

Analytical Descriptions of Cosmic Ray Transport via the Fokker-Planck Equation

by

Brock Klippenstein

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Department of Physics & Astronomy
University of Manitoba
Winnipeg

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Abstract

When energetic particles such as cosmic rays travel through space, their motion is driven by large-scale magnetic fields. In addition, these magnetic fields are highly turbulent which leads to the particles being scattered in a nearly random manner. Understanding these interactions is an important fundamental problem, but also has a wide range of applications including diffusive shock acceleration and improving our understanding of the magnetic turbulence which is partially present in fusion reactors. Due to the desire for analytical results and decent computational efficiency, this thesis studies this motion based on a statistical approach. In particular, the Fokker-Planck equation is solved to obtain the distribution function in phase-space.

As there is no exact solution to the Fokker-Planck equation, the results of this thesis consist of a nearly pure analytical solution of a simplified version of the equation. Furthermore, a semi-analytical method is developed which allows for a near-perfect approximation which is still significantly quicker than standard numerical solvers. The benefit of this method is that it is flexible in the sense that it can handle more complex versions of the Fokker-Planck equation. Lastly, one of the most important quantities when dealing with cosmic ray scattering is referred to as the diffusion coefficient. In the general case of a spatially varying mean magnetic field, obtaining the exact expression has been controversial in the past. For the final result of this thesis, an exact form of the diffusion coefficient is derived using the most natural definition.

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Arguably my most memorable moment occurred a year into the program when FGS said that my two 4000 level courses wouldn't count and so I needed to take two more courses. However, the support I received to resolve this issue and make it so I didn't need to take more courses couldn't have been any better. Some of this support included: The whole 314 people coming to my desk to watch me reply to the first email from FGS. Susan arguing with FGS for the whole day. Janette reading through FGS's guidelines to try to find something that would help me. Gerald, who couldn't have possibly been any

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1

Introduction

More than a century ago, it was well documented that there was an abundance of ionizing radiation present at ground level. It was originally believed that this high-energy radiation came from radioactive elements in the Earth's crust. In 1909, Theodor Wulf measured the rate of this radiation at the top of the Eiffel tower. Although less of this radiation was detected there than at ground level, it was still higher than expected. This brought into question the relationship between radiation and altitude. Victor Hess sought to resolve this issue with significantly higher altitudes in 1911 and 1912 when they ascended up to 5 km in a balloon. The findings showed that at a high enough altitude, there is more radiation than there is on the ground; thus concluding that the origin is somewhere in space. Additionally, through the production of the same experiments at night, it was found that the source is something other than the Sun. This work led to Hess jointly winning the Physics Nobel Prize in 1936. The term *cosmic ray* was later coined since the radiation had a cosmic origin, and it was initially believed that the radiation was mostly γ -rays. Although later determined that cosmic rays consist of roughly more than 80% protons, 10% helium nuclei, 1% electrons, and 1% γ -rays, [Pérez-Peraza, 2013], the name cosmic rays stuck. Observations of a strong correlation between solar flares and enhanced fluxes of high-energy particles detected at Earth led [Forbush, 1946] to identify the Sun as an additional source of energetic particles, which are now referred to as *solar energetic particles*. Even to this day, it is not known how these particles obtain their high energy. The prominent theory, known as *diffusive shock*

acceleration, states that particles travel back and forth through a shock front multiple times to obtain their high energies, as first introduced by [Krymsky, 1977, Axford et al., 1977, Bell, 1978b, Bell, 1978a, Blandford and Ostriker, 1978]. The aim of this thesis is to study the motion of these energetic particles. The remainder of this chapter outlines the physics which is relevant to this process.

In addition to the solar energetic particles, [Biermann, 1951, Biermann, 1979] found that the Sun produces particles that have lower energies. It was argued that through observing the tails of comets, there must be particles flowing radially outward from the Sun, which is now referred to as the solar wind. This theory was later refined by [Parker, 1958] who calculated that the particles travel in a spiral. It then immediately followed that the magnetic field was in the shape of an Archimedean spiral, which is now commonly referred to as the Parker spiral. The first observational evidence supporting the existence of the Parker spiral was provided by [Ness and Wilcox, 1964].

The findings of [Ness and Wilcox, 1964] also indicated that the Parker spiral alone does not adequately describe the interplanetary magnetic field, as significant fluctuations were also observed. Further support for the fluctuations was later found by [Hedgcock, 1975, Burlaga and Turner, 1976]. These findings motivate the use of referring to the interplanetary magnetic field as the superposition of a background mean component and a turbulent component. As discussed in [Ghanbari and Florinski, 2023], there are two main generators of the turbulence. The first is a result of the convection process within the outer layers of the Sun [Brun et al., 2004]. The other mechanism which creates turbulence is solar wind shearing. Theoretical evidence of this is given by [Roberts et al., 1992], where simulations were employed to show that when a fast stream in the solar wind overtakes a slower stream, a shock is produced which results in fluctuations of the physical quantities. This theory was later observationally supported by [Crooker et al., 1999].

Although the magnetic turbulence cannot be described exactly, theories have been proposed in the past that provide adequate approximations. One of the simplest models of turbulence is known as the *slab model* and dates back to [Jokipii, 1966]. One of the defining features of the slab model is that it assumes spatial dependence only along the direction of the mean magnetic field. Another well-known model is referred to as the *2D turbulence model* [Matthaeus et al., 1990]. This model is opposite to

the slab model in the sense that it depends on the perpendicular position to the mean magnetic field. Together, the slab and 2D turbulence models were employed in [Bieber et al., 1994] to present agreement with theoretical and observational results of solar energetic particles. Further details of these models, along with generalizations, can be found in [Arendt, 2025].

Thanks to the turbulent component of the magnetic field, the trajectories of electrically charged particles become chaotic and cannot be exactly described. Understanding how magnetic turbulence affects the scattering of charged particles in the solar system has been an active field of research dating back to [Meyer et al., 1956], and continues to be extensively studied to this day. When describing energetic charged particle motion through turbulent magnetic fields theoretically, there are two methods which are commonly used.

- The first method begins with determining the turbulence. One way this can be done is by solving the magnetohydrodynamic equations. The downside of this is that the calculations are very computationally expensive. Another option is to use statistical information about the turbulence, such as the power spectrum, in combination with random numbers. Once the magnetic field has been created, the equation of motion can be solved to obtain the resulting particle trajectory. In order to obtain meaningful and smooth results, this method must be performed on a significant number of particles. This method is employed in, for example, [Qin et al., 2002, Reville et al., 2008, Tautz, 2010].
- An alternative approach is to work with the distribution function. This function can be obtained by solving a transport equation, such as the *Fokker-Planck equation*, which can be derived from the relativistic Vlasov equation [Vlasov, 1938]. Within this method, information about the turbulence is contained in the *Fokker-Planck coefficients* which are often approximated by relatively simple functions. While these transport equations are in general still too complicated to analytically solve exactly, there are many analytical results that can be derived. This approach is considered in [Kourakis, 2000, Kourakis and Grecos, 2003].

While this thesis investigates the scattering effect from a theoretical standpoint, it is important to

note that theoretical insights are equally valuable from an observational perspective. This is because studying the scattering effect of solar energetic particles observationally generally follows a four-step approach (see [Palmer, 1982, Bieber et al., 1994] and the references therein for more detail related to each step):

1. Measure the particles of a chosen energy range. While it is possible to obtain the data on Earth, it is far more common to use space probes such as Helios I and II, or IMP 4 and 5, see for example [Ma Sung and Earl, 1978, Bieber et al., 1980, Beeck and Wibberenz, 1986]. At a minimum, the probes are able to count how many particles are detected per unit time. It is also common for the detectors to spin, thus allowing for the direction of the motion to be determined as well.
2. Employ a theoretical model. The model used is up to the choice of the observer, although one model that is employed extensively is the focused Fokker-Planck equation. Alternatively, it is possible to employ a more general model by simply assuming the particle motion is diffusive [Christon, 1981]. However, the downside here is that multiple space probes are required rather than a single probe if the Fokker-Planck equation is employed.
3. Select at least one profile, such as the intensity profile, and perform a comparison between the profiles derived from observational data and those predicted by the theoretical model. As shown in [Palmer, 1982], results are obtainable if only the intensity profile is used; however, the results are far more reliable if the anisotropy profile is also considered.
4. Determine which set of parameters result in the best agreement between theory and observations. These parameters represent the inferred physical quantities. Although the desired parameters vary by author, the *parallel mean free path* is the most common quantity. One example that emphasizes the importance of this quantity is the *Palmer consensus* [Palmer, 1982], which relates this quantity to particle energy.

Studying how charged particles travel through magnetic turbulence has a wide range of applications. As mentioned above, the prominent theory describing how cosmic rays obtain their high energy is referred

to as diffusive shock acceleration. This process can be modeled using the *Parker transport equation*, which is dependent on the *diffusion coefficient*, see [Zank et al., 2000b, Ferrand et al., 2014]. Determining the diffusion coefficient is a central problem within energetic particle transport theory. Another effect that employs the Parker transport equation is known as *solar modulation*, see [Shen and Qin, 2018, Ngobeni et al., 2022], which explains how the cycle of solar activity affects the intensity of cosmic rays which reach the inner solar system.

Additionally, studying particle transport enhances our understanding of the mechanisms by which particles are ejected from the Sun. This is because measurements of solar energetic particles are inherently influenced by their ejection process. An expression for the particle ejection mechanism, including parameters such as the so-called coronal diffusion time, was derived by [Reid, 1964]. This expression was later applied by, for example, [Bieber et al., 1980, Lang et al., 2024], who used the observational approach outlined above to determine the relevant parameters.

Furthermore, discrepancies between theory and observations of particle transport, which were brought forth by the *Palmer consensus* [Palmer, 1982], have driven significant improvements in our understanding of magnetic turbulence, [Matthaeus et al., 1990, Bieber et al., 1994]. Moreover, certain properties of the magnetic turbulence within the interplanetary magnetic field are also present in laboratory experiments [Robinson and Rusbridge, 1971, Boffetta and Ecke, 2012].

The structure of this thesis is outlined in the following.

In Chapter 2, we introduce a statistical approach commonly used to describe energetic particle transport known as the Fokker-Planck equation. The exact structure of the Fokker-Planck equation is dependent on several assumptions which lead to different terms and different forms for some coefficients. Moreover, the solution of the Fokker-Planck equation naturally leads to the computation of expectation values. Since subsequent chapters will contain many of the same expectation values, this chapter will contain broader information concerning these values.

As no version of the space-dependent Fokker-Planck equation has ever been solved exactly, Chapter 3 attempts to derive an exact analytical solution of the simplest form. The solution is given in terms of Airy functions, and constants which follow from the initial and boundary conditions. The solution is

then employed to obtain analytical expressions for certain expectation values.

Chapter 4 derives a method known as the subspace method to solve the standard form of the focused Fokker-Planck equation with isotropic scattering. The method is first applied with a low dimension which allows for decent and simple analytical results. Then, a higher dimension is used to solve the Fokker-Planck equation significantly faster than a purely numerical method, while maintaining near-perfect accuracy. Next, Chapter 5 is similar, except the conservative form is explored.

When the mean magnetic field is constant, there are different identities which lead to simpler calculations when analytically computing an important quantity known as the parallel diffusion coefficient. The assumption that these identities held when the mean field was not constant led to serious controversy, along with no results derived from the most natural definition. It is thus the goal of Chapter 6 to provide the exact result for the parallel diffusion coefficient derived from this natural definition.

A final summary of the thesis is given in Chapter 7. We also outline two avenues for future work which naturally follow from the work presented in this thesis.

2

The Fokker-Planck Equation

As discussed in the previous chapter, there are two options when theoretically studying the scattering of particles in turbulent magnetic fields. In this thesis, we adopt the statistical approach, with an emphasis on obtaining analytical results. This chapter begins with a discussion of the Fokker-Planck equation. Furthermore, since the following chapters share an overlapping interest in expectation values, we introduce these concepts at the end of the present chapter.

2.1 Formulation of the Fokker-Planck Equation

The general form of the Fokker-Planck equation is lengthy and far too complicated to obtain analytical results. This motivates the simplification of the general equation with reasonable assumptions and the negligence of insignificant terms. These assumptions are justified in the next subsection, then the subsequent subsections outline the main ideas behind the derivation of the Fokker-Planck equation, followed by a general discussion.

2.1.1 Main Assumptions

The derivation of the Fokker-Planck equation that will be seen in this thesis depends on five main assumptions.

1. The first important assumption is the diffusion approximation, which assumes that the particle motion is diffusive at late times. This assumption stems from observations that the particles have a very nearly isotropic pitch-angle distribution. For solar energetic particles, these observations can be found in [Palmer, 1982, Bieber et al., 1994] and the references therein, and [Schlickeiser, 2002]. As for cosmic rays originating outside the solar system, validation of the diffusion approximation is presented in [Zhang et al., 2023].
2. Another assumption is that the electric field is negligible. To understand why, we decompose the electric and magnetic field as

$$\vec{F} = \vec{F}_0 + \delta\vec{F} \quad (2.1)$$

where $\vec{F}_0$ represents the mean field and $\delta\vec{F}$ is the turbulent component. In the following, we justify the negligence of both of these components for the electric field.

For the mean field, we follow the argument described in [Landau et al., 2013, Borovsky, 2016]. We begin with Ohm's law which states

$$\vec{J} = \sigma \vec{E}'_0 \quad (2.2)$$

where $\vec{J}$ is the current, σ is the conductivity, $\vec{E}'_0$ is the electric field, and the prime denotes that we are in the rest frame of the plasma¹. We also use the well-known transformation between the rest frame of the plasma and the lab frame:

$$\vec{E}'_0 = \vec{E}_0 + \frac{\vec{v}}{c} \times \vec{B}_0, \quad (2.3)$$

where we have assumed that $\vec{v}$, the velocity of the plasma, is non-relativistic. The combination of these two identities leads to

$$\vec{J} = \sigma \left[\vec{E}_0 + \frac{\vec{v}}{c} \times \vec{B}_0 \right]. \quad (2.4)$$

¹In the setting of the solar system, this plasma is the solar wind.

As plasmas are highly conductive, the limiting value $\sigma \rightarrow \infty$ implies

$$\vec{E}_0 = -\frac{\vec{v}}{c} \times \vec{B}_0. \quad (2.5)$$

Analysis of the solar wind by [Borovsky, 2016] shows that the speed of the solar wind is at most $10^3 km/s$, and thus $\vec{E}_0 \approx 0$.

As for the turbulence, we summarize the reasoning given in [Schlickeiser, 2002] which allows us to neglect the fluctuations in the electric field as well. The key observation here is that magnetic fluctuations are primarily in the low-frequency range [Matthaeus and Goldstein, 1982]. The resulting dispersion relation in combination with Faraday's law results in

$$\delta E = \frac{v_A}{c} \delta B \quad (2.6)$$

where v_A is the Alfén speed, and δE and δB denote the magnitude of the fluctuation vectors. Since the Alfén speed is significantly smaller than the speed of light, we can also reasonably neglect the turbulent electric field.

3. As found in [Marsch, 2006], the gyroradius of high energy particles is significantly smaller than the collisional mean free path. In other words, particles are able to travel large distances before collisions become noticeable. Consequently, it is assumed that the motion discussed here is collisionless.
4. It is assumed that the spatial variation of the mean magnetic field is weak. In the setting of the solar wind, this is justified in, for example, [Effenberger and Litvinenko, 2014].
5. We also assume relativistic speeds since we deal with energetic particles.

2.1.2 Brief Derivation

In the following, we outline the ideas behind the derivation of the focused one-dimensional Fokker-Planck equation. The exact derivation can be found in [Roelof, 1969, Schlickeiser, 2002].

The first step is to choose a coordinate system in which the mean magnetic field points in the z -direction. We then define μ to be the pitch-angle cosine, or $\mu = v_z/v$ where v_z is the z -component of the velocity and v is the speed of the particle. Consequently, in one-spatial dimension, the distribution function is given as $f(z, \mu, t)$. The starting point is the *source-free Vlasov equation* [Vlasov, 1938] which states that

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} + \dot{v}_z \frac{\partial f}{\partial v_z} = 0 \quad (2.7)$$

where the dot indicates a time derivative. The first term quantifies the temporal evolution of the distribution function. Advection of the particle is represented by the second term. Lastly, the third term accounts for any forces acting on the particle, which in this instance is the force due to the magnetic field.

Alternatively, we would like to rewrite Eq. (2.7) in terms of μ . Using the relation $v_z = v\mu$ therein yields

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} + \dot{\mu} \frac{\partial f}{\partial \mu} = 0. \quad (2.8)$$

The next step is to decompose the distribution function according to

$$f = \langle f \rangle + \delta f \quad (2.9)$$

where $\langle f \rangle$ is the ensemble average and the term δf accounts for the fluctuations in the distribution function, which are assumed to be small. In the following, the aim is to derive a differential equation which solves for $\langle f \rangle$. In order to achieve this, the force term $\dot{\mu}$ is similarly decomposed as

$$\dot{\mu} = \langle \dot{\mu} \rangle + \delta \dot{\mu} \quad (2.10)$$

with $\langle \dot{\mu} \rangle$ arising from the mean magnetic field and $\delta \dot{\mu}$ from the turbulence. Using these decompositions, we derive two differential equations. The first is obtained by taking the ensemble average of Eq. (2.8) which yields

$$\frac{\partial \langle f \rangle}{\partial t} + v\mu \frac{\partial \langle f \rangle}{\partial z} + \langle \dot{\mu} \rangle \frac{\partial \langle f \rangle}{\partial \mu} + \left\langle \delta \dot{\mu} \frac{\partial \delta f}{\partial \mu} \right\rangle = 0. \quad (2.11)$$

For the second equation, placing the above decompositions into Eq. (2.8) and only keeping terms which are linear in the fluctuations results in

$$\frac{\partial \delta f}{\partial t} + v\mu \frac{\partial \delta f}{\partial z} + \delta \dot{\mu} \frac{\partial \langle f \rangle}{\partial \mu} = 0. \quad (2.12)$$

The remaining steps can be found in [Schlickeiser, 2002]. The main idea is to solve Eq. (2.12) for δf , which is given in terms of $\langle f \rangle$. Upon placing this result into Eq. (2.11), the Fokker-Planck equation is obtained:

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu} \left[D_{\mu\mu} \frac{\partial f}{\partial \mu} \right] - \frac{v}{2L(z)} (1 - \mu^2) \frac{\partial f}{\partial \mu} \quad (2.13)$$

where we have renamed $\langle f \rangle \rightarrow f$ for simpler notation. Moreover, $D_{\mu\mu}(\mu)$ is the pitch-angle scattering diffusion coefficient and $L(z)$ is the focusing parameter; more on both of these terms below. It should be noted that the crucial assumption used to simplify the term involving $D_{\mu\mu}$ is the diffusion approximation. Moreover, without the assumption of the electric field being negligible, Eq. (2.13) would contain additional terms such as a momentum diffusion term.

The solution of Eq. (2.13) has been compared with observations in numerous studies. For examples demonstrating agreement in the setting of solar energetic particles, see [Palmer, 1982, Bieber et al., 1994].

2.1.3 Description of the Terms

The first term on the right-hand side of Eq. (2.13) describes the pitch-angle scattering of the particle, where the diffusion coefficient depends on the turbulence. It is defined by

$$D_{\mu\mu}(\mu, t) = \int_0^t d\tau \langle \dot{\mu}(\tau) \dot{\mu}(0) \rangle \quad (2.14)$$

where again, $\langle \cdot \rangle$ denotes the ensemble average. Determining an accurate analytical form for this coefficient has been an active field of research dating back to [Jokipii, 1966]. It should be noted that a standard assumption is that the above expression quickly converges to a function which only depends

on μ , see [Qin and Shalchi, 2009]. That is, we can approximate

$$D_{\mu\mu}(\mu, t) \approx D_{\mu\mu}(\mu) = \lim_{t \rightarrow \infty} D_{\mu\mu}(\mu, t). \quad (2.15)$$

A relatively complicated form for this late time limit is also given in [Qin and Shalchi, 2009] which depends on the strength of the turbulence compared to the mean field. As direct measurements have shown that the turbulence is roughly the same strength as the mean field, [Hedgecock, 1975, Burlaga and Turner, 1976], the expression provided by [Qin and Shalchi, 2009] can be simplified to

$$D_{\mu\mu}(\mu) = D(1 - \mu^2) \quad (2.16)$$

for a constant D dependent on several factors such as the strength of the magnetic field. When this approximation is employed, the particles are said to undergo isotropic scattering. By solving a simplified version of Eq. (2.13), [Lasuik and Shalchi, 2017] demonstrated that the solution is not very dependent on the pitch-angle scattering diffusion coefficient. In particular, they compared the solution for isotropic scattering and for a constant $D_{\mu\mu}$, and found only minor differences between the two solutions. This motivates Chapter 3, where we show that the constant coefficient allows us to obtain an analytical solution in terms of Airy functions.

While other forms of the pitch-angle scattering diffusion coefficient can be found in the literature, a property that almost always holds is

$$D_{\mu\mu}(\mu = \pm 1) = 0. \quad (2.17)$$

This condition is required to show that Eq. (2.13) describes diffusive motion at late times, as shown in [Shalchi, 2020]. It should be noted that the above condition evidently does not hold if $D_{\mu\mu}$ is constant. However, the calculations done in [Shalchi, 2020] can be extended to the more general condition of

$$\left[D_{\mu\mu} \frac{\partial f}{\partial \mu} \right]_{\mu=\pm 1} = 0. \quad (2.18)$$

This implies that if the coefficient is constant, then a necessary condition is that the derivative of the

solution vanishes at the boundaries.

The last term of Eq. (2.13) is referred to as the focusing term, with $L(z)$ known as the focusing parameter. This term is induced by the spatial-variation of the mean magnetic field. The presence of the focusing term induces a directional bias in the pitch-angle distribution. For instance, in the solar wind, this focusing effect guides particles away from the Sun.

The focusing parameter is given by

$$L(z) = - \left(\frac{1}{B_{0z}} \frac{dB_{0z}}{dz} \right)^{-1} \quad (2.19)$$

where B_{0z} is the z -component of the mean magnetic field. As the interplanetary magnetic field takes the shape of an Archimedean spiral, now known as the Parker spiral [Parker, 1958], the exact form of $L(z)$ is complicated. However, [Ng and Wong, 1979] show that in terms of the solution of Eq. (2.13), the simplest approximation of $L(z)$ being a constant is sufficient. In this instance, Eq. (2.19) is easily solved to obtain

$$B_{0z}(z) \propto e^{-z/L}. \quad (2.20)$$

Throughout this thesis, the focusing parameter will always be taken to be a constant.

2.1.4 Conservative Form

The form of the Fokker-Planck equation given by Eq. (2.13) is hereafter referred to as the *standard form* of the Fokker-Planck equation. It can be verified that for certain expressions of the pitch-angle scattering diffusion coefficient, such as for isotropic scattering, the solution of Eq. (2.13) becomes independent of μ at late times. Another property concerns the temporal evolution of the total number of particles. We define

$$N = \frac{1}{2} \int_{-\infty}^{\infty} dz \int_{-1}^1 d\mu f \quad (2.21)$$

which for one particle statistics, should always be one. However, this is not the case. To see this, we begin with the time derivative of the above expression:

$$\frac{dN}{dt} = \frac{1}{2} \int_{-\infty}^{\infty} dz \int_{-1}^1 d\mu \left[-v\mu \frac{\partial f}{\partial z} + \frac{\partial}{\partial \mu} \left(D_{\mu\mu} \frac{\partial f}{\partial \mu} \right) - \frac{v}{2L} (1 - \mu^2) \frac{\partial f}{\partial \mu} \right] \quad (2.22)$$

where we used Eq. (2.13) therein. It is easy to see that the first term is zero since the z -integral is easy to evaluate, and we assume the distribution function vanishes as $z \rightarrow \pm\infty$. Likewise, the second term vanishes because of Eq. (2.18). Next, performing integration by parts on the last term leads to

$$\frac{dN}{dt} = -\frac{v}{2L} \int_{-\infty}^{\infty} dz \int_{-1}^1 d\mu \mu f. \quad (2.23)$$

Since the right-hand side is generally non zero, we conclude that N varies with time.

The absence of a conservation law for the number of particles motivates the use of the *conservative form* of the Fokker-Planck equation. The relation given by Eq. (2.20) along with conservation of magnetic flux within a perfectly conducting plasma show that the surface area of a flux tube must grow proportionally to

$$A(z) \propto e^{z/L}. \quad (2.24)$$

This motivates the transformation

$$\tilde{f}(z, \mu, t) = f(z, \mu, t) e^{z/L}. \quad (2.25)$$

Placing this transformation into Eq. (2.13) along with straightforward calculations leads to

$$\frac{\partial \tilde{f}}{\partial t} + v\mu \frac{\partial \tilde{f}}{\partial z} = \frac{\partial}{\partial \mu} \left[D_{\mu\mu} \frac{\partial \tilde{f}}{\partial \mu} \right] - \frac{v}{2L} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \tilde{f} \right]. \quad (2.26)$$

Upon comparing the conservative form with the standard form, we see that the only difference lies with the last term: in conservative form, the derivative acts on $(1 - \mu^2)\tilde{f}$ rather than solely on f . This change allows one to perform the steps above, and show that the number of particles is constant within

the conservative form. However, this form does not result in a pitch-angle isotropization process, see [Beeck and Wibberenz, 1986, Shalchi and Danos, 2013].

For the remainder of the thesis, whether we use the standard or the conservative form will be made clear in text. Hence, when dealing with the conservative form, $\tilde{f}$ will be denoted simply as f .

2.1.5 Fourier Transform

Throughout this thesis, the first step towards solving the Fokker-Planck equation is to always employ the Fourier transform

$$f(z, \mu, t) = \int_{-\infty}^{\infty} dk F_k(\mu, t) e^{ikz}, \quad (2.27)$$

with an inverse given as

$$F_k(\mu, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz f(z, \mu, t) e^{-ikz}. \quad (2.28)$$

In Fourier space, we have

$$\frac{\partial F_k}{\partial t} + ivk\mu F_k = \frac{\partial}{\partial \mu} \left[D_{\mu\mu} \frac{\partial F_k}{\partial \mu} \right] - \frac{v}{2L} (1 - \mu^2) \frac{\partial F_k}{\partial \mu} \quad (2.29)$$

for the standard form, and

$$\frac{\partial F_k}{\partial t} + ivk\mu F_k = \frac{\partial}{\partial \mu} \left[D_{\mu\mu} \frac{\partial F_k}{\partial \mu} \right] - \frac{v}{2L} \frac{\partial}{\partial \mu} [(1 - \mu^2) F_k] \quad (2.30)$$

for the conservative form. The benefit of the Fourier transform is that F_k only depends on μ and t , which clearly renders solving the equation easier. Additionally, we discuss certain expectation values in Section 2.2 and show that the corresponding expressions have the potential of being elegant when written in terms of the solution in Fourier space.

Another important function to consider is the spatially-integrated solution defined as

$$F(\mu, t) = \int_{-\infty}^{\infty} dz f(z, \mu, t) \quad (2.31)$$

which can be seen to differ from F_0 by a factor of 2π . As we establish in Section 2.2, the simplified forms of Eqs. (2.29) and (2.30) for $k = 0$ are sufficient when dealing with expectation values that are independent of z .

2.1.6 Initial Condition

An initial condition must also be specified. We suppose the initial pitch-angle cosine is μ_0 , and we choose a coordinate system in which the particle begins at $z = 0$. The corresponding distribution is thus

$$f(z, \mu, t = 0) = 2\delta(z)\delta(\mu - \mu_0) \quad (2.32)$$

with the Dirac delta $\delta(x)$. The factor of 2 is included so that, using the normalization condition given in Eq. (2.21), we obtain

$$N(t = 0) = \int_{-\infty}^{\infty} dz \int_{-1}^1 d\mu \delta(z)\delta(\mu - \mu_0) = 1 \quad (2.33)$$

which follows directly from the defining property of the Dirac delta, see [Zwillinger, 2012].

To determine the initial condition in Fourier space, we place Eq. (2.32) into Eq. (2.28) to derive

$$F_k(\mu, t = 0) = \frac{1}{\pi}\delta(\mu - \mu_0). \quad (2.34)$$

The initial condition described by Eq. (2.32), and in particular the μ -dependence, is the simplest choice. When analyzing observational data, the initial condition is often given in terms of a Gaussian distribution, or a similar distribution, see [Palmer, 1982, Bieber et al., 1994].

2.2 Important Quantities

We define the expectation value of a quantity $A(z, \mu, t)$ as

$$\langle A \rangle(t) = \frac{1}{2} \int_{-1}^1 d\mu \int_{-\infty}^{\infty} dz A(z, \mu, t) f(z, \mu, t). \quad (2.35)$$

This definition results in an expression which is dependent on μ_0 . In some instances, we want the μ_0 -average as well, in which case we define the expectation value as

$$\langle A \rangle(t) = \frac{1}{4} \int_{-1}^1 d\mu_0 \int_{-1}^1 d\mu \int_{-\infty}^{\infty} dz A(z, \mu, t) f(z, \mu, t). \quad (2.36)$$

There are several expectation values which are useful within certain theories of particle transport. As some of the expectation values appear in multiple of the upcoming chapters, we elect to introduce them here, as well as perform preliminary calculations to avoid repeating them later in the thesis.

2.2.1 Characteristic Function

We define the characteristic function to be

$$\langle e^{ikz} \rangle = \frac{1}{4} \int_{-1}^1 d\mu_0 \int_{-1}^1 d\mu \int_{-\infty}^{\infty} dz f e^{ikz}. \quad (2.37)$$

This quantity is important because the theory of magnetic turbulence makes use of what is referred to as *Corrsin's approximation*, see [Corrsin, 1959, Tautz and Shalchi, 2010]. The physical interpretation of Corrsin's approximation is that the exponential function due to the Fourier transform and the magnetic field components are uncorrelated.

Since the distribution function is real, it is easy to see that $\langle e^{-ikz} \rangle$ is the complex conjugate of $\langle e^{ikz} \rangle$. Hence, we elect to consider this alternate value which, when combined with Eq. (2.28), is given as

$$\langle e^{-ikz} \rangle = \frac{\pi}{2} \int_{-1}^1 d\mu_0 \int_{-1}^1 d\mu F_k. \quad (2.38)$$

From this equality, we see that the characteristic function is simply a scalar multiple of the μ_0 and μ -average of F_k .

2.2.2 Moments

Moments of the form $\langle z^n \mu^m \rangle$, for integers n and m , can be used to write an exact solution to the Fokker-Planck equation [Malkov, 2017]. Additionally, certain moments such as $\langle z^n \rangle$ for $n = 1, 2$ are used to define a diffusion coefficient, which we explain further at the end of this chapter. While the general case allows for arbitrary n and m , in this thesis we restrict our attention to moments where either $n = 0$ or $m = 0$.

2.2.2.1 The Moments $\langle \mu^n \rangle$

We are interested in the μ_0 -dependent moment and thus we define

$$\langle \mu^n \rangle = \frac{1}{2} \int_{-1}^1 d\mu \int_{-\infty}^{\infty} dz \mu^n f. \quad (2.39)$$

The z -integral can be recognized in Eq. (2.28) with $k = 0$. We therefore have

$$\langle \mu^n \rangle = \pi \int_{-1}^1 d\mu \mu^n F_0. \quad (2.40)$$

As mentioned above, this expression shows that in order to obtain the moments $\langle \mu^n \rangle$, it is sufficient to solve Eqs. (2.29) or (2.30) with $k = 0$.

2.2.2.2 The Moments $\langle z^n \rangle$

The moments $\langle z^n \rangle$ are given as

$$\langle z^n \rangle = \frac{1}{2} \int_{-1}^1 d\mu \int_{-\infty}^{\infty} dz z^n f. \quad (2.41)$$

Replacing f with the Fourier transform given by Eq. (2.27) yields

$$\langle z^n \rangle = \frac{1}{2} \int_{-1}^1 d\mu \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dz z^n F_k e^{ikz}. \quad (2.42)$$

Notice that

$$z^n e^{ikz} = (-i)^n \frac{\partial^n}{\partial k^n} e^{ikz}, \quad (2.43)$$

which implies

$$\langle z^n \rangle = (-i)^n \frac{1}{2} \int_{-1}^1 d\mu \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dk F_k \frac{\partial^n}{\partial k^n} e^{ikz}. \quad (2.44)$$

Next, we rewrite the k -integral using integration by parts to obtain

$$\int_{-\infty}^{\infty} dk F_k \frac{\partial^n}{\partial k^n} e^{ikz} = F_k \frac{\partial^{n-1}}{\partial k^{n-1}} e^{ikz} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dk \left[\frac{\partial}{\partial k} F_k \right] \left[\frac{\partial^{n-1}}{\partial k^{n-1}} e^{ikz} \right]. \quad (2.45)$$

By assuming reasonable decay of the solution in Fourier space, that is $F_{k \rightarrow \pm\infty} = 0$, this simplifies to

$$\int_{-\infty}^{\infty} dk F_k \frac{\partial^n}{\partial k^n} e^{ikz} = - \int_{-\infty}^{\infty} dk \left[\frac{\partial}{\partial k} F_k \right] \left[\frac{\partial^{n-1}}{\partial k^{n-1}} e^{ikz} \right]. \quad (2.46)$$

If we repeat this step $n - 1$ more times, we then derive

$$\int_{-\infty}^{\infty} dk F_k \frac{\partial^n}{\partial k^n} e^{ikz} = (-1)^n \int_{-\infty}^{\infty} dk e^{ikz} \frac{\partial^n}{\partial k^n} F_k. \quad (2.47)$$

We return to Eq. (2.44) and now find

$$\langle z^n \rangle = i^n \frac{1}{2} \int_{-1}^1 d\mu \int_{-\infty}^{\infty} dk \frac{\partial^n}{\partial k^n} F_k \int_{-\infty}^{\infty} dz e^{ikz}. \quad (2.48)$$

A standard definition for the Dirac delta is given in terms of the Fourier transform (see [Zwillinger, 2012])

$$2\pi\delta(a) = \int_{-\infty}^{\infty} dx e^{iax}. \quad (2.49)$$

We thus have

$$\langle z^n \rangle = i^n \pi \int_{-1}^1 d\mu \int_{-\infty}^{\infty} dk \delta(k) \frac{\partial^n}{\partial k^n} F_k. \quad (2.50)$$

Now thanks to the Dirac delta, the k -integral is straightforward, leaving us with

$$\langle z^n \rangle = i^n \pi \left[\frac{\partial^n}{\partial k^n} \int_{-1}^1 d\mu F_k \right]_{k=0} \quad (2.51)$$

which is an expression for the μ_0 -dependent moment $\langle z^n \rangle$.

2.2.2.3 Connections Between the Moments

As shown in [Malkov, 2017, Shalchi, 2017], an infinite system of differential equations can be derived for the moments $\langle z^n \mu^m \rangle$. For example,

$$\frac{d}{dt} \langle z \rangle = \frac{1}{2} \int_{-1}^1 d\mu \int_{-\infty}^{\infty} dz z \frac{\partial f}{\partial t}. \quad (2.52)$$

If we use the conservative form of the Fokker-Planck equation given in Eq. (2.26), then it is easy to see that

$$\frac{d}{dt} \langle z \rangle = -\frac{1}{2} \int_{-1}^1 d\mu \int_{-\infty}^{\infty} dz z v \mu \frac{\partial f}{\partial z}. \quad (2.53)$$

By integrating by parts and assuming that f vanishes at $z = \pm\infty$,

$$\int_{-\infty}^{\infty} dz z v \mu \frac{\partial f}{\partial z} = - \int_{-\infty}^{\infty} dz v \mu f. \quad (2.54)$$

We therefore have

$$\frac{d}{dt} \langle z \rangle = v \langle \mu \rangle \quad (2.55)$$

corresponding to the classical expectation that the time derivative of the position is the velocity. If a similar calculation is performed with the standard form instead, then a similar result is obtained with an additional term.

2.2.3 Velocity Correlation Function

Next, we consider the velocity correlation function given by

$$V_{zz}(t) = v^2 \langle \mu \mu_0 \rangle = \frac{1}{4} v^2 \int_{-1}^1 d\mu_0 \int_{-1}^1 d\mu \int_{-\infty}^{\infty} dz \mu_0 \mu f. \quad (2.56)$$

We discuss why this quantity is of interest in more detail below; however, it is briefly mentioned here that the velocity correlation function can be used to compute a diffusion coefficient. The above expression can be rewritten as

$$V_{zz}(t) = \frac{1}{2} v^2 \int_{-1}^1 d\mu_0 \mu_0 \left[\frac{1}{2} \int_{-1}^1 d\mu \int_{-\infty}^{\infty} dz \mu f \right]. \quad (2.57)$$

By comparing the term in brackets with Eq. (2.39), we see

$$V_{zz}(t) = \frac{\pi}{2} v^2 \int_{-1}^1 d\mu_0 \mu_0 \int_{-1}^1 d\mu \mu F_0 \quad (2.58)$$

corresponding to the spatially-integrated solution averaged over μ_0 and μ .

2.2.4 Velocity-Position Correlation Function

The final expectation value that is considered here is the velocity-position correlation function defined by

$$\langle \mu_0 \mu e^{-ikz} \rangle = \frac{1}{4} \int_{-1}^1 d\mu_0 \int_{-1}^1 d\mu \int_{-\infty}^{\infty} dz \mu_0 \mu f e^{-ikz}. \quad (2.59)$$

This quantity is of particular interest within the theory of perpendicular diffusion as it is used to calculate a diffusion coefficient [Shalchi, 2010]. Simplifying the given expression is similar to that of the characteristic function. The z -integral is again found in Eq. (2.28) implying

$$\langle \mu_0 \mu e^{-ikz} \rangle = \frac{\pi}{2} \int_{-1}^1 d\mu_0 \int_{-1}^1 d\mu \mu_0 \mu F_k \quad (2.60)$$

which is a scalar multiple of the μ_0 and μ -average of $\mu_0 \mu F_k$.

2.2.5 Parallel Diffusion Coefficient

One of the most important quantities in cosmic ray transport theory is the *parallel diffusion coefficient* $\kappa_{\parallel}$. In this section, we discuss its significance and how it is defined. The parallel diffusion coefficient can be defined in several ways, which all become equivalent when the focusing effect is neglected, as assumed throughout this section. We will revisit this discussion in Chapter 6, where the inclusion of focusing is addressed.

Provided that the particle undergoes a pitch-angle isotropization process, the parallel diffusion coefficient completely describes the solution of the Fokker-Planck equation in the late time limit. It is shown in [Jokipii, 1966, Earl, 1974, Shalchi, 2020] that for late times,

$$\frac{\partial f}{\partial t} = \kappa_{\parallel} \frac{\partial^2 f}{\partial z^2} \quad (2.61)$$

which corresponds to a standard diffusion equation. Examples for when pitch-angle isotropization occurs are when $D_{\mu\mu} = D(1 - \mu^2)$, see [Shalchi, 2024]. It also happens when $D_{\mu\mu} = \text{const}$, which we show in Chapter 3.

It is also briefly mentioned that the parallel diffusion coefficient is closely related to the *parallel mean free path*, $\lambda_{\parallel}$, which quantifies how far a particle is expected to travel before experiencing significant scattering due to the turbulence. The relation between these two quantities is given as

$$\lambda_{\parallel} = 3 \frac{\kappa_{\parallel}}{v}. \quad (2.62)$$

Observational work is most often aimed at determining the parallel mean free path. In particular, the *Palmer consensus*, see [Palmer, 1982, Bieber et al., 1994], investigates the relationship between particle energy and parallel mean free path.

The most natural definition of the parallel diffusion coefficient is given in terms of the mean square displacement

$$\kappa_{\parallel}^{msd}(t) = \frac{1}{2} \frac{d}{dt} [\langle z^2 \rangle - \langle z \rangle^2]. \quad (2.63)$$

While this definition leads to a time-dependent result, here we only consider the late time limit

$$\kappa_{\parallel}^{msd} = \frac{1}{2} \lim_{t \rightarrow \infty} \frac{d}{dt} [\langle z^2 \rangle - \langle z \rangle^2]. \quad (2.64)$$

Another definition, see [Taylor, 1922, Green, 1951, Kubo, 1957], is

$$\kappa_{\parallel}^{TGK} = \int_0^{\infty} d\tau V_{zz}(\tau) \quad (2.65)$$

with the velocity correlation function V_{zz} given in Eq. (2.58). To proceed, we assume that the scattering is stationary. That is, the correlation function $\langle \mu(t_1) \mu(t_2) \rangle$ only depends on the absolute time difference between t_1 and t_2 . Then, the Taylor-Green-Kubo formula states that

$$\kappa_{\parallel}^{msd} = \kappa_{\parallel}^{TGK}. \quad (2.66)$$

The final definition that we consider here is given as

$$\kappa_{\parallel}^{Earl} = \frac{v^2}{8} \int_{-1}^1 d\mu \frac{(1 - \mu^2)^2}{D_{\mu\mu}}. \quad (2.67)$$

With the assumption that the motion becomes purely diffusive at late times, it is shown in [Earl, 1974] that this expression is equivalent to the other two, and thus we have

$$\kappa_{\parallel 0} = \kappa_{\parallel}^{msd} = \kappa_{\parallel}^{TGK} = \kappa_{\parallel}^{Earl}. \quad (2.68)$$

where we have introduced the simpler notation $\kappa_{\parallel 0}$ to denote the parallel diffusion coefficient when focusing is neglected.

Of these three expressions, it is evident that $\kappa_{\parallel}^{Earl}$ is the simplest to compute, provided we use a simple form of the pitch-angle scattering diffusion coefficient. For example, if we have isotropic scattering, that

is, $D_{\mu\mu} = D(1 - \mu^2)$, then evaluating the integral for $\kappa_{\parallel}^{Earl}$ is trivial, and we obtain

$$\kappa_{\parallel 0} = \frac{v^2}{6D}. \quad (2.69)$$

The relation given by Eq. (2.62) allows for the immediate computation of

$$\lambda_{\parallel 0} = \frac{v}{2D} \quad (2.70)$$

which corresponds to the parallel mean free path for isotropic scattering, and focusing neglected.

Throughout the rest of this thesis, $\kappa_{\parallel 0}$ and $\lambda_{\parallel 0}$ denote the parallel diffusion coefficient and parallel mean free path, respectively, without focusing. On the other hand, if focusing is present, then we use superscripts such as TGK to denote the method used to compute the quantity.

2.3 Numerical Solution of the Fokker-Planck Equation

Although this thesis focuses on techniques that improve the efficiency of solving the Fokker-Planck equation compared to standard numerical approaches, it is necessary to compare the proposed methods with the numerical method to assess their accuracy and efficiency. For the numerical solution, we aim to employ a simple yet reliable method. The simplest method is known as the *explicit Euler method*. The problem here is that the explicit Euler method has the potential to be seriously unreliable. Consequently, we elect to use the slightly more complicated, but more reliable, method known as the *implicit Euler method*. An excellent example illustrating the contrast between the reliability of the two methods is given by [Ascher and Greif, 2011].

It is important to note that there are more advanced methods for solving differential equations numerically [LeVeque, 2007, Ascher and Greif, 2011]. One example is known as the Crank-Nicolson method. While in general these higher order methods are better, it was found in [Shalchi, 2024] that the implicit Euler method is the best for the version of the Fokker-Planck equation that we consider here.

The implicit Euler method involves formulating a linear system of algebraic equations. The following

subsections detail the derivation of this system, its numerical solution, and the numerical treatment of the initial condition. The analysis focuses on the standard form of the focused Fokker–Planck equation, given by Eq. (2.13), though the ideas presented can be readily adapted to the conservative form if needed. Throughout this section, we employ concepts that are fundamental within the theory of numerical analysis, and can be found in introductory textbooks such as [Ascher and Greif, 2011, Burden and Faires, 2011].

2.3.1 Deriving the Governing Equation

The first step towards numerically solving the Fokker–Planck equation is to render the variables dimensionless. If we assume that either $D_{\mu\mu} = D(1 - \mu^2)$ or $D_{\mu\mu} = D$, then with the transformations

$$z \rightarrow \frac{D}{v}z \quad , \quad t \rightarrow Dt, \quad (2.71)$$

we obtain

$$\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu} \left[\tilde{D}_{\mu\mu} \frac{\partial f}{\partial \mu} \right] - \xi(1 - \mu^2) \frac{\partial f}{\partial \mu}, \quad (2.72)$$

where we have defined $\tilde{D}_{\mu\mu} = D_{\mu\mu}/D$ and introduced the focusing-to-scattering ratio

$$\xi = \frac{v}{2DL}. \quad (2.73)$$

With these transformations, each of the quantities in Eq. (2.72) is now dimensionless. Rather than solving this equation directly, we elect to work in Fourier space via the transformation given in Eq. (2.27). In this instance, we have

$$\frac{\partial F_k}{\partial t} + ik\mu F_k = \tilde{D}_{\mu\mu} \frac{\partial^2 F_k}{\partial \mu^2} + \tilde{D}'_{\mu\mu} \frac{\partial F_k}{\partial \mu} - \xi(1 - \mu^2) \frac{\partial F_k}{\partial \mu}. \quad (2.74)$$

To proceed, we discretize the time interval $[0, T]$ into N equispaced points denoted by $\{t_k\}_{k=0}^{N-1}$, and

the interval $[-1, 1]$ into M equispaced points denoted by $\{\mu_k\}_{k=1}^M$. Next, we introduce the notation

$$F_k^{m,n} = F_k(\mu = \mu_m, t = t_n). \quad (2.75)$$

We can then approximate the time derivative via the *forward difference formula*:

$$\frac{\partial F_k^{m,n}}{\partial t} = \frac{F_k^{m,n+1} - F_k^{m,n}}{h_t} \quad (2.76)$$

where h_t is the distance between t_n and t_{n-1} . As for the first and second order μ -derivatives, we employ the *centered difference*, yielding

$$\begin{aligned} \frac{\partial F_k^{m,n}}{\partial \mu} &= \frac{F_k^{m+1,n+1} - F_k^{m-1,n+1}}{2h_\mu} \\ \frac{\partial^2 F_k^{m,n}}{\partial \mu^2} &= \frac{F_k^{m+1,n+1} - 2F_k^{m,n+1} + F_k^{m-1,n+1}}{h_\mu^2} \end{aligned} \quad (2.77)$$

where h_μ is the distance between μ_m and μ_{m-1} . Upon placing the derivatives into Eq. (2.74) and rearranging,

$$\begin{aligned} &\left[-\frac{h_t}{h_\mu^2} \tilde{D}_{\mu\mu}(\mu_m) + \frac{h_t}{2h_\mu} (-\tilde{D}'_{\mu\mu}(\mu_m) + \xi[1 - \mu_m^2]) \right] F_k^{m+1,n+1} + \left[1 + ikh_t\mu_m + 2\frac{h_t}{h_\mu^2} \tilde{D}_{\mu\mu}(\mu_m) \right] F_k^{m,n+1} \\ &+ \left[-\frac{h_t}{h_\mu^2} \tilde{D}_{\mu\mu}(\mu_m) - \frac{h_t}{2h_\mu} (-\tilde{D}'_{\mu\mu}(\mu_m) + \xi[1 - \mu_m^2]) \right] F_k^{m-1,n+1} = F_k^{m,n}. \end{aligned} \quad (2.78)$$

This system of equations can be summarized via the matrix equation

$$\mathbf{A} \vec{F}_k^{n+1} = \vec{F}_k^n \quad (2.79)$$

where we have used the M -dimensional vector

$$\vec{F}_k^n = \begin{pmatrix} F_k^{1,n} \\ F_k^{2,n} \\ \vdots \\ F_k^{M,n} \end{pmatrix}. \quad (2.80)$$

We have also used the $M \times M$ matrix $\mathbf{A}$ whose entries are given by

$$\begin{aligned} A_{m,m-1} &= -\frac{h_t}{h_\mu^2} \tilde{D}_{\mu\mu}(\mu_m) - \frac{h_t}{2h_\mu} (-\tilde{D}'_{\mu\mu}(\mu_m) + \xi[1 - \mu_m^2]) \\ A_{m,m} &= 1 + ikh_t\mu_m + 2\frac{h_t}{h_\mu^2} \tilde{D}_{\mu\mu}(\mu_m) \\ A_{m,m+1} &= -\frac{h_t}{h_\mu^2} \tilde{D}_{\mu\mu}(\mu_m) + \frac{h_t}{2h_\mu} (-\tilde{D}'_{\mu\mu}(\mu_m) + \xi[1 - \mu_m^2]) \end{aligned} \quad (2.81)$$

provided $m \neq 1$ and $m \neq M$. These edge cases, which correspond to the values $\mu = \pm 1$, require additional work as $A_{1,0}$ and $A_{M,M+1}$ are not defined. How these special cases are handled depends on the form of the pitch-angle scattering diffusion coefficient. As a result, these cases are split into the two following subsections. After we discuss these edge cases, we return to the general ideas for obtaining the numerical solution.

2.3.1.1 Constant Scattering

If $\tilde{D} = 1$, then we can simplify Eq. (2.74) around the boundaries by using $\tilde{D}' = 0$, and the fact that the last term vanishes. We thus have

$$\frac{\partial F_k}{\partial t} + ik\mu F_k = \frac{\partial^2 F_k}{\partial \mu^2}. \quad (2.82)$$

The boundary condition given in Eq. (2.18) implies that the solution is constant around these points. Within the discretized grid, this can be summarized with the following relations

$$F_k^{0,n+1} = F_k^{1,n+1} \quad \text{and} \quad F_k^{M+1,n+1} = F_k^{M,n+1}. \quad (2.83)$$

First we consider the boundary at $\mu = -1$. The time derivative can again be approximated via Eq. (2.76). As for the second-order μ -derivative, we use Eq. (2.83) in Eq. (2.77). By placing these approximate derivatives in Eq. (2.82),

$$\frac{F_k^{1,n+1} - F_k^{1,n}}{h_t} - ikF_k^{1,n+1} = \frac{F_k^{2,n+1} - F_k^{1,n+1}}{h_\mu^2}, \quad (2.84)$$

which can be rewritten as

$$\left[1 - ikh_t + \frac{h_t}{h_\mu^2}\right] F_k^{1,n+1} - \frac{h_t}{h_\mu^2} F_k^{2,n+1} = F_k^{1,n}. \quad (2.85)$$

At this point, the matrix entries are easily read as

$$\begin{aligned} A_{1,1} &= 1 - ikh_t + \frac{h_t}{h_\mu^2} \\ A_{1,2} &= -\frac{h_t}{h_\mu^2}. \end{aligned} \quad (2.86)$$

Regarding the boundary at the point $\mu = 1$, we use an identical process to that above, except here the second-order μ -derivative follows from the second relation in Eq. (2.83) applied to Eq. (2.77). Here, Eq. (2.82) is given by

$$\frac{F_k^{M,n+1} - F_k^{M,n}}{h_t} + ikF_k^{M,n+1} = \frac{-F_k^{M,n+1} + F_k^{M-1,n+1}}{h_\mu^2}, \quad (2.87)$$

or,

$$\left[1 + ikh_t + \frac{h_t}{h_\mu^2}\right] F_k^{M,n+1} - \frac{h_t}{h_\mu^2} F_k^{M-1,n+1} = F_k^{M,n}. \quad (2.88)$$

The matrix entries are thus

$$\begin{aligned} A_{M,M-1} &= -\frac{h_t}{h_\mu^2} \\ A_{M,M} &= 1 + ikh_t + \frac{h_t}{h_\mu^2}. \end{aligned} \quad (2.89)$$

With Eqs. (2.81), (2.86) and (2.89), the matrix $\mathbf{A}$ is completely defined for the case of $\tilde{D}_{\mu\mu} = 1$.

2.3.1.2 Isotropic Scattering

When $\tilde{D}_{\mu\mu} = 1 - \mu^2$, there is no boundary condition imposed on the solution F_k . Consequently, this requires a different process from that used in the previous case. We can first simplify Eq. (2.74) by using the fact that $\tilde{D}_{\mu\mu}(\pm 1) = 0$, resulting in

$$\frac{\partial F_k}{\partial t} + ik\mu F_k = \tilde{D}'_{\mu\mu} \frac{\partial F_k}{\partial \mu}. \quad (2.90)$$

For now we assume $\mu = -1$. Rather than using the centered difference given in Eq. (2.77), we elect to use the forward difference. This is given as

$$\left. \frac{\partial F_k}{\partial \mu} \right|_{\mu=-1} = \frac{F_k^{2,n+1} - F_k^{1,n+1}}{h_\mu}. \quad (2.91)$$

With this derivative and that given in Eq. (2.76), Eq. (2.90) along with simple algebra shows that the matrix entries are

$$\begin{aligned} A_{1,1} &= 1 - ikh_t + 2\frac{h_t}{h_\mu} \\ A_{1,2} &= -2\frac{h_t}{h_\mu}. \end{aligned} \quad (2.92)$$

On the other hand, if $\mu = 1$, then we need to use the backward difference:

$$\left. \frac{\partial F_k}{\partial \mu} \right|_{\mu=1} = \frac{F_k^{M,n+1} - F_k^{M-1,n+1}}{h_\mu}. \quad (2.93)$$

Combining this with Eqs. (2.76) and (2.90) yields

$$\begin{aligned} A_{M,M-1} &= -2\frac{h_t}{h_\mu} \\ A_{M,M} &= 1 + ikh_t + 2\frac{h_t}{h_\mu}. \end{aligned} \quad (2.94)$$

The entries of the matrix $\mathbf{A}$ are now completely described by Eqs. (2.81), (2.92) and (2.94).

2.3.2 Solving the Governing Equation

Thanks to Eq. (2.81), which holds in all cases, the matrix $\mathbf{A}$ is determined once the entries of the first and last rows are specified. These rows are given either by Eqs. (2.86) and (2.89) in the constant scattering case, or by Eqs. (2.92) and (2.94) in the isotropic scattering case. The next step is to solve Eq. (2.79) inductively, provided the initial vector $\vec{F}_k^0$ is given.

The simplest method of solving Eq. (2.79) is through inverting $\mathbf{A}$ to obtain

$$\vec{F}_k^{n+1} = \mathbf{A}^{-1} \vec{F}_k^n. \quad (2.95)$$

The problem with this method is the time complexity. Inverting the $M \times M$ matrix requires $\mathcal{O}(M^3)$ operations. Next, performing the matrix-vector product takes $\mathcal{O}(M^2)$ time, which must be repeated N times to obtain the solution at the desired time T . Overall, this method has time complexity $\mathcal{O}(M^3 + NM^2)$.

Because $\mathbf{A}$ has non zero entries only on the main, sub, and super diagonal, we can significantly improve upon the above method. Rather than solving Eq. (2.79) through matrix inversion, it can be solved using *Thomas' algorithm*, which can be found in, for example, [Ford, 2014]. This algorithm can solve Eq. (2.79) in $\mathcal{O}(M)$ time. Since the system must be solved N times to reach the final time T , the overall time complexity is $\mathcal{O}(MN)$.

2.3.3 Numerical Approximation of the Initial Condition

The methods outlined throughout this section show that once the initial vector $\vec{F}_k^0$ is specified, Eq. (2.79) can be inductively solved until the solution at the desired time is obtained. As the Dirac delta cannot be implemented numerically, we require an alternative initial condition. This can be done by

approximating the Dirac delta with a narrow Gaussian [Olver et al., 2024]. That is, we approximate

$$F_k(\mu, t = 0) = \frac{1}{\sqrt{2\pi^3\epsilon}} e^{-\frac{(\mu-\mu_0)^2}{2\epsilon^2}} \quad (2.96)$$

where ϵ is taken to be a small positive number². The factor in front of the exponential ensures that the integral

$$\int_{-1}^1 d\mu F_k(\mu, t = 0) \quad (2.97)$$

is roughly the same whether Eq. (2.34) or Eq. (2.96) is employed, provided ϵ is sufficiently small.

Due to differing initial conditions, it is important to point out that there is bound to be slight discrepancies between any analytical solution and the numerical solution for early times. Fortunately, these differences quickly become negligible at later times.

2.3.4 Error Analysis

To conclude the analysis of the numerical solution, we derive a bound on the difference between the numerical solution and the exact solution. To begin, we define the difference to be

$$\vec{E}_k^n = \vec{F}_k^n - \vec{F}_k^{n*} \quad (2.98)$$

where $\vec{F}_k^{n*}$ is the exact solution. The error corresponding with the forward difference is first order in h_t , and the error for the centered difference is second order in h_μ [Ascher and Greif, 2011]. It then follows that using the exact solution in Eq. (2.79) yields

$$\mathbf{A}\vec{F}_k^{n+1*} = \vec{F}_k^{n*} + \vec{\Gamma}^{n+1} \quad (2.99)$$

where

$$\|\vec{\Gamma}^{n+1}\|_2 \leq C_1(h_t + h_\mu^2) \quad (2.100)$$

²The numerical calculations throughout this thesis use a value of $\epsilon = 10^{-3}$.

for the standard 2-norm of a vector $\|\cdot\|_2$ and a constant C_1 dependent on the coefficients of Eq. (2.74), but independent of h_t, h_μ and n . Upon combining Eqs. (2.79) and (2.99), we have

$$\vec{E}_k^{n+1} = \mathbf{A}^{-1} \vec{E}_k^n + \mathbf{A}^{-1} \vec{\Gamma}^{n+1}. \quad (2.101)$$

Next, we can recursively bound the error at the $(n+1)^{\text{th}}$ step:

$$\|\vec{E}_k^{n+1}\|_2 \leq \|\mathbf{A}^{-1}\|_2 [\|\vec{E}_k^n\|_2 + \|\vec{\Gamma}^{n+1}\|_2]. \quad (2.102)$$

Numerical evidence suggests that $\|\mathbf{A}^{-1}\|_2 \leq C_2 \approx 1$. Therefore, inductively applying the above relation results in

$$\|\vec{E}_k^{N-1}\|_2 \leq \|\vec{E}_k^0\|_2 + NC_2(h_t + h_\mu^2). \quad (2.103)$$

Provided the numerical initial condition is sufficiently accurate, we have that $\|\vec{E}_k^0\|_2$ is negligible. We thus find that the error is $\mathcal{O}(h_t + h_\mu^2)$.

3

Analytical Solution of the Space-Dependent Cosmic Ray Fokker-Planck Equation Using Airy Functions

3.1 Introduction

The unfocused spatially-integrated Fokker-Planck equation is given by

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial \mu} \left[D_{\mu\mu} \frac{\partial F}{\partial t} \right] \quad \text{and} \quad \left[D_{\mu\mu} \frac{\partial F}{\partial \mu} \right]_{\mu=\pm 1} = 0 \quad (3.1)$$

where $F(\mu, t)$ is the probability of the particle having pitch-angle μ at a time t . Eq. (3.1) is simply the general diffusion equation, which would lead one to believe that, depending on the choice of the pitch-angle scattering diffusion coefficient, the solution should be relatively simple. Indeed, as seen in the literature such as in [Lasuik and Shalchi, 2017], this is in fact the case. In particular, they obtain a relatively simple result for three different forms of $D_{\mu\mu}$, including if it is taken to be a constant.

The simplicity of the results obtained in [Lasuik and Shalchi, 2017] motivate us to attempt to derive

a similar result for the simplest form of the space-dependent Fokker-Planck equation:

$$\frac{\partial f}{\partial t} + v\mu f = D \frac{\partial^2 f}{\partial \mu^2} \quad \text{and} \quad \frac{\partial f}{\partial \mu} \Big|_{\mu=\pm 1} = 0. \quad (3.2)$$

In this chapter, we derive an analytical solution of Eq. (3.2) using Airy functions. Our solution is then applied to obtain several important expectation results.

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3.2 The Solution of the Fokker-Planck Equation

We take the Fourier transform of the solution of Eq. (3.2) to obtain the differential equation

$$\frac{\partial F_k}{\partial t} + iv\mu k F_k = D \frac{\partial^2 F_k}{\partial \mu^2} \quad \text{and} \quad \frac{\partial F_k}{\partial \mu} \Big|_{\mu=\pm 1} = 0. \quad (3.3)$$

One may recognize this partial differential equation as having the same form as the one-dimensional Schrödinger equation with a linear potential. See, for example, [Giese et al., 2014, Martin et al., 2014, Soto-Eguibar and Moya-Cessa, 2015] for techniques of solving the latter equation. The main difference here is that we have the boundary conditions given in Eq. (3.3) and thus, previously known methods cannot be identically applied here.

The observation of Eq. (3.3) being separable leads us to employ the ansatz

$$F_k(\mu, t) = \sum_n a_n g_n(\mu) e^{-\omega_n^2 D t} \quad (3.4)$$

which depends on the undetermined functions $g_n(\mu)$, the constants a_n , and the eigenvalues ω_n . The functions $g_n(\mu)$ can be determined by placing this series into Eq. (3.3), giving us

$$\frac{d^2 g_n}{d\mu^2} - \left(i \frac{vk}{D} \mu - \omega_n^2 \right) g_n = 0. \quad (3.5)$$

We aim to simplify this differential equation with the transformation

$$x = c\mu - d\omega_n^2 \quad (3.6)$$

where we have defined the two constants c and d as

$$c = \left(i \frac{vk}{D}\right)^{1/3} \quad \text{and} \quad d = \frac{1}{c^2} = \left(-i \frac{D}{vk}\right)^{2/3}. \quad (3.7)$$

After straightforward calculations, we find the ordinary differential equation

$$\frac{d^2 g_n}{dx^2} - x g_n = 0. \quad (3.8)$$

The latter equation is known as the *Airy differential equation*. The two independent solutions are the Airy functions of the first kind $\text{Ai}(x)$ and second kind $\text{Bi}(x)$. Therefore, our solution can be written as

$$g_n(\mu) = a'_n \text{Ai}(c\mu - d\omega_n^2) + b_n \text{Bi}(c\mu - d\omega_n^2) \quad (3.9)$$

with the coefficient b_n which will be determined below. Notice that by scaling the solution, we can assume the coefficient $a'_n = 1$ since we have the constant a_n in our ansatz given in Eq. (3.4). With our solution, the boundary conditions given by Eq. (3.3) turn into

$$\left. \frac{dg_n}{d\mu} \right|_{\mu=\pm 1} = 0. \quad (3.10)$$

Therefore we obtain the following equations

$$\begin{aligned} 0 &= \text{Ai}'(-c - d\omega_n^2) + b_n \text{Bi}'(-c - d\omega_n^2) \\ 0 &= \text{Ai}'(c - d\omega_n^2) + b_n \text{Bi}'(c - d\omega_n^2). \end{aligned} \quad (3.11)$$

Here we have used the derivatives of the two Airy functions $\text{Ai}'(x)$ and $\text{Bi}'(x)$, respectively. As we can

see, the constant b_n can be easily obtained using either of the boundary conditions. This results in

$$b_n = -\frac{\text{Ai}'(c - d\omega_n^2)}{\text{Bi}'(c - d\omega_n^2)} = -\frac{\text{Ai}'(-c - d\omega_n^2)}{\text{Bi}'(-c - d\omega_n^2)}. \quad (3.12)$$

Completely determining our solution is now a matter of finding the eigenvalues by using the conditions given by Eqs. (3.11), and we can use the initial condition given in Eq. (2.34) to determine the constants a_n .

Evidently, properties of Airy functions are important here. A survey concerning the properties we will use throughout the chapter can be found in [Olver et al., 2024] and [Vallée and Soares, 2010].

3.2.1 The Initial Condition

By Eq. (2.34), we have

$$\frac{1}{\pi}\delta(\mu - \mu_0) = \sum_n a_n g_n. \quad (3.13)$$

The idea behind determining the constants a_n is to use the fact that Eq. (3.5) can be rewritten in terms of a Sturm-Liouville eigenvalue problem, which is a fundamental concept in differential equations. See, for example, [Simmons, 2016] for properties of the Sturm-Liouville problem, including what we use here. To rewrite Eq. (3.5), we begin with defining the operator

$$\mathcal{L} = \frac{d^2}{d\mu^2} - i\frac{vk}{d}\mu. \quad (3.14)$$

Eq. (3.5) is then given as

$$\mathcal{L}g_n = -\omega_n^2 g_n \quad (3.15)$$

which is the general form of a Sturm-Liouville problem. A fundamental property of Sturm-Liouville problems is that the eigenfunctions $\{g_n\}_n$ form an orthogonal set. That is, we have

$$\int_{-1}^1 d\mu g_n g_m = 0 \quad \text{if} \quad n \neq m. \quad (3.16)$$

It then follows that the constant a_m can be determined by multiplying Eq. (3.13) with g_m and integrating over all μ to obtain

$$a_m = \frac{1}{\pi} \frac{g_m(\mu_0)}{\int_{-1}^1 d\mu g_m^2}. \quad (3.17)$$

The remaining step is to evaluate the integral in the denominator. For this, we consider the identity found in [Vallée and Soares, 2010]:

$$\begin{aligned} \int d\mu F_{Airy}(c\mu - d\omega_m^2) G_{Airy}(c\mu - d\omega_m^2) &= \left(\mu - \frac{d\omega_m^2}{c} \right) F_{Airy}(c\mu - d\omega_m^2) G_{Airy}(c\mu - d\omega_m^2) \\ &\quad - \frac{1}{c} F'_{Airy}(c\mu - d\omega_m^2) G'_{Airy}(c\mu - d\omega_m^2) \end{aligned} \quad (3.18)$$

where F_{Airy} and G_{Airy} are both functions in the set $\{\text{Ai}, \text{Bi}\}$. By Eq. (3.9), we have

$$g_m^2 = \text{Ai}(c\mu - d\omega_m^2)^2 + 2b_m \text{Ai}(c\mu - d\omega_m^2) \text{Bi}(c\mu - d\omega_m^2) + b_m^2 \text{Bi}(c\mu - d\omega_m^2)^2. \quad (3.19)$$

This implies that simplifying Eq. (3.17) is a matter of applying the identity given in Eq. (3.18) three times, which leads to

$$\begin{aligned} \int_{-1}^1 d\mu g_m^2 &= \left[\left(\mu - \frac{d\omega_m^2}{c} \right) \text{Ai}(c\mu - d\omega_m^2)^2 - \frac{1}{c} \text{Ai}'(c\mu - d\omega_m^2)^2 \right]_{-1}^1 \\ &\quad + 2b_m \left[\left(\mu - \frac{d\omega_m^2}{c} \right) \text{Ai}(c\mu - d\omega_m^2) \text{Bi}(c\mu - d\omega_m^2) - \frac{1}{c} \text{Ai}'(c\mu - d\omega_m^2) \text{Bi}'(c\mu - d\omega_m^2) \right]_{-1}^1 \\ &\quad + b_m^2 \left[\left(\mu - \frac{d\omega_m^2}{c} \right) \text{Bi}(c\mu - d\omega_m^2)^2 - \frac{1}{c} \text{Bi}'(c\mu - d\omega_m^2)^2 \right]_{-1}^1. \\ &= \left[\mu - \frac{d\omega_m^2}{c} \right] \left[\text{Ai}(c\mu - d\omega_m^2) + b_m \text{Bi}(c\mu - d\omega_m^2) \right]^2 \Big|_{-1}^1 \\ &\quad - \frac{1}{c} \left[\text{Ai}'(c\mu - d\omega_m^2) + b_m \text{Bi}'(c\mu - d\omega_m^2) \right]^2 \Big|_{-1}^1 \end{aligned} \quad (3.20)$$

where obtaining the second equality is done by regrouping the terms. Notice that the last term of the last equality is zero by Eq. (3.11). Moreover, the first term is given in terms of g_m , and thus we have

$$\int_{-1}^1 d\mu g_m^2 = \left(1 - \frac{d\omega_m^2}{c}\right) g_m(1)^2 + \left(1 + \frac{d\omega_m^2}{c}\right) g_m(-1)^2. \quad (3.21)$$

It is briefly pointed out that $g_m(\mu = \pm 1)$ can be further simplified by first combining Eqs. (3.9) and (3.12) to obtain

$$g_m(\pm 1) = \frac{\text{Ai}(\pm c - d\omega_m^2)\text{Bi}'(\pm c - d\omega_m^2) - \text{Bi}(\pm c - d\omega_m^2)\text{Ai}'(\pm c - d\omega_m^2)}{\text{Bi}'(\pm c - d\omega_m^2)}. \quad (3.22)$$

Notice that the numerator is given as the Wronskian of Ai and Bi. By [Olver et al., 2024], we can further simplify this to

$$g_m(\pm 1) = \frac{1}{\pi \text{Bi}'(\pm c - d\omega_m^2)}. \quad (3.23)$$

Finally, we combine Eqs. (3.17), (3.21), and (3.23) to derive the complete result

$$a_m = \pi c \frac{g_m(\mu_0)}{\frac{c-d\omega_m^2}{\text{Bi}'(c-d\omega_m^2)^2} + \frac{c+d\omega_m^2}{\text{Bi}'(-c-d\omega_m^2)^2}}. \quad (3.24)$$

With this result, we can see that the last step to obtain a complete analytical solution is to find the eigenvalues.

3.2.2 The Boundary Conditions

Here, we use the boundary conditions given in Eq. (3.11) to determine the eigenvalues. We begin by combining the two equations in Eq. (3.11) to obtain the new condition

$$\frac{\text{Bi}'(c - d\omega_m^2)}{\text{Ai}'(c - d\omega_m^2)} = \frac{\text{Bi}'(-c - d\omega_m^2)}{\text{Ai}'(-c - d\omega_m^2)}. \quad (3.25)$$

The eigenvalues are solely dependent on the dimensionless quantity

$$\alpha = \left(\frac{vk}{D} \right)^{1/3}, \quad (3.26)$$

as with this definition, we can write the constants c and d as

$$c = \alpha e^{i\pi/6} \quad \text{and} \quad d = \frac{1}{\alpha^2} e^{-i\pi/3}. \quad (3.27)$$

From here, we can see that if we transform $k \rightarrow -k$, then $c \rightarrow -c$ and $d \rightarrow d$, and hence the right-hand and left-hand sides of Eq. (3.25) simply switch with each other. This implies that the eigenvalues for $-k$ are identical to those corresponding to $+k$. To simplify our calculations, we will therefore assume that k is positive.

Unfortunately, it is seemingly not possible to obtain exact expressions for the eigenvalues. The next best option is to approximately solve Eq. (3.25) through employing approximations of the relevant functions. One downside is that this method gives rise to several cases, depending on the arguments in Eq. (3.25). For example, if z is a large complex number, the following asymptotic approximation, see [Olver et al., 2024],

$$\text{Ai}'(z) = -\frac{z^{1/4} e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}} \quad (3.28)$$

is accurate, as seen in Figure 3.1, as long as z is not near the negative real line. As the restriction for the asymptotic expansion of the Airy function of the second kind is significantly stricter, see [Olver et al., 2024], we are motivated to remove that function from Eq. (3.25) using the connection formula given in [Olver et al., 2024]:

$$\text{Bi}(z) = i\text{Ai}(z) - 2ie^{i5\pi/3}\text{Ai}(ze^{-i2\pi/3}). \quad (3.29)$$

Placing this into Eq. (3.25) results in

$$\frac{\text{Ai}'([c - d\omega^2]e^{-i2\pi/3})}{\text{Ai}'(c - d\omega^2)} = \frac{\text{Ai}'([-c - d\omega^2]e^{-i2\pi/3})}{\text{Ai}'(-c - d\omega^2)}. \quad (3.30)$$

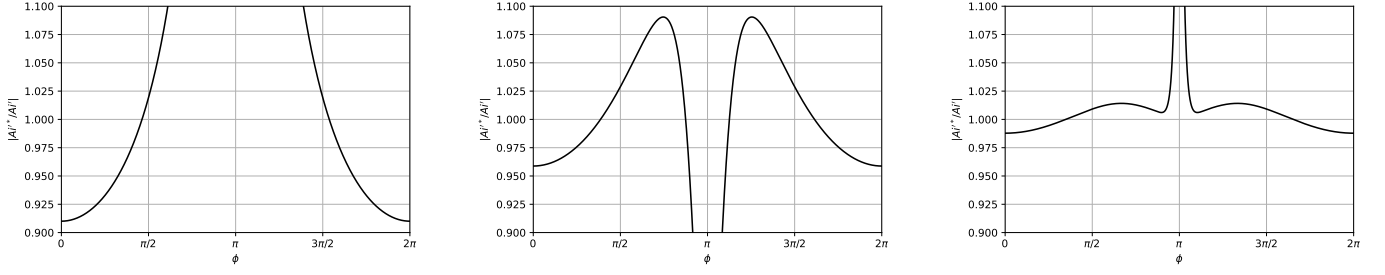


Figure 3.1: These figures show the absolute value of the ratio of $\text{Ai}'(z)$ with respect to the approximation provided by Eq. (3.28). Here, we use $z = re^{i\phi}$ where on the left panel $r = 1$, in the middle panel $r = 2$, and the right panel $r = 5$.

As Figure 3.1 shows, it is important to understand where the arguments in Eq. (3.30) are in the complex plane. A calculation shows that for real eigenvalues,

$$|\pm c - d\omega^2|^2 = \alpha^2 + \frac{\omega^4}{\alpha^4}, \quad (3.31)$$

and if we consider the extreme cases of $\omega^2 = 0$ and $\omega^2 \rightarrow \infty$, we see that

$$\begin{aligned} \phi_+ &:= \text{phase}(c - d\omega^2) \in \left[\frac{\pi}{6}, \frac{2\pi}{3}\right), \\ \phi_- &:= \text{phase}(-c - d\omega^2) \in \left(\frac{2\pi}{3}, \frac{7\pi}{6}\right], \end{aligned} \quad (3.32)$$

which implies

$$\begin{aligned} \text{phase}\left([c - d\omega^2] e^{-i2\pi/3}\right) &\in \left[\frac{3\pi}{2}, 2\pi\right), \\ \text{phase}\left([-c - d\omega^2] e^{-i2\pi/3}\right) &\in \left(0, \frac{\pi}{2}\right]. \end{aligned} \quad (3.33)$$

These properties lead us to categorize the eigenvalues in different cases according to the following:

- **Case 0:** For small wavenumbers, the smallest eigenvalue, and only the smallest, corresponds to small arguments in Eq. (3.25). This allows us to use Taylor expansions here.
- **Case 1:** For every other real eigenvalue, the arguments in Eq. (3.25) are large. For this case,

we assume the arguments are not near the negative real line. We thus use Eq. (3.28). After straightforward calculations, Eq. (3.25) is reduced to a new equation involving trigonometric functions. Simplifying these new functions depends on what quadrant of the complex plane $c - d\omega_n^2$ lies in.

- **Case 1.a:** In this instance, $c - d\omega_n^2$ lies in the first quadrant. After simplifying further, we derive approximations of the eigenvalues using a technique known as best least squares approximations.
- **Case 1.b:** This case is similar to case 1.a, the only difference being that $c - d\omega_n^2$ belongs instead to the second quadrant.
- **Case 1.c:** This scenario corresponds to the largest real eigenvalues. The argument $c - d\omega_n^2$ is again in the second quadrant. However, rather than using the best least squares method to approximate the eigenvalues, we use Taylor expansions based on the assumption that α^3/ω_n^2 is small.
- **Case 2:** Here, the arguments of Eq. (3.25) are again large. However, $-c - d\omega_n^2$ is near the negative real line. As a result, we require a different form of the asymptotic approximation compared to the one used in case 1. Otherwise, we use a similar approach as used in that case.
- **Case 3:** At various values of α , a so-called ray of complex eigenvalues forms. Along each ray, numerical evidence suggests that as α increases, the value $c - d\omega_n^2$ converges to a constant.

3.2.2.1 Case 0

To determine the eigenvalues within this case, we require $\pm c - d\omega_m^2$ to be small. By Eq. (3.31), we assume

$$\alpha^2 + \frac{\omega^4}{\alpha^4} < 1. \quad (3.34)$$

Rearranging this inequality leads to

$$\omega^4 < \alpha^4 - \alpha^6. \quad (3.35)$$

Eq. (3.34) implies that we must have $\alpha < 1$. Using Eq. (3.35), and performing a straightforward calculation shows that ω^2 is bounded by

$$\omega^2 \leq \frac{4}{27}. \quad (3.36)$$

As a result, these eigenvalues are well approximated by a Taylor expansion centered around $\omega^2 = 0$. A concern one may have here is that the number of eigenvalues we can determine is limited by the order of the expansions. However, numerical evidence suggests that for any value of α , there is at most one small eigenvalue.

To proceed, we begin with setting

$$g_{\pm}(\omega^2) = \text{Ai}'(\pm c - d\omega^2)\text{Bi}'(\mp c - d\omega^2). \quad (3.37)$$

Hence, Eq. (3.25) can be written as

$$g_+(\omega^2) = g_-(\omega^2). \quad (3.38)$$

The second-order expansion of $g_{\pm}$ is given by

$$\begin{aligned} g_{\pm}(\omega^2) &= \text{Ai}'(\pm c)\text{Bi}'(\mp c) - d \left[\text{Ai}''(\pm c)\text{Bi}'(\mp c) + \text{Ai}'(\pm c)\text{Bi}''(\mp c) \right] \omega^2 \\ &\quad - \frac{d^2}{2} \left[\text{Ai}'''(\pm c)\text{Bi}'(\mp c) + 2\text{Ai}''(\pm c)\text{Bi}''(\mp c) + \text{Ai}'(\pm c)\text{Bi}'''(\mp c) \right] \omega^4. \end{aligned} \quad (3.39)$$

Now we can write Eq. (3.38) as

$$\frac{d^2}{2}C_2\omega^4 - dC_1\omega^2 + C_0 = 0 \quad (3.40)$$

with

$$\begin{aligned} C_2 &= \text{Ai}'''(c)\text{Bi}'(-c) + 2\text{Ai}''(c)\text{Bi}''(-c) + \text{Ai}'(c)\text{Bi}'''(-c) \\ &\quad - \text{Ai}'''(-c)\text{Bi}'(c) - 2\text{Ai}''(-c)\text{Bi}''(c) - \text{Ai}'(-c)\text{Bi}'''(c), \\ C_1 &= \text{Ai}''(c)\text{Bi}'(-c) + \text{Ai}'(c)\text{Bi}''(-c) - \text{Ai}''(-c)\text{Bi}'(c) - \text{Ai}'(-c)\text{Bi}''(c), \\ C_0 &= \text{Ai}'(c)\text{Bi}'(-c) - \text{Ai}'(-c)\text{Bi}'(c). \end{aligned} \quad (3.41)$$

As we have $|\alpha| < 1$, the quantities C_n can be simplified through Taylor expansions centered at $c = 0$. Therefore, with the aid of the Airy differential equation, we compute

$$\begin{aligned} C_2 &= \frac{8}{3} \frac{c^3}{\pi}, \\ C_1 &= \frac{2}{\pi} c, \\ C_0 &= -\frac{4}{15\pi} c^5, \end{aligned} \tag{3.42}$$

where we calculated up to the lowest nonzero term.

Placing these simplifications into Eq. (3.40) yields

$$\frac{2}{3} \omega^4 - \omega^2 + \frac{2}{15} \alpha^6 = 0. \tag{3.43}$$

We can easily solve this to derive

$$\omega^2 = \frac{3}{4} \left[1 \pm \sqrt{1 - \frac{16}{45} \alpha^6} \right]. \tag{3.44}$$

Since we are interested in small eigenvalues here, we take the negative sign to obtain

$$\omega_0^2 = \frac{3}{4} \left[1 - \sqrt{1 - \frac{16}{45} \alpha^6} \right]. \tag{3.45}$$

If we take a first-order Taylor series of the square root, we can simplify the result to

$$\omega_0^2 = \frac{2}{15} \alpha^6. \tag{3.46}$$

The parameter α can be replaced by using Eq. (3.26) allowing us to write

$$\omega_0^2 = \frac{2}{15} \frac{v^2 k^2}{D^2} = \frac{\kappa_{\parallel 0}}{D} k^2 \tag{3.47}$$

where we have used the parallel diffusion coefficient defined in Section 2.2.5 for the case of $D_{\mu\mu}$ being a

constant.

3.2.2.2 Case 1

Now we assume that the arguments in Eq. (3.30) are large and not near the negative real line. As previously mentioned, the asymptotic approximation of Ai' given by Eq. (3.28) holds given these assumptions. With Eq. (3.28) and some algebra, Eq. (3.30) becomes

$$e^{\frac{8}{3}(c-d\omega^2)^{3/2}-\frac{8}{3}[(c-d\omega^2)e^{-i2\pi/3}]^{3/2}} = e^{\frac{8}{3}(-c-d\omega^2)^{3/2}-\frac{8}{3}[(-c-d\omega^2)e^{-i2\pi/3}]^{3/2}}. \quad (3.48)$$

It then follows that

$$(c-d\omega^2)^{3/2} - [(c-d\omega^2)e^{-i2\pi/3}]^{3/2} = (-c-d\omega^2)^{3/2} - [(-c-d\omega^2)e^{-i2\pi/3}]^{3/2} + 3in\pi \quad (3.49)$$

for some integer n . Next, we write $\pm c - d\omega^2$ in their polar form. That is,

$$\pm c - d\omega^2 = re^{i\phi_{\pm}}, \quad (3.50)$$

where both values have the same modulus according to Eq. (3.31). We use this form of $\pm c - d\omega^2$ along with algebra to obtain

$$r^{3/2} \left[-2 \sin\left(\frac{3}{4}(\phi_+ + \phi_-)\right) \sin\left(\frac{3}{4}(\phi_+ - \phi_-)\right) + i2 \cos\left(\frac{3}{4}(\phi_+ + \phi_-)\right) \sin\left(\frac{3}{4}(\phi_+ - \phi_-)\right) \right] = i\frac{3}{2}n\pi \quad (3.51)$$

where we have used the trigonometric identities

$$\cos \alpha + \cos \beta = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (3.52)$$

and

$$\sin \alpha - \sin \beta = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right). \quad (3.53)$$

We can see that the real component on the left side must be zero, implying

$$\sin\left(\frac{3}{4}(\phi_+ + \phi_-)\right) = 0 \quad \text{or} \quad \sin\left(\frac{3}{4}(\phi_+ - \phi_-)\right) = 0. \quad (3.54)$$

If the second condition holds, we have

$$\phi_+ - \phi_- = \frac{4}{3}m\pi \quad (3.55)$$

for some integer m . However, by Eq. (3.32), we thus obtain

$$-1 \leq \frac{4}{3}m < 0 \quad (3.56)$$

which cannot hold for any integer. We then must have that the first condition of Eq. (3.54) holds, resulting in

$$\frac{3}{4}(\phi_+ + \phi_-) = m\pi. \quad (3.57)$$

With Eq. (3.32) again, this implies

$$\frac{15}{24} \leq m < \frac{33}{24} \quad (3.58)$$

which shows that $m = 1$. Placing this into Eq. (3.51) and simplifying results in

$$\frac{1}{\sqrt{2}}r^{3/2} \left[\sin \phi_+ \sqrt{1 + \cos \phi_+} + \cos \phi_+ \sqrt{1 - \cos \phi_+} \right] = \frac{3}{4}n\pi \quad (3.59)$$

where we have used the trigonometric identities

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \quad (3.60)$$

and

$$\cos(2\alpha) = 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha. \quad (3.61)$$

Further simplifications of Eq. (3.59) depends on the quadrant of the complex plane that $c - d\omega^2$ belongs to. We proceed with simplifying these two cases independently.

Case 1.a First, we assume $c - d\omega^2$ is in the first quadrant. Thus, if we define γ to be the ratio of the imaginary component of $c - d\omega^2$ with its real component, we have $\gamma = \tan \phi_+$, and so

$$\begin{aligned}\sin \phi_+ &= \sin(\tan^{-1} \gamma) = \frac{\gamma}{\sqrt{1 + \gamma^2}}, \\ \cos \phi_+ &= \cos(\tan^{-1} \gamma) = \frac{1}{\sqrt{1 + \gamma^2}}.\end{aligned}\tag{3.62}$$

With these expressions, Eq. (3.59) becomes

$$\frac{r^{3/2}}{(1 + \gamma^2)^{3/4}} \left[\gamma \sqrt{\sqrt{1 + \gamma^2} + 1} + \sqrt{\sqrt{1 + \gamma^2} - 1} \right] = \frac{3\sqrt{2}}{4} n\pi.\tag{3.63}$$

One can square both sides of this equation twice to derive a quartic polynomial equation in terms of ω^2 . However, this results in the expression for ω being far too complicated. Alternatively, we can obtain an approximate value for ω here through polynomial approximations. Approximating functions with polynomials is an elementary topic in approximation theory, and can be found in most numerical analysis books such as [Burden and Faires, 2011]. However, we first need to rearrange Eq. (3.63). We define u to be the real component of $c - d\omega_m^2$ and v to be the imaginary component of $c - d\omega^2$. Next, by using the fact that $\gamma = v/u$, we have

$$\Omega(\omega^2) := v\sqrt{r + u} + u\sqrt{r - u} = \frac{3\sqrt{2}}{4} n\pi.\tag{3.64}$$

This can be rewritten

$$\Omega(\omega^2) = \alpha^{3/2} g\left(\frac{\omega^2}{\alpha^3}\right)\tag{3.65}$$

with g defined as

$$g(x) = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}x\right) \sqrt{\sqrt{1+x^2} + \frac{\sqrt{3}}{2} - \frac{1}{2}x} + \left(\frac{\sqrt{3}}{2} - \frac{1}{2}x\right) \sqrt{\sqrt{1+x^2} - \frac{\sqrt{3}}{2} + \frac{1}{2}x}. \quad (3.66)$$

Here, we restrict ω^2/α^3 to the interval

$$\frac{1}{\sqrt{3}} \leq x := \frac{\omega^2}{\alpha^3} \leq \sqrt{3} \quad (3.67)$$

where the upper bound is to ensure $c - d\omega^2$ is in the first quadrant. Moreover, the lower bound is derived from the fact that we want $-c - d\omega^2$ to not be near the negative real line. The best least squares degree two approximation of g on this interval is given by

$$p_2(x) = d_0 + d_1x + \frac{1}{2}d_2x^2 \quad (3.68)$$

for

$$\begin{aligned} d_0 &= 1.071601023079209, \\ d_1 &= 1.2777618882511206 \\ d_2 &= -0.30618590060929951. \end{aligned} \quad (3.69)$$

We demonstrate that this provides an accurate approximation in Figure 3.2.

Eq. (3.64) can then be approximately solved through solving

$$p_2(x) = \frac{3\sqrt{2}}{4\alpha^{3/2}}n\pi \quad (3.70)$$

leading to

$$x = -\frac{d_1}{d_2} \pm \sqrt{\left(\frac{d_1}{d_2}\right)^2 - 4\left(\frac{d_0}{d_2} - \frac{3\sqrt{2}}{4d_2\alpha^{3/2}}n\pi\right)}. \quad (3.71)$$

Notice that only certain values of n correspond to the desired interval of $1/\sqrt{3} \leq x \leq \sqrt{3}$. To determine

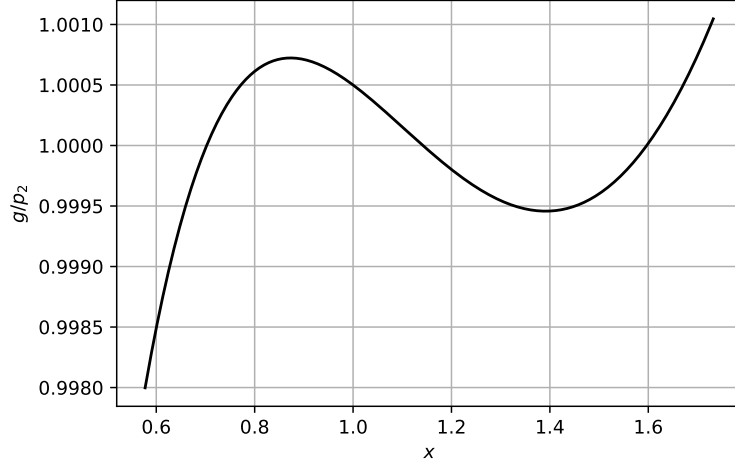


Figure 3.2: This plot shows $g(x)/p_2(x)$ for $1/\sqrt{3} \leq x \leq \sqrt{3}$. The functions $g(x)$ and $p_2(x)$ are given by Eqs. (3.66) and (3.68), respectively.

which values we want, we first rearrange Eq. (3.71) to get

$$\frac{1}{\sqrt{3}} + \frac{d_1}{d_2} \leq \pm \sqrt{\left(\frac{d_1}{d_2}\right)^2 - 4\left(\frac{d_0}{d_2} - \frac{3\sqrt{2}}{4d_2\alpha^{3/2}}n\pi\right)} \leq \sqrt{3} + \frac{d_1}{d_2}. \quad (3.72)$$

As $\sqrt{3} + d_1/d_2 < 0$, we only keep the negative solution. Moreover, we can further rearrange to show that

$$0.52767\alpha^{3/2} \approx \frac{d_2}{3\sqrt{2}\pi} \left[\frac{1}{3} + \frac{2}{\sqrt{3}} \frac{d_1}{d_2} + 4 \frac{d_0}{d_2} \right] \alpha^{3/2} \leq n \leq \frac{d_2}{3\sqrt{2}\pi} \left[3 + 2\sqrt{3} \frac{d_1}{d_2} + 4 \frac{d_0}{d_2} \right] \alpha^{3/2} \approx 0.84794\alpha^{3/2}. \quad (3.73)$$

Hence, if n is in this range, we have

$$\omega_n^2 = -\alpha^3 \left[\frac{d_1}{d_2} + \sqrt{\left(\frac{d_1}{d_2}\right)^2 - 4\left(\frac{d_0}{d_2} - \frac{3\sqrt{2}}{4d_2\alpha^{3/2}}n\pi\right)} \right]. \quad (3.74)$$

Case 1.b Now we consider the case where $c - d\omega^2$ is in the second quadrant. The calculations here are very similar compared to the previous case, however, here we have

$$\sin \phi_+ = -\sin(\tan^{-1} \gamma) = -\frac{\gamma}{\sqrt{1 + \gamma^2}} \quad (3.75)$$

and

$$\cos \phi_+ = -\cos(\tan^{-1} \gamma) = -\frac{1}{\sqrt{1+\gamma^2}} \quad (3.76)$$

where this time

$$\gamma = \tan(\phi_+ - \pi). \quad (3.77)$$

As in the previous case, we can rearrange Eq. (3.59) to

$$\Omega(\omega^2) = -\frac{3\sqrt{2}}{4}n\pi \quad (3.78)$$

where Ω is defined in Eq. (3.64). Next, we again write Ω in terms of g as defined in Eq. (3.65). As before we want to use polynomial approximations to solve this equation, however, we can only approximate g with an accurate polynomial on a bounded interval. Hence, we first assume $x = \omega^2/\alpha^3 \in (\sqrt{3}, 5)$, then we deal with $x \in [5, \infty)$ in Case 1.c.

For $x \in (\sqrt{3}, 5)$, the best least squares degree two polynomial of $g(x)$ is given by

$$q_2(x) = c_0 + c_1x + \frac{1}{2}c_2x^2 \quad (3.79)$$

with

$$\begin{aligned} c_0 &= 1.4610279573803782, \\ c_1 &= 0.8715466281663558, \\ c_2 &= -0.086145399451625789. \end{aligned} \quad (3.80)$$

Figure 3.3 illustrates that this quadratic polynomial provides an accurate approximation.

We now solve Eq. (3.78) by approximating

$$q_2(x) = -\frac{3\sqrt{2}}{4\alpha^{3/2}}n\pi. \quad (3.81)$$

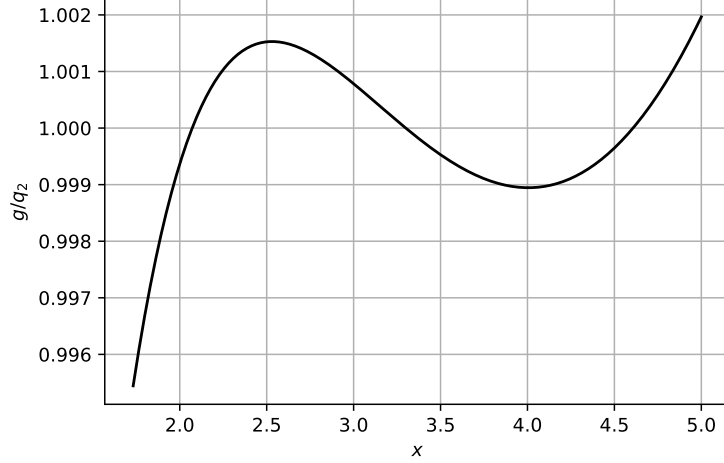


Figure 3.3: This plot shows $g(x)/q_2(x)$ for $\sqrt{3} < x < 5$. The functions $g(x)$ and $p_2(x)$ are given by Eqs. (3.66) and (3.79), respectively.

This is a quadratic equation which can be easily solved to obtain

$$x = -\frac{c_1}{c_2} \pm \sqrt{\left(\frac{c_1}{c_2}\right)^2 - 4\left(\frac{c_0}{c_2} + \frac{3\sqrt{2}}{4c_2\alpha^{3/2}}n\pi\right)}. \quad (3.82)$$

As we require $x \in (\sqrt{3}, 5)$, we have

$$\sqrt{3} + \frac{c_1}{c_2} < \pm \sqrt{\left(\frac{c_1}{c_2}\right)^2 - 4\left(\frac{c_0}{c_2} + \frac{3\sqrt{2}}{4c_2\alpha^{3/2}}n\pi\right)} < 5 + \frac{c_1}{c_2}. \quad (3.83)$$

As $5 + c_1/c_2 < 0$, the positive solution is neglected. We can further show that

$$-1.4231\alpha^{3/2} \approx -\frac{c_2}{3\sqrt{2}\pi} \left[25 + 10\frac{c_1}{c_2} + 4\frac{c_0}{c_2} \right] \alpha^{3/2} < n < -\frac{c_2}{3\sqrt{2}\pi} \left[3 + 2\sqrt{3}\frac{c_1}{c_2} + 4\frac{c_0}{c_2} \right] \alpha^{3/2} \approx -0.8527\alpha^{3/2}. \quad (3.84)$$

To summarize, if n is in this range, we take

$$\omega_n^2 = -\alpha^3 \left[\frac{c_1}{c_2} + \sqrt{\left(\frac{c_1}{c_2}\right)^2 - 4\left(\frac{c_0}{c_2} + \frac{3\sqrt{2}}{4c_2\alpha^{3/2}}n\pi\right)} \right]. \quad (3.85)$$

Case 1.c Now we derive the largest real eigenvalues. In other words, we deal with the case of $x = \omega^2/\alpha^3 \geq 5$. To derive an approximate analytical form of these eigenvalues, we begin with squaring

both sides of Eq. (3.78) and perform simple algebra to derive

$$(1 + x^2)^{3/2} - x^3 + 3x = \frac{9}{8\alpha^3} n^2 \pi^2. \quad (3.86)$$

Next, we take a second-order Taylor expansion of the first term to approximate

$$(1 + x^2)^{3/2} = x^3 \left(\frac{1}{x^2} + 1 \right)^{3/2} \approx x^3 + \frac{3}{2}x + \frac{3}{8} \frac{1}{x}. \quad (3.87)$$

We place this into Eq. (3.86) and simplify to obtain the quadratic equation

$$x^2 - \frac{n^2 \pi^2}{4\alpha^3} x + \frac{1}{12} = 0 \quad (3.88)$$

which is solved by

$$x = \frac{n^2 \pi^2}{8\alpha^3} \pm \sqrt{\left(\frac{n^2 \pi^2}{8\alpha^3} \right)^2 - \frac{1}{12}}. \quad (3.89)$$

To ensure that $x \geq 5$, we only keep the positive solution. Thus, we have

$$\omega_n^2 = \frac{n^2 \pi^2}{8} + \sqrt{\left(\frac{n^2 \pi^2}{8} \right)^2 - \frac{1}{12}} \alpha^6 \quad (3.90)$$

where $x \geq 5$ implies that we need

$$\frac{100 + \frac{1}{3}}{5\pi^2} \alpha^3 < n^2. \quad (3.91)$$

We note that if

$$\left(\frac{n^2 \pi^2}{8} \right)^2 \gg \frac{1}{12} \alpha^6 \quad (3.92)$$

we can simplify Eq. (3.90) to

$$\omega_n^2 = \frac{n^2 \pi^2}{4} - \frac{\alpha^6}{3n^2 \pi^2}. \quad (3.93)$$

which follows from a first-order Taylor expansion.

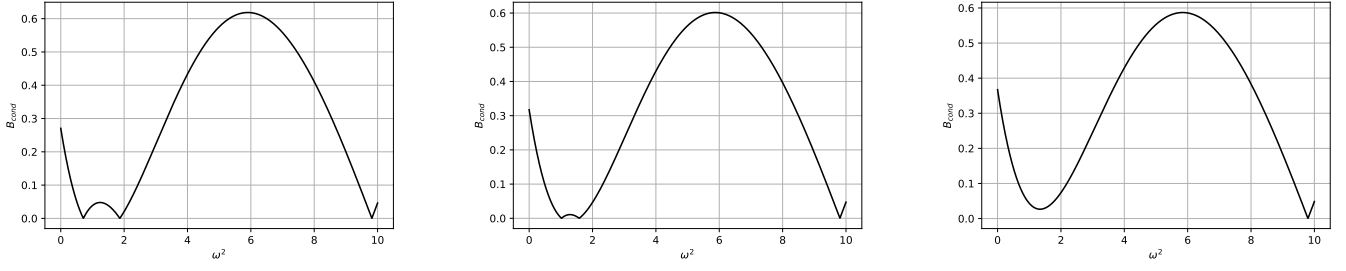


Figure 3.4: These plots show Eq. (3.94) for $\alpha^3 = 2, 2.2$, and 2.4 , respectively from left to right. By observing the first graph, one would intuitively expect that for $\alpha^3 = 2.2$, the first two roots would both increase to higher values. However, we see instead that the smaller root increases while the larger root decreases. Then, for $\alpha^3 = 2.4$, both of the small roots become complex, which we discuss later.

3.2.2.3 Case 2

Now we deal with the case where $-c - d\omega^2$ is near the real line. As we shall see, the solution we derive here will oscillate as α increases, both in terms of the expressions for the eigenvalues and even if there are eigenvalues which fall into this category. We demonstrate this oscillatory nature via Figure 3.4. To understand the plots, we rearrange Eq. (3.25) and define

$$B_{cond}(\omega^2) = \text{Ai}'(c - d\omega^2)\text{Bi}'(-c - d\omega^2) - \text{Ai}'(-c - d\omega^2)\text{Bi}'(c - d\omega^2) \quad (3.94)$$

so that the eigenvalues are roots of this function.

From Eqs. (3.32) and (3.33), we can see that $-c - d\omega^2$ is the only problematic argument in Eq. (3.30). Hence, we remove this argument using the identity

$$\text{Ai}'(z) = -e^{-i4\pi/3}\text{Ai}'(ze^{-i2\pi/3}) - e^{i4\pi/3}\text{Ai}'(ze^{i2\pi/3}) \quad (3.95)$$

from [Olver et al., 2024]. Thanks to Eq. (3.32), we can see that neither of $(-c - d\omega^2)e^{\pm i2\pi/3}$ is near the negative real line. Therefore, if we apply Eq. (3.95) to $z = -c - d\omega^2$, and place that into Eq. (3.30), we obtain the new equation

$$\frac{\text{Ai}'(c - d\omega^2)}{\text{Ai}'([c - d\omega^2]e^{-i2\pi/3})} + e^{i4\pi/3} \frac{\text{Ai}'([-c - d\omega^2]e^{i2\pi/3})}{\text{Ai}'([-c - d\omega^2]e^{-i2\pi/3})} = -e^{-i4\pi/3}. \quad (3.96)$$

Now we solve this equation using Eq. (3.28). After several straightforward calculations we find

$$e^{-\frac{4}{3}(c-d\omega^2)^{3/2}+i3\pi/2} + e^{\frac{4}{3}(-c-d\omega^2)^{3/2}+i\pi/2} = -1. \quad (3.97)$$

We again write the arguments in their polar form $\pm c - d\omega^2 = re^{i\phi_{\pm}}$. Through placing this into Eq. (3.97) and considering the real and imaginary components separately, we obtain

$$e^{-\frac{4}{3}r^{3/2}\cos(\frac{3}{2}\phi_+)} \sin\left(\frac{4}{3}r^{3/2}\sin\left(\frac{3}{2}\phi_+\right)\right) + e^{\frac{4}{3}r^{3/2}\cos(\frac{3}{2}\phi_-)} \sin\left(\frac{4}{3}r^{3/2}\sin\left(\frac{3}{2}\phi_-\right)\right) = 1 \quad (3.98)$$

and

$$-e^{-\frac{4}{3}r^{3/2}\cos(\frac{3}{2}\phi_+)} \cos\left(\frac{4}{3}r^{3/2}\sin\left(\frac{3}{2}\phi_+\right)\right) + e^{\frac{4}{3}r^{3/2}\cos(\frac{3}{2}\phi_-)} \cos\left(\frac{4}{3}r^{3/2}\sin\left(\frac{3}{2}\phi_-\right)\right) = 0. \quad (3.99)$$

The simplest way for the second equation to hold is if

$$\cos\left(\frac{3}{2}\phi_+\right) + \cos\left(\frac{3}{2}\phi_-\right) = 0 \quad \text{and} \quad \sin\left(\frac{3}{2}\phi_+\right) - \sin\left(\frac{3}{2}\phi_-\right) = 0 \quad (3.100)$$

which is also confirmed numerically. If we divide the first condition in Eq. (3.98) by the exponential factor in the first term and use Eq. (3.100), we derive

$$2 \sin\left(\frac{4}{3}r^{3/2}\sin\left(\frac{3}{2}\phi_+\right)\right) = e^{\frac{4}{3}r^{3/2}\cos(\frac{3}{2}\phi_+)}. \quad (3.101)$$

Using the fact that $\tan \phi_+$ is the ratio of the imaginary and real components of $c - d\omega^2$, we have

$$\phi_+ = \tan^{-1}\left(\frac{1 + \sqrt{3}x}{\sqrt{3} - x}\right) \quad (3.102)$$

for $x = \omega^2/\alpha^3$. Moreover, Eq. (3.31) can be rewritten as

$$r^2 = \alpha^2(1 + x^2). \quad (3.103)$$

As $-c - d\omega^2$ is near the negative real line, it follows from Eq. (3.27) that $x \approx 1/\sqrt{3}$. This in turn allows us to simplify Eq. (3.101) with a second-order Taylor expansion centered at $x = 1/\sqrt{3}$. The expansion results in the quadratic equation

$$\Omega_2 \left(\frac{\omega^2}{\alpha^3} - \frac{1}{\sqrt{3}} \right)^2 + \Omega_1 \left(\frac{\omega^2}{\alpha^3} - \frac{1}{\sqrt{3}} \right) + \Omega_0 = 0 \quad (3.104)$$

where we have defined $\beta = (4/3)^{7/4} \alpha^{3/2}$ and

$$\begin{aligned} \Omega_0 &= 1 - 2 \sin \beta, \\ \Omega_1 &= -\frac{3^{1/4}}{\sqrt{2}} \alpha^{3/2} - \left(\frac{4}{3} \right)^{1/4} \alpha^{3/2} \cos \beta, \\ \Omega_2 &= \frac{1}{2} \alpha^{3/2} \left[2\sqrt{3} \alpha^{3/2} - \left(\frac{3}{4} \right)^{3/4} + \frac{4}{\sqrt{3}} \alpha^{3/2} \sin \beta + \left(\frac{3}{4} \right)^{1/4} \cos \beta \right]. \end{aligned} \quad (3.105)$$

The solution of this is

$$\omega^2 = \alpha^3 \left[\frac{1}{\sqrt{3}} - \Omega_1 \pm \sqrt{\Omega_1^2 - \Omega_2 \Omega_0} \right]. \quad (3.106)$$

In order to ensure the expansion used above is accurate, we need $\alpha^{3/2} |x - 1/\sqrt{3}|$ to be small, and hence we require

$$\alpha^{3/2} \left| x - \frac{1}{\sqrt{3}} \right| < 1 \quad (3.107)$$

in order to ensure the resulting eigenvalue is an accurate approximation.

3.2.2.4 Case 3

As illustrated in Figure 3.4, as α increases, the two smallest real eigenvalues converge into each other, and this produces a so-called ray of complex eigenvalues. We can determine for which values of α a ray originates through determining when the discriminant of Eq. (3.104) is zero. Numerically, we find that the first five rays begin at $\alpha_0^3 \approx 2.26, 28, 83, 170, 280$. We visualize the first five rays in Figure 3.5. Note that for each complex eigenvalue, its complex conjugate is also an eigenvalue.

With Figure 3.5 in mind, in order to derive an analytical approximation, consider a ray originating

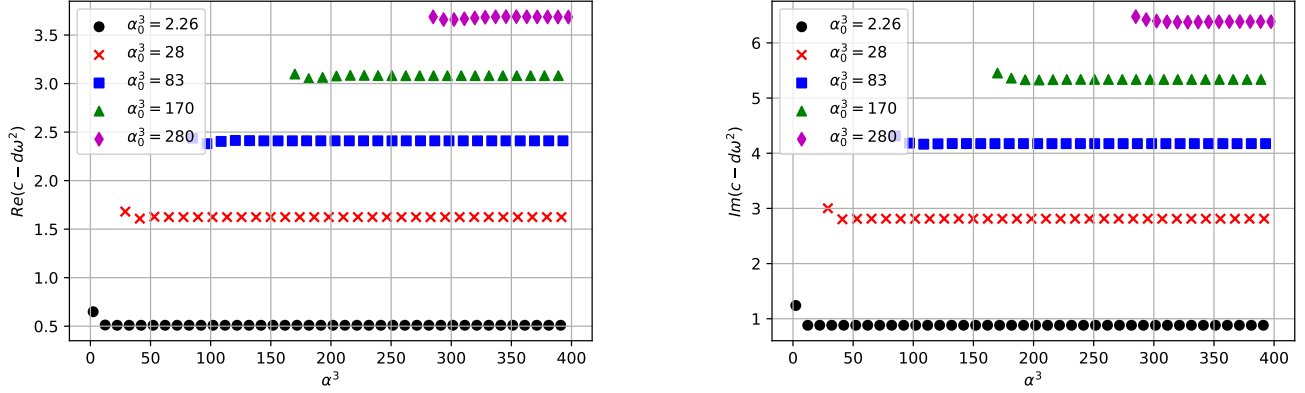


Figure 3.5: We illustrate the relation between $c - d\omega^2$ and α^3 for the first five rays of complex eigenvalues. The left plot shows the real component and the right plot shows the imaginary component.

at α_0 . We can then approximate

$$c - d\omega^2 = L_{\alpha_0^3} = L_{r_{\alpha_0^3}} + iL_{i_{\alpha_0^3}} \quad (3.108)$$

where $L_{r_{\alpha_0^3}}$ and $L_{i_{\alpha_0^3}}$ are the real and imaginary limits of $c - d\omega^2$ on the ray beginning at α_0^3 . In other words, we set $c - d\omega^2$ to be independent of α . We thus have

$$\omega^2 = c^3 - c^2 L_{\alpha_0^3}. \quad (3.109)$$

For the first five rays, we can verify numerically that

$$L_{r_{2.3}} = 0.50939648582374 + i0.88230059464375, \quad (3.110)$$

$$L_{r_{28}} = 1.62409879108991 + i2.81302162267893, \quad (3.111)$$

$$L_{r_{83}} = 2.41004960558751 + i4.17432836564291, \quad (3.112)$$

$$L_{r_{170}} = 3.08165427005540 + i5.33757986315356, \quad (3.113)$$

$$L_{r_{280}} = 3.68561116489002 + i6.38475667642645. \quad (3.114)$$

It should be noted that alternatively, we can find the corresponding complex conjugated eigenvalue.

The approach is similar, except here we have the limiting values

$$-c - d\bar{\omega}^2 = M_{\alpha_0^3} \quad (3.115)$$

with the first five limits being

$$M_{r_{2.3}} = -1.0187929716474, \quad (3.116)$$

$$M_{r_{28}} = -3.2481975821798, \quad (3.117)$$

$$M_{r_{83}} = -4.8200992111784, \quad (3.118)$$

$$M_{r_{170}} = -6.1633073545484, \quad (3.119)$$

$$M_{r_{280}} = -7.3721794624661. \quad (3.120)$$

It is interesting to note that these values are the zeros of Ai' , which can be found in [Olver et al., 2024].

3.2.2.5 The Eigenvalues Summarized

We summarize the results for the eigenvalues in Table 3.1. By observing the different cases, one can see that the values of n summed over in Eq. (3.4) depend on the value of α . For example, if α is small, we have that the eigenvalues are given by Eqs. (3.47) and (3.93) which state

$$\omega_0^2 = \frac{\kappa_{||0}}{D} k^2, \quad \omega_{n \geq 1}^2 = \frac{n^2 \pi^2}{4} - \frac{\alpha^6}{3n^2 \pi^2}. \quad (3.121)$$

Since our expressions for the eigenvalues are approximations, when computing the solution, it is necessary to refine our results numerically. One option for this is to use Newton's method with an initial guess as our derived approximation.

ω^2	Range	Eq.
$\frac{\kappa_{\parallel 0}}{D} k^2$	$\alpha^6 < \frac{45}{16}$	(3.47)
$-\alpha^3 \left[\frac{d_1}{d_2} + \sqrt{\left(\frac{d_1}{d_2}\right)^2 - 4 \left(\frac{d_0}{d_2} - \frac{3\sqrt{2}}{4d_2\alpha^{3/2}} n\pi\right)} \right]$	$0.52767\alpha^{3/2} \leq n \leq 0.84794\alpha^{3/2}$	(3.74)
$-\alpha^3 \left[\frac{c_1}{c_2} \pm \sqrt{\left(\frac{c_1}{c_2}\right)^2 - 4 \left(\frac{c_0}{c_2} + \frac{3\sqrt{2}}{4c_2\alpha^{3/2}} n\pi\right)} \right]$	$-1.4231\alpha^{3/2} < n < -0.8527\alpha^{3/2}$	(3.85)
$\frac{n^2\pi^2}{4} - \frac{\alpha^6}{3n^2\pi^2}$	$\frac{100+\frac{1}{3}}{5\pi^2}\alpha^3 < n^2$	(3.93)
$\alpha^3 \left[\frac{1}{\sqrt{3}} - \Omega_1 \pm \sqrt{\Omega_1^2 - 4\Omega_2\Omega_0} \right]$	$\alpha^{3/2} \left \frac{\omega^2}{\alpha^3} - \frac{1}{\sqrt{3}} \right < 1$	(3.106)
$\omega^2 = c^3 - c^2 L_{\alpha_0^3}$	$\alpha^3 \geq \alpha_0^3$	(3.109)

Table 3.1: This table gives an analytical approximation for each eigenvalue depending on the parameter $\alpha^3 = vk/D$. See the corresponding equation for more information about each case, including the definition of the constants.

The analytical solution is now completely described by Eqs. (3.4), (3.9), (3.12), (3.24), and Table 3.1.

It should be noted that it is possible that Table 3.1 does not include all of the eigenvalues. While it is seemingly not possible to prove that we have indeed considered each case, extensive numerical calculations show that it is unlikely we have missed any eigenvalues. Moreover, in Section 3.2.5, we demonstrate excellent accuracy of our analytical solution via comparisons with the numerical solution.

3.2.3 The μ -Integrated Solution

For certain applications, such as analyzing the intensity profile of a solar event, the μ -dependence of the solution is irrelevant. This motivates the use of the μ -integrated solution, which is given by

$$\begin{aligned} M(z, t) &= \frac{1}{2} \int_{-1}^1 d\mu \ f(\mu, z, t) \\ &= \frac{1}{2} \int_{-\infty}^{\infty} dk \ e^{ikz} \int_{-1}^1 d\mu \ F_k(\mu, t) \end{aligned} \quad (3.122)$$

where we have employed Eq. (2.27). With [Olver et al., 2024],

$$\begin{aligned} \int_{-1}^1 d\mu \ F_{Airy}(c\mu - d\omega_m^2) &= \frac{\pi}{c} \left[F_{Airy}(c - d\omega_m^2) \text{Hi}'(c - d\omega_m^2) - F'_{Airy}(c - d\omega_m^2) \text{Hi}(c - d\omega_m^2) \right] \\ &\quad - \frac{\pi}{c} \left[F_{Airy}(-c - d\omega_m^2) \text{Hi}'(-c - d\omega_m^2) - F'_{Airy}(-c - d\omega_m^2) \text{Hi}(-c - d\omega_m^2) \right], \end{aligned} \quad (3.123)$$

where F_{Airy} is in the set $\{\text{Ai}, \text{Bi}\}$, and Hi is known as the *Scorer function* which can be found in [Olver et al., 2024]. It then follows from superposition that

$$\int_{-1}^1 d\mu \ g_n(\mu) = \frac{\pi}{c} \left[\text{Hi}'(c - d\omega_n^2) g_n(1) - \text{Hi}'(-c - d\omega_n^2) g_n(-1) \right] \quad (3.124)$$

where g_n is defined via Eq. (3.9), and we invoked the boundary conditions in Eq. (3.11) to simplify our result. By using Eq. (3.23), this can be written as

$$\int_{-1}^1 d\mu \ g_n(\mu) = \frac{1}{c} \left[\frac{\text{Hi}'(c - d\omega_n^2)}{\text{Bi}'(c - d\omega_n^2)} - \frac{\text{Hi}'(-c - d\omega_n^2)}{\text{Bi}'(-c - d\omega_n^2)} \right]. \quad (3.125)$$

Upon placing this result into our ansatz given in Eq. (3.4), we obtain

$$\int_{-1}^1 d\mu \ F_k(\mu, t) = \sum_n \frac{a_n}{c} \left[\frac{\text{Hi}'(c - d\omega_n^2)}{\text{Bi}'(c - d\omega_n^2)} - \frac{\text{Hi}'(-c - d\omega_n^2)}{\text{Bi}'(-c - d\omega_n^2)} \right] e^{-\omega_n^2 D t}. \quad (3.126)$$

Now it only remains to place this expression into Eq. (3.122) to derive the result for $M(z, t)$:

$$M(z, t) = \frac{1}{2} \int_{-\infty}^{\infty} dk e^{ikz} \sum_n \frac{a_n}{c} \left[\frac{\text{Hi}'(c - d\omega_n^2)}{\text{Bi}'(c - d\omega_n^2)} - \frac{\text{Hi}'(-c - d\omega_n^2)}{\text{Bi}'(-c - d\omega_n^2)} \right] e^{-\omega_n^2 Dt}. \quad (3.127)$$

Recall that the values of n depend on the wavenumber, as can be seen in Table 3.1.

The next step is to verify the accuracy of this analytical expression by comparing it with the numerical solution. We do this in Section 3.2.5, but first derive an asymptotic approximation of Eq. (3.127) in the following section.

3.2.4 Evaluations of the Analytical Solution

If one attempts to plot the solutions directly from what is outlined in the previous sections, then they will succeed for certain eigenvalues. However, for most of the eigenvalues, specifically the larger ones, rounding errors result in a 0/0-type error. The cause of this problem is rooted in the exponential nature of Airy functions. To avoid this, one must use the appropriate asymptotic expansion. Here, we derive forms of our solution that do not have the aforementioned issues for both the function $f(z, \mu, t)$, and the μ -integrated function $M(z, t)$. We briefly point out that an alternative partial solution to this problem is to increase the number of significant figures. For example, in Python, the package *mpmath* can be used, [The mpmath development team, 2023].

Numerical evidence suggests that the aforementioned problem is not present for the complex eigenvalues. We therefore only use the asymptotic expansion for real eigenvalues. In this case, we have that if ω_n^2 is real, then by Eq. (3.32), $\text{phase}(c\mu - d\omega_n^2) \in [\pi/6, 7\pi/6]$ for any value of μ . Additionally, when z is near the negative real line, numerical evidence suggests that none of the relevant Airy functions are too large. Thus, we elect to only use the asymptotic expansion if $\text{phase}(c\mu - d\omega_n^2) \in [\pi/6, 9\pi/10]$. In this range, the two following approximations hold (see, e.g., [Olver et al., 2024]):

$$\text{Ai}(z) = \frac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}z^{1/4}} \quad \text{and} \quad \text{Ai}'(z) = -\frac{z^{1/4}e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}}. \quad (3.128)$$

Again, we aim to remove the explicit Bi dependence due to the restrictive asymptotic approximation,

which can be done using Eq. (3.29).

To continue, we introduce the notation $z = c\mu - d\omega_n^2$, $z_0 = c\mu_0 - d\omega_n^2$, and $z_{\pm} = \pm c - d\omega_n^2$. Then by combining Eq. (3.128) with Eq. (3.29), straightforward calculations result in

$$\frac{\text{Ai}'(z_+)}{\text{Bi}'(z_+)} = \frac{e^{-\frac{2}{3}z_+^{3/2}}}{ie^{-\frac{2}{3}z_+^{3/2}} - 2e^{\frac{2}{3}z_+^{3/2}}}. \quad (3.129)$$

Furthermore, one can similarly show that

$$g_n(\mu) = \text{Ai}(z) - \frac{\text{Ai}'(z_+)}{\text{Bi}'(z_+)}\text{Bi}(z) = -\frac{e^{-\frac{2}{3}z^{3/2}}}{\sqrt{\pi}z^{1/4}} \left(\frac{e^{\frac{4}{3}z_+^{3/2}} + e^{\frac{4}{3}z^{3/2}}}{i - 2e^{\frac{4}{3}z_+^{3/2}}} \right). \quad (3.130)$$

Note that by Eq. (3.25), we can replace $z_+ \rightarrow z_-$ in Eqs. (3.129) and (3.130). After straightforward rearrangements, we therefore have that Eq. (3.24) is asymptotically

$$a_n = -\frac{c}{4\sqrt{\pi}} \frac{\frac{z_+^{1/2}}{z_0^{1/4}} \left(e^{\frac{2}{3}z_+^{3/2} - \frac{2}{3}z_0^{3/2}} + e^{\frac{2}{3}z_0^{3/2} - \frac{2}{3}z_+^{3/2}} \right) \left(i - 2e^{\frac{4}{3}z_+^{3/2}} \right) e^{-\frac{2}{3}z_+^{3/2}}}{c - d\omega_n^2 + (c + d\omega_n^2) \left[\frac{z_+^{1/4}}{z_-^{1/4}} e^{\frac{2}{3}z_-^{3/2} - \frac{2}{3}z_+^{3/2}} \frac{i - 2e^{\frac{4}{3}z_+^{3/2}}}{i - 2e^{\frac{4}{3}z_-^{3/2}}} \right]^2}. \quad (3.131)$$

3.2.4.1 The Solution $f(z, \mu, t)$

Recall from Eq. (3.4) that our solution is given by

$$F_k(\mu, t) = \sum_n a_n g_n(\mu) e^{-\omega_n^2 D t}. \quad (3.132)$$

It follows from Eqs. (3.130) and (3.131) that the asymptotic expansion of $a_n g_n$ is

$$a_n g_n \approx \frac{c}{4\pi} \frac{\frac{z_+^{1/2}}{z_0^{1/4}} z_+^{1/4}}{c - d\omega_n^2 + (c + d\omega_n^2) \left[\frac{z_+^{1/4}}{z_-^{1/4}} e^{\frac{2}{3}z_-^{3/2} - \frac{2}{3}z_+^{3/2}} \frac{i - 2e^{\frac{4}{3}z_+^{3/2}}}{i - 2e^{\frac{4}{3}z_-^{3/2}}} \right]^2} \left(e^{\frac{2}{3}z_+^{3/2} - \frac{2}{3}z_0^{3/2}} + e^{\frac{2}{3}z_0^{3/2} - \frac{2}{3}z_+^{3/2}} \right) \left(e^{\frac{2}{3}z_+^{3/2} - \frac{2}{3}z^{3/2}} + e^{\frac{2}{3}z^{3/2} - \frac{2}{3}z_+^{3/2}} \right). \quad (3.133)$$

One can observe that the argument in each exponential is either not too large, or the real component is negative, as suggested numerically. In either case, there is no risk of encountering an indeterminate

form.

3.2.4.2 The μ -Integrated Solution

By Eq. (3.126), the μ -integrated solution in Fourier space can be written as

$$\int_{-1}^1 d\mu F_k(\mu, t) = \sum_n \frac{1}{c} \tilde{F}_{k,n} e^{-\omega_n^2 D t} \quad (3.134)$$

with

$$\tilde{F}_{k,n} = a_n \left[\frac{\text{Hi}'(z_+)}{\text{Bi}'(z_+)} - \frac{\text{Hi}'(z_-)}{\text{Bi}'(z_-)} \right]. \quad (3.135)$$

We proceed similarly to the previous case, except here we also need the asymptotic approximation

$$\text{Hi}'(z) = \frac{1}{\pi z^2} \quad (3.136)$$

which is found in [Olver et al., 2024]. Placing Eqs. (3.29), (3.128), (3.131), (3.136) into Eq. (3.135) results in

$$\tilde{F}_{k,n} \approx \frac{c}{2\pi} \frac{\frac{z_+^{1/2}}{z_0^{1/4}} \left(e^{\frac{2}{3}z_+^{3/2} - \frac{2}{3}z_0^{3/2}} + e^{\frac{2}{3}z_0^{3/2} - \frac{2}{3}z_+^{3/2}} \right)}{c - d\omega_n^2 + (c + d\omega_n^2) \left[\frac{z_+^{1/4}}{z_-^{1/4}} e^{\frac{2}{3}z_-^{3/2} - \frac{2}{3}z_+^{3/2}} \frac{i - 2e^{\frac{4}{3}z_+^{3/2}}}{i - 2e^{\frac{4}{3}z_-^{3/2}}} \right]^2} \left[\frac{1}{z_+^{9/4}} - \frac{\left(i - 2e^{\frac{4}{3}z_+^{3/2}} \right) e^{\frac{2}{3}z_-^{3/2} - \frac{2}{3}z_+^{3/2}}}{z_-^{9/4} \left(i - 2e^{\frac{4}{3}z_-^{3/2}} \right)} \right] \quad (3.137)$$

where similar to Eq. (3.133), we avoid any of the aforementioned issues.

3.2.5 Comparisons with the Numerical Solution

It is obviously good to verify the accuracy of any given theory. However, verifying the accuracy of the analytical solutions discussed here have additional motivation for two reasons. First, it is possible that Section 3.2.2 does not cover every possible case for the eigenvalues. Although extensive numerical evidence suggests that we have indeed considered each case, there will always be underlying uncertainty. Second, the approximations used in Section 3.2.4 are far from rigorously proven. Hence, this section aims to provide additional confirmation of our approximations.

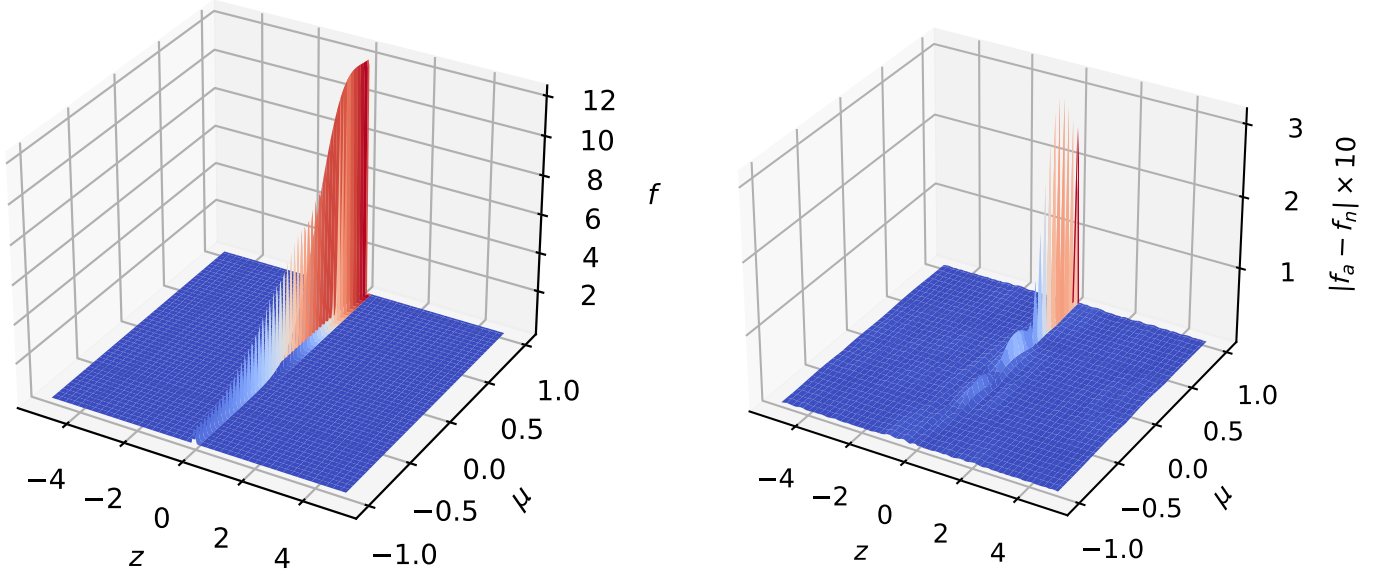


Figure 3.6: Distribution functions at the time $Dt = 0.2$ for an initial value of $\mu_0 = 0.5$. The left panel shows our analytical solution, and the right panel shows the absolute difference between our analytical solution and the numerical solution.

To demonstrate the accuracy of our analytical findings, we compare our analytical solution with the numerical solution. For the analytical solution, we derive $F_k(\mu, t)$ as described throughout the previous sections, where we terminate the series given in Eq. (3.4) when $|\omega_n^2 Dt| = 10$. Refer to Section 2.3 for details about the numerical method.

In the following Figures 3.6-3.8, we show our analytical solution at various times, with $\mu_0 = 0.5$ as an example. Additionally, next to each graph of the analytical solution, we have the absolute difference between the two solution methods. One can see that our analytical solution agrees with the numerical solution at each given value of Dt , up to the issue at early times due to different initial conditions.

We also compare the μ -integrated solution for the numerical method and our analytical method, given by Eq. (3.127), in Figs. 3.9-3.11. For integrating the numerical solution, we use the simple rectangle method. As one can see, our analytical form of the μ -integrated solution agrees with the one obtained numerically regardless of the value of Dt , up to the mentioned problems in Section 2.3.3 associated with comparing the solutions at early times.

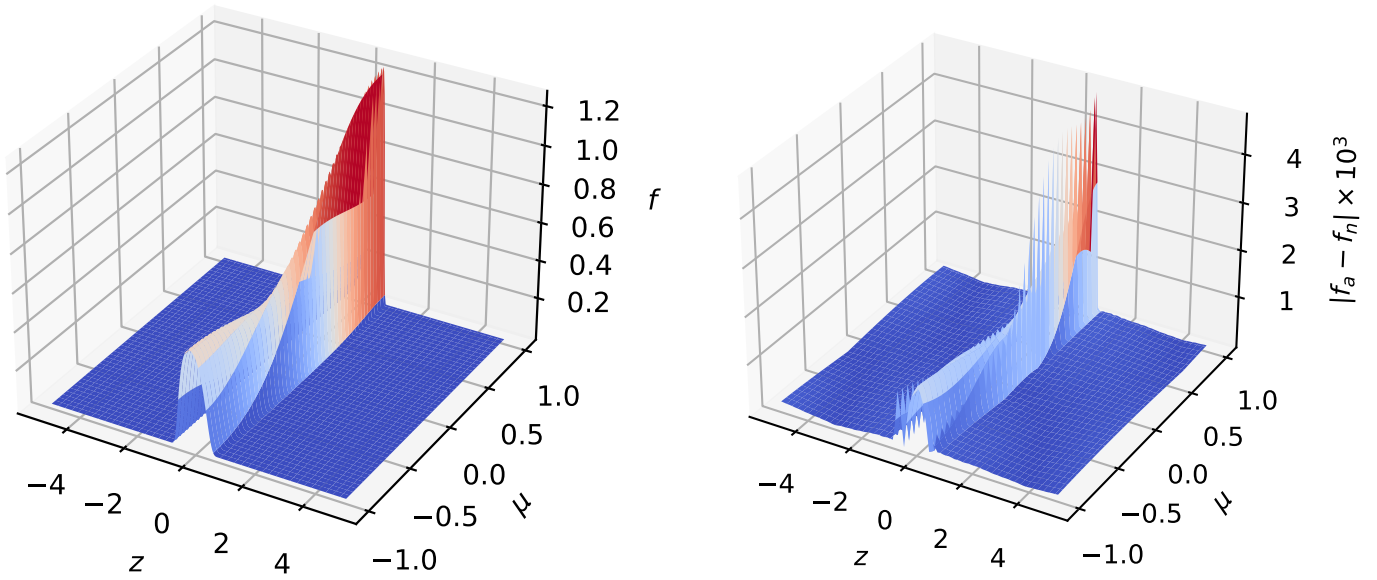


Figure 3.7: Caption is as in Figure 3.6, but here the time is $Dt = 1$

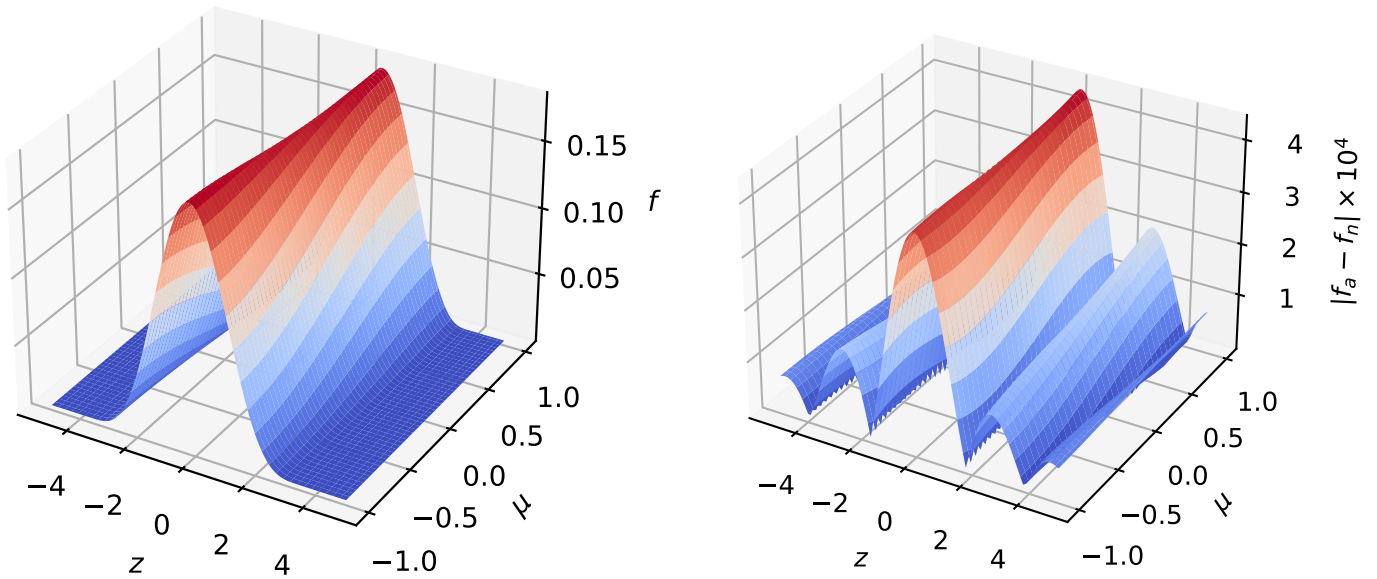


Figure 3.8: Caption is as in Figure 3.6 but the time is $Dt = 5$

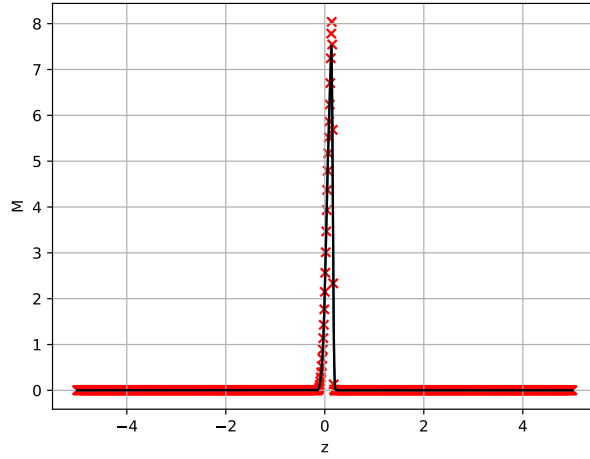


Figure 3.9: The μ -integrated solution as a function of z . The solid black line is the numerical solution, whereas the red crosses represent the analytical solution. These results were obtained for $Dt = 0.2$ and $\mu_0 = 0.5$.

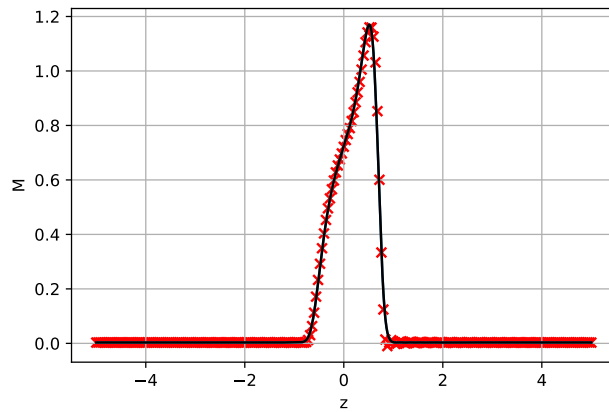


Figure 3.10: Caption is as in Fig. 3.9 but we have used $Dt = 1$.

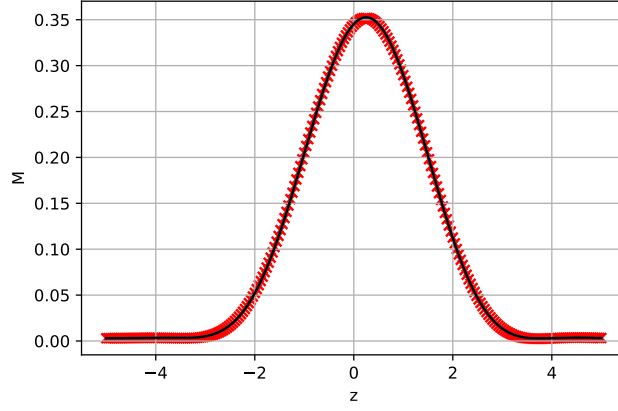


Figure 3.11: Caption is as in Fig. 3.9 but we have used $Dt = 5$.

3.2.6 Small Wavenumbers

Eq. (3.3) was solved in [Lasuik and Shalchi, 2017] for the case of $k = 0$. The solution¹ is

$$F_0(\mu, t) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos(n\pi\mu_0) \cos(n\pi\mu) e^{-(n\pi)^2 Dt} + \frac{1}{\pi} \sum_{n=0}^{\infty} \sin(\pi[n + 1/2]\mu_0) \sin(\pi[n + 1/2]\mu) e^{-(n+1/2)^2 Dt}. \quad (3.138)$$

As a final test for our analytical solution, we show that

$$\lim_{k \rightarrow 0} F_k(\mu, t) = F_0(\mu, t) \quad (3.139)$$

where F_k is our analytical solution. Notice that since $\alpha \propto k^{1/3}$, the above limit is equivalent to $\alpha \rightarrow 0$.

Given that we are interested in small α here, we can write our solution as

$$F_k(\mu, t) = \sum_{n=0}^{\infty} a_n g_n(\mu) e^{-\omega_n^2 Dt} \quad (3.140)$$

where g_n is defined by Eq. (3.9) and a_n is given by Eq. (3.24). Moreover, the eigenvalues in this case

¹Note that their actual solution differs from the stated result by a factor of 2π due to a scaling of the initial condition.

are given in Eq. (3.121). From Eq. (3.27) it follows that

$$c\mu - d\omega_0^2 = \alpha e^{i\pi/6}\mu - \frac{2}{15}\alpha^4 e^{-i\pi/3} \quad (3.141)$$

and

$$c\mu - d\omega_{n \geq 1}^2 = \alpha e^{i\pi/6}\mu - \left[\frac{n^2\pi^2}{4\alpha^2} - \frac{\alpha^4}{3n^2\pi^2} \right] e^{-i\pi/3} \quad (3.142)$$

which shows that if we want to consider the limit $\alpha \rightarrow 0$, we can simply set $\alpha = 0$ for the zeroth term.

With this in mind, we consider the limit $\alpha \rightarrow 0$ of Eq. (3.9) to obtain

$$\lim_{\alpha \rightarrow 0} g_0(\mu) = \text{Ai}(0) - \frac{\text{Ai}'(0)}{\text{Bi}'(0)} \text{Bi}(0), \quad (3.143)$$

which is independent of μ . Combining this result with Eq. (3.23) gives

$$\lim_{\alpha \rightarrow 0} g_0(\mu) = \frac{1}{\pi \text{Bi}'(0)}. \quad (3.144)$$

We then place this into Eq. (3.24) to obtain

$$\lim_{\alpha \rightarrow 0} a_0 = \frac{1}{2} \text{Bi}'(0). \quad (3.145)$$

It then follows from the two previous expressions that

$$\lim_{\alpha \rightarrow 0} a_0 g_0(\mu) = \frac{1}{2\pi}. \quad (3.146)$$

As seen in Eq. (3.138), we have agreement with the constant term.

On the other hand, Eq. (3.142) shows that for $n \geq 1$, the asymptotic expansion must be used, which is given in Eq. (3.133). This result can be simplified further by noticing that Eq. (3.142) shows

$c\mu - d\omega_{n\geq 1}^2 \approx -d\omega_{n\geq 1}^2$. We can therefore use a Taylor expansion to obtain

$$(c\mu - d\omega_n^2)^{3/2} = -\frac{\omega_n^3}{|\alpha|^3} + i\frac{3}{2}\text{sgn}(\alpha)\omega_n\mu \quad (3.147)$$

where we have used the sgn function given by $\text{sgn}(x) = 1$ if $x \geq 0$ and -1 otherwise. With this expansion, Eq. (3.133) simplifies to

$$a_n g_n = \frac{c}{4\pi} \frac{z_+^{1/2}}{z_0^{1/4} z^{1/4}} \frac{\left(e^{i\text{sgn}(\alpha)\omega_n(1-\mu_0)} + e^{-i\text{sgn}(\alpha)\omega_n(1-\mu_0)}\right) \left(e^{i\text{sgn}(\alpha)\omega_n(1-\mu)} + e^{-i\text{sgn}(\alpha)\omega_n(1-\mu)}\right)}{c - d\omega_n^2 + (c + d\omega_n^2) \left(\frac{z_+^{1/4}}{z_-^{1/4}} e^{-i2\text{sgn}(\alpha)\omega_n} \frac{i-2e^{\frac{4}{3}z_+^{3/2}}}{i-2e^{\frac{4}{3}z_-^{3/2}}}\right)^2}. \quad (3.148)$$

As a next step, we aim to simplify the denominator. As it is not sufficient to simply take the zeroth order approximation, we must be more careful with the calculations. We begin with noting that by Eq. (3.147), the exponential terms involving $z_{\pm}^{3/2}$ are effectively zero, and thus the last fraction in the denominator is simply one. Additionally, by Eq. (3.121),

$$\lim_{\alpha \rightarrow 0} \omega_{n\geq 1} = \frac{n\pi}{2}, \quad (3.149)$$

implying that

$$e^{-i2\text{sgn}(\alpha)\omega_n} = \pm 1. \quad (3.150)$$

Moreover, with a Taylor expansion, it is straightforward to show that

$$\left(\frac{z_+^{1/4}}{z_-^{1/4}}\right)^2 = 1 - i\frac{\alpha^3}{\omega_n^2} = 1 - \frac{c}{d\omega_n^2}. \quad (3.151)$$

It then follows that the denominator of Eq. (3.148) is asymptotically

$$c - d\omega_n^2 + (c + d\omega_n^2) \left(1 - \frac{c}{d\omega_n^2}\right) = c - \frac{c^2}{d\omega_n^2} = c \quad (3.152)$$

where in the last step, we have used the fact that Eq. (3.27) shows c is significantly larger than the

other term. Eq. (3.148) is thus simplified to

$$a_n g_n = \frac{1}{4\pi} \frac{z_+^{1/2}}{z_0^{1/4} z^{1/4}} \left(e^{i \operatorname{sgn}(\alpha) \omega_n (1-\mu_0)} + e^{-i \operatorname{sgn}(\alpha) \omega_n (1-\mu_0)} \right) \left(e^{i \operatorname{sgn}(\alpha) \omega_n (1-\mu)} + e^{-i \operatorname{sgn}(\alpha) \omega_n (1-\mu)} \right). \quad (3.153)$$

Unlike the previous steps with simplifying the denominator, simplifying the term involving the fraction is much simpler. Here, we take the zeroth order approximation and write $z^n \approx (-d\omega_n^2)^n$, which renders the fraction equal to one. This gives us

$$a_n g_n = \frac{1}{4\pi} \left(e^{i \operatorname{sgn}(\alpha) \omega_n (1-\mu_0)} + e^{-i \operatorname{sgn}(\alpha) \omega_n (1-\mu_0)} \right) \left(e^{i \operatorname{sgn}(\alpha) \omega_n (1-\mu)} + e^{-i \operatorname{sgn}(\alpha) \omega_n (1-\mu)} \right). \quad (3.154)$$

Now we use the identity

$$2 \cos z = e^{iz} + e^{-iz} \quad (3.155)$$

to derive

$$a_n g_n = \frac{1}{\pi} \cos \left(\frac{n\pi}{2} (1-\mu_0) \right) \cos \left(\frac{n\pi}{2} (1-\mu) \right) \quad (3.156)$$

where we have used Eq. (3.149) and the fact that $\cos \theta$ is an even function, and thus the factor $\operatorname{sgn}(\alpha)$ can be dropped. Finally, we inductively apply the trigonometric identities

$$\cos \left(\frac{\pi}{2} - \theta \right) = \sin \theta \quad \text{and} \quad \sin \left(\frac{\pi}{2} - \theta \right) = \cos \theta \quad (3.157)$$

to obtain

$$a_{2n} g_{2n} = \frac{1}{\pi} \cos (n\pi \mu_0) \cos (n\pi \mu) \quad (3.158)$$

which agrees with the first series of Eq. (3.138). As for the odd terms, we again apply the above identities inductively to show

$$a_{2n+1} g_{2n+1} = \frac{1}{\pi} \sin ([n+1/2]\pi \mu_0) \sin ([n+1/2]\pi \mu) \quad (3.159)$$

which agrees with the second series in Eq. (3.138).

3.3 Expectation Values

We now employ our analytical solution to compute the expectation value of a given function $A = A(z, \mu, t)$. It should be noted that several expectation values of interest are independent of z . This implies that rather than working with the solution of the space-dependent Fokker-Planck equation, the spatially-integrated equation is sufficient. As this form was extensively investigated in [Lasuik and Shalchi, 2017], in the following, we elect to only consider expectation values which are space-dependent.

The motivation for considering the expectation values discussed throughout this thesis, as well as supplementary general calculations, is provided in Section 2.2.

3.3.1 The Characteristic Function

The characteristic function is given by

$$\langle e^{-ikz} \rangle = \frac{\pi}{2} \int_{-1}^1 d\mu_0 \int_{-1}^1 d\mu F_k(\mu, t). \quad (3.160)$$

By Eqs. (3.4) and (3.24), we have

$$\begin{aligned} \int_{-1}^1 d\mu_0 \int_{-1}^1 d\mu F_k(\mu, t) &= \pi c \sum_n \frac{e^{-\omega_n^2 Dt}}{\frac{c-d\omega_n^2}{\text{Bi}'(c-d\omega_n^2)^2} + \frac{c+d\omega_n^2}{\text{Bi}'(-c-d\omega_n^2)^2}} \int_{-1}^1 d\mu_0 \int_{-1}^1 d\mu g_n(\mu_0) g_n(\mu) \\ &= \frac{\pi}{c} \sum_n \frac{e^{-\omega_n^2 Dt}}{\frac{c-d\omega_n^2}{\text{Bi}'(c-d\omega_n^2)^2} + \frac{c+d\omega_n^2}{\text{Bi}'(-c-d\omega_n^2)^2}} \left[\frac{\text{Hi}'(c-d\omega_n^2)}{\text{Bi}'(c-d\omega_n^2)} - \frac{\text{Hi}'(-c-d\omega_n^2)}{\text{Bi}'(-c-d\omega_n^2)} \right]^2 \end{aligned} \quad (3.161)$$

where evaluating the integrals was a matter of invoking Eq. (3.125) twice. The final result for the characteristic function is then obtained by placing this result into Eq. (3.160):

$$\langle e^{-ikz} \rangle = \frac{\pi^2}{2c} \sum_n \frac{\left[\text{Hi}'(c-d\omega_n^2) \text{Bi}'(-c-d\omega_n^2) - \text{Hi}'(-c-d\omega_n^2) \text{Bi}'(c-d\omega_n^2) \right]^2}{(c-d\omega_n^2) \text{Bi}'(-c-d\omega_n^2)^2 + (c+d\omega_n^2) \text{Bi}'(c-d\omega_n^2)^2} e^{-\omega_n^2 Dt}. \quad (3.162)$$

We now aim to simplify our result for small wavenumbers, i.e. $|\alpha| \ll 1$, and late times. As $e^{-\omega_0^2 Dt}$

decays slower than every other term, we only consider the first term ($n = 0$) of Eq. (3.162):

$$\langle e^{-ikz} \rangle \approx \frac{\pi^2}{2c} \frac{[\text{Hi}'(c - d\omega_0^2)\text{Bi}'(-c - d\omega_0^2) - \text{Hi}'(-c - d\omega_0^2)\text{Bi}'(c - d\omega_0^2)]^2}{(c - d\omega_0^2)\text{Bi}'(-c - d\omega_0^2)^2 + (c + d\omega_0^2)\text{Bi}'(c - d\omega_0^2)^2} e^{-\omega_0^2 Dt}. \quad (3.163)$$

In the case of small wavenumbers, ω_0^2 is given by Eq. (3.47) and, thus, we have

$$\pm c - d\omega_0^2 = \pm \alpha e^{i\pi/6} - \frac{2}{15} \alpha^4 e^{-i\pi/3}, \quad (3.164)$$

implying that we can approximate Bi' and Hi' with first order Taylor expansions [Olver et al., 2024]

$$\text{Bi}'(z) = \text{Bi}'(0) \quad \text{and} \quad \text{Hi}'(z) = \frac{3^{-1/3} \Gamma(\frac{2}{3})}{\pi} + \frac{1}{\pi} z \quad (3.165)$$

where $\Gamma(z)$ is the gamma function. Consequently, we find that the coefficient in front of the exponential in Eq. (3.163) is nearly one, and thus

$$\langle e^{-ikz} \rangle \approx e^{-\kappa_{\parallel 0} k^2 t} \quad (3.166)$$

where we have rewritten the eigenvalue in terms of the parallel diffusion coefficient. On the other hand, the characteristic function of a diffusion equation has the same form, see for example [Shalchi, 2009]. This is indeed what we expect since the solution $f(z, \mu, t)$ becomes diffusive in late times.

3.3.2 Velocity-Position Correlations

Now we consider the velocity-position correlation function. In Fourier space, this is given as

$$\langle \mu_0 \mu e^{-ikz} \rangle = \frac{\pi}{2} \int_{-1}^1 d\mu_0 \int_{-1}^1 d\mu \mu F_k(\mu, t). \quad (3.167)$$

If F_{Airy} is in the set $\{Ai, Bi\}$, then with the transformation $z = c\mu - d\omega_n^2$, we have

$$\int_{-1}^1 d\mu \mu F_{Airy}(c\mu - d\omega_n^2) = \frac{1}{c^2} \int_{-c-d\omega_n^2}^{c-d\omega_n^2} dz z F_{Airy}(z) + \frac{d\omega_n^2}{c^2} \int_{-c-d\omega_n^2}^{c-d\omega_n^2} dz F_{Airy}(z). \quad (3.168)$$

With the Airy differential equation given by Eq. (3.8), we obtain

$$\int_{-c-d\omega_n^2}^{c-d\omega_n^2} dz z F_{Airy}(z) = F'_{Airy}(c - d\omega_n^2) - F'_{Airy}(-c - d\omega_n^2). \quad (3.169)$$

Therefore, by invoking superposition along with the boundary conditions provided by Eq. (3.11), we can see that the first integral on the right-hand side of Eq. (3.168) vanishes. Thus, we have

$$\int_{-1}^1 d\mu \mu g_n(\mu) = \frac{d\omega_n^2}{c} \int_{-1}^1 d\mu g_n(\mu) \quad (3.170)$$

which we have already evaluated in Eq. (3.160). To compute Eq. (3.167), it only remains to put everything together. The idea for this is identical to that used for the characteristic function. We thus derive

$$\begin{aligned} \langle \mu_0 \mu e^{-ikz} \rangle &= \frac{\pi^2 c}{2} \sum_n \frac{1}{\frac{c-d\omega_n^2}{Bi'(c-d\omega_n^2)^2} + \frac{c+d\omega_n^2}{Bi'(-c-d\omega_n^2)^2}} \left[\int_{-1}^1 d\mu \mu g_n(\mu) \right]^2 e^{-\omega_n^2 D t} \\ &= \frac{\pi^2 d^2}{2c} \sum_n \frac{1}{\frac{c-d\omega_n^2}{Bi'(c-d\omega_n^2)^2} + \frac{c+d\omega_n^2}{Bi'(-c-d\omega_n^2)^2}} \left[\int_{-1}^1 d\mu g_n(\mu) \right]^2 \omega_n^4 e^{-\omega_n^2 D t} \\ &= \frac{\pi^2 d^2}{2c^3} \sum_n \frac{[Hi'(c - d\omega_n^2)Bi'(-c - d\omega_n^2) - Hi'(-c - d\omega_n^2)Bi'(c - d\omega_n^2)]^2}{(c - d\omega_n^2)Bi'(-c - d\omega_n^2)^2 + (c + d\omega_n^2)Bi'(c - d\omega_n^2)^2} \omega_n^4 e^{-\omega_n^2 D t}. \end{aligned} \quad (3.171)$$

This result can be compared with Eq. (3.162) to see that the final result between the two expectation values have a similar structure.

3.3.3 The Moments $\langle z^n \rangle$

The moments $\langle z^n \rangle$ can be expressed as

$$\langle z^n \rangle = i^n \pi \left[\frac{\partial^n}{\partial k^n} \int_{-1}^1 d\mu F_k(\mu, t) \right]_{k=0}. \quad (3.172)$$

In what follows, we shall use this formula for $n = 1$ and $n = 2$, respectively. One could proceed by applying Eq. (3.126). Unfortunately, this results in a series of long and complicated calculations. Therefore, as seen below, we will proceed differently.

For the moment $\langle z \rangle$, we first consider the solution given by Eq. (3.4) for small wavenumbers:

$$\int_{-1}^1 d\mu F_k = \sum_{n=0}^{\infty} a_n e^{\omega_n^2 D t} \int_{-1}^1 d\mu g_n(\mu). \quad (3.173)$$

This implies

$$\begin{aligned} \frac{\partial}{\partial k} \int_{-1}^1 d\mu F_k &= \frac{\partial}{\partial k} \left[a_0 e^{-\omega_0^2 D t} \int_{-1}^1 d\mu g_0(\mu) \right] \\ &+ \sum_{n=1}^{\infty} \int_{-1}^1 d\mu g_n(\mu) \frac{\partial}{\partial k} (a_n e^{-\omega_n^2 D t}) \\ &+ \sum_{n=1}^{\infty} a_n e^{-\omega_n^2 D t} \frac{\partial}{\partial k} \int_{-1}^1 d\mu g_n(\mu). \end{aligned} \quad (3.174)$$

For the first term, we expand the relevant functions using Taylor expansions, found in [Olver et al., 2024], up to third-order² in α to derive

$$\lim_{k \rightarrow 0} \frac{\partial}{\partial k} \left[a_0 e^{-\omega_0^2 D t} \int_{-1}^1 d\mu g_0(\mu) \right] = -i \frac{v}{2\pi D} \left[\mu_0 - \frac{1}{3} \mu_0^3 \right] \quad (3.175)$$

where a_0 is found in Eq. (3.24) and the integral is given by Eq. (3.125).

As for the second term in Eq. (3.174), by combining the fact that F_0 corresponds to the spatially-

²By definition of α , this is equivalent to first order in k .

integrated solution of Eq. (3.2) and the normalization constraint for this solution, we have that

$$\lim_{k \rightarrow 0} \int_{-1}^1 d\mu \ g_{n \geq 1}(\mu) = 0. \quad (3.176)$$

This implies that the middle term in Eq. (3.174) vanishes.

The last term in Eq. (3.174) is the most complicated one. Because $|c\mu - d\omega_{n \geq 1}^2|$ approaches infinity as $k \rightarrow 0$ and the exponential nature of Airy functions, it is too complicated to handle it directly. Thus, we evaluate it using a different approach. It begins with noting that if $n = 2m$, then as shown in Section 3.2.6,

$$\lim_{k \rightarrow 0} a_n = C_m \cos(m\pi\mu_0) \quad (3.177)$$

for some constant C_m . On the other hand, if $n = 2m + 1$, then

$$\lim_{k \rightarrow 0} a_n = D_m \sin((m + 1/2)\pi\mu_0). \quad (3.178)$$

Moreover, we have that the term

$$\frac{\partial}{\partial k} \int_{-1}^1 d\mu \ g_n(\mu) \quad (3.179)$$

is independent of μ_0 . Therefore, upon placing it all together, we have that for $n = 1$, Eq. (3.172) yields

$$\langle z \rangle = \frac{v}{2D} \left[\mu_0 - \frac{1}{3}\mu_0^3 \right] + \sum_{m=1}^{\infty} C'_m \cos(m\pi\mu_0) e^{-m^2\pi^2 Dt} + \sum_{m=0}^{\infty} D'_m \sin((m + 1/2)\pi\mu_0) e^{-(m+1/2)^2\pi^2 Dt} \quad (3.180)$$

where C'_m and D'_m are constants independent of μ_0 and the eigenvalues are found by setting $k = 0$ in Eq. (3.121). It remains to determine the constants C'_m and D'_m , which is done by first noting that at $t = 0$, $\langle z \rangle = 0$ as our initial condition states that the particle is originally at $z = 0$. Therefore, Eq. (3.180) becomes

$$-\frac{v}{2D} \left[\mu_0 - \frac{1}{3}\mu_0^3 \right] = \sum_{m=1}^{\infty} C'_m \cos(m\pi\mu_0) + \sum_{m=0}^{\infty} D'_m \sin((m + 1/2)\pi\mu_0) \quad (3.181)$$

which is a Fourier series for the term on the left-hand side. Thus, we can find C'_m and D'_m by using

orthogonality of the cosine and sine functions. First, we multiply Eq. (3.181) by $\cos(n\pi\mu_0)$ and integrate over μ_0 to derive

$$C'_n = 0. \quad (3.182)$$

Alternatively, C'_n can easily be seen to be zero by considering a parity argument. Next, we repeat as above except multiplying both sides by $\sin((n + 1/2)\pi\mu_0)$ to obtain

$$D'_n = -(-1)^n \frac{2}{\pi^4(n + 1/2)^4} \frac{v}{D}. \quad (3.183)$$

Therefore, Eq. (3.180) can be written as

$$\langle z \rangle = \frac{v}{2D} \left[\mu_0 - \frac{1}{3}\mu_0^3 \right] - \frac{2}{\pi^4} \frac{v}{D} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m + 1/2)^4} \sin((m + 1/2)\pi\mu_0) e^{-(m+1/2)^2\pi^2 Dt}. \quad (3.184)$$

The above expression is exact for all times. In what follows, we simplify the result in early and late times. It is evident that we have

$$\langle z \rangle (t \rightarrow \infty) = \frac{v}{2D} \left[\mu_0 - \frac{1}{3}\mu_0^3 \right]. \quad (3.185)$$

As for early times, we aim to show $\langle z \rangle \propto t$. To begin, we rewrite Eq. (3.184) as

$$\begin{aligned} \langle z \rangle = & \frac{v}{2D} \left[\mu_0 - \frac{1}{3}\mu_0^3 \right] - \frac{2}{\pi^4} \frac{v}{D} \sum_{m=0}^N \frac{(-1)^m}{(m + 1/2)^4} \sin((m + 1/2)\pi\mu_0) e^{-(m+1/2)^2\pi^2 Dt} \\ & - \frac{2}{\pi^4} \frac{v}{D} \sum_{m=N+1}^{\infty} \frac{(-1)^m}{(m + 1/2)^4} \sin((m + 1/2)\pi\mu_0) e^{-(m+1/2)^2\pi^2 Dt} \end{aligned} \quad (3.186)$$

for some large integer N . For the first series, we can approximate

$$e^{-(m+1/2)^2\pi^2 Dt} \approx 1 - (m + 1/2)^2\pi^2 Dt \quad (3.187)$$

using a Taylor expansion with the assumption that the argument in the exponential is small. We can also approximate the second series in Eq. (3.186) using this Taylor expansion since the terms decay via

$1/m^4$, and thus each term is effectively zero with or without the Taylor expansion in Eq. (3.187). Thus, we can simplify Eq. (3.184) to

$$\begin{aligned} \langle z \rangle (t \ll 1) = & \frac{v}{2D} \left[\mu_0 - \frac{1}{3} \mu_0^3 \right] - \frac{2}{\pi^4} \frac{v}{D} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1/2)^4} \sin((m+1/2)\pi\mu_0) \\ & + \frac{2}{\pi^2} vt \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1/2)^2} \sin((m+1/2)\pi\mu_0). \end{aligned} \quad (3.188)$$

By Eqs. (3.181)-(3.183), we have that the first series and the first term cancel, which simplifies the result to

$$\langle z \rangle (t \ll 1) = \frac{2}{\pi^2} vt \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1/2)^2} \sin((m+1/2)\pi\mu_0). \quad (3.189)$$

Now notice that the Fourier series of μ_0 over $[-1, 1]$ is given by

$$\mu_0 = \frac{2}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1/2)^2} \sin((m+1/2)\pi\mu_0) \quad (3.190)$$

which can be easily verified. Thus, we have

$$\langle z \rangle (t \ll 1) = \mu_0 vt. \quad (3.191)$$

The physical interpretation of this result is that initially, the magnetic field does not have time to affect the particle, and thus the particle experiences a brief period of no net force.

It does not seem feasible to obtain a simple exact result for $\langle z^2 \rangle$, and thus we only derive a result for the late time limit. By combining Eq. (3.172) with Eq. (3.126) and keeping only the largest term for late times, we have

$$\langle z^2 \rangle = \lim_{k \rightarrow 0} \pi \frac{Da_0}{c} \left[\frac{\text{Hi}'(c - d\omega_0^2)}{\text{Bi}'(c - d\omega_0^2)} - \frac{\text{Hi}'(-c - d\omega_0^2)}{\text{Bi}'(-c - d\omega_0^2)} \right] t e^{-\omega_0^2 D t} \frac{\partial^2}{\partial k^2} \omega_0^2. \quad (3.192)$$

As we are interested in the case of the smallest eigenvalue and small wavenumbers, we have that by Eq. (3.47),

$$\frac{\partial^2}{\partial k^2} \omega_0^2 = 2 \frac{\kappa_{\parallel 0}}{D}. \quad (3.193)$$

Moreover, with linear Taylor expansions, we can simplify

$$\lim_{k \rightarrow 0} \left[\frac{\text{Hi}'(c - d\omega_0^2)}{\text{Bi}'(c - d\omega_0^2)} - \frac{\text{Hi}'(-c - d\omega_0^2)}{\text{Bi}'(-c - d\omega_0^2)} \right] = \frac{2c}{\pi \text{Bi}'(0)} \quad (3.194)$$

where we have used Eq. (3.165). Moreover, by Eq. (3.145), we have

$$\lim_{k \rightarrow 0} a_0 = \frac{1}{2} \text{Bi}'(0). \quad (3.195)$$

Placing these results into Eq. (3.192) implies

$$\langle z^2 \rangle (t \rightarrow \infty) = 2\kappa_{\parallel 0} t. \quad (3.196)$$

This is again indeed what we expect for diffusion in late times [Shalchi, 2009].

3.4 Fast Analytical - Numerical Hybrid Solution

The big downside with Airy functions is that numerically computing them can take a considerable amount of time, as Table 3.2 illustrates. Hence, in this section we develop a variation of the *subspace method*, which we cover in significantly more detail in Chapters 4 and 5. We will see that the most natural choice when using the subspace method is to employ orthogonal polynomials. The problem here is that we have the condition which requires the derivative to vanish at the boundaries. This renders orthogonal polynomials of no use in this instance. We elect to instead use a Fourier series.

We begin with writing the solution of Eq. (3.3) as the Fourier expansion

$$F_k(\mu, t) = C_0(t) + \sum_{n=1}^{\infty} C_n(t) \cos(n\pi\mu) + \sum_{n=0}^{\infty} D_n(t) \sin((n + 1/2)\pi\mu). \quad (3.197)$$

Evidently, determining the solution is a matter of deriving an expression for each $C_n(t)$ and $D_n(t)$. The idea here is to use Eq. (3.3) and orthogonality of trigonometric functions. It should be noted that the term $ivk\mu F_k$ renders this process more difficult, which motivates us to also write μ in terms of its Fourier

transform

$$\mu = \sum_{n=0}^{\infty} (-1)^n \frac{2}{(n+1/2)^2 \pi^2} \sin((n+1/2)\pi\mu). \quad (3.198)$$

Combining Eq. (3.197) with (3.198) results in

$$\begin{aligned} \mu F_k = & \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1/2)^2 \pi^2} \left[2C_0 \sin((n+1/2)\pi\mu) \right. \\ & + \sum_{m=1}^{\infty} C_m (\sin((n+m+1/2)\pi\mu) + \sin((n-m+1/2)\pi\mu)) \\ & \left. + \sum_{m=0}^{\infty} D_m (\cos((n-m)\pi\mu) - \cos((n+m+1)\pi\mu)) \right] \end{aligned} \quad (3.199)$$

where we have used the identities

$$2 \sin \alpha \cos \beta = \sin(\alpha + \beta) + \sin(\alpha - \beta) \quad (3.200)$$

and

$$2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta). \quad (3.201)$$

Placing Eq. (3.199) into (3.3) results in

$$\begin{aligned} & \dot{C}_0 + \sum_{n=1}^{\infty} \dot{C}_n \cos(n\pi\mu) + \sum_{n=0}^{\infty} \dot{D}_n \sin((n+1/2)\pi\mu) \\ & + ivk \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1/2)^2 \pi^2} \left[2C_0 \sin((n+1/2)\pi\mu) \right. \\ & + \sum_{m=1}^{\infty} C_m (\sin((n+m+1/2)\pi\mu) + \sin((n-m+1/2)\pi\mu)) \\ & \left. + \sum_{m=0}^{\infty} D_m (\cos((n-m)\pi\mu) - \cos((n+m+1)\pi\mu)) \right] \\ & = -D \sum_{n=1}^{\infty} n^2 \pi^2 C_n \cos(n\pi\mu) - D \sum_{n=0}^{\infty} (n+1/2)^2 \pi^2 D_n \sin((n+1/2)\pi\mu) \end{aligned} \quad (3.202)$$

where the dot represents a time derivative. In what follows, we use the orthogonality identities

$$\int_{-1}^1 d\mu \cos(n\pi\mu) \cos(m\pi\mu) = \delta_{nm} \quad (3.203)$$

and

$$\int_{-1}^1 d\mu \sin((n+1/2)\pi\mu) \cos((m+1/2)\pi\mu) = \delta_{nm} \quad (3.204)$$

where δ_{nm} is the Kronecker delta. To continue, we integrate Eq. (3.202) with respect to μ over $[-1, 1]$ to obtain

$$\dot{C}_0 = -ivk \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1/2)^2 \pi^2} D_n. \quad (3.205)$$

Next, multiplying (3.202) by $\cos(\ell\pi\mu)$ and integrating with respect to μ over $[-1, 1]$ results in

$$\dot{C}_\ell = -ivk \sum_{m=0}^{\infty} \left[(-1)^{m+\ell} \frac{1}{(m+\ell+1/2)^2 \pi^2} + (-1)^{m-\ell} \frac{1}{(m-\ell+1/2)^2 \pi^2} \right] D_m - D \ell^2 \pi^2 C_\ell. \quad (3.206)$$

Finally, we multiply Eq. (3.202) by $\sin((\ell+1/2)\pi\mu)$ and integrate with respect to μ over $[-1, 1]$ to derive

$$\begin{aligned} \dot{D}_\ell = & -ivk(-1)^\ell \frac{2}{(\ell+1/2)^2 \pi^2} C_0 - ivk \sum_{m=1}^{\infty} \left[\frac{(-1)^{\ell-m}}{(\ell-m+1/2)^2 \pi^2} + \frac{(-1)^{m+\ell}}{(m+\ell+1/2)^2 \pi^2} \right] C_m \\ & - D(\ell+1/2)^2 \pi^2 D_\ell. \end{aligned} \quad (3.207)$$

We can write Eqs. (3.205)-(3.207) as the following linear system:

$$\dot{\vec{C}} = \mathbf{M} \vec{C} \quad (3.208)$$

with

$$\vec{C} = \begin{pmatrix} C_0 \\ D_0 \\ C_1 \\ D_1 \\ C_2 \\ D_2 \\ \vdots \end{pmatrix} \quad (3.209)$$

and the entries of the matrix $\mathbf{M}$ are obtained from Eqs. (3.205)-(3.207).

The differential equation (3.208) has the formal solution

$$\vec{C}(t) = e^{\mathbf{M}t} \vec{C}(0) \quad (3.210)$$

where we have used the matrix exponential. We determine the initial values by using the initial condition provided by Eq. (2.34) implying

$$\frac{1}{\pi} \delta(\mu - \mu_0) = C_0(0) + \sum_{n=1}^{\infty} C_n(0) \cos(n\pi\mu) + \sum_{n=0}^{\infty} D_n(0) \sin((n + 1/2)\pi\mu). \quad (3.211)$$

By orthogonality, we have

$$\begin{aligned} C_0(0) &= \frac{1}{2\pi}, \\ C_{n \geq 1}(0) &= \frac{1}{\pi} \cos(n\pi\mu_0), \\ D_n(0) &= \frac{1}{\pi} \sin((n + 1/2)\pi\mu_0). \end{aligned} \quad (3.212)$$

In order for this method to be practical, we must truncate our expansion of F_k . Here, we elect to include the first ten terms. While a brief demonstration of the effectiveness of the 10-dimensional

subspace approximation is provided in this chapter, subsequent chapters present a significantly more detailed justification for this approximation. Figures 3.12-3.14 illustrate the solution of Eq. (3.2) using the Airy function method at various values of Dt and an initial value of $\mu_0 = 0.5$. For each given time, we also include the absolute difference between the Airy function method and the 10-dimensional subspace method. From these figures, we can see that the subspace method provides an excellent approximation.

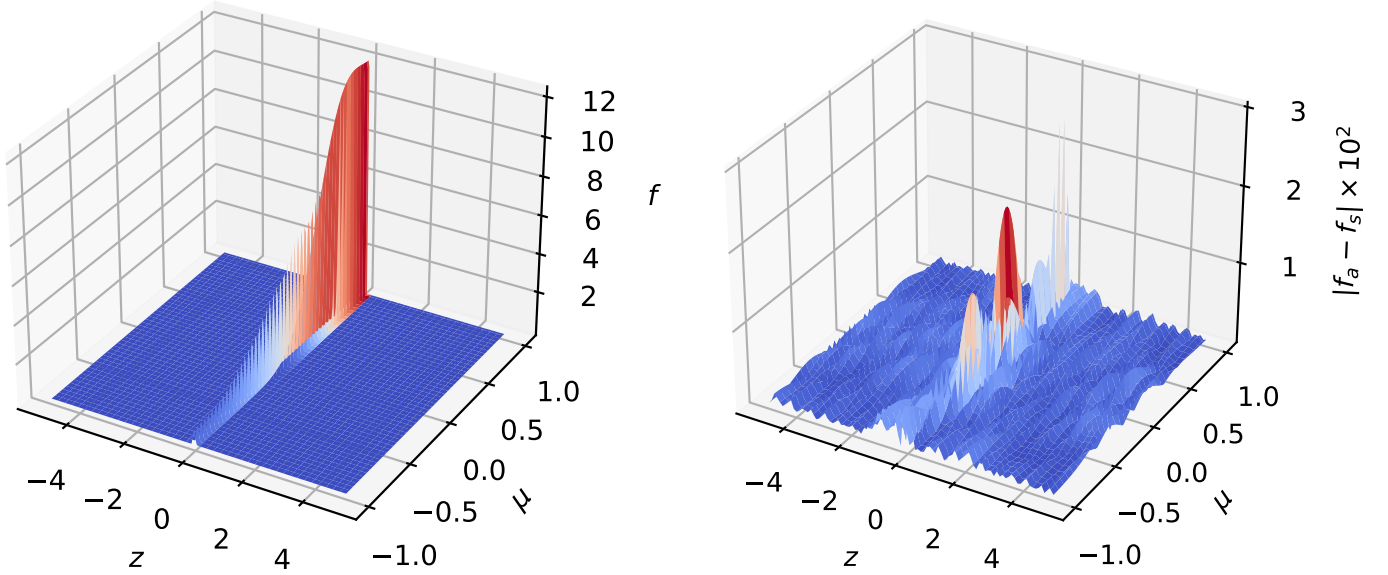


Figure 3.12: Distribution functions at the time $Dt = 0.2$ for an initial value of $\mu_0 = 0.5$. The left panel shows our analytical solution and the right panel illustrates the absolute difference between the Airy function and the 10-dimensional subspace methods.

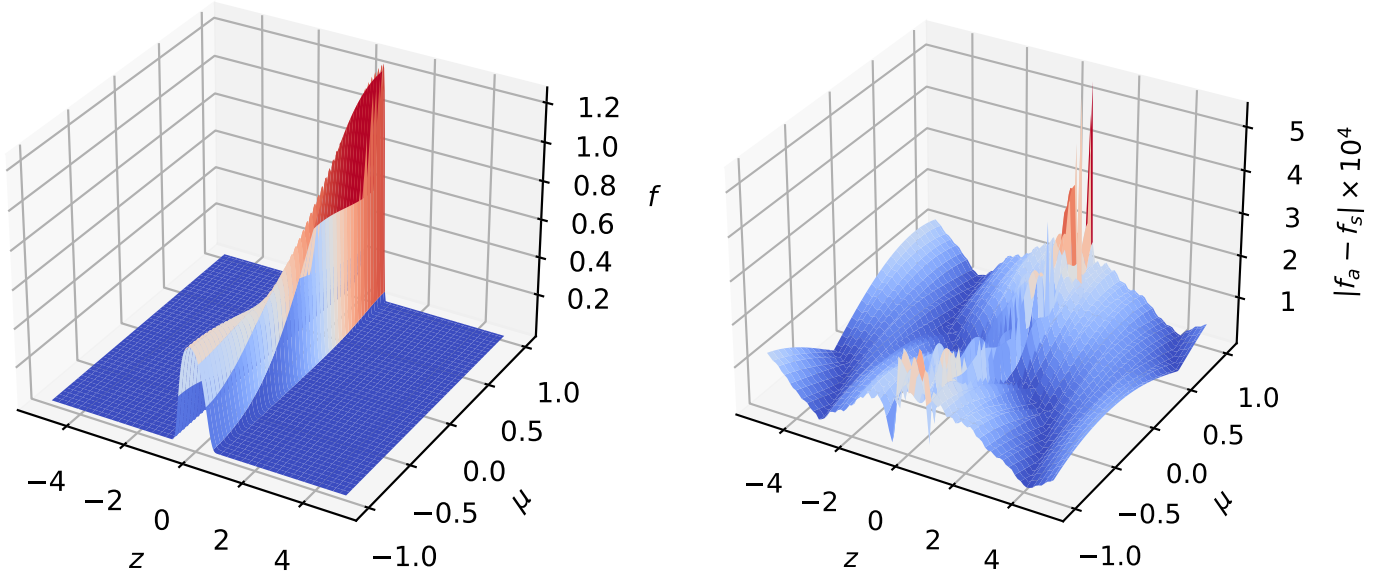


Figure 3.13: Caption is as in Figure 3.12 but the time is $Dt = 1$

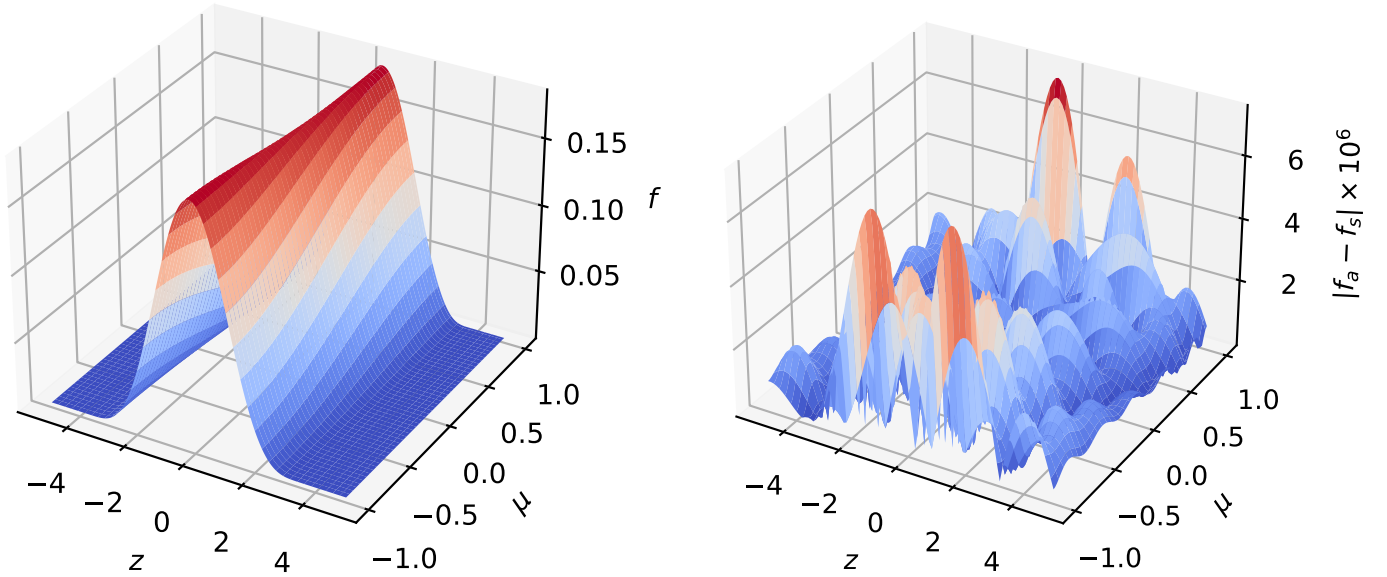


Figure 3.14: Caption is as in Figure 3.12 but the time is $Dt = 5$

We also compare the time taken to compute the two solutions in Table 3.2. For both solutions, we compute the solution in Fourier space up to the wavenumber such that $|F_k| \approx e^{-10}$, which is determined with the eigenvalues found using the Airy function method. Moreover, we compute each F_k for 50 equispaced values of μ for both methods. It should be noted that here we only consider the time required

to compute the solution in Fourier space, and neglect the time to compute the Fourier transform integral.

The following table illustrates several facts. First, we see that for larger values of Dt , the Airy function method is comparable to the subspace method. However, if we compute the solution for decreasing values of Dt , the subspace method becomes significantly quicker. Moreover, computing the solution becomes significantly quicker, regardless of the method, when dealing with later times. This is what we expect as when the value of Dt increases, Eq. (3.4) shows that the solution decays significantly faster. This in turn implies that the solution in Fourier space is required for fewer wavenumbers.

Dt	Airy Method	Subspace Method
10	11.54	2.31
5	32.10	6.30
2	191.02	25.51
1	767.73	71.48
0.5	3815.06	206.17

Table 3.2: For various values of Dt , we compare the times in seconds to compute the solution using the 10-dimensional subspace approximation to the time to compute the solution using the Airy function method.

Orthogonality of sine and cosine also allows us to derive a relatively simple form for the μ -integrated solution:

$$\begin{aligned}
M(z, t) &= \frac{1}{2} \int_{-1}^1 d\mu f(z, \mu, t) \\
&= \frac{1}{2} \int_{-\infty}^{\infty} dk e^{ikz} \int_{-1}^1 d\mu F_k(\mu, t) \\
&= \int_{-\infty}^{\infty} dk e^{ikz} C_0(k, t).
\end{aligned} \tag{3.213}$$

In Figs. 3.15-3.17 we compare the μ -integrated solution $M(z, t)$ for the two methods and again demonstrate very good agreement.

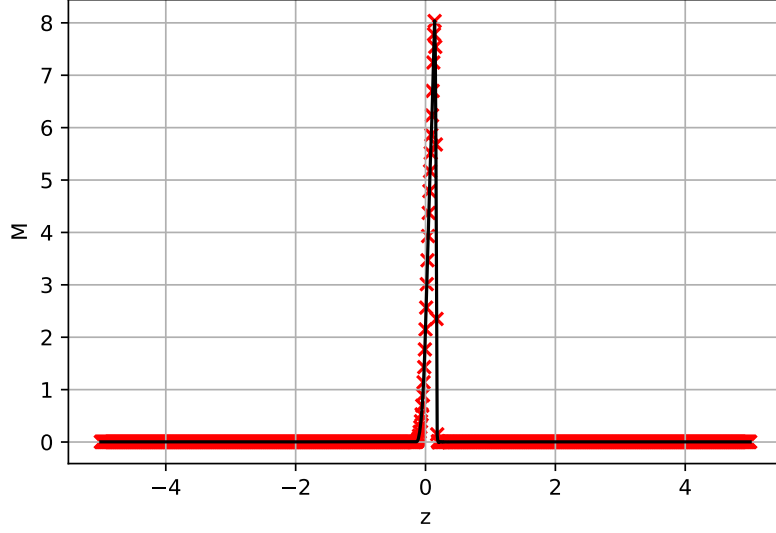


Figure 3.15: The solid black line is our analytical solution using Airy functions, whereas the red crosses represent the subspace approximation. Here we have used $Dt = 0.2$ and $\mu_0 = 0.5$.

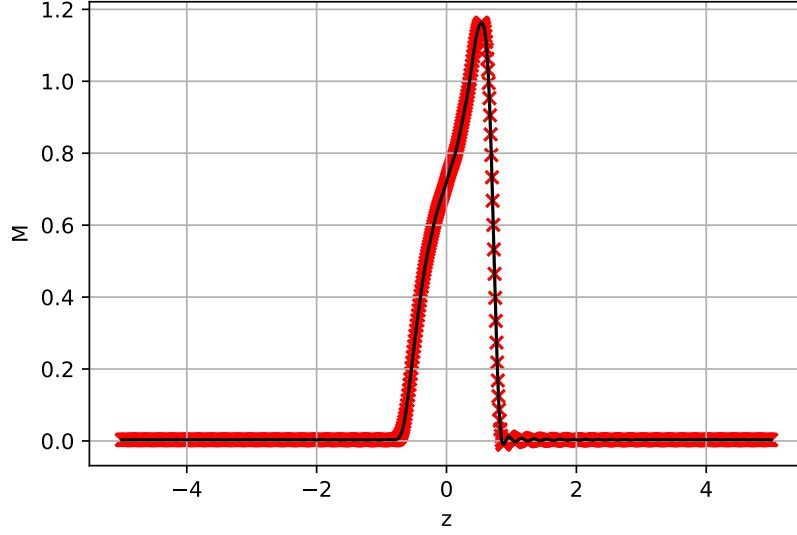


Figure 3.16: Caption is as in Fig. 3.15 but we have used $Dt = 1$.

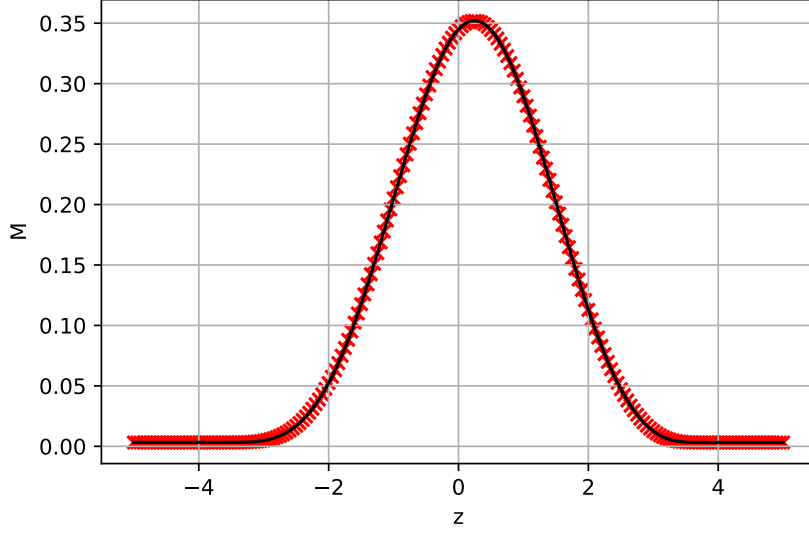


Figure 3.17: Caption is as in Fig. 3.15 but we have used $Dt = 5$.

3.5 Conclusion

In this chapter, we derived an analytical solution of the space-dependent Fokker-Planck equation. Deriving an analytical solution is motivated by the fact that there is yet to be an exact analytical solution of even the simplest form of the space-dependent Fokker-Planck equation. It is shown in [Lasuik and Shalchi, 2017] that indeed, there is a simple and exact analytical solution of the spatially-integrated Fokker-Planck equation, depending on the form of the pitch-angle scattering diffusion coefficient.

This project was a partial success. We succeeded in showing that the solution in Fourier space is given in terms of Airy functions. Additionally, constants which depend on the initial condition are present, which we show have a simple form. On the other hand, the expressions for the eigenvalues, which depend on the boundary conditions, are far from satisfactory. As Table 3.1 illustrates, there are several cases, some of which are quite unpleasant. This leaves the door open for future work that has a great potential to improve our solution.

Once we obtained our solution, we computed several expectation values. Some of these values have

simple forms, in particular in late times. Although we were able to verify that our analytical solution is accurate by comparing with the numerical solution, we found that the computations of our solution are very expensive. This suggests that rather than attempting to derive a complicated exact solution, it might be more efficient to focus on a much cleaner approximate solution. We briefly saw this idea in the form of a variation of the subspace method, and found that it provides a nearly exact approximation while still being computationally cheaper. In the following chapters, we explore this concept in significantly more detail.

4

Subspace Approximations to the Focused Transport Equation

I. The Standard Form

4.1 Introduction

Expanding functions in a series consisting of Legendre polynomials has been common practice for over two centuries. These polynomials were first introduced by Adrien-Marie Legendre in 1782, when they expressed the multipole expansion of the gravitational or electromagnetic potential as a series of Legendre polynomials. Another famous example is found in the solution of the Schrödinger equation in the setting of the hydrogen atom. Use of Legendre polynomial expansions to solve the Fokker-Planck equation can be found in [Zank et al., 2000a, Lasuik and Shalchi, 2019, Shalchi, 2024]. The latter two articles developed a framework known as the *subspace approximation method*.

When using the subspace method, there are two options. The first is to use a low dimension which suffers in terms of accuracy, but has the potential to produce very simple results. The second route is to employ a higher dimension, one where analytical results are no longer feasible. However, this method can provide a near-perfect approximation which is still significantly quicker than pure numerical

solutions. Both of these methods have been employed in the past when focusing is not present. See [Lasuik and Shalchi, 2019] for the first method, and [Shalchi, 2024] for the second.

Both of the above papers considered the Fokker-Planck equation with isotropic scattering and without focusing. In this chapter, we seek to extend both of those results to when the focusing effect is included. That is, we work with

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = D \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial f}{\partial \mu} \right] - \frac{v}{2L} (1 - \mu^2) \frac{\partial f}{\partial \mu}. \quad (4.1)$$

The content of this chapter can be found in [Klippenstein and Shalchi, 2025c]. Some of the content including the figures throughout this chapter are copied from the cited article in The Open Journal of Astrophysics, which is licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0), permitting reuse with appropriate attribution.

4.2 Fundamentals of the Subspace Method

In Fourier space, Eq. (4.1) is given as

$$\frac{\partial F_k}{\partial t} + ivk\mu \frac{\partial F_k}{\partial z} = D \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial F_k}{\partial \mu} \right] - \frac{v}{2L} (1 - \mu^2) \frac{\partial F_k}{\partial \mu}. \quad (4.2)$$

We attempt to solve Eq. (4.2) via the expansion

$$F_k(\mu, t) = \sum_{n=0}^{N-1} C_n(t) P_n(\mu) \quad (4.3)$$

where we have used the *Legendre Polynomials* $P_n(\mu)$, the currently unknown functions of time $C_n(t)$, and an integer N . The main portion of this method comes down to the derivation of expressions for the functions C_n , which follows from Eq. (4.2) along with properties of Legendre polynomials. Additionally, choosing the value N is up to the user of this method. We discuss certain values of interest in later sections.

The first step to determine the unknown functions C_n is to place the above expansion into Eq. (4.2)

yielding

$$\sum_{n=0}^{N-1} \dot{C}_n P_n + iv\mu k \sum_{n=0}^{N-1} C_n P_n = D \sum_{n=0}^{N-1} C_n \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial P_n}{\partial \mu} \right] - \frac{v}{2L} (1 - \mu^2) \sum_{n=0}^N C_n \frac{\partial P_n}{\partial \mu} \quad (4.4)$$

where $\dot{C}_n$ denotes the time derivative of the coefficient C_n . We aim to derive a system of differential equations for the functions C_n . The first step is to remove the derivatives of the polynomials along with any explicit μ -dependence from Eq. (4.4). This can be done through applying several properties of Legendre polynomials, all of which can be found in [Abramowitz and Stegun, 1964] and [Olver et al., 2024]. We have:

$$\begin{aligned} \mu P_n &= \frac{n+1}{2n+1} P_{n+1} + \frac{n}{2n+1} P_{n-1} \\ \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial P_n}{\partial \mu} \right] &= -n(n+1) P_n \\ (1 - \mu^2) \frac{dP_n}{d\mu} &= n \frac{n+1}{2n+1} (P_{n-1} - P_{n+1}). \end{aligned} \quad (4.5)$$

‘ Placing each of these relations into Eq. (4.4) yields

$$\begin{aligned} \sum_{n=0}^{N-1} \dot{C}_n P_n &= -ivk \sum_{n=0}^{N-1} C_n \left[\frac{n+1}{2n+1} P_{n+1} + \frac{n}{2n+1} P_{n-1} \right] - D \sum_{n=0}^{N-1} C_n n(n+1) P_n \\ &\quad + \frac{v}{2L} \sum_{n=0}^{N-1} C_n \frac{n(n+1)}{2n+1} (P_{n+1} - P_{n-1}). \end{aligned} \quad (4.6)$$

We now multiply this equation by the Legendre polynomial P_m , integrate over all μ , and use the orthogonality relation found in [Abramowitz and Stegun, 1964]

$$\int_{-1}^1 d\mu P_n(\mu) P_m(\mu) = \frac{2}{2m+1} \delta_{nm} \quad (4.7)$$

for the Kronecker delta δ_{nm} . With this, we derive

$$\begin{aligned}\dot{C}_m = & -Dm(m+1)C_m - ivk \left[\frac{m}{2m-1}C_{m-1} + \frac{m+1}{2m+3}C_{m+1} \right] \\ & + \frac{v}{2L} \left[\frac{m(m-1)}{2m-1}C_{m-1} - \frac{(m+1)(m+2)}{2m+3}C_{m+1} \right].\end{aligned}\quad (4.8)$$

This leads to a system of N coupled ordinary differential equations. We can summarize this with the following linear system:

$$\dot{\vec{C}} = \mathbf{M}\vec{C} \quad (4.9)$$

where we have used

$$\vec{C}(t) = \begin{pmatrix} C_0(t) \\ C_1(t) \\ \vdots \\ C_{N-1}(t) \end{pmatrix} \quad \text{and} \quad \dot{\vec{C}}(t) = \begin{pmatrix} \dot{C}_0(t) \\ \dot{C}_1(t) \\ \vdots \\ \dot{C}_{N-1}(t) \end{pmatrix}, \quad (4.10)$$

and the tridiagonal matrix $\mathbf{M}$ whose entries of the m^{th} row are given by the coefficients of Eq. (4.8).

The first several entries are

$$\mathbf{M} = \begin{pmatrix} 0 & -\frac{1}{3}ivk - \frac{v}{3L} & 0 & \dots \\ -ivk & -2D & -\frac{2}{5}ivk - \frac{3v}{5L} & \ddots \\ 0 & -\frac{2}{3}ivk + \frac{v}{3L} & -6D & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \quad (4.11)$$

corresponding to an $N \times N$ matrix. Eq. (4.9) is a well-known linear system of differential equations with the formal solution

$$\vec{C}(t) = e^{\mathbf{M}t}\vec{C}(0) \quad (4.12)$$

where we have used the matrix exponential. The column vector $\vec{C}(0)$ is determined using the initial

conditions. In combination with expansion (4.3), this gives us

$$\frac{1}{\pi} \delta(\mu - \mu_0) = \sum_{n=0}^{N-1} C_n(t=0) P_n(\mu). \quad (4.13)$$

Multiplying this equation by P_m and invoking Eq. (4.7) results in

$$C_m(t=0) = \frac{2m+1}{2\pi} P_m(\mu_0). \quad (4.14)$$

By combining Eqs. (4.3), (4.12) and (4.14), we have completely determined our N -dimensional subspace approximations.

Orthogonality of the Legendre polynomials also allows us to derive a pleasant form for the μ -integrated distribution function defined by

$$M(z, t) = \frac{1}{2} \int_{-1}^1 d\mu f(z, \mu, t). \quad (4.15)$$

Using Eq. (2.27) therein yields

$$M(z, t) = \frac{1}{2} \int_{-\infty}^{\infty} dk e^{ikz} \int_{-1}^1 d\mu F_k(\mu, t). \quad (4.16)$$

The μ -integral can be evaluated using Eqs. (4.3) and (4.7) which give

$$M(z, t) = \int_{-\infty}^{\infty} dk e^{ikz} \int_{-1}^1 d\mu C_0(k, t). \quad (4.17)$$

To continue, we must specify the value of N . The first option is to take N to be at most three. This leads to the possibility of a simple approximation of the solution, albeit not necessarily very accurate. Alternatively, we can let N be much larger than three. In this case, Eq. (4.12) cannot be simplified analytically. However, if N is sufficiently large, yet not too large, then the resulting expansion given by Eq. (4.3) can provide a very accurate, yet computationally cheap approximation.

4.3 Analytical Results of the Subspace Method

Provided N is sufficiently small in Eq. (4.3), then it is possible to derive simple analytical results. It was shown in [Lasuik and Shalchi, 2019] that if the subspace method is used to solve the Fokker-Planck equation without focusing, then $N = 1$ results in an inaccurate approximation. If we take $N = 2$, then we have a simple, yet decent approximation. They also showed that $N = 3$ provides only slightly more accuracy than $N = 2$, however, the formula for the eigenvalues is too complicated in the general case¹. These findings motivate us to primarily consider the $N = 2$ case here. As we shall see, one downside with the 2-dimensional approximation is that some expectation values are independent of the focusing parameter, which disagrees with previous and well-accepted theories. Consequently, we will also apply the 3-dimensional subspace method to derive certain results which avoid the aforementioned problem.

4.3.1 2-Dimensional Subspace Approximation

In the case of the 2-dimensional subspace approximation, the expansion provided by Eq. (4.3) is simplified to

$$F_k(\mu, t) = C_0(t) + \mu C_1(t) \quad (4.18)$$

which follows from the fact that the first two Legendre polynomials are given by $P_0(\mu) = 1$ and $P_1(\mu) = \mu$. Additionally, Eq. (4.9) becomes

$$\begin{pmatrix} \dot{C}_0 \\ \dot{C}_1 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3}ivk - \frac{v}{3L} \\ -ivk & -2D \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \end{pmatrix}. \quad (4.19)$$

The *ansatz*

$$C_0(t) = b_0 e^{\omega t} \quad \text{and} \quad C_1(t) = b_1 e^{\omega t} \quad (4.20)$$

¹The expression is manageable if we consider the case of small wavenumbers

leads to the matrix equation

$$\omega \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3}ivk - \frac{v}{3L} \\ -ivk & -2D \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \quad (4.21)$$

corresponding to a simple eigenvalue problem. Alternatively, this can be written as

$$\begin{pmatrix} -\omega & -\frac{1}{3}ivk - \frac{v}{3L} \\ -ivk & -2D - \omega \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = 0. \quad (4.22)$$

The characteristic equation of this system is

$$\omega^2 + 2D\omega + \frac{1}{3}v^2k^2 - i\frac{v^2k}{3L} = 0. \quad (4.23)$$

This quadratic equation has the two solutions

$$\omega_{\pm} = -D \pm \sqrt{D^2 - \frac{1}{3}v^2k^2 + i\frac{v^2k}{3L}}. \quad (4.24)$$

These two values for ω can be used in the ansatz given by Eq. (4.20). The general solution for the function $C_0(t)$ is, therefore, given in terms of the superposition

$$C_0(t) = b_+e^{\omega_+t} + b_-e^{\omega_-t}. \quad (4.25)$$

By combining Eqs. (4.19) and (4.25), we also obtain

$$C_1(t) = \frac{i3}{vk - i\frac{v}{L}} \left(\omega_+ b_+ e^{\omega_+t} + \omega_- b_- e^{\omega_-t} \right). \quad (4.26)$$

Eq. (4.14) states that

$$C_0(0) = \frac{1}{2\pi} \quad \text{and} \quad C_1(0) = \frac{3}{2\pi}\mu_0. \quad (4.27)$$

Together, Eqs. (4.25)-(4.27) imply the linear system for b_+ and b_- :

$$b_+ + b_- = \frac{1}{2\pi} \quad (4.28)$$

and

$$\frac{i3}{vk - i\frac{v}{L}} (\omega_+ b_+ + \omega_- b_-) = \frac{3\mu_0}{2\pi}. \quad (4.29)$$

Eqs. (4.28) and (4.29) can be combined to determine the parameters we seek. After some straightforward algebra we find

$$\begin{aligned} b_+ &= -\frac{1}{2\pi} \frac{\omega_- + iv\mu_0 k + \frac{v\mu_0}{L}}{\omega_+ - \omega_-}, \\ b_- &= \frac{1}{2\pi} \frac{\omega_+ + iv\mu_0 k + \frac{v\mu_0}{L}}{\omega_+ - \omega_-}. \end{aligned} \quad (4.30)$$

These formulas can be used in Eqs. (4.25) and (4.26) and our coefficients $C_0(t)$ and $C_1(t)$ are completely determined. Our solution for the Fourier-transformed distribution function as provided by Eq. (4.18) can, therefore, be written as

$$F_k(\mu, t) = \left[1 + \frac{3i\mu\omega_+}{vk - i\frac{v}{L}} \right] b_+ e^{\omega_+ t} + \left[1 + \frac{3i\mu\omega_-}{vk - i\frac{v}{L}} \right] b_- e^{\omega_- t} \quad (4.31)$$

where the the eigenvalues $\omega_{\pm}$ are provided by Eq. (4.24) and the coefficients $b_{\pm}$ are given by Eq. (4.30).

4.3.2 2-Dimensional Subspace Expectation Values

In the following, we apply the solution resulting from the 2-dimensional subspace approximation to obtain various expectation values. See Section 2.2 for motivation behind considering each of the following values, along with more details considering any calculations.

4.3.2.1 The Characteristic Function

The characteristic function is given by

$$\langle e^{-ikz} \rangle = \frac{\pi}{2} \int_{-1}^1 d\mu \int_{-1}^1 d\mu_0 F_k(\mu, t) \quad (4.32)$$

If we average Eq. (4.18) over all μ , we find

$$\frac{1}{2} \int_{-1}^1 d\mu F_k(\mu, t) = C_0(t) \quad (4.33)$$

where we have used orthogonality of the Legendre polynomials as given by Eq. (4.7). Using the result provided by Eq. (4.25) therein yields

$$\langle e^{-ikz} \rangle = \pi \int_{-1}^1 d\mu_0 [b_+ e^{\omega_+ t} + b_- e^{\omega_- t}] \quad (4.34)$$

After using Eq. (4.30) we can solve the remaining integral to obtain

$$\langle e^{-ikz} \rangle = \frac{\omega_+}{\omega_+ - \omega_-} e^{\omega_- t} - \frac{\omega_-}{\omega_+ - \omega_-} e^{\omega_+ t} \quad (4.35)$$

where the parameters ω_+ and ω_- are given by Eq. (4.24). In the following we consider special cases with the aim to simplify this result.

4.3.2.1.1 Small Wavenumbers

It is convenient to use the short notation

$$\Omega := \sqrt{D^2 - \frac{1}{3}v^2 k^2 + i \frac{v^2 k}{3L}} \quad (4.36)$$

so that Eq. (4.24) can be written as

$$\omega_{\pm} = -D \pm \Omega. \quad (4.37)$$

We suppose that small wavenumbers is equivalent to

$$\left| -v^2 k^2 + i \frac{v^2 k}{L} \right| \ll 3D^2. \quad (4.38)$$

After Taylor-expanding the square root given by Eq. (4.36), we find up to second order the approximation

$$\Omega = D + i \frac{v^2}{6DL} k + \left(\frac{v^4}{36L^2 D^3} - \frac{v^2}{3D} \right) \frac{1}{2} k^2. \quad (4.39)$$

For isotropic pitch-angle scattering and without focusing, the parallel spatial diffusion coefficient defined in Eq. (2.2.5) is given by

$$\kappa_{\parallel 0} = \frac{v^2}{6D}. \quad (4.40)$$

Therefore, our result can be written as

$$\Omega = D + i \frac{\kappa_{\parallel 0}}{L} k - \kappa_{\parallel 0} \left(1 - \frac{v^2}{12L^2 D^2} \right) k^2. \quad (4.41)$$

Using this expansion in Eq. (4.24) yields

$$\omega_+ = i \frac{\kappa_{\parallel 0}}{L} k - \kappa_{\parallel 0} \left(1 - \frac{v^2}{12L^2 D^2} \right) k^2 \quad (4.42)$$

as well as

$$\omega_- = -\frac{v}{\lambda_{\parallel 0}} - i \frac{\kappa_{\parallel 0}}{L} k + \kappa_{\parallel 0} \left(1 - \frac{v^2}{12L^2 D^2} \right) k^2 \quad (4.43)$$

where we have used the parallel mean free path without focusing given in Eq. (2.70) which states

$$\lambda_{\parallel 0} = \frac{v}{2D}. \quad (4.44)$$

We can further simplify the result given by Eq. (4.43) by only keeping the lowest order terms for both the real and imaginary components to obtain

$$\omega_- = -\frac{v}{\lambda_{\parallel 0}} - i\frac{\kappa_{\parallel 0}}{L}k. \quad (4.45)$$

It should be noted that without focusing, the coefficient of k^2 in Eq. (4.42) reduces down to simply $-\kappa_{\parallel 0}$. This motivates us to rewrite Eq. (4.42) as

$$\omega_+ = -\kappa_{\parallel}^{msd}k^2 + i\frac{\kappa_{\parallel 0}}{L}k \quad (4.46)$$

where we have defined the parallel diffusion coefficient with focusing

$$\kappa_{\parallel}^{msd} = \kappa_{\parallel 0} \left(1 - \frac{\lambda_{\parallel 0}^2}{3L^2} \right). \quad (4.47)$$

It has not yet been definitively established; however, as we will show below, the resulting parallel diffusion coefficient corresponds exactly to the coefficient obtained from the mean square displacement definition. This result can be interpreted to imply that the focusing effect leads to a reduction in the parallel diffusion coefficient. As we shall see in Chapter 6 where we compute the diffusion coefficient exactly, the result given by Eq. (4.47) has the correct structure.

By placing the approximated expressions of the eigenvalues given by Eqs. (4.45) and (4.46) into Eq. (4.35), we obtain

$$\langle e^{-ikz} \rangle = \frac{i\frac{\kappa_{\parallel 0}}{L}k - \kappa_{\parallel}^{msd}k^2}{2\Omega} e^{-vt/\lambda_{\parallel 0} - i\kappa_{\parallel 0}kt/L} + \frac{\frac{v}{\lambda_{\parallel 0}} + i\frac{\kappa_{\parallel 0}}{L}k}{2\Omega} e^{i\kappa_{\parallel 0}kt/L - \kappa_{\parallel}^{msd}k^2t}. \quad (4.48)$$

Since we have used already the condition given by Eq. (4.38), we can further simplify our result by noticing that the first exponential decays significantly quicker than the second. This leads to

$$\langle e^{-ikz} \rangle \approx e^{i\kappa_{\parallel 0}kt/L - \kappa_{\parallel}^{msd}k^2t} \quad (4.49)$$

where we have also used the fact that the fraction is nearly one which follows from Eqs. (4.41) and (4.44).

4.3.2.1.2 Large Wavenumbers

Now we consider the limit of large wavenumbers. We can formally describe this by assuming

$$v^2 k^2 \gg \left| 3D^2 + i \frac{v^2 k}{L} \right|. \quad (4.50)$$

In this case, Eq. (4.41) becomes

$$\begin{aligned} \Omega &= i \frac{v|k|}{\sqrt{3}} + \frac{v \operatorname{sgn}(k)}{2\sqrt{3}L} - i \frac{\sqrt{3}D^2}{2v|k|} + i \frac{v}{8\sqrt{3}L^2|k|} \\ &= i \frac{v|k|}{\sqrt{3}} + \frac{v \operatorname{sgn}(k)}{2\sqrt{3}L} \end{aligned} \quad (4.51)$$

where we have kept only the lowest order terms in each the real and imaginary component. Moreover, we have used the sign function defined by $\operatorname{sgn}(x) = 1$ if $x \geq 0$ and -1 otherwise. The eigenvalues are then given as

$$\omega_{\pm} = -D \pm \left(i \frac{v|k|}{\sqrt{3}} + \frac{v \operatorname{sgn}(k)}{2\sqrt{3}L} \right). \quad (4.52)$$

Using this in Eq. (4.35) yields

$$\begin{aligned} \langle e^{-ikz} \rangle &= \frac{1}{\frac{1}{2kL} + i} \left[\left(-\frac{\sqrt{3}}{2} \frac{D}{v|k|} + \frac{1}{4kL} + i \frac{1}{2} \right) e^{-\left(\frac{v \operatorname{sgn}(k)}{2\sqrt{3}L} + i \frac{v|k|}{\sqrt{3}} \right) t} \right. \\ &\quad \left. + \left(\frac{\sqrt{3}}{2} \frac{D}{v|k|} + \frac{1}{4kL} + i \frac{1}{2} \right) e^{-\left(\frac{v \operatorname{sgn}(k)}{2\sqrt{3}L} + i \frac{v|k|}{\sqrt{3}} \right) t} \right] e^{-Dt}. \end{aligned} \quad (4.53)$$

Unfortunately, this result cannot be further simplified while retaining information pertaining to the effect of the focusing. However, at the very least we can recover the result obtained in

[Lasuik and Shalchi, 2019] if we take the limiting cases of $|k|, L \rightarrow \infty$:

$$\begin{aligned}\langle e^{-ikz} \rangle &= \frac{1}{2} \left[e^{i\frac{vk}{\sqrt{3}}t} + e^{-i\frac{vk}{\sqrt{3}}t} \right] e^{-Dt} \\ &= \cos\left(\frac{vk}{\sqrt{3}}t\right) e^{-Dt}.\end{aligned}\tag{4.54}$$

The latter result is easy to understand. It corresponds to the unperturbed orbit but damped due to pitch-angle scattering.

4.3.2.1.3 Early and Late Times

The characteristic function can be used to extract information on the μ_0 and μ -averaged distribution defined by

$$\tilde{M}(z, t) = \frac{1}{2} \int_{-1}^1 d\mu_0 M(z, t) \tag{4.55}$$

where M is given in Eq. (4.15). To see how, we can place Eq. (2.27) into this formula and rearrange the order of the integrals to obtain

$$\tilde{M}(z, t) = \frac{1}{4} \int_{-\infty}^{\infty} dk e^{ikz} \left[\int_{-1}^1 d\mu_0 \int_{-1}^1 d\mu F_k(\mu, t) \right]. \tag{4.56}$$

The term in the brackets is present in Eq. (4.32). Using this observation yields

$$\tilde{M}(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \langle e^{-ikz} \rangle e^{ikz}. \tag{4.57}$$

As this is too complicated to evaluate in general, in the following we derive simple results for early and late times.

We first consider the initial time $t = 0$. Here, Eq. (4.35) shows that the characteristic function is one. This then implies

$$\tilde{M}(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikz} = \delta(z) \tag{4.58}$$

where we have evaluated the integral with Eq. (2.49). This result simply reproduces our initial condition.

On the other hand, if we consider the late-time limit, we see that by Eq. (4.24), the factor $e^{\omega_{\pm}(k)t}$ is effectively zero, unless the wavenumber is small. This motivates using the approximation given by Eq. (4.49) in Eq. (4.57) to obtain

$$M(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{i\kappa_{\parallel 0} kt/L - \kappa_{\parallel}^{msd} k^2 t + ikz}. \quad (4.59)$$

This is a well-known integral which evaluates to

$$M(z, t) = \frac{1}{\sqrt{4\pi\kappa_{\parallel}^{msd}t}} e^{-(z + \kappa_{\parallel 0} t/L)^2 / (4\kappa_{\parallel}^{msd}t)}. \quad (4.60)$$

This has the form of a normal, or Gaussian, distribution, and thus the mean and mean square displacement are easily read off to be

$$\begin{aligned} \langle z \rangle (t \rightarrow \infty) &= -\frac{\kappa_{\parallel 0} t}{L} \\ \left[\langle z^2 \rangle - \langle z \rangle^2 \right] (t \rightarrow \infty) &= 2\kappa_{\parallel}^{msd} t. \end{aligned} \quad (4.61)$$

These two results can be combined to obtain the second moment:

$$\langle z^2 \rangle (t \rightarrow \infty) = \frac{\kappa_{\parallel 0}^2}{L^2} t^2. \quad (4.62)$$

These are well-known results in the theory of adiabatic focusing, see [Shalchi, 2008]. The above findings correspond to the solution of a diffusion-convection equation. The mean of the Gaussian distribution moves with the speed $v_c = -\kappa_{\parallel 0}/L$ whereas the width is given by $2\kappa_{\parallel}^{msd}t$. It should also be noted that the second relation in Eq. (4.61) demonstrates why the parallel diffusion coefficient defined in Eq. (4.47) is declared to be from the definition in terms of the mean square displacement.

4.3.2.2 The Velocity Correlation Function

The velocity correlation function is given as

$$V_{zz}(t) = v^2 \langle \mu_0 \mu \rangle = \frac{\pi}{2} v^2 \int_{-1}^1 d\mu_0 \mu_0 \int_{-1}^1 d\mu \mu F_0(\mu, t). \quad (4.63)$$

It follows from orthogonality of the polynomials given by Eq. (4.7) that

$$\begin{aligned} \int_{-1}^1 d\mu \mu F_0(\mu, t) &= \int_{-1}^1 d\mu P_1(\mu)^2 C_1(k=0, t) \\ &= \frac{2}{3} C_1(k=0, t). \end{aligned} \quad (4.64)$$

We see from Eqs. (4.24), (4.26) and (4.30) that

$$C_1(k=0, t) = \frac{3}{2\pi} \mu_0 e^{-2Dt}. \quad (4.65)$$

Next, we combine the two previous results and place them into Eq. (4.63) to obtain

$$\begin{aligned} V_{zz}(t) &= \frac{1}{2} v^2 \int_{-1}^1 d\mu_0 \mu_0^2 e^{-2Dt} \\ &= \frac{1}{3} v^2 e^{-2Dt}. \end{aligned} \quad (4.66)$$

Using this quantity, we can compute the parallel diffusion coefficient via the TGK formulation, see Eq. (2.65):

$$\begin{aligned} \kappa_{\parallel}^{TGK} &= \int_0^{\infty} dt \langle V_z(t) V_z(0) \rangle \\ &= \frac{v^2}{6D}. \end{aligned} \quad (4.67)$$

This result is very surprising since it is independent of the focusing parameter. However, this is merely a consequence of the 2-dimensional subspace approximation. This issue is resolved later with the 3-dimensional approximation.

4.3.2.3 The Moment $\langle \mu \rangle$

In the following we compute the mean pitch-angle cosine $\langle \mu \rangle$ by employing the 2-dimensional subspace approximation. This quantity is obtained via

$$\langle \mu \rangle = \pi \int_{-1}^1 d\mu \mu F_0(\mu, t). \quad (4.68)$$

The integral is evaluated in Eq. (4.64) and gives us

$$\langle \mu \rangle = \frac{2}{3} \pi C_1(k=0, t). \quad (4.69)$$

Next, this quantity is stated in Eq. (4.65), and results in

$$\langle \mu \rangle = \mu_0 e^{-2Dt}. \quad (4.70)$$

Again, we see that within the 2-dimensional subspace approximation, adiabatic focusing has no effect on this result. However, we again show that this is not the case for the 3-dimensional approximation.

4.3.2.4 The Moments $\langle z^n \rangle$

The moments $\langle z^n \rangle$, when expressed in Fourier space, are given by

$$\langle z^n \rangle = i^n \pi \left[\frac{\partial^n}{\partial k^n} \int_{-1}^1 d\mu F_k \right]_{k=0}. \quad (4.71)$$

It then follows from orthogonality of Legendre polynomials that

$$\langle z^n \rangle = i^n 2\pi \left[\frac{\partial^n}{\partial k^n} C_0(k, t) \right]_{k=0}. \quad (4.72)$$

As a first example we consider $n = 1$. Using Eq. (4.25) in Eq. (4.72) yields

$$\langle z \rangle = i2\pi \left[e^{\omega_+ t} \frac{d}{dk} b_+ + b_+ e^{\omega_+ t} t \frac{d}{dk} \omega_+ + e^{\omega_- t} \frac{d}{dk} b_- + b_- e^{\omega_- t} t \frac{d}{dk} \omega_- \right]_{k=0}. \quad (4.73)$$

To simplify this result, it helps to first decompose it into the following two terms:

$$\langle z \rangle = i2\pi \left[e^{\omega_+ t} \frac{d}{dk} b_+ + e^{\omega_- t} \frac{d}{dk} b_- \right]_{k=0} + i2\pi \left[b_+ e^{\omega_+ t} t \frac{d}{dk} \omega_+ + b_- e^{\omega_- t} t \frac{d}{dk} \omega_- \right]_{k=0}. \quad (4.74)$$

From Eqs. (4.24) and (4.30), we can compute

$$\begin{aligned} \left. \frac{d\omega_{\pm}}{dk} \right|_{k=0} &= \pm i \frac{v^2}{6DL} \\ \omega_+(k=0) &= 0 \\ \omega_-(k=0) &= -2D \\ \left. \frac{db_{\pm}}{dk} \right|_{k=0} &= \mp \frac{i}{8\pi D^2} \left[\frac{1}{3} \frac{v^2}{L} + 2Dv\mu_0 - \frac{1}{3} \frac{v^3\mu_0}{DL^2} \right] \\ b_+(k=0) &= \frac{1}{2\pi} \frac{2DL - v\mu_0}{2DL} \\ b_-(k=0) &= \frac{1}{2\pi} \frac{v\mu_0}{2DL}. \end{aligned} \quad (4.75)$$

Placing these into Eq. (4.74) gives

$$\langle z \rangle = -\frac{1}{4D^2} \left[\frac{1}{3} \frac{v^2}{L} + 2Dv\mu_0 - \frac{1}{3} \frac{v^3\mu_0}{DL^2} \right] \left[e^{-2Dt} - 1 \right] - \frac{v^2}{12D^2 L^2} \left[2DL - v\mu_0 - v\mu_0 e^{-2Dt} \right] t. \quad (4.76)$$

This result is the exact value of $\langle z \rangle$ provided by the 2-dimensional subspace approximation. In the remainder of this section, we simplify this result for early and late times.

For early times, we can employ the Taylor expansion

$$e^{-2Dt} = 1 - 2Dt \quad (4.77)$$

with the intention of obtaining an expression that is first order in time. Thus,

$$\langle z \rangle (Dt \ll 1) = \frac{1}{2D} \left[\frac{1}{3} \frac{v^2}{L} + 2Dv\mu_0 - \frac{1}{3} \frac{v^3\mu_0}{DL^2} \right] t - \frac{v^2}{12D^2L^2} [2DL - 2v\mu_0] t. \quad (4.78)$$

From here, it can be seen that each term in the brackets on the right cancel with a term in the brackets on the left. The only remaining term is

$$\langle z \rangle (Dt \ll 1) = v\mu_0 t. \quad (4.79)$$

This result is identical to that of Eq. (3.191) which corresponds to constant scattering and focusing neglected. Again, we see that in very early times, the particle experiences a brief period of no net forces acting on it.

On the other hand, if we consider $t \rightarrow \infty$, we first notice that the term involving the exponential in Eq. (4.76) is effectively zero, which gives

$$\langle z \rangle (t \rightarrow \infty) = -\frac{v^2}{12D^2L^2} [2DL - v\mu_0] t \quad (4.80)$$

where we have also neglected terms independent of time. Alternatively, we can use Eqs. (4.40) and (4.44) to rewrite this as

$$\langle z \rangle (t \rightarrow \infty) = -\frac{\kappa_{||0}}{L} \left[1 - \frac{\lambda_{||0}\mu_0}{L} \right] t. \quad (4.81)$$

By averaging this over all μ_0 , we obtain agreement with this result and Eq. (4.61).

Now we consider the second moment in Eq. (4.72). As this case is evidently far more complex than the previous one, we only consider the late-time limit here. Eqs. (4.24) and (4.25) show that here we can simplify Eq. (4.72) to

$$\langle z^2 \rangle = -2\pi \left[\frac{\partial^2}{\partial k^2} (b_+ e^{\omega_+ t}) \right]_{k=0}. \quad (4.82)$$

Keeping only the largest term allows us to further simplify:

$$\langle z^2 \rangle = -2\pi \left[b_+ e^{\omega_+ t} \left(\frac{d}{dk} \omega_+ \right)^2 t^2 \right]_{k=0}. \quad (4.83)$$

Using Eq. (4.75) therein yields

$$\langle z^2 \rangle = \left(\frac{v^2}{6DL} \right)^2 \left[1 - \frac{v\mu_0}{2DL} \right] t^2. \quad (4.84)$$

We again use Eqs. (4.40) and (4.44) to rewrite this as

$$\langle z^2 \rangle = \frac{\kappa_{\parallel 0}^2}{L^2} \left[1 - \frac{\lambda_{\parallel 0} \mu_0}{L} \right] t^2. \quad (4.85)$$

Once more, we get agreement with Eq. (4.62) if we average the preceding equation over all μ_0 .

4.3.3 3-Dimensional Subspace Approximation

Here, we derive analytical results using the three-dimensional subspace approximation. For the case $N = 3$, Eq. (4.9) becomes

$$\begin{pmatrix} \dot{C}_0 \\ \dot{C}_1 \\ \dot{C}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3}ivk - \frac{v}{3L} & 0 \\ -ivk & -2D & -\frac{2}{5}ivk - \frac{3v}{5L} \\ 0 & -\frac{2}{3}ivk + \frac{v}{3L} & -6D \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix}. \quad (4.86)$$

The ansatz

$$C_n(t) = \tilde{b}_n e^{\omega t} \quad \text{for } n = 0, 1, 2 \quad (4.87)$$

leads to the matrix equation

$$\omega \begin{pmatrix} \tilde{b}_0 \\ \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3}ivk - \frac{v}{3L} & 0 \\ -ivk & -2D & -\frac{2}{5}ivk - \frac{3v}{5L} \\ 0 & -\frac{2}{3}ivk + \frac{v}{3L} & -6D \end{pmatrix} \begin{pmatrix} \tilde{b}_0 \\ \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix} \quad (4.88)$$

corresponding to a simple eigenvalue problem. Alternatively, this can be written as

$$\begin{pmatrix} -\omega & -\frac{1}{3}ivk - \frac{v}{3L} & 0 \\ -ivk & -2D - \omega & -\frac{2}{5}ivk - \frac{3v}{5L} \\ 0 & -\frac{2}{3}ivk + \frac{v}{3L} & -6D - \omega \end{pmatrix} \begin{pmatrix} \tilde{b}_0 \\ \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix} = 0. \quad (4.89)$$

Non-trivial solutions of the latter equation are obtained by setting the determinant of this 3×3 matrix equal to zero. We find

$$\omega^3 + 8D\omega^2 + \left[12D^2 + \frac{3}{5}v^2k^2 + \frac{v^2}{5L^2} - i\frac{3v^2k}{5L}\right]\omega + 2Dv^2\left[k^2 - i\frac{k}{L}\right] = 0 \quad (4.90)$$

corresponding to a cubic equation for ω . It is possible to solve this analytically using the cubic formula; however, the resulting expressions are far too complicated to be of any use.

It now remains to use the initial conditions to determine $\tilde{b}_n$ in Eq. (4.87). We can use Eq. (4.14) to easily derive

$$\begin{aligned} C_0(t=0) &= \frac{1}{2\pi}, \\ C_1(t=0) &= \frac{3\mu_0}{2\pi}, \\ C_2(t=0) &= \frac{5}{4\pi}(3\mu_0^2 - 1). \end{aligned} \quad (4.91)$$

For convenience we write Eq. (4.86) as

$$\begin{pmatrix} \dot{C}_0 \\ \dot{C}_1 \\ \dot{C}_2 \end{pmatrix} = \begin{pmatrix} 0 & M_{01} & 0 \\ M_{10} & M_{11} & M_{12} \\ 0 & M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix}. \quad (4.92)$$

Now we assume

$$C_0 = \tilde{b}_0 e^{\omega_0 t} + \tilde{b}_1 e^{\omega_1 t} + \tilde{b}_2 e^{\omega_2 t} \quad (4.93)$$

where $\tilde{b}_0$, $\tilde{b}_1$, and $\tilde{b}_2$ are unknown constants. From Eq. (4.92), we obtain

$$C_1 = \frac{1}{M_{01}} \dot{C}_0 = \frac{1}{M_{01}} [\omega_0 \tilde{b}_0 e^{\omega_0 t} + \omega_1 \tilde{b}_1 e^{\omega_1 t} + \omega_2 \tilde{b}_2 e^{\omega_2 t}]. \quad (4.94)$$

Eq. (4.92) also shows that

$$\begin{aligned} C_2 &= \frac{1}{M_{12}} [\dot{C}_1 - M_{10} C_0 - M_{11} C_1] \\ &= \frac{1}{M_{12}} \left[\left(\frac{\omega_0^2}{M_{01}} - M_{10} - \frac{M_{11} \omega_0}{M_{01}} \right) \tilde{b}_0 e^{\omega_0 t} + \left(\frac{\omega_1^2}{M_{01}} - M_{10} - \frac{M_{11} \omega_1}{M_{01}} \right) \tilde{b}_1 e^{\omega_1 t} \right. \\ &\quad \left. + \left(\frac{\omega_2^2}{M_{01}} - M_{10} - \frac{M_{11} \omega_2}{M_{01}} \right) \tilde{b}_2 e^{\omega_2 t} \right]. \end{aligned} \quad (4.95)$$

Combining Eqs. (4.93)-(4.95) with Eq. (4.91) leads to the three equations

$$\begin{aligned} \frac{1}{2\pi} &= \tilde{b}_0 + \tilde{b}_1 + \tilde{b}_2, \\ \frac{3\mu_0}{2\pi} M_{01} &= \omega_0 \tilde{b}_0 + \omega_1 \tilde{b}_1 + \omega_2 \tilde{b}_2, \\ \frac{5}{4\pi} (3\mu_0^2 - 1) M_{12} &= \left(\frac{\omega_0^2}{M_{01}} - M_{10} - \frac{M_{11} \omega_0}{M_{01}} \right) \tilde{b}_0 + \left(\frac{\omega_1^2}{M_{01}} - M_{10} - \frac{M_{11} \omega_1}{M_{01}} \right) \tilde{b}_1 \\ &\quad + \left(\frac{\omega_2^2}{M_{01}} - M_{10} - \frac{M_{11} \omega_2}{M_{01}} \right) \tilde{b}_2. \end{aligned} \quad (4.96)$$

Using the first two equations, the third line of Eq. (4.96) can be rewritten as

$$\frac{5}{4\pi}(3\mu_0^2 - 1)M_{12}M_{01} + \frac{1}{2\pi}M_{10}M_{01} + \frac{3\mu_0}{2\pi}M_{11}M_{01} = \omega_0^2\tilde{b}_0 + \omega_1^2\tilde{b}_1 + \omega_2^2\tilde{b}_2. \quad (4.97)$$

The first two lines of Eq. (4.96) and Eq. (4.97) are summarized by a linear system with a Vandermonde matrix. Solving similar systems has been of interest in the literature such as [Turner, 1966]. Here, we have the solution

$$\begin{aligned} \tilde{b}_0 &= \frac{\omega_1\omega_2 - 3M_{01}(\omega_1 + \omega_2)\mu_0 + \frac{5}{2}M_{12}M_{01}(3\mu_0^2 - 1) + M_{10}M_{01} + 3M_{11}M_{01}\mu_0}{2\pi(\omega_0 - \omega_1)(\omega_0 - \omega_2)}, \\ \tilde{b}_1 &= \frac{\omega_0\omega_2 - 3M_{01}(\omega_0 + \omega_2)\mu_0 + \frac{5}{2}M_{12}M_{01}(3\mu_0^2 - 1) + M_{10}M_{01} + 3M_{11}M_{01}\mu_0}{2\pi(\omega_1 - \omega_0)(\omega_1 - \omega_2)}, \\ \tilde{b}_2 &= \frac{\omega_0\omega_1 - 3M_{01}(\omega_0 + \omega_1)\mu_0 + \frac{5}{2}M_{12}M_{01}(3\mu_0^2 - 1) + M_{10}M_{01} + 3M_{11}M_{01}\mu_0}{2\pi(\omega_2 - \omega_0)(\omega_2 - \omega_1)}, \end{aligned} \quad (4.98)$$

corresponding to formulas for the three needed coefficients.

4.3.4 3-Dimensional Subspace Expectation Values

Previously, we employed the 2-dimensional subspace approximation to compute several expectation values. In the following we determine several of the same quantities but employ the 3-dimensional subspace approximation. This is particularly important because several expectation values are independent of the focusing effect when the 2-dimensional approach is employed. We shall see that one advantage of using the extra dimension is that each result contains the focusing length.

4.3.4.1 The Characteristic Function

Eq. (4.33) was derived without specifying the dimension of the subspace approximation. Thus, we start from there and derive

$$\begin{aligned}
\langle e^{-ikz} \rangle &= \pi \int_{-1}^1 d\mu_0 C_0(t) \\
&= \pi \int_{-1}^1 d\mu_0 [\tilde{b}_0 e^{\omega_0 t} + \tilde{b}_1 e^{\omega_1 t} + \tilde{b}_2 e^{\omega_2 t}] \\
&= \frac{\omega_1 \omega_2 - \frac{1}{3} v^2 k^2 + i \frac{1}{3} \frac{v^2 k}{L}}{(\omega_0 - \omega_1)(\omega_0 - \omega_2)} e^{\omega_0 t} + \frac{\omega_0 \omega_2 - \frac{1}{3} v^2 k^2 + i \frac{1}{3} \frac{v^2 k}{L}}{(\omega_1 - \omega_0)(\omega_1 - \omega_2)} e^{\omega_1 t} + \frac{\omega_0 \omega_1 - \frac{1}{3} v^2 k^2 + i \frac{1}{3} \frac{v^2 k}{L}}{(\omega_2 - \omega_0)(\omega_2 - \omega_1)} e^{\omega_2 t}.
\end{aligned} \tag{4.99}$$

Comparing this expression with the one derived using 2-dimensions in Eq. (4.35) is a prime example of how much more complicated the results are in this extra dimension, especially considering that the eigenvalues are also significantly more complicated.

In what follows, we want to simplify this result for small wave numbers and late times. To do this, we write $\omega = a + bk + ck^2$ with the constants a, b, c independent of k , and place that into Eq. (4.90) to obtain

$$\begin{aligned}
&a^3 + 8Da^2 + 12D^2a + \frac{v^2}{5L^2}a + \left(3a^2b + 16Dab + 12D^2b + \frac{1}{5} \frac{v^2}{L^2}b - i \frac{3}{5} \frac{v^2}{L}a - i 2 \frac{Dv^2}{L} \right) k \\
&+ \left(3ab^2 + 3a^2c + 8Db^2 + 16Dac + 12D^2c + \frac{3}{5} v^2 a + 2Dv^2 + \frac{1}{5} \frac{v^2}{L^2}c - i \frac{3}{5} \frac{v^2}{L}b \right) k^2 = 0
\end{aligned} \tag{4.100}$$

where we neglect terms higher order than quadratic in k . The equation found here has to be satisfied for any value of k . Therefore, the terms without a factor of k must be zero and we find

$$a^3 + 8Da^2 + 12D^2a + \frac{v^2}{5L^2}a = 0. \tag{4.101}$$

From this condition we can derive the following three values for the constant a :

$$a = 0 \quad \text{or} \quad a = -4D \pm \sqrt{4D^2 - \frac{v^2}{5L^2}}. \quad (4.102)$$

Since we are interested in the late-time limit, we can see that the term with the slowest decay in Eq. (4.99) corresponds to the eigenvalue with $a = 0$. Henceforth, we set $a = 0$ in Eq. (4.100) to simplify it to

$$\left(12D^2b + \frac{1}{5}\frac{v^2}{L^2}b - i2\frac{Dv^2}{L}\right)k + \left(8Db^2 + 12D^2c + 2Dv^2 + \frac{1}{5}\frac{v^2}{L^2}c - i\frac{3}{5}\frac{v^2}{L}b\right)k^2 = 0. \quad (4.103)$$

Now, from the linear term, we easily obtain the expression

$$b = i10\frac{\frac{v^2}{DL}}{60 + \frac{v^2}{D^2L^2}}. \quad (4.104)$$

If we place this into the quadratic term of Eq. (4.103), we obtain

$$c = -\frac{40\left(900\frac{v^2}{D} - 25\frac{v^4}{D^3L^2} + \frac{v^6}{D^5L^4}\right)}{\left(60 + \frac{v^2}{D^2L^2}\right)^3}. \quad (4.105)$$

By combining Eqs. (4.104) and (4.105), we derive an expression for one eigenvalue, which we denote by ω_0 :

$$\omega_0 = -\frac{40\left(900\frac{v^2}{D} - 25\frac{v^4}{D^3L^2} + \frac{v^6}{D^5L^4}\right)}{\left(60 + \frac{v^2}{D^2L^2}\right)^3}k^2 + i10\frac{\frac{v^2}{DL}}{60 + \frac{v^2}{D^2L^2}}k. \quad (4.106)$$

Since the factor $e^{\omega_0 t}$ has significantly slower decay than the other exponential factors in Eq. (4.99), we can simplify it to

$$\langle e^{-ikz} \rangle(t \rightarrow \infty) = \frac{\omega_1\omega_2 - \frac{1}{3}v^2k^2 + i\frac{1}{3}\frac{v^2k}{L}}{(\omega_0 - \omega_1)(\omega_0 - \omega_2)}e^{\omega_0 t}. \quad (4.107)$$

Next, we use the fact that $\omega_0 \rightarrow 0$ as $k \rightarrow 0$ once more to simplify the factor

$$\frac{\omega_1\omega_2 - \frac{1}{3}v^2k^2 + i\frac{1}{3}\frac{v^2k}{L}}{(\omega_0 - \omega_1)(\omega_0 - \omega_2)} \approx 1. \quad (4.108)$$

If we assume further that the focusing effect is weak, that is the parameter

$$\xi = \frac{v}{2DL} \quad (4.109)$$

is small, we can additionally simplify Eqs. (4.104) and (4.105) to

$$b = i \frac{v^2}{6DL} \quad \text{and} \quad c = -\frac{v^2}{6D} + \frac{7v^4}{540D^3L^2}, \quad (4.110)$$

resulting in

$$\langle e^{-ikz} \rangle (t \rightarrow \infty) = e^{-\kappa_{\parallel 0} k^2 t + \frac{7}{15} \frac{\kappa_{\parallel 0}^2 k^2}{DL^2} t + i \frac{\kappa_{\parallel 0}}{L} k t}. \quad (4.111)$$

Here, we have used the parallel diffusion coefficient without focusing as given by Eq. (4.40). Following the same idea applied to Eq. (4.46), we can interpret this result as providing the parallel diffusion coefficient with focusing:

$$\kappa_{\parallel}^{msd} = \kappa_{\parallel 0} \left(1 - \frac{7}{15} \frac{\kappa_{\parallel 0}}{DL^2} \right). \quad (4.112)$$

Or, we can use the parallel mean free path without focusing, given in Eq. (4.44), to write this as

$$\kappa_{\parallel}^{msd} = \kappa_{\parallel 0} \left(1 - \frac{14}{45} \frac{\lambda_{\parallel 0}^2}{L^2} \right). \quad (4.113)$$

This result can be compared with Eq. (4.47) which is based on the 2-dimensional subspace approximation. Due to $14/45 \approx 1/3$ the results for the parallel diffusion coefficient do not change significantly between the 2 and 3-dimensional subspace approximation. Moreover, in Chapter 6 we show that this expression for $\kappa_{\parallel}^{msd}$ is correct in the weak focusing limit.

4.3.4.2 The Velocity Correlation Function

In Eqs. (4.63) and (4.64), we derived

$$V_{zz}(t) = \frac{\pi}{3} v^2 \int_{-1}^1 d\mu_0 \mu_0 C_1(k=0, t). \quad (4.114)$$

It follows from Eqs. (4.94) and (4.98) that

$$C_1(k=0, t) = -3 \left[\frac{\omega_2(k=0)\mu_0 + 2D\mu_0 + \frac{1}{2}\frac{v}{L}(3\mu_0^2 - 1)}{2\pi [\omega_1(k=0) - \omega_2(k=0)]} e^{\omega_1(k=0)t} + \frac{\omega_1(k=0)\mu_0 + 2D\mu_0 + \frac{1}{2}\frac{v}{L}(3\mu_0^2 - 1)}{2\pi [\omega_2(k=0) - \omega_1(k=0)]} e^{\omega_2(k=0)t} \right] \quad (4.115)$$

where we have also used the fact that $\omega_0(k=0) = 0$ by Eq. (4.106). Placing this expression into Eq. (4.114) yields

$$V_{zz}(t) = -\frac{v^2}{3} \left[\frac{\omega_2(k=0) + 2D}{\omega_1(k=0) - \omega_2(k=0)} e^{\omega_1(k=0)t} + \frac{\omega_1(k=0) + 2D}{\omega_2(k=0) - \omega_1(k=0)} e^{\omega_2(k=0)t} \right]. \quad (4.116)$$

We can see that since $k=0$, the other two eigenvalues are easy to find in Eq. (4.90) and are

$$\begin{aligned} \omega_1(k=0) &= -4D - \sqrt{-4D^2 - \frac{v^2}{5L^2}}, \\ \omega_2(k=0) &= -4D + \sqrt{-4D^2 - \frac{v^2}{5L^2}}. \end{aligned} \quad (4.117)$$

The next step is to derive the parallel diffusion coefficient via the Taylor-Green-Kubo formula:

$$\begin{aligned} \kappa_{\parallel}^{TGK} &= \int_0^{\infty} dt V_{zz}(t) \\ &= \frac{v^2}{3} \left[\frac{\omega_2(k=0) + 2D}{\omega_1(k=0) - \omega_2(k=0)} \frac{1}{\omega_1(k=0)} + \frac{\omega_1(k=0) + 2D}{\omega_2(k=0) - \omega_1(k=0)} \frac{1}{\omega_2(k=0)} \right] \\ &= -\frac{v^2}{3} \frac{\omega_1(k=0) + \omega_2(k=0) + 2D}{\omega_1(k=0)\omega_2(k=0)}. \end{aligned} \quad (4.118)$$

Now we place Eq. (4.117) into this result to obtain

$$\kappa_{\parallel}^{TGK} = \frac{v^2}{3} \frac{6D}{12D^2 + \frac{v^2}{5L^2}}. \quad (4.119)$$

This is an exact result for the 3-dimensional subspace approximation, but we can simplify this if we assume the parameter ξ is small:

$$\kappa_{\parallel}^{TGG} = \kappa_{\parallel 0} \left(1 - \frac{1}{15} \frac{\lambda_{\parallel 0}^2}{L^2} \right) \quad (4.120)$$

where we have used the parallel diffusion coefficient and parallel mean free path without focusing given in Eqs. (4.44) and (4.40). Unlike the corresponding result within 2-dimensions, given in Eq. (4.67), we see that the result derived here depends on the focusing effect. Moreover, we can compare this result with the diffusion coefficient given in Eq. (4.113) and notice that they are not the same. This discrepancy is not due to the subspace approximation. Rather, this is in fact expected when focusing is present, which we see in more detail in Chapter 6.

4.3.4.3 The Moment $\langle \mu \rangle$

To determine the expectation value $\langle \mu \rangle$, we employ Eqs. (4.69) and (4.115) to obtain

$$\begin{aligned} \langle \mu \rangle = & - \left[\frac{\omega_2(k=0)\mu_0 + 2D\mu_0 + \frac{v}{2L}(3\mu_0^2 - 1)}{\omega_1(k=0) - \omega_2(k=0)} e^{\omega_1(k=0)t} \right. \\ & \left. + \frac{\omega_1(k=0)\mu_0 + 2D\mu_0 + \frac{v}{2L}(3\mu_0^2 - 1)}{\omega_2(k=0) - \omega_1(k=0)} e^{\omega_2(k=0)t} \right] \end{aligned} \quad (4.121)$$

where the eigenvalues are given in Eq. (4.117). We can simplify this result for the early and late-time limit. First if $t = 0$, then

$$\langle \mu \rangle (t = 0) = \mu_0. \quad (4.122)$$

Moreover, we also have

$$\langle \mu \rangle (t \rightarrow \infty) = 0 \quad (4.123)$$

regardless of the initial pitch angle cosine value μ_0 . This is what we have expected due to the pitch-angle isotropization process.

4.3.4.4 The Moments $\langle z^n \rangle$

As a final application, we compute the moments $\langle z \rangle$ and $\langle z^2 \rangle$. Due to the complexity of the expressions, we elect to only compute the late-time limit. The first step is to notice that by Eqs. (4.106) and (4.117), we have

$$C_0(t \rightarrow \infty) = \tilde{b}_0 e^{\omega_0 t} \quad (4.124)$$

where we have also used the definition of C_0 given in Eq. (4.93). We place this expression in Eq. (4.72) to derive

$$\langle z \rangle (t \rightarrow \infty) = i2\pi \left[\tilde{b}_0 t e^{\omega_0 t} \frac{d}{dk} \omega_0 \right]_{k=0} \quad (4.125)$$

where we kept only the largest term. From Eq. (4.106), the derivative is easy to calculate. Additionally, $\tilde{b}_0(k=0)$ is given by Eqs. (4.98) and (4.117). Here, we have

$$\begin{aligned} \tilde{b}_0(k=0) &= \frac{1}{2\pi} + \frac{-\frac{\lambda_{\parallel 0}}{L}\mu_0 + \frac{\lambda_{\parallel 0}^2}{6L^2}(3\mu_0^2 - 1)}{2\pi \left(1 - \frac{\lambda_{\parallel 0}^2}{15L^2}\right)} = \frac{1}{2\pi} + \frac{-\frac{\lambda_{\parallel 0}}{L}\mu_0 + \frac{\lambda_{\parallel 0}^2}{6L^2}(3\mu_0^2 - 1) + \frac{\lambda_{\parallel 0}^2}{15L^2}}{2\pi} \\ \frac{d\omega_0}{dk} \Big|_{k=0} &= i \frac{\frac{\kappa_{\parallel 0}}{L}}{1 + \frac{\lambda_{\parallel 0}^2}{15L^2}} = i \frac{\kappa_{\parallel 0}}{L} \left(1 - \frac{\lambda_{\parallel 0}^2}{15L^2}\right) \end{aligned} \quad (4.126)$$

where we have used the parallel mean free path and parallel diffusion coefficient given in Eqs. (4.40) and (4.44), both defined without focusing. The last expression on both lines is obtained from assuming ξ is small and Taylor-expanding. Placing it all together yields

$$\langle z \rangle (t \rightarrow \infty) = -\frac{\kappa_{\parallel 0}}{L} \left[1 - \frac{\lambda_{\parallel 0}}{L}\mu_0 + \frac{\lambda_{\parallel 0}^2}{6L^2}(3\mu_0^2 - 1) \right] t \quad (4.127)$$

which would be the same as Eq. (4.80), corresponding to the 2-dimensional subspace approximation, if we neglected the third order term.

Next, we consider the second moment. At late times, we keep only the largest term, which has the

same form as when 2-dimensions were used, and is given in Eq. (4.83). This results in

$$\langle z^2 \rangle (t \rightarrow \infty) = -2\pi \left[\tilde{b}_0 t^2 e^{\omega_0 t} \left(\frac{d}{dk} \omega_0 \right)^2 \right]_{k=0}. \quad (4.128)$$

We can see that this expression is very similar to the first moment given in Eq. (4.125). In fact, we have

$$\langle z^2 \rangle (t \rightarrow \infty) = it \langle z \rangle \frac{d\omega_0}{dk} \Big|_{k=0}. \quad (4.129)$$

This allows for the easy third-order computation of

$$\langle z^2 \rangle (t \rightarrow \infty) = \frac{\kappa_{\parallel 0}^2}{L^2} \left[1 - \frac{\lambda_{\parallel 0}}{L} \mu_0 \right] t^2 \quad (4.130)$$

which gives us agreement with the 2-dimensional approximation given in Eq. (4.85).

4.4 Efficient and Accurate Subspace Approximations

In the following, we discuss the N -dimensional subspace approximation. This approach is discussed in detail in [Shalchi, 2024] for the Fokker-Planck equation without adiabatic focusing. In the current section, we extend this method to include the focusing effect. Furthermore, we shall demonstrate that the 10-dimensional subspace approximation provides a very accurate, but also very efficient, approximation in most cases. This approach has to be understood as a semi-analytical method because some steps have to be performed numerically.

Within the N -dimensional subspace approximation, we employ the expansion in Legendre polynomials as given by Eq. (4.3). The time-dependent functions $C_n(t)$ therein are obtained via numerically solving the matrix equation provided by Eq. (4.12). The initial conditions $C_n(0)$ in the latter equation are obtained via Eq. (4.14). The first several entries of the matrix $\mathbf{M}$ in our matrix equation are provided by Eq. (4.11) where the rest of the matrix elements are given as the coefficient of C_n in Eq. (4.8).

To test the validity of various subspace approximations, we also solve the focused transport equation

numerically. See Section 2.3 for details including the transformations which render the variables dimensionless. In this case, the solution is completely described by the dimensionless parameter $\xi = v/(2DL)$.

In the following subsections, we shall compare the subspace approximation method for 2, 3, and 10-dimensions with the numerical solution. We provide results for $\xi = 0.1$, 0.5, and 2.5 corresponding to different strengths of the focusing effect. Moreover, we also compare the computation times needed to obtain the solution with the aforementioned methods. We conclude this section with stating the limitations of this method, which are present if the parameter ξ is too big.

4.4.1 The Parameter $\xi = 0.1$

First, we consider the case $\xi = 0.1$. Figures 4.1-4.3 illustrate the μ -integrated solution, defined in Eq. (4.15), at the different times $t = 0.1$, 1, and 5, respectively. Moreover, we use an initial value of $\mu_0 = 0.5$ as an example.

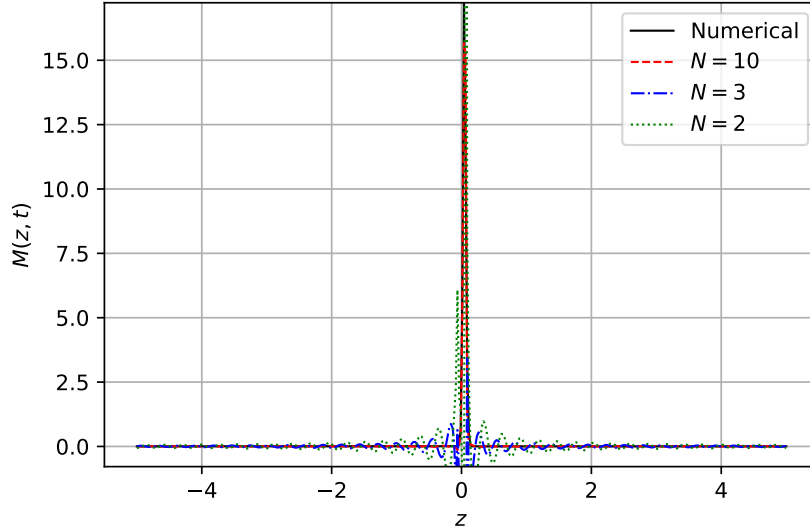


Figure 4.1: The graph compares the μ -integrated solution of the subspace method for various dimensions and the pure numerical solution at the time $t = 0.1$ for an initial value of $\mu_0 = 0.5$. For the focusing parameter we set $\xi = 0.1$.

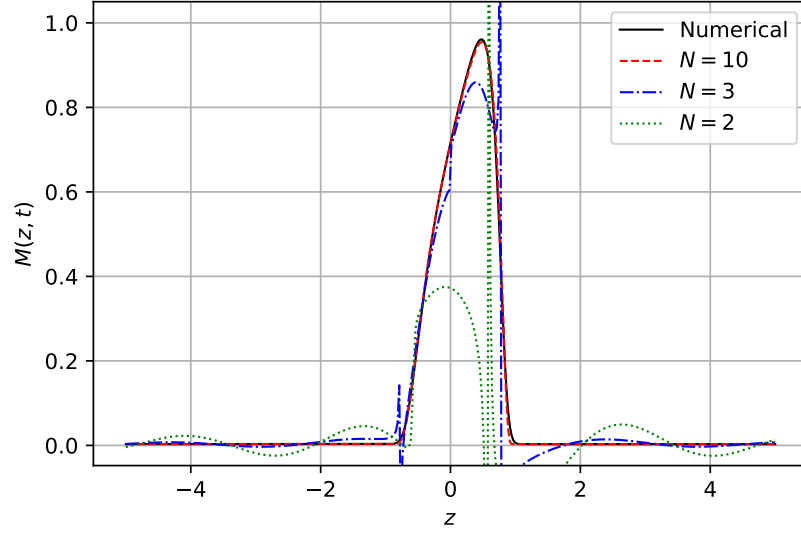


Figure 4.2: Caption is as in Figure 4.1 except that we have used $t = 1$.

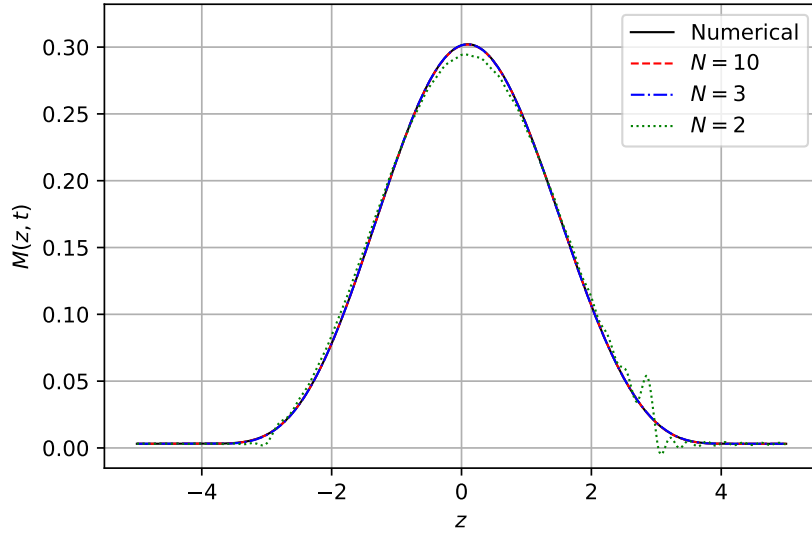


Figure 4.3: Caption is as in Figure 4.1 except that we have used $t = 5$.

Next, we examine the μ -averaged characteristic function obtained via

$$\langle e^{-ikz} \rangle (\mu_0, t) = \frac{1}{2} \int_{-1}^1 d\mu \int_{-\infty}^{\infty} dz f(z, \mu, t) e^{-ikz}. \quad (4.131)$$

Note that the difference between this quantity and the one defined in Eq. (4.32) is that here, we do not average over μ_0 . It follows from Section 2.2 that

$$\langle e^{-ikz} \rangle(\mu_0, t) = 2\pi C_0(k, t). \quad (4.132)$$

Figures 4.4-4.6 demonstrate the accuracy of the subspace method for the real component of the characteristic function. We keep the wavenumber constant and vary time.

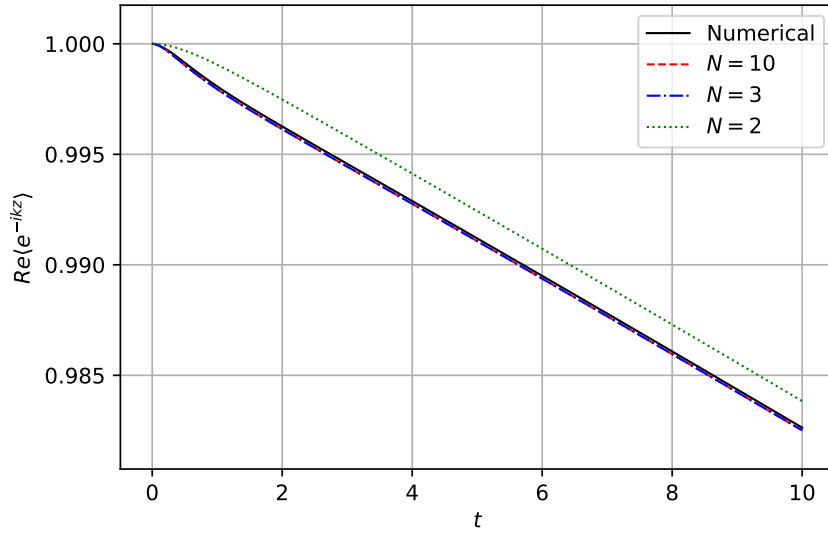


Figure 4.4: Comparison of the μ -averaged characteristic function between the subspace method for various dimensions and the numerical solution. We have used an initial value of $\mu_0 = 0$ and for the focusing parameter we have set $\xi = 0.1$. Here we have kept the dimensionless wavenumber $k \rightarrow vk/D = 0.1$ constant and varied time.

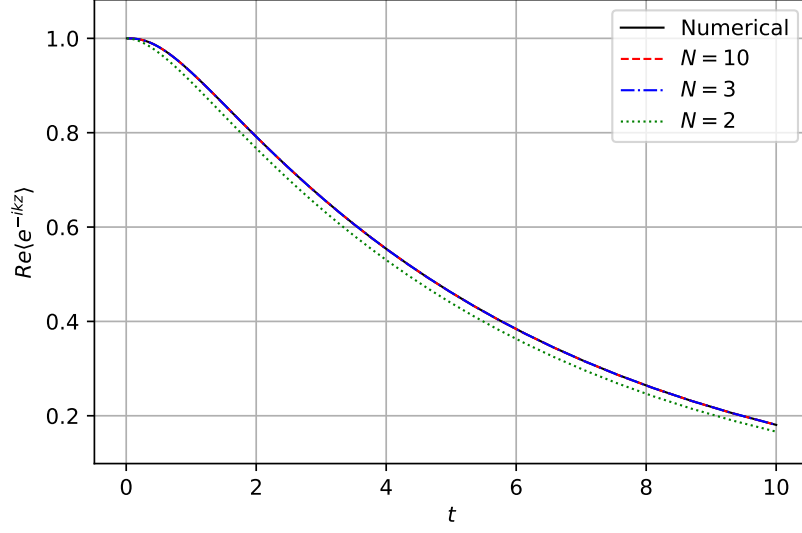


Figure 4.5: Caption is as in Figure 4.4 except that we have used $k = 1$.

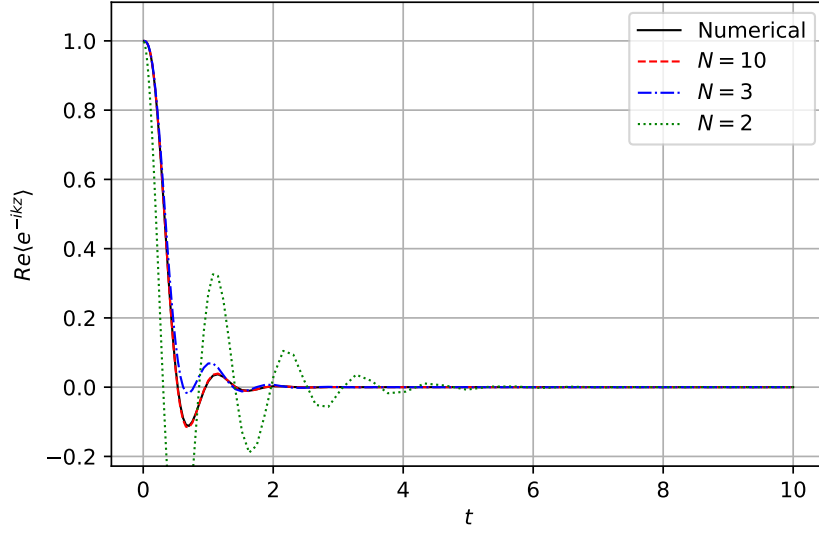


Figure 4.6: Caption is as in Figure 4.4 except that we have used $k = 10$.

The final comparisons are given in Figures 4.7 and 4.8 where we illustrate the first and second moments $\langle z \rangle$ and $\langle z^2 \rangle$, respectively. Recall the definition of these moments in Eq. (4.72).

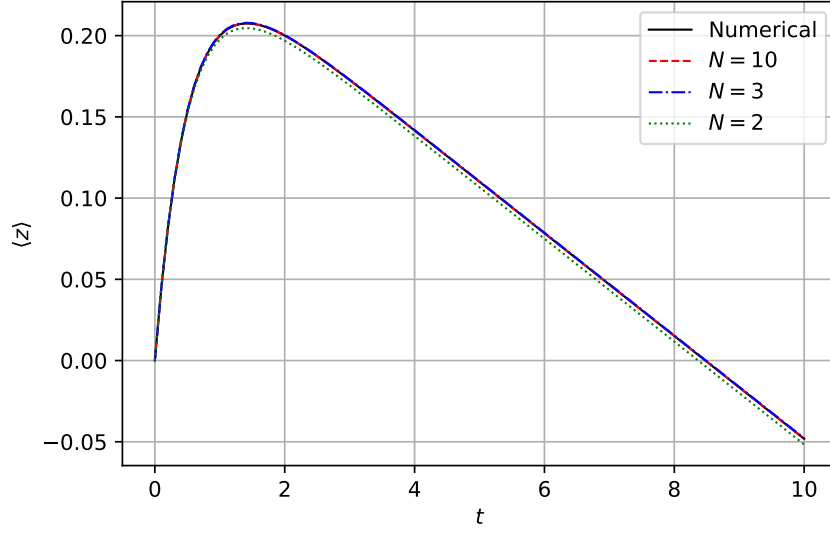


Figure 4.7: Comparison of the mean position $\langle z \rangle$ as a function of time for the N -dimensional subspace method and the numerical solution. Here, we have used $\mu_0 = 0.5$ and $\xi = 0.1$.

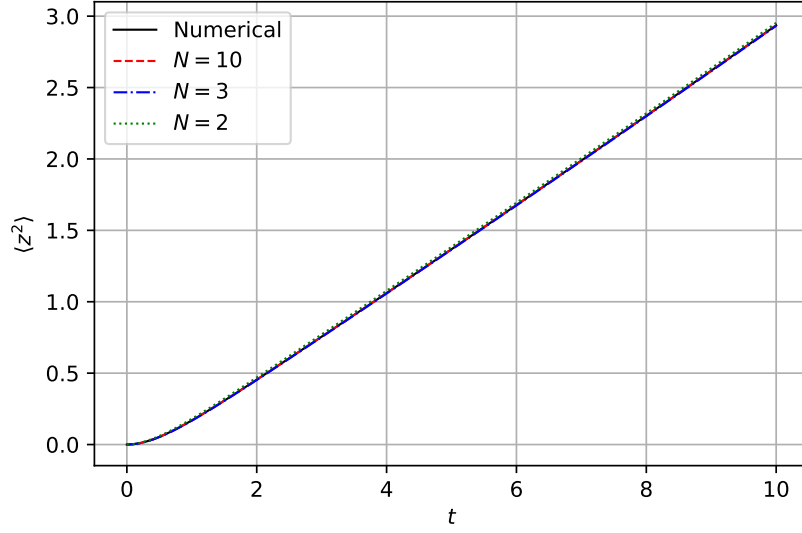


Figure 4.8: Comparison of the second moment $\langle z^2 \rangle$ as a function of time for the N -dimensional subspace method and the numerical solution. Here, we have used $\mu_0 = 0.5$ and $\xi = 0.1$.

4.4.2 The Parameter $\xi = 0.5$

To show that the 10-dimensional subspace approximation works for larger values of ξ , we reproduce each plot in the previous section, except here we have $\xi = 0.5$.

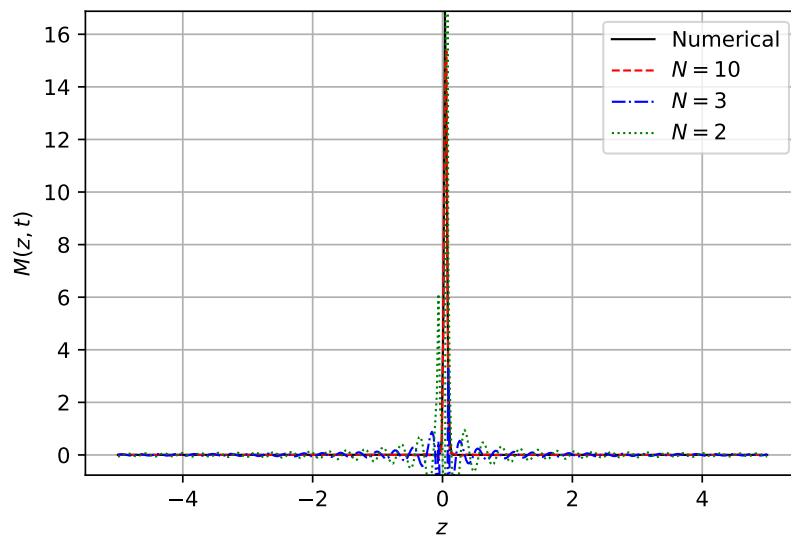


Figure 4.9: This graph compares the μ -integrated solution of the subspace method for various dimensions and the numerical solution at the time $t = 0.1$ for an initial value of $\mu_0 = 0.5$. Here we have set $\xi = 0.5$.

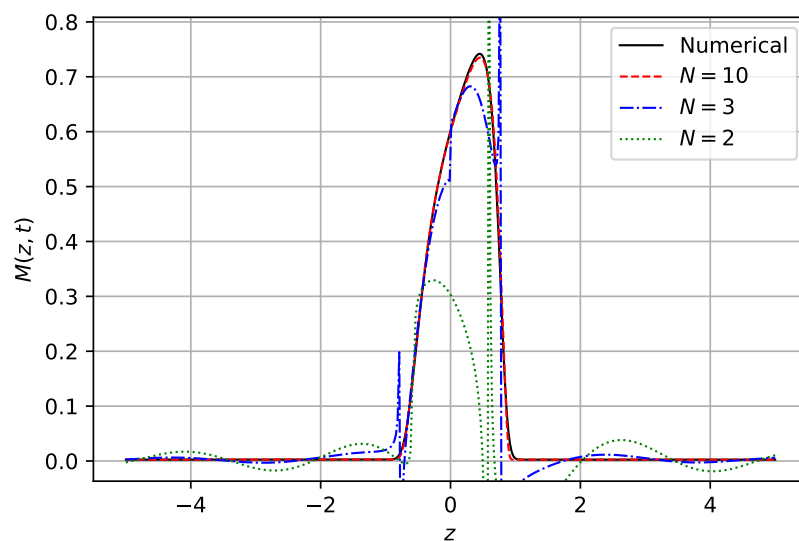


Figure 4.10: Caption is as in Figure 4.9 except that we have used $t = 1$.

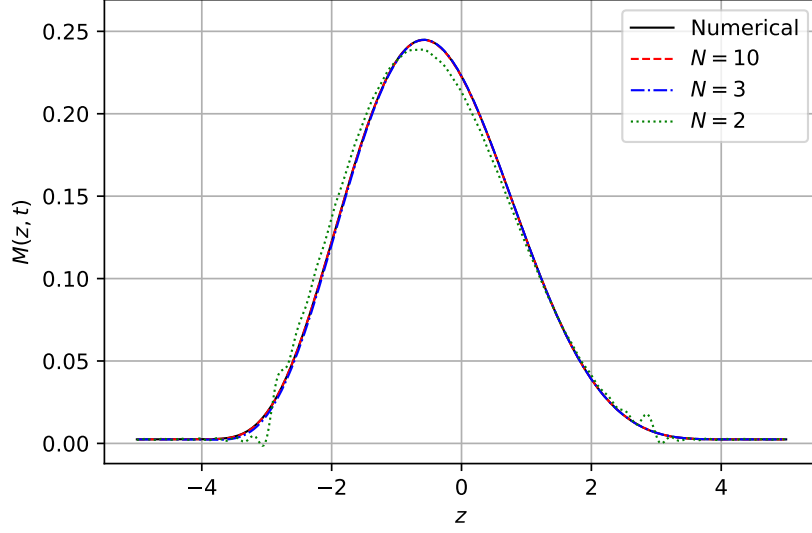


Figure 4.11: Caption is as in Figure 4.9 except that we have used $t = 5$.

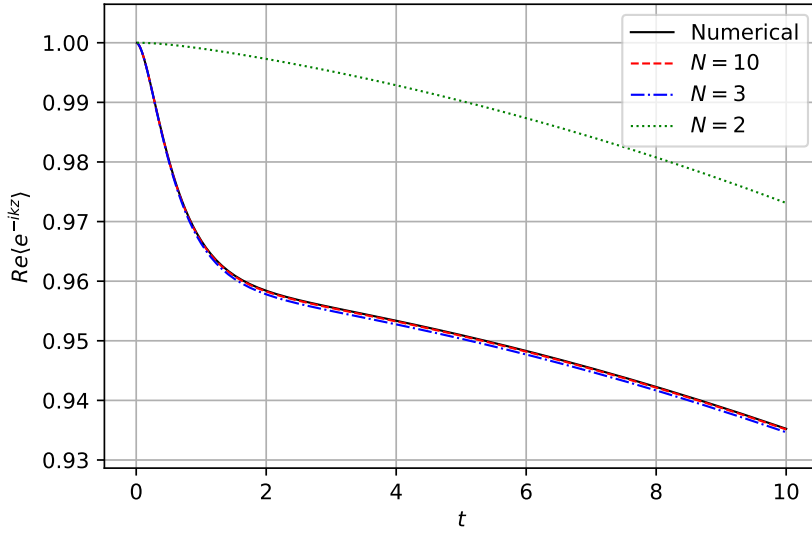


Figure 4.12: Comparison of the μ -averaged characteristic function between the subspace method for various dimensions and the numerical solution. We have used an initial value of $\mu_0 = 0$ and set $\xi = 0.5$. We kept the dimensionless wave number $k \rightarrow vk/D = 0.1$ constant and varied time.

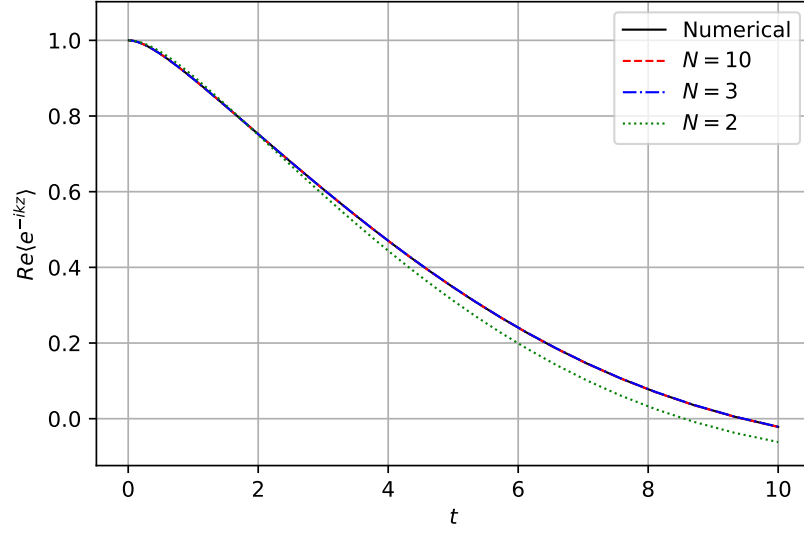


Figure 4.13: Caption is as in Figure 4.12 but we have set $k = 1$.

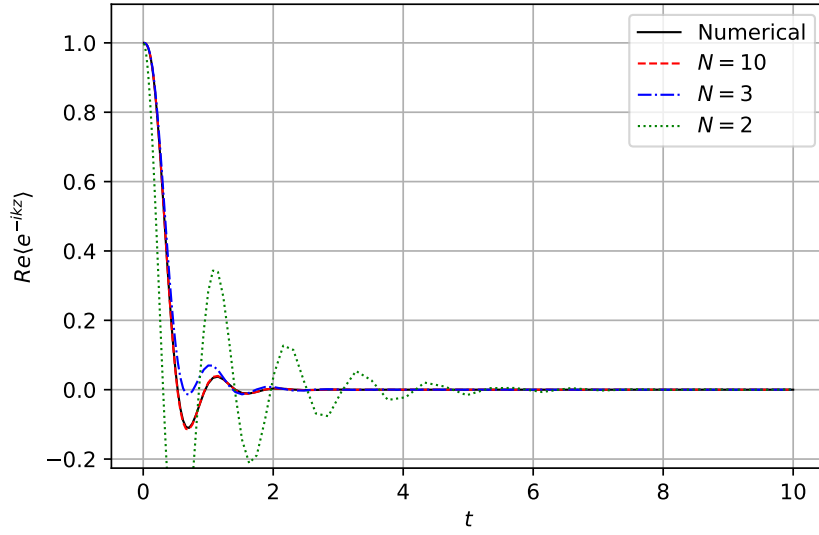


Figure 4.14: Caption is as in Figure 4.12 but we have set $k = 10$.

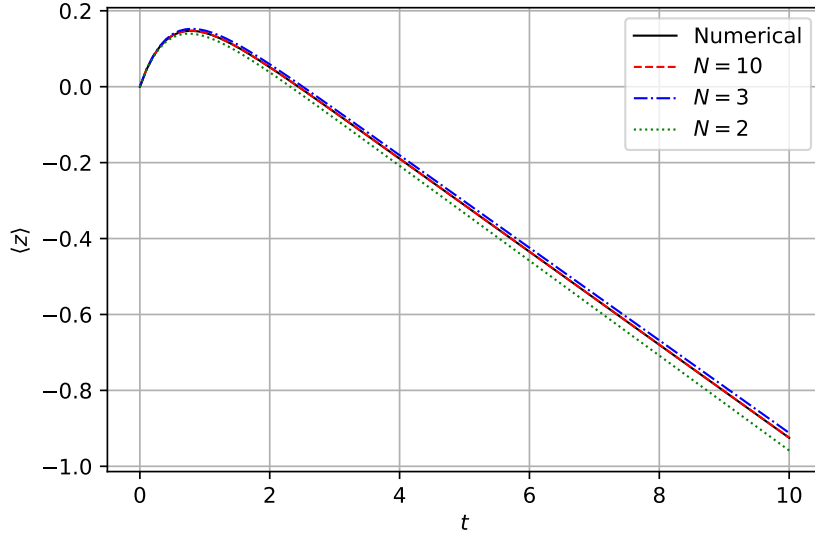


Figure 4.15: Comparison of the mean position $\langle z \rangle$ as a function of time for the N -dimensional subspace method and the numerical solution. Here, we have set $\mu_0 = 0.5$ and $\xi = 0.5$.

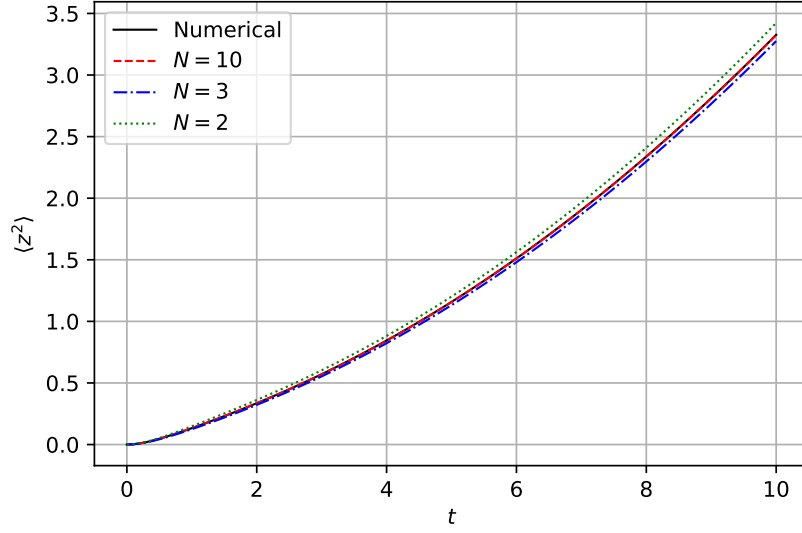


Figure 4.16: Comparison of the second moment $\langle z^2 \rangle$ as a function of time for the N -dimensional subspace method and the numerical solution. Here, we have used $\mu_0 = 0.5$ and $\xi = 0.5$.

4.4.3 The Parameter $\xi = 2.5$

As a final set of tests, we demonstrate that if we take $\xi = 2.5$, then the 10-dimensional subspace method still provides an accurate approximation. The difference here is that we do not include the case of $N = 2$ in the following plots. This is discussed in further detail below, but a brief explanation is that the expansion given by Eq. (2.27) has severe convergence issues.

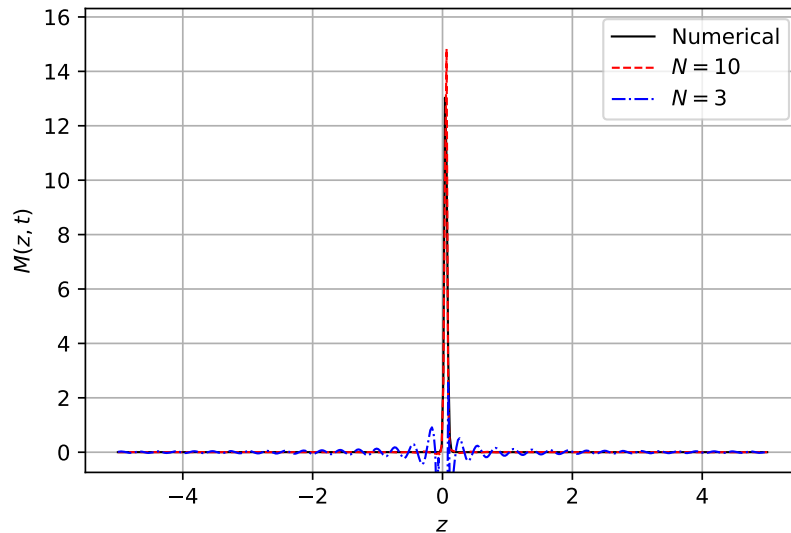


Figure 4.17: This graph compares the μ -integrated solution of the subspace method for various dimensions and the numerical solution at the time $t = 0.1$ for an initial value of $\mu_0 = 0.5$. Here we have set $\xi = 2.5$.

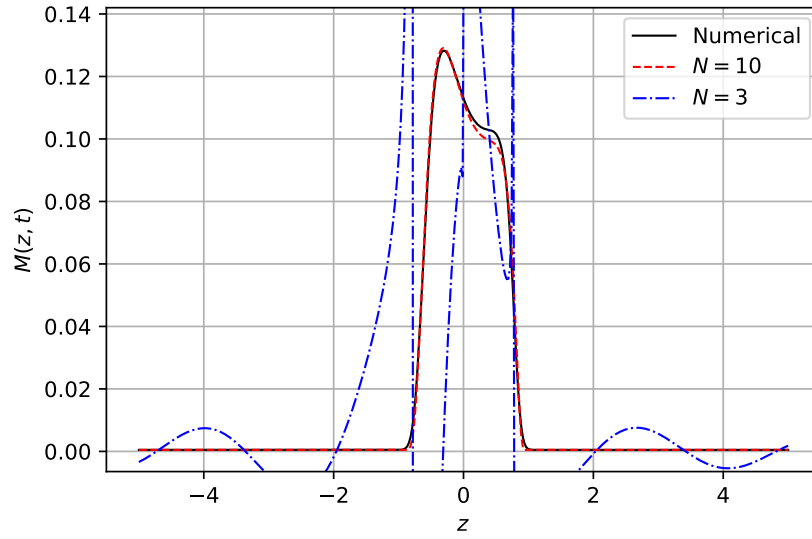


Figure 4.18: Caption is as in Figure 4.17 but we have used $t = 1$.

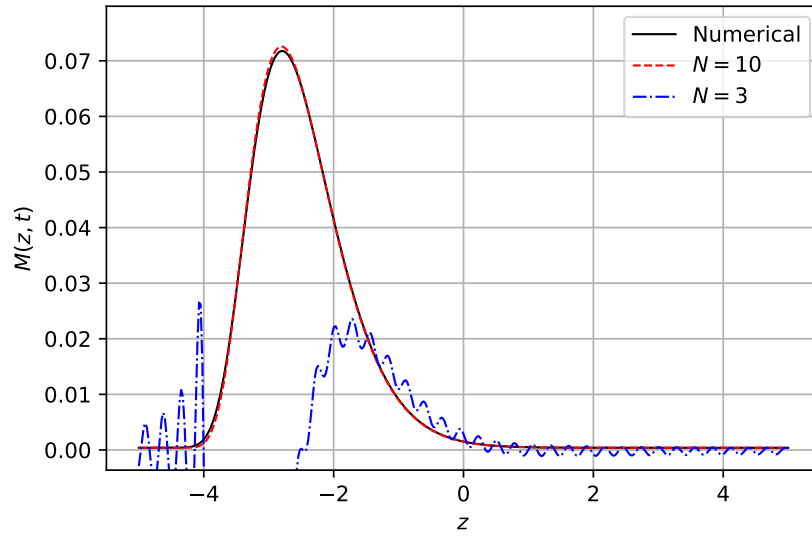


Figure 4.19: Caption is as in Figure 4.17 but we have used $t = 5$.

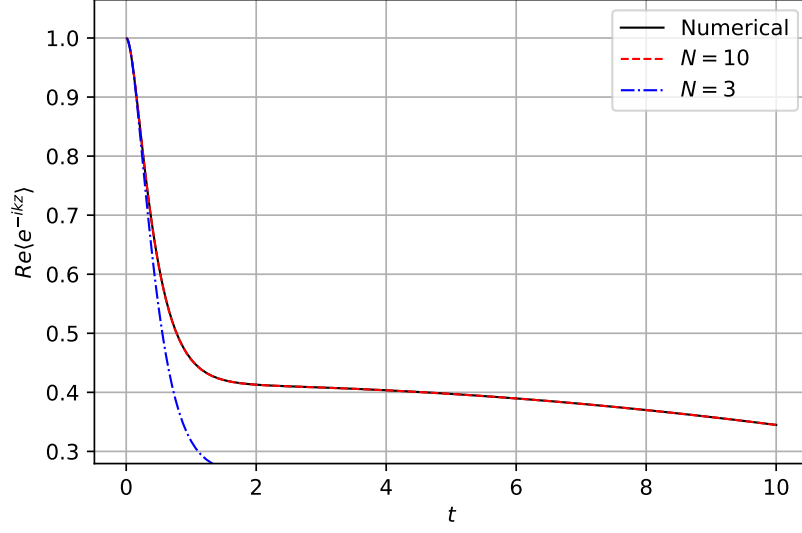


Figure 4.20: Comparison of the μ -averaged characteristic function between the subspace method for various dimensions and the numerical solution. Here we have used an initial pitch-angle cosine of $\mu_0 = 0$ and for the focusing parameter we set $\xi = 2.5$. We have kept the dimensionless wave number $k \rightarrow vk/D = 0.1$ constant and varied time.

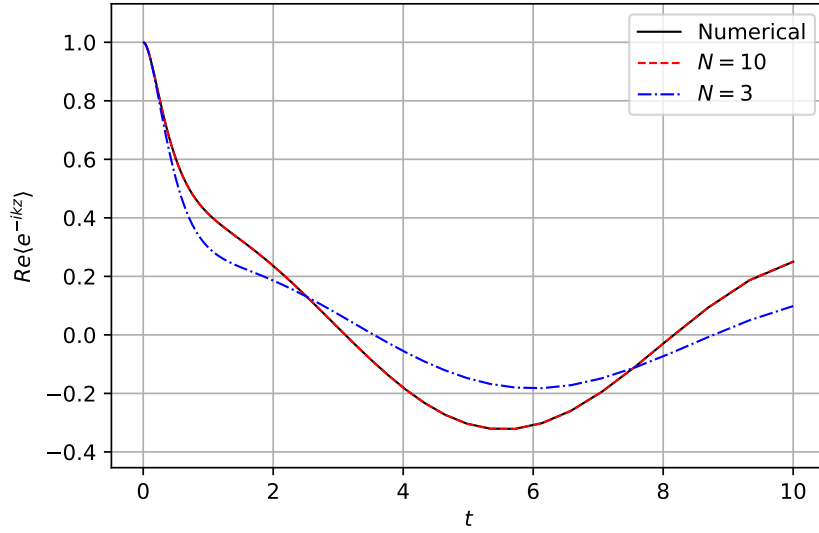


Figure 4.21: Caption is as in Figure 4.20 but we have set $k = 1$.

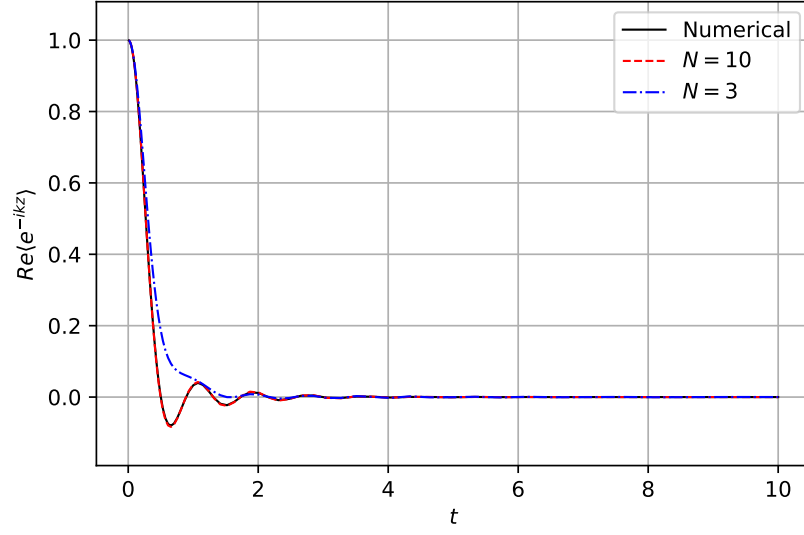


Figure 4.22: Caption is as in Figure 4.20 but we have set $k = 10$.

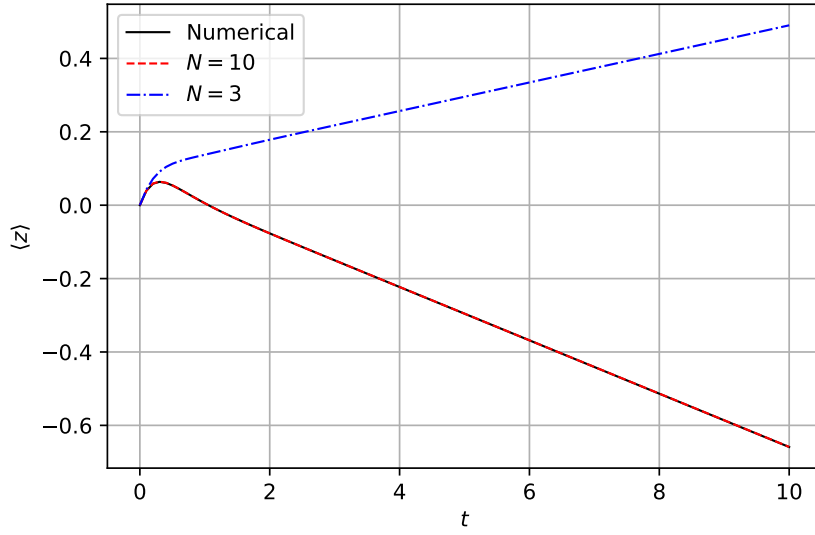


Figure 4.23: Comparison of the mean position $\langle z \rangle$ as a function of time for the N -dimensional subspace method and the numerical solution. Here, we have set $\mu_0 = 0.5$ and $\xi = 2.5$.

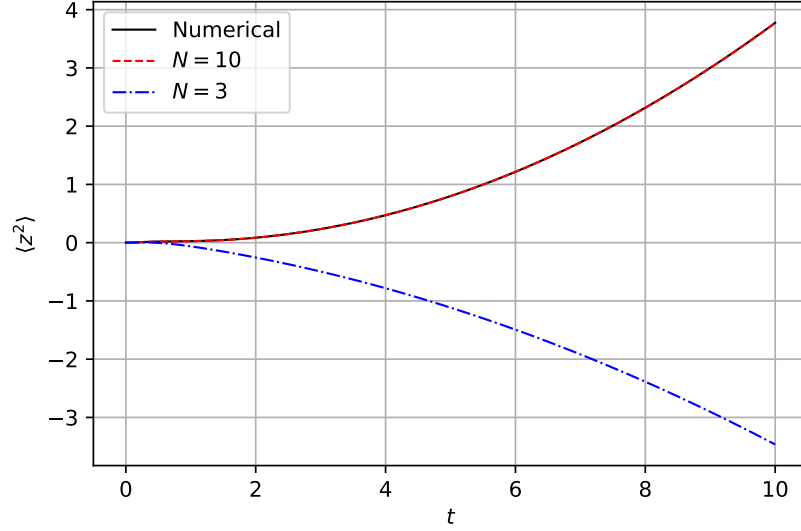


Figure 4.24: Comparison of the moment $\langle z^2 \rangle$ with respect to time for the N -dimensional subspace method and the numerical solution. Here, we have used $\mu_0 = 0.5$ and $\xi = 2.5$.

4.4.4 Time Comparisons

In the following, we compare the time needed to compute the solution of Eq. (4.2) using the N -dimensional subspace approximation with the time to compute the solution numerically with the method discussed in Section 2.3. Tables 4.1-4.3 show the computational times for the different methods. We have only considered the time to compute each F_k ; in other words, computing the Fourier transform integral is not included. As the tables demonstrate, numerically computing the solution for early times is marginally quicker, but for any other time, the subspace method is significantly faster. The cause of the numerical solution being quicker for early times is due to the fact that in this case, very few time steps are required. Additionally, just as Table 3.2 illustrates, here we see that the solution is faster at late times since it is sufficient to compute the solution in Fourier space for fewer wavenumbers. The following tables also show that the parameter ξ does not affect the time required to compute the solution, regardless of the method used.

It is important to note that each runtime was calculated using *Python* on the same computer. We also point out that for the subspace methods, we compute the solution for 100 values of μ , whereas for

the numerical solution we must use 2000 values to ensure the solution is accurate.

Dt	2D	3D	10D	Pure Numerical
0.1	1112.7	1739.3	6790.3	6116.4
1	34.7	54.9	213.4	1811.3
5	3.0	4.8	18.4	778.8

Table 4.1: For different values of Dt , we compare the time in seconds to compute the solution of Eq. (4.2) using the subspace approximation for various values of N with the numerical solution. Here, we have used $\xi = 0.1$.

Dt	2D	3D	10D	Pure Numerical
0.1	1125.3	1757.1	6841.1	6113.9
1	35.3	55.7	217.3	1809.1
5	3.0	4.8	19.0	775.8

Table 4.2: Caption is as in Table 4.1, except that we have used $\xi = 0.5$.

Dt	3D	10D	Pure Numerical
0.1	1767.6	6865.0	6102.3
1	57.2	220.5	1804.2
5	4.8	19.1	779.0

Table 4.3: Caption is as in Table 4.1, except that we have used $\xi = 2.5$.

4.4.5 Limitations

It is evident that we need to truncate the Fourier transform integral given by Eq. (2.27) at some wavenumber. By Eqs. (4.3) and (4.12), we observe that we require all of the eigenvalues of the matrix $\mathbf{M}$ to have a negative real component. As we will now show, this is not always the case.

As a first example, we consider the 2-dimensional subspace approximation. We consider Eq. (4.24)

and explore the limit $k \rightarrow \infty$. We assume that $L > 0$ and first notice that

$$Re \left\{ \sqrt{D^2 - \frac{1}{3}v^2k^2 + i\frac{v^2k}{3L}} \right\} = \left[\left(D^2 - \frac{1}{3}v^2k^2 \right)^2 + \frac{v^4k^2}{9L^2} \right]^{1/4} \cos \left[\frac{1}{2} \tan^{-1} \left(\frac{\frac{v^2k}{3L}}{D^2 - \frac{1}{3}v^2k^2} \right) + \frac{1}{2}\pi \right] \quad (4.133)$$

where Re denotes the real component. One can easily see here that in the limit $k \rightarrow \infty$, the indeterminate form $\infty \cdot 0$ arises. Evaluating the limit is then a matter of applying l'Hôpital's rule. After a tedious yet straightforward calculation, we derive

$$\lim_{k \rightarrow \infty} Re \left\{ \sqrt{D^2 - \frac{1}{3}v^2k^2 + i\frac{v^2k}{3L}} \right\} = \frac{v}{2\sqrt{3}L}. \quad (4.134)$$

As a result, we can see

$$\lim_{k \rightarrow \infty} Re \{ \omega_+ \} = -D + \frac{v}{2\sqrt{3}L} \quad (4.135)$$

which is positive if

$$\frac{v}{DL} > 2\sqrt{3}. \quad (4.136)$$

Similarly, if $L < 0$, we have that ω_- has a positive real component for large wavenumbers if

$$-\frac{v}{DL} > 2\sqrt{3}. \quad (4.137)$$

Therefore, there is a limit for the possible values of the parameter ξ .

Next, we consider a given $N \geq 3$. As it turns out, if ξ is large enough, we have the same problem as above. The downside here is that analytical calculations are not possible in this case, and thus the following paragraphs are based off numerical evidence.

The first important observation is that if $\{\omega_n(k)\}_{n=1}^N$ is the set of eigenvalues of the matrix $\mathbf{M}$ associated with the N -dimensional subspace approximation, then we have that for each $1 \leq n \leq N$,

$$\lim_{k \rightarrow \infty} Re \{ \omega_n(k) \} = \tilde{\omega}_n. \quad (4.138)$$

In other words, the real component of each eigenvalue converges to some finite value as k goes to infinity.

The next observation is that if we do not take N to be large enough, there are values of n such that $\tilde{\omega}_n$ is positive. Figure 4.25 shows the relation between the parameter ξ and the smallest value of N , denoted by N^* , which does not give rise to this issue. It should be noted that we do not claim that for a given ξ , the N^* -dimensional subspace method guarantees a good approximation. Instead, we claim that if one wants a good approximation, then they must take N to be at least N^* . For example, Figure 4.25 shows that for $\xi = 2.5$, $N = 3$ does not encounter this problem. However, we saw above that for $\xi = 2.5$, $N = 3$ does not give a good approximation.

The main takeaway from the above observations is that if one employs the 10-dimensional subspace approximation, they must proceed with caution if a large value of ξ is present. However, it is highly unlikely to pose a practical issue in the context of the solar wind, as the analysis by [Ma Sung and Earl, 1978] revealed that ξ never exceeded a value of one throughout 30 events.

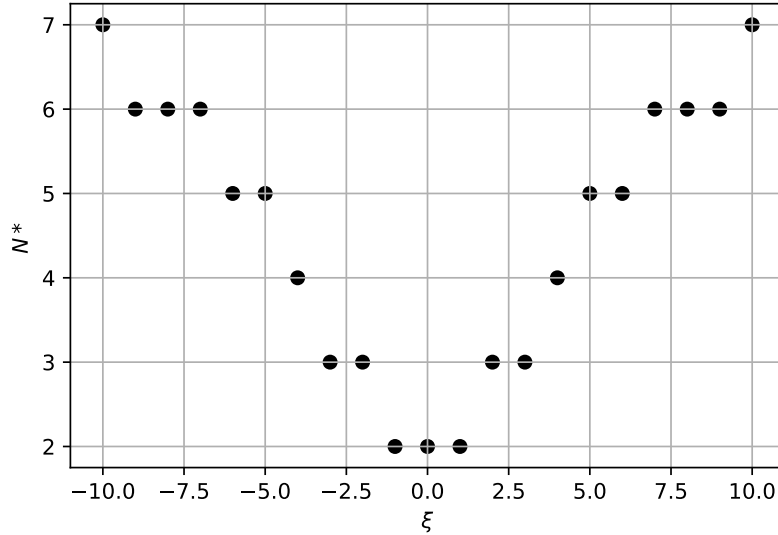


Figure 4.25: The relation between ξ and N^* , which is the minimum dimension one must use for the subspace method to produce a good approximation.

There is another inconvenience that arises from the fact that the real component of the eigenvalues does not decay to zero. That is the uncertainty of knowing when to truncate the Fourier transform integral. For example, by solving Eq. (4.2) numerically, it can be seen that the magnitude of F_k decays

exponentially as the wave number increases. This in turn leads to naturally truncating the Fourier transform integral when F_k is deemed to be sufficiently small. As for the subspace method, however, we do not achieve this decay, and thus deciding when to truncate the Fourier transform integral is not clear. Nevertheless, in our results for the μ -integrated solutions which appear in the sections above, we truncate the Fourier transform integral for the subspace method at the same wavenumber as we did for the numerical solution. Judging from the figures we produced, this choice of truncation is seen to be sufficient.

4.5 Conclusion

In this chapter, we considered the standard Fokker-Planck equation with focusing. An approximate solution was derived using the N -dimensional subspace approximation. Two different applications of this method were explored, which we summarize in the following paragraphs.

The first application was to employ the 2 and 3-dimensional subspace methods with the intention of obtaining simple results. The downside with these dimensions is that the resulting solution of Eq. (4.1) is far from accurate, as demonstrated in [Lasuik and Shalchi, 2019]. However, as we see in Section 4.4, we still get a good approximation for the expectation values. It was also shown in [Lasuik and Shalchi, 2019] that the sole difference between 2 and 3-dimensions is a slight increase in accuracy for the expectation values, provided focusing is neglected. On the other hand, when the focusing effect is taken into account, we found that there can, in fact, be a substantial difference between the two dimensions. For example, in 2-dimensions, we found that the resulting value for $\kappa_{\parallel}^{TGK}$ was independent of the focusing parameter, which is not what we expect. Fortunately, this issue was resolved when we employed the 3-dimensional subspace approximation.

The next application we considered was to derive a semi-analytical method which can solve the Fokker-Planck equation significantly quicker than the pure numerical solution. This portion of the chapter is an extension of the work by [Shalchi, 2024] who showed that the 10-dimensional subspace approximation produces what we sought when focusing is neglected. Here, we extended those results

to include the focusing effect. We also found that caution must be taken when using a subspace approximation. Specifically, we found that for larger values of ξ , there is potential to run into convergence issues which render the solution completely invalid. Thankfully, the parameter $\xi = v/(2DL)$ is almost always small in the setting of the solar wind. This implies that this issue will almost never be present within 10-dimensions.

5

Subspace Approximation to the Focused Transport Equation

II. The Modified Form

5.1 Introduction

In the previous chapter, we considered the standard form of the focused Fokker-Planck equation with isotropic scattering. Analytical approximate results were derived using the 2 and 3-dimensional subspace approximation. Moreover, we showed that provided the focusing is not too strong, the 10-dimensional subspace method can be employed to obtain an accurate yet significantly quicker approximation compared to a purely numerical approach.

In this chapter, we repeat the above steps, except here we deal with the conservative form of the Fokker-Planck equation. That is, we have

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = D \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial f}{\partial \mu} \right] - \frac{v}{2L} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) f \right]. \quad (5.1)$$

The contents of this chapter can be found in [Klippenstein and Shalchi, 2025b]. Some of the content including the figures throughout this chapter are copied from the cited article in The Open Journal of

Astrophysics, which is licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0), permitting reuse with appropriate attribution.

5.2 Fundamentals of the Subspace Method

In Fourier space, Eq. (5.1) is given as

$$\frac{\partial F_k}{\partial t} + iv\mu k F_k = \frac{\partial}{\partial \mu} \left[D_{\mu\mu} \frac{\partial F_k}{\partial \mu} \right] - \frac{v}{2L} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) F_k \right]. \quad (5.2)$$

As with the standard form of the Fokker-Planck equation, we expand the function F_k in Legendre polynomials $P_n(\mu)$ so that

$$F_k(\mu, t) = \sum_{n=0}^{N-1} D_n(t) P_n(\mu). \quad (5.3)$$

This method revolves around determining the unknown functions D_n . Additionally, we also need to choose the value of N , which we discuss later. With this expansion, Eq. (5.2) turns into

$$\sum_{n=0}^{N-1} \dot{D}_n P_n + iv\mu k \sum_{n=0}^{N-1} D_n P_n = D \sum_{n=0}^{N-1} D_n \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial P_n}{\partial \mu} \right] - \frac{v}{2L} \sum_{n=0}^{N-1} D_n \frac{\partial}{\partial \mu} \left[(1 - \mu^2) P_n \right]. \quad (5.4)$$

The explicit μ dependence and derivatives can be removed from the second and third terms by using Eq. (4.5). As for the last term, we employ, see [Abramowitz and Stegun, 1964],

$$\frac{d}{d\mu} \left[(1 - \mu^2) P_n \right] = (n - 1)\mu P_n - (n + 1)P_{n+1}. \quad (5.5)$$

Upon using the given identities, we can rewrite Eq. (5.4) as

$$\begin{aligned} \sum_{n=0}^{N-1} \dot{D}_n P_n + ivk \sum_{n=0}^{N-1} D_n \left[\frac{n+1}{2n+1} P_{n+1} + \frac{n}{2n+1} P_{n-1} \right] = & -D \sum_{n=0}^{N-1} D_n n(n+1) P_n \\ & - \frac{v}{2L} \sum_{n=0}^{N-1} D_n \left[-\frac{(n+1)(n+2)}{2n+1} P_{n+1} + \frac{(n-1)n}{2n+1} P_{n-1} \right]. \end{aligned} \quad (5.6)$$

We aim to continue by making use of the orthogonality relation of Legendre polynomials, found in, for example [Abramowitz and Stegun, 1964], given by

$$\int_{-1}^1 d\mu P_n P_m = \frac{2}{2m+1} \delta_{nm}, \quad (5.7)$$

where $\delta_{nm} = 1$ for $n = m$ and zero otherwise. Multiplying Eq. (5.6) by P_m , integrating over all μ , and using the orthogonality relation yields

$$\begin{aligned} \dot{D}_m = & -Dm(m+1)D_m - ivk \left[\frac{m}{2m-1} D_{m-1} + \frac{m+1}{2m+3} D_{m+1} \right] \\ & - \frac{v}{2L} \left[-\frac{m(m+1)}{2m-1} D_{m-1} + \frac{m(m+1)}{2m+3} D_{m+1} \right]. \end{aligned} \quad (5.8)$$

This corresponds to an infinite linear system of coupled differential equations for the functions D_n . We can summarize this system using the matrix equation

$$\dot{\vec{D}}(t) = \mathbf{T} \vec{D}(t) \quad (5.9)$$

where we have used the column vector

$$\vec{D}(t) = \begin{pmatrix} D_0(t) \\ D_1(t) \\ \vdots \\ D_{N-1}(t) \end{pmatrix} \quad (5.10)$$

and the (m, n) entry of the matrix $\mathbf{T}$ is given as the coefficient of D_n in Eq. (5.8). The formal solution of this equation can be written as

$$\vec{D}(t) = e^{\mathbf{T}t} \vec{D}(0) \quad (5.11)$$

where we have used the matrix exponential. As for determining $\vec{D}(0)$, we combine the initial condition given by Eq. (2.34) with the expansion given by Eq. (5.3) to obtain

$$\frac{1}{\pi}\delta(\mu - \mu_0) = \sum_{n=0}^{N-1} D_n(0)P_n(\mu). \quad (5.12)$$

Employing the orthogonality relation given by Eq. (5.7) yields

$$D_m(t=0) = \frac{2m+1}{2\pi}P_m(\mu_0). \quad (5.13)$$

Eqs. (5.3), (5.11) and (5.13) completely describe the solution corresponding to the N -dimensional subspace approximation. The structure of this solution is identical to that of the subspace approximation for the standard Fokker-Planck equation discussed in the previous chapter, with the only difference being the matrix $\mathbf{T}$.

The remaining step is to select a value for N . In the following, we set N to a small value to derive analytical approximations of the solution along with certain expectation values. After that, we again show that taking $N = 10$ results in a quick yet accurate approximation of the solution.

5.3 Analytical Results of the Subspace Method

In this section, we use the subspace method outlined in the previous section with $N = 2$. The benefit of using a small value for N is that it is possible to derive analytical results. However, we will derive certain expectation results which are unexpectedly independent of the focusing constant L . This requires us to also consider $N = 3$, as this dimension avoids the aforementioned problem.

5.3.1 2-Dimensional Subspace Approximation

By only keeping the first two terms of the expansion in Eq. (5.3),

$$F_k(\mu, t) = D_0(t) + D_1(t)\mu. \quad (5.14)$$

Setting $m = 0$ in Eq. (5.8) yields

$$\dot{D}_0 = -i\frac{1}{3}vkD_1, \quad (5.15)$$

and similarly $m = 1$ results in

$$\dot{D}_1 = -ivkD_0 - 2DD_1 + \frac{v}{L}D_0. \quad (5.16)$$

Combining Eqs. (5.15) and (5.16) results in the linear system given by

$$\begin{pmatrix} \dot{D}_0 \\ \dot{D}_1 \end{pmatrix} = \begin{pmatrix} 0 & -i\frac{1}{3}vk \\ -ivk + \frac{v}{L} & -2D \end{pmatrix} \begin{pmatrix} D_0 \\ D_1 \end{pmatrix}. \quad (5.17)$$

To solve this, we require the eigenvalues of Eq. (5.17). It is straightforward to show they solve

$$\phi^2 + 2D\phi + \frac{1}{3}v^2k^2 + i\frac{1}{3}\frac{v^2k}{L} = 0 \quad (5.18)$$

and are given by

$$\phi_{\pm} = -D \pm \sqrt{D^2 - \frac{1}{3}v^2k^2 - i\frac{1}{3}\frac{v^2k}{L}}. \quad (5.19)$$

Now, by superposition, we suppose that D_0 is given by

$$D_0(t) = c_+e^{\phi_+t} + c_-e^{\phi_-t} \quad (5.20)$$

for constants c_+ and c_- . Placing Eq. (5.20) into Eq. (5.15) gives

$$D_1(t) = i\frac{3}{vk} [\phi_+c_+e^{\phi_+t} + \phi_-c_-e^{\phi_-t}]. \quad (5.21)$$

To determine $c_{\pm}$, we use the initial conditions provided by Eq. (5.13) and find

$$\begin{aligned} c_+ + c_- &= \frac{1}{2\pi} \\ i\frac{3}{vk}(\phi_+c_+ + \phi_-c_-) &= \frac{3\mu_0}{2\pi}. \end{aligned} \quad (5.22)$$

After some straightforward algebra, one can show that

$$\begin{aligned} c_+ &= -\frac{1}{2\pi} \frac{\phi_- + iv\mu_0 k}{\phi_+ - \phi_-} \\ c_- &= \frac{1}{2\pi} \frac{\phi_+ + iv\mu_0 k}{\phi_+ - \phi_-}. \end{aligned} \quad (5.23)$$

The coefficients given by Eqs. (5.20) and (5.21) can be used in Eq. (5.14) to write

$$F_k(\mu, t) = \left(1 + i3\frac{\phi_+ \mu}{vk}\right) c_+ e^{\phi_+ t} + \left(1 + i3\frac{\phi_- \mu}{vk}\right) c_- e^{\phi_- t} \quad (5.24)$$

where the coefficients $c_{\pm}$ are given by Eq. (5.23) and the eigenvalues $\phi_{\pm}$ are provided by Eq. (5.19).

The next step is to apply this approximate solution to obtain approximate expectation values.

5.3.2 2-Dimensional Subspace Expectation Values

Here we employ Eq. (5.24) to compute several expectation values. Refer to Section 2.2 for the motivation behind each value along with preliminary calculations.

5.3.2.1 The Characteristic Function

The characteristic function is given as

$$\langle e^{-ikz} \rangle = \frac{\pi}{2} \int_{-1}^1 d\mu_0 \int_{-1}^1 d\mu F_k(\mu, t). \quad (5.25)$$

Thanks to orthogonality of Legendre polynomials, we have

$$\frac{1}{2} \int_{-1}^1 d\mu F_k(\mu, t) = D_0(t). \quad (5.26)$$

Using Eq. (5.20) therein yields

$$\langle e^{-ikz} \rangle = \pi \int_{-1}^1 d\mu_0 \left[c_+ e^{\phi_+ t} + c_- e^{\phi_- t} \right]. \quad (5.27)$$

With Eq. (5.23), the μ_0 -integral is easy to evaluate and we derive

$$\langle e^{-ikz} \rangle = \frac{\phi_+}{\phi_+ - \phi_-} e^{\phi_- t} - \frac{\phi_-}{\phi_+ - \phi_-} e^{\phi_+ t}. \quad (5.28)$$

Up to the value of the eigenvalues, we see that this expression is identical to that of Eq. (4.35) corresponding to the standard form. In the following paragraphs, we aim to simplify this result for the extremal values of the wavenumbers.

5.3.2.1.1 Small Wavenumbers

First we consider small wavenumbers and assume

$$\left| \frac{1}{3} v^2 k^2 + i \frac{1}{3} \frac{v^2 k}{L} \right| \ll D^2. \quad (5.29)$$

We define the shorthand notation

$$\phi_{\pm} = -D \pm \Phi \quad (5.30)$$

for

$$\Phi = \sqrt{D^2 - \frac{1}{3} v^2 k^2 - i \frac{1}{3} \frac{v^2 k}{L}}. \quad (5.31)$$

Taylor-expanding this quantity up to second order in wavenumber yields

$$\Phi = D - i \frac{\kappa_{\parallel 0}}{L} k - \kappa_{\parallel}^{msd} k^2 \quad (5.32)$$

where we have used the parallel diffusion coefficient $\kappa_{\parallel 0}$ without focusing defined in Section 2.2.5 given by

$$\kappa_{\parallel 0} = \frac{v^2}{6D} \quad (5.33)$$

and we have defined $\kappa_{\parallel}^{msd}$ as

$$\kappa_{\parallel}^{msd} = \kappa_{\parallel 0} \left(1 - \frac{1}{3} \frac{\lambda_{\parallel 0}^2}{L^2} \right) \quad (5.34)$$

which can be interpreted as the 2-dimensional approximation of the parallel diffusion coefficient with focusing. Here we have used the parallel mean free path without focusing

$$\lambda_{\parallel 0} = \frac{v}{2D}. \quad (5.35)$$

It should be noted that the resulting parallel diffusion coefficient obtained here is the same as that obtained using the 2-dimensional subspace approximation on the standard form of the Fokker-Planck equation. This is indeed what we expect as this is shown to hold in general in [Danos et al., 2013]. With the approximation described above, we can write

$$\phi_+ = -i \frac{\kappa_{\parallel 0}}{L} k - \kappa_{\parallel}^{msd} k^2 \quad (5.36)$$

and

$$\phi_- = -\frac{v}{\lambda_{\parallel 0}} + i \frac{\kappa_{\parallel 0}}{L} k. \quad (5.37)$$

Therefore, we can write Eq. (5.28) as

$$\langle e^{-ik_{\parallel} z} \rangle = \frac{-i \frac{\kappa_{\parallel 0}}{L} k - \kappa_{\parallel}^{msd} k^2}{2\Phi} e^{-vt/\lambda_{\parallel 0} + i\kappa_{\parallel 0} kt/L} + \frac{\frac{v}{\lambda_{\parallel 0}} - i \frac{\kappa_{\parallel}^{msd}}{L} k}{2\Phi} e^{-i\kappa_{\parallel 0} kt/L - \kappa_{\parallel} k^2 t}. \quad (5.38)$$

We can further simplify by noticing that the first term decays much quicker than the second term, thus the first term can be neglected. Moreover, it follows from Eq. (5.32) that the coefficient in front of the second exponential is close to one. Placing it together results in

$$\langle e^{-ikz} \rangle = e^{-i\kappa_{\parallel 0} kt/L - \kappa_{\parallel}^{msd} k^2 t} \quad (5.39)$$

which is identical to the result for the standard form, see Eq. (4.49).

5.3.2.1.2 Large Wavenumbers

Now we consider the case of large wavenumbers. That is, we assume

$$v^2 k^2 \gg \left| -3D^2 + i \frac{v^2 k}{L} \right|. \quad (5.40)$$

This implies that we can approximate Eq. (5.31) as

$$\begin{aligned} \Phi &= i \frac{v|k|}{\sqrt{3}} - \frac{v \operatorname{sgn}(k)}{2\sqrt{3}L} - i \frac{\sqrt{3}D^2}{2v|k|} \\ &= -\frac{v \operatorname{sgn}(k)}{2\sqrt{3}L} + i \frac{v|k|}{\sqrt{3}} \end{aligned} \quad (5.41)$$

where we have the sgn function defined by $\operatorname{sgn}(x) = 1$ if $x \geq 0$ and -1 otherwise. This result was simplified further by keeping only the largest term for both the real and imaginary components. We then obtain

$$\phi_+ = -D - \frac{v \operatorname{sgn}(k)}{2\sqrt{3}L} + i \frac{v|k|}{\sqrt{3}}, \quad (5.42)$$

$$\phi_- = -D + \frac{v \operatorname{sgn}(k)}{2\sqrt{3}L} - i \frac{v|k|}{\sqrt{3}}, \quad (5.43)$$

and

$$\phi_+ - \phi_- = 2\Phi = -\frac{v \operatorname{sgn}(k)}{\sqrt{3}L} + i \frac{2v|k|}{\sqrt{3}}. \quad (5.44)$$

Eq. (5.28) therefore simplifies to

$$\begin{aligned} \langle e^{-ikz} \rangle &= \frac{1}{-\frac{1}{2kL} + i} \left[\left(-\frac{\sqrt{3}}{2} \frac{D}{v|k|} - \frac{1}{4kL} + i \frac{1}{2} \right) e^{-\left(\frac{v \operatorname{sgn}(k)}{2\sqrt{3}L} + i \frac{v|k|}{\sqrt{3}} \right) t} \right. \\ &\quad \left. + \left(\frac{\sqrt{3}}{2} \frac{D}{v|k|} - \frac{1}{4kL} + i \frac{1}{2} \right) e^{-\left(\frac{v \operatorname{sgn}(k)}{2\sqrt{3}L} - i \frac{v|k|}{\sqrt{3}} \right) t} \right] e^{-Dt}. \end{aligned} \quad (5.45)$$

Although this cannot be simplified further, we can take the limiting case of $L, |k| \rightarrow \infty$ to obtain

$$\langle e^{-ikz} \rangle = \cos\left(\frac{vk}{\sqrt{3}}t\right) e^{-Dt} \quad (5.46)$$

which recovers the result obtained by [Lasuik and Shalchi, 2019] when focusing is neglected.

5.3.2.2 The Velocity Correlation Function

The velocity correlation function can be obtained from

$$V_{zz}(t) = \frac{\pi v^2}{2} \int_{-1}^1 d\mu_0 \mu_0 \int_{-1}^1 d\mu \mu F_0. \quad (5.47)$$

By the expansion given in Eq. (5.14) along with orthogonality of Legendre polynomials,

$$\int_{-1}^1 d\mu \mu F_0(\mu, t) = \frac{2}{3} D_1(k=0, t). \quad (5.48)$$

Due to the factor of k in the denominator of Eq. (5.21), we must be careful when computing $D_1(k=0, t)$.

We first compute $\phi_{\pm c_{\pm}}$ up to the lowest order in k that provides a non-zero result. By considering Eqs. (5.23), (5.36) and (5.37), we see that for small wavenumbers

$$\begin{aligned} \phi_{+c_+} &= -i \frac{\kappa_{\parallel 0}}{2\pi L} k \\ \phi_{-c_-} &= i \frac{\kappa_{\parallel 0}}{2\pi L} k - i \frac{v\mu_0}{2\pi} k. \end{aligned} \quad (5.49)$$

Placing these expressions into Eq. (5.21) and simplifying yields

$$D_1(k=0, t) = \frac{3}{2\pi v} \left[\frac{\kappa_{\parallel 0}}{L} - \frac{\kappa_{\parallel 0}}{L} e^{-vt/\lambda_{\parallel 0}} + v\mu_0 e^{-vt/\lambda_{\parallel 0}} \right] \quad (5.50)$$

where we have also used $\phi_+(k=0) = 0$ and $\phi_-(k=0) = -v/\lambda_{\parallel 0}$. Using this expression in Eq. (5.47) and evaluating the μ_0 -integral results in

$$V_{zz}(t) = \frac{v^2}{3} e^{-vt/\lambda_{\parallel 0}}. \quad (5.51)$$

From here, we easily obtain a parallel diffusion coefficient via the Taylor-Green-Kubo formula:

$$\begin{aligned} \kappa_{\parallel}^{TGK} &= \int_0^{\infty} dt V_{zz}(t) \\ &= \kappa_{\parallel 0}. \end{aligned} \quad (5.52)$$

Likewise with the standard form of the Fokker-Planck equation, the 2-dimensional approximation leads to $\kappa_{\parallel}^{TGK}$ being independent of focusing, which is known to be incorrect, see [Shalchi, 2011]. However, we again resolve this issue when we consider the 3-dimensional approximation.

5.3.2.3 The Moment $\langle \mu \rangle$

The first moment of μ is given as

$$\langle \mu \rangle = \pi \int_{-1}^1 d\mu \mu F_0(\mu, t). \quad (5.53)$$

The integral is evaluated in Eq. (5.48) and implies

$$\langle \mu \rangle = \frac{2}{3} \pi D_1(k=0, t). \quad (5.54)$$

Next, $D_1(k=0)$ is established in Eq. (5.50). We thus have

$$\langle \mu \rangle = \frac{\kappa_{\parallel 0}}{vL} + \left(\mu_0 - \frac{\kappa_{\parallel 0}}{vL} \right) e^{-vt/\lambda_{\parallel 0}}. \quad (5.55)$$

At late times, we see

$$\langle \mu \rangle (t \rightarrow \infty) = \frac{\kappa_{\parallel 0}}{vL} \quad (5.56)$$

which agrees with our expectations that the distribution function does not undergo a pitch-angle isotropization process. The formula derived here gives us the so-called *coherent speed*

$$v_c(t \rightarrow \infty) = v \langle \mu \rangle (t \rightarrow \infty) = \frac{\kappa_{\parallel 0}}{L} \quad (5.57)$$

as found, for instance, in [Shalchi and Danos, 2013] when focusing is weak.

5.3.2.4 The Moments $\langle z^n \rangle$

The remaining expectation values calculated here are the moments $\langle z \rangle$ and $\langle z^2 \rangle$. The exact expression for the 2-dimensional approximate value of $\langle z \rangle$ resulting from the standard form of the Fokker-Planck equation is given in Eq. (4.76) which is a lengthy expression. As we expect a similar result here, we elect to simplify the result by only considering the late-time limit.

The moments $\langle z^n \rangle$ are given by

$$\langle z^n \rangle = i^n \pi \left[\frac{\partial^n}{\partial k^n} \int_{-1}^1 d\mu F_k \right]_{k=0}. \quad (5.58)$$

Using the expansion given in Eq. (5.3) and orthogonality of Legendre polynomials, we have

$$\langle z^n \rangle = i^n 2\pi \left[\frac{\partial^n}{\partial k^n} D_0(k, t) \right]_{k=0}. \quad (5.59)$$

Since we are interested in the late time limit, we first aim to simplify the expression for D_0 for small wavenumbers. It follows from Eqs. (5.20), (5.36) and (5.37) that here,

$$D_0 = c_+ e^{\phi_+ t}. \quad (5.60)$$

In the remainder of this section, the goal is to evaluate Eq. (5.59) at late times for $n = 1$ and $n = 2$.

It is easy to see from Eqs. (5.59) and (5.60) that if we keep only the largest term, we have

$$\langle z \rangle (t \rightarrow \infty) = i 2\pi \left[c_+ t e^{\phi_+ t} \frac{\partial \phi_+}{\partial k} \right]_{k=0}. \quad (5.61)$$

From Eqs. (5.23) and (5.36), it follows that

$$\begin{aligned}\phi_+(k=0) &= 0 \\ \left. \frac{\partial \phi_+}{\partial k} \right|_{k=0} &= -i \frac{\kappa_{\parallel 0}}{L} \\ c_+(k=0) &= \frac{1}{2\pi}.\end{aligned}\tag{5.62}$$

Placing these values into Eq. (5.61) results in

$$\langle z \rangle (t \rightarrow \infty) = \frac{\kappa_{\parallel 0}}{L} t.\tag{5.63}$$

This result can also be compared with that of the standard form of the Fokker-Planck equation found in Eq. (4.61). If we average over all μ_0 , then both forms give an equal in magnitude, yet opposite result.

We can also verify the relation given in Eq. (2.55) which states that we expect

$$\frac{d}{dt} \langle z \rangle = v \langle \mu \rangle.\tag{5.64}$$

Indeed, if we take the time derivative of the above expression, we recover Eq. (5.57).

Now we consider Eq. (5.59) with $n = 2$, again in the late-time limit. In this case,

$$\langle z^2 \rangle (t \rightarrow \infty) = -2\pi \left[c_+ t^2 e^{\phi_+ t} \left(\frac{\partial \phi_+}{\partial k} \right)^2 \right]_{k=0}.\tag{5.65}$$

The relevant quantities are all given in Eq. (5.62) and so we have

$$\langle z^2 \rangle (t \rightarrow \infty) = \frac{\kappa_{\parallel 0}^2}{L^2} t^2\tag{5.66}$$

which is in perfect agreement with the μ_0 -averaged late-time limit resulting from the standard form.

It is interesting to note that while these results are independent of μ_0 , the two moments derived using the standard form of the Fokker-Planck equation were dependent on μ_0 . As we shall see in the upcoming

sections, this is also present within the 3-dimensional approximation. However, it is also important to point out that the μ_0 -independence is not a general result, but rather only a late-time limiting result.

5.3.3 3-Dimensional Subspace Approximation

Now we repeat the above calculations, except here we employ the 3-dimensional approximation. In this case, Eq. (5.9) is given as

$$\begin{pmatrix} \dot{D}_0 \\ \dot{D}_1 \\ \dot{D}_2 \end{pmatrix} = \begin{pmatrix} 0 & -i\frac{1}{3}vk & 0 \\ -ivk + \frac{v}{L} & -2D & -i\frac{2}{5}vk - \frac{v}{5L} \\ 0 & -i\frac{2}{3}vk + \frac{v}{L} & -6D \end{pmatrix} \begin{pmatrix} D_0 \\ D_1 \\ D_2 \end{pmatrix}. \quad (5.67)$$

Next we employ the ansatz

$$D_n(t) = \tilde{c}_n e^{\phi t} \quad \text{for } n = 0, 1, 2. \quad (5.68)$$

Just as in the previous cases, ϕ can be determined by finding the eigenvalues. That is,

$$\begin{pmatrix} -\phi & -i\frac{1}{3}vk & 0 \\ -ivk + \frac{v}{L} & -2D - \phi & -i\frac{2}{5}vk - \frac{v}{5L} \\ 0 & -i\frac{2}{3}vk + \frac{v}{L} & -6D - \phi \end{pmatrix} \begin{pmatrix} \tilde{c}_0 \\ \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix} = 0. \quad (5.69)$$

The corresponding characteristic polynomial is

$$\phi^3 + 8D\phi^2 + \left[12D^2 + \frac{3}{5}v^2k^2 + \frac{v^2}{5L^2} + i\frac{3v^2k}{5L}\right]\phi + 2Dv^2\left[k^2 + i\frac{k}{L}\right] = 0 \quad (5.70)$$

which is a cubic equation for the eigenvalues. Due to the complexity, we do not attempt to solve this equation exactly, but later it will be solved for small wavenumbers.

The remaining step is to determine $\tilde{c}_n$ in Eq. (5.68). By Eq. (5.13),

$$\begin{aligned} D_0(t=0) &= \frac{1}{2\pi}, \\ D_1(t=0) &= \frac{3\mu_0}{2\pi}, \\ D_2(t=0) &= \frac{5}{4\pi}(3\mu_0^2 - 1), \end{aligned} \tag{5.71}$$

where we have used the degree-2 Legendre polynomial $P_2(x) = (3x^2 - 1)/2$. For simpler notation, we write Eq. (5.67) in the shorter form of

$$\begin{pmatrix} \dot{D}_0 \\ \dot{D}_1 \\ \dot{D}_2 \end{pmatrix} = \begin{pmatrix} 0 & T_{01} & 0 \\ T_{10} & T_{11} & T_{12} \\ 0 & T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} D_0 \\ D_1 \\ D_2 \end{pmatrix}. \tag{5.72}$$

Superpositions allows us to assume

$$D_0 = \tilde{c}_0 e^{\phi_0 t} + \tilde{c}_1 e^{\phi_1 t} + \tilde{c}_2 e^{\phi_2 t}. \tag{5.73}$$

We then readily obtain

$$D_1 = \frac{1}{T_{01}} \dot{D}_0 = \frac{1}{T_{01}} [\phi_0 \tilde{c}_0 e^{\phi_0 t} + \phi_1 \tilde{c}_1 e^{\phi_1 t} + \phi_2 \tilde{c}_2 e^{\phi_2 t}] \tag{5.74}$$

from Eq. (5.72). Similarly, Eq. (5.72) also shows that

$$\begin{aligned} D_2 &= \frac{1}{T_{12}} [\dot{D}_1 - T_{10} D_0 - T_{11} D_1] \\ &= \frac{1}{T_{12}} \left[\left(\frac{\phi_0^2}{T_{01}} - T_{10} - \frac{T_{11} \phi_0}{T_{01}} \right) \tilde{c}_0 e^{\phi_0 t} + \left(\frac{\phi_1^2}{T_{01}} - T_{10} - \frac{T_{11} \phi_1}{T_{01}} \right) \tilde{c}_1 e^{\phi_1 t} \right. \\ &\quad \left. + \left(\frac{\phi_2^2}{T_{01}} - T_{10} - \frac{T_{11} \phi_2}{T_{01}} \right) \tilde{c}_2 e^{\phi_2 t} \right]. \end{aligned} \tag{5.75}$$

Combining Eqs. (5.73)-(5.75) with Eq. (5.71) leads to the three equations

$$\begin{aligned}
\frac{1}{2\pi} &= \tilde{c}_0 + \tilde{c}_1 + \tilde{c}_2, \\
\frac{3\mu_0}{2\pi} T_{01} &= \phi_0 \tilde{c}_0 + \phi_1 \tilde{c}_1 + \phi_2 \tilde{c}_2, \\
\frac{5}{4\pi} (3\mu_0^2 - 1) T_{12} &= \left(\frac{\phi_0^2}{T_{01}} - T_{10} - \frac{T_{11}\phi_0}{T_{01}} \right) \tilde{c}_0 + \left(\frac{\phi_1^2}{T_{01}} - T_{10} - \frac{T_{11}\phi_1}{T_{01}} \right) \tilde{c}_1 \\
&\quad + \left(\frac{\phi_2^2}{T_{01}} - T_{10} - \frac{T_{11}\phi_2}{T_{01}} \right) \tilde{c}_2.
\end{aligned} \tag{5.76}$$

The last equation can be simplified thanks to the first two equations:

$$\frac{5}{4\pi} (3\mu_0^2 - 1) T_{12} T_{01} + \frac{1}{2\pi} T_{10} T_{01} + \frac{3\mu_0}{2\pi} T_{11} T_{01} = \phi_0^2 \tilde{c}_0 + \phi_1^2 \tilde{c}_1 + \phi_2^2 \tilde{c}_2. \tag{5.77}$$

As with the standard form, here we have derived a system of equations which can be summarized with a Vandermonde matrix. This can again be solved using the inverse of Vandermonde matrices as given in [Turner, 1966], resulting in

$$\begin{aligned}
\tilde{c}_0 &= \frac{\phi_1\phi_2 - 3T_{01}(\phi_1 + \phi_2)\mu_0 + \frac{5}{2}T_{12}T_{01}(3\mu_0^2 - 1) + T_{10}T_{01} + 3T_{11}T_{01}\mu_0}{2\pi(\phi_0 - \phi_1)(\phi_0 - \phi_2)}, \\
\tilde{c}_1 &= \frac{\phi_0\phi_2 - 3T_{01}(\phi_0 + \phi_2)\mu_0 + \frac{5}{2}T_{12}T_{01}(3\mu_0^2 - 1) + T_{10}T_{01} + 3T_{11}T_{01}\mu_0}{2\pi(\phi_1 - \phi_0)(\phi_1 - \phi_2)}, \\
\tilde{c}_2 &= \frac{\phi_0\phi_1 - 3T_{01}(\phi_0 + \phi_1)\mu_0 + \frac{5}{2}T_{12}T_{01}(3\mu_0^2 - 1) + T_{10}T_{01} + 3T_{11}T_{01}\mu_0}{2\pi(\phi_2 - \phi_0)(\phi_2 - \phi_1)}.
\end{aligned} \tag{5.78}$$

The final 3-dimensional approximate solution is

$$F_k(\mu, t) = D_0(t)P_0(\mu) + D_1(t)P_1(\mu) + D_2(t)P_2(\mu). \tag{5.79}$$

Here, the constants D_n are given in Eqs. (5.73)-(5.75), the constants $\tilde{c}_n$ are given in Eq. (5.78), and the eigenvalues solve Eq. (5.70).

5.3.4 3-Dimensional Subspace Expectation Values

The next step is to employ the 3-dimensional approximate solution derived in the previous section and obtain several expectation values. The following expectation values are similar to those encountered for the standard Fokker-Planck equation in the 3-dimensional approximation in the sense that the results are far too complicated, unless we assume that the wavenumbers are small.

5.3.4.1 The Characteristic Function

The characteristic function in terms of the Fourier transformed solution is

$$\langle e^{-ikz} \rangle = \frac{\pi}{2} \int_{-1}^1 d\mu_0 \int_{-1}^1 d\mu F_k. \quad (5.80)$$

It then follows from the orthogonality of Legendre polynomials that

$$\langle e^{-ikz} \rangle = \pi \int_{-1}^1 d\mu_0 D_0 \quad (5.81)$$

where D_0 is defined in Eq. (5.73) which depends on the constants $\tilde{c}_n$ given in Eq. (5.78). Evaluating the μ_0 -integral is then trivial and we derive

$$\langle e^{-ikz} \rangle = \left[\frac{\phi_1\phi_2 - \frac{1}{3}v^2k^2 - i\frac{v^2k}{3L}}{(\phi_0 - \phi_1)(\phi_0 - \phi_2)} e^{\phi_0 t} + \frac{\phi_0\phi_2 - \frac{1}{3}v^2k^2 - i\frac{v^2k}{3L}}{(\phi_1 - \phi_0)(\phi_1 - \phi_2)} e^{\phi_1 t} + \frac{\phi_0\phi_1 - \frac{1}{3}v^2k^2 - i\frac{v^2k}{3L}}{(\phi_2 - \phi_0)(\phi_2 - \phi_1)} e^{\phi_2 t} \right]. \quad (5.82)$$

The next step is to simplify the above expression for small wavenumbers. We obtain an expression for the eigenvalues via a similar approach to that used when using the standard form of the Fokker-Planck equation. The eigenvalues are first written as

$$\phi = \alpha + \beta k + \gamma k^2 \quad (5.83)$$

for the constants α , β and γ that are independent of k . By placing this expression in Eq. (5.70) and

neglecting terms higher than quadratic in k , we have

$$\begin{aligned} & \alpha^3 + 8D\alpha^2 + 12D^2\alpha + \frac{v^2}{5L^2}\alpha + \left(3\alpha^2\beta + 16D\alpha\beta + 12D^2\beta + \frac{v^2}{5L^2}\beta + i\frac{3v^2}{5L}\alpha + i2\frac{Dv^2}{L}\right)k \\ & + \left(3\alpha\beta^2 + 3\alpha^2\gamma + 8D\beta^2 + 16D\alpha\gamma + 12D^2\gamma + \frac{3}{5}v^2\alpha + 2Dv^2 + \frac{v^2}{5L^2}\gamma + i\frac{3v^2}{5L}\beta\right)k^2 = 0. \end{aligned} \quad (5.84)$$

The constant term must be zero, immediately resulting in the following three values:

$$\alpha = 0 \quad \text{or} \quad \alpha = -4D \pm \sqrt{4D^2 - \frac{v^2}{5L^2}}. \quad (5.85)$$

To continue, we denote the eigenvalue associated with $\alpha = 0$ as ϕ_0 . Since the exponential involving ϕ_0 will have the slowest decay, we can approximate Eq. (5.82) as

$$\langle e^{-ikz} \rangle \approx \frac{\phi_1\phi_2 - \frac{1}{3}v^2k^2 - i\frac{v^2k}{3L}}{(\phi_0 - \phi_1)(\phi_0 - \phi_2)} e^{\phi_0 t}. \quad (5.86)$$

Furthermore, we can see that for small wavenumbers,

$$\frac{\phi_1\phi_2 - \frac{1}{3}v^2k^2 - i\frac{v^2k}{3L}}{(\phi_0 - \phi_1)(\phi_0 - \phi_2)} \approx \frac{\phi_1\phi_2}{\phi_1\phi_2} = 1 \quad (5.87)$$

which allows us to further simplify the expression to

$$\langle e^{-ikz} \rangle \approx e^{\phi_0 t}. \quad (5.88)$$

This implies that we only need to determine ϕ_0 , and thus we can set $\alpha = 0$ in Eq. (5.84). The resulting expression is

$$\left(12D^2\beta + \frac{v^2}{5L^2}\beta + i2\frac{Dv^2}{L}\right)k + \left(8D\beta^2 + 12D^2\gamma + 2Dv^2 + \frac{v^2}{5L^2}\gamma + i\frac{3v^2}{5L}\beta\right)k^2 = 0. \quad (5.89)$$

The linear term is easy to solve and we obtain

$$\beta = -i \frac{2 \frac{Dv^2}{L}}{12D^2 + \frac{v^2}{5L^2}}. \quad (5.90)$$

Finally, the quadratic term in Eq. (5.89) is also trivial and gives

$$\gamma = -\frac{8D\beta^2 + 2Dv^2 + i\frac{3v^2}{5L}\beta}{12D^2 + \frac{v^2}{5L^2}}. \quad (5.91)$$

Even with the limiting case of small wavenumbers, the value

$$\phi_0 = \beta k + \gamma k^2 \quad (5.92)$$

is still somewhat lengthy. We thus attempt to simplify the result further in the case of weak focusing; that is when $\xi = v/(2DL)$ is small. Up to second order in ξ ,

$$\beta = -i\frac{1}{3}v\xi \quad (5.93)$$

and

$$\gamma = -\frac{v^2}{6D} \left(1 - \frac{14}{15}\xi^2\right) = -\kappa_{\parallel 0} \left(1 - \frac{14}{45}\xi^2\right). \quad (5.94)$$

Placing these simplified values into Eq. (5.92) and then in Eq. (5.88) yields

$$\langle e^{-ikz} \rangle = e^{-\kappa_{\parallel 0} \left(1 - \frac{14}{45}\xi^2\right) k^2 t - i\frac{1}{3}v\xi k t} \quad (5.95)$$

corresponding to our simplified expression for the characteristic function with small wavenumbers and at late times. Comparing this result with that obtained using the standard form of the Fokker-Planck equation given in Eq. (4.111) shows that the only difference is the sign of the imaginary component. In particular, if we interpret the coefficient of the quadratic term to be the negative of the parallel diffusion coefficient, we obtain the same result.

5.3.4.2 The Velocity Correlation Function

Similar to the 2-dimensional approximation, the velocity correlation function is

$$V_{zz}(t) = \frac{\pi}{3} v^2 \int_{-1}^1 d\mu_0 \mu_0 D_1(k=0, t). \quad (5.96)$$

To determine $D_1(k=0, t)$, we first need the eigenvalues, which were determined for $k=0$ in Eq. (5.85).

We have

$$\begin{aligned} \phi_0(k=0) &= 0 \\ \phi_1(k=0) &= -4D - \sqrt{4D^2 - \frac{v^2}{5L^2}} \\ \phi_2(k=0) &= -4D + \sqrt{4D^2 - \frac{v^2}{5L^2}}. \end{aligned} \quad (5.97)$$

This implies that Eq. (5.74) reduces down to

$$D_1(k=0, t) = \left[\frac{\phi_0 \tilde{c}_0}{T_{01}} e^{\phi_0 t} + \frac{\phi_1 \tilde{c}_1}{T_{01}} e^{\phi_1 t} + \frac{\phi_2 \tilde{c}_2}{T_{01}} e^{\phi_2 t} \right]_{k=0}. \quad (5.98)$$

It is necessary to proceed with caution as Eq. (5.72) shows $T_{01}(k=0) = 0$. However, thankfully, Eq. (5.78) shows that every term in $\phi_n \tilde{c}_n$, except for one term, has a factor of T_{01} . As for the other term, it has a factor of ϕ_0 which is given in Eq. (5.92) for small k . We can then write

$$\begin{aligned} D_1(k=0, t) &= \left[i \frac{3\beta}{2\pi v} + \frac{i \frac{3\beta}{v} \phi_2 - 3(\phi_0 + \phi_2) \mu_0 + \frac{5}{2} T_{12} (3\mu_0^2 - 1) + T_{10} + 3T_{11} \mu_0}{2\pi(\phi_1 - \phi_2)} e^{\phi_1 t} \right. \\ &\quad \left. + \frac{i \frac{3\beta}{v} \phi_1 - 3(\phi_0 + \phi_1) \mu_0 + \frac{5}{2} T_{12} (3\mu_0^2 - 1) + T_{10} + 3T_{11} \mu_0}{2\pi(\phi_2 - \phi_1)} e^{\phi_2 t} \right]_{k=0} \\ &= i \frac{3\beta}{2\pi v} + \frac{i \frac{3\beta}{v} \phi_2(k=0) - 3\phi_2(k=0) \mu_0 - \frac{v}{2L} (3\mu_0^2 - 1) + \frac{v}{L} - 6D\mu_0}{2\pi(\phi_1(k=0) - \phi_2(k=0))} e^{\phi_1(k=0)t} \\ &\quad + \frac{i \frac{3\beta}{v} \phi_1(k=0) - 3\phi_1(k=0) \mu_0 - \frac{v}{2L} (3\mu_0^2 - 1) + \frac{v}{L} - 6D\mu_0}{2\pi(\phi_2(k=0) - \phi_1(k=0))} e^{\phi_2(k=0)t} \end{aligned} \quad (5.99)$$

for the eigenvalues given in Eq. (5.97). Placing this into Eq. (5.96) yields

$$V_{zz}(t) = \frac{1}{3}v^2 \left[\frac{-\phi_2(k=0) - 2D}{\phi_1(k=0) - \phi_2(k=0)} e^{\phi_1(k=0)t} + \frac{-\phi_1(k=0) - 2D}{\phi_2(k=0) - \phi_1(k=0)} e^{\phi_2(k=0)t} \right]. \quad (5.100)$$

This result can be used to obtain a parallel diffusion coefficient via the Taylor-Green-Kubo formulation:

$$\begin{aligned} \kappa_{\parallel}^{TGK} &= \int_{-\infty}^{\infty} dt V_{zz}(t) \\ &= \frac{1}{3}v^2 \left[\frac{\phi_2(k=0) + 2D}{\phi_1(k=0)(\phi_1(k=0) - \phi_2(k=0))} + \frac{\phi_1(k=0) + 2D}{\phi_2(k=0)(\phi_2(k=0) - \phi_1(k=0))} \right] \\ &= \frac{2\frac{v^2}{D}}{12 + \frac{v^2}{5D^2L^2}}. \end{aligned} \quad (5.101)$$

This is an exact result according to the 3-dimensional approximation and is identical to that of the standard form of the Fokker-Planck equation. As a next step, this result could be simplified for small values of ξ . However, as the result here is identical to the standard form, this simplification is given in Eq. (4.120).

5.3.4.3 The Moment $\langle \mu \rangle$

The moment $\langle \mu \rangle$ is

$$\langle \mu \rangle = \pi \int_{-1}^1 d\mu \mu F_0(\mu, t). \quad (5.102)$$

By orthogonality of Legendre polynomials, we have

$$\langle \mu \rangle = \frac{2}{3}\pi D_1(k=0, t) \quad (5.103)$$

for $D_1(k=0, t)$ given in Eq. (5.99). The exact result is far too complicated, and thus we simplify it at late times. By Eq. (5.97) it is easy to see that the factors of $e^{\phi_{1,2}(k=0)t}$ decay, and thus we are left with

$$\langle \mu \rangle (t \rightarrow \infty) = i \frac{\beta}{v} = \frac{2\frac{v}{DL}}{12 + \frac{v^2}{5D^2L^2}} \quad (5.104)$$

where β is given in Eq. (5.90). If we assume further that ξ is small, then up to third order

$$\langle \mu \rangle (t \rightarrow \infty) = \frac{\kappa_{\parallel 0}}{vL} - \frac{3\kappa_{\parallel 0}^3}{5v^3L^3}. \quad (5.105)$$

If this result was truncated after the first order, it would produce the same result found in Eq. (5.56) corresponding to the 2-dimensional approximation. Recall that Eq. (5.56) agreed with that found in [Shalchi and Danos, 2013] up to first-order in the weak focusing limit. With this 3-dimensional result, the agreement is up to third-order.

5.3.4.4 The Moments $\langle z^n \rangle$

As two final expectation values, we consider $\langle z \rangle$ and $\langle z^2 \rangle$. It was shown in the case of the 2-dimensional approximation that

$$\langle z^n \rangle = i^n 2\pi \left[\frac{\partial^n}{\partial k^n} D_0(k, t) \right]_{k=0}. \quad (5.106)$$

Thanks to Eqs. (5.73) and (5.97), we see that at late times

$$D_0(|k| \ll 1, t \rightarrow \infty) = \tilde{c}_0 e^{\phi_0 t}. \quad (5.107)$$

By only keeping the largest term, Eq. (5.106) for $n = 1$ is

$$\langle z \rangle (t \rightarrow \infty) = i2\pi \left[\tilde{c}_0 t e^{\phi_0 t} \frac{\partial}{\partial k} \phi_0 \right]_{k=0}. \quad (5.108)$$

Since we only care about small wavenumbers here, the relevant quantities are found in Eqs. (5.78), (5.92) and (5.97), which are summarized as

$$\begin{aligned} \tilde{c}_0(k=0) &= \frac{1}{2\pi} \\ \phi_0(k=0) &= 0 \\ \left. \frac{\partial}{\partial k} \phi_0 \right|_{k=0} &= \beta = -i \frac{2 \frac{Dv^2}{L}}{12D^2 + \frac{v^2}{5L^2}}. \end{aligned} \quad (5.109)$$

Placing these values into Eq. (5.108) yields

$$\langle z \rangle (t \rightarrow \infty) = \frac{2 \frac{v^2}{DL}}{12 + \frac{v^2}{5D^2L^2}} t. \quad (5.110)$$

Once more we assume ξ is small, allowing the result to be simplified to

$$\langle z \rangle (t \rightarrow \infty) = \left[\frac{\kappa_{\parallel 0}}{L} - \frac{3\kappa_{\parallel 0}^3}{5v^2L^3} \right] t. \quad (5.111)$$

By comparing this result with Eq. (5.105), we see that

$$\frac{d}{dt} \langle z \rangle (t \rightarrow \infty) = v \langle \mu \rangle (t \rightarrow \infty) \quad (5.112)$$

confirming the relation given in Eq. (2.55). The above expression can also be compared with the 2-dimensional result given in Eq. (5.63). If we truncate the above expansion at second order, then we obtain agreement.

Next, we consider Eq. (5.106) for $n = 2$ at late times. This can be simplified to

$$\langle z^2 \rangle (t \rightarrow \infty) = -2\pi \left[\tilde{c}_0 t^2 e^{\phi_0 t} \left(\frac{\partial}{\partial k} \phi_0 \right)^2 \right]_{k=0} \quad (5.113)$$

where each of the quantities needed is given in Eq. (5.109). We thus have

$$\langle z^2 \rangle (t \rightarrow \infty) = \left(\frac{2 \frac{v^2}{DL}}{12 + \frac{v^2}{5D^2L^2}} \right)^2 t^2. \quad (5.114)$$

If we assume that ξ is small, we find

$$\langle z^2 \rangle (t \rightarrow \infty) = \frac{\kappa_{\parallel 0}^2}{L^2} \left[1 - \frac{6\kappa_{\parallel 0}^2}{5v^2L^2} \right] t^2. \quad (5.115)$$

If the lowest order term is the only term kept, then we recover Eq. (5.66) which is derived using the 2-dimensional approximation.

5.4 Efficient and Accurate Subspace Approximations

Similar to when dealing with the standard form of the Fokker-Planck equation, we now develop the N -dimensional subspace method corresponding to a semi-analytical/semi-numerical approach. It was shown in [Shalchi, 2024] and in Chapter 4 that the 10-dimensional subspace approximation provides the best balance in terms of accuracy and speed. In this section, we show once more that the 10-dimensional subspace approximation is the best to use here.

To verify the accuracy of the subspace method, we also solve Eq. (5.1) numerically. See Section 2.3 for the method we use here. In order to render the variables dimensionless, we again employ the transformations

$$t \rightarrow Dt \quad \text{and} \quad z \rightarrow \frac{Dz}{v} \quad (5.116)$$

transforming Eq. (5.1) into

$$\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial f}{\partial \mu} \right] - \xi \frac{\partial}{\partial \mu} [(1 - \mu^2)f] \quad (5.117)$$

with the dimensionless parameter ξ .

In the following, we compare the subspace method with the numerical method for $\xi = 0.1, 0.5$ and 2.5 . As can be seen, for each value of ξ , the 10-dimensional subspace approximation is indistinguishable from the exact solution obtained using the numerical method.

5.4.1 The Parameter $\xi = 0.1$

As a first consideration, we set $\xi = 0.1$. The Figures 5.1 - 5.3 compare the μ -integrated solution obtained from the above mentioned methods for a fixed initial value $\mu_0 = 0.5$, and at the times $t = 0.1, 1$, and 5 .

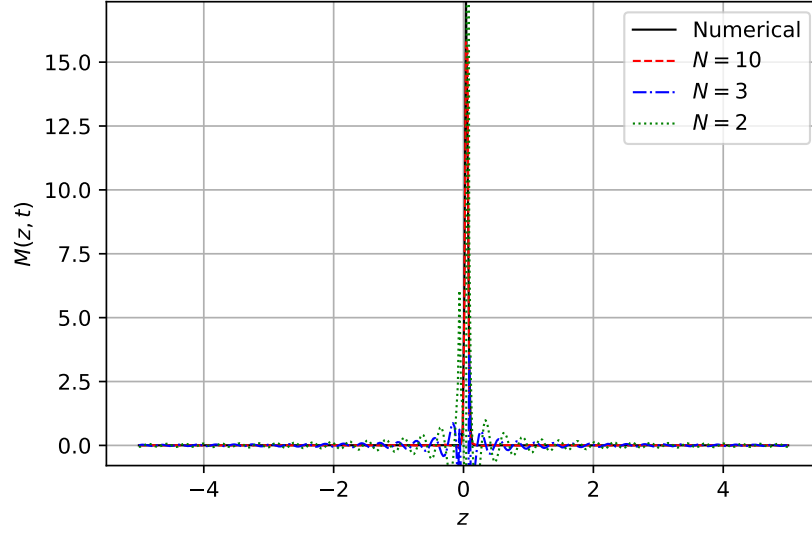


Figure 5.1: The graph compares the μ -integrated solution of the subspace method for various dimensions and the numerical solution at the time $t = 0.1$ for an initial value $\mu_0 = 0.5$. We set the parameter $\xi = 0.1$.

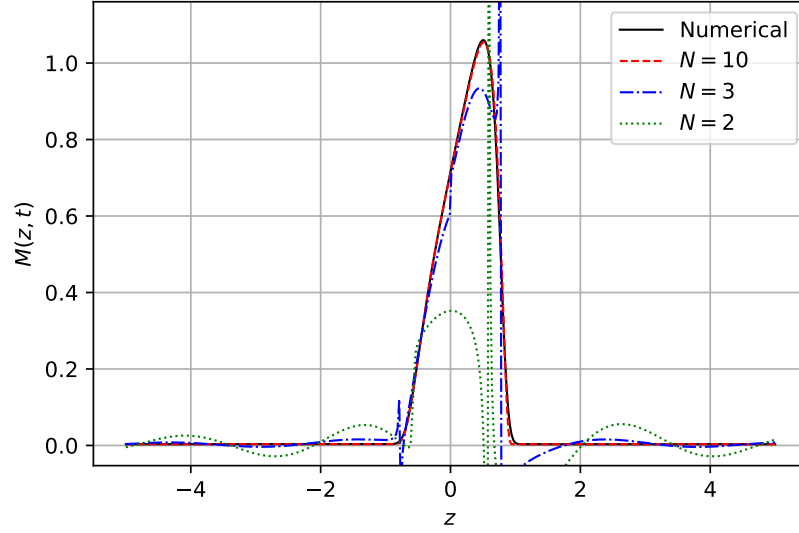


Figure 5.2: Caption as in Figure 5.1 except that we have used $t = 1$.

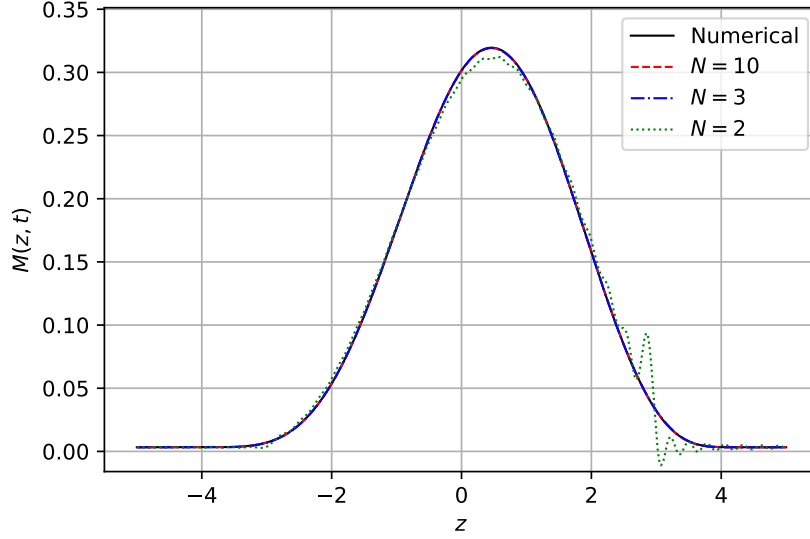


Figure 5.3: Caption is as in Figure 5.1 except that we have used $t = 5$.

We also compare the subspace method with the numerical solution in terms of various expectation values. Rather than computing a given expectation value directly from the definition given in Eqs. (2.35) and (2.36), the computations can be significantly sped up by using analytical work outlined in Section 5.3.2.

The first expectation value we consider is the μ_0 -dependent characteristic function defined as

$$\langle e^{-ikz} \rangle(\mu_0, t) = \frac{1}{2} \int_{-1}^1 d\mu \int_{-\infty}^{\infty} dz f(z, \mu, t) e^{-ikz}. \quad (5.118)$$

It follows from the same argument in Eqs. (5.25) and (5.26) that

$$\langle e^{-ikz} \rangle(\mu_0, t) = 2\pi D_0(k, t). \quad (5.119)$$

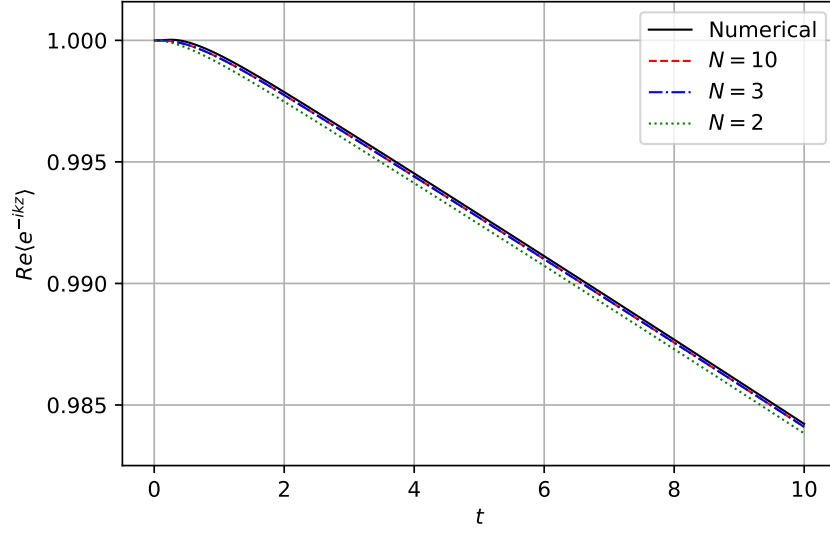


Figure 5.4: Comparison of the μ -averaged characteristic function between the subspace method for various dimensions and the numerical solution. We use an initial value $\mu_0 = 0$ and $\xi = 0.1$ for the focusing parameter. Here we have kept the dimensionless wavenumber $k \rightarrow vk/D = 0.1$ constant and varied time.

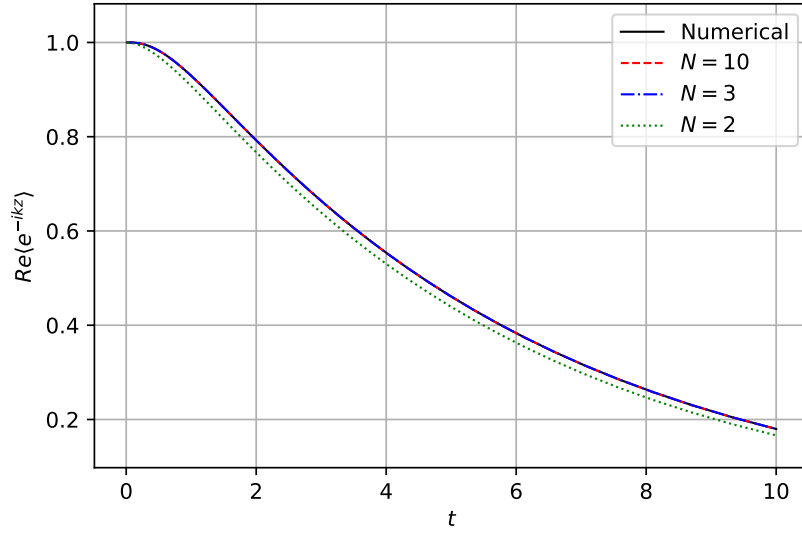


Figure 5.5: Caption is as in Figure 5.4 except that we have used $k = 1$.

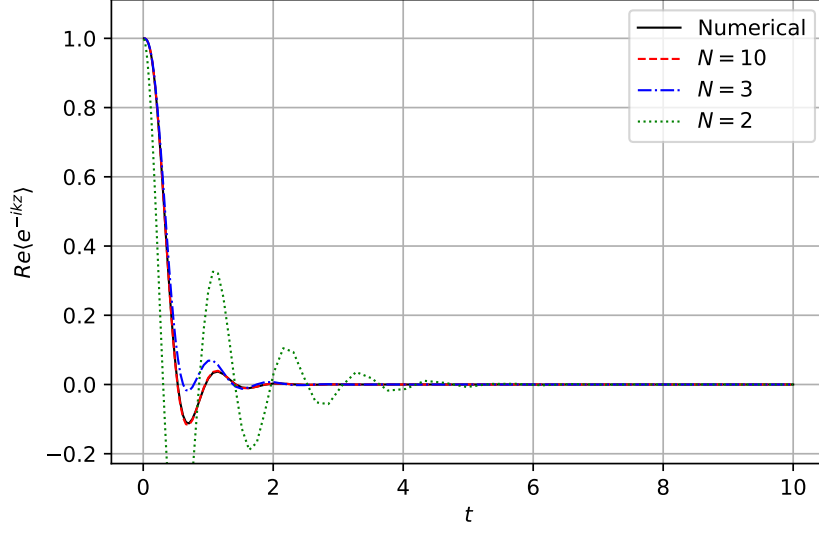


Figure 5.6: Caption is as in Figure 5.4 except that we have used $\tilde{k} = 10$.

The final expectation values considered here are the moments $\langle z \rangle$ and $\langle z^2 \rangle$. These values are computed via Eq. (5.59), where the k -derivative is evaluated numerically.

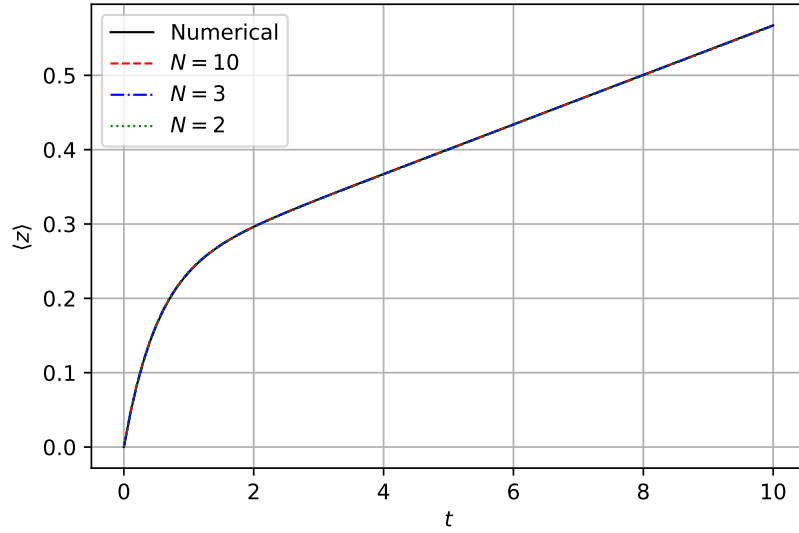


Figure 5.7: Comparison of the moment $\langle z \rangle$ as a function of time for the N -dimensional subspace method and the numerical solution. Here, we have used $\mu_0 = 0.5$ and $\xi = 0.1$.

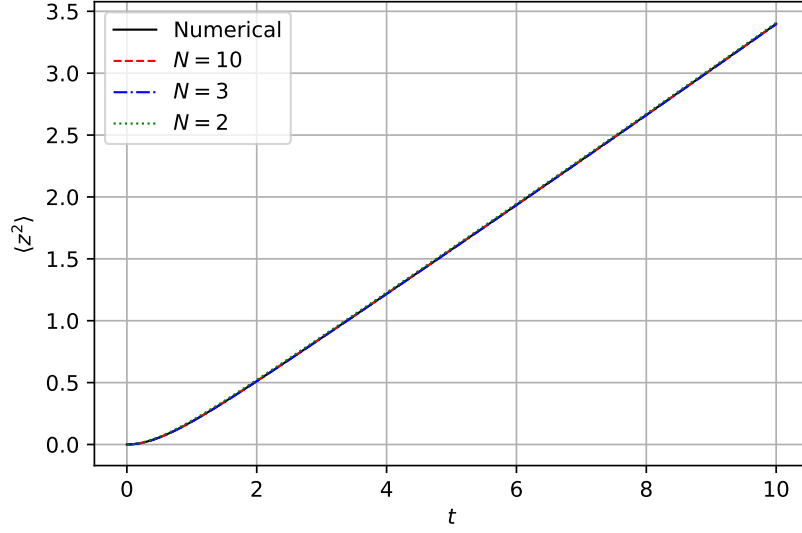


Figure 5.8: Comparison of the moment $\langle z^2 \rangle$ as a function of time for the N -dimensional subspace method and the numerical solution. Here, we have used $\mu_0 = 0.5$ and $\xi = 0.1$.

5.4.2 The Parameter $\xi = 0.5$

We plot the same graphs as in the previous section, except here we use $\xi = 0.5$.

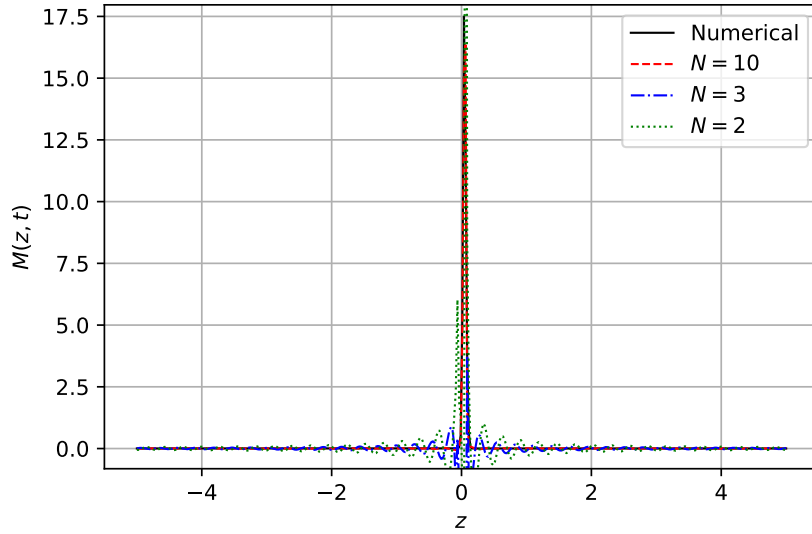


Figure 5.9: The graph compares the μ -integrated solution of the subspace method for various dimensions and the numerical solution at the time $t = 0.1$ for an initial value of $\mu_0 = 0.5$. We have set $\xi = 0.5$.

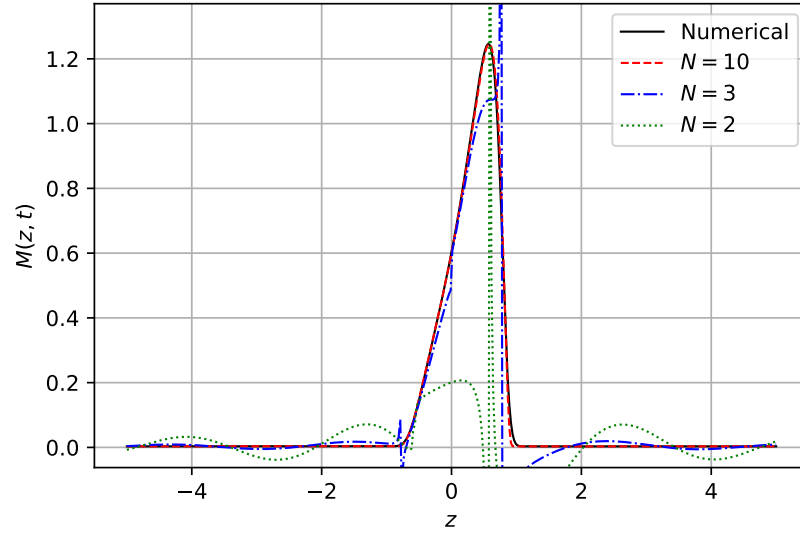


Figure 5.10: Caption is as in Figure 5.9 that we have used $t = 1$.

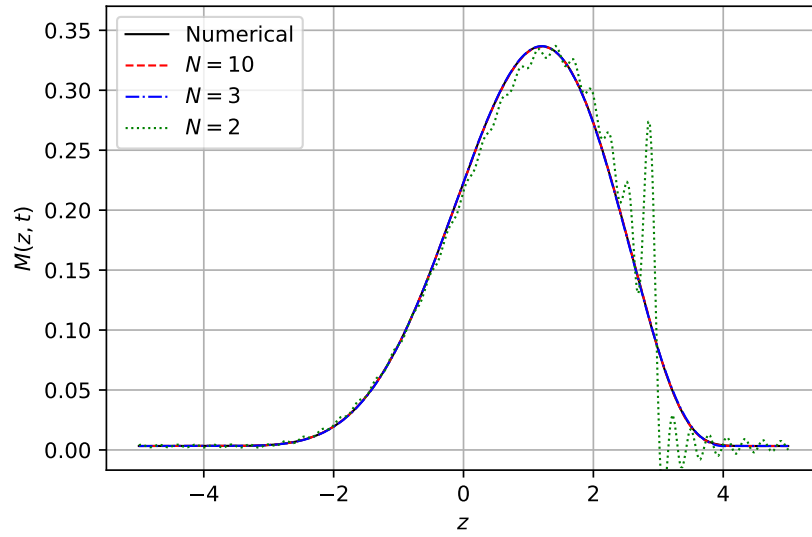


Figure 5.11: Caption is as in Figure 5.9 that we have used $t = 5$.

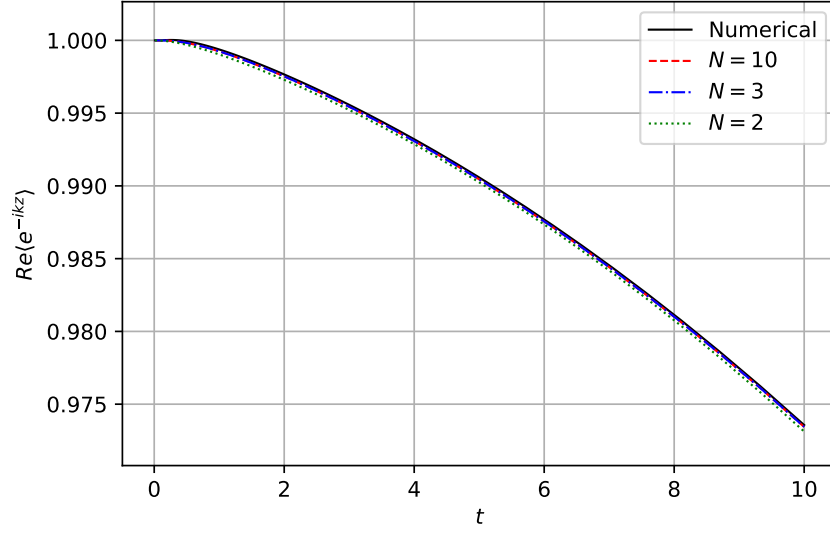


Figure 5.12: Comparison of the μ -averaged characteristic function between the subspace method for various dimensions and the numerical solution. We have used an initial value of $\mu_0 = 0$ and set $\xi = 0.5$. Furthermore, we kept the dimensionless wavenumber $k \rightarrow vk/D = 0.1$ constant and varied time.

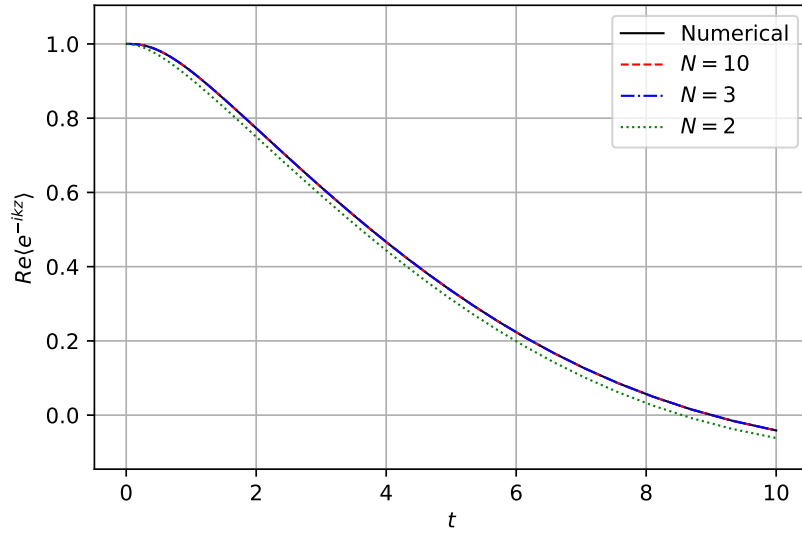


Figure 5.13: Caption is as in Figure 5.12 but we have set $k = 1$.

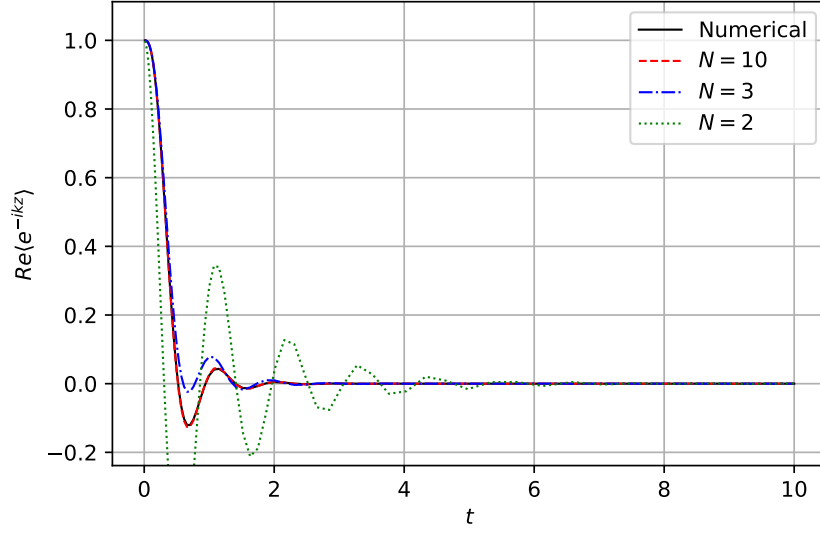


Figure 5.14: Caption is as in Figure 5.12 but we have set $k = 10$.

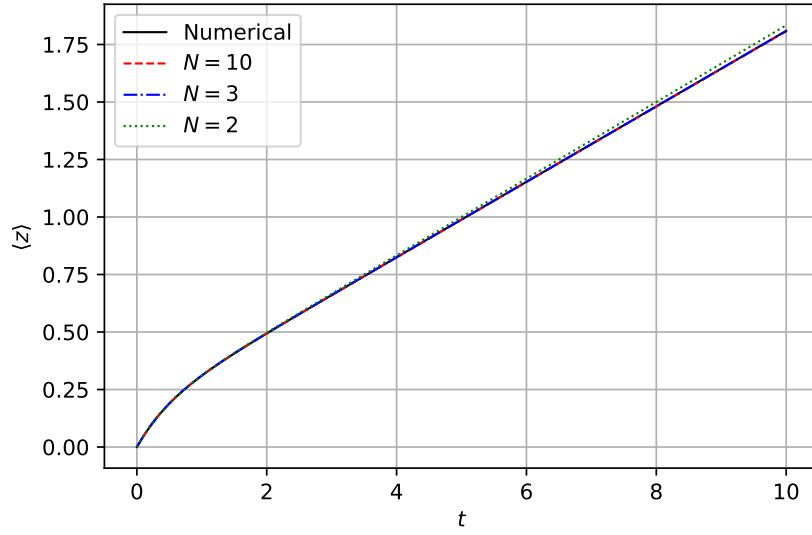


Figure 5.15: Comparison of the moment $\langle z \rangle$ as a function of time for the N -dimensional subspace method and the numerical solution. Here, we have used $\mu_0 = 0.5$ and $\xi = 0.5$.

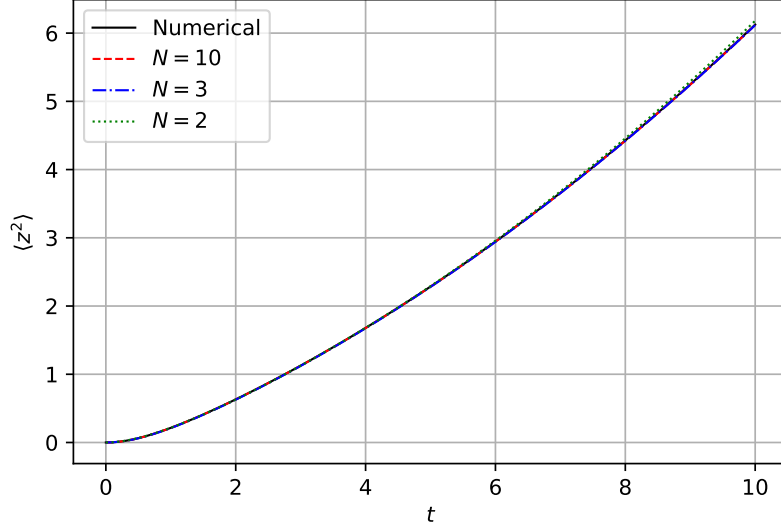


Figure 5.16: Comparison of the moment $\langle z^2 \rangle$ as a function of time for the N -dimensional subspace method and the numerical solution. Here, we have used $\mu_0 = 0.5$ and $\xi = 0.5$.

5.4.3 The Parameter $\xi = 2.5$

Finally, we again repeat the previous two sections, except this time we set $\xi = 2.5$. Moreover, the 2-dimensional subspace approximation is not included in the following figures. This is because the limitations of the N -dimensional subspace approximation within this form are identical to those of the standard form. See Section 4.4.5 for more details.

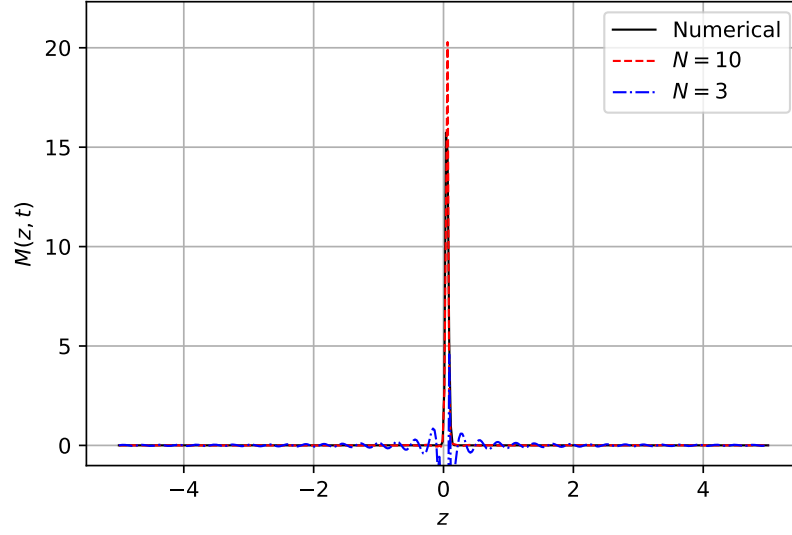


Figure 5.17: The graph compares the μ -integrated solution of the subspace method for various dimensions and the numerical solution at the time $t = 0.1$ for an initial value of $\mu_0 = 0.5$. We have set $\xi = 2.5$.

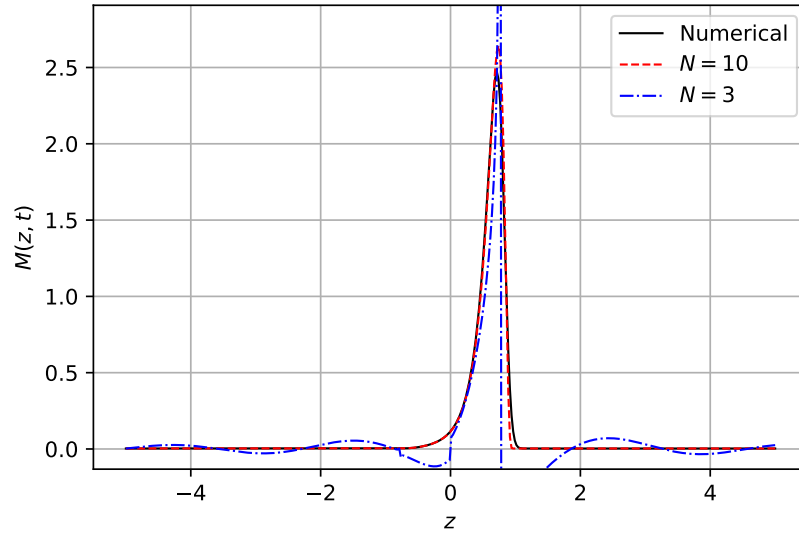


Figure 5.18: Caption is as in Figure 5.17 but we have used $t = 1$.

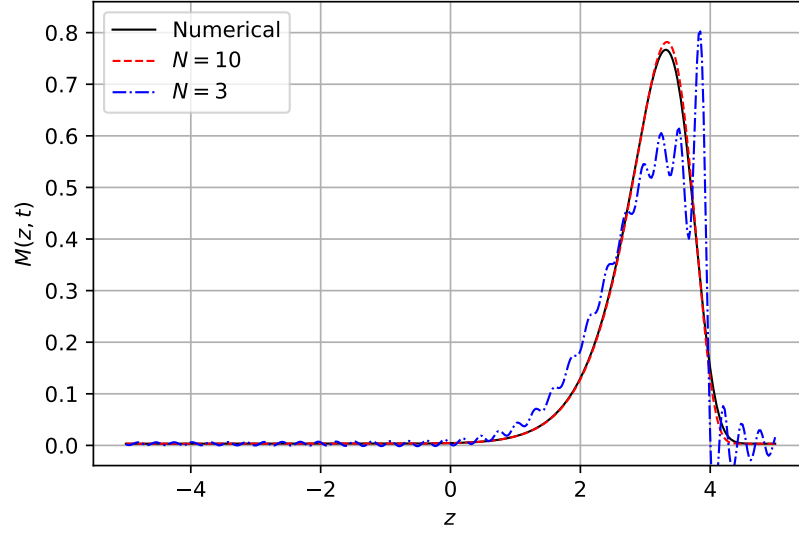


Figure 5.19: Caption is as in Figure 5.17 but we have used $t = 5$.

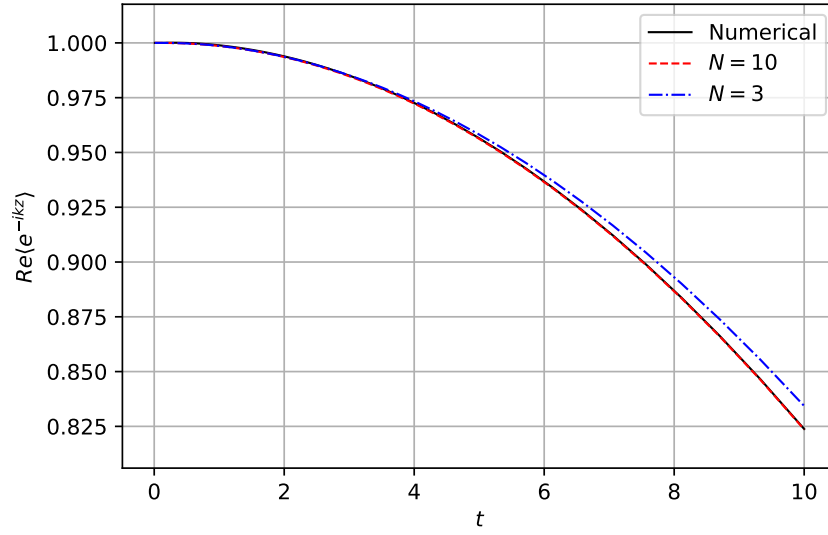


Figure 5.20: Comparison of the μ -averaged characteristic function between the subspace method for various dimensions and the numerical solution. We have used an initial pitch-angle cosine of $\mu_0 = 0$ and for the focusing parameter we used $\xi = 2.5$. We have kept the dimensionless wavenumber constant at $k \rightarrow vk/D = 0.1$ and varied time.

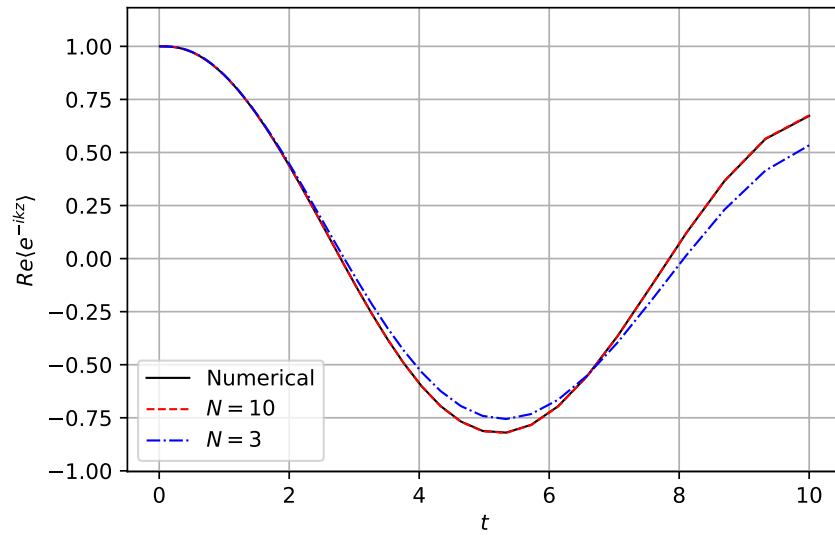


Figure 5.21: Caption is as in Figure 5.20 but we have used $k = 1$.

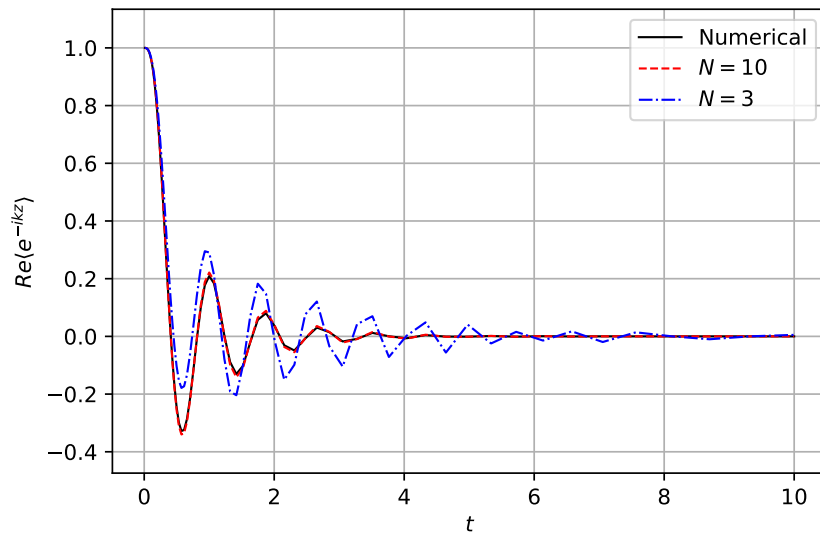


Figure 5.22: Caption is as in Figure 5.20 but we have used $k = 10$.

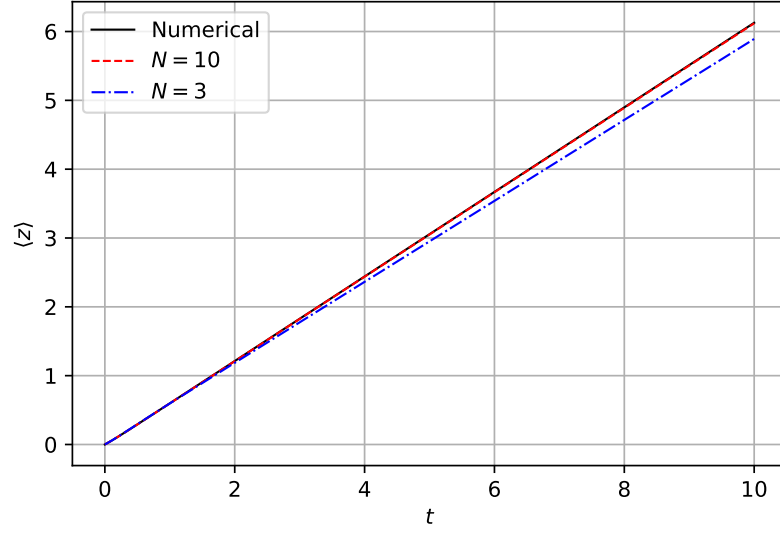


Figure 5.23: Comparison of the moment $\langle z \rangle$ as a function of time t for the N -dimensional subspace method and the numerical solution. Here, we have used $\mu_0 = 0.5$ and $\xi = 2.5$.

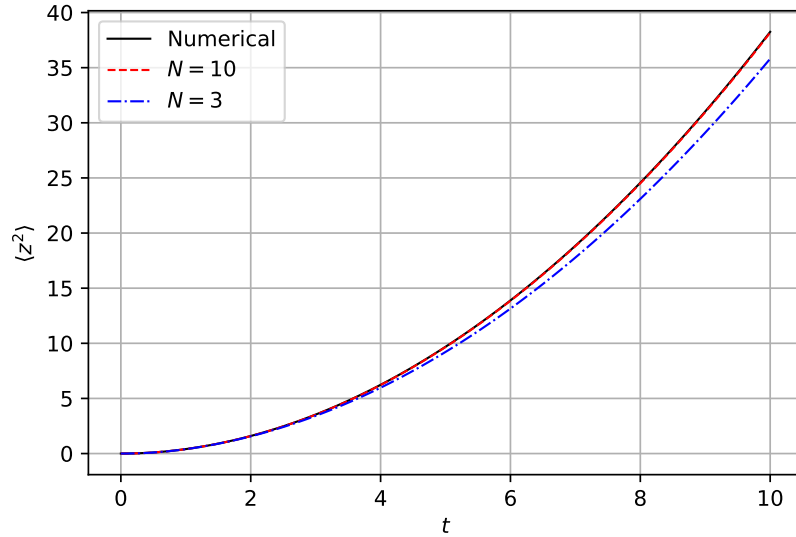


Figure 5.24: Comparison of the moment $\langle z^2 \rangle$ as a function of time for the N -dimensional subspace method and the numerical solution. Here, we have used $\mu_0 = 0.5$ and $\xi = 2.5$.

An interesting observation that can be seen in the figures throughout this section is that the 3-dimensional subspace approximation is decently accurate when $\xi = 2.5$. On the other hand, when dealing with the standard form of the Fokker-Planck equation and $\xi = 2.5$, Section 4.4.3 shows that

the 3-dimensional subspace approximation is far from accurate. This suggests that when applied to the conservative form of the Fokker-Planck equation, the subspace method demonstrates improved performance compared to its application to the standard form. However, this comparison is not delved into rigorously here.

5.4.4 Time Comparisons

The final comparison we provide between the subspace method and the numerical method is the computational complexity of the methods. To perform this comparison, we solve the Fokker-Planck equation in Fourier space for a required set of wavenumbers which ensure an accurate solution. When calculating the computational complexity, we only consider the time needed to compute the solution in Fourier space and neglect the time required to numerically solve the Fourier transform integral. The following Tables 5.1-5.3 demonstrate the computational complexity of the numerical method compared to the subspace method for various dimensions. For completeness, we reproduce these results for the values $\xi = 0.1$, $\xi = 0.5$, and $\xi = 2.5$.

The following tables show several facts. First, as we expected, as the dimension of the subspace method is decreased, the subspace method becomes faster. Additionally, for small values of Dt , the numerical method is actually quicker than the 10-dimensional subspace method. This is due to the fact that for small values of Dt , very few time steps are required to ensure an accurate numerical solution. On the other hand, when Dt increases, significantly more time steps are needed, which can cause the numerical method to take a considerable amount of time relative to the subspace method.

Dt	2D	3D	10D	Numerical
0.1	1127.9	1777.1	6807.8	6117.3
1	34.9	55.2	215.6	1812.0
5	3.1	4.8	18.5	778.7

Table 5.1: For different values of Dt , we compare the time in seconds to compute the solution using the subspace approximation for various dimensions with the numerical solution. Here, we have $\xi = 0.1$.

Dt	2D	3D	10D	Numerical
0.1	1103.4	1758.5	6771.8	6077.8
1	34.8	54.9	214.1	1800.4
5	3.0	4.7	18.4	773.7

Table 5.2: Caption is as in Table 5.1, except that here we have used $\xi = 0.5$.

Dt	3D	10D	Numerical
0.1	1730.2	6759.3	6066.9
1	56.3	216.3	1810.3
5	4.9	19.2	778.0

Table 5.3: Caption is as in Table 5.1, except that here we have used $\xi = 2.5$.

5.4.5 Limitations

In Section 4.4.5, we discuss how the N -dimensional subspace approximation can experience convergence issues when solving the standard form of the Fokker-Planck equation. As it turns out, the exact same issues are present when dealing with the conservative form. In particular, if we reproduce Figure 4.25 when solving the conservative form, we obtain an identical figure. It should be noted that in that section, we also mention the work of [Ma Sung and Earl, 1978] which revealed that ξ never exceeded a value of one throughout 30 solar events.

5.5 Conclusion

This chapter was based on the conservative form of the Fokker-Planck equation with isotropic scattering. The subspace method has been employed in the past such as in [Lasuik and Shalchi, 2019] and [Shalchi, 2024] when focusing is neglected. Moreover, the results of this chapter are very similar to those of Chapter 4 when dealing with the standard form of the focused Fokker-Planck equation. The following two paragraphs outline the main results of this chapter.

The first result of this chapter concerned an analytical approximation of the solution along with certain expectation values. As the complexity of the solution grows significantly with an increase of the dimension, the desire for analytical results constrains us to use at most a 3-dimensional approximation. We began with the 2-dimensional subspace approximation. While the resulting expressions were very simple, they were often inaccurate and sometimes even independent of the strength of the focusing effect. These problems motivated the use of the 3-dimensional subspace approximation, which was able to avoid the aforementioned problems. However, this extra dimension brought forth the new problem of the expressions being very lengthy, even if limiting cases such as small wavenumbers were considered.

We also sought a higher dimensional subspace approximation which would be very accurate while still being significantly quicker than a purely numerical solution. Similar to Chapter 4 and [Shalchi, 2024], we found that the 10-dimensional approximation provided an excellent balance of speed and accuracy. One downside with this method is that one must proceed with caution, as there is a chance convergence issues arise, specifically when dealing with large values of $\xi = v/(2DL)$. However, as found in [Ma Sung and Earl, 1978], this is very unlikely to ever cause an issue in practical applications.

6

The Parallel Diffusion Coefficient of Energetic Particles in a Spatially Varying Magnetic Field

6.1 Introduction

Thanks to the Taylor-Green-Kubo formula [Taylor, 1922, Green, 1951, Kubo, 1957] along with a result given in [Earl, 1974], one of the most well-known results in cosmic ray transport is that the parallel diffusion coefficient in late-times is equivalently given by

$$\kappa_{\parallel} = \frac{1}{2} \lim_{t \rightarrow \infty} \frac{d}{dt} [\langle z^2 \rangle - \langle z \rangle^2] = v^2 \int_0^{\infty} dt \langle \mu(t) \mu(0) \rangle = \frac{v^2}{8} \int_{-1}^1 d\mu \frac{(1 - \mu^2)^2}{D_{\mu\mu}(\mu)} \quad (6.1)$$

provided the mean magnetic field is constant. It is then natural to wonder if these identities hold when the focusing effect is present. As discussed in Section 2.2.5, the identity from [Earl, 1974] depends on the assumption of purely diffusive motion at late times. Moreover, the Taylor-Green-Kubo formula requires the scattering to be stationary. It turns out that neither of these assumptions are valid when focusing is present.

Originally, it was assumed that Eq. (6.1) did in fact hold if focusing was present. However, the

publication of contradicting results by [Beeck and Wibberenz, 1986] and [Shalchi, 2011], along with heavy criticism by [Litvinenko, 2012], motivated [Danos et al., 2013] to investigate this matter further. They came to the conclusion that Eq. (6.1) is generally not true.

The goal of this chapter is to derive an analytical expression for the parallel diffusion coefficient using the first equality of Eq. (6.1). It is shown in [Danos et al., 2013] that the result is identical regardless of whether we use the standard form of the Fokker-Planck equation or the conservative form. For convenience, we elect to work with the conservative form with isotropic scattering. That is,

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = D \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial f}{\partial \mu} \right] - \frac{v}{2L} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) f \right]. \quad (6.2)$$

It should be noted that [Wang and Qin, 2020] previously obtained an analytical result correct up to second order in the parameter $\xi = v/(2DL)$, whereas in this work, we will derive the exact result for any value of ξ .

This chapter is based on [Shalchi and Klippenstein, 2025].

6.2 Problem Reformulation and Analytical Setup

We seek to compute the parallel diffusion coefficient via

$$\kappa_{\parallel}^{msd} = \frac{1}{2} \lim_{t \rightarrow \infty} \frac{d}{dt} \left[\langle z^2 \rangle - \langle z \rangle^2 \right]. \quad (6.3)$$

The process of computing this coefficient defined in terms of the mean square displacement is very complicated. Consequently, we first perform several preliminary computations which will later simplify the calculation of the moments.

6.2.1 Transforming the Fokker-Planck Equation

The first step is to invoke the transformation

$$z' = z - \langle z \rangle. \quad (6.4)$$

As we now show, this allows us to effectively take $\langle z \rangle = 0$ in Eq. (6.3). To see this, we first can easily notice that

$$\langle z' \rangle = \langle z - \langle z \rangle \rangle = 0. \quad (6.5)$$

Next,

$$\begin{aligned} \langle z'^2 \rangle - \langle z' \rangle^2 &= \langle z^2 - 2 \langle z \rangle z + \langle z \rangle^2 \rangle \\ &= \langle z^2 \rangle - 2 \langle z \rangle^2 + \langle z \rangle^2 \\ &= \langle z^2 \rangle - \langle z \rangle^2. \end{aligned} \quad (6.6)$$

The calculations in Eqs. (6.5) and (6.6) make use of the fact that the operator $\langle \cdot \rangle$ is easily seen to be linear, and the fact that the solution is normalized. It then follows from Eq. (6.6) that we can alternatively deal with $f(z', \mu, t)$. It is important to note that in this case, Eq. (6.2) is altered. In particular, we have the variable transformation $(z, \mu, t) \rightarrow (z', \mu, t)$, which implies

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{\partial z'}{\partial z} \frac{\partial}{\partial z'} = \frac{\partial}{\partial z'} \\ \frac{\partial}{\partial \mu} &= \frac{\partial \mu}{\partial \mu} \frac{\partial}{\partial \mu} = \frac{\partial}{\partial \mu} \\ \frac{\partial}{\partial t} &= \frac{\partial z'}{\partial t} \frac{\partial}{\partial z'} + \frac{\partial t}{\partial t} \frac{\partial}{\partial t} = -\frac{d\langle z \rangle}{dt} \frac{\partial}{\partial z'} + \frac{\partial}{\partial t}. \end{aligned} \quad (6.7)$$

Recall the connection formula between $d\langle z \rangle / dt$ and $v\langle \mu \rangle$ given in Eq. (2.55). Placing Eq. (2.55) and the above relations into Eq. (6.2) and renaming $z' \rightarrow z$ for simplicity results in

$$\frac{\partial f}{\partial t} + v(\mu - \langle \mu \rangle) \frac{\partial f}{\partial z} = D \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial f}{\partial \mu} \right] - \frac{v}{2L} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) f \right]. \quad (6.8)$$

This new form of the Fokker-Planck equation is dependent on the moment $\langle \mu \rangle$. While there is no analytical form in general, the late-time value is given in [Shalchi and Danos, 2013]:

$$\lim_{t \rightarrow \infty} \langle \mu \rangle = \coth \xi - \frac{1}{\xi} \quad (6.9)$$

where we have used the focusing-to-scattering ratio ξ . As mentioned above, the benefit of this transformation is that we now have

$$\kappa_{\parallel}^{msd} = \frac{1}{2} \lim_{t \rightarrow \infty} \frac{d}{dt} \langle z^2 \rangle. \quad (6.10)$$

In the following, we derive an analytical expression for the second moment in the late-time limit. We shall see that even after the above transformation, the required computations are still complicated.

6.2.2 Taylor-Expanding the Solution F_k

In Fourier space, Eq. (6.10) is given as

$$\frac{\partial F_k}{\partial t} + i v k (\mu - \langle \mu \rangle) F_k = D \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial F_k}{\partial \mu} \right] - \frac{v}{2L} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) F_k \right]. \quad (6.11)$$

Recall that Eq. (2.51) states

$$\langle z^n \rangle = i^n \pi \left[\frac{\partial^n}{\partial k^n} \int_{-1}^1 d\mu F_k(\mu, t) \right]_{k=0}, \quad (6.12)$$

and since we seek an analytical form of the second moment, it is sufficient to consider the Taylor expansion

$$F_k(\mu, t) = \alpha(\mu, t) + \beta(\mu, t)k + \gamma(\mu, t)k^2 \quad (6.13)$$

where we have the currently unknown functions α , β and γ . It is easy to see that these unknown functions are related to the first three moments according to

$$1 = \pi \int_{-1}^1 d\mu \alpha(\mu, t) \quad (6.14)$$

$$\langle z \rangle = i\pi \int_{-1}^1 d\mu \beta(\mu, t) = 0 \quad (6.15)$$

$$\langle z^2 \rangle = -2\pi \int_{-1}^1 d\mu \gamma(\mu, t). \quad (6.16)$$

Differential equations for each of the unknown functions can be derived by placing the expansion given by Eq. (6.13) into Eq. (6.11). By considering the resulting equations up to zeroth, first, and second order in k , we derive

$$\frac{\partial \alpha}{\partial t} = D \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \alpha}{\partial \mu} \right] - \frac{v}{2L} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \alpha \right], \quad (6.17)$$

$$\frac{\partial \beta}{\partial t} + iv(\mu - \langle \mu \rangle) \alpha = D \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \beta}{\partial \mu} \right] - \frac{v}{2L} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \beta \right], \quad (6.18)$$

$$\frac{\partial \gamma}{\partial t} + iv(\mu - \langle \mu \rangle) \beta = D \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \gamma}{\partial \mu} \right] - \frac{v}{2L} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \gamma \right]. \quad (6.19)$$

Now we take the time derivative of Eq. (6.16) to obtain

$$\frac{d}{dt} \langle z^2 \rangle = -2\pi \int_{-1}^1 d\mu \frac{\partial \gamma}{\partial t}. \quad (6.20)$$

The integrand is the first term on the left-hand side of Eq. (6.19). We thus have

$$\frac{d}{dt} \langle z^2 \rangle = i2\pi v \int_{-1}^1 d\mu (\mu - \langle \mu \rangle) \beta \quad (6.21)$$

where both terms on the right-hand side of Eq. (6.19) can easily be seen to be zero when integrated over all μ .

6.3 Analytical Solution of the Functions α and β

In the previous section, it was established that determining the parallel diffusion coefficient is a matter of solving Eqs. (6.17) and (6.18) for β . In this section, we solve these new differential equations. Since the parallel diffusion coefficient is defined in terms of a late-time limit, we begin with simplifying Eqs. (6.17) and (6.18) in the limit $t \rightarrow \infty$.

6.3.1 The Late-Time Limit

By Eq. (6.13), we see that $\alpha = F_0$, which corresponds to the spatially-integrated solution of the Fokker-Planck equation. In [Shalchi and Danos, 2013] it is shown that the late-time solution is independent of time and given by

$$\alpha(\mu, t \rightarrow \infty) = C e^{\xi\mu} \quad (6.22)$$

for some constant $C = C(\xi)$. The constant C can be determined via the normalization constraint given in Eq. (6.14). The integral is straightforward to evaluate and so we obtain

$$\alpha(\mu, t \rightarrow \infty) = \frac{\xi}{2\pi \sinh \xi} e^{\xi\mu}. \quad (6.23)$$

The next quantity to consider is $\beta(\mu, t \rightarrow \infty)$. We claim that this is also independent of time. Although we cannot prove this rigorously, Eq. (6.15) suggests that this should hold. Additionally, we illustrate that our final result is indeed correct by comparing with the numerical solution near the end of the chapter. By placing this assumption along with Eq. (6.23) into Eq. (6.18), we have

$$\frac{iv\xi}{2\pi \sinh \xi} (\mu - \langle \mu \rangle) e^{\xi\mu} = D \frac{d}{d\mu} \left[(1 - \mu^2) \frac{d\beta}{d\mu} \right] - \frac{v}{2L} \frac{d}{d\mu} [(1 - \mu^2)\beta]. \quad (6.24)$$

By integrating this equation over $[-1, \mu]$, we easily derive

$$\frac{iv\xi}{2\pi \sinh \xi} \int_{-1}^{\mu} d\mu' (\mu' - \langle \mu \rangle) e^{\xi\mu'} = D(1 - \mu^2) \frac{d\beta}{d\mu} - \frac{v}{2L} (1 - \mu^2) \beta. \quad (6.25)$$

The integral is straightforward to evaluate using integration by parts:

$$\begin{aligned}
\int_{-1}^{\mu} d\mu' (\mu' - \langle \mu \rangle) e^{\xi \mu'} &= \left[\frac{e^{\xi \mu'}}{\xi} \left(\mu' - \coth \xi + \frac{1}{\xi} \right) \right]_{-1}^{\mu} - \frac{1}{\xi} \int_{-1}^{\xi} d\mu' e^{\xi \mu'} \\
&= (\mu - \coth \xi) \frac{e^{\xi \mu}}{\xi} + (1 + \coth \xi) \frac{e^{-\xi}}{\xi}
\end{aligned} \tag{6.26}$$

where we used the expression for $\langle \mu \rangle$ in the late-time limit given in Eq. (6.9). Placing this result into Eq. (6.25) yields

$$\frac{d\beta}{d\mu} - \xi \beta = \frac{iv}{2\pi D \sinh \xi (1 - \mu^2)} \left[(\mu - \coth \xi) e^{\xi \mu} + (1 + \coth \xi) e^{-\xi} \right] \tag{6.27}$$

where we also performed simple rearrangements.

6.3.2 General Form of the Function $\beta(\mu)$

The next step towards obtaining the parallel diffusion coefficient is to solve Eq. (6.27). As this is a first-order linear differential equation, it can be solved using the *method of integrating factors*. We begin with multiplying both sides by $e^{-\xi \mu}$ and observing that

$$\frac{d}{d\mu} [\beta e^{-\xi \mu}] = e^{-\xi \mu} \frac{d\beta}{d\mu} - \xi e^{-\xi \mu} \beta, \tag{6.28}$$

allowing us to rewrite Eq. (6.27) as

$$\frac{d}{d\mu} [\beta e^{-\xi \mu}] = \frac{iv}{2\pi D \sinh \xi (1 - \mu^2)} \left[(\mu - \coth \xi) + (1 + \coth \xi) e^{-\xi(1+\mu)} \right]. \tag{6.29}$$

The right-hand side can be rewritten if we define:

$$d_1 = \frac{iv}{2\pi D \sinh \xi} \quad (6.30)$$

$$d_2 = -\frac{iv \coth \xi}{2\pi D \sinh \xi} \quad (6.31)$$

$$d_3 = \frac{iv(1 + \coth \xi)e^{-\xi}}{2\pi D \sinh \xi}. \quad (6.32)$$

These constants allow us to write

$$\frac{d}{d\mu} [\beta e^{-\xi\mu}] = d_1 \frac{\mu}{1 - \mu^2} + \frac{d_2}{1 - \mu^2} + d_3 \frac{e^{-\xi\mu}}{1 - \mu^2}. \quad (6.33)$$

We then see that solving for β is now a matter of integrating both sides of this equation. To proceed, we consider each term on the right-hand side individually. For the first term, we can see that

$$\int d\mu \frac{\mu}{1 - \mu^2} = -\frac{1}{2} \ln(1 - \mu^2) + C_1 \quad (6.34)$$

where for this integral, and the following integrals, C_i denotes a constant of integration. The second term is also simple when considering the partial fraction decomposition:

$$\begin{aligned} \int d\mu \frac{1}{1 - \mu^2} &= \frac{1}{2} \int d\mu \left[\frac{1}{1 - \mu} + \frac{1}{1 + \mu} \right] \\ &= \frac{1}{2} [-\ln(1 - \mu) + \ln(1 + \mu)] + C_2. \end{aligned} \quad (6.35)$$

As for the third term of Eq. (6.33), the first step is again to consider the partial fraction decomposition to obtain

$$\int d\mu \frac{e^{-\xi\mu}}{1 - \mu^2} = \frac{1}{2} \int d\mu \frac{e^{-\xi\mu}}{1 - \mu} + \frac{1}{2} \int d\mu \frac{e^{-\xi\mu}}{1 + \mu}. \quad (6.36)$$

The transformations $x = \xi(1 - \mu)$ and $z = -\xi(1 + \mu)$ can be employed on the first and second integral, respectively. This yields

$$\int d\mu \frac{e^{-\xi\mu}}{1 - \mu^2} = -\frac{1}{2}e^{-\xi} \int dx \frac{e^x}{x} + \frac{1}{2}e^{\xi} \int dz \frac{e^z}{z}. \quad (6.37)$$

Both integrals are instances of the *exponential integral function* ei . The fundamental property of this function is, see [Abramowitz and Stegun, 1964],

$$\frac{d}{dx} \text{ei}(x) = \frac{e^x}{x}. \quad (6.38)$$

We therefore have

$$\int d\mu \frac{e^{-\xi(1+\mu)}}{1 - \mu^2} = -\frac{1}{2}e^{-\xi} \text{ei}(\xi(1 - \mu)) + \frac{1}{2}e^{\xi} \text{ei}(-\xi(1 + \mu)) + C_3. \quad (6.39)$$

Placing Eqs. (6.34), (6.35) and (6.39) into Eq. (6.33) yields

$$\beta(\mu) = \frac{1}{2}e^{\xi\mu} \left[(d_2 - d_1) \ln(1 + \mu) - (d_2 + d_1) \ln(1 - \mu) + d_3 e^{\xi} \text{ei}(-\xi(1 + \mu)) - d_3 e^{-\xi} \text{ei}(\xi(1 - \mu)) + 2\tilde{C} \right] \quad (6.40)$$

where $\tilde{C}$ is a new integration constant. Upon determining the value of this constant with Eq. (6.15), we will have the complete expression for β .

6.3.3 Determining the Constant $\tilde{C}$

To find the value for the constant $\tilde{C}$, we combine Eqs. (6.15) and (6.40) and decompose the integral according to

$$0 = \int_{-1}^1 d\mu \beta = \frac{1}{2}I_1 + \frac{1}{2}I_2 + \frac{1}{\xi} (e^{\xi} - e^{-\xi}) \tilde{C} \quad (6.41)$$

for

$$\begin{aligned} I_1 &= \int_{-1}^1 d\mu \left[(d_2 - d_1) e^{\xi\mu} \ln(1 + \mu) - (d_2 + d_1) e^{\xi\mu} \ln(1 - \mu) \right], \\ I_2 &= d_3 \int_{-1}^1 d\mu \left[e^{\xi(1+\mu)} \text{ei}(-\xi(1 + \mu)) - e^{\xi(\mu-1)} \text{ei}(-\xi(\mu - 1)) \right]. \end{aligned} \quad (6.42)$$

Rewriting I_1 results in

$$I_1 = \frac{ive^\xi}{\pi D(e^{2\xi} - 1) \sinh \xi} \int_{-1}^1 d\mu \left[-e^{\xi(1+\mu)} \ln(1 + \mu) + e^{\xi(\mu-1)} \ln(1 - \mu) \right]. \quad (6.43)$$

By using the transformation $\mu \rightarrow -\mu$ on the second term, we obtain

$$I_1 = -\frac{i2ve^\xi}{\pi D(e^{2\xi} - 1) \sinh \xi} \int_{-1}^1 d\mu \ln(1 + \mu) \sinh(\xi(1 + \mu)). \quad (6.44)$$

Next, integration by parts leads to

$$I_1 = -\frac{i2ve^\xi}{\pi D\xi(e^{2\xi} - 1) \sinh \xi} \left[\ln(1 + \mu) \cosh(\xi(1 + \mu)) \Big|_{-1}^1 - \int_{-1}^1 d\mu \frac{\cosh(\xi(1 + \mu))}{1 + \mu} \right]. \quad (6.45)$$

The remaining integral is given by the *hyperbolic cosine integral function*:

$$\text{chi}(z) = \gamma + \ln z + \int_0^z dt \frac{\cosh t - 1}{t} \quad (6.46)$$

where γ is known as the *Euler-Mascheroni constant*, and is given by

$$\gamma = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \frac{1}{k} - \ln n \right] \approx 0.5772. \quad (6.47)$$

It is easy to see that

$$\frac{d}{dx} \text{chi}(x) = \frac{\cosh x}{x}, \quad (6.48)$$

and so consequently, we have

$$I_1 = -\frac{i2ve^\xi}{\pi D\xi(e^{2\xi} - 1) \sinh \xi} \left[\ln 2 \cosh(2\xi) - \text{chi}(2\xi) + \lim_{\epsilon \rightarrow 0^+} (\text{chi}(\xi\epsilon) - \ln \epsilon) \right] \quad (6.49)$$

where the limit is required as the relevant functions are not defined at zero. To evaluate this limit, we notice that as $x \rightarrow 0$, the integral in Eq. (6.46) is zero, and so

$$\lim_{\epsilon \rightarrow 0^+} (\text{chi}(\xi\epsilon) - \ln \epsilon) = \lim_{\epsilon \rightarrow 0^+} (\gamma + \ln(\xi\epsilon) - \ln \epsilon) = \gamma + \ln \xi. \quad (6.50)$$

The complete result for I_1 is then

$$I_1 = -\frac{i2ve^\xi}{\pi D\xi(e^{2\xi} - 1) \sinh \xi} [\ln 2 \cosh(2\xi) - \text{chi}(2\xi) + \gamma + \ln \xi]. \quad (6.51)$$

As for I_2 , we begin with considering the first term given by

$$I = \int_{-1}^1 d\mu e^{\xi(1+\mu)} \text{ei}(-\xi(1+\mu)). \quad (6.52)$$

The transformation $z = -\xi(1+\mu)$ renders

$$I = -\frac{1}{\xi} \lim_{\epsilon \rightarrow 0^+} \int_{-\xi\epsilon}^{-2\xi} dz e^{-z} \text{ei}(z) \quad (6.53)$$

where we use the small positive number ϵ as again, some of the functions are undefined at zero. Next, with integration by parts, we have

$$\begin{aligned} I &= -\frac{1}{\xi} \lim_{\epsilon \rightarrow 0^+} \left[-e^{-z} \text{ei}(z) \Big|_{-\xi\epsilon}^{-2\xi} + \int_{-\xi\epsilon}^{-2\xi} dz \frac{1}{z} \right] \\ &= \frac{1}{\xi} \lim_{\epsilon \rightarrow 0^+} \left[e^{2\xi} \text{ei}(-2\xi) - e^{\xi\epsilon} \text{ei}(-\xi\epsilon) - \ln(2\xi) + \ln(\xi\epsilon) \right]. \end{aligned} \quad (6.54)$$

It remains to take the limit $\epsilon \rightarrow 0$. With the series definition of the exponential integral function, given

in [Masina, 2019], we have

$$\lim_{\epsilon \rightarrow 0^+} (\ln(\xi\epsilon) - e^{\xi\epsilon} \text{ei}(-\xi\epsilon)) = \lim_{\epsilon \rightarrow 0^+} (\ln(\xi\epsilon) - \gamma - \ln(\xi\epsilon)) = -\gamma. \quad (6.55)$$

It follows that

$$\int_{-1}^1 d\mu e^{\xi(1+\mu)} \text{ei}(-\xi(1+\mu)) = \frac{1}{\xi} [e^{2\xi} \text{ei}(-2\xi) - \ln(2\xi) - \gamma]. \quad (6.56)$$

By an identical argument, we can evaluate the integral of the second term in I_2 to obtain

$$\int_{-1}^1 d\mu e^{\xi(\mu-1)} \text{ei}(-\xi(\mu-1)) = \frac{1}{\xi} [-e^{-2\xi} \text{ei}(2\xi) + \ln(2\xi) + \gamma]. \quad (6.57)$$

We therefore have

$$I_2 = \frac{d_3}{\xi} [e^{2\xi} \text{ei}(-2\xi) + e^{-2\xi} \text{ei}(2\xi) - \ln(4\xi^2) - 2\gamma]. \quad (6.58)$$

By combining Eqs. (6.41), (6.51), and (6.58), we derive the following expression

$$\tilde{C} = \frac{iv}{4\pi D \sinh^3 \xi} \left[\ln 2 \cosh(2\xi) - \text{chi}(2\xi) - \frac{1}{2} e^{2\xi} \text{ei}(-2\xi) - \frac{1}{2} e^{-2\xi} \text{ei}(2\xi) + \ln(2\xi^2) + 2\gamma \right]. \quad (6.59)$$

Upon placing this result into Eq. (6.40), we have derived the exact analytical expression for β .

6.4 Parallel Diffusion Coefficient

Given the previous calculations, we are finally able to directly calculate the exact form of the parallel diffusion coefficient. After finding the final result, we simplify the expression by considering the extreme cases of the strength of focusing. Additionally, we compare this new result with the numerical solution along with other forms of this coefficient.

6.4.1 Deriving the Exact Form of the Parallel Diffusion Coefficient

With the function β given by Eqs. (6.40) and (6.59), we can now compute the parallel diffusion coefficient in late-times with the use of Eq. (6.21). It follows from Eqs. (6.10), (6.15) and (6.21) that

$$\kappa_{\parallel}^{msd} = i\pi v \int_{-1}^1 d\mu \mu \beta. \quad (6.60)$$

We evaluate this integral using the same method that was applied to determine the constant $\tilde{C}$. Hence, we first write

$$\kappa_{\parallel}^{msd} = \frac{i\pi v}{2} \left[\tilde{I}_1 + \tilde{I}_2 + \frac{4\tilde{C}}{\xi^2} (\xi \cosh \xi - \sinh \xi) \right] \quad (6.61)$$

for

$$\begin{aligned} \tilde{I}_1 &= -\frac{i2ve^{\xi}}{\pi D(e^{2\xi} - 1) \sinh \xi} \int_{-1}^1 d\mu \mu \ln(1 + \mu) \cosh(\xi(1 + \mu)), \\ \tilde{I}_2 &= d_3 \int_{-1}^1 d\mu \mu \left[e^{\xi(1+\mu)} \text{ei}(-\xi(1 + \mu)) - e^{\xi(\mu-1)} \text{ei}(-\xi(\mu - 1)) \right]. \end{aligned} \quad (6.62)$$

With integration by parts, the first integral can be written as

$$\begin{aligned} \tilde{I}_1 &= -\frac{i2ve^{\xi}}{\pi D(e^{2\xi} - 1) \sinh \xi} \left[\frac{1}{\xi} \mu \ln(1 + \mu) \sinh(\xi(1 + \mu)) - \frac{1}{\xi^2} \ln(1 + \mu) \cosh(\xi(1 + \mu)) \right]_{-1}^1 \\ &\quad + \frac{i2ve^{\xi}}{\pi D(e^{2\xi} - 1) \sinh \xi} \int_{-1}^1 d\mu \left(\frac{1}{\xi} \sinh(\xi(1 + \mu)) - \frac{1}{\xi} \frac{\sinh(\xi(1 + \mu))}{1 + \mu} - \frac{1}{\xi^2} \frac{\cosh(\xi(1 + \mu))}{1 + \mu} \right). \end{aligned} \quad (6.63)$$

The remaining integrals are straightforward to evaluate as the first is $\cosh(\xi(1 + \mu))/\xi^2$, the second is given in terms of the *hyperbolic sine integral function*

$$\text{shi}(z) = \int_0^z dt \frac{\sinh t}{t}, \quad (6.64)$$

and the third is given in terms of the hyperbolic cosine integral function. We therefore have

$$\begin{aligned}\tilde{I}_1 = & \frac{i2ve^\xi}{\pi D(e^{2\xi} - 1) \sinh \xi} \left[\frac{1}{\xi^2} \cosh(2\xi) - \frac{1}{\xi} \text{shi}(2\xi) - \frac{1}{\xi^2} \text{chi}(2\xi) - \frac{1}{\xi} \ln 2 \sinh(2\xi) + \frac{1}{\xi^2} \ln 2 \cosh(2\xi) \right] \\ & - \frac{i2ve^\xi}{\pi D(e^{2\xi} - 1) \sinh \xi} \lim_{\epsilon \rightarrow 0^+} \left[\frac{1}{\xi^2} - \frac{1}{\xi} \text{shi}(\xi\epsilon) - \frac{1}{\xi^2} \text{chi}(\xi\epsilon) + \frac{1}{\xi} \ln \epsilon \sinh(\xi\epsilon) + \frac{1}{\xi^2} \ln \epsilon \right].\end{aligned}\quad (6.65)$$

The limit $\epsilon \rightarrow 0^+$ can be evaluated in three parts. First, from Eq. (6.46), we have

$$\lim_{\epsilon \rightarrow 0^+} [-\text{chi}(\xi\epsilon) + \ln \epsilon] = \lim_{\epsilon \rightarrow 0^+} [-\gamma - \ln(\xi\epsilon) + \ln \epsilon] = -\gamma - \ln \xi. \quad (6.66)$$

Additionally, it is easy to see that the portion of the limit involving $\ln \epsilon \sinh(\xi\epsilon)$ exhibits an indeterminate form. We thus proceed with l'Hopital's rule to derive

$$\begin{aligned}\lim_{\epsilon \rightarrow 0^+} \ln \epsilon \sinh(\xi\epsilon) &= \lim_{\epsilon \rightarrow 0^+} \frac{\ln \epsilon}{\frac{1}{\sinh(\xi\epsilon)}} \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{\sinh(\xi\epsilon)^2}{\cosh(\xi\epsilon)} \\ &= 0.\end{aligned}\quad (6.67)$$

Moreover, we can see from the definition given in Eq. (6.64) that $\text{shi}(0) = 0$. Placing it all together yields

$$\tilde{I}_1 = \frac{iv}{\pi D\xi \sinh^2 \xi} \left[\frac{1}{\xi} \cosh(2\xi) - \text{shi}(2\xi) - \frac{1}{\xi} \text{chi}(2\xi) - \ln 2 \sinh(2\xi) + \frac{1}{\xi} \ln 2 \cosh(2\xi) + \frac{1}{\xi} \ln \xi + \frac{1}{\xi} \gamma - \frac{1}{\xi} \right]. \quad (6.68)$$

To evaluate $\tilde{I}_2$, we first evaluate

$$\tilde{I} = \int_{-1}^1 d\mu \mu e^{\xi(1+\mu)} \text{ei}(-\xi(1+\mu)). \quad (6.69)$$

Using the transformation $z = -\xi(1+\mu)$, we have

$$\tilde{I} = \frac{1}{\xi} \lim_{\epsilon \rightarrow 0^+} \int_{-\xi\epsilon}^{-2\xi} dz \left(\frac{1}{\xi} z + 1 \right) e^{-z} \text{ei}(z) = \lim_{\epsilon \rightarrow 0^+} \left[\frac{1}{\xi^2} \int_{-\xi\epsilon}^{-2\xi} dz z e^{-z} \text{ei}(z) + \frac{1}{\xi} \int_{-\xi\epsilon}^{-2\xi} dz e^{-z} \text{ei}(z) \right]. \quad (6.70)$$

To evaluate the first integral, integration by parts leads to

$$\begin{aligned}
\int_{-\xi\epsilon}^{-2\xi} dz z e^{-z} \text{ei}(z) &= -(1+z)e^{-z} \text{ei}(z) \Big|_{-\xi\epsilon}^{-2\xi} + \int_{-\xi\epsilon}^{-2\xi} dz \frac{1+z}{z} \\
&= -(1-2\xi)e^{2\xi} \text{ei}(-2\xi) + \ln(2\xi) - 2\xi + (1-\xi\epsilon)e^{\xi\epsilon} \text{ei}(-\xi\epsilon) - \ln(\xi\epsilon) + \xi\epsilon.
\end{aligned} \tag{6.71}$$

Notice that the second integral in Eq. (6.70) is the negative of Eq. (6.52), which is evaluated in Eq. (6.54). We therefore arrive at

$$\begin{aligned}
\tilde{I} &= -\frac{1}{\xi^2}(1-2\xi)e^{2\xi} \text{ei}(-2\xi) + \frac{1}{\xi^2} \ln(2\xi) - 2\frac{1}{\xi} - \frac{1}{\xi}e^{2\xi} \text{ei}(-2\xi) + \frac{1}{\xi} \ln(2\xi) \\
&\quad + \lim_{\epsilon \rightarrow 0^+} \left[\frac{1}{\xi^2} \text{ei}(-\xi\epsilon) - \frac{1}{\xi^2} \ln(\xi\epsilon) + \frac{1}{\xi} \text{ei}(-\xi\epsilon) - \frac{1}{\xi} \ln(\xi\epsilon) \right].
\end{aligned} \tag{6.72}$$

The limit is evaluated by applying Eq. (6.55) twice, which results in

$$\int_{-1}^1 d\mu \mu e^{\xi(1+\mu)} \text{ei}(-\xi(1+\mu)) = -\frac{1}{\xi^2}(1-\xi)e^{2\xi} \text{ei}(-2\xi) + \frac{1}{\xi^2} \ln(2\xi) + \frac{1}{\xi} \ln(2\xi) - 2\frac{1}{\xi} + \frac{1}{\xi^2}\gamma + \frac{1}{\xi}\gamma. \tag{6.73}$$

Similarly, we have

$$\int_{-1}^1 d\mu \mu e^{\xi(\mu-1)} \text{ei}(-\xi(\mu-1)) = \frac{1}{\xi^2}(1+\xi)e^{-2\xi} \text{ei}(2\xi) - \frac{1}{\xi^2} \ln(2\xi) + \frac{1}{\xi} \ln(2\xi) - 2\frac{1}{\xi} - \frac{1}{\xi^2}\gamma + \frac{1}{\xi}\gamma. \tag{6.74}$$

By placing Eqs. (6.73) and (6.74) into Eq. (6.62), we derive

$$\tilde{I}_2 = d_3 \left[-\frac{1}{\xi^2}(1-\xi)e^{2\xi} \text{ei}(-2\xi) - \frac{1}{\xi^2}(1+\xi)e^{-2\xi} \text{ei}(2\xi) + \frac{2}{\xi^2} \ln(2\xi) + \frac{2}{\xi^2}\gamma \right]. \tag{6.75}$$

Upon combining Eqs. (6.61), (6.68) and (6.75), we arrive at the complete, yet lengthy expression

$$\begin{aligned}
\kappa_{\parallel}^{msd} = & -\frac{v^2}{2D\xi \sinh^2 \xi} \left[\frac{1}{\xi} \cosh(2\xi) - \text{shi}(2\xi) - \frac{1}{\xi} \text{chi}(2\xi) - \ln 2 \sinh(2\xi) \right. \\
& \left. + \frac{1}{\xi} \ln 2 \cosh(2\xi) + \frac{1}{\xi} \ln \xi + \frac{1}{\xi} \gamma - \frac{1}{\xi} \right] \\
& - \frac{v^2 [1 + \coth \xi] e^{-\xi}}{4D \sinh \xi} \left[-\frac{1}{\xi^2} (1 - \xi) e^{2\xi} \text{ei}(-2\xi) - \frac{1}{\xi^2} (1 + \xi) e^{-2\xi} \text{ei}(2\xi) + \frac{2}{\xi^2} \ln(2\xi) + \frac{2}{\xi^2} \gamma \right] \\
& + \frac{i2\pi v}{\xi^2} (\xi \cosh \xi - \sinh \xi) \tilde{C}
\end{aligned} \tag{6.76}$$

with $\tilde{C}$ given in Eq. (6.59). Fortunately, this result can be simplified significantly. Combining and canceling similar terms immediately leads to

$$\begin{aligned}
\kappa_{\parallel}^{msd} = & \frac{v^2}{2D\xi \sinh^3 \xi} \left[\frac{1}{2} e^{\xi} \text{ei}(-2\xi) + \frac{1}{2} e^{-\xi} \text{ei}(2\xi) + \cosh \xi \text{chi}(2\xi) + \sinh \xi \text{shi}(2\xi) \right] \\
& - \frac{v^2}{2D\xi^2 \sinh^2 \xi} \left[2 \sinh^2 \xi + \xi \ln(4\xi^2) \coth \xi + 2\gamma \xi \coth \xi \right].
\end{aligned} \tag{6.77}$$

By invoking two identities involving the integral functions, the term in the first brackets can be further simplified. To derive these identities, we first note an alternative form of the exponential integral function given by [Masina, 2019]:

$$\text{ei}(z) = \gamma + \ln |z| + \int_0^z dt \frac{e^t - 1}{t}. \tag{6.78}$$

By placing

$$e^t = \cosh t + \sinh t \tag{6.79}$$

into Eq. (6.78) and using Eqs. (6.46) and (6.64) therein yields

$$\text{ei}(2\xi) = \text{chi}(2\xi) + \text{shi}(2\xi). \tag{6.80}$$

On the other hand, we have

$$\text{ei}(-2\xi) = \gamma + \ln(2\xi) + \int_0^{-2\xi} dt \frac{\cosh t - 1}{t} + \int_0^{-2\xi} dt \frac{\sinh t}{t}. \tag{6.81}$$

Since $\cosh t$ is an even function and $\sinh t$ is an odd function, we have

$$\text{ei}(-2\xi) = \text{chi}(2\xi) - \text{shi}(2\xi). \quad (6.82)$$

The combination of Eqs. (6.80) and (6.82) leads to

$$\frac{1}{2}e^\xi \text{ei}(-2\xi) + \frac{1}{2}e^{-\xi} \text{ei}(2\xi) = \cosh(\xi)\text{chi}(2\xi) - \sinh(\xi)\text{shi}(2\xi). \quad (6.83)$$

Plugging this relation into Eq. (6.77) and performing straightforward simplifications results in

$$\kappa_{\parallel}^{msd} = \frac{6\kappa_{\parallel 0} \cosh \xi}{\xi \sinh^3 \xi} [\text{chi}(2\xi) - \ln(2\xi) - \gamma] - \frac{6\kappa_{\parallel 0}}{\xi^2} \quad (6.84)$$

where we have used the parallel diffusion coefficient without focusing $\kappa_{\parallel 0}$, which for isotropic scattering is given by Eq. (2.69). This expression is our exact result for the parallel diffusion coefficient with focusing.

6.4.2 Limiting Values of the Focusing-to-Scattering Ratio

Thanks to Eq. (6.84), the final result for the parallel diffusion coefficient was significantly simplified from Eq. (6.76). However, this form still contains the special function chi , so we aim to further simplify the result for the limiting values of ξ .

6.4.2.1 Small Values

Here, we simplify Eq. (6.84) for $\xi \ll 1$ using Taylor expansions. Although we want the result to be up to the second order in ξ , it is necessary to take a higher-order expansion of the individual functions in Eq. (6.84) as some of them are proportional to $1/\xi$ around $\xi = 0$. This leads to the need of the following

sixth-order expansions:

$$\begin{aligned}
\sinh(z) &= z + \frac{1}{6}z^3 + \frac{1}{120}z^5 \\
\coth(z) &= \frac{1}{z} + \frac{1}{3}z - \frac{1}{45}z^3 + \frac{2}{945}z^5 \\
\frac{1}{\sinh(z)} &= \frac{1}{z} - \frac{1}{6}z + \frac{7}{360}z^3 - \frac{31}{15120}z^5 \\
\chi(z) &= \gamma + \ln z + \frac{1}{4}z^2 + \frac{1}{96}z^4 + \frac{1}{4320}z^6.
\end{aligned} \tag{6.85}$$

The first three are well-known and can be computed easily, while the last expansion can be found in [Abramowitz and Stegun, 1964]. With these expansions, our analytical result for small values of ξ is given by

$$\begin{aligned}
\kappa_{\parallel}^{msd} &= -6\kappa_{\parallel 0} \left[\frac{1}{\xi^4} - \frac{1}{3} \frac{1}{\xi^2} + \frac{1}{15} - \frac{2}{189}\xi^2 + \frac{1}{675}\xi^4 \right] \left[\xi^2 + \frac{1}{3}\xi^4 + \frac{2}{45}\xi^6 - \xi^2 - \frac{1}{2}\xi^4 - \frac{13}{270}\xi^6 \right] \\
&= \kappa_{\parallel 0} \left[1 - \frac{14}{45}\xi^2 \right].
\end{aligned} \tag{6.86}$$

This result agrees with the one obtained in [Wang and Qin, 2020], who only consider the result for weak focusing.

6.4.2.2 Large Values

Now we repeat the above steps, except for the case of $\xi \gg 1$. We use the following facts for large values of positive z :

$$\begin{aligned}
\cosh(z) &= \frac{1}{2}e^z \\
\sinh(z) &= \frac{1}{2}e^z \\
\chi(z) &= \frac{1}{2}\text{ei}(z) \\
\text{ei}(z) &= \frac{e^z}{z} \left(1 + \frac{1}{z} \right).
\end{aligned} \tag{6.87}$$

The first three approximations follow simply from that fact that for large positive values of z , $e^z \gg e^{-z}$, and the last approximation is given in [Masina, 2019]. Together, we have

$$\kappa_{\parallel}^{msd} = -\frac{24\kappa_{\parallel 0}}{\xi^2 e^{2\xi}} \left[\xi \ln(2\xi) + \gamma\xi - \frac{1}{8\xi} e^{2\xi} \right]. \quad (6.88)$$

We can further simplify the result if we take $\xi \rightarrow \infty$ and neglect all but the largest term to derive

$$\kappa_{\parallel}^{msd} = \frac{3\kappa_{\parallel 0}}{\xi^3} \quad (6.89)$$

corresponding to the asymptotic approximation of our analytical expression for the parallel diffusion coefficient.

6.4.3 Numerical Confirmation

In this subsection, we compare our analytical form of the parallel diffusion coefficient in late-times with results obtained numerically. Before we validate our analytical result, we should first point out that the derivation did not require any information about the initial velocity. This falls in line with the standard assumption that the particles forget their initial velocity after a while. In Figures 6.1 - 6.3, we confirm this assumption by solving the Fokker-Planck equation with sharp initial conditions to numerically obtain the time-dependent parallel diffusion coefficient. The figures also confirm that the parallel diffusion coefficient becomes time-independent in late-times. Rather than using a purely numerical method to obtain the numerical results, we employ the 10-dimensional subspace method outlined in Chapter 5.

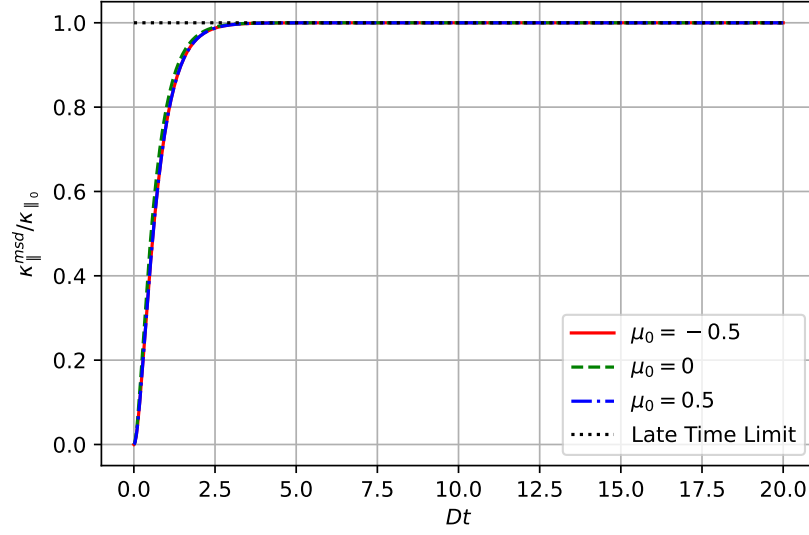


Figure 6.1: The parallel diffusion coefficient given as a function of time, which is obtained by using the subspace method given in Chapter 5. Here, we set $\xi = 0$ and have used $\kappa_{||_0} = v^2/(6D)$ which is the parallel diffusion coefficient without focusing. The asymptote denoted by the late-time limit is our analytical result for the parallel diffusion coefficient given by Eq. (6.84).

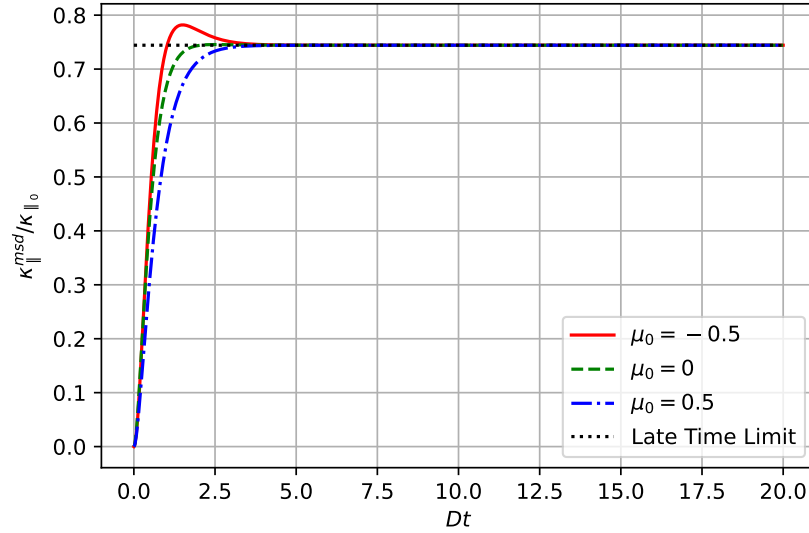


Figure 6.2: Caption as in Figure 6.1, except here $\xi = 1$.

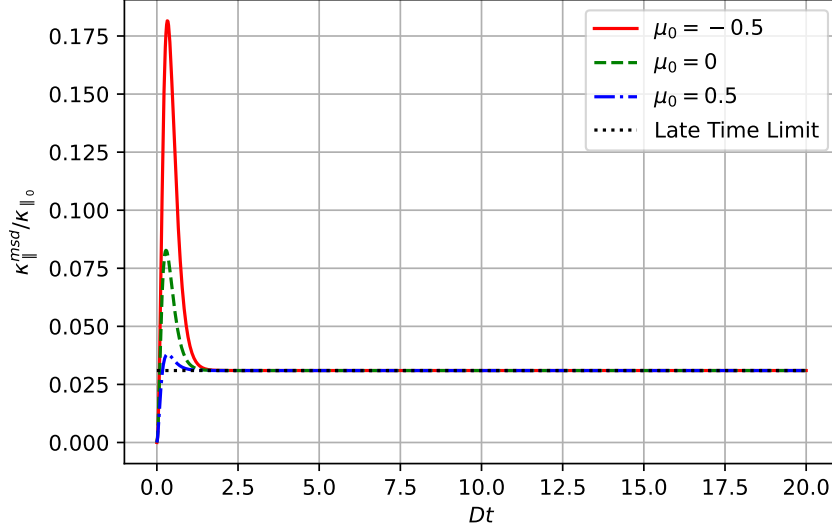


Figure 6.3: Caption as in Figure 6.1, except here $\xi = 5$.

Figures 6.1 - 6.3 also demonstrate that our analytical result is correct for the given values of ξ . To prove that the accuracy extends to all values of ξ , we also plot Eq. (6.84) as a function of ξ and compare it with the numerical results in Figure 6.4. To confirm the result proven by [Danos et al., 2013] which states that the parallel diffusion coefficient defined in terms of the mean square displacement is the same for both forms of the Fokker-Planck equation, we provide the numerical solution for both forms. It should be noted that for the standard Fokker-Planck equation, we define an expectation value via

$$\langle A \rangle = \frac{\int_{-\infty}^{\infty} dz \int_{-1}^1 d\mu A f}{\int_{-\infty}^{\infty} dz \int_{-1}^1 d\mu f} \quad (6.90)$$

which effectively normalizes the solution. Again, we see excellent agreement between the different methods.

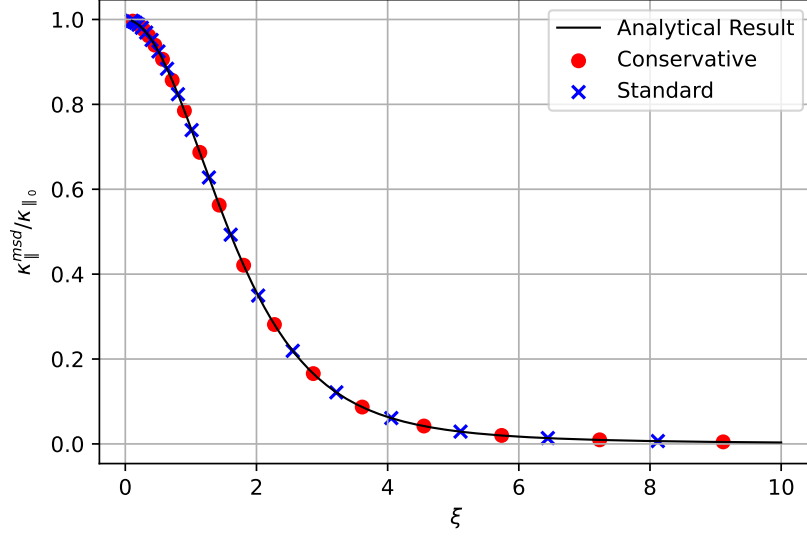


Figure 6.4: The parallel diffusion coefficient is shown as a function of the parameter ξ . The analytical result is given by Eq. (6.84), and the results from the two different forms of the Fokker-Planck equation are obtained via the subspace method.

6.4.4 Comparison with Previous Results

The new result for the parallel diffusion coefficient derived here is compared with the previous result given in [Beeck and Wibberenz, 1986]. It should be noted they do not derive the result, but rather state the result derived in the thesis by G. Green in 1983, which is seemingly inaccessible. As a result, the method is not clearly stated in [Beeck and Wibberenz, 1986].

We also compare our result with the one derived by [Shalchi, 2011]. In that work, the definition

$$\kappa_{||}^{TGK} = \int_0^\infty dt \langle \mu(t) \mu(0) \rangle \quad (6.91)$$

was employed. This result was derived more than twenty years after that by Green, which is due to the fact that Eq. (6.1) was initially believed to extend to the case where focusing is included, which rendered using the definition given by Eq. (6.91) redundant.

See Table 6.1 for a comparison of the analytical expressions between each expression in both the general case of ξ , and the limiting case of small values. In the weak focusing limit, we see that that our result

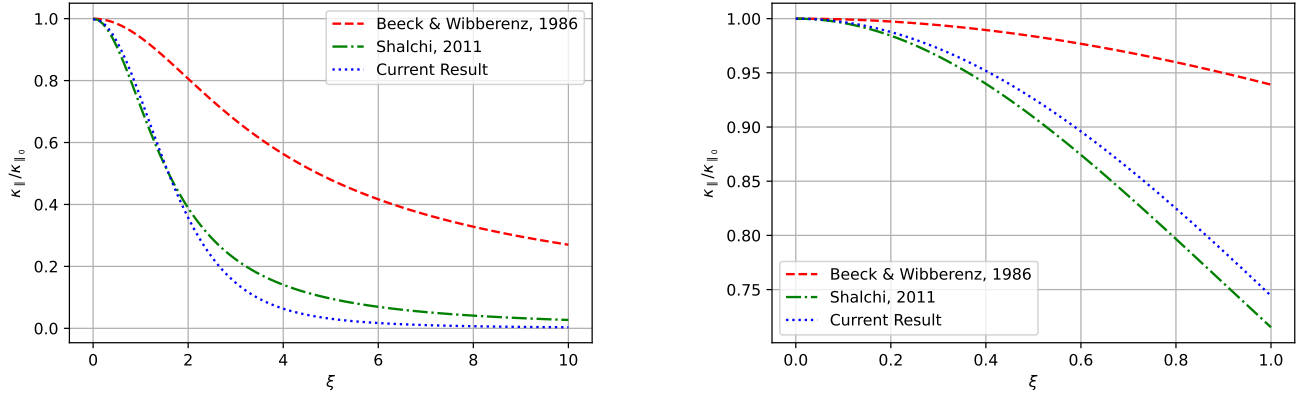


Figure 6.5: The current result for the parallel diffusion coefficient is compared with the results previously obtained by [Beeck and Wibberenz, 1986] and [Shalchi, 2011] as a function of ξ . The left plot is given for ξ over $[0, 10]$, whereas the plot on the right is zoomed in on $[0, 1]$.

is similar to that of [Shalchi, 2011], but not too similar to that stated in [Beeck and Wibberenz, 1986]. This can be further seen in Figure 6.5.

Literature	Exact Form of the Diffusion Coefficient	Weak Focusing Limit
[Beeck and Wibberenz, 1986]	$\frac{\kappa_{ }^*}{\kappa_{ 0}} = \frac{3}{\xi} \left[\coth \xi - \frac{1}{\xi} \right]$	$\frac{\kappa_{ }^*}{\kappa_{ 0}} = 1 - \frac{3}{45} \xi^2$
[Shalchi, 2011]	$\frac{\kappa_{ }^{T GK}}{\kappa_{ 0}} = \frac{3}{\xi^3} [\xi - \tanh \xi]$	$\frac{\kappa_{ }^{T GK}}{\kappa_{ 0}} = 1 - \frac{18}{45} \xi^2$
Current Result	$\frac{\kappa_{ }^{msd}}{\kappa_{ 0}} = \frac{6 \cosh \xi}{\xi \sinh^3 \xi} [\text{chi}(2\xi) - \ln(2\xi) - \gamma] - \frac{6}{\xi^2}$	$\frac{\kappa_{ }^{msd}}{\kappa_{ 0}} = 1 - \frac{14}{45} \xi^2$

Table 6.1: Comparisons of the different analytical forms for the parallel diffusion coefficient derived in the literature. The middle column provides the exact result, and the last column corresponds to the limit of $\xi \ll 1$. The current result for weak focusing was first derived in [Wang and Qin, 2020]. The parallel diffusion coefficient present in [Beeck and Wibberenz, 1986] is denoted by $\kappa_{||}^*$ since their method is unknown.

Recall that we also derived an expression for the parallel diffusion coefficient using the subspace approximation. Within the 3-dimensional approximation, our result is given by Eq. (4.113) which states

$$\frac{\kappa_{||}^{msd}}{\kappa_{||0}} = 1 - \frac{14}{45} \xi^2 \quad (6.92)$$

which is exactly the result derived throughout this chapter in the weak focusing limit.

6.5 Conclusion

This chapter concerns analytical expressions for one of the most important quantities when dealing with cosmic ray transport, namely the parallel diffusion coefficient. This quantity appears in the Parker transport equation which is used to study the process of diffusive shock acceleration [Li et al., 2012, Guo et al., 2021], as well as solar modulation [Shen and Qin, 2018, Ngobeni et al., 2022].

The Taylor-Green-Kubo formula [Taylor, 1922, Green, 1951, Kubo, 1957] combined with an identity given in [Earl, 1974] gives rise to simpler methods to analytically compute the parallel diffusion coefficient when the mean magnetic field is constant. Originally, it was believed that these results extend to when the mean magnetic field is spatially varying. As a result, Green found the earliest analytical result for the parallel diffusion coefficient in this case in 1983. It was not until more than 20 years later when [Shalchi, 2011] derived a contradictory result to that of Green. This led to further investigations in [Danos et al., 2013] who realized that the above-mentioned identities do not hold when the mean magnetic field is not constant.

The goal of this chapter was to derive an analytical expression for the parallel diffusion coefficient in late-times using the definition based on the mean square displacement. This definition is the most natural as this, by definition, quantifies the rate of change of the diffusion of the particles. Our final result is exact for any value of the parameter $\xi = v/(2DL)$ and is given by Eq. (6.84). We showed that our result agrees with [Wang and Qin, 2020], who considered the same problem, except only in the weak focusing limit. We also compared our result with the previous results by [Beeck and Wibberenz, 1986] and [Shalchi, 2011] and found that while none of the results agree with each other, our result is similar to that by [Shalchi, 2011].

Conclusion and Outlook

In this thesis, we analytically investigated the motion of electrically charged energetic particles through magnetized plasma. These particles are driven by large-scale magnetic fields and scattered by small-scale turbulent magnetic fields. Applications of understanding this motion include the theories of solar processes such as solar modulation [Shen and Qin, 2018, Ngobeni et al., 2022] and coronal mass ejections [Reid, 1964, Bieber et al., 1980], diffusive shock acceleration [Zank et al., 2000b, Ferrand et al., 2014], and the theory of magnetic turbulence [Matthaeus et al., 1990, Bieber et al., 1994] which is also present in laboratory settings [Robinson and Rusbridge, 1971, Boffetta and Ecke, 2012], including in the setting of nuclear fusion reactors. Moreover, since the analysis of observational data requires implementing a theoretical model, the results of this work contribute to improving both the accuracy and efficiency of observational studies.

Given the turbulent nature of the magnetic fields, exact analytical results are unattainable. Moreover, numerical solutions of the equation of motion are highly computationally expensive. These issues motivated the use of a statistical approach consisting of solving the Fokker-Planck equation. Although working with the Fokker-Planck equation is significantly simpler than working with the equation of motion, this model is still highly complex. As derived results are difficult, and in most cases not possible, a numerical method is often employed. The aim of this thesis was to improve our understanding of the Fokker-Planck equation by deriving closed-form expressions where possible, and explore more efficient

numerical solutions when explicit solutions are not feasible.

The results of Chapters 3 - 6 are briefly summarized in the following paragraphs.

The elusiveness of an exact analytical solution of the Fokker-Planck equation motivated the work of Chapter 3, where we investigated the simplest version of the Fokker-Planck equation. In this instance, the corresponding equation in Fourier space had the same structure as the Schrödinger equation with a linear potential. This allowed us to express the solution in terms of a series of Airy functions, and constants which depend on the boundary condition and the initial condition. Although the general form of our solution was too complicated to be useful in the general case, the consideration of special cases, such as the late time limit, allowed for simple and exact expressions for certain expectation values. These expectation values are crucial in other aspects of understanding this scattering. For example, thanks to Corrsin's approximation, the characteristic function is used in the theory of magnetic turbulence.

Chapters 4 and 5 derived simple approximate analytical results of the focused Fokker-Planck equation via the subspace method. It was shown in [Lasuik and Shalchi, 2019] that if focusing is neglected, then the 2-dimensional subspace approximation provides the best balance between simplicity and accuracy. However, we found that the 2-dimensional approximation is insufficient as certain results were found to be independent of the focusing effect, which is well-documented to be incorrect. While this issue was resolved with the use of the 3-dimensional approximation, the results were too complicated in the general case. We also sought to solve the focused Fokker-Planck equation significantly quicker than a standard numerical solver. As shown in [Shalchi, 2024] when focusing is neglected, we found that the 10-dimensional subspace approximation, a method which utilizes a Legendre polynomial expansion, provided a quick but also nearly exact approximation for both the standard and conservative forms. Given that the equation considered in Chapter 4 is often utilized when analyzing observational data, see [Bieber et al., 1994] and the references therein, the subspace method allows for a significantly quicker analysis.

Lastly, Chapter 6 revisited the parallel diffusion coefficient when the background magnetic field is spatially-varying. Thanks to the Taylor-Green-Kubo formulation [Taylor, 1922, Green, 1951, Kubo, 1957], computing the parallel diffusion coefficient is simplified when focusing is neglected.

It was originally believed that this result extended to the case of including focusing. After the work by [Danos et al., 2013], it became clear that this formulation is not true in general. The main result of this chapter was a new expression of the parallel diffusion coefficient defined naturally in terms of the mean square displacement. Even in recent work on solar energetic particles such as [Lang et al., 2024, Ogunjobi and Sivla, 2024], the parallel diffusion coefficient is defined without focusing. Consequently, obtaining an exact result in terms of the most natural definition allows for more consistency concerning how this quantity is defined.

We outline two ideas for future work which naturally follow from the work done here.

The first future project discussed here seeks to extend the subspace method to the case of a more general form of the pitch-angle scattering diffusion coefficient. As shown in Chapters 4 and 5, significant analytical progress can be made within the subspace method when we assume isotropic scattering. If we relax this assumption, which is equivalent to intermediate strength turbulence, we obtain a more general form of the pitch-angle scattering diffusion coefficient; see [Shalchi et al., 2009]. A similar expression is applied in recent work including [Lang et al., 2024, Ogunjobi and Sivla, 2024]. With this general form, it is unfeasible to derive an analytical expression for the governing matrix; which, one might think renders the subspace method not as useful. However, preliminary calculations have shown that in this instance, the subspace method is still significantly faster than the implicit Euler method. This work would require a more extensive analysis, as it seems the minimum dimension required to ensure an accurate approximation is inversely proportional with the turbulence-to-mean field ratio. A further avenue for this project would be to consider the resulting pitch-angle scattering diffusion coefficient when the focusing effect is included. As shown in [Tautz et al., 2014, Florinski, 2024], this case yields an even more complicated form.

The analytical work in Chapter 3 exploits the relation between the simplified Fokker-Planck equation and the Airy differential equation. As it turns out, we can include 3-dimensional diffusion while maintaining this structure. To see how, the resulting Fokker-Planck equation in this case is given by Eq. (3.3) with the added term

$$-D_{\perp}(\mu) \left(k_x^2 + k_y^2 \right) F_k \quad (7.1)$$

for the perpendicular diffusion coefficient $D_{\perp}$ [Schlickeiser, 2002]. Since an accurate approximation is $D_{\perp} \propto |\mu|$ [Shalchi, 2005], it is necessary to consider the positive and negative values of μ separately. In both cases, it is straightforward to see that again the solution can be given in terms of a series of Airy functions. Evidently the general case will again be far too complicated to be useful, but preliminary calculations show that in certain cases, such as small wavenumbers, the analytical results are manageable. Applications of 3-dimensional diffusion arise in solar modulation [Potgieter and Ferreira, 1999] and diffusive shock acceleration [Aerdker et al., 2024].

Bibliography

- Abramowitz, M. and Stegun, I. (1964). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. U.S. Government Printing Office.
- Aerdker, S., Merten, L., Tjus, J. B., Walter, D., Effenberger, F., and Fichtner, H. (2024). Numerical modeling of time dependent diffusive shock acceleration. *Journal of Cosmology and Astroparticle Physics*, 2024(01):068.
- Arendt, V. (2025). *Detailed test-particle simulations of energetic particles interacting with magnetized plasmas*. PhD thesis, Univesity of Manitoba.
- Ascher, U. M. and Greif, C. (2011). *A First Course on Numerical Methods*. SIAM.
- Axford, W., Leer, E., and Skadron, G. (1977). The acceleration of cosmic rays by shock waves. In *International Cosmic Ray Conference*, volume 11. Springer.
- Beeck, J. and Wibberenz, G. (1986). Pitch angle distributions of solar energetic particles and the local scattering properties of the interplanetary medium. *Astrophysical Journal*, 311:437–450.
- Bell, A. (1978a). The acceleration of cosmic rays in shock fronts–I. *Monthly Notices of the Royal Astronomical Society*, 182(2):147–156.
- Bell, A. (1978b). The acceleration of cosmic rays in shock fronts–II. *Monthly Notices of the Royal Astronomical Society*, 182(3):443–455.

- Bieber, J., Earl, J., Green, G., Kunow, H., Müller-Mellin, R., and Wibberenz, G. (1980). Interplanetary pitch angle scattering and coronal transport of solar energetic particles: New information from Helios. *Journal of Geophysical Research: Space Physics*, 85(A5):2313–2323.
- Bieber, J. W., Matthaeus, W. H., Smith, C. W., Wanner, W., Kallenrode, M.-B., and Wibberenz, G. (1994). Proton and electron mean free paths: The Palmer consensus revisited. *Astrophysical Journal*, 420:294–306.
- Biermann, L. (1951). Kometenschweife und solare korpuskularstrahlung. *Zeitschrift für Astrophysik*, Vol. 29, p. 274, 29:274.
- Biermann, L. F. (1979). Solar corpuscular radiation and the interplanetary gas. In *A Source Book in Astronomy and Astrophysics, 1900–1975*, pages 147–148. Harvard University Press.
- Blandford, R. D. and Ostriker, J. P. (1978). Particle acceleration by astrophysical shocks. *Astrophysical Journal*, 221:L29–L32.
- Boffetta, G. and Ecke, R. E. (2012). Two-dimensional turbulence. *Annual review of fluid mechanics*, 44(1):427–451.
- Borovsky, J. E. (2016). Relativity and the solar wind: The maxwell-equation origins of the solar-wind motional electric field. *Journal of Electromagnetic Analysis and Applications*, 8(8):133–151.
- Brun, A. S., Miesch, M. S., and Toomre, J. (2004). Global-scale turbulent convection and magnetic dynamo action in the solar envelope. *The Astrophysical Journal*, 614(2):1073.
- Burden, R. L. and Faires, J. D. (2011). *Numerical Analysis*. Brooks/Cole, Boston, 9th edition.
- Burlaga, L. and Turner, J. (1976). Microscale ‘Alfvén waves’ in the solar wind at 1 AU. *Journal of Geophysical Research*, 81(1):73–77.
- Christon, S. P. (1981). On the origin of the MeV energy nucleon flux associated with CIRs. *Journal of Geophysical Research: Space Physics*, 86(A11):8852–8868.

- Corrsin, S. (1959). Atmospheric diffusion and air pollution. In Frenkiel, F. and Sheppard, P., editors, *Advances in Geophysics*, volume 6, pages 161–164. Academic, New York.
- Crooker, N., Gosling, J., Bothmer, V., Forsyth, R., Gazis, P., Hewish, A., Horbury, T., Intriligator, D., Jokipii, J., Kóta, J., et al. (1999). CIR morphology, turbulence, discontinuities, and energetic particles. *Space Science Reviews*, 89(1):179–220.
- Danos, R. J., Fiege, J. D., and Shalchi, A. (2013). Numerical analysis of the Fokker–Planck equation with adiabatic focusing: Isotropic pitch-angle scattering. *The Astrophysical Journal*, 772(1):35.
- Earl, J. A. (1974). The diffusive idealization of charged particle transport in random magnetic fields. *Astrophysical Journal*, 193:231–242.
- Effenberger, F. and Litvinenko, Y. E. (2014). The diffusion approximation versus the telegraph equation for modeling solar energetic particle transport with adiabatic focusing. I. Isotropic pitch-angle scattering. *The Astrophysical Journal*, 783(1):15.
- Ferrand, G., Danos, R. J., Shalchi, A., Safi-Harb, S., Edmon, P., and Mendygral, P. (2014). Cosmic ray acceleration at perpendicular shocks in supernova remnants. *The Astrophysical Journal*, 792(2):133.
- Florinski, V. (2024). Particle scattering in the presence of moderate to strong adiabatic focusing. *Journal of Geophysical Research: Space Physics*, 129(5):e2024JA032579.
- Forbush, S. E. (1946). Three unusual cosmic-ray increases possibly due to charged particles from the Sun. *Physical Review*, 70(9-10):771.
- Ford, W. (2014). *Numerical Linear Algebra with Applications: Using MATLAB and Octave*. Academic Press.
- Ghanbari, K. and Florinski, V. (2023). Simulation of solar wind turbulence near corotating interaction regions: Superposed epoch analysis of simulations and observations. *The Astrophysical Journal*, 943(2):87.

- Giese, E., Zeller, W., Kleinert, S., Meister, M., Tamma, V., Roura, A., and Schleich, W. (2014). The interface of gravity and quantum mechanics illuminated by Wigner phase space. In *Atom Interferometry*, pages 171–236. IOS Press.
- Green, M. S. (1951). Brownian motion in a gas of noninteracting molecules. *Journal of Chemical Physics*, 19:1036–1046.
- Guo, F., Giacalone, J., and Zhao, L. (2021). Shock propagation and associated particle acceleration in the presence of ambient solar-wind turbulence. *Frontiers in Astronomy and Space Sciences*, 8:644354.
- Hedgecock, P. (1975). Measurements of the interplanetary magnetic field in relation to the modulation of cosmic rays. *Solar Physics*, 42:497–527.
- Jokipii, J. R. (1966). Cosmic-ray propagation. I. Charged particles in a random magnetic field. *Astrophysical Journal*, 146:480.
- Klippenstein, B. and Shalchi, A. (2025a). Analytical solution of the space-dependent cosmic ray Fokker-Planck equation using Airy functions. *Physica Scripta*, 100:075614.
- Klippenstein, B. and Shalchi, A. (2025b). Subspace approximation to the focused transport equation. II. The modified form. *The Open Journal of Astrophysics*, 8:13.
- Klippenstein, B. and Shalchi, A. (2025c). Subspace approximations to the focused transport equation of energetic particles. I. The standard form. *The Open Journal of Astrophysics*, 8:23.
- Kourakis, I. (2000). Kinetic theory for a test-particle in magnetized plasma. *Physica Scripta*, 2000(T84):215–216.
- Kourakis, I. and Grecos, A. (2003). Plasma diffusion and relaxation in a magnetic field. *Communications in Nonlinear Science and Numerical Simulation*, 8(3-4):547–551.
- Krymsky, G. F. (1977). Regular mechanism of charged particle acceleration on the shock wave front. In *Doklady Akademii Nauk*, volume 234, pages 1306–1308.

- Kubo, R. (1957). Statistical-mechanical theory of irreversible processes. *Journal of the Physical Society of Japan*, 12:570–586.
- Landau, L. D., Bell, J. S., Kearsley, M., Pitaevskii, L., Lifshitz, E., and Sykes, J. (2013). *Electrodynamics of continuous media*, volume 8. elsevier.
- Lang, J., Strauss, R., Engelbrecht, N., van den Berg, J., Dresing, N., Ruffolo, D., and Bandyopadhyay, R. (2024). A detailed survey of the parallel mean free path of solar energetic particle protons and electrons. *The Astrophysical Journal*, 971(1):105.
- Lasuik, J. and Shalchi, A. (2017). Solutions of the cosmic ray velocity diffusion equation. *Advances in Space Research*, 60:1532–1539.
- Lasuik, J. and Shalchi, A. (2019). Subspace approximations to the cosmic ray Fokker–Planck equation. *Monthly Notices of the Royal Astronomical Society*, 485(2):1635–1650.
- LeVeque, R. J. (2007). *Finite difference methods for ordinary and partial differential equations: steady-state and time-dependent problems*. SIAM.
- Li, G., Shalchi, A., Ao, X., Zank, G., and Verkhoglyadova, O. (2012). Particle acceleration and transport at an oblique CME-driven shock. *Advances in Space Research*, 49(6):1067–1075.
- Litvinenko, Y. E. (2012). Effects of non-isotropic scattering, magnetic helicity, and adiabatic focusing on diffusive transport of solar energetic particles. *The Astrophysical Journal*, 752(1):16.
- Ma Sung, L. and Earl, J. A. (1978). Interplanetary propagation of flare-associated energetic particles. *Astrophysical Journal*, 222:1080–1096.
- Malkov, M. (2017). Exact solution of the Fokker-Planck equation for isotropic scattering. *Physical Review D*, 95(2):023007.
- Marsch, E. (2006). Kinetic physics of the solar corona and solar wind. *Living Reviews in Solar Physics*, 3(1):1–100.

- Masina, E. (2019). Useful review on the exponential-integral special function. *arXiv preprint arXiv:1907.12373*.
- Matin, L., Bouzari, H., and Ahmadi, F. (2014). Solving Schrödinger equation specializing to the stark effect in linear potential by the canonical function method. *Journal of Theoretical and Applied Physics*, 8:1–6.
- Matthaeus, W. H. and Goldstein, M. L. (1982). Measurement of the rugged invariants of magnetohydrodynamic turbulence in the solar wind. *Journal of Geophysical Research: Space Physics*, 87(A8):6011–6028.
- Matthaeus, W. H., Goldstein, M. L., and Roberts, D. A. (1990). Evidence for the presence of quasi-two-dimensional nearly incompressible fluctuations in the solar wind. *Journal of Geophysical Research: Space Physics*, 95(A12):20673–20683.
- Meyer, P., Parker, E., and Simpson, J. (1956). Solar cosmic rays of February, 1956 and their propagation through interplanetary space. *Physical Review*, 104(3):768.
- Ness, N. F. and Wilcox, J. M. (1964). Solar origin of the interplanetary magnetic field. *Physical Review Letters*, 13(15):461.
- Ng, C. and Wong, K.-Y. (1979). Solar particle propagation under the influence of pitch-angle diffusion and collimation in the interplanetary magnetic field. In *16th International Cosmic Ray Conference*, volume 5, page 252.
- Ngobeni, M. D., Potgieter, M. S., Aslam, O. P. M., Bisschoff, D., Ramokgaba, I. I., and Ndiitwani, D. C. (2022). Simulations of the solar modulation of Helium isotopes constrained by observations. *Advances in Space Research*, 69:2330–2344.
- Ogunjobi, O. and Sivla, W. T. (2024). Modulation of cosmic ray ground-level enhancements by solar wind stream interface: a case study. *EGUsphere*, 2024:1–19.

- Olver, F. W. J., Daalhuis, A. B. O., Lozier, D. W., Schneider, B. I., Boisvert, R. F., Clark, C. W., Miller, B. R., Saunders, B. V., Cohl, H. S., and McClain, M. A. (2024). NIST digital library of mathematical functions. <https://dlmf.nist.gov/>.
- Palmer, I. (1982). Transport coefficients of low-energy cosmic rays in interplanetary space. *Reviews of Geophysics*, 20(2):335–351.
- Parker, E. N. (1958). Dynamics of the interplanetary gas and magnetic fields. *Astrophysical Journal*, 128:664.
- Pérez-Peraza, J. A. (2013). *Homage to the Discovery of Cosmic Rays, the Meson-Muon and Solar Cosmic Rays*. Nova Publishers.
- Potgieter, M. and Ferreira, S. (1999). The importance of perpendicular diffusion in the heliospheric modulation of cosmic ray electrons. *Advances in Space Research*, 23(3):463–466.
- Qin, G., Matthaeus, W. H., and Bieber, J. W. (2002). Perpendicular transport of charged particles in composite model turbulence: Recovery of diffusion. *Astrophysical Journal*, 578:L117–L120.
- Qin, G. and Shalchi, A. (2009). Pitch-angle diffusion coefficients of charged particles from computer simulations. *The Astrophysical Journal*, 707(1):61.
- Reid, G. C. (1964). A diffusive model for the initial phase of a solar proton event. *Journal of Geophysical Research*, 69(13):2659–2667.
- Reville, B., O’Sullivan, S., Duffy, P., and Kirk, J. G. (2008). The transport of cosmic rays in self-excited magnetic turbulence. *Monthly Notices of the Royal Astronomical Society*, 386(1):509–515.
- Roberts, D. A., Goldstein, M. L., Matthaeus, W. H., and Ghosh, S. (1992). Velocity shear generation of solar wind turbulence. *Journal of Geophysical Research: Space Physics*, 97(A11):17115–17130.
- Robinson, D. and Rusbridge, M. (1971). Structure of turbulence in the zeta plasma. *The Physics of Fluids*, 14(11):2499–2511.

- Roelof, E. (1969). VII. Propagation of solar cosmic rays in the interplanetary magnetic field. *Lectures in high-energy astrophysics*, page 111.
- Schlickeiser, R. (2002). *Cosmic Ray Astrophysics*. Springer, Berlin.
- Shalchi, A. (2005). Time-dependent transport and subdiffusion of cosmic rays. *Journal of Geophysical Research: Space Physics*, 110(A9).
- Shalchi, A. (2008). Analytical forms of the cosmic ray parallel mean free path with adiabatic focusing. *Journal of Physics G: Nuclear and Particle Physics*, 36:025202.
- Shalchi, A. (2009). *Nonlinear Cosmic Ray Diffusion Theories*. Astrophysics and Space Science Library. Springer, Berlin.
- Shalchi, A. (2010). A unified particle diffusion theory for cross-field scattering: subdiffusion, recovery of diffusion, and diffusion in three-dimensional turbulence. *Astrophysical Journal*, 720:L127–L130.
- Shalchi, A. (2011). Improved analytical description of parallel diffusion with adiabatic focusing. *The Astrophysical Journal*, 728(2):113.
- Shalchi, A. (2017). Analytical forms of the first 14 moments of the cosmic ray Fokker–Planck equation. *Journal of Plasma Physics*, 83(6):905830603.
- Shalchi, A. (2020). Perpendicular transport of energetic particles in magnetic turbulence. *Space Science Reviews*, 216:23.
- Shalchi, A. (2024). Transport of energetic particles in turbulent space plasmas: pitch-angle scattering, telegraph, and diffusion equations. *Frontiers in Astronomy and Space Sciences*, 11:1385820.
- Shalchi, A. and Danos, R. (2013). On the different analytical results obtained for the parallel diffusion coefficient of cosmic particles with adiabatic focusing. *The Astrophysical Journal*, 765(2):153.
- Shalchi, A. and Klippenstein, B. (2025). The parallel diffusion coefficient of energetic particles in a spatially varying magnetic field. *The Astrophysical Journal*, 990(2):128.

- Shalchi, A., Skoda, T., Tautz, R. C., and Schlickeiser, R. (2009). Analytical description of nonlinear cosmic ray scattering: Isotropic and quasilinear regimes of pitch-angle diffusion. *Astronomy & Astrophysics*, 507:589–595.
- Shen, Z.-N. and Qin, G. (2018). Modulation of galactic cosmic rays in the inner heliosphere over solar cycles. *The Astrophysical Journal*, 854(2):137.
- Simmons, G. F. (2016). *Differential Equations with Applications and Historical Notes*. CRC Press.
- Soto-Eguibar, F. and Moya-Cessa, H. M. (2015). Solution of the Schrödinger equation for a linear potential using the extended Baker-Campbell-Hausdorff formula. *Applied Mathematics & Information Sciences*, 9:175–180.
- Tautz, R. and Shalchi, A. (2010). On the widespread use of the Corrsin hypothesis in diffusion theories. *Physics of Plasmas*, 17(12).
- Tautz, R., Shalchi, A., and Dosch, A. (2014). Pitch-angle scattering of energetic particles with adiabatic focusing. *The Astrophysical Journal*, 794(2):138.
- Tautz, R. C. (2010). A new simulation code for particle diffusion in anisotropic, large-scale and turbulent magnetic fields. *Computer Physics Communications*, 181:71–77.
- Taylor, G. I. (1922). Diffusion by continuous movements. *Proceedings of the London Mathematical Society*, 20:196–212.
- The mpmath development team (2023). *mpmath: a Python library for arbitrary-precision floating-point arithmetic (version 1.3.0)*. <http://mpmath.org/>.
- Turner, L. R. (1966). Inverse of the Vandermonde matrix with applications. Technical report, National Aeronautics and Space Administration.
- Vallée, O. and Soares, M. (2010). *Airy Functions and Applications to Physics*. World Scientific Publishing Company.

- Vlasov, A. A. (1938). On vibration properties of electron gas. *Journal of Experimental and Theoretical Physics*, 8:291–318.
- Wang, J. and Qin, G. (2020). The invariance of the diffusion coefficient with iterative operations of the charged particle transport equation. *The Astrophysical Journal*, 899(1):39.
- Zank, G., Lu, J., Rice, W., and Webb, G. (2000a). Transport of energetic charged particles in a radial magnetic field. Part 1. Large-angle scattering. *Journal of plasma physics*, 64:507–541.
- Zank, G. P., Rice, W. K. M., and Wu, C. C. (2000b). Particle acceleration and coronal mass ejection driven shocks: A theoretical model. *Journal of Geophysical Research*, 105:25079–25095.
- Zhang, M., Maalal, N., and Pogorelov, N. (2023). Pitch-angle distribution of TeV cosmic rays in the lism. In *Proceedings of the 38th International Cosmic Ray Conference*, volume 328.
- Zwillinger, D. (2012). *CRC Standard Mathematical Tables and Formulae*. CRC Press, Boca Raton, FL, 31st edition.