AN INVESTIGATION OF SUBHARMONIC OSCILLATIONS IN NONLINEAR FEEDBACK SYSTEMS

A Thesis

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Master of Science

by

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ABSTRACT

The object of this thesis was to investigate the following two problems in forced nonlinear feedback systems:

1) Are any approximation methods available for determining which periodic solution will occur for a given set of initial conditions of the system?

2) Can two subharmonics exist simultaneously in the system?

Several new techniques for the study of nonlinear systems are explained and investigated for application to the above problems.

It was concluded from the investigation that a numerical solution of the nonlinear differential equation was the only method available for determining which periodic solution would result for a given set of initial conditions. The investigation also proved that two subharmonics can exist simultaneously in a nonlinear feedback system.

PREFACE

The object of this thesis was to find if there are any methods available for determining which of the possible periodic solutions will occur for a given set of initial conditions in a forced nonlinear feedback system. Only feedback systems which can be reduced to a form having one nonlinear element in the forward path are considered. An answer was also sought to the question: can two subharmonics exist simultaneously in a nonlinear feedback system? An exact solution of the nonlinear differential equation is required for both of these questions. However, the answers to these problems were sought mainly by new approximation techniques for nonlinear systems. Since the older methods, such as the perturbation method, are not generally suited to the analysis of feedback systems, they were avoided unless they were found to be particularly useful.

In Chapter 1 a short discussion on the differences between the response of linear and nonlinear systems is given. Chapter 2 deals with the problem of determining the periodic oscillation resulting from a given set of initial conditions. Hayashi's method for determining the transient response of forced nonlinear systems is given in detail with an example. Since not too much success was achieved in determining the periodic response resulting from a given set of initial conditions, it was decided to simplify the problem by determining regions, in the input amplitude versus input frequency plane, in which a given subharmonic oscillation could exist. Chapter 3 deals with this problem and in particular presents Oldenburger's stability criterion for subharmonic oscillations. In Chapter 4 the special case of relay systems is considered, and Gille's method for determining regions in which a subharmonic oscillation can exist is given. Chapter 5 discusses the problem of the simultaneous occurrence of two subharmonics. The possibility of producing a triple-input describing function for the study of this phenomenon is considered. An approximation method by Atherton for the response of a nonlinearity to multiple inputs by the use of a modified nonlinearity concept is considered in Chapter 6 for its possible application to subharmonic oscillations. Chapter 7 consists of a discussion of the approximation methods that have been considered and on the work done in answering the original problems.

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CHAPTER 1

INTRODUCTION

When a periodic force is applied to a linear system, the resulting response is obtained by a superposition of the transient and steady state components. The former is due to the free oscillations of the system, while the latter is related to the forced oscillation which arise from the action of the external force. Since the free oscillation is normally damped out after a sufficiently long period of time, only the forced oscillation having the same frequency as that of the external force would be observed. Thus, as far as linear systems are concerned, the forced oscillation is uniquely determined once the system and the external force are given, and it is not affected by the initial conditions of the system. In nonlinear systems the theorem of superposition no longer applies and the system can possess a wide variety of periodic oscillations in addition to those which have the same period as the external force. The response of a system is termed subharmonic if its frequency is a proper rational fraction (i.e. 1/2, 1/2...1/n) of the forcing function frequency, and it is called the subharmonic of order n.

Although much work has been done on subharmonics, the problem can not be considered solved. The tedious solution by the perturbation, iteration (7)[1], or similar methods results in little better understanding of the subharmonic behavior, and is not suited to the analysis of the performance of feedback systems. New approximation methods are constantly being sought.

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[1] The bracketed numbers refer to reference in the bibliography.

The purpose of this thesis was to examine the techniques available for the analysis of subharmonic oscillations in nonlinear feedback systems to determine:

(1) For a given set of initial conditions of the system, can one predict which among a variety of possible periods the output of the system will exhibit?

(2) Can the output of the system consist of two or more subharmonics simultaneously?

A method developed by Hayashi for second-order nonlinear systems will be used to study the subharmonic transient response. Since not much success was achieved in the study of the transient response, it was decided to divide the input amplitude versus input frequency plane into regions in which a given subharmonic can and cannot exist. This problem is studied by a method developed by Oldenburger for the stability of subharmonic oscillations. A method by Gille for relay systems is studied with the view of extending the method to more general piecewise-linear systems.

The problem of two subharmonics existing simultaneously is analysed, and the possibility of the use of a triple-input describing function to study this phenomenon is considered. A method developed by Atherton for determining the response of a nonlinearity to several inputs is presented. The use of this method to study the simultaneous occurrence of two subharmonics is considered.

CHAPTER 2

TRANSIENT RESPONSE OF A NONLINEAR FEEDBACK SYSTEM

For a sinusoidal input of given magnitude and frequency, often a number of periodic solutions are possible in some nonlinear feedback systems. The one that will be excited depends on the initial conditions of the system. A method for studying the transient response of a non-linear second-order differential equation has been developed by Hayashi and will be considered here.

2.1 HAYASHI'S METHOD

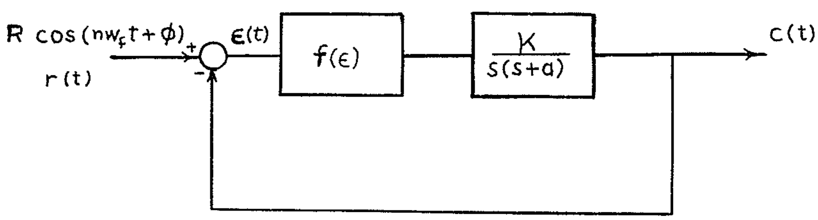

Figure 2.1

Nonlinear Feedback System

Consider the feedback system shown in Figure 2.1. From this system the following nonlinear differential equation can be derived[1].

$$\ddot{\epsilon}(t) + a\dot{\epsilon}(t) + K f(\epsilon) = \ddot{r}(t) + a\dot{r}(t) = w_f^2 B \cos(nw_f t) \qquad (2.1)^2$$

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[1] The notation $\dot{\epsilon}(t)$ denotes a derivative of the function with respect to the parameter inside the brackets.

[2] The letter w is used throughout this text in place of the Greek symbol omega, ω.

where $B = -n^2 R\,(1 + a^2/[n^2 w_f^2])^{1/2}$

$\Theta = \tan^{-1}\,(1/n w_f)$

$\Theta = -\phi$

Now making the following change in the time scale

$$\tau = w_f t$$

then equation 2.1 becomes

$$\ddot{\epsilon}(\tau) + \frac{a}{w_f}\dot{\epsilon}(\tau) + \frac{K}{w_f^2} f(\epsilon) = B\cos(n\tau) \tag{2.2}$$

The equation has now been put into the form of the equation Hayashi investigated for subharmonic oscillations. Hayashi assumed that the coefficient of the $\dot{\epsilon}(\tau)$ term, a/w_f, was a comparatively small quantity. For an odd symmetric nonlinearity Hayashi approximated the steady-state solution of equation 2.2 for the subharmonic of order n by

$$\epsilon(\tau) = x\sin(\tau) + y\cos(\tau) + W\cos(n\tau) \tag{2.3}$$

Hayashi then stated that Mandelstam and Papalexi (17) have shown that W may be approximated by

$$W = B/(1-n^2) \tag{2.4}$$

This approximation is said to be legitimate in the case when the nonlinearity is small, but is a good approximation even when the departure from linearity is large.

Hayashi uses equation 2.3 as a basis for his study of the transient response of subharmonic oscillations. He assumed the transient solution of equation 2.2 to be

$$\epsilon(\tau) = x(\tau)\sin(\tau) + y(\tau)\cos(\tau) + W\cos(n\tau) \tag{2.5}$$

At the equilibrium points $x(\tau)$ and $y(\tau)$ are constants. Therefore, $\dot{x}(\tau)$ and $\dot{y}(\tau)$ are zero at the equilibrium points.

Equation 2.5 was then substituted into equation 2.2. If $x(\tau)$ and $y(\tau)$ are assumed to be slowly varying functions, then $\ddot{x}(\tau)$ and $\ddot{y}(\tau)$ are small and may be omitted in the resulting equation. The sine and cosine terms of the resulting equation were then rearranged into groups according to their frequency. For this equation to be satisfied at all times the coefficient of any sinusoidal component on the left hand side of the equation must be set equal to the coefficient of the identical sinusoidal component of the right hand side of the equation. Thus, the equation can be reduced to a number of simultaneous nonlinear equations from which the sinusoidal terms can be eliminated.

Since the subharmonic of order n is of main interest, it was assumed that the solution of the simultaneous equations resulting from the sine (τ) the cosine (τ) terms was a good approximation to the simultaneous solution of the entire set of equations. From these two equations, two equations of the following form were derived:

$$dy/d\tau = f_1(x,y,w_f) = Y(x,y) \tag{2.6}$$

$$dx/d\tau = f_2(x,y,w_f) = X(x,y) \tag{2.7}$$

A steady state oscillation occurs when:

$$dy/d\tau = dx/d\tau = 0 \tag{2.8}$$

The parameter, time, can now be eliminated if equation 2.7 is divided into equation 2.6

$$\frac{dy/d\tau}{dx/d\tau} = \frac{dy}{dx} = \frac{Y(x,y)}{X(x,y)} \tag{2.9}$$

The problem has now been reduced to the plotting of a state-space to determine the equilibrium points of equation 2.2.

2.2. AN EXAMPLE

Consider a nonlinear feedback system incorporating a cubic non-linearity in the forward path - see Figure 2.2, page 8.

Hayashi's method was applied for the one-third subharmonic for this system. The equation for the system can be written in the form of equation 2.2, viz:

$$\ddot{\epsilon}(\tau) + \dot{\epsilon}(\tau)/W + \epsilon^3(\tau)/(10W^2) = B\cos(3\tau) \qquad (2.10)$$

Equation 2.5 was substituted into equation 2.10, and then $\ddot{x}(\tau)$ and $\ddot{y}(\tau)$ were assumed to be small enough to be omitted from the resulting equation. From the coefficients of the sine (τ) and cosine (τ) components in the equation, the following two equations were derived.

$$2\dot{x} + \dot{y}/W = y - x/W - 3[2yW^2 + W(y^2-x^2) + y^3 + yx^2]/(40W^2) \qquad (2.11)$$

$$\dot{x}/W - 2\dot{y} = x + y/W - 3[2xW^2 - 2xyW + x(x^2+y^2)]/(40W^2) \qquad (2.12)$$

From equation 2.11 and 2.12, $dy/d\tau$ and $dx/d\tau$ can be solved for. Then, with the aid of a digital computer, a state-plane can be plotted by calculating dy/dx for specified **x** and y.

The state-plane was plotted for an input magnitude of 4 and frequency of 18 rps - see Graph 2.1, page 9. From this graph it can be seen that there are three stable equilibrium points for the one-third subharmonic. The three equilibrium points are approximately 120 degrees apart on a circle of radius 22.5.

The system was simulated on an analogue computer - see Figure 2.4 - to determine experimentally the initial conditions which result in a subharmonic response. It should be noted that the initial conditions by the Hayashi method are not the same as the initial conditions set on the analogue computer:

$$\epsilon(0) = R\cos(\phi) - C(0) = y(0) + W \qquad (2.13)$$

Note that: $\dot{\epsilon}(t)\big|_{t=0^+} = W\dot{\epsilon}(\tau)\big|_{\tau=0^+}$

$$\dot{\epsilon}(t)\big|_{t=0^+} = -3WR\sin(\phi) - \dot{C}(0)$$

Therefore $\dot{\epsilon}(\tau)\big|_{\tau=0^+} = -3R\sin(\phi) - \dot{C}(0)/W = x(0) + \dot{y}(0) \qquad (2.14)$

It was assumed that since ϕ was small (approximately 3 degrees), the term $3R\sin(\phi)$ could be neglected. Since W, approximately 9R/8, is approximately equal to R cos(ϕ), it was assumed that the two cancel one another in equation 2.13. The equations 2.13 and 2.14 can now be written in the form

$$y(o) = -c(o) \tag{2.15}$$

$$x(o) = -\dot{y}(o) - \frac{\dot{c}(t)}{w}\Big|_{t=o^+} \tag{2.16}$$

Since there was no information available about $\dot{y}(o)$, for comparison purposes it was assumed that $\dot{y}(o)$ could also be omitted from the equation. The regions of initial conditions which gave rise to the one-third subharmonic were determined experimentally and plotted as -c(o) versus $-\dot{c}(o)/w$ - see Graph 2.2, page 10 - for comparison with the state-plane obtained by Hayashi's method.

Comparison of Graphs 2.1 and 2.2 show that although the two graphs agree in basic form, the two differ quite substantially in details. The amplitude of the subharmonic was found experimentally to be approximately 23 as compared to 22.5 by Hayashi's method. Thus, the steady state approximation to the solution of Hayashi's equation appears to give reasonable results.

2.3 DISCUSSION OF HAYASHI'S METHOD

Hayashi's method was the only method found which could be used to determine which periodic solution a set of initial conditions would produce. However, the method can not be considered very satisfactory for the following reasons:

(1) Hayashi's method is limited to second-order nonlinear systems.

(2) The amount of work associated with the use of Hayashi's method makes the method of little use for a detailed analysis of a system. The complexity of the equations for Y(x,y) and X(x,y) rapidly increases as the power of the equation for the nonlinearity is increased. The plotting of the state-plane is very time consuming, and it must be replotted if any

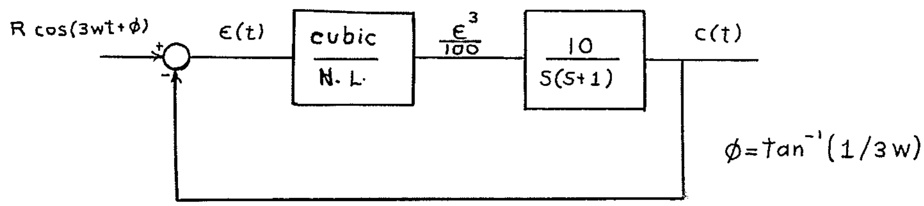


Figure 2.2

Cubic Nonlinearity Feedback System

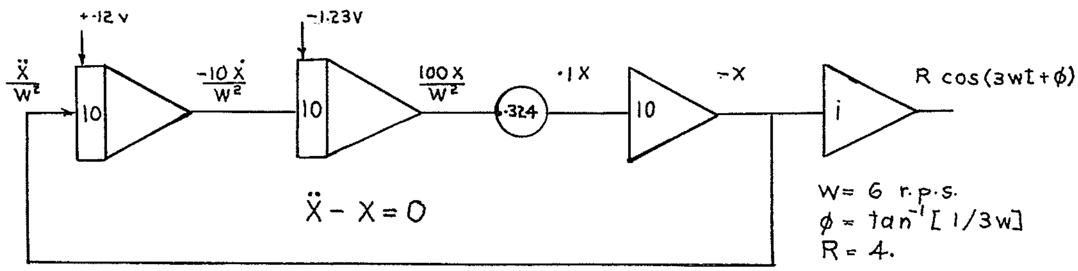


Figure 2.3

Analogue Computer Circuit For $R\cos(3wt+\phi)$

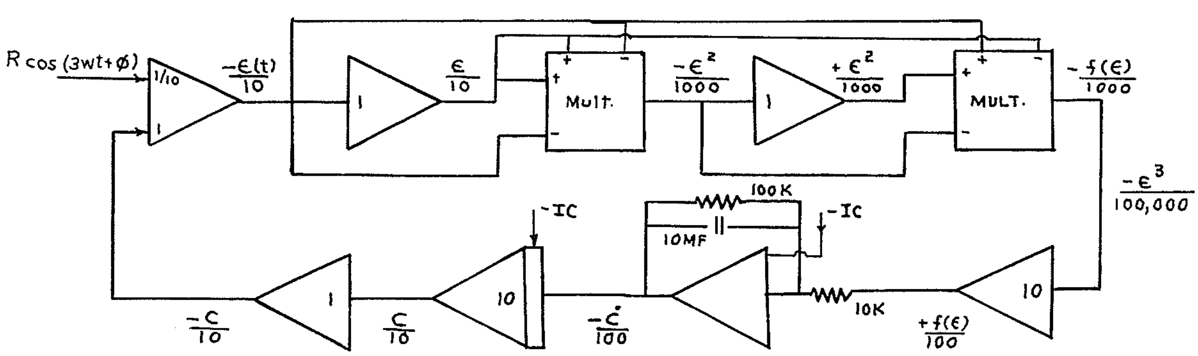
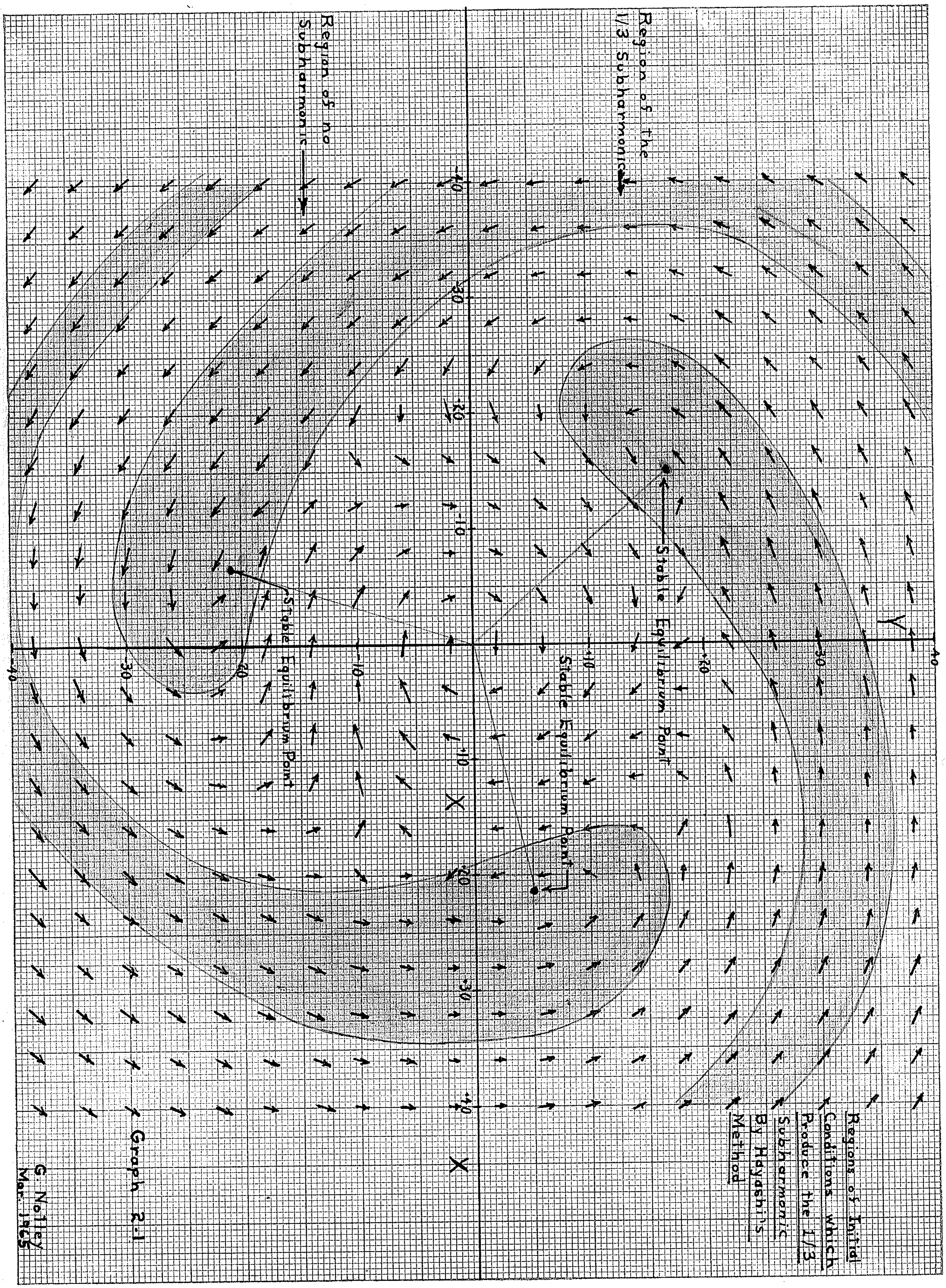
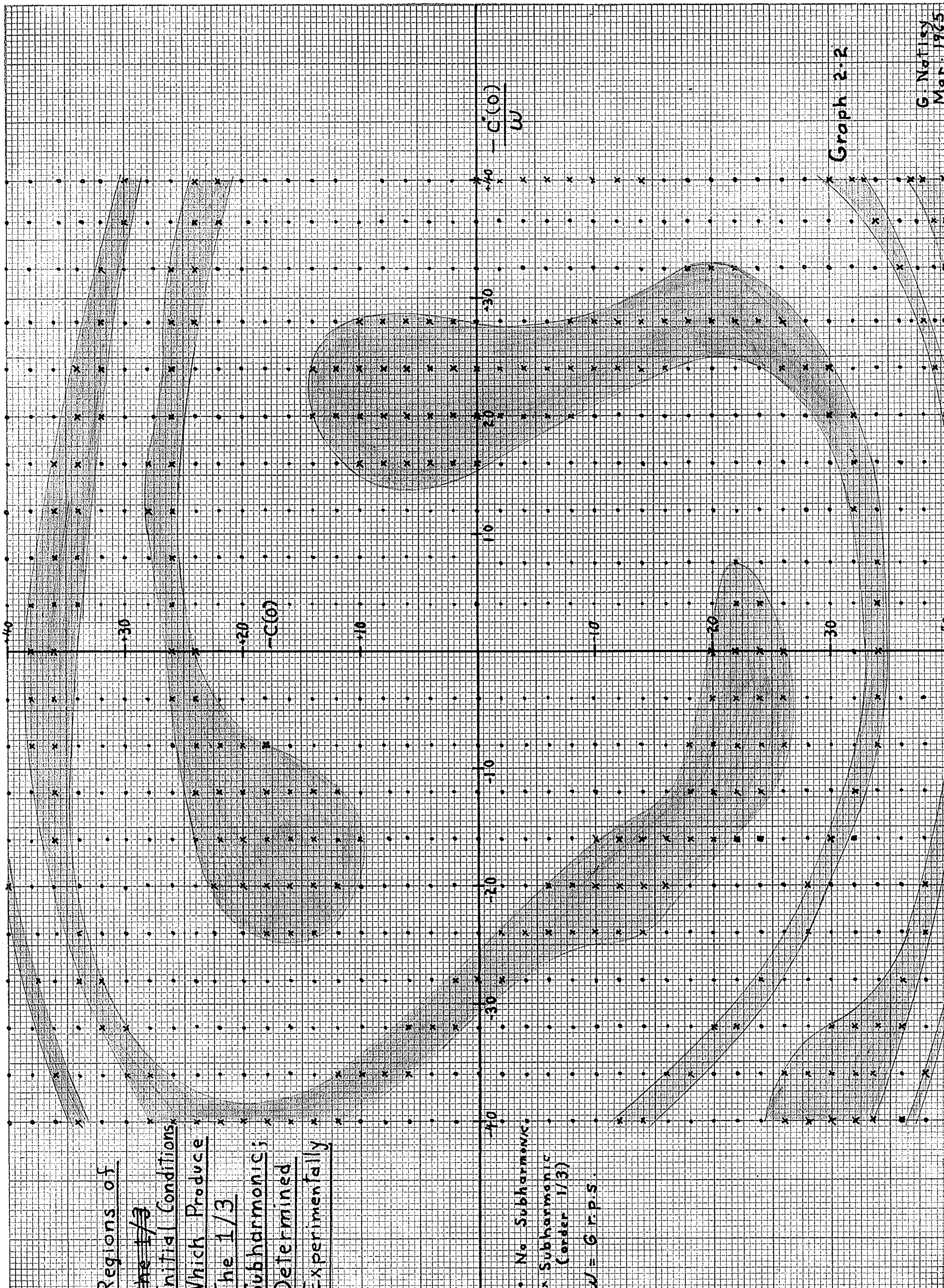


Figure 2.4

Analogue Simulation of Nonlinear System


Region of the 1/3 Subharmonic
Region of no Subharmonic
Stable Equilibrium Point
Stable Equilibrium Point
Stable Equilibrium Point
X
Y
Regions of Initial Conditions which Produce the 1/3 Subharmonic By Hayashi's Method
Graph 2.1
G. Notley
Mar. 1965


Graph 2.2
G. Notley
Mar. 1965
Regions of ~~the 1/3~~ Initial Conditions Which Produce The 1/3 Subharmonic; Determined Experimentally
• No Subharmonic
× Subharmonic (order 1/3)
ω = 6 r.p.s.
C(0)
−Ċ(0)/ω
40
30
20
10
10
20
30
40

of the system parameters, such as frequency, are changed. For all the work that is required, the method appears, from Graph 2.1 and 2.2, to give results that are not very accurate.

(3) The initial conditions given by Hayashi's method are not the true initial conditions of the system. As seen by the example assumptions must be made to find a direct relation between the two.

(4) The method can only be used for systems which have just one possible subharmonic. Although the method could be applied to systems with more than one subharmonic, the results would be meaningless since any given state-plane would neglect the rest of the subharmonics. The individual state-planes could not be superimposed on one another since there would be no way of determining which plane applied for a given initial condition.

The problem of determining the periodic solutions of a feedback system for a given set of initial conditions appears to require an exact solution of the nonlinear differential equation for the system. The differential equation could be solved by one of the numerical techniques given by Cunningham (7), but the methods are not suited to a general analysis of a feedback system. The method would require a digital computer, and it would be necessary to solve the equation every time a change in one of the parameters of the system was made.

CHAPTER 3

REGIONS OF SUBHARMONIC RESPONSE OF A FEEDBACK SYSTEM CONTAINING ONE SINGLE-VALUED NONLINEARITY

The problem of determining regions, on the input amplitude versus input frequency plane, in which the subharmonic of order n exists, for a general single-valued, odd-function nonlinearity will be considered in this chapter. A method outlined by Oldenburger (11) for the study of subharmonic oscillations will be investigated here.

3.1 THE PROBLEM

By the use of the perturbation method or iteration procedure, the regions in which a subharmonic may exist can be determined from the differential equation for the system. However, all constants - such as gains and time constants - must be specified for the system before either of these methods can be used. This procedure is not suited to the study of feedback systems since it generally gives no information on the stability of the solution or the effect of varying one of the parameters of the system. In addition these methods are generally rather time consuming and make assumptions about the equation - such as that the system is quasi-linear - which may or may not be completely valid. For the above reasons a more general procedure was desired.

3.2 OLDENBURGER'S METHOD

Output of the Nonlinear Element: This method is restricted to feedback systems incorporating one, single-valued, odd-function nonlinear element - see Figure 3.1.

Consider a nonlinear element with an input x_1 as shown in Figure 3.1 where

$$x_1 \text{ is } A \sin\left((kwt) + \phi\right) + B \sin (wt) \qquad (3.1)$$

and k is an odd integer

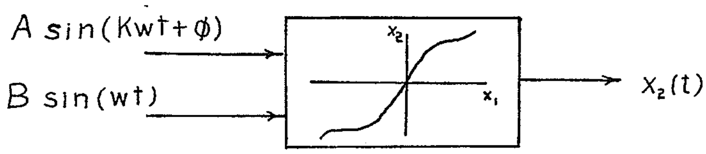


Figure 3.1

Odd - Function Nonlinearity

The output of the nonlinear element $x_2(t)$ can be written as a function of the input $x_i(t)$ using the integral representation of Rice (12). Thus

$$x_2 = \int_{-\infty}^{\infty} F(ju)\exp(jux_1)\,du \tag{3.2}$$

where $\quad j$ is $\sqrt{-1}$

and $\quad F(ju)$ is the Fourier transform of the nonlinear input-output function $f(x_1)$.

By the use of equations 3.1 and 3.2 the following equations are derived in Appendix A, where $G_1(u,w)$ and $G_2(u,w)$ are the portions of ~~the~~ $\exp(jux_1)$ which contribute to ~~the~~ the sine and cosine terms respectively of the fundamental component in the output.

$$G_1(u,w) = j\left\{\sum_{n=0}^{\infty}\epsilon_n(-1)^n J_n(Au)\left[J_{nK-1}(Bu) - J_{nK+1}(Bu)\right]\cos(n\theta)\right\} \tag{3.3}$$

$$G_2(u,w) = j\left\{\sum_{n=0}^{\infty}\epsilon_n(-1)^{n+1} J_n(Au)\left[J_{nK-1}(Bu) + J_{nK+1}(Bu)\right]\sin(n\theta)\right\} \tag{3.4}$$

where $\quad J_i(u)$ is the Bessel function of order i and modulus u.

and $\quad \epsilon_n$ is the Neumann factor

$= 1\ ,\ n = 0$

$= 2\ ,\ n = 1, 2, 3.....$

By the use of these expressions the fundamental part of the output may be written as (see Appendix A):

$$(x_2)_f = \int_{-\infty}^{\infty} F(ju)\,G_1(u,w)\sin(wt)\,du + \int_{-\infty}^{\infty} F(ju)\,G_2(u,w)\cos(wt)\,du \tag{3.5}$$

Thus, Oldenburger has succeeded in obtaining an expression for the output of the nonlinearity with two harmonically related inputs.

Equilibrium Points and Stability Criterion: Having obtained a method for determining the output of the nonlinearity, Oldenburger now considers the problem of determining equilibrium points and the stability of the equilibrium points.

Consider a system as shown in Figure 3.2 incorporating a single-valued, odd-function nonlinear element, and a linear element possessing a transfer function $G(jw)$. In the following the assumption that only the subharmonic appears in the output will be implicit. Thus the characteristic of the linear element must be that of a low-pass filter, and only the component of the output of the nonlinearity at the subharmonic frequency need be considered.

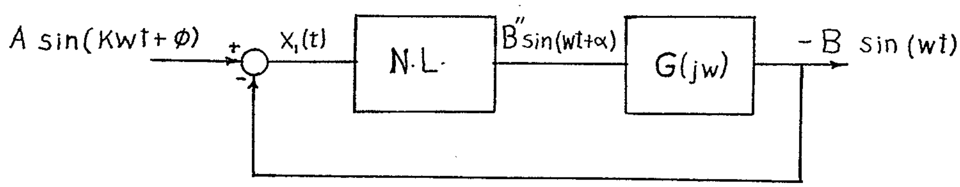


Figure 3.2

Feedback System

The output of the nonlinearity can be represented by a complex number B', with real and imaginary parts Re B' and Im B' respectively, where the magnitude of B' is B'', and $\tan^{-1}[\mathrm{Im}B'/\mathrm{Re}B']$ is α. Now

$$\mathrm{Re}\ B' \text{ is } \int_{-\infty}^{\infty} F(ju)\, G_1(u,w)\, du \tag{3.6}$$

$$\mathrm{Im}\ B' \text{ is } \int_{-\infty}^{\infty} F(ju)\, G_2(u,w)\, du \tag{3.7}$$

To sustain a continuous oscillation of frequency w the following relation must hold:

$$B' = -B/G(jw) \tag{3.8}$$

where B' is a function of B, A, ϕ, and the parameters of the nonlinear element. Solutions of equation 3.8 can be found graphically from intersections of curves representing B' and $B/G(jw)$ plotted on the complex plane.

Not all intersections represent stable solutions and thus use will be made of the incremental Nyquist diagram (13) and equivalent linearization about a point to determine the stability of the oscillation at the equilibrium points. This method is explained more completely in Appendix B.

3.3 DISCUSSION OF OLDENBURGER'S METHOD

Oldenburger gave a very complicated procedure for determining the output of the nonlinear element subject to an input consisting of two harmonically related sinusoids. However, to determine equilibrium points he assumed that only the subharmonic appeared at the output of the system due to the filtering action of the linear block. Therefore, he only considered the subharmonic component at the output of the nonlinearity. This is basically a describing function approach to the problem and could have been handled by West's dual-input describing function (14). The subharmonic output by Oldenburger's method was an infinite series of very complex integrals involving Bessel functions. In general these integrals are very difficult to evaluate.

The stability criterion is an extension of Loeb's criterion for the stability of limit cycles in autonomous systems. This criterion states that if the vector product $\widehat{dG}/dw \times \widehat{dN}/dE$,

where: 1) $\widehat{dG}/dw$ is the vector lying in the direction of increasing frequency along the Nyquist locus of the linear elements at the equilibrium point.

2) $\widehat{dN}/dE$ is the vector lying in the direction of increasing amplitude along the critical locus at the equilibrium point.

is out of the page, the oscillation is stable. This is only an approximate stability criterion for small disturbances in the system.

3.4 EXAMPLE USING THE DUAL-INPUT DESCRIBING FUNCTION

Consider the feedback system shown in Figure 3.2, page 14, where the nonlinearity is $x^3/100$ and the linear block is $10/(s(s+1))$. Assume, as

Oldenburger did, that the output of the system consists of only the 1/3 subharmonic. Then,

$$x_1(t) \text{ is } A\cos(3wt + \phi) + B\cos(wt) \tag{3.9}$$

The describing function for the subharmonic component is:

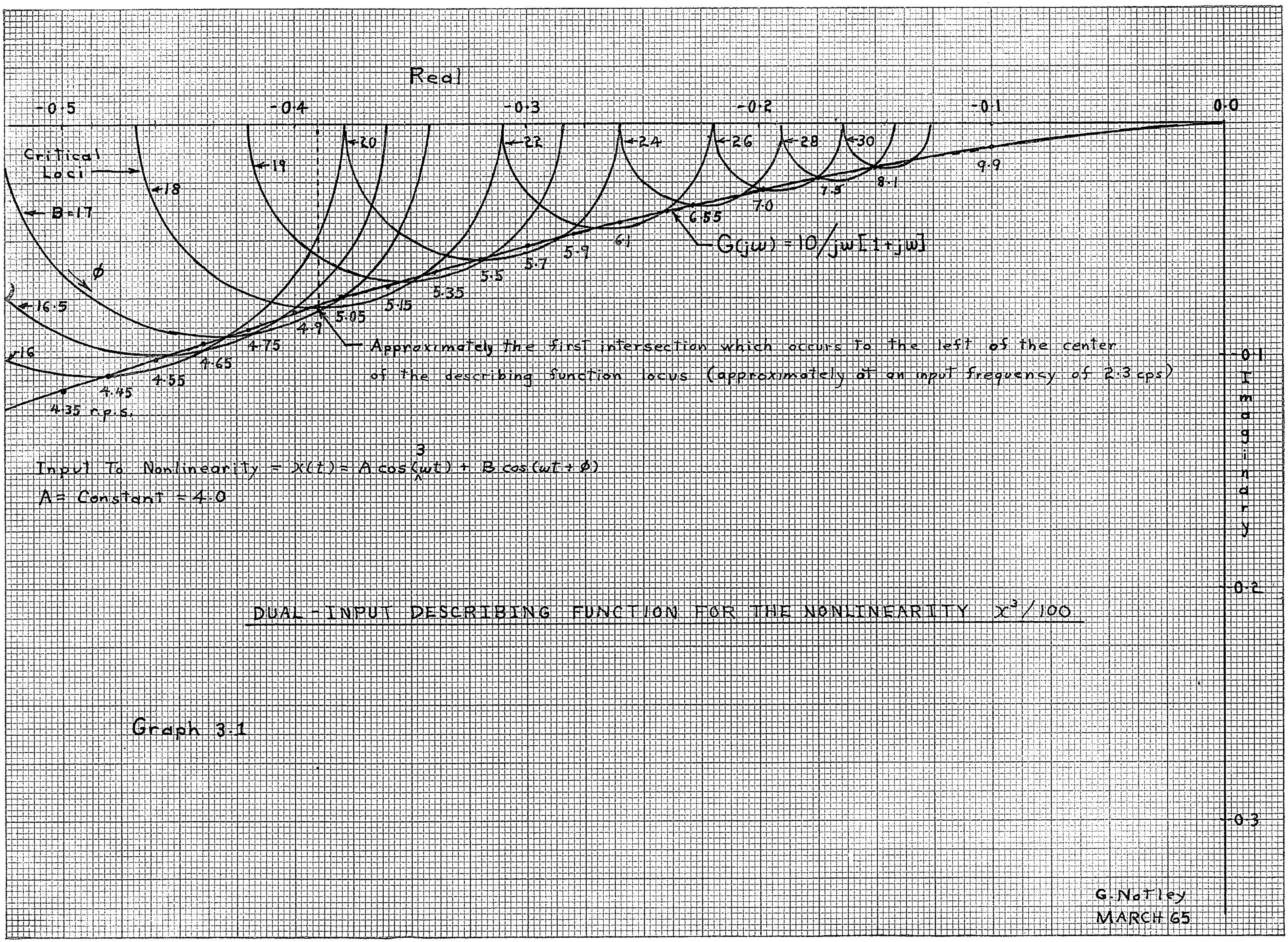
$$D.F. = 3.\{(2A^2 + B^2) + ABe^{j\phi}\}/400 \tag{3.10}$$

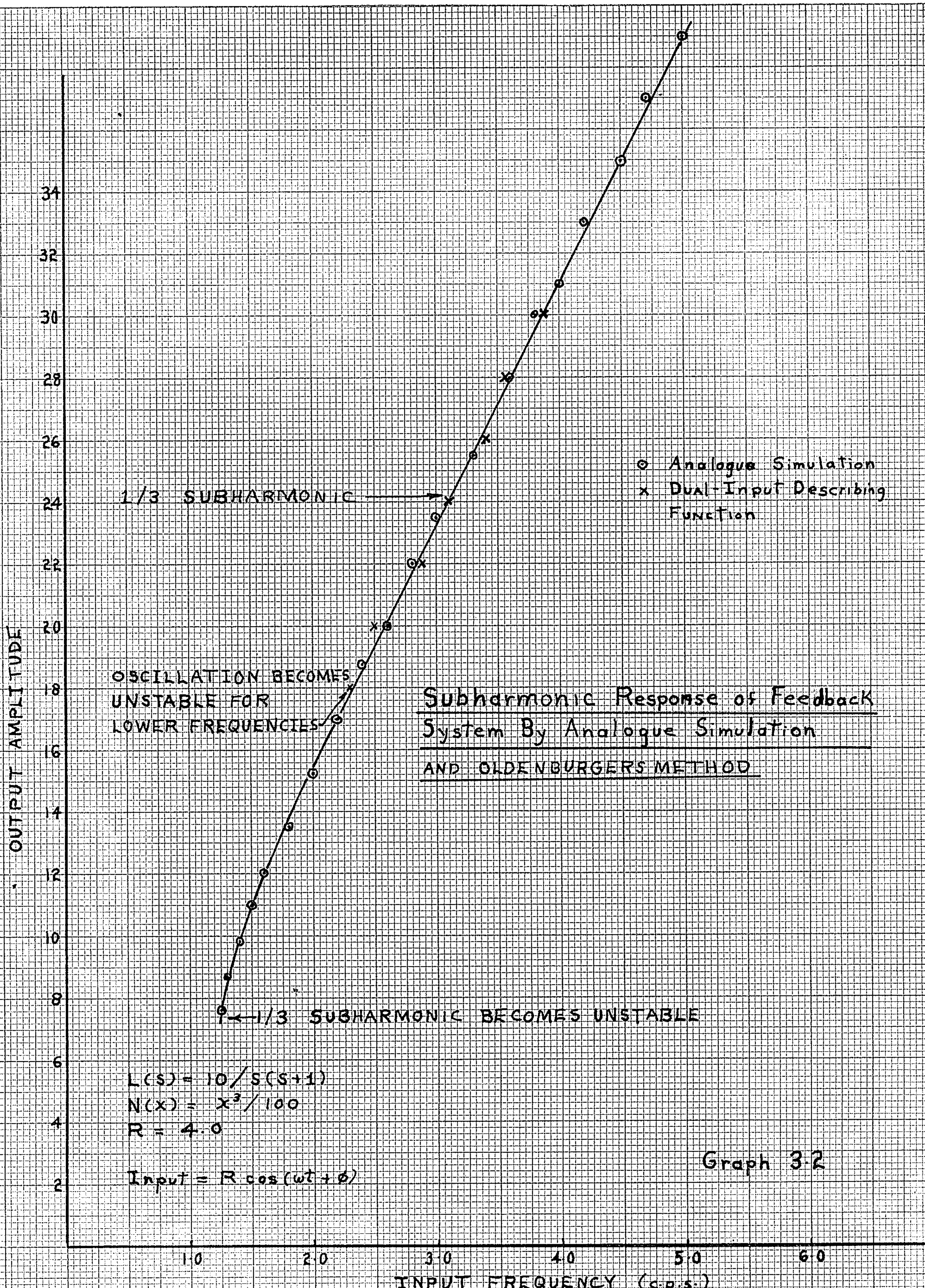
Note that for a constant A and B this forms a circle as ϕ is varied through 360 degrees. To determine equilibrium points plot the critical loci, which are also circles, and G(jw) on the complex plane - see Graph 3.1.

The direction in which the vector $\widehat{dN}/dB$ lies at the equilibrium points must be determined in order to apply Loeb's stability criterion. The vector must point in the direction that the equilibrium point would move on the describing function locus if B was increased incrementally in magnitude. Therefore, the phase relation between A and B should not change. This would indicate that the direction to move at the equilibrium point is perpendicular to the curve of constant B. By applying Loeb's stability criterion to Graph 3.1, it can be seen that any equilibrium point to the left of its loci's center is stable while those to the right are unstable.

By using the above fact, it can be seen from Graph 3.1, page 17, that a stable subharmonic begins at an input frequency of 2.3 cps with an amplitude of eighteen. There is no upper limit at which frequency the subharmonic will no longer be stable.

The system was simulated on an analogue computer and it was found that the predicted and the actual results agree closely for the higher frequency range - see Graph 3.2, page 18. The lower frequency range did not agree with the predicted values very closely; the subharmonic oscillation became unstable at 1.2 cps., not 2.3 cps. as predicted. This discrepancy can be explained by the fact that at the lower frequencies the output contained a rather large component at the forcing frequency. The input to the system and the forcing function frequency component of the input to the nonlinearity are not the same due to the forcing function frequency component fed back


Real
-0.5
-0.4
-0.3
-0.2
-0.1
0.0
Critical Loci
B=17
18
19
20
22
24
26
28
30
16.5
16
φ
4.35 r.p.s.
4.45
4.55
4.65
4.75
4.9
5.05
5.15
5.35
5.5
5.7
5.9
6.1
6.55
7.0
7.5
8.1
9.9
G(jω) = 10/jω[1+jω]
Approximately the first intersection which occurs to the left of the center of the describing function locus (approximately at an input frequency of 2.3 cps)
Imaginary
-0.1
-0.2
-0.3
Input To Nonlinearity = x(t) = A cos(ωt) + B cos(ωt + φ)
A = Constant = 4.0
DUAL-INPUT DESCRIBING FUNCTION FOR THE NONLINEARITY x³/100
Graph 3.1
G. Notley
MARCH 65



G. NOTLEY
MARCH 65

from the output of the system. Thus, the indicated equilibrium point was not a valid equilibrium point for the assumed input to the system.

3.5 CONCLUSIONS

Although Oldenburger's method appears to be a new approach to the problem of subharmonic oscillations, closer inspection reveals that it is based on two well-known nonlinear techniques; the dual-input describing function and Loeb's stability criterion.

The main drawback of Oldenburger's method is the amount of data which must be handled for determining equilibrium points by the dual-input describing function method. To be strictly correct in determining the equilibrium points for a given input, the input frequency component which is fed back to the input of the nonlinearity from the output of the system must be taken into account. Although this correction can be made for each equilibrium point, it requires a great deal of work. Due to the low-pass characteristic of the linear elements, this correction is often small and may be omitted.

Oldenburger's stability criterion provides a very useful method for determining the regions in which subharmonic oscillations exists. The regions are not strictly correct if the assumption that only the subharmonic exists at the output is made. However, the flexibility of the method to changes in the linear elements makes the method very attractive for feedback systems.

The paper by Oldenburger can be criticized for presenting a useful idea in a complicated form, adding nothing to the original idea, which could have been explained using conventional techniques associated with feedback systems.

CHAPTER 4

SUBHARMONIC RESPONSE OF RELAY CONTROL SYSTEMS TO SINUSOIDAL INPUTS

In this chapter a method, developed by Gille (4), for determining regions in which subharmonic oscillations are possible in relay feedback systems, is studied. Although most of this work has been done by Gille and his associates, a paper by Fleishman (5) investigating the same subject suggests Gille's method. These methods will be studied with the thought of extending them to more general piecewise-linear systems.

4.1 FLEISHMAN'S METHOD

Consider a feedback system which has a relay in the forward path as shown in Figure 4.1. Assume the relay is symmetric with negligible dead zone and no hysteresis.

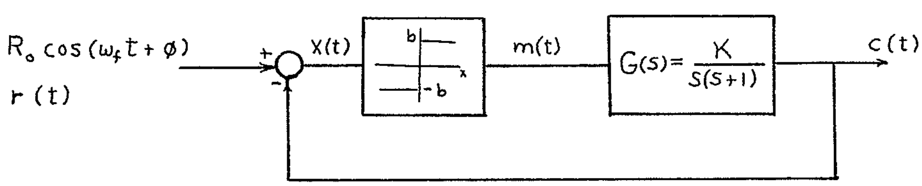


Figure 4.1

Nonlinear System Containing a Relay in The Forward Path

The relay characteristic is given by:

$$m(t) = b \quad x > o$$

$$= -b \quad x < o$$

From the linear block the following equation can be obtained:

$$Km(t) = L(c) \tag{4.1}$$

in which $L(c)$ is a linear operation on $c(t)$ which can be determined from $G(s)$. The following equation can be derived for the system:

$$L(x) + Kb\ \mathrm{sgn}(x) = L(r) \tag{4.2}$$

Assume that $x(t)$ consists of two components, viz:

$$x(t) = x_a(t) + x_f(t) \tag{4.3}$$

Now equation 4.2 can be written as follows since $L(x)$ is a linear operation.

$$L(x_a) + L(x_f) + Kb\ \mathrm{sgn}(x) = L(r) \tag{4.4}$$

Therefore, one possible solution to equation 4.4 can be written as

$$L(x_a) = -Kb\ \mathrm{sgn}(x) \tag{4.5}$$

$$L(x_f) = L(r) \tag{4.6}$$

From equation 4.6 obviously there results:

$$x_f(t) = r(t) = R_o \cos(w_f t + \phi) \tag{4.7}$$

It is desired to find the forced periodic solutions for equation 4.2; therefore $x_a(t)$ must be periodic with a period of $2\pi/w_f$. If $x_a(t)$ is to be periodic, the relay must commutate at particular values of $x_a(t)$ which are determined by the period of the oscillation. The problem is now reduced to solving equations 4.5 and 4.6 under the constraint that the relay commutates at the points determined by equation 4.5 for periodic solutions for $x_a(t)$.

The method may easily be extended to subharmonic oscillations by requiring that the period of $x_a(t)$ be $2\pi n/w_f$, where n, an integer, is the order of the subharmonic solution.

Fleishman then proceeded to solve equations 4.5 and 4.6, but his technique was to find particular solutions for the system being considered.

4.2 GILLE'S METHOD

Gille's method essentially begins where Fleishman's method ended. The method does not go through the above argument but resorts to a graphical argument which is essentially the same.

Consider the same system as Fleishman does - see Figure 4.1, page 20. The following discussion will start with the equations developed by Fleishman instead of the beginning of the graphical argument of Gille. The point that Fleishman appears to have missed is that an identical equation to equation 4.5 could have been obtained if the system had been autonomous. Periodic oscillations for this autonomous system can be determined by the well known Hamel locus method (6).

Figure 4.2, page 23, shows the Hamel locus for the linear block shown in Figure 4.1. Also shown in Figure 4.2 is the path s(t) that the autonomous system is assumed to traverse if the relay commutated at point A. The time required for the system to traverse the curve s(t) to point B, after having commutated at point A, is π/w_f seconds.

The Hamel locus gives all the possible solutions to equation 4.5. It is now necessary to make the relay commutate such that the commutation points assumed in deriving the Hamel locus occur for the forced system. Note that at a fixed instant of time, say for t = o, then:

$$x(o) = x_a(o) + R_o \cos(\phi)$$

$$\dot{x}(o) = \dot{x}_a(o) - R_o \sin(\phi)$$

which is a closed path in the $\dot{x}$, x plane as ϕ is varied through 360° - see Figure 4.3, page 23. By proper choice of units for x, this closed path is a circle.

As can be seen from Figure 4.3, x(t) intersects the commutation line twice - at C and D. Therefore it is possible to get the relay to commutate at the point A on the Hamel locus. Once the relay has commutated, x(t) is just the sum of s(t) and $R_o \cos(w_f t + \phi)$ and is represented by the path P_f shown in Figure 4.3.

It should be noted from Figure 4.3 that unless R_o is equal to or greater than x_o the relay cannot be made to commutate at point A, and no

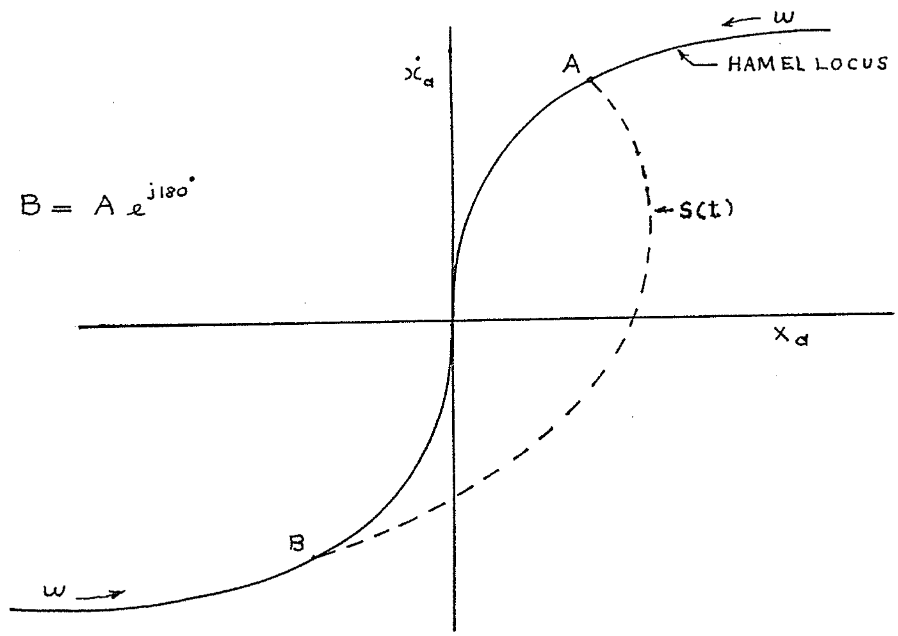


Figure 4.2

Hamel Locus of $1/s(s+1)$

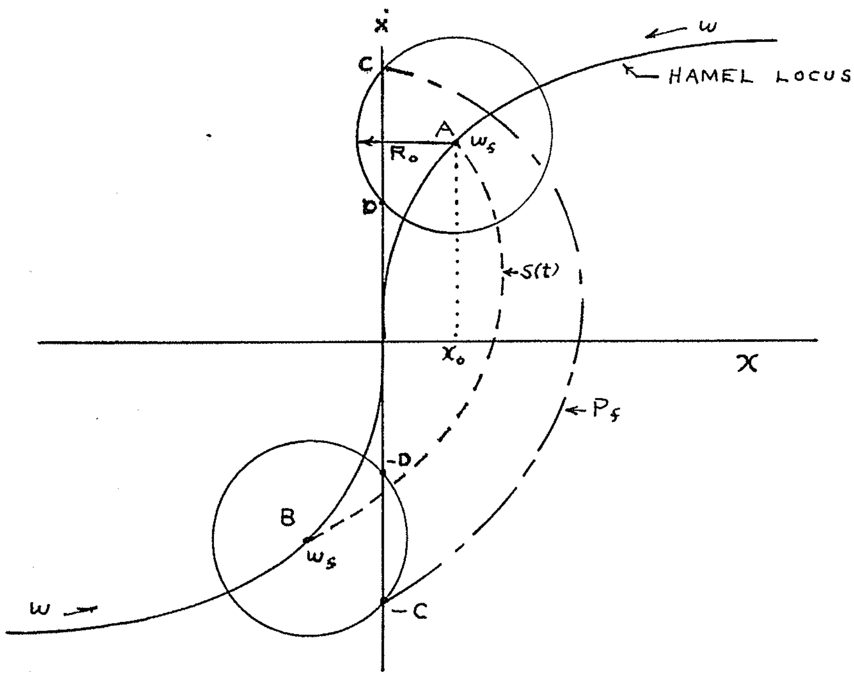


Figure 4.3

Generation of Forced Oscillations

forced periodic oscillation exists in the system. For an input greater than x_o - see Figure 4.3 - there are two points at which the relay may commutate, indicating that two amplitudes for the oscillation are possible in the system. Tsypkin (15) has shown that the smaller amplitude of oscillation is unstable. If the input is larger than the minimum allowable amplitude, F_f, an intersection with the commutation line can always be obtained by adjusting the phase of the input. F_f can be determined as a function of frequency and plotted on input magnitude versus the logarithm of the input frequency axes thus dividing the plane into two regions: $F < F_f$ in which no forced oscillations are possible and $F > F_f$ in which forced oscillations may exist - see Figure 4.4., page 26.

The method can easily be extended to determine the regions in which the subharmonic of order n is possible. Fleishman's equation still applies but now the period of $x_a(t)$ is $2\pi n/w_f$. However, the Hamel locus is graduated in frequency and is independent of the frequency of the input. For an input frequency w_f, the point on the Hamel locus at which the relay must commutate is the point at the frequency w_f/n. The argument then proceeds in exactly the same manner as for the forced oscillations to insure that the relay commutates at this point. Since the Hamel locus has not changed, the minimum input amplitude to produce a forced oscillation at a frequency w_f is the same as for producing, for an input of frequency nw_f, a subharmonic of order n. Therefore the curve of F_n for the subharmonic of order n is the same as for the forced oscillation except shifted log(n) to the right - see Graph 4.2, page 28.

Gille's method can be extended to relays which have either hysteresis or dead zone (8) or which are asymmetric (9). In the case where the relay is asymmetric or has dead zone the method becomes much more difficult.

4.3 AN EXAMPLE

Consider the control system shown in Figure 4.5. It was assumed that the relay was symmetric with negligible dead zone and a hysteresis width of 0.4. The equations for the Hamel locus are

$$\dot{x} = -\pi/2w - \text{Tanh}\ (\pi/2w)$$

$$x = -\text{Tanh}\ (\pi/2w)$$

From these equations the Hamel locus can be drawn - see Graph 4.1, page 27. The minimum input amplitude to produce a forced oscillation is

$$F_f = \left| x(w_f) + 0.2 \right| \qquad (4.8)$$

where $x(w_f)$ is the value of x on the Hamel locus in the third quadrant which produces an oscillation at a frequency w_f. Graph 4.2 shows the curve of F_f versus $\log(w_f)$ to produce forced oscillations in the system. To determine similar curves for a subharmonic of order n the curve for the forced oscillations was shifted log(n) to the right - see Graph 4.2, page 28.

4.4 GILLE'S SECOND CONDITION FOR THE EXISTENCE OF A SUBHARMONIC OSCILLATION

Gille's second condition for the existence of a subharmonic oscillation will further limit the region in which a subharmonic will exist, but the exact regions which are eliminated are difficult to determine. The second condition is that the locus P_n, which is the sum of $x_a(t)$ and $x_f(t)$, must not intersect the commutation line within a half-period. Gille refers to this condition as "premature commutation". The possibility of this can best be seen by an example.

Consider a relay system in which the linear elements have the Hamel locus shown in Figure 4.6. Assume that a subharmonic oscillation of order n exists in the system. From Figure 4.6 it can be seen that ultimately P_n appears to arrive at the correct point, D, to insure the existence of the subharmonic. However, before this P_n intersects the commutation line at E which would cause the relay to commutate at this point. Once the relay had commutated, the assumed solution for $x_a(t)$ from equation 4.5 would cease to apply and the commutation point D would never occur.

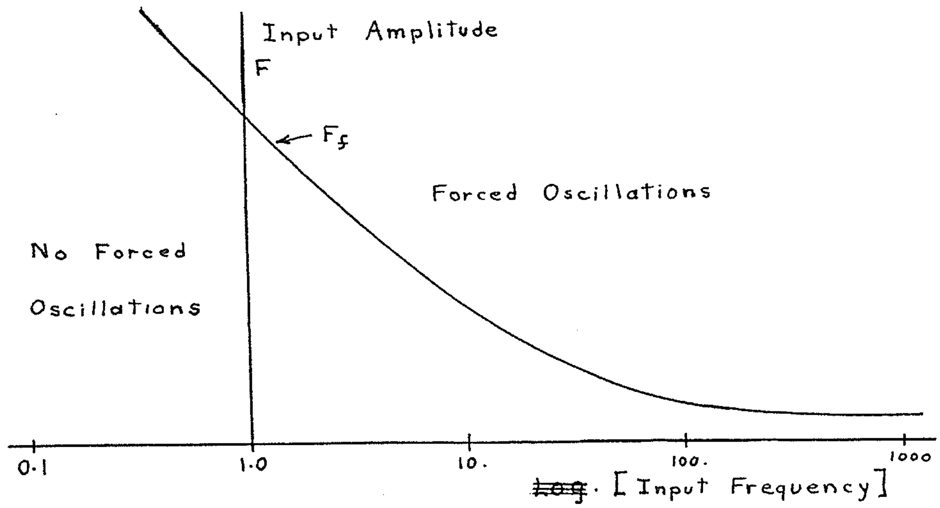


Figure 4.4

Synchronization Threshold For $L(s) = 1/s(s+1)$

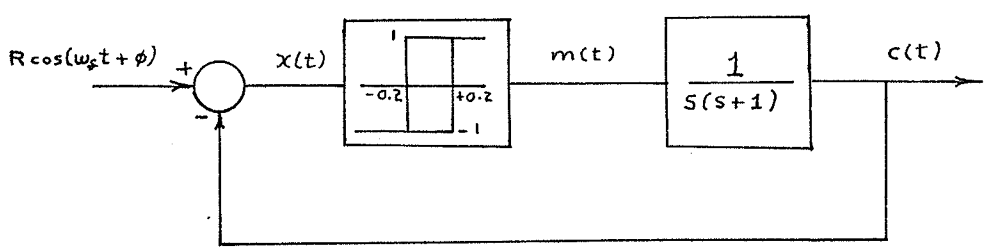
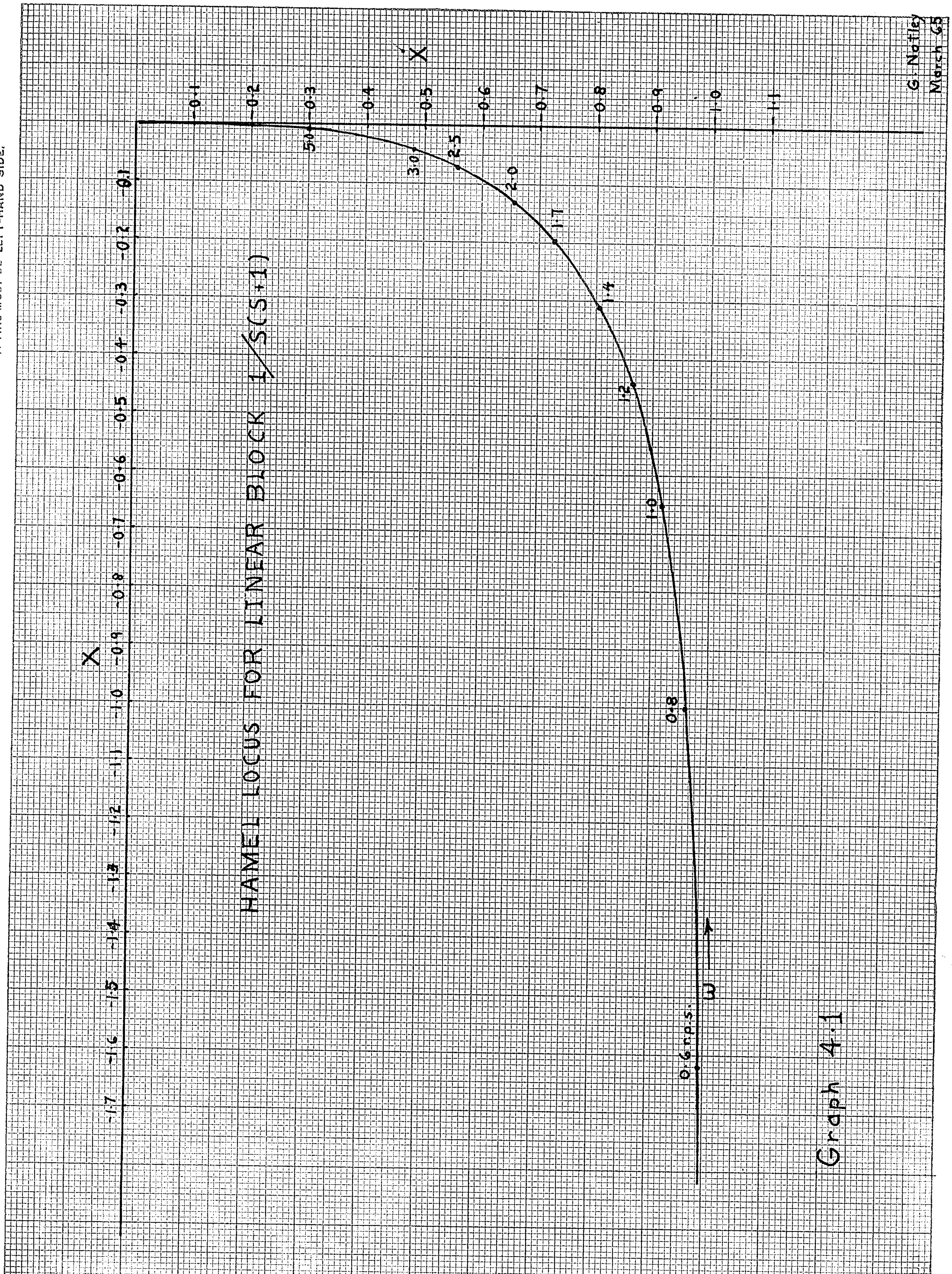
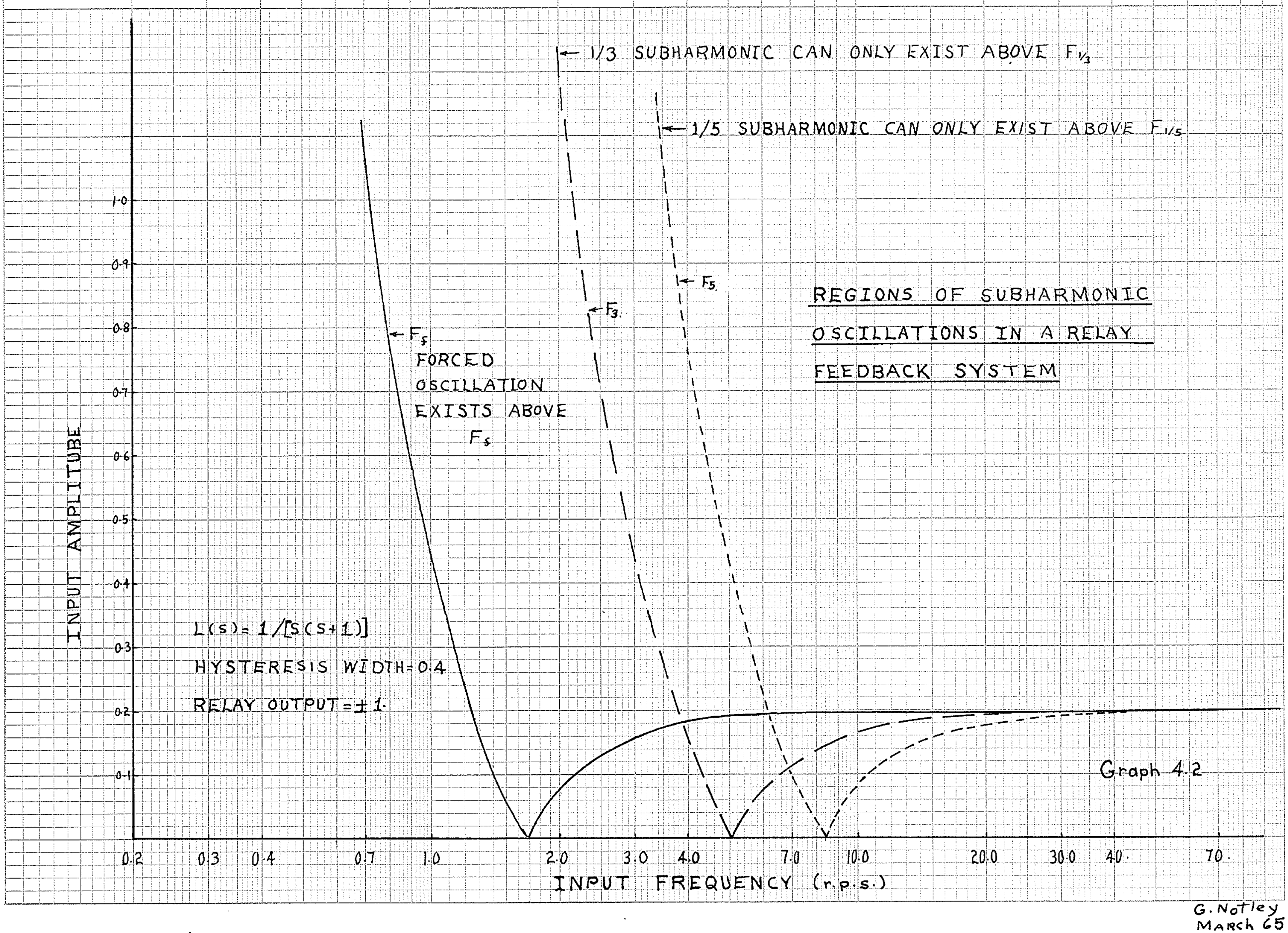


Figure 4.5

Feedback System with a Relay with Hysteresis


HAMEL LOCUS FOR LINEAR BLOCK 1/S(S+1)
Graph 4.1
G. Notley
March 65
X
X
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1.0
-1.1
-1.2
-1.3
-1.4
-1.5
-1.6
-1.7
5.0
3.0
2.5
2.0
1.7
1.4
1.2
1.0
0.8
0.6 r.p.s.
ω



Logarithmic
10 to the inch

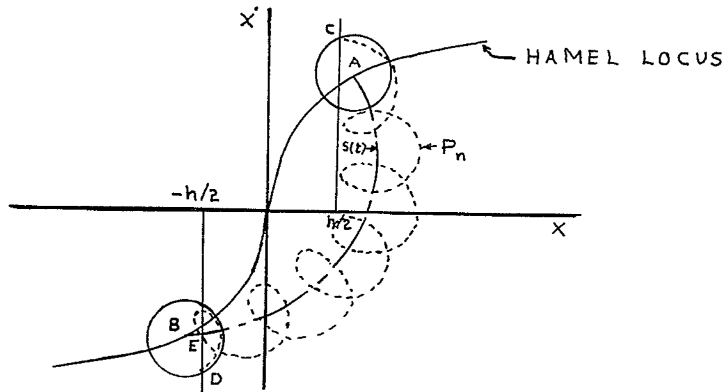


Figure 4.6

Commutation Within Half-Period

A detailed study of this condition is not easy as it requires knowledge of the exact shape of the locus P_n which can be very complicated. One general conclusion can be arrived at by studying Figure 4.6. Since the input frequency is w_f and the frequency of the n-th-order subharmonic is w_f/n, in the same time interval n times as many revolutions are performed by the vector representing the input than the vector representing $x_a(t)$. Thus, a commutation within a half-period is most likely to occur when the following conditions are met simultaneously: the order n of the subharmonic is high, and the w_f and w_f/n points do not lie too far from each other on the Hamel locus. For the case of a regular system the second condition generally occurs for high frequencies.

4.5 EXTENSION OF GILLE'S METHOD TO A MORE GENERAL PIECEWISE-LINEAR SYSTEM

Since Gille's method is so easy to apply, it is desirable to see if the method can be extended to a more general piecewise-linear, nonlinearity. For a relay system the input to the relay affects only the commutation points, thus, the input has no effect on the output except in so far as it changes the relay commutation points. For a general nonlinearity, a change in the input to the nonlinearity would immediately change the output of the system. Therefore, the autonomous response of the general piecewise-linear system can not be subtracted from the input to determine the response of the forced system.

4.6 DISCUSSION

Gille assumed that a necessary and sufficient condition for the existence of a subharmonic oscillation is that there are no commutations of the relay between the half period commutations. All of Gille's articles started from this assumption, and they proved that this was a sufficient condition. However, the idea that this is a necessary condition was never questioned. No other paper has been found investigating this condition, and a direct solution of the equations, assuming this type of oscillation, appears impossible for even the simplest system. Correspondence with J. Paquet, one of Gille's co-workers, indicates that they have proved that this type of oscillation was possible in the case where the relay was asymmetric. A paper dealing with this subject will be published in the spring. When this paper is published it may be found that subharmonic oscillations are possible in regions in which Gille's method indicates none are possible.

Otherwise Gille's method cannot be criticized much in that it predicts large regions in which no subharmonics are possible. Although the method does not divide the plane into regions where subharmonics exist and do not exist, it does give a good idea of the regions in which a subharmonic is most likely to occur. If a table of Hamel loci is available, the effect of changing the linear elements is not difficult to determine. Unfortunately, it does not seem that the method can be extended to more general piecewise-linear systems.

CHAPTER 5

SIMULTANEOUS OCCURRENCE OF TWO SUBHARMONICS

The problem of determining whether two or more subharmonic oscillations can exist simultaneously in a nonlinear feedback system is considered in this chapter. The possibility of extending, to deal with this problem, one of the methods for determining the existence of subharmonic oscillations in nonlinear feedback systems will be considered.

5.1 THE PROBLEM

Most feedback control systems, linear or nonlinear, have linear elements which act as low-pass filters. The filtering out of the higher frequency components tends to produce an output that is close to sinusoidal, as is assumed by the describing function method. Thus, a subharmonic response in a system often appears quite sinusoidal at the output of the system. However, the output contains all the harmonics of the subharmonic, no matter how small they are, since the nonlinearity is responding at the subharmonic frequency. If the system is responding at the one-ninth subharmonic, assuming an odd-function nonlinearity, the third harmonic of the one-ninth subharmonic will generally also be present at the output of the system. The third harmonic of the one-ninth subharmonic is the one-third subharmonic, indicating that two harmonically related subharmonics oscillations can exist simultaneously in some systems.

The problem thus reduces to that of determining whether two subharmonics which are not harmonically related can exist simultaneously. Two subharmonics are considered to be non-harmonically related if the higher frequency subharmonic is not a harmonic of the lower frequency subharmonic, for example the one-third and one-fifth subharmonics. Consider the feedback system shown in Figure 5.1. Assume that the output consists of two subharmonics, viz: the $1/m$ and $1/n$ subharmonic. The output is therefore a periodic oscillation with a frequency of w/mn, or it can be considered to be a $1/mn$ subharmonic oscillation. The feedback loop returns this output to the input of the non-linearity; therefore $x(t)$ must be periodic with the same frequency as the output. The output of the nonlinearity consists of a component at the fundamental frequency, the frequency of the $1/mn$ subharmonic, and all the harmonics of this component. The only way the given output of the system can occur for the given input to the nonlinearity is if the magnitude of the fundamental frequency component in the output of the nonlinearity is zero. Thus, the occurrence of two non-harmonically related subharmonics is ~~in~~ the special case of a subharmonic oscillation in which the output of the nonlinear element has no component at the fundamental frequency of the output of the system.

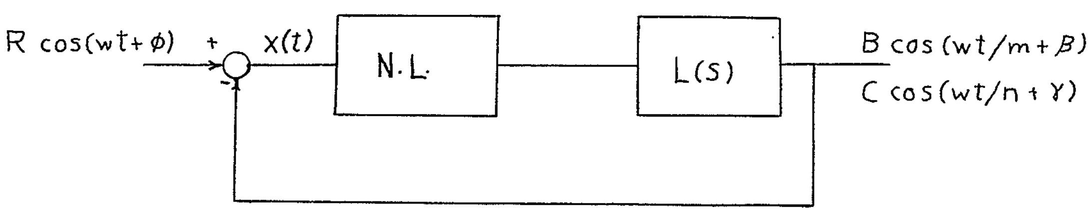


Figure 5.1

Feedback System

5.2 POLYNOMIAL NONLINEARITY

At first glance it appears that the problem of the simultaneous occurrence of two non-harmonically related subharmonics could be handled by a triple-input describing function. Immediately one tends to reject this approach as a poor one because of the amount of data it would be necessary to handle. The describing function would be a function of five parameters. Another objection to this method is that given a possible equilibrium point in a plane in which some of the parameters of the describing function are fixed it is necessary to check that this same equilibrium point exists in the other planes where these parameters are allowed to vary. This correlation of equilibrium points is difficult to do even for the dual-input describing function where the describing function is a function of only three parameters. It was this problem that made Oldenburger (11) assume that only the subharmonic existed at the output of the system.

The triple-input describing function approach would fail if the condition, that the output of the nonlinearity contains no component at the frequency of the periodic input to it, is not taken into account. Therefore, it would be necessary to take a Fourier analysis of the output to determine the magnitude of the w/mn frequency component. If this component was not zero then the resulting describing function for the assumed values of the parameters of the input would be inapplicable. For a polynomial nonlinearity an equation can be determined for the 1/mn subharmonic component present in the output and can be set to zero.

For example assume that

$$f(X) \text{ is } X^5 \qquad (5.1)$$

and

$$X \text{ is } R\cos(\tau+\alpha) + B\cos(\tau/3+\beta + C\cos(\tau/5+\gamma) \qquad (5.2)$$

Equation 5.2 is substituted into equation 5.1, and by the proper algebraic manipulation the one-fifteenth subharmonic component is determined. The one-fifteenth subharmonic is:

$$SH_{1/15} = 5\,BC\,\{[6C\,B^2\cos(2\gamma-B) + 2C^2B\cos(2\beta-3\gamma) + 4\,C^3\cos(2\gamma-\beta) + 6RBC\cos(2\beta+2\gamma-\alpha) + 4\,C^2R\cos(\alpha-\beta-3\gamma) + 12\,R^2C\cos(2\gamma-\beta)]\cos(\;/15) + [6C\,B^2\sin(2\gamma-\beta) + 2\,C^2B\sin(2\beta-3\gamma) + 4C^3\sin(2\gamma-\beta) + 6RBC\sin(2\beta+2\gamma-\alpha) + 4C^2R\sin(\alpha-\beta-3\gamma) + 12\,R^2C\sin(2\gamma-\beta)]\sin(\;/15)\}$$

The sine and cosine components must simultaneously be zero to produce a zero one-fifteenth subharmonic. Obviously this condition could only be obtained by trial and error or by some numerical method.

If a triple-input describing function method was attempted, a Fourier analysis of the output of the non-linearity by a numerical method would be required. A digital computer would be essential to perform these calculations. Some sort of stability criteria would be necessary to determine if the solution could exist in the system. Possibly Oldenburger's stability criterion could be used, but it would require the describing function loci and not just one isolated equilibrium point.

5.3 PIECEWISE LINEAR SYSTEM

The general piecewise-linear feedback system could be handled in the same manner as the polynomial nonlinearity system. A numerical method would still be required to calculate the triple-input describing function. However, the relay system is a rather special piecewise-linear system and will be considered here.

The only way the output of the relay can have a zero fundamental component is if the relay has commutation points between half periods. The existence of these "premature commutation" points violates the assumption by Gille, therefore his method is not applicable to this problem. As previously mentioned, Paquet has indicated that a periodic oscillation with commutation points between half periods is possible for relay systems containing an asymmetric relay. However, this paper has not been published yet.

It was suggested that the problem be attempted from the opposite direction. That is, determine a relay output that has a fundamental component of zero magnitude, and from this synthesize a relay feedback system which will produce this relay output. By performing a Fourier analysis of the output of the relay the magnitudes and phases of the two subharmonic components, which are assumed to exist at the output of the system, can be calculated. A low-pass filter with the largest cut-off rate practical can be connected in series with the relay to filter out all the higher harmonics in the relay output. The magnitudes and phases of the two subharmonics at the output of the filter can be calculated since they are known at the input to the filter. It now remains to calculate a linear block which will produce the calculated output when the output of the filter is passed through it.

5.4 ATTEMPTED SYNTHESIS OF A RELAY SYSTEM EXHIBITING THE SIMULTANEOUS OCCURRENCE OF TWO NON-HARMONICALLY RELATED SUBHARMONICS

In general, only subharmonics of odd-order will exist in a feedback system which incorporates a symmetrical nonlinearity. However, certain systems which contain only symmetrical nonlinearities can exhibit stable subharmonics of even order (16).

It was assumed that for the system being synthesized, incorporating a symmetrical relay, one or both of the non-harmonically related subharmonics were of even-order. Due to the even-order subharmonic components, the input to the relay, $x(t)$, is neither an odd-function, $x(-\tau)$ does not equal $-x(\tau)$, nor an antiperiodic function, $x(\tau)$ does not equal $-x(\tau + T/2)$ where T is the period of the input to the relay. Therefore, in general a d.c. component will exist in the system. The input to the relay has the form

$$x(\tau) = D + A\cos(\tau + \alpha) + B\cos\left(\frac{\tau}{m} + \beta\right) + C\cos\left(\frac{\tau}{n} + \gamma\right) \qquad (5.3)$$

where m is an even integer and n is either an odd or an even integer. The locations of the zero crossings, the relay commutation points, of this equation are functions of six independent variables. Since, in general, for the given input the output of the relay has no half-period symmetry, each commutation point in the time interval $o < \tau < T$ yields one independent equation for determining the input to the relay.

If it is assumed that the relay output has two commutation points per period, then it is impossible for the relay output to have fundamental component of zero magnitude - see Appendix D. A periodic relay output can not be constructed with an odd number of commutations per period - see Figure 5.2. For a relay output with four commutation points per half period, it appears that it is possible for the output of the relay to have a fundamental component of zero magnitude - see Appendix D. However no actual relay output was found which had a fundamental component of zero magnitude.

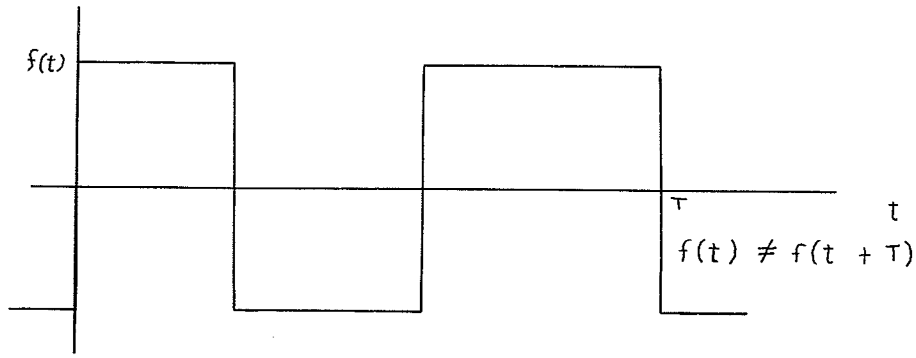


Figure 5.2

Relay Output with Three Commutations per Period

It was decided to assume that the output of the system contained subharmonics of odd-order. The input to the relay is then an antiperiodic function - see Appendix D - therefore only odd harmonics of the fundamental frequency exist at the output of the relay. Since the input to the relay is an antiperiodic function, an independent equation for the calculation of the magnitudes and phases of the input components is obtained at each commutation point in the interval $o < \tau < T/2$. The input to the relay has the form

$$x(\tau) = A\left[\cos(\tau + \alpha) + \frac{B}{A}\cos\left(\frac{\tau}{m} + \beta\right) + \frac{C}{A}\cos\left(\frac{\tau}{n} + \gamma\right)\right] \qquad (5.4)$$

where m and n are odd integers. In this equation there are five independent variables which determine the location of the relay commutation points.

A relay output with an even number of commutation points per half-period cannot be an antiperiodic function; therefore an even number of commutation points per half period cannot occur for the assumed output. A relay output which has one commutation point per half-period and is an antiperiodic function, is the trivial case of a symmetrical square wave, and it cannot have a zero fundamental component. It was found that a relay output with three commutation points per half-period could not have a fundamental component of zero magnitude - see Appendix D. By trial and error, a relay output with five commutations per half-period was constructed which had a fundamental component of zero magnitude - see Appendix D.

If it is assumed that the output of the system consists of the third and fifth order subharmonics, the input to the relay has the form

$$x(\tau) = A\left[\cos(\tau + \alpha) + \frac{B}{A}\cos\left(\frac{\tau}{3} + \beta\right) + \frac{C}{A}\cos\left(\frac{\tau}{5} + \gamma\right)\right] \qquad (5.5)$$

By the use of this equation and the constructed relay output which has a fundamental component of zero magnitude, five independent nonlinear equations can be derived for $x(\tau)$. These five equations must be solved simultaneously to determine the five independent variables in equation 5.5. Once the variables in equation 5.5. have been determined, $x(\tau)$ must be checked to be sure that it has no other commutation points. The number of zero crossings that $x(\tau)$ has is not at all apparent and it is a function of the relative magnitudes of the subharmonics. The solution of the five simultaneous equations from the commutation points would be difficult requiring a numerical technique since all of the equations are nonlinear.

The low-pass filter connected in series with the relay would be designed to pass the two subharmonics and to effectively block all the higher frequency components in the output of the relay. The two subharmonics components at the output of the chosen filter could then be determined. Since the input to the linear block is now known and the output of the system has been calculated previously, the gain and phase-shift of the two subharmonic components being passed through the linear block can be determined. From these gains and phase-shifts, four independent equations can be derived for the linear block; however, the equations are nonlinear. A form for the linear block which would possibly satisfy the required gain and phase-shift conditions could be determined from a rough Nyquist locus plot. The solution of the four simultaneous nonlinear equations for the gains and time constants of the linear block would require a numerical technique.

Although the original idea has merit, the actual calculations that result involve a great deal of work. The solution obtained would satisfy the differential equation but would in no way indicate if the solution was stable and could, therefore, be obtained experimentally.

5.5 CONCLUSIONS

Two subharmonics can exist simultaneously in a feedback system if they are harmonically related. Theoretically two apparently non-harmonically related subharmonics can exist at the output of the system. However, this is just the special case of the output of the nonlinearity containing no component at the fundamental frequency of the periodic input to the nonlinearity. Whether or not this type of oscillation could actually exist is difficult to determine since the problem appears to require a direct solution of the nonlinear equation or the use of a triple-input describing function which would require a vast amount of work to obtain the desired results. Another objection to the triple-input describing function method is that the method assumes that only two subharmonics exist in the output. However, the higher harmonics of the output of the nonlinearity are also present, and they may not be effectively filtered by the linear elements since their frequencies are not much greater than the frequencies of the two subharmonics.

The idea of synthesizing a relay system from a relay output with a fundamental component of zero magnitude would result in a formidable amount of work with no guarantee that the desired oscillation could be observed experimentally.

CHAPTER 6

ATHERTON'S MODIFIED NONLINEAR CHARACTERISTICS

Atherton has developed a method, which is presented here, for obtaining the response of a nonlinearity to several uncorrelated inputs. By making certain approximations, the concept of a modified nonlinearity is then introduced. To determine the response of a particular input x, the nonlinear characteristic is first modified in turn by each of the input signals; then the input x is applied to this modified characteristic. The application of this concept to subharmonic oscillations is investigated.

6.1 RESPONSE OF NONLINEAR CHARACTERISTIC TO SEVERAL INPUTS

Consider a single-valued nonlinearity with several uncorrelated inputs, Figure 6.1. An output component originating from one of these inputs depends on, among other factors, not only the magnitude of this particular input but also the magnitude of all other inputs. Additional output terms are present besides these fundamental components, such as harmonics and cross-modulation products. For a system with two inputs the transform method (18) may be used to evaluate the various output terms; but, in general, the expressions are very complex. Solutions using the same technique for more than two inputs appear virtually impossible.

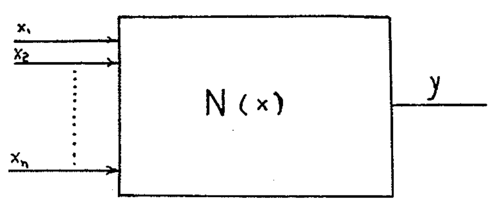


Figure 6.1

Multiple Input Nonlinearity

The autocorrelation function of a signal containing a periodic component $A \cos(wt + \phi)$ has a periodic component $\frac{A^2}{2} \cos(w\tau)$. By studing the autocorrelation function of a signal, the amplitude of the periodic components of the signal (1) can be determined.

It can be shown (1) that the autocorrelation function $\Psi(\tau)$ of the output of a nonlinearity $n(x)$ having an input consisting of two uncorrelated sinusoidal components of amplitudes A and B is given by

$$\Psi(\tau) = \sum_{s_1=0}^{\infty} \sum_{s_2=0}^{\infty} E_{s_1} E_{s_2} \alpha^2_{s_1 s_2} \cos(s_1 W_A \tau) \cos(s_2 W_B \tau) \tag{6.1}$$

where E_s is the Neumann factor

and
$$\alpha_{S_1 S_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} N(jw)\, J_{S_1}(Aw) J_{S_2}(Bw)\, dw \tag{6.2}$$

where $N(jw)$ is the Laplace transform of the nonlinearity $n(x)$.

$J_i(m)$ is the Bessel function of order i and modulus m.

When the input consists of a sinusoidal signal, $x = A \cos(wt)$, together with Gaussian noise y of r.m.s. magnitude σ and autocorrelation function $\phi(\tau)$, equation 6.1 becomes

$$\Psi(\tau) = \sum_{S=0}^{\infty} \sum_{K=0}^{\infty} E_s \alpha^2_{SK} \frac{\phi^K(\tau)}{K!} \cos(sw\tau) \tag{6.3}$$

where
$$\alpha_{SK} = \frac{1}{2\pi} \int_{-\infty}^{\infty} N(jw)\, (jw)^K \exp[-\sigma^2 w^2/2 j s]\, J_s(Aw)\, dw \tag{6.4}$$

Alternatively (2) the coefficients, α_{SK}, may be obtained from the expression

$$\alpha_{SK} = \frac{1}{\sigma^K} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} n(x+y)\, H_K(y/\sigma)\, T_s(x/A)\, r(x)\, q(y)\, dx\, dy \tag{6.5}$$

where

T_s is the Chebyshev polynomial of order s.

H_k is the Hermite polynomial of order k.

$r(x)$ is the amplitude probability-density distribution of the sinusoidal input signal.

$q(y)$ is the amplitude probability-density distribution of the Gaussian input signal.

The use of the equation 6.5 is generally to be preferred as it may be extended for use with more than two input signals and signals possessing other than sinusoidal and Gaussian amplitude probability density distributions. Moreover, the use of equation 6.5 shows how the various output terms are formed if the integral is considered to be divided into two separate integrals.

$$n(\gamma,s) = \int_{-\infty}^{\infty} n(\gamma+x)\, T_s(x/A)\, r(x)\, dx \qquad (6.6a)$$

$$\alpha_{SK} = \frac{1}{\sigma^K} \int_{-\infty}^{\infty} n(y,s)\, H_K(y/\sigma)\, q(y)\, dy \qquad (6.6b)$$

or alternatively

$$n(\gamma,K) = \int_{-\infty}^{\infty} n(\gamma+y)\, H_K(y/\sigma)\, q(y)\, dy \qquad (6.7a)$$

$$\alpha_{SK} = \frac{1}{\sigma^K} \int_{-\infty}^{\infty} n(x,K)\, T_s(x/A)\, r(x)\, dx \qquad (6.7b)$$

These equations show that the evaluation of α_{SK} may be considered as a two-stage process in which the nonlinearity is first modified by one input signal, and then the response of the modified nonlinearity to the other input signal is determined. In equation 6.7a, $n(\gamma,k)$ is referred to as the k^{th}-modified nonlinearity.

In a quasilinear analysis, all the output terms from the nonlinearity except those given by the coefficients α_{01} and α_{10}, assuming two inputs, are neglected. The "0"-modified characteristic for other than sinusoidal and Gaussian input signals can be evaluated from the expression

$$n(\gamma,o) = \int_{-\infty}^{\infty} n(\gamma,u)\, p(u)du \qquad (6.8)$$

where $p(u)$ is the amplitude probability-density distribution of the signal.

As previously stated, extension of equation 6.5 to n input signals is possible, in which case n integrals are involved to determine the gain of a specific input.

For an example of an on-off characteristic modified by a sinusoid of peak amplitude A see Appendix C.

Atherton extends the modified characteristic concept to double-valued nonlinearities which have characteristics that are independent of the form of the input signal.

6.2 DISCUSSION OF THE MODIFIED NONLINEARITY METHOD

The modified nonlinear characteristic method is an approximation method; and, like most approximation methods, its accuracy is difficult to determine. The "O"-modified characteristic concept neglects all the components in the output which result from cross-modulation products.

In order to determine α_{01} and α_{10} a numerical method is necessary to evaluate the integrals since, in general, the modified characteristic is available only in graphical or very complex analytic form.

It can be shown (19) that a signal containing a sinusoidal component with a phase shift, $A \cos(wt + \phi)$, has an autocorrelation function which contains a periodic component, $\frac{A^2}{2} \cos(w\tau)$, which has no phase-shift. Thus, the autocorrelation function of a signal contains no information on the phase relationship of the components of the signal. Therefore, the phase-shift of a signal through a nonlinearity can not be determined. This is a serious drawback in feedback systems since it prevents the determination of oscillations in a closed loop.

Atherton gives an example of an on-off nonlinearity in which he uses his method to calculate the phase shift through the nonlinearity. However, for a modified on-off nonlinear characteristic, it was found that a sinusoidal signal created an "operating range" on the modified relay characteristic - see Appendix C. Rather arbitrarily he speaks of the relay commutating at the end of the operating range. He then determines the phase of the signal $A \cos(\phi)$ which will cause the relay to commutate. This phase angle is then considered to be the phase-shift through the nonlinearity of the sinusoidal signal being applied to the modified characteristic. Whether or not the concept of the relay commutating at the end of the "operating range" has any meaning is difficult to determine. However, for a more general nonlinearity the concept of commutating at the end of an "operating range" has no meaning, and could not be used to determine the phase shift of the sinusoidal signal being passed through the nonlinearity.

6.3 CONCLUSIONS

Atherton's method was presented as a means of determining ~~determing~~ the response of a single-valued nonlinearity with several inputs. However, Atherton never takes into account the "phase-shift" of the inputs when passed through the nonlinearity. In effect, Atherton replaces the nonlinearity by a real-valued gain for each input. The phase-shift of the subharmonic through the nonlinearity is essential to determine if the subharmonic can be sustained around the closed loop. This fact eliminates this method for the study of subharmonics in feedback systems.

The modified nonlinearity concept is useful for a quantitative study of a nonlinear system with more than one input. The effect of the other inputs on the gain of the nonlinearity for one of the inputs can roughly be determined by examining the modified characteristic.

In the quasilinear analysis, assumed by Atherton, all higher harmonics and cross-modulation products are assumed negligible. This in effect reduces Atherton's method to a describing function approach to the problem. Since the calculation of the gain of the nonlinearity for a particular input requires a numerical integration, and the method in no way takes into account the phases of the inputs or the components of the output of the nonlinearity, no great advantage can be seen to the use of this method over a multiple-input describing function. Atherton's method has, however, the advantage that the inputs to the nonlinearity do not have to be sinusoidal, but they must be uncorrelated.

CHAPTER 7

DISCUSSION AND CONCLUSIONS

In general it can be said that the results of this thesis are rather negative both for the new techniques for the analysis of subharmonic oscillations in nonlinear feedback systems, and the original problems being investigated. The techniques for the analysis of subharmonic oscillations tend to be of very limited application or are just modifications of the dual-input describing function.

The study of the transient response of a nonlinear system can be considered a failure. Hayashi's method is too limited in application, being applicable only to second-order nonlinear systems with only one possible subharmonic oscillation, and it requires too much work to plot the state-plane for results which are of questionable accuracy. The only apparent solution to the problem of determining the transient response appears to be a numerical solution of the nonlinear differential equation, but the resulting work would be excessive.

The dividing of the input amplitude versus input frequency plane into regions in which a given subharmonic may or may not exist can be accomplished by Oldenburger's method. For a given input the frequency range over which a stable subharmonic exists can be determined by Oldenburger's method. Although Oldenburger's stability criterion is a very useful one for feedback systems, his article makes it appear that his ideas are completely original. In effect, Oldenburger's method is just the dual-input describing function method used to determine the subharmonic equilibrium points, and the stability of these equilibrium points determined by an extension of Loeb's stability criterion to subharmonic oscillations. No justification can be seen for Oldenburger's method for determining the output of the nonlinearity since his method gave the output in a very complicated form involving an infinite series. The same information could have been obtained from a dual-input describing function for the nonlinearity since Oldenburger assumed that the linear elements filter out all the components of the output of the nonlinearity except the subharmonic. The vast amount of data that must be manipulated when using a dual-input describing function appears to be the major draw-back of Oldenburger's method.

Gille's method, for determining the regions of subharmonic oscillations in the input amplitude versus input frequency plane for relay systems, is a new technique for the study of subharmonic oscillations in nonlinear systems. Unfortunately it does not seem that the method can be extended to more general piecewise-linear systems. The method actually only predicts regions where a given subharmonic cannot occur or is most likely to occur. The concept of "premature commutation" limits the usefulness of the results obtained by Gille's method. The only criticism of Gille's method is that he assumes that subharmonic oscillations are only possible if the relay has one commutation per half period. This fact was never questioned in any of Gille's articles, but Paquet has said that subharmonic oscillations with more than one commutation per half period are possible in systems with an asymmetric relay.

The problem of the simultaneous occurrence of two subharmonics in a nonlinear feedback system has been answered theoretically. Since the system is nonlinear, all the harmonics of the subharmonic oscillations are present in the system, and some of these harmonics may also be subharmonics of the input. Thus two harmonically related subharmonics can exist simultaneously in a nonlinear system. The case of two "non-harmonically" related subharmonics occurs when the output of the nonlinearity has no component at the fundamental frequency of the periodic subharmonic response of the system. Whether or not this type of oscillation can exist is very difficult to determine theoretically. A triple -input describing function appears to require the manipulation of too much data to be of any use.

Although Atherton's modified nonlinear characteristic appears as though it would be very useful for the study of the simultaneous occurrence of several subharmonics, the method is of little use for the study of feedback systems. The method completely neglects the phase-shift of the inputs when passed through the nonlinearity. The nonlinearity is replaced by a real-valued gain for each input; thus it is of no use for harmonically related inputs in feedback systems. The modified nonlinear characteristic is only used as a means of determining the gain of a particular input to the nonlinearity in the presence of other inputs. It does not give the time response of the nonlinearity resulting from the multiple inputs. The actual gain of a given input requires the evaluation of an integral which contains this modified nonlinear characteristic. Generally a numerical technique is required to evaluate this integral.

In conclusion it may be said that the investigation of subharmonic oscillations still requires the "grinding-out" of solutions. The dual-input describing function technique is probably the best method available for the study of the subharmonic performance of feedback systems although it requires the manipulation of a large amount of data.

APPENDIX A

DERIVATION OF EQUATIONS 3.3, 3.4 AND 3.5

Consider equation 3.2 for the output of the single-valued odd-function nonlinearity.

$$x_2 = \int_{-\infty}^{\infty} F(ju)\ \exp(jux_1)du \tag{A.1}$$

with

$$x_1 = A\ \sin(kwt + B\ \sin(wt) \tag{A.2}$$

If $f(x_1)$ is an odd-function of x_1, then $F(ju)$ will be an odd function of u. Substituting equation A.2 into equation A.1 produces

$$x_2 = \int_{-\infty}^{\infty} F(ju)\ \exp(juA\sin(Kwt+\phi))\ \exp(juB\sin(wt))du \tag{A.3}$$

In (20) it is shown that the exponential may be expanded as

$$\exp[ju\sin(\theta)] = \cos(u\sin(\theta)) + j\sin(u\sin(\theta))$$

$$= \sum_{n=0}^{\infty} \epsilon_n J_{2n}(u)\cos(2n\theta) + 2j\sum_{n=0}^{\infty} J_{2n+1}(u)\sin((2n+1)\theta) \tag{A.4}$$

where $J_i\ (u)$ is the Bessel function of order i and modulus u

ϵ_n is the Neumann factor

$= 1,\ n = 0$

$= 2,\ n = 1,\ 2,\ 3......$

If the product of the two exponentials in equation A.3 is defined as G(u,w) then:

$$G(u,w) = \exp[juA\ \sin(Kwt+\theta)]\ \exp(juB\ \sin(wt))$$

This may be expanded according to equation A.4 to yield

$$G(u,w)=\left\{\sum_{n=0}^{\infty}\epsilon_n J_{2n}(Au)\cos[2n(Kwt+\phi)]+2j\sum_{n=0}^{\infty}J_{2n+1}(Au)\sin[(2n+1)(Kwt+\phi)]\right\}\left\{\sum_{m=0}^{\infty}\epsilon_m J_{2m}(Bu)\cos(2m(wt))+2j\sum_{m=0}^{\infty}J_{2m+1}(Bu)\sin[(2m+1)wt]\right\} \quad (A.5)$$

As the function F(ju) is odd, only those parts of equation A.5 which are odd contribute towards the integral of equation A.3. Thus, the odd portion of G(u,w) may be defined as G_{odd} (u,w) and is given as

$$G_{odd}(u,w)=2j\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\epsilon_n J_{2n}(Au)J_{2m+1}(Bu)\cos[2n(Kwt+\phi)]\sin[(2m+1)wt]\right.$$
$$\left.+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\epsilon_n J_{2m}(Bu)J_{2n+1}(Au)\cos(2mwt)\sin[(2n+1)(Kwt+\phi)]\right\} \quad (A.6)$$

Only the fundamental component in the output of the nonlinear element is of interest. Thus, after expanding the trigonometric products those terms contributing towards the fundamental are extracted to yield

$$G_f(u,w)=j\sum_{n=0}^{\infty}\epsilon_n(-1)^n J_n(Au)J_{nK+1}(Bu)\sin(wt-n\phi)$$
$$-\sum_{n=0}^{\infty}\epsilon_n(-1)^n J_n(Au)J_{nK-1}(Bu)\sin(wt+n\phi) \quad (A.7)$$

where $G_f(u,w)$ is the portion of $G_{odd}(u,w)$ contributing to the fundamental output. The portion $G_f(u,w)$ may be split into two parts $G_1(u,w)$ and $G_2(u,w)$, the former giving the sine component and the latter giving the cosine component. These are given by

$$G_1(u,w)=j\left\{\sum_{n=0}^{\infty}\epsilon_n(-1)^{n+1}J_n(Au)[J_{nK-1}(Bu)-J_{nK+1}(Bu)]\cos(n\phi)\right\} \quad (A.8)$$

$$G_2(u,w)=j\left\{\sum_{n=0}^{\infty}\epsilon_n(-1)^{n+1}J_n(Au)[J_{nK-1}(Bu)+J_{nK+1}(Bu)]\sin(n\phi)\right\} \quad (A.9)$$

By the use of ~~Using~~ these expressions, the fundamental part of the output may be written as

$$(x_2)_f=\int_{-\infty}^{\infty}F(ju)G_1(u,w)\sin(wt)du+\int_{-\infty}^{\infty}F(ju)G_2(u,w)\cos(wt)du \quad (A.10)$$

where $(x_2)_f$ is the fundamental component of the output.

This is the desired result.

APPENDIX B

EQUIVALENT LINEARIZATION ABOUT A POINT

Oldenburger determines the stability of the equilibrium points by the use of the incremental Nyquist diagram and equivalent linearization about a point. This method assumes that at any equilibrium point the system is linear for incremental disturbances in the system. Thus, the criterion only indicates the stability of the system for small disturbances, and the effect of a large disturbance in the system can not be predicted.

The method requires a Nyquist locus G(jw) for the linear elements, and the critical locus N(E) for the nonlinearity. The critical locus N(E) is a plot of -1/(describing function) as a function of the magnitude, E, of the input to the nonlinearity. Equilibrium points for the system occur at the intersection of the two curves, Figure B.1.

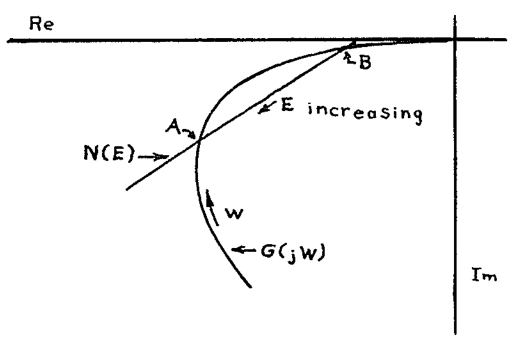


Figure B.1

Linearization about Equilibrium Point

Now consider equilibrium point B; assume the system behaves as a linear system in an incremental region around the equilibrium point.

Assume also that E increases incrementally due to a disturbance. The equilibrium point would move along N(E) in the direction of increasing E. However, this moves the equilibrium point into the interior of the Nyquist locus indicating that the oscillation would increase in amplitude. The increase in amplitude would move the equilibrium point further from B, thus it is concluded that this equilibrium point is unstable.

Now examining the equilibrium point A it can be seen that an incremental increase in E would move the equilibrium point away from the interior of the Nyquist locus. Thus, the oscillation would tend to die out, decreasing the magnitude of E and moving the equilibrium point back to A. This indicates the oscillation is stable for small disturbances.

Oldenburger (13) gives a mathematical justification of the above argument. For determining the stability of limit cycles in autonomous systems this stability criterion is known as Loeb's criterion. However, it should be noted that this stability criterion only applies for small disturbances in the system and is, even then, only approximate.

APPENDIX C

CALCULATION OF THE MODIFIED CHARACTERISTIC FOR A ON-OFF CHARACTERISTIC SUBJECT TO A SINUSOIDAL INPUT

Consider the sinusoid A cos(wt) as an input to the on-off characteristic n(x) where

$$n(x) = h \qquad x > o$$
$$= -h \qquad x < o$$

The "O" - modified nonlinearity is given by equation

$$n(\Upsilon, o) = \int_{-\infty}^{\infty} (\Upsilon + x)\ p(x)\ dx \tag{C.1}$$

For a sinusoid the amplitude probability-density distribution, p(x), is

$$p(x) = 1/\pi\ (A^2 - x^2) \qquad -A \leq x \leq A$$
$$= o \qquad |x| > A$$

where $\quad x \quad$ is $\ A\cos(wt)$

Now if $-A \leq \Upsilon \leq +A$, then the output of the relay is

$$n(\Upsilon + x) = -h \qquad (\Upsilon + x) \leq o$$
$$= h \qquad (\Upsilon + x) \geq o$$

where the relay commutates at $x = -\Upsilon$. Equation C.1 can now be evaluated.

$$n(\Upsilon, O) = \int_{-\infty}^{-A} (-h)\, o\, dx + \int_{-A}^{-\Upsilon} \frac{(-h)\, dx}{\pi (A^2 - x^2)} + \int_{-\Upsilon}^{A} \frac{h\, dx}{\pi (A^2 - x^2)} + \int_{A}^{\infty} h \times O\, dx$$

Thus $\qquad n(\Upsilon, o) = +\frac{2h}{\pi} \sin^{-1}(\Upsilon/A) \qquad -A \leq \Upsilon \leq A$

Now if $\Upsilon > A$, then

$$n(\Upsilon + x) = -h \qquad x \leq -\Upsilon$$
$$= +h \qquad x \geq -\Upsilon$$

$$n(\Upsilon, o) = \int_{-\infty}^{-\Upsilon} (-h)\, O\, dx + \int_{-\Upsilon}^{-A} h \cdot O\, dx + \int_{-A}^{A} \frac{h\, dx}{\pi (A^2 - x^2)} + \int_{A}^{\infty} h \times O\, dx$$

$$n(\Upsilon, o) = +h \qquad \Upsilon > A$$

Similarly for $\Upsilon < A$

$$n(\Upsilon, o) = -h$$

Therefore

$$
\begin{aligned}
n(\Upsilon, o) &= -h & \Upsilon < -A \\
&= \frac{2h}{\pi} \sin^{-1} \Upsilon/A & |\Upsilon| < A \\
&= h & \Upsilon > A
\end{aligned}
$$

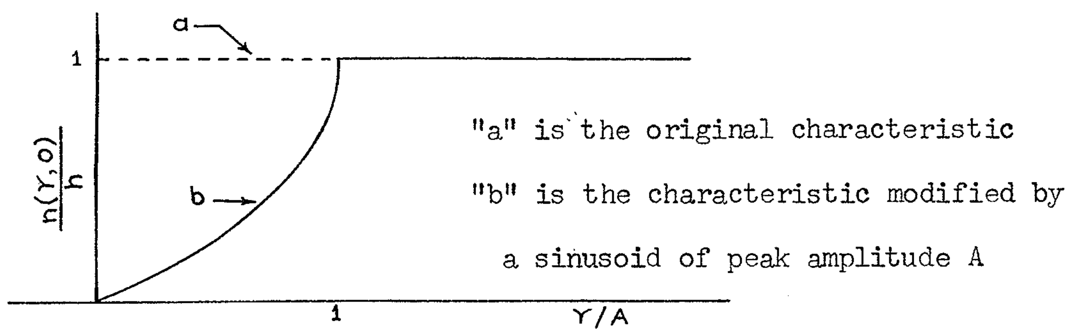


Figure C.1

"O" -Modified On-Off Characteristic

The sinusoidal input has produced an "operating range" from $-A$ to $+A$ for another input to the relay.

APPENDIX D

RELAY OUTPUT WITH A FUNDAMENTAL COMPONENT OF ZERO MAGNITUDE

Most periodic waveforms can be represented by a Fourier series of the form

$$f(\tau) = \frac{a_o}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nw_1\tau) + b_n \cos(nw_1\tau)\right] \qquad (D.1)$$

The coefficients of the Fourier series may be determined by the following equations

$$a_n = \frac{2}{T}\int_o^T f(\tau)\cos(nw_1\tau)d\tau \qquad n = o,1,2,\ldots \qquad (D.2)$$

$$b_n = \frac{2}{T}\int_o^T f(\tau)\sin(nw_1\tau)d\tau \qquad n = 1,2,3,\ldots \qquad (D.3)$$

where T is the period of $f(\tau)$, and w_1 is $2\pi/T$.

1) Relay Output with Two Commutations Per Period

Consider the relay output shown in Figure D.1. This waveform has a period of 2π seconds and w_1 is unity.

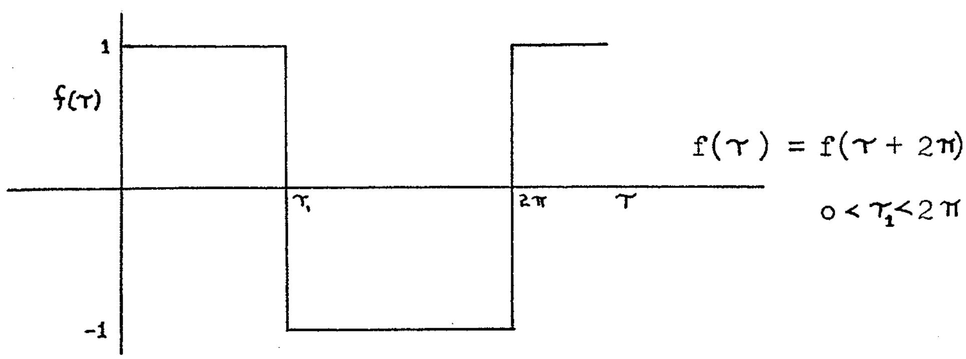


Figure D.1

Relay Output with Two Commutations Per Period

The magnitude of the fundamental component of $f(\tau)$ may be determined from the coefficients a_1 and b_1 of the Fourier series.

From equation D.2 and D.3 these coefficients may be calculated as follows:

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} f(\tau)\cos(\tau)d\tau = \frac{1}{\pi}\left[\int_0^{\tau_1}\cos(\tau)d\tau - \int_{\tau_1}^{2\pi}\cos(\tau)d\tau\right]$$
$$= \frac{2}{\pi}\left[\sin(\tau_1)\right] \tag{D.4}$$

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} f(\tau)\sin(\tau)d\tau = \frac{1}{\pi}\left[\int_0^{\tau_1}\sin(\tau)d\tau - \int_{\tau_1}^{2\pi}\sin(\tau)d\tau\right]$$
$$= -\frac{2}{\pi}\left[\cos(\tau_1) - 1\right] \tag{D.5}$$

For a fundamental component of zero magnitude, a_1 and b_1 must simultaneously be zero. Therefore

$$a_1 = \frac{2}{\pi}\left[\sin(\tau_1)\right] = 0 \tag{D.6}$$

$$b_1 = -\frac{2}{\pi}\left[\cos(\tau_1) - 1\right] = 0 \tag{D.7}$$

The only solution that simultaneously satisfies equations D.6 and D.7 is when τ_1 is zero (modulus 2π) ~~2π~~. However, $f(\tau)$ is then a constant d.c. output. Therefore, no relay output with two commutations per period exists that has a fundamental component of zero amplitude.

2) Relay Output with Four Commutations Per Period

Consider the relay output shown in Figure D.2 which has four commutations per period.

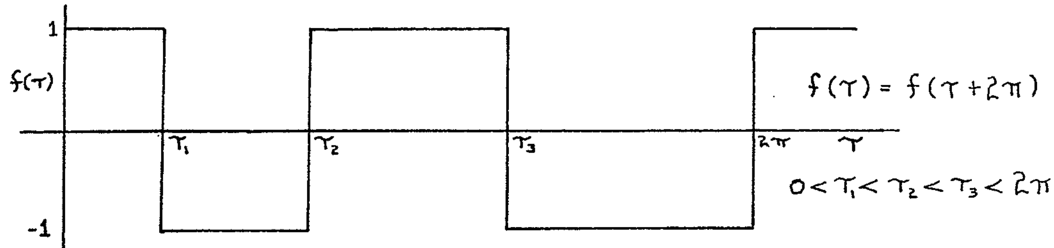


Figure D.2

Relay Output with Four Commutations per Period

The coefficients of the fundamental component in the Fourier series of the periodic waveform in Figure D.2 are

$$a_1 = \frac{2}{\pi}\left[\sin(\tau_1) - \sin(\tau_2) + \sin(\tau_3)\right] \qquad \text{(D.8)}$$

$$b_1 = -\frac{2}{\pi}\left[\cos(\tau_1) - \cos(\tau_2) + \cos(\tau_3) - 1\right] \qquad \text{(D.9)}$$

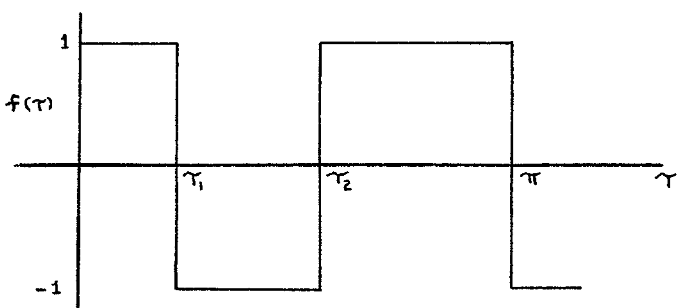
By proper~~ly~~ adjustment of ~~adjusting~~ the positions of the equilibrium points, it appears that a_1 and b_1 could be simultaneously set to zero. However, no solutions were obtained that had a period of 2π. Therefore, although it appears as though a fundamental component of zero maghitude is possible for this relay output, none could be found.

3) Relay Output with Three Commutations Per Half-Period

Consider the relay shown in Figure D.3 which has three commutation points per half-period. The Fourier coefficients for the fundamental component of the waveform in Figure D.3 are:

$$a_1 = \frac{4}{\pi}\left[\sin(\tau_1) - \sin(\tau_2)\right] \qquad \text{(D.10)}$$

$$b_1 = -\frac{4}{\pi}\left[\cos(\tau_1) - \cos(\tau_2) - 1\right] \qquad \text{(D.11)}$$



$f(\tau + 2\pi) = f(\tau)$

$f(\tau + \pi) = -f(\tau)$

$0 < \tau_1 < \tau_2 < \pi$

Figure D.3

Relay Output With Three Commutations per Half-Period

For a fundamental component of zero magnitude a_1 and b_1 must be simulanteously zero. Note that a_1 is zero if

$$\tau_2 = \pi - \tau_1 \qquad \text{(D.12)}$$

Substituting this into equation D.11 yields

$$b_1 = \frac{-4}{\pi} \left[\cos(\tau_1) - \cos(\pi - \tau_1) - 1 \right] = \frac{-4}{\pi}\left[2 \cos(\tau_1) - 1\right]$$

Then b_1 is zero when

τ_1 is $\cos^{-1}(1/2) = \pi/3$ radians

τ_2 is $2\pi/3$

A relay output with these commutation points is actually a symmetric, periodic square wave with a frequency of 3 r.p.s. Thus, no relay output with three commutations per half-period has a fundamental component of zero magnitude.

4) Relay Output With Five Commutations Per Half-Period

Consider the relay output shown in Figure D.4 which has five commutations per half-period. The coefficients of the fundamental frequency components in its Fourier series are:

$$a_1 = \frac{4}{\pi} \left[\sin(\tau_1) - \sin(\tau_2) + \sin(\tau_3) - \sin(\tau_4) \right] \quad (D.13)$$

$$b_1 = \frac{-4}{\pi} \left[\cos(\tau_1) - \cos(\tau_2) + \cos(\tau_3) - \cos(\tau_4) - 1 \right] \quad (D.14)$$

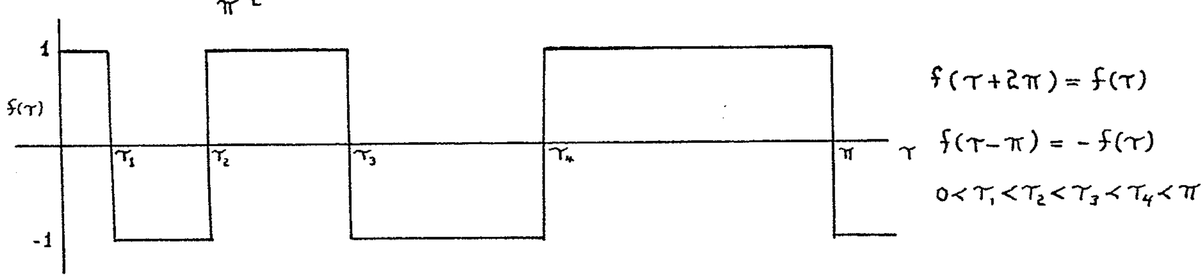


Figure D.4

Relay Output With Five Commutations Per Half-Period

By trial and error it was found that the equations D.13 and D.14 were simultaneously zero for the following set of commutation points.

$$\tau_1 = 30° \quad = \quad .52 \text{ radians}$$
$$\tau_2 = 70° \quad = \quad 1.22 \text{ radians}$$
$$\tau_3 = 113° \ 50' = 1.99 \text{ radians}$$
$$\tau_4 = 151° \ 40' = 2.65 \text{ radians}$$

The coefficients of some of the harmonics of the fundamental were calculated and are listed below

Fundamental	$a_1 = -4(.00013)/\pi$	$b_1 = 4(.00043)/\pi$
3rd Harmonic	$a_1 = +4(.06)/\pi$	$b_3 = 4(.30)/\pi$
5th Harmonic	$a_5 = +4(.08)/\pi$	$b_5 = 4(.90)/\pi$
7th Harmonic	$a_7 = +4(.003)/\pi$	$b_7 = 4(.28)/\pi$

Thus it is possible to construct a relay output with five commutations per half period which as a fundamental component of zero magnitude.

5) Relay Input Composed of Odd-Order Subharmonics is an Antiperiodic Function

Consider an input, $x(\tau)$, to the relay of the form

$$x(\tau) = A \cos(\tau+\alpha) + B \cos\left(\frac{\tau}{m} + \beta\right) + C \cos\left(\frac{\tau}{n} + \gamma\right) \qquad (D.15)$$

where m and n are odd relatively prime integers.

The input to the relay is a periodic function with a period T of $2mn\pi$ seconds. Note that, since m and n are odd integers, the product mn is also an odd integer. Consider the input after a half of a period

$$x(\tau+\pi) = A \cos(\tau + mn\pi + \alpha) + B \cos\left(\frac{\tau}{m} + n\pi + \beta\right) + C \cos\left(\frac{\tau}{n} + m\pi + \gamma\right)$$

$$= A \cos\left[(\tau+ \alpha) + mn\pi\right] + B \cos\left[\left(\frac{\tau}{m} + \beta\right) + n\pi\right] + C \cos\left[\left(\frac{\tau}{n} + \gamma\right) + m\pi\right] \qquad (D.16)$$

Equation D.16 can be rewritten in~~to~~ the form of equation D.17 by using the trigonometric relation

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

$$x(\tau+\pi) = A\cos(mn\pi)\cos(\tau+\alpha) + B\cos(n\pi)\cos\left(\frac{\tau}{m}+\beta\right) + C\cos(m\pi)\cos\left(\frac{\tau}{n}+\gamma\right) \quad (D.17)$$

Note that since m,n, and mn are odd integers, then

$$\cos(m\pi) \text{ is } \cos(n\pi) = \cos(mn\tau) = \cos(\pi) = -1$$

Therefore $x(\tau+\pi)$ is $-\left[A\cos(\tau+\gamma) + B\cos\left(\frac{\tau}{m}+\beta\right) + C\cos\left(\frac{\tau}{n}+\gamma\right)\right]$

is $-x(\tau)$

Therefore $x(\tau)$ is an antiperiodic function.

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