

MAXIMUM ENTROPY MODELS
FOR
REPAIRABLE SYSTEMS
AND
SINGLE SERVER QUEUEING SYSTEMS.

By

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TO MY MOTHER.

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ABSTRACT

This thesis deals with an information theoretic approach in obtaining the approximation of :

- i) the first passage time distribution to a certain state j for two-unit paralleled redundant systems; and
- ii) The performance distribution of a single server systems.

The raison d'etre of redundancy and repair maintenance is to improve/maintain the reliability/availability of a system. When a operating component fails, a stand-by component is put into operation, and a redundant configuration continue to function. Failure of an operating component thus causes a stand-by component to be more susceptible to failure because the latter component is now under-load. Examples of these two-unit paralleled redundant system can be found in various industries (Hanley & Kumamoto, Chapter 8). Abundant literature dealing with multi-unit repairable systems is available, for example Osaki & Nakagawa (1976), Lie et. al. (1977), and Reddy & Grosh (1984) have compiled detailed bibliographies of papers concerning multi-unit repairable systems. Most of the papers included in these bibliogra-

phies obtained Laplace-Stieltjes (L-S) transforms of the quantities, such as the reliability and availability of the system in $(0, t]$. In order to obtain above quantities, inversions of the L-S transforms are required. Generally, it is mathematically difficult to obtaining the inversion of these L-S transforms; therefore, approximation methods are desirable. The objective of this thesis, in this part, is to approximate the first passage time distribution of two unit paralleled redundant systems using the Maximum Entropy Principle (MEP) when the information is available in the form of first (or first two) moment(s) of the repair time distribution. It has been shown that the MEP provides a very good approximation for first passage time distribution for the above two unit system when the repair time distribution is either Gamma or Weibull. It has also been shown that the percentage inaccuracy reduces to less than 1% as $t \geq 40$ for the first passage time distribution.

It has been demonstrated by Shore (1982) that an approximation of a single server queueing system can be obtained using Shannon's entropy when the information is available in the form of the first four moments of the service time distribution. Shore's results are only applicable under light load conditions, that is for $\rho < 0.1$. Our results are equally applicable for heavy load conditions, that is $0.0 \leq \rho < 1.0$, as well as light load condition. In this thesis, we consider models in which we apply two other measures of en-

tropy known as Renyi's (1961) Parametric (1967). It has been shown that Renyi's entropy consistently provides a better approximation to busy period distribution and waiting distribution than those obtained using the Shannon or Parametric models. Parametric entropy provides a better approximation than Shannon's but inferior to those obtained using Renyi's model. Our results are obtained under the assumption that the only information available is in the form of first moment of service time distribution. This assumption is more plausible than Shore's because the available information rarely extend beyond the first two moments.

Finally, the concluding remarks and recommendations for further research are provided.

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Chapter I

INTRODUCTION

A great break-through in Information Sciences was provided by Shannon (1948). He proposed

$$S = -k \sum_{i=1}^n p_i(x) \ln p_i(x)$$

as a measure of information or uncertainty for any probability density function $p_i(x)$.

The corresponding case for a continuous random variable x over $(0, \infty)$ and probability density function (p.d.f.) $f(x)$ is given as follows :

$$S = -k \int_0^{\infty} f(x) \ln f(x) dx$$

where k is an arbitrary constant.

Jaynes (1957) while interpreting entropy, stated :

"Just as in applied statistics the crux of a problem is often the devising of some method of sampling that avoids bias, our problem is that of finding a probability assignment which avoids bias, while agreeing with whatever information is given. The great advance provided by information theory lies in the discovery that there is a unique unambiguous criterion for the 'amount of uncertainty' represented by a discrete probability distribution, which agrees with our intuitive notions that a broad distribution represents more uncertainty than does a sharply peaked one, and satisfies all other conditions which make it reasonable."

However, the above statement of Jaynes is equally applicable in case of continuous probability distribution.

Since the pioneering works of Shannon and Jaynes, the concept of entropy has been applied in wide range of applications in the area of business and economics (Herniter(1973), Bass(1974), Kapur, Bector & Kumar(1984), Arizoni(1984), Theil(1969)), behaviour sciences (Shore(1979 a,b), Ramakrishna(1973), Takatsuji(1975), Dynner(1984)), computer modelling (Ferdinard(1970)), system simulation (Chan(1971)), production line decision making (Tribus & Fitts(1968)), stock market analysis (Cozzolio & Zahn (1973)), entropy spectral analysis (Shore(1981)), insurance (Berliner & Lev(1984)), and physical engineering sciences (Tribus(1962)). Kapur (1983) and Tribus (1979) provided detailed bibliographies dealing with this concept. The justification for the use of this concept has been provided by several authors, such as Good (1963), Shore & Johnson (1980), Clough (1964), Thomas (1979) and Kapur (1983).

In the present thesis, applications of entropy to the repairable systems and single server queueing systems have been studied. The present chapter provides (i) notations & symbols, used in this thesis, (ii) various definitions and basic concepts of maximum entropy principle, (iii) an overview of, the problems considered in the thesis, (iv) an overview of the repairable systems and single server queueing systems, and (v) summary of the thesis.

1.1 NOTATIONS AND SYMBOLS

<u>Symbols</u>	<u>Definitions</u>
$B(t)=B(T)=B(x)$	Inverse function of Mill's ratio.
$B'(t)$	First derivative of $B(t)$.
b_i	i th moment of busy period duration.
$c_i; i=1,2$	Langrangian multipliers of Parametric entropy.
C. D. F.	Cumulative distribution function
$f(x)$	Continuous probability density function.
$f^b(t)$	p.d.f. for busy period.
$F^w(t)$	C.D.F. for waiting time.
$F^r(t)$	C.D.F. for residence time.
$G(z)=G(y)=G(t)$	Repair time distribution.
$\overline{G}(t)$	$1 - G(t)$
$g(s)$	Laplace-Stieltjes transform $G(t)$.
$g'(s)$	First derivative of $g(s)$.
$g''(0)$	Second derivative of $g(s)$ at $s=0$.
g_k	k th moment of service time.
H_{ij} $i=0,1; j=0,1,2.$	First passage time distribution from state i to j .
H_{ij}^k $k=1,2; i=0,1;$ $j=0,1,2.$	Information-theoretic approximation to H_{ij} when the first k moments are specified.
h_{ij} $i=0,1; j=0,1,2.$	L-S transform of $H_{ij}(x)$.

I	$\{0, 1, 2, \dots\}$
i.i.d.	Independent identically distributed.
k	Arbitrary constant.
$L-S$	Laplace-Stieltjes Transform.
l_{ij}^k	k th moment of H_{ij} around zero.
$k=1,2; i=0,1;$ $j=0,1,2.$	
$M[\cdot]$	Value of Mill's ratio.
m_i	i th moment of the system.
MEP	Maximum Entropy Principle.
$p_i(x)$	i th discrete probability density function.
p.d.f.	Probability density function
P.D.F.	Probability distribution function.
P_{α_1}	Parametric measure of entropy
$Q_{ij}(t)$	The probability that after making a
$i=0,1; j=0,1,2.$	transition into state i the process next
	makes transition into state j in $(0,t]$.
$Q_1^{2,1}(t)$	The recurrence time transition probability for state 1 via state 2.
$q_{ij}(t)$	L-S transform of $Q_{ij}(t)$
$q_1^{2,1}(t)$	L-S transform of $Q_1^{2,1}(t)$.
R	$1/(a-1)$
H_{α_1}	Renyi's measure of entropy of order α_1 .
r_i	i th moment of residence service time.

S	Shannon's measure of entropy.
s	L-S transform variable.
t, x	Random variables.
T	Inverse value of Mill's ratio.
w_1	First moment of waiting time in queue.
u	Average service rate in queue.
λ	Average arrival rate in queue.
γ	Lagrangian multipliers of Shannon's entropy.
a	Gamma distribution parameter
α_1	Renyi's entropy parameter.
α_2	Parametric measure parameter.
$\beta_i ; \beta_i > 0;$ $i=0,1.$	Failure rate of the system in state i .
ρ	Queueing system's utility factor.
$*$	Stieltjes Convolution.

1.2 DEFINITIONS AND THE BASIC CONCEPTS

This section is devoted to definitions and basic concepts of some technical terms used in this thesis.

Cumulative distribution function (C.D.F.), $F(x)$. It is the probability that a component has failed by time x and is given by :

$$\begin{aligned} F(x) &= \text{prob} (X \leq x) \\ &= \int_0^x f(u) du. \end{aligned}$$

Survivor Function. Survivor function, $\bar{F}(x)$, is defined as a probability that a component has not failed up to time x ,

$$\begin{aligned} \bar{F}(x) &= \text{prob} (X > x) \\ &= 1 - F(x) \\ &= \int_x^{\infty} f(u) du. \end{aligned}$$

Entropy. Entropy is a measure of the uncertainty of or lack of information about a system. Shannon (1949) provided the following expression for the measure of entropy of a system :

$$S = -k \sum_{i=1}^n p_i(x) \ln p_i(x)$$

where k is an arbitrary constant and $P(x) = (p_1(x), p_2(x), \dots, p_n(x))$ is the discrete probability distribution.

Maximum Entropy Principle (MEP). MEP states that by maximizing entropy subject to given constraints one can obtain the unbiased encoding of one's information concerning the system.

Stochastic Process. A Stochastic Process is a collection of random variables that are all defined on the same probability space and indexed by a real parameter.

Reliability. Reliability is a probability that the system will operate for a given time t under a given environment and conditions.

Availability. Availability is a probability that the system will be available for use (operable condition) at any arbitrary instant time under the given condition.

Regenerative Process. A stochastic process $\{ Y(t); t > 0 \}$ is a regenerative process if there exist an epoch, say s_1 , such that the continuation of the process beyond s_1 is a probability replica of the process beginning at time zero.

Renewal Process. Suppose that we have a population of components and their failure time, T , is a continuous random variable with p.d.f. $f(x)$. We start with a new component at time zero. This component fails at time T_1 . Let it be replaced immediately by a new component with failure-time T_2 . Then the second failure will occur after a total time $T_1 + T_2$. The failure-time of r th component used is T_r , and the r th failure occurs at time S_r , where

$$S_r = T_1 + T_2 + \dots + T_r .$$

If $\{T_1, T_2, \dots, T_r\}$ are independent identically distributed (i.i.d.) random variables, all with p.d.f. $f(x)$, we call this system an (ordinary) renewal process.

Poisson Process of rate λ . If the above distribution is exponential with p.d.f. $\lambda \exp(-\lambda x)$.

Two other processes namely modified renewal process and equilibrium process differ slightly from the ordinary renewal process.

The Modified Renewal Process. The modified renewal process is a process in which T_1 have p.d.f. $f_1(x)$ and $T_2, T_3, \dots$ have the same p.d.f. $f(x)$. That is we have all the conditions for an ordinary renewal process, except that the time from the origin to the first failure has a different distribution from the other failure-rate.

The Equilibrium renewal process. The equilibrium renewal process is a modified renewal process, where $T_1 = F(x)/u$, with $F(x)$ being the survivor function and u being the mean failure time.

Markovian Renewal Process (MRP). Let the (X, Y) -process be any two-dimensional stochastic process $\{(X_n, Y_n); n \in I\}$ such that for all $i, j \in S$.

$$\Pr \{ Y_0 = k \} = a_k, \quad k \in S, \quad (a_k \geq 0, \sum_{k \in S} a_k = 1)$$

and

$$\begin{aligned} \Pr \{ Y_{n+1} = j, X_n \leq x \mid Y_0, X_0, \dots, X_{n-1}, Y_n = i \} \\ = \Pr \{ Y_{n+1} = j, X_n \leq x \mid Y_n = i \} = Q_{ij}(x); \quad x \geq 0 \end{aligned}$$

Let us define the functions

$$U_{nk}(t) = \begin{cases} 1 & \text{if } Y = k, X_0 + \dots + X_{n-1} \leq t, n \in I \\ 0 & \text{otherwise} \end{cases}$$

for $k \in I$ and $t \geq 0$, and

$$N_k(t) = \sum_{n=0}^t U_{nk}(t); \quad k \in S, t \geq 0.$$

Then the process $\{(N_k(t); k \in S); t \geq 0\}$ is called a Markovian Renewal Process.

M/M/1 Queueing System. M/M/1 is a queueing system in which the arrival and service rate distribution are Poisson, and the system has only one server to provides services.

M/G/1 Queueing System. M/G/1 is a queueing system in which an arrival rate of the system is poisson distributed, service time distribution is general, and there is only one server available in the system.

G/G/1 Queueing System. G/G/1 is a queueing system with general distribution of arrival and service rate and with only one server.

Residence Time. Residence time is the total time a customer spends in the system.

Waiting Time. Waiting time is the interval period of time between the arrival time and the time at which service begins.

Busy Period. If the system is empty and a customer arrives at time t_1 , and if t_2 is the next time at which the system is empty, then the period between t_1 and t_2 is known as a busy period of the system.

First Passage Time. The first passage time from state i to state j is the elapsed time from the epoch at which the process makes a transition into state j .

1.3 MAXIMUM ENTROPY PRINCIPLE

The Maximum Entropy Principle (MEP) is based on the ancient role of wisdom that one should acknowledge the limitations of one's knowledge, that one should use only the knowledge one has and completely avoid any knowledge which is not available. Therefore, according to MEP when partial information about a random variable is provided, we should choose a probability distribution which is consistent with the given information but has otherwise maximum uncertainty associated with it.

Generally, the available information is in the form of moments of an associated random variable. This information rarely extends beyond the first two moments (Thomas(1979)). In other words, we wish to find a distribution that maximizes the entropy of the system, subject to a set of first moment constraints reflecting the known information about x . This involves solving a maximization problem using one of the following models :

1.3.1 Shannon's Entropy

$$\text{MAX : } S = - \int_a^b f(x) \ln f(x) dx$$

(1.3.1.1)

subject to

$$\int_a^b x^i f(x) dx = m_i, \quad i=1, 2, \dots, r.$$

$$\int_a^b f(x) dx = 1$$

(1.3.1.2)

where (a,b) are known.

If in (1.3.1.2), $r=1$ and $(a,b) = (0,\infty)$ then the solution to (1.3.1.2) is given as (Appendix A) :

$$f(x) = 1/m_1 \exp[-x/m_1], \quad x \geq 0, \quad m_1 \geq 0$$

(1.3.1.3)

On the other hand if in (1.3.1.2), $r=2$ and $(a,b) = (0,\infty)$, then (refer to Appendix B for the detail derivation) :

$$f(x) = A(\gamma_1, \gamma_2) \exp [-\gamma_1 x - \gamma_2 x^2]$$

(1.3.1.4)

where

$$A(\gamma_1, \gamma_2) = \sqrt{(2\gamma_1)} B(x)$$

$$\gamma_1 = T\sqrt{2\gamma_2}$$

$$\gamma_2 = \frac{[B'(x)]^2}{2\{B(x) m_1\}^2}$$

(1.3.1.5)

and

$B(x)$ is the inverse function of Mill's ratio,

$B'(x)$ is the first derivative of $B(x)$,

T is the value of Mill's ratio which is determined by using:

$$\{m_2(m_1)^{-2}\} - 1 = F(T) \quad (1.3.1.6)$$

where $2m_1^2 \geq m_2$

The selected values of $F(T)$ and corresponding values are provided in Table 2.16.

For the cases where $2m_1^2 < m_2$, the MEP solution does not exist for (1.3.1.2), Dowson & Wragg (1973).

1.3.2 Renyi's Entropy

$$\text{MAX : } H_{\alpha_1} = \frac{1}{1-\alpha_1} \ln \left\{ \frac{\int_a^b f^{\alpha_1}(x) dx}{\int_a^b f(x) dx} \right\}$$

subject to

$$\int_a^b x^i f(x) dx = m_i, \quad i=1, 2, \dots, r.$$

$$\int_a^b f(x) dx = 1$$

(1.3.2.1)

where (a,b) are known.

If in (1.3.2.1), $r=1$, $\int_a^b f(x)dx = 1$ and $(a,b) = (0,\infty)$ then solution to (1.3.2.1) is given as (Appendix G) :

$$f(x) = (E + F x)^{1/(1-a_1)}$$

(1.3.2.2)

where

$$E = \left[\frac{(R+2) m_1}{(R+1)} \right]^{-1/R} ; \quad R < -2.$$

$$F = - \frac{R+1}{R+1} \cdot \frac{E}{R+1}.$$

$$a_1 = 1 + 1/R$$

In (1.3.2.1), as a_1 tends to 1, Renyi's entropy gives Shannon's entropy (refer to Appedix I for detail).

1.3.3 Parametric Entropy

$$\begin{aligned} \text{MAX : } P_{\alpha_2} &= - \int_a^b f(x) \ln f(x) dx \\ &+ 1/a_2 \int_a^b (1+a_2 f(x)) \ln(1+a_2 f(x)) dx - 1 \end{aligned}$$

subject to

$$\int_a^b x^i f(x) dx = m_i, \quad i=1, 2, \dots, r.$$

$$\int_a^b f(x) dx = 1$$

(1.3.3.1)

where (a,b) is known.

If $r=1$ in (1.3.3.1) and $(a,b)=(0,\infty)$ then the solution to (1.3.3.1) is as follows (Appendix H) :

$$f(x) = \frac{\exp(-c_1 - c_2 x)}{1 - a_2 \exp(-c_1 - c_2 x)} \quad (1.3.3.2)$$

where

$$c_2 = 1/m_1$$

$$\exp(-c_1) = \frac{1 - \exp(-a_2/m_1)}{a_2}$$

In (1.3.3.1), as a_2 tends to 0, Parametric entropy gives Shannon's entropy (Appendix I).

1.4 OVERVIEW OF THE PROBLEMS

This thesis deals with two problems :

1) Two Unit Paralleled Redundant System.

In the study of stochastic behavior of two unit paralleled redundant system with repair maintenance and under the assumptions that (i) the failure time distribution is exponential, and (ii) the repair time distribution is Gamma or Weibull, it is of interest to know the following :

- a) what is the mean first passage time to a certain state j ; and
- b) what is the limiting probability that the process will be in a certain state j .

where state j ($j=0,1,2$) corresponds to the number of units under repair,

The method to determine the exact values of the first passage time distribution involves the inversion of L-S transforms as given in Nagakawa & Osaki (1975). Generally, the process of inversion of these L-S transforms is cumbersome. Therefore, approximate solutions are desirable. The purpose of this study is to use the Maximum Entropy Principle (MEP) to obtain the approximate first passage time distribution.

2) Single Server Queueing Systems.

In single server queueing system, one is interested in knowing the following :

- i) Busy period distribution ;
- ii) Waiting time distribution; and
- iii) Residence time distribution.

Shore (1982) has provided a method to obtain the approximate value of distributions using Shannon's entropy. His results are only valid for light load conditions ($\rho < 0.1$) and also when the available information is in the form of the first four moments of the service time distribution. In this thesis, we improve the results obtained by Shore by using two other entropy measures, namely Renyi's model and the Parametric model. The results obtained are valid not only for light load but equally for heavy load condition ($0.0 \leq \rho < 1.0$).

1.5 REPAIRABLE SYSTEMS

An equipment may be unreliable due to various reasons such as infant mortalities, random failures, and wear-out. High reliability of each component of an equipment does not ensure high system reliability. There are several methods of improving system reliability. These include (i) introduction of redundancy -- which means more units are available for performing the system functions than required, and (ii) repair and maintenance.

There are two major types of redundancies: paralleled redundancy and stand-by redundancy. A paralleled redundant system with n components is one in which all the components operate simultaneously but the system operation requires at least one operable component. System failure occurs only when all the components fail. A stand-by redundant system with n components is one in which the components operate one at a time and when an operating component fails, a unit from the spares pool is switched on.

When the cost of a single unit is considerable and/or when the system is required to be used for a long period of time, it may be worthwhile to introduce a repair facility so that the failed units may be repaired and put back into operation. In one part of this thesis, we are mainly concerned with systems in which there is one repair facility and the system is comprised of two units operating as stand-by redundant system.

Reddy and Grosh (1984), Srinivasan and Subramanian (1980) have compiled detailed bibliographies of the literature dealing with this area. Initial attempts in the study of 2-unit systems can be found in Epstein & Hosford (1960), where they discussed a system where both the failure and repair time distributions are exponential. Gavar (1963) further generalized the results of Epstein & Hosford. He assumed the failure time distribution as exponential and repair time distribution as general. He also derived the availability of the system using the supplementary variable techniques. Gnedenko (1970) and Buzacott (1971) have discussed several variation of such a system using renewal theoretical approach. Srinivasan (1968) and Arora (1976) have used Markov and Renewal processes to analyse two unit systems. Barrow & Proschan (1965) provided a summary of many related problems and discussed the possibility of analysis of such systems. Nakagwa & Osaki (1975) obtained of such a system using a modified regeneration point technique. The model proposed by Nakagwa & Osaki is described as follows :

Consider a paralleled system, consisting of two identical units and a repair facility. Both units commence operation at time $t=0$. Upon failure, a unit is taken up for repair provided that the repair facility is free. It is assumed that the repair facility is always open, repairs are perfect and the repair discipline is on First Come First Serve

(FCFS) basis. If the two units fail simultaneously, there will be a queue until the repair facility is free. It is also assumed that the failure rates in state k ($k=0,1$) are β_k , and are time independent. State k , where k ($k=0,1,2$) corresponds to the number of inoperable units, that is the number of units under repair or waiting for repair.

Let $Q_{ij}(t)$ denotes the mass function of the probability that after making a transition into state i , the process next makes a transition into state j , in an amount of time less than or equal to t .

Using the Markov Renewal Process theory (MRP), one can obtain :

$$\begin{aligned} Q_{01}(t) &= 1 - \exp(-\beta_0 t) \\ Q_{10}(t) &= \int_0^t \exp(-\beta_1 t) dG(t) \\ Q_{12}(t) &= \int_0^t \beta_1 \exp(-\beta_1 t) \bar{G}(t) dt \end{aligned} \tag{1.5.1}$$

where

$$\bar{G}(t) = 1 - G(t)$$

$G(t)$ is the arbitrary repair time distribution.

However, it is important to obtain the mass function $Q_{21}(t)$ explicitly because state 2 is not a regeneration point and $Q_{21}(t)$ depends on the previous record of how long the repair lasted.

Let $Q_1^2(t)$ be the recurrence time distribution for state 1 via state 2 which has the mass function :

$$Q_1^2(t) = \int_0^t [1 - \exp\{-\beta_1 t\}] dG(t) \quad (1.5.2)$$

We define the Laplace-Stieltjes (LS) transform $q_{ij}(s)$ and $q_1^2(s)$ of the mass function $Q_{ij}(t)$ and $Q_1^2(t)$, respectively, and obtain :

$$q_{01}(s) = \beta_0 / (s + \beta_0) \quad (1.5.3)$$

$$q_{10}(s) = g(s + \beta_1) \quad (1.5.4)$$

$$q_{12}(s) = \beta_1 [1 - g(s + \beta_1)] [s + \beta_1]^{-1} \quad (1.5.5)$$

$$q_1^2(s) = g(s) - g(s + \beta_1) \quad (1.5.6)$$

where $g(s)$ is the L-S transform of $G(t)$.

Now if we define $H_{ij}(t)$ and l_{ij} be the first-passage time distribution and the mean first-passage time from state i to j , ($i=0,1$; $j=0,1,2$), respectively.

Let $H_{11}(t)$ be the recurrence time distribution for state 1 and we have :

$$H_{11}(t) = Q_{10}(t) * Q_{01}(t) + Q_1^2(t) \quad (1.5.7)$$

Taking the L-S transform on both sides of (1.5.3), (1.5.5), (1.5.6) and (1.5.7) :

$$h_{11}(s) = g(s) - s g(s + \beta_1) (s + \beta_0)^{-1} \quad (1.5.8)$$

The mean first-passage time l_{11} is obtained from $h_{11}(s)$:

$$(d/ds) h_{11}(s) = g'(s) -$$

$$(s + \beta_0)[sg'(s + \beta_1) + g(s + \beta_1) - sg(\beta_1 + s)]\{s + \beta_0\}^{-2}$$

at $s = 0$,

$$(d/ds)h_{11}(s) = -l_{11}$$

$$= g'(0) - \beta_0 g(\beta_1) [\beta_0]^{-2}$$

$$= g'(0) - g(\beta_1) \beta_0^{-1}$$

or

$$- l_{11} = -(1/\beta_0) - g(\beta_1) \beta_0$$

$$l_{11} = 1/u + g(\beta_1)\beta_0 \quad (1.5.9)$$

where

$$1/u = \int_0^{\infty} t \, dG(t), \text{ the mean repair time.}$$

The first-passage time distribution $H_{12}(t)$ is given by the renewal equation :

$$H_{12}(t) = Q_{12}(t) + Q_{10}(t) * Q_{01}(t) * H_{12}(t) \quad (1.5.10)$$

Similarly,

$$H_{10}(t) = Q_{10}(t) + Q_{12}(t) * H_{10}(t) \quad (1.5.11)$$

Let us consider the first passage time distribution represented by $H_{0j}(t)$ ($j=0, 2$), then we have

$$H_{0j}(t) = Q_{01}(t) * H_{1j}(t), \quad j = 0, 2. \quad (1.5.12)$$

Taking the L-S transforms on both sides of (1.5.10) - (1.5.12), we have

$$h_{00}(s) = \frac{\beta_0 \, q(s+\beta_1)}{[s+\beta_0][1-g(s)+g(s+\beta_1)]} \quad (1.5.13)$$

$$h_{02}(s) = \frac{\beta_0 \beta_1 [1-g(s+\beta_1)]}{[s+\beta_1][s+\beta_0(1-g(s+\beta_1))]} \quad (1.5.14)$$

The mean first-passage time are :

$$\begin{aligned} l_{00}^1 &= -\frac{d}{ds} h_{00}(s=0) \\ &= 1/\beta_0 + [u g(\beta_1)]^{-1} \end{aligned} \tag{1.5.15}$$

$$\begin{aligned} l_{02}^1 &= -\frac{d}{ds} h_{02}(s=0) \\ &= 1/\beta_1 + [\beta_0 \{1-g(\beta_1)\}]^{-1} \end{aligned} \tag{1.5.16}$$

1.6 SINGLE SERVER QUEUEING SYSTEMS

Since the early development of the Theory of Queues by A. K. Erlang in the study of telephone traffic problem, queueing theory has had extensive applications in industry. Kleinrock (1976), Meantzs (1975) studied its applications to performance modeling and analysis of computer systems. The concept of queueing theory has been applied in wide range of applications in the area of Data Processing (Krell & Arimino(1982)), Inventory (Graves(1982), Steudel(1977)), Human behaviour (McKeown(1979), Gillian(1979)), Business & Economics (Brown(1979), Grassmann(1979), Adiri & Yechiali(1974)), Simulation (Huang & Clayton(1983), Rasmussen(1984), Adam(1983), Iglehart & Shedler(1974)), Computer(Sockut(1978)) and Scheduling (Harrison(1975a,b)).

Consider the $M/M/1$ queueing system in which customers arrive independently with exponentially distributed interarrival times at an average rate λ from an infinite customer pool; they wait in an infinite capacity queue; they are served independently by a single server with exponentially distributed service times at an average rate μ ; and they return to the customer pool.

The probability density function for the $M/M/1$ queueing system busy period length is given as follows (Kleinrock(1975), pp 213-214) :

$$f(x) = [x\sqrt{\lambda/u}]^{-1} \exp[-(\lambda+u)x] I_1(2x\sqrt{\lambda u}) \quad (1.6.1)$$

where I_1 is the modified Bessel function of the first kind (order one).

In the M/M/1 queueing system, the corresponding cumulative distribution function of residence time is given (Kleinrock(1975), p 202) as :

$$F(x) = 1 - \exp(-u(1-\rho)x), \quad x > 0 \quad (1.6.2)$$

Similarly, the cumulative distribution function of waiting time can be written (Kleinrock(1975), p 203) as :

$$F(x) = 1 - \rho \exp(-u(1-\rho)x), \quad x > 0 \quad (1.6.3)$$

Shore(1982) proposed a method to approximate the above distributions (1.5.1, 1.5.2, 1.5.3) through the use of MEP. However, his results are only good for light load condition, $\rho < 0.1$. As this utility factor increases, the absolute percentage error increases exponentially. Shore's models and formulations can be described as follows :

Consider the probability density $f^b(x)$ for the length of the busy period. In general, the moments $b = \int_0^{\infty} dx \, x \, f(x)$ can be expressed in terms of β and the service time moments $g_1, g_2, \dots, g_n$. For b_1 the explicit formula is given as (Kleinrock(1975), p 213) :

$$b_1 = g_1[1-\beta g_1]^{-1} \quad (1.6.4)$$

where $\rho = \lambda g_1$

If g_1 is known, then b_1 can be determined. Using (1.3.1.3), the result of the maximum entropy solution for $f^b(x)$ can be written as :

$$f^b(x) = 1/b_1 \exp(x/b_1) \quad (1.6.5)$$

where $b_1 = m_1$

(1.6.5) provides a good light-load approximation of the busy period in the case of M/M/1 queue.

The first moment r_1 of the residence time probability density can be expressed in terms of the service time moment g_1 by :

$$r_1 = \frac{\lambda g_2}{2(1-\rho)} + \frac{\rho}{\lambda} \quad (1.6.6)$$

For the M/M/1 system, this reduces to

$$r_1 = \frac{\rho}{\lambda(1-\rho)}. \quad (1.6.7)$$

Setting $r_1 = m_1$, then (1.3.1.3) gives the exact M/M/1 solution.

Let $f^w(x)$ be the waiting time probability density with moments w_i . Then w_1 can be expressed in terms of g_2 and can be written as :

$$w_1 = \frac{g_2}{2(1-\rho)}. \quad (1.6.8)$$

In the M/M/1 case, it becomes

$$w_1 = \frac{\rho}{u(1-\rho)}. \quad (1.6.9)$$

Similarly, by setting $w_1 = m_1$ in (1.3.1.3), we can obtain :

$$f^w(x) = (u/\rho)(1-\rho)\exp(-u(1-\rho)x/\rho) \quad (1.6.10)$$

The exact M/M/1 result reported by (Kleingrock(1975), p 203) has the same mean w_1 , but (1.6.10) lacks the impulse term at $x=0$ that results from the finite probability that the system is empty when a customer arrives.

1.7 SUMMARY OF THE THESIS

In Chapter 2, the approximate first passage time distributions of two unit redundant repairable systems are obtained using Shannon's measure of entropy. The repair time distribution for the two unit system is assumed to be gamma distributed.

Chapter 3 provides the first passage time distribution of two unit repair system when the repair time distribution is given by the Weibull distribution. It is shown in Chapters 2 and 3 that MEP provides a very good approximation to the actual first passage time distribution.

Chapter 4 deals with the single server queueing system. In this chapter, we generalized the results obtained by Shore by using two other entropy models, namely Renyi's model and the Parametric model. It is shown in this chapter that Renyi's entropy provide a consistently a better approximation results for busy period distribution and waiting time distribution than Parametric and Shannon's entropy. However, Shannon's entropy does lead to an exact residence time distribution. In this chapter, we also show the relationship between the mean service time, traffic intensity and the optimum values of parameters values a_1 , in the case of Renyi's measure, and a_2 , in the case of Parametric measure, for busy period distribution, residence time distribution and waiting time distribution.

In Chapter 5, we provide concluding remarks and recommendations for further research work.

Chapter II

TWO UNIT PARALLELED REDUNDANT SYSTEM WITH GAMMA REPAIR TIME DISTRIBUTION.

This chapter deals with the information-theoretic approach to obtain an approximation of quantities, such as first passage time and mean first passage time distribution, for two unit paralleled redundant systems. The repair time distribution of these systems is represented by the Gamma distribution. The results obtained by Nakagawa & Osaki (1975), with the application of a modified Markov Renewal Process, are compared with the ones obtained by using the Maximum Entropy Principle. It is proved, empirically, that Shannon's Entropy provides a very close approximation when the information is available in the form of the first two moments of repair time distribution. The details of the Nakagawa and Osaki models are provided in section 1.5, and the definitions and the notations used in this chapter are provided in section 1.3.

2.1 THE RECURRENCE TIME FOR STATE 1 : FIRST-PASSAGE TIME ($H_{11}(T)$) AND THE MEAN FIRST-PASSAGE TIME (L_{11})

In this section, we consider the recurrence first passage time and mean first passage time for state 1. The System is in the state 1 when only one unit is operable and the other unit is being repaired or waiting for repair.

Using (1.5.7),

$$H_{11}(t) = Q_{10}(t) * Q_{01}(t) + Q_{1^2}(t) \quad (2.1.1)$$

Combining (2.1.1) and (1.5.1), we can obtain :

$$\begin{aligned} Q_{10}(t) * Q_{01}(t) &= \int_0^t e^{-\beta_1 y} \cdot dG(y) \cdot (1 - e^{-\beta_0(t-y)}) \\ &= \int_0^t e^{-\beta_1 y} \cdot dG(y) \\ &\quad - \int_0^t e^{-\beta_0(t-y) - \beta_1 y} \cdot dG(y) \end{aligned} \quad (2.1.2)$$

Assuming the repair time distribution $G(y)$ is represented by Gamma, then, $g(y, \theta, \alpha)$, can be obtained as follows :

$$g(y, \theta, \alpha) = \frac{\exp(-y\theta) y^{\alpha-1}}{\Gamma \alpha}$$

Therefore,

$$\begin{aligned} \int_0^t e^{-\beta_1 y} \cdot dG(y) &= \int_0^t \frac{e^{-\beta_1 y} \cdot e^{-y\theta} y^{\alpha-1} \cdot \theta^\alpha}{\Gamma \alpha} \cdot dy \\ &= \frac{\theta^\alpha}{(\theta + \beta_1)^\alpha} \cdot \frac{1}{\Gamma \alpha} \int_0^t e^{-(\beta_1 + \theta)y} y^{\alpha-1} (\beta_1 + \theta)^\alpha \cdot dy \end{aligned}$$

$$\text{let } z = (\beta_1 + \theta)y \quad ; \quad dz = (\beta_1 + \theta)dy$$

$$\text{when } y=0, z=0 \quad ; \quad y=t, z=(\beta_1 + \theta)t$$

Therefore,

$$\begin{aligned} \int_0^t e^{-\beta_1 y} \cdot dG(y) &= \frac{\theta^\alpha}{(\beta_1 + \theta)^\alpha} \cdot \frac{1}{\Gamma \alpha} \cdot \int_0^{(\beta_1 + \theta)t} e^{-z} \cdot \left(\frac{z}{\beta_1 + \theta}\right)^{\alpha-1} \cdot (\beta_1 + \theta)^{\alpha-1} \frac{dz}{(\beta_1 + \theta)} \\ &= \frac{\theta^\alpha}{(\beta_1 + \theta)^\alpha} \cdot \frac{1}{\Gamma \alpha} \int_0^{(\beta_1 + \theta)t} e^{-z} \cdot z^{\alpha-1} \cdot dz \end{aligned}$$

(2.1.3)

And now,

$$\begin{aligned} \int_0^t e^{-\beta_0(t-y) - \beta_1 y} \cdot dG(y) &= e^{-\beta_0 t} \int_0^t e^{-(\beta_1 - \beta_0)y} \cdot dG(y) \quad ; \quad \beta_1 \geq \beta_0 \\ &= e^{-\beta_0 t} \int_0^t e^{-(\beta_1 - \beta_0)y} \cdot e^{-y\theta} \cdot \frac{1}{\Gamma \alpha} \cdot \\ &\quad \cdot y^{\alpha-1} \cdot \theta^\alpha \cdot dy \quad ; \quad \beta_1 \geq \beta_0 \end{aligned}$$

$$\begin{aligned} &= \theta^\alpha \cdot e^{-\beta_0 t} \cdot \frac{1}{\Gamma \alpha} \int_0^t e^{-(\theta + \beta_1 - \beta_0)y} \cdot \\ &\quad \cdot y^{\alpha-1} \cdot (\theta + \beta_1 - \beta_0)^\alpha \cdot dy \quad ; \end{aligned}$$

$$\beta_1 \geq \beta_0$$

let

$$z = (\theta + \beta_1 - \beta_0)y$$

then

$$dz = (\theta + \beta_1 - \beta_0)dy$$

$$\text{and } y=0, z=0; y=t, z=(\theta + \beta_1 - \beta_0)t$$

$$\begin{aligned} & \int_0^t e^{-\beta_0(t-y) - \beta_1 y} \cdot dG(y) \\ &= \frac{\theta^\alpha \cdot e^{-\beta_0 t}}{(\theta + \beta_1 - \beta_0)^\alpha} \cdot \frac{1}{\Gamma(\alpha)} \int_0^{(\theta + \beta_1 - \beta_0)t} e^{-z} \cdot \left(\frac{z}{\theta + \beta_1 - \beta_0} \right)^{\alpha-1} \cdot \frac{dz}{(\theta + \beta_1 - \beta_0)} \\ & \quad \cdot (\theta + \beta_1 - \beta_0)^\alpha \\ &= \frac{\theta^\alpha \cdot e^{-\beta_0 t}}{(\theta + \beta_1 - \beta_0)^\alpha} \cdot \frac{1}{\Gamma(\alpha)} \int_0^{(\theta + \beta_1 - \beta_0)t} e^{-z} \cdot z^{\alpha-1} \cdot dz \end{aligned}$$

(2.1.4)

Combining (2.1.2 - 2.1.4),

$$\begin{aligned} Q_{10}(t) * Q_{01}(t) &= \frac{\theta^\alpha}{(\beta_1 + \theta)^\alpha} \cdot \frac{1}{\Gamma(\alpha)} \int_0^{(\beta_1 + \theta)t} e^{-z} \cdot z^{\alpha-1} \cdot dz \\ & \quad - \frac{\theta^\alpha \cdot e^{-\beta_0 t}}{(\theta + \beta_1 - \beta_0)^\alpha} \cdot \frac{1}{\Gamma(\alpha)} \int_0^{(\theta + \beta_1 - \beta_0)t} e^{-z} \cdot z^{\alpha-1} \cdot dz \end{aligned}$$

(2.1.5)

We know from (1.5.2),

$$\begin{aligned} Q_{1,2,1}(t) &= \int_0^t (1 - e^{-\beta \cdot y}) \cdot dG(y) \\ &= \int_0^t dG(y) - \int_0^t e^{-\beta \cdot y} \cdot dG(y) \end{aligned}$$

(2.1.6)

From the Gamma distribution, we have

$$\int_0^t dG(y) = \int_0^t \frac{e^{-y\theta} \cdot y^{\alpha-1} \cdot \theta^{\alpha}}{\Gamma_{\alpha}} dy$$

Let $y = z$ then $dy = dz$

if $y = 0, z=0; y=t, z=t$

$$\begin{aligned} \int_0^t dG(u) &= \int_0^{\theta t} \frac{e^{-z} \cdot \left(\frac{z}{\theta}\right)^{\alpha-1} \cdot \theta^{\alpha}}{\Gamma_{\alpha}} \cdot \frac{dz}{\theta} \\ &= \int_0^{\theta t} \frac{e^{-z} \cdot z^{\alpha-1}}{\Gamma_{\alpha}} \cdot dz \end{aligned}$$

(2.1.7)

From (2.1.5) and (2.1.7),

$$\begin{aligned}
H_{11}(t) &= \frac{\theta^\alpha}{(\beta_1 + \theta)^\alpha} \int_0^{(\beta_1 + \theta)t} \frac{e^{-z} \cdot z^{\alpha-1}}{\Gamma(\alpha)} dz - \frac{\theta^\alpha \cdot e^{-\beta_0 t}}{(\theta + \beta_1 - \beta_0)^\alpha} \int_0^{(\theta + \beta_1 - \beta_0)t} \frac{e^{-z} \cdot z^{\alpha-1}}{\Gamma(\alpha)} dz \\
&\quad + \int_0^{\theta t} \frac{e^{-z} \cdot z^{\alpha-1}}{\Gamma(\alpha)} dz - \frac{\theta^\alpha}{(\beta_1 + \theta)^\alpha} \int_0^{(\beta_1 + \theta)t} \frac{e^{-z} \cdot z^{\alpha-1}}{\Gamma(\alpha)} dz \\
&= \int_0^{\theta t} \frac{e^{-z} \cdot z^{\alpha-1}}{\Gamma(\alpha)} dz \\
&\quad - \frac{\theta^\alpha \cdot e^{-\beta_0 t}}{(\theta + \beta_1 - \beta_0)^\alpha} \int_0^{(\theta + \beta_1 - \beta_0)t} \frac{e^{-z} \cdot z^{\alpha-1}}{\Gamma(\alpha)} dz
\end{aligned}
\tag{2.1.8}$$

Similarly,

$$H_{00}(t) = Q_{01}(t) * H_{10}(t) \tag{2.1.9}$$

$$H_{02}(t) = Q_{01}(t) * H_{12}(t) \tag{2.1.10}$$

It is evident from (2.1.9) and (2.1.10) that the values of $H_{00}(t)$ and $H_{02}(t)$ cannot be obtained in a closed form. Let $H_{11}^1(t)$ denote as first-passage time distribution for MEP solution when the only information available is in the form of the first moment (l_{11}^1) and $h_{11}^1(t)$ as the density of $H_{11}^1(t)$.

From (1.3.1.3), we have

$$h_{11}^1(t) = 1/l_{11}^1 \exp(-x/l_{11}^1)$$

where l_{11}^{-1} is the mean recurrence time of state 1,

$$\begin{aligned} H_{11}^{-1}(t) &= \int_0^\infty h_{11}^{-1}(t) \, dx \\ &= 1 - \exp(-t/l_{11}^{-1}) \end{aligned} \quad (2.1.11)$$

Rewriting (1.5.9),

$$l_{11}^{-1} = 1/u + g(\beta_1) / \beta_0 \quad (2.1.12)$$

In the above (2.1.12), we need to determine $g(\beta_1)$. Since $g(\beta_1)$ is L-S transform of $G(t)$; therefore,

$$\begin{aligned} g(\beta_1) &= \int_0^\infty \exp(-\beta_1 y) \, dG(y) \\ &= \frac{1}{\Gamma_\alpha} \int_0^\infty e^{-\beta_1 y} \cdot e^{-y\theta} \cdot y^{\alpha-1} \cdot \theta^\alpha \cdot dy \\ &= \frac{1}{\Gamma_\alpha} \int_0^\infty e^{-(\beta_1+\theta)y} \cdot y^{\alpha-1} \cdot dy \end{aligned}$$

The above expression can be rewritten to represent the complete Gamma function,

$$\begin{aligned} g(\beta_1) &= \frac{\theta^\alpha}{(\beta_1+\theta)^\alpha} \cdot \frac{1}{\Gamma_\alpha} \int_0^\infty e^{-(\beta_1+\theta)y} \cdot y^{\alpha-1} \cdot (\beta_1+\theta)^\alpha \cdot dy \\ &= \frac{\theta^\alpha}{(\beta_1+\theta)^\alpha} \end{aligned} \quad (2.1.13)$$

Substituting (2.1.11) in (2.1.9),

$$\begin{aligned} l_{11}^1 &= 1/u + g(\beta_1)/\beta_0 \\ &= 1/u + (\theta/\beta_1 + \theta)^\alpha/\beta_0 \end{aligned}$$

Similarly, if the information is available in the form of first two moments (that is l_{11}^1 and l_{11}^2), then using MEP the distribution $f(t)$ is given by (refer to section 1.3.1).

$$H_{11}^2(t) = A(\gamma_1, \gamma_2) \exp [-\gamma_1 x - \gamma_2 x^2] \quad (2.1.14)$$

where γ_1, γ_2 are the Langrangian multipliers and can be determined through the procedure given in Appendix C. From Appendix D we can write l_{11}^2 as follows :

$$\begin{aligned} l_{11}^2 &= g''(0) - 2g'(\beta_1)/\beta_0 \\ &\quad + 2g(\beta_1)/(\beta_0)^2 \end{aligned} \quad (2.1.15)$$

By selecting any combination of values of parameters $\alpha, \beta_0, \beta_1, \theta$, the exact value of $H_{11}(t)$, (2.1.8), can be determined. The corresponding approximate distributions ($H_{11}^1(t)$ and $H_{11}^2(t)$) can be determined using (2.1.9) and (2.1.12) respectively.

The detailed steps involved in determining $H_{11}^2(t)$ are given in Appendix C.

2.2 THE FIRST-PASSAGE TIME AND THE MEAN FIRST-PASSAGE TIME FROM STATE 0 TO STATE 2, ($H_{02}(x)$ AND L_{02})

Let $H_{02}^i(x)$ represents the first passage time from state 0 to 2 in the entropy formulation when the information is available in the form of first i moments.

Let l_{02} represents the mean first-passage time from state 0 to state 2 for $H_{02}(x)$.

From appendix E, we have:

$$l_{02}^1 = 1/\beta_1 + \{\beta_0[1-g(\beta_1)]\}^{-1} \quad (2.2.1)$$

and

$$\begin{aligned} l_{02}^2 = & 2g'(\beta_1)[1-\beta_0g'(\beta_1)]\{\beta_0[1-g(\beta_1)]^2\}^{-1} \\ & + 2/\beta_1^2 + 2g'(\beta_1)/\{\beta_1[1-g(\beta_1)]\} \\ & + 2[1-\beta_0g'(\beta_1)]^2/\{\beta_0[1-g(\beta_1)]\}^2 \\ & + 2[1-\beta_0g'(\beta_1)]/\{\beta_0\beta_1[1-g(\beta_1)]\} \end{aligned} \quad (2.2.2)$$

Proceeding in a similar fashion as section 2.1, we can obtain :

$$H_{02}^1(x) = 1 - \exp(-t/l_{02}^1) \quad (2.2.3)$$

and

$$H_{02}^2(x) = A(\gamma_1, \gamma_2) \exp[-\gamma_1 x - \gamma_2 x^2] \quad (2.2.4)$$

2.3 THE RECURRENCE TIME FOR STATE 0 : FIRST-PASSAGE TIME (H₀₀(X) AND MEAN FIRST-PASSAGE TIME (L₀₀)).

Let $H_{00}^i(x)$ represents the first recurrence passage time of the system for state 0 in the entropy formulation when the information is available in the form of first i moments.

Let l_{00} represents the mean first recurrence time for state 0 in $H_{00}(x)$.

From Appendix F, we have

$$l_{00}^1 = 1/\beta_0 + 1/\{ug(\beta_1)\} \quad (2.3.1)$$

$$l_{00}^2 = 2/\beta_0^2 + 2/\{\beta_0 ug(\beta_1)\} \\ + g''(0)/g(\beta_1) - 2[-g'(\beta_1) - 1/u]\{u[g(\beta_1)]\}^{-1} \quad (2.3.2)$$

Proceeding in a similar fashion as section 2.1, we can obtain :

$$H_{00}^1(x) = \exp(-t/l_{00}^1) \quad (2.3.3)$$

and

$$H_{00}^2(x) = A(\gamma_1, \gamma_2) \exp [-\gamma_1 t - \gamma_2 t^2] \quad (2.3.4)$$

2.4 MEASURE OF ACCURACY

In this chapter, the percentage difference between the exact distribution and its approximation is considered as a measure of accuracy. In case of (2.1.7) and the entropy approximation (2.1.9 and 2.1.14), the measure of accuracy is given as follows :

$$\text{DIFF}_{i11} = \left| \frac{H_{11}(x) - H_{11i}(x)}{H_{11}(x)} \right| 100, \quad i=1, 2, x>0.$$

The absolute percentage difference between results obtained, using MEP, when the information is provided in the form of (i) first two moments and (ii) the only first moment is given as follows :

$$\text{DIFF} = \left| \frac{H_{11}^2(x) - H_{11}^1(x)}{H_{11}^2(x)} \right| 100, \quad x>0$$

As stated in section 2.1, the exact solution of $H_{00}(x)$ and $H_{02}(x)$ cannot be obtained. Therefore, it is assumed that the results obtained using MEP, when first two moments are provided, will lead to a better approximation of the actual distribution than when only first moment is given.

The following expression is used as a relative measure of accuracy for $H_{0i}^1(t)$:

$$\text{DIFF}_{0i} = \left| \frac{H_{0i}^2(x) - H_{0i}^1(x)}{H_{0i}^2(x)} \right| 100, \quad i=0, 2, x>0$$

2.5 RESULTS AND ACCURACY

The exact distribution for the recurrence time of state 1 has been determined using (2.1.8). Similarly, the values of $H_{11}^1(t)$ and $H_{11}^2(t)$ are computed using (2.1.11) and (2.1.14). The measures of accuracy given in section 2.4 are used. Table 2.1, 2.4, 2.7, 2.10 and 2.13 provide the values of exact distributions $H_{11}(t)$, $H_{11}^1(t)$, $H_{11}^2(t)$, differences DIFF111, DIFF211 and DIFF. For various values of parameters $(\alpha, \beta_0, \beta_1, \theta)$ of Gamma distribution and l_{11}^1 and l_{11}^2 , the above distribution are computed. The corresponding figures of these tables are given in Figure 2.1, 2.4, 2.7, 2.10 and 2.13, respectively. One can observe from these tables that, depending upon the combination of parameters, the values of DIFF111, DIFF211 and DIFF can become insignificant. The percentage differences between $H_{11}(t)$ and $H_{11}^2(t)$ also become insignificant for values of t larger than 25 (Gupta et.al., 1985). Therefore, we suggest that the consideration of an MEP distribution on the basis of higher order moments may not contribute any further significant information. The exact solution of $H_{00}(t)$, $[H_{02}(t)]$, can not be obtained; therefore, the differences between $H_{00}^1(t)$ and $H_{00}^2(t)$ $[H_{02}^1(t), H_{02}^2(t)]$ are considered as a measure of accuracy. The values of these differences, for various values of t are provided in Tables 2.2, 2.5, 2.8, 2.11 and 2.14 [2.3, 2.6, 2.9, 2.12 and 2.15]. Once again it can be noted from these tables that these differences become insignificant for val-

ues of t larger than 25. We justify the application for MEP to approximate $H_{00}(t)$ and $H_{02}(t)$ on the basis of the results obtained for $H_{11}(t)$ as its approximations become insignificant; therefore, we assume that the differences between $H_{00}(t)$, $H_{02}(t)$ and their approximations would also become insignificant for values of t larger than 25.

α θ β_0 β_1
 1 1 1 1

RESULTS OF $H_{11}(t)$

T	L_{11}^1		L_{11}^2		MEAN	VARI
	1.5	3.5	1	1		
T	$H_{11}(t)$	$H_{11}^1(t)$	$H_{11}^2(t)$	DIFF111	DIFF211	DIFF
0.5	0.154818	0.283469	0.205692	83.097874	32.860250	27.4376
1.0	0.399576	0.486583	0.400835	21.774679	0.315030	17.6224
1.5	0.603527	0.632121	0.572427	4.737787	5.152933	9.4433
2.0	0.747645	0.736403	0.712273	1.503683	4.731179	3.2768
2.5	0.842568	0.811124	0.817907	3.731871	2.926922	0.8362
3.0	0.902905	0.864665	0.891861	4.235209	1.223092	3.1453
3.5	0.940517	0.903028	0.939849	3.986008	0.071017	4.0775
4.0	0.963704	0.930517	0.968710	3.443758	0.519396	4.1045
4.5	0.977905	0.950213	0.984797	2.831816	0.704700	3.6396
5.0	0.986570	0.964326	0.993108	2.254631	0.662724	2.9846
5.5	0.991843	0.974438	0.997087	1.754783	0.528725	2.3243
6.0	0.995049	0.981684	0.998853	1.343078	0.382366	1.7489
6.5	0.996995	0.986876	0.999580	1.014961	0.259220	1.2872
7.0	0.998177	0.990596	0.999857	0.759447	0.168272	0.9348
7.5	0.998894	0.993262	0.999955	0.563832	0.106160	0.6738
8.0	0.999329	0.995172	0.999987	0.415993	0.065786	0.4838
8.5	0.999593	0.996541	0.999996	0.305372	0.040339	0.3468
9.0	0.999753	0.997521	0.999999	0.223250	0.024593	0.2484
9.5	0.999850	0.998224	1.000000	0.162665	0.014950	0.1779
10.0	0.999909	0.998727	1.000000	0.118194	0.009076	0.1274
10.5	0.999945	0.999088	1.000000	0.085686	0.005507	0.0913
11.0	0.999967	0.999347	1.000000	0.062001	0.003340	0.0654
11.5	0.999980	0.999532	1.000000	0.044792	0.002026	0.0468
12.0	0.999988	0.999665	1.000000	0.032318	0.001229	0.0336
12.5	0.999993	0.999760	1.000000	0.023292	0.000745	0.0240
13.0	0.999995	0.999828	1.000000	0.016771	0.000452	0.0172
13.5	0.999997	0.999877	1.000000	0.012067	0.000274	0.0123
14.0	0.999998	0.999912	1.000000	0.008676	0.000166	0.0088
14.5	0.999999	0.999937	1.000000	0.006235	0.000101	0.0063
15.0	0.999999	0.999955	1.000000	0.004479	0.000061	0.0045

Table 2.1

α θ β_0 β_1
 1 1 1 1

RESULTS OF THE GAMMA DISTRIBUTION MODEL ---- $H_{02}(t)$

L_{02}^1 L_{02}^2 MEAN VAR1
 3 16 1 1

T	$H_{02}^1(t)$	$H_{02}^2(t)$	DIFF02
0.5	0.153518	0.133915	12.7695
1.0	0.283469	0.252740	10.8403
1.5	0.393469	0.357616	9.1122
2.0	0.486583	0.449689	7.5821
2.5	0.565402	0.530095	6.2446
3.0	0.632121	0.599938	5.0912
3.5	0.688597	0.660284	4.1116
4.0	0.736403	0.712149	3.2935
4.5	0.776870	0.756488	2.6236
5.0	0.811124	0.794191	2.0876
5.5	0.840120	0.826082	1.6709
6.0	0.864665	0.852914	1.3590
6.5	0.885441	0.875370	1.1375
7.0	0.903028	0.894063	0.9928
7.5	0.917915	0.909541	0.9123
8.0	0.930517	0.922290	0.8841
8.5	0.941184	0.932735	0.8977
9.0	0.950213	0.941246	0.9436
9.5	0.957856	0.948146	1.0137
10.0	0.964326	0.953709	1.1009
10.5	0.969803	0.958171	1.1993
11.0	0.974438	0.961731	1.3041
11.5	0.978363	0.964556	1.4112
12.0	0.981684	0.966785	1.5177
12.5	0.984495	0.968536	1.6212
13.0	0.986876	0.969903	1.7199
13.5	0.988891	0.970965	1.8127
14.0	0.990596	0.971786	1.8989
14.5	0.992040	0.972417	1.9761
15.0	0.993262	0.972899	2.0501

Table 2.3

RESULTS OF THE GAMMA DISTRIBUTION MODEL ---- $H_{00}(t)$

L_{00}^1 L_{00}^2 MEAN VAR1
 3 16 1 1

T	$H_{00}^1(t)$	$H_{00}^2(t)$	DIFF00
0.5	0.153518	0.133915	12.7695
1.0	0.283469	0.252740	10.8403
1.5	0.393469	0.357616	9.1122
2.0	0.486583	0.449689	7.5821
2.5	0.565402	0.530095	6.2446
3.0	0.632121	0.599938	5.0912
3.5	0.688597	0.660284	4.1116
4.0	0.736403	0.712149	3.2935
4.5	0.776870	0.756488	2.6236
5.0	0.811124	0.794191	2.0876
5.5	0.840120	0.826082	1.6709
6.0	0.864665	0.852914	1.3590
6.5	0.885441	0.875370	1.1375
7.0	0.903028	0.894063	0.9928
7.5	0.917915	0.909541	0.9123
8.0	0.930517	0.922290	0.8841
8.5	0.941184	0.932735	0.8977
9.0	0.950213	0.941246	0.9436
9.5	0.957856	0.948146	1.0137
10.0	0.964326	0.953709	1.1009
10.5	0.969803	0.958171	1.1993
11.0	0.974438	0.961731	1.3041
11.5	0.978363	0.964556	1.4112
12.0	0.981684	0.966785	1.5177
12.5	0.984495	0.968536	1.6212
13.0	0.986876	0.969903	1.7199
13.5	0.988891	0.970965	1.8127
14.0	0.990596	0.971786	1.8989
14.5	0.992040	0.972417	1.9781
15.0	0.993262	0.972899	2.0501

Table 2.2

α 0 β_0 β_1
 2 1 0.2 0.5

RESULTS OF THE GAMMA DISTRIBUTION MODEL --- $H_{1,1}(t)$

		$L_{1,1}^1$	$L_{1,1}^2$	MEAN	VARI		
		4.22222	34.1481	2	2		
T	$H_{1,1}(t)$	$H_{1,1}^1(t)$	$H_{1,1}^2(t)$	DIFF111	DIFF211	DIFF	
0.5	0.015984	0.111678	0.106968	598.706931	569.236064	4.21792	
1.0	0.083453	0.210884	0.203047	152.697118	143.305918	3.71639	
1.5	0.187802	0.299011	0.289283	59.216526	54.036719	3.25331	
2.0	0.303410	0.377296	0.366629	24.351763	20.835965	2.82730	
2.5	0.412951	0.446838	0.435950	8.206208	5.569354	2.43688	
3.0	0.508321	0.508614	0.498033	0.057811	2.023878	2.08049	
3.5	0.587507	0.563491	0.553594	4.087748	5.772448	1.75650	
4.0	0.651640	0.612240	0.603281	6.046346	7.421128	1.46326	
4.5	0.703073	0.655544	0.647684	6.760104	7.878092	1.19904	
5.0	0.744347	0.694012	0.687335	6.762218	7.659305	0.96215	
5.5	0.777731	0.728184	0.722717	6.370625	7.073638	0.75085	
6.0	0.805070	0.758540	0.754267	5.779576	6.310431	0.56342	
6.5	0.827789	0.785506	0.782378	5.107912	5.485744	0.39817	
7.0	0.846954	0.809460	0.807409	4.426903	4.669124	0.25344	
7.5	0.863352	0.830739	0.829679	3.777421	3.900208	0.12761	
8.0	0.877557	0.849642	0.849480	3.180966	3.199460	0.01910	
8.5	0.889991	0.866434	0.867071	2.646904	2.575263	0.07359	
9.0	0.900966	0.881350	0.882689	2.177220	2.028615	0.15191	
9.5	0.910717	0.894601	0.896544	1.769623	1.556219	0.21725	
10.0	0.919423	0.906372	0.908827	1.419535	1.152479	0.27090	
10.5	0.927225	0.916828	0.919708	1.121341	0.810762	0.31410	
11.0	0.934236	0.926116	0.929339	0.869147	0.524169	0.34800	
11.5	0.940549	0.934367	0.937859	0.657214	0.285991	0.37368	
12.0	0.946241	0.941697	0.945390	0.480195	0.089950	0.39213	
12.5	0.951379	0.948208	0.952042	0.333244	0.069679	0.40427	
13.0	0.956020	0.953992	0.957913	0.212045	0.198033	0.41095	
13.5	0.960214	0.959130	0.963091	0.112806	0.299667	0.41294	
14.0	0.964005	0.963695	0.967655	0.032222	0.378585	0.41094	
14.5	0.967434	0.967749	0.971674	0.032568	0.438287	0.40559	
15.0	0.970535	0.971351	0.975212	0.084031	0.481817	0.39745	

Table 2.4

α 0 β_0 β_1
 2 1 0.2 0.5

RESULTS OF THE GAMMA DISTRIBUTION MODEL ---- $H_{00}(t)$

	L_{00}^1	L_{00}^2	MEAN	VARI
	9.5	137	2	2
T	$H_{00}^1(t)$	$H_{00}^2(t)$	DIFF00	
0.5	0.051271	0.030152	41.1896	
1.0	0.099912	0.060654	39.2929	
1.5	0.146060	0.091438	37.3974	
2.0	0.189842	0.122435	35.5072	
2.5	0.231379	0.153575	33.6265	
3.0	0.270787	0.184786	31.7596	
3.5	0.308174	0.215998	29.9105	
4.0	0.343644	0.247137	28.0835	
4.5	0.377296	0.278133	26.2825	
5.0	0.409222	0.308916	24.5115	
5.5	0.439512	0.339416	22.7743	
6.0	0.468248	0.369567	21.0746	
6.5	0.495512	0.399304	19.4159	
7.0	0.521377	0.428564	17.8014	
7.5	0.545916	0.457291	16.2342	
8.0	0.569197	0.485428	14.7171	
8.5	0.591285	0.512924	13.2526	
9.0	0.612240	0.539732	11.8430	
9.5	0.632120	0.565809	10.4903	
10.0	0.650982	0.591117	9.1962	
10.5	0.668876	0.615621	7.9619	
11.0	0.685853	0.639293	6.7887	
11.5	0.701960	0.662108	5.6772	
12.0	0.717240	0.684047	4.6279	
12.5	0.731737	0.705095	3.6410	
13.0	0.745491	0.725242	2.7153	
13.5	0.758540	0.744481	1.8535	
14.0	0.770920	0.762811	1.0518	
14.5	0.782665	0.780236	0.3103	
15.0	0.793308	0.796762	0.3721	

Table 2.5

RESULTS OF THE GAMMA DISTRIBUTION MODEL ---- $H_{02}(t)$

	L_{02}^1	L_{02}^2	MEAN	VARI
	11	225.2	2	2
T	$H_{02}^1(t)$	$H_{02}^2(t)$	DIFF02	
0.5	0.044437	0.040872	8.02320	
1.0	0.086899	0.080243	7.66007	
1.5	0.127475	0.118161	7.30664	
2.0	0.166247	0.154672	6.96287	
2.5	0.203297	0.189821	6.62868	
3.0	0.238700	0.223652	6.30400	
3.5	0.272529	0.256208	5.98876	
4.0	0.304856	0.287531	5.68287	
4.5	0.335746	0.317662	5.38625	
5.0	0.365264	0.346640	5.09879	
5.5	0.393469	0.374503	4.82039	
6.0	0.420422	0.401289	4.55094	
6.5	0.446176	0.427034	4.29033	
7.0	0.470787	0.451774	4.03843	
7.5	0.494303	0.475544	3.79514	
8.0	0.516775	0.498376	3.56031	
8.5	0.538248	0.520304	3.33381	
9.0	0.558767	0.541358	3.11551	
9.5	0.578374	0.561571	2.90526	
10.0	0.597110	0.580970	2.70292	
10.5	0.615013	0.599586	2.50835	
11.0	0.632121	0.617447	2.32138	
11.5	0.648468	0.634579	2.14138	
12.0	0.664083	0.651009	1.96968	
12.5	0.679016	0.666762	1.80463	
13.0	0.693279	0.681864	1.64657	
13.5	0.706909	0.696339	1.49534	
14.0	0.719933	0.710209	1.35076	
14.5	0.732379	0.723497	1.21269	
15.0	0.744271	0.736226	1.08095	

Table 2.6

α θ β_0 β_1
 2 2 2 2

RESULTS OF THE GAMMA DISTRIBUTION MODEL ---- $H_{11}(t)$

	L_{11}^1	L_{11}^2	MEAN	VARI		
	1.125	1.75	1	0.5		
T	$H_{11}(t)$	$H_{11}^1(t)$	$H_{11}^2(t)$	DIFF111	DIFF211	DIFF
0.2	0.020292	0.162872	0.073998	702.621934	264.657759	54.5667
0.4	0.105293	0.299216	0.161833	184.175617	53.698234	45.9143
0.6	0.235758	0.413354	0.260868	75.329680	10.650710	36.8899
0.8	0.379154	0.508902	0.366936	34.220241	3.222571	27.8965
1.0	0.513606	0.588888	0.474843	14.657531	7.547104	19.3661
1.2	0.628822	0.655846	0.579122	4.297569	7.903706	11.6985
1.4	0.722164	0.711899	0.674844	1.421354	6.552523	5.2052
1.6	0.795015	0.758823	0.758308	4.552436	4.617173	0.0678
1.8	0.850421	0.798103	0.827437	6.152005	2.702669	3.6754
2.0	0.891783	0.830987	0.881825	6.817439	1.116692	6.1178
2.2	0.922239	0.858514	0.922470	6.909796	0.025081	7.4496
2.4	0.944431	0.881558	0.951323	6.657173	0.729848	7.9139
2.6	0.960469	0.900849	0.970779	6.207427	1.073420	7.7627
2.8	0.971986	0.916998	0.983241	5.657345	1.157891	7.2239
3.0	0.980213	0.930517	0.990823	5.069964	1.082416	6.4810
3.2	0.986063	0.941833	0.995205	4.485505	0.927065	5.6668
3.4	0.990208	0.951307	0.997610	3.928602	0.747501	4.8673
3.6	0.993136	0.959238	0.998865	3.413247	0.576823	4.1311
3.8	0.995198	0.965877	0.999486	2.946250	0.430880	3.4797
4.0	0.996646	0.971434	0.999778	2.529672	0.314241	2.9177
4.2	0.997662	0.976087	0.999909	2.162541	0.225232	2.4406
4.4	0.998372	0.979982	0.999964	1.842053	0.159466	2.0391
4.6	0.998868	0.983242	0.999987	1.564401	0.111954	1.7030
4.8	0.999214	0.985972	0.999995	1.325327	0.078152	1.4223
5.0	0.999455	0.988256	0.999998	1.120496	0.054349	1.1882

Table 2.7

α 0 β_0 β_1
2 2 2 2

RESULTS OF THE GAMMA DISTRIBUTION MODEL ---- $H_{00}(t)$

L_{00}^1 L_{00}^2 MEAN VARI
4.5 38.5 1 0.5

T	$H_{00}^1(t)$	$H_{00}^2(t)$	DIFF00
0.2	0.043471	0.041114	5.42250
0.4	0.085053	0.080646	5.18180
0.6	0.124827	0.118651	4.94771
0.8	0.162872	0.155184	4.72017
1.0	0.199263	0.190298	4.49910
1.2	0.234072	0.224043	4.28443
1.4	0.267368	0.256469	4.07609
1.6	0.299216	0.287624	3.87400
1.8	0.329680	0.317554	3.67808
2.0	0.358820	0.346303	3.48823
2.2	0.386693	0.373915	3.30439
2.4	0.413354	0.400430	3.12645
2.6	0.438856	0.425891	2.95433
2.8	0.463250	0.450335	2.78793
3.0	0.486583	0.473800	2.62716
3.2	0.508902	0.496322	2.47192
3.4	0.530250	0.517937	2.32211
3.6	0.550671	0.538679	2.17764
3.8	0.570204	0.558581	2.03840
4.0	0.588888	0.577674	1.90428
4.2	0.606759	0.595988	1.77520
4.4	0.623854	0.613554	1.65103
4.6	0.640205	0.630400	1.53168
4.8	0.655846	0.646553	1.41704
5.0	0.670807	0.662040	1.30700

Table 2.8

RESULTS OF THE GAMMA DISTRIBUTION MODEL ---- $H_{02}(t)$

L_{02}^1 L_{02}^2 MEAN VARI
1.16667 2.27778 1 0.5

T	$H_{02}^1(t)$	$H_{02}^2(t)$	DIFF02
0.2	0.157540	0.124659	20.8712
0.4	0.290260	0.240097	17.2823
0.6	0.402072	0.345849	13.9833
0.8	0.496270	0.441691	10.9978
1.0	0.575627	0.527619	8.3401
1.2	0.642483	0.603834	6.0155
1.4	0.698806	0.670709	4.0207
1.6	0.746256	0.728759	2.3446
1.8	0.786231	0.778609	0.9693
2.0	0.819908	0.820959	0.1262
2.2	0.848279	0.856551	0.9751
2.4	0.872181	0.886143	1.6007
2.6	0.892318	0.910482	2.0357
2.8	0.909262	0.930287	2.3101
3.0	0.923574	0.946230	2.4531
3.2	0.935614	0.958926	2.4916
3.4	0.945757	0.968928	2.4499
3.6	0.954303	0.976723	2.3494
3.8	0.961502	0.982733	2.2081
4.0	0.967567	0.987317	2.0412
4.2	0.972676	0.990776	1.8606
4.4	0.976981	0.993358	1.6763
4.6	0.980607	0.995265	1.4948
4.8	0.983662	0.996658	1.3212
5.0	0.986236	0.997665	1.1508

Table 2.9

α θ β_0 β_1

2.5 0.5 0.1 0.2

RESULTS OF THE GAMMA DISTRIBUTION MODEL ---- $H_{11}(t)$

L_{11}^1 L_{11}^2 MEAN VAR1

9.31201 152.04 5 10

T	$H_{11}(t)$	$H_{11}^1(t)$	$H_{11}^2(t)$	DIFF111	DIFF211	DIFF
3.0	0.116065	0.275421	0.242026	137.298584	108.526104	12.1250
3.5	0.162627	0.313301	0.277622	92.649702	70.710674	11.3880
4.0	0.212945	0.349200	0.311925	63.985719	46.481353	10.6743
4.5	0.265070	0.383222	0.344962	44.573910	30.139719	9.9840
5.0	0.317359	0.415466	0.376757	30.913696	18.716495	9.3170
5.5	0.368520	0.446025	0.407339	21.031440	10.533891	8.6734
6.0	0.417603	0.474985	0.436734	13.740847	4.581068	8.0532
6.5	0.463966	0.502432	0.464969	8.290762	0.216300	7.4563
7.0	0.507219	0.528444	0.492074	4.184518	2.986006	6.8825
7.5	0.547178	0.553096	0.518075	1.081471	5.318778	6.3318
8.0	0.583812	0.576459	0.543003	1.259439	6.990171	5.8038
8.5	0.617200	0.598601	0.566884	3.013449	8.152237	5.2985
9.0	0.647499	0.619585	0.589750	4.310915	8.918710	4.8154
9.5	0.674911	0.639473	0.611628	5.250734	9.376410	4.3543
10.0	0.699665	0.658320	0.632548	5.909185	9.592745	3.9149
10.5	0.722000	0.676183	0.652538	6.345908	9.620792	3.4968
11.0	0.742153	0.693111	0.671628	6.608045	9.502808	3.0996
11.5	0.760351	0.709155	0.689845	6.733180	9.272717	2.7229
12.0	0.776805	0.724360	0.707220	6.751458	8.957915	2.3662
12.5	0.791713	0.738770	0.723779	6.687159	8.580602	2.0291
13.0	0.805250	0.752426	0.739551	6.559882	8.158795	1.7112
13.5	0.817575	0.765369	0.754563	6.385452	7.707100	1.4118
14.0	0.828829	0.777635	0.768844	6.176636	7.237329	1.1305
14.5	0.839135	0.789260	0.782418	5.943702	6.758989	0.8668
15.0	0.848604	0.800277	0.795314	5.694877	6.279675	0.6201
15.5	0.857328	0.810718	0.807557	5.436709	5.805401	0.3899
16.0	0.865392	0.820613	0.819172	5.174372	5.340868	0.1756
16.5	0.872866	0.829991	0.830185	4.911916	4.889686	0.0234
17.0	0.879812	0.838879	0.840620	4.652468	4.454570	0.2076
17.5	0.886284	0.847302	0.850501	4.398402	4.037490	0.3775
18.0	0.892330	0.855285	0.859851	4.151484	3.639810	0.5338
18.5	0.897988	0.862850	0.868692	3.912986	3.262399	0.6771
19.0	0.903296	0.870020	0.877048	3.683784	2.905725	0.8078
19.5	0.908282	0.876815	0.884940	3.464437	2.569929	0.9266
20.0	0.912975	0.883255	0.892388	3.255254	2.254899	1.0340
20.5	0.917397	0.889358	0.899413	3.056347	1.960320	1.1306
21.0	0.921570	0.895142	0.906035	2.867677	1.685718	1.2169
21.5	0.925512	0.900624	0.912273	2.689089	1.430501	1.2934
22.0	0.929239	0.905819	0.918144	2.520340	1.193989	1.3606
22.5	0.932767	0.910743	0.923668	2.361126	0.975437	1.4192
23.0	0.936107	0.915409	0.928861	2.211096	0.774059	1.4695
23.5	0.939273	0.919831	0.933740	2.069872	0.589042	1.5121
24.0	0.942275	0.924022	0.938321	1.937057	0.419561	1.5475
24.5	0.945122	0.927994	0.942620	1.812247	0.264790	1.5760
25.0	0.947825	0.931759	0.946650	1.695036	0.123906	1.5982

Table 2.10

α	θ	β_0	β_1
2.5	0.5	0.1	0.2

RESULTS OF THE GAMMA DISTRIBUTION MODEL ---- $H_{00}(t)$

L_{00}^1	L_{00}^2	MEAN	VARI
21.5955	699.166	5	10

T	$H_{00}^1(t)$	$H_{00}^2(t)$	DIFF00
3.0	0.129700	0.077271	40.4237
3.5	0.149619	0.090438	39.5545
4.0	0.169082	0.103673	38.6849
4.5	0.188099	0.116969	37.8153
5.0	0.206681	0.130320	36.9462
5.5	0.224837	0.143721	36.0779
6.0	0.242579	0.157165	35.2108
6.5	0.259914	0.170646	34.3453
7.0	0.276852	0.184157	33.4818
7.5	0.293403	0.197693	32.6207
8.0	0.309575	0.211247	31.7623
8.5	0.325376	0.224812	30.9071
9.0	0.340816	0.238383	30.0554
9.5	0.355903	0.251952	29.2077
10.0	0.370645	0.265514	28.3643
10.5	0.385049	0.279062	27.5256
11.0	0.399123	0.292589	26.6919
11.5	0.412875	0.306090	25.8637
12.0	0.426313	0.319558	25.0414
12.5	0.439443	0.332987	24.2252
13.0	0.452272	0.346370	23.4156
13.5	0.464808	0.359702	22.6128
14.0	0.477057	0.372976	21.8173
14.5	0.489026	0.386186	21.0294
15.0	0.500720	0.399328	20.2494
15.5	0.512147	0.412393	19.4776
16.0	0.523313	0.425378	18.7144
16.5	0.534223	0.438276	17.9600
17.0	0.544883	0.451082	17.2148
17.5	0.555299	0.463791	16.4791
18.0	0.565477	0.476357	15.7530
18.5	0.575422	0.488896	15.0370
19.0	0.585139	0.501282	14.3312
19.5	0.594634	0.513550	13.6359
20.0	0.603912	0.525697	12.9514
20.5	0.612977	0.537717	12.2777
21.0	0.621835	0.549607	11.6153
21.5	0.630490	0.561361	10.9642
22.0	0.638947	0.572977	10.3247
22.5	0.647210	0.584451	9.6969
23.0	0.655285	0.595778	9.0810
23.5	0.663174	0.606956	8.4772
24.0	0.670883	0.617980	7.8855
24.5	0.678415	0.628850	7.3061
25.0	0.685776	0.639560	6.7392

Table 2.11

α	θ	β_0	β_1
2.5	0.5	0.1	0.2

RESULTS OF THE GAMMA DISTRIBUTION MODEL ---- $H_{02}(t)$

L_{02}^1	L_{02}^2	MEAN	VARI
22.5809	939.185	5	10

T	$H_{02}^1(t)$	$H_{02}^2(t)$	DIFF02
3.0	0.124408	0.113761	8.55865
3.5	0.143583	0.131577	8.36158
4.0	0.162338	0.149080	8.16712
4.5	0.180682	0.166272	7.97524
5.0	0.198625	0.183160	7.78594
5.5	0.216174	0.199747	7.59922
6.0	0.233339	0.216037	7.41507
6.5	0.250129	0.232036	7.23348
7.0	0.266550	0.247747	7.05445
7.5	0.282612	0.263174	6.87796
8.0	0.298323	0.278323	6.70400
8.5	0.313689	0.293197	6.53258
9.0	0.328719	0.307800	6.36367
9.5	0.343419	0.322137	6.19727
10.0	0.357798	0.336211	6.03336
10.5	0.371862	0.350026	5.87194
11.0	0.385617	0.363587	5.71299
11.5	0.399072	0.376897	5.55650
12.0	0.412232	0.389961	5.40245
12.5	0.425103	0.402782	5.25084
13.0	0.437693	0.415364	5.10165
13.5	0.450007	0.427710	4.95487
14.0	0.462052	0.439825	4.81048
14.5	0.473832	0.451712	4.66847
15.0	0.485355	0.463374	4.52883
15.5	0.496625	0.474816	4.39152
16.0	0.507649	0.486041	4.25656
16.5	0.518431	0.497051	4.12390
17.0	0.528977	0.507852	3.99355
17.5	0.539292	0.518446	3.86548
18.0	0.549381	0.528836	3.73967
18.5	0.559249	0.539026	3.61611
19.0	0.568902	0.549020	3.49479
19.5	0.578342	0.558819	3.37567
20.0	0.587576	0.568429	3.25875
20.5	0.596608	0.577851	3.14401
21.0	0.605442	0.587083	3.03142
21.5	0.614083	0.596145	2.92097
22.0	0.622534	0.605024	2.81264
22.5	0.630800	0.613728	2.70641
23.0	0.638885	0.622260	2.60226
23.5	0.646793	0.630623	2.50016
24.0	0.654528	0.638819	2.40011
24.5	0.662094	0.646852	2.30208
25.0	0.669494	0.654724	2.20605

Table 2.12

α θ β_0 β_1
 2.5 3.1 2 3

RESULTS OF THE GAMMA DISTRIBUTION MODEL ---- $H_{11}(t)$

	L_{11}^1	L_{11}^2	MEAN	VAR1
	0.898507	1.07802	0.806452	0.260146

T	$H_{11}(t)$	$H_{11}^1(t)$	$H_{11}^2(t)$	DIFF111	DIFF211	DIFF
0.2	0.024486	0.199558	0.080831	714.988035	230.111270	59.4950
0.4	0.143862	0.359293	0.188948	149.748977	31.339877	47.4112
0.6	0.323626	0.487151	0.319494	50.528973	1.276864	34.4159
0.8	0.504496	0.589494	0.461791	16.848256	8.464761	21.6632
1.0	0.655275	0.671414	0.601811	2.462951	8.159013	10.3666
1.2	0.768588	0.736986	0.726189	4.111707	5.516567	1.4651
1.4	0.848532	0.789473	0.825925	6.960116	2.664189	4.6173
1.6	0.902655	0.831485	0.898123	7.884448	0.502025	8.0143
1.8	0.938274	0.865114	0.945302	7.797365	0.749040	9.2692
2.0	0.961249	0.892031	0.973134	7.200757	1.236457	9.0919
2.2	0.975849	0.913577	0.987955	6.381297	1.240523	8.1413
2.4	0.985027	0.930824	0.995080	5.502746	1.020505	6.9031
2.6	0.990750	0.944628	0.998171	4.655214	0.749060	5.6681
2.8	0.994297	0.955678	0.999382	3.884034	0.511443	4.5731
3.0	0.996487	0.964523	0.999811	3.207643	0.333558	3.6586
3.2	0.997835	0.971603	0.999947	2.628913	0.211695	2.9173
3.4	0.998664	0.977270	0.999987	2.142298	0.132448	2.3245
3.6	0.999174	0.981806	0.999997	1.738245	0.082388	1.8528
3.8	0.999488	0.985437	0.999999	1.405820	0.051213	1.4778
4.0	0.999681	0.988343	1.000000	1.134180	0.031903	1.1795

Table 2.13

α θ β_0 β_1
 2.5 3.1 2 3

RESULTS OF THE GAMMA DISTRIBUTION MODEL ---- $H_{00}(t)$

L_{00}^1	L_{00}^2	MEAN	VARI
4.88025	44.6085	0.806452	0.260146

T	$H_{00}^1(t)$	$H_{00}^2(t)$	DIFF00
0.2	0.040153	0.037237	7.26148
0.4	0.078694	0.073213	6.96499
0.6	0.115687	0.107964	6.67574
0.8	0.151195	0.141528	6.39368
1.0	0.185277	0.173941	6.11877
1.2	0.217991	0.205237	5.85095
1.4	0.249391	0.235450	5.59016
1.6	0.279531	0.264614	5.33634
1.8	0.308460	0.292761	5.08942
2.0	0.336227	0.319922	4.84934
2.2	0.362880	0.346129	4.61603
2.4	0.388462	0.371411	4.38941
2.6	0.413017	0.395797	4.16940
2.8	0.436586	0.419315	3.95591
3.0	0.459209	0.441994	3.74888
3.2	0.480924	0.463860	3.54820
3.4	0.501766	0.484938	3.35380
3.6	0.521772	0.505255	3.16558
3.8	0.540974	0.524835	2.98344
4.0	0.559406	0.543701	2.80729

Table 2.14

RESULTS OF THE GAMMA DISTRIBUTION MODEL ---- $H_{02}(t)$

L_{02}^1	L_{02}^2	MEAN	VARI
0.946162	1.49524	0.806452	0.260146

T	$H_{02}^1(t)$	$H_{02}^2(t)$	DIFF02
0.2	0.190534	0.151885	20.2847
0.4	0.344765	0.289868	15.9230
0.6	0.469609	0.413148	12.0229
0.8	0.570667	0.521472	8.6205
1.0	0.652469	0.615079	5.7306
1.2	0.718686	0.694630	3.3471
1.4	0.772285	0.761119	1.4459
1.6	0.815673	0.815771	0.0121
1.8	0.850793	0.859951	1.0763
2.0	0.879222	0.895074	1.8029
2.2	0.902235	0.922535	2.2500
2.4	0.920062	0.943651	2.4747
2.6	0.935941	0.959619	2.5299
2.8	0.948146	0.971494	2.4625
3.0	0.958026	0.980180	2.3125
3.2	0.966023	0.986428	2.1122
3.4	0.972497	0.990848	1.8869
3.6	0.977737	0.993922	1.6554
3.8	0.981979	0.996026	1.4305
4.0	0.985413	0.997442	1.2207

Table 2.15

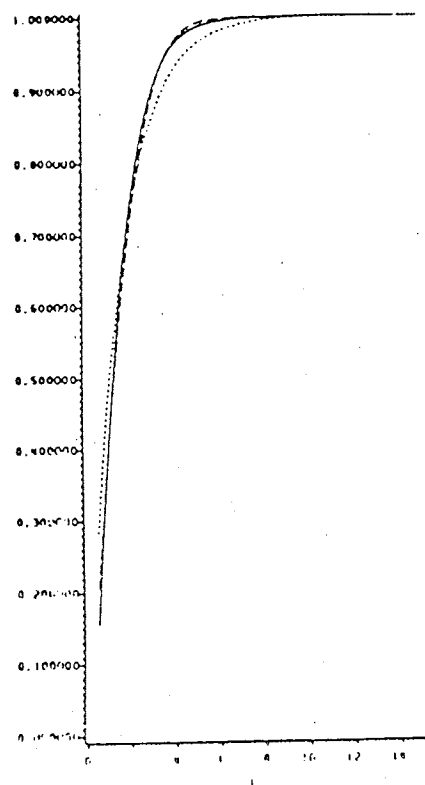


Figure 2.1

GRAPHICAL REPRESENTATION OF $H_{11}(t)$ GIVEN IN TABLE 1

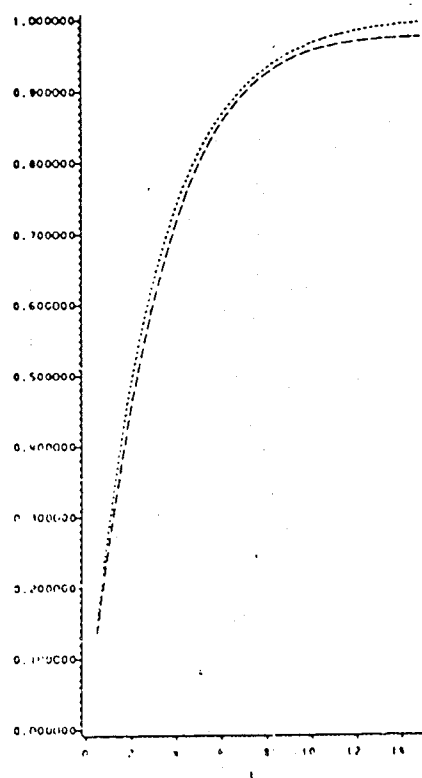


Figure 2.2

GRAPHICAL REPRESENTATION OF H_{∞} GIVEN IN TABLE 2

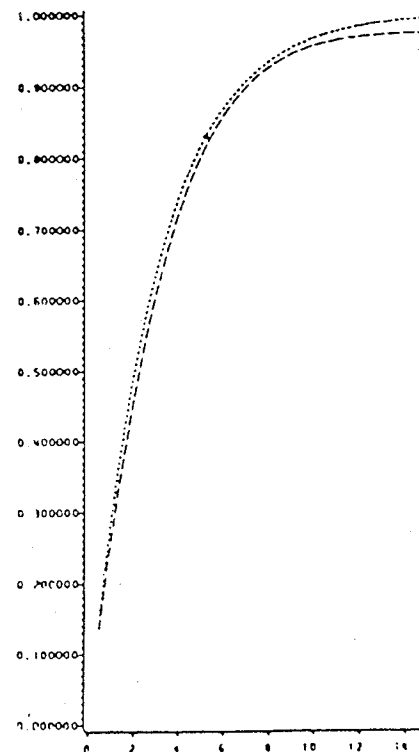


Figure 2.3

GRAPHICAL REPRESENTATION OF $H_{o2}(t)$ GIVEN IN TABLE 3

— $H_{ij}(t)$

- - - $H_{ij}^2(t)$

... $H_{ij}^1(t)$

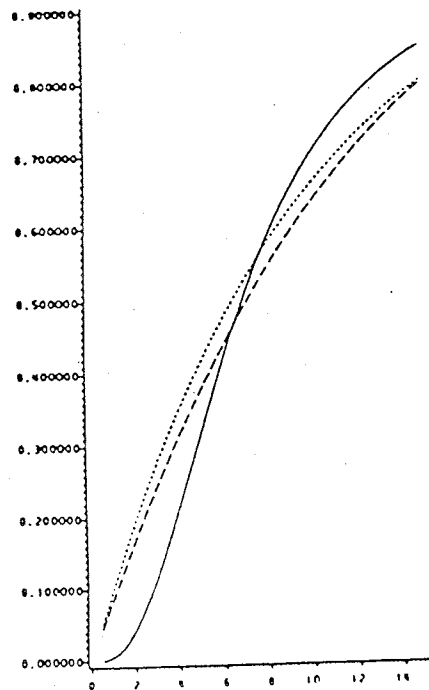


Figure 2.4

GRAPHICAL REPRESENTATION OF $H_{11}(t)$ GIVEN IN TABLE 4

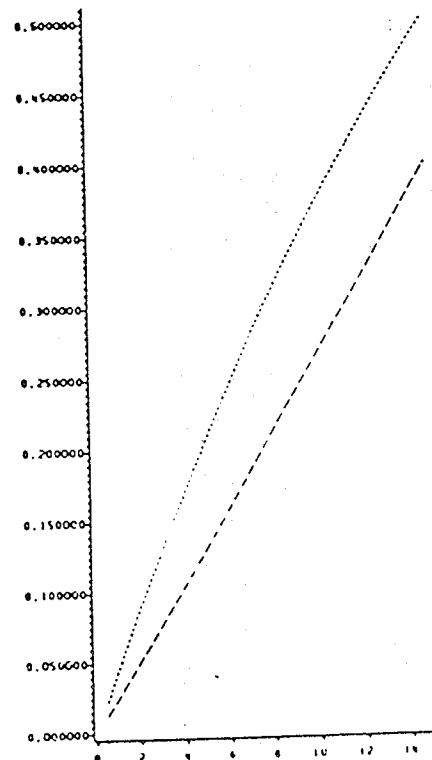


Figure 2.5

GRAPHICAL REPRESENTATION OF $H_{00}(t)$ GIVEN IN TABLE 5

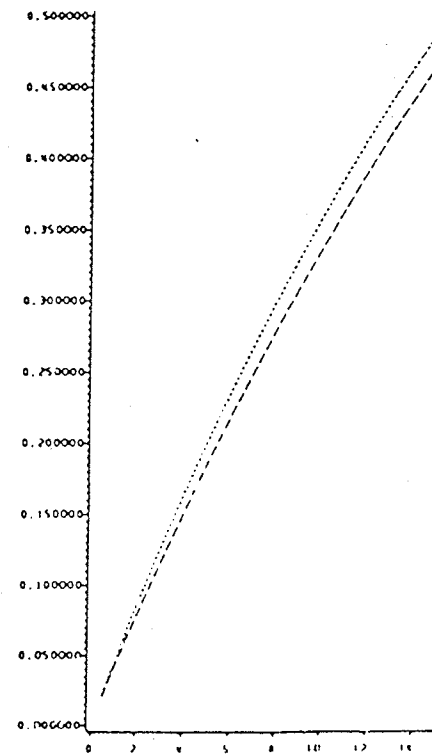


Figure 2.6

GRAPHICAL REPRESENTATION OF $H_{02}(t)$ GIVEN IN TABLE 6

— $H_{ij}(t)$

--- $H_{ij}^2(t)$

..... $H_{ij}^1(t)$

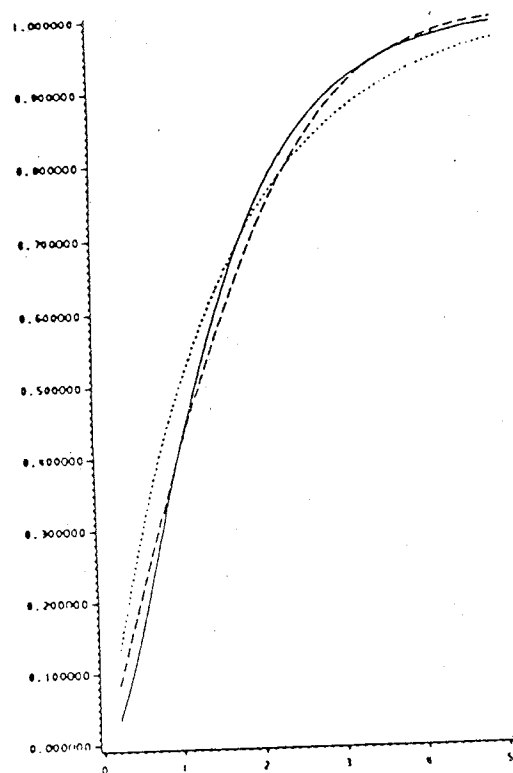


Figure 2.7

GRAPHICAL REPRESENTATION OF $H_{11}(t)$ GIVEN IN TABLE 7

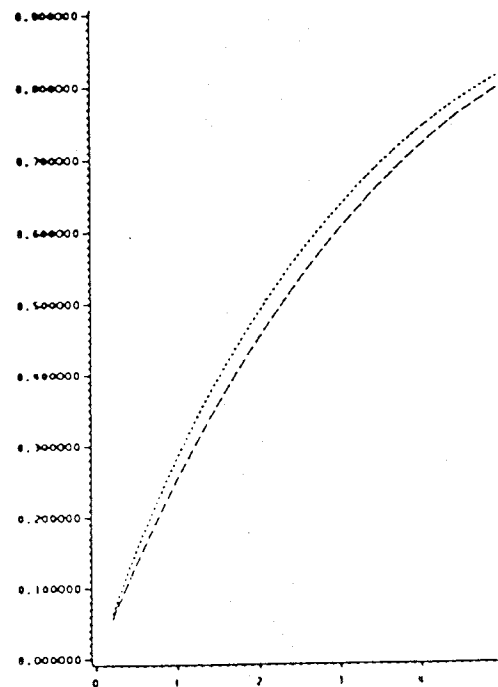


Figure 2.8

GRAPHICAL REPRESENTATION OF $H_{\infty}(t)$ GIVEN IN TABLE 8

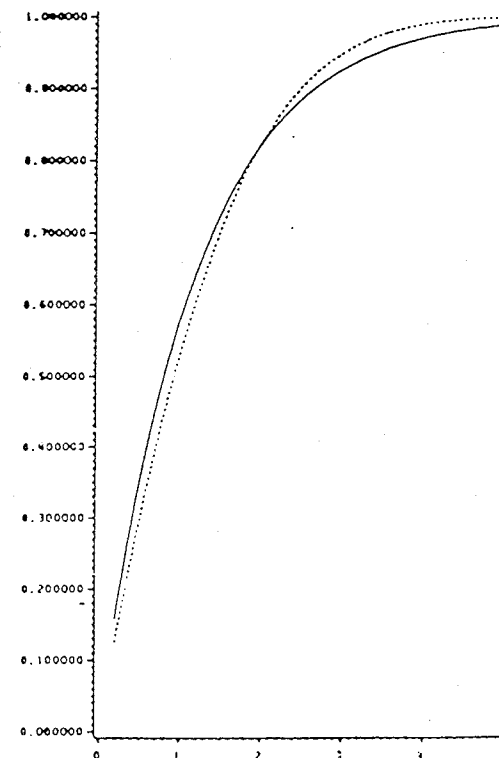


Figure 2.9

GRAPHICAL REPRESENTATION OF $H_{02}(t)$ GIVEN IN TABLE 9

— $H_{ij}(t)$

--- $H_{ij}^2(t)$

..... $H_{ij}^1(t)$

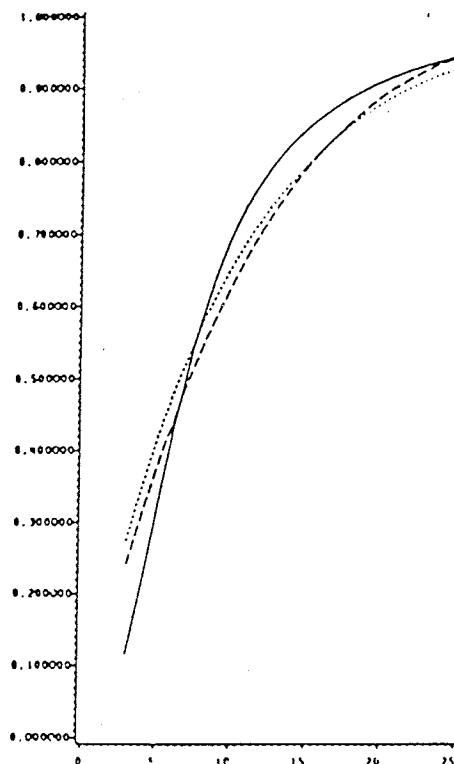


Figure 2.10

GRAPHICAL REPRESENTATION OF $H_{11}(t)$ GIVEN IN TABLE 10

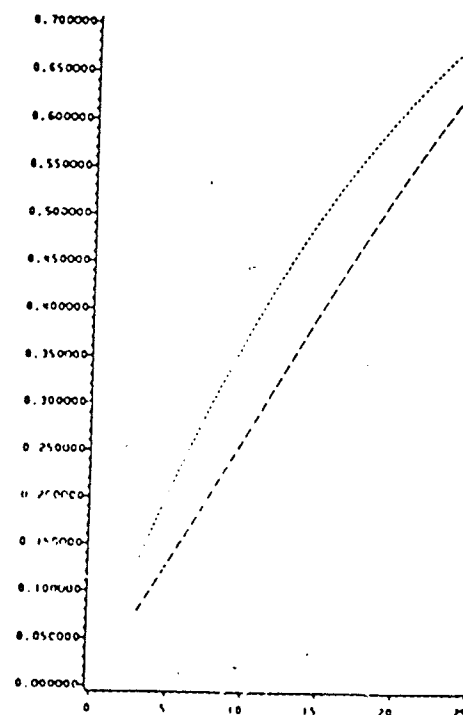


Figure 2.11

GRAPHICAL REPRESENTATION OF $H_{00}(t)$ GIVEN IN TABLE 11

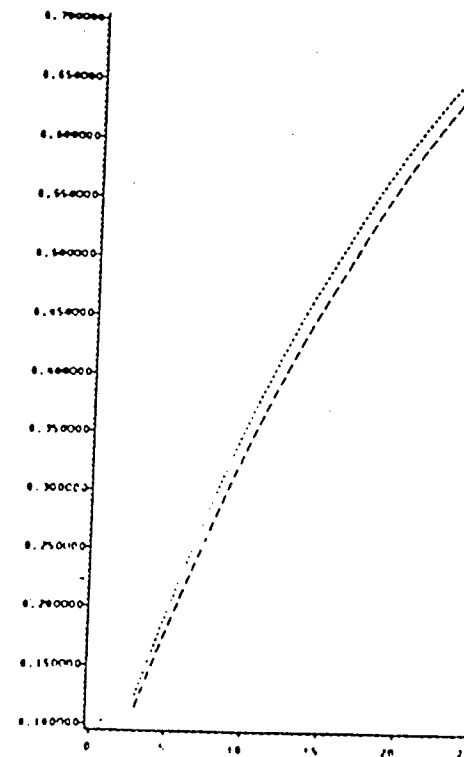


Figure 2.12

GRAPHICAL REPRESENTATION OF $H_{02}(t)$ GIVEN IN TABLE 12

— $H_{ij}(t)$ --- $H_{ij}^2(t)$ $H_{ij}^1(t)$

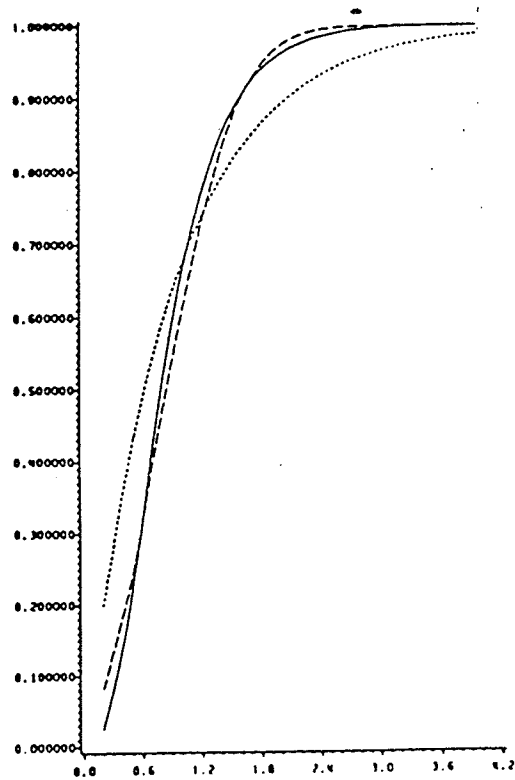


Figure 2.13
GRAPHICAL REPRESENTATION OF $H_{11}(t)$ GIVEN IN TABLE 13

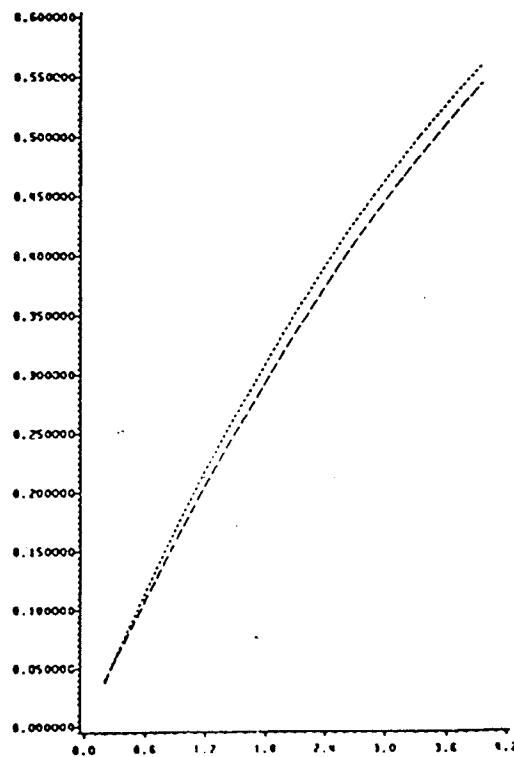


Figure 2.14
GRAPHICAL REPRESENTATION OF $H_{\infty}(t)$ GIVEN IN TABLE 14

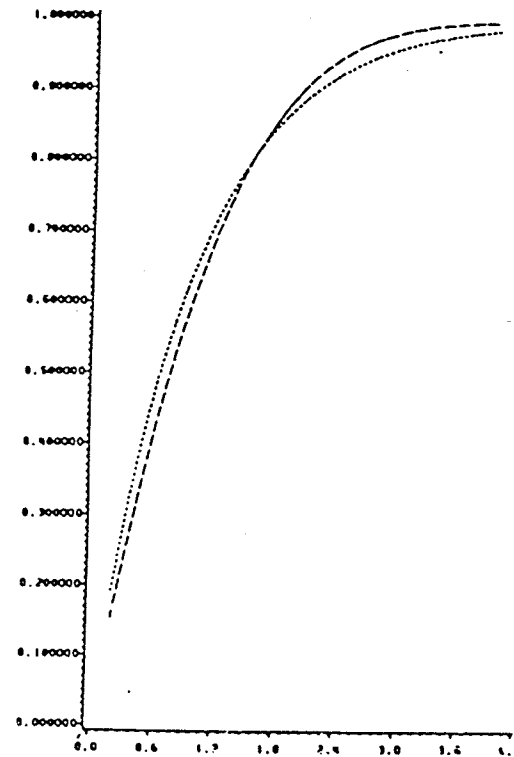


Figure 2.15
GRAPHICAL REPRESENTATION OF $H_{02}(t)$ GIVEN IN TABLE 15

— $H_{ij}(t)$ --- $H_{ij}^2(t)$ $H_{ij}^1(t)$

Chapter III

TWO UNIT PARALLELED REDUNDANT SYSTEM WITH WEIBULL REPAIR TIME DISTRIBUTION.

3.1 FIRST-PASSAGE TIME DISTRIBUTION AND THE MEAN FIRST- PASSAGE TIME OF RECURRENCE TIME FOR STATE 1, $H_{11}(T)$ AND L_{11} .

In this chapter, we compute the first passage time distribution and mean first passage time of two unit repairable systems when the repair is characterized by the Weibull distribution. The result obtained by Nakagawa and Osaki are compared with the results obtained by using MEP. From (1.3.9), we obtain the actual value of $H_{11}(t)$:

$$H_{11}(t) = Q_{10}(t) * Q_{01}(t) + Q_{1^2_1}(t) \quad (3.1.1)$$

From (2.1.1), we have

$$\begin{aligned} Q_{10}(t) * Q_{01}(t) &= \int_0^t e^{-\beta_1 u} \cdot dG(u) \\ &\quad - \int_0^t e^{-\beta_0(t-u) - \beta_1 u} \cdot dG(u) \end{aligned} \quad (3.1.2)$$

and also from (2.1.5)

$$Q_{1^2_1}(t) = \int_0^t dG(u) - \int_0^t e^{-\beta_1 u} \cdot dG(u) \quad (3.1.3)$$

Therefore,

$$H_{11}(t) = \int_0^t dG(u) - \int_0^t e^{-\beta_0(t-u) - \beta_1 u} dG(u) \quad (3.1.4)$$

Now, given a Weibull density function :

$$f(x, c, \theta) = c \cdot \theta \cdot x^{c-1} \cdot e^{-\theta x^c};$$

$$c > 0, \theta > 0, x \geq 0 \quad (3.1.5)$$

$$\int_0^t dG(t) = \int_0^t c \cdot \theta \cdot x^{c-1} \cdot e^{-\theta x^c} dx$$

$$= 1 - \exp(-\theta t^c), \quad t \geq 0, \theta > 0, c > 0$$

(3.1.6)

which is the Weibull probability distribution function.

Now,

$$\int_0^t e^{-\beta_0(t-u) - \beta_1 u} dG(u)$$

$$= e^{-\beta_0 t} \cdot c \cdot \theta \cdot \int_0^t e^{-(\beta_1 - \beta_0)u} \cdot u^{c-1} \cdot e^{-\theta u^c} du;$$

$$\beta_1 \geq \beta_0$$

(3.1.7)

$$H_{11}(t) = (1 - \exp(-\theta t^c))$$

$$- c \cdot \theta \cdot e^{-\beta_0 t} \int_0^t e^{-(\beta_1 - \beta_0)u} \cdot u^{c-1} \cdot e^{-\theta u^c} du;$$

$$\beta_1 \geq \beta_0$$

(3.1.8)

Now, let $H_{11}(t)$ denote the MEP distribution of $H_{11}(t)$ when first i moments are provided.

From (1.2.3), we have

$$h_{11}^{-1}(t) = 1/l_{11}^{-1} \exp(-t/l_{11}^{-1})$$

(3.1.9)

where l_{11}^{-1} is the mean recurrent passage time of state 1.

From (1.3.11), we have

$$H_{11}^{-1}(t) = 1 - \exp(-t/l_{11}^{-1})$$

(3.1.10)

from (1.5.9),

$$l_{11}^{-1} = 1/u + g(\beta_1)\beta_0^{-1}$$

(3.1.11)

and

$$\begin{aligned} g(\beta_1) &= \int_0^{\infty} \exp(-\beta_1 u) dG(u) \\ &= \int_0^{\infty} \exp(-\beta_1 u) \cdot c \cdot \theta \cdot u^{c-1} \cdot \exp(-\theta u^c) \cdot du \end{aligned}$$

(3.1.12)

$$g'(\beta_1) = -c \theta \int_0^{\infty} \exp(-\beta_1 u) \cdot u^c \cdot \exp(-\theta u^c) du$$

(3.1.13)

By selecting any combination of values of parameter c , β_0 , β_1 , and θ the exact value of $H_{11}(t)$ can be determined. The

corresponding approximate distributions of $H_{1,1}^1(t)$ and $H_{1,1}^2(t)$ can be determined using (2.1.9) and (2.1.12) respectively.

3.2 THE FIRST-PASSAGE TIME DISTRIBUTION AND THE MEAN FIRST-PASSAGE TIME FROM STATE 0 TO STATE2, $H_{02}(T)$ AND L_{02} .

From (2.1.3), we have :

$$H_{02}^1(t) = 1 - \exp(-t/l_{02}^1) \quad (3.2.1)$$

and from Appendix E, we have

$$l_{02}^1 = 1/\beta_1 + 1/\{\beta_0 g(\beta_1)\} \quad (3.2.2)$$

where $g(\beta_1)$ is given in (3.1.5).

Again from Appendix E,

$$\begin{aligned} l_{02}^2 = & 2g'(\beta_1)/\{\beta_1[1-g(\beta_1)]\} + 2/\beta_1^2 \\ & + 2[1-\beta_0 g'(\beta_1)]/\{\beta_0 \beta_1 (1-g(\beta_1))\} \\ & + 2g'(\beta_1)[1-\beta_0 g'(\beta_1)]/\{\beta_0 [1-g(\beta_1)]^2\} \\ & + 2[1-\beta_0 g'(\beta_1)]^2/\{\beta_0 [1-g(\beta_1)]\}^2 \end{aligned} \quad (3.2.3)$$

From chapter 2 (2.1.14),

$$H_{02}^2(t) = A(\gamma_1, \gamma_2) \exp [-\gamma_1 t - \gamma_2 t^2] \quad (2.3.4)$$

can be determined.

3.3 THE FIRST-PASSAGE TIME TIME DISTRIBUTION AND THE MEAN FIRST-PASSAGE TIME FROM THE RECURRENCE STATE 0, $H_{00}(T)$ AND L_{00} .

The expressions for l_{00}^1 and l_{00}^2 for Weibull repair time distributions are obtained using similar procedure as given in section 2.3. Appendix F provides the detailed derivations.

$$l_{00}^1 = 1/\beta_0 + 1/\{ug(\beta_1)\} \quad (3.3.1)$$

$$l_{00}^2 = 2/\beta^2 + 2/\{\beta_0 ug(\beta_1)\} + g''(0)/g(\beta_1) - 2[-g'(\beta_1)-1/u]/\{u[g(\beta_1)]^2\} \quad (3.3.2)$$

Using (3.3.1) and (3.3.2) in (2.3.3) and (2.3.4), respectively, we can obtain $H_{00}^1(t)$ and $H_{00}^2(t)$.

3.4 RESULTS AND ACCURACY

Since the exact distribution for the recurrence time of state 1 could be determined using (2.1.7), the percentage differences between this distribution and $H_{11}^1(t)$ [$H_{11}^2(t)$] for various values of t are considered as a measure of accuracy. The values of these differences with respect to $H_{11}^1(t)$ and $H_{11}^2(t)$ are given in Tables 3.1, 3.4, 3.7. After a relatively short time, depending upon the combination of parameters, these differences become insignificant. Similarly, the percentage differences between $H_{11}^1(t)$ and $H_{11}^2(t)$ become insignificant for values of t larger than 45 (Gupta et. al., 1985). Therefore, as suggested in chapter II, the consideration of MEP distribution on the basis of higher order moments may not contribute any further significant information. The exact solution of $H_{00}(t)$, [$H_{02}(t)$], can not be obtained; therefore, the difference between $H_{00}^1(t)$ and $H_{00}^2(t)$ [$H_{02}^1(t)$, $H_{02}^2(t)$] are considered as a measure of accuracy. The values of these differences, for various values of t are provided in Tables 3.2, 3.5 and 3.8 [3.3, 3.6 and 3.9]. Once again, it may be noted from these tables that these differences become insignificant for values of t larger than 45. We justify the application of the MEP to approximation $H_{00}(t)$ and $H_{02}(t)$ on the basis of the results that the differences obtained for $H_{11}(t)$ and its approximations become insignificant for t larger than 45; therefore, we assume the differences between $H_{00}(t)$, $H_{02}(t)$ and their approximation would also become insignificant for values of t larger than 45.

RESULTS OF WEIBULL DISTRIBUTION, H11

θ	β_0	β_1	C	MEAN	VAR1	L11	V11
0.5	0.3	1	1	2	4	3.1111	7.20985
T	EQ6	H111	H112	DIFF1	DIFF2		
0.1	0.003047	0.031632	0.026179	938.275465	759.299009		
0.2	0.011435	0.062263	0.051866	444.500730	353.573759		
0.3	0.024166	0.091926	0.077063	280.395922	218.892051		
0.4	0.040391	0.120650	0.101773	198.707713	151.972702		
0.5	0.059390	0.148465	0.126001	149.982379	112.157343		
1.0	0.177766	0.274888	0.240025	54.635005	35.022989		
2.5	0.526475	0.552274	0.515678	4.900393	2.050745		
5.0	0.825175	0.799542	0.794456	3.106366	3.722664		
7.5	0.932571	0.910250	0.923881	2.393567	0.931819		
10.0	0.972518	0.959816	0.975482	1.306006	0.304798		
15.0	0.994818	0.991945	0.998342	0.288827	0.354213		
20.0	0.998922	0.998385	0.999937	0.053708	0.101665		
25.0	0.999766	0.999676	0.999999	0.008953	0.023292		
30.0	0.999948	0.999935	1.000000	0.001316	0.005172		
35.0	0.999989	0.999987	1.000000	0.000151	0.001150		
45.0	0.999999	0.999999	1.000000	0.000005	0.000057		

Table 3.1

RESULTS OF WEIBULL DISTRIBUTION, H00

θ	β_0	β_1	C	MEAN	VAR1	L001	V00
0.5	0.3	1	1	2	4	9.33339	63.112
T	H001	H002	DIFF00				
0.1	0.010657	0.008582	19.4707				
0.2	0.021200	0.017115	19.2705				
0.3	0.031632	0.025599	19.0712				
0.4	0.041951	0.034034	18.8728				
0.5	0.052461	0.042420	18.6754				
1.0	0.101602	0.083617	17.7017				
2.5	0.234982	0.199916	14.9226				
5.0	0.414747	0.369992	10.7908				
7.5	0.552271	0.511895	7.3108				
10.0	0.657479	0.628012	4.4818				
15.0	0.799538	0.794540	0.6252				
20.0	0.882679	0.894011	1.2838				
25.0	0.931338	0.949001	1.8965				
30.0	0.959815	0.977135	1.8045				
35.0	0.976482	0.990457	1.4312				
45.0	0.991944	0.998663	0.6773				

Table 3.2

RESULTS OF WEIBULL DISTRIBUTION, H02

θ	β_0	β_1	C	MEAN	VAR1	L021	V02
0.5	0.3	1	1	2	4	5.99998	129.333
T	H021	H022	DIFF02				
0.1	0.016529	0.014599	11.6749				
0.2	0.032784	0.029019	11.4833				
0.3	0.048771	0.043262	11.2934				
0.4	0.064493	0.057331	11.1052				
0.5	0.079956	0.071226	10.9187				
1.0	0.153519	0.138149	10.0119				
2.5	0.340761	0.315044	7.5467				
5.0	0.565403	0.541284	4.2658				
7.5	0.713497	0.699703	1.9332				
10.0	0.811126	0.807868	0.4016				
15.0	0.917916	0.926629	0.9492				
20.0	0.964327	0.974485	1.0534				
25.0	0.984496	0.991928	0.7548				
30.0	0.993562	0.997679	0.4446				
35.0	0.997072	0.999394	0.2329				
45.0	0.999447	0.999970	0.0522				

Table 3.3

RESULTS OF WEIBULL DISTRIBUTION, H11

θ	β_0	β_1	C	MEAN	VAR1	L11	V11
0.6	0.5	1	1	1.6667	2.7778	2.41667	3.65276
T	EQ6	H111	H112	DIFF1	DIFF2		
0.1	0.004189	0.040535	0.028867	867.714538	589.167463		
0.2	0.015613	0.079427	0.057446	408.716352	267.935410		
0.3	0.032771	0.116742	0.085717	256.236628	161.563569		
0.4	0.054406	0.152545	0.113659	180.383479	108.910055		
0.5	0.079470	0.186896	0.141254	135.178743	77.747460		
1.0	0.230479	0.338862	0.273428	47.025351	18.634500		
2.5	0.630585	0.644590	0.599237	2.220953	4.497124		
5.0	0.905622	0.873684	0.893447	3.526709	1.344436		
7.5	0.976066	0.955106	0.981828	2.147458	0.590237		
10.0	0.993846	0.984044	0.998053	0.986259	0.423293		
12.5	0.998394	0.994329	0.999871	0.407134	0.147909		
15.0	0.999575	0.997985	0.999995	0.159107	0.041999		
30.0	1.000000	0.999996	1.000000	0.000388	0.000000		
40.0	1.000000	1.000000	1.000000	0.000000	0.000000		

Table 3.4

RESULTS OF WEIBULL DISTRIBUTION, H00

θ	β_0	β_1	C	MEAN	VAR1	L001	V00
0.6	0.5	1	1	1.6667	2.7778	6.44444	33.0124
T	H001	H002	DIFF00				
0.1	0.153970	0.013355	13.2663				
0.2	0.030558	0.026566	13.0645				
0.3	0.045485	0.039634	12.8643				
0.4	0.060182	0.052559	12.6657				
0.5	0.074653	0.065345	12.4687				
1.0	0.143732	0.127192	11.5079				
2.5	0.321541	0.293025	8.8687				
5.0	0.539693	0.511231	5.2739				
7.5	0.687701	0.669639	2.6264				
10.0	0.788118	0.781747	0.8083				
12.5	0.853246	0.859095	0.3326				
15.0	0.902469	0.911119	0.9584				
30.0	0.990488	0.996590	0.6161				
40.0	0.997985	0.999760	0.1779				

Table 3.5

RESULTS OF WEIBULL DISTRIBUTION, H02

θ	β_0	β_1	C	MEAN	VAR1	L021	V02
0.6	0.5	1	1	1.6667	2.7778	1.2	13.64
T	H021	H022	DIFF02				
0.1	0.023528	0.020015	14.9335				
0.2	0.046503	0.039720	14.5860				
0.3	0.068937	0.059199	14.2426				
0.4	0.090844	0.078213	13.9033				
0.5	0.112234	0.097006	13.5680				
1.0	0.211872	0.186546	11.9538				
2.5	0.448569	0.413847	7.7406				
5.0	0.695924	0.676679	2.7654				
7.5	0.832323	0.832374	0.0061				
10.0	0.907538	0.918400	1.1969				
12.5	0.949013	0.962734	1.4458				
15.0	0.971884	0.984046	1.2513				
30.0	0.999210	0.999980	0.0767				
40.0	0.999927	1.000000	0.0073				

Table 3.6

RESULTS OF WEIBULL DISTRIBUTION, H11

θ	β_0	β_1	C	MEAN	VAR1	L11	V11
1	1	1	0.85	1.08796	1.6523	1.59879	1.75947
T	EQ6	H111	H112	DIFF1	DIFF2		
0.1	0.012573	0.060631	0.047385	385.246293	276.889348		
0.2	0.040806	0.117587	0.093449	188.160179	129.160179		
0.3	0.078321	0.171089	0.138171	118.444625	76.415103		
0.4	0.121425	0.221347	0.181531	82.290393	49.499737		
0.5	0.167638	0.258558	0.223515	60.200920	33.332208		
1.0	0.399683	0.464992	0.412599	16.340285	3.231748		
2.5	0.814060	0.790637	0.783973	2.877422	3.695983		
5.0	0.973706	0.956167	0.977003	1.801244	0.338630		
7.5	0.995537	0.990823	0.998845	0.473543	0.332260		
10.0	0.999112	0.998079	0.999973	0.103467	0.086164		
12.5	0.999804	0.999598	1.000000	0.020658	0.019548		
16.0	0.999974	0.999955	1.000000	0.001891	0.002615		
17.5	0.999989	0.999982	1.000000	0.000632	0.001131		
20.0	0.999997	0.999996	1.000000	0.000082	0.000287		

Table 3.7

RESULTS OF WEIBULL DISTRIBUTION, H00

θ	β_0	β_1	C	MEAN	VAR1	L001	V00
1	1	1	0.85	1.08796	1.6523	3.12976	9.17807
T	H001	H002	DIFF00				
0.1	0.031446	0.030396	3.3397				
0.2	0.061904	0.059902	3.2330				
0.3	0.091403	0.088544	3.1285				
0.4	0.119975	0.116345	3.0262				
0.5	0.147649	0.143329	2.9260				
1.0	0.273497	0.266780	2.4562				
2.5	0.550124	0.542775	1.3358				
5.0	0.797611	0.795640	0.2471				
7.5	0.908950	0.910715	0.1941				
10.0	0.959039	0.961870	0.2952				
12.5	0.981573	0.984084	0.2558				
16.0	0.993977	0.995492	0.1524				
17.5	0.996270	0.997410	0.1144				
20.0	0.998322	0.998991	0.0670				

Table 3.8

RESULTS OF WEIBULL DISTRIBUTION, H02

θ	β_0	β_1	C	MEAN	VAR1	L021	V02
1	1	1	0.85	1.08796	1.6523	3.0443	7.09313
T	H021	H022	DIFF02				
0.1	0.030315	0.027318	15.4621				
0.2	0.063585	0.054069	14.9656				
0.3	0.093845	0.080259	14.4770				
0.4	0.123127	0.105894	13.9963				
0.5	0.151463	0.130980	13.5237				
1.0	0.279985	0.248394	11.2831				
2.5	0.560100	0.527607	5.8012				
5.0	0.806488	0.802542	0.4893				
7.5	0.914874	0.927206	1.3480				
10.0	0.962553	0.976393	1.4378				
12.5	0.983527	0.993279	0.9915				
16.0	0.994782	0.999072	0.4312				
17.5	0.996912	0.999634	0.2830				
20.0	0.998598	0.999930	0.1334				

Table 3.9

θ	β_0	β_1	c
1.0	0.5	2.0	1.0

COMPARISON OF ACTUAL AND APPROXIMATE VALUE --- H11

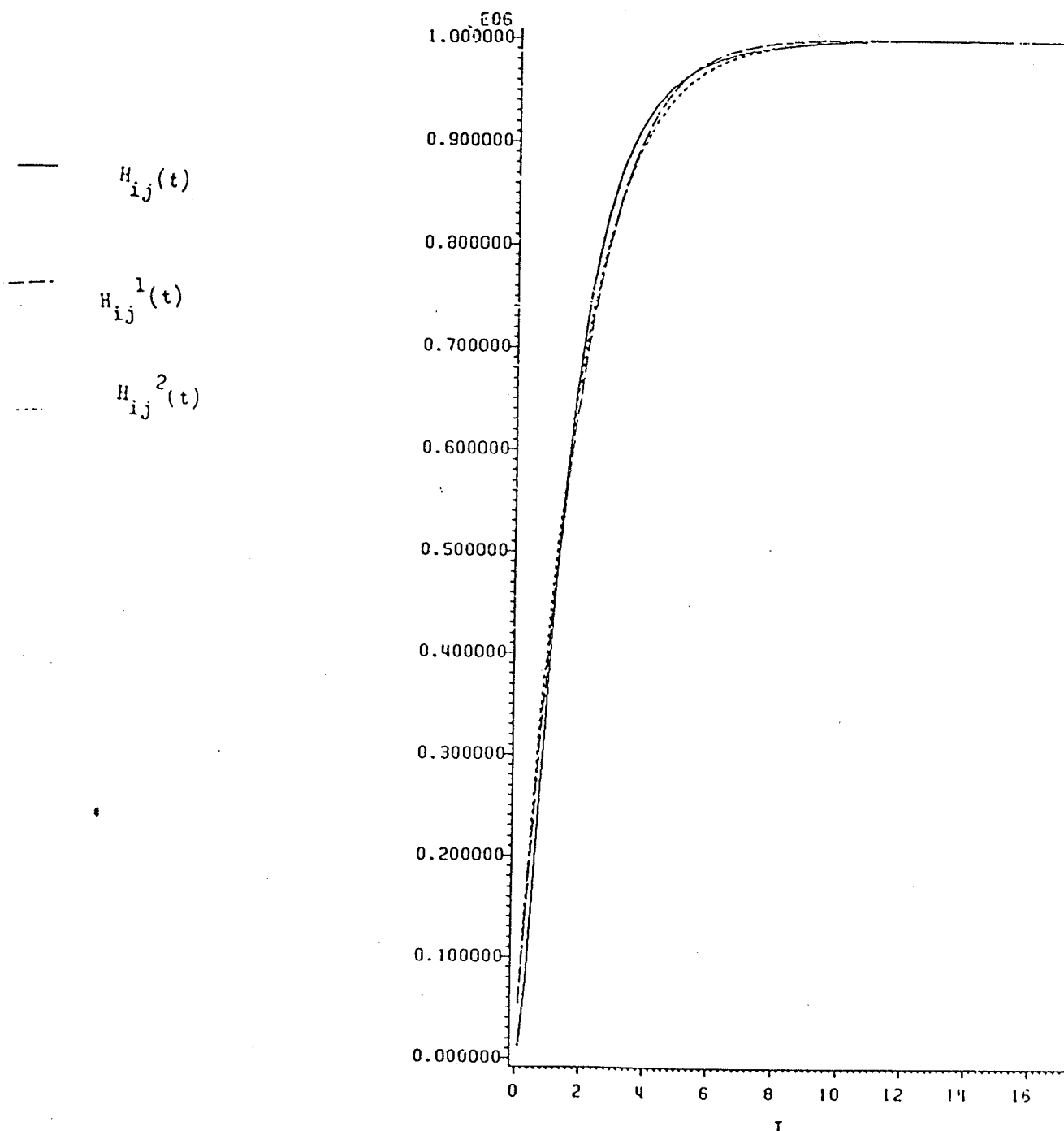


Figure 3.1

θ	β_0	β_1	c
1.0	0.5	2.0	1.0

COMPARISON OF H00.

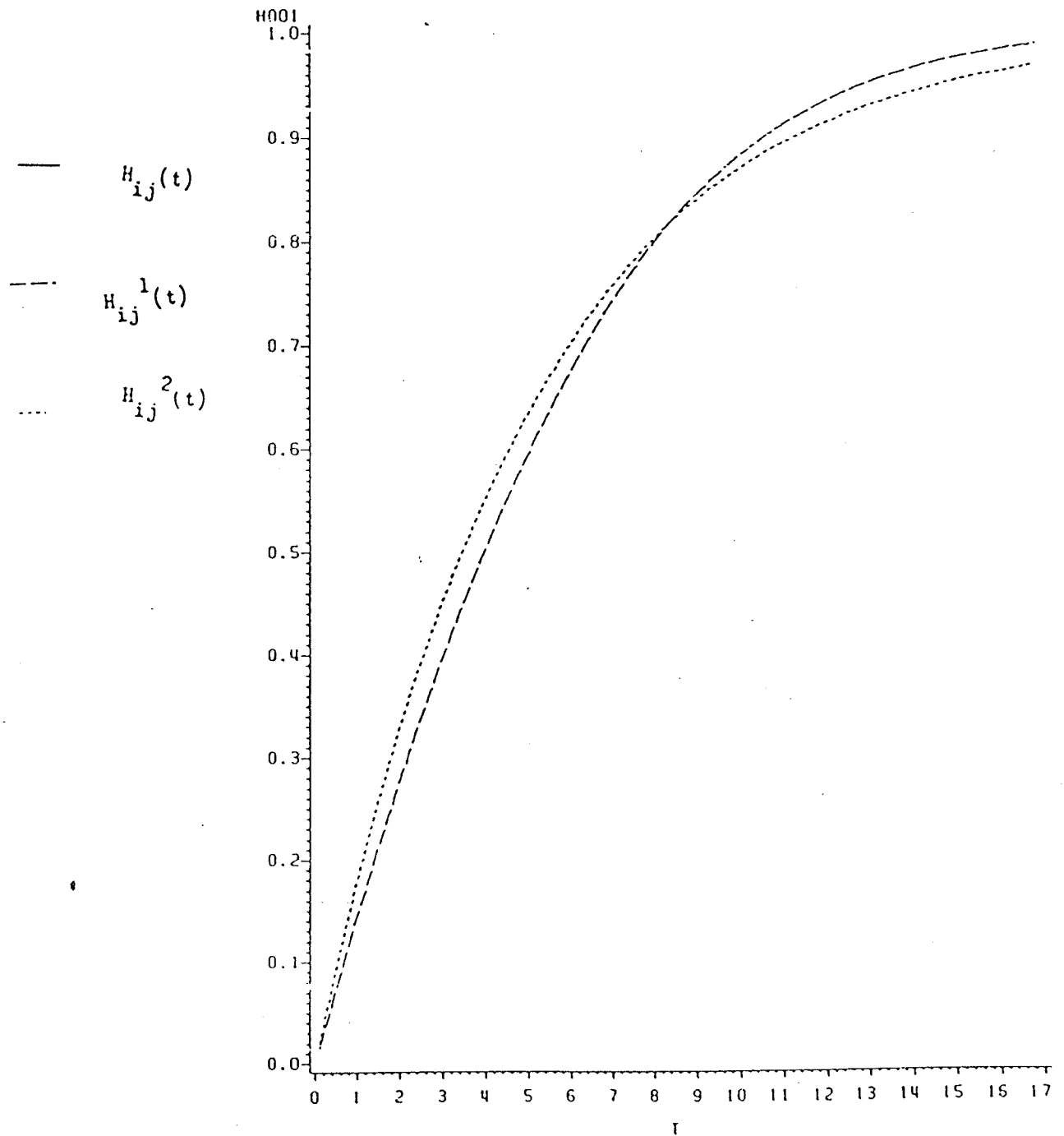


Figure 3.2

θ	B_0	B_1	c
1.0	0.5	2.0	1.0

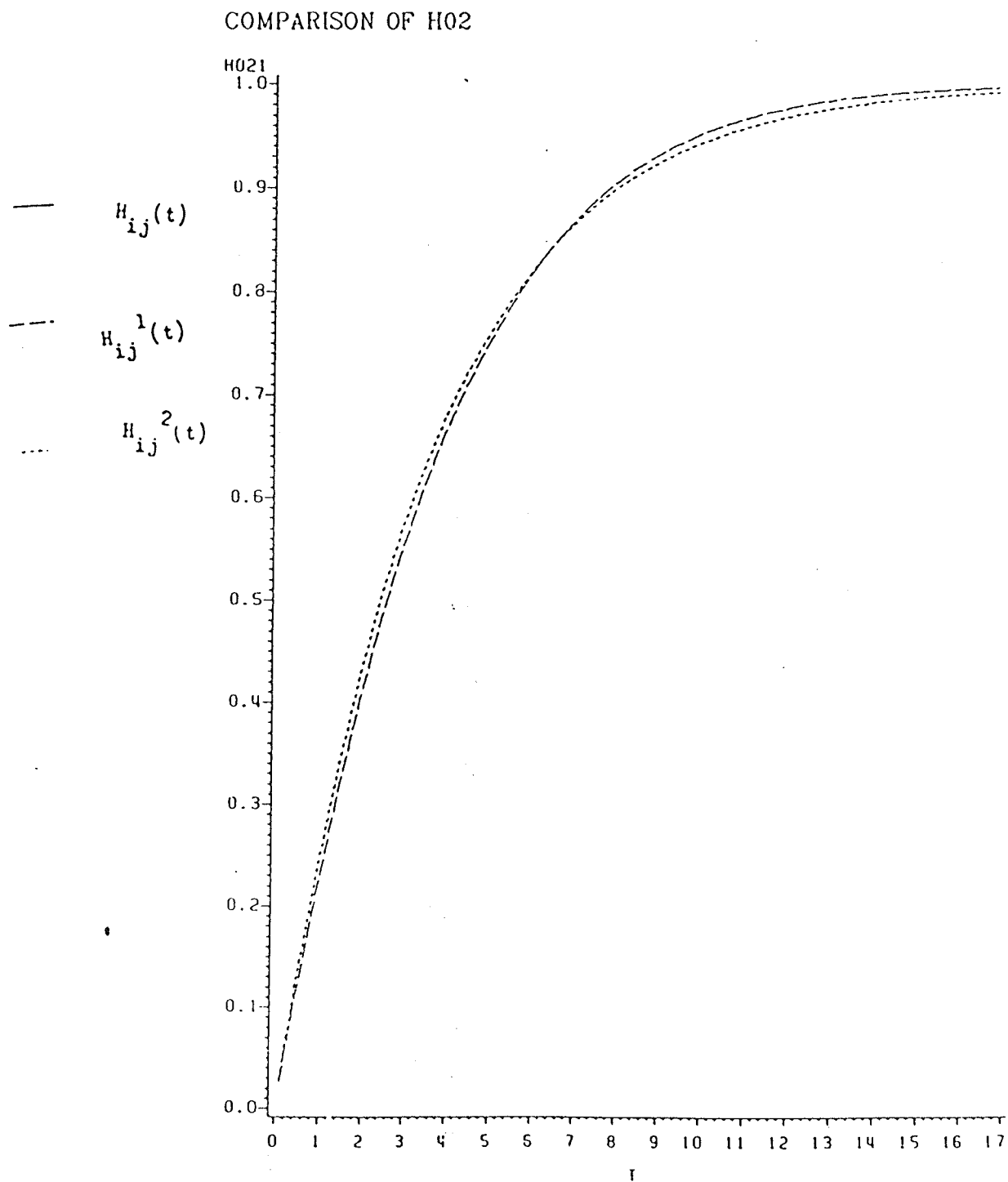


Figure 3.3

θ	β_0	β_1	c
1.0	1.0	2.0	2.0

COMPARISON OF ACTUAL AND APPROXIMATE VALUE --- H11

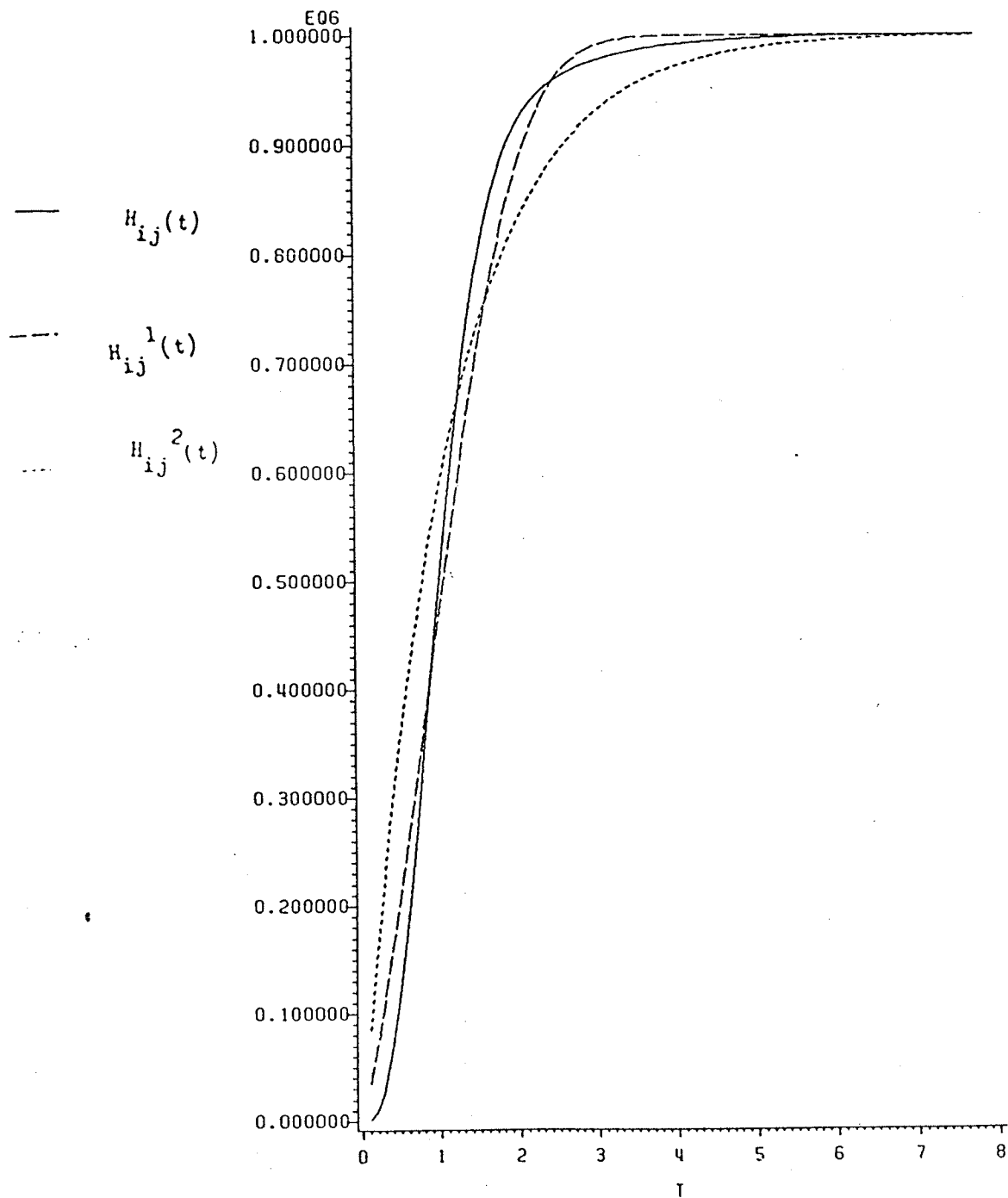


Figure 3.4

θ	β_0	β_1	c
1.0	1.0	2.0	2.0

COMPARISON OF H00.

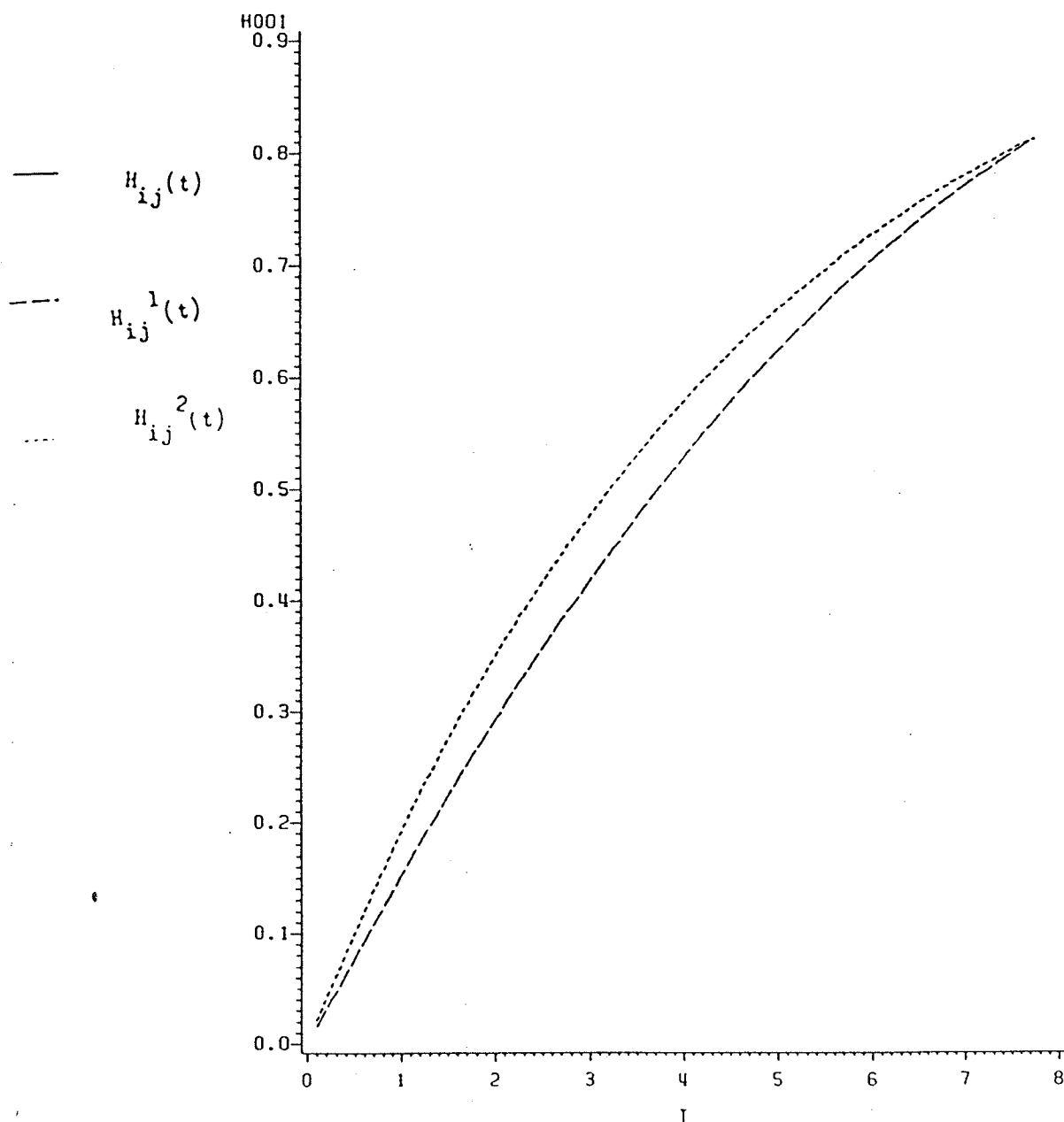


Figure 3.5

θ	β_0	β_1	c
1.0	1.0	2.0	2.0

COMPARISON OF H02

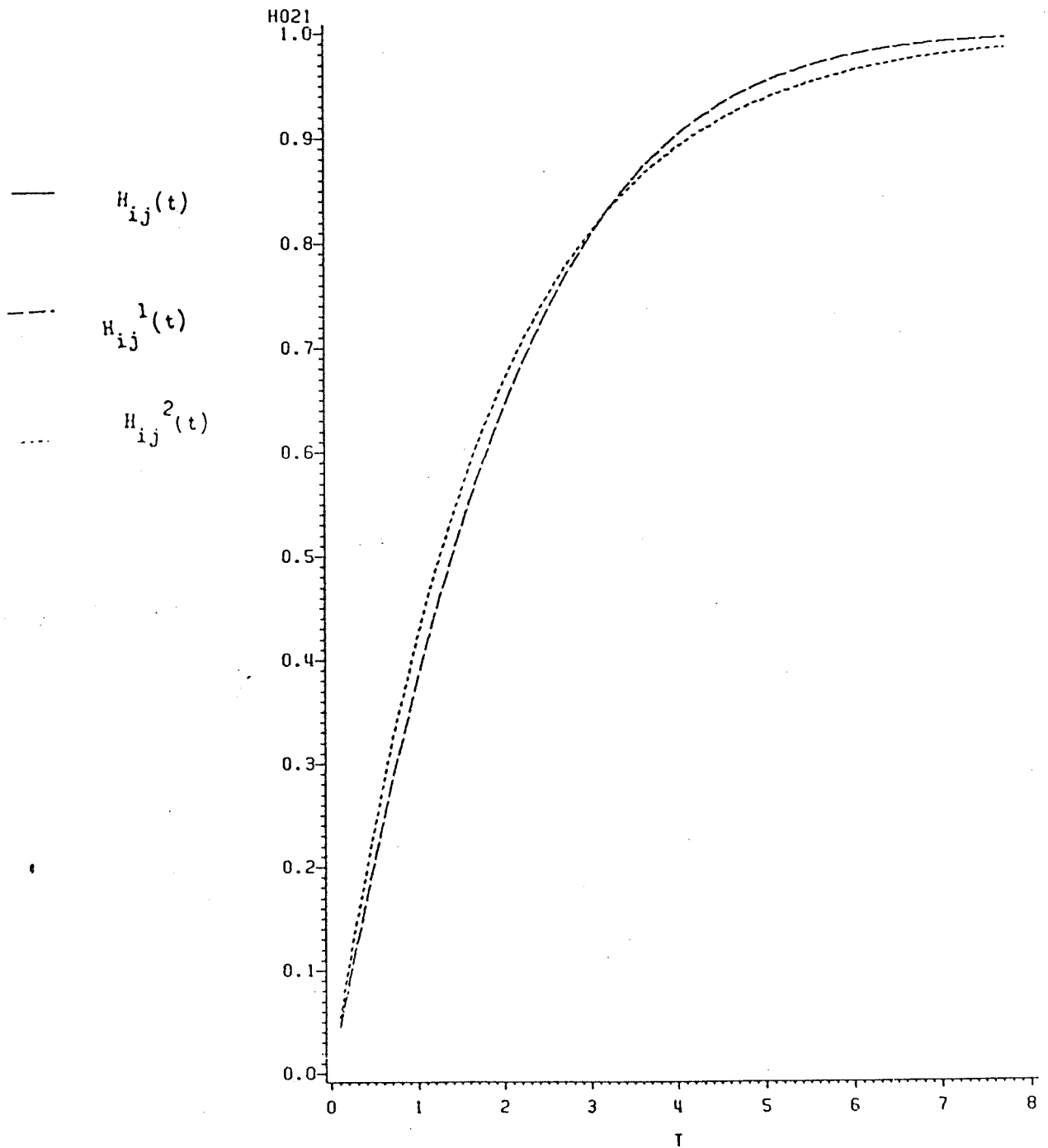


Figure 3.6

θ	β_0	β_1	c
1	1	2	4

COMPARISON OF ACTUAL AND APPROXIMATE VALUE --- H11

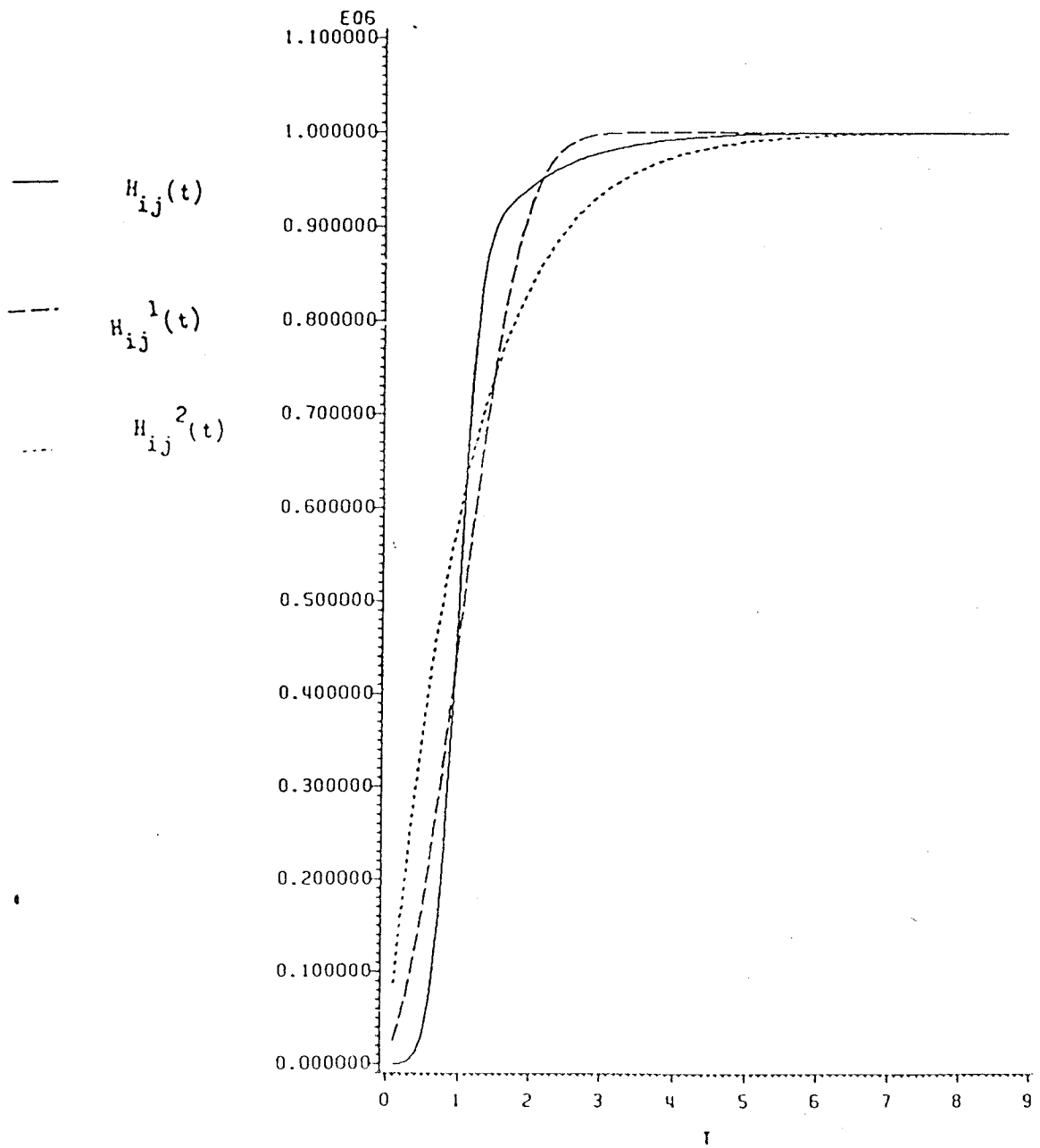


Figure 3.7

θ	β_0	β_1	c
1	1	2	4

COMPARISON OF H00.

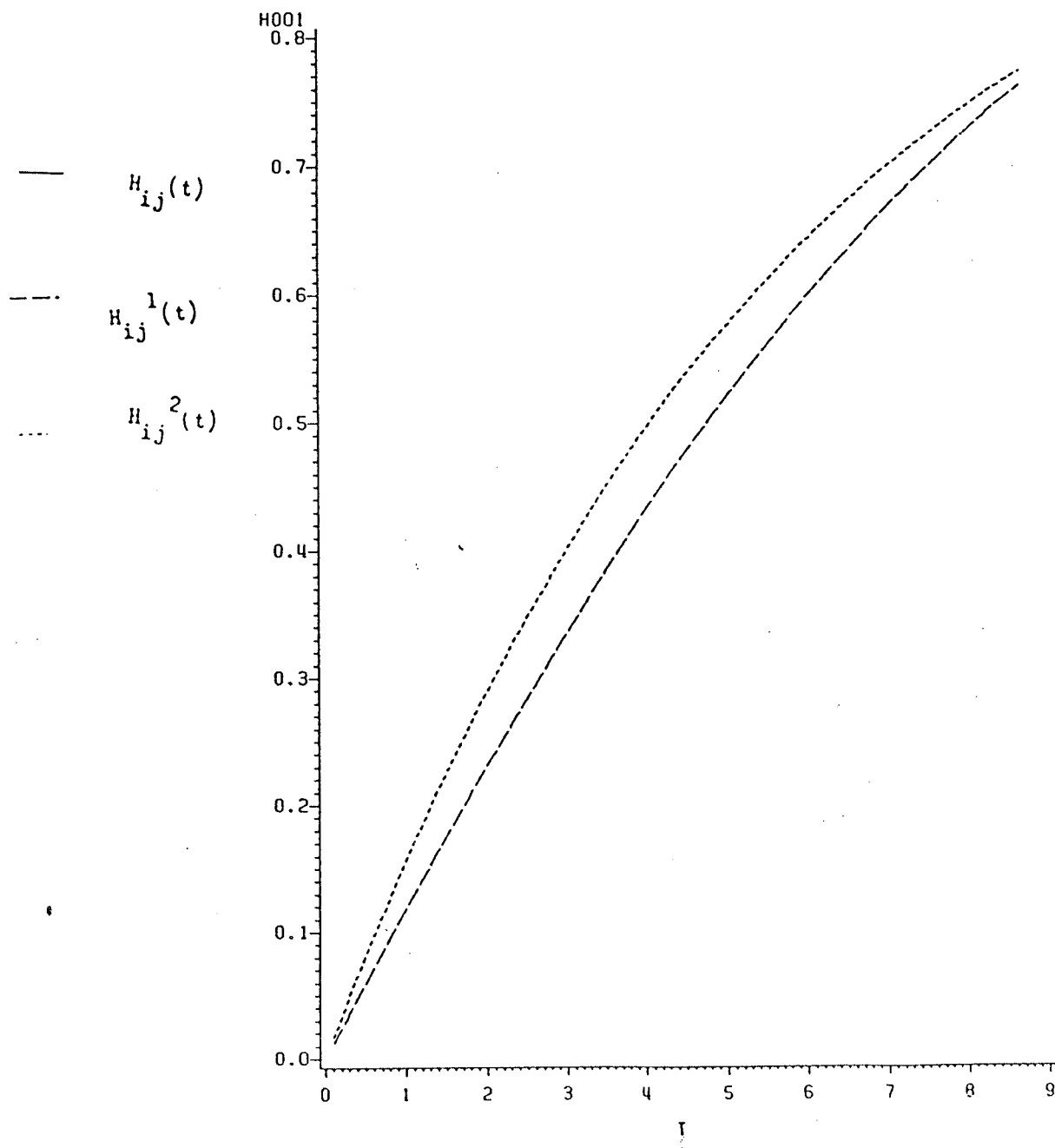


Figure 3.8

θ	β_0	β_1	c
1	1	2	4

COMPARISON OF H02

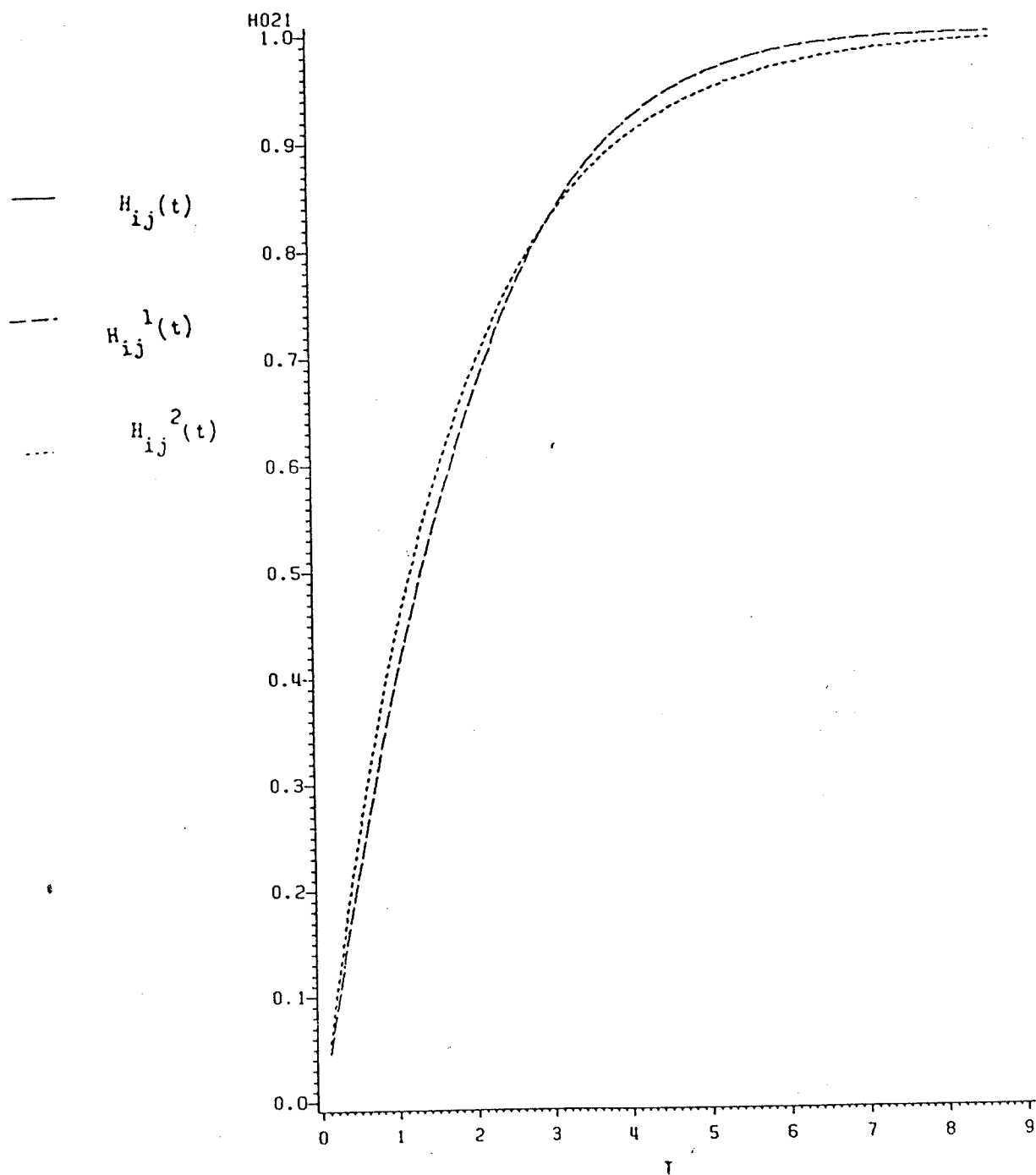


Figure 3.9

PARAMETERS OF WEIBULL DISTRIBUTION AND
CORRESPONDING MEAN AND VARIANCE OF THE
SYSTEM

θ	β_0	β_1	C	<u>Repair Time</u>		(1) L_{11}	V_{11}	(1) L_{02}	V_{02}	(1) L_{00}	V_{00}
				Mean	Var.						
1	1	1	4	0.9064	0.0646	1.0923	0.3534	1.7284	2.1925	5.8752	21.9694
1	0.5	2.0	1.0	1.0000	1.0000	1.6667	2.3333	3.4999	10.2499	5.0000	17.0003
0.5	0.3	1.0	1.0	2.0000	4.0000	3.1111	7.2098	5.9999	29.3333	9.3334	63.1120
0.6	0.5	1.0	1.0	1.6667	2.7778	3.4167	3.6528	4.2000	13.6400	6.4444	33.0124
1.0	1.0	2.0	2.0	0.8862	0.2146	1.1284	0.4847	1.8195	2.4674	4.6603	14.3914
1.0	1.0	1.0	0.85	1.0879	1.6523	1.5988	1.7595	3.0443	7.0931	3.1298	9.1781

Chapter IV

INFORMATION THEORETIC APPROXIMATION FOR SINGLE SERVER QUEUEING SYSTEMS.

In this chapter, we consider queueing systems in which the customers arrive with independent, exponentially or generally distributed inter-arrival times at an average rate (λ) from an infinite capacity pool; they wait in an infinite capacity queue; they are served independently by a single server with exponentially or generally distributed service times at an average rate μ ; and they return to the customer pool.

Shore (1982) applied MEP in determining the approximate performance distributions such as queueing length, busy period length, numbers served during busy period and waiting time distribution of a single server queueing systems. He provided the approximate values of performance distributions under the assumption that the information is available in the form of moments of up to 4th order. However, in most real life situations, the information available in the form of moments rarely extend beyond first two moments (Thomas, 1979).

Shore used Shannon's (1948) measure of entropy in obtaining the approximate performance distributions. Shore's re-

sults are appropriate approximation of true results (that is percentage error $< 15\%$), under the assumption that the only first moment is provided, when the traffic intensity ρ , is less than 0.1. However, as ρ approaches 1, the percentage error increases exponentially. In this chapter, we propose the application of two other measures of entropy, Parametric (Kapur, 1983) and Renyi (1963), to approximate performance distributions of a single server queue system when only the first moment is provided. These measures of entropy were explained in section 1.3. We will show that the results obtained by the application of these measures are consistently better than those obtained by Shore using Shannon's entropy. Furthermore, in this chapter, we will deal with the relationship between the first moment and the value of the parameter α_1 , in the case of Renyi, and parameter α_2 , in the case of Parametric, under various values of traffic intensity.

4.1 SINGLE SERVER QUEUING MODEL

In M/M/1, the p.d.f. of busy period length is given as (Kleinrock (1975), pp 213-214) :

$$f^b(x) = [x \sqrt{\lambda/u}]^{-1} \exp[-(\lambda+u)x] I_1(2x\sqrt{\lambda u}) \quad (4.1.1)$$

The corresponding C.D.F. of residence time is given as (Kleinrock (1975), p202) :

$$F^r(x) = 1 - \exp(-u(1-\rho)x), \quad x > 0$$

(4.1.2)

Similarly, the corresponding C.D.F. of waiting time can be written as (Kleinrock (1975), p203) :

$$F^w(x) = 1 - \rho \exp(-u(1-\rho)x), \quad x > 0 \quad (4.1.3)$$

Using the two Renyi and Parametric measures of entropy, we obtain approximate performance distributions and compare our results with the results obtained by Shore (1979 a, b) for the above three performance distributions.

Renyi's Measure

Using Equation (1.3.2.2), the p.d.f. of busy period length is given as :

$$f(x) = (E + F x)^{1/(1-a_1)}$$

where

$$E = \left[\frac{(R+2) m_1}{(R+1)} \right]^{-1/R} ; \quad R < -2.$$

$$F = - \frac{R+1}{R+1} \cdot E$$

$$a_1 = 1 + 1/R$$

$$m = g_1/(1-\rho)$$

(4.1.4)

The corresponding C.D.F. can be written as (ref to Appendix G) :

$$F(x) = [1/\{b(R+1)\}] [(a + b x)^{R+1} - a^{R+1}] ; R < -2$$

where

$$a = 1/[m(R+2)/(R+1)]^{\frac{1}{R}}$$

$$b = -a^{R+1}/(R+1)$$

(4.1.5)

In M/M/1 queueing systems, the waiting time distribution can be obtained by substituting for the value of m in (4.1.5) as follow :

$$m = \rho/u(1-\rho)$$

(4.1.6)

Similarly, the value of m for residence time distribution can be written as :

$$m = \rho/\lambda(1-\rho)$$

(4.1.7)

Parametric Measure

Using Equation (1.3.3.2), the p.d.f. of busy period length is given as follows :

$$f(x) = \frac{\exp(-c_1 - c_2 x)}{1 - a_2 \exp(-c_1 - c_2 x)} \quad (4.1.8)$$

where

$$c_2 = 1/m_1$$

$$\exp(-c_1) = \frac{1 - \exp(-a_2/m_1)}{P}$$

$$m = g_1 / (1 - \rho) \quad (4.1.9)$$

The corresponding C.D.F. is given (ref to Appendix H) as :

$$F(x) = (1/a_2 \gamma_1) [\ln(1 - a_2 \exp(-\gamma_0 - \gamma_1 x)) - \ln(1 - a_2 \exp(-\gamma_0))]$$

where

$$\gamma_1 = 1/m$$

$$\gamma_0 = - \ln([1 - \exp(-a_2 \gamma_1)] / a_2) \quad (4.1.10)$$

Similarly, the m values given in (4.1.6) and (4.1.7) are used for residence time distribution and waiting time distribution, respectively.

4.2 RESULTS

Using the above three models, namely Shannon's measure, Renyi's measure and the Parametric measure, we can approximate the performance distributions a single server queueing system when the available information is a mean value (service time moment), m , and traffic intensity, ρ .

In the case of the busy period distribution, the MEP model does not discriminate between $M/M/1$, $M/G/1$ or $G/G/1$ queueing systems as shown by Shore(1979). Therefore, the results presented in this chapter concerning the busy period distribution are equally applicable in all the three types of queueing systems.

Table 4.1 provides the probability density function of the busy period when the traffic intensity ρ varies from 0.1 to 0.9 and value of m (mean service time moment) varies from 0.5 to 55.

The mean percentage absolute deviation was used as the measure of accuracy for obtaining the best fit of the approximate with the exact performance distribution. Table 4.1 shows for a given value of m and various values of ρ , the corresponding optimum value of parameter a_1 , in the case of Renyi's measure, and parameter a_2 , in the case of Parametric measure. Similarly, Tables 4.2 and 4.3 provide the approximate waiting time and residence time distributions, respectively. The values of parameter a_1 and a_2 are the

same throughout in Table 4.1 - 4.3. However, these values are the optimum values of the parameters a_1 and a_2 , (reflecting the smallest value of mean percentage absolute deviation between the exact and the approximate) for busy period distribution only. Tables 4.4 and 4.5 provide the approximate waiting time and residence time distributions for various values of m and ρ . The value of a_1 and a_2 in Table 4.4 and 4.5 are the optimum values for waiting time and residence time distributions, respectively. The purpose of providing Table 4.2 and 4.3 is to show the level of inaccuracy which one may have to face, should one decide to use the same parameter values as busy period distribution for approximating waiting time and residence time distributions. These inaccuracies can be determined by comparing Table 4.2 and 4.3 with 4.5 and 4.4, respectively.

Figures 4.1 - 4.6 show the probability density function of busy period for values of parameter given in Table 4.1. Figures 4.7 - 4.12 provide the waiting time distribution for values of m and ρ given in Tables 4.2 and 4.4. Similarly, Figures 4.13 - 4.18 show the residence time distribution for values of m and ρ given in Tables 4.3 and 4.5.

It can be seen in Table 4.1 and Figures 4.1 - 4.6 that Renyi's measure of entropy provides a better approximation to the exact distribution than did Shannon's measure as proposed by Shore (1979). The approximation suggested by the parametric model is also consistently better than Shannon's

model but inferior to Renyi's model. In most cases, for Renyi's model, the maximum percentage difference error (the percentage difference between exact and approximation) is less than 10% when ρ tends to 1 (Gupta et. al, 1985), whereas the corresponding error for Shannon's entropy goes beyond 300%. The parametric model provides the maximum percentage difference around 60%.

Similarly, Tables 4.2 and 4.4 and Figures 4.7 - 4.12 provide the comparison of waiting time distribution when Shannon's, Renyi' and Parametric model are used.

The Figures 4.7 - 4.12 show the exact waiting time distribution given in (4.1.3) and the approximate distribution obtained :

- i) when the values of parameters of the three models were optimized for waiting time distribution and;
- ii) when the values of the parameters were optimized for busy period distribution only.

It can be seen from these figures that Renyi's model consistently provides a better approximation than Shannon's or parametric model. Additionally, from these figures one can measure the amount of inaccuracy that will occur if one chooses to use the same values of parameters as used for the Busy period distribution to determine the value of the waiting time distribution.

Tables 4.3 and 4.5 provide the approximate residence time distribution. The corresponding values of residence time distribution are shown in Figures 4.13 - 4.18. In the case of residence time distribution, Shannon's entropy leads to the exact distribution; therefore, Renyi's and Parametric entropy are compared with Shannon's model. In Figures 4.13 - 4.18, show the results,

- i) When the values of Renyi's and Parametric models were optimized for the waiting time distribution and ;
- ii) when the values of parameters were optimized for the busy period distribution.

Again, it can be seen from these graphs that Renyi's model consistently provides a better approximation than the Parametric model. Also, from these Figures 4.13 - 4.18 one can measure the amount of inaccuracy that will occur if one chooses to use the same value of parameters as in Busy period distribution to determine the value of residence time distribution.

Figures 4.20 and 4.19 show the relationship between parameters a_1 , in case of Renyi's model, a_2 , in case of Parametric model, and m for various values of traffic intensity for the busy period distribution, respectively. Similarly, Figures 4.22 and 4.21 show the relationship between parameters a_1 and a_2 and m for various values of traffic intensity ρ for the residence time distribution and Figures 4.24 and

4.23 show the relationship for the waiting time distribution. These figures give the optimum values for the busy period, waiting time and residence time distributions. The respective figures can be used to determine optimum values of the parameters of performance distributions when the values of m and ρ are provided.

4.3 CONCLUSION

This chapter provides the approximate performance distributions of a single server queueing system when the mean service time and traffic intensity are provided by using three MEP models, namely Shannon's, Renyi's and Parametric entropy. It is shown that Renyi's model provides a better approximation than Shannon's or the Parametric model (except for residence time where Shannon's model gives the exact distribution). The relationship between the optimum values of parameters a_1 and a_2 for busy period, residence time and waiting time distributions are provided.

Busy Period Distribution.

Absolute percentage difference between exact with approximation distribution.

m	Traffic intensity	a_1	a_2	Arrival time	Service time	Renyi's measure			Parametric's measure			Shannon's measure		
						Average	Maximum	Minimum	Average	Maximum	Minimum	Average	Maximum	Minimum
0.5	0.1	1.0945	0.1000	0.2222	2.2222	0.0004	0.4681	0.1493	0.0073	3.2009	1.3990	0.0212	9.601	4.449
	0.2	1.1695	0.2500	0.5000	2.5000	0.3792	0.6074	0.5013	1.7929	4.7337	3.0040	0.1636	19.201	9.599
	0.3	1.2375	0.4200	0.8571	2.8571	0.6755	1.5820	1.1038	2.6722	8.7458	4.3881	0.3675	29.617	15.763
	0.4	1.2933	0.5620	1.3333	3.3333	1.0527	2.3908	1.9739	2.3233	9.5892	5.9602	0.3801	45.380	23.422
	0.5	1.3413	0.7500	2.0000	4.0000	0.7906	3.8457	2.9865	1.5987	12.9569	7.4385	0.0874	65.840	33.339
	0.6	1.3831	0.9700	3.0000	5.0000	0.0791	5.6487	3.8896	0.0637	16.6023	9.1046	0.2628	93.145	46.869
	0.7	1.4202	1.3000	4.6667	6.6667	0.1063	7.8310	4.9518	0.2748	26.4325	10.0622	0.8465	131.893	66.734
	0.8	1.4484	2.2000	8.0000	10.0000	0.2771	11.7581	7.0829	0.2974	64.8356	9.7804	0.4398	194.453	99.953
	0.9	1.4717	2.6000	18.0000	20.0000	0.0778	22.4226	11.4278	0.6983	25.2346	15.8915	4.2473	330.473	174.925
1.0	0.1	1.0909	0.1830	0.1111	1.1111	0.0001	0.1270	0.0507	0.0149	1.2171	0.6502	0.0212	9.800	4.414
	0.2	1.1667	0.4700	0.2500	1.2500	0.0003	0.5712	0.2357	1.1580	2.6069	1.6182	0.1636	19.600	8.628
	0.3	1.2273	0.6000	0.4286	1.4286	0.0042	1.5961	0.6881	0.0094	5.0037	2.4633	0.1877	29.400	12.695
	0.4	1.2882	1.0200	0.6667	1.6667	0.7339	2.7010	1.5351	0.7922	6.2472	2.6300	0.3801	39.198	16.733
	0.5	1.3378	1.4000	1.0000	2.0000	1.7429	4.3779	2.8770	0.6799	8.3395	3.5720	0.0874	48.995	20.996
	0.6	1.3876	1.8000	1.5000	2.5000	4.1715	8.4990	5.1195	0.0012	12.2247	4.4623	0.2628	58.790	26.046
	0.7	1.4202	2.5000	2.3333	3.3333	5.8707	8.3451	6.8220	0.1872	26.4735	5.9208	0.8465	68.579	33.156
	0.8	1.4505	3.0000	4.0000	5.0000	4.0934	10.4081	8.4107	0.1884	26.5333	11.8408	0.4398	91.160	45.710
	0.9	1.4762	4.2000	9.0000	10.0000	0.0443	14.9809	8.3107	0.3519	41.8302	21.7858	0.8221	177.836	77.452
6.0	0.1	1.0909	1.2000	0.0185	0.1852	0.0000	0.0057	0.0020	0.2935	0.3686	0.3412	8.0344	9.967	8.995
	0.2	1.1667	2.6000	0.0417	0.2083	0.0000	0.0282	0.0099	0.0003	0.1266	0.0472	16.0296	19.933	17.976
	0.3	1.2304	4.2000	0.0714	0.2381	0.0010	0.0850	0.0474	0.4226	1.3490	0.7931	23.9683	29.900	26.938
	0.4	1.2857	5.8000	0.1111	0.2778	0.0001	0.1900	0.0675	0.0034	1.0712	0.3973	31.8216	39.867	35.871
	0.5	1.3333	7.6300	0.1667	0.3333	0.0001	0.4095	0.1470	0.0063	1.9159	1.0829	39.5365	49.833	44.755
	0.6	1.3759	10.0000	0.2500	0.4167	0.5858	1.2155	0.7842	0.0410	3.0645	1.7895	47.0055	59.800	53.555
	0.7	1.4115	12.7000	0.3889	0.5556	0.0045	1.7761	0.6179	0.0433	6.3060	4.0572	53.9732	69.766	62.183
	0.8	1.4464	16.6000	0.6667	0.8333	2.9930	5.5692	3.7992	0.0630	10.6208	7.7126	59.6599	79.732	70.370
	0.9	1.4762	23.0000	1.5000	1.6667	7.6499	12.6541	9.4738	0.6589	18.9700	14.6231	60.2728	89.697	76.772
15.0	0.1	1.0909	3.2000	0.0074	0.0741	0.0000	0.0010	0.0004	0.2715	0.3196	0.2935	9.2054	9.987	9.595
	0.2	1.1667	6.5000	0.0167	0.0833	0.0000	0.0048	0.0017	0.0041	0.1308	0.0612	18.4043	19.973	19.188
	0.3	1.2308	10.3000	0.0286	0.0952	0.0043	0.0178	0.0089	0.2052	0.6161	0.3948	27.5938	29.960	28.778
	0.4	1.2857	14.4500	0.0444	0.1111	0.0000	0.0334	0.0116	0.0057	0.9064	0.4198	36.7693	39.947	38.363
	0.5	1.3333	19.0000	0.0667	0.1333	0.0000	0.0739	0.0257	0.0070	1.0639	0.4920	45.9222	49.933	47.940
	0.6	1.3745	24.7000	0.1000	0.1667	0.0813	0.2975	0.2130	0.0097	1.7373	0.8943	55.0351	59.920	57.503
	0.7	1.4132	31.5000	0.1556	0.2222	1.3461	1.4590	1.3798	0.0484	4.2778	2.0831	64.0673	69.907	67.039
	0.8	1.4444	41.5000	0.2667	0.3333	0.0004	1.0149	0.3670	0.0630	7.8310	4.2240	72.8935	79.893	76.505
	0.9	1.4739	57.0000	0.6000	0.6667	0.8890	4.4425	2.1699	0.2863	18.9111	12.1567	80.8667	89.879	85.682

Table 4.1 (continued)

Busy Period Distribution.

Absolute percentage difference between exact with approximation distribution.

m	Traffic intensity	a_1	a_2	Arrival time	Service time	Renyi's measure			Parametric's measure			Shannon's measure		
						Average	Maximum	Minimum	Average	Maximum	Minimum	Average	Maximum	Minimum
30.0	0.1	1.0909	6.2000	0.0037	0.0370	0.0000	0.0003	0.0001	0.0251	0.0363	0.0312	9.6013	9.993	9.797
	0.2	1.1613	13.0000	0.0083	0.0417	0.9044	0.9516	0.9281	0.0590	0.1322	0.0940	19.2010	19.987	19.594
	0.3	1.2294	21.5000	0.0143	0.0476	0.3164	0.3386	0.3280	1.9647	2.3170	2.1360	28.7984	29.980	29.389
	0.4	1.2857	29.5000	0.0222	0.0556	0.0000	0.0086	0.0030	1.4766	2.0913	1.7734	38.3922	39.973	39.184
	0.5	1.3333	38.0000	0.0333	0.0667	0.0000	0.0192	0.0066	0.0070	0.6002	0.2498	47.9803	49.967	48.977
	0.6	1.3752	48.7000	0.0500	0.0833	0.1491	0.1786	0.1577	0.0068	1.5278	0.7228	57.5584	59.960	58.766
	0.7	1.4115	62.7000	0.0778	0.1111	0.1023	0.2320	0.1823	0.0316	2.2419	1.0484	67.1163	69.953	68.548
	0.8	1.4444	81.3000	0.1333	0.1667	0.0001	0.2845	0.1000	0.0159	5.2924	2.5085	76.6234	79.947	78.313
	0.9	1.4739	114.0000	0.3000	0.3333	0.8888	1.8794	1.2071	0.2492	14.0338	7.3526	85.9216	89.940	88.007
45.0	0.1	1.0909	8.8000	0.0025	0.0247	0.0000	0.0002	0.0001	0.5877	0.5970	0.5926	9.7339	9.996	9.865
	0.2	1.1667	20.0000	0.0056	0.0278	0.0000	0.0006	0.0002	0.6572	0.7309	0.6932	19.4671	19.991	19.729
	0.3	1.2299	31.9000	0.0095	0.0317	0.2031	0.2126	0.2081	1.6558	1.8756	1.7636	29.1993	29.987	29.593
	0.4	1.2841	43.5000	0.0148	0.0370	0.4995	0.5259	0.5132	0.7503	1.1140	0.9277	38.9298	39.982	39.457
	0.5	1.3333	57.5000	0.0222	0.0444	0.0000	0.0087	0.0030	0.5662	1.2820	0.9139	48.6579	49.978	49.319
	0.6	1.3745	73.0000	0.0333	0.0556	0.2599	0.2982	0.2819	0.0066	1.0548	0.4840	58.3814	59.973	59.180
	0.7	1.4115	95.0000	0.0519	0.0741	0.1667	0.2322	0.2064	0.0587	3.0724	1.4919	68.0961	69.969	69.038
	0.8	1.4444	120.0000	0.0889	0.1111	0.0001	0.1314	0.0458	0.0340	5.2580	2.7098	77.7882	79.964	78.889
	0.9	1.4739	168.0000	0.2000	0.2222	0.8888	1.3073	1.0133	0.1723	10.5006	5.3493	87.3887	89.960	88.708
50.0	0.1	1.0909	10.5000	0.0022	0.0222	0.0000	0.0002	0.0000	0.1352	0.1475	0.1412	9.7604	9.996	9.878
	0.2	1.1681	22.5000	0.0050	0.0250	0.2461	0.2530	0.2495	0.9547	1.0319	0.9926	19.5203	19.992	19.756
	0.3	1.2353	35.2000	0.0086	0.0286	1.0742	1.1105	1.0921	1.4117	1.5985	1.5034	29.2794	29.988	29.634
	0.4	1.2857	48.0000	0.0133	0.0333	0.0000	0.0032	0.0011	0.4170	0.7254	0.5677	39.0371	39.984	39.511
	0.5	1.3333	63.0000	0.0200	0.0400	0.0000	0.0070	0.0024	0.0003	0.3667	0.1599	48.7929	49.980	49.388
	0.6	1.3745	81.5000	0.0300	0.0500	0.2654	0.2982	0.2842	0.0058	0.6864	0.3045	58.5450	59.976	59.263
	0.7	1.4115	105.0000	0.0467	0.0667	0.1772	0.2322	0.2104	0.0052	2.3228	0.9937	68.2898	69.972	69.136
	0.8	1.4444	135.0000	0.0800	0.1000	0.0001	0.1073	0.0373	0.0444	3.0430	1.4925	78.0165	79.968	79.002
	0.9	1.4739	186.0000	0.1800	0.2000	0.8888	1.2180	0.9846	0.1386	9.8368	4.9146	87.6691	89.964	88.844
55.0	0.1	1.0905	12.0000	0.0020	0.0202	0.0538	0.0552	0.0545	0.5503	0.5726	0.5613	9.7822	9.996	9.889
	0.2	1.1661	23.9000	0.0045	0.0227	0.0964	0.0995	0.0980	0.1535	0.1979	0.1752	19.5639	19.993	19.778
	0.3	1.2353	38.5000	0.0078	0.0260	1.0775	1.1105	1.0938	1.2107	1.3724	1.2902	29.3449	29.989	29.667
	0.4	1.2857	53.0000	0.0121	0.0303	0.0000	0.0026	0.0009	0.6452	0.9380	0.7887	39.1249	39.985	39.556
	0.5	1.3333	69.0000	0.0182	0.0364	0.0000	0.0058	0.0020	0.1226	0.6245	0.3799	48.9032	49.982	49.443
	0.6	1.3745	90.0000	0.0273	0.0455	0.2696	0.2982	0.2859	0.0024	1.0926	0.5150	58.6785	59.978	59.330
	0.7	1.4115	114.0000	0.0424	0.0606	0.1851	0.2322	0.2134	0.0137	1.6681	0.7390	68.4478	69.975	69.215
	0.8	1.4444	148.9000	0.0727	0.0909	0.0000	0.0892	0.0310	0.0123	3.2269	1.3885	78.2021	79.971	79.095
	0.9	1.4739	205.5000	0.1636	0.1818	0.8888	1.1519	0.9638	0.0935	9.1324	4.4238	87.8952	89.967	88.954

Table 4.1

Waiting Time distribution using optimum
parameters of Busy Period Distribution.

Absolute percentage difference between exact with
approximation distribution.

m	Traffic intensity	a_1	a_2	Arrival time	Service time	Renyi's measure			Parametric's measure			Shannon's measure		
						Average	Minimum	Maximum	Average	Minimum	Maximum	Average	Minimum	Maximum
0.5	0.1	1.0945	0.1000	0.0222	0.2222	53.8084	26.8513	97.5545	53.7754	26.3940	97.5751	55.9522	28.5318	97.8047
	0.2	1.1695	0.2500	0.1000	0.5000	48.7705	22.6283	96.9212	47.7255	20.9120	96.8248	53.5190	26.1919	97.5370
	0.3	1.2375	0.4200	0.2571	0.8571	42.4073	17.7427	95.9621	40.3859	15.0126	95.6607	50.7615	23.6937	97.1950
	0.4	1.2933	0.5620	0.5333	1.3333	34.6147	12.3207	94.5100	32.7382	9.5663	94.1012	47.5989	21.0204	96.7428
	0.5	1.3413	0.7500	1.0000	2.0000	24.6272	6.1552	92.1095	22.6116	3.3311	91.2833	43.9163	18.1529	96.1166
	0.6	1.3831	0.9700	1.8000	3.0000	12.4313	0.0818	87.7115	12.4373	0.0874	86.0417	39.5410	15.0695	95.1925
	0.7	1.4202	1.3000	3.2667	4.6667	17.1000	0.2196	78.2268	16.7363	0.4093	72.9695	34.1925	11.7446	93.6910
	0.8	1.4484	2.2000	6.4000	8.0000	37.5165	0.6021	58.0435	47.8821	0.3294	82.5731	27.3564	8.1487	90.8260
	0.9	1.4717	2.6000	16.2000	18.0000	81.4899	28.6403	167.5829	86.7629	28.5327	208.1470	17.8499	4.2474	83.1937
1.0	0.1	1.0909	0.1830	0.0111	0.1111	71.4150	50.2370	98.7743	71.6105	50.4115	98.7894	73.3279	52.8173	98.8956
	0.2	1.1667	0.4700	0.0500	0.2500	66.7154	44.0598	98.4530	66.2557	43.2945	98.4209	71.1944	49.8757	98.7593
	0.3	1.2273	0.6000	0.1286	0.4286	60.8890	36.8631	98.0031	61.7037	37.5923	98.0685	68.6763	46.5429	98.5846
	0.4	1.2882	1.0200	0.2667	0.6667	52.5390	27.3429	97.2505	52.0899	26.6392	97.1610	65.6542	42.7354	98.3526
	0.5	1.3378	1.4000	0.5000	1.0000	41.3048	15.7516	96.0184	40.6445	15.0818	95.7640	61.9499	38.3439	98.0296
	0.6	1.3876	1.8000	0.9000	1.5000	23.3759	0.0343	93.4360	25.6346	1.8747	93.2913	57.2822	33.2229	97.5490
	0.7	1.4202	2.5000	1.6333	2.3333	21.0927	0.2196	88.5292	18.9686	0.6032	86.2523	51.1716	27.1742	96.7585
	0.8	1.4505	3.0000	3.2000	4.0000	48.8565	3.3649	74.6168	38.5647	1.0290	72.1304	42.6931	19.9205	95.2153
	0.9	1.4762	4.2000	8.1000	9.0000	127.5804	8.6482	187.9294	104.1940	9.9073	154.2664	29.6112	11.0621	90.8676
6.0	0.1	1.0909	1.2000	0.0019	0.0185	94.1100	88.6766	99.7945	94.1310	88.7157	99.7952	94.6612	89.7012	99.8150
	0.2	1.1667	2.6000	0.0083	0.0417	92.6745	86.0336	99.7400	92.6722	86.0341	99.7397	94.0459	88.5612	99.7919
	0.3	1.2304	4.2000	0.0214	0.0714	90.6488	82.3575	99.6609	90.5518	82.2039	99.6560	93.2700	87.1372	99.7623
	0.4	1.2857	5.8000	0.0444	0.1111	87.6263	76.9739	99.5384	87.5810	76.9560	99.5334	92.2612	85.3084	99.7228
	0.5	1.3333	7.6300	0.0833	0.1667	82.8750	68.7207	99.3361	82.9523	69.0206	99.3303	90.8960	82.8733	99.6675
	0.6	1.3759	10.0000	0.1500	0.2500	74.6079	54.8654	98.9581	74.7248	55.5401	98.9338	88.9443	79.4706	99.5847
	0.7	1.4115	12.7000	0.2722	0.3889	59.0856	30.2177	98.1682	59.7824	32.6454	98.1036	85.9219	74.3807	99.4471
	0.8	1.4464	16.6000	0.5333	0.6667	30.9718	0.3772	95.7560	29.5900	0.2901	95.5982	80.6032	65.9347	99.1729
	0.9	1.4762	23.0000	1.3500	1.5000	128.3260	3.5008	187.1428	95.3439	0.4039	134.3169	68.6478	49.1810	98.3593
15.0	0.1	1.0909	3.2000	0.0007	0.0074	97.5772	95.2789	99.9177	97.5699	95.2651	99.9175	97.8132	95.7328	99.9260
	0.2	1.1667	6.5000	0.0033	0.0167	96.9557	94.0884	99.8959	96.9531	94.0847	99.8958	97.5487	95.2248	99.9167
	0.3	1.2308	10.3000	0.0086	0.0286	96.0600	92.3830	99.8641	96.0417	92.3529	99.8632	97.2113	94.5796	99.9048
	0.4	1.2857	14.4500	0.0178	0.0444	94.7024	89.8182	99.8150	94.6715	89.7744	99.8133	96.7663	93.7328	99.8890
	0.5	1.3333	19.0000	0.0333	0.0667	92.4991	85.7001	99.7338	92.5006	85.7430	99.7322	96.1521	92.5725	99.8668
	0.6	1.3745	24.7000	0.0600	0.1000	88.6008	78.5243	99.5855	88.5212	78.5019	99.5771	95.2499	90.8849	99.8336
	0.7	1.4132	31.5000	0.1089	0.1556	80.2319	63.4945	99.2513	80.5498	64.4615	99.2449	93.7946	88.2049	99.7782
	0.8	1.4444	41.5000	0.2133	0.2667	59.3969	27.6869	98.3427	59.0299	28.8879	98.2184	91.0510	83.2927	99.6677
	0.9	1.4739	57.0000	0.5400	0.6000	59.5985	1.9539	113.6035	52.0573	0.2953	92.9806	83.9316	71.3689	99.3375

Table 4.2 (continued)

Waiting Time distribution using optimum
parameters of Busy Period Distribution.

Absolute percentage difference between exact with
approximation distribution.

m	Traffic Intensity	a_1	a_2	Arrival time	Service time	Renyi's measure			Parametric's measure			Shannon's measure		
						Average	Minimum	Maximum	Average	Minimum	Maximum	Average	Minimum	Maximum
30.0	0.1	1.0909	6.2000	0.0004	0.0037	98.7771	97.6061	99.9589	98.7775	97.6069	99.9589	98.8978	97.8408	99.9630
	0.2	1.1613	13.0000	0.0017	0.0083	98.4724	97.0146	99.9484	98.4563	96.9837	99.9479	98.7623	97.5775	99.9583
	0.3	1.2294	21.5000	0.0043	0.0143	98.0019	96.1041	99.9322	97.9511	96.0075	99.9304	98.5888	97.2410	99.9524
	0.4	1.2857	29.5000	0.0089	0.0222	97.2881	94.7284	99.9075	97.2370	94.6344	99.9055	98.3586	96.7958	99.9445
	0.5	1.3333	38.0000	0.0167	0.0333	96.1284	92.5054	99.8668	96.1177	92.4965	99.8660	98.0388	96.1795	99.9334
	0.6	1.3752	48.7000	0.0300	0.0500	94.0191	88.4941	99.7916	94.0495	88.5866	99.7914	97.5642	95.2695	99.9167
	0.7	1.4115	62.7000	0.0544	0.0778	89.6281	80.2477	99.6311	89.5785	80.2818	99.6241	96.7866	93.7906	99.8890
	0.8	1.4444	81.3000	0.1067	0.1333	77.5599	58.1058	99.1690	77.5301	58.6553	99.1405	95.2790	90.9664	99.8336
	0.9	1.4739	114.0000	0.2700	0.3000	36.6748	0.2121	96.6571	34.5223	0.2953	96.2062	91.1019	83.4297	99.6677
45.0	0.1	1.0909	8.8000	0.0002	0.0025	99.1822	98.3965	99.9726	99.1870	98.4061	99.9727	99.2632	98.5548	99.9753
	0.2	1.1667	20.0000	0.0011	0.0056	98.9674	97.9780	99.9653	98.9601	97.9639	99.9650	99.1721	98.3771	99.9722
	0.3	1.2299	31.9000	0.0029	0.0095	98.6584	97.3767	99.9548	98.6313	97.3247	99.9538	99.0554	98.1495	99.9683
	0.4	1.2841	43.5000	0.0059	0.0148	98.1871	96.4622	99.9386	98.1595	96.4107	99.9376	98.9002	97.8478	99.9630
	0.5	1.3333	57.5000	0.0111	0.0222	97.3910	94.9230	99.9112	97.3637	94.8761	99.9100	98.6840	97.4284	99.9556
	0.6	1.3745	73.0000	0.0200	0.0333	95.9703	92.1910	99.8616	95.9676	92.2024	99.8609	98.3622	96.8060	99.9445
	0.7	1.4115	95.0000	0.0363	0.0519	92.9330	86.3996	99.7539	92.7721	86.1586	99.7457	97.8318	95.7862	99.9260
	0.8	1.4444	120.0000	0.0711	0.0889	84.5059	70.5846	99.4455	84.7624	71.3451	99.4434	96.7933	93.8098	99.8890
	0.9	1.4739	168.0000	0.1800	0.2000	42.8050	0.5071	97.7668	42.3055	0.2457	97.5864	93.8438	88.3412	99.7782
50.0	0.1	1.0909	10.5000	0.0002	0.0022	99.2635	98.5555	99.9753	99.2624	98.5535	99.9753	99.3366	98.6983	99.9778
	0.2	1.1681	22.5000	0.0010	0.0050	99.0676	98.1732	99.9687	99.0605	98.1597	99.9684	99.2544	98.5380	99.9750
	0.3	1.2353	35.2000	0.0026	0.0086	98.7753	97.6040	99.9587	98.7699	97.5942	99.9585	99.1491	98.3326	99.9714
	0.4	1.2857	48.0000	0.0053	0.0133	98.3573	96.7919	99.9445	98.3470	96.7735	99.9441	99.0092	98.0601	99.9667
	0.5	1.3333	63.0000	0.0100	0.0200	97.6468	95.4160	99.9200	97.6464	95.4197	99.9199	98.8141	97.6811	99.9600
	0.6	1.3745	81.5000	0.0180	0.0300	96.3623	92.9405	99.8755	96.3410	92.9133	99.8742	98.5234	97.1180	99.9500
	0.7	1.4115	105.0000	0.0327	0.0467	93.6116	87.6795	99.7785	93.5055	87.5292	99.7728	98.0439	96.1940	99.9334
	0.8	1.4444	135.0000	0.0640	0.0800	85.9557	73.2456	99.5008	85.8596	73.3174	99.4871	97.1030	94.3976	99.9001
	0.9	1.4739	186.0000	0.1620	0.1800	47.3804	3.2660	97.9893	47.3276	5.9400	97.8480	94.4171	89.3896	99.8004
55.0	0.1	1.0905	12.0000	0.0002	0.0020	99.3305	98.6866	99.9776	99.3263	98.6785	99.9774	99.3966	98.8159	99.9798
	0.2	1.1661	23.9000	0.0009	0.0045	99.1547	98.3433	99.9716	99.1523	98.3388	99.9715	99.3219	98.6698	99.9773
	0.3	1.2353	38.5000	0.0023	0.0078	98.8855	97.8186	99.9625	98.8831	97.8144	99.9624	99.2260	98.4827	99.9740
	0.4	1.2857	53.0000	0.0048	0.0121	98.5046	97.0779	99.9495	98.4921	97.0547	99.9490	99.0985	98.2342	99.9697
	0.5	1.3333	69.0000	0.0091	0.0182	97.8569	95.8217	99.9273	97.8631	95.8373	99.9274	98.9208	97.8885	99.9636
	0.6	1.3745	90.0000	0.0164	0.0273	96.6848	93.5587	99.8868	96.6512	93.5055	99.8852	98.6557	97.3745	99.9546
	0.7	1.4115	114.0000	0.0297	0.0424	94.1713	88.7393	99.7986	94.1717	88.7807	99.7971	98.2181	96.5297	99.9394
	0.8	1.4444	148.9000	0.0582	0.0727	87.1574	75.4660	99.5462	86.9856	75.3591	99.5313	97.3581	94.8834	99.9092
	0.9	1.4739	205.5000	0.1473	0.1636	51.5770	10.4638	98.1715	50.7694	11.4868	98.0183	94.8926	90.2649	99.8185

Table 4.2

Residence time distribution using optimum
parameters of Busy Period Distribution.

Absolute percentage difference between exact with approximation distribution.											
m	Traffic intensity	a_1	a_2	Arrival time	Service time	Renyi's measure			Parametric's measure		
						Average	Maximum	Minimum	Average	Maximum	Minimum
0.5	0.1	1.0945	0.1000	0.2222	2.2222	2.3513	11.3971	6.0678	2.9912	10.4594	5.9233
	0.2	1.1695	0.2500	0.5000	2.5000	4.8283	25.0043	12.7956	7.1536	28.9180	15.1991
	0.3	1.2375	0.4200	0.8571	2.8571	7.7988	43.9548	21.3790	11.3766	54.7027	26.2096
	0.4	1.2933	0.5620	1.3333	3.3333	11.0150	68.5472	31.4057	14.5025	81.0994	35.7477
	0.5	1.3413	0.7500	2.0000	4.0000	14.6589	103.1865	43.8628	18.1093	124.4609	48.7551
	0.6	1.3831	0.9700	3.0000	5.0000	18.8472	155.6100	59.9728	21.6380	190.3411	64.3716
	0.7	1.4202	1.3000	4.6667	6.6667	23.7979	245.1163	82.3109	25.7742	328.4457	88.1877
	0.8	1.4484	2.2000	8.0000	10.0000	28.8406	392.2674	110.4450	32.4915	993.6201	150.4970
	0.9	1.4717	2.6000	18.0000	19.9999	34.3470	700.7904	151.7964	34.2345	1375.6378	174.3234
1.0	0.1	1.0909	0.1830	0.1111	1.1111	5.4687	10.9882	7.9895	5.0988	9.6252	7.1072
	0.2	1.1667	0.4700	0.2500	1.2500	11.6029	24.6893	17.4438	13.1297	27.2788	19.1814
	0.3	1.2273	0.6000	0.4286	1.4286	18.1076	41.0808	28.0973	16.7436	36.4624	25.0209
	0.4	1.2882	1.0200	0.6667	1.6667	26.8797	66.8957	43.6248	28.1085	72.3267	45.4134
	0.5	1.3378	1.4000	1.0000	2.0000	36.6424	102.0688	62.7397	37.7288	114.9768	65.7433
	0.6	1.3876	1.8000	1.5000	2.5000	50.3400	167.8118	93.6381	46.9446	173.7150	88.8407
	0.7	1.4202	2.5000	2.3333	3.3333	62.8143	253.8767	127.2918	60.5703	324.1167	132.5862
	0.8	1.4505	3.0000	4.0000	5.0000	78.6307	430.5126	181.2526	68.4281	482.4740	165.6402
	0.9	1.4762	4.2000	9.0000	10.0000	97.5733	900.3104	276.3952	82.1919	1103.4827	246.9946
6.0	0.1	1.0909	1.2000	0.0185	0.1852	9.9492	11.0910	10.5120	9.5693	10.6810	10.1139
	0.2	1.1667	2.6000	0.0417	0.2083	22.0967	24.9485	23.4964	22.0917	25.1105	23.5611
	0.3	1.2304	4.2000	0.0714	0.2381	37.1597	42.6336	39.8319	38.3533	44.6997	41.4144
	0.4	1.2857	5.8000	0.1111	0.2778	56.7298	66.4819	61.4561	56.8515	68.3062	62.3131
	0.5	1.3333	7.6300	0.1667	0.3333	82.6352	99.6677	90.8087	80.8835	101.4145	90.5096
	0.6	1.3759	10.0000	0.2500	0.4167	100.0000	150.8821	134.5236	100.0000	156.7521	134.8737
	0.7	1.4115	12.7000	0.3889	0.5556	100.0000	231.2754	199.6068	100.0000	242.9610	197.7812
	0.8	1.4464	16.6000	0.6667	0.8333	100.0000	413.1055	333.2331	100.0000	432.1748	314.7091
	0.9	1.4762	23.0000	1.5000	1.6667	100.0000	981.9772	666.0767	100.0000	1037.2468	574.9910
15.0	0.1	1.0909	3.2000	0.0074	0.0741	10.6345	11.1035	10.8677	10.9594	11.4586	11.2069
	0.2	1.1667	6.5000	0.0167	0.0833	23.7987	24.9794	24.3847	23.8757	25.1444	24.5032
	0.3	1.2308	10.3000	0.0286	0.0952	40.5253	42.8227	41.6630	41.0808	43.7024	42.3728
	0.4	1.2857	14.4500	0.0444	0.1111	62.4626	66.5928	64.5018	63.1621	68.1186	65.5923
	0.5	1.3333	19.0000	0.0667	0.1333	92.5272	99.8670	96.1363	91.9487	101.0662	96.3866
	0.6	1.3745	24.7000	0.1000	0.1667	100.0000	149.0062	142.1553	100.0000	154.0773	144.6124
	0.7	1.4132	31.5000	0.1556	0.2222	100.0000	237.5597	223.0681	100.0000	240.4327	219.7099
	0.8	1.4444	41.5000	0.2667	0.3333	100.0000	398.6710	364.0736	100.0000	436.0930	375.0735
	0.9	1.4739	57.0000	0.6000	0.6667	100.0000	903.0140	761.5157	100.0000	1033.5948	753.0851

Table 4.3 (continued)

Residence time distribution using optimum

parameters of Busy Period Distribution.

Absolute percentage difference between exact with approximation distribution.

approximation distribution.											
Traffic m intensity	α_1	α_2	Arrival time	Service time	Renyi's measure			Parametric's measure			
					Average	Maximum	Minimum	Average	Maximum	Minimum	
30.0	0.1	1.0909	6.2000	0.0037	0.0370	10.8712	11.1078	10.9892	10.8360	11.0793	10.9571
	0.2	1.1613	13.0000	0.0083	0.0417	23.2366	23.8001	23.5174	24.5105	25.1557	24.8314
	0.3	1.2294	21.5000	0.0143	0.0476	41.2041	42.3530	41.7758	44.7051	46.1508	45.4225
	0.4	1.2857	29.5000	0.0222	0.0556	64.5238	66.6297	65.5701	67.4575	70.1240	68.7774
	0.5	1.3333	38.0000	0.0333	0.0667	96.1676	99.9336	98.0348	96.4017	101.1516	98.7444
	0.6	1.3752	48.7000	0.0500	0.0833	100.0000	150.2507	146.7002	100.0000	150.5464	145.8204
	0.7	1.4115	62.7000	0.0778	0.1111	100.0000	232.3007	225.0863	100.0000	238.5911	227.7468
	0.8	1.4444	81.3000	0.1333	0.1667	100.0000	399.3344	381.0783	100.0000	416.4765	385.4486
	0.9	1.4739	114.0000	0.3000	0.3333	100.0000	906.0428	827.4023	100.0000	1041.7293	865.6856
45.0	0.1	1.0909	8.8000	0.0025	0.0247	10.9507	11.1089	11.0297	10.2925	10.4449	10.3685
	0.2	1.1667	20.0000	0.0056	0.0278	24.5934	24.9931	24.7927	25.4586	25.9075	25.6822
	0.3	1.2299	31.9000	0.0095	0.0317	41.7658	42.5398	42.1516	44.5729	45.5255	45.0468
	0.4	1.2841	43.5000	0.0148	0.0370	64.3780	65.7654	65.0688	66.7711	68.5039	67.6318
	0.5	1.3333	57.5000	0.0222	0.0444	97.4230	99.9556	98.6822	99.2498	102.5322	100.8759
	0.6	1.3745	73.0000	0.0333	0.0556	100.0000	149.1712	146.8129	100.0000	150.4279	147.2396
	0.7	1.4115	95.0000	0.0519	0.0741	100.0000	232.3864	227.5201	100.0000	243.4945	235.8251
	0.8	1.4444	120.0000	0.0889	0.1111	100.0000	399.5563	387.1558	100.0000	401.4488	381.4666
	0.9	1.4739	168.0000	0.2000	0.2222	100.0000	907.0568	852.5864	100.0000	988.4025	872.5752
50.0	0.1	1.0909	10.5000	0.0022	0.0222	10.9664	11.1089	11.0375	11.1230	11.2727	11.1977
	0.2	1.1681	22.5000	0.0050	0.0250	24.9450	25.3104	25.1273	25.8725	26.2844	26.0778
	0.3	1.2353	35.2000	0.0086	0.0286	43.6940	44.4318	44.0618	44.2824	45.1306	44.7046
	0.4	1.2857	48.0000	0.0133	0.0333	65.3710	66.6446	66.0054	66.3177	67.8580	67.0833
	0.5	1.3333	63.0000	0.0200	0.0400	97.6766	99.9601	98.8126	97.5187	100.3796	98.9375
	0.6	1.3745	81.5000	0.0300	0.0500	100.0000	149.1794	147.0535	100.0000	151.6681	148.7488
	0.7	1.4115	105.0000	0.0467	0.0667	100.0000	232.4038	228.0137	100.0000	241.0012	234.2014
	0.8	1.4444	135.0000	0.0800	0.1000	100.0000	399.6006	388.3978	100.0000	413.3485	394.1675
	0.9	1.4739	186.0000	0.1800	0.2000	100.0000	907.2600	857.8503	100.0000	978.0227	874.0343
55.0	0.1	1.0905	12.0000	0.0020	0.0202	10.9188	11.0476	10.9831	11.6026	11.7452	11.6738
	0.2	1.1661	23.9000	0.0045	0.0227	24.5448	24.8703	24.7072	24.8861	25.2421	25.0636
	0.3	1.2353	38.5000	0.0078	0.0260	43.7615	44.4328	44.0962	44.0441	44.8084	44.4248
	0.4	1.2857	53.0000	0.0121	0.0303	65.4876	66.6466	66.0651	66.8003	68.2140	67.5033
	0.5	1.3333	69.0000	0.0182	0.0364	97.8849	99.9637	98.9195	97.1486	99.7275	98.4285
	0.6	1.3745	90.0000	0.0273	0.0455	100.0000	149.1862	147.2509	100.0000	152.6883	149.9967
	0.7	1.4115	114.0000	0.0424	0.0606	100.0000	232.4177	228.4190	100.0000	234.9247	229.0117
	0.8	1.4444	148.9000	0.0727	0.0909	100.0000	399.6369	389.4208	100.0000	416.0351	398.2638
	0.9	1.4739	205.5000	0.1636	0.1818	100.0000	907.4265	862.2167	100.0000	991.7756	893.1620

Table 4.3

WAITING TIME DISTRIBUTION

Absolute percentage difference between exact with approximation distribution.

m	Traffic intensity		Arrival Service		approximation distribution.									
	α_1	α_2	time	time	Renyi's measure			Parametric's measure			Shannon's measure			
					Average	Minimum	Maximum	Average	Minimum	Maximum	Average	Minimum	Maximum	
0.5	0.1	1.4876	7.5000	0.0222	0.2222	6.6659	0.4500	67.3476	1.9846	0.0004	18.1208	55.9522	28.5318	97.8047
	0.2	1.4748	4.0625	0.1000	0.5000	8.8723	0.1397	78.3963	3.8970	0.0268	35.4343	53.5190	26.1919	97.5370
	0.3	1.4494	2.2500	0.2571	0.8571	13.8291	1.5280	85.9679	8.1288	0.0157	68.0111	50.7615	23.6937	97.1950
	0.4	1.4124	1.5313	0.5333	1.3333	18.3524	3.1502	89.5666	11.6414	0.0539	81.7904	47.5989	21.0204	96.7428
	0.5	1.3604	1.0157	1.0000	2.0000	22.0001	4.7121	91.3282	15.7214	0.0253	88.1007	43.9163	18.1529	96.1166
	0.6	1.3053	1.0039	1.8000	3.0000	22.8031	5.0127	91.5535	12.1717	0.0114	85.4719	39.5410	15.0695	95.1925
	0.7	1.2395	0.5078	3.2667	4.6667	22.1404	4.7728	90.8738	16.4327	0.0266	89.2467	34.1925	11.7446	93.6910
	0.8	1.1646	0.2578	6.4000	8.0000	19.5142	3.8820	88.6301	17.2191	1.3891	88.0754	27.3564	8.1487	90.8260
	0.9	1.0815	0.1289	16.2000	18.0000	13.9979	2.3486	81.5937	11.9891	0.5856	80.8842	17.8499	4.2474	83.1937
1.0	0.1	1.4937	18.5000	0.0111	0.1111	6.7750	0.0773	67.9787	2.2595	0.0287	16.6697	73.3279	52.8173	98.8956
	0.2	1.4871	9.2500	0.0500	0.2500	10.2397	0.0253	79.2932	4.9935	0.0661	37.3268	71.1944	49.8757	98.7593
	0.3	1.4699	6.1250	0.1286	0.4286	15.4089	0.1223	88.4376	8.2258	0.0430	60.2450	68.6763	46.5429	98.5846
	0.4	1.4520	4.2500	0.2667	0.6667	18.1895	0.1368	91.0214	12.3508	0.0208	79.6130	65.6542	42.7354	98.3526
	0.5	1.4211	3.0625	0.5000	1.0000	22.2718	0.0697	92.9623	16.2051	0.0003	88.0602	61.9499	38.3439	98.0296
	0.6	1.3810	2.0625	0.9000	1.5000	25.0886	1.0381	93.7276	21.0197	0.0348	92.1079	57.2822	33.2229	97.5490
	0.7	1.3306	1.5313	1.6333	2.3333	25.5300	1.6719	93.6578	20.9674	0.0295	92.4636	51.1716	27.1742	96.7585
	0.8	1.2614	1.0157	3.2000	4.0000	23.8444	1.7567	92.6338	19.8810	0.1468	91.7751	42.6931	19.9205	95.2153
	0.9	1.1594	0.5156	8.1000	9.0000	18.7261	1.3425	88.7576	15.9597	0.0177	88.0904	29.6112	11.0621	90.8676
6.0	0.1	1.4984	190.5000	0.0019	0.0185	10.0702	0.0481	76.6551	1.9821	0.0261	11.3193	94.6612	89.7012	99.8150
	0.2	1.4938	98.5000	0.0083	0.0417	24.9884	1.6923	92.0188	4.3179	0.1156	23.7453	94.0459	88.5612	99.7919
	0.3	1.4877	65.5000	0.0214	0.0714	35.0614	6.7301	95.2046	7.3610	0.0999	40.7656	93.2700	87.1372	99.7623
	0.4	1.4878	49.5000	0.0444	0.1111	26.5814	0.2655	94.3869	11.1705	0.1083	59.6863	92.2612	85.3084	99.7228
	0.5	1.4847	38.5000	0.0833	0.1667	26.5795	0.4546	94.5572	15.8128	0.2300	78.1615	90.8960	82.8733	99.6675
	0.6	1.4758	30.5000	0.1500	0.2500	29.4537	0.4254	95.5689	20.6928	0.2400	88.3918	88.9443	79.4706	99.5847
	0.7	1.4638	23.5000	0.2722	0.3889	30.8358	0.2498	95.9516	25.3055	0.1462	93.3197	85.9219	74.3807	99.4471
	0.8	1.4396	17.2500	0.5333	0.6667	31.7163	0.1351	96.1883	28.7464	0.0904	95.2539	80.6032	65.9347	99.1729
	0.9	1.3738	10.7500	1.3500	1.5000	31.0320	0.1050	95.9387	28.9718	0.1159	95.4408	68.6478	49.1810	98.3593
15.0	0.1	1.4996	475.5000	0.0007	0.0074	8.2351	0.0796	67.2345	3.8404	0.7559	14.5615	97.8132	95.7328	99.9260
	0.2	1.4984	306.5000	0.0033	0.0167	16.7773	0.2183	87.9419	3.7441	0.0768	19.7537	97.5487	95.2248	99.9167
	0.3	1.4938	205.5000	0.0086	0.0286	41.8831	12.8778	96.2075	6.3856	0.1074	33.4066	97.2113	94.5796	99.9048
	0.4	1.4878	154.5000	0.0178	0.0444	54.6904	25.9377	97.7038	9.8515	0.2024	50.9013	96.7663	93.7328	99.8890
	0.5	1.4878	121.5000	0.0333	0.0667	46.0646	12.2255	97.2452	14.3608	0.0181	71.3386	96.1521	92.5725	99.8668
	0.6	1.4878	98.5000	0.0600	0.1000	34.6213	0.2384	96.5639	19.6368	0.2693	85.2653	95.2499	90.8849	99.8336
	0.7	1.4845	79.5000	0.1089	0.1556	32.9198	0.2378	96.3544	25.1275	0.2414	92.1673	93.7946	88.2049	99.7782
	0.8	1.4727	61.5000	0.2133	0.2667	34.8973	0.0897	96.8120	30.1129	0.6151	95.2827	91.0510	83.2927	99.6677
	0.9	1.4420	42.5000	0.5400	0.6000	34.9442	0.6652	96.8220	32.8919	0.2807	96.2783	83.9316	71.3689	99.3375

Table 4.4 (continued)

Absolute percentage difference between exact with
approximation distribution.

m	Traffic intensity		a ₁	a ₂	Arrival time	Service time	approximation distribution.								
	Renyi's measure						Parametric's measure			Shannon's measure					
	Average	Minimum					Maximum	Average	Minimum	Maximum	Average	Minimum	Maximum		
30.0	0.1	1.4980	935.5000	0.0004	0.0037	44.2028	20.9681	95.5392	6.3934	3.1307	17.4187	98.8978	97.8408	99.9630	
	0.2	1.4992	714.5000	0.0017	0.0083	16.8918	0.0653	87.9939	3.3254	0.0295	17.0286	98.7623	97.5775	99.9583	
	0.3	1.4969	476.5000	0.0043	0.0143	42.1042	12.8244	96.2541	5.6927	0.0654	29.1616	98.5888	97.2410	99.9524	
	0.4	1.4938	357.5000	0.0089	0.0222	55.1438	26.2310	97.7531	8.8253	0.1344	45.0477	98.3586	96.7958	99.9445	
	0.5	1.4878	284.5000	0.0167	0.0333	67.2911	42.4619	98.6153	13.0105	0.1559	64.5301	98.0388	96.1795	99.9334	
	0.6	1.4878	231.5000	0.0300	0.0500	59.3688	28.7579	98.2695	18.2950	0.4348	81.9154	97.5642	95.2695	99.9167	
	0.7	1.4878	189.5000	0.0544	0.0778	46.3808	6.4851	97.6933	24.2153	0.4724	91.0847	96.7866	93.7906	99.8890	
	0.8	1.4860	151.5000	0.1067	0.1333	35.9237	0.4915	96.9730	30.0356	0.4964	95.0190	95.2790	90.9664	99.8336	
	0.9	1.4644	110.5000	0.2700	0.3000	39.7348	0.7324	97.5033	34.4241	0.5490	96.5241	91.1019	83.4297	99.6677	
45.0	0.1	1.4998	1376.5000	0.0002	0.0025	9.9448	0.1496	75.7663	8.1515	4.7818	19.4339	99.2632	98.5548	99.9753	
	0.2	1.4992	1154.5000	0.0011	0.0056	23.4315	0.0271	91.7164	3.1140	0.0806	15.9896	99.1721	98.3771	99.9722	
	0.3	1.4984	770.5000	0.0029	0.0095	33.3919	3.5819	95.0821	5.3315	0.1086	27.3324	99.0554	98.1495	99.9683	
	0.4	1.4938	578.5000	0.0059	0.0148	66.3755	42.1061	98.4987	8.2727	0.0917	42.2476	98.9002	97.8478	99.9630	
	0.5	1.4878	461.5000	0.0111	0.0222	76.4969	57.3015	99.0746	12.2571	0.3114	61.1909	98.6840	97.4284	99.9556	
	0.6	1.4878	378.5000	0.0200	0.0333	70.7480	46.9721	98.8435	17.4296	0.0091	79.4204	98.3622	96.8060	99.9445	
	0.7	1.4878	312.5000	0.0363	0.0519	61.2701	30.0409	98.4583	23.4703	0.1755	90.0576	97.8318	95.7862	99.9260	
	0.8	1.4877	252.5000	0.0711	0.0889	43.0934	0.2730	97.7013	29.6923	0.3435	94.7672	96.7933	93.8098	99.8890	
	0.9	1.4758	188.5000	0.1800	0.2000	40.2907	0.1357	97.6003	34.8569	0.4108	96.5862	93.8438	88.3412	99.7782	
50.0	0.1	1.4980	1532.5000	0.0002	0.0022	56.3060	33.4571	97.2599	8.5055	5.1344	19.7733	99.3366	98.6983	99.9778	
	0.2	1.4996	1308.5000	0.0010	0.0050	15.5343	0.0954	85.9719	3.0606	0.0810	15.6868	99.2544	98.5380	99.9750	
	0.3	1.4984	872.5000	0.0026	0.0086	36.6433	6.8636	95.5586	5.2422	0.1408	26.8762	99.1491	98.3326	99.9714	
	0.4	1.4997	655.5000	0.0053	0.0133	35.5105	1.7916	77.0423	8.1346	0.1004	41.4929	99.0092	98.0601	99.9667	
	0.5	1.4878	523.5000	0.0100	0.0200	78.5136	60.6926	99.1668	12.0633	0.2284	60.1729	98.8141	97.6811	99.9600	
	0.6	1.4878	428.5000	0.0180	0.0300	73.2447	51.1490	98.9587	17.2194	0.2441	79.0101	98.5234	97.1180	99.9500	
	0.7	1.4878	354.5000	0.0327	0.0467	64.5518	35.4851	98.6118	23.2784	0.0591	89.9005	98.0439	96.1940	99.9334	
	0.8	1.4878	288.5000	0.0640	0.0800	47.4903	5.0348	97.9184	29.5592	0.3794	94.6367	97.1030	94.3976	99.9001	
	0.9	1.4760	215.5000	0.1620	0.1800	43.6289	0.4620	97.8242	34.9541	0.2620	96.6233	94.4171	89.3896	99.8004	
55.0	0.1	1.4999	1704.5000	0.0002	0.0020	8.2469	0.2377	66.0161	8.6291	5.2892	19.7788	99.3966	98.8159	99.9798	
	0.2	1.4996	1464.5000	0.0009	0.0045	16.2164	0.0169	87.1158	3.0138	0.0729	15.4352	99.3219	98.6698	99.9773	
	0.3	1.4984	977.5000	0.0023	0.0078	39.5747	9.9353	95.9510	5.1607	0.1075	26.3758	99.2260	98.4827	99.9740	
	0.4	1.4969	733.5000	0.0048	0.0121	52.8432	22.9950	97.5876	8.0131	0.1292	40.8630	99.0985	98.2342	99.9697	
	0.5	1.4878	586.5000	0.0091	0.0182	80.2097	63.5839	99.2423	11.8910	0.1619	59.2708	98.9208	97.8885	99.9636	
	0.6	1.4878	480.5000	0.0164	0.0273	75.3484	54.7189	99.0529	17.0131	0.1038	78.3476	98.6557	97.3745	99.9546	
	0.7	1.4878	397.5000	0.0297	0.0424	67.3209	40.1490	98.7375	23.1001	0.1179	89.7104	98.2181	96.5297	99.9394	
	0.8	1.4878	324.5000	0.0582	0.0727	51.5395	11.7546	98.1068	29.4533	0.4629	94.5730	97.3581	94.8834	99.9092	
	0.9	1.4816	244.5000	0.1473	0.1636	38.7422	0.0038	97.4542	34.9644	0.6249	96.5867	94.8926	90.2649	99.8185	

Table 4.4

RESIDENCE TIME DISTRIBUTION

m	Traffic intensity	a_1	a_2	Arrival time	Service time	Absolute percentage difference between exact with approximation distribution.					
						Renyi's measure			Parametric's measure		
						Average	Maximum	Minimum	Average	Maximum	Minimum
0.5	0.1	1.0003	0.0000010	0.2222	2.2222	0.0093	0.0367	0.0210	0.0000	0.0001	0.0001
	0.2	1.0003	0.0000010	0.5000	2.5000	0.0093	0.0367	0.0210	0.0000	0.0001	0.0001
	0.3	1.0003	0.0000010	0.8571	2.8571	0.0093	0.0367	0.0210	0.0000	0.0001	0.0001
	0.4	1.0003	0.0000010	1.3333	3.3333	0.0093	0.0367	0.0211	0.0000	0.0001	0.0001
	0.5	1.0003	0.0000010	2.0000	4.0000	0.0093	0.0368	0.0211	0.0000	0.0001	0.0001
	0.6	1.0003	0.0000010	3.0000	5.0000	0.0093	0.0368	0.0211	0.0000	0.0001	0.0001
	0.7	1.0003	0.0000010	4.6667	6.6667	0.0093	0.0368	0.0211	0.0000	0.0001	0.0001
	0.8	1.0003	0.0000010	8.0000	10.0000	0.0094	0.0368	0.0211	0.0000	0.0001	0.0001
	0.9	1.0003	0.0000010	18.0000	19.9999	0.0097	0.0374	0.0215	0.0000	0.0001	0.0001
1.0	0.1	1.0005	0.0000010	0.1111	1.1111	0.0287	0.0520	0.0396	0.0000	0.0000	0.0000
	0.2	1.0005	0.0000010	0.2500	1.2500	0.0287	0.0521	0.0396	0.0000	0.0000	0.0000
	0.3	1.0005	0.0000010	0.4286	1.4286	0.0287	0.0521	0.0397	0.0000	0.0000	0.0000
	0.4	1.0005	0.0000010	0.6667	1.6667	0.0288	0.0521	0.0397	0.0000	0.0000	0.0000
	0.5	1.0005	0.0000010	1.0000	2.0000	0.0288	0.0522	0.0397	0.0000	0.0000	0.0000
	0.6	1.0005	0.0000010	1.5000	2.5000	0.0289	0.0522	0.0398	0.0000	0.0000	0.0000
	0.7	1.0005	0.0000010	2.3333	3.3333	0.0289	0.0524	0.0399	0.0000	0.0000	0.0000
	0.8	1.0005	0.0000010	4.0000	5.0000	0.0290	0.0524	0.0399	0.0000	0.0000	0.0000
	0.9	1.0005	0.0000010	9.0000	10.0000	0.0290	0.0524	0.0399	0.0000	0.0000	0.0000
6.0	0.1	1.0005	0.0000010	0.0185	0.1852	0.0503	0.0554	0.0528	0.0000	0.0000	0.0000
	0.2	1.0005	0.0000010	0.0417	0.2083	0.0503	0.0554	0.0528	0.0000	0.0000	0.0000
	0.3	1.0005	0.0000010	0.0714	0.2381	0.0504	0.0554	0.0529	0.0000	0.0000	0.0000
	0.4	1.0005	0.0000010	0.1111	0.2778	0.0504	0.0554	0.0529	0.0000	0.0000	0.0000
	0.5	1.0005	0.0000010	0.1667	0.3333	0.0504	0.0555	0.0529	0.0000	0.0000	0.0000
	0.6	1.0005	0.0000010	0.2500	0.4167	0.0505	0.0555	0.0530	0.0000	0.0000	0.0000
	0.7	1.0005	0.0000010	0.3889	0.5556	0.0505	0.0555	0.0530	0.0000	0.0000	0.0000
	0.8	1.0005	0.0000010	0.6667	0.8333	0.0505	0.0556	0.0530	0.0000	0.0000	0.0000
	0.9	1.0005	0.0000010	1.5000	1.6667	0.0507	0.0558	0.0532	0.0000	0.0000	0.0000
15.0	0.1	1.0007	0.0000010	0.0074	0.0741	0.1072	0.1105	0.1088	0.0000	0.0000	0.0000
	0.2	1.0007	0.0000010	0.0167	0.0833	0.1072	0.1105	0.1088	0.0000	0.0000	0.0000
	0.3	1.0007	0.0000010	0.0286	0.0952	0.1071	0.1105	0.1088	0.0000	0.0000	0.0000
	0.4	1.0007	0.0000010	0.0444	0.1111	0.1072	0.1106	0.1088	0.0000	0.0000	0.0000
	0.5	1.0007	0.0000010	0.0667	0.1333	0.1072	0.1106	0.1089	0.0000	0.0000	0.0000
	0.6	1.0007	0.0000010	0.1000	0.1667	0.1073	0.1107	0.1090	0.0000	0.0000	0.0000
	0.7	1.0007	0.0000010	0.1556	0.2222	0.1074	0.1108	0.1091	0.0000	0.0000	0.0000
	0.8	1.0007	0.0000010	0.2667	0.3333	0.1074	0.1108	0.1091	0.0000	0.0000	0.0000
	0.9	1.0007	0.0000010	0.6000	0.6667	0.1074	0.1108	0.1091	0.0000	0.0000	0.0000

Table 4.5 (continued)

Absolute percentage difference between exact with
approximation distribution.

approximation of distribution											
Traffic m intensity	α_1	α_2	Arrival time	Service time	Renyi's measure			Parametric's measure			
					Average	Maximum	Minimum	Average	Maximum	Minimum	
30.0	0.1	1.0008	0.0000010	0.0037	0.0370	0.1202	0.1221	0.1211	0.0000	0.0001	0.0000
	0.2	1.0008	0.0000010	0.0083	0.0417	0.1202	0.1221	0.1212	0.0000	0.0001	0.0000
	0.3	1.0008	0.0000010	0.0143	0.0476	0.1202	0.1221	0.1211	0.0000	0.0001	0.0000
	0.4	1.0008	0.0000010	0.0222	0.0556	0.1202	0.1221	0.1211	0.0000	0.0001	0.0000
	0.5	1.0008	0.0000010	0.0333	0.0667	0.1202	0.1221	0.1212	0.0000	0.0000	0.0000
	0.6	1.0008	0.0000010	0.0500	0.0833	0.1202	0.1221	0.1212	0.0000	0.0001	0.0000
	0.7	1.0008	0.0000010	0.0778	0.1111	0.1203	0.1222	0.1213	0.0000	0.0001	0.0000
	0.8	1.0008	0.0000010	0.1333	0.1667	0.1205	0.1224	0.1214	0.0000	0.0000	0.0000
	0.9	1.0008	0.0000010	0.3000	0.3333	0.1204	0.1223	0.1214	0.0000	0.0000	0.0000
45.0	0.1	1.0009	0.0000010	0.0025	0.0247	0.1023	0.1035	0.1029	0.0000	0.0001	0.0000
	0.2	1.0009	0.0000010	0.0056	0.0278	0.1023	0.1036	0.1030	0.0000	0.0001	0.0000
	0.3	1.0009	0.0000010	0.0095	0.0317	0.1023	0.1036	0.1029	0.0000	0.0001	0.0000
	0.4	1.0009	0.0000010	0.0148	0.0370	0.1023	0.1036	0.1029	0.0000	0.0001	0.0000
	0.5	1.0009	0.0000010	0.0222	0.0444	0.1023	0.1036	0.1029	0.0000	0.0001	0.0000
	0.6	1.0009	0.0000010	0.0333	0.0556	0.1023	0.1036	0.1030	0.0000	0.0001	0.0000
	0.7	1.0009	0.0000010	0.0519	0.0741	0.1025	0.1037	0.1031	0.0000	0.0001	0.0000
	0.8	1.0009	0.0000010	0.0889	0.1111	0.1026	0.1038	0.1032	0.0000	0.0001	0.0000
	0.9	1.0009	0.0000010	0.2000	0.2222	0.1028	0.1041	0.1035	0.0000	0.0000	0.0000
50.0	0.1	1.0010	0.0000010	0.0022	0.0222	0.1021	0.1033	0.1027	0.0000	0.0003	0.0000
	0.2	1.0010	0.0000010	0.0050	0.0250	0.1022	0.1034	0.1028	0.0000	0.0002	0.0000
	0.3	1.0010	0.0000010	0.0086	0.0286	0.1022	0.1034	0.1028	0.0000	0.0002	0.0000
	0.4	1.0010	0.0000010	0.0133	0.0333	0.1022	0.1034	0.1028	0.0000	0.0002	0.0000
	0.5	1.0010	0.0000010	0.0200	0.0400	0.1022	0.1034	0.1028	0.0000	0.0002	0.0000
	0.6	1.0010	0.0000010	0.0300	0.0500	0.1022	0.1034	0.1028	0.0000	0.0001	0.0000
	0.7	1.0010	0.0000010	0.0467	0.0667	0.1023	0.1035	0.1029	0.0000	0.0002	0.0000
	0.8	1.0010	0.0000010	0.0800	0.1000	0.1025	0.1037	0.1031	0.0000	0.0002	0.0000
	0.9	1.0010	0.0000010	0.1800	0.2000	0.1028	0.1040	0.1034	0.0000	0.0003	0.0000
55.0	0.1	1.0011	0.0000010	0.0020	0.0202	0.1943	0.1959	0.1951	0.0000	0.0002	0.0000
	0.2	1.0011	0.0000010	0.0045	0.0227	0.1943	0.1960	0.1952	0.0000	0.0001	0.0000
	0.3	1.0011	0.0000010	0.0078	0.0260	0.1944	0.1960	0.1952	0.0000	0.0001	0.0000
	0.4	1.0011	0.0000010	0.0121	0.0303	0.1944	0.1960	0.1952	0.0000	0.0002	0.0000
	0.5	1.0011	0.0000010	0.0182	0.0364	0.1944	0.1960	0.1952	0.0000	0.0002	0.0000
	0.6	1.0011	0.0000010	0.0273	0.0455	0.1944	0.1960	0.1952	0.0000	0.0001	0.0000
	0.7	1.0011	0.0000010	0.0424	0.0606	0.1944	0.1960	0.1952	0.0000	0.0004	0.0000
	0.8	1.0011	0.0000010	0.0727	0.0909	0.1947	0.1964	0.1955	0.0000	0.0002	0.0000
	0.9	1.0011	0.0000010	0.1636	0.1818	0.1951	0.1967	0.1959	0.0000	0.0004	0.0000

Table 4.5

Exact —————

Renyi's measure - - - - -

Parametric measure - - - - -

Shannon's measure - - - - -

COMPARISON OF BUSY PERIOD, 0.4(0.5)

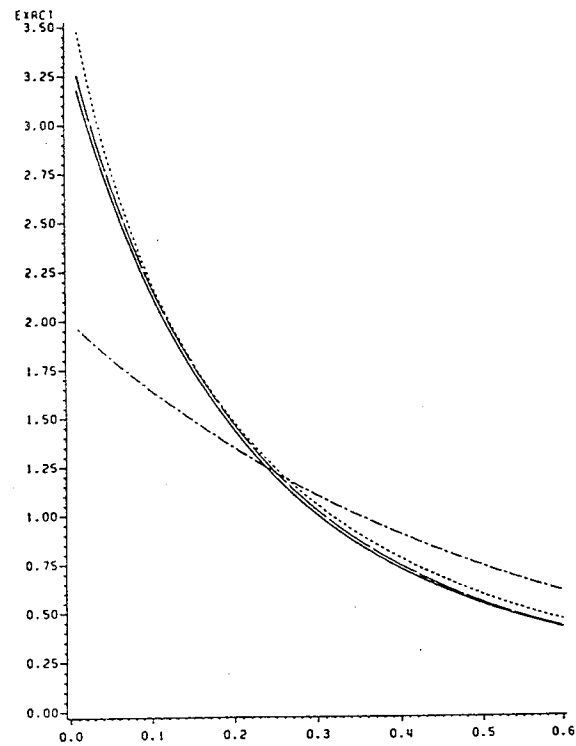


Figure 4.2

COMPARISON OF BUSY PERIOD, 0.1(0.5)

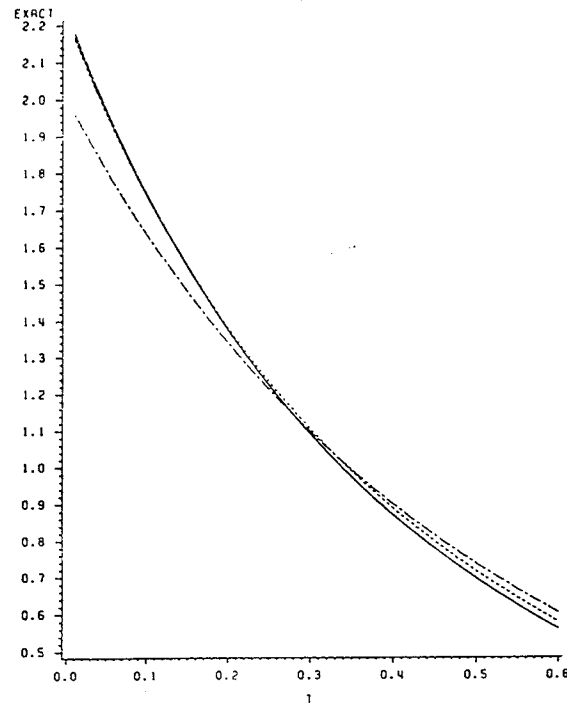


Figure 4.1

COMPARISON OF BUSY PERIOD, 0.9(0.5)

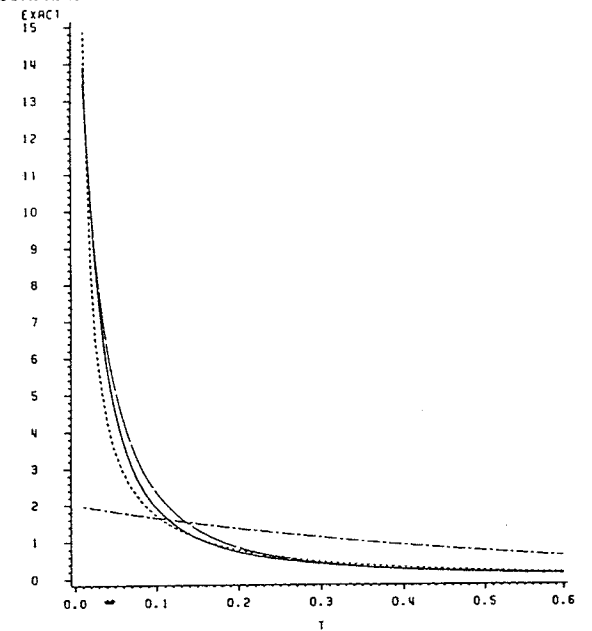


Figure 4.3

Exact

Renyi's measure

Parametric measure

Shannon's measure

COMPARISON OF BUSY PERIOD, 0.4(15)

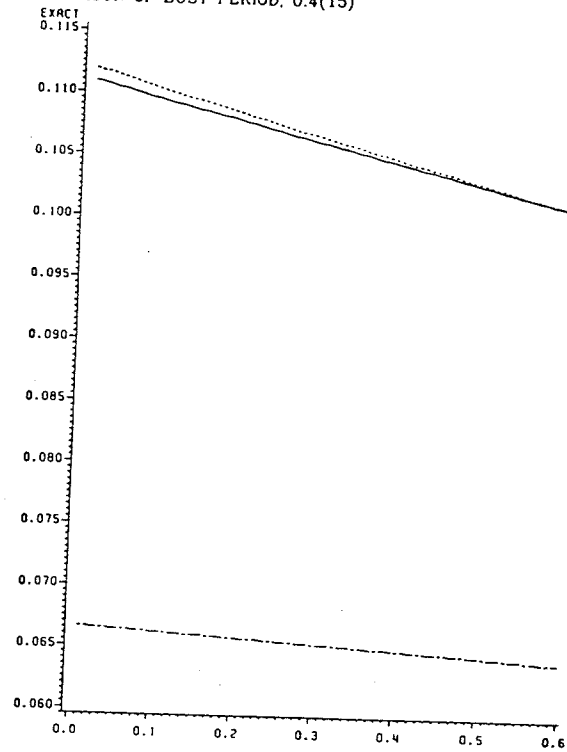


Figure 4.4

COMPARISON OF BUSY PERIOD, 0.9(15)

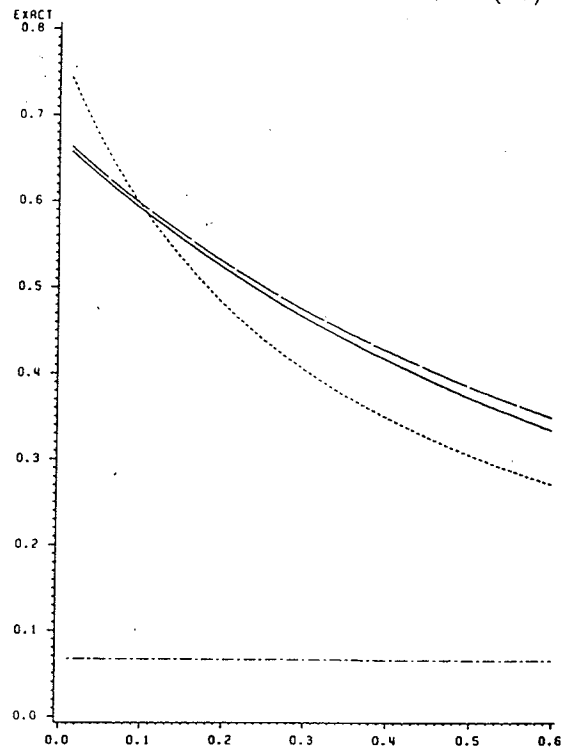


Figure 4.5

COMPARISON OF BUSY PERIOD, 0.9(55)

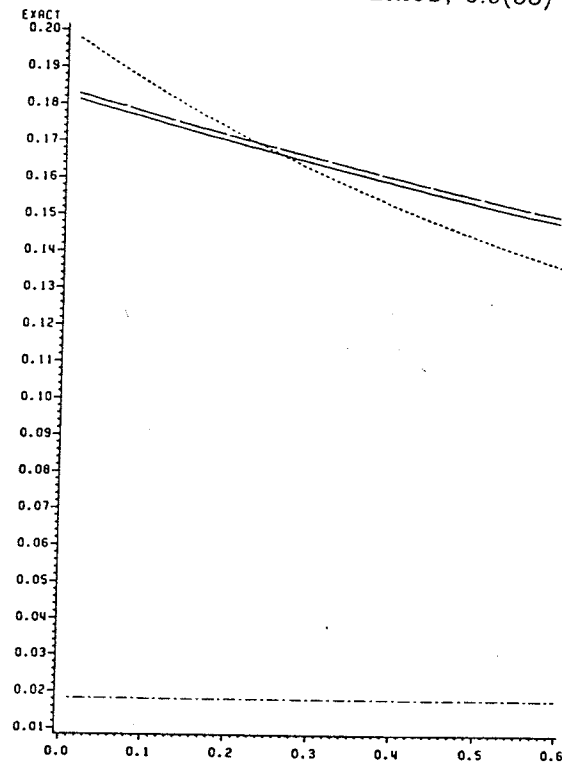


Figure 4.6

Exact —————

Renyi's Measure - - - - -

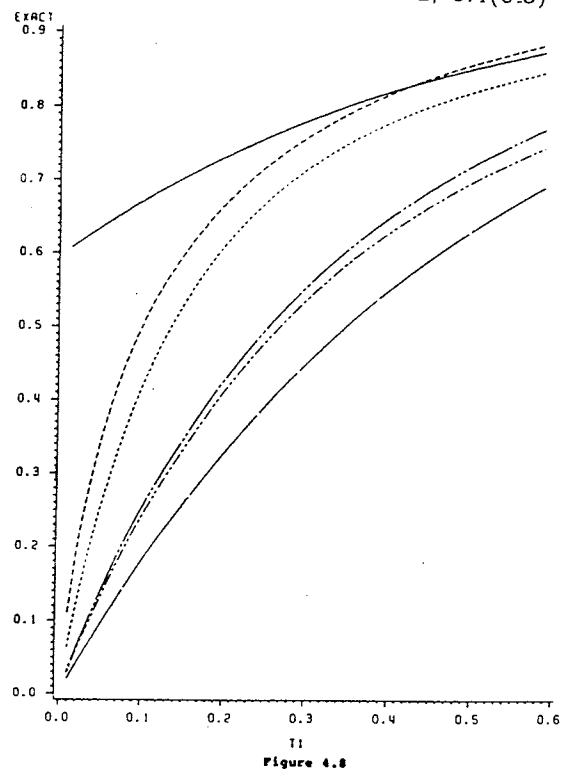
Parametric Measure - - - - -

Renyi's measure with Busy Period parameters - - - - -

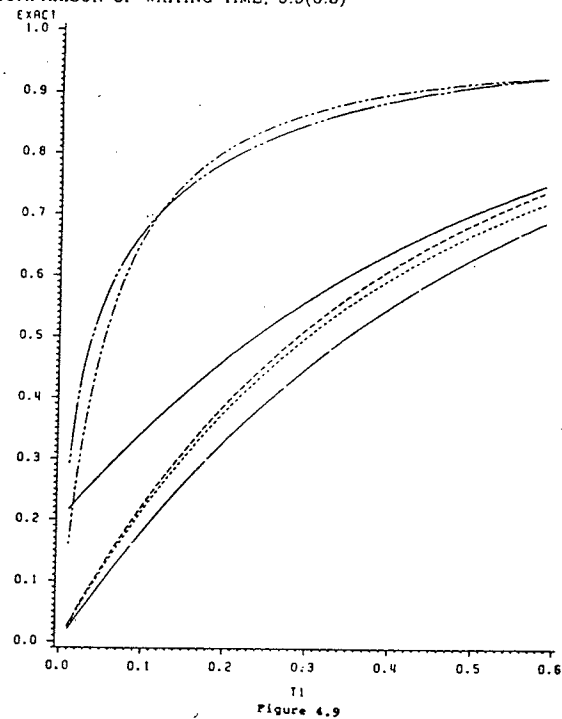
Parametric measure with Busy Period parameters - - - - -

Shannon's measure - - - - -

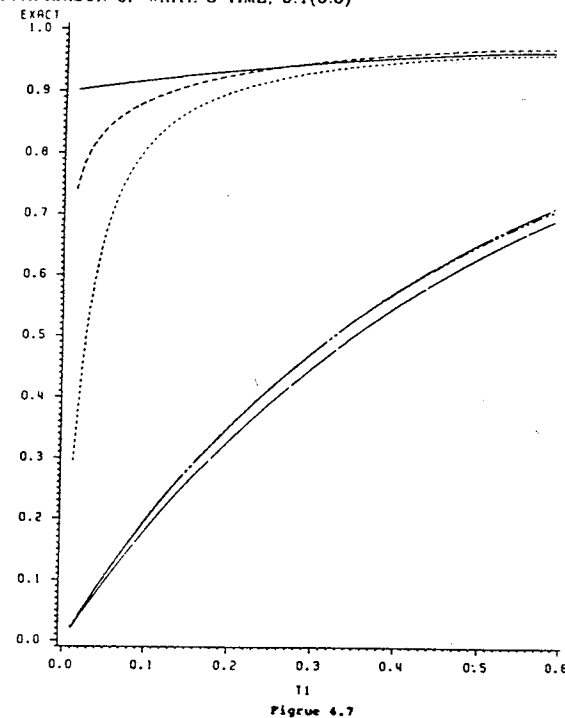
COMPARISON OF WAITING TIME, 0.4(0.5)



COMPARISON OF WAITING TIME, 0.9(0.5)



COMPARISON OF WAITING TIME, 0.1(0.5)



Exact	—————	Renyi's measure with Busy Period parameters	—————
Renyi's Measure	-----	Parametric measure with Busy Period parameters	-----
Parametric Measure	-----	Shannon's measure	-----

COMPARISON OF WAITING TIME, 0.4(15)

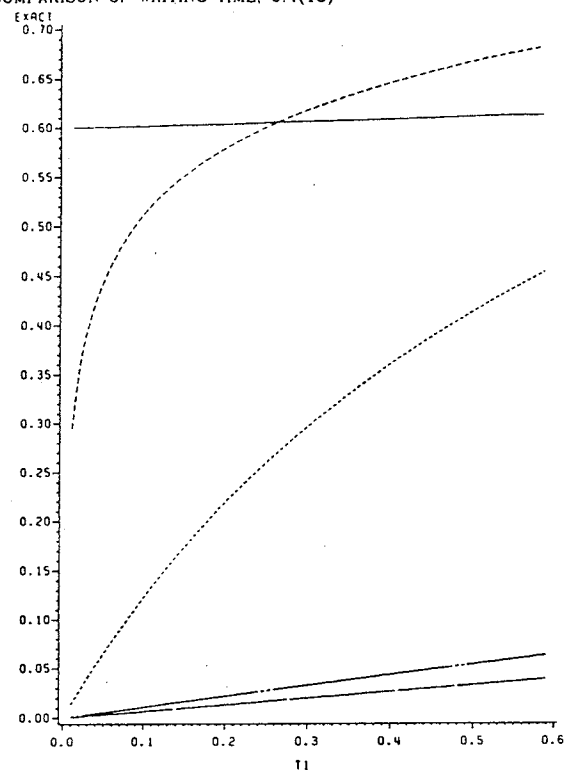


Figure 4.10

COMPARISON OF WAITING TIME, 0.9(15)

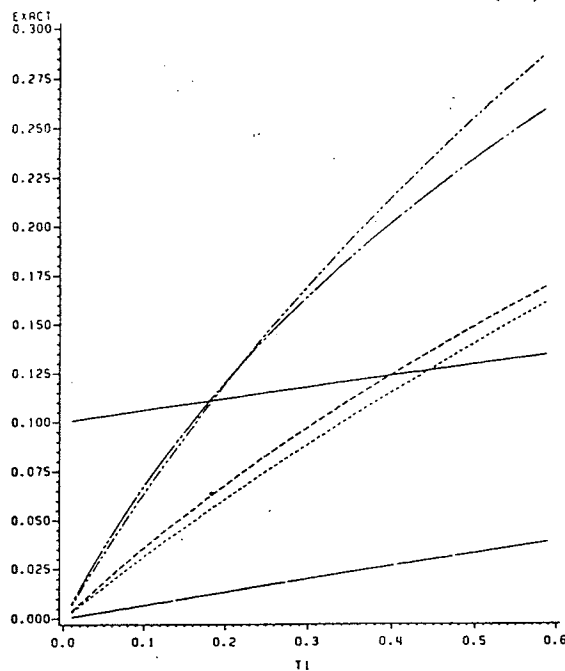


Figure 4.11

COMPARISON OF WAITING TIME, 0.9(55)

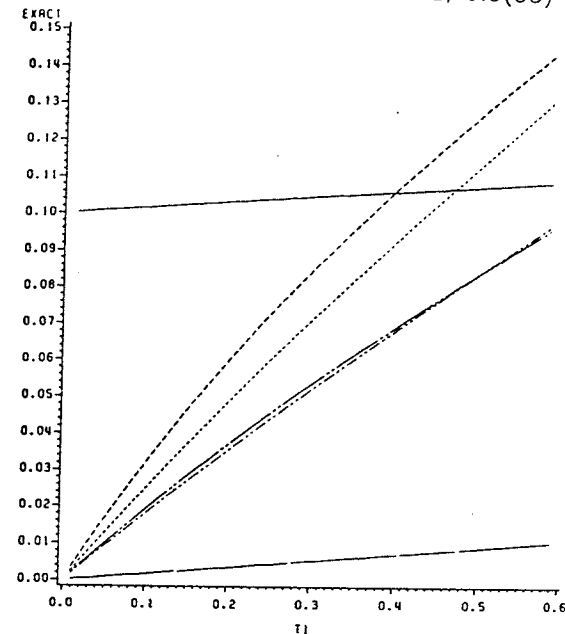


Figure 4.12

Exact
 Renyi's measure
 Parametric measure

COMPARISON OF THE EXACT AND APPROXIMATION MODELS OF RESIDENCE TIME DISTRIBUTION

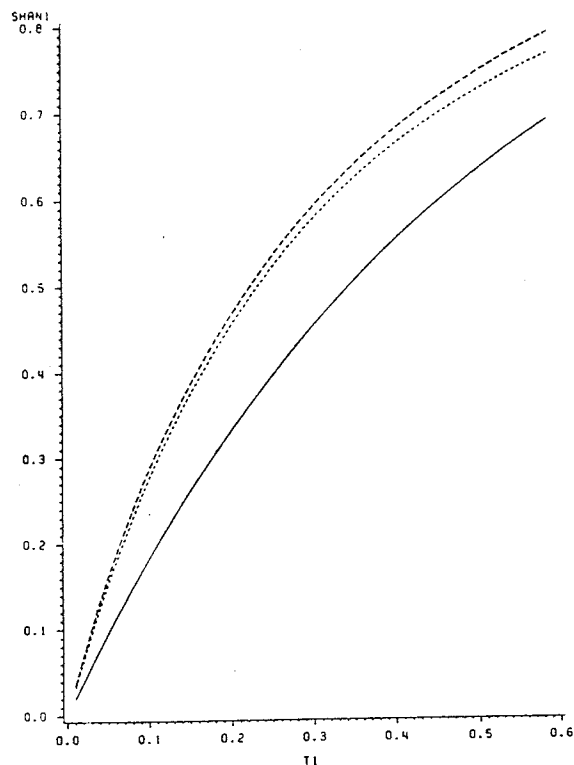


Figure 4.14

COMPARISON OF THE EXACT AND APPROXIMATION MODELS OF RESIDENCE TIME DISTRIBUTION

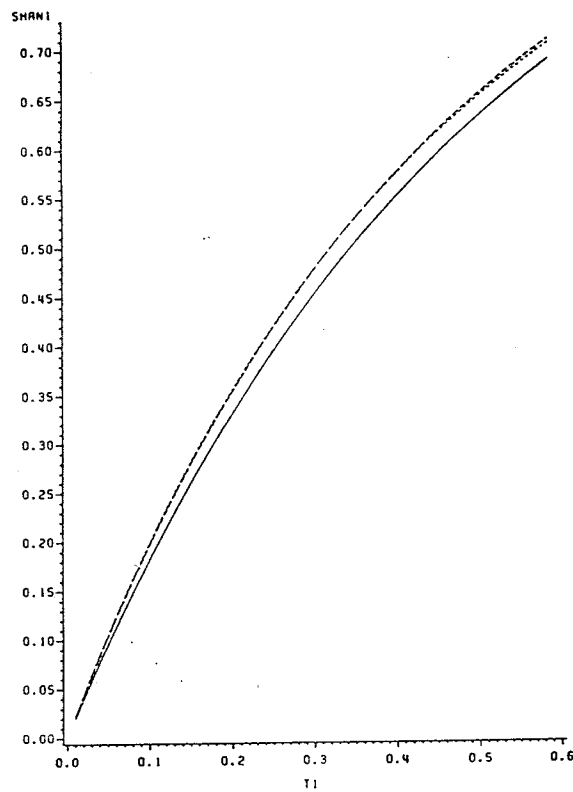


Figure 4.13

COMPARISON OF THE EXACT AND APPROXIMATION MODELS OF RESIDENCE TIME DISTRIBUTION

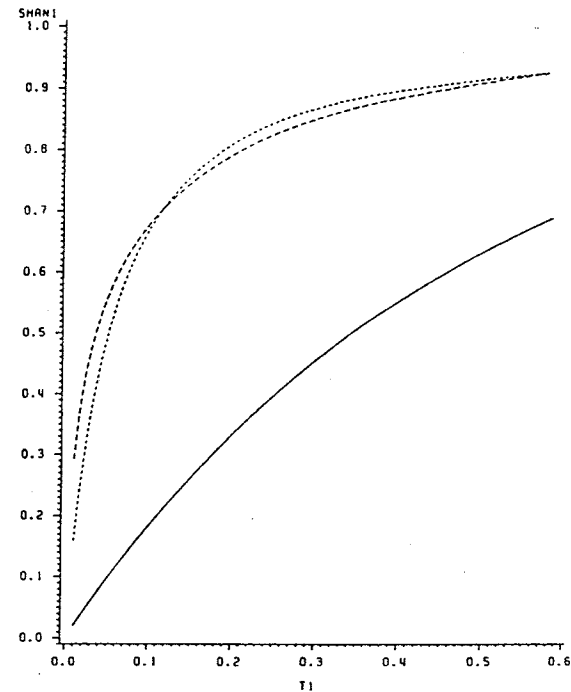


Figure 4.15

Exact _____
 Renyi's measure - - - - -
 Parametric measure - - - - -

COMPARISON OF THE EXACT AND APPROXIMATION MODELS OF RESIDENCE TIME DISTRIBUTION

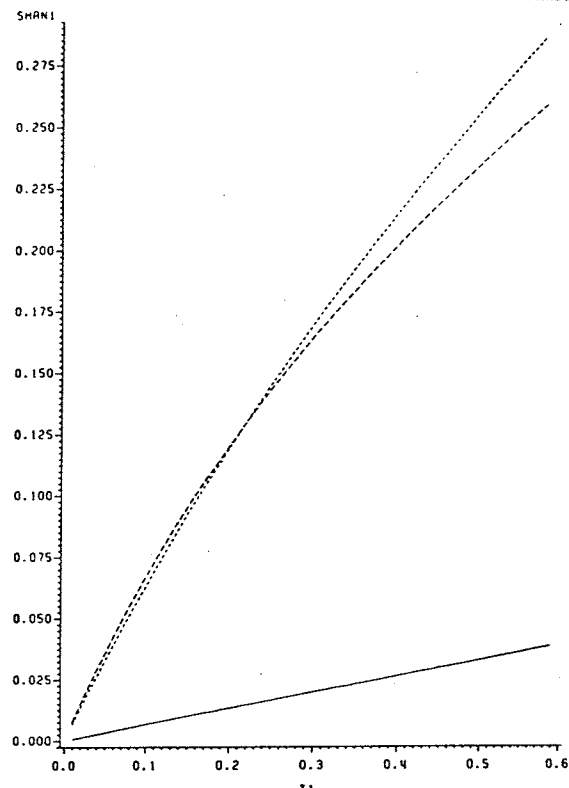


Figure 4.17

COMPARISON OF THE EXACT AND APPROXIMATION MODELS OF RESIDENCE TIME DISTRIBUTION

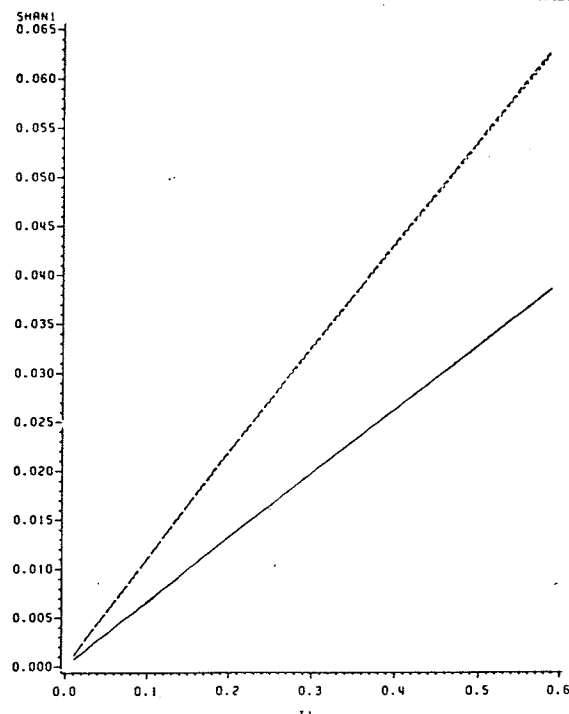


Figure 4.16

COMPARISON OF THE EXACT AND APPROXIMATION MODELS OF RESIDENCE TIME DISTRIBUTION

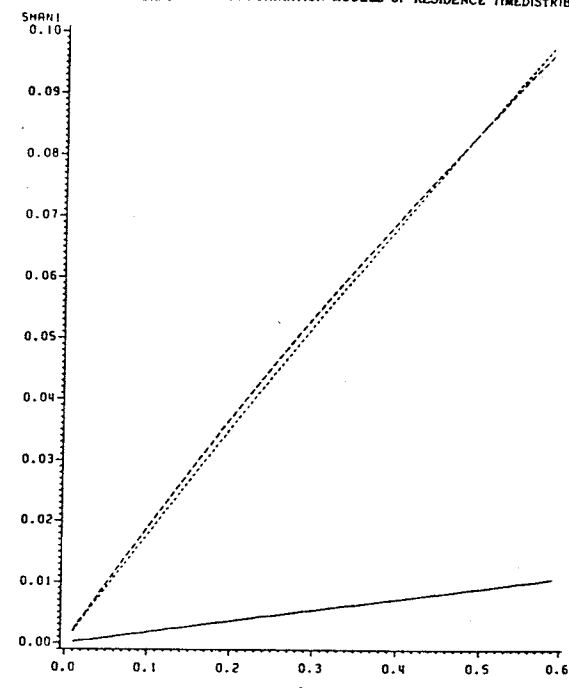


Figure 4.18

RELATIONSHIP BETWEEN M AND PARAMETER α_2 FOR VARIOUS VALUES OF ρ .

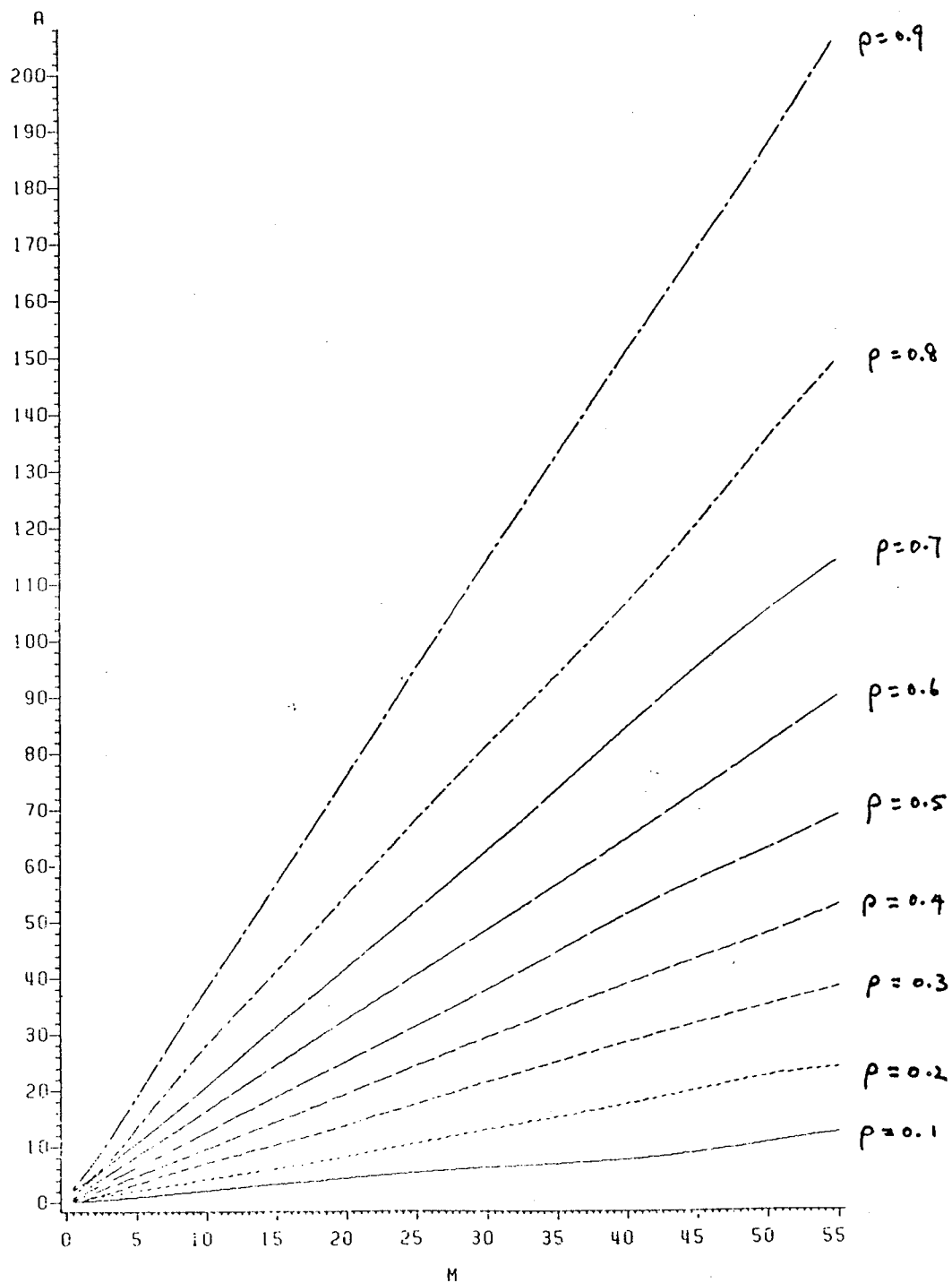


Figure 4.19

RELATIONSHIP BETWEEN M AND PARAMETER α , FOR VARIOUS VALUES

OF ρ .

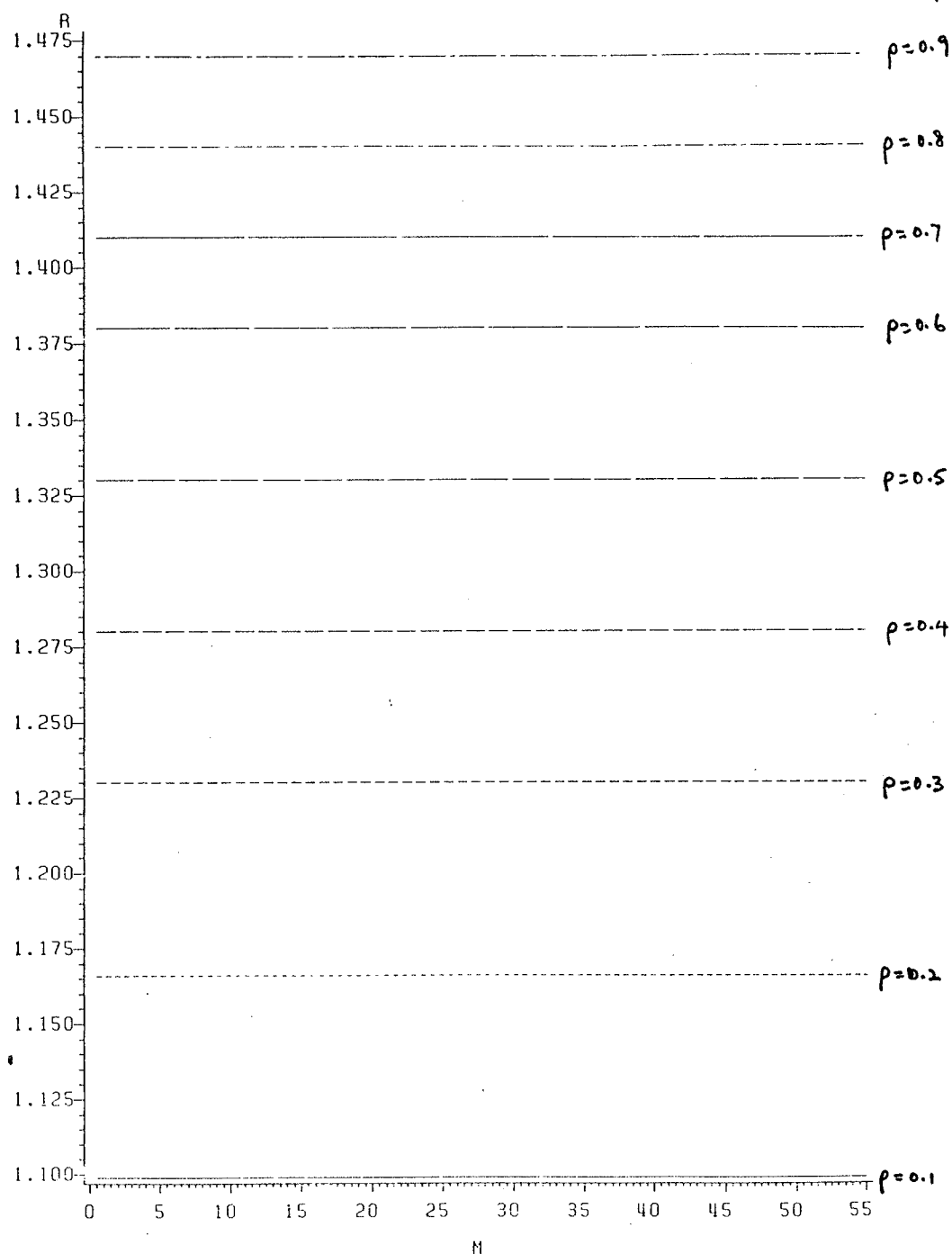


Figure 4.20

RELATIONSHIP BETWEEN M AND PARAMETER α_2 FOR VARIOUS VALUES OF ρ .

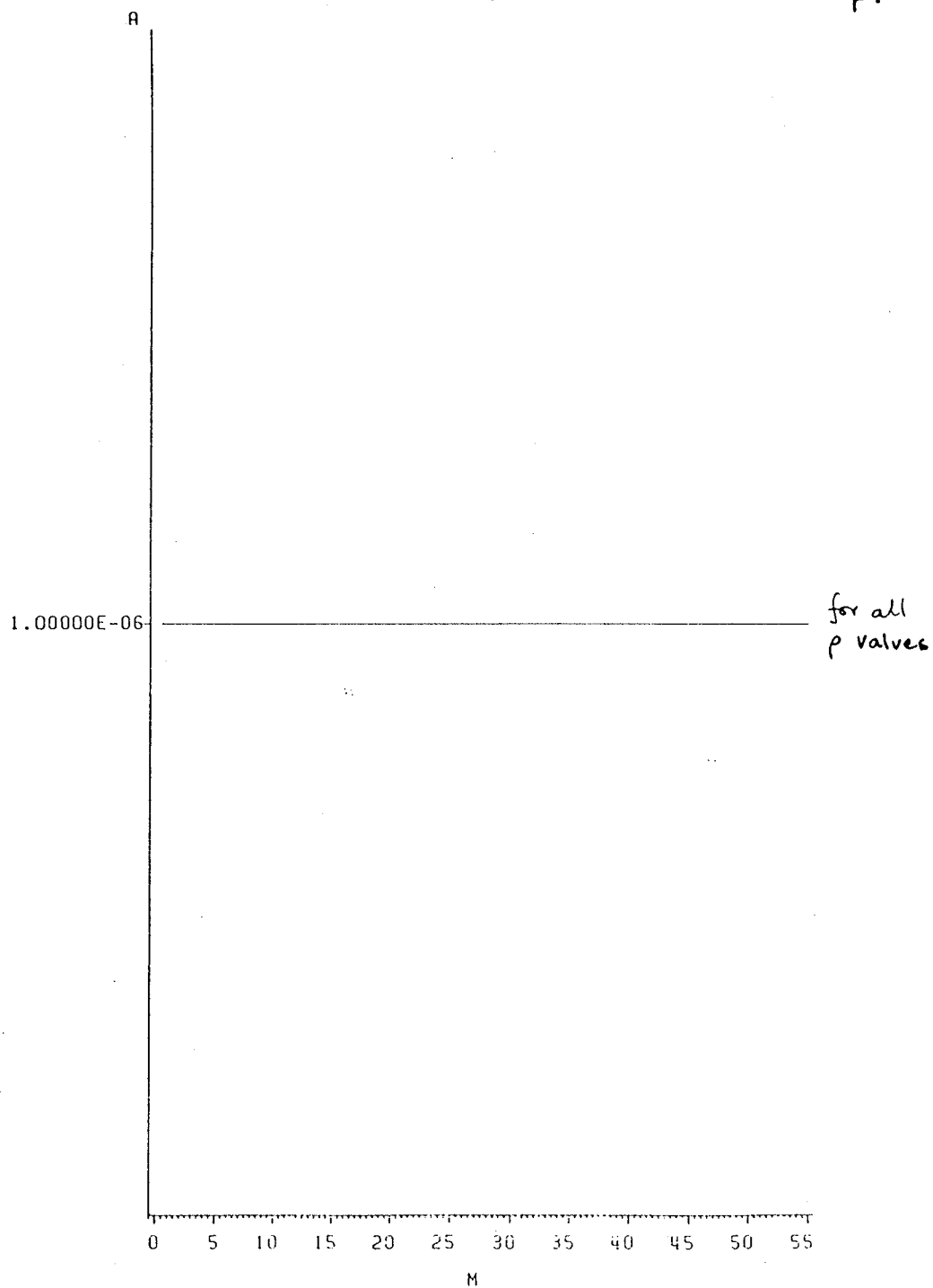


Figure 4.21

RELATIONSHIP BETWEEN M AND PARAMETER α , FOR VARIOUS VALUES OF ρ .

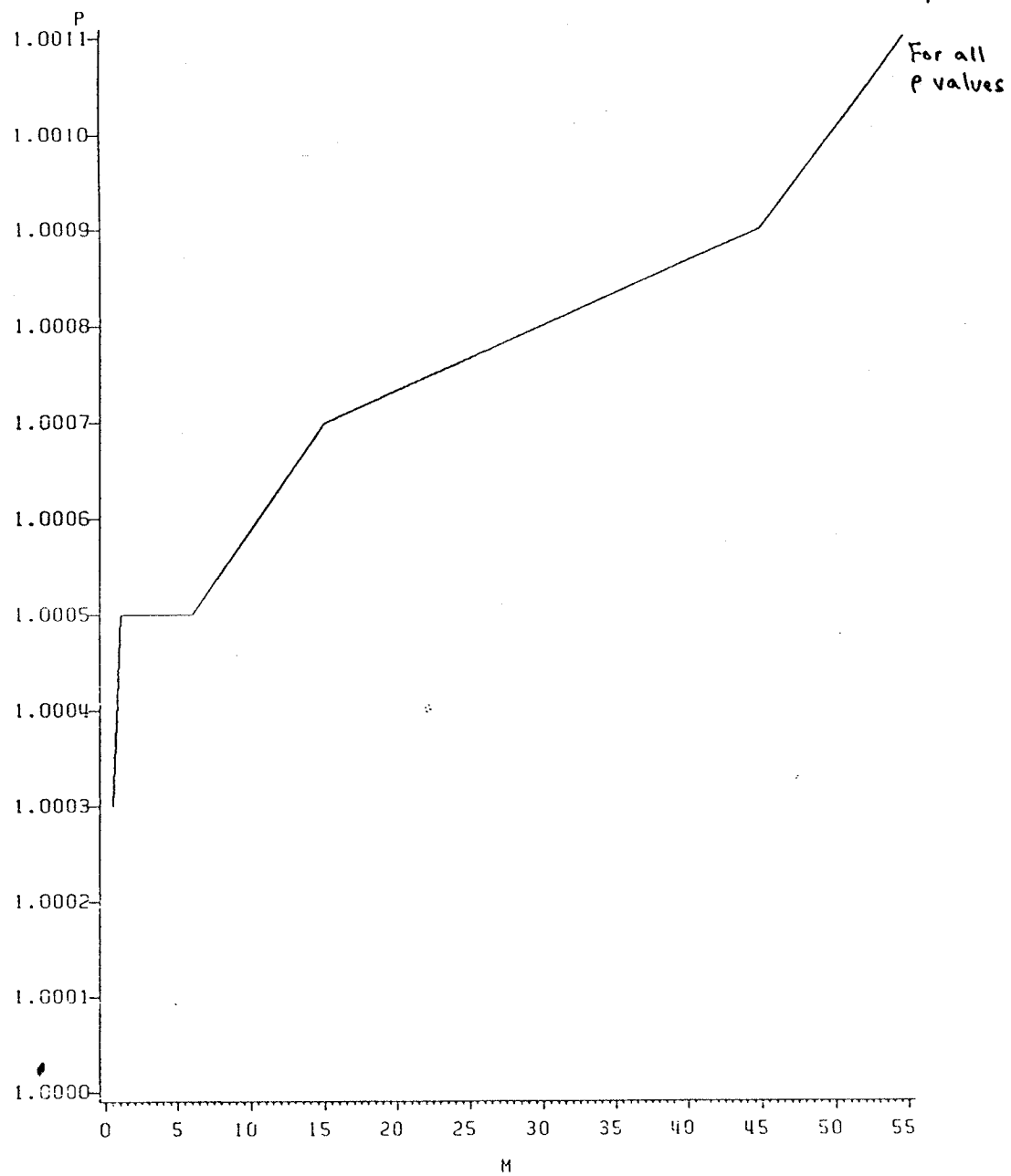


Figure 4.22

RELATIONSHIP BETWEEN M AND PARAMETER α_1 FOR VARIOUS VALUES OF ρ .

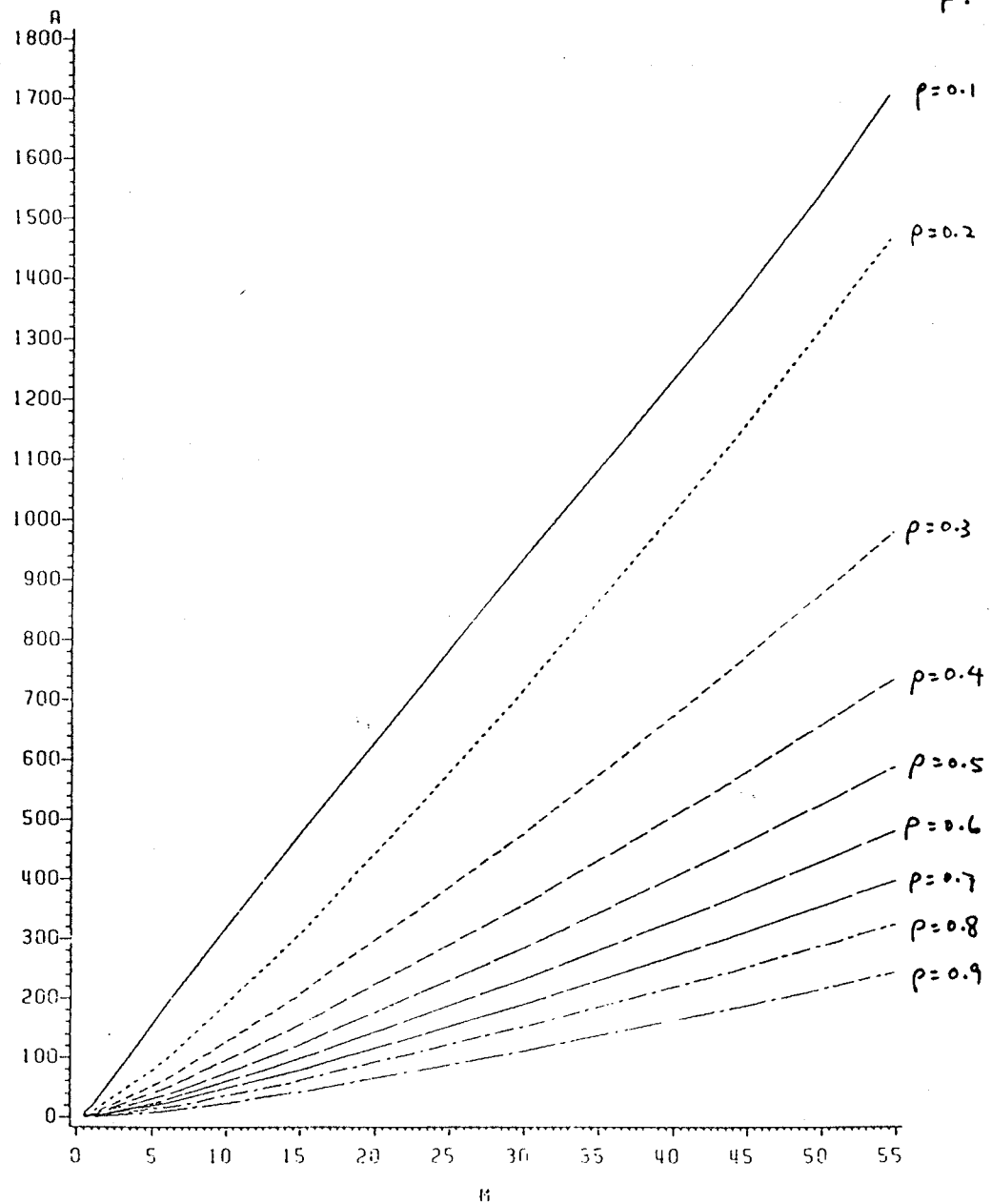


Figure 4.23

RELATIONSHIP BETWEEN M AND PARAMETER α , FOR VARIOUS VALUES OF ρ .

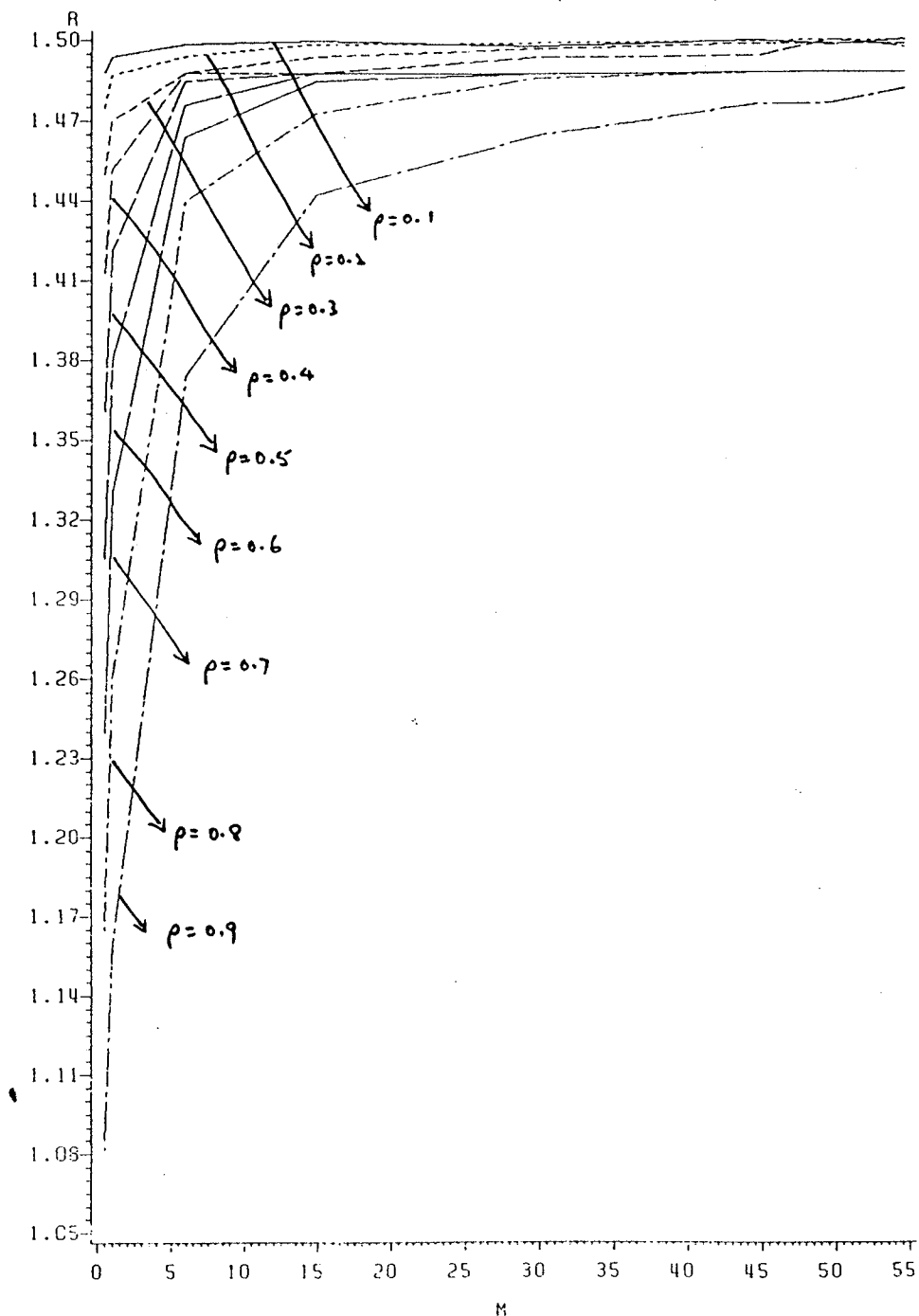


Figure 4.24

Chapter V

CONCLUDING REMARKS AND RECOMMENDATIONS FOR FURTHER WORK.

Concluding Remarks.

In the present thesis, we have provided an MEP approach to obtain approximate operating characteristics for two unit paralleled redundant systems when the repair time distribution is either Gamma and Weibull. These results are obtained when the only information is available in the form of first moment (or first two moments) of repair time distribution. It has been shown that MEP provides a very good approximation to the operating characteristics of this system.

In case of single server queueing systems, we have shown that the use of Renyi's entropy in obtaining performance distribution consistently provide better results. Our analysis assumes that the only available information is in the form of the first moment of service time distribution. Our results are valid for $0.0 \leq \rho < 1.0$, in contrast to the results obtained by Shore which are valid only under light load condition $\rho < 0.1$ and when the information available is in the form of first four moments of service time distribution.

Recommendations for Further Work.

1. In the present thesis, the repair time distribution for a two unit paralleled redundant system is considered to be either Weibull or Gamma. It is proposed that the information theoretic approximation of operating characteristics be obtained when the repair time distributions are such as Log-normal, Laplace and Beta.
2. We have used Shannon's measure for obtaining the approximation of operating characteristics of the above system. It is proposed that Renyi's measure and the Parametric measure should also be applied.
3. This thesis deals with two unit paralleled redundant systems. It will be interesting to obtain the approximate operating characteristics of multi-unit redundant systems.
4. In single service queueing systems, to obtain the information-theoretic approximation for performance distributions, we assumed the information is available in the form of first moment only. It is proposed that the results be extended when the information is available in the form of first two or more moments.
5. In single server queueing systems, we have used three entropy models, namely Shannon's, Renyi's and Para-

metric. It is proposed that the measure given by Kapur (1967) be also used to obtain the approximate performance distributions. Kapur's measure may give more general results since it is characterized by two parameters as compared to Renyi's and Parametric which have only one parameter each.

6. In case prior information is available, then it is proposed that the Minimum Cross Entropy (Shore, 1981) concept be applied to the problems dealt in this thesis and proposed in 1 - 5 above.
7. It is proposed that MEP be applied to obtain the approximate performance distribution of multi-server queueing systems.

REFERENCES

- Adam, N. R., "Achieving a confidence interval for parameters estimated by simulation", Management Science, Vol 29, (August, 1984), pp 856-66.
- Adiri, I. & Yechiali, O., "Optimal Priority-Purchasing and Pricing Decisions in Non-Monopoly and Monopoly Queues", Operations Research, Vol 22, (S'1974), pp 1051-66.
- Arora, J. R., "Reliability of a 2-unit Standby Redundant System with Constrained Repair Time", IEEE Trans. Rel., R-25, (1976), pp 203-205.
- Arisono, I & Ohta, H., "Marketing Structure Analysis and Price Decision of New Brand --- An Analysis of Brand Selection Behaviour Based on Entropy Model", (submitted for publication to Naval Research Logistics Quarterly).
- Atkinson, Ken, An Introduction to Numerical Analysis, John Wiley & Son, 1978.
- Barlow, R. E. & Proschan, F., Mathematical Theory of Reliability, New York : John Wiley, 1965.
- Barrow, D. F. & Cohen, A. C., "On Some Functions Involving Mill's ratio", The Annal of Math. Stat., Vol. 25, (1954), pp 405-408.
- Bass, F. M., "The Theory of Stochastic Preference and Brand Switching", Journal of Marketing Reserch., Vol XI, (Feb 1974), pp 1-20.
- Benes, V. E., Mathematical Theory of Connecting Networks and Telephone Traffic New York : Academic (1965).
- Brown, D.J., "Modelling the Use of the Manager's Time", Omega, No 1, (1979), pp 55-60.
- Buzacott, J. A., "Availability of Priority Standby Redundant Systems", IEEE Trans. Rel., R-20, pp 60-63.
- Chan, M., "System Simulation and Maximum Entropy", Operations Research, Vol 19, (1971), pp 1751-1753.
- Clough, D. J., "Application of the Principle of Maximizing Entropy in Formulation of Hypotheses", CORS Journal, 2 (1964), pp 53-70.

- Cozzolino, J. M. & Zahner, M. J., "The Maximum-Entropy Distribution of the Future Market Price of a Stock", Operations Research, Vol. 21, (1973), pp 1200-1211.
- Davies, B., "Integral Transforms and Their Applications", Springer-Verlag, 1978.
- Dowson, D. C. & Wragg, A., "Maximum-Entropy Distribution having Prescribed First and Second Moments", IEEE Trans. Rel., Vol 19, (1973), pp 689-693.
- Dyer, D. & Chiou, P., "An Information-Theoretic Approach to Incorporating Prior Information in Binomial Sampling", COMM. Stat. - Theor. Meth., 10(17), (1984), pp 2051-2081.
- Epstein, B. & Hosford, J., "Reliability of Some Two-unit Redundant Systems", Sixth Symp. Rel. -- Qualt. Contr., (1960), pp 469-476.
- Ferdinand, A. E., "A Statistical Mechanics Approach to Systems Analysis", IBM J. Res. Develop., Vol 14, (1970), pp 539-547.
- , "A Theory of General Complexity", Int. J. General Syst., Vol 1, (1974), pp 19-33.
- Gnedenko, B. V. ; Belyayer, Y. K. & Solovyev, A. D., The Mathematical Methods of Reliability Theory, (translated by R. E. Barlow), New York : Academic Press, 1969.
- Good, I. J., "Maximum Entropy for Hypothesis Formulation", Ann. Math. Stat., Vol 34, (1963), pp 911-930.
- Grassmann, W. K., "Economic Service Rate", Journal of Operations Research, Vol 30, (Fall'1979), pp 149-5.
- Graves, S. C., "The Application of Queueing Theory to Continuous Perishable Inventory Systems", Management Science, Vol 28, (August, 1982), pp 400-6.
- Green, L & Kolesan, P., "The Feasibility of One-Officer Patrol in New York City", Management Science, (August 1984)
- Gupta, Y., Dharmadhikari, A. & Chow, W. S., "Application of Maximum Entropy Principle in Paralleled Repairable Systems", Submitted for publication.
- , -----, & -----, "On the Use of Maximum Entropy Principle in Two-unit into Repairable Systems", Submitted for publication.

- , -----, ----- & C. R. Bector, "Information-Theoretic Approximations for Single Server Queueing Systems", Submitted for publication
- Harrison, J. M., "Priority Queueing With Discounted Linear Cost", Operations Research, Vol 23, (March 1975), pp 260-9.
- , "Dynamic Scheduling of a Multiclass Queue : Discount Optimality", Operations Research, Vol 23, (March 1975), pp 270-82.
- Henley, E. J. & Kumamoto, H., Reliability Engineering and Risk Assessment, New Jersey : Prentice-Hall, Inc., 1981.
- Herniter, J. D., "An Entropy Model of Brand Purchase Behaviour", Journal of Marketing Reserach, Vol X, (Nov., 1973), pp 361-375.
- Huang, P. Y. & Clayton, E. R., "A Q-Gert Network Simulation Model for the Selection of Job Shop Scheduling Rules", Rev. Bus. Econ. Res., Vol 19, (Fall'1983), pp 1-12.
- Iglehart, D. & Shedler, G., "Simulation of Response Time in Finite Capacity Open Network of the Queueing", Operations Research Vol 26, (Spring'1978), pp 896-914.
- Jaynes, E. T., "Information Theory and Statistical Mechanics", Phys. rev., Vol 106, (1957), pp 620-630.
- , "Probability Theory in Science and Engineering", Field Res. Lab., Cocony Mobil Oil Co., Inc., Collquium Lecturs in Pure and Applied Science, No 4, (1958).
- , "New Engineering Applications of the Information Theory", In Proceedings of the First Symposium on Engineering Applications of Random Function Theory and Probability, J. L. Boganoff, Ed., New York : Wiley (1963), pp 163-203.
- , "Prior Probabilities", IEEE Trans. Syst. Sci. Cybern., Vol SSc-4, (1968), pp 227-241.
- , "Probability Theory in Science and Engineering", unpublished lectured notes (available from Physic Department, Washington University, St. Louis, Mo), (1972).
- Kapur, J. N., "Generalized Entropies of order α and type β ", Seminar 4, pp 78-79, (1967).
- , "Twenty-five Years of Maximum-Entropy Principle", Journal of Math. and Phys. Sc., Vol 17, no 2 (April, 1983), pp 103-156.

-----, "Maximum-Entropy Models in Science and Engineering" (1985).

Kapur, J. N., C. R. Bector & U. Kumar, "A Generalization of the Entropy Model for Brand Purchase Behaviour", Naval Reserch Logistics Quarterly, Vol 31, (1984), pp 183-194.

Kleinrock, L., Queueing Systems, Vol I : Theory, New York : John Wiley (1975), p215 and pp 214-215.

Krell, B. E. & Armino, M., "Queueing Theory Applied to Data Processing Network", Interfaces, Vol 12, (August 1982), pp 21-33.

Lie, C. H., Hwang, C. L. & Tillman, F. A., "Availability of Maintained Systems : A State-of-the-Art Survey", AIIE Trans., (Sept, 1977), pp 247-257.

Mckeown, P. G., "Application of Queueing Analysis to the New York State Child Abuse and Maltreatment Register Telephone Reporting System", Interfaces, Vol 9, (May 1979), pp 20-5.

Nakagawa, T. & Osaki, S. "Stochastic Behaviour of Two-unit Paralleled Redundant System with Repair Maintenance", Microelectronics and Reliability, Vol 14, (1975), pp 457-461.

Noonan, J. P.; Tzanneo & Costello, T., "On the Inverse Problem of Entropy Maximization", IEEE Trans. Inform. Theory, Vol. It-22, Jan (1976), pp 120-123.

Osaki, S., "Renewal Theoretic Aspects of Two-unit Redendant System", IEEE Trans. Rel., R-19, (1970), pp 105-110.

Osaki, S. & Nakagawa, T., "Bibliograph for Reliability and Availability of Stochastic Systems", IEEE Trans. Rel., Vol. R-25, 4(Oct, 1976), pp 284-286.

Ramakrishna, B. S., "The Efficiency of Languages", Science Today, (August, 1973), pp 13-20.

Ramakrishna Rau, B., "The Exact Analysis of Models of Program Reference Strings", Stanford Elec. Lab., Stanford Uni., Stanford, Ca. Tech. Rep 124 (SU-SEL-77-003), (1976, Dec.).

Rasmussen, E. H., "Queueing Simulation", Byte, Vol 9, (March 1984), pp 9-157.

Reddy, P. & Grosh, D. R., "Standby Redundancy in Reliability ---- A Review", IEEE Trans. Rel. (Send for publication, 1984).

- Renyi, A., "On the Measure of Information and Entropy", Proceedings 4th Berkeley Symposium on Mathematics, Statistics and Probability, (1961), pp 547-61.
- Shannon, C. E., "A Mathematical Theory of Communication", Bell System Technical Journal, Vol 27, (1948), pp 279-423, 623-656
- Shore, J. E. "Derivation of Equilibrium and Time-Dependent Solution to M/M/ //N and M/M/ Queueing Systems using Entropy Maximization", AFIPS - Conference Preceedings, Vol 47, (1978), pp 483-487.
- , "Accuracy of an Information-Theoretic, Light-load Approximation for the M/M/1 Busy Period Probability Density", Noval Reserch Laboratory, NRL Memorandum Report 4037, (July, 11, 1979a).
- , "Information Theoretic Approximations for M/G/1 and G/G/1 Queueing Systems", Noval Research Laboratory, NRL Memorandum Report 4047, (July, 18, 1979b).
- , "Properties of Cross-Entropy Minimization", IEEE Trans. on Information Theory, Vol IT-27, No 4, (July, 1981).
- Shore, J. E., Johnson, R. W., "Axiomatic Derivation of the Principle of Maximum Entropy and the Principle of Minimum Cross-Entropy", IEEE Trans. on Inf. Theory, Vol IT-26, No 1, (Jan. 1980), pp 26-37, Comment, Vol IT-29, No 6, (Nov. 1983), pp 942-943.
- Sockut, G. H., "Performance Model for Computer Data-Base Reorganization Performed Concurrently With Usage", Operations Research, Vol 26, (Spring'1979), pp 789-804.
- Srinivansan, V. S., "A Standby Redundant Model with Non-Instantaneous Switchover", IEEE Trans. on Rel. R-17, (1976), pp 175-178.
- Srinivasan, S. K. & Subramanian, R., Probablistic Analysis of Redundant Systems, New York : Springer-Verlag, 1980.
- Steudel, H. J. & Wu, S. M., "Time Series Approach to Queueing Systems with Applications for Modeling Job-Shop In-Process Inventory", Management Science, Vol 23, (March 1977), pp 745-55, Discussion Vol 24, (July 1978), pp 1199-1205.
- Takatsuji, M., "An Information-Theoretic Approach to a System of Interacting Elements", Biol. Cybern., Vol 17, pp 207-210.

Theil, H., "On the use of Information Theory Concepts in the Analysis of Financial Statements", Management Science, Vol 15, no 9, (1969), pp 459-480.

Thomas, M. S., "A Generalized Maximum Entropy Principle", Operations Research, Vol 27, no 6(Nov-Dec 1979), pp 1188-1196.

Tribus, M., "The use of the Maximum Entropy Estimate in Reliability Engineering", Recent Development in Information and Decision Process, Edited by R. E. Mcclol and P. Gray. New York : McMilliam, 1962.

-----, Rational Descriptions, Decisions, and Designs, New York : Pergamon (1969).

-----, "Thirty Year of Information Theory", In Maximum Entropy Formulism, edited by R. D. Levine and M. Tribus, (1979), pp 1-9.

Tribus. M. & Fitts, G. "The Widget Problem revisited", IEEE Trans. Syst. Sci. Cybern., Vol. SSC-4, (1968), pp 241-248.

Verma, R. S., "Generalization of Renyi's Entropy of Order α ", Journal of Mathematical Sciences, Vol 1, (1966), pp 34-48.

Appendix A

DERIVATION OF $H_{1,1}^{-1}(x)$ FUNCTION.

In this appendix the probability distribution $H_{1,1}^{-1}(x)$ is obtained using Shannon's measure of uncertainty and the only information is available in the form of first moment.

Consider :

$$\text{Max } S : - \int_0^{\infty} f(x) \ln f(x) dx \quad (A1)$$

subject to

$$\int_0^{\infty} f(x) dx = 1 \quad (A2)$$

$$\int_0^{\infty} x f(x) dx = m_1 \quad (A3)$$

where m_1 is the known mean service time.

The Langrangian equation of above system is :

$$\begin{aligned} L(f(x), \gamma_0, \gamma_1) = & - \int_0^{\infty} f(x) \ln f(x) dx \\ & - (\gamma_0 - 1) \left[\int_0^{\infty} f(x) dx - 1 \right] - \gamma_1 \left[\int_0^{\infty} x f(x) dx - m_1 \right] \end{aligned}$$

where γ_0, γ_1 are Langrangian multipliers.

$$dL/df(x) = -1 - \ln f(x) - (\gamma_0 - 1) - \gamma_1 x = 0$$

$$\ln f(x) = -\gamma_0 - \gamma_1 x$$

$$f(x) = \exp(-\gamma_0 - \gamma_1 x)$$

(A4)

Using (A2) and (A4), we have

$$\int_0^{\infty} f(x) dx = 1 = \int_0^{\infty} \exp(-\gamma_0 - \gamma_1 x) dx$$

$$1 = \exp(-\gamma_0) \int_0^{\infty} \exp(-\gamma_1 x) dx$$

$$= \exp(-\gamma_0) \left[\frac{\exp(-\gamma_1 x)}{-\gamma_1} \right] \Big|_0^{\infty}$$

$$1 = \gamma_1 \exp(-\gamma_0)$$

$$\gamma_1 = \exp(-\gamma_0)$$

(A5)

Now, using (A3) and (A4), and get

$$\int_0^{\infty} x f(x) dx = m_1 = \int_0^{\infty} x \exp(-\gamma_0 - \gamma_1 x) dx$$

$$m_1 = \exp(-\gamma_0) \int_0^{\infty} x \exp(-\gamma_1 x) dx$$

$$\text{let } u = x \quad ; \quad dv = \exp(-\gamma_1 x) dx$$

and

$$du = dx \quad ; \quad v = -\gamma_1 \exp(-\gamma_1 x)$$

$$m_1 = \exp(-\gamma_0) \int_0^{\infty} x \exp(-\gamma_1 x) dx$$

$$= \exp(-\gamma_0) [-x \exp(-\gamma_1 x) / \gamma_1$$

$$+ (1/\gamma_1) \int_0^{\infty} \exp(-\gamma_1 x) dx]$$

$$= - \exp(-\gamma_0) [\exp(-\gamma_1 x)/(\gamma_1)^2]$$

$$m_1 = \exp(-\gamma_1)/\gamma_1^2$$

(A6)

substituting (A5) in (A6),

$$m_1 = \gamma_1/\gamma_1^2 = 1/\gamma_1$$

or

$$\gamma_1 = 1/m_1$$

(A7)

From (A4),

$$f(x) = \exp(-\gamma_0 - \gamma_1 x)$$

$$= \gamma_1 \exp(-\gamma_1 x)$$

$$= (1/m_1) \exp(-x/m_1)$$

(A8)

Appendix B

DERIVATION OF $H_{1,1}^2(x)$ FUNCTION.

In this appendix the probability distribution $H_{1,1}^2(x)$ is obtained using Shannon's measure of uncertainty and the only information is available in the form of first two moments.

Consider :

$$\text{Max } S : - \int_0^{\infty} f(x) \ln f(x) dx \quad (B1)$$

subject to

$$\int_0^{\infty} f(x) dx = 1 \quad (B2)$$

$$\int_0^{\infty} xf(x)dx = m_1 \quad (B3)$$

$$\int_0^{\infty} x^2 f(x)dx = m_2 \quad (B4)$$

The Langrangian equation of above system is :

$$\begin{aligned} L(f(x), \gamma_0, \gamma_1, \gamma_2) = & \int_0^{\infty} f(x) \ln f(x) dx \\ & - (\gamma_0 - 1) \left[\int_0^{\infty} f(x) dx - 1 \right] \\ & - \gamma_1 \left[\int_0^{\infty} xf(x) dx - m_1 \right] \end{aligned}$$

$$- \gamma_2 \left[\int_0^{\infty} x^2 f(x) dx - m_2 \right]$$

then,

$$dL/df(x) = -1 - \ln f(x) - (\gamma_0 - 1) - \gamma_1 x - \gamma_2 x^2 = 0$$

$$\ln f(x) = -\gamma_0 - \gamma_1 x - \gamma_2 x^2$$

$$f(x) = \exp(-\gamma_0 - \gamma_1 x - \gamma_2 x^2)$$

(B5)

Using (B2) with (B5), we have

$$\int_0^{\infty} \exp(-\gamma_0 - \gamma_1 x - \gamma_2 x^2) dx = 1$$

(B6)

or

$$1/A(\gamma_1, \gamma_2) = \int_0^{\infty} \exp(-\gamma_1 x - \gamma_2 x^2) dx$$

where

$$1/A(\gamma_1, \gamma_2) = \exp(\gamma_0)$$

(6.1)

Then, (B5) can be written as :

$$H_{1,1}^2(t) = A(\gamma_1, \gamma_2) \exp(-\gamma_1 x - \gamma_2 x^2)$$

(B7)

Using (B6) to obtain $A(\gamma_1, \gamma_2)$,

$$1/A(\gamma_1, \gamma_2) = \exp(\gamma_0)$$

$$= \int_0^{\infty} \exp(-\gamma_1 x - \gamma_2 x^2) dx$$

Since we need the distribution to be in a proper form, we assume $\gamma_2 > 0$, so that the function is a truncated normal distribution.

Now,

$$\begin{aligned} 1/A(\gamma_1, \gamma_2) &= \int_0^{\infty} e^{-\gamma_2 \left[x^2 + \frac{\gamma_1}{\gamma_2} x + \frac{\gamma_1^2}{4\gamma_2^2} \right] - \frac{\gamma_1^2}{4\gamma_2}} dx ; x \geq 0 \\ &= e^{-\frac{\gamma_1^2}{4\gamma_2}} \int_0^{\infty} e^{-\gamma_2 \left(x + \frac{\gamma_1}{2\gamma_2} \right)^2} dx ; x \geq 0 \end{aligned}$$

Let

$$x + \gamma_1/(2\gamma_2) = z ; \quad dx = dz$$

$$x=0 ; \quad z = \gamma_1/(2\gamma_2)$$

$$x = \infty ; \quad z = \infty$$

$$1/A(\gamma_1, \gamma_2) = e^{-\frac{\gamma_1^2}{4\gamma_2}} \int_0^{\infty} e^{-\gamma_2 z^2} dz ; \quad z \geq \frac{\gamma_1}{2\gamma_2}$$

(B8)

Again, let

$$\gamma_2 z^2 = (1/2)y^2$$

then

$$y^2 = 2\gamma_2 z^2 \text{ ie. } y = z \sqrt{2\gamma_2}$$

$$z = \gamma_1/(2\gamma_2) ; \text{ then } y = \gamma_1/\sqrt{2\gamma_2}$$

$$2\gamma_2 dz = dy$$

$$dz = dy/\sqrt{2\gamma_2}$$

$$\begin{aligned} 1/A(\gamma_1, \gamma_2) &= e^{\frac{\gamma_1^2}{4\gamma_2}} \int_{\frac{\gamma_1}{\sqrt{2\gamma_2}}}^{\infty} e^{-\frac{1}{2}y^2} \cdot \frac{dy}{\sqrt{2\gamma_2}} \quad ; \quad y \geq \frac{\gamma_1}{\sqrt{2\gamma_2}} \\ &= \frac{1}{\sqrt{2\gamma_2}} e^{\frac{\gamma_1^2}{4\gamma_2}} \cdot \sqrt{2\pi} \cdot \frac{1}{\sqrt{2\pi}} \int_{\frac{\gamma_1}{\sqrt{2\gamma_2}}}^{\infty} e^{-\frac{1}{2}y^2} dy \quad y \geq \frac{\gamma_1}{\sqrt{2\gamma_2}} \\ &= \frac{1}{\sqrt{2\gamma_2}} \left[\phi\left(\frac{\gamma_1}{\sqrt{2\gamma_2}}\right) \right]^{-1} \cdot \left[1 - \Phi\left(\frac{\gamma_1}{\sqrt{2\gamma_2}}\right) \right] \end{aligned}$$

where $\phi(\gamma_1/\sqrt{2\gamma_2})$ is a standard Normal density function

and $\Phi(\gamma_1/\sqrt{2\gamma_2})$ is a standard Normal distribution function.

Therefore,

$$\begin{aligned} A(\gamma_1, \gamma_2) &= (2\gamma_2)^{\frac{1}{2}} \cdot \phi\left(\frac{\gamma_1}{\sqrt{2\gamma_2}}\right) \cdot \left[1 - \Phi\left(\frac{\gamma_1}{\sqrt{2\gamma_2}}\right) \right]^{-1} \\ &= (2\gamma_2)^{\frac{1}{2}} \cdot \phi(t) \cdot \left[1 - \Phi(t) \right]^{-1} \end{aligned}$$

Where

$$t = \frac{\gamma_1}{\sqrt{2\gamma_2}}$$

(B9)

or otherwise,

$$1/A(\gamma_1, \gamma_2) = (2\gamma_2)^{-\frac{1}{2}} \cdot e^{-\frac{\gamma_1^2}{4\gamma_2}} \cdot \int_{\frac{\gamma_1}{\sqrt{2\gamma_2}}}^{\infty} e^{-\frac{1}{2}y^2} \cdot dy$$

$$1/A(\gamma_1, \gamma_2) = M[\gamma_1/\sqrt{2\gamma_2}]/\sqrt{2\gamma_2}$$

where $M[\gamma_1/\sqrt{2\gamma_2}]$ is known as Mill's ratio.

or

$$\begin{aligned} A(\gamma_1, \gamma_2) &= \sqrt{2\gamma_2}/M[\gamma_1/\sqrt{2\gamma_2}] \\ &= \sqrt{2\gamma_2} B(t) \end{aligned} \tag{B10}$$

where $B(t)$ is the inverse function of Mill's ratio and $t = \gamma_1/\sqrt{2\gamma_2}$

Equating (B9) to (B10), we have

$$\begin{aligned} A(\gamma_1, \gamma_2) &= \sqrt{2\gamma_2} \cdot \phi(t) \cdot [1 - \Phi(t)]^{-1} \\ &= \sqrt{2\gamma_2} \cdot B(t) \end{aligned} \tag{B10.1}$$

Thus, $A(\gamma_1, \gamma_2)$ is known.

Using (B10.1) and (B6.1),

$$\exp(-\gamma_0) = \sqrt{2\gamma_2} B(t) \tag{B11}$$

From (B6),

$$\exp(\gamma_0) = \int_0^\infty \exp(-\gamma_1 x - \gamma_2 x^2) dx$$

Taking derivative w.r.t. γ_1 :

$$\frac{d}{dr_1} \exp(r_0) = \int_0^{\infty} -x \exp(-r_1 x - r_2 x^2) dx \quad (B12)$$

Multiplying both side by $\exp(-r_0)$, then we have

$$e^{-r_0} \left(\frac{d}{dr_1} e^{r_0} \right) = - \int_0^{\infty} x e^{-r_1 x - r_2 x^2} dx = -m_1 \quad (B13)$$

But,

$$\frac{d}{dr_1} e^{r_0} = \frac{d}{dr_1} \left[\sqrt{2r_2} \left(B\left(\frac{r_1}{\sqrt{2r_2}}\right) \right)^{-1} \right]$$

From (B11) ; therefore we can obtain

$$\begin{aligned} \frac{d}{dr_1} e^{r_0} &= \sqrt{2r_2} \frac{d}{dr_1} \left[B\left(\frac{r_1}{\sqrt{2r_2}}\right) \right]^{-1} \\ &= \sqrt{2r_2} (-1) \left[B\left(\frac{r_1}{\sqrt{2r_2}}\right) \right]^{-2} \cdot \frac{d}{dr_1} \left[B\left(\frac{r_1}{\sqrt{2r_2}}\right) \right] \\ &= - \sqrt{2r_2} \left[B\left(\frac{r_1}{\sqrt{2r_2}}\right) \right]^{-2} B'\left(\frac{r_1}{\sqrt{2r_2}}\right) \cdot \frac{1}{\sqrt{2r_2}} \end{aligned} \quad (B14)$$

Using B(11) and B(14),

$$\begin{aligned} e^{-r_0} \frac{d}{dr_1} e^{r_0} &= - (2r_2)^{\frac{1}{2}} B\left(\frac{r_1}{\sqrt{2r_2}}\right) \frac{(2r_2)^{-\frac{1}{2}}}{\sqrt{2r_2}} \left[B\left(\frac{r_1}{\sqrt{2r_2}}\right) \right]^{-2} \\ &\quad \cdot B'\left(\frac{r_1}{\sqrt{2r_2}}\right) \end{aligned}$$

$$= -B'(\frac{\gamma_1}{\sqrt{2}\gamma_2}) \cdot [B(\frac{\gamma_1}{\sqrt{2}\gamma_2}) \cdot \sqrt{2}\gamma_2]^{-1}$$

(B15)

Using B(15) and B(13), we have :

$$m_1 = \frac{B'(t)}{B(t)} \cdot \frac{1}{\sqrt{2}\gamma_2}$$

(B16)

Where

$$t = \frac{\gamma_1}{\sqrt{2}\gamma_2}$$

Now, from Equation (B12),

$$\frac{d^2 e^{\gamma_0}}{d\gamma_1^2} = \int_0^\infty x^2 e^{-(\gamma_1 x + \gamma_2 x^2)} dx$$

Again, multiplying both sides by $e^{-\gamma_0}$, we have

$$e^{-\gamma_0} \frac{d^2 e^{\gamma_0}}{d\gamma_1^2} = \int_0^\infty x^2 e^{-(\gamma_0 + \gamma_1 x + \gamma_2 x^2)} dx = m_2$$

(B17)

Since we have,

$$e^{\gamma_0} = (2\gamma_2)^{-\frac{1}{2}} [B(t)]^{-1}$$

therefore,

$$\begin{aligned} \frac{d}{d\gamma_1} e^{\gamma_0} &= (2\gamma_2)^{-\frac{1}{2}} (-1) [B(t)]^{-2} B'(t) \cdot \frac{1}{\sqrt{2}\gamma_2} \\ &= -B'(t) [(B(t))^2 \cdot 2\gamma_2]^{-1} \end{aligned}$$

and,

$$\begin{aligned}
\frac{d^2}{dx_1^2} e^{x_0} &= -\frac{1}{2x_2} \frac{d}{dx_1} \frac{B'(t)}{[B(t)]^2} \\
&= -\frac{1}{2x_2} \left[B'(t) (-2) (B(t))^{-3} B'(t) \frac{1}{\sqrt{2x_2}} \right. \\
&\quad \left. + \frac{1}{[B(t)]^2} B''(t) \frac{1}{\sqrt{2x_2}} \right] \\
&= -\frac{1}{2x_2 \sqrt{2x_2}} \cdot \frac{1}{[B(t)]^2} \left[-\frac{2[B'(t)]^2}{B(t)} + B''(t) \right] \\
&\hspace{15em} (B18)
\end{aligned}$$

Substituting (B18) into (B17),

$$\begin{aligned}
e^{-x_0} \frac{d^2}{dx_0^2} \cdot e^{x_0} &= -\sqrt{2x_2} \cdot B(t) \cdot \frac{1}{2x_2 \sqrt{2x_2} [B(t)]^2} \cdot \\
&\quad \cdot \left\{ -\frac{2[B'(t)]^2}{B(t)} + B''(t) \right\}
\end{aligned}$$

or,

$$\begin{aligned}
m_2 &= -\frac{1}{2x_2 B(t)} \left[-\frac{2[B'(t)]^2}{B(t)} + B''(t) \right] \\
&= 2 \left[\frac{B'(t)}{B(t) \sqrt{2x_2}} \right]^2 - \frac{B''(t)}{2x_2 B(t)}
\end{aligned}$$

and from (B16),

$$m_2 = 2m_1^2 - \frac{B''(t)}{2x_2 B(t)}$$

or,

$$m_2 - m_1^2 - m_1^2 = - \frac{B''(t)}{2 \gamma_2 B(t)}$$

$$\delta^2 - m_1^2 = - \frac{B''(t)}{2 \gamma_2 B(t)}$$

or,

$$\gamma_2 = \frac{B''(t)}{2 B(t)} \cdot \frac{1}{m_1^2 - \delta^2}$$

(B19)

From (B16), we can obtain,

$$m_1 = \frac{B'(t)}{B(t) \sqrt{2 \gamma_2}}$$

therefore,

$$m_2 = \frac{[B'(t)]^2}{[B(t)]^2 \cdot 2 \gamma_2}$$

or,

$$m_1^2 \gamma_2 = \frac{[B'(t)]^2}{2 [B(t)]^2}$$

or,

$$\gamma_2 = \frac{[B'(t)]^2}{2 [B(t) m_1]^2}$$

(B20)

Substituting (B19) into (B20), we can obtain,

$$\frac{B''(t)}{2 B(t)} \cdot \frac{m_1^2}{m_1^2 - \sigma^2} = \frac{[B'(t)]^2}{2 [B(t)]^2}$$

$$\frac{m_1^2 - \sigma^2}{m_1^2} = \frac{B(t)}{[B'(t)]^2} \cdot B''(t)$$

$$\frac{\sigma^2}{m_1^2} = 1 - \frac{B(t) B''(t)}{[B'(t)]^2} \quad (B21)$$

or,

$$\frac{\sigma^2}{m_1^2} = F(t) \quad (B22)$$

where,

$$t = \frac{\sigma_1}{\sqrt{2}\sigma_2}$$

and

$$F(t) = 1 - \frac{B(t) B''(t)}{[B'(t)]^2}$$

It has been shown by Barrow & Cohen (1954) that the function $F(t)$ is a continuous and monotonically increasing from 0 to 1 as t increases from $-\infty$ to ∞ .

If $\sigma^2 < m_1^2$, we can derive a unique value of t from (B22) and we can also determine γ_1, γ_2 by using (B8) and (B16). Using (B10) the value of $A(\gamma_1, \gamma_2)$ can be determine.

If $\sigma^2 = m_1^2$, the system gives $t = \infty, \gamma_2 = 0$ and it is a degenerate case, the truncated normal distribution reduces to an exponential distribution.

If $\sigma^2 > m_1^2$, we can not find a value of t, γ_1, γ_2 ; and therefore no maximum entropy solution of the form (B7) exists.

Appendix C

PROCEDURE OF FINDING $H_{1,1}^2(X)$ FUNCTION.

From Appendix E, we have

$$l_{1,1}^1 = m_1 = 1/u + g(\gamma_1)/\gamma_0$$

and

$$\begin{aligned} l_{1,1}^2 &= d^2/ds^2 h_{1,1}(0) \\ &= g''(0) - 2g'(\gamma_1)/\gamma_0 + 2g(\gamma_1)/\gamma_0^2 \end{aligned}$$

To obtain the $H_{1,1}^2(t)$, we need to following procedure :

1) For given values of $l_{1,1}^1$, $l_{1,1}^2$ and parameters c , θ , γ_0 , and γ_1 ensure that $F(t)$ value exists. That is, it satisfies the condition $l_{1,1}^2 < (l_{1,1}^1)^2$

2) If $l_{1,1}^2 < (l_{1,1}^1)^2$ condition is satisfied, then Mill's ratio $B(x)$ is given by :

$$\begin{aligned} B(x) &= \sigma^2/m_1^2 \\ &= m_2/m_1^2 - 1 \\ &= l_{1,1}^2/(l_{1,1}^1)^2 - 1 \end{aligned}$$

The values of $B(x)$, for a given value of x , can be determine by using Appendix J.

$$\text{Since } X = \gamma_1/\sqrt{2\gamma_2}$$

and

$$l_{11}' = B'(x)/[B(x)\sqrt{2\gamma_2}]$$

Therefore,

$$\gamma_2 = (1/2)[B'(x)/\{B(x)l_{11}'\}]^2$$

and

$$\gamma_1 = x\sqrt{2\gamma_2}$$

Therefore,

$$A(\gamma_1, \gamma_2) = \sqrt{2\gamma_2} B(x)$$

and the value of $H_{11}^2(x)$ can be obtained using the following expression :

$$H_{11}^2(x) = \int_0^x A(\gamma_1, \gamma_2) \exp(-\gamma_1 x - \gamma_2 x^2) dx$$

Appendix D

DERIVATION OF L_{11}^1 AND L_{11}^2

From (1.5.8),

$$h_{11}(s) = g(s) - \frac{s}{s+\beta_0} g(s+\beta_1)$$

Therefore,

$$\frac{d}{ds} h_{11}(s) = g'(s) - \frac{(s+\beta_0)[g(s+\beta_1) + s g'(s+\beta_1)]}{(s+\beta_0)^2}$$

$$- \frac{s g(s+\beta_1)}{(s+\beta_0)^2}$$

and,

(D1)

$$\frac{d^2}{ds^2} h_{11}(s) = g''(s)$$

$$- \left[(s+\beta_0)^2 \{ g(s+\beta_1) + s g'(s+\beta_1) \right.$$

$$+ (s+\beta_0) \cdot [g'(s+\beta_1) + g'(s+\beta_1)$$

$$+ s g''(s+\beta_1)] - g(s+\beta_1)$$

$$- s g'(s+\beta_1) \} - 2(s+\beta_0) \cdot$$

$$\cdot \{ (s+\beta_0) [g(s+\beta_1) + s g'(s+\beta_1)]$$

$$- s g(s+\beta_1) \} \cdot \frac{1}{(s+\beta_0)^4}$$

(D2)

Since $-\frac{d}{ds} h_{11}(0) = l_{11}'$

therefore,

$$l_{11}' = -g(0) + \frac{g(\beta_1)}{\beta_0} \quad (D3)$$

And also because

$$\frac{d^2}{ds^2} h_{11}(0) = l_{11}''$$

therefore,

$$\begin{aligned} l_{11}'' &= g''(0) - \frac{\beta_0^2 [g(\beta_1) + 2\beta_0 g'(\beta_1)]}{\beta_0^4} \\ &\quad + \frac{\beta_0^2 g(\beta_1) - 2\beta_0^2 g(\beta_1)}{\beta_0^4} \\ &= g''(0) - \frac{2\beta_0 g'(\beta_1) - 2g(\beta_1)}{\beta_0^2} \end{aligned}$$

or,

$$l_{11}'' = g''(0) - \frac{2g'(\beta_1)}{\beta_0} + \frac{2g(\beta_1)}{\beta_0^2}$$

(D4)

Appendix E

DERIVATION OF L_{02}^1 AND L_{02}^2

We know from (1.5.13),

$$h_{02}(s) = \frac{\beta_0 \beta_1}{(s + \beta_1)} \cdot \frac{1 - g(s - \beta_1)}{s + \beta_0(1 - g(s + \beta_1))}$$

therefore,

$$\begin{aligned} \frac{d}{ds} h_{02}(s) &= \frac{\beta_0 \beta_1}{(s + \beta_1)} \cdot \left\{ [s + \beta_0(1 - g(s + \beta_1))] \cdot \right. \\ &\quad \cdot (-g'(s + \beta_1)) - [1 - g(s + \beta_1)] \cdot \\ &\quad \cdot [1 + \beta_0(-g'(s + \beta_1))] \left. \right\} \cdot \\ &\quad \cdot (s + \beta_0[1 - g(s + \beta_1)])^{-4} \end{aligned}$$

or,

$$\begin{aligned} \frac{d}{ds} h_{02}(0) &= \beta_0 \left\{ \beta_1 [1 - g(\beta_1)] (-g'(\beta_1)) \right. \\ &\quad \left. - (1 - g(\beta_1)) \cdot [1 + \beta_0(-g'(\beta_1))] \right\} \cdot \\ &\quad \cdot \{ \beta_0 [1 - g(\beta_1)] \}^2 \\ &\quad + \frac{1 - g(\beta_1)}{\beta_0 [1 - g(\beta_1)]} \cdot \left(-\frac{\beta_0 \beta_1}{\beta_1^2} \right) \end{aligned}$$

$$\begin{aligned}
&= - \frac{\beta_0 [1 - g(\beta_1)] g'(\beta_1)}{\beta_0^2 (1 - g(\beta_1))^2} - \frac{1 - g(\beta_1)}{1 - g(\beta_1)} \cdot \frac{1}{\beta_1} \\
&\quad - \frac{\beta_0 (1 - g(\beta_1)) [1 - \beta_0 g'(\beta_1)]}{\beta_0^2 [1 - g(\beta_1)]^2} \\
&= - \frac{1}{\beta_0 (1 - g(\beta_1))} - \frac{1}{\beta_1}
\end{aligned} \tag{E1}$$

therefore ,

$$\begin{aligned}
h_{02}' &= - \frac{d}{ds} h_{02}(s) \\
&= \frac{1}{\beta_1} + \frac{1}{\beta_0 [1 - g(\beta_1)]}
\end{aligned} \tag{E2}$$

Similarly, we have

$$\begin{aligned}
\frac{d^2}{ds^2} h_{02}(s) &= \frac{d}{ds} [*] \left\{ \frac{\beta_0 \beta_1}{(s + \beta_1)} \left[(s + \beta_0) [1 - g(s + \beta_1)] \cdot \right. \right. \\
&\quad \left. \left. \cdot (-g'(s + \beta_1) - [1 - g(s + \beta_1)] [1 - \beta_0 g'(s + \beta_1)]) \right] \right\} \\
&\quad \left\{ s + \beta_0 [1 - g(s + \beta_1)] \right\}^2 - \\
&\quad \frac{d}{ds} [**] \left\{ \frac{1 - g(s + \beta_1)}{s + \beta_0 [1 - g(s + \beta_1)]} \cdot \frac{\beta_0 \beta_1}{(s + \beta_1)^2} \right\}
\end{aligned}$$

and,

$$\frac{d}{ds} [*] = -\frac{\beta_0 \beta_1}{(s+\beta_1)^2} \left\{ -\frac{[s+\beta_0(1-g(s+\beta_1))] g'(s+\beta_1)}{[s+\beta_0(1-g(s+\beta_1))]^2} \right.$$

$$- \frac{[1-g(s+\beta_1)] \cdot [1-\beta_0 g'(s+\beta_1)]}{[s+\beta_0(1-g(s+\beta_1))]^2}$$

$$+ \frac{\beta_0 \beta_1}{(s+\beta_1)} \cdot \frac{1}{\{s+\beta_0(1-g(s+\beta_1))\}^4} \cdot \left\{ [s+\beta_0(1-g(s+\beta_1))]^2 \right.$$

$$[-g''(s+\beta_1) \cdot (s+\beta_0(1-g(s+\beta_1))) - g'(s+\beta_1) \cdot$$

$$(1-\beta_0 g'(s+\beta_1)) - [1-g(s+\beta_1)] \cdot [-\beta_0 g''(s+\beta_1)]$$

$$- [-g'(s+\beta_1)] \cdot [1-\beta_0 g'(s+\beta_1)] -$$

$$2 \{s+\beta_0(1-g(s+\beta_1))\} \cdot [1-\beta_0 g'(s+\beta_1)] \cdot$$

$$\cdot [\{s+\beta_0(1-g(s+\beta_1))\} \cdot [-g'(s+\beta_1)]$$

$$- [1-g(s+\beta_1)] \cdot [1-\beta_0 g'(s+\beta_1)] \}$$

and,

$$\frac{d}{ds} [**] = -\frac{2\beta_0 \beta_1}{(s+\beta_1)^3} \cdot \frac{(1-g(s+\beta_1))}{[s+\beta_0(1-g(s+\beta_1))]} +$$

$$\frac{\beta_0 \beta_1}{(s + \beta_1)^2} \cdot \left\{ [s - \beta_0 [1 - g(s + \beta_1)]] \cdot (-g'(s + \beta_1)) \right. \\ \left. - (1 - \beta_0 g'(s + \beta_1)) \cdot (1 - g(s + \beta_1)) \right\} \cdot \\ \left\{ s + \beta_0 (1 - g(s + \beta_1)) \right\}^{-2}.$$

Since,

$$h_{02}'' = \frac{d^2}{ds^2} h_{02}(0)$$

$$= \frac{d}{ds} [*](0) - \frac{d}{ds} [**](0)$$

therefore,

$$\frac{d}{ds} [*](0) = -\frac{\beta_0 \beta_1}{\beta_1} \cdot \frac{1}{\{\beta_0 [1 - g(\beta_1)]\}^2} \cdot \left[-\beta_0 (1 - g(\beta_1)) \cdot \right. \\ \left. \cdot g'(\beta_1) - [1 - g(\beta_1)] \cdot [1 - \beta_0 g'(\beta_1)] \right] \\ - \frac{\beta_0 \beta_1}{\beta_1} \cdot \frac{1}{\beta_0^4 [1 - g(\beta_1)]^4} \cdot \left\{ \beta_0^2 (1 - g(\beta_1))^2 \right. \\ \left. \cdot [-g''(\beta_1) \beta_0 [1 - g(\beta_1)] - g'(\beta_1) \cdot (1 - \beta_0 g'(\beta_1))] \right. \\ \left. - (1 - g'(\beta_1)) \cdot (-\beta_0 g''(\beta_1)) - (-g'(\beta_1)) \cdot \right.$$

$$\cdot (1 - \beta_0 g'(\beta_1)) \} - 2 \{ \beta_0 (1 - g(\beta_1)) \cdot (1 - \beta_0 g'(\beta_1)) \} \cdot$$

$$\cdot [-\beta_0 (1 - g(\beta_1)) g'(\beta_1) - [1 - g(\beta_1)] \cdot [1 - \beta_0 g'(\beta_1)]] \cdot$$

$$= \frac{g'(\beta_1)}{\beta_1 (1 - g(\beta_1))} + \frac{[1 - \beta_0 g'(\beta_1)]}{\beta_0 \beta_1 [1 - g(\beta_1)]} + \frac{2}{\beta_0^2} \frac{[1 - \beta_0 g'(\beta_1)]}{[1 - g(\beta_1)]^2}$$

$$+ \frac{2}{\beta_0} \frac{g'(\beta_1) [1 - \beta_0 g'(\beta_1)]}{[1 - g(\beta_1)]^2}$$

and

$$\frac{d}{ds} [**] (0) = - \frac{2\beta_0 \beta_1}{\beta_0^3} \cdot \frac{(1 - g(\beta_1))}{\beta_0 (1 - g(\beta_1))} + \frac{\beta_0 \beta_1}{\beta_1^2 [\beta_0 (1 - g(\beta_1))]^2}$$

$$\cdot [-\beta_0 (1 - g(\beta_1)) g'(\beta_1) - (1 - \beta_0 g'(\beta_1)) (1 - g(\beta_1))]$$

$$= - \frac{2}{\beta_1^2} - \frac{g'(\beta_1)}{\beta_1 (1 - g(\beta_1))} - \frac{(1 - \beta_0 g'(\beta_1))}{\beta_0 \beta_1 (1 - g(\beta_1))}$$

Therefore,

$$l_{02}^2 = \frac{d}{ds} [*] (0) - \frac{d}{ds} [**] (0)$$

or,

$$\begin{aligned} l_{02}^2 &= \frac{2 g'(\beta_1)}{\beta_1 [1 - g(\beta_1)]} + \frac{2 [1 - \beta_0 g'(\beta_1)]}{\beta_0 \beta_1 [1 - g(\beta_1)]} \\ &+ \frac{2}{\beta_1^2} + \frac{2 g'(\beta_1) [1 - \beta_0 g'(\beta_1)]}{\beta_0 [1 - g(\beta_1)]^2} \\ &+ \frac{g'(\beta_1)}{\beta_1 [1 - g(\beta_1)]} + \frac{2 [1 - \beta_0 g'(\beta_1)]^2}{\beta_0^2 [1 - g(\beta_1)]^2} \end{aligned}$$

(E 3)

Appendix F

DERIVATION OF L_{00}^1 AND L_{00}^2

We know from (1.5.14),

$$h_{00}(s) = \frac{\beta_0}{(s+\beta_0)} \cdot \frac{g(s+\beta_1)}{1-g(s)+g(s+\beta_1)} \quad (F1)$$

therefore,

$$\begin{aligned} \frac{d}{ds} h_{00}(s) &= -\frac{\beta_0}{(s+\beta_0)^2} \cdot \frac{g(s+\beta_1)}{1-g(s)+g(s+\beta_1)} \\ &\quad + \frac{\beta_0}{(s+\beta_0)} \cdot \frac{1}{[1-g(s)+g(s+\beta_1)]^2} \cdot \\ &\quad \cdot \left\{ [1-g(s)+g(s+\beta_1)] \cdot g'(s+\beta_1) \right. \\ &\quad \left. - (-g'(s) + g'(s+\beta_1) \cdot g(s+\beta_1)) \right\} \end{aligned} \quad (F2)$$

Since,

$$\begin{aligned} L_{00}^1 &= -\frac{d}{ds} h_{00}(0) \\ &= \frac{\beta_0}{\beta_0^2} \cdot \frac{g(\beta_1)}{1-g(0)-g(\beta_1)} + \frac{1}{\beta_0} \end{aligned}$$

$$\frac{g(\beta_1)g'(\beta_1) + g'(0)g(\beta_1) - g(\beta_1)g'(\beta_1)}{[g(\beta_1)]^2}$$

$$= \frac{1}{\beta_0} - \frac{g'(0)}{g(\beta_1)} \quad (F3)$$

Similarly,

$$\begin{aligned} \frac{d^2}{ds^2} h_{00}(s) &= \frac{d}{ds} [*] \left[-\frac{\beta_0}{(s+\beta_0)^2} \cdot \frac{g(s+\beta_1)}{1-g(s)+g(s+\beta_1)} \right] \\ &+ \frac{d}{ds} [**] \left[\frac{\beta_0}{(s+\beta_0)} \cdot \left\{ \frac{g'(s+\beta_1) - g(s)g'(s+\beta_1)}{[1-g(s)+g(s+\beta_1)]^2} \right. \right. \\ &\left. \left. + \frac{g'(s)g(s+\beta_1)}{[1-g(s)+g(s+\beta_1)]^2} \right\} \right] \end{aligned}$$

Since

$$\begin{aligned} \frac{d}{ds} [*] &= \frac{2\beta_0}{(s+\beta_0)^3} \cdot \frac{g(s+\beta_1)}{1-g(s)+g(s+\beta_1)} - \frac{\beta_0}{(s+\beta_0)^2} \cdot \\ &\left[\left\{ (1-g(s)+g(s+\beta_1))g'(s+\beta_1) - g(s+\beta_1) \cdot \right. \right. \\ &\left. \left. \cdot \{-g'(s) + g'(s+\beta_1)\} \right\} \cdot [1-g(s)+g(s+\beta_1)]^{-2} \right] \quad (F4) \end{aligned}$$

And

$$\frac{d}{ds} [**] = -\frac{\beta_0}{(s+\beta_0)^2} \cdot \frac{g'(s+\beta_1) - g(s)g'(s+\beta_1) + g'(s)g(s+\beta_1)}{[1-g(s)+g(s+\beta_1)]^2}$$

$$\begin{aligned}
& + \frac{\beta_0}{(s+\beta_0)} \left\{ [1-g(s)+g(s+\beta_1)] \cdot [g''(s+\beta_1) + g'(s)g'(s+\beta_1) \right. \\
& \quad - g(s)g''(s+\beta_1) + g''(s)g(s+\beta_1) + g'(s)g'(s+\beta_1)] \\
& \quad - 2[g'(s+\beta_1) - g(s)g'(s+\beta_1) + g'(s)g(s+\beta_1)] \cdot \\
& \quad \left. [1-g(s)+g(s+\beta_1)] \cdot [-g'(s) + g'(s+\beta_1)] \right\} \cdot \\
& \quad \cdot [1-g(s)+g(s+\beta_1)]^{-4} .
\end{aligned}$$

(F5)

Since

$$\begin{aligned}
l_{00}^2 &= \frac{d^2}{ds^2} h_{00}(0) \\
&= \frac{d}{ds} [*](0) + \frac{d}{ds} [**](0)
\end{aligned}$$

and from (F4), we can obtain :

$$\begin{aligned}
\frac{d}{ds} [*](0) &= \frac{2\beta_0}{\beta_0^3} \cdot \frac{g(\beta_1)}{1-g(0)+g(\beta_1)} \\
&\quad - \frac{\beta_0}{\beta_0^2} \cdot \frac{1}{[1-g(0)+g(\beta_1)]} \cdot \left\{ [1-g(0)+g(\beta_1)] \cdot \right. \\
&\quad \left. \cdot g'(\beta_1) - g(\beta_1)[-g'(0) + g'(\beta_1)] \right\} \\
&= \frac{2}{\beta_0^2} - \frac{1}{\beta_0} \frac{g'(0)}{g(\beta_1)}
\end{aligned}$$

(F6)

$$\frac{\beta_0 \beta_1}{(s + \beta_1)^2} \cdot \left\{ [s - \beta_0 [1 - g(s + \beta_1)]] \cdot (-g'(s + \beta_1)) \right. \\ \left. - (1 - \beta_0 g'(s + \beta_1)) \cdot (1 - g(s + \beta_1)) \right\} \cdot \\ \{ s + \beta_0 (1 - g(s + \beta_1)) \}^{-2}.$$

Since,

$$h_{02}^2 = \frac{d^2}{ds^2} h_{02}(0) \\ = \frac{d}{ds} [*](0) - \frac{d}{ds} [**](0)$$

therefore,

$$\frac{d}{ds} [*](0) = -\frac{\beta_0 \beta_1}{\beta_1} \cdot \frac{1}{\{\beta_0 [1 - g(\beta_1)]\}^2} \cdot [-\beta_0 (1 - g(\beta_1)) \cdot g'(\beta_1) - [1 - g(\beta_1)] \cdot [1 - \beta_0 g'(\beta_1)]] \\ - \frac{\beta_0 \beta_1}{\beta_1} \cdot \frac{1}{\beta_0^4 [1 - g(\beta_1)]^4} \cdot \left\{ \beta_0^2 (1 - g(\beta_1))^2 \cdot [-g''(\beta_1) \beta_0 [1 - g(\beta_1)] - g'(\beta_1) \cdot (1 - \beta_0 g'(\beta_1))] \right. \\ \left. - (1 - g'(\beta_1)) \cdot (-\beta_0 g''(\beta_1)) - (-g'(\beta_1)) \cdot \right.$$

Appendix G

DERIVATION OF C.D.F. OF RENYI'S ENTROPY.

In this appendix the cumulative distribution function, $F(y)$, is obtained using Renyi's measure of uncertainty and the only information is available in the form of first moment. It is assumed that the given system is continuous distribution over range $(0, \infty)$ and $\int f(x) dx = 1$.

$$\text{Max } H_{\alpha_1} = \frac{1}{1-\alpha_1} \ln \int_0^{\infty} f^{\alpha_1}(x) dx \quad (G1)$$

subject to

$$\int_0^{\infty} f(x) dx = 1 \quad (G2)$$

$$\int_0^{\infty} x f(x) dx = m_1 \quad (G3)$$

where m_1 is the mean service time distribution.

The Langrangian equation of above system is :

$$L(f(x), \gamma_0, \gamma_1) =$$

$$\begin{aligned} & 1/(1-\alpha_1) [\ln \int_0^{\infty} f^{\alpha_1}(x) dx] - \gamma_0 (\int_0^{\infty} f(x) dx - 1) \\ & - \gamma_1 (\int_0^{\infty} x f(x) dx - m_1) \end{aligned}$$

where γ_0, γ_1 are Langrangian multipliers.

$$\frac{dL}{df(x)} = \frac{1}{1-\alpha_1} \frac{\alpha_1 f^{\alpha_1-1}(x)}{\int_0^\infty f^{\alpha_1}(x) dx} - \alpha_0 - \alpha_1 x = 0 \quad (G4)$$

Let

$$N = \frac{1}{1-\alpha_1} \cdot \frac{\alpha_1}{\int_0^\infty f^{\alpha_1}(x) dx} \quad (G5)$$

Using (G5) in (G4), we can obtain :

$$N \cdot f^{\alpha_1-1}(x) - \alpha_0 - \alpha_1 x = 0 \quad (G6)$$

or,

$$f^{\alpha_1-1}(x) = \frac{\alpha_0}{N} - \frac{\alpha_1}{N} x \quad (G7)$$

Let

$$\frac{\alpha_0}{N} = n_1 \quad \text{and} \quad -\frac{\alpha_1}{N} = n_2, \quad \text{and (G7) becomes}$$

$$f^{\alpha_1-1}(x) = n_1 + n_2 x$$

or,

$$f(x) = (n_1 + n_2 x)^{\frac{1}{\alpha_1-1}} \quad (G8)$$

Using (G2) and (G8), we can obtain :

$$1 = \int_0^\infty (n_1 + n_2 x)^{\frac{1}{\alpha_1-1}} dx \quad (G9)$$

$$\text{let } y = n_1 + n_2 x \quad ; \quad dy = n_2 dx$$

$$\text{and } x = 0, y = n_1 \quad ; \quad x = \infty, y = \infty$$

Therefore, (G9) can be rewritten as:

$$1 = \frac{1}{n_2} \int_{n_1}^{\infty} y^{\frac{1}{\alpha_1-1}} dy \quad (G10)$$

Let $p = \frac{1}{\alpha_1-1}$, then (G10) can be written as:

$$1 = \frac{1}{n_2} \int_{n_1}^{\infty} y^p dy$$

or,

$$1 = \frac{1}{n_2} \left. \frac{y^{p+1}}{p+1} \right|_{n_1}^{\infty} \quad (G11)$$

For (G11) to define, we assume that $p < -1$;

therefore, (G11) can be rewritten as:

$$1 = \frac{1}{n_2} \left(- \frac{n_1^{p+1}}{p+1} \right)$$

or,

$$n_2 = - \frac{n_1^{p+1}}{p+1} \quad (G12)$$

Using (G3) and (G8), we have

$$\int_0^{\infty} x (n_1 + n_2 x)^{\frac{1}{\alpha_1-1}} dx = m_1 \quad (G13)$$

$$\text{let } p = \frac{1}{\alpha_1 - 1}$$

$$\text{and } u = n_1 + n_2 x, \quad du = n_2 dx$$

$$x = \frac{u - n_1}{n_2} \quad ; \quad \text{As } x=0 \text{ implies } u = n_1$$

$$\text{and } x = \infty \text{ implies } u = \infty.$$

Therefore, (G13) becomes

$$\int_{n_1}^{\infty} \frac{u - n_1}{n_2} \cdot u^p \cdot \frac{1}{n_2} \cdot du = m_1$$

or,

$$\frac{1}{n_2} \left[- \frac{n_1 \cdot u^{p+1}}{p+1} \Big|_{n_1}^{\infty} + \frac{u^{p+2}}{p+2} \Big|_{n_1}^{\infty} \right] = m_1 \quad (G14)$$

Let assume $p < -2$; therefore, (G14) becomes

$$\frac{1}{n_2} \left[\frac{n_1^{p+2}}{p+1} - \frac{n_1^{p+2}}{p+2} \right] = m_1$$

or,

$$\frac{1}{n_2} \cdot \frac{n_1^{p+2}}{(p+1)(p+2)} = m_1 \quad (G15)$$

From (G12), we have

$$n_2 = - \frac{n_1^{p+1}}{p+1} \quad (G15.1)$$

or

$$n_2^2 = \frac{n_1^{2p+2}}{(p+1)^2} \quad (G16)$$

Substituting (G16) into (G15), we can obtain:

$$\frac{(P+1)^2}{n_1^{2P+2}} \cdot \frac{n_1^{P+2}}{(P+1)(P+2)} = m_1$$

or,

$$n_1 = \left[\frac{P+2}{P+1} \cdot m_1 \right]^{-\frac{1}{P}} ; P < -2$$

(G17)

Using (G12) and (G17) in (G8); therefore, the probability density function, $f_{\alpha_1}(x)$, of the Renyi's function can be written as:

$$f_{\alpha_1}(x) = \left\{ \left[\frac{P+2}{P+1} \cdot m_1 \right]^{-\frac{1}{P}} - \frac{n_1^{P+1}}{P+1} x \right\}^P ;$$

$P < -2$
(G18)

And the corresponding cumulative probability function, $F_{\alpha_1}(y)$, can be written as:

$$F_{\alpha_1}(y) = \int_0^y f_{\alpha_1}(x) dx$$

$$= \int_0^y (n_1 + n_2 x)^{\frac{1}{P}} dx ; P < -2$$

(G19)

Let $(n_1 + n_2 x) = u$, $n_2 dx = du$

and $x=0$, $u=n_1$; $x=y$, $u=n_1 + n_2 y$

Therefore, (G19) can be rewritten as:

$$F_{\alpha}(y) = \frac{1}{n_2} \int_{n_1}^{n_1 + n_2 y} u^P du$$

$$= \frac{1}{n_2} \left. \frac{u^{P+1}}{P+1} \right|_{n_1}^{n_1 + n_2 y} ; P < -2$$

Therefore ,

$$F_{\alpha}(y) = \frac{1}{n_2 (P+1)} \cdot \left[(n_1 + n_2 y)^{P+1} - n_1^{P+1} \right] ;$$

$$P < -2$$

(G 20)

where n_1 , n_2 are given in (G17) and (G15.1), respectively.

Appendix H

DERIVATION OF C.D.F. OF PARAMETRIC ENTROPY.

In this appendix the cumulative distribution function, $F(y)$, is obtained using Parametric entropy of uncertainty and the only information is available in the form of first moment. It is assumed that the system given system is a continuous distribution over range $(0, \infty)$.

$$\text{Max } P_{\alpha_L} = - \int_0^{\infty} f(x) \ln f(x) dx + \frac{1}{\alpha_2} \int_0^{\infty} (1 + \alpha_2 f(x)) \ln(1 + \alpha_2 f(x)) dx - 1 \quad (\text{H1})$$

subject to

$$\int_0^{\infty} f(x) dx = 1 \quad (\text{H2})$$

$$\int_0^{\infty} x f(x) dx = m_1 \quad (\text{H3})$$

where m_1 is the known mean service time.

The Langrangian equation of above system is :

$$\begin{aligned} L(f(x), \gamma_0, \gamma_1) = & - \int_0^{\infty} f(x) \ln f(x) dx + \frac{1}{\alpha_2} \int_0^{\infty} (1 + \alpha_2 f(x)) \ln(1 + \alpha_2 f(x)) dx \\ & - 1 - \gamma_0 \left(\int_0^{\infty} f(x) dx - 1 \right) - \gamma_1 \left(\int_0^{\infty} x f(x) dx - m_1 \right) \end{aligned}$$

Where γ_0, γ_1 are the Langrangian multipliers.

Therefore,

$$\frac{dL}{df(x)} = -\ln f(x) - 1 + \frac{1}{\alpha_2} [\alpha_2 \ln(1 + \alpha_2 f(x)) + \alpha_2] - \gamma_0 - \gamma_1 x = 0$$

or,

$$f(x) = \frac{e^{-\gamma_0 - \gamma_1 x}}{1 - \alpha_2 e^{-\gamma_0 - \gamma_1 x}} \quad (H4)$$

Using (H2) and (H4), we have

$$1 = \int_0^{\infty} \frac{e^{-\gamma_0 - \gamma_1 x}}{1 - \alpha_2 e^{-\gamma_0 - \gamma_1 x}} dx \quad (H5)$$

or,

$$e^{-\gamma_0} = \frac{1 - e^{-\alpha_2 \gamma_1}}{\alpha_2} \quad (H6)$$

Using (H3) and (H4), we have:

$$m_1 = \int_0^{\infty} x \frac{e^{-\gamma_0 - \gamma_1 x}}{1 - \alpha_2 e^{-\gamma_0 - \gamma_1 x}} dx \quad (H7)$$

Using integral by part, (H7) can be written as:

$$m_1 = \frac{x}{\alpha_2 \gamma_1} \ln(1 - \alpha_2 e^{-\gamma_0 - \gamma_1 x}) \Big|_0^{\infty} - \frac{1}{\alpha_2 \gamma_1} \int_0^{\infty} \ln(1 - \alpha_2 e^{-\gamma_0 - \gamma_1 x}) dx$$

or,

$$m_1 = -\frac{1}{\alpha_2 \gamma_1} \int_0^{\infty} \ln(1 - \alpha_2 e^{-\gamma_0 - \gamma_1 x}) dx \quad (H8)$$

Let

$$I = \int_0^{\infty} \ln(1 - \alpha_2 e^{-\gamma_0 - \gamma_1 x}) dx$$

then,

$$\frac{dI}{d\alpha_2} = - \int_0^{\infty} \frac{e^{-\gamma_0 - \gamma_1 x}}{1 - \alpha_2 e^{-\gamma_0 - \gamma_1 x}} \cdot dx \quad (H9)$$

From (H5), R.H.S. of (H9) equal to 1.

Therefore,

$$\frac{dI}{d\alpha_2} = -1$$

or

$$I = - \int d\alpha_2 + C$$

Where C is a constant.

or,

$$I = -\alpha_2 + C$$

As $\alpha_2 = 0$ implies $C = 0$; therefore,

$$I = -\alpha_2$$

Therefore, (H8) can be written as:

$$m_1 = -\frac{1}{\alpha_2 \gamma_1} (-\alpha_2)$$

or

$$\gamma_1 = \frac{1}{m_1} \quad (H11)$$

And (H6) becomes

$$e^{-\gamma_0} = \frac{1 - e^{-\frac{\alpha_2}{m_1}}}{\alpha_2} \quad (H12)$$

And cumulative probability function, $F_{\alpha_2}(y)$, is

$$\begin{aligned} F_{\alpha_2}(y) &= \int_0^y \frac{e^{-\gamma_0 - \gamma_1 x}}{1 - \alpha_2 e^{-\gamma_0 - \gamma_1 x}} \cdot dx \\ &= \frac{1}{\alpha_2 \gamma_1} [\ln(1 - \alpha_2 e^{-\gamma_0 - \gamma_1 y}) - \ln(1 - \alpha_2 e^{-\gamma_0})] \end{aligned}$$

Appendix I

EXAMINING OF SHANNON'S ENTROPY AS A SPECIAL CASE OF RENYI'S AND PARAMETRIC ENTROPY.

Renyi's Entropy (H_{α_1})

$$\text{Max } H_{\alpha_1} : \frac{1}{1-\alpha_1} \ln \left[\frac{\int_0^{\infty} f^{\alpha_1}(x) dx}{\int_0^{\infty} f(x) dx} \right] \quad (I1)$$

We want to show that as α_1 tends to 1,
 H_{α_1} tends to S (Shannon's Entropy).

Proof :

Assume that $\int_0^{\infty} f(x) dx = 1$, using L'Hospital Rule.

$$\begin{aligned} \lim_{\alpha_1 \rightarrow 1} \frac{d H_{\alpha_1}}{d \alpha_1} &= \lim_{\alpha_1 \rightarrow 1} - \frac{1}{1-\alpha_1} \cdot \frac{f^{\alpha_1}(x) \ln f(x)}{\int_0^{\infty} f^{\alpha_1}(x) dx} \\ &= - f(x) \ln f(x) \\ &= S \end{aligned}$$

Therefore, as α_1 tends to 1 H_{α_1} gives S .

Parametric Entropy (P_{α_2})

$$\text{Max } P_{\alpha_2} : - \int_0^{\infty} f(x) \ln f(x) dx + \frac{1}{\alpha_2} \int_0^{\infty} (1 + \alpha_2 f(x)) \ln(1 + \alpha_2 f(x)) dx - 1 \quad (I2)$$

We want to show that as α_2 tends to 0, P_{α_2} gives S (Shannon's Entropy).

Proof :

Using L'Hospital Rule,

$$\begin{aligned} \lim_{\alpha_2 \rightarrow 0} \frac{d P_{\alpha_2}}{d \alpha_2} &= \lim_{\alpha_2 \rightarrow 0} \int_0^{\infty} [f(x) \ln(1 + \alpha_2 f(x)) + \alpha_2] dx \\ &= \int_0^{\infty} f(x) dx \\ &= 1 \end{aligned}$$

Therefore, as α_2 tends to 0, $\text{Max } P_{\alpha_2}$ becomes

$$\begin{aligned} \text{Max } P_{\alpha_2} &= - \int_0^{\infty} f(x) \ln f(x) dx + 1 - 1 \\ &= - \int_0^{\infty} f(x) \ln f(x) dx \\ &= S \\ &= \text{Max } S \text{ (Shannon's Entropy).} \end{aligned}$$

Appendix J

VALUES OF INVERSE MILL'S RATIO FUNCTION

T	B(B'(t))	B1(B'(t))	B11(B'(t))	FCN(F(T))
-6.0	0.00000001	0.00000004	0.00000021	0.02777778
-5.9	0.00000001	0.00000006	0.00000037	0.02872738
-5.8	0.00000002	0.00000011	0.00000065	0.02972651
-5.7	0.00000004	0.00000020	0.00000111	0.03077869
-5.6	0.00000006	0.00000035	0.00000188	0.03188774
-5.5	0.00000011	0.00000059	0.00000315	0.03305783
-5.4	0.00000019	0.00000100	0.00000523	0.03429352
-5.3	0.00000032	0.00000168	0.00000859	0.03559979
-5.2	0.00000054	0.00000279	0.00001396	0.03698214
-5.1	0.00000090	0.00000458	0.00002244	0.03844656
-5.0	0.00000149	0.00000743	0.00003568	0.03999968
-4.9	0.00000244	0.00001195	0.00005612	0.04164877
-4.8	0.00000396	0.00001901	0.00008731	0.04340188
-4.7	0.00000637	0.00002994	0.00013434	0.04526787
-4.6	0.00001014	0.00004665	0.00020444	0.04725657
-4.5	0.00001598	0.00007193	0.00030769	0.04937881
-4.4	0.00002494	0.00010975	0.00045795	0.05164664
-4.3	0.00003854	0.00016570	0.00067401	0.05407336
-4.2	0.00005894	0.00024757	0.00098087	0.05667372
-4.1	0.00008926	0.00036599	0.00141135	0.05946404
-4.0	0.00013383	0.00053536	0.00200773	0.06246236
-3.9	0.00019867	0.00077483	0.00282349	0.06568858
-3.8	0.00029197	0.00110956	0.00392502	0.06916461
-3.7	0.00042483	0.00157204	0.00539305	0.07291444
-3.6	0.00061200	0.00220357	0.00732356	0.07696430
-3.5	0.00087289	0.00305586	0.00982797	0.08134262
-3.4	0.00123263	0.00419248	0.01303212	0.08608009
-3.3	0.00172340	0.00569020	0.01707386	0.09120956
-3.2	0.00238573	0.00764002	0.02209879	0.09676581
-3.1	0.00326998	0.01014764	0.02825407	0.10278537
-3.0	0.00443784	0.01333321	0.03568014	0.10930602
-2.9	0.00596366	0.01733018	0.04450056	0.11636631
-2.8	0.00793573	0.02228302	0.05481038	0.12400490
-2.7	0.01045719	0.02834376	0.06666377	0.13225970
-2.6	0.01364658	0.03566733	0.08006196	0.14116699
-2.5	0.01763783	0.04440566	0.09494275	0.15076032
-2.4	0.02257963	0.05470095	0.11117290	0.16106942
-2.3	0.02863411	0.06667837	0.12854470	0.17211900
-2.2	0.03597477	0.08043867	0.14677783	0.18392760
-2.1	0.04478363	0.09605119	0.16552692	0.19650647
-2.0	0.05524786	0.11354805	0.18439481	0.20985861
-1.9	0.06755578	0.13291977	0.20295078	0.22397790
-1.8	0.08189258	0.15411305	0.22075233	0.23884858
-1.7	0.09843592	0.17703069	0.23736862	0.25444487
-1.6	0.11735162	0.20153399	0.25240345	0.27073100
-1.5	0.13878975	0.22744722	0.26551576	0.28766146
-1.4	0.16288121	0.25456398	0.27643574	0.30518167
-1.3	0.18973503	0.28265493	0.28497546	0.32322881
-1.2	0.21943654	0.31147625	0.29103350	0.34173303
-1.1	0.25204630	0.34077827	0.29459361	0.36061881
-1.0	0.28759997	0.37031371	0.29571817	0.37980644
-0.9	0.32610889	0.39984501	0.29453765	0.39921361
-0.8	0.36756142	0.42915054	0.29123737	0.41875704
-0.7	0.41192475	0.45802932	0.28604300	0.43835403
-0.6	0.45914713	0.48630437	0.27920600	0.45792387

(continued)

VALUES OF INVERSE MILL'S RATIO FUNCTION

T	B(B(t))	B1(B'(t))	B11(B''(t))	FCN(F(T))
-0.5	0.50916043	0.51382456	0.27099012	0.47738920
-0.4	0.56188270	0.54046525	0.26165954	0.49667710
-0.3	0.61722085	0.56612783	0.25146930	0.51571998
-0.2	0.67507317	0.59073843	0.24065784	0.53445633
-0.1	0.73533174	0.61424595	0.22944194	0.55283118
0.0	0.79788455	0.63661976	0.21801360	0.57079635
0.1	0.86261747	0.65784714	0.20653869	0.58831060
0.2	0.92941584	0.67793064	0.19515698	0.60533952
0.3	0.99816596	0.69688550	0.18398316	0.62185535
0.4	1.06875616	0.71473727	0.17310866	0.63783660
0.5	1.14107776	0.73151958	0.16260389	0.65326772
0.6	1.21502575	0.74727213	0.15252072	0.66813857
0.7	1.29049933	0.76203899	0.14289499	0.68244395
0.8	1.36740226	0.77586713	0.13374897	0.69618302
0.9	1.44564309	0.78880516	0.12509372	0.70935885
1.0	1.52513526	0.80090231	0.11693114	0.72197781
1.1	1.60579716	0.81220764	0.10925589	0.73404915
1.2	1.68755201	0.82276938	0.10205698	0.74558447
1.3	1.77032782	0.83263442	0.09531919	0.75659733
1.4	1.85405719	0.84184799	0.08902427	0.76710285
1.5	1.93867715	0.85045337	0.08315183	0.77711733
1.6	2.02412897	0.85849173	0.07768024	0.78665794
1.7	2.11035791	0.86600205	0.07258716	0.79574247
1.8	2.19731301	0.87302105	0.06785013	0.80438905
1.9	2.28494690	0.87958322	0.06344689	0.81261597
2.0	2.37321552	0.88572085	0.05935572	0.82044149
2.1	2.46207793	0.89146409	0.05555560	0.82788372
2.2	2.55149613	0.89684103	0.05202644	0.83496044
2.3	2.64143481	0.90187780	0.04874907	0.84168909
2.4	2.73186118	0.90659866	0.04570541	0.84808660
2.5	2.82274478	0.91102613	0.04287841	0.85416942
2.6	2.91405733	0.91518107	0.04025210	0.85995341
2.7	3.00577256	0.91908278	0.03781155	0.86545383
2.8	3.09786606	0.92274915	0.03554287	0.87068533
2.9	3.19031512	0.92619670	0.03343313	0.87566192
3.0	3.28309863	0.92944073	0.03147034	0.88039702
3.1	3.37619697	0.93249536	0.02964341	0.88490336
3.2	3.46959184	0.93537366	0.02794207	0.88919312
3.3	3.56326624	0.93808771	0.02635684	0.89327782
3.4	3.65720430	0.94064867	0.02487896	0.89716842
3.5	3.75139124	0.94306688	0.02350036	0.90087531
3.6	3.84581326	0.94535189	0.02221360	0.90440831
3.7	3.94045749	0.94751253	0.02101182	0.90777674
3.8	4.03531191	0.94955697	0.01988872	0.91098938
3.9	4.13036529	0.95149280	0.01883846	0.91405457
4.0	4.22560711	0.95332702	0.01785569	0.91698014
4.1	4.32102755	0.95506614	0.01693549	0.91977351
4.2	4.41661740	0.95671618	0.01607329	0.92244168
4.3	4.51236803	0.95828273	0.01526493	0.92499126
4.4	4.60827137	0.95977097	0.01450653	0.92742846
4.5	4.70431981	0.96118573	0.01379455	0.92975916
4.6	4.80050624	0.96253148	0.01312570	0.93198891
4.7	4.89682398	0.96381238	0.01249697	0.93412294
4.8	4.99326672	0.96503230	0.01190556	0.93616616
4.9	5.08982856	0.96619484	0.01134891	0.93812321

(continued)

VALUES OF INVERSE MILL'S RATIO FUNCTION

T	B (B(t))	B1 (B'(t))	B11 (B''(t))	FCN (F(T))
5.0	5.18650393	0.96730336	0.01082462	0.93999851
5.1	5.28328758	0.96836098	0.01033052	0.94179614
5.2	5.38017457	0.96937062	0.00986455	0.94352011
5.3	5.47716024	0.97033501	0.00942486	0.94517398
5.4	5.57424019	0.97125667	0.00900972	0.94676113
5.5	5.67141027	0.97213798	0.00861749	0.94828505
5.6	5.76866656	0.97298115	0.00824665	0.94974908
5.7	5.86600534	0.97378824	0.00789570	0.95115678
5.8	5.96342312	0.97456122	0.00756361	0.95250950
5.9	6.06091657	0.97530190	0.00724914	0.95381016
6.0	6.15848256	0.97601208	0.00695171	0.95505773
6.1	6.25611809	0.97669323	0.00666958	0.95625917
6.2	6.35382020	0.97734591	0.00639554	0.95745821
6.3	6.45158633	0.97797231	0.00613370	0.95862519
6.4	6.54941402	0.97857428	0.00588682	0.95973797
6.5	6.64730222	0.97916241	0.00571901	0.96034873
6.6	6.74524511	0.97971387	0.00546372	0.96160388
6.7	6.84323899	0.98021862	0.00503683	0.96412659
6.8	6.94129806	0.98079198	0.00525541	0.96207781
6.9	7.03939148	0.98123121	0.00465444	0.96597018