lambda_S$, choosing the next coordinate not in $\lambda_1,\ldots,\lambda_S,\alpha_j$ and so on. We then apply the same argument to the set $y_{\beta_1},\ldots,y_{\beta_r}$. We finally arrive at a set of r algebraically independent coordintes of (x) + (y). Hence dim Γ_1 + $\Gamma_2 \geq r$. ⊠

Suppose that F is algebraically closed. There is a natural embedding of the space $S_A^n(F)$ x $S_A^n(F)$ in the space $S_A^{2n}(F)$ which is defined as follows. Let (x) and (y) be algebraically free realizations of Γ_1 and Γ_2 where $(\Gamma_1,\Gamma_2) \in S_A^n(F)$ x $S_A^n(F)$. Since F is algebraically closed it follows that F(x) and F(y) are linearly disjoint over F. If V_1 is the locus of (x) over F and V_2 the locus of (y) over F then V_1 x V_2 is irreducible over F and (x,y) is a generic point. A proof of this can be found in [1] page 83. Let Γ_3 be the type of (x,y). Then Γ_3 is invariant of our choice of free realizations of Γ_1 and Γ_2. We identify (Γ_1,Γ_2) with the point Γ_3 and thus embed $S_A^n(F)$ x $S_A^n(F)$ in $S_A^{2n}(F)$. The subspace topology inherited by $S_A^n(F)$ x $S_A^n(F)$ from the space $S_A^{2n}(F)$ we call the <u>open topology</u> on the product space.

<u>Theorem 2.29</u>

Suppose that F is algebraically closed. Then the operation + is a

continuous function from $S_A^n(F)$ x $S_A^n(F)$ to $S_A^n(F)$ where both spaces have the open topology.

Proof

If defines a basic open set in $S_A^n(F)$ we must show that the inverse under + in $S_A^n(F)$ x $S_A^n(F)$ is an open set in the open topology on that space. Consider the following formula:

$\pi = \exists v_1,\ldots,v_n(\psi(v_1,\ldots,v_n) \wedge v_{n+1} + v_{2n+1} = v_1 \wedge,\ldots,\wedge v_{2n} + v_{3n} = v_n)$.

This is an F-formula and the latter is an algebraically closed field so by Tarski's Theorem there is a quantifier free formula $\psi_1(v_{n+1},\ldots,v_{2n},v_{2n+1},\ldots,v_{3n})$ such that $F \models \psi_1 \equiv \pi$. The formula ψ_1 defines a basic open set in $S_A^n(F)$ and so a basic open set of the space $S_A^n(F)$ x $S_A^n(F)$ with the open topology. We claim that the set of solutions of ψ_1 in $S_A^n(F)$ x $S_A^n(F)$ is the inverse of + in this space of the open set ψ. For consider points Γ_1 and Γ_2 in $S_A^n(F)$ such that $\Gamma_1 + \Gamma_2 = \Gamma_3 \in \psi$. Letting (x) and (y) be linearly disjoint sequences in $\mathfrak{N}^n$ of type Γ_1 and Γ_2 respectively we see that the sequence (x) + (y) = (z) in $\mathfrak{N}^n$ is a solution of ψ. Hence the sequence (x,y) is a solution of π so it is a solution of ψ_1 since $\mathfrak{N}$ is an elementary extension of F so that $\mathfrak{N} \models \pi \equiv \psi_1$. By the definition of the embedding of $S_A^n(F)$ x $S_A^n(F)$ in $S_A^n(F)$ we see that (Γ_1,Γ_2) is mapped to the type Γ of the sequence (x,y). Thus (Γ_1,Γ_2) is in the open set in $S_A^n(F)$ x $S_A^n(F)$ defined by ψ_1. Suppose on the other hand that (Γ_1,Γ_2) is in the open set defined by ψ_1. If (x) and (y) are linearly disjoint realizations of Γ_1 and Γ_2 it follows that the sequence (x,y) is a solution of ψ_1.

By definition of π and the fact that $\mathfrak{Q} \models \pi \equiv \psi_1$ it follows that (x) + (y) is a solution of and so by definition of the operation + on $S_A^n(F)$ we have that $\Gamma_1 + \Gamma_2 \in \psi$. This proves the theorem. ⊠

7.2 If F is an algebraically closed field and $\Gamma \in S_A^n(F)$ then we define $\lambda\Gamma$ for $\lambda \in F$ by choosing a realization (x) of Γ in $\mathfrak{Q}^n$ and letting $\lambda\Gamma$ be the type of (λx). This definition is invariant of our choice of realization for let (y) be any other such realization. Then there is an automorphism σ of leaving F fixed such that $\sigma(x) = (y)$. Then $\sigma(\lambda x) = \lambda\sigma(x) = \lambda(y) = (\lambda y)$ so that (λx) and (λy) are conjugate points of $\mathfrak{Q}^n$ over F so that they realize the same open type in $S_A^n(F)$.

Lemma 2.23

Suppose that F is algebraically closed and $\Gamma_1, \Gamma_2 \in S_A^n(F)$. Let λ_1, λ_2 F. Then $\lambda_1(\Gamma_1 + \Gamma_2) = \lambda_1\Gamma_1 + \lambda_1\Gamma_2$ and $(\lambda_1 + \lambda_2)\Gamma_1 = \lambda_1\Gamma_1 + \lambda_2\Gamma_1$. Also $(\lambda_1\lambda_2)\Gamma_1 = \lambda_1(\lambda_2\Gamma_1)$.

Proof

Suppose that (x) and (y) are linearly disjoint realizations of the types Γ_1 and Γ_2 respectively. We claim that the (λx) and (λy) are also linearly disjoint over F. But this is immediate by assumption and the fact that $F(\lambda x) = F(x)$ and $F(\lambda y) = F(y)$ since $\lambda_1 \in F$. Now $\Gamma_1 + \Gamma_2$ is the type of (x) + (y) and $\lambda_1(\Gamma_1 + \Gamma_2)$ is then the type of $\lambda_1[(x) + (y)] = \lambda_1(x) + \lambda_1(y)$. Since $(\lambda_1 x)$ and $(\lambda_1 y)$ are

linearly disjoint over F it follows that $\lambda_1\Gamma_1 + \lambda_2\Gamma_2$ is the type of $\lambda_1(x) + \lambda_1(y)$. Hence $\lambda_1(\Gamma_1 + \Gamma_2) = \lambda_1\Gamma_1 + \lambda_1\Gamma_2$. The other assertions follow similarly. ☒

It follows from the above lemma that if F is algebraically closed the space $S_A^n(F)$ is an F-linear monoid.

Theorem 2.30

Let F be algebraically closed. If $\Gamma \in S_A^n(F)$ has dimension r then for any point $\lambda \in F$ the type $\lambda\Gamma$ has dimension r.

Proof

Let $\bar{x}_i$ be a realization of Γ in $\mathfrak{U}^n$. Say $x_1,\ldots,x_r$ are algebraically independent coordinates of $\bar{x}_i$ over F. Suppose $\lambda \in F$ and that $\sum a_\alpha M_\alpha(\lambda\bar{x}_j) = 0$ is an algebraic relation on $\lambda x_1,\ldots,\lambda x_r$. Let d_α be the degree of M_α. Then $\sum a_\alpha M_\alpha(\lambda\bar{x}_j) = \sum a_\alpha \lambda^{d_\alpha} M_\alpha(\bar{x}_j)$. It follows from the relation $\sum(a_\alpha \lambda^{d_\alpha})M_\alpha(\bar{x}_j) = 0$ that $a\lambda^{d_\alpha} = 0$ since if $x_1,\ldots,x_r$ are algebraically independent the monomials $M_\alpha(x_j)$ are linearly independent over F. Hence $a_\alpha = 0$ for each α. Hence $x_1,\ldots, x_r$ is a sequence of algebraically independent points and thus x_i has dimension $\geq r$. Now if x_j is another coordinate let $p(x_i v_0)$ be the minimal polynomial of x_j over $F(x_1,\ldots,x_r)$. Then $p(x_i v_0) = \sum a_i(\bar{x}_i)v_0{}^i$. Let $p_1(\bar{x}_i v_0)$ be the polynomial $\sum a_i(\bar{x}_i)\lambda^{-1}v_0{}^i$. Since $\lambda \in F$ this polynomial is defined over $F(x_1,\ldots,x_r)$ and clearly λx_j is a root. Thus $\dim(\lambda\bar{x}_i) = \dim(\bar{x}_i)$. By definition of $\lambda\Gamma$ this implies that $\lambda\Gamma$ has dimension r. ☒

Theorem 2.31

If F is algebraically closed and $\lambda \in F$ then the map Γ ---> $\lambda\Gamma$ is continuous in the open topology on $S_A^n(F)$.

Proof

Let γ be an open F-formula. Define ψ_1 to be the formula $\exists \bar{v}_i(\psi(\bar{v}_i^1) \wedge v_1^1 = v_1 \wedge, \ldots, \wedge v_n^1 = \lambda v_n)$. Let $\theta(v_i)$ be an open formula such that $F \models \psi_1 \equiv \theta$. Then if β_λ is the map Γ ---> $\lambda\Gamma$ we claim that $\theta = \beta_\lambda^{-1}(\psi)$. For on the one hand if $\Gamma \in \theta$ then any realization (z) of Γ in $\mathfrak{M}^n$ is a solution of θ so since $\mathfrak{M} \models \psi_1 \equiv \theta$ there is (x) in ψ such that $\lambda(z) = (x)$. By definition the type of (x) is $\lambda\Gamma$ so we have $\lambda\Gamma \in \psi$. Thus $\beta_\lambda(\theta) \subseteq \psi$. Suppose then that $\lambda\Gamma \in \psi$. Then there is a realization (z) of Γ in $\mathfrak{M}^n$ such that $\lambda(z) \in \psi$. Clearly $\mathfrak{M} \models \psi_1(z)$ so that $\mathfrak{M} \models \theta(z)$ and thus $\Gamma \in \theta$. This proves the theorem. ⊠

Note that we can define another unary operation on the space $S_A^n(F)$ in terms of the operation - on $\mathfrak{M}^n$. But a point $\Gamma \in S_A^n(F)$ has an additive inverse in $S_A^n(F)$ iff it is a point of F^n.

7.3 Theorem 2.32

If π is a regular map from F^r to F^n where F is an algebraically closed field then π lifts to an open continuous map $\pi^* : S_A^r(F)$ ---> $S_A^n(F)$. That is the following diagram commutes.

$$\begin{array}{ccc} F^r & \dashrightarrow & F^n \\ \downarrow i & & \downarrow i \\ S_A^r(F) & \dashrightarrow & S_A^n(F) \end{array}$$

The map π^* is unique.

In particular projection lifts to an F-linear monoid homorphism.

Proof

Since π is regular it is defined by a set of polynomials $(f_1,\ldots, f_n)$. Given $\Gamma \in S_A^r(F)$ choose a realization (x) of Γ in Ω^r. We define $\pi^*(\Gamma)$ to be the type of the sequence $(f_1(x),\ldots,f_n(x))$. This definition is invariant of our choice of representative of for suppose that (y) is also a realization of Γ in Ω^n. Then (x) and (y) are conjugate points over F so there is an automorphism σ of Ω leaving F fixed such that $\sigma(x) = (y)$. Each polynomial is fixed by σ so that $\sigma(f_i(x)) = f_i(y)$. Thus $(f_1(x),\ldots,f_n(x))$ and $(f_1(y),\ldots, f_n(y))$ are conjugate points over F so have the same open type. Thus π^* is well defined. From this definition we can easily prove, using Chevalley's Theorem (which we proved in a previous section), that π^* is open and also continuous. For example let us prove that π^* is continuous. Let φ be an open formula in $S_A^r(F)$. Then the open formula $\varphi_1 = \varphi(f_1,\ldots,f_n)$ defines an open subset of $S_A^n(F)$. We claim that $\pi^{*-1}(\varphi) = \varphi_1$. For suppose that $\Gamma \in \varphi_1$. Then any realization (x) of Γ in Ω^r has the property that $\Omega \models \varphi_1(x)$ or in other words that $\Omega \models \varphi(f_1(x),\ldots,f_n(x))$ (here as above Ω is a universal domain extending F). Hence $\pi(x)$ is in φ in Ω and by definition this means that $\pi^*(\Gamma) \in \varphi$. Thus $\pi^*(\varphi_1) \subseteq \varphi$. On the other hand suppose $\pi^*(\Gamma) \in \varphi$. Let (x) be a realization of Γ in Ω^r; then $\pi(x) \in \varphi$ or $\Omega \models \varphi_1(x)$. Thus $\Gamma \in \varphi_1$. We next show that π^* is unique. Here we need the fact that F^i is dense in the space $S_A^i(F)$. It follows from this that if $\Gamma \in S_A^r(F)$ we can choose a net

$\{(a_i)\}_{i\in I}$ in F^r which converges to Γ. Suppose f is another continuous map which lifts π. Then $\{f(a_i)\}_{i\in I}$ converges to the point $f(\Gamma)$ and also $\{\pi^*(a_i)\}_{i\in I}$ converges to $\pi^*(\Gamma)$. But $\{f(a_i)\}_{i\in I}$ and $\{\pi^*(a_i)\}_{i\in I}$ are both equal to the net $\{\pi(a_i)\}_{i\in I}$ in F^n since π^* and f are extensions of π on F^r. Hence $f(\Gamma) = \pi^*(\Gamma)$ and thus $f = \pi^*$. Finally the assertions concerning the projection follow easily from the definition of the map lifting the projection. ☒

Theorem 2.33

Suppose that F is an algebraically closed field and F_1 is an algebraically closed extension of F in a universal domain $\mathfrak{U}$ extending F_1. Then the induced map $i^* : S_A^n(F_1) \longrightarrow S_A^n(F)$ is an F-linear monoid homomorphism continuous in the open topologies on these spaces.

Proof

The continuity of the induced map is a general model-theoretic fact which in any case admits a simple direct proof. Let Γ_1 and $\Gamma_2 \in S_A^n(F_1)$. Let (x) and (y) be realizations of Γ_1 and Γ_2 respectively in the space $\mathfrak{U}^n$. Then if (x) and (y) are linearly disjoint over F_1 the type $\Gamma_1 + \Gamma_2$ is defined to be the type Γ_3 over F_1 of the sequence (x) + (y). Now (x) and (y) are linearly disjoint over $F \leq F_1$ and the type of (x) over F is $i^*(\Gamma_1)$ and the type of (y) over F is $i^*(\Gamma_2)$. Thus $i^*(\Gamma_1) + 1^*(\Gamma_2)$ is the type Γ of (x) + (y) over F. But this is the reduct of the type Γ_3 that is $\Gamma = i^*(\Gamma_3)$. This means $i^*(\Gamma_1 + \Gamma_2) = i^*(\Gamma_1) + i^*(\Gamma_2)$. Since the origin is in F this means i^* is a monoid homomorphism. The linearity of

this map with respect to F follows by a similar argument. ⊠

7.4 Having proved the basic results concerning the monoid structure of the Stone space in the above sections as well as the fundamental mapping theorem we continue in this and the succeeding subsection to study linear maps and subspaces and some particular examples which shed light on the nature of the linear operations.

Theorem 2.34

Suppose that $L_1(x_i),\ldots,L_m(x_i)$ is a set of linearly independent linear polynomials defined over F in n variables. Then the equations $L_1 = 0,\ldots,L_m = 0$ define a linear subspace of $S_A^n(F)$ which is a linear variety of dimension n - m.

Proof

By the corollary on page 91 of [43] the above equations define a linear variety in the affine space $\mathfrak{A}^n$ over F of dimension n - m, and so a linear variety L of the same dimension in $S_A^n(F)$. Thus we need only prove that the variety L is a linear submonoid. Let Γ_1,Γ_2 $S_A^n(F)$ with Γ_1, Γ_2 $\in$ L and say λ_1, λ_2 F. Choose linearly disjoint representatives (x) and (y) of Γ_1 and Γ_2 respectively in affine space. Then (x) and (y) are in L in $\mathfrak{A}^n$ by definition. The linear equations defining the variety L^n in $\mathfrak{A}^n$ define a linear subspace L_1 of $\mathfrak{A}^n$ and clearly L_1 = L. So $\lambda_1(x) + \lambda_2(y) \in L$. By definition of the operations in $S_A^n(F)$ this means $\lambda_1\Gamma_1 + \lambda_2\Gamma_2 \in L$. ⊠

By earlier results the converse of the above theorem is not true. For let Γ be a point of $S_A^n(F)$ of dimension ≥ 1. Let M be the F-linear submonoid generated by Γ. Then every point in M has dimension ≥ 1 so it is not definable as a constructible set much less as a linear variety. We do have the following weaker result.

Theorem 35

Suppose that X is a closed linear subspace of $S_A^n(F)$ with the property that if $\Gamma \in X$ then the algebraic locus of Γ is in X. Then X is a linear variety.

Proof

Let L be the set of all algebraic points of X. Then as the linear structure of $S_A^n(F)$ extends that of F^n the set L is a linear subspace of F^n so it is defined by a set of linear equations which by Theorem 2.34 define a linear variety L_1 in $S_A^n(F)$. Suppose that $\Gamma \in X$. Then if $\dim \Gamma \geq 1$ the point Γ is the limit of the algebraic points in its locus. For let Ψ be an open F-formula with $\Gamma \in U_\Psi$. If Ψ is written in disjunctive normal form then Γ is a solution of one of the disjuncts, say the formula $f_1 = 0 \wedge, \ldots, \wedge f_n = 0 \wedge \neg f_{n+1} = 0$, $\ldots, \wedge \neg f_k = 0$. This can be rewritten in the form $f_1 = 0 \wedge, \ldots, \wedge f_n = 0 \wedge \neg (F_{n+1} = 0 \vee, \ldots, \vee f_k = 0)$. Let $f = \prod_{i=n+1}^{k} f_i$. Say that (x) is a realization of Γ in affine space. Then (x) is a solution of our disjunct so $f(x) \neq 0$. We can now apply the assertion on page 49 of [1] to obtain an algebraic specialization of (x), say (x_0), such that $f(x_0) \neq 0$. Then as (x_0) is a specialization of (x) we have $f_i(x_0) = 0$ for $1 \leq i \leq n$ and so (x_0) is a solution of our disjunct

and so of ψ. By definition (x_0) is in the locus of Γ so our claim is proven. As L_1 is closed this implies that $\Gamma \varepsilon \tilde{L} = L_1$ and so $X \subseteq L_1$. Now X is closed and $L \subseteq X$ so $\tilde{L} \subseteq X$. Thus $X = L_1$. ∎

Theorem 2.36

A linear transformation $\pi : F^n \dashrightarrow F^m$ lifts to a map $\hat{\pi}$ from $S_A^n(F)$ to $S_A^m(F)$ which is linear on $S_A^n(F)$. Any continuous open linear map $\hat{\pi} : S_A^n(F) \dashrightarrow S_A^m(F)$ is the extension of a linear transformation $\pi : F^n \dashrightarrow F^m$.

Proof

A linear transformation π is a regular map given by linear polynomials : $\pi = (L_1,\ldots,L_m)$. Thus π lifts to a map from $S_A^n(F)$ to $S_A^m(F)$ (as was shown in 7.3). We need only prove that this map $\hat{\pi}$ is linear. Let Γ_1, Γ_2 $S_A^n(F)$ and λ_1, $\lambda_2 \varepsilon F$. Choose linearly disjoint representatives (x) and (y) of Γ_1 and Γ_2 respectively. Then $\lambda_1\Gamma_1 + \lambda_2\Gamma_2$ is the type of $\lambda_1(x) + \lambda_2(y)$. So $\hat{\pi}(\lambda_1\Gamma_1 + \lambda_2\Gamma_2)$ = the type of $\pi(\lambda_1(x) + \lambda_2(y))$ = 1 $\pi(x) + \lambda_2\pi(y)$. We have $\pi(x) = (L_1(x),\ldots,L_m(x))$ so this sequence is in F(x) and $\pi(y) = (L_1(y),\ldots,L_m(y))$ is a sequence in F(y). Thus the sequences $(L_i(x))$ and $(L_i(y))$ are linearly disjoint since F(x) and F(y) are linearly disjoint over F. So by definition the type of $\pi(\lambda_1(x) + \lambda_2(y))$ is $\lambda_1\hat{\Gamma}_1 + \lambda_2\hat{\Gamma}_2$ where $\hat{\Gamma}_1$ is the type of (x) and $\hat{\Gamma}_2$ that of (y). As $\hat{\Gamma}_1 = \hat{\pi}(\Gamma_1)$ and $\hat{\Gamma}_2 = \hat{\pi}(\Gamma_2)$ be definition we obtain the identity: $\hat{\pi}(\lambda_1\Gamma_1 + \lambda_2\Gamma_2) = \lambda_1\hat{\pi}(\Gamma_1) + \lambda_2\hat{\pi}(\Gamma_2)$. That is, $\hat{\pi}$ is a linear map.

Suppose on the other hand that $\hat{\pi}$ is an open continuous linear map of $S_A^n(F)$ into $S_A^m(F)$. Then the restriction π of $\hat{\pi}$ to F^n is a linear transformtion of F^n into F^m. For as each point $(z) \in S_A^n(F) \cap F^n$ is isolated it is an open set so that $\pi(z)$ is open and thus isolated. But the only isolated points of $S_A^m(F)$ are algebraic points so $\pi(z) \in F^m$. Hence π is a regular map. But $\hat{\pi}$ is a continuous extension of π to $S_A^n(F)$ and so by Theorem 2.32 must be the regular extension of . ⊠

Note that in the above proof we only used the condition that π be open to prove that F^n was mapped into F^m. We can replace this condition by an algebraic assumption, namely that $\hat{\pi}(0) = 0$. For let $(x) \in F$ and assume this latter condition. Then $\hat{\pi}((x) + (-x)) = \hat{\pi}(x) + \hat{\pi}(-(x))$ and also $\pi((x) - (x)) = \hat{\pi}(0) = 0$ so $\hat{\pi}(x) + \hat{\pi}(-(x)) = 0$. If $\dim \hat{\pi}(x) \geq 1$ there is no element of $S_A^m(F)$ which satisfies this equation so we must have (x) F^m.

The above theorem yields a strong form of Chevalley's theorem for linear maps.

Corollary 2.8

Suppose that $\hat{\pi}$: $S_A^n(F)$ ----> $S_A^m(F)$ is a continuous linear map such that either $\hat{\pi}$ is open or $\hat{\pi}(0) = 0$. Then the image under π of any linear variety is a linear variety.

Proof

The restriction of $\hat{\pi}$ to F^n is a linear transformation π into F^m so that image of $L \cap F^n$, which is a linear subspace, is a linear

subspace L_1 of F^m. By Theorem 2.34 the equations of this subspace define a linear variety $\hat{L}_1$ in $S_A^m(F)$. We claim that $\hat{\pi}(L) = \hat{L}_1$. First note that each point Γ of L is either in F^n or is a limit of such points (by the proof of Theorem 2.35). Since $\pi(L \cap F^n) = L_1$ we have continuity and the fact that $\hat{L}_1$ is closed that $\hat{\pi}(\Gamma) \in L_1$. But by Chevalley's Theorem we have that $\hat{\pi}(L)$ is closed and $L_1 \subseteq \hat{\pi}(L)$ so $L_1 \subseteq \hat{\pi}(L) \subseteq \hat{L}_1$, and thus $\hat{\pi}(L) = \hat{L}_1$ as $\hat{L}_1$ is the closure of L_1. ⊠

7.5 In the following few results and examples we examine the properties of the linear operations on $S_A^n(F)$ in some detail. We would like to obtain a complete description of the structure of this semigroup.

Lemma 2.24

Suppose that $\Gamma_1 \in S_A^n(F)$ and Γ is the type of the generic point of affine n-space over F. Then $\Gamma_1 + \Gamma = \Gamma$.

Proof

Let $\langle x_1,\ldots,x_n\rangle$ be a sequence of algebraically independent elements and $\langle y_1,\ldots,y_n\rangle$ be a realization of Γ_1 free over F with respect to (x). If (x) + (y) is not a sequence of algebraically independent elements over F there is a least m such that the sequence $x_1 + y_1,\ldots,x_{m-1} + y_{m-1}$ is algebraically independent and $x_m + y_m$ is algebraic over this sequence. Let f be an irreducible polynomial with $f(x_1 + y_1,\ldots,x_{m-1} + y_{m-1}, x_m + y_m) = 0$. As our sequence is minimal with the property we stipulate above we can expand this resulting polynomial by the binomial formula and multiply out all

terms to obtain a polynomial $f_1(x_1,\ldots,x_{m-1},y_1,\ldots,y_{m-1},y_m,x_m) = 0$ in which all the x_i and y_j appear in some nonzero term. For there is clearly a symmetry among the x_i and y_j in this regard so if any such element does not appear in our resulting polynomial then we could omit an $x_i + y_i$ from our original polynomial. Choose a maximal set of algebraically independent elements, say $y_1,\ldots,y_r$ among the y_j. The above polynomial then tells us that x_m is algebraic over $F(x_1,\ldots,x_{m-1},y_1,\ldots,y_r)$ and this is impossible by our original assumption that (x) and (y) are free of F. ⊠

Corollary 2.9

A structure $S_A^n(F)$ is not a cancellation semigroup.

Proof

Let $\Gamma_1 \neq \Gamma_2$ and say both points are not equal to the type Γ of the generic point of the affine space. Then by the above lemma we have $\Gamma + \Gamma_1 =$ and $\Gamma + \Gamma_2 = \Gamma$. ⊠

Theorem 2.37

There are subgroups of $S_A^n(F)$ not contained in F^n.

Proof

We give a typical example for a particular n from which it will be clear how to generalize. Consider the sequences (x,x+2) and (y,y-2) of type Γ_1 and Γ_2 respectively. Let S be the subsemigroup generated by Γ_1 and Γ_2. Then each point of S has dimension at least 1 (by Theorem 2.29). We claim that S is a group. In fact the zero of S is the

type of (x,x) (x algebraically independent over F). For let $\Gamma_0 \in S$. Then Γ_0 has a representation $n\Gamma_1 + m\Gamma_2$ (for integers n,m in Z). The type of $n\Gamma_1$ is represented by (z,z+2n) (z transcendental over F) and similarly that of $m\Gamma_2$ by a sequence $(z_1, z_1 - 2m)$ and so the type of $(z_2, z_2 + 2(n-m))$ is Γ_0. Adding (x,x) (assuming the sequences are free) gives the sequence $(z_2 + z, z_2 + x + 2(n-m))$ and the type of this is the type of $(z, z + 2(n-m))$ hence equals Γ_0. This analysis also shows us the inverse of any point: For Γ_0 it is the type of $(z_3, z_3 + 2(m-n))$. Hence S is a group. ⊠

We now present two examples illustrating properties of the operations on $S_A^n(F)$.

Example 2.7

Suppose that F is algebraically closed and has characteristic 2. Consider the sequence $(x, ax^2 + bx + c)$ where $a,b,c \neq 0$. Let Γ be the type of this sequence over F. The $\Gamma+\Gamma$ has the type of $(x + y, a(x + y)^2 + b(x + y) + 2c) = ((x + y), a(x + y)^2 + b(x + y))$ and this sequence has the type Γ_1 of the sequence $(z, az^2 + bz)$. Here x,y,z are all independent indeterminates over F. Clarly $\Gamma + \Gamma_1 = \Gamma$ and $\Gamma_1 + \Gamma_1 = \Gamma_1$ so the elements Γ, Γ_1 constitute a subgroup of $S_A^n(F)$ of the order of the characteristic of F. ⊠

Example 2.8

Let Γ be the type of the generic point of the hyperbola $xy = 1$, defined over F. Then $\Gamma + \Gamma$ is the type of the generic point of S^2.

Proof

A generic point of the hyperbola is given by $(x,\frac{1}{x})$ where x is transcendental over F. Let y be algebraically independent with respect to x. Then + is the type of $(x,\frac{1}{x}) + (y,\frac{1}{y}) = (x + y,\frac{x+y}{xy})$. Suppose that $\frac{x + y}{xy}$ is algebraic over $F(x + y)$. Then it is a solution of a polynomial

$$\frac{F_n(x + y)}{G_n(x + y)} \quad \frac{y + x^n}{xy} \quad + \ldots + \frac{F_0(x + y)}{G_0(x + y)} = 0$$

with rational coefficients. Multiply the above relation by $(xy)^n$; we obtain a new relation: $\frac{F_n(x + y)}{G_n(x + y)} (y + x)^n +,\ldots,+ (xy)^n \frac{F_0(x + y)}{G_0(x + y)} = 0.$

Hence $- \frac{F_n(x + y)}{G_n(x + y)} (y + x)^n = (xy) \frac{F_{n-1}(x + y)}{G_{n-1}(x + y)} +,\ldots,+ (xy)^n \frac{F_0(x + y)}{G_0 (x + y)}$

is obtained via subtraction. Now the left hand expression is divisible by (xy) so this factor must also divide $F_n(x + y) (y + x)^n$. But if we expand both polynomial factors via the Binomial Theorem and multiple we obtain a leading term of the form cy^{n+r} for some r and $C \neq F$. As this term is not divisible by x we have a contradiction ☒

A basic problem suggested by the above examples is the compatibility of algebraic relations: When do two algebraic relations blow up to the type of the generic point of affine space? For example if we are dealing with curves in which the second coordinate is a rational function or polynomial in the first coordinate must the second coordinate by a polynomial or rational function in the sum of the first coordinates in order that the sequence not blow up? More general algebraic relations are more complicated.

The final result of this section shows that if $F \leq F_1$ where F_1 is algebraically closed and X is a set of types in $S_A^n(F_1)$ defined over the field F then the submonoid generated by X is also defined over F.

Theorem 2.38

Let F_1 be an algebraically closed field extending F. If Γ_1 and Γ_2 are points of $S_A^n(F_1)$ then Γ_1 and Γ_2 defined over F implies that $\Gamma_1 + \Gamma_2$ is defined over F.

Proof

Choose algebraically free generic points (x) and (y) of the varieties of Γ_1 and Γ_2. Then as both Γ_1 and Γ_2 are defined over F the points (x) and (y) are also generic over F. Then (x) + (y) is regular over F as F is algebraically closed so the variety V_0 of this sequence is absolutely irreducible over F and so is an absolutely irreducible variety over F_1. Further (x) + (y) and F_1 are free over F. For both (x) and (y) are separately free over F_1 with respect to F and as they are free over F_1 with respect to each other we deduce that (x,y) is free over F with respect to F_1 (in fact linearly disjoint over F). Thus F(x,y) has this property and (x) + (y) is a sequence in the latter field. This proves our claim. Hence (x) + (y) is a generic point of V_0 over F_1. Thus $\Gamma_1 + \Gamma_2$ is the type of generic point of V_0 and so is the extension of the generic point of V_0 over F. ⊠

A question not touched on in the study of the semigroup structure of the Stone space in this section is that of analytic properties. There is

an extensive theory of integrals on topological semigroups and it is natural to ask whether integrals linked to the geometry of the space arise from the semigroup structure. Such integrals in classical algebraic geometry are useful in the classification of surfaces.

Chapter 3: The Fundamental Configurations of Linear Projective Geometry

3.0 Introduction

In this chapter we study the fundamental configurations of linear projective geometry and some related figures. A major theme of the chapter is the central position of the Desargue Configuration in projective geometry. We begin by giving an axiomatic proof of the extension of this theorem to three dimensional geometry. Next we demonstrate the similarity in the basic geometric theories of the Desargue and Pappus configurations. It is shown in [33] that the Pappus Theorem implies the Desargue Theorem. We prove that certain other fundamental results of projective geometry also imply the Desargue Theorem.

The second major theme of the chapter is the relationship of a theorem to the dual theorem. We give new proofs that the Desargue and Pappus theorems are equivalent to the dual theorems. In the case of the Harmonic and Involution constructions the implication of the dual constructions is demonstrated under an additional assumption.

3.1 The Desargue Theorem

The following theorem is proven in [35] by an analytic argument. We give a purely geometric proof. Suppose that xyzw is a space tetrad. A face of the tetrad is a plane determined by three points of the tetrad.

Theorem 3.7 (Perspective Tetrad Theorem)

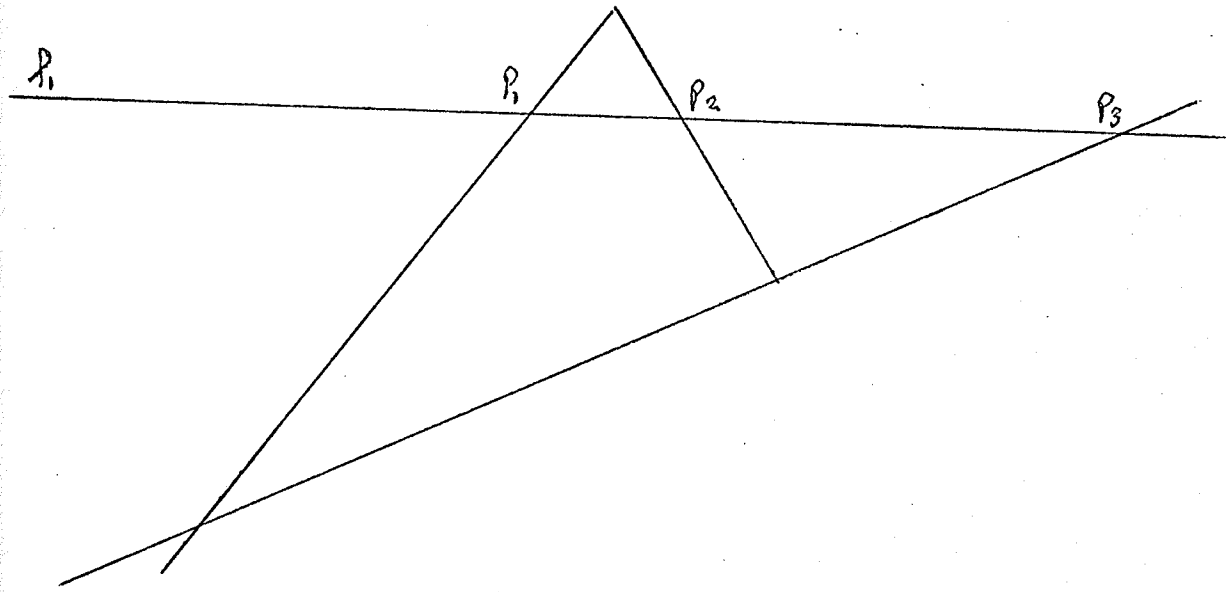
Suppose that xyzw and $x^1y^1z^1w^1$ are two general tetrads in space. Then xyzw and $x^1y^1z^1w^1$ are in pespective iff the corres-

ponding faces of the tetrads meet in a plane in four general lines.

Proof

Suppose that $xyzw$ and $x_1y_1z_1w_1$ are in perspective. Then the corresponding triangles of each pair of corresponding faces are in perspective. Therefore the corresponding sides of each pair of triangles meet on a line. As there are four pairs of corresponding faces there are four such lines. Choose one of the lines ℓ_1, determined by points P_1, P_2, P_3 that are the intersections of the sides of the corresponding triangles of the faces.

Figure 1



Now each of the other triangles of a given tetrad shares a side with the triangle chosen above. Hence each of the other three lines pass through exactly one of the points P_1, P_2, P_3. This proves that all four lines meet in pairs since the choice of ℓ_1 was arbitrary. Thus the lines are coplanar and the lines are in general position. Assume on the other hand that corresponding faces of the tetrads meet in a plane in four general lines. Suppose that the planes of xyz and $x^1y^1z^1$ meet in the plane S in a line ℓ_1. The planes of xyw and $x^1y^1w^1$ also meet an S in a line ℓ_2 that is distinct from ℓ_1 since the four lines are in general position. Hence $\ell_1 \cdot \ell_2 = P_1$. Now P_1 lies in the three planes xyz, $x^1y^1z^1$ and xyw so is the unique intersection point. But xyz and xyw meet on xy so P_1 lies on the line xy. In the same way it lies on x^1y^1 so $P_1 = (xy) \cdot (x^1y^1)$. By a repetition of this argument we can show that xz and x^1z^1 meet on ℓ_1 and zy and z^1y^1 meet on ℓ_1. An application of the Desargue Theorem shows that xyz and $x^1y^1z^1$ are in perspective through some point P. In the same way we can prove that the triangles xyw and $x^1y^1w^1$ are in perspective. But xx^1 and yy^1 meet at P so that ww^1 must also pass through P. Consequently the tetrads are in perspective. ☒

Let S^n denote n-dimensional projective space. An <u>m-gon</u> in S^n consists of m linearly independent points. A <u>face</u> of an m-gon is the subspace generated by a set of m-1 points of the m-gon. In order to generalize the Perspective Tetrad Theorem to higher dimensional projective space we need the following lemma.

Lemma

Suppose that $n>3$. Two n-gons in S^n are in perspective iff the corresponding faces of the n-gons meet in general position on some n-2 dimensional subspace.

The proof of the above lemma is a direct generalization of the proof of the Desargue Theorem in S^3. We can utilize this lemma as in the proof of Theorem 3.1 to prove the following generalization of the Perspective Tetrad Theorem.

Theorem 3.2

Suppose that $n>3$. Two n+1-gons in S^n are in perspective iff the corresponding faces of the n+1-gons meet in general position on an n-1 dimensional subspace of S^n.

The Principle of Duality in projective geometry was discovered by Gergonne and perfected by Steiner in [40]. It should be noted that this principle is an empirical hypothesis that requires verification in specific cases. A natural question that is related to this principle is whether a given theorem is logically equivalent to the dual theorem. It was suggested by H.F. Baker in [3] that such an equivalence should hold in theorems regarding self-dual figures. In [33] there is an indication of a proof of this equivalence for the Desargue Theorem. The following is a detailed proof.

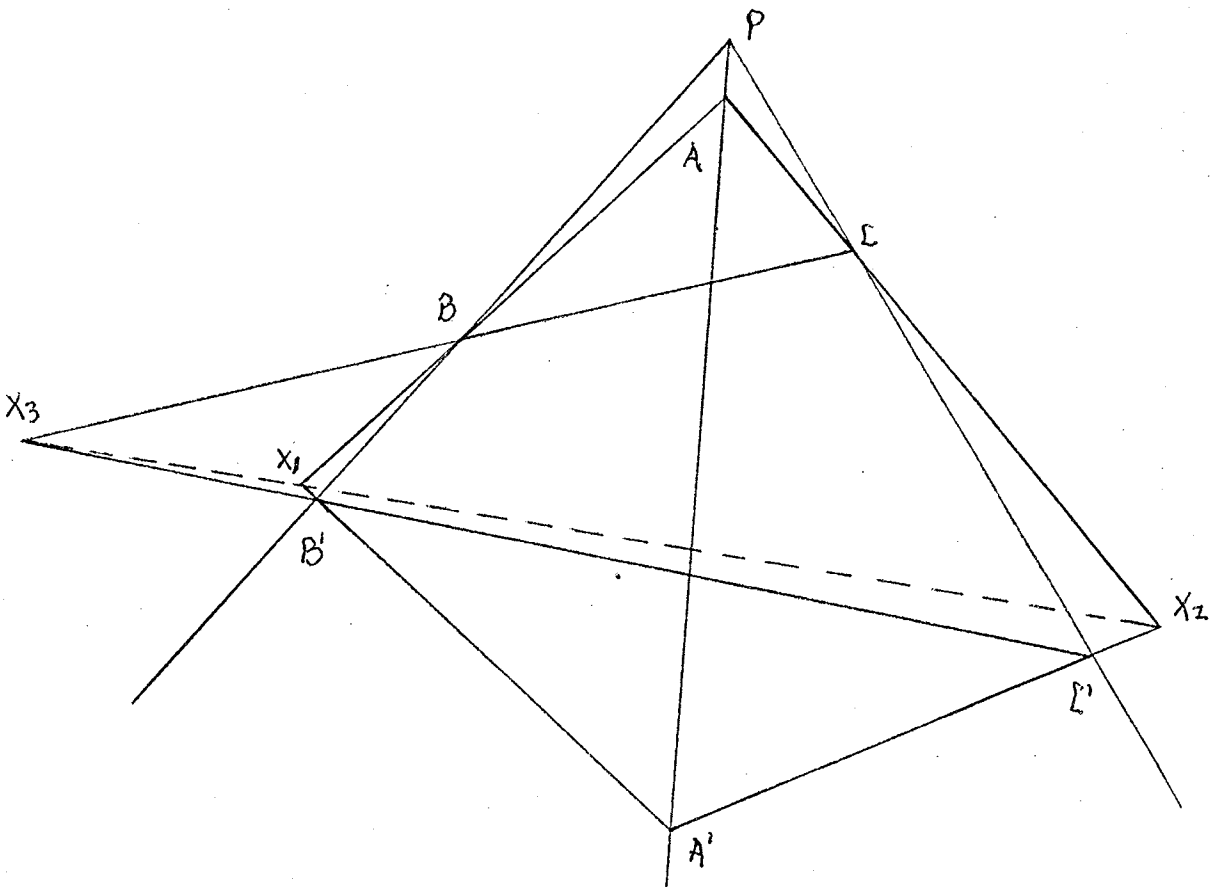
Theorem 3.3

The Desargue Theorem is equivalent to the Dual Desargue Theorem.

Proof

We prove first that the Desargue Theorem implies the Dual Desargue Theorem. Suppose that corresponding sides of the two triangle ABC and $A^1B^1C^1$ meet in line. Let $AB \cdot A^1B^1 = X_1$, $AC \cdot A^1C^1 = X_2$ and $BC \cdot B^1C^1 = X_3$. Then the two triangles CC^1X_2 and BB^1X_1 are in perspective through the point X_3.

Figure 2



The line X_1X_2 passes through X_3 by assumption. Furthermore $X_3=BC \cdot B^1C^1$. This proves the contention. It follows from the

Desargue Theorem that the points $P_1 = BB^1 \cdot CC^1$, $P_2 = X_1B \cdot X_2C$ and $P_3 = X_1B^1 \times X_2C^1$ are collinear. Now by definition $P_2 = A$ and $P_3 = A^1$ and so AA^1 passes through P_1. Hence ABC and $A^1B^1C^1$ are in perspective through the point of P_1. Assume now the Dual Desargue Theorem and suppose that the triangles are in perspective through the point P. As before let $X_1 = AB \cdot A^1B^1$ and $X_2 = AC \cdot A^1C^1$. The corresponding sides of the triangles X_1BB^1 and X_2CC^1 meet in line. For $BB^1 \cdot CC^1 = P$ and $X_1B \cdot X_2C = A$ and $X_1B^1 \cdot X^2C^1 = A^1$. Hence by the Dual Desargue Theorem the lines BC and B^1C^1 meet at a point X_3 on the line X_1X_2. Therefore the corresponding sides of the triangles ABC and $A^1B^1C^1$ meet in line. ⊠

We conclude this section of the chapter with a presentation of two results that are applications of the Desargue Theorem and the Dual Desargue Theorem. The first result is a geometric proof of a condition that is equivalent to the perspectivity of a pair of triads on two distinct lines. This result is one of two equivalences proven by analytic methods in [35]. A geometric proof of the second equivalent can be found in [12]. Another geometric approach to the first equivalent can be found in [3].

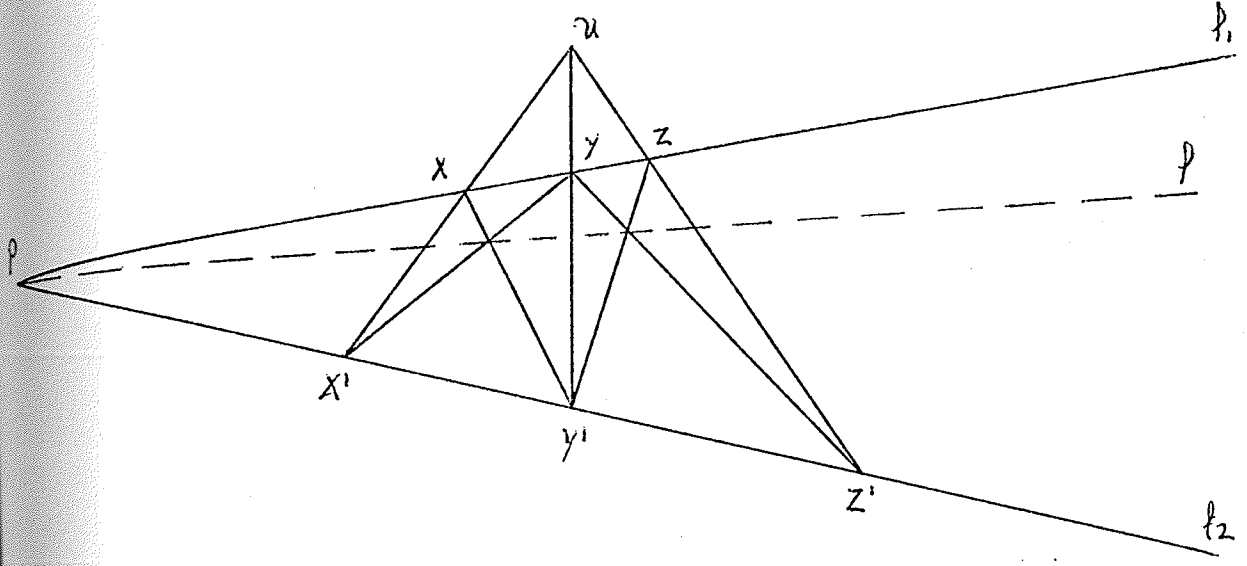
Theorem 3.4

Suppose that ℓ_1 and ℓ_2 are lines meeting at a point P. Let x,y,z and x^1,y^1,z^1 be triads of distinct points on ℓ_1 and ℓ_2 such that no point is equal to P. Let ℓ be the Pappus line determined by the two triads. Then x,y,z and x^1,y^1,z^1, are in perspective iff ℓ passes through P.

Proof

Suppose first that the triads are in perspective from a point U.

Figure 3



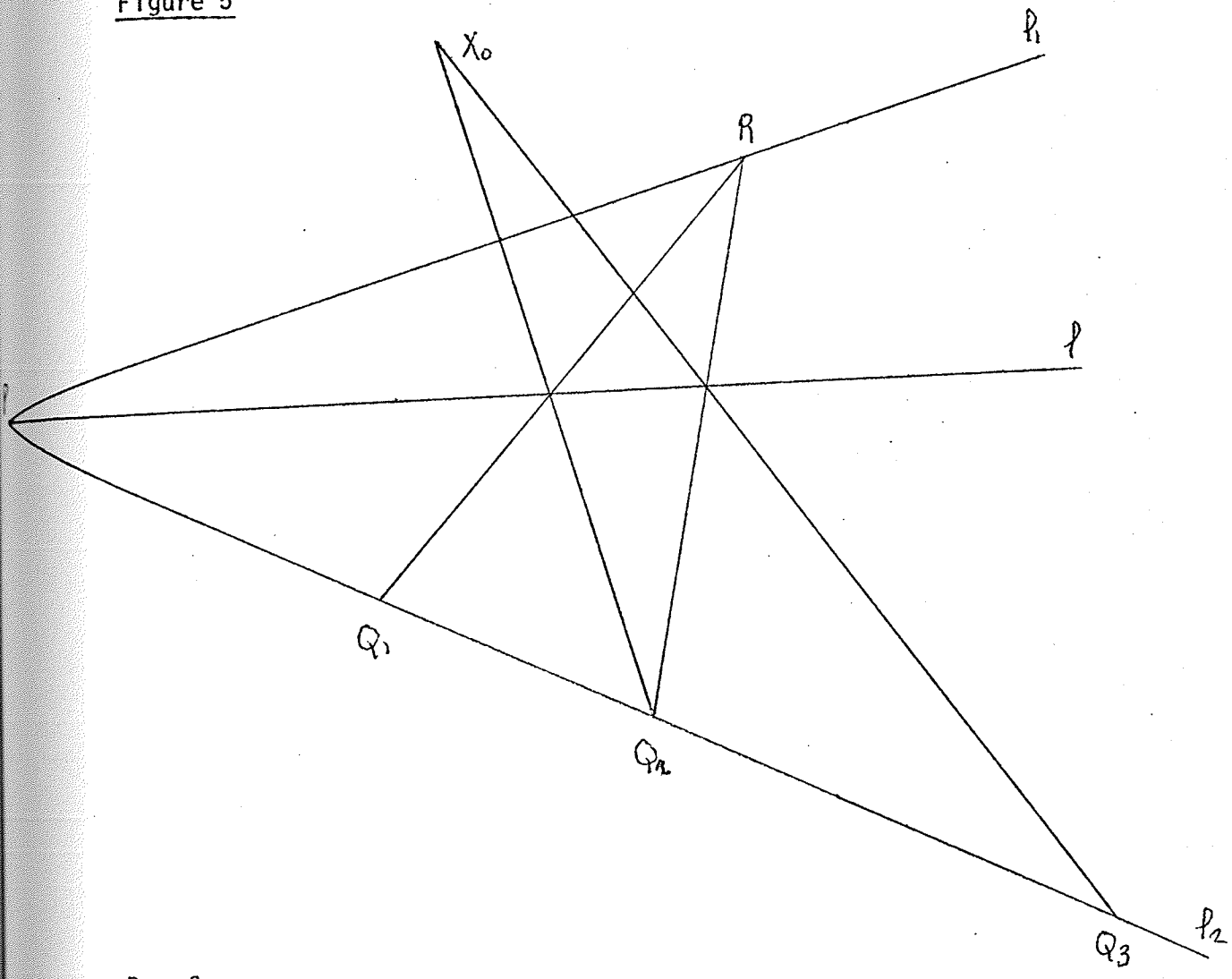
Then the triangles XZY^1 and X^1Z^1Y are in perspective through the point U so that if $S_1 = XY^1 \cdot X^1Y$ and $P_2 = ZY^1 \cdot Z^1Y$ the line P_1P_2 passes through P by the Desargue Theorem. But P_1P_2 is the Pappus line of the two triads. Suppose on the other hand that the Pappus line ℓ of the two triads passes through P. This means that corresponding sides of the two triangles XZY^1 and X^1Z^1Y meet in line. Hence by the Dual Desargue Theorem the triangles XZY^1 and X^1Z^1Y are perspective through some point U. This means that the two triads are in perspective through U. ☒

Theorem 3.5

Suppose that ℓ_1 and ℓ_2 are lines that meet at a point P. Let

R be a point of ℓ_1 not equal to P and Q_1, Q_2, Q_3 a triad of points on ℓ_2 such that no Q_i is equal to P. Suppose that ℓ is a line through P. Then if $P_1 = RQ_1 \cdot \ell$ and $P_2 = RQ_2 \cdot \ell$ the point $X_0 = (Q_2P_1) \cdot (Q_3P_2)$ lies on a line through R that does not depend on the choice of ℓ.

Figure 5



Proof

Let ℓ_0 be another line through P and define $X_1 = \ell \cdot Q_1R$, $y = \ell_0 \cdot Q_1R$, $X_1 = \ell \cdot Q_2R$ and $Y_1 = \ell_0 \cdot Q_2R$. Then the

triangles XYQ_2 and $X_1Y_1Q_3$ are in perspective through the point P. Now XY and X_1Y_1 intersect at the point R and YQ_2 and Y_1Q_3 intersect at a point X_o that does not depend on the position of O. |x|

The examples above illustrate the central position of the Desargue Configuration in the structure of the theory of projective geometrical relations. We will return to the study of this question in the third section of this chapter.

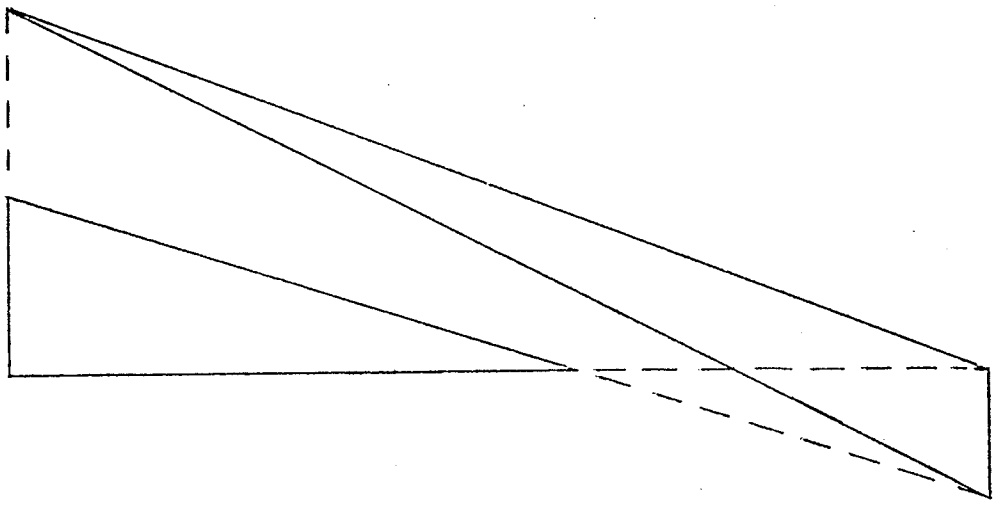
3.2 The Pappus Theorem

The geometrical relation defined in the following definition was first noticed by Graves. A discussion of this relation can be found in [9].

Definition 3.1

A triangle ABC is in second perspective to a triangle $A_1B_1C_1$ if the sides of ABC pass through the vertices of $A_1B_1C_1$.

Figure 6

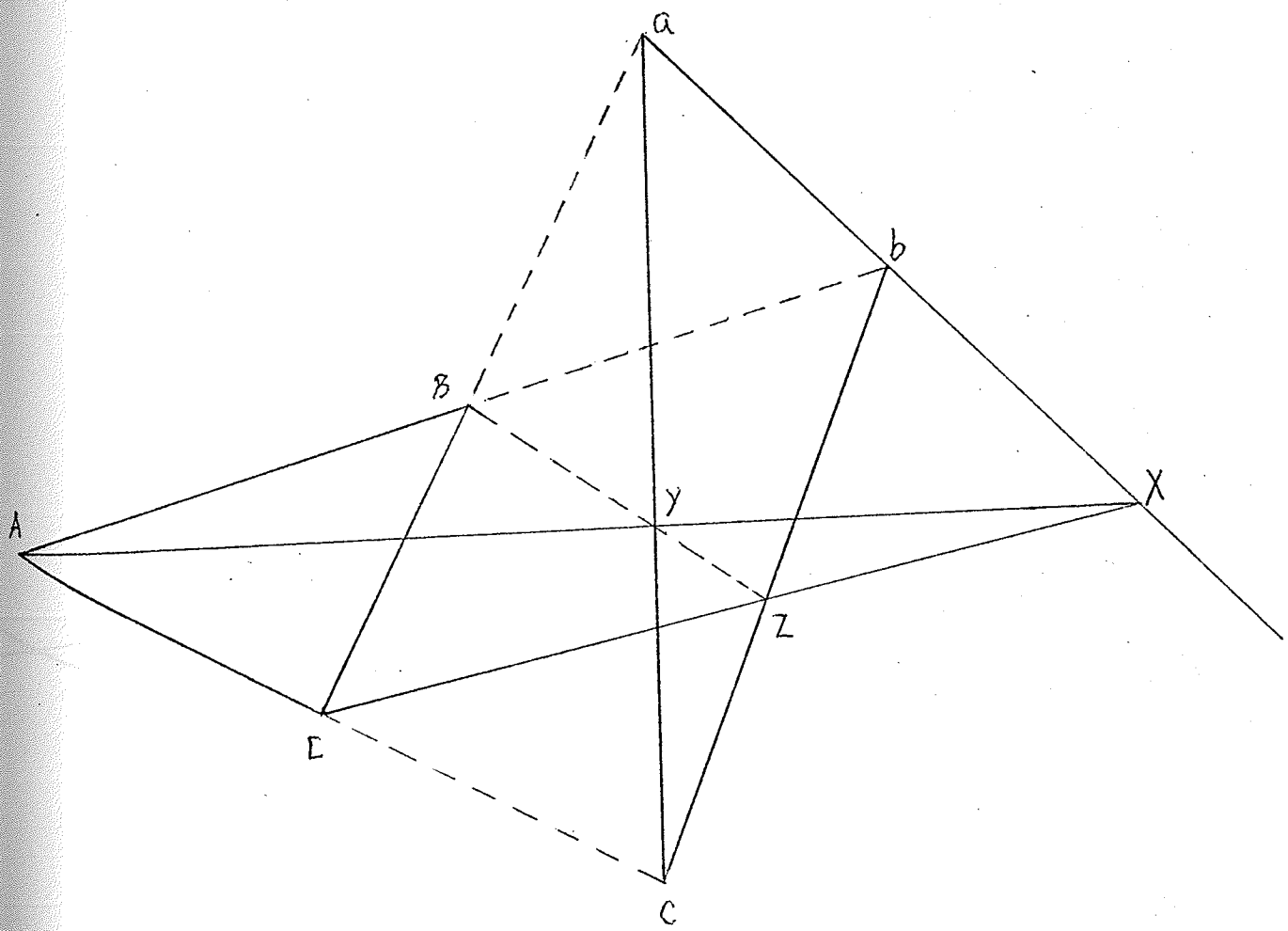


We utilize the above definition to formulate the following geometric principle.

The Second Perspective Theorem

Suppose that the triangle ABC and abc are in the same plane and ABC is in second perspective to abc. Assume that a is on CB, b is on AB and c is on AC. Let X be a point of ab not equal to a or b and let y = AX . ac and Z = CX . bc. Then the point B lies on the line yz.

Figure 7



The above theorem can be verified as an empirical fact. In regard to the relation of this configurational fact to projective geometric theory we have the following result.

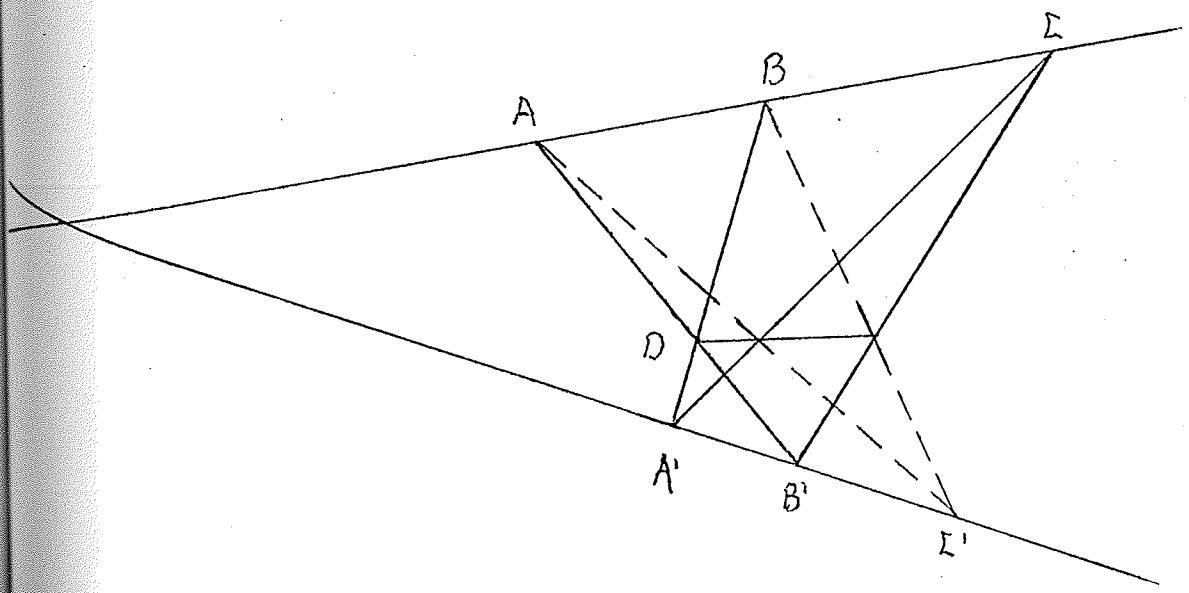
Theorem 3.6

The Second Perspective Theorem is equivalent to the Pappus Theorem.

Proof

Assume the Second Perspective Theorem. Given lines ℓ_1 and ℓ_2 and triads A,B,C and A^1,B^1,C^1 on ℓ_1 and ℓ_2 respectively we let $D = AB^1 \cdot A^1B$. The triangles ABD and $A^1B^1C^1$ are in second perspective. For B^1 lies on AD, A^1 lies on BD and C is a point of AB. The point C^1 lies on the line A^1B^1 and C^1 is not equal to A^1 or B^1. Let $y = AC^1 \cdot A^1C$ and $Z = BC^1 \cdot B^1C$. By the Second Perspective Theorem the line yz passes through the vertex D. But this is the conclusion of the Pappus Theorem.

Figure 8



Assume the Pappus Theorem and suppose that the triangle ABC is in second perspective with the triangle abc. Suppose that a lies on CB, b lies on AB and c lies on AC. Let X be a point on the line ab not equal to a or b. Consider the triads A,C,c and a,b,X. By the Pappus Theorem the points Ab . aC, AX . ac and CX . bc are in line. As the point B is equal to Ab . aC it follows that if y = AX . ac and Z = CX . bc then the line yz passes through the vertex B. This is the conclusion of the Second Perspective Theorem and completes the proof of the equivalence. ☒

The equivalence of the Pappus Theorem and the Second Perspective Theorem can be applied to provide a very simple proof that the Pappus Theorem is equivalent to the Dual Pappus Theorem. The proof to be given avoids the use as in [3] of the three dimensional form of the Pappus Theorem. In this regard note that the second perspective configuration is a self-dual figure.

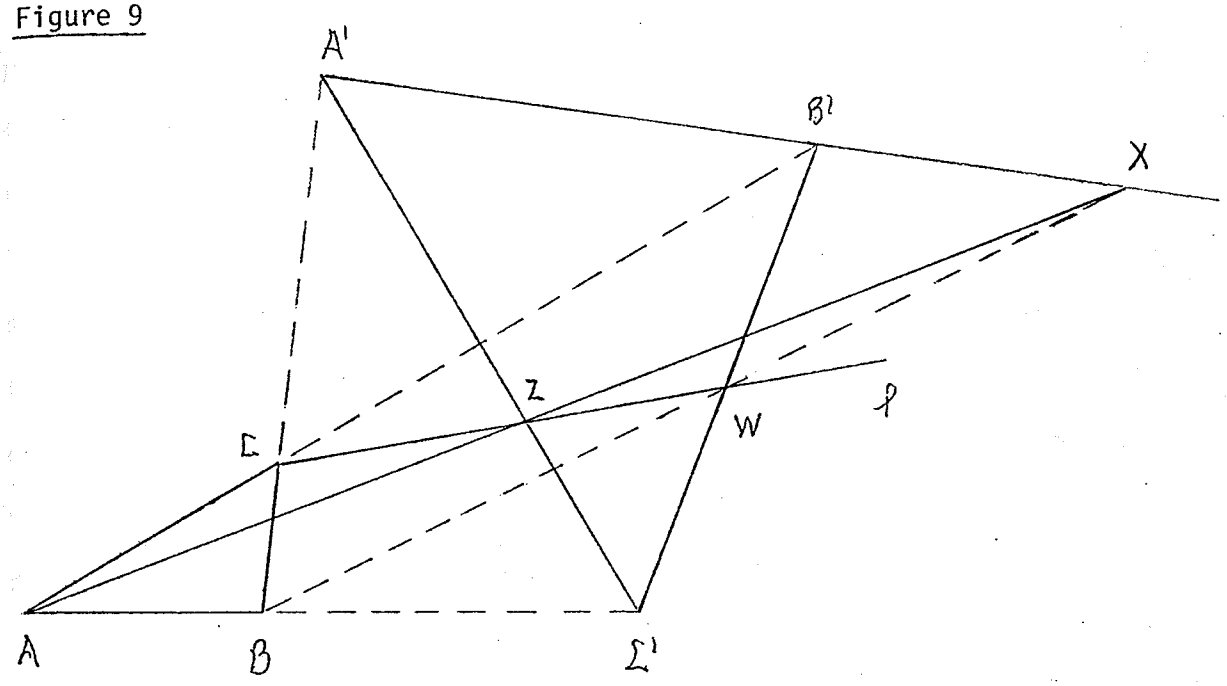
Theorem 3.7

The Pappus Theorem is equivalent to the Dual Pappus Theorem.

Proof

It suffices to prove that the Second Perspective Theorem is equivalent to the Dual Second Perspective Theorem. Assume the Second Perspective Theorem.

Figure 9



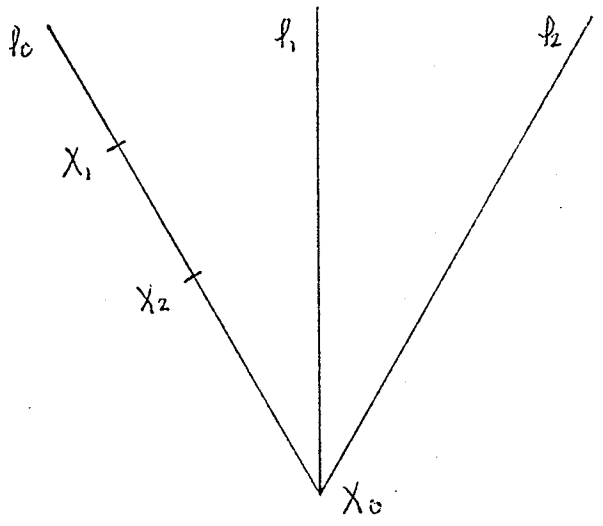
Suppose that the triangle ABC is in second perspective to the triangle $A^1B^1C^1$ so that A^1 lies on BC, B^1 lies on AC and C^1 lies on AB. Let ℓ be a line through the vertex C and let $z = \ell \cdot A^1C^1$. Let $X = Az \cdot A^1B^1$ and let $w = BX \cdot B^1C^1$. By the Second Perspective Theorem the line zw passes through the vertex C. But $Cz = \ell$ and so $\ell = zw$. Thus $\ell \cdot B^1C^1 = w$. Therefore the lines $A(\ell \cdot A^1C^1)$ and $B(\ell \cdot B^1C^1)$ intersect on A^1B^1. This is the conclusion of the Dual Second Perspective Theorem. Assume the Dual Second Perspective Theorem and suppose the triangles ABC and $A^1B^1C^1$ are as above. Let X be a point of A^1B^1 not equal to A^1 or B^1. Let $Z = AX \cdot A^1C^1$ and let $w = CZ \cdot B^1C^1$. By the Dual Second Perspective Theorem the lines AZ and Bw meet on A^1B^1. But $AZ \cdot A^1B^1 = X$ so that $w = BX \cdot B^1C^1$ and thus zw passes through the vertex C. This is the conclusion of the Second Perspective Theorem. ☒

A self-dual configuration is a linkage of the point geometry of the plane and the dual line geometry. In [41] it is shown that by the use of the Desargue configuration one can construct an incidence preserving correspondence of the points and lines of the plane. As shown in [35] and [3] such a correspondence is closely linked with the geometry of a conic section. In the final portion of this section of the chapter we utilize the Pappus Theorem to construct another incidence preserving correspondence of the points and lines of the plane. It was shown above that both the Desargue Theorem and the Pappus Theorem can be formulated as self-dual triangle relations. The direct use of the Pappus Theorem to construct a point and line correspondence thus completes the picture of the analogy of the structure of the geometrical theory of the Desargue and Pappus configurations.

Definition 3.2

A 3-3 configuration in a plane S consists of three distinct lines ℓ_0, ℓ_1, ℓ_2 and three distinct points X_0, X_1, X_2 such that X_1 and X_2 lie on ℓ_0 and the three lines pass through X_0.

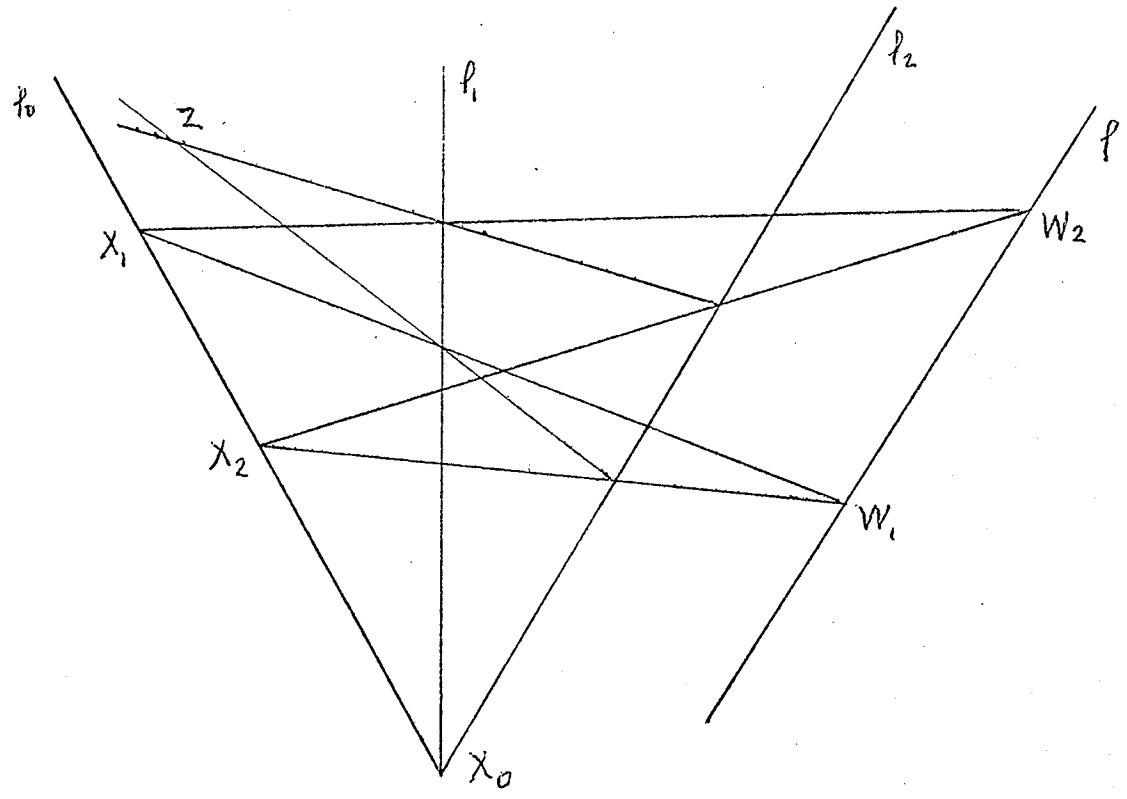
Figure 10



Theorem 3.8

Suppose that the lines ℓ_0, ℓ_1, ℓ_2 and the points X_0, X_1, X_2 constitute a 3-3 configuration in a plane S. Let ℓ be another distinct line in S. There is a unique point Z such if W is any point on ℓ then the line $(X_1W \cdot \ell_1)(X_2W \cdot \ell_2)$ passes through the point Z.

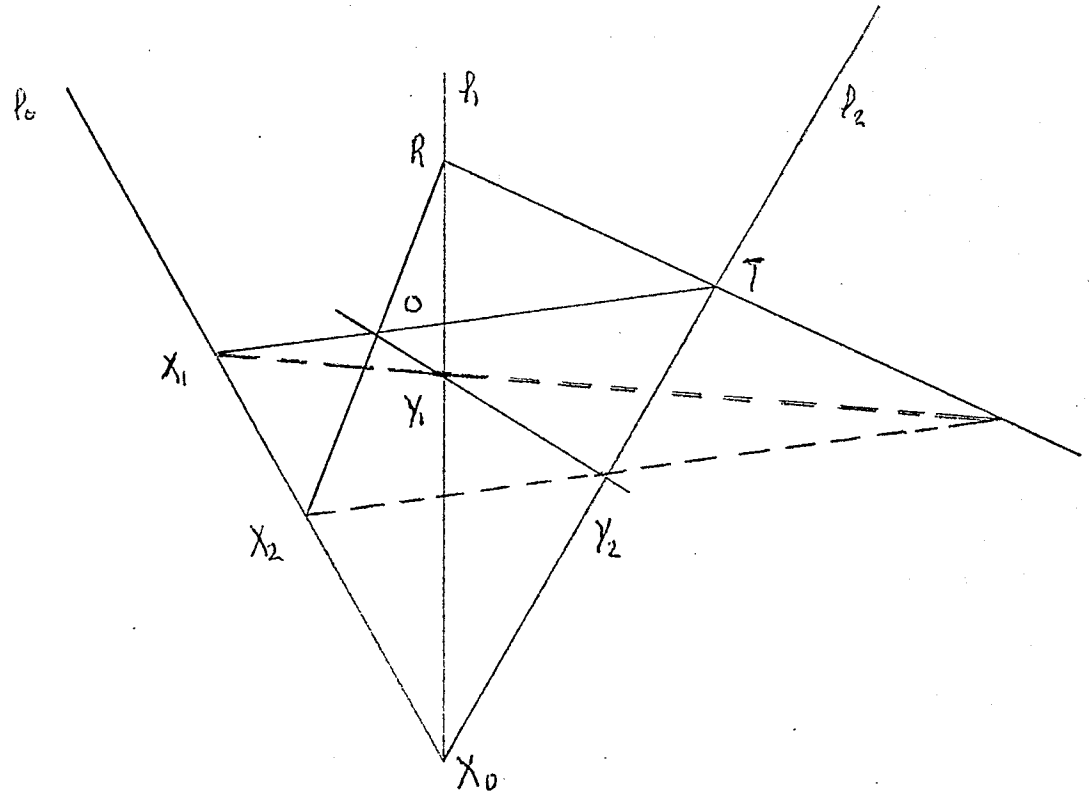
Figure 11



Proof

The proof of the theorem proceeds by a reduction to the Second Perspective Theorem. Let $R = \ell \cdot \ell_1$ and $T = \ell \cdot \ell_2$. Define the point D to be $X_1T \cdot X_2R$. Then D is the intersection point.

Figure 12



To prove this note that the triangle X_1X_2D is in second perspective with the triangle X_0RT. Thus if X is a point on ℓ not equal to R or T and if $Y_1 = \ell_1 . X_1X$ and $Y_2 = \ell_2 . X_2X$ then it follows from the Second Perspective Theorem that the line Y_1Y_2 passes through the point D. This proves the theorem. ☒

Note that the 3-3 configuration with a line and the corresponding pencil constitutes a self-dual configuration. Suppose that D is the center of a pencil. Let $T = X_1D . \ell_2$ and $R = X_2D . \ell_1$. If ℓ is a line through the point D let $Y_1 = \ell . \ell_1$ and $Y_2 = \ell . \ell_2$. Then by the Dual Second Perspective Theorem the lines X_1Y_1 and X_2Y_2 meet on RT. This proves the dual of the above theorem. The following result is an application of the dual form of Theorem 3.8.

Theorem 3.9

Suppose that the lines ℓ_0, ℓ_1, ℓ_2 and the points X_0, X_1, X_2 in a plane S constitute a 3-3 configuration. Let S_1 be a pencil of lines in S. To each line ℓ of S_1 there corresponds a pencil of lines with center X_ℓ. The set of points X_ℓ for ℓ in S_1 are in line.

Proof

If ℓ is a line in the pencil S_1 then it was shown in the Proof of Theorem 8 that the center of the corresponding pencil is defined as follows. Let $R_1 = \ell \cdot \ell_1$ and $R_2 = \ell \cdot \ell_2$. Then the point $X_\ell = X_1R_2 \cdot X_2R_1$ is the center of the corresponding pencil. It is now an immediate deduction from the dual of Theorem 3.8 that the points X are in line. ☒

3.3 The Harmonic and Involution Constructions

It is shown in [41] that the harmonic and involution constructions can be obtained as traces of a quadrangle on a line. In [3] a construction is introduced in regard to the addition of coordinates which we will call the addition construction. This construction can be obtained as the third trace of a quadrangle on a line. We begin this section by showing how each construction can be obtained as a trace of a quadrangle in order of complexity. The three cases that are possible are: (i) Two diagonal points of the quadrangle lie on the line, (ii) One diagonal point of the quadrangle lies on the line, (iii) No diagonal point of the quadrangle lies on the line.

We first give an explicit statement of each of the trace constructions. All lines and points are in a fixed plane.

Theorem 3.10 (Harmonic Construction)

Suppose that A,B,C are points of a line ℓ and a and b are lines through A and B respectively. Let c be a line through C and define $X_1 = a \cdot c$ and $X_2 = b \cdot C$. If $X = a \cdot b$ and $y = (aX_2) \cdot (bX_1)$ then the point $P = xy \cdot \ell$ is invariant of the choice of lines.

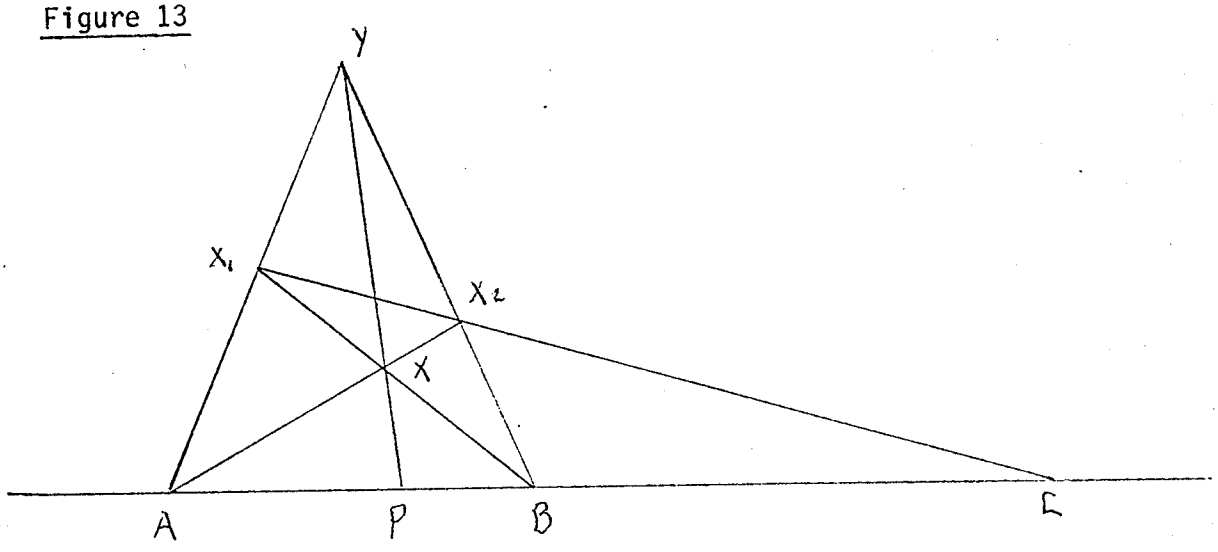
Theorem 3.11 (Addition Construction)

Suppose that A,B,C,D are points of a line ℓ and a,b,c are lines through A,B,C respectively. Suppose that d_1 and d_2 are lines through D such that $b \cdot c$ lies on d_1 and $a \cdot c$ lies on d_2. Then the point $P = (b \cdot d_2)(a \cdot d_1) \cdot \ell$ is invariant of the choice of lines.

Theorem 3.12 (Involution Construction)

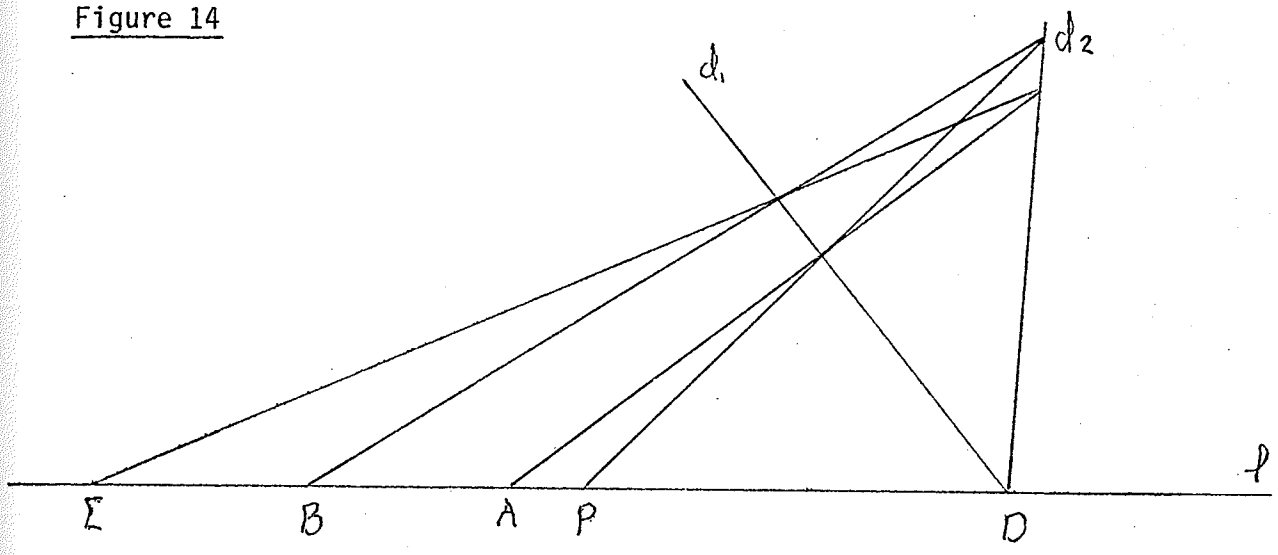
Suppose that A,B,C,D,E are points of a line ℓ and a,b,c,d,e are lines through A,B,C,D,E respectively such that $a \cdot c$ and $b \cdot d$ lie on the line e. Then the point $P = (a \cdot d)(b \cdot c) \cdot \ell$ is invariant of the choice of lines.

Figure 13



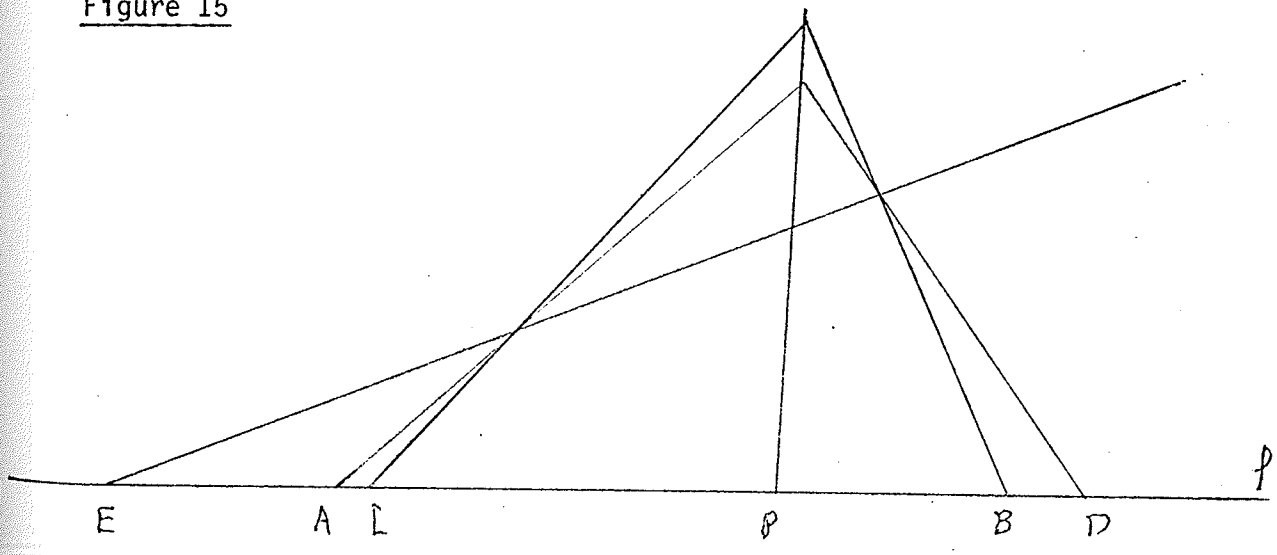
In the case of the Harmonic Construction the points of the quadrangle are X_1, X_2, X, Y so that A and B are diagonal points. The points P and C are intersections of the remaining two lines of the figure.

Figure 14



In the case of the Addition Construction the points of the quadrangle are c . b, a . d_1, b . d_2, c . a. The point D is a diagonal point and the other four points are intersections of l with the quadrilateral configuration.

Figure 15



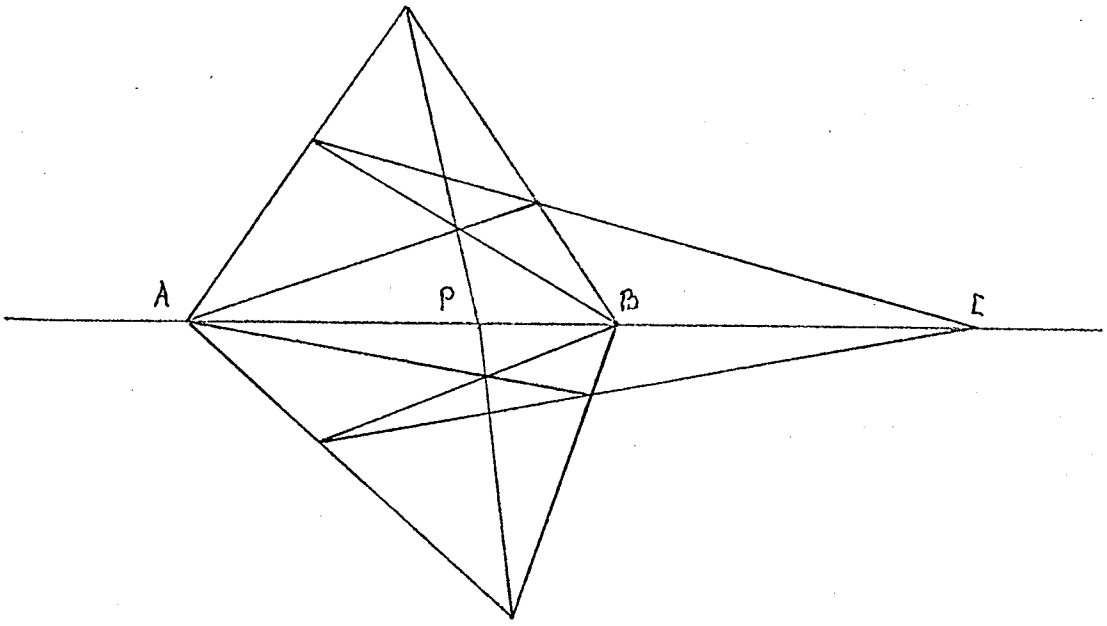
In the case of the Involution Construction the points of the quadrangle are a . c, b . d, b . c, a . d.

The usual proofs of the invariance of the Harmonic and Involution Constructions as in [3] are not uniform in regard to method. There is one common aspect and that is the use of three dimensional methods. It will be shown later in this section of the chapter that this use of three dimensional geometry to prove the invariance of the three trace constructions is essential. The following proofs use a new and uniform method.

Proof of the Invariance of the Harmonic Construction

As in [3] we will prove that two instances of the construction in different planes yield the same point. The general proof then follows from this.

Figure 16

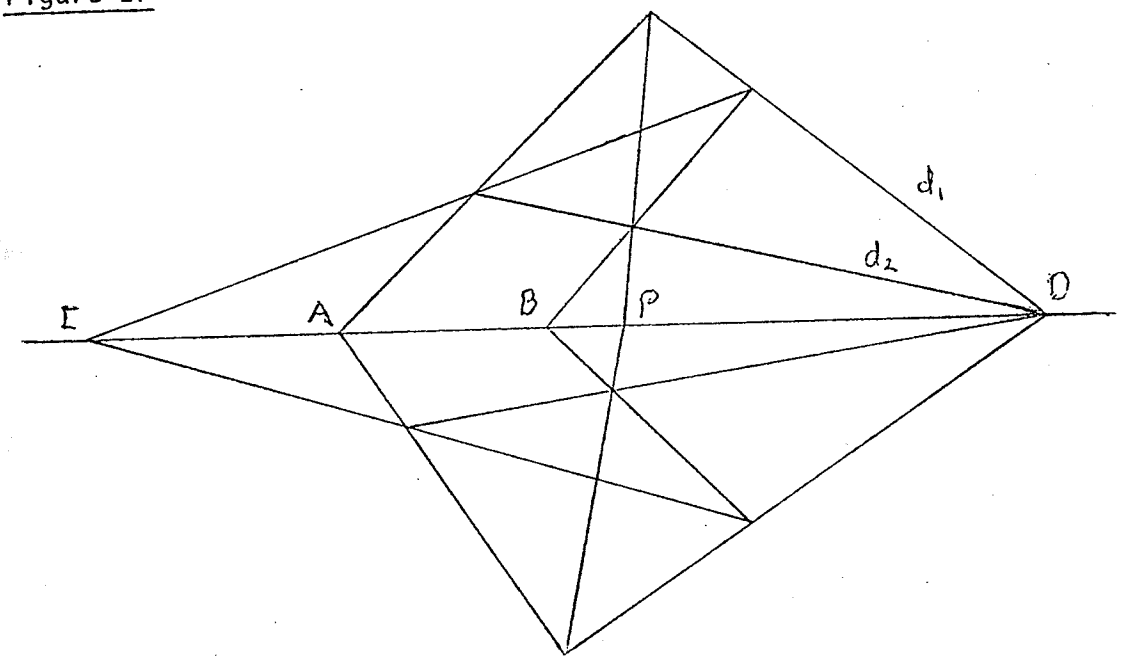


We agree in this and the next two proofs to let corresponding points and lines of the second construction be denoted by the corresponding dashed letter. Note that the triangles XX_1X_2 and $X^1\ X_1{}^1\ X_2{}^1$ are in perspective through some point Q. The triangles $X_1\ X_2Y$ and $X_1{}^1X_2{}^1Y^1$ are also in perspective. The triangles XX_1X_2 and X_1X_2Y have a side in common as do the triangles $X^1X_1{}^1X_2{}^1$ and $X_1{}^1X_2{}^1Y^1$. Consequently X_1X_2Y and $X_1{}^1X_2{}^1Y^1$ are also in perspective through the point Q. The lines XX^1 and YY^1 thus meet at Q and hence these lines are coplanar. If S is the plane of the two lines then S and ℓ meet at one point and thus $XY \cdot \ell = X^1Y^1 \cdot \ell$. ☒

In the next two proofs we will point out a chain of perspective triangles that allow us to deduce the conclusion as in the above argument.

Proof of the Invariance of the Addition Construction

Figure 17



We note the following chain of perspective triangles:

(i) $(a \cdot d_1)(b \cdot d_1)(a \cdot d_2)$ is perspective with

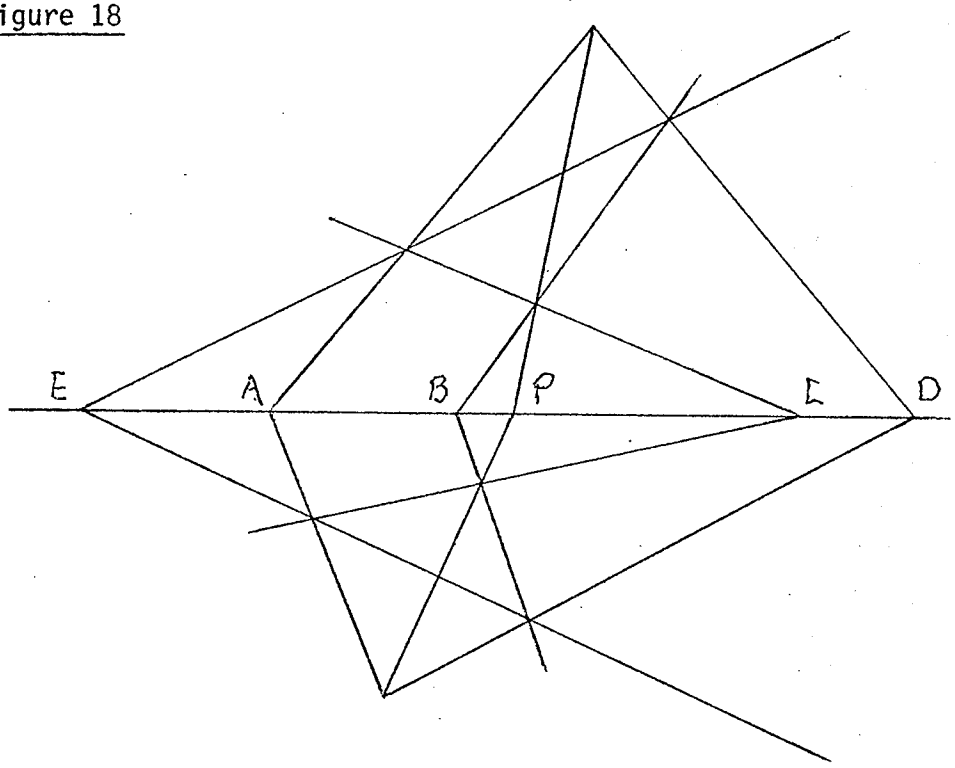
$(a^1 \cdot d_1{}^1)(b^1 \cdot d_1{}^1)(a^1 \cdot d_2{}^1)$

(ii) $(a \cdot d_2)(b \cdot d_1)(b \cdot d_2)$ is perspective with

$(a^1 \cdot d_2{}^1)(b^1 \cdot d_1{}^1)(b^1 \cdot d_2{}^1)$ ⊠

Proof of the Invariance of the Involution Construction

Figure 18



We note the following chain of perspective triangles:

(i) $(a \cdot d)(b \cdot d)(c \cdot a)$ is perspective with

$(a^1 \cdot d^1)(b^1 \cdot d^1)(c^1 \cdot a^1)$

(ii) $(b \cdot d)(c \cdot a)(c \cdot b)$ is perspective with

$(b^1 \cdot d^1)(c^1 \cdot a^1)(c^1 \cdot b^1)$. ⊠

A relation among a set of points on a line that is determined by an invariant line construction in a plane containing the line we call an invariant relation.

Theorem 3.13

An invariant point relation is projectively invariant.

Proof

Suppose that ℓ is a line and points $P_1, \ldots, P_n$ are in some invariant relation on ℓ. Assume that the lines ℓ and ℓ_1 are in perspective from a point Q. Let S_1 be a plane through ℓ that does not contain ℓ_1 and let S_2 be a plan through ℓ_1 that does not contain ℓ. In the plane S_1 construct an instance of the invariant line construction that determines the relation of $P_1, \ldots, P_n$. If we project this construction from Q onto the plane S_2 the projected figure is an instance of the invariant line construction and so the projections of the points $P_1, \ldots, P_n$ are in the given invariant relation. ⊠

It follows from the above general theorem that the three invariant trace relations are projectively invariant. Other proofs for specific cases can be found in [3] and [35]. Note that the above proof depends on the use of three dimensional geometry. Consequently it does not dualize to the case of pencils for in three dimensional space the dual of a point is a plane so that projection from a line cannot be extended. This results in an asymmetry in the theory.

Definition 3.3

Let S be a plane. Then S satisfies the invariant condition if every invariant relation among points on lines in S is also projectively invariant.

For purposes of exposition we now give an explicit statement of each of the duals of the three trace theorems.

The Dual Harmonic Construction

Suppose that ℓ_1, ℓ_2, ℓ_3 are lines of a plane pencil and P is a point of ℓ_1. Let S_1 and S_2 be lines through P not equal to ℓ_1. The point $z=(S_1 \cdot \ell_2)(S_2 \cdot \ell_3) \cdot (S_1 \cdot \ell_3)(S_2 \cdot \ell_2)$ lies on a line of the pencil that is independent of the construction.

The Dual Addition Construction

Suppose that $\ell_1, \ell_2, \ell_3, \ell_4$ are lines of a plane pencil and P is a point of ℓ_1. Let S_1 and S_2 be lines through P not equal to ℓ_1. The point $z=(S_1 \cdot \ell_2)(S_2 \cdot \ell_4) \cdot (S_1 \cdot \ell_4)(S_2 \cdot \ell_3)$ lies on a line of the pencil that is independent of the construction.

The Dual Involution Construction

Suppose that $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$ are lines of a plane pencil and P is a point of ℓ_1. Let S_1 and S_2 be lines through P not equal to ℓ_1. The point $z = (S_1 \cdot \ell_2)(S_2 \cdot \ell_5) \cdot (S_1 \cdot \ell_4)(S_2 \cdot \ell_3)$ lies on a line of the pencil that is independent of the construction.

We can now state and prove the following three theorems.

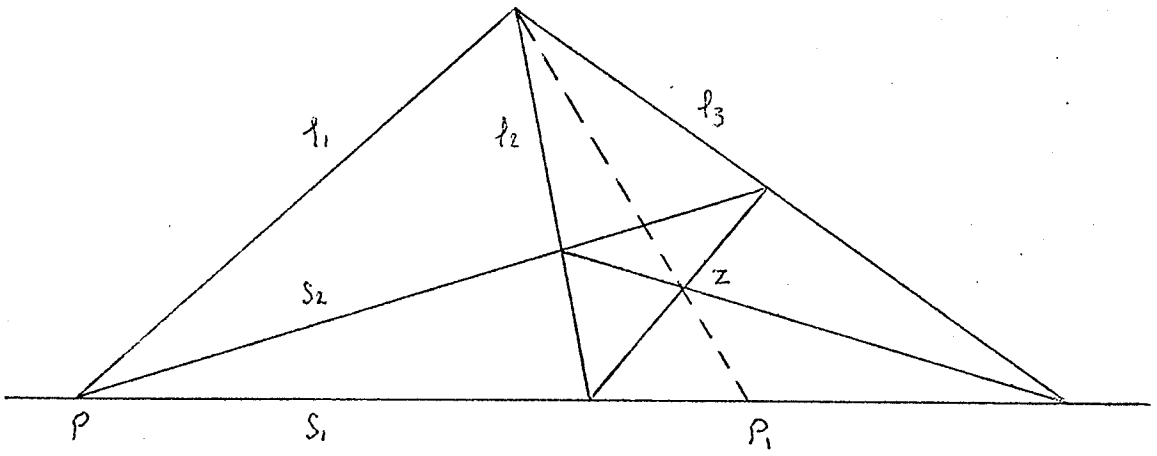
Theorem 3.14

Assume the invariant condition. Then the Harmonic Construction implies the Dual Harmonic Construction.

Proof

Suppose that ℓ_1, ℓ_2, ℓ_3 are three lines of a plane pencil and P is a point of ℓ_1.

Figure 19



Assume that we have completed an instance of the dual harmonic construction with lines S_1 and S_2 through the point P. Let ℓ be the pencil line through z and define P_1 to be $\ell \cdot S_1$. If $A = \ell_2 \cdot S_1$, $B = \ell_3 \cdot S_1$ and $C = P$ then it is immediate that the point P_1 is the constructed point of the Harmonic Construction on the line S_1. Suppose now that Q is another point of ℓ_1 and R_1 and R_2 are lines through Q not equal to ℓ_1. With these lines we can complete another instance of

the dual harmonic construction. As above the line $\ell 1$ so constructed intersected with R_1 provides an instance of the Harmonic Construction. It follows from the invariant condition that the line $\ell 1$ must be ℓ. ☒

The proofs of the next two theorems use the same method as that of the above argument. That is we interpret an instance of a dual trace constructions as an instance of the corresponding trace construction and then apply the invariant condition to prove that the line obtained is independent of the particular construction. The proofs will therefore be given more briefly.

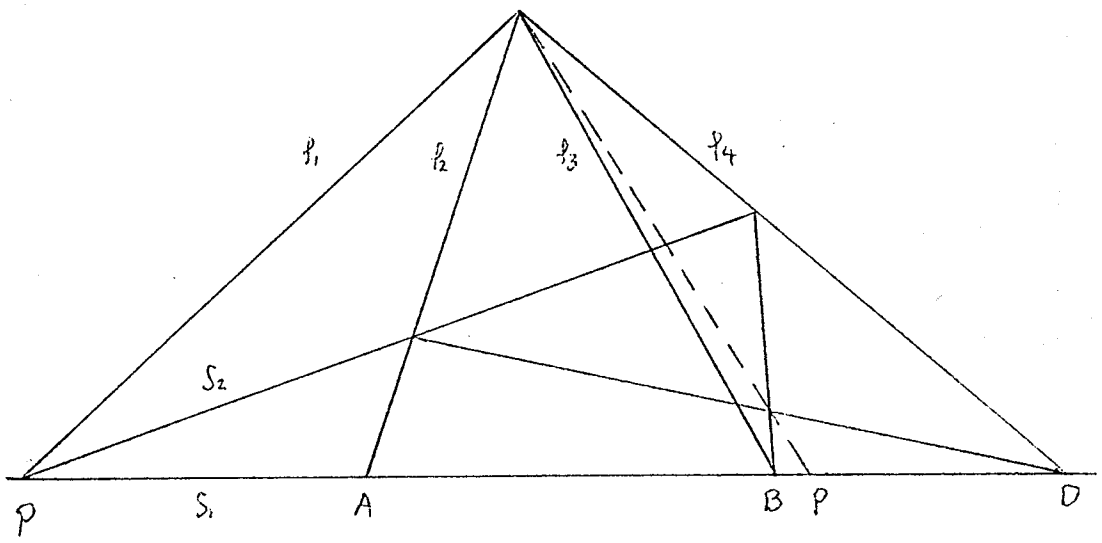
Theorem 3.15

Assume the invariant condition. Then the Addition Construction implies the Dual Addition Construction.

Proof

Let $\ell_1, \ell_2, \ell_3, \ell_4$ be lines of a plane pencil and suppose P is a point of ℓ_1.

Figure 20



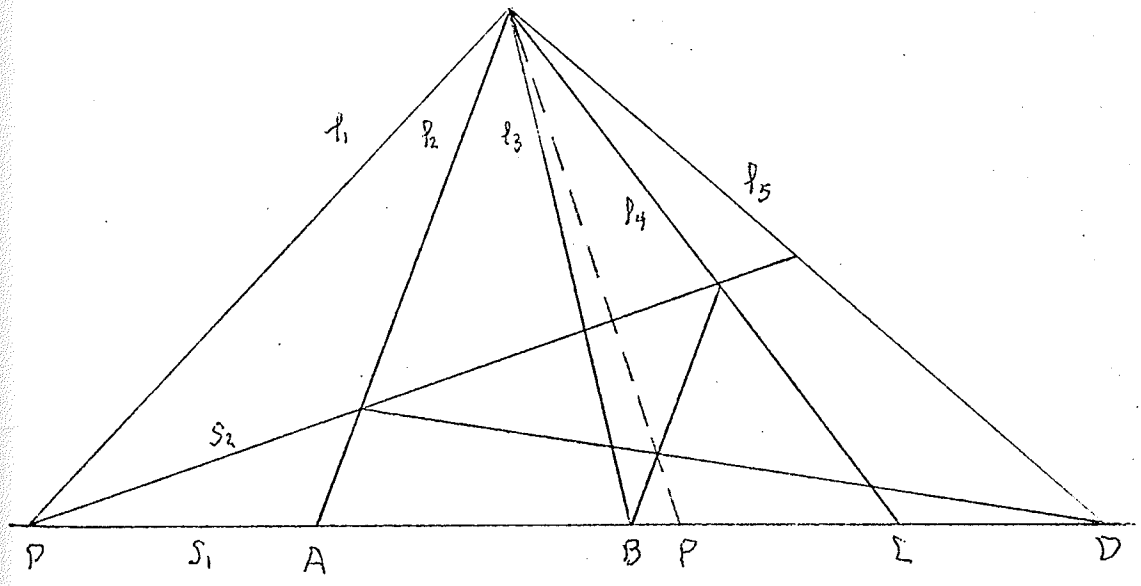
Let S_1 and S_2 be lines through P not equal to ℓ_1. let $A = \ell_2 \cdot S_1$, $B = \ell_3 \cdot S_1$, $C = P$ and $D = \ell_4 \cdot S_1$. If an instance of the dual addition construction is completed using the lines S_1 and S_2 then it is immediate that the configuration on the line S_1 with points A,B,C,D and the constucted line constitute an instance of the Addition Construction. ☒

Theorem 3.16

Assume the invariant condition. Then the Involution Construction implies the Dual Involution Construction.

Proof

Figure 21



Let ℓ_1, ℓ_2, ℓ_3, ℓ_4, ℓ_5 be lines of a plane pencil and suppose that P is a point of ℓ_1. Let S_1 and S_2 be lines through P not

equal to ℓ_1. Suppose that with the lines S_1 and S_2 we have completed an instance of the dual involution construction. We define on S_1 the points $A = \ell_2 \cdot S_1$, $B = \ell_3 \cdot S_1$, $C = \ell_4 \cdot S_1$, $D = \ell_5 \cdot S_1$ and $E = P$. With the points so defined it is immediate that the configuration is an instance of the Involution Construction. ⊠

The objective of the next portion of this section of the chapter is to prove that the assumption of the invariance of the dual trace constructions implies the Desargue Theorem. For purposes of proof we state the following three propositions.

First Three Line Theorem

Let ℓ_1, ℓ_2, ℓ_3 be three concurrent lines and A a point of ℓ_1 and B a point of ℓ_2. There is a fixed point z on the line BA such that if S is any point of ℓ_1 and $R = SB \cdot \ell_3$ the line $S(AR \cdot \ell_2)$ passes through z.

Second Three Line Theorem

Let ℓ_1, ℓ_2, ℓ_3 be three concurrent lines and A a point not on any of these lines. Suppose that B is a point on ℓ_2. There is a fixed point Z on the line AB such that for any point S on ℓ_1 if $R = SB \cdot \ell_3$ then the line $S(RA \cdot \ell_2)$ passes through the point z.

Third Three Line Theorem

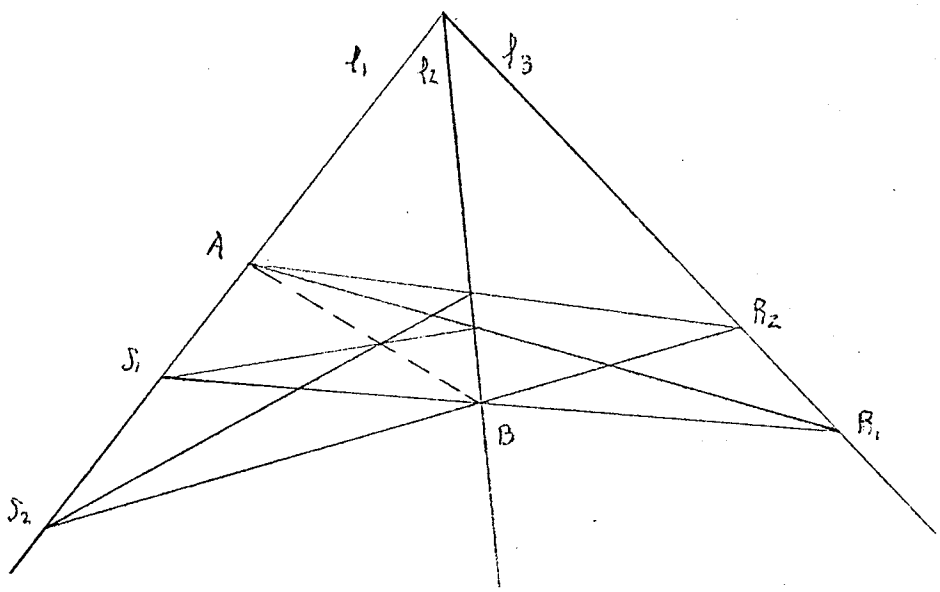
Let ℓ_1, ℓ_2, ℓ_3 be three concurrent lines and A and B two distinct points not on these lines. There is a fixed point z on the line AB such that for any point S on ℓ_1 if $R = SB \cdot \ell_3$ then the line $S(RA \cdot \ell_2)$ passes through the point z.

Theorem 3.17

The Dual Harmonic Construction implies the First Three Line Theorem.

Proof

Figure 22



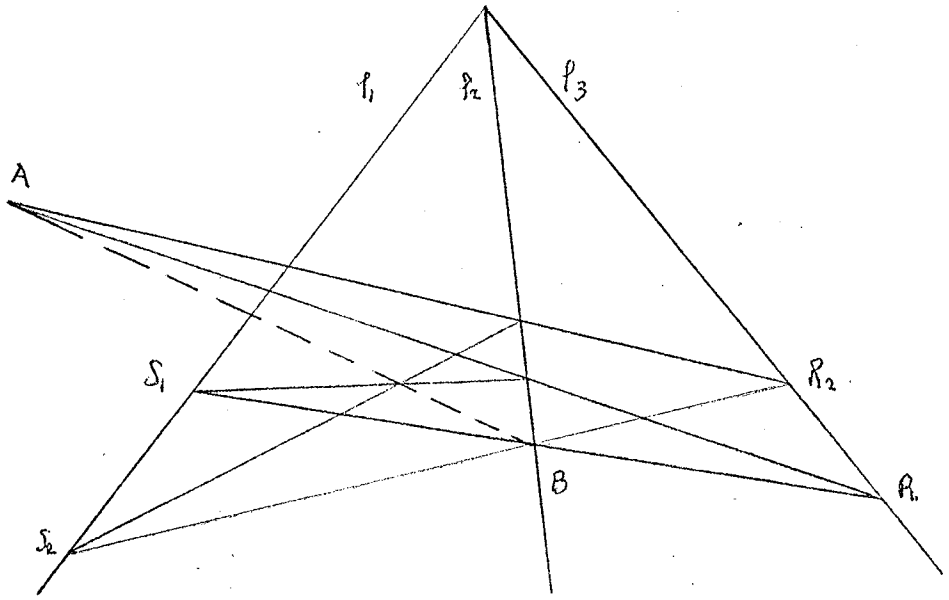
Given a point S_1 on ℓ_1 the point R_2 can serve as the point P of the Dual Harmonic Construction. If $AR_2 \cdot \ell_2 = B_1$ the $z = AB \cdot S_1B_1$ lies on the invariant line ℓ of the Dual Harmonic Construction. With another choice of point S_2 on ℓ_1 the same line is produced and so the line S_2 ($R_2A \cdot \ell_2$) must pass through $Z = AB \cdot \ell$. ☒

Theorem 3.18

The Dual Addition Construction implies the Second Three Line Theorem.

Proof

Figure 23



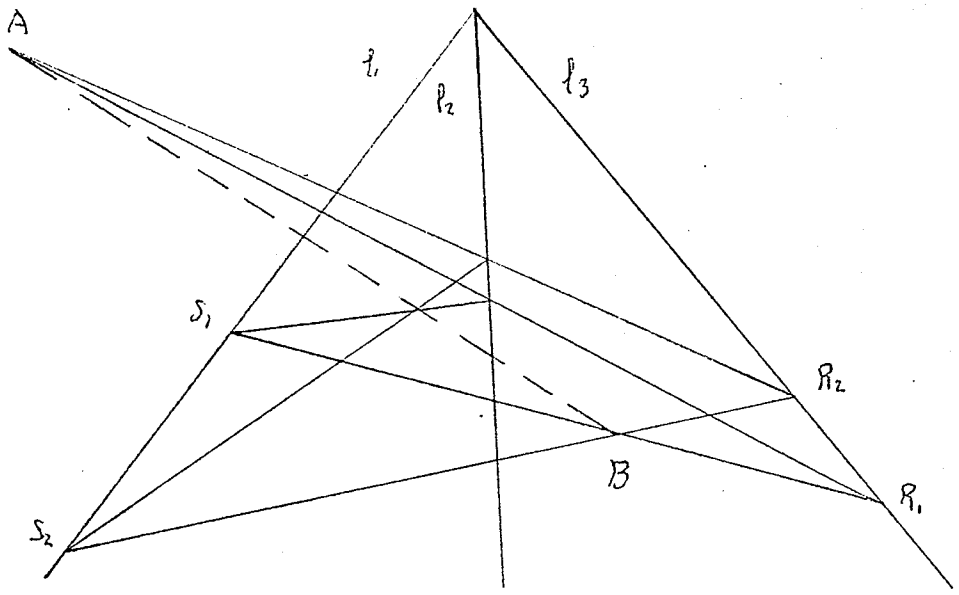
Given a point S_1 on ℓ_1 we define the point R_2 to be the point P of the Dual Addition Construction. We define ℓ_4^1 to be the line AQ where $Q = \ell_1 . \ell_2$. Then z lies on the line AB and so is independent of the choice of the point S_1. ☒

Theorem 3.19

The Dual Involution Construction implies the Third Three Line Theorem.

Proof

Figure 24



Given a point S_1 on ℓ_1 we define $R_2 = P$ and the line A $(\ell_1 . \ell_2)$ to be ℓ_5^1 and B$(\ell_1 . \ell_2)$ to be ℓ_2^1 of the Dual Involution Construction. It is clear how to complete the interpretation and the proof. ☒

There are three possible cases of the Desargue Theorem. We list them as follows.

First Case of the Desargue Theorem

The intersection points of two corresponding sides are on lines of perspective.

Second Case of the Desargue Theorem

The intersection point of a pair of corresponding sides is on a line of perspective. No other intersection point of corresponding sides is on a line of perspective.

Third Case of the Desargue Theorem

The intersection points of corresponding sides are not on lines of perspective.

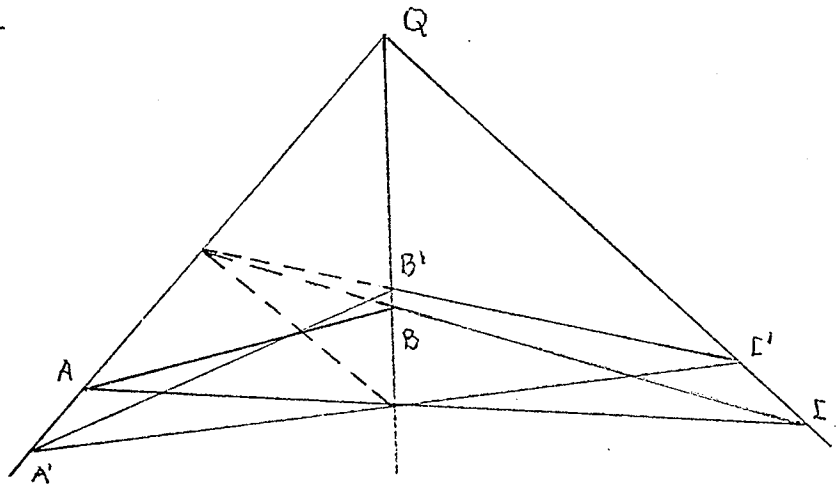
Theorem 3.20

The First Three Line Theorem implies the First Case of the Desargue Theorem.

Proof

Suppose that the triangles ABC and $A^1B^1C^1$ are in perspective from a point Q.

Figure 25



Assume that the two sides AC and A^1C^1 meet on BB^1 and the two sides BC and B^1C^1 meet on AA^1. We let $\ell_1 = AQ$ and $\ell_2 = BQ$ and $\ell_3 = CQ$ for the three lines of the first three line configuration. For the points A and B we take $BC \cdot B^1C^1$ and $AC \cdot A^1C^1$ respectively. It follows from the First Three Line Theorem that $AB \cdot A^1B^1$ is on the line $(AC \cdot A^1C^1)(BC \cdot B^1C^1)$. ☒

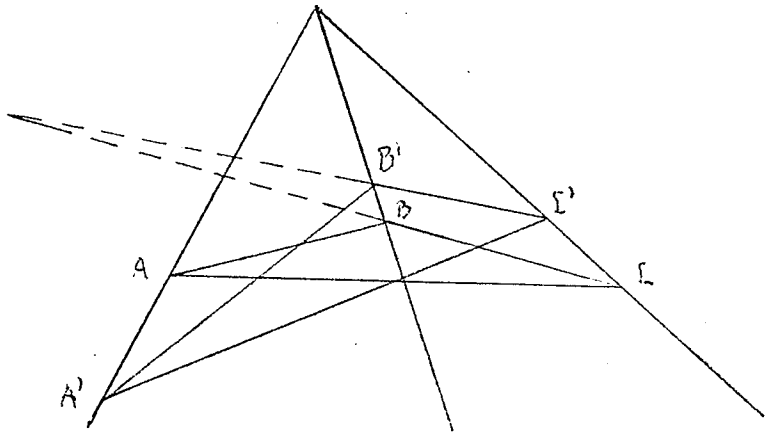
Theorem 3.21

The Second Three Line Theorem implies the Second Case of the Desargue Theorem.

Proof

Suppose that the triangles ABC and $A^1B^1C^1$ are in perspective from Q.

Figure 26



Suppose that the sides AC and A^1C^1 meet on BB^1. For the ℓ_1 take the line AQ, for ℓ_2 the line BQ and for ℓ_3 the line CQ. For the point A in the second three line configuration take BC . B^1C^1 and for the point B take AC . A^1C^1. It follows from the Second Three Line Theorem that the points AB . A^1B^1, AC . A^1C^1 and BC . B^1C^1 are in line. ☒

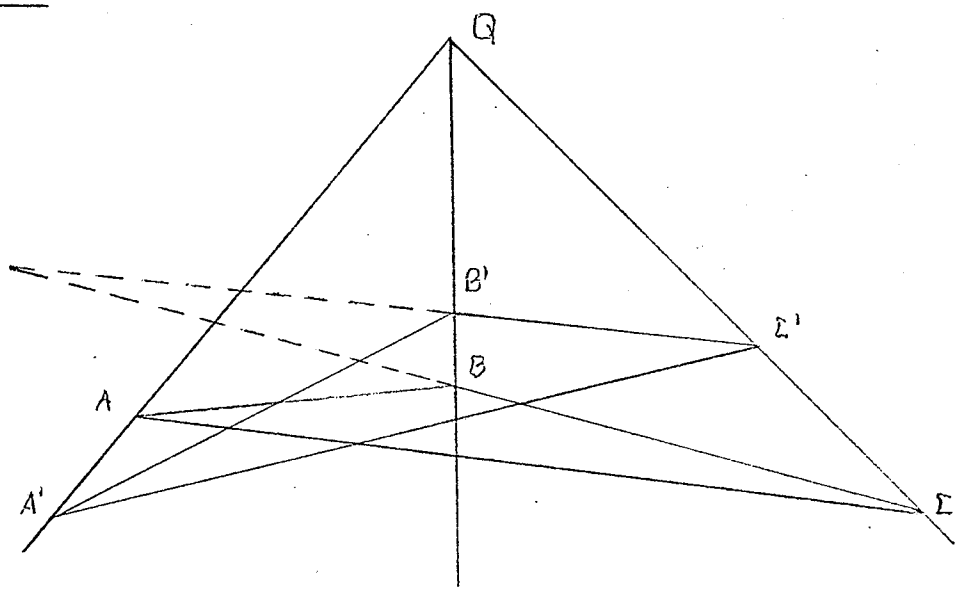
Theorem 3.22

The Third Three Line Theorem implies the Third Case of the Desargue Theorem.

Proof

Suppose that ABC and $A^1B^1C^1$ are triangles in perspective from a point Q.

Figure 27



We assume that no pair of corresponding sides meet on a line of perspective. For the interpretation take ℓ_1 to be AQ, ℓ_2 to be BQ and ℓ_3 to be CQ. Take for the point A the point BC . B^1C^1 and for the point B the point AC . A^1C^1. Then it follows from the Third Three Line Theorem that the points AB . A^1B^1, AC . A^1C^1 and BC . B^1C^1 are in line. ⊠

The three theorems proven above imply together the following theorem.

Theorem 3.23

The assumption of the invariance of the three dual trace constructions implies the Desargue Theorem.

The proof of the above theorem can be dualized. As the Desargue Theorem is equivalent to the Dual Desargue Theorem this implies the following theorem.

Theorem 3.24

The assumption of the invariance of the three trace constructions implies the Desargue Theorem.

A result of [3] shows that the Desargue Theorem does not follow from the propositions of incidence in the plane. This fact together with Theorem 3.24 imply the following theorem.

Theorem 3.25

The invariance of the three trace constructions does not follow from the propositions of incidence in the plane.

3.4 Invariant Constructions

The following four line constructions are constructions using six, seven, eight and nine points. These constructions continue the sequence of constructions utilizing three, four and five points initiated by the Harmonic, Addition and Involution Constructions. The existence of these four new invariant constructions demonstrates that not all invariant constructions with more than five points reduce to the constructions on three, four and five points.

Theorem 3.26 (Six Point Construction)

Let F,A,B,C,E,D be points of a line ℓ in a plane S. Assume there

are lines f,a,b,c,e and d_1, d_2 through the respective points such that: (i) b . d_1 and a . d_2 lie on f, (ii) c . a lies on d_1 and e . b lies on d_2. Then the point P = (c . d_2)(e . d_1) . ℓ is invariant.

Theorem 3.27 (Seven Point Construction)

Let H,A,C,B,D,F,G be points of a line ℓ in a plane S. Assume there are lines h,a,b,c,d,f,g through the respective points such that c . f and a . d lie on h. If g and b pass through c . d then the point P = (g . a) (b . f) . ℓ is invariant.

Theorem 3.28 (Eight Point Construction)

Let G,A,B,C,D,E,F,H be points on a line ℓ in a plane S and assume that g,b,c,d,e and a_1, a_2 are lines through the corresponding points such that b,d,a_1 meet on g and c,e,a_2 meet on g. Suppose that the line f passes through F and b . e and h passes through H and a_2 . d. The point (h . a_1)(c . f) . ℓ = P is invariant.

Theorem 3.29 (Nine Point Construction)

Let F,A,C,E,G,H,I,B,D be points on a line ℓ in a plane S and assume that f,a,c,e,g,h,i,b,d are lines through the respective points such that a,c,e meet on f and b,d,i meet on f. If g passes through a . i and h passes through b . e then the point (g . d)(c . h) . ℓ = P is invariant.

The following diagrams illustrate the above constructions.

Figure 28 (Six Point)

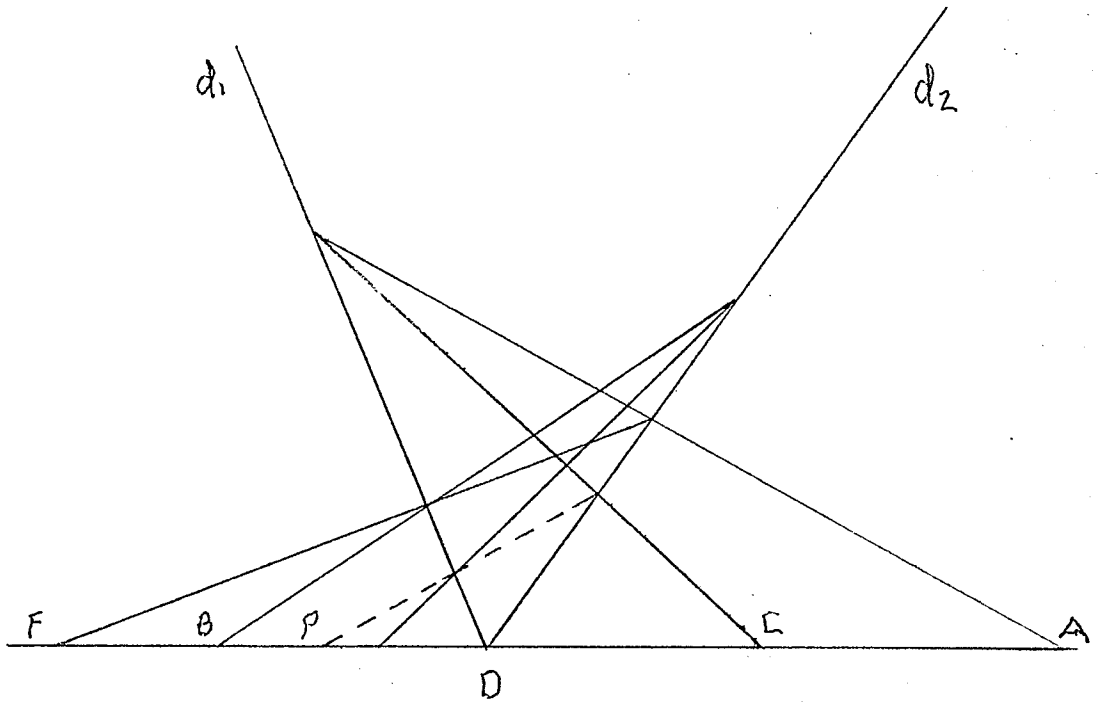


Figure 29 (Seven Point)

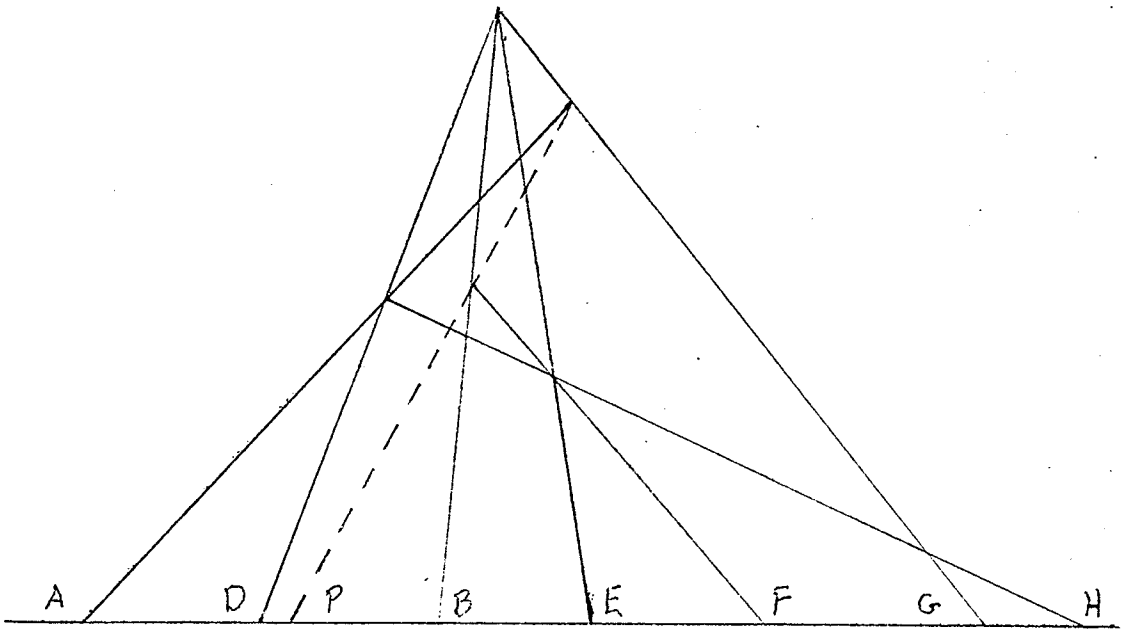


Figure 30 (Eight Point)

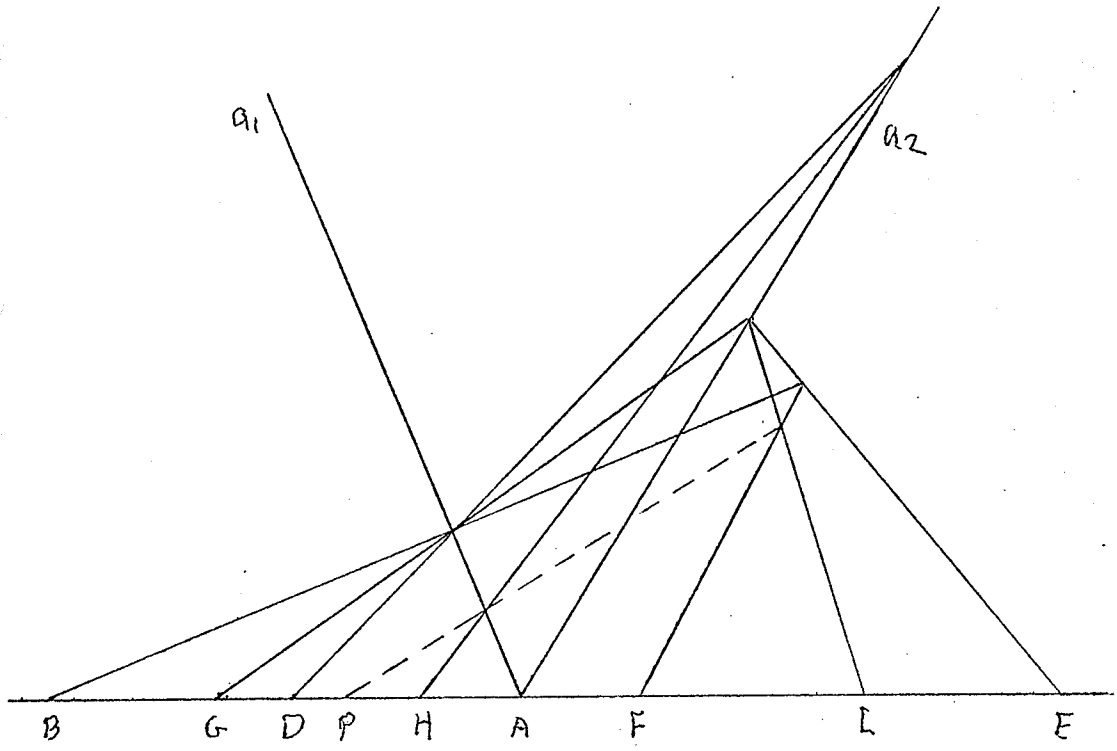
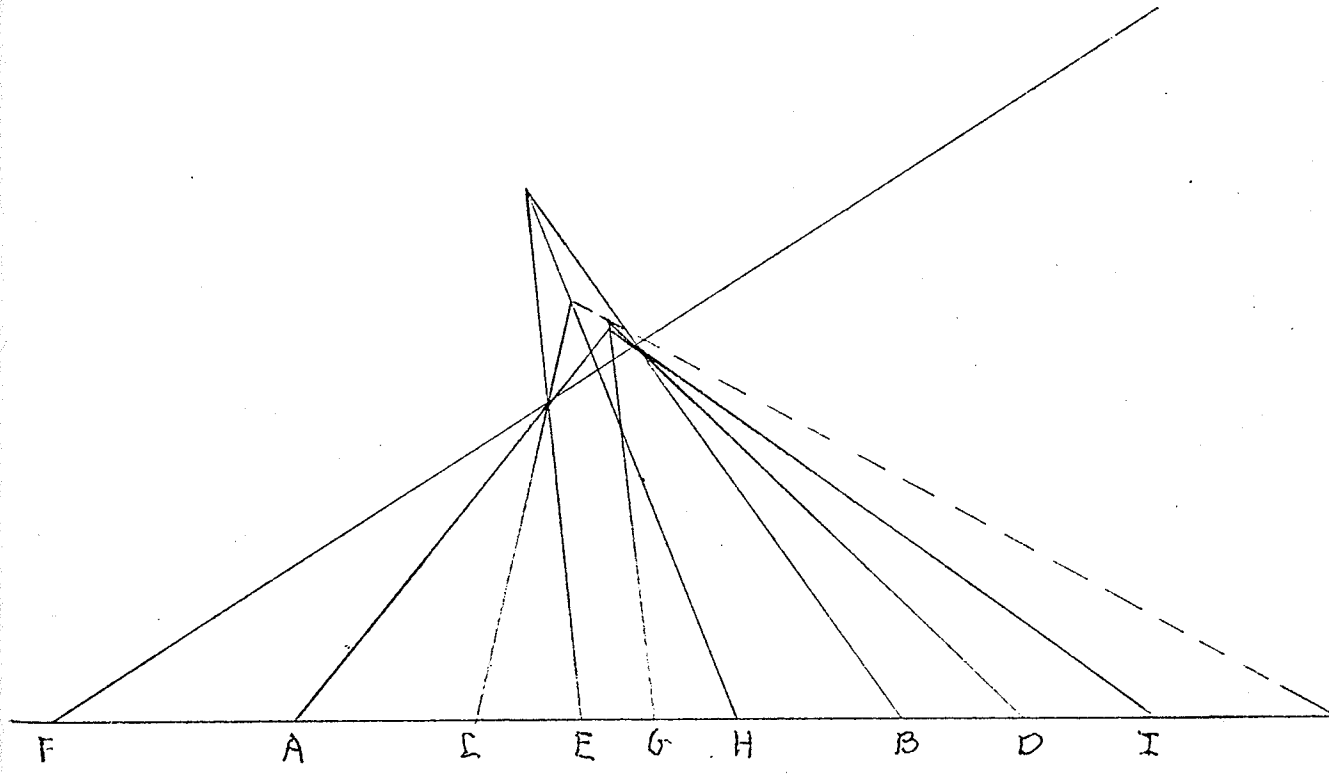


Figure 31 (Nine Point)



Note that in the sequence of invariant constructions that begins with the Harmonic Construction the constructions alternate between having a single line or two lines at the center of the configuration. The constructions beginning with an even number of points utilize two lines.

We now give proofs of the above theorems in the given order. The method of proof in each case is the method of triangle chains introduced above. In each of the proofs we assume that we have two instances of the particular construction in two distinct planes S_1 and S_2. Lines through points on the line that have upper case labels are labelled with the corresponding lower case letters in the plane S_1 and in S_2 with corresponding dashed letters. In each proof we display a chain of perspective triangles.

(1) (Six Point)

We display a chain of perspective triangles.

(i) $(c \cdot d_2)(a \cdot d_2)(c \cdot f)$ is perspective with
$(c^1 \cdot d_2{}^1)(a^1 \cdot d_2{}^1)(c^1 \cdot f^1)$

(ii) $(c \cdot f)(a \cdot d_2)(c \cdot d_1)$ is perspective with
$(c^1 \cdot f^1)(a^1 \cdot d_2{}^1)(c^1 \cdot d_1{}^1)$

(iii) $(c \cdot d_1)(a \cdot d_2)(b \cdot f)$ is perspective with
$(c^1 \cdot d_1{}^1)(a^1 \cdot d_2{}^1)(b^1 \cdot f^1)$

(iv) $(a \cdot d_2)(b \cdot f)(b \cdot d_2)$ is perspective with
$(a^1 \cdot d_2)(b^1 \cdot f^1)(b^1 \cdot d_2{}^1)$

(v) $(b \cdot f)(b \cdot d_2)(e \cdot d_1)$ is perspective with
$(b_1 \cdot f^1)(b^1 \cdot d_2{}^1)(e^1 \cdot d_1{}^1)$.

Thus the lines $(c \cdot d_2)(c^1 \cdot d_2{}^1)$ and $(e \cdot d_1)(e^1 \cdot d_1{}^1)$ are coplanar.

(2) (Seven Point)

We display the following chain of perspective triangles.

(i) $(a \,.\, g)(c \,.\, a)(b \,.\, d)$ is perspective with

$(a^1 \,.\, g^1)(c^1 \,.\, a^1)(b^1 \,.\, d^1)$

(ii) $(c \,.\, a)(b \,.\, d)(a \,.\, h)$ is perspective with

$(c^1 \,.\, a^1)(b^1 \,.\, d^1)(a^1 \,.\, h^1)$

(iii) $(a \,.\, h)(b \,.\, d)(c \,.\, h)$ is perspective with

$(a^1 \,.\, h^1)(b^1 \,.\, d^1)(c^1 \,.\, h^1)$

(iv) $(c \,.\, h)(b \,.\, d)(b \,.\, f)$ is perspective with

$(c^1 \,.\, h^1)(b^1 \,.\, d^1)(b^1 \,.\, f^1)$.

Thus the lines $(a \,.\, g)(a^1 \,.\, g^1)$ and $(b \,.\, f)(b^1 \,.\, f^1)$ are coplanar.

(3) (Eight Point)

We display the following chain of perspective triangles.

(i) $(h.a_1)(b.a_1)(d.h)$ is perspective with

$(h^1.a_1{}^1)(b^1.a_1{}^1)(d^1.h^1)$

(ii) $(b.a_1)(d.h)(g.a_2)$ is perspective with

$(b^1.a_1{}^1)(d^1.h^1)(g^1.a_2{}^1)$

(iii) $(g.a_2)(b.f)(b.a_1)$ is perspective with

$(g^1.a_2{}^1)(b^1.f^1)(b^1.a_1{}^1)$

(iv) $(g.a_2)(b.f)(f.c)$ is perspective with

$(g^1.a_2{}^1)(b^1.f^1)(f^1.c^1)$.

Thus the lines $(h.a_1)(h^1.a_1{}^1)$ and $(f.c)(f^1.c^1)$ are coplanar.

(4) (Nine Point)

We display the following chain of perspective triangles.

(i) $(c.h)(b.e)(c.e)$ is perspective with

$(c^1.h^1)(b^1.e^1)(c^1.e^1)$

(ii) $(c.e)(b.e)(f.b)$ is perspective with

$(c^1.e^1)(b^1.e^1)(f^1.b^1)$

(iii) $(c.e)(f.b)(i.g)$ is perspective with

$(c^1.e^1)(f^1.b^1)(i^1.g^1)$

(iv) $(f.b)(i.g)(g.d)$ is perspective with

$(f^1.b^1)(i^1.g^1)(g^1.d^1)$.

Thus the lines $(c.h)(c^1.h^1)$ and $(g.d)(g^1.d^1)$ are coplanar.

We next state and prove the duals of the theorems proved above. The method of proof is an extension of the method used earlier to prove the duals of the Harmonic, Addition and Involution Constructions. In these cases the dual line construction cannot be interpreted in the configuration. Instead the configuration must be completed and the proof reduced to a new line construction. The latter are proven using the triangle chain method.

The following is the list of the pencil constructions that are proven in this section to be invariant.

Theorem 3.30 (Six Line Construction)

Let F,A,B,C,D,E be lines of a pencil. From some point f of F draw lines X and y. Let $a=X.A$, $d_1=y.D$, $d_2=X.D$ and $b=y.B$. Further let $e=ad_1.E$ and $c=bd_2.C$. If $p=cd_1.cd_2$ then the line P of the pencil through p is invariant.

Theorem 3.31 (Seven Line Construction)

Let H,A,C,B,D,F,G be lines of a pencil and from a point h of H draw lines x and y. Let a=x.A, d=x.D, c=y.C and f=y.F. Further let g=cd.G and b=cd.B. If p=ag.bf and P is the line of the pencil through p then P is invariant.

Theorem 3.32 (Eight Line Construction)

Let G,A,B,C,D,E,F,H be lines of a pencil and from a point g of G draw lines x and y that meet A at a_1 and a_2 respectively. Let d=X.D, b=X.B, e=y.E and c=y.C. If f=be.F and h=da_2.H then the point p=fc.ha_1 lies on a fixed line P of the pencil.

Theorem 3.33 (Nine Line Construction)

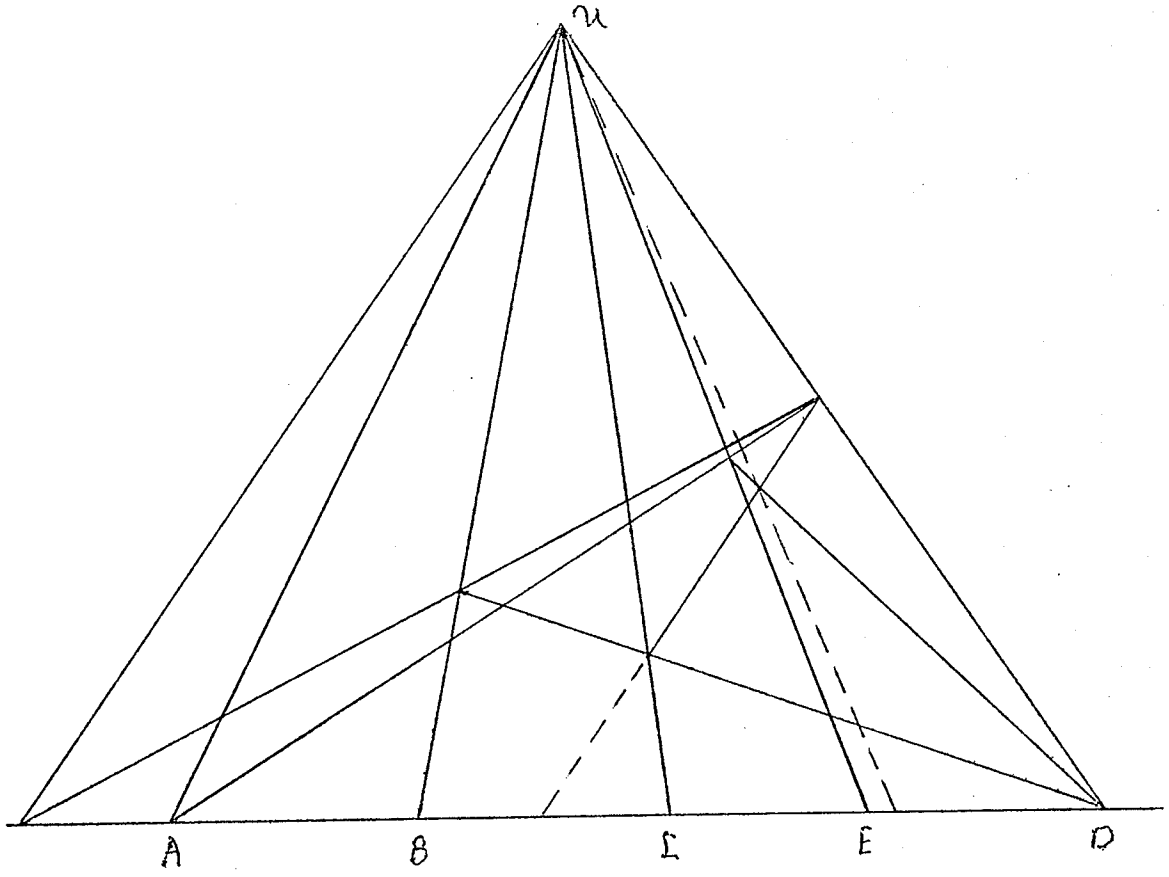
Let F,A,C,E,G,H,B,I,D be lines of a pencil and from a point f of F draw lines x and y. Let a=x.A, e=x.E, c=x.C, b=y.B, i=y.I and d=y.D. Further let g=G.a and h=be.H. If p=gd.ch then the line P of the pencil through p is invariant.

We next give proofs for the above theorems in the above order of presentation. In each proof we assume that we are given two instances of the pencil construction in two distinct planes that pass through a particular line of the construction.

(1) (Six Line)

Let the center point of the pencil be U. We let the base line for the two constructions be the line x of the pencil construction. Corresponding points of the second construction are denoted by dashed letters.

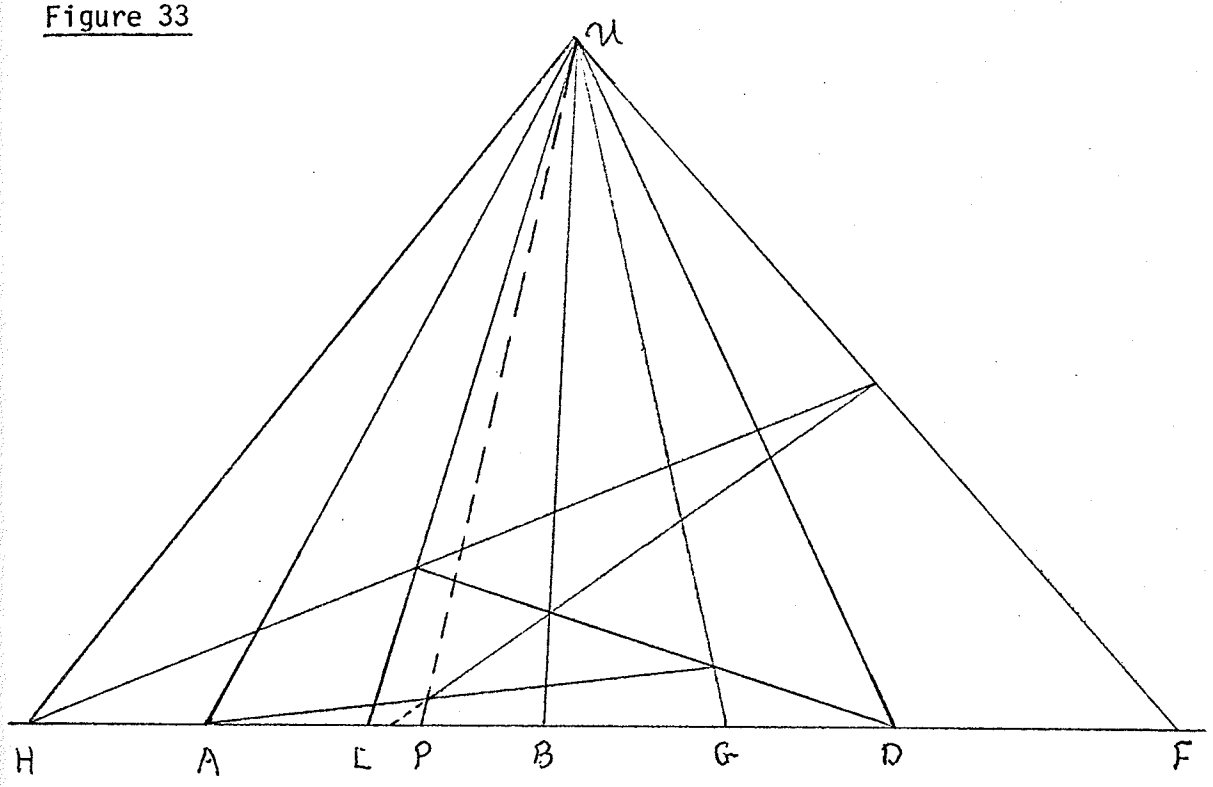
Figure 32



Let Z=y.C. We first prove that cd_1 and $c^1d_1{}^1$ meet on the line X. To demonstrate this we show that the triangles bcd, and $b^1c^1d_1{}^1$ are in perspective. Note that cbz,bzU and zUd_1 yield a chain of triangles and we thus conclude that the lines bb^1, cc^1 and $d_1d_1{}^1$ are concurrent. By the Desargue Theorem we conclude that cd_1 and $c^1d_1{}^1$ meet on the line X. To complete the proof of the theorem note that epd_1 and ed_1U yield the required triangle chain. The proof can now be completed be an application of the invariant condition.

(2) (Seven Line)

Figure 33



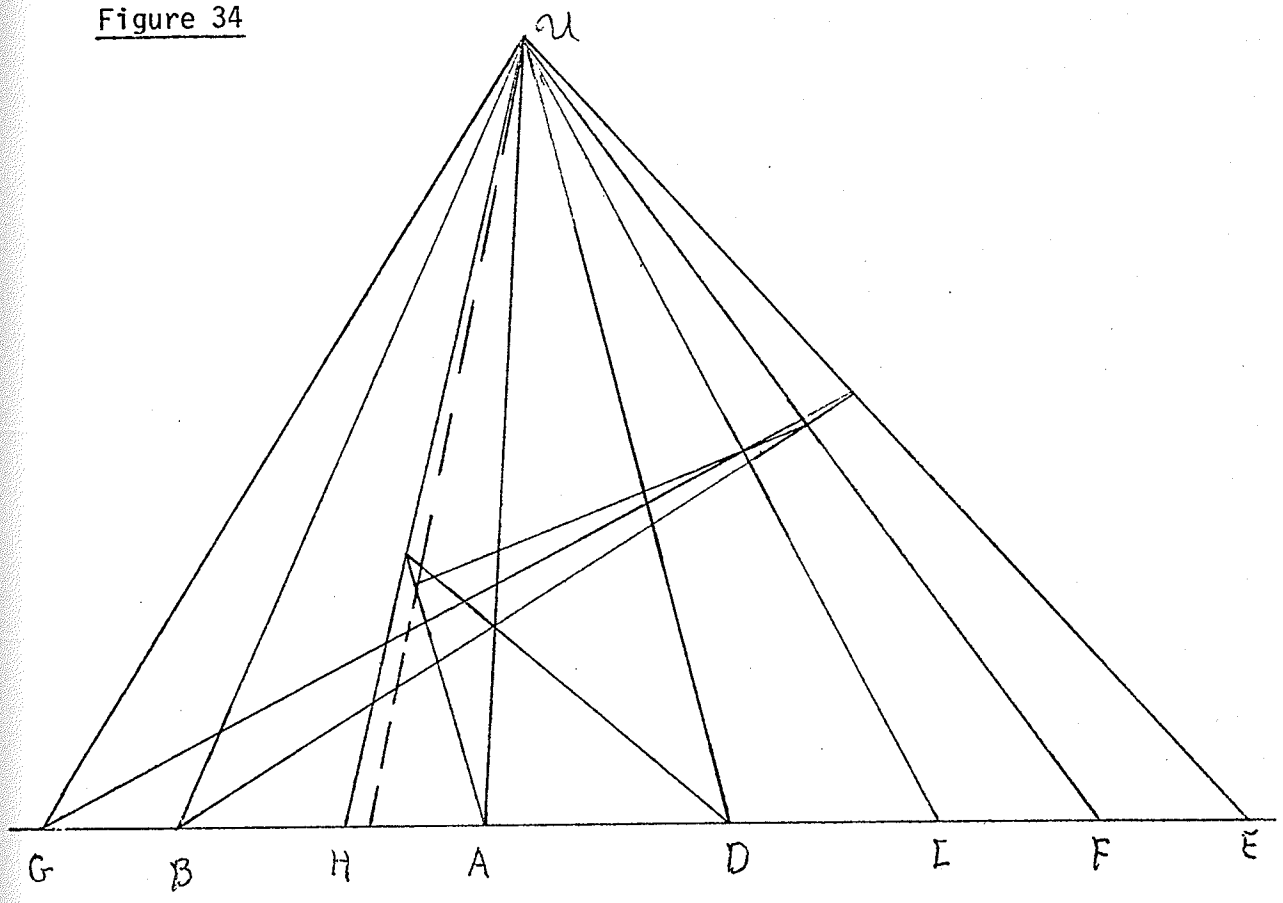
We assume the conventions of the previous proof and also that the center of the pencil is the point U.As in the six line construction a preliminary fact concerning the construction must be proven. In this case we have to demonstrate that the lines bf and b^1f^1 meet on the line x which is taken to be the base line. To prove this we show that the triangles cbf and $c^1b^1f^1$ are in perspective. This perspectivity follows from the triangle chain Ucf and Ucb. To complete the proof note that the triangle chain Ubg and pbg connects the points U and p. We can now proceed as in the previous case.

3. (Eight Line)

We assume the conventions of the previous proofs and also that the

center of the pencil is the point U.

Figure 34



We let the base line be the line x. Let a=A.y. The proof consists of the following three steps.

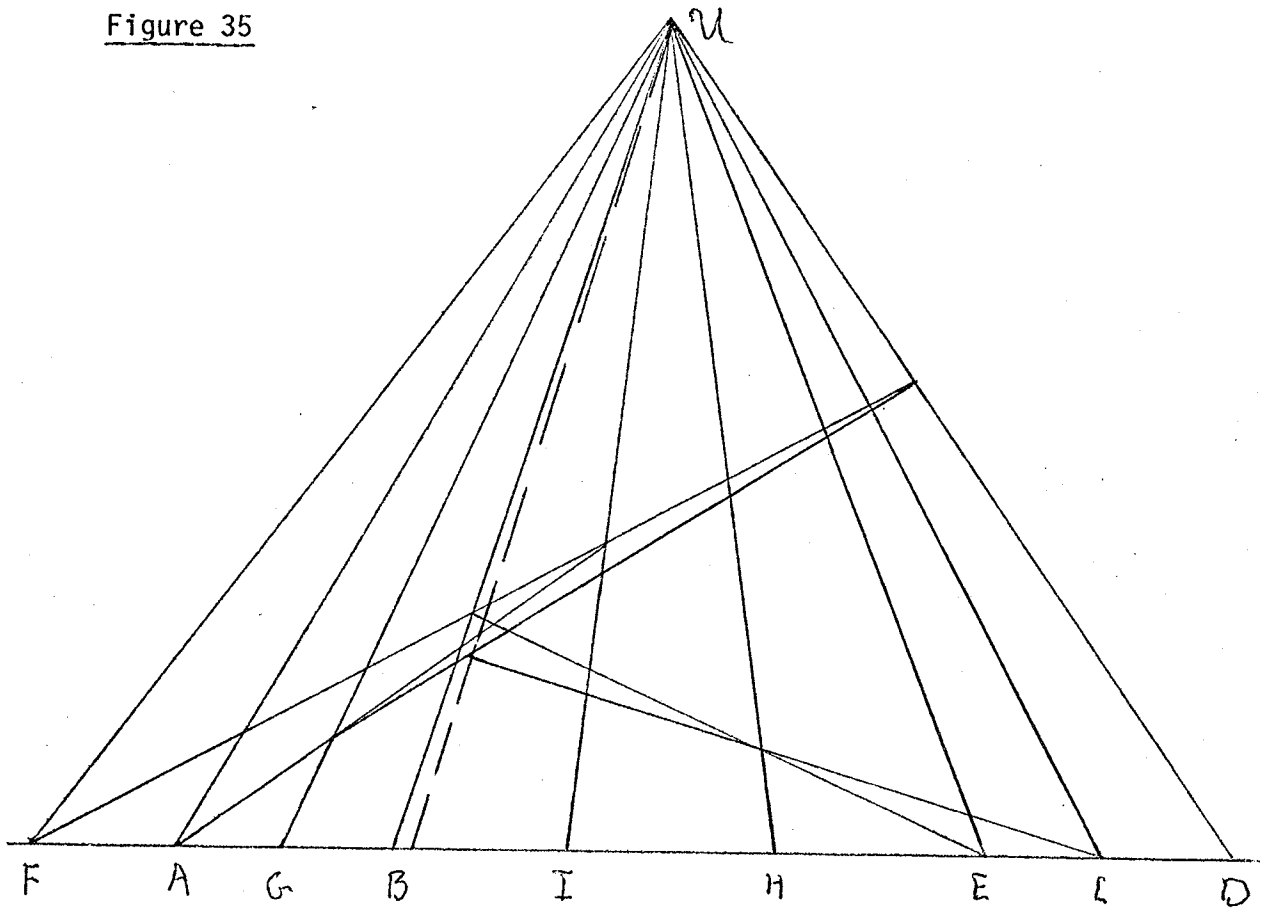
(i) Uce and Uef constitute a triangle chain and so Ucf and $U^1c^1f^1$ are perspective. Thus cf and c^1f^1 meet on the line X.

(ii) a_2ac and Uac constitute a triangle chain and so the triangles Ua_2c and $U^1a_2{}^1c^1$ are in perspective. Thus Ua_2 and $U^1a_2{}^1$ meet on the line X.

(iii) pa_2h and ha_2U constitute a triangle chain and this demonstrates that the line P is invariant.

4. (Nine Line)

Figure 35



We follow the conventions of the previous proofs. Assume that the center of the pencil is the point U and let e be the point hb.dg. The line x is the base line for the construction. The proof consists of the following three steps.

(i) giU and Uid constitute a triangle chain and so the lines gd and g^1d^1 meet on the line X.

(ii) bed and bdU constitute a triangle chain and so the lines eU and e^1U^1 meet on the line X.

(iii) peh and ehU constitute a triangle chain and so the lines pU and p^1U^1 meet on the line X. Hence the line P is invariant.

Chapter 4: The Model Theory of Universal Classes

4.0 Introduction

In this chapter we derive some results in the model theory of universal classes. For the most part we will assume that the class has the amalgamation property. As was demonstrated in [13] universal classes having the amalgamation property have many of the properties of the class of fields of some fixed characteristic. Thus one of the goals of this chapter is to extend key results of algebraic geometry to the general context of open formulas defined over structures in universal classes with the amalgamation property. A second goal of the chapter is to prove analogues of basic theorems of first order model theory for the model theory of universal classes with the amalgamation property.

4.1 Existentially Closed Structures and a Two Cardinal Theorem

We begin this section with some results concerning the atomic diagrams of structures in universal classes.

Definition 4.1

Suppose that Σ is a universal class. If $A \in \Sigma$ has the property that for all extensions B of A in Σ the atomic diagram of B equals that of A then the atomic diagram D of A is a maximal Σ-diagram.

Theorem 4.1

Suppose that Σ is a universal class and D is a maximal Σ-diagram. Then $\Sigma_D = \Sigma_T$ for some complete theory T.

Proof

Suppose that A Σ has atomic diagram D. Let T = Th(A).

Claim

$\Sigma_D = \Sigma_T$. Suppose first that $B \in \Sigma$ is a D-structure. We want to show that $T \cup D_A(B)$ is a consistent theory. If Γ is the open type of any finite subset of B then $\Gamma \in D$ and so Γ is realized in A. Therefore the consistency of $T \cup D_A(B)$ follows immediately from the Compactness Theorem. Suppose next that C is a model of T and $(x_1,\ldots,x_n)$ is a sequence in C. Let Γ be the open type of this sequence. If S is any finite subset of Γ then $C \models \exists v_1,\ldots,\exists v_n \wedge S$ and so $A \models \exists v_1,\ldots,\exists v_n \wedge S$. It follows that Γ is consistent with $T \cup D_A(A)$ and so some extension $B \in \Sigma$ of A realizes Γ. By the maximality of D it follows that Γ is realized in A. Thus $D_A F(C) \subseteq D$ and therefore $C \in \Sigma_D$. ⊠

The following two theorems are analogues of results for types in first order model theory.

Theorem 4.2

Suppose that Σ is a universal class and D is a maximal Σ-diagram with $|D| \geq w$. Then $|D| = w$ or $|D| = 2^w$.

Proof

By definition if D is a maximal Σ-diagram there is $A \in \Sigma$ such that $D_A F(A) = D$. Then $|D| \geq w$ if A is not maximal in Σ. We assume that $|D| > w$. Then for some $n < w$ we must have that the set of n-types in

has power >w. For simplicity assume that n = 1. The argument is similar in the other cases. Since L is countable there is some formula $\psi_1(v_0)$ in L_A such that ψ_1 contains > w D-types. We claim there is $\theta(v_0)$ in L_A such that both $\psi_1 \wedge \theta$ and $\psi_1 \wedge \theta$ contains > w D-types. For a given formula θ let π_θ be the side of the splitting of the D 1-types in ψ_1 of power $\leq$ w. Then there is no more than one D-type in ψ_1 not in $\cup\pi_\theta$ as θ ranges over all L_A-formulas. For if $\Gamma_1 \neq \Gamma_2$ let $\theta \in \Gamma_1$ and $\neg\theta \in \Gamma_2$. Then at least one of Γ_1, Γ_2 is in π_θ. But $\cup\pi_\theta$ has power $\leq$ w. Contradiction. It is now clear that we can proceed to construct a full binary tree of formulas such that a finite segment of each branch has a solution in A. It follows by the compactness theorem that if X is the family of (incomplete) L_A-types formed from this tree and $\{c_i\}$ $i < 2^w$ is a set of new constants then $Th(\Sigma) \cup D_A(A) \cup \{\Gamma_i(c_i) : i < 2^w\}$ (where X is ordered in a sequence $\{\Gamma_i\} : i < 2^w$) is a consistant theory. Hence it has a model $B \geq A$ in Σ. Since D is maximal we have $D_A^F(B) = D$ but by construction $|D_A^F(B)| = 2^w$. Hence $|D| = 2^w$. ☒

Theorem 4.3

If Σ is a universal class and D is a maximal Σ-diagram such that $|D| = w$ then D is atomistic.

Proof

Let $\psi(v_i)$ be an L_A-formula consistent with D. If ψ is not an atom then we can split $\psi(A)$ into two sets by a formula θ, where A is such that $D_A^F(A) = D$. For a formula θ exists such that both $\psi \wedge \theta$ and $\psi \wedge \neg\theta$ are consistent with D so these formulas both have

solutions in A. We can proceed now to construct a full binary tree of formulas of L_A such that each initial segment of each branch has a solution in A. As in the proof of lemma 4.1 the existence of such a tree implies $|D| = 2^w$. Hence our procedure must terminate at some finite stage and so there is an atom ψ_1 such that $\psi_1 \dashrightarrow \psi$. ⊠

The following result is an analogue for universal classes with the amalgamation property of the Prime Model Theorem of first order model theory. For a proof of the latter see [6]. This analogue of the Prime Model Theorem improves a result of [13]. For a more general study of this topic see [4]. The techniques of that paper follow the classical line by the application of an omitting types theorem. In contrast the method of proof of the following theorem is direct.

Theorem 4.4

Suppose that Σ is a universal class with the amalgamation property and D is a maximal Σ-diagram. There is a prime existentially closed D-structure A in Σ if D is atomistic.

Proof

Let C be a countable set $\{c_i : i < w\}$ of distinct new constants. Let S be the set of all formulas formed by replacing variables in formulas in L_A by constants from C. Then S is countable so that it can be ordered in a sequence $\{\psi_i : i < w\}$. In forming S we consider only L_A-formulas consistent with D. We construct a structure A in Σ_D = the class of all D-structures by induction as follows: Let $\psi(v_i)$ be the first formula in S without constant symbols from C. Let $\theta(v_i)$ be

some Σ-atom such that θ ---> ψ. Let A_0 be generated by a solution $(x_1,\ldots,x_n)$. We interpret c_i to be the name of x_i. Suppose that the construction has proceeded to level m so that we have $A_m \geq A_{m-1} \geq,\ldots,\geq A_0$ defined and generators $x_1,\ldots,x_k$ such that (i) each A_r is generated by an initial segment of this sequence (ii) $(x_1, \ldots,x_k)$ is a solution of some D-atom $\theta_1(v_1,\ldots,v_k)$. Let $\psi_1(v_1,\ldots,v_r)$ be the first formula in S with constants among $c_1,\ldots,c_k$ which is consistent with A_m under our interpretation and has not been previously selected in our construction. Say $\psi_1(v_1, \ldots, v_r) = \psi_2(c_1,\ldots,c_k, v_1,\ldots,v_r)$ for an L_A-formula ψ_2. Then the formula $\theta_1(v_1,\ldots,v_k)\wedge\psi_2(v_1,\ldots,v_k, v_{k+1},\ldots,v_{k+r})$ is consistent with D so there is a Σ-atom θ_2 such that θ_2 ---> $\theta_1\wedge\psi_2$, and θ_2 is consistent with D. Let $(z_1, \ldots,z_k,\ldots,z_{k+r})$ be a solution in $B\{\Sigma_D$ and let B_1 be generated in B by this sequence. Then $(z_1,\ldots,z_k)$ is a solution of θ_1 so we can define an embedding $f:A_m$ ---> B_1 with $f(x_i) = z_i$ for $1 \leq i \leq k$. It follows that we can replace B_1 by an isomorphic structure with A_m as an actual substructure and with $(z_1,\ldots,z_k,\ldots, z_{k+r}) = (x_1, \ldots,x_k,z_{k+1},\ldots,z_{k+r})$. Let $A_{m+1} = B_1$ and interpret $c_{k+1},\ldots,c_{k+r}$ by $z_{k+1}, \ldots,z_{k+r}$ respectively. We have that our generators are a solution of a D-atom and $A_{m+1} \models \psi_1(z_{k+1},\ldots,z_{k+r})$. We thus define a chain $\{A_i\}$ $i < w$.

Claim

Let $A = UA_i$. Then A is a prime existentially closed D-structure.

We first prove that A is existentially closed in Σ_D. Suppose that $\psi(a_1,\ldots,a_n,v_i)$ has a solution in an extension C of A. By AP we know that C is then a D-structure. Here the a_i's are the names of all the elements of A which appear in the L_A-formula $\psi(v_r,v_i)$. We have a canonical sequence of generators for A defined above in our construction, say $\{x_i\}i < w$. Let $a_i = t_i(x_1,\ldots,x_r)$ (that is, we choose r such that each a_i is in the substructure generated by $x_1,\ldots,x_r$). Let m be the least integer such that $x_1,\ldots,x_r$ is in A_m. We replace a_i in by the term $t_i(c_1,\ldots,c_r)$ to form the formula $\psi(t_1(c_j),\ldots,t_n(c_j),v_i)$. Now this formula with c_j interpreted to be the element x_j is consistent with A_m since it is consistent with A. It is also in the sequence S. So there is some $k \geq m$ such that at level k the formula selected would be $\psi(t_1(c_j),\ldots,t_n(c_j),v_i)$. Hence this formula has a solution in A_{k+1}. Hence $\psi(a_1,\ldots,a_nv_i)$ has a solution in A.

The fact that A is prime existentially closed in Σ_D follows by our construction since the sequence of generators $\{x_i\}$ i < w can be segmented so that each segment is a solution of a D-atom. This atom has a solution in every existentially closed D-structure, and each atom with an initial sequence of points isomorphic to a segment of $\{x_i\}$ i < w has a solution over this initial sequence. This follows by AP. Hence we can construct an embedding of A into an existentially closed D-structure by induction. ⊠

Corollary 4.1

Suppose that Σ is a universal class with AP and D is a maximal Σ-diagram with $|D| = w$. Then there is a prime existentially closed D-structure.

It was shown in [13] that if Σ is a universal class with the amalgamation property then the existentially closed structures in Σ have many of the properties of the class of models of complete first order theories. The above theorem and the next provide additional evidence for this conclusion. A result of [11] shows that if T is a model complete theory then the class of models of T coincides with the class of existentially closed structures in Σ_T. Thus if T is a theory with a model companion one would expect that analogues of first order model theory results could be found. In [39] an analogue of the w-categoricity theorem is proven in this case without the assumption of the amalgamation property. For the definition of generalized principal type refer to definition 3.35 of [13].

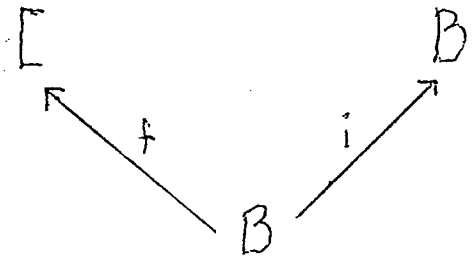
Theorem 4.5 (The Categoricity Theorem)

Suppose that Σ is a universal class with the amalgamation property. Let D be a maximal Σ-diagram. There is a unique countable existentially closed D-structure iff each $p \in D$ is a generalized principal type.

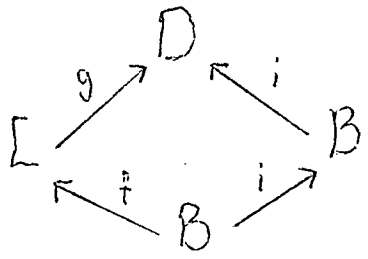
Proof

Suppose first that there is a unique countable existentially closed D-structure A. Since Σ has the amalgamation property it follows by a result of [13] that each $B \in \Sigma$ has an existentially closed extension C

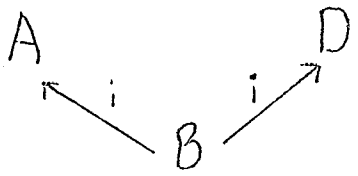
and it is easy to see that we can assume for countable B that C is also countable. Since A realizes only a countable set of types in D it follows that D is countable. Hence by Corollary 4.1 the structure A is a prime existentially closed D-structure. By a result of [14] every open type realized in A must be a generalized principal type. Since $D=D_A^F(A)$ every type in D is a generalized principal type. Assume next that every $\Gamma \in$ D is a generalized principal type. Let A be any countable existentially closed D-structure and suppose that B is a finitely generated substructure of A. Say B is generated by $b_1,\ldots,b_n$. Suppose that $\Gamma(b_i,v)$ is a type over B. Then $\Gamma(x_i,v)$ is a type in D so by assumption there is a formula $\psi(y_j,x_iv)$ which is an $L(\sum)$-atom such that any solution (z_j,r_i,S) is such that (r_i,S) has type $\Gamma(x_iv)$. Consider the formula $\psi(y_j,b_i,v)$. As $\psi(y_j,x_i,v)$ is an $L(\sum')$ formula consistent with $\sum$ let (a_j,c_i,d) be a solution in a structure C generated by the elements of this sequence. Then (c_i,d) has open type $\Gamma(x_i,v)$ so that (c_i) has the same open type as (b_i). Therefore there is an isomorphism f that embeds B in the structure C in such a way that (b_i) is mapped onto (c_i). We thus obtain a diagram



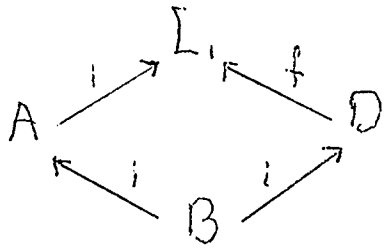
We can apply the amalgamation property to complete this diagram to obtain the following commutative diagram:



Now $C \models \psi(a_j, c_i d)$ so that $D \models \psi(g(a_j), g(c_i), g(d))$. By the commutativity of the diagram we deduce that $D \models \psi(g(a_j), b_i, g(d))$ and so the formula $\psi(y, b_i, v)$ is consistent over B. The formula must also be consistent over A. For if D is an extension of B in which $\psi(y_j, b_i, v)$ has a solution (a_j, d) we obtain the diagram



We can amalgamate this diagram over B to obtain the following commutative diagram:



Then C_1 is an extension of A and $C_1 \models \psi(f(a_j), b_i, f(d))$. As A is existentially closed A must already contain a solution of $\psi(y_j, b_i, v)$. If $A \models \psi(c_j, b_i, d_1)$ then (b_i, d_i) must have type $\Gamma(x_i, v)$ so that d_1 is a realization of $\Gamma(b_i, v)$. It follows that A is weakly L_A-saturated. By a result of [13] all countable weakly L_A-saturated D-structures are isomorphic. Thus there is a unique countable existentially closed D-structure. ⊠

The following example shows that the diagram condition in the above theorem cannot be improved. Thus the analogue for existentially closed structures of the w-categoricity theorem is not obtained as a direct translation of the first order theory. The example also demonstrates that a universal class with the amalgamation property can have a unique countable existentially closed structure but have no model companion.

Example 4.1

Let $L = \{U,V\} \cup \{c_i\}_{i<w} \cup \{f\}$. T says: (i) U,V are infinite disjoint sets that contain all elements (ii) $U(c_i)$ for all $i<w$ (iii) $c_i \neq c_j$ if $i \neq j$ (iv) f is the identity on U and maps V into U in such a way that no c_i is in the image set (v) f is one-one on V. Note that A is existentially closed in $\sum_T$ if V is infinite and all elements of U not equal to some c_i are in the image of V under f. It is therefore clear that the existentially closed spectrum of T is categorical in all infinite powers. But there are an infinite number of open 1-types in the unique maximal $\sum_T$ diagram. Clearly $\sum_T$ has the amalgamation property and since the class of existentially closed structures in $\sum_T$ cannot be axiomatized it follows by a result of [11] that T has no model companion. ☒

The next result shows that there is an extensive class of theories that satisfy the diagram condition of Theorem 4.5. This theorem is a direct analogue of the corresponding first order theorem.

Theorem 4.6

Let $\sum$ be a universal class with the amalgamation property and

suppose that D is a maximal Σ-diagram such that for each n<w the set of open types in D in n variables is finite. Then each type in D is principal.

Proof

Let $\Gamma \in D$ and suppose that Γ is not principal. Let $\{\pi_i\}_{i<w}$ be an ordering of Γ. As Γ is not principal for each k<w there is a j<w such that $\bigwedge_{i\leq k} \pi_i \wedge \neg\pi_j$ is consistent with D. For each k<w let δ_k be such a formula and let Γ_k be a type in D that contains δ_k. As there are a finite number of types in D in each n<w number of variables all the types Γ_k are contained in some finite subset $\Gamma_1,\ldots,\Gamma_m$. There are an infinite number of distinct δ_j so some type Γ_r contains an infinite number of δ_j. As Γ_r is a maximally consistent set of open formulas it follows that Γ_r contains all π_i and also some formula $\neg\pi_j$. Contradiction. ⊠

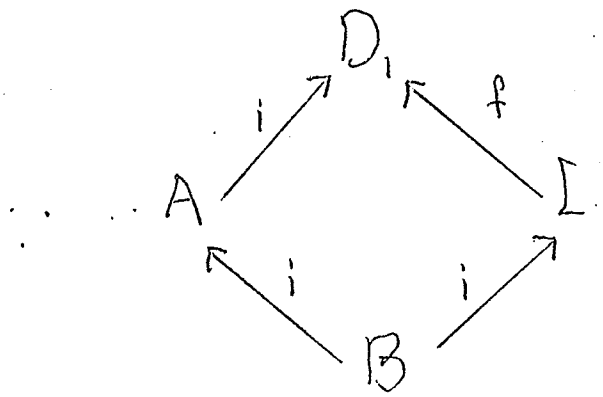
In the final portion of this section we prove three results concerned with the Loweheim-Skolem properties of structures in universal classes. The first result is a version of the Downward Lowenheim-Skolem Theorem.

Theorem 4.7

Suppose that Σ is a universal class with the amalgamation property and D is a maximal Σ-diagram. If $A \in \Sigma_D$ is existentially closed and B is an elementary substructure of A then B is existentially closed.

Proof

Suppose that $\psi(v_i) \in L_A^n(B)$ and for some $C \geq B$ in Σ_D there are elements $X_1,\ldots,X_n$ in C such that $C \models \psi(X_i)$. We can amalgamate C and A over B to obtain the following communitative diagram:



Since the diagram is commutative the map f transforms the formula ψ onto itself and thus $D_1 \models \psi(f(X_1),\ldots,f(X_n))$. Now D_1 is an extension of A and A is existentially closed and so $A \models \exists\, v_i \psi(v_i)$. As B is an elementary substructure of A it follows that $B \models \exists\, v_i \psi(v_i)$. This implies that B is existentially closed. ⊠

Theorem 4.7 can be applied in the proof of a Two Cardinal Theorem for universal classes with the amalgamation property satisfying a natural algebraic stability condition.

Theorem 4.8

Suppose that Σ is an L_A-(w)-stable universal class with the amalgamation property and $A \in \Sigma$ is existentially closed. If $\psi(v)$ is an open $L(\Sigma)$ formula such that $w \leq |\psi(A)| < |A|$ then for all $\lambda > |\psi(A)|$ there is an existentially closed structure B in Σ such that $|B| = \lambda$ and $|\psi(B)| = |\psi(A)|$.

Proof

Suppose first that $\lambda < |A|$. Let B be an elementary substructure of A of cardinality λ such that $\mathfrak{P}(B) = \mathfrak{P}(A)$. By Theorem 4.7 the structure B is existentially closed and so B satisfies the required conditions. Assume next that $|A| < \lambda$. In this case we can utilize the results of [13] to prove the assertion by a slight modification of the proof of Lemma 2 of [15]. ☒

The above theorem is a generalization of a result of [29]. It is natural to ask to what extent the theorem can be strengthened.

Definition 4.2

Suppose that Σ is an L_A-(w)-stable universal class with the amalgamation property. Then Σ has atomic existential closures if for each $A \in \Sigma$ if x is an element of the existential closure of A then x realizes a principal open type over A.

Lemma 4.1

Suppose that Σ is a universal class and $A \in \Sigma$. If $X \subseteq A$ is an infinite set of L_A-indiscernibles then there is an extension B of A in Σ and a set Y of L_A-indiscernibles extending X in B such that $|Y \sim X| \geq w$ and each element Z of $Y \sim X$ has the same open type over A.

Proof

Let $\lambda = |A|$. By use of the Compactness Theorem we can find an extension C of A in Σ that contains a set Z of L_A-indiscernibles extending X and having cardinality $(2^{\lambda})^+$. There are at most 2^{λ} open types over A so that some infinite subset Y of Z must realize a single

open type over A. Let B be the substructure of C generated by A u Y. Then Y and B satisfy the requirements. ☒

The following theorem allows a considerable improvement of Theorem 4.8 with the additional assumption that Σ has atomic existential closures. Note that the existence of an existential closure for L_A-(w)-stable universal classes with the amalgamation property is proven in [13]. If T is a complete full and w-stable theory then Σ_T has atomic existential closures. The proof of the next theorem thus provides a new proof for a two cardinal property of first order model theory. This property was first demonstrated in [26] in a somewhat stronger form for stable theories.

Theorem 4.9

Suppose that Σ is an L_A-(w)-stable universal class with the amalgamation property having atomic existential closures. Let A be an existentially closed structure in Σ. If $\psi(v)$ is an open $L(\Sigma)$ formula such that $|\psi(A)| < |A|$ then there is an existentially closed proper extension C of A in Σ such that $\psi(A) = \psi(C)$.

Proof

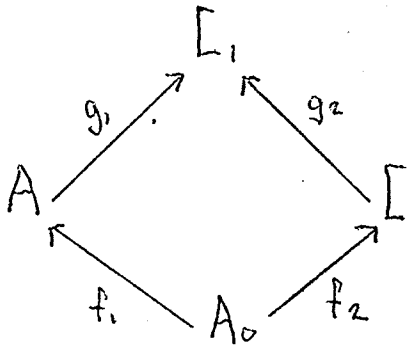
If $\psi(v)$ is an algebraic formula then A contains all solutions since A is existentially closed. Thus any extension of A in Σ satisfies the requirements. Otherwise we must have that $|\psi(A)| > w$ since A is existentially closed. Thus A must have uncountable cardinality. Let λ be a regular cardinal such that $|\psi(A)| < \lambda \leq |A|$. It follows by Theorem 4.14 of [13] that A contains a set X of L_A-indiscernibles of cardinality λ. Let B be an extension of A in Σ containing a set Y of

L_A-indiscernibles extending X and having the properties of Lemma 4.1. Let $c \in Y \sim X$. We define the structure C to be the existential closure of A(c).

Claim

$\psi(A) = \psi(C)$. For suppose not. Let $X \in \psi(C) \sim (A)$. Then there is an $L_A(A(c))$ atom $\theta(X_1,\ldots,X_n,C,V)$ such that X is a solution of this atom. We assume that each X_i for $i < n$ is an element of A and $\theta(v_1,\ldots,v_n,w,v)$ is an $L_A(\Sigma)$ formula. Let Γ be the open type of the sequence $< x_1,\ldots,x_n,c>$. Then $Th(\Sigma) \cup \Gamma(\bar{x}_1,\ldots,\bar{x}_n,c) \vdash \theta(\bar{x}_1,\ldots,\bar{x}_n,c,v) \longrightarrow \psi(v)$. It follows by the Compactness Theorem that there is a sentence ψ_0 of $Th(\Sigma)$ and a sentence $\psi_1(x_1,\ldots, x_n,c)$ of $\Gamma(x_1,\ldots,x_n,c)$ such that ψ_0, $\psi_1(x_1,\ldots, x_n,c) \vdash \theta(x_1,\ldots,x_n,c,v) \longrightarrow \psi(v)$. By the Deduction Theorem (see for example [38]) this implies that $\psi_0 \vdash \psi_1(x_1,\ldots,x_n,c) \longrightarrow (\theta(x_1,\ldots,x_n,c,v) \longrightarrow \psi(v)$. As c is a constant not in the language $L(\Sigma)$ the latter implication implies that $Th(\Sigma) \vdash \forall w(\psi_1(\bar{x}_1 \ldots,\bar{x}_n,w) \longrightarrow \forall v(\theta(x_1,\ldots,x_n,w,v) \longrightarrow \psi(v)))$. Let B be the substructure of A generated by the elements $x_1,\ldots,x_n$. Let Γ_1 be the open B-type of the elements $Y \sim X$. For each formula $\theta_1(v)$ of Γ_1 let $X_{\theta 1}$ be the set of all elements of X that are not solutions of θ_1. As the set $Y \sim X$ is infinite and each element $y \in Y \sim X$ is a solution of θ_1 it follows from Lemma 4.18 of [13] and the fact that Y is a set of L_A-indiscernibles that $X_{\theta 1}$ is finite. Let $X_0 = \cup\{X_{\theta 1}: \theta_1 \in \Gamma_1\}$. The type Γ_1 is countable and so X_0 must be countable. Let $X_1 = X \sim X_0$. Then X_1 has cardinality λ.

Clearly each element z of X_1 is a realization of Γ_1. Since c is an element of Y X and $C \models \Psi_1(x_1,\ldots,x_n,c)$ the formula $\Psi_1(\vec{x}_1,\ldots,\vec{x}_n,v)$ is an element of Γ_1. It follows that if $z \in X_1$ then $A \models \Psi_1(x_1,\ldots,x_n,z)$. Let A_0 be the substructure of C generated by the element c and suppose that $Z \in X_1$. Define f_1 to be the identity map embedding A_0 in C and f_2 to be the embedding of A_0 in A mapping c to z. We can amalgamate the resulting diagram over A_0 to obtain the following commutative diagram:



We can assume that g_1 is the identity map. We have that $C \models \theta(x_1,\ldots,x_n,c,x)$ and so by the commutativity of the diagram it follows that $C_1 \models \theta(x_1,\ldots,x_n,z,g_2(x))$. Since A is existentially closed and C_1 is an extension of A in Σ there must be an element y in A such that $A \models \theta(x_1,\ldots,x_n,z,y)$. By the implications proven above the latter implication implies that y is a solution of the formula $\Psi(v)$. We conclude that each element z of the set X_1 has a solution to the formula $\theta(x_1,\ldots,x_n,z,v)$ in the set $\Psi(A)$. Suppose next that y is some point of $\Psi(A)$. If z is an element of $Y \sim X$ then z is not a solution of the formula $\theta(x_1,\ldots,x_n,w,y)$. For suppose that $C \models \theta(x_1,\ldots,x_n,z,y)$. Since $c \in Y \sim X$ and all elements of the set $Y \sim X$ have the same open A-type it follows that $C \models \theta(x_1,\ldots,x_n,c,y)$. But we have also that $C \models \theta(x_1,\ldots,x_n,c,x)$

and the element x is not in the set (A). Since $\theta(x_1,\ldots,x_n,c,v)$ is an $L_A(A(c))$-atom we deduce that $D_A(A(c)) \vdash \theta(x_1,\ldots,x_n,c,v)$ $\dashrightarrow \neg\, v=y$. Contradiction. The set $Y \sim X$ is infinite and so by Lemma 4.18 of [13] the number of solutions of the formula $\theta(x_1,\ldots, x_n,w,y)$ in the set X_1 must be finite. For each element y in $\Psi(A)$ let X_y be the set of all solutions to the formula $\theta(x_1,\ldots, x_n,w,y)$ in the set X_1. Define X_2 to be the set $U\{X_y : y \in \Psi(A)\}$. Then the cardinality of X_2 is not greater than the cardinality of $\Psi(A)$. But by what was shown above we must have $X_1 = X_2$. This contradicts the fact that the set X_1 has cardinality λ. ☒

Corollary 4.2

Suppose that Σ is an L_A-(w)-stable universal class with the amalgamation property and having atomic existential closures. If $\Psi(v)$ is an $L(\Sigma)$ formula and $A \in \Sigma$ is an existentially closed structure in Σ such that $|\Psi(A)| < |A|$ then for each cardinal $\lambda > |A|$ there is an existentially closed structure C in Σ such that $|C| = \lambda$ and $\Psi(A) = \Psi(C)$.

The next result is a Two Cardinal Theorem for open formulas in general universal classes. In the proof stability properties are replaced by the use of a theorem of set theory.

Theorem 4.10

Suppose that Σ is a universal class and A,B are structures in Σ satisfying the following conditions: (i) B is an infinite extension of A (ii) $\Psi(v_0) \in L_A(A)$ has the property that $\Psi(A) = \Psi(B)$ (iii) for all

$L_A(A)$-terms t the set $\{x \in \Psi(A) : x = t(b_i)$ for some sequence b_i in $B - A\}$ has power $< w$. Then there is a countable substructure A_0 of A and for each $\lambda \geq w$, there is an extension $\Gamma_\lambda \geq A_0$ in Σ such that $\Psi(C_\lambda) = \Psi(A_0)$ and Γ_λ has cardinality λ.

Proof

Let $U(v_0)$ be a new unary predicate symbol. Interpret this predicate in B by the structure A. Let $\langle B_0, A_0\rangle$ be a countable elementary substructure of (B,A). Then the structures A_0, B_0 satisfy conditions i-ii-iii in the statement of our theorem. For each condition can be expressed by a sentence or set of sentences in $\langle B,A\rangle$ and these sentences are true in $\langle B_0, A_0\rangle$. Well order the elements of $B_0 - A_0$ in a sequence x_i $i < w$. Let $_i(v_j) : i < w$ be a listing of all open formulas over A_0. We construct a decreasing sequence of infinite sets I_i $i < w$ by induction as follows: Let I^1 and I^2 be those sequences in proper order from x_i $i < w$ such that $B_0 \models$ 1 (x_{r_i}) or $B_0 \models$ $_1(x_{r_i})$ respectively. By the Ramsey Theorem there is a homogeneous subset I_0 contained in x_i $i < w$, with $|I_0| = w$. Applying this procedure repeatedly we obtain a sequence of infinite sets which at level n is homogeneous (in the proper order) for n. By the compactness theorem there is a structure $B_1 \geq A_0$ in which is generated by an EM-set y_i $i < w$ (for the language $L_A(A_0)$) of the type constructed above. For each $\geq w_1$ let C be the stretch of B_1 by the EM-set y_i $i < w$. That is, C is generated by $A_0 \cup y$ $<$ where y $<$ is an EM-set extending y_i $i < w$.

Claim

$\forall \lambda \geq w_1$ we have $\tau(C_\lambda) = \tau(A_0)$. For suppose not. Then there is $\lambda > w$ such that some element $x \in C_\lambda - A_0$ is a solution of τ. Since C_λ is generated by $A_0 \cup \{y_\alpha\}\ \alpha < \lambda$ it follows that $x = t(y_{\alpha 1},\ldots,y_{\alpha n})$ for some A_0-term $t(v_1,\ldots,v_n)$ and elements $y_{\alpha 1},\ldots,y_{\alpha n}$ where we can assume that $\alpha_1 <,\ldots,< \alpha_n$. Hence $C_\lambda \models \tau(t(y_1,\ldots,y_n))$. Since $\{y_\alpha\}\alpha < \lambda$ is an EM-set over A_0 it follows that $B_1 \models \tau(t(y_1,\ldots, y_n))$. Now the formula $\tau(t(v_i)) \in L_A(A_0))$ and so $\tau(t(v_i)) = \tau_k(v_i)$ for some $k < w$. By the construction above the set I_k is homogeneous for τ_k and since $\{y_i\}\ i < w$ was selected as an EM-set whose type was determined by the sequence $\{I_i\}\ i < w$ it follows that for all $x_{r1},\ldots,x_{rn}$ with $x_{ri} \in I_k$ and $r_1 <,\ldots,< r_n$ we have $B_0 \models \tau(t(x_{r1},\ldots, x_{rn}))$. By condition (iii) applied to the term t there are only a finite number $a_1,\ldots,a_m$ of points of $\tau(A_0)$ that are in the image under t of sequences in $B_0 - A_0$. Now $x \notin A_0$ so that $C_\lambda \models \neg t(y_{\alpha 1},\ldots,y_{\alpha n}) = a_1 \wedge,\ldots, \wedge \neg t(y_{\alpha 1},\ldots, y_{\alpha n}) = a_m \wedge \neg t(y_{\alpha 1},\ldots,y_{\alpha n}))$. But each formula $(t(v_i) = a_j) = \tau_{rj}(v_i) \in L_A(A_0)$ for some $r_j < w$ so if $r = \max r_j$ then $B_0 \models \neg\tau_{rj}(x_{k1},\ldots,x_{kn})$ for all properly ordered sequences $\langle x_{k1},\ldots,x_{kn}\rangle$ in I_r. And so be definition of $a_1,\ldots,a_m$ we have $B_0 \models \neg\tau(t(x_{k1},\ldots,x_{kn}))$. But again $\{y_\alpha\}\alpha < \lambda$ is an EM-set whose type is determined by $\{I_i\}\ i < w$ and so $C_\lambda \models \exists\tau(t(y_1,\ldots,y_n))$. Contradiction. This proves the theorem. ☒

Example 4.2

Let $L = \{U_0\}$. T says that U_0 is an infinite set and that $\neg U_0$ has exactly one element. Let $\Sigma = \Sigma_T$. Let B be any structure in Σ with both U_0 and $\neg U_0$ nonempty and let A be $U_0(B)$. Then both conditions (ii) and (iii) are satisfied but Σ has no two cardinal model for U_0. ⊠

Example 4.3

Let $L = \{U_0, f\}$. T says that both U_0 and $\neg U_0$ are infinite and f is a one-one onto function from $\neg U_0$ to U_0 but is the identity on U_0. Let $\Sigma = \Sigma_T$. Then there are structures in Σ satisfying conditions (i) and (ii) but Σ has no two cardinal model for U_0. ⊠

2. Amalgamation and Quanitifier Reduction

Example 4.1 provides a simple example of a universal class with the amalgamation property that has no quantifier eliminable model completion. The first example of such a universal class was the theory of groups axiomatized so that all subalgebras are subgroups. The proof in [11] that the theory of groups has no model companion actually shows by direct calculation that the class of existentially closed groups is not closed under ultrapowers. It was shown in [13] by an elementary argument that if T admits elimination of quantifiers then the class of models of T coincides with the class of existentially closed structures in Σ_T. Thus the theory of groups provides a natural example of a universal class with the amalgamation property that has no quanitifier eliminable model completion and the proof of this is elementary. A second proof that the

theory of groups has no model companion can be found in [16].

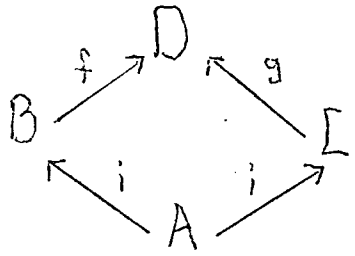
As noted in [6] a theory admits elimination of quantifiers if it is submodel complete. A very simple example of [6] shows that there is a complete theory that is model complete but not submodel complete. In this regard we have the following theorem.

Theorem 4.11

A model complete theory T such that $\sum_T$ has the amalgamation property is submodel complete.

Proof

Suppose that B,C are models of T and A is a substructure of B and C. If we amalgamate the resulting inclusion diagram over A we obtain the following commutative diagram:



We can assume that D is a model of T. Let $\psi(a_i)$ be an L(A) sentence. By the commutativity of the above diagram this sentence is transformed to a sentence $\psi(c_i)$ in L(D) defined over f(B) and g(C). By model completeness we have f(B) < D and g(C) < D and so $\psi(c_i)$ is valid in f(B) iff it is valid in g(C). Since f and g are isomorphisms it follows that $\psi(a_i)$ is valid in B iff it is valid in C. Thus $T \cup D_A(A)$ is a complete theory. ⊠

If T is model complete then as shown in [20] the class of models of

T has the amalgamation property. The above result thus gives us an exact measure of how far a model complete theory is from being quantifier eliminable. The next theorem is proven in [34] by a different method.

Theorem 4.12

If the theory T is submodel complete then $\sum_T$ has the amalgamation property.

Proof

Let B,C be models of T and suppose that $A < B$ and $A < C$. By submodel completeness $T_U D_A(A) = T_0$ is a complete theory and $T_1 = T_U D_A(B)$ and $T_2 = T_U D_A(C)$ are extensions of T_0 such that $L(T_0) = L(T_1) \cap L(T_2)$. It follows by the Joint Consistency Theorem (as proven for example in [6]) that $T_1 \cup T_2$ is consistent and so it has a model D. Let D_1 be the reduct of D to L(T). It is clear how to define isomorphisms of B and C into D_1 to obtain the required commutative diagram. ⊠

It is noted in [6] that a theory T is submodel complete if for all finite substructures A of models of T the theory $T_U D_A(A)$ is complete. This suggests the following definition.

Definition 4.3

Suppose that T is a theory and S is a set of open L(T) types. Then S has the splitting property if each $\Gamma \varepsilon S$ is contained in a unique closed L(T) type.

Theorem 4.13

If T is a theory then T is submodel complete iff the set of all open L(T) types has the splitting property.

The following example shows that for a given theory T the set of open n types in L(T) can have the splitting property but for some $m \neq n$ the open m types in L(T) may not have the splitting property.

Example 4.4

Let $L = \{U,V\} \cup R$. T says: (i) U,V are infinite disjoint sets (ii) $\forall x(U(x) \vee V(x))$ (iii) R has first coordinate in U and second coordinate in V (iv) if x is in U then R(xy) has exactly two solutions (v) if y is in V then R(xy) has exactly two solutions. It is clear that the set of open L(T) 1-types has the splitting property. But note that the formula $\exists y(R(xy) \wedge R(zy))$ splits the open type generated by the open atom $U(x) \wedge U(z)$. ⊠

The notions of E_n^0 and Π_n^0 formulas are defined in [6].

Definition 4.4

Suppose that T is a theory and ψ is an L(T) formula. Then ψ is a Δ^n-formula if there is a formula ψ_1 that is a Boolean combination of E_n^0 and Π_n^0 formulas such that $T \vdash \psi \leftrightarrow \psi_1$. If a sentence is a Δ^n-formula it is a Δ^n-sentence.

I do not know who first defined Δ^n-formulas but the notion was introduced in a logic course given by Herbert Enderton.

Definition 4.5

Suppose that T is a theory. A Δ^n-type is $L^m(T)$ is a maximal consistent set of Δ^n-formulas in $L^m(T)$. A set S of Δ^n-types in $L^m(T)$ has the splitting property if each $\Gamma \in S$ is contained in a unique closed $L^m(T)$ type.

The following result generalizes the fourth lemma on page 83 of [38] and the proof is very similar. If $\mathfrak{A}$ is a model of a theory T then the Δ^n-theory of $\mathfrak{A}$ is the set of Δ^n-sentences in L(T) valid in $\mathfrak{A}$.

Lemma 4.2

Suppose that T is a theory and φ is an L(T) sentence. Let n be a number greater than zero. Suppose that whenever $\mathfrak{A}$ and $\mathfrak{B}$ are models of T having the same Δ^n-theory then $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$. There is a Δ^n sentence φ_1 such that $T \vdash \varphi \leftrightarrow \varphi_1$.

Proof

If φ is not consistent with T there is no difficulty in finding φ_1 so assume that the theory $T \cup \{\varphi\}$ has models. Let B be the set of Δ^n-sentences that are consequences of $T \cup \{\varphi\}$. The set B is nonempty since there are valid Δ^n-sentences in L(T). Suppose that $T \cup B$ does not imply φ. Then there is a model $\mathfrak{A}$ of $T \cup B \cup \{\neg\varphi\}$. Let X be the set of all Δ^n-sentences valid in $\mathfrak{A}$.

Claim

$T \cup X \vdash \neg\varphi$. For if not let $\mathfrak{B}$ be a model of $T \cup X$ such that $\mathfrak{B} \models \varphi$. But X is a complete set of Δ^n-sentences so that $\mathfrak{A}$ and $\mathfrak{B}$ have the

same Δ^n-theory. By the Compactness Theorem a finite subset X_1 of X exists such that $T \cup X_1 \vdash \neg\psi$. Let θ be the conjunction of the sentences in X_1. Then $T \vdash \theta \dashrightarrow \neg\psi$ and so $T \vdash \neg\theta \vee \neg\psi$. The latter implication shows that $T \vdash \psi \dashrightarrow \neg\theta$. The formula $\neg\theta$ is equivalent to a disjunction of Δ^n-sentences. And so $\neg\theta$ is equivalent to a sentence θ_1 in the set B. But $\mathfrak{A}$ is a model of TUB and $\mathfrak{A} \models \theta$. We conclude that $TUB \vdash \psi$ and so again by the Compactness Theorem some finite subset B_1 of B has the property that $T \cup B_1 \vdash \psi$. If ψ_1^* is the conjunction of the sentence of B_1 then ψ_1^* is equivalent to a Δ^n-sentence ψ_1. By definition of B we have that $T \vdash \psi \dashleftarrow\dashrightarrow \psi_1$. ⊠

Theorem 4.14 (The Quantifier Reduction Theorem)

Suppose that T is a theory and for some number n the set of Δ^n-types in $L^m(T)$ has the splitting property. If $\psi \in L^m(T)$ there is a Δ^n-formula ψ_1 such that $T \vdash \psi \dashleftarrow\dashrightarrow \psi_1$.

Proof

Let $c_1,\ldots,c_m$ be new constants and let $\theta = \psi(c_1,\ldots,c_m)$. Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are models of T in the language $L(T) \cup \{c_1,\ldots,c_m\}$ and $\mathfrak{A}$ and $\mathfrak{B}$ have the same Δ^n-theory. If $(a_1,\ldots,a_m)$ and $(b_1,\ldots,b_m)$ are the interpretations of $(c_1,\ldots,c_m)$ in $\mathfrak{A}$ and $\mathfrak{B}$ respectively then (a_i) and (b_i) have the same Δ^n-type Γ in the language $L(T) \cup \{c_1,\ldots,c_m\}$ and so those sequences have the same Δ^n-type Γ_1 in L(T). Thus $\mathfrak{A} \models \theta$ iff $\mathfrak{B} \models \theta$ for otherwise would split the Δ^n-type Γ_1 into two closed $L^m(T)$ types. If $n > 0$ the above lemma implies that there is a Δ^n-sentence $\theta 1(c1,\ldots,cm)$ such that

$T \vdash \theta \longleftrightarrow \theta_1$. If we let $\psi_1 = \theta_1(v_1, \ldots, v_m)$ (assuming there is no conflict with quantifiers) then $T \vdash \psi \longleftrightarrow \psi_1$. If $n = 0$ the lemma of [38] can be applied to obtain the same result. ☒

The proofs of the lemmas required to prove the above theorem have a marked syntactical character. In two important special cases the use of these lemmas can be avoided and the result proved by a direct argument. We will restrict the following proofs to the case $n = 0$. For the sake of clarity we give complete statements of the special cases.

Theorem 4.15

Suppose that T is a complete w-categorical theory such that for a given $n < w$ the set of open $L^n(T)$ types has the splitting property. If $\psi \in L^n(T)$ then there is an open formula $\psi_1 \in L^n(T)$ such that $T \vdash \psi \longleftrightarrow \psi_1$.

Proof

Since T is a complete w-categorical theory it follow from results of [6] that T is atomistic and has only a finite number of inequivalent formulas in n variables. Given $\psi \in L^n(T)$ let θ_1 be an atom such that $T \vdash \theta_1 \longrightarrow \psi$. If we let $\psi_1 = \neg\theta_1 \wedge \psi$ then again there is an atom θ_2 such that $T \vdash \theta_2 \longrightarrow \psi_1$. Clearly θ_1 and θ_2 are not equivalent and so by a continuation of this procedure we obtain a finite set of atoms $\theta_1, \ldots, \theta_k$ such that $T \vdash \theta_1 \vee \ldots \vee \theta_k \longrightarrow \psi$ and $\neg \bigvee_{i \leq k} \theta_i \wedge \psi$ is not consistent with T. As T is a complete theory we must have that $T \vdash \bigvee_{i \leq k} \theta_i \vee \neg\psi$ and hence

$T \vdash \psi \longleftrightarrow \bigvee_{i \leq K} \theta_i$. Now suppose that θ is an $L^n(T)$ atom and Γ is the type generated by θ. Let $\bar{\Gamma}$ be the open type determined by Γ and suppose that $\bar{\Gamma}$ is ordered in a sequence $\{\pi_i\}$ $i < w$. If for each $n < w$ the formula $\bigwedge_{i \leq n} \pi_i \wedge \neg\theta$ is consistent with T then it follows by the Compactness Theorem that the set of formulas $\bar{\Gamma} \cup \{\neg\theta\}$ is consistent with T and so has an extension to a closed $L^n(T)$ type Γ_1. But then θ splits $\bar{\Gamma}$ into two closed types contrary to assumption. Therefore there is an open formula δ in Γ such that $T \vdash \delta \longleftrightarrow \theta$. If we choose such a formula δ_i for each θ_i it follows that $T \vdash \bigvee_{i \leq K} \delta_i \longleftrightarrow \bigvee_{i \leq K} \theta_i$. If we let $\bigvee_{i \leq K} \delta_i = \psi_1$ then the two equivalences proven above imply that $T \vdash \psi \longleftrightarrow \psi_1$ and ψ_1 is an open formula. ⊠

The proof of the second special case resembles that of Theorem 4.15. We need some preliminary discussion before stating the theorem.

Suppose that T is a complete theory and $\mathcal{A}$ is a model of T. If ψ is a first order formula defined over $\mathcal{A}$ the rank of ψ is defined as in [6]. For a type Γ in the Stone space $S(\mathcal{A})$ the rank is defined to be the rank of the formula of least rank in Γ. The following lemma has a simple verification based on the definitions.

Lemma 4.3

Suppose that T is a complete theory and $\mathcal{A}$ is a model of T. If $\psi \in L(\mathcal{A})$ has degree 1 then ψ contains a unique type $\Gamma \in S(\mathcal{A})$ such that rank ψ = rank Γ.

There is an alternate method for obtaining a rank that makes no use of the results of [6]. For simplicity assume that L contains no function symbols. If T is a complete theory then as in [13] we can construct a

full expansion T^* of T and expansions $\mathfrak{A}^*$ for models $\mathfrak{A}$ of T to models of T^*. If $\mathfrak{A}^*$ is a model of T^* and $\psi^* \in L(\mathfrak{A}^*)$ there is a formula $\psi \in L(\mathfrak{A})$ such that $\mathfrak{A}^* \models \psi \leftrightarrow \psi^*$. If follows from this that if $\Gamma \in S(\mathfrak{A})$ then Γ has a unique extension to a type $\Gamma^* \in S(\mathfrak{A}^*)$. Consequently the rank on types defined in [13] for a complete full theory induces a rank on $S(\mathfrak{A})$. If we define rank for formulas as in [13] then the induced rank on formulas and types over models of T has the properties mentioned above.

Theorem 4.16

Suppose that T is a complete theory such that T is w-stable and Σ_T has the amalgamation property. Let $\mathfrak{A}$ be the prime model of T. If the set of types in $S_A^m(\mathfrak{A})$ has the splitting property then given $\psi \in L^m(T)$ there is an extension T_1 of T by a principal type and a formula $\psi_1 \in L_A^m(T_1)$ such that $T_1 \vdash \psi \leftrightarrow \psi_1$.

Proof

Note that since T is w-stable the universal class Σ_T is L_A-(w)stable. By assumption Σ_T has the amalgamation property so by the results of [13] there is a rank defined on a pen formulas and types over structures in Σ_T. Let $\psi \in L^m(T)$. Split ψ over $\mathfrak{A}$ into degree 1 sets $\psi_1, \ldots, \psi_n$ having rank equal to the rank of ψ. By Lemma 4.3 each formula ψ_i contains a unique type Γ_i in $S^m(\mathfrak{A})$ such that rank Γ_i = rank ψ_i. Let $\bar{\Gamma}_i$ be the open type determined by Γ_i. Choose θ_i in $L_A^m(\mathfrak{A})$ such that θ_i isolates $\bar{\Gamma}_i$ in the subspace of $S_A^m(\mathfrak{A})$ whose elements are the open types having rank

greater than or equal to the rank of $\bar{\Gamma}_i$. The formulas θ_i can be chosen so that for each $i < n$ we have $\mathcal{A} \models \theta_i \dashrightarrow \psi_i$. For suppose not. If $i < n$ is such that no such θ_i can be found order $\bar{\Gamma}_i$ in a sequence $\{\pi_j\}_{j<w}$. For each $j < w$ let δ_j be the formula $\bigwedge_{r \leq j} \pi_r \wedge \theta_i$. Each formula δ_j isolates $\bar{\Gamma}_i$ in the subspace of $S_A^m(\mathcal{A})$ whose points are the types with rank greater than or equal to the rank of $\bar{\Gamma}_i$. Hence for no $j < w$ is δ_j contained in ψ_i. Consequently for each $j < w$ we have that $\mathcal{A} \models \exists x\ (\delta_j(x) \wedge \psi_i(x))$. By definition of the formulas δ_j and the Compactness Theorem it follows that the set of formulas $\{\delta_j(x) \wedge \neg\psi_i(x)\}_{j<w}$ is consistent over $\mathcal{A}$. Hence this set of formulas has a completion to a type $\Gamma \in S(\mathcal{A})$. Let $\hat{\Gamma}$ be the open type determined by Γ. By definition of the formula δ_j and the type Γ it follows that $\hat{\Gamma}_i \subseteq \hat{\Gamma} \subseteq \Gamma$. But $\neg\psi_i \in \Gamma$ and so Γ is not equal to Γ_i. Contradiction. We can therefore assume that for each $i < n$ we have that $\mathcal{A} \models \theta_i \dashrightarrow \psi_i$. As $\theta_i \in \bar{\Gamma}_i$ the type Γ_i contains θ_i. Now ψ_i has degree 1 and so contains only one type of rank equal to that of ψ_i. Consequently the formula $\psi_i \wedge \neg\theta_i$ must be of rank less than that of ψ_i. Let γ_1 be the open formula $\bigvee_{i \leq n} \theta_i$. Then $\psi \wedge \neg\gamma_1$ has rank less than rank of ψ. We can repeat this procedure to obtain a sequence $\gamma_1, \ldots, \gamma_k$ of open formulas such that $\mathcal{A} \models \gamma_i \dashrightarrow \psi$ for each i and $\psi \bigvee_{i \leq n+1} \neg\gamma_i$ has rank less than $\psi \wedge \neg \bigvee_{i \leq n} \gamma_i$ for each n. We finally obtain a set γ_m such that $\psi \wedge \neg \bigvee_{i \leq m} \gamma_i$ is an algebraic formula and is thus representable by an open $L^m(\mathcal{A})$ formula γ_{m+1}. It follows that $\mathcal{A} \models \psi \dashleftarrow\dashrightarrow \bigvee_{i \leq m+1} \gamma_i$.

Now $\bigvee_{i \leq m+1} \gamma_i$ is an open $\mathfrak{A}$ formula with say the parameters $(a_1,\ldots,a_k)$. Since $\mathfrak{A}$ is a prime model of T it is an atomic model and so $(a_1,\ldots,a_k)$ realizes a principal L(T) type. Let T_1 be the extension of T by this principal type. Then $T_1 \vdash \psi \dashrightarrow \bigvee_{i \leq m+1} \gamma_i$.

⊠

The fact that universal classes with the amalgamation property do not in general have quantifier eliminable model completions means that one cannot prove any strong form of the Chevalley Theorem of algebraic geometry for these classes. This is true even for projections of open formulas. For if arbitrary projections of open formulas are definable by open formulas then all formulas over models have quantifier eliminable equivalents in the same language. The next theorem shows that projections of open formulas can be definable under special conditions. To state the theorem we need some additional terminology.

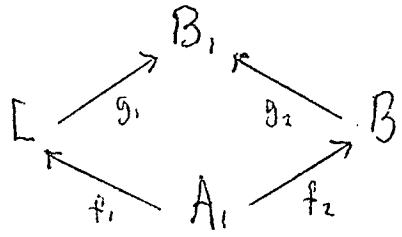
Suppose that $\sum$ is a universal class and $A \in \sum$. Let $\psi(v_1,\ldots, v_n, w_1,\ldots,w_m)$ be an $L_A{}^{n+m}(A)$-formula with all free variables among the v_i and w_j. We write the formula in the abbreviated form $\psi(v,w)$. The first and second block of variables are called the first and second coordinate of ψ. The projection of ψ on the second coordinate is the solution set in A of the formula $\exists v \psi(v,w)$. An open type Γ in $S_A{}^m(A)$ is a type through the second coordinate of ψ if there is an extension B of A in $\sum$ and a sequence (x,y) in B^{n+m} such that $B \models \psi(x,y)$ and (y) has open type Γ over A. The following two lemmas are required to prove the next theorem.

Lemma 4.4

Let Σ be a universal class with the amalgamation property and suppose that $A \in \Sigma$ and $\psi(v,w) \in L_A^{n+m}(A)$ is an algebraic formula. Then all open types over A through the second coordinate of ψ are algebraic.

Proof

Suppose not. Then there is a type $\Gamma \in S_A^m(A)$ through the second coordinate of ψ that is not algebraic. By definition of algebraic type it follows that there is an extension B of A in Σ such that B^m contains an infinite set $\{(c^i)\}\ i < w$ of distinct solutions of Γ. Since Γ is a type through the second coordinate of ψ there is an extension C of A in Σ and a sequence (x,y) in C^{n+m} such that $C \models \psi(x,y)$ and (y) has open type Γ over A. Let $A_1 = A(y)$. We form a chain of structures $\{B_i\}\ i < w$ in Σ as follows. Let f_1 be the identity map of A_1 into C and let f_2 be the natural embedding of A_1 into B that sends (y) onto (c^1). We can amalgamate the resulting diagram over A_1 to obtain the following commutative diagram:



We can assume the map g_2 is the identity. Since $C \models \psi(x,y)$ we have that $B_1 \models \psi(g_1(x),g_1(y))$ so that by commutativity of the diagram we deduce that $B_1 \models \psi(g_1(x), c^1)$. If we now assume that the chain has been constructed up to B_r in such a way that for each

sequence $(c^1),\ldots,(c^r)$ there is a sequence (z^i) in B_r such that $B_r \models \psi(z^i,c^i)$ then we can repeat the above construction with B_r in place of B. Having completed the chain $\{B_i\}\, i < w$ let $C_1 = \bigcup B_i$. Since Σ is a universal class it is closed under unions of chains so that $C_1 \in \Sigma$. Furthermore for each sequence (c^i) in C_1 there is a sequence (z^i) such that $C_1 \models \psi(z^i,c^i)$. Thus ψ is not algebraic over A. Contradiction. It follows that Γ must be algebraic. ⊠

For the next lemma and the proof of the theorem that follows note that the results of [13] apply also to types and formulas in several variables. We will thus refer in the proofs to the required results given in [13] for the single variable case.

Lemma 4.5

Suppose that Σ is a universal class with the amalgamation property and $A \in \Sigma$. Let $\psi(v,w) \in L_A{}^{n+m}(A)$ be a strongly minimal set such that the set of all open types through the second coordinate of ψ has the splitting property. Then any open type through the second coordinate of ψ has rank not greater than one.

Proof

First note that there is at most one type through the second coordinate of ψ in $S_A{}^m(A)$ that is not algebraic. For ψ is a strongly minimal set so that there is a unique nonalgebraic open type Γ in ψ. If Γ_1 is an algebraic open type in ψ then by lemma 2.7 of [13] there is an $L_A{}^{n+m}(A)$-algebraic formula θ that generates the type Γ_1. By Lemma 4.4 each open type through the second coordinate of θ

is algebraic. It follows that if Γ_2 is an open type through the second coordinate of ψ and B is an extension of A with a sequence (x,y) in B^{n+m} such that $\psi(x,y)$ is valid in B and (y) has open type Γ_2 over A then (x,y) has open type Γ over A if Γ_2 is not algebraic. Suppose then that Γ_2 is an open type through the second coordinate of ψ such that Γ_2 has rank greater than one. For every open formula $\theta \in L_A^m(A)$ such that Γ_2 is a type in θ there is some nonalgebraic type Γ_θ in θ with Γ_θ not equal to Γ_2. If not then every type in θ not equal to Γ_2 is algebraic so that Γ_2 must have rank equal to one. Let S be the set of all finite conjunctions of formulas in Γ_2. Given $\delta \in S$ the type Γ_δ is not an open type through the second coordinate of ψ. This follows by what we showed above. Thus the formula $\delta \wedge \forall v \neg \psi(v,w)$ is consistent over A. It follows by the Compactness Theorem that $\Gamma_2 \cup \{\forall v \neg \psi(v,w)\}$ is consistent over A. Thus Γ_2 splits over A which contradicts the assumption that the set of open types through the second coordinate of has the splitting property. ☒

Let Σ be a universal class and $A \in \Sigma$. If $\psi(v,w) \in L_A^{n+m}(A)$ is a stongly minimal set call the unique type in ψ of rank one the type of ψ. If Γ is the type of ψ and (x,y) is a realization of Γ in an extension B of A in Σ call the type of (y) over A in $S_A^m(A)$ the projection of Γ on the second coordinate. Note that we have shown in the proof of the above lemma the stronger result that if the projection Γ_1 of the type of ψ on the second coordinate does not split over A then assuming the amalgamation property is valid in Σ the rank of Γ_1 is not greater than one.

Theorem 4.17

Suppose that Σ is a universal class with the amalgamation property and $A \in \Sigma$ is existentially closed. Let $\psi(v,w)$ be a strongly minimal $L_A^{n+m}(A)$-formula such that the projection on the second coordinate of the type of ψ does not split over A. Then the projection of ψ on the second coordinate is finite or definable by a strongly minimal open A formula.

Proof

Suppose that the projection X of ψ on the second coordinate is not finite. For each finite subset $(x^1),\ldots,(x^r)$ of A^m the set of formulas $\{\neg(x^1) = (w),\ldots,\neg(x^r) = (w), \exists v\, \psi(v,w)\}$ is therefore consistent over A. Thus the set Γ_0 of all such sentences is consistent over A and is therefore contained in a complete A-type Γ_1. The type Γ_1 is a type through the second coordinate of ψ so that the open part Γ of Γ_1 is an open type through the second coordinate of ψ. Furthermore Γ is not an algebraic type. For if so then Γ is generated by an $L_A^m(A)$-algebraic formula θ. Since A is existentially closed there is a solution (x) of θ in A. But Γ contains the formula $\neg(x) = (w)$. And so Γ has rank greater than zero. By Lemma 4.5 it follows that Γ must have rank equal to one. Let θ be a formula that isolates Γ in the subspace of $S_A^m(A)$ consisting of all open types having rank greater than zero. By Theorem 4.44 of [13] the formula θ must be strongly minimal since it is minimal over A.

Claim

The set $Y = \theta(A)$ X is finite. For suppose not. Let S be the set of all finite conjunctions of formulas in Γ. If $\delta \in S$ then the formula $\delta \wedge \theta$ is a member of Γ so that $\delta \wedge \theta$ has an infinite number of solutions in A. For this formula cannot be algebraic so that for each finite subset $(x^1),\ldots,(x^r)$ in A^m the formula $\neg (x^1) = (w) \wedge,\ldots,\wedge \neg (x^r) = (w) \wedge (\delta \wedge \theta)(w)$ must have a solution in the existentially closed structure A. Since θ is strongly minimal the formula $\delta \wedge \theta$ must have a solution in the set Y. This implies that the formula $\delta \wedge \theta \wedge \forall v \psi(v,w)$ has a solution in A. By the Compactness Theorem it follows that $\Gamma \cup \{\forall v \neg \psi(v,w)$ is consistent over A. But this means that Γ splits over A contrary to assumption. Thus the set Y must be finite. Note next that the set X $\theta(A)$ is also finite. For if not the formulas $\psi(v,w) \wedge \theta(w)$ and $\psi(v,w) \wedge \neg \theta(w)$ have infinite disjoint solution sets. This contradicts the assumption that the formula $\psi(v,w)$ is strongly minimal. Thus by a slight alteration of the formula θ we obtain a strongly minimal open formula that has the set X as solution set in A. ⊠

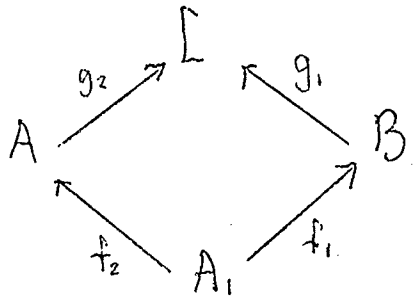
The following proposition is another application of Lemma 4.4. For the definition of algebraically closed structures see [13].

Proposition 4.1 (The Weak Nullstellensatz)

Suppose that Σ is a universal class with the amalgamation property and $A \in \Sigma$ is algebraically closed. If ψ is an open algebraic A-formula then ψ has a solution in A.

Proof

If ψ is an open formula in one variable it has a solution in A since A is algebraically closed. Suppose then that ψ is a formula is n+1 variables and the assertion is true for all open algebraic formulas in n variables. Write ψ in the form $\psi(v,w)$ where w is a single variable. By lemma 4.3 all open types through the second coordinate of ψ are algebraic. Let $\Gamma(w)$ be a type through the second coordinate of ψ. Then Γ is generated by an open algebraic formula θ defined over A. Since θ is a formula in one variable it has a solution x in A. Let $\psi_1(v)$ be the formula $\psi(v,\vec{x})$. Then ψ_1 is algebraic over A if it has any solution in extensions of A. Since Γ is an open type through the second coordinate of $\psi(v,w)$ there is an extension B of A and a sequence (z,y) in B^{n+1} such that $B \models \psi(z,y)$ and y has open type Γ over A. Let A_1 be the substracture generated by x in A. The elements x and y have the same open $L(\sum)$ types so that we can embed A_1 in B by a map f_1. Let f_2 be the identity embedding of A_1 in A. We can amalgamate these maps over A_1 to obtain the following commutative diagram:



We can assume that the map g_2 is the identity. Then since $B \models \psi(x,y)$ it follows that $C \models \psi(g_1(z),g_1(y))$. By the commutativity of the diagram we deduce that $C \models \psi(g_1(z),x)$. Thus $C \models \psi_1(g_1(z))$ so that ψ_1 has solutions in extensions of A. By the induction hypothesis it follows that ψ_1 has a solution in A. This

means that $\exists \psi(v,w)$ has a solution in A. ☒

Theorem 4.17 is a localized quantifier elimination theorem for universal classes with the amalgamation property. The question in the large for such classes is the following: Given a universal class Σ with the amalgamation property when is there a theory T such that $\Sigma = \Sigma_T$ and T admits elimination of quantifiers? The following theorem shows that for universal classes with the amalgamation property the above problem is equivalent to the model companion problem for Σ.

Theorem 4.18

Suppose that Σ is a universal class with the amalgamation property. There is a theory T such that $\Sigma = \Sigma_T$ and T admits elimination of quantifiers if and only if Σ has a model companion.

Proof

Suppose that Σ has a model companion. Then according to the definition of model companion given in [20] there is a model complete theory T such that $\Sigma = \Sigma_T$. By assumption Σ has the amalgamation property so that by Theorem 4.11 it follows that T is submodel complete. Thus by a result of [6] the theory T admits elimination of quantifiers. The other direction is immediate from the definitions. ☒

The above result suggests that in the context of universal classes with the amalgamation property the theory of model companions may have a character distinct from the general theory. A measure of the difference is the variance in the submodel and model completeness criteria.

The following lemma is proven in [20] by a diagram argument.

Lemma 4.6

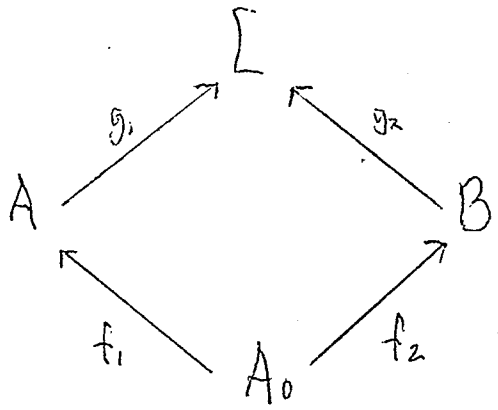
Suppose that Σ is a universal class and $A \in \Sigma$ is existentially closed. If B is an extension of A in Σ then there is an elementary extension C of A such that C is an extension of B.

Lemma 4.7

Suppose that Σ is a universal class with the amalgamation property and D is a maximal Σ-diagram. If Γ is an open type in D there is a unique closed $L(\Sigma)$ type $\vec{\Gamma}$ such if A is an existentially closed D structure and $\langle x_1,\ldots,x_n\rangle$ is a sequence in A^n that realizes Γ then $\langle x_1,\ldots,x_n\rangle$ realizes the type $\vec{\Gamma}$.

Proof

Suppose not. Then there is an open type $\Gamma(v_1,\ldots,v_n)$ in D and an $L(\Sigma)$ formula $\psi(v_1,\ldots,v_n)$ such that both $\Gamma \cup \{\psi\}$ and $\Gamma \cup \{\neg\psi\}$ are realized in D-structures. Let A be a D-structure and $\langle a_1,\ldots,a_n\rangle$ a sequence in A^n such that $\langle a_1,\ldots,a_n\rangle$ realizes Γ and $A \models \psi(a_1, \ldots,a_n)$. Similarly let B be a D-structure and $\langle b_1,\ldots,b_n\rangle$ a sequence in B^n such that $\langle b_1,\ldots,b_n\rangle$ realizes Γ and $B \models \neg\psi(b_1, \ldots,b_n)$. Let A_0 be the substructure of A generated by the elements $a_1,\ldots,a_n$. Define the map f_1 to be the identity embedding of A_0 in A and the map f_2 to be the embedding of A_0 in B such that for each $i < n$ we have $f_2(a_i) = b_i$. We can amalgamate the resulting diagram over A_0 to obtain the following communtative diagram:



It follows by the communtativity of the diagram that for each $i < n$ we have $g_1f_1(a_i) = g_2f_2(a_i) = c_i$. Let $A_1 = g_1(A)$ and $B_1 = g_2(B)$. Then the structures A_1 and B_1 are existentially closed and $A_1 \models \psi(c_1,\ldots,c_n)$ and $B_1 \models \neg\psi(c_1,\ldots,c_n)$. We construct elementary chains $\{A_i\}$ $i < w$ and $\{B_i\}$ $i < w$ as follows. By Lemma 4.6 there is an elementary extension A_2 of A_1 such that A_2 is an extension of C. In the same way there is an elementary extension B_2 of B_1 that is an extension of A_2. We repeat this alternating construction to produce the elementary chains $\{A_i\}$ $i < w$ and $\{B_i\}$ $i < w$. Let $C_1 = UA_i = U\ B_i$. Then C_1 is an elementary extension of A_1 and B_1 so that $C_1 \models \psi(c_1,\ldots,c_n)$ and $C_1 \models \neg\psi(c_1,\ldots,c_n)$. Contradiction. ⊠

The following theorem is proven in [20] in the general case.

Theorem 4.19 (The Model Companion Theorem)

Suppose that Σ is a universal class with the amalgamation property having a unique maximal Σ-diagram D. Then Σ has a unique model companion if and only if the class of existentially closed structures in Σ is axiomatizable.

Proof

If Σ has a model companion T it follows from Theorem 4.11 that T admits elimination of quantifiers. The argument of Theorem 3.39 of [13] then shows that the models of T are the existentially closed structures in Σ. Thus if a model companion exists it is unique. Suppose that the existentially closed structures in Σ are axiomatizable by a theory T. Since every structure in Σ is contained in an existentially closed structure it follows that $\Sigma = \Sigma_T$. Thus D is the set of open types consistent with T. By Lemma 4.7 it follows that the set of open T types has the splitting property. Thus by Theorem 4.13 and a result of [6] we deduce that T admits the elimination of quantifiers. In particular T is model complete and consequently T is the model companion of . ☒

Lemma 4.8

Suppose that Σ is a universal class with the amalgamation property. Then there is a unique maximal Σ-diagram if and only if Σ has the joint embedding property.

Proof

If Σ has the joint embedding property then it follows from Definition 3.14 of [13] that there is a unique maximal Σ-diagram. Suppose then that Σ has a unique maximal Σ-diagram D. By Theorem 3.16 of [13] there is a complete theory T such that Σ_T is the class of all D-structures. Since the diagram D is unique the class of all D-structures is equal to Σ. Thus $\Sigma = \Sigma_T$. By Theorem 1.11 of [13] it follows

that Σ_T has the joint embedding property and so Σ has the joint embedding property. ☒

It follows from Lemma 3.15 and Theorem 3.16 of [13] that if Σ is a universal class with the amalgamation property then Σ decomposes into universal classes that have unique maximal Σ-diagrams. Each such factor of a universal class with the amalgamation property also has the amalgamation property. There is thus no loss of generality in the restriction that a universal class have a unique maximal diagram.

Definition 4.6

Suppose that Σ is a universal class with the amalgamation property and D is a maximal Σ-diagram. Then D is atomic if each type in D is generated by an open $L(\Sigma)$ atom.

Lemma 4.9

Suppose that Σ is a universal class with the amalgamation property having a unique maximal Σ-diagram D. If D is atomic then the class of existentially closed structures in Σ is axiomatizable.

Proof

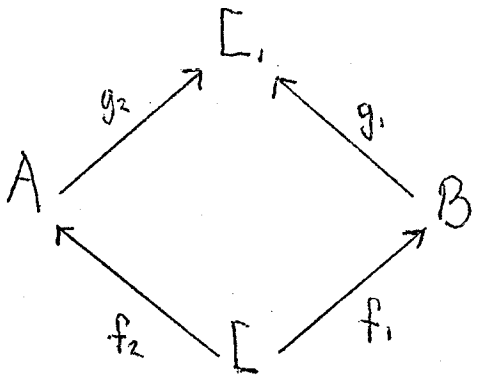
We define a set of sentences T as follows. If $\psi(v_1,\ldots,v_n)$ is an open $L(\Sigma)$ atom then add to T the sentence $\exists v_1,\ldots, \exists v_n \psi(v_1, \ldots,v_n)$. For a given open $L(\Sigma)$ atom $\psi(v_1,\ldots, v_n)$ let $\Gamma(v_1, \ldots, v_n,v_{n+1})$ be an open type in D such that $\psi \in \Gamma$. Since D is atomic there is an open $L(\Sigma)$ atom $\psi_1(v_1,\ldots, v_{n+1})$ such that ψ_1 generates Γ. We add to T the sentence

$$\forall v_1,\ldots, \forall v_n \exists v_{n+1}(\psi_1(v_1,\ldots,v_{n+1}) \vee \neg\psi(v_1,\ldots,v_n)$$

Let $T_1 = Th(\Sigma)UT$.

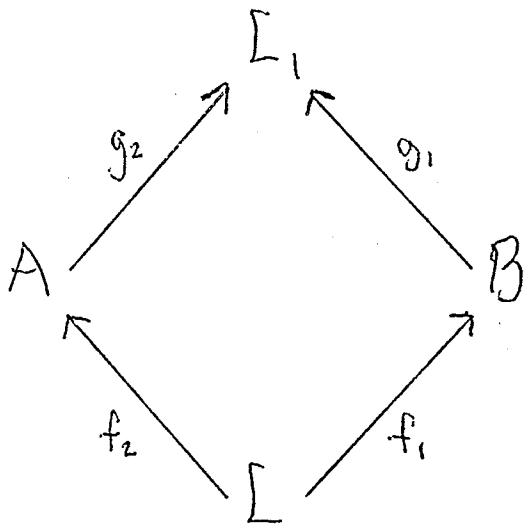
Claim

The theory T_1 is an axiomatization of the class of existentially closed structures in Σ. Suppose first that A is an existentially structure in Σ. Let $\psi(v_1,\ldots,v_n)$ be an $L(\Sigma)$ atom. Since ψ is consistent with Σ there is a structure B in Σ and a sequence $\langle b_1,\ldots,b_n\rangle$ in B^n such that $B \models \psi(b_1,\ldots,b_n)$. By Lemma 4.8 the universal class Σ has the joint embedding property so the structures A and B have a common extension C in Σ. As A is existentially closed and $C \models \psi(b_1,\ldots,b_n)$ it must be that $A \models \exists v_1,\ldots,v_n \psi(v_1,\ldots,v_n)$. Assume next that θ is a sentence in T of the form $\forall v_1,\ldots,\forall v_n \exists v_{n+1}(\psi_1(v_1,\ldots,v_{n+1}) \rightarrow \psi(v_1,\ldots,v_n))$. Since the formula ψ_1 is an $L(\Sigma)$ atom there is a structure B in Σ and a sequence $\langle b_1,\ldots,b_{n+1}\rangle$ in B such that $B \models \psi_1(b_1,\ldots,b_{n+1})$. By the definition of T it follows that $B \models \psi(b_1,\ldots,b_n)$. Suppose that there is a sequence $\langle a_1,\ldots,a_n\rangle$ in A^n such that $A \models \psi(a_1,\ldots,a_n)$. Let C be the substructure of A generated by the elements $a_1, \ldots, a_n$. Since ψ is an $L(\Sigma)$ atom the sequences $\langle b_1,\ldots,b_n\rangle$ and $\langle a_1,\ldots,a_n\rangle$ realize the same open type in D. There is thus an isomorphism f_1 of C into B such that for each $i < n$ we have $f_1(a_1) = b_i$. Let f_2 be the identity isomorphism that maps C into A. We can amalgamate the resulting diagram over C to obtain the following commutative diagram:



We can assume that the map g_2 is the identity. By the commutativity of the diagram it follows that $C_1 \models \psi_1(a_1,\ldots,a_n,g_1(b_{n+1}))$. Since A is existentially closed and the open A-formula $\psi_1(a_1,\ldots,a_n,v)$ has a solution in an extension of A in Σ the formula must have a solution in A. Thus we conclude that $A \models \forall v_1,\ldots,\forall v_n,\exists v_{n+1}$ $(\psi_1(v_i,\ldots,v_{n+1}) \vee \neg\psi(v_1,\ldots,v_n))$. Therefore A is a model of the theory T_1. Suppose then that A is a model of the theory T_1. Let $\psi(v_1,\ldots,v_n)$ be an open A-formula and suppose that B is an extension of A in Σ that contains a solution $\langle b_1,\ldots,b_n\rangle$ of the formula ψ. If ψ contains no constants representing elements of A that are not the interpretation of constants from $L(\Sigma)$ then let $\theta(v_1,\ldots,v_n)$ be an $L(\Sigma)$ atom that generates an open type Γ in D such that $\psi \in \Gamma$. Such an atom exists since the diagram D is atomic. Since A is a model of T it follows that $A \models \exists v_1,\ldots,\exists v_n\ \theta(v_1,\ldots,v_n)$. Thus $A \models \exists v_1,\ldots,\exists v_n\ \psi(v_1,\ldots,v_n)$ and consequently the formula has a solution in A. Assume therefore that the formula ψ is equal to $\psi_1(\bar{c}_1,\ldots,\bar{c}_m,v_1,\ldots,v_n)$ where the constants c_i are all the new constants in ψ. Let Γ be the open type of the sequence $\langle c_1,\ldots,c_m,b_1\rangle$ in B. Then $\Gamma \in D$ and since D is atomic there is an $L(\Sigma)$ atom θ that generates Γ. Similarly the open

type of the sequence $\langle c_1,\ldots,c_m\rangle$ is generated by an open $L(\Sigma)$ atom θ. Since A is a model of T it follows that $A \models \forall v_1,\ldots,\forall v_m \exists v_{n+1}\ (\theta\ (v_1,\ldots,v_{m+1})\ \vee \neg\ \theta\ (v_1,\ldots,v_m))$. Let c_{m+1} be an element of A such that $A \models \theta_1\ (c_1,\ldots,\ c_{m+1})$. Let C be the substructure of A generated by the elements $c_1,\ldots,\ c_{m+1}$ in A. Since θ_1 is an $L(\Sigma)$ atom the sequences $\langle c_1,\ldots,c_{m+1}\rangle$ and $\langle c_1,\ \ldots,c_m,b_1\rangle$ realize the same open $L(\Sigma)$ type. Thus there is an isomorphism f_1 of C into B such that $f_1(c_1) = C_1,\ldots,f_1(c_m) = c_m$ and $f_1(c_{m+1}) = b_1$. Let f_2 be the identity isomorphism of C into A. The resulting diagram can be amalgamated over C to obtain the following commutative diagram:



We can assume that g_2 is the identity map. From the commutativity of the diagram we deduce that $C_1 \models \psi_1(c_1,\ldots,c_m,c_{m+1},g_2\ (b_2),\ldots,g_2(b_n))$. We can repeat this argument n times to obtain a solution of ψ in A. Therefore the structure A is existentially closed. ☒

Theorem 4.20 (The Characterization Theorem)

Suppose that Σ is a universal class with the amalgamation property having a unique maximal Σ-diagram D. Then Σ has an w-categorical model companion if and only if the diagram D is atomic.

Proof

Suppose first that Σ has an w-categorical model companion T. It follows from Theorem 4.18 that T must admit the elimination of quantifiers. A result of [6] implies that all closed (T)-types are generated by open atoms. Since $\Sigma = \Sigma_T$ each type in D is an open (T)-type and so D is atomic. Next assume that D is atomic. It follows from Lemma 4.9 that the class of existentially closed structures in Σ is axiomatizable and so by Theorem 4.19 the universal class Σ has a model companion T. As was demonstrated in the proof of Theorem 4.19 the theory T axiomatizes the class of existentially closed structures in Σ. By Theorem 4.5 the latter class is w-categorical and so T is an w-categorical theory. ☒

The above result can be applied to prove the following theorem of [20]. Other proofs of the general form of this theorem can be found in [21] and [39].

Theorem 4.21

Suppose that Σ is a universal class with the amalgamation property. If there is an w-categorical theory T such that $\Sigma = \Sigma_T$ then Σ has an w-categorical model companion.

Proof

Let A be the countable model of T and define T_1 to be the theory Th(A). Then T_1 is complete and $\Sigma_{T1} =$. As each $C \in \Sigma_{T1}$ is in Σ we must show that if $B \in \Sigma$ then $D_A(B) \cup T_1$ is a consistent theory. By the Compactness Theorem it suffices to consider the finitely generated structures B in Σ. By assumption $\Sigma = \Sigma_T$ so there is a countable model C of T such that B is a substructure of C. Since T is w-categorical the structures C and A are isomorphic so that B is a substructure of a model of T_1. By Theorem 1.11 of [13] it follows that Σ has the joint embedding property and so by Lemma 4.8 the universal class Σ has a unique maximal Σ-diagram D.

Claim

The diagram D is atomic. Let Γ be an open type in D. If Γ is not atomic then for each conjunction of formulas in there is a formula in Γ whose negation is consistent in Σ with the conjunction of formulas. Order Γ in a sequence $\{\theta_i\}$ $i < w$. For each $n < w$ we define a formula ψ_n as follows. Let θ_j be a formula such the $\theta_1 \wedge \neg \theta_j$ is consistent in Σ. Assume that ψ_n has been defined for all $n < m + 1$ and that $\psi_m = \bigwedge_{i \leq s} \theta_i \wedge \neg \theta_r$. Let θ_k be a formula in Γ such that $\bigwedge_{i \leq r} \theta_i \wedge \neg \theta_k$ is consistent in Σ. The formula $\psi_{m+1} = \bigwedge_{i \leq r} \theta_i \wedge \neg \theta_k$. The formulas $\{\psi_n\}$ $n < w$ are pairwise inequivalent relative to the theory T. This is clear from the construction and the fact that $\Sigma = \Sigma_T$. This contradicts a property of w-categorical theories (see for example [6]). If follows that for some $n < w$ the formula $\bigwedge_{i \leq n} \theta_i$ is an open $L(\Sigma)$ atom. Thus the

diagram D is atomic. The conclusion follows from Theorem 4.20. ☒

3. The Principle of Counting Constants

In this section we give a complete description of the possible correspondences of two strongly minimal sets. The result obtained is a generalization of a fundamental correspondence theorem of algebraic geometry. In the course of the proof of the main result we also generalize an irreducibility property of the product of algebraic varieties. This latter property was earlier generalized to the product of constructible sets defined over fields.

We assume throughout the section that $\mathfrak{A}$ is a fixed countable existentially closed structure in a universal class $\mathfrak{K}$ with the amalgamation property. The structure $\mathfrak{A}$ is to be considered as a substructure of a fixed structure $\mathfrak{C}$ that is w_1-saturated. Such an arrangement is similar to the use of universal domains in algebraic geometry. As also in algebraic geometry the sets that are considered can have several free variables.

Definition 4.7

Suppose that φ_1 is an open $L^n(\mathfrak{A})$ formula and φ_2 is an open $L^m(\mathfrak{A})$ formula. A correspondence of φ_1 and φ_2 is defined by an open $L^{n+m}(\mathfrak{A})$ formula Π if for each (x) in φ_1 the solution set of $\Pi(\bar{x},y)$ is a subset of φ_2 and for each (y) in φ_2 the solution set of $\Pi(x,\bar{y})$ is a subset of φ_1.

The above definition is a generalization of the definition of correspondence in algebraic geometry. Basic results and references in

this subject can be found in [42]. The following definition is also a generalization of a standard notion of algebraic geometry.

Definition 4.8

Suppose that Π is a correspondence of open formulas Ψ_1 and Ψ_2. Then Π is an (n,m)-correspondence if given (x) in Ψ_1 the solution set of $\Pi(\vec{x},y)$ has at most m elements and given (y) in Ψ_2 the solution set of $\Pi(x,\vec{y})$ has at most n elements.

The following result is an immediate consequence of the Compactness Theorem.

Theorem 4.22

Suppose that Π is a correspondence of open formulas Ψ_1 and Ψ_2. If the solution sets of $\Pi(\vec{x},w)$ and $\Pi(v,\vec{y})$ are finite for each choice of (x) in Ψ_1 and (y) in Ψ_2 the for some numbers n and m Π is an (n,m)correspondence.

Definition 4.9

Suppose that Ψ_1 is an open $L^n(\mathfrak{A})$ formula and Ψ_2 is an open $L^m(\mathfrak{A})$ formula. The product $\Psi_1 x \Psi_2$ of Ψ_1 and Ψ_2 is the formula $\Psi_1(v_0,\ldots,v_{n-1}) \wedge \Psi_2(v_n,\ldots,v_{n+m})$.

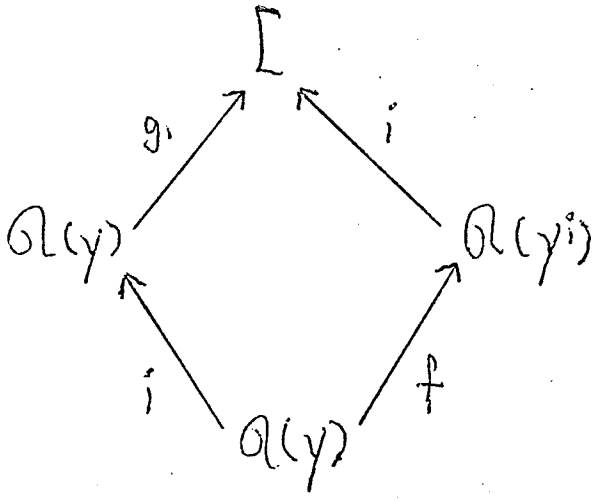
If Ψ_1 and Ψ_2 are open $L(\mathfrak{A})$ formulas the solution set of the product formula $\Psi_1 x \Psi_2$ is the direct product of the solution sets of Ψ_1 and Ψ_2.

Theorem 4.23

Suppose that Ψ_1 and Ψ_2 are open strongly minimal $L(\mathfrak{A})$ formulas. Let (x) be an independent point of Ψ_1 and (y) an independent point of Ψ_2. Then (x) is algebraic over (y) iff (y) is algebraic over (x).

Proof

Suppose that Π is an open $L(\mathfrak{A})$ formula such that $\Pi(x,v)$ is algebraic and has (y) as a solution. Then (x) is a solution of the formula $\Psi_1(v) \wedge \Pi(v,y) = \theta(v,\vec{y})$. If $\theta(v,\vec{y})$ is not algebraic then all but some finite subset of Ψ_1 is in the solution set since Ψ_1 is strongly minimal. Let n be the number of solutions of $\Pi(\vec{x},v)$. Choose a set of n+1 distinct independent points $(y^1),\ldots,(y^{n+1})$ of Ψ_2. For each point (y^i) the formulas $\theta(v,y^i)$ must have all but a finite subset of Ψ_1 as solutions. Since (y^i) is an independent point of Ψ_2 there is an $\mathfrak{A}$ isomorphism $f: \mathfrak{A}(y) \dashrightarrow \mathfrak{A}(y^i)$ that maps (y) onto (y^i). We can amalgamate the maps f and the identity isomorphism over $\mathfrak{A}$ to obtain a structure C and maps $g_1: \mathfrak{A}(y) \dashrightarrow C$ and $g_2: \mathfrak{A}(y^i) \dashrightarrow C$ with g_2 the identity map such that the following diagram commutes:



Note that C can be taken to be a substructure of $\mathfrak{J}$. Now $\mathfrak{A}$ is existentially closed so that ψ_1 has an infinite number of solutions in $\mathfrak{A}$. Thus $\theta(v,y)$ has an infinite solution set in $\mathfrak{A}(y)$ so that $\theta(v,y^i) = \theta(v,g_1(y))$ has an infinite solution set in C and hence in $\mathfrak{J}$. For each $i < n+1$ let X_i be the solution set of $\theta(v,y^i)$. Since each X_i has a finite complement the intersection $\bigcap_{i \leq n+1} X_i$ is infinite and so must contain an independent point (z) of ψ_1. Now the point (z) is a solution of each formula $\pi(v,\bar{y}^i)$ so that (y^i) is in the solution set of $\pi(\bar{z},v)$ for each $i < n+1$. Consequently the points (x) and (z) have solution sets for the formulas $\pi(\bar{x},v)$ and $\pi(\bar{z},v)$ of different cardinalities. But (x) and (z) have the same type over $\mathfrak{A}$ and so by an amalgamation argument the solution sets of $\pi(\bar{x},v)$ and $\pi(\bar{z},v)$ must have the same cardinality. Hence $\pi(v,\bar{y})$ is an algebraic formula and (x) is algebraic over (y). ⊠

Definition 4.10

Suppose that ψ_1 and ψ_2 are open strongly minimal $L(\mathfrak{A})$ formulas and (x) is an independent point of ψ_1 and (y) is an independent point of ψ_2 that is not algebraic over (x). Then (x) and (y) are algebraically free over $\mathfrak{A}$.

The above definition is justified by the previous theorem. The concept thus defined is related to the notion of linear disjointness in algebraic geometry. See for example [1].

Theorem 4.24

Suppose that ψ_1 and ψ_2 are open strongly minimal $L(\mathfrak{A})$

formulas and (x) in ψ_1 and (y) in ψ_2 are algebraically free. The open type Γ of the sequence (x,y) does not depend on the choice of (x) and (y).

Proof

Suppose that (z) in ψ_1 and (w) in ψ_2 are algebraically free. Then (x) and (z) have the same open type over $\mathfrak{A}$ so there is an $\mathfrak{A}$ isomorphism f: $\mathfrak{A}(x) \longrightarrow \mathfrak{A}(z)$ that maps (x) to (z). The point (y) is not algebraic over (x) and so it has the unique type Γ_1 in ψ_2 of rank one over $\mathfrak{A}(x)$. In the same way (w) realizes the unique type Γ_2 in ψ_2 of rank one. The isomorphism f induces a map $\hat{f}$ from types to types that preserves the rank of types and so $\hat{f}(\Gamma_1) = \Gamma_2$. We can thus lift f to an isomorphism f_1: $\mathfrak{A}(x,y) \longrightarrow \mathfrak{A}(z,w)$ that maps (x,y) to (z,w). As f_1 is an $\mathfrak{A}$ isomorphism this implies that (x,y) and (z,w) have the same open $\mathfrak{A}$ type. ⊠

The following correspondence theorem is required in the calculation of the rank of the type generated by algebraically free sequences and also in the proof of the main result.

Theorem 4.25

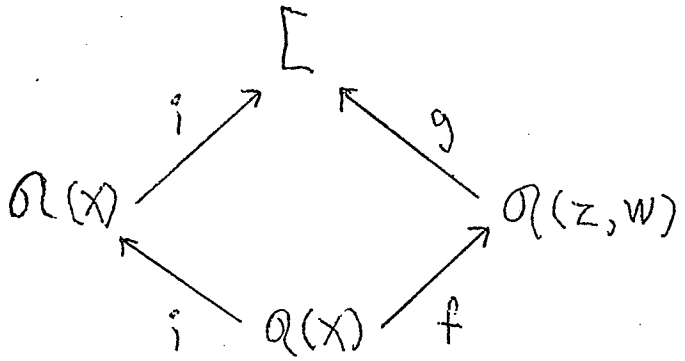
Suppose that ψ_1 and ψ_2 are open strongly minimal $L(\mathfrak{A})$ formulas and R is a correspondence of ψ_1 and ψ_2. Let (x) be an independent point of ψ_1. If $R(\bar{x},v)$ is algebraic then the correspondence has rank one.

Proof

Suppose that R has rank greater than one. Then there is a countable

extension $\mathfrak{A}_1$ of $\mathfrak{A}$ in $\mathfrak{F}$ and a sequence of open $\mathfrak{A}_1$ formulas $\{\theta_i\}_{i<\omega}$ such that the solution sets of the θ_i are infinite disjoint subsets of R. Note that since $\mathcal{Y}_1$ is strongly minimal our assumptions imply that R(x,v) is finite for all but a finite subset of $\mathcal{Y}_1$. For if not it follows by the compactness theorem that some independent point (z) of $\mathcal{Y}_1$ has an infinite solution set. By an argument similar to that of the proof of Theorem 4.20 this yields a contradiction. If a finite number of points of $\mathcal{Y}_1$ have infinite solution sets we can split these points off $\mathcal{Y}_1$ and replace R by a correspondence which differs from R only by a set of rank one. So assume that for all points of $\mathcal{Y}_1$ the solution set of R(x,v) is a finite set. The set θ_i splits R into the two infinite sets θ_i and $\neg\theta_i$. If the projection onto $\mathcal{Y}_1$ of one of these two sets is finite it would follow from our above assumption that the set itself is finite. Consequently the respective projections X and Y in $\mathcal{Y}_1$ must both be infinite. There are two possibilities: (1) Only one projection contains an independent point (2) Both projections contain independent points. Suppose that the first case holds and say X contains no independent point. Let $\{\Pi_i\}_{i<\omega}$ be an ordering of the open $\mathfrak{A}$ type of the independent point of $\mathcal{Y}_1$ over $\mathfrak{A}$. For each $n<\omega$ let $\mathfrak{b}_n(v,w)$ be the formula $\theta_i(v,w) \wedge \bigwedge_{i\leq n} \Pi_i(v)$. Since X is infinite for each $n<\omega$ there is (z^n) in X and (y^n) in $\mathcal{Y}_2$ such that (z^n,y^n) is a solution of $\mathfrak{b}_n$. Hence the set of formulas $\{\mathfrak{b}_n\}_{n<\omega}$ is consistent over $\mathfrak{A}$ so has a realization (x,y) in $\mathfrak{F}$. Then (z,y) is a solution of θ_i and (z) must be an independent point of $\mathcal{Y}_1$. Consequently case two must hold. Let $(x) \in X$ and $(z) \in Y$ be independent points of $\mathcal{Y}_1$. Since (x) and (z) have the same open $\mathfrak{A}$ type there is an $\mathfrak{A}$

isomorphism f: $\mathfrak{A}(x)$----> $\mathfrak{N}(z)$ that maps (x) to (z). Let (w) be a point of ψ_2 such that (x,w) is a solution of $\neg\theta_1$. Then f is an isomorphism of $\mathfrak{N}(x)$ into $\mathfrak{A}(x,w)$ so we can amalgamate f and the identity map over to obtain the following communtative diagram:



We can assume $C \preceq \mathfrak{F}$ and $\mathfrak{A}(x)$ embeds in C by the identity map. Now $C \models \neg\theta_1(g(z),g(w))$ and by commutativity of the diagram $g(z)=(x)$. So $(x,g(w))$ is a solution of $\neg\theta_i$. Thus an independent point of ψ_1 has solutions in all the disjoint sets θ_i. This contradicts the assumption that the solution set of $R(\vec{x},v)$ is finite. ☒

Corollary 4.3

Let ψ_1 and ψ_2 be open strongly minimal $L(\mathfrak{A})$ formulas and suppose that R is a (n,1)-correspondence of ψ_1 and ψ_2. Then R has degree one.

Proof

If not then R can be split into two infinite disjoint sets θ_1 and θ_2 that cover R. As in the above proof if (x) is an independent point of ψ_1 then (x) has solutions in $\theta_1(x,v)$ and $\theta_2(x,v)$.

Contradiction. ⊠

Theorem 4.26

Assume that Σ is L_A-(w)-stable. Suppose that ψ_1 and ψ_2 are open strongly minimal $L(\mathfrak{A})$ formulas. The product $\psi_1 \times \psi_2$ of ψ_1 and ψ_2 has rank two and degree one. If (x) in ψ_1 and (y) in ψ_2 are algebraically free then (x,y) is an independent point of $\psi_1 \times \psi_2$.

Proof

First note that $\psi_1 \times \psi_2$ must have rank at least two. For say $\psi_1 \in L^n(\mathfrak{A})$. Let $\{(x^i)\}_{i<w}$ be an infinite sequence of distinct elements of ψ_1. Then for each $i < w$ we define θ_i to be the formula $(v_1 = \bar{x}_1^i \wedge \ldots \wedge v_n = \bar{x}_n^i) \wedge \psi_2(w)$. The solution sets of the θ_i are infinite and disjoint. Assume that (z,y) is a point of $\psi_1 \times \psi_2$. There are three possibilities: (1) (z) and (y) are algebraic points (2) (z) is an independent point of ψ_1 and (y) is algebraic over (z) (3) (z) and (y) are algebraically free. In the first case the type of (z,y) is algebraic. Consider the second case. Let Γ_1 be the type of (z,y) in the open $\mathfrak{A}$ language. There is then some formula θ that isolates Γ_1 in the subspace of $S_A^{n+m}(\mathfrak{A})$ consisting of types of rank greater than or equal to the rank of Γ_1 (here $\psi_2 \in L^m(\mathfrak{A})$). As $\psi_1 \times \psi_2$ is in Γ_1 the solution set of θ is a subset of the solution set of $\psi_1 \times \psi_2$. Thus θ is a correspondence of ψ_1 and ψ_2. The formula θ can also be chosen so that $\theta(zv)$ is an algebraic formula. For (y) is algebraic over $\mathfrak{A}(z)$ so (y) is a solution of an $L_A(\mathfrak{A}(z))$-algebraic formula

$\theta_1(z,v)$. This is proven in [13]. The formula θ_1 is in the type Γ_1 and so $\theta \wedge \theta_1$ also isolates Γ_1 in the subspace of $S_A^{n+m}(\mathfrak{A})$ consisting of points whose ranks are greater than or equal to the rank of Γ_1. It follows that θ satisfies the assumptions of Theorem 4.26. Hence the rank of θ is one and the rank of Γ_1 must also be one. The types of all solutions of $\mathcal{Y}_1 \times \mathcal{Y}_2$ of the type of case two above are therefore of rank one. If (x) in $\mathcal{Y}_1$ and (y) in $\mathcal{Y}_2$ are algebraically free then by Theorem 4.25 the type of (x,y) is uniquely determined. It is therefore of rank two and cannot split. The remainder of the assertion of the theorem is now immediate. ⊠

The above theorem generalizes a result of algebraic geometry (see for example [1]). In chapter 2 a generalization is proven for constructible sets defined over fields. We can now prove the main result of this section.

Theorem 4.27 (Principle of Counting Constants)

Assume that Σ is L_A-(w)-stable. Suppose that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are open strongly minimal $L(\mathfrak{A})$ formulas and that R is a correspondence of $\mathcal{Y}_1$ and $\mathcal{Y}_2$. If (x) is an independent point of $\mathcal{Y}_1$ and n is the rank of $R(\bar{x},v)$ then the rank of R is equal to n+1.

Proof

Since by Theorem 4.26 the product $\mathcal{Y}_1 \times \mathcal{Y}_2$ has rank two and a correspondence of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ is contained in the product it follows that R has rank one or two.

Case 1 R has rank one

Let (x) be an independent point of Ψ_1. If R(x,v) is not an algebraic formula then the solution set is infinite. By an argument similar to that of the proof of Theorem 4.23 the formula R(z,v) cannot be algebraic for an independent point of Ψ_1. We can use the method of Theorem 4.26 to define an infinite set of infinite disjoint open subsets of R. This contradicts our assumption that R has rank one.

Case 2 R has rank two

Let (x) be an independent point of Ψ_1. If R(x,v) is an algebraic formula then R has rank one by Theorem 4.25. Thus R(x,v) is infinite so it has rank one. ☒

Corollary 4.3

Assume that Σ is L_A-(w)-stable. Suppose that Ψ_1 and Ψ_2 are open strongly minimal $L(\mathfrak{A})$ formulas and R is a degree one correspondence of Ψ_1 and Ψ_2 having rank one. Then R is an (n,m)-correspondence.

The theorem proven above generalizes the Principle of Counting Constants in algebraic geometry. See [42] for a proof of this result as well as some applications. The following examples illustrate the theorem.

Example 4.5

Let $L = \{U,V\} \cup \{R\}$. T says: (i) U,V are disjoint, infinite sets that cover the universe (ii) R is a one-one onto relation of U and V with first coordinate in U. Then U,V are both strongly minimal and R is a

strongly minimal correspondence. ⊠

Example 4.6

Let $L = \{U,V\} \cup \{R\}$. T says: (i) U,V are disjoint, infinite sets that cover the universe (ii) each point of U is connected by R to each point of V. Then U,V are strongly minimal and R is a rank two correspondence of degree one. In fact R is equivalent to $\mathcal{Y}_1 \times \mathcal{Y}_2$. ⊠

The following example demonstrates that the assumption of the amalgamation property is essential in the above theorem.

Example 4.7

Let $L = \{U,V,W\} \cup \{R_1,R_2,R_3\}$. T says: (i) U,V,W, are disjoint, infinite sets that cover the universe (ii) R_1 is a one-one relation of U onto V (iii) R_2 defines an infinite equivalence relation with infinite equivalence classes on W (iv) R_3 is a one-one relation of W onto R_1. The U and V are strongly minimal but R_3 is a one-one correspondence having rank two. Note that $\sum_T$ does not have the amalgamation property. ⊠

Footnotes

(1) The openness condition in this theorem was suggested by R. Venkataraman.

(2) This result generalizes Lemma 4.2 in Chapter 1 of [19] in one direction.

(3) A field theoretic proof of this result was first given by W. Whiteley in [46].

(4) This splitting theorem proven in [13] generalizes a result for varieties in [1].

(5) It is noted on page 68 of [34] that the set theoretic projection of a complex variety is defined by an open formula. The same argument is valid for any algebraically closed field. For a discussion of the set theoretic projection see page 88 of [27].

(6) In [8] Chevalley proves more generally that homeomorphic constructible sets in the Zaristi topology have equal dimension.

(7) The reader can consult Chapter 3 of [1] for the classical results concerning absolutely irreducible varieties.

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