UNIVERSITY OF MANITOBA

A GENERALIZED COVERING PROBLEM

by

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ABSTRACT

This thesis presents a new problem in combinatorial design theory which is a generalization of the well-known problem of covering designs. The problem is defined as follows. Let there be a set of objects, of cardinality V, called varieties. Define a K-set to be a subset containing exactly K of these varieties. A (T,K,L,V) design is defined to be a collection of K-sets such that every possible L-set must intersect at least one of these K-sets in at least T varieties. A minimal (T,K,L,V) design is one which contains the smallest possible number of K-sets. The number of K-sets in a minimal design is denoted by B(T,K,L,V). The problem is to determine the value of B(T,K,L,V) and the structure of the minimal designs.

The sub-problem in which T=L is the well-known problem of covering designs. The subcase in which T=K is also an established problem which has been posed by P. Turan. A summary of the known results in these areas is given.

In this thesis, The value of B(2,3,3,V) is determined for all values of V, as is the structure of the corresponding minimal designs. The value of B(T,K,L,V) is also investigated for the remaining cases in which T=2, K=3 or 4, and $L \leq 5$. The values of B(T,K,L,V) are determined for most values of V in these cases.

A computer algorithm is presented which was used to establish some additional results, and a table of known values of B(T,K,L,V) is given.

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CHAPTER 1 - INTRODUCTION

1.1 Statement of the problem

Combinatorial problems, according to M.Hall Jr. [5], are primarily concerned with the arrangement of a given set of objects according to a specified set of rules. The central problem may be to determine whether or not such an arrangement exists, or to determine the number of ways in which the arrangement can be done, or to find out how large or how small such an arrangement can be.

The problem which is discussed in this thesis belongs to the class of problems known as combinatorial designs. Design theory first arose from J. Steiner's work in algebraic geometry in 1853 [17]. Since then, the field has flourished, and the number of different design problems is constantly growing. The many branches of design theory now include Steiner systems, coverings, packings, block designs, including balanced incomplete block designs of many different types, (r,λ) designs, and many others. These designs have found applications in the design of statistical experiments, in the construction of error-correcting codes, in optimization problems of various kinds, and in many other areas, including such things as games and puzzles.

The difficulty of constructing these designs varies greatly from problem to problem. Some classes of designs are constructed relatively easily, while others are found only with a large amount of time and effort. With the proliferation of modern high-speed computers, many designs can now be constructed which could not have been found before. Not only can programs be written which will find relatively large designs by using exhaustive search techniques, but the production of a reasonable number of actual designs may enable a design theorist to detect a pattern or trend, and thereby produce a general result which might not have been discovered otherwise. The solution of many problems also requires that they be broken down into a large number of relatively small subcases, which may then be treated individually. Many such problems could not easily be solved without the aid of a computer, either to generate and keep track of the subcases, or to process the large number of resulting smaller problems. (The recent proof of the four-colour conjecture is an excellent example.) It is becoming increasingly rare to find a paper on design theory in which a computer has not been used, either directly or indirectly, to obtain some of the results. In this thesis, many of the results which are obtained or referred to were obtained with the aid of a computer program.

In this thesis, a new and fairly general type of combinatorial design is defined. It is actually a generalization of several known classes of designs, the most important of which is the class known as <u>covering</u> designs. This provides not only several new, previously unstudied classes of designs, but also gives a common framework within which all designs of this type may be studied. The problem is defined as follows.

<u>Definitions</u> Let there be a set of V objects called <u>varieties</u>. A <u>K-set</u> is defined to be a subset of the varieties of cardinality K. A <u>(T,K,L,V) design</u> is a collection of K-sets, often referred to as <u>blocks</u>, such that every possible L-set must intersect one of these K-sets in at least T varieties. Clearly, the parameters must satisfy the conditions $T \leq K \leq V$ and $T \leq L \leq V$. If a K-set intersects an L-set in T or more varieties, then that K-set is said to <u>cover</u> the L-set. A <u>minimal</u> (T,K,L,V) design is a (T,K,L,V) design containing the smallest possible number of blocks. <u>B(T,K,L,V)</u> denotes the number of blocks in a minimal (T,K,L,V) design.

For example, let the varieties be the integers 1 to 7. The three blocks (1 2 3), (3 4 5), and (5 6 7) form a (2,3,4,7) design, since any 4-set on the 7 varieties must intersect one of these 3-sets in at least two varieties. It is not a minimal (2,3,4,7) design, however, since the blocks

(1 2 3) and (4 5 6) also form a (2,3,4,7) design. But a single block cannot form such a design (the 4-set containing the 4 unused varieties would not be covered), and so this design is minimal and B(2,3,4,7)=2.

Standard terminology and notation will be employed in the remainder of this thesis. In several sections, graph-theoretic terms will be employed, and the reader is referred to Harary [30] for definitions of these terms. In particular, the terms "bipartite", "tripartite", and "N-partite" will be used frequently. In some cases, a design or graph will be referred to as simply "partite" or "non-partite". In these cases, the number of partitions will be clear from the context.

1.2 Overview of the thesis

In Chapter 2, several special subcases of the general problem are discussed, and the interrelationships between them are described. Several of these subcases are well known problems, and a summary of the work which has been done on these problems is given. Several small results that concern the problem as a whole are also presented here.

Chapter 3 investigates (2,2,L,V) designs, which is one of these well-known subcases, known as Turan's problem. An independent proof of the value of B(2,2,L,V) is given, but several other, more important results concerning other types of (2,2,L,V) designs are also given. These results provide the foundation for the following chapters.

Chapters 4 and 5 investigate (T,K,K,V) designs, which is the only special subcase of the problem which has not previously been studied to any great extent. In Chapter 4 the case in which T=2 and K=3 is discussed, and Chapter 5 considers the case T=2 and K=4. W. H. Mills, in a summary of covering designs [9], noted that, in that problem, the case K=4 was much more difficult than the case K=3. He stated "In part this is because . . . and in part it is because 4 is a bigger number than 3." It may certainly be stated that, in any design theory problem, if 4 is a bigger number than 3, then 5 is a <u>much</u> bigger number than 4. This is certainly true in the current problem, and (2,5,5,V) designs will be mentioned only briefly.

In Chapter 6, similar results are presented concerning (T,K,L,V) designs in general. The value of B(T,K,L,V) is determined for the majority of the sets of parameters in which T=2, K=3 or 4, and L≤5.

In Chapter 7, several algorithms are described which were used to determine B(T,K,L,V) for some sets of parameters, and the feasibility of using computer algorithms to determine minimal designs for other, larger, sets of parameters is discussed. Finally, an appendix is provided containing tables of the known values of $B(T,K,L,V)$ for $V \le 16$.

CHAPTER 2 - BACKGROUND

2.1 Special subcases

In this section, several subcases of the general problem are presented, and the relationships between them are demonstrated. In the following section, a summary of the known results concerning these subcases will be given. First, a duality may be established between designs by the following theorem.

Definition The complement of a K-set is the set of V-K elements which do not appear in the K-set. The complement of a (T,K,L,V) design is obtained by complementing every K-set in the design.

Theorem 2.1.1 The complement of a (T,K,L,V) design is a (V-K-L+T,V-K,V-L,V) design.

Proof Consider a (T,K,L,V) design. Any L-set must intersect one of the K-sets in at least T varieties. Therefore there are at most K+L-T distinct varieties contained in these two sets, and at least V-K-L+T varieties which are contained in neither of them. Now consider an arbitrary (V-L)-set, and its complement, which is an

L-set. One of the K-sets in the design will intersect this L-set in at least T varieties. The complement of this K-set will intersect the (V-L)-set in exactly those varieties which are contained in neither the K-set nor the L-set. There must be at least V-K-L+T such varieties. Therefore, by complementing every K-set, a design will be formed which has the required parameters.

Corollary B(T,K,L,V) = B(V-K-L+T,V-K,V-L,V)

All of the special subcases of the problem in which two or more of the parameters are equal will now be considered. Several of these are trivial subcases, and these are treated in the following lemmas.

Lemma 2.1.1 If V=T, V=K, or V=L, B(T,K,L,V)=1.

Proof If V=K, there is only one possible K-set. If V=L, any K-set will form a design. If V=T, this implies that T=K=L=V.

Lemma 2.1.2 If T=K=L, then B(T,K,L,V) = $\binom{V}{K}$,

and the design consists of every possible K-set.

The remaining subcases occur when T=K, T=L, or K=L. If T=L, then the design will be a collection of K-sets which includes every T-set at least once. Such a (T,K,T,V) design is known as a <u>covering</u> design. The problem of covering designs is a well known one, and it has been extensively studied in recent years. A summary of the work which has been done in this field is contained in section 2.2.1. A special case of covering designs occurs when every T-set appears <u>exactly</u> once in the design. Such a design is known as a "tactical configuration" or a <u>Steiner system</u>. This problem is a very old one, and has received a great deal of study. This subcase of the problem also includes all balanced incomplete block designs with $\lambda=1$, as well as many other similar designs.

If T=K, then the problem becomes equivalent to a problem first posed by Paul Turan in 1941. Such a (T,T,L,V) design will be referred to as a <u>Turan design</u>. This subcase is a particularly difficult one, and relatively little is known about it. The original problem posed by Turan and a survey of the known results on this topic is presented in section 2.2.2.

These two subcases are actually complements of each other. Thus every result concerning covering designs will provide a corresponding result on Turan designs, and conversely. This duality is stated in the following lemma.

<u>Lemma 2.1.3</u> The complement of a covering design is a Turan design, and vice versa.

Proof $B(T,K,T,V)=B(V-K,V-K,V-T,V)=B(T',T',L',V)$

The subcase in which K=L is a relatively new problem. Such a (T,K,K,V) design will be referred to as a symmetric design. It was actually a problem of this type which led to the formulation of the general (T,K,L,V) problem. Recently Bernhard Neumann posed the following problem. What is the minimum number of 6-sets on 16 varieties which will intersect every possible 6-set in at least 4 varieties? In the terminology of this thesis, such a design would be a minimal (4,6,6,16) design. In attempting to solve this problem, the general symmetric problem for any T, K, and V was considered, which, in turn, led to the formulation of the general problem on which this thesis is based. However, the value of B(4,6,6,16) is still unknown. The subcase of the problem in which L=K will receive the majority of the attention in the remainder of this thesis.

Lemma 2.1.4 The complement of a symmetric design is a symmetric design.

Proof $B(T,K,K,V)=B(V-2K+T,V-K,V-K,V)=B(T',K',K',V)$

The only remaining subcase of the overall problem is the most general case in which the four parameters are distinct. This subcase will also be considered in the following chapters.

2.2 Survey of results

In this section, the historical background of the problem will be presented, and a summary of the results which have been obtained will be given. As noted in the last section, the two subcases of the problem which have been studied are those concerning covering designs and Turan designs. The results that have been obtained in these areas are summarized separately below. The topic of coverings is also closely related to Steiner systems, balanced incomplete block designs, and other related combinatorial designs. Even a brief survey of these areas would be much too large to be included here. Instead, the reader is referred to the bibliography for further information in these areas.

2.2.1 Covering designs

In 1853, J. Steiner [17] posed the following problem. For what integer N is it possible to form triples, out of N given elements, in such a way that every pair of elements appears in exactly one triple? Such a system is now referred to as a Steiner triple system, or an exact covering of pairs by triples. Steiner then went on to formulate the problems of exact coverings of triples by quadruples, quadruples by quintuples, etc.

In 1859, M. Reiss [14] proved that Steiner triple systems can be constructed if and only if the number of elements, N, is congruent tc 1 or 3 (modulo 6). This result was anticipated by Woolhouse in 1844 [42], and was also proved independently by Mcore in 1896 [13]. Moore also generalized the problem to include exact coverings of T-sets by K-sets for any T<K, and called such designs "tactical configurations". Such a design on V varieties will be denoted by S(T,K,V). He also showed that the following divisibility conditions are necessary for the existence of any S(T,K,V).

$$\binom{K-i}{T-i} \,\Big|\, \binom{V-i}{T-i} \quad \text{for all } 0 \le i \le T.$$

These divisibility conditions were proved to be sufficient as well as necessary by Hanani between 1960 and 1963 [6,7,8] for some values of T and K. Of particular interest here, he showed that Steiner systems S(T,K,V) exist for

T=2,K=4 if and only if V=1 or 4 (modulo 12),

T=2,K=5 if and only if V=1 or 5 (modulo 20),

and T=3,K=4 if and only if V=2 or 4 (modulo 6) .

The first results concerning minimal coverings other than Steiner systems were found by Fort and Hedlund in 1958 [3]. They solved the problem of covering pairs by triples and determined that

$$B(2,3,2,V) = \left\lceil \frac{V}{3} \left\lceil \frac{V-1}{2} \right\rceil \right\rceil \quad \text{for all V.}$$

The first important results concerning coverings in general were given by J. Schonheim in 1964 [15]. In this paper, the following important lower bound was established.

$$B(T,K,T,V) \geq L(T,K,V) = \left\lceil \frac{V}{K} \left\lceil \frac{V-1}{K-1} \cdots \left\lceil \frac{V-T+1}{K-T+1} \right\rceil \cdots \right\rceil \right\rceil .$$

This is a very good lower bound, and it can be achieved in many cases, including $B(2,3,2,V)$ as shown above. Schonheim also proved the important result that if a Steiner system $S(T,K,V)$ exists, then $B(T,K,L,V+1)=L(T,K,V+1)$. This theorem, in conjunction with the results of Hanani, shows that

$B(2,4,2,V) = L(2,4,V)$ for $V=1,2,4$ or 5 (modulo 12),

$B(2,5,2,V) = L(2,5,V)$ for $V=1,2,5$ or 6 (modulo 20),

and $B(3,4,3,V) = L(3,4,V)$ for $V=2,3,4$ or 5 (modulo 6).

Since 1964, the problem of coverings has received a great deal of study, principally by Mills [9,10,11,12], Stanton, Kalbfleisch, and Mullin [16], and Gardner [4], among others. The cases for which many results are known are as follows.

The problem of covering pairs by quadruples, $B(2,4,2,V)$, has been solved for all values of V by Mills. He has shown that

$B(2,4,2,V) = L(2,4,V)$ for all $V \neq 7,9,10,$ or 19,

$B(2,4,2,V) = L(2,4,V)+1$ for $V=7,9,$ or 10,

and $B(2,4,2,19) = L(2,4,2,19)+2$.

The problem of covering triples by quadruples has also been solved by Mills for most values of V. In this case, it is known that

$$B(3,4,3,V) = L(3,4,V) \quad \text{for all } V \neq 7 \text{ (modulo 12)},$$

and $B(3,4,3,7) = L(3,4,3,7)+1$.

There are many results known concerning $B(2,5,2,V)$ as well, although these are not nearly so complete as the results in the previous two cases. The results of Schonheim and Hanani show that

$$B(2,5,2,V) = L(2,5,V) \text{ for } V=1,2,5, \text{ or } 6 \text{ (modulo 20)}.$$

Gardner [4] has shown that

$$B(2,5,2,V) = L(2,5,V) \text{ for } V=10,14,17,18,30,94,97,98 \text{ (modulo 100)}$$
$$(V \neq 17,30,94,110,114,130,194,210,230),$$

$$B(2,5,2,V) = L(2,5,V)+1 \text{ for } V=13,93 \text{ (mod 100) } (V \geq 293)$$

and $B(2,5,2,V) > L(2,5,V)$ for $V=13$ (modulo 20).

In addition, the value of $B(2,5,2,V)$ is known for many specific values of V. Some of these are listed in Table 2.2.1.1.

V	B(2,5,2,V)	L(2,5,V)	Result due to
12	9	8	
13	10	8	Stanton, Kalbfleisch,
14	12	12	and Mullin
15	13	12	
20	21	20	
17	16	14	Stanton, Kingsley
18	18	18	
19	19	19	
23	28	28	Gardner
38,39,			
54,70,			
and	L(2,5,V)		
others			

Table 2.2.1.1

The values of B(2,K,2,V) for all small values of V, including the values of B(2,5,2,V) when V≤11, are given by the following results, which are due to Stanton, Kalbfleisch, and Mullin [16], and Mills [9].

$$
\begin{aligned}
B(2,K,2,V) &= 3 \text{ if } K < V \le 3K/2,\\
&= 4 \text{ if } 3K/2 < V \le 5K/3,\\
&= 5 \text{ if } 5K/3 < V \le 9K/5,\\
&= 6 \text{ if } 9K/5 < V \le 2K,\\
&= 7 \text{ if } 2K < V \le 7K/3 \text{ and } 7K-3V \ne 1,\\
&= 8 \text{ if } 2K < V \le 7K/3 \text{ and } 7K-3V = 1.
\end{aligned}
$$

There are also a very few, widely scattered, results which concern other, larger, classes of covering designs. These results are summarized in [9]. Also, since covering designs may be obtained by complementing Turan designs, the results given in the following section will provide results on covering designs whose parameters are of the form (V-L,V-T,V-L,V). Appendix II contains tables of the known values of B(T,K,T,V) for $V \leq 16$.

2.2.2 Turan designs

In 1941, P. Turan posed the following problem on hypergraphs. Consider an r-graph whose vertices are denoted by 1,...,v and whose edges are r-subsets of these v vertices. What is the maximal number of edges in such an r-graph which does not contain a complete r-subgraph on L vertices? For example, if r=3 and L=4, then not all of the edges 123, 124, 134, and 234 may occur. This problem is equivalent to the determination of B(T,T,L,V) since it is clear that the edges which are <u>not</u> included in such a maximal graph must form a minimal (r,r,L,v) design.

Turan determined the answer to this question for r=2 (see Chapter 3), but very little is known for $r \geq 3$. He did, however, give the following conjectures (which have been reformulated in terms of B(T,K,L,V) designs).

<u>Conjecture I</u> Let there be 3n varieties, and partition them into three subsets, X, Y, and Z, each of cardinality n. The minimal (3,3,4,3n) design will then consist of all

possible blocks of the form

(x x x), (y y y), (z z z), (x x y), (y y z), and (z z x),

where x, y, and z denote elements of X, Y, and Z, respectively. This would give the result

$$B(3,3,4,3n) = 3\binom{n}{3} + 3n\binom{n}{2} .$$

<u>Conjecture II</u> Let there be 2n vertices and partition them into two sets X and Y of cardinality n. The minimal (3,3,5,2n) design will consist of all possible blocks of the form

(x x x), (y y y), (x x y), and (y y x),

where x and y denote elements of X and Y, respectively. This would give

$$B(3,3,5,2n) = 2\binom{n}{3} + 2n\binom{n}{2} .$$

Conjecture I was generalized to include all values of V by P.H.Dirksen and R.G.Stanton [1], by requiring only that the vertices be partitioned into three sets as evenly as possible. The conjecture was also proved for all V subject to the condition that, in some minimal (3,4,3,V) design, the set of blocks which do not include some variety, x, form a (3,4,3,V-1) design. This condition has not, however, been proved.

The only other results which are known concerning Turan designs are those which can be obtained from complementing the results on covering designs which were given previously.

2.3 Small results

In this section, several lemmas are presented which concern small (T,K,L,V) designs.

Lemma 2.3.1 $B(T,K,L,V)=1$ if and only if $T \le K+L-V$.

Proof A single K-set will form a (T,K,L,V) design if and only if it is not possible to choose L varieties without including T or more of the varieties that appear in the K-set. This will occur if and only if $L>(V-K)+(T-1)$, which gives the stated result.

Lemma 2.3.2 $B(1,K,L,V) = \lceil (V-L+1)/K \rceil$.

Proof In order to form a (1,K,L,V) design it is only necessary to include at least $V-(L-1)$ varieties in the design, in order that no L-set exists which does not intersect any of the K-sets. This can clearly be done in the stated number of blocks.

Lemma 2.3.3 There exists a minimal (T,K,L,V) design which either contains every variety, or else contains no repeated varieties.

Proof Suppose some variety, x, occurs more than once in a minimal (T,K,L,V) design, and that a second variety, y, does not occur. Select one of the sets containing x, call it set A, and change the x in this set to a y. The resulting design must also be a minimal (T,K,L,V) design for the following reasons.

If there exists some L-set which was covered in the original design, but is not covered in the modified design, then that L-set must contain x, must not contain y, must contain exactly T-1 other varieties from set A, and must not be covered by any other K-set in the design. But if such an L-set existed, then that L-set, with the x replaced by a y, would not have been covered in the original design. Since this design was assumed to be a (T,K,L,V) design, this is a contradiction.

Therefore any minimal design may be altered in this way until it contains every variety, or until no repeated varieties remain.

<u>Corollary</u> If $B(T,K,L,V) \geq V/K$ then there exists a minimal design which contains every variety.

<u>Corollary</u> If $B(T,K,L,V) < V/K$ then there exists a minimal design which consists of distinct varieties.

<u>Lemma 2.3.4</u> $B(T,K,L,V) \leq V/K$ if and only if

$$\frac{V-L}{K+1-T} < \left\lfloor \frac{V}{K} \right\rfloor .$$

<u>Proof</u> Select $N \leq \lfloor V/K \rfloor$ K-sets which contain no repeated

elements. If a (T,K,L,V) design exists on N blocks, then it may be constructed in this way, by the previous lemma. The resulting design will be a (T,K,L,V) design if and only if it is not possible to choose an L-set which intersects each K-set in no more than T-1 varieties, which occurs when

$$L > N(T-1)+(V-NK)$$
$$\text{or } L-V > N(T-K-1)$$
$$\text{or } V-L < N(K+1-T)$$
$$\text{or } \frac{V-L}{K+1-T} < N \text{ .} \qquad \text{Q.E.D.}$$

Corollary If $\frac{V-L}{K+1-T} < \left\lfloor \frac{V}{K} \right\rfloor$ then

$$B(T,K,L,V) = \left\lceil \frac{V-L+1}{K+1-T} \right\rceil \text{ .}$$

Note: The extra +1 in the numerator is required since the number of blocks must be greater than this quantity, not greater than or equal to it.

Lemma 2.3.5 If $N \geq V/K$ and $L>(T-1)N$ then $B(T,K,L,V) \leq N$.

Proof Form a design on N blocks which contains each variety at least once. Suppose there exists an L-set which is not covered by this design. Such an L-set can intersect each block in at most T-1 varieties, and hence can contain at most N(T-1) varieties. But $L>N(T-1)$ and therefore such an L-set cannot exist.

<u>Lemma 2.3.6</u> $B(2,K,L,V) \neq L$.

<u>Proof</u> If $\lceil V/K \rceil < L$ then $B(2,K,L,V)<L$ by the preceeding lemma. If $\lceil V/K \rceil \geq L$ then suppose there exists a design on $\lceil V/K \rceil$ blocks. By a previous lemma, such a design may be constructed which contains every variety at least once. Such a design can contain at most $K-1$ repeated varieties and therefore every block must contain at least one variety which does not appear in any other block. Therefore it is possible to form an L-set which intersects each block in such a design in at most 1 variety. This is a contradiction. Therefore $B(2,K,L,V) > \lceil V/K \rceil \geq L$.

CHAPTER 3 - (2,2,L,V) DESIGNS

3.1 Introduction

In this chapter, several results concerning (2,2,L,V) designs will be presented. As mentioned earlier, the subcase in which T=K is equivalent to a problem first stated by Turan in 1941 [19]. This particular case, in which T=K=2, was first solved by Turan himself, and since then several additional proofs have appeared [2,18].

The results of this chapter will be needed in order to investigate B(2,K,L,V) designs in Chapters 4, 5, and 6. In addition to a determination of the structure of the minimal (2,2,L,V) designs, and an independent proof of the value of B(2,2,L,V), several necessary results will be obtained about other, non-minimal, (2,2,L,V) designs.

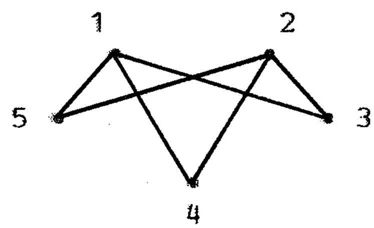
3.2 Minimal (2,2,L,V) designs

It will be convenient throughout this chapter to reformulate the problem as a graph-theoretical problem, as follows.

Definition The equivalent graph of a (2,2,L,V) design is a graph on V vertices, corresponding to the V varieties in the design, such that two vertices are connected by an edge if and only if the corresponding pair of varieties does not appear in the design.

An example of a design and its equivalent graph is shown in Figure 3.2.1 .

1	2
3	4
3	5
4	5



A (2,2,3,5) design and its equivalent graph

Figure 3.2.1

Since every L-set of the V varieties contains at least one pair which is present in the design, the following lemma is immediate.

Lemma 3.2.1 A graph is the equivalent graph of a (2,2,L,V) design if and only if it contains no complete subgraphs on L vertices.

It is also clear that the minimal design will correspond to the equivalent graph containing the maximum number of edges. The structure of this maximal graph will be determined by induction, beginning with the following lemma.

Lemma 3.2.2 The equivalent graph of the minimal (2,2,3,V) design is a complete bipartite graph on two sets of $\lceil V/2 \rceil$ and $\lfloor V/2 \rfloor$ vertices.

Proof Let v^1 be a vertex of maximum valence, m, in the graph. Let A denote the set of m vertices which are connected to v^1, and B denote the remaining vertices (Figure 3.2.2).

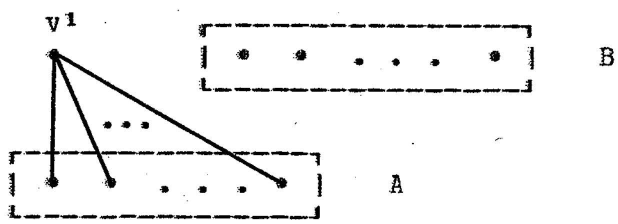


Figure 3.2.2

Since the graph can contain no triangles, there can be no edges within set A, and so all of the remaining vertices must be incident with a vertex in set B. Since the maximum valence is m there can be at most m(V-m) edges in the graph. Also, there can be exactly m(V-m) edges only if no edges occur between two vertices in set B, in which case each vertex in B will be connected to each vertex in A, giving a complete bipartite graph, as required.

To achieve this maximum without violating the assumption that m is the maximum valence of any vertex in the graph, it is also necessary that $m \geq V/2$. However, $m(V-m)$ is maximized when $m=|V/2|$, which satisfies this condition and completes the proof.

<u>Corollary</u>

$$B(2,2,3,V) = \binom{\lfloor \frac{V}{2} \rfloor}{2} + \binom{\lceil \frac{V}{2} \rceil}{2} = \left\lceil \frac{V^2 - 2V}{4} \right\rceil .$$

An analogous result for any L may now be proved as follows.

<u>Theorem 3.2.1</u> The equivalent graph of the minimal (2,2,L,V) design is a complete (L-1)-partite graph.

<u>Proof</u>

Suppose that the theorem is true for all $L \leq L'$, and consider the minimal $(2,2,L'+1,V)$ design and its equivalent graph. Again, let v^1 be a vertex of maximum valence, m, and let A and B denote the sets of vertices connected to v^1 and not connected to v^1, respectively (Figure 3.2.2).

Since no vertex may be incident with more than m edges, the maximum number of edges outside of set A is $m(V-m)$. This maximum is achieved only when each vertex in B is connected to each vertex in A, and no edges occur within set B.

The edges within set A can contain no complete subgraphs on L' edges, else this subgraph, in conjunction with v^1, would form a complete subgraph on L'+1 vertices, which is forbidden. Thus the maximum number of edges within A occurs when this subgraph is a complete (L'-1)-partite graph, by the induction hypothesis. This gives a complete L'-partite graph overall, as required.

It is clear, from symmetry, that the maximum number of edges will occur when the vertices are partitioned into the L-1 sets as evenly as possible. (That is, every set has cardinality $\lceil V/(L-1) \rceil$ or $\lfloor V/(L-1) \rfloor$.) If a detailed proof is desired, please see the proof given in Appendix I. This gives the following corollary.

<u>Corollary</u> The minimal (2,2,L,V) design can be formed by partitioning the varieties into L-1 disjoint sets, as evenly as possible, and including all possible pairs from within each of these sets.

<u>Corollary</u>

$$B(2,2,L,V) = (L-1-r)\binom{\frac{V-r}{L-1}}{2} + r\binom{\frac{V-r}{L-1}+1}{2},$$

$$= \frac{V^2-(L-1)V}{2(L-1)} + \frac{r(L-1-r)}{2(L-1)},$$

where r=V (modulo L-1) and $0 \le r \le L-2$.

Since the maximum possible value of $r(L-1-r)/2(L-1)$ is $(L-1)/8$, a simpler formula can be obtained for $L \le 8$, as follows.

$$B(2,2,L,V) = \left\lceil \frac{V^2 - (L-1)V}{2(L-1)} \right\rceil \qquad (L \le 8).$$

3.3 Non-minimal (2,2,L,V) designs

In the last section, it was seen that the optimal method of constructing (2,2,L,V) designs involved partitioning the varieties into L-1 disjoint sets, and including all possible pairs from within each of these sets. The equivalent graph for such a design is an (L-1)-partite graph, and the design itself will be called an (L-1)-partite design.

In Chapters 4, 5, and 6, (2,K,L,V) designs will be constructed using a similar technique. In order to establish that these designs are minimal, it will be necessary to consider the class of (2,2,L,V) designs which are <u>not</u> constructed in this manner.

This section, then, will discuss the following question. What is the number of blocks in the smallest (2,2,L,V) design that is not (L-1)-partite?

3.3.1 Non-minimal (2,2,3,V) designs

Consider the graph with the maximum number of edges that contains no triangles, and is not bipartite. (Since such a graph must contain a cycle with an odd number of vertices, there must be at least 5 vertices.) Again, let v^1 be a vertex of maximum valence, m, and let sets A and B denote the vertices connected to v^1 and not connected to v^1, respectively. There can be no edges within set A, else there would be a triangle, and so, since the graph is not bipartite, there must be an edge within set B (Figure 3.3.1).

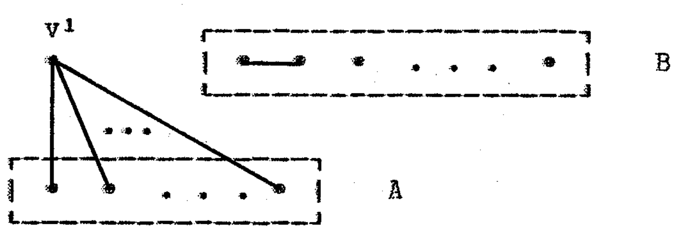


Figure 3.3.1

Note that the presence of this edge is a necessary condition but is not a sufficient condition to guarantee a non-bipartite graph. There must also be at least two vertices in set A, or else the maximum valence in the graph would be 1 and the graph would be bipartite, and each endpoint of the edge in B must be connected to at least one vertex in A, or else one of these endpoints could be considered to be in the same partition as the vertices in A and the graph would again be bipartite. If both of these additional conditions are met, however, then the graph contains a 5-cycle and must therefore be non-bipartite. These conditions are easily met for $V \geq 5$ in the construction below.

Consider the edge in B and its two end points. Both of these vertices cannot be connected to any given vertex in set A, since that would form a triangle, and so the maximum number of edges between these two vertices and set A is m. All remaining edges in the graph must be incident with the remaining (V-m-3) vertices in set B. Since the maximum valence is m, there are at most m(V-m-3) other edges. Thus the total number of edges is at most

$$\begin{aligned} & m+m(V-m-3)+m+1 \\ = \; & m(V-m-1)+1 \, . \end{aligned}$$

This is maximized when $m = \lceil (V-1)/2 \rceil$, giving a total of at most

$$\frac{V^2-2V+4}{4} \text{ (V even)} \quad \text{or} \quad \frac{V^2-2V+5}{4} \text{ (V odd)}$$

edges. Furthermore, this number of edges is easily achieved without violating the original assumption that m is the maximum valence in the graph. Each end point of the edge in B must be connected to at least one vertex in A, else the other end point would be $(m+1)$-valent. This poses no problem. Each vertex in A is connected to v^1 as well as all but one vertex in B. Thus no vertex in A will be more than m-valent as long as $m \geq (V-1)/2$, but since the maximum was obtained with $m = \lceil (V-1)/2 \rceil$, this condition is satisfied. This proves

<u>Theorem 3.3.1</u> The minimum cardinality of a non-bipartite $(2,2,3,V)$ design $(V \geq 5)$ is

$$\binom{V}{2} - \left\lfloor \frac{V^2-2V+4}{4} \right\rfloor = \left\lceil \frac{V^2-4}{4} \right\rceil .$$

<u>3.3.2 Non-minimal (2,2,L,V) Designs</u>

The results of the previous section may now be generalized as follows.

<u>Lemma 3.3.1</u> A graph on V vertices which contains no complete subgraphs on L vertices, and is not $(L-1)$-partite, can be constructed if and only if $V \geq L+2$.

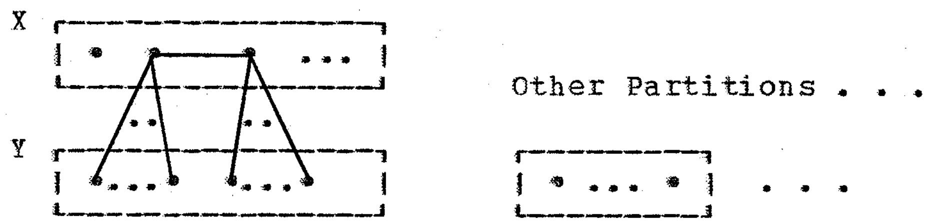
<u>Proof</u> Since the only subgraph on L vertices or less which is not $(L-1)$-partite is the complete graph on L vertices, which is forbidden, it is clear that $V>L$. Suppose $V=L+1$ and v^1 is the vertex of minimum valence in the graph. First, suppose that v^1 is $(L-1)$-valent, and that only the edge between v^1 and v^2 is missing. Since the subgraph on the L vertices

other than v^1 cannot be complete, there must be another missing edge, say between v^3 and v^4. But this graph would then be (L-1)-partite, with partitions (v^1,v^2),(v^3,v^4),etc. Therefore v^1 is at most (L-2)-valent. Suppose the edges between v^1 and v^2 and between v^1 and v^3 are missing. Again, there must be an additional missing edge not involving v^1. If this edge is the edge between v^2 and v^3, then the graph is (L-1)-partite with partitions (v^1,v^2,v^3),(v^4),etc. If this edge involves only one of v^2 and v^3, say the edge between v^2 and v^4, then the graph is (L-1)-partite with partitions (v^1,v^3),(v^2,v^4),etc. If this edge involves neither v^2 nor v^3, say the edge between v^4 and v^5, then the graph is (L-1)-partite with partitions (v^1,v^2),(v^4,v^5),etc. Thus such a graph must always be (L-1)-partite. Therefore $V \geq L+2$.

For all $V \geq L+2$, a suitable graph may be constructed as follows. Form a complete (L-3)-partite graph from the first V-5 vertices. Connect the remaining 5 vertices to form a 5-cycle, and connect each of them to all of the other L-3 vertices. The resulting graph contains no complete sub-graphs on L vertices, and is not (L-1)-partite.

Theorem 3.3.2 The structure of the equivalent graph of some minimal, non-partite (2,2,L,V) design is as follows. It is a complete (L-1)-partite graph with the following two modifications. First, there is a single edge within one of the partitions (X). Second, each vertex in one of the other partitions (Y) is connected to only one of the two endpoints of this edge (Figure 3.3.2) .

In addition, partition X will contain at least three vertices, partition Y will contain at least two vertices, and each endpoint of the edge in X will be connected to at least one vertex in Y. These conditions are necessary to guarantee that the graph is non-partite.



(Other edges omitted for clarity)

Figure 3.3.2

Proof The equivalent graph of some minimal non-bipartite (2,2,3,V) design has this structure, as shown in the previous section. The theorem may now be proved by induction, as follows.

Suppose that the theorem is true for minimal, non-partite, (2,2,L-1,V) designs, and consider a minimal, non-partite, (2,2,L,V) design, and its equivalent graph. Define m, v^1, A, and B as in the last section. Since the graph is not (L-1)-partite, there are two cases to be considered. Either A is not (L-2)-partite, or else A is (L-2)-partite and there is an edge within B.

If A is not (L-2)-partite, then the maximum number of edges within A will occur when A has the required structure, by the induction hypothesis. The maximum number of edges outside of A will occur when each vertex in B is connected to each vertex in A, as before. This gives the overall graph the required structure.

Now suppose that A is (L-2)-partite, and that there is an edge within set B. The set of vertices consisting of v^1 and B will contain at least three vertices, as required. Let the partitions of set A be P1, P2, ...,Pn and let their cardinalities be m1, m2, ...,mn, where $m1 \le m2 \ldots \le mn$.

Consider the vertices in B other than the two endpoints of the special edge, and consider the edges incident with these vertices. Since no vertex may be more than m-valent, the maximum number of edges incident with these vertices will be achieved when each of them is connected to every vertex in set A.

Now consider the remaining subgraph consisting of the two endpoints of the edge in B, and set A. Suppose that every possible edge is present in this subgraph, that is, A is a complete partite graph, and each endpoint is connected to every vertex in A. There will be exactly (m1)(m2)...(mn) complete subgraphs on L vertices in this graph. Since such subgraphs are forbidden, at least one edge must be missing from each of these complete subgraphs. If an edge from P1 to one of the endpoints of the edge in B is removed, an edge will now be missing from exactly (m2)...(mn) of these complete subgraphs. Furthermore, this is the largest number of complete subgraphs which share a common edge. By removing an edge from each vertex in P1 to an endpoint of the edge in B, no complete subgraphs will remain, and the minimum number of edges (m1) will have been removed.

There is one additional restriction, however. At least one edge must be removed from each endpoint of the edge in B, otherwise one of the endpoints would be (m+1)-valent, which is contrary to the assumption that the maximum valence is m. Therefore there can be at most 2m-2 edges between these endpoints and set A, and the construction above cannot be used for m1=1. In this case, however, it is always possible to select another partition, P, containing exactly two vertices, and to remove one edge from each vertex in P to a distinct endpoint of the edge in B, resulting in a graph with the maximum number of edges and the required structure. There must always be a partition containing exactly two vertices, for the following reasons.

If there is no partition containing exactly two vertices, then either all of the partitions contain only one vertex, or some partition contains more than two vertices. If all of the partitions contain only one vertex, then m=L-2 and the number of edges in the graph is at most $V(L-2)/2$. However, it is easy to show that a larger non-partite graph can, in fact, be constructed. The construction used in Lemma 3.3.1 yields a graph in which every vertex has valence $\geq L-1$, for example. Thus m cannot equal L-2 in the maximal graph. If there is a partition containing $p>2$ vertices, on the other hand, then perform the following operation. Remove one vertex from this partition, and add it to P1, giving a partition containing exactly two vertices. This operation will increase the number of edges within A by $(2p-2)-p = p-2$, while still allowing the maximum number of edges $(2m-2)$ between A and the endpoints of the edge in B. Thus a larger non-partite graph has been formed, and therefore the original graph could not have been the maximal graph.

Therefore the maximum number of edges occurs when A is a complete (L-2)-partite graph, and every vertex in some partition P, containing at least two vertices, is connected to only one endpoint of the edge in B. This gives the overall graph the required structure and proves the theorem.

<u>Corollary</u> The maximum number of edges will occur when the varieties are partitioned as evenly as possible, subject to the restriction that at least one partition (partition X) must contain at least three vertices.

Proof The number of edges in a maximal non-partite graph which is formed from a partite graph by the construction above is $N-n+1$, where n is the cardinality of partition Y(which must be at least two), and N is the number of edges in the corresponding complete partite graph. As shown in appendix I, N is maximized when the vertices are partitioned as evenly as possible. For $V\geq 2L-1$, at least one partition will contain at least three vertices, and the rest will contain at least two vertices, which satisfies all conditions. For $V\leq 2L-2$, the value of N is maximized when one partition contains exactly three vertices, and the rest are split evenly into partitions of sizes two and one. Since $V\geq L+2$, by Lemma 3.3.1, there will always be at least one partition of size two, as required. The value of n is minimized when partition Y is chosen to be the smallest partition containing at least two vertices.

Suppose that the vertices are initially partitioned in this way so as to maximize N. Now suppose that the vertices are re-grouped in order to attempt to increase the number of edges in the graph. Since the vertices were initially partitioned as evenly as possible, such a re-grouping can always be done by a series of operations in which e vertices are moved from a partition containing r vertices into a partition containing s vertices, where $r\leq s$. Such an operation can decrease n by at most e (if r was formerly the size of the smallest partition and $(r-e)>1$). However, such an operation must also decrease N by exactly

$$(rs)-(r-e)(s+e) = e^2 + (s-r)e \geq e^2 \geq e .$$

Therefore the number of edges in the graph cannot be increased by such operations, and the maximum number of edges will occur when the vertices are partitioned evenly.

Corollary The minimum cardinality of a non-partite (2,2,L,V) design is

$$B(2,2,L,V) + \left\lfloor \frac{V}{L-1} \right\rfloor - 1 \qquad \text{for } V \geq 2L-1,$$

$$\text{and } B(2,2,L,V) + 2 \qquad \text{for } L+2 \leq V \leq 2L-2 .$$

Proof For $V \geq 2L-1$, the equivalent graph of the minimal (2,2,L,V) design is a complete partite graph with the vertices partitioned as evenly as possible. The equivalent graph of a minimal non-partite design may be formed from the same partite graph by adding one edge to one of the partitions, and deleting one edge from each vertex in the smallest partition to one of the endpoints of this new edge, giving a design with the stated size.

For $V \leq 2L-2$, the equivalent graph of the minimal design is a complete partite graph with all partitions having cardinalities 2 or 1. To form the minimal non-partite design, however, the size of one of these partitions must be increased to 3 vertices, and another one decreased from 2 to

1 vertex. This will decrease the number of edges by 1. One edge is then added to the graph, and two edges removed, giving two less edges in all. This gives the stated result.

3.4 Summary

The value of B(2,2,L,V) has been determined for all L and V, and it has been shown that the minimal design is an (L-1)-partite design in which the varieties are partitioned as evenly as possible. The cardinalities of the minimal non-partite (2,2,L,V) designs have also been determined for all L and V. The resulting formulas for L=3, 4, and 5 will be used in subsequent chapters. For convenience, these formulas have be re-written in a simplified form, and are summarized in Table 3.4.1.

L	B(2,2,L,V)	Minimal Non-partite
3	$\left\lceil \frac{V^2-2V}{4} \right\rceil$	$\left\lfloor \frac{V^2-4}{4} \right\rfloor$
4	$\left\lceil \frac{V^2-3V}{6} \right\rceil$	$\left\lfloor \frac{V^2-V-6}{6} \right\rfloor$
5	$\left\lceil \frac{V^2-4V}{8} \right\rceil$	$\left\lfloor \frac{V^2-2V-7}{8} \right\rfloor$

Table 3.4.1

CHAPTER 4 - (2,3,3,V) DESIGNS

4.1 Preliminary results

A (2,3,3,V) design is a set of n triples which intersect every possible triple in at least a pair. It is worthwhile to consider the 3n pairs that are contained in the triples of such a design, rather than the triples themselves. Clearly these 3n pairs, or perhaps a subset of these pairs, must form a (2,2,3,V) design. If it were possible to take the minimal (2,2,3,V) design, and then find a set of triples which contained each pair in this design exactly once, the result would clearly be the minimal (2,3,3,V) design.

In general, the set of all T-sets contained in any (T,K,L,V) design must form a (T,T,L,V) design. This gives the following lower bound on B(T,K,L,V).

Lemma 4.1.1

$$B(T,K,L,V) \geq \left\lceil \frac{B(T,T,L,V)}{\binom{K}{T}} \right\rceil .$$

In the current case, it has been shown that

$$B(2,2,3,V) = \left\lceil \frac{V^2-2V}{4} \right\rceil ,$$

and so $B(2,3,3,V) \geq \left\lceil \frac{1}{3} \left\lceil \frac{V^2 - 2V}{4} \right\rceil \right\rceil$.

This is a reasonably good bound, and, as will be shown, it can actually be obtained for all V = 2, 4, or 6 (mod 12).

The following two sections will determine B(2,3,3,V) for all V by considering the (2,2,3,V) design that must always be present, and the way in which this design can be covered by triples. First, in section 4.2, only bipartite (2,2,3,V) designs will be considered. In the remaining section, non-bipartite (2,2,3,V) designs will be used.

4.2 Bipartite designs

As shown in Chapter 3, minimal (2,2,3,V) designs are bipartite, that is, they consist of all possible pairs from each of two disjoint subsets of the varieties. Consider the problem of covering such a design by triples in order to create a (2,3,3,V) design. No triple can cover pairs from both subsets of the varieties, and so there must be two independent sets of triples, each covering all the pairs of one of the two subsets, in other words, two independent (2,2,3,V) designs. Thus

$$B(2,3,3,V) \leq B(2,3,2,s) + B(2,3,2,V-s)$$

for any s between 0 and V.

The minimal (2,2,3,V) design will occur when the varieties are partitioned evenly. However, the minimal (2,3,3,V) design will not necessarily result from the same partitioning, since some sets of pairs are covered by triples more efficiently than others. The behaviour of this inequality must be investigated for all V and s.

The value of B(2,3,2,V) is known for all V. It was shown by Fort and Hedlund in 1958 [3] that

$$B(2,3,2,V) = \left\lceil \frac{V}{3} \left\lceil \frac{V-1}{2} \right\rceil \right\rceil .$$

This gives the bound

$$B(2,3,3,V) \le \left\lceil \frac{s}{3} \left\lceil \frac{s-1}{2} \right\rceil \right\rceil + \left\lceil \frac{V-s}{3} \left\lceil \frac{V-s-1}{2} \right\rceil \right\rceil .$$

Since this function behaves differently for each residue class of V(mod 12) and s(mod 6), it is convenient to look at particular residue classes of V and s. Let $V=12a+x$ and let $s=6b+y$, where $0 \le x \le 11$ and $0 \le y \le 5$. The bound becomes

$$24a^2 + 12b^2 - 24ab + K^1 a + K^2 b + K^3 ,$$

where K^1, K^2, and K^3 are functions of x and y, as follows.

$$K^1 = 4\left\lceil \frac{x-y-1}{2} \right\rceil + 2(x-y)$$

$$K^2 = 2\left\lceil \frac{y-1}{2} \right\rceil + 2y - x - 2\left\lceil \frac{x-y-1}{2} \right\rceil$$

$$K^3 = \left\lceil \frac{x-y}{3} \left\lceil \frac{x-y-1}{2} \right\rceil \right\rceil + \left\lceil \frac{y}{3} \left\lceil \frac{y-1}{2} \right\rceil \right\rceil$$

For any particular value of a, and fixed values of x and y, the bound is a quadratic function of b which is maximized when

$$24b - 24a + K^2 = 0$$

$$\text{or} \quad b = a - \frac{K^2}{24} \quad .$$

However, b must be integral, and so the minimum is obtained when b=a+e, where ϵ is the nearest integer to $(-K^2)/24$.

Substituting a+e for b, and then (V-x)/12 for a, the bound becomes simply

$$\frac{(V-x)^2}{12} + c^1\frac{(V-x)}{12} + c^2$$

where $c^1 = K^1 + K^2$ and $c^2 = 12e^2 + K^2e + K^3$.

The values of K^1, K^2, K^3, e, c^1, and c^2 are easily computed for all x and y by a simple APL program, and the results are given in Table 4.2.1. For each V, the optimum value for s, within a particular residue class, will be 6(a+e)+y, and this value is also shown in the table.

V = 12A+0

y	K^1	K^2	K^3	e	Best s	c^1	c^2	Best
0	0	0	0	0	6A+0	0	0	
1	-6	4	1	0	6A+1	-2	1	*
2	-8	8	2	0	6A+2	0	2	
3	-14	12	3	0	6A+3	-2	3	
4	-16	16	6	-1	6A-2	0	2	
5	-22	20	9	-1	6A-1	-2	1	*

V = 12A+1

y	K^1	K^2	K^3	e	Best s	c^1	c^2	Best
0	2	-1	0	0	6A+0	1	0	*
1	0	1	0	0	6A+1	1	0	*
2	-6	7	2	0	6A+2	1	2	
3	-8	9	2	0	6A+3	1	2	
4	-14	15	5	-1	6A-2	1	2	
5	-16	17	7	-1	6A-1	1	2	

V = 12A+2

y	K^1	K^2	K^3	e	Best s	c^1	c^2	Best
0	8	-4	1	0	6A+0	4	1	
1	2	0	0	0	6A+1	2	0	*
2	0	4	1	0	6A+2	4	1	
3	-6	8	2	0	6A+3	2	2	
4	-8	12	4	0	6A+4	4	4	
5	-14	16	6	-1	6A-1	2	2	

V = 12A+3

y	K^1	K^2	K^3	e	Best s	c^1	c^2	Best
0	10	-5	1	0	6A+0	5	1	*
1	8	-3	1	0	6A+1	5	1	*
2	2	3	1	0	6A+2	5	1	*
3	0	5	1	0	6A+3	5	1	*
4	-6	11	4	0	6A+4	5	4	
5	-8	13	5	-1	6A-1	5	4	

Table 4.2.1

V= 12A+4

y	K^1	K^2	K^3	e	Best s	c^1	c^2	Best
0	16	-8	3	0	6A+0	8	3	
1	10	-4	1	0	6A+1	6	1	*
2	8	0	2	0	6A+2	8	2	
3	2	4	1	0	6A+3	6	1	*
4	0	8	3	0	6A+4	8	3	
5	-6	12	5	0	6A+5	6	5	

V= 12A+5

y	K^1	K^2	K^3	e	Best s	c^1	c^2	Best
0	18	-9	4	0	6A+0	9	4	
1	16	-7	3	0	6A+1	9	3	
2	10	-1	2	0	6A+2	9	2	*
3	8	1	2	0	6A+3	9	2	*
4	2	7	3	0	6A+4	9	3	
5	0	9	4	0	6A+5	9	4	

V= 12A+6

y	K^1	K^2	K^3	e	Best s	c^1	c^2	Best
0	24	-12	6	1	6A+6	12	6	
1	18	-8	4	0	6A+1	10	4	
2	16	-4	4	0	6A+2	12	4	
3	10	0	2	0	6A+3	10	2	*
4	8	4	4	0	6A+4	12	4	
5	2	8	4	0	6A+5	10	4	

V= 12A+7

y	K^1	K^2	K^3	e	Best s	c^1	c^2	Best
0	26	-13	7	1	6A+6	13	6	
1	24	-11	6	0	6A+1	13	6	
2	18	-5	5	0	6A+2	13	5	
3	16	-3	4	0	6A+3	13	4	*
4	10	3	4	0	6A+4	13	4	*
5	8	5	5	0	6A+5	13	5	

Table 4.2.1 (continued)

	y	K^1	K^2	K^3	e	Best s	c^1	c^2	Best
	0	32	-16	11	1	6A+6	16	7	
	1	26	-12	7	1	6A+7	14	7	
V=	2	24	-8	7	0	6A+2	16	7	
12A+8	3	18	-4	5	0	6A+3	14	5	*
	4	16	0	6	0	6A+4	16	6	
	5	10	4	5	0	6A+5	14	5	*

	y	K^1	K^2	K^3	e	Best s	c^1	c^2	Best
	0	34	-17	12	1	6A+6	17	7	*
	1	32	-15	11	1	6A+7	17	8	
V=	2	26	-9	8	0	6A+2	17	8	
12A+9	3	24	-7	7	0	6A+3	17	7	*
	4	18	-1	7	0	6A+4	17	7	*
	5	16	1	7	0	6A+5	17	7	*

	y	K^1	K^2	K^3	e	Best s	c^1	c^2	Best
	0	40	-20	17	1	6A+6	20	9	
	1	34	-16	12	1	6A+7	18	8	*
V=	2	32	-12	12	1	6A+8	20	12	
12A+10	3	26	-8	8	0	6A+3	18	8	*
	4	24	-4	9	0	6A+4	20	9	
	5	18	0	8	0	6A+5	18	8	*

	y	K^1	K^2	K^3	e	Best s	c^1	c^2	Best
	0	42	-21	19	1	6A+6	21	10	*
	1	40	-19	17	1	6A+7	21	10	*
V=	2	34	-13	13	1	6A+8	21	12	
12A+11	3	32	-11	12	0	6A+3	21	12	
	4	26	-5	10	0	6A+4	21	10	*
	5	24	-3	10	0	6a+5	21	10	*

Table 4.2.1 (continued)

For each residue class of V, it can be seen that one or more of the six bounds is the smallest for every $V \geq 3$. These are indicated by a * in the table. Most of the bounds will appear twice in the table since if s' gives a particular bound, so must V-s', and these two values will usually be in different residue classes.

The minimal (2,3,3,V) designs which contain bipartite (2,2,3,V) designs have thus been determined. These minimal designs consist of two independent, minimal, (2,3,2,V) designs on subsets of s and V-s varieties. A summary of the bounds for each residue class of V, as well as the optimum value(s) for s, is given in Table 4.2.2. The bound has been re-written in terms of V alone by replacing x by the appropriate value, and simplifying. The optimum value for s has also been rewritten in terms of V.

For V=2, 4, or 6 (mod 12), the upper bound is equal to the lower bound of Lemma 4.1.1, and is therefore the actual value of B(2,3,3,V).

V (mod 12)	upper bound	for s=
0	$(V^2-2V+12)/12$	$(V+2)/2$
1	$(V^2-V)/12$	$(V+1)/2$
2	$(V^2-2V)/12$	$V/2$
3	$(V^2-V+6)/12$	$(V+1)/2$ or $(V+3)/2$
4	$(V^2-2V+4)/12$	$(V+2)/2$
5	$(V^2-V+4)/12$	$(V+1)/2$
6	$(V^2-2V)/12$	$V/2$
7	$(V^2-V+6)/12$	$(V+1)/2$
8	$(V^2-2V+12)/12$	$(V+2)/2$
9	$(V^2-V+12)/12$	$(V+1)/2$ or $(V+3)/2$
10	$(V^2-2V+16)/12$	$V/2$ or $(V+4)/2$
11	$(V^2-V+10)/12$	$(V+1)/2$ or $(V+3)/2$

Table 4.2.2

4.3 Non-bipartite designs

The bounds in Table 4.2.2 reflect the smallest possible (2,3,3,V) designs which contain bipartite (2,2,3,V) designs. If a design does not contain a bipartite (2,2,3,V) design, then it must contain a non-bipartite (2,2,3,V) design. By Theorem 3.3.1 the minimum cardinality of such a non-bipartite design is

$$\left\lfloor \frac{V^2-4}{4} \right\rfloor ,$$

and therefore the number of blocks in any (2,3,3,V) design which contains it must be at least

$$\left\lceil \frac{1}{3} \left\lfloor \frac{V^2-4}{4} \right\rfloor \right\rceil = \left\lceil \frac{V^2-5}{12} \right\rceil \quad (V \text{ odd})$$

$$\text{or } \left\lceil \frac{V^2-4}{12} \right\rceil \quad (V \text{ even}) .$$

But this is greater than or equal to the upper bounds in Table 4.2.2 for every $V \geq 3$. Therefore those bounds are the best possible, and the following theorem is proved.

Theorem 4.3.1

$$B(2,3,3,V) = \left\lceil \frac{V^2-2V}{12} \right\rceil \quad (V=2,\ 4,\ \text{or } 6 \pmod{12})$$

$$\left\lceil \frac{V^2-2V}{12} \right\rceil + 1 \quad (V=0,\ 8,\ \text{or } 10 \pmod{12})$$

$$\left\lceil \frac{V^2-V}{12} \right\rceil \quad (V=1,\ 3,\ 5,\ \text{or } 7 \pmod{12})$$

$$\left\lceil \frac{V^2-V}{12} \right\rceil+1 \quad (V=9 \text{ or } 11 \pmod{12}) .$$

In every case, the minimal design consists of two disjoint (2,3,2,V) designs. The exact value of B(2,3,3,V) and the optimum partitioning(s) of the varieties for every V are given in Table 4.2.2.

CHAPTER 5 - (2,4,4,V) DESIGNS

5.1 Introduction

In this chapter, the structure of (2,4,4,V) designs will be investigated, and the value of B(2,4,4,V) will be determined for most values of V. The approach to this problem will be similar to that used in the last chapter.

From Lemma 4.1.1, any (2,4,4,V) design must contain a (2,2,4,V) design, and therefore

$$B(2,4,4,V) \geq \left\lceil \frac{1}{6} B(2,2,4,V) \right\rceil = \left\lceil \frac{1}{6} \left\lceil \frac{V^2-3V}{6} \right\rceil \right\rceil .$$

For all V=3 or 12 (mod 36) this bound becomes simply

$$\frac{V^2-3V}{36} .$$

For these values of V, the minimal (2,2,4,V) design consists of all possible pairs from each of three disjoint subsets of the varieties, each having cardinality 1 or 4 (mod 12). It is known [15] that such sets of pairs can be covered exactly by quadruples, and therefore this bound can actually be attained for all V=3 or 12 (mod 36).

In general, the minimal (2,2,4,V) designs are tripartite, that is, they contain all possible pairs from three disjoint subsets of the varieties. Clearly, such a tripartite design can be covered by quadruples by using three independent (2,4,2,V) designs. The minimal designs that can be obtained in this manner will be determined in section 5.2. In section 5.3, several special cases that are not covered by the general results of the previous section will be discussed. In section 5.4, it will be shown that these designs are the smallest possible designs that contain tripartite (2,2,4,V) designs. Finally, in section 5.5, designs that do not contain tripartite (2,2,4,V) designs will be considered.

5.2 The upper bound

As shown in Chapter 3, the minimal (2,2,4,V) designs are tripartite, that is, they consist of all possible pairs from each of three disjoint subsets of the varieties. One method of covering such a tripartite design by quadruples in order to form a (2,4,4,V) design is to combine three independent (2,4,2,V) designs. This gives the following bound.

$$B(2,4,4,V) \leq B(2,4,2,r)+B(2,4,2,s)+B(2,4,2,V-r-s) ,$$

for any $r \geq 0$, $s \geq 0$, and $r+s \leq V$.

The behaviour of this inequality may easily be investigated, since the value of B(2,4,2,V) is known for all values of V. Mills [10,11] has shown that

$$B(2,4,2,V) = \left\lceil \frac{V}{4} \left\lceil \frac{V-1}{3} \right\rceil \right\rceil$$

for all values of V with the exception of 7, 9, 10, and 19. These four special cases will be ignored for the rest of this section. Their effect on the following results will be discussed in detail in section 5.3.

Since this bound will behave differently according to the residue classes of r and s (mod 12), and V (mod 36), it is convenient to let $r=12a+x$, $s=12b+y$, and $V=36c+z$, where $0 \le x \le 11$, $0 \le y \le 11$, and $0 \le z \le 35$. The bound can then be rewritten as follows.

$$B(2,4,4,V) \le 108c^2 + 24a^2 + 24b^2 + 24ab - 72ac - 72bc$$
$$+ K^1 a + K^2 b + K^3 c + K^4 ,$$

where K^1, K^2, K^3, and K^4 are functions of x, y, and z, as follows.

$$K^1 = 3\left\lceil \frac{x-1}{3} \right\rceil - 3\left\lceil \frac{z-x-y-1}{3} \right\rceil + 2x + y - z$$

$$K^2 = 3\left\lceil \frac{y-1}{3} \right\rceil - 3\left\lceil \frac{z-x-y-1}{3} \right\rceil + 2y + x - z$$

$$K^3 = 9\left\lceil \frac{z-x-y-1}{3} \right\rceil + 3z - 3x - 3y$$

$$K^4 = \left\lceil \frac{x}{4}\left\lceil \frac{x-1}{3} \right\rceil \right\rceil + \left\lceil \frac{y}{4}\left\lceil \frac{y-1}{3} \right\rceil \right\rceil + \left\lceil \frac{z-x-y}{4}\left\lceil \frac{z-x-y-1}{3} \right\rceil \right\rceil$$

For a fixed value of V, and fixed values of x and y, the bound is a quadratic function of both of the variables a and b. There is a single minimum which occurs when

$$48a + 24b - 72c + K^1 = 0$$
$$\text{and } 48b + 24a - 72c + K^2 = 0$$

or when $a=c+e^1$ and $b=c+e^2$, where

$$e^1 = \frac{K^2 - 2K^1}{72} \text{ and } e^2 = \frac{K^1 - 2K^2}{72} .$$

This minimum, however, will normally occur for non-integral values of a and b, and, for the purposes of this construction, a and b must be integers. The integral values of a and b which minimize the bound will be one of the four possible sets of values that can be obtained by letting

$$e^1 = \left\lceil \frac{K^2 - 2K^1}{72} \right\rceil \text{ or } \left\lfloor \frac{K^2 - 2K^1}{72} \right\rfloor$$

$$\text{and } e^2 = \left\lceil \frac{K^1 - 2K^2}{72} \right\rceil \text{ or } \left\lfloor \frac{K^1 - 2K^2}{72} \right\rfloor .$$

(This result is proved in Appendix I.)

Substituting $c+e^1$ for a, and $c+e^2$ for b, and $(V-z)/36$ for c, the bound becomes simply

$$\left(\frac{V-z}{36}\right)^2 + c^1\frac{(V-z)}{36} + c^2 ,$$

where $c^1 = K^1 + K^2 + K^3$

and $c^2 = 24{e^1}^2 + 24{e^2}^2 + 24e^1e^2 + K^2e^2 + K^1e^1 + K^4$.

It should be noted that only the constant term c^2 depends on the values of e^1 and e^2. Thus the optimum values for e^1 and e^2 are easily determined, and these values are independent of the magnitude of V.

With the aid of a simple APL program, the best bound may be determined for any residue class of V (mod 36). First, the set of all possible values for x and y must be generated, and redundant cases rejected. For example, if V=36c+4, one possible partitioning of the varieties is into subsets of 12c, 12c+1, and 12c+3 varieties. Thus the three cases in which x=0 and y=1, x=0 and y=3, and x=1 and y=3, are equivalent, and only one of them need be considered.

For each pair of values for x and y, K^1, K^2, K^3, and K^4 are determined, and the values of the coefficients c^1 and c^2 are found for the optimum settings of e^1 and e^2. Since there are slightly over 1000 individual cases to be considered, a complete table of results is too large to be presented here. As an example, the results for the case V=36c+4 are shown in Table 5.2.1.

In this case, four different partitionings of the varieties will give the optimum bound, and these are marked by a * in the table. The partitioning that involves a subset of size 12c-1 clearly cannot be used when c=0, but otherwise all of these four cases will give the optimum bound for all values of V in this residue class.

x	y	V-r-s (mod 12)	c^1	c^2	for r/s/V-r-s =	Best
0	0	4	7	1	12c/12c/12c+4	*
0	1	3	7	1	12c/12c+1/12c+3	*
0	2	2	10	2	12c/12c+2/12c+2	
0	5	11	10	3	12c/12c+5/12c-1	
0	6	10	7	4	12c/12c+6/12c-2	
0	7	9	7	5	12c/12c+7/12c-3	
0	8	8	10	7	12c/12c+8/12c-8	
1	1	2	7	1	12c+1/12c+1/12c+2	*
1	4	11	7	1	12c+1/12c+4/12c-1	*
1	5	10	7	4	12c+1/12c+5/12c-2	
1	6	9	7	4	12c+1/12c+6/12c-3	
1	7	8	7	5	12c+1/12c+7/12c-4	
2	3	11	10	2	12c+2/12c+3/12c-1	
2	4	10	7	3	12c+2/12c+4/12c-2	
2	5	9	10	5	12c+2/12c+5/12c-3	
2	6	8	10	5	12c+2/12c+6/12c-4	
2	7	7	7	8	12c+2/12c+7/12c-5	
3	3	10	7	3	12c+3/12c+3/12c-2	
3	4	9	7	3	12c+3/12c+4/12c-3	
3	5	8	10	5	12c+3/12c+5/12c-4	
3	6	7	7	7	12c+3/12c+6/12c-5	
4	4	8	7	3	12c+4/12c+4/12c-4	
4	5	7	7	7	12c+4/12c+5/12c-5	
4	6	6	7	7	12c+4/12c+6/12c-6	
5	5	6	10	9	12c+5/12c+5/12c-6	
6	11	11	10	3	12c+6/12c-1/12c-1	
7	10	11	7	5	12c+7/12c-2/12c-1	
8	9	11	10	7	12c+8/12c-3/12c-1	
8	10	10	7	8	12c+8/12c-2/12c-2	
9	9	10	7	9	12c+9/12c-3/12c-2	

Table 5.2.1

For most residue classes of V, the x and y values which give the smallest value for c^2 also give the smallest value for c^1. This is convenient, since it always gives a bound that is optimal for all values of V in that particular residue class. For example, referring to Table 5.2.1, all of the cases in which $c^2=1$ (the minimum value for c^2) give $c^1=7$ (the minimum value for c^1). The single exception occurs when $V \equiv 18 \pmod{36}$. In this case, there is one set of values for x and y ($x=6, y=6$) which gives the smallest value for c^2, but does not give the smallest value for c^1. This means that there is one partitioning of the varieties (12c+6/12c+6/12c+6) which gives the optimum bound <u>only</u> when $c=0$, that is, only when $V=18$.

The optimal results for each residue class of V are summarized in Table 5.2.2. It is convenient to obtain the bound in the form

$$(V^2+pV+q)/36$$

where $p=c^1-2z$ and $q=z^2-c^1z+36c^2$.

The values of p and q are given in the table as well as the values of c^1 and c^2. The optimal partitionings are also given in a condensed form. For example, "0/2/-1" will be used in place of "12c/12c+2/12c-1". The special case for $V=18$ is also noted.

V (mod 36)	c^1	c^2	p	q	optimum partitioning(s)
0	-3	1	-3	36	1/1/-2
1	1	0	-1	0	0/0/1 1/1/-1
2	2	0	-2	0	0/1/1
3	3	0	-3	0	1/1/1
4	7	1	-1	24	0/0/4 0/1/3 1/1/2 1/4/-1(1)
5	8	1	-2	21	0/1/4 1/1/3
6	9	1	-3	18	1/1/4
7	13	2	-1	30	0/3/4 1/2/4 1/3/3 4/4/-1(1)
8	14	2	-2	24	0/4/4 1/3/4
9	15	2	-3	18	1/4/4
10	19	3	-1	18	2/4/4 3/3/4
11	20	3	-2	9	3/4/4
12	21	3	-3	0	4/4/4
13	25	5	-1	24	3/4/6 4/4/5
14	26	5	-2	12	4/4/6
15	27	6	-3	36	4/4/7
16	31	7	-1	12	4/6/6
17	32	8	-2	33	4/6/7
18	33	9	-3	54	4/7/7 [6/6/6](2)
19	37	10	-1	18	6/6/7
20	38	11	-2	36	6/7/7
21	39	12	-3	54	7/7/7
22	43	14	-1	42	6/6/10 6/7/9 7/7/8
23	44	15	-2	57	6/7/10 7/7/9
24	45	16	-3	72	7/7/10
25	49	18	-1	48	6/9/10 7/8/10 7/9/9
26	50	19	-2	60	6/10/10 7/9/10
27	51	20	-3	72	7/10/10
28	55	22	-1	36	8/10/10 9/9/10
29	56	23	-2	45	9/10/10
30	57	24	-3	54	10/10/10
31	61	27	-1	42	12/9/10 13/8/10 13/9/9 10/10/11
32	62	28	-2	48	12/10/10 13/9/10
33	63	29	-3	54	13/10/10
34	67	32	-1	30	12/12/10 12/13/9 13/13/8 13/10/11
35	68	33	-2	33	12/13/10 13/13/9

(1)-not applicable for V<36   (2)-only for V=18

Table 5.2.2

5.3 Special cases

As stated in the last section, the value of B(2,4,2,V) is usually given by the formula

$$B(2,4,2,V) = \left\lceil \frac{V}{4} \left\lceil \frac{V-1}{3} \right\rceil \right\rceil .$$

However, for the four cases in which V=7, 9, 10, or 19, the actual value of B(2,4,2,V) is greater than the value given by this formula. For V=7, 9, or 10 the actual value is greater by 1 (5, 8, and 9 instead of 4, 7, and 8), and for V=19 the value is greater by 2 (31 instead of 29).

The bounds obtained in the last section are correct for most values of V. However, all of the cases in which an optimum partitioning involves a subset of 7, 9, 10, or 19 varieties must be treated separately. There are 30 such cases to be considered. These occur when V=15, 17 to 36, 51, 53 to 57, 59, 60, and 63.

In two of these cases, when V=18 or 34, not all of the optimum partitionings involve one of the special cases. For V=34, the partitioning 13/13/8 can be used, and for V=18, the special partitioning 6/6/6 can be used. The predicted bound can therefore still be achieved in these two cases.

In the other 28 cases the bound given in Table 5.2.2 cannot be achieved. For these cases, a very simple program was used to determine the best design which could be constructed from three (2,4,2,V) designs. The results are given in Table 5.3.1.

V	Predicted bound	Actual bound	Optimum partitioning(s)
15	6	7	4/5/6 3/6/6 4/4/7
17	8	9	5/6/6 4/6/7
18	9	9	6/6/6
19	10	11	6/6/7
20	11	12	6/6/8
21	12	14	6/7/8 6/6/9
22	14	15	6/8/8 6/6/10
23	15	17	7/8/8 6/8/9 6/7/10 6/6/11 4/6/13
24	16	18	8/8/8 6/8/10 6/6/12
25	18	19	6/6/13
26	19	21	8/8/10 6/10/10 6/8/12 6/7/13
27	20	22	6/8/13
28	22	24	8/10/10 8/8/12 6/10/12 7/8/13 6/9/13
29	23	25	8/8/13 6/10/13
30	24	27	10/10/10 8/10/12 6/12/12 8/9/13 7/10/13 6/11/13 4/13/13
31	27	28	8/10/13 6/12/13
32	28	29	6/13/13
33	29	31	10/10/13 8/12/13 7/13/13
34	32	32	8/13/13
35	33	34	10/12/13 9/13/13
36	34	35	10/13/13
51	69	71	16/16/19
53	76	77	16/16/21
54	78	79	16/16/22
55	83	84	16/18/21
56	85	86	16/18/22
57	87	90	16/19/22 16/16/25
59	95	96	16/21/22
60	97	98	16/22/22
63	107	109	19/22/22 16/22/25

<u>Table 5.3.1</u>

5.4 Overlap

In sections 5.2 and 5.3 the smallest (2,4,4,V) designs which could be constructed from three (2,4,2,V) designs were determined. Any design constructed in this manner must clearly contain a tripartite (2,2,4,V) design. The question to be considered in this section is as follows. Are these designs the smallest possible (2,4,4,V) designs which contain tripartite (2,2,4,V) designs?

Consider the three subsets of the varieties in such a design, and the three independent sets of pairs which must be covered. Each set of pairs could be covered by its own independent set of quadruples, which would give a design of the type constructed in the preceding sections. If this is not the case, however, then there must be at least one quadruple in the design which covers pairs from two of the subsets of the varieties. Such a quadruple must consist of two independent pairs, one from each of two subsets of the varieties. Let such a quadruple be called a split quadruple.

Lemma 5.4.1 The minimal (2,4,4,V) design which contains a tripartite (2,2,4,V) design can be constructed using at most one split quadruple.

Proof The two pairs in a split quadruple are completely independent. They may be separated from one another without affecting the integrity of the design. Suppose that a design contains 2 or more split quadruples. At least two of the pairs in these quadruples must then be from the same subset of the varieties. The pairs in the split quadruples can therefore be re-grouped in order to place these two pairs in the same quadruple, thereby removing one of the split quadruples. This process can be continued until at most one split quadruple remains, without affecting the size of the design. Q.E.D.

The minimal designs which contain no split quadruples have already been determined in the last two sections. Any design which contains one split quadruple consists of one independent (2,4,2,V) design, and two overlapping (2,4,2,V) designs. That is, two designs, each of which consists of a set of quadruples plus one pair. Every such design will be exactly one quadruple smaller than a corresponding design formed from three non-overlapping (2,4,2,V) designs. Thus the bounds already determined can be improved if and only if two of the three independent, minimal, (2,4,2,V) designs that form the optimal solution(s) can be overlapped. To determine whether or not this can be done, two questions must be answered.

First, for what values of V can the minimal (2,4,2,V) design be constructed using E(2,4,2,V)-1 quadruples and one pair? Second, are there any cases in the last two sections in which an optimal partitioning involves two or more of these values of V?

5.4.1 General results

Consider a minimal (2,4,2,V) design in which two varieties can be removed from a single quadruple without affecting the integrity of the design. If this is the case, five of the six pairs in that quadruple must also appear in other blocks. That is, there must be at least five repeated pairs.

In addition, every variety must appear at least once with each of the other V-1 varieties, and therefore must appear in at least $\lceil (V-1)/3 \rceil$ blocks. Therefore, if V=0(mod 3) each variety must occur in at least one repeated pair, and if V=2(mod 3) each variety must occur in at least two repeated pairs. Thus, in addition to the five repeated pairs involving the varieties in the special quadruple, there must be at least k(V-4)/2 repeated pairs involving the other V-4 varieties, where k depends on the residue class of V-1 (mod 3).

For each residue class of V (mod 12), the exact number of repeated pairs in any minimal (2,4,2,V) design is easily calculated. In Table 5.4.1.1, the number of repeated pairs in the minimal designs is shown, along with the minimum number required by the calculations above. In all but the case V=2 or 5 (mod 12), there are not enough repeated pairs to enable two varieties to be deleted from a single block.

However, for V=2 or 5 (mod 12) a suitable design <u>can</u> be constructed, as follows. Begin with a minimal (2,4,2,V-1) design. Since V-1=1 or 4 (mod 12), this design will contain every pair exactly once (a Steiner system). Only the V-1 pairs involving the remaining variety have yet to be covered. This requires (V-2)/3 quadruples and one pair. This gives a minimal (2,4,2,V) design which consists of B(2,4,2,V)-1 quadruples and one pair, as required.

V (mod 12)	repeated pairs	minimum required
0,6	V/2	(V+6)/2
3,9	(V+3)/2	(V+6)/2
1,4	0	5
7,10	3	5
8,11	V	V+1
2,5	V+3	V+1

<u>Table 5.4.1.1</u>

<u>5.4.2 The special cases</u>

The arguments presented above do not apply to the four special cases in which V=7, 9, 10, or 19. These cases are treated separately below.

<u>V=19</u>

In this case, B(2,4,2,19)=31. Mills [11] has shown that a minimal (2,4,2,19) design can be constructed using 30 quadruples and one pair.

<u>V=7</u> B(2,4,2,7)=5

Suppose that there are four quadruples which cover all but one of the pairs. Each variety must still occur at least twice. Since there are only 16 varieties in all, at least 5 of them occur exactly twice. Without loss of generality, suppose that the missing pair contains neither of the varieties 1 or 2, and that both 1 and 2 occur exactly twice. This forces the quadruples

1 2 3 4
1 5 6 7
2 5 6 7

This clearly cannot be completed with only one more quadruple, since neither 3 nor 4 has yet appeared with 5, 6, or 7.

<u>V=9</u> B(2,4,2,9)=8

Suppose there are seven quadruples which cover all but one of the pairs. Every variety must occur at least three times, and therefore, since there are 28 varieties in the quadruples, 8 varieties occur exactly 3 times, and the remaining variety occurs exactly 4 times. Without loss of generality, suppose that the missing pair is 1 2 and that 1 occurs three times. The first three quadruples must then be

1 3 4 5		1 3 4 5		1 3 4 5
1 3 6 7	or	1 3 4 6	or	1 3 6 7
1 4 8 9		1 7 8 9		1 3 8 9
(1)		(2)		(3)

Case (2) can be eliminated, since both 3 and 4 would have to occur twice more, which would give a contradiction. In case (3), 3 must occur a fourth time, and therefore 2 must only occur three times. This implies that the three quadruples involving 2 must have one of these three structures as well. It is easily seen that the quadruples involving 2 must have the same structure as case (1). Thus only case (1) need be considered.

In case (1), one of 3 or 4 must occur exactly three times, forcing the quadruple

2 3 8 9 .

One of 8 and 9 must occur exactly three times, forcing

5 6 7 8 .

One of 6 and 7 must occur exactly three times, forcing

2 4 6 9 .

Since 2, 5, and 7 have appeared only twice, the last quadruple must be

2 5 7 x .

But none of the pairs 4 7, 5 9, or 7 9 have appeared, and therefore the design cannot be completed.

<u>V=10</u> B(2,4,2,10)=9

Suppose that all but one pair can be covered by 8 quadruples. Every variety must occur at least three times, and therefore at most two varieties can occur more than three times. If a variety occurs in a repeated pair, then either it occurs more than three times, or it does not appear with every other variety. Only the two varieties in the missing pair do not occur with every other variety. Therefore at most 4 varieties occur in repeated pairs, and at least 6 varieties appear with every other variety exactly once. Let 1 be such a variety, giving the quadruples

```
1 2 3 4
1 5 6 7
1 8 9 10
```

Let 2 be another such variety, giving the quadruples

```
2 5 6 7         2 5 6 8
2 8 9 10   or   2 7 9 10
  (1)             (2)
```

In case (1), six varieties have appeared in repeated pairs, and so this case can be rejected. In case (2) 5, 6, 9, and 10 have appeared in repeated pairs. Therefore 3 must appear exactly three times. But this cannot be done without a repeated pair involving 7. Therefore this case can be rejected as well.

5.4.3 Conclusion

The results of this section can be summarized as follows.

Theorem 5.4.1 A minimal (2,4,2,V) design can be constructed using B(2,4,2,V)-1 quadruples and one pair if and only if V=2 or 5 (mod 12) or V=19.

By consulting Tables 5.3.1 and 5.2.2, it can be seen that there are no values of V in which an optimum partitioning involves two such values of V. Therefore, the designs constructed in sections 5.2 and 5.3 are the smallest possible (2,4,4,V) designs that contain tripartite (2,2,4,V) designs.

5.5 Non-partite designs

The bounds obtained in the preceding sections give the cardinalities of the minimal (2,4,4,V) designs which contain tripartite (2,2,4,V) designs. If a (2,4,4,V) design does not contain a tripartite (2,2,4,V) design, then it must contain a non-tripartite (2,2,4,V) design. In Chapter 3, it was shown that such a non-tripartite design must contain at least

$$\left\lfloor \frac{V^2-V-6}{6} \right\rfloor$$

blocks, and therefore any (2,4,4,V) design which contains it must include at least

$$Q = \left\lceil \frac{1}{6} \left\lfloor \frac{V^2-V-6}{6} \right\rfloor \right\rceil$$

$$= \left\lceil \frac{V^2-V-6}{36} \right\rceil \qquad \text{blocks.}$$

This quantity Q is greater than or equal to the bounds established in the previous sections for most values of V, thereby establishing that these bounds are, in fact, the exact values of B(2,4,4,V). However, for all values of V congruent to 7, 22, 25, 28, 31, or 34 (modulo 36), as well as 22 of the 30 special values of V, the value of Q is less than the bounds given in Tables 5.2.2 and 5.3.1. Therefore, for these values of V, Q provides only a lower bound on the value of B(2,4,4,V). These cases are summarized in Tables 5.5.1 and 5.5.2, below.

In one small case, the value of B(2,4,4,V) is easily established. For V=7, the value of Q is 1. However, it is clear that B(2,4,4,7) cannot be 1, and therefore B(2,4,4,7)= 2, as given by the bound in Table 5.2.2.

5.6 Summary

For all values of V except those in Tables 5.5.1 and 5.5.2, the minimal (2,4,4,V) design is a tripartite design consisting of three independent (2,4,2,V) designs. For these values of V, the value of B(2,4,4,V) and the partitioning of the varieties which will give the minimal design are given in either Table 5.2.2 or 5.3.1.

For each value of V in Tables 5.5.1 and 5.5.2, an upper bound and a lower bound is given. These bounds differ by at most three.

V	Lower bound (Q)	Upper bound from Table 5.3.1	Difference
15	6	7	1
17	8	9	1
19	10	11	1
20	11	12	1
32	28	29	1
33	30	31	1
34	31	32	1
35	33	34	1
55	83	84	1
57	89	90	1
59	95	96	1
21	12	14	2
22	13	15	2
24	16	18	2
25	17	19	2
27	20	22	2
29	23	25	2
31	26	28	2
23	14	17	3
26	18	21	3
28	21	24	3
30	24	27	3

Table 5.5.1

V (mod 36)	Lower bound (Q)	Upper bound from Table 5.2.2	Difference
7*	$\frac{V^2-V-6}{36}$	$\frac{V^2-V+30}{36}$	1
22	$\frac{V^2-V+6}{36}$	$\frac{V^2-V+42}{36}$	1
25	$\frac{V^2-V+12}{36}$	$\frac{V^2-V+48}{36}$	1
28	$\frac{V^2-V}{36}$	$\frac{V^2-V+36}{36}$	1
31	$\frac{V^2-V+6}{36}$	$\frac{V^2-V+42}{36}$	1
34	$\frac{V^2-V-6}{36}$	$\frac{V^2-V+30}{36}$	1

* - except for V=7

<u>Table 5.2.2</u>

CHAPTER 6 - OTHER (2,K,L,V) DESIGNS

6.1 Introduction

The methods which were used in Chapters 4 and 5 to determine B(2,3,3,V) and B(2,4,4,V), for most values of V, may also be applied to any (2,K,L,V) design in which K=3 or 4. In this chapter, the results which can be obtained in this manner will be presented for (2,3,4,V), (2,4,3,V), (2,3,5,V), and (2,4,5,V) designs. No details concerning the methods used will be presented, only the results themselves, since Chapters 3 and 4 provide adequate examples of these techniques.

In general, any (2,K,L,V) design must contain a (2,2,L,V) design, which may be either a partite design, or a non-partite design. Any partite design may be covered by K-sets, in order to form a (2,K,L,V) design, by using L-1 independent (2,K,2,V) designs, which gives the following upper bound.

$$B(2,K,L,V) \le \min_{\{v\}} \left[\sum_{i=1}^{L-1} B(2,K,2,v_i) \right] \quad \left(\sum_{i=1}^{L-1} v_i = V \right)$$

For K=3, any partite (2,2,L,V) design <u>must</u> be covered by independent (2,3,2,V) designs since no overlap is possible.

Therefore, for K=3, the above bound gives the size of the smallest possible design which contains a partite (2,2,L,V) design. This is also true for K=4, unless at least two of the partitions in some optimum partitioning have cardinalities which are congruent to 2 or 5 (mod 12), or equal to 19 (see section 5.4). Since any two (2,4,2,V) designs with such cardinalities could be overlapped, this would allow the bound to be decreased by $\lfloor N/2 \rfloor$, if there were N such partitions. This situation is unlikely to arise, however, since the (2,4,2,V) designs for which it occurs are not particularly good ones, and so it would be rare to find two of them in an optimum partitioning. There is, in fact, no known case in which this occurs. The values of B(2,3,2,V) and B(2,4,2,V) are known for all values of V, and therefore the size of the smallest possible (2,K,L,V) design which contains a partite (2,2,L,V) design may be determined for K=3 or K=4.

Similar techniques can not easily be applied when K=5 since the problem of overlapping designs is not so simply solved in this case, and since relatively little is known about (2,5,2,V) designs.

If a (2,K,L,V) design contains no partite (2,2,L,V) designs, then it must contain a non-partite (2,2,L,V) design. The minimum size of such a design may be calculated by the formula given in Chapter 3, and used to obtain a lower bound on the size of any (2,K,L,V) design which does not contain a partite (2,2,L,V) design. If this lower bound is greater than the upper bound on B(2,K,L,V) which is given by the formula presented above, then that upper bound is the actual value of B(2,K,L,V). Otherwise, the non-partite lower bound provides a good lower bound on B(2,K,L,V).

The remaining four classes of designs in which $L \le 5$ have been examined in this way, and the results are presented here. Although the techniques used can be applied to classes of designs in which $L \ge 6$, it does not seem worthwhile to do so. The calculation of the general upper bound becomes quite tedious, even for L=5, and the number of cases which must be examined grows extremely rapidly as L becomes large. However, any specific (2,K,L,V) design may easily be investigated. A fairly simple computer search will provide the size of the minimal partite design, and the optimum partitioning, although the size of this search also grows rapidly as L increases. The lower bound is also very easily obtained. This will determine either the exact value of B(2,K,L,V), or at least a fairly good pair of bounds, for any specific set of parameters in which T=2 and K=3 or 4. For this reason, no attempt has been made to obtain a general solution for any class of designs in which $L \ge 6$.

6.2 (2,3,4,V) designs

In this case, the upper bound is given by

$$B(2,3,4,V) \le B(2,3,2,r)+B(2,3,2,s)+B(2,3,2,V-r-s).$$

The behaviour of this function depends on the residue classes of r and s (modulo 6) and V (modulo 18). If $V=18A+a$, $r=6B+b$, and $s=6C+c$, the bound becomes

$$B(2,3,4,V) \le \frac{(V-a)^2}{18} + c^1\frac{(V-a)}{18} + c^2$$

$$= \frac{V^2+pV+q}{18},$$

where c^1, c^2, p, and q depend on a. The values of these coefficients for each residue class are given in Table 6.2.1, along with the values of b and c which are used to obtain them. The lower bound on the size of a non-partite design in this case is

$$\left\lceil \frac{1}{3} \left\lfloor \frac{V^2-V-6}{6} \right\rfloor \right\rceil = \left\lceil \frac{V^2-V-6}{18} \right\rceil$$

This lower bound is greater than or equal to the values given by the bounds in Table 6.2.1, with the exception of three small cases. For V=12, 14, or 16, a lower bound is obtained which is exactly one less than the upper bound. For all other values of V, Table 6.2.1 gives the value of B(2,3,4,V).

a	c^1	c^2	p	q	b/c/a-b-c
0	-2	1	-2	18	0/1/-1 1/1/-2
1	-1	1	-3	20	1/1/-1
2	2	0	-2	0	0/1/1
3	3	0	-3	0	1/1/1
4	6	1	-2	10	0/1/3 1/1/2
5	7	1	-3	8	1/1/3
6	10	2	-2	12	0/3/3 1/2/3
7	11	2	-3	8	1/3/3
8	14	3	-2	6	2/3/3
9	15	3	-3	0	3/3/3
10	18	5	-2	10	3/3/4
11	19	6	-3	20	3/3/5
12	22	8	-2	24	3/3/6 3/4/5
13	23	9	-3	32	3/3/7 3/5/5
14	26	11	-2	30	3/5/6 3/4/7 4/5/5
15	27	12	-3	36	3/5/7 5/5/5
16	30	14	-2	28	3/6/7 5/5/6 4/5/7
17	31	15	-3	32	3/7/7 5/5/7

Table 6.2.1

6.3 (2,3,5,V) designs

In this case, the upper bound is

$B(2,3,5,V) \le B(2,3,2,r) + B(2,3,2,s) + B(2,3,2,t) + B(2,3,2,V-r-s-t)$

and its behaviour depends on the residue classes of r, s, and t (modulo 6), and V (modulo 24). For V=24A+a, the bound becomes

$$B(2,3,5,V) \le \frac{(V-a)^2}{24} + c^1\frac{(V-a)}{24} + c^2$$

$$= \frac{V^2+pV+q}{24},$$

where the coefficients c^1, c^2, p, and q, depend on the residue class of V. The values of these coefficients for each residue class are given in Table 6.3.1 along with the optimum partitioning(s).

a	c^1	c^2	p	q	r/s/t/V-r-s-t
0	-4	2	-4	48	1/1/1/-3 1/1/-1/-1
1	-1	1	-3	26	1/1/1/-2 1/1/0/-1
2	0	1	-4	28	1/1/1/-1
3	3	0	-3	0	1/1/1/0
4	4	0	-4	0	1/1/1/1
5	7	1	-3	14	3/1/1/0 2/1/1/1
6	8	1	-4	12	3/1/1/1
7	11	2	-3	20	3/3/1/0 3/2/1/1
8	12	2	-4	16	3/3/1/1
9	15	3	-3	18	3/3/3/0 3/3/2/1
10	16	3	-4	12	3/3/3/1
11	19	4	-3	8	3/3/3/2
12	20	4	-4	0	3/3/3/3
13	23	6	-3	14	4/3/3/3
14	24	7	-4	28	5/3/3/3
15	27	9	-3	36	6/3/3/3 5/4/3/3
16	28	10	-4	48	7/3/3/3 5/5/3/3
17	31	12	-3	50	7/4/3/3 6/5/3/3 5/5/4/3
18	32	13	-4	60	7/5/3/3 5/5/5/3
19	35	15	-3	56	7/6/3/3 7/5/4/3 6/5/5/3 5/5/5/4
20	36	16	-4	64	7/7/3/3 7/5/5/3 5/5/5/5
21	39	18	-3	54	7/7/4/3 7/6/5/3 7/5/5/4 6/5/5/5
22	40	19	-4	60	7/7/5/3 7/5/5/5
23	43	21	-3	44	7/7/6/3 7/7/5/4 7/6/5/5

Table 6.3.1

The non-partite lower bound in this case is

$$\left\lceil \frac{1}{3} \left\lfloor \frac{V^2-2V-7}{8} \right\rfloor \right\rceil = \left\lceil \frac{V^2-2V-11}{24} \right\rceil .$$

This non-partite bound is greater than or equal to the upper bounds given in Table 6.3.1 except for the small cases noted here. The lower bound for V=19 is exactly two smaller than the upper bound. The lower bounds for the following values of V are exactly one smaller than the corresponding upper bound: 7,15,16,17,18,20,21,22,23,43. The bounds in Table 6.3.1 are the exact values of B(2,3,5,V) for all other values of V.

6.4 (2,4,3,V) designs

In this case, the upper bound is

$$B(2,4,3,V) \leq B(2,4,2,s)+B(2,4,2,V-s) ,$$

and its behaviour depends on the residue classes of s (modulo 12) and V (modulo 24). For V=24A+a, the bound becomes

$$B(2,4,3,V) \leq \frac{(V-a)^2}{24} + c^1\frac{(V-a)}{24} + c^2$$

$$= \frac{V^2+pV+q}{24} ,$$

where, as usual, the coefficients depend on the residue class of V. The values of the coefficients for each residue class are given in Table 6.4.1, along with the optimum partitionings.

a	c^1	c^2	p	q	s/V-s
0	0	0	0	0	0/0 1/-1
1	1	0	-1	0	0/1
2	2	0	-2	0	1/1
3	6	1	0	15	0/3 1/2 4/-1
4	7	1	-1	12	0/4 1/3
5	8	1	-2	9	1/4
6	12	2	0	12	2/4 3/3
7	13	2	-1	6	3/4
8	14	2	-2	0	4/4
9	18	4	0	15	3/6 4/5
10	19	4	-1	6	4/6
11	20	5	-2	21	4/7
12	24	6	0	0	6/6
13	25	7	-1	12	6/7
14	26	8	-2	24	7/7
15	30	10	0	15	6/9 7/8
16	31	11	-1	24	6/10 7/9
17	32	12	-2	33	7/10
18	36	14	0	12	8/10 9/9
19	37	15	-1	18	9/10
20	38	16	-2	24	10/10
21	42	19	0	15	12/9 13/8 10/11
22	43	20	-1	18	12/10 13/9
23	44	21	-2	21	13/10

Table 6.4.1

In this case, the non-partite lower bound is

$$\left\lceil \frac{1}{6} \left\lfloor \frac{V^2-4}{4} \right\rfloor \right\rceil = \left\lceil \frac{V^2-4}{24} \right\rceil .$$

This non-partite bound is greater than or equal to the bound in Table 6.4.1 for all values of V. However, it is less than the actual upper bound obtainable in some of the special cases which occur when the optimum partitioning from the table will result in a sub-design on 7, 9, 10, or 19 varieties. In these cases, the upper bound of Table 6.4.1 cannot actually be achieved. There are 15 such cases, and they are summarized in Table 6.4.2. In this table, the predicted upper bound, the actual upper bound and the partitioning which gives it, and the lower bound are given for each value of V.

V	Predicted bound	Actual bound	Optimum partitions	Lower bound
11	5	6	7/4 6/5	5
13	7	8	6/7	7
14	8	9	6/8	8
15	10	11	6/9 7/8	10
16	11	12	6/10 8/8	11
17	12	14	4/13 6/11 7/10 8/9	12
18	14	15	6/12 8/10	14
19	15	16	6/13	15
20	16	18	7/13 8/12 10/10	17
22	20	21	9/13 10/12	20
23	21	22	10/13	22
35	49	51	16/19	51
37	56	57	16/21	57
38	58	59	16/22	60
41	68	70	16/25 19/22	70

Table 6.4.2

6.5 (2,4,5,V) designs

In this case, the upper bound is

$$B(2,4,5,V) \leq B(2,4,2,r)+B(2,4,2,s)+B(2,4,2,t)+B(2,4,2,V-r-s-t)$$

and its behaviour depends on the residue classes of r, s, and t (modulo 12) and V (modulo 48). (This generates a very large number of cases to be considered.) For V=48A+a, the bound becomes

$$B(2,4,5,V) \le \frac{(V-a)^2}{48} + c^1\frac{(V-a)}{24} + c^2$$

$$= \frac{V^2+pV+q}{48},$$

where the coefficients depend on the residue class of V. Table 6.5.1 lists the general bounds for each residue class by giving the four coefficients, and the partitioning(s) which achieve these coefficients.

a	c^1	c^2	p	q	r/s/t/V-r-s-t
0	-3	1	-3	48	1/1/1/-3 1/1/0/-2
1	-2	1	-4	51	1/1/1/-2
2	2	0	-2	0	1/1/1/-1 1/1/0/0
3	3	0	-3	0	1/1/1/0
4	4	0	-4	0	1/1/1/1
5	8	1	-2	33	4/1/1/-1 4/1/0/0 3/1/1/0 2/1/1/1
6	9	1	-3	30	4/1/1/0 3/1/1/1
7	10	1	-4	27	4/1/1/1
8	14	2	-2	48	4/4/1/-1 4/4/0/0 4/3/1/0 4/2/1/1 3/3/1/1
9	15	2	-3	42	4/4/1/0 4/3/1/1
10	16	2	-4	36	4/4/1/1
11	20	3	-2	45	4/4/4/-1 4/4/3/0 4/4/2/1 4/3/3/1
12	21	3	-3	36	4/4/4/0 4/4/3/1
13	22	3	-4	27	4/4/4/1
14	26	4	-2	24	4/4/4/2 4/4/3/3
15	27	4	-3	12	4/4/4/3
16	28	4	-4	0	4/4/4/4
17	32	6	-2	33	6/4/4/3 5/4/4/4
18	33	6	-3	18	6/4/4/4

Table 6.5.1

a	c^1	c^2	p	q	r/s/t/V-r-s-t
19	34	7	-4	51	7/4/4/4
20	38	8	-2	24	6/6/4/4
21	39	9	-3	54	7/6/4/4
22	40	10	-4	84	7/7/4/4 [6/6/6/4](1)
23	44	11	-2	45	7/6/6/4
24	45	12	-3	72	7/7/6/4 [6/6/6/6](1)
25	46	13	-4	99	7/7/7/4 [7/6/6/6](1)
26	50	14	-2	48	7/7/6/6
27	51	15	-3	72	7/7/7/6
28	52	16	-4	96	7/7/7/7
29	56	18	-2	81	10/7/6/6 9/7/7/6 8/7/7/7
30	57	19	-3	102	10/7/7/6 9/7/7/7
31	58	20	-4	123	10/7/7/7
32	62	22	-2	96	10/10/6/6 10/9/7/6
					10/8/7/7 9/9/7/7
33	63	23	-3	114	10/10/7/6 10/9/7/7
34	64	24	-4	132	10/10/7/7
35	68	26	-2	93	10/10/9/6 10/10/8/7 10/9/9/7
36	69	27	-3	108	10/10/10/6 10/10/9/7
37	70	28	-4	123	10/10/10/7
28	74	30	-2	72	10/10/10/8 10/10/9/9
39	75	31	-3	84	10/10/10/9
40	76	32	-4	96	10/10/10/10
41	80	35	-2	81	13/10/10/8 13/10/9/9
					12/10/10/9 11/10/10/10
42	81	36	-3	90	13/10/10/9 12/10/10/10
43	82	37	-4	99	13/10/10/10
44	86	40	-2	72	13/13/10/8 13/13/9/9 13/12/10/9
					13/11/10/10 12/12/10/10
45	87	41	-3	78	13/13/10/9 13/12/10/10
46	88	42	-4	84	13/13/10/10
47	92	45	-2	45	13/13/13/8 13/13/12/9
					13/13/11/10 13/12/12/10

(1) - only valid for $V \leq 47$

Table 6.5.1 - continued

The non-partite lower bound for this class of designs is

$$\left\lceil \frac{1}{6} \left\lfloor \frac{V^2-2V-7}{8} \right\rfloor \right\rceil = \left\lceil \frac{V^2-2V-8}{48} \right\rceil .$$

This non-partite bound is not quite as good as the non-partite bounds in the previous cases, although it is still greater than or equal to the upper bound for the majority of the values of V. There are eleven residue classes (modulo 48) for which the lower bound is smaller than the upper bound for every value of V in that residue class. The lower bound is exactly one less than the upper bound for all V congruent to 8, 11, 23, 26, 29, 38, 41, 44, or 47 (modulo 48), and is exactly two less than the upper bound for all V congruent to 32 or 35 (modulo 48). In addition, the lower bound is one less than the upper bound for the two small cases V=22 and V=24.

There are also 42 special cases in which the partitioning(s) given in Table 6.5.1 will result in a sub-design on 7, 9, 10, or 19 varieties. These 42 cases have larger upper bounds than those given by the formulas in Table 6.5.1. In 37 of these cases, the lower bound is less than the upper bound. Table 6.5.2 gives the predicted upper bound, the actual upper bound, the optimum partitioning(s), and the lower bound for each of these 42 special cases. Note that many of these cases have a large number of optimum partitionings. For V=36, for example, there are ten different ways to partition the varieties which will all result in a minimal partite design.

V	Pred. Bound	Act. Bound	Lower Bound	Optimum Partitions
19	7	8	7	7/4/4/4 6/6/4/3 6/5/4/4
21	9	10	9	7/6/4/4 6/6/6/3 6/6/5/4
23	11	12	10	7/6/6/4 6/6/6/5
25	13	14	12	7/6/6/6
26	14	15	13	8/6/6/6
27	15	17	14	9/6/6/6 8/7/6/6
28	16	18	15	10/6/6/6 8/8/6/6
29	18	20	17	13/6/6/4 11/6/6/6 10/7/6/6
				9/8/6/6 8/8/7/6
30	19	21	18	12/6/6/6 10/8/6/6 8/8/8/6
31	20	22	19	13/6/6/6
32	22	24	20	13/7/6/6 12/8/6/6 10/10/6/6
				10/8/8/6 8/8/8/8
33	23	25	22	13/8/6/6
34	24	27	23	13/9/6/6 13/8/7/6 12/10/6/6
				12/8/8/6 10/10/8/6 10/8/8/8
35	26	28	24	13/10/6/6 13/8/8/6
36	27	30	26	13/13/6/4 13/11/6/6 13/9/8/6
				13/10/7/6 13/8/8/7 12/12/6/6
				12/10/8/6 12/8/8/8
				10/10/10/6 10/10/8/8
37	28	31	27	13/12/6/6 13/10/8/6 13/8/8/8
38	30	32	29	13/13/6/6
39	31	34	30	13/13/7/6 13/12/8/6
				13/10/10/6 13/10/8/8
40	32	35	32	13/13/8/6
41	35	37	34	13/13/9/6 13/13/8/7
				13/12/10/6 13/12/8/8
				13/10/10/8
42	36	38	35	13/13/10/6 13/13/8/8
43	37	40	37	13/13/13/4 13/13/11/6
				13/13/10/7 13/13/9/8
				13/12/12/6 13/12/10/8
				13/10/10/10
44	40	41	39	13/13/12/6 13/13/10/8
45	41	42	41	13/13/13/6
46	42	44	42	13/13/13/7 13/13/12/8
				13/13/10/10
48	46	47	46	13/13/13/9 13/13/12/10
49	47	48	48	13/13/13/10

Table 6.5.2

V	Pred. Bound	Act. Bound	Lower Bound	Optimum Partitions
67	89	91	91	19/16/16/16
69	96	97	97	21/16/16/16
70	98	99	99	22/16/16/16
71	103	104	102	21/18/16/16
72	105	106	105	22/18/16/16
73	107	110	108	25/16/16/16 22/19/16/16
74	112	113	111	22/18/18/16
75	114	116	114	22/21/16/16
76	116	118	117	22/22/16/16
77	122	123	121	22/21/18/16
78	124	125	124	22/22/18/16
79	126	129	127	25/22/16/16 22/22/19/16
81	134	135	134	22/22/21/16
82	136	137	137	22/22/22/16
85	146	148	147	25/22/22/16 22/22/22/19

Table 6.5.2 - continued

CHAPTER 7 - Computer searches

7.1 Introduction

In this chapter, several versions of an algorithm are presented which will determine the value of B(T,K,L,V) for any set of parameters. This algorithm uses the well-known technique of backtracking. The term backtracking was coined by D.H.Lehmer in 1950, and in 1960 Walker[20] gave a formalized definition of the technique. This type of algorithm has been used in the solution of a very large number of problems in combinatorics and graph theory, and is by far the most common search technique.

In general, a backtrack algorithm builds up a sequence of items (c(1) c(2)...c(n)) which are taken from a given set of items C. In the algorithm presented here, C is the set of all possible K-sets on V varieties. At each stage, the algorithm must use the partial solution (c(1) c(2)..c(k)) to determine a set S of all candidates for c(k+1). The first of these items is then added to the sequence, and the process is repeated. If S contains no items, then the algorithm must backtrack and choose the next possibility for c(k). If no possibilities remain for c(k), then the algorithm must backtrack still further, and so on.

It is helpful to think of this algorithm as traversing a tree of possibilities. The root of this tree is the empty sequence. The nodes at the first level consist of all the possible sequences of length one, and so on. The search proceeds down one side of the tree before it begins to backtrack. For this reason, it is sometimes called a "depth first" search.

Section 7.2 describes the basic structure of the back-track algorithm for determining B(T,K,L,V) as well as some details concerning efficient implementation.

Any backtrack algorithm, if implemented in its simplest form, is inherently inefficient. A good deal of work has been done on improving the efficiency of backtrack algorithms, and there are a number of techniques which are commonly used. Among these are the following. "Preclusion" consists of immediately backtracking whenever it becomes certain that the current sequence cannot lead to a solution. "Branch merging" or "isomorphism rejection" entails removing from the search tree any sub-tree which is equivalent (or isomorphic) to some other sub-tree. "Search re-arrangement" involves choosing the next item in such a way that the set S is kept as small as possible early in the search. The "branch and bound" technique is used when the solution is to be minimal in some sense, and it involves rejecting any sequence which is "larger" than the best solution which has been obtained thus far. All of these techniques have been considered, and a discussion of the enhancements which were made to the basic algorithm is contained in section 7.3.

In the remaining section, a few conclusions are reached concerning the practicality of using such a search technique.

7.2 The basic algorithm

The basic structure of the algorithm is as follows.

[1] Find the next L-set which has not been covered. If there are no uncovered L-sets, go to [5].

[2] If the design already contains the maximum number of blocks, return to the last level immediately.

[3] Generate every K-set which covers this L-set. For each of these K-sets:

- [3A] Add it to the design.
- [3B] Flag every L-set which is covered by this K-set to indicate that it has been covered.
- [3C] Apply this algorithm to attempt to complete the design.
- [3D] Remove the flag from every L-set covered by this K-set.
- [3E] If a solution was found on the level immediately below, return to the previous level immediately.

[4] All possibilities have been considered. Return to the previous level.

[5] A solution has been found. Record this solution. Reset the maximum number of blocks to one less than the number contained in this solution. Return to the previous level.

The most time-consuming steps in this algorithm are steps [3], [3B], and [3D]. The task of generating all of the K-sets (or L-sets) which intersect a given set in T or more elements consumes the major portion of time in this algorithm. In order to save unnecessary effort, this operation should be done as few times as possible. Note that in steps [3B] and [3D] the same L-sets are affected. Therefore a list of these L-sets should be generated once, and maintained until it is no longer needed. In general, whenever any change is made to the status of an L-set (or K-set), that change should be recorded so that it may be removed with as little effort as possible at a later time.

It is also necessary, as mentioned above, to keep track of the status of every L-set (and in subsequent versions, every K-set as well). In order to do this, an efficient method is required for locating the entry in a table that corresponds to a particular set. To accomplish this, an algorithm is used which converts a set into an index number which can then be used to access the appropriate entry of a table. It is also necessary to convert the index number back into the set itself in a number of cases (in step [1], for example).

To do this, an ordering of the sets must first be defined. It is most convenient to use the natural ordering in which (1 2 3...) is at one extreme and (...K-1 K) is at the other. For example, the 3-sets on 5 varieties would be ordered as shown in Table 7.2.1. The corresponding index numbers that will be used is also shown in this table.

set	index
1 2 3	9
1 2 4	8
1 2 5	7
1 3 4	6
1 3 5	5
1 4 5	4
2 3 4	3
2 3 5	2
2 4 5	1
3 4 5	0

Table 7.2.1

Notice that the number of sets beginning with each variety is easily determined by a simple formula. The number of sets which contain a given second variety is also easily computed once the first variety is known, and so on. In general, if n varieties have already been chosen, then there are exactly

$$\binom{V-x}{K-n-1}$$

sets which contain x as the next element. A simple algorithm can be obtained which will use this fact in order to transform a set into an index number, of the reverse. First, a table of the binomial coefficients is required as shown in Table 7.2.2. The entry in row R and column C will be denoted TABLE(R,C) and the values in the table will be given in general by

$$\mathrm{TABLE}(R,C) = \binom{C-2+R}{R}.$$

The algorithms themselves are as follows. S(I) will denote the Ith element of the set. V and K are the number of varieties and the size of the set, respectively. N refers to the index number of the set. Finally, ":=" is the assignment operator.

Row	Col. 1	2	3	4	5
1	0	1	2	3	4
2	0	1	3	6	10
3	0	1	4	10	20
4	0	1	5	15	35

TABLE

Table 7.2.2

Algorithm I Convert a set S to an index N

```
COL := V+1-S(K)
N := TABLE(1,COL)
FOR I := K TO 2 BY -1
     COL := COL+S(I)-S(I-1)-1
     N := N+TABLE(K+2-I,C)
END
```

Algorithm II Convert an index N to a set S

```
COL := V+1-K
FOR ROW := K TO 1 BY -1
     WHILE TABLE(ROW,COL)>N
          COL := COL-1
     S(K+1-ROW) := V+2-ROW+COL
     N := N-TABLE(ROW,COL)
END
```

Each of these algorithms can be optimized somewhat and implemented in 10 machine language instructions on a PDP11. This makes them very efficient. The reader may wish to try these algorithms by converting the set (1 3 4 6) with V=7 into its index number (23) and back again.

The basic algorithm as described here was implemented as efficiently as possible in assembler language to run on a PDP11/45 computer. However, as might be expected, the basic algorithm is much too inefficient to allow minimal designs to be found for any but the smallest of parameters. In order for this algorithm to be at all useful, several improvements had to be made, and these are discussed in the following section.

7.3 Enhancements

7.3.1 Preclusion

The first, and simplest, enhancement which can be made is of the type known as "preclusion". The number of L-sets which are covered by a given K-set is easily computed by the formula

$$C = \sum_{i=T}^{K} \binom{K}{i}\binom{V-K}{L-i}.$$

This quantity is computed once at the beginning of the program, and is stored for later use. Also defined at the beginning of the program is a depth limit called LIMIT. This quantity determines the maximum number of blocks which may be placed in the design. It is set initially to a large number which depends on the amount of available space for the required tables and stacks. Whenever a solution is found, LIMIT is decreased to one less than the number of blocks in the solution. These two quantities may be combined to give MAXPOS, the maximum possible number of additional L-sets which could be covered by adding enough blocks to reach the LIMIT. For example, if there were already B blocks in the design, the value of MAXPOS would be C times LIMIT-B. This variable may easily be maintained by simply subtracting C from it whenever a block is added to the design, and adding C to it whenever the program backtracks. It must also be adjusted whenever LIMIT is changed, which may require a multiplication, but this happens very infrequently.

A count is also maintained of the number of L-sets which have not yet been covered. This variable, NLEFT, is easily maintained by decrementing it whenever an L-set is flagged as covered and incrementing it when that flag is removed.

Thus the two quantities NLEFT and MAXPOS may be maintained at the cost of only a few additions and subtractions. By comparing the two, it can be determined whether or not is is possible for the design to be completed in the maximum number of blocks. If it is not possible, the algorithm backtracks immediately. This seemingly trivial addition often has a dramatic effect on the size of the search tree. In one case, the number of nodes in the tree was reduced from 16,663,323 to 211,305 by this enhancement alone.

It is natural to ask whether or not any other enhancements of this type would be effective. Unfortunately, the answer seems to be no. The problem of determining whether the design can be completed in a given number of blocks is, in fact, a smaller but more general version of the overall problem, and as such, it is very difficult. This bound on the number of blocks required may seem to be a trivial one, but it seems to be the only one which can be calculated easily enough to justify its use.

7.3.2 Isomorphism rejection

The greatest improvements in the speed of any backtrack algorithm will result from the inclusion of some form of isomorphism testing. However, in this area more than any other, the overhead introduced must be carefully weighed against the resulting reduction in the size of the search.

If a certain K-set X has been added to the design at level L, and it has subsequently been determined that the resulting sub-design cannot be completed, then X need no longer be considered as a possible choice on level L, or any lower level of the search tree, until the search backtracks to a higher level. Therefoe, the following changes can be made to the basic algorithm.

1) A status flag is kept for each K-set which indicates whether or not that K-set has been rejected.

2) Before adding a K-set to the design, this flag is tested, and if it is set then the K-set is skipped.

3) After each K-set has been tried, its rejection flag is set, and a record of the K-sets which have been rejected in this way is kept.

4) After every K-set has been tried, and before returning

to the next level, all of these flags are reset.

This change produces a moderate improvement. However, step 3) can be extended by rejecting not only the K-set itself, but also all those K-sets which would result in an isomorphic sub-design. That is, if the partial design to this point is (c(1) c(2)...c(k)) and a K-set x has just been tried without success, then all K-sets s such that (c(1)... c(k) s) is isomorphic to (c(1)...c(k) x) may be rejected. Such K-sets will be called <u>equivalent</u> to x.

If a complete isomorphism test of this nature is done on every level, this will have the effect of eliminating <u>all</u> duplication of isomorphic subtrees. In practice, however, the cost of such an isomorphism search is far too great. Instead, some simple technique must be found which will detect some, but not necessarily all, of the K-sets which are equivalent to a given K-set. The simplified test which was adopted is as follows.

<u>Definition</u> Two varieties are said to be <u>trivially isomorphic</u> if every block in the design contains either both of them or neither of them. Two blocks are <u>trivially equivalent</u> if one of them can be transformed into the other by replacing varieties with trivially isomorphic varieties.

In the modified algorithm, a table is kept which indicates which varieties are trivially isomorphic to any given variety. When a K-set is added to the design, this table is first updated, and then an attempt is made to complete the design. When this attempt is complete, the table is restored to its former status. Following this, the K-set and any other K-sets which are trivially equivalent to it are rejected. A record is also kept of the K-sets which were rejected. When every possible K-set has been tried, all of the rejection flags are reset before the program returns to the previous level.

This simple form of isomorphism testing results in a dramatic reduction of the size of the search tree, with only a moderate increase in overhead. Table 7.3.1 provides a comparison of the three versions of the algorithm discussed to this point. The number of nodes in the search tree and the approximate execution time, in seconds, are given for each algorithm in a number of different cases.

				Basic alg.		Preclusion		Isomorphism	
T	K	L	V	nodes	time	nodes	time	nodes	time
2	3	2	8	16,663,323	15,600	211,305	193.1	7,224	7.2
2	3	3	9	5,126,742	18,000	2,567,524	9,000	10,173	43.4
3	3	4	7	324,467	319.3	80,234	78.5	7,786	8.1
2	4	2	8	432,061	587.9	38,920	52.2	5,710	10.1
3	3	5	8	958,842	678.6	130,275	247.1	2,335	5.5
3	4	4	8	283,972	1016.3	88,183	312.2	8,676	34.1
3	4	5	9	859,644	8,400	845,825	9,200	1,471	15.9
2	5	2	9	747,682	432.4	145,192	295.9	4,062	12.5

Table 7.3.1

7.3.3 Other techniques

Two other changes were made to the basic algorithm to improve its efficiency, both of which enable the user to restrict the search in some way. First, the initial value of LIMIT was placed under direct control of the user. Some searches may begin by finding a solution containing as many as 50 blocks, and then the size of the solution is reduced one block at a time until the minimal design, which may contain only 4 or 5 blocks, is found. If such a search is limited to, say, 10 levels from the very beginning, a substantial saving in time will often result. In many cases, a fairly good upper bound on the number of blocks in the minimal design is known, enabling the user to effectively restrict the depth of the search.

The second modification allows the user to specify an initial set of blocks. Often, theoretical methods can be used to determine that a set of blocks with a certain structure must appear in the minimal design. If the search can be started from this point, the result will be obtained much more rapidly.

The only technique which was not used is that of "search re-arrangement". The most obvious improvement of this kind could be made in step [1]. Instead of choosing any L-set which has not yet been covered, choose the one which is covered by the smallest number of non-rejected K-sets. This will decrease the number of branches in the tree at each point, and therefore will improve the speed of the search. However, the cost of keeping track of the number of non-rejected K-sets which covers each L-set is prohibitive. To do this, whenever a K-set is rejected, all of the L-sets which are covered by that K-set must be determined. This is much too time-consuming, and the amount of overhead that is involved more than makes up for any reduction in the size of the tree.

7.4 Conclusions

The algorithm described in the previous sections was implemented in assembler language on a PDP11/45 computer. A considerable amount of time and effort was spent in optimizing the algorithm, and implementing it as efficiently as possible. Despite this, the number of cases which could be solved in a reasonable amount of time was disappointing. The size of the search grows so rapidly as the parameters increase that even a very efficient backtrack search soon runs into difficulties. The addition of isomorphism testing, which resulted in a drastic reduction of the search times, as shown in Table 7.3.1, nevertheless allowed only a relatively small number of cases to be solved which could not have been solved by the previous versions.

This does not mean that computer search techniques will be of little use in determining the value of B(T,K,L,V) for additional sets of parameters. In many cases, sufficient theoretical results may be obtained to allow an efficient search program to be written for a restricted set of parameters. Any general program which is capable of handling any values of T, K, L, and V, however, will necessarily be fairly limited in scope.

APPENDIX I - SUPPLEMENTARY PROOFS

In this appendix, several supplementary proofs are presented. The results of these theorems have been used in previous chapters, but the proofs were not considered suitable for inclusion in the chapters themselves.

In Chapter 3, it was stated that the minimal (2,2,L,V) design occurred when the varieties were partitioned as evenly as possible. This result is proved by the theorem below.

<u>Theorem</u> A complete L-partite graph on v vertices has the maximum number of edges when the L subsets of the vertices all have cardinality $\lfloor v/L \rfloor$ or $\lceil v/L \rceil$.

<u>Proof</u> Let the cardinalities of the L subsets be

$$n(1),\ n(2),\ \ldots\ ,n(L) \quad \text{where} \quad \sum_{i=1}^{L} n(i) = v.$$

The number of edges in the complement of the graph is then

$$N = \sum_{i=1}^{L} \binom{n(i)}{2} = \sum_{i=1}^{L} \frac{n(i)^2}{2} - \sum_{i=1}^{L} \frac{n(i)}{2} .$$

The number of edges in the partite graph will be maximized when this quantity N is minimized. Suppose that N is minimized, but that the caradinalities of two of the subsets, say n(L) and n(L-1), differ by at least two.

Now let n(1) through n(L-2) remain fixed and let n(L-1) vary. n(L) will now be a function of n(L-1). The number of edges becomes a quadratic function of n(L-1) which is minimized when

$$n(L-1) + n(L)\frac{d\ n(L)}{d\ n(L-1)} - \frac{1}{2} - \frac{1}{2}\frac{d\ n(L)}{d\ n(L-1)}$$

$$= n(L-1) - n(L) = 0.$$

Since n(L-1) and n(L) must be integral, the minimum occurs either when they are equal (if their sum is even), or when they differ by exactly 1 (if their sum is odd). But it was assumed that N was minimized and that n(L) and n(L-1) differed by at least 2. This is a contradiction and therefore the cardinalities of any two subsets can differ by at most 1. This can only happen when $n(i)=\lceil v/L \rceil$ or $\lfloor v/L \rfloor$, for all i. Q.E.D.

The following theorem was used in section 5.2 to aid in establishing the upper bound on B(2,4,4,V).

<u>Theorem</u>

Let $f(x,y) = ax^2 + by^2 + cxy + dx + ey + f \quad (a>0, b>0)$.

Let the function have a single minimum for $x=x^1$, $y=y^1$.

Then if $|c| \le a$ and $|c| \le b$, the minimal value of the function for integers x and y will occur when $x = \lceil x^1 \rceil$ or $\lfloor x^1 \rfloor$ and $y = \lceil y^1 \rceil$ or $\lfloor y^1 \rfloor$.

Proof For a fixed value of y, the function becomes a simple quadratic function of x with a single minimum when $2ax+cy+d=0$ or $x=(-cy-d)/2a$. The minimum value when $y=y^1$ occurs when $x=x^1$, and so the minimum for $y=y^1+t$ will occur for $x=x^1-(c/2a)t$. Since $|c| \le a$, this implies that the minimum occurs when $|x-x^1| \le |t|/2$. Similarly, the minimum value when $x=x^1+t$ will occur when $|y-y^1| \le |t|/2$.

Consider the behaviour of the function when $x = \lceil x^1 \rceil$ or $\lfloor x^1 \rfloor$. The minimum value in either case will occur for $|y-y^1| \le 1/2$, and therefore the minimum value which can be obtained for an integral value of y will occur when $y = \lceil y^1 \rceil$ or $\lfloor y^1 \rfloor$. Similarly, if $y = \lceil y^1 \rceil$ or $\lfloor y^1 \rfloor$, the integral value of x which minimizes the function will be either $\lceil x^1 \rceil$ or $\lfloor x^1 \rfloor$.

Now consider an arbitrary integral point (x^*, y^*). Without loss of generality, assume that

$$|x^* - x^1| \le |y^* - y^1| \ .$$

The minimum value of the function when $x=x^*$ will occur when

$$| y-y^1 | \le |x^* - x^1|/2 \le |y^* - y^1|/2.$$

Therefore either $(x^*, \lceil y^1 \rceil)$ or $(x^*, \lfloor y^1 \rfloor)$ will be at least as close to the minimum as (x^*, y^*), and therefore will give at least as small a function value. But, as shown earlier, either $x = \lceil x^1 \rceil$ or $\lfloor x^1 \rfloor$ will give the minimum function value when $y = \lceil y^1 \rceil$ or $\lfloor y^1 \rfloor$, and therefore these values will give the overall minimum value of the function for integers x and y. Q.E.D.

APPENDIX II - TABLES OF B(T,K,L,V)

In this appendix, tables of values of B(T,K,L,V) are given for $V \leq 16$. In order to reduce the size of the tables and the number of duplications, only those sets of parameters in which $V \geq K+L$ are given. Any design in which $V<K+L$ is the complement of one of these designs. The trivial cases in which either only 1 K-set, or all possible K-sets, form the minimal design are also omitted.

The following codes are used in the tables to indicate the source of several of the results. Entries which are unmarked were either determined by a computer search, are part of a known family of designs (these will be noted), or were small enough to find easily by hand.

(1) - a covering number with $V \leq 7K/3$ [9]

(2) - Stanton, Kalbfleisch, Mullin [16]

(3) - Johnson, Krieger [see 9]

T=2,K=2

K	L	V=5	6	7	8	9	10	11	12	13	14	15	16	Remarks
	3	4	6	9	12	16	20	25	30	36	42	49	56	
	4		3	5	7	9	12	15	18	22	26	30	35	
	5			3	4	6	8	10	12	15	18	21	24	
	6				3	4	5	7	9	11	13	15	18	
	7					3	4	5	6	8	10	12	14	
	8						3	4	5	6	7	9	11	
	9							3	4	5	6	7	8	Turan
2	10								3	4	5	6	7	designs
	11									3	4	5	6	
	12										3	4	5	
	13											3	4	
	14												3	

T=2,K=3

K	L	V=6	7	8	9	10	11	12	13	14	15	16	Remarks
	2	6	7	11	12	17	19	24	26	33	35	43	Fort,Hed.
	3	2	4	5	7	8	10	11	13	14	18	19	chap. 4
	4		2	3	3	5	6	7-8	9	10-11	12	13-14	chap. 6
	5			3	3	3	4	4	6	7	8-9	9-10	chap. 6
3	6				2	3	3	4	4	5	5	7	
	7					2	3	3	4	4	5	5	
	8						2	3	3	4	4	5	
	9							2	3	3	4	4	
	10								2	3	3	4	
	11									2	3	3	
	12										2	3	
	13											2	

T=2,K=4

K	L	V=7	8	9	10	11	12	13	14	15	16	Remarks
4	2	5	6	8	9	11	12	13	18	19	20	Mills
	3		2	4	4	6	6	7-8	8-9	10-11	11-12	chap. 6
	4		2	2	3	3	3	5	5	6-7	7	chap. 5
	5			2	2	3	3	3	4	4	4	chap. 6
	6				2	2	3	3	3	4	4	
	7					2	2	3	3	3	4	
	8						2	2	3	3	3	
	9							2	2	3	3	
	10								2	2	3	
	11									2	2	
	12										2	

T=2,K=5

K	L	V=8	9	10	11	12	13	14	15	16	Remarks
5	2	4	5	6	7	9	10	12	13		(1) and (2)
	3		2	2	4	4	5	5-6	6-7	7-8	
	4			2	2	3	3	3	3	5	
	5			2	2	2	3	3	3	3	
	6				2	2	2	3	3	3	
	7					2	2	2	3	3	
	8						2	2	2	3	
	9							2	2	2	
	10								2	2	
	11									2	

T=2,K=6

K	L	V 9	10	11	12	13	14	15	16	Remarks
6	2	3	4	6	6	7	7	10	10	(1)
	3		2	2	2	4	4	4	5	
	4		2	2	2	3	3	3	3	
	5				2	2	2	3	3	
	6				2	2	2	2	3	
	7					2	2	2	2	
	8						2	2	2	
	9							2	2	
	10								2	

T=2,K=7,8

K	L	V 10	11	12	13	14	15	16	Remarks
7	2	3	4	5	6	6	7	8	(1)
	3		2	2	2	2	4	4	
	4			2	2	2	2	3	
	5				2	2	2	2	
	6					2	2	2	
	7					2	2	2	
	8						2	2	
	9							2	
8	2		3	3	4	5	6	6	(1)
	3			2	2	2	2	2	
	4				2	2	2	2	
	5					2	2	2	
	6						2	2	
	7							2	
	8							2	

T=2,K=9,10,11,12,13

K	L	V 12	13	14	15	16	Remarks
9	2	3	3	4	4	5	(1)
	3		2	2	2	2	
	4			2	2	2	
	5				2	2	
	6					2	
10	2		3	3	3	4	(1)
	3			2	2	2	
	4				2	2	
	5					2	
11	2			3	3	3	(1)
	3				2	2	
	4					2	
12	2				3	3	(1)
	2					2	
13	2					3	

T=3

K	L	V	B(T,K,L,V)	K	L	V	B(T,K,L,V)
3	4	7	12 (2)	4	7	11	3
3	4	8	20 (3)	4	7	12	3
3	4	9	30 (3)	4	8	12	3
3	4	10	45 (3)	4	8	13	3
3	5	8	8 (3)	4	9	13	3
3	5	9	12 (3)	4	9	14	3
3	5	10	20 (3)	4	9	15	4
3	6	9	7	4	9	16	4
3	7	10	6	4	10	14	3
3	8	11	5	4	10	15	3
3	9	12	4	4	10	16	4
3	9	13	≥6	4	11	15	3
3	10	13	4	4	11	16	3
3	10	14	6	4	12	16	3
3	11	14	4	5	3	9	12 (3)
3	11	15	5	5	5	10	2
3	11	16	≥7	5	6	11	2
3	12	15	4	5	7	12	2
3	12	16	5	5	8	13	2
3	13	16	4	5	9	14	2
4	3	8	14	5	10	15	2
4	4	8	6	5	11	16	2
4	5	9	5	6	6	12	2
4	6	10	4				

T=4

K	L	V	B(T,K,L,V)	K	L	V	B(T,K,L,V)
5	10	15	3	7	8	15	2
5	11	16	3	7	9	16	2
6	10	16	3	8	8	16	2
7	7	14	2				

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