FREE VIBRATION ANALYSIS OF A VARIABLE THICKNESS, FLEXIBLE CYLINDRICAL TANK PARTIALLY FILLED WITH FLUID

BY

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A Thesis
Submitted to the Faculty of Graduate Studies
in Partial Fulfillment of the Requirements
for the Degree of

MASTER OF SCIENCE

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ABSTRACT

The work here is concerned with the free vibration analysis of axially non-uniform, flexible cylindrical storage tanks, partially filled with fluid. The potential flow theory is employed to model the fluid while the tank is formulated based on Flugge's thin shell theory. The coupled partial differential equations for this fluid-structure interaction problem subject to appropriate boundary and continuity conditions, are solved exactly for the case of a constant thickness tank. Using the constant thickness solution as the basis for analyzing a variable thickness tank which is discretized into a series of constant thickness elements, a general procedure via the transfer matrix approach, is suggested. Parametric studies involving a variable thickness tank whose wall thickness is linearly-varying and an equivalent constant thickness tank are performed. The resulting eigensolutions of natural frequencies and their associated free vibration mode shapes demonstrate both the versatility and accuracy of the proposed technique.

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CHAPTER 1

LITERATURE SURVEY AND PROPOSED WORK

1.1 Introduction

Due to the popularity of circular cylindrical storage tanks for holding liquid, the dynamic charateristics of these fluid-loaded structures has been extensively studied. Over the past two decades, their effects due to the hydrodynamic fluid-structure interactions have become an active area of engineering research. An excellent and a very thorough literature review of the subject is outlined in Rammerstorfer et. al. [1]. Both analytical methods (Stillman [2], Jain [3], Veletsos and Yang [4], Fisher [5], Nash et. al. [6], Parkus [7], Yamaki and Tani [8], Haroun and Tayel [9], Goncalves and Batista [10], Tedesco et. al. [11], and Gupta and Hutchinson [12, 13]) and numerical methods (Lakis and Paidoussis [14], Epstein [15], Haroun and Housner [16], Balendra et. al. [17],) are widely used, with the latter for more complex situations and/or loadings. Several experimental studies have also been carried out (Clough et. al. [18, 19], Housner and Haroun [20], Kana [21], Manos and Clough [22], and Eberle et. al. [23]). Except for certain numerical techniques based on the finite element method, most of the analysis methods developed so far are valid only for tanks of constant thickness. But in practise, many of the storage tanks have variable wall thickness, in an effort to achieve a better distribution of strength and weight, and sometimes to satisfy architectural and other functional requirements. The research here is concerned with a free vibration analysis of an axially non-uniform, flexible cylindrical tank partially filled with fluid.

1.2 Literature Survey

The basic equations which describe the behavior of a thin elastic shells were originally derived by Love in 1888 [24, 25]. These equations, together with the assumptions upon which they are based, are commonly referred to as Love's first approximation. Love's first approximation to the theory of thin elastic shells is based upon the following postulates: i) the shell has a small thickness compared to other dimensions, ii) the deflections of the shell are small, iii) the transverse normal stress is negligible, and iv) normals to the reference surface of the shell remain normal to it and undergo no change in length after deformation.

The first assumption sets the stage for the entire theory. Indeed, as will be noted, the rest of Love's postulates seem to be appropriate only to thin shells and are, therefore, consequences of this first postulate. The second assumption permits us to refer all derivations and calculations to the original configuration of the shell and, together with Hooke's law, assures us that the resulting theory will be a linear elastic one. The two remaining hypotheses of the Love theory deal with the constitutive equations of thin elastic shells and represent the most significant features of the first approximation [26].

Love's equations can be written, after the substitution of the strain-displacement relations, as following [27]:

$$L_j\{u_1, u_2, u_3\} + q_j = \rho_s t_z \frac{\partial^2 u_j}{\partial t^2} \qquad (j = 1, 2, 3) \tag{1}$$

where L_j is a coupled function of displacements, q_j is external load, ρ_s is density of shell, and t_z is the thickness of the shell. Setting $q_j = 0$ and recognizing that at a natural frequency, every point in the elastic system moves harmonically. We may assume that:

$$u_j(\alpha_1, \alpha_2, t) = U_j(\alpha_1, \alpha_2)e^{i\omega t} \tag{2}$$

where α_1, α_2 are $2-D$ curvilinear surface coordinates. All of the functions $U_j(\alpha_1, \alpha_2)$ together constitute a natural mode of vibration. Boundary conditions can in general be written as:

$$B_k\{u_1, u_2, u_3\} = 0 \tag{3}$$

where $k = 1, 2, ..., N$ and where N is the total number of boundary conditions. Solutions of above equations have N unknown coefficients. Substitution of these solutions into the separate boundary conditions yields a homogeneous set of N equations. The determinant of these equations furnishes the so-called characteristic equation. The roots of the characteristic equation are the natural frequencies.

An improved first order approximation theory for thin shells was given by J.L. Sanders in 1959 [28]. In his theory, all strains vanish for small rigid-body motions. The dynamic counterpart of the equations proposed for the analysis of cylinders was presented by Donnell in 1938. Donnell's formulation was based on the assumptions that the expressions for the changes in curvature and twist of the cylinder are the same as those of a flat plate and that the effects of the transverse shearing-stress resultant on the equilibrium of forces in the circumferential direction are negligible [26]. This approach was also developed independently by Mushtari in 1938 [29]. Vlasov used the same approach to generalize for any geometry of shells, and gave general equations similar to those of Love in 1944 [30].

In this chapter, we shall briefly explain typical theories of circular cylindrical shells, that is, those developed by Donnell, Flugge and Sanders. The Donnell theory is based on the following assumptions:

(1) the shell is sufficiently thin, i.e., $t_z/r_0 << 1$, $t_z/H << 1$ (where: H is height of the shell),

(2) the strains ε are sufficiently small, $\varepsilon << 1$ and Hooke's law holds,

(3) straight lines normal to the undeformed middle surface remain straight and normal to the deformed middle surface with their length unchanged,

(4) the normal stress acting in the direction normal to the middle surface may be neglected in comparison with the stresses acting in the direction parallel to the middle surface,

(5) displacements U and V are infinitesimal, while W is of the same order as the shell thickness,

(6) the derivatives of W are small, but their squares and products are of the same order as the strain here considered, and

(7) curvature changes are small and the influences of U and V are negligible so that they can be represented by linear functions of W only.

Donnell's theory has a deficiency inherent to the shallow shell approximation. It is not applicable to the analysis of deformations of a cylinder in which the magnitude of its in-plane displacement is of the same order as that of the deflection. The bending deformations of a long cylinder with the circumferential wave number n less than four is a good example [31].

The modified Flugge theory is based on the assumptions (1)-(4) stated in the previous section and includes the following hypotheses:

(1) in deriving expressions for stress resultants, we retain terms with orders up to $(t_z/r_0)^2$ from unity,

(2) the rotations are moderately small but the effect of their product and squares on the mid-surface strains will be considered, and

(3) the curvature changes are small enough to allow linearized expressions for the bending moment.

The modified Flugge equations for the finite deformation of the cylindrical shell represent a set of three coupled nonlinear partial differential equations in U, V, and W [31].

According to the Sanders-Koiter theory for finite deformations of thin shells, the finite deformations of non-shallow shells with small strains and moderately small rotations are considered but emphases are placed on deriving simplified basic equations through rational reasonings, rather than the exact ones [32]. Thus, a set of Sanders equations for finite deformations of a circular cylindrical shell will be more complex than that of the Donnell theory but somewhat simpler than that of the modified Flugge theory. Since its generality makes it directly applicable to non-shallow shells with any geometric configuration, the theory is likely to find favour in the future, especially in structural analysis using the finite element method.

A knowledge of the free vibration characteristics of thin elastic shells filled with fluid is important both to our general understanding of the fundamentals of the behavior of a shell and to the practical application of such structures. In connection with the latter, the natural frequencies of shell structures must be known in order to avoid the destructive effect of resonance with nearby rotating or oscillating equipment (such as storage tanks under earthquake loading, fuel and oxidizer tanks of missiles, launch vehicles and marine turbines, etc.).

The first papers dealing with vibrations of cylindrical shells containing liquid were published by Lord Rayleigh [33] and Nikolai [34]. A pioneering solution for the impulsive pressure on harmonically excited, rigid vertical dams was developed by Westergaad in 1933 [35]. Hoskins and Jacobson gave the first report on analytical and experimental observations of rigid rectangular and cylindrical tanks under a simulated horizontal earthquake excitation in 1934 [36]. An overview on classical historical solutions including the *old* problem of sloshing for a rigid open circular channel was presented by Lamb in 1945 [37]. Jacobson and Ayre gave the first approximate solution for a rigid cylindrical tank omitting the modified Bessel functions [38]. In 1955, Y.Y. Yu [39] presented the general problem of free vibrations of thin cylindrical shells that is to be investigated on the basis of a set of differential equations which are derived in a similar manner as Donnell [40] obtained his equations for the bending and buckling of cylindrical shells. At the end of the fifties, elasticity of the tank wall and the tank bottom and nonlinear sloshing effects were taken into account by the spaceflight industry (eg, Schmitt [41] and Miles [42]) for development of design procedures for liquid filled shells in aerospace vehicles. The method of solution they used was the added mass concept, where the dynamically activated fluid was represented by an additional mass attached to the tank wall. A summary of the investigations of dynamic behavior of fuel tanks in flying objects can be found in the work of Cooper [43], Abramson [44, 45], and Rapport [46].

Recently, the field of analysis of liquid storage tank has become an active area of engineering research. Numerous researches, both theoretical and experimental, have been conducted on this area. The fundamental scientific findings of these liquid-solid interaction problems were published by Housner [47, 48, 49], which allow one to estimate the dynamic loads of rigid tanks resting on rigid foundations. This simple procedure is based on the easily usable response spectrum method developed by Biot [50] and Housner et al [51]. Housner splits the hydrodynamic pressure into two parts:

(i) *impulsive* pressure, caused by the movement of the liquid together with the rigid shell, and

(ii) *convective* pressure, caused by the free surface movement (sloshing) of the liquid.

The procedure presented by Housner for estimating the overturning moment and the shear forces in the tank wall was incorporated by the US Atomic Energy Commission in the TID-7024 regulations [52]. An extended application of Housner's concept in the sense of a practical design rule is given by Epstein [15]. At approximately the same time, Bauer [53] presented the exact solution for all hydrodynamic effects. The classical Rayleigh-Ritz method has been used by Lindholm, Kana and Abramson [54], Baron and Skalak [55], Arya, Thakkar and Goyal [56], Stillman [2], Gupta and Hutchinson [12, 13], Goncalves and Batista [10], Jain [3], and Fischer [5] to obtain a set of equations describing the behaviour of the vibrating shell. Solutions based on various analytical methods were presented by Chu [57], Saleme and Liber [58], Bauer and Siekmann [59], Kondo [60], Parkus [7], Haroun and Tayel [9], Chiba, Yamaki and Tani [61], Yamaki and Tani [8], Nash, Shaabran and Mouzakis [6], Tedesco, Kostem and Kalnins [11]. For closely related problems, investigations, which deal with the sloshing of the liquid in the cylindrical shell, were conducted by Miles [62], Bauer, Hsu and Wang [63], Bauer [64] and Bauer, Wang and Chen [65]. The parametric instablity problem under longitudinal excitation has been studied by Kana and Craig [66], Pih and Wu [67], Obraztsova [68], Obraztsova and Shklyarchuk [69] using an energy method. The solutions utilizing the finite element method were reported by Lakis and Paidoussis [14], Komatsu [70], Balendra, Ang, Paramasivam and Lee [17, 71], Haroun and Tayel [72], Fujita [73] and Shimizu, Yamamoto and Kawano [74].

The concluding list summarizes the main topics of research in recent years [1]:

- investigation of the shell-liquid interaction problem,
- investigation of different kinds of tank failure with respect to a three-dimensional earthquake excitation to find the most critical mode of tank failure,
- investigations of unanchored liquid storage tanks to find solutions for the maximum axial membrane force in the tank wall and the maximum dynamic loads, and
- investigations of the dynamic stability behavior of liquid storage tanks.

1.3 Proposed Research and Future Work

In this present work, a free vibration analysis of an axially non-uniform tank partially filled with liquid is presented [75]. The tank is considered to be clamped at the base and free at the top. The scope of the present investigation is not only concerned with the frequencies and mode shapes but also the effects of a variable thickness of the cylindrical storage tank walls and height of the fluid. It is an analytical procedure, using Flugge's shell theory [76] to model the thin-walled, circular cylindrical shell and the potential flow theory for the liquid which is assumed to be inviscid and incompressible. The shell thickness variation can be arbitrary and the variation is restricted only to axially non-uinformity. The constant thickness solution of a partially-filled tank is first computed. Next, a variable thickness tank is discretized into elements, each of constant thickness. The results of the constant thickness shell are employed for each of these elements, which are then assembled together using continuity conditions. To minimize computational effort, a transfer matrix approach is adopted. Several examples are solved to illustrate the procedure and accuracy. Reliable computer programs are developed to calculate the natural frequencies and corresponding mode shapes.

Future research can focus on solving the following areas:

- free vibration of cylindrical tanks filled with viscous fluid,
- response vibration of storage tanks loaded by earthquake excitation,
- flexible ground motion modelled as a soil-liquid-shell interaction problem, and
- vibration of tanks with confined fluid.

CHAPTER 2

COMPUTING THE SOLUTION FOR CONSTANT WALL THICKNESS

2.1 Equations of Motion

2.1.1 System Defintions

A thin, circular cylindrical tank of mean radius r_0, height H and an axially-varying wall thickness t_z as depicted in *Figure* 1 is studied. The tank is considered fixed to a rigid ground and filled with stationary liquid to a height h. Its material is assumed to be homogeneous and isotropic. The quantities r, θ, and z denote radial, circumferential, and axial coordinates respectively while U, V, W represent their corresponding displacement components of a point in the middle surface of the shell.

2.1.2 Hydrodynamic Fluid Loading

The exact mathematical procedure for describing fluid motion is extremely complex. Therefore, the following simplifying assumptions are employed here: (1) inviscid fluid (2) incompressible fluid (3) small displacements, velocities and slopes (4) irrotational flow field and (5) homogeneous fluid. As the flow is assumed to be irrotational, there exists a velocity potential ϕ that will satisfy the Laplace equation which in cylindrical coordinates is:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{4}$$

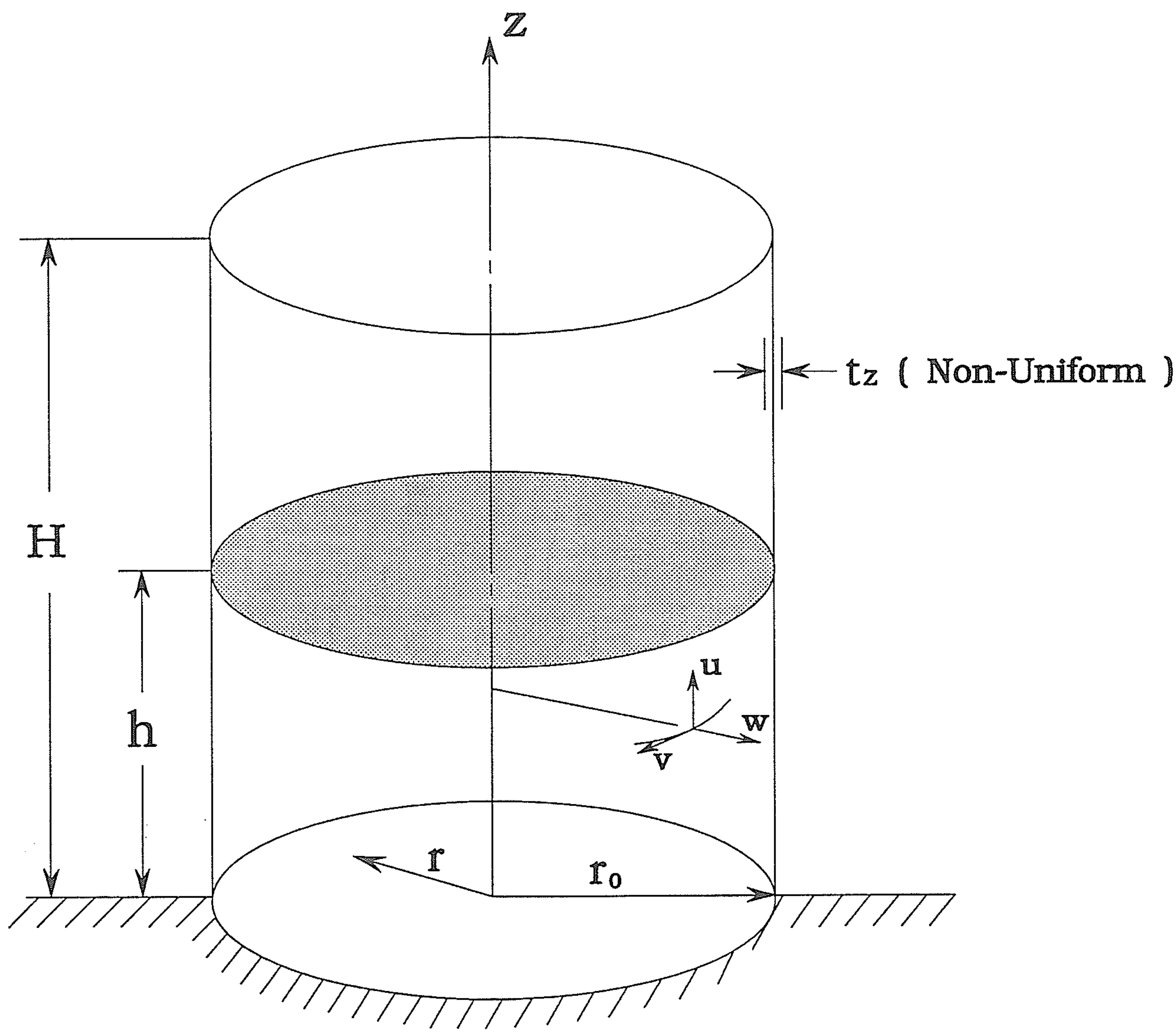


Figure 1 Shell-liquid system and coordinate system.

The boundary conditions for the flow analysis are given by:

At the tank bottom $z = 0$, the velocity component of liquid normal to the rigid base slab is zero (or the liquid velocity in the z-direction is zero),

$$\frac{\partial \phi}{\partial z} = 0 \tag{5}$$

The liquid adjacent to the wall of the elastic shell $r = r_0$ must move radially at the same velocity as the shell, i.e.

$$\frac{\partial \phi}{\partial r} = \frac{\partial W}{\partial t} \tag{6}$$

At the free fluid surface $z = h$, imposing the condition that the fluid particles must stay on the surface, it follows that (or the linearized liquid free surface may be expressed in the form):

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \tag{7}$$

For small amplitude vibrations of the shell, one obtains the solution of *Equation* (4) in the form:

$$\phi(r,\theta,z,t) = \varphi(r,\theta,z)e^{i\omega t} \tag{8}$$

where ω is the natural frequency for free vibration and $\varphi(r,\theta,z)$ is the amplitude of vibration of the fluid. Substituting *Equation* (8) into *Equation* (4) yields,

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{9}$$

Because of cyclic variation of the velocity potential in circumference, using separation of variables the solution of *Equation* (9) obviously has the form:

$$\varphi(r,\theta,z) = R(r)Z(z)\cos n\theta \tag{10}$$

Substituting *Equation* (10) into *Equation* (9) with the appropriate boundary conditions, the velocity potential can be expressed as:

$$\phi(r,\theta,z,t) = \sum_{m=1}^{\infty} A_{nm} I_n(\lambda_m r)\cos n\theta \cos \lambda_m z e^{i\omega t} \tag{11}$$

where I_n is the modified Bessel function of the first kind of order n. Consider the boundary condition given by *Equation* (6) and introduce a forcing function of the form:

$$W(\theta,z,t) = W_0(z)\cos n\theta e^{i\omega t} \tag{12}$$

then we have:

$$A_{nm} = \frac{2i\omega}{I_n'(\lambda_m r_0)(2\lambda_m^2 h + \lambda_m \sin 2\lambda_m h)^{1/2}} \int_0^h W_0(z)\cos \lambda_m z dz \tag{13}$$

Using boundary condition *Equation* (7) we can get:

$$\omega^2 = -g\lambda_m \tan \lambda_m h \qquad (m = 1, 2, 3, ...) \tag{14}$$

The hydrodynamic pressure in the fluid field $P(r,\theta,z,t)$ is given by:

$$P(r,\theta,z,t) = -\rho_f \frac{\partial \phi}{\partial t} \tag{15}$$

At $r = r_0$ we have:

$$P(r_0, \theta, z, t) = 2\rho_f \omega^2 \cos n\theta e^{i\omega t} \sum_{m=1}^{\infty} \frac{I_n(\lambda_m r_0)\cos\lambda_m z}{I_n'(\lambda_m r_0)(2\lambda_m^2 h + \lambda_m \sin 2\lambda_m h)^{1/2}}$$

$$\int_0^h W_0(z) \cos\lambda_m z dz \tag{16}$$

Note that ρ_f is the mass density of fluid and ω, λ_m are the natural frequency and eigenvalue of the coupled liquid-elastic system. They are related by *Equation* (14). It may be observed that the hydrodynamic pressure contains the unknown shell displacement $W_0(z)$ and eigenquantities ω, λ_m. This is a classical example of a fluid-structure interaction problem and to obtain solutions, it is necessary to solve the coupled system.

2.1.3 Thin Shell Motion

The hydrodynamic pressure can be obtained if we know $W_0(z)$, but to evaluate $W_0(z)$ we have to solve the motion of the shell. The displacements in the cylindrical coordinate system are expressed as:

$$U(\theta, z, t) = U_0(z) \cos n\theta e^{i\omega t} \tag{17}$$

$$V(\theta, z, t) = V_0(z) \sin n\theta e^{i\omega t} \tag{18}$$

$$W(\theta, z, t) = W_0(z) \cos n\theta e^{i\omega t} \tag{19}$$

where n denotes the number of circumferential waves and ω denotes the natural frequency of the coupled liquid-solid system. The tank is modelled using the well-known Flugge shell equations. The resulting equations of motion for a fluid-loaded tank are:

$$\left[\frac{\partial^2}{\partial z^2}+(\frac{1-\nu}{2r_0^2})(1+\frac{t_z^2}{12r_0^2})\frac{\partial^2}{\partial\theta^2}\right]U+\frac{1+\nu}{2r_0}\frac{\partial^2 V}{\partial z\partial\theta}+$$

$$\left[\frac{\nu}{r_0}\frac{\partial}{\partial z}-\frac{t_z^2}{12r_0}\frac{\partial^3}{\partial z^3}+\frac{(1-\nu)t_z^2}{24r_0^3}\frac{\partial^3}{\partial z\partial\theta^2}\right]W-\frac{\rho_s(1-\nu^2)}{E}\frac{\partial^2 U}{\partial t^2}=0 \quad (20)$$

$$\frac{1+\nu}{2r_0}\frac{\partial^2 U}{\partial z\partial\theta}+\left[\frac{1}{r_0^2}\frac{\partial^2}{\partial\theta^2}+\frac{1-\nu}{2}(1+\frac{t_z^2}{4r_0^2})\frac{\partial^2}{\partial z^2}\right]V+$$

$$\left[\frac{1}{r_0^2}\frac{\partial}{\partial\theta}-\frac{(3-\nu)t_z^2}{24r_0^2}\frac{\partial^3}{\partial\theta\partial z^2}\right]W-\frac{\rho_s(1-\nu^2)}{E}\frac{\partial^2 V}{\partial t^2}=0 \quad (21)$$

$$\left[\frac{\nu}{r_0}\frac{\partial}{\partial z}-\frac{t_z^2}{12r_0}\frac{\partial^3}{\partial z^3}+\frac{(1-\nu)t_z^2}{24r_0^3}\frac{\partial^3}{\partial z\partial\theta^2}\right]U+\left[\frac{1}{r_0^2}\frac{\partial}{\partial\theta}-\frac{(3-\nu)t_z^2}{24r_0^2}\frac{\partial^3}{\partial\theta\partial z^2}\right]V+$$

$$\left[\frac{1}{r_0}^2+\frac{t_z^2}{12r_0^4}+\frac{t_z^2}{12}\frac{\partial^4}{\partial z^4}+\frac{t_z^2}{6r_0^2}\frac{\partial^4}{\partial z^2\partial\theta^2}+\frac{t_z^2}{12r_0^4}\frac{\partial^4}{\partial\theta^4}+\frac{t_z^2}{6r_0^4}\frac{\partial^2}{\partial\theta^2}\right]W+$$

$$\frac{\rho_s(1-\nu^2)}{E}\frac{\partial^2 W}{\partial t^2}=\frac{(1-\nu^2)P}{Et_z} \quad (22)$$

where r, θ, z are the cylindrical coordinates, t is time and the physical characteristics of the shell are defined by the mean radius r_0, wall thickness t_z, density ρ_s, Young's modulus E, Poisson's ratio ν, and the hydrodynsmic pressure P.

Substituting *Equations* (17)-(19) into *Equations* (20)-(22) yields, a set of three coupled ordinary differential equations in U_0, V_0, W_0:

$$\frac{d^2U_0(z)}{dz^2} + \left[\frac{\rho_s(1-\nu^2)\omega^2}{E} - \frac{(1-\nu)n^2}{2r_0^2}(1+\frac{t_z^2}{12r_0^2})\right] U_0(z) +$$

$$\left[\frac{(1+\nu)n}{2r_0}\right] \frac{dV_0(z)}{dz} - \frac{t_z^2}{12r_0}\frac{d^3W_0(z)}{dz^3} + \left[\frac{\nu}{r_0} - \frac{(1-\nu)t_z^2n^2}{24r_0^3}\right]\frac{dW_0(z)}{dz} = 0 \tag{23}$$

$$\left[-\frac{(1+\nu)n}{2r_0}\right]\frac{dU_0(z)}{dz} + \left[\frac{1-\nu}{2}(1+\frac{t_z^2}{4r_0^2})\right]\frac{d^2V_0(z)}{dz^2} + \left[\frac{\rho_s(1-\nu^2)\omega^2}{E} - \frac{n^2}{r_0^2}\right] V_0(z) +$$

$$\left[\frac{(3-\nu)nt_z^2}{24r_0^2}\right]\frac{d^2W_0(z)}{dz^2} - \frac{n}{r_0^2}W_0(z) = 0 \tag{24}$$

$$-\frac{t_z^2}{12r_0}\frac{d^3U_0(z)}{dz^3} + \left[\frac{\nu}{r_0} - \frac{(1-\nu)t_z^2n^2}{24r_0^3}\right]\frac{dU_0(z)}{dz} + \left[-\frac{(3-\nu)nt_z^2}{24r_0^2}\right]\frac{d^2V_0(z)}{dz^2} + \frac{n}{r_0^2}V_0(z) +$$

$$\frac{t_z^2}{12}\frac{d^4W_0(z)}{dz^4} - \frac{n^2t_z^2}{6r_0^2}\frac{d^2W_0(z)}{dz^2} + \left[\frac{1}{r_0^2} + \frac{t_z^2}{12r_0^4} + \frac{n^4t_z^2}{12r_0^4} - \frac{n^2t_z^2}{6r_0^4} - \frac{\rho_s(1-\nu^2)\omega^2}{E}\right] W_0(z)$$

$$= \sum_{m=1}^{\infty} f_m \cos\lambda_m z \tag{25}$$

where:

$$f_m = 2\frac{\rho_f(1-\nu^2)\omega^2}{Et_z}\frac{I_n(\lambda_m r_0)}{I_n'(\lambda_m r_0)(2\lambda_m^2 h + \lambda_m \sin 2\lambda_m h)^{1/2}}\int_0^h W_0(z)\cos\lambda_m z dz \tag{26}$$

The free vibration solution of a partially-filled, constant thickness tank is discussed in this section. Although there has been extensive investigation for the constant thickness tank, it is necessary to present its solution method here, since it is required for solving the variable-thickness tank. The procedure for the analysis of a partially-filled tank involves computing the homogeneous solution of the differential equations in *Equations* (20)-(22) by solving the empty tank (the *dry* solution), obtaining the particular solution by solving the wet part of the tank (the *wet* solution) and finally, applying appropriate boundary and continuity conditions at the top and bottom and the dry part-wet part interface of the tank. It should be noted that in the formulation of Haroun and Tayel [9], considerable simplifications are obtained by invoking axisymmetry which is not assumed here.

(a) *The dry solution*. For the prescribed structural boundary conditions, the mid-surface displacements of the shell is given by *Equations* (17)-(19). Introducing:

$$U_0(z) = Ae^{\sigma z}; \qquad V_0(z) = Be^{\sigma z}; \qquad W_0(z) = Ce^{\sigma z} \tag{27}$$

in which A, B, C are constants and σ is an unknown to be determined. Substituting *Equation* (27) into *Equations* (23)-(25), we may drop the exponential factor, resulting in three ordinary linear equations for the constants A, B, C which can be expressed as,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{28}$$

Let:

$$[\tilde{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{29}$$

For a nontrivial solution of these equations, the determinant of the coefficent matrix is set equal to zero, leading to:

$$| \tilde{A} | = 0 \tag{30}$$

Equation (30) can be written as,

$$C_{11}(\sigma^2)^4 + C_{22}(\sigma^2)^3 + C_{33}(\sigma^2)^2 + C_{44}(\sigma^2) + C_{55} = 0 \tag{31}$$

in which the coefficients $C_{11}, C_{22}, C_{33}, C_{44}, C_{55}$ are constants which depend on the frequency ω. *Equation* (31) is an 8^{th} order polynomial in term of σ(i.e 8 roots for σ). Actually, from *Equation* (31), we can see that is the 4^{th} order polynomial in σ^2 as only even powers are present, thus the roots σ^2 are complex conjugate pairs. Therefore, the 8 roots σ_j (j=1,...8) are given by:

$$U_0 = \sum_{j=1}^{8} A_j e^{\sigma_j z}; \qquad V_0 = \sum_{j=1}^{8} B_j e^{\sigma_j z}; \qquad W_0 = \sum_{j=1}^{8} C_j e^{\sigma_j z} \tag{32}$$

Since the constants A, B, C are not independent, we can write, $A_j = \alpha_j C_j$ and $B_j = \beta_j C_j$, and the independent constants are reduced to $C_j (j = 1...8)$ that is:

$$U_0 = \sum_{j=1}^{8} \alpha_j C_j e^{\sigma_j z}; \qquad V_0 = \sum_{j=1}^{8} \beta_j C_j e^{\sigma_j z}; \qquad W_0 = \sum_{j=1}^{8} C_j e^{\sigma_j z} \tag{33}$$

where α_j, β_j are complex numbers derived from the coefficients of *Equation* (28) and expressed explicitly in the form:

$$\alpha_j = (a_{12}a_{23} - a_{13}a_{22})/(a_{11}a_{22} - a_{12}a_{21})$$

$$\beta_j = (a_{13}a_{21} - a_{11}a_{23})/(a_{11}a_{22} - a_{12}a_{21}) \tag{34}$$

Finally, the homogeneous solution becomes,

$$U(\theta,z,t)=\sum_{j=1}^{8}\alpha_j C_j e^{\sigma_j z}\cos n\theta e^{i\omega t} \tag{35}$$

$$V(\theta,z,t)=\sum_{j=1}^{8}\beta_j C_j e^{\sigma_j z}\sin n\theta e^{i\omega t} \tag{36}$$

$$W(\theta,z,t)=\sum_{j=1}^{8} C_j e^{\sigma_j z}\cos n\theta e^{i\omega t} \tag{37}$$

(b) *The wet solution.* Considering the wet part of tank, the complete solution for the three displacement components is given by

$$\tilde{U}(\theta,z,t)=[\sum_{j=1}^{8}\tilde{\alpha}_j\tilde{C}_j e^{\sigma_j z}+\sum_{m=1}^{\infty}G_{1m}\sin\lambda_m z]\cos n\theta e^{i\omega t} \tag{38}$$

$$\tilde{V}(\theta,z,t)=[\sum_{j=1}^{8}\tilde{\beta}_j\tilde{C}_j e^{\sigma_j z}+\sum_{m=1}^{\infty}G_{2m}\cos\lambda_m z]\sin n\theta e^{i\omega t} \tag{39}$$

$$\tilde{W}(\theta,z,t)=[\sum_{j=1}^{8}\tilde{C}_j e^{\sigma_j z}+\sum_{m=1}^{\infty}G_{3m}\cos\lambda_m z]\cos n\theta e^{i\omega t} \tag{40}$$

in which G_{1m}, G_{2m}, G_{3m} are bounded coefficients. Substitution of *Equations* (38)-(40) into *Equations* (20)-(22) leads to the following relation:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{Bmatrix} G_{1m} \\ G_{2m} \\ G_{3m} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f_m \end{Bmatrix} \tag{41}$$

Solving *Equation* (41) yields, we have:

$$G_{1m} = f_m(b_{12}b_{23} - b_{13}b_{22})/\Delta \tag{42}$$

$$G_{2m} = -f_m(b_{11}b_{23} - b_{13}b_{21})/\Delta \tag{43}$$

$$G_{3m} = f_m(b_{11}b_{22} - b_{12}b_{21})/\Delta \tag{44}$$

where Δ is the determinant defined by,

$$\Delta = \mid \tilde{B} \mid \tag{45}$$

with matrix $[\tilde{B}]$ given by,

$$[\tilde{B}] = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{46}$$

Let

$$B_1 = (b_{12}b_{23} - b_{13}b_{22})/\Delta \tag{47}$$

$$B_2 = (b_{13}b_{21} - b_{11}b_{23})/\Delta \tag{48}$$

$$B_3 = (b_{11}b_{22} - b_{12}b_{21})/\Delta \tag{49}$$

and

$$f_m = P_m \sum_{j=1}^{8} C_j T_{jm} \tag{50}$$

in which:

$$P_m = 2\frac{\rho_f(1-\nu^2)\omega^2}{Et_z} \frac{I_n(\lambda_m r_0)}{I_n'(\lambda_m r_0)(2\lambda_m^2 h + \lambda_m \sin 2\lambda_m h)^{1/2}} \tag{51}$$

$$T_{jm} = \int_0^h e^{\sigma_j z} \cos \lambda_m z dz \tag{52}$$

then *Equations* (43)-(45) become:

$$G_{1m} = B_1 P_m \sum_{j=1}^{8} C_j T_{jm} \tag{53}$$

$$G_{2m} = B_2 P_m \sum_{j=1}^{8} C_j T_{jm} \tag{54}$$

$$G_{3m} = B_3 P_m \sum_{j=1}^{8} C_j T_{jm} \tag{55}$$

At this stage the solution of the *wet part* is solved. It remains to apply the boundary conditions.

2.1.4 System Boundary Conditions

The analysis of a partially-filled tank is realized by combining the dry solution with the wet solution. This is done by enforcing the necessary boundary conditions at the top and bottom of the tank and compatibility at the interface of the dry-wet solutions. There are altogether 16 unknown coefficients in the system which are solvable from the 4 boundary conditions at each end of the tank and the 8 continuity conditions at the interface. Setting the determinant of this 16×16 matrix to zero yields the frequency equation. The boundary conditions of the tank and the continuity conditions at the interface are:

i) at the base $z = 0$:

$$\tilde{U}(\theta, 0, t) = \tilde{V}(\theta, 0, t) = \tilde{W}(\theta, 0, t) = \frac{\partial \tilde{W}(\theta, 0, t)}{\partial z} = 0 \tag{56}$$

ii) at the upper rim $z = H$:

$$N_z(\theta, H, t) = N_{z\theta}(\theta, H, t) = M_z(\theta, H, t) = Q_z(\theta, H, t) = 0 \tag{57}$$

iii) at the interface $z = h$:

$$\begin{aligned}
\tilde{U}(\theta, h, t) &= U(\theta, h, t) \\
\tilde{V}(\theta, h, t) &= V(\theta, h, t) \\
\tilde{W}(\theta, h, t) &= W(\theta, h, t) \\
\frac{\partial \tilde{W}(\theta, h, t)}{\partial z} &= \frac{\partial W(\theta, h, t)}{\partial z} \\
\tilde{N}_z(\theta, h, t) &= N_z(\theta, h, t) \\
\tilde{N}_{z\theta}(\theta, h, t) &= N_{z\theta}(\theta, h, t) \\
\tilde{M}_z(\theta, h, t) &= M_z(\theta, h, t) \\
\tilde{Q}_z(\theta, h, t) &= Q_z(\theta, h, t)
\end{aligned} \tag{58}$$

where from shell theory, the stress resultants are defined as,

$$N_z = D[\frac{\partial U}{\partial z} + \frac{\nu}{r_0}(\frac{\partial V}{\partial \theta} + W)] \tag{59}$$

$$N_{z\theta} = \frac{D(1-\nu)}{2}[\frac{\partial V}{\partial z} + \frac{1}{r_0}\frac{\partial U}{\partial \theta}] \tag{60}$$

$$M_z = K[\frac{\partial^2 W}{\partial z^2} + \frac{\nu}{r_0^2}\frac{\partial^2 W}{\partial \theta^2}] \tag{61}$$

$$Q_z = \frac{\partial M_z}{\partial z} + \frac{1}{r_0}\frac{\partial M_{z\theta}}{\partial \theta} \tag{62}$$

in which:

$$K = \frac{Et_z^3}{12(1-\nu^2)}; \qquad D = \frac{Et_z}{1-\nu^2} \tag{63}$$

For free vibration analysis, we use *Equations* (35)-(37), *Equations* (38)- (40) and *Equations* (59)-(62) with 16 boundary conditions and we get:

$$[\tilde{K}]\{\tilde{C}\} = \{0\} \tag{64}$$

where:

$$[\tilde{K}] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \ldots & k_{116} \\ k_{21} & k_{22} & k_{23} & \ldots & k_{216} \\ k_{31} & k_{32} & k_{33} & \ldots & k_{316} \\ \vdots & \vdots & \vdots & & \vdots \\ k_{161} & k_{162} & k_{163} & \ldots & k_{1616} \end{bmatrix} \tag{65}$$

$$\{\tilde{C}\} = \left\langle \tilde{C}_1 \quad \tilde{C}_2 \quad \ldots \quad \tilde{C}_8 \quad C_1 \quad C_2 \quad \ldots \quad C_8 \right\rangle^T \tag{66}$$

The frequency equation is then obtained by requiring the determinant of the matrix $[\tilde{K}]$ to be zero. That is,

$$| \tilde{K} | = 0 \tag{67}$$

To solve *Equation* (67) for natural frequencies of the coupled fluid-structure system ω, we note that *Equation* (67) is now expressed only in terms of the two eigenquantities ω, λ_m. The second equation relating these two eigenquantities are given by *Equation* (14), and this permits the natural frequencies to be computed via an iterative procedure. Having obtained the natural frequencies, their corresponding mode shapes can be easily calculated.

2.2 Computer Implementation

A computer program is available for the determination of these natural frequencies and associated mode shapes for given geometric and elastic input parameters of the liquid-shell system. The program, which has been implemented on the PC-486, is written by using the Maple software.

2.3 Numerical Simulation

To verify and demonstrate the applicability of the proposed method, several examples of cylindrical storage tanks with different fluid loading conditions are solved. For the case of constant thickness tanks where published results are readily available, a comparison of the solutions obtained is made. Having verified its accuracy, the method is then extended to handle the analysis of the non-uniform tanks. The information listed in *Table* 1 is employed in the simulation study:

Table 1 *Simulation parameters and material properties.*

Short Tank Tall Tank	$H/r_0 = 0.67$ $H/r_0 = 3.00$
Wall Thickness Variation	$\xi = t_b/t_t = \begin{cases} 1 & \text{(Constant Thickness)} \\ 1-5 & \text{(Linear Thickness)} \end{cases}$
Fluid Depth-Tank Height Variation	$h/H = \begin{cases} 0.0 & \text{(Empty Tank)} \\ 0.4-0.8 & \text{(Partially Filled Tank)} \\ 1.0 & \text{(Completely Filled Tank)} \end{cases}$
Material Properties Steel Water	$\begin{cases} E = 2.07 \times 10^8 \mathrm{KPa} \\ \rho_s = 7840 \mathrm{kg/m^3} \\ \nu = 0.3 \end{cases}$ $\rho_f = 1000 \mathrm{kg/m^3}$

In order to compare the results of the current analysis with the results of Haroun and Tayel [9], the same basic data for their tanks is selected. Hence, the two distinct tank types have the following values: for short tanks, $r_0 = 18.29$ m (60 ft), $H = 12.19$ m (40 ft) and for tall tanks, $r_0 = 7.32$ m (24.0 ft), $H = 21.95$ m (72 ft). The tank thickness is 2.54 cm (1 in) and is assumed to be filled with water. The fluid loading conditions correspond to three situations: an empty tank, a completely filled tank and a partially-filled tank with the fluid depth to tank height ratio, $h/H = 0.6$.

The natural frequency results are tabulated in *Tables* 2-4 for the first three lowest modes. As shown, excellent agreement with Haroun and Tayel is obtained even though different shell theories have been used in the modelling of the tank. Our formulation is based on Flugge's theory while their model is derived using Novozhilov's theory. The axial and radial displacement mode shapes corresponding to the fundamental natural frequency are presented in *Figures* 2-4. Observe that they agree almost perfectly with Haroun and Tayel's work. In computing the frequencies and mode shapes, it is found that typically, 20 iterations are required to achieve convergence while in the series summation, the number of terms used is 10.

Table 2 *Natural frequencies of empty tanks with constant thickness.*

Tank Type	Mode Number	Natural Frequency (Hz)	
		This Study	Haroun & Tayel [9]
Short	1	44.40	44.40
	2	44.70	44.71
	3	44.76	44.77
Tall	1	57.72	57.72
	2	108.89	108.97
	3	111.01	111.04

Table 3 *Natural frequencies of completely-filled tanks with constant thickness.*

Tank Type	Mode Number	Natural Frequency (Hz)	
		This Study	Haroun & Tayel [9]
Short	1	6.16	6.27
	2	11.69	11.77
	3	15.03	15.10
Tall	1	6.70	6.75
	2	17.94	17.99
	3	25.72	25.79

Table 4 Natural frequencies of a partially-filled tanks with constant thickness ($h/H = 0.6$).

Tank Type	Mode Number	Natural Frequency (Hz)	
		This Study	Haroun & Tayel [9]
Short	1	8.71	8.74
	2	15.34	15.39
	3	19.57	19.64
Tall	1	10.87	10.96
	2	25.89	25.91
	3	34.68	34.74

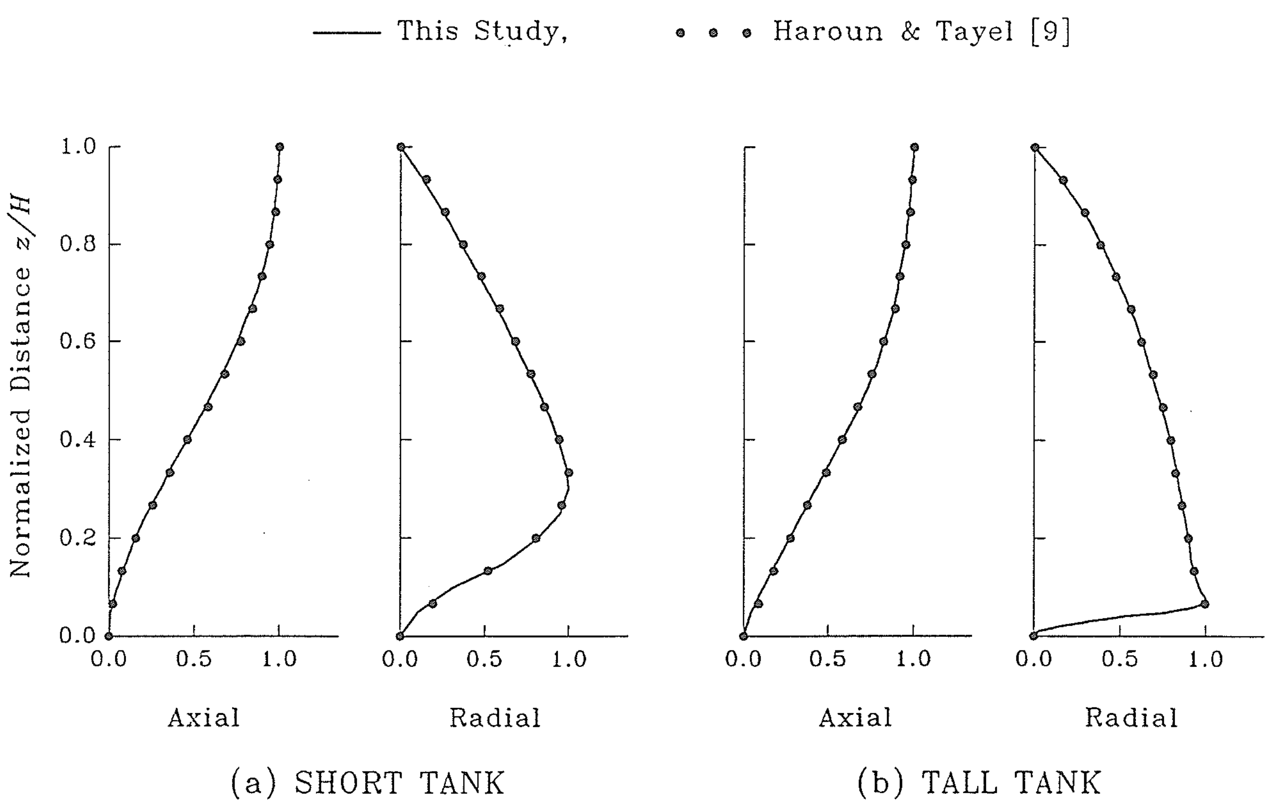


Figure 2 Mode shapes for empty, constant thickness tanks.

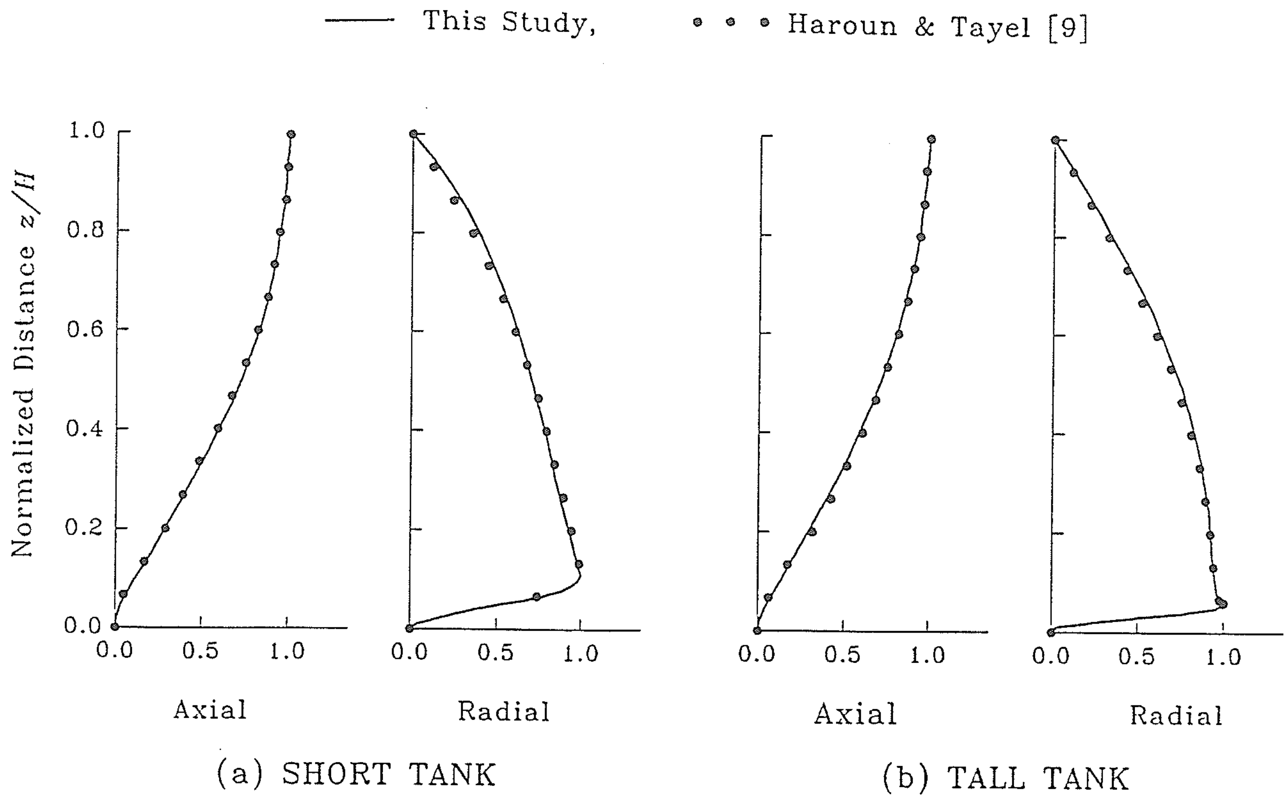


Figure 3 Mode shapes for completely-filled, constant thickness tanks.

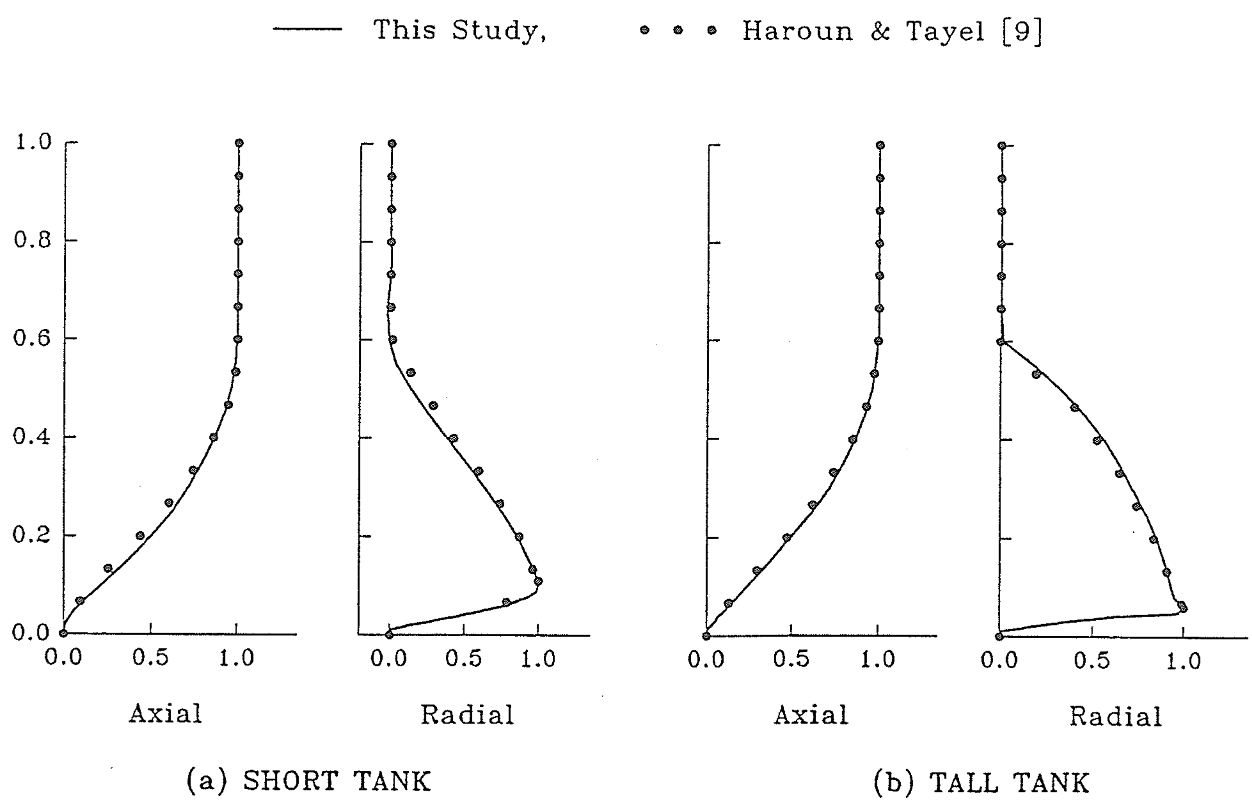


Figure 4 Mode shapes for partially-filled, constant thickness tanks ($h/H = 0.6$).

CHAPTER 3

COMPUTING THE SOLUTION FOR VARIABLE WALL THICKNESS

To formulate the variable thickness solution, the shell can be discretized into a number of elements, each of constant thickness. The results of the constant thickness shell obtained in the previous section are employed for each of these elements, which are then assembled together using continuity conditions. Naturally, the more elements used in the discretization, the more accurate will be the solution. However, if the mesh is too fine, the problem may become computationally difficult to handle. If a shell is divided into n elements, one would have to be contend with solving a $8n \times 8n$ matrix. One approach to overcome this difficulty is to use the transfer matrix technique. Using the conditions of force equilibrium and continuity at an element interface, the transfer relations for forces and displacements can be derived and assembled together for the overall analysis.

3.1 Solution for Variable Wall Thickness for an Empty Tank

Recall displacements and stress given by *Equations* (35)-(37) and *Equations* (59)-(62), these equations are expresed as,

$$U(\theta, z, t) = \sum_{j=1}^{8} \alpha_j C_j e^{\sigma_j z} \cos n\theta e^{i\omega t} \tag{68}$$

$$V(\theta, z, t) = \sum_{j=1}^{8} \beta_j C_j e^{\sigma_j z} \sin n\theta e^{i\omega t} \tag{69}$$

$$W(\theta, z, t) = \sum_{j=1}^{8} C_j e^{\sigma_j z} \cos n\theta e^{i\omega t} \tag{70}$$

$$\frac{\partial W(\theta,z,t)}{\partial z} = \sum_{j=1}^{8} C_j \sigma_j e^{\sigma_j z} \cos n\theta e^{i\omega t} \tag{71}$$

$$N_z = D[\frac{\partial U}{\partial z} + \frac{\nu}{r_0}(\frac{\partial V}{\partial \theta} + W)] \tag{72}$$

$$N_{z\theta} = \frac{D(1-\nu)}{2}[\frac{\partial V}{\partial z} + \frac{1}{r_0}\frac{\partial U}{\partial \theta}] \tag{73}$$

$$M_z = K[\frac{\partial^2 W}{\partial z^2} + \frac{\nu}{r_0^2}\frac{\partial^2 W}{\partial \theta^2}] \tag{74}$$

$$Q_z = \frac{\partial M_z}{\partial z} + \frac{1}{r_0}\frac{\partial M_{z\theta}}{\partial \theta} \tag{75}$$

Using *Equations* (33) for the first three equations and the subsequent equations become:

$$U(z) = \sum_{j=1}^{8} \alpha_j C_j e^{\sigma_j z} \tag{76}$$

$$V(z) = \sum_{j=1}^{8} \beta_j C_j e^{\sigma_j z} \tag{77}$$

$$W(z) = \sum_{j=1}^{8} C_j e^{\sigma_j z} \tag{78}$$

$$\frac{\partial W(z)}{\partial z} = \sum_{j=1}^{8} C_j \sigma_j e^{\sigma_j z} \tag{79}$$

$$N_z = D\sum_{j=1}^{8} C_j e^{\sigma_j z}(\alpha_j \sigma_j + \frac{\nu}{r_0}\beta_j n + \frac{\nu}{r_0}) \tag{80}$$

$$N_{z\theta} = \frac{D(1-\nu)}{2}\sum_{j=1}^{8} C_j e^{\sigma_j z}(\beta_j \sigma_j - \frac{\alpha_j n}{r_0}) \tag{81}$$

$$M_z = K\sum_{j=1}^{8} C_j e^{\sigma_j z}(\sigma_j^2 - \frac{\nu n^2}{r_0^2}) \tag{82}$$

$$Q_z = K\sum_{j=1}^{8} C_j e^{\sigma_j z}(\sigma_j^3 - \frac{\nu n^2}{r_0^2}\sigma_j) \tag{83}$$

Equations (76)-(83) can be written in a matrix form:

$$\{R\} = [P]\{C\} \tag{84}$$

where

$$\{R\} = \left\langle U \quad V \quad W \quad \frac{\partial W}{\partial z} \quad N_z \quad N_{z\theta} \quad M_z \quad Q_z \right\rangle^T \tag{85}$$

$$[P] = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{18} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{28} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{38} \\ \vdots & \vdots & \vdots & & \vdots \\ p_{81} & p_{82} & p_{83} & \cdots & p_{88} \end{bmatrix} \tag{86}$$

and

$$\{C\} = \langle C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5 \quad C_6 \quad C_7 \quad C_8 \rangle^T \tag{87}$$

$\{R\}$ is called the state vector and $[P]$ is a matrix relating the coefficients of unknown constants $\{C\}$. For a shell of constant thickness the constants $C_1, ..., C_8$ in the above equation can be determined by using the boundary conditions at the two ends. Application of these boundary conditions yields eight simultaneously homogeneous equations which can be written in the form:

$$[P]\{C\} = \{0\} \tag{88}$$

As shown in *Figure* 5, a three-element discretization of the tank is adopted. The procedure is as follows:

For element [1]

at $z = z_0$:

$$\{R(z_0)\} = [P(z_0, t_{z_1})]\{C^1\} \tag{89}$$

at $z = z_1$:

$$\{R(z_1)\} = [P(z_1, t_{z_1})]\{C^1\} \tag{90}$$

substituting *Equation* (89) to *Equation* (90), we obtain:

$$\{R(z_1)\} = [P(z_1, t_{z_1})][P(z_0, t_{z_1})]^{-1}\{R(z_0)\} \tag{91}$$

For element [2]

similarly,

at $z = z_1$:

$$\{R(z_1)\} = [P(z_1, t_{z_2})]\{C^2\} \tag{92}$$

at $z = z_2$:

$$\{R(z_2)\} = [P(z_2, t_{z_2})]\{C^2\} \tag{93}$$

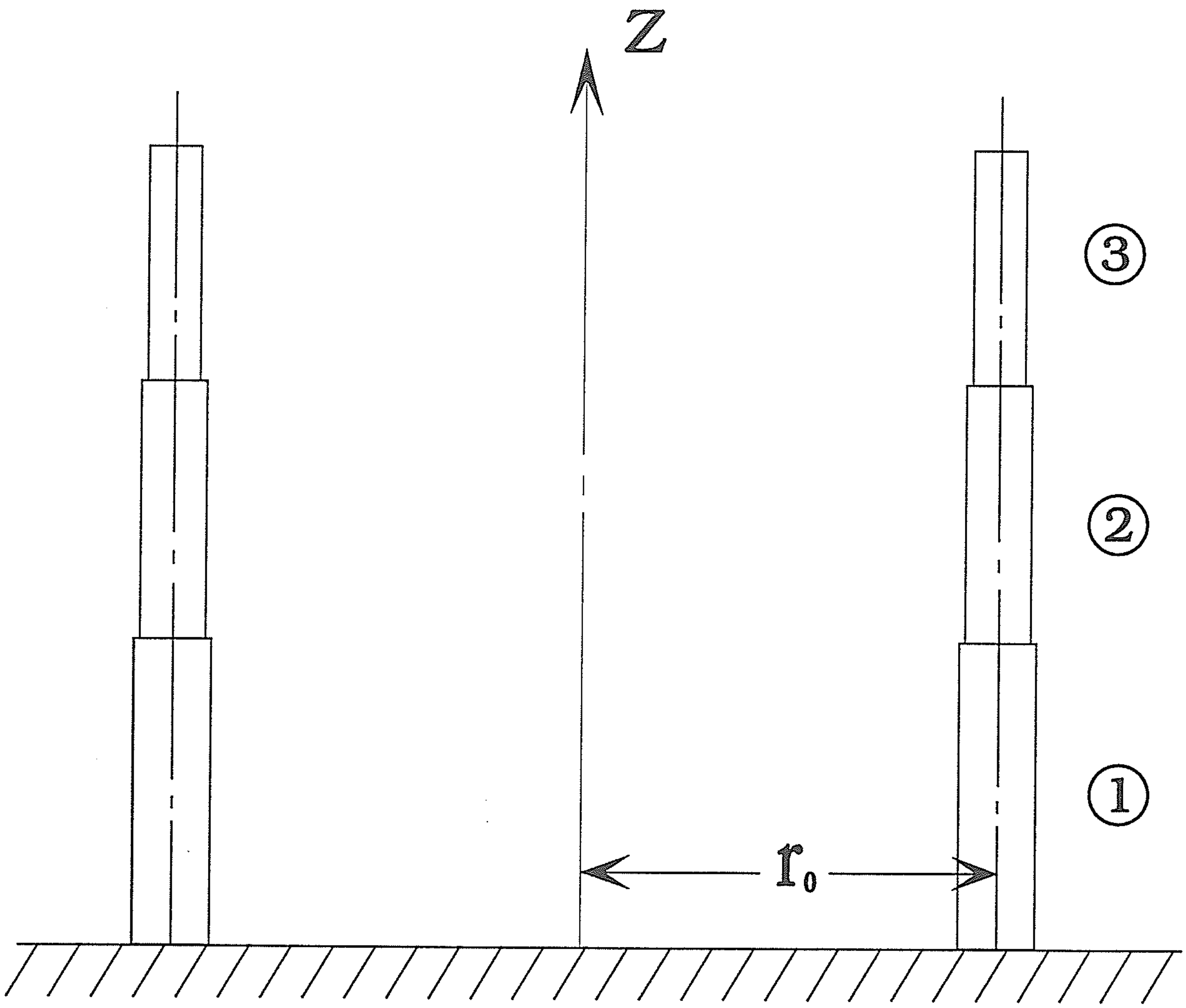


Figure 5 A three-element discretization of a tapered cylindrical tank.

substituting *Equation* (92) to *Equation* (93), we get:

$$\{R(z_2)\} = [P(z_2, t_{z_2})][P(z_1, t_{z_2})]^{-1}\{R(z_1)\} \tag{94}$$

For element [3]

at $z = z_2$:

$$\{R(z_2)\} = [P(z_2, t_{z_3})]\{C^3\} \tag{95}$$

at $z = z_3$:

$$\{R(z_3)\} = [P(z_3, t_{z_3})]\{C^3\} \tag{96}$$

From *Equations* (95)-(96), we have:

$$\{R(z_3)\} = [P(z_3, t_{z_3})][P(z_2, t_{z_3})]^{-1}\{R(z_2)\} \tag{97}$$

and using *Equations* (91),(94) and (97), the final expression is:

$$\begin{aligned}\{R(z_3)\} =&[P(z_3, t_{z_3})][P(z_2, t_{z_3})]^{-1}[P(z_2, t_{z_2})][P(z_1, t_{z_2})]^{-1}\\ &[P(z_1, t_{z_1})][P(z_0, t_{z_1})]^{-1}\{R(z_0)\}\end{aligned} \tag{98}$$

Similarly, the same method will be carried on to $n \to \infty$. In this way, we can get the following general expression:

$$\begin{aligned}\{R(z_n)\} =&[P(z_n, t_{z_n})][P(z_{n-1}, t_{z_n})]^{-1}[P(z_{n-1}, t_{z_{n-1}})][P(z_{n-2}, t_{z_{n-1}})]^{-1}\ldots\ldots\\ &[P(z_2, t_{z_2})][P(z_1, t_{z_2})]^{-1}[P(z_1, t_{z_1})][P(z_0, t_{z_1})]^{-1}\{R(z_0)\}\end{aligned} \tag{99}$$

The above expression can be written as:

$$\{R(z_n)\} = [S]\{R(z_0)\} \tag{100}$$

where $[S]$ is the resultant matrix of the products of a chain of matrices. $\{R(z_n)\}$ and $\{R(z_0)\}$ are the state vectors corresponding to node n and node 0 respectively. If D_z are the displacement and F_z are the forces and stresses, *Equation* (100) becomes:

$$\begin{Bmatrix} D_{z_n} \\ F_{z_n} \end{Bmatrix} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \begin{Bmatrix} D_{z_0} \\ F_{z_0} \end{Bmatrix} \tag{101}$$

where S_1, S_2, S_3 and S_4 are 4×4 submatrices. Using the boundary conditions at node 0 and node n the above equations are solved. At the bottom of the tank, the node 0 is fixed, therefore all displacement would be zero. At the top, the node n is free, all forces/stresses would be zero. Hence for such a case *Equation* (101) reduces to:

$$\begin{Bmatrix} D_{z_n} \\ 0 \end{Bmatrix} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \begin{Bmatrix} 0 \\ F_{z_0} \end{Bmatrix} \tag{102}$$

By partition the above matrix equation we can get:

$$\{D_{z_n}\} = [S_2]\{F_{z_0}\} \tag{103}$$

$$\{0\} = [S_4]\{F_{z_0}\} \tag{104}$$

From *Equation* (104) we can get the frequency equation is thus given by,

$$| S_4 | = 0 \tag{105}$$

Solving *Equation* (105) yields the natural frequencies, from which the associated mode shapes of the empty tank can be easily obtained.

3.2 Solution for Variable Wall Thickness for a Completely Filled Tank

For the case of a completely filled tank, we can use the same precedure as before and the results of Chapter 2 to get the following expression (for an arbitrary element):

$$\{\tilde{R}\} = [\tilde{P}]\{\tilde{C}\} \tag{106}$$

where:

$$\{\tilde{R}\} = \left\langle \tilde{U} \quad \tilde{V} \quad \tilde{W} \quad \frac{\partial \tilde{W}}{\partial z} \quad \tilde{N}_z \quad \tilde{N}_{z\theta} \quad \tilde{M}_z \quad \tilde{Q}_z \right\rangle^T \tag{107}$$

$$[\tilde{P}] = \begin{bmatrix} \tilde{p}_{11} & \tilde{p}_{12} & \tilde{p}_{13} & \cdots & \tilde{p}_{18} \\ \tilde{p}_{21} & \tilde{p}_{22} & \tilde{p}_{23} & \cdots & \tilde{p}_{28} \\ \tilde{p}_{31} & \tilde{p}_{32} & \tilde{p}_{33} & \cdots & \tilde{p}_{38} \\ \vdots & \vdots & \vdots & & \vdots \\ \tilde{p}_{81} & \tilde{p}_{82} & \tilde{p}_{83} & \cdots & \tilde{p}_{88} \end{bmatrix} \tag{108}$$

and

$$\{\tilde{C}\} = \left\langle \tilde{C}_1 \quad \tilde{C}_2 \quad \tilde{C}_3 \quad \tilde{C}_4 \quad \tilde{C}_5 \quad \tilde{C}_6 \quad \tilde{C}_7 \quad \tilde{C}_8 \right\rangle^T \tag{109}$$

Similarly, we can obtain the final matrix equation as:

$$\{\tilde{R}(z_n)\} = [\tilde{S}]\{\tilde{R}(z_0)\} \tag{110}$$

where $[\tilde{S}]$ is the resultant matrix of the products of a chain of matrices. The above equation can also be written as:

$$\left\{ \begin{array}{c} \tilde{D}_{z_n} \\ \tilde{F}_{z_n} \end{array} \right\} = \begin{bmatrix} \tilde{S}_1 & \tilde{S}_2 \\ \tilde{S}_3 & \tilde{S}_4 \end{bmatrix} \left\{ \begin{array}{c} \tilde{D}_{z_0} \\ \tilde{F}_{z_0} \end{array} \right\} \tag{111}$$

Imposing the boundary conditions at two ends, we have the frequency equation:

$$\mid \tilde{S}_4 \mid = 0 \tag{112}$$

Solving *Equation* (112) yields the natural frequencies, and the associated mode shapes of the completely filled tank.

3.3 Solution for Variable Wall Thickness for a Partially-filled Tank

For partially-filled tank solution, as shown in *Figure* 6, a six-element discretization of the tank is adopted. The formulation involves specializing the results for the constant wall tank to compute the dry solution, the wet solution and the total solution.

(a) *The dry solution.* The empty tank solution of the previous section can be expressed as following,

$$\begin{aligned} \{R(z_n)\} =& [P(z_n, t_{z_n})][P(z_{n-1}, t_{z_n})]^{-1}[P(z_{n-1}, t_{z_{n-1}})][P(z_{n-2}, t_{z_{n-1}})]^{-1} \ldots\ldots \\ & [P(z_{m+2}, t_{z_{m+2}})][P(z_{m+1}, t_{z_{m+2}})]^{-1}[P(z_{m+1}, t_{z_{m+1}})][P(z_m, t_{z_{m+1}})]^{-1} \\ & \{R(z_m)\} \end{aligned} \tag{113}$$

or in matrix form:

$$\begin{Bmatrix} D_{d_{z_n}} \\ F_{d_{z_n}} \end{Bmatrix} = \begin{bmatrix} S_{d_1} & S_{d_2} \\ S_{d_3} & S_{d_4} \end{bmatrix} \begin{Bmatrix} D_{d_{z_m}} \\ F_{d_{z_m}} \end{Bmatrix} \tag{114}$$

(b) *The wet solution.* In a similar fashion, solution of the wet part using m elements, is given by:

$$\begin{aligned} \{\tilde{R}(z_m)\} =& [\tilde{P}(z_m, t_{z_m})][\tilde{P}(z_{m-1}, t_{z_m})]^{-1}[\tilde{P}(z_{m-1}, t_{z_{m-1}})][\tilde{P}(z_{m-2}, t_{z_{m-1}})]^{-1} \ldots\ldots \\ & [\tilde{P}(z_2, t_{z_2})][\tilde{P}(z_1, t_{z_2})]^{-1}[\tilde{P}(z_1, t_{z_1})][\tilde{P}(z_0, t_{z_1})]^{-1}\{\tilde{R}(z_0)\} \end{aligned} \tag{115}$$

or written in matrix form:

$$\begin{Bmatrix} \tilde{D}_{w_{z_m}} \\ \tilde{F}_{w_{z_m}} \end{Bmatrix} = \begin{bmatrix} \tilde{S}_{w_1} & \tilde{S}_{w_2} \\ \tilde{S}_{w_3} & \tilde{S}_{w_4} \end{bmatrix} \begin{Bmatrix} \tilde{D}_{w_{z_0}} \\ \tilde{F}_{w_{z_0}} \end{Bmatrix} \tag{116}$$

(c) *The total solution.* Invoking continuity at the element interface, the total solution is,

$$\begin{aligned} \{R(z_n)\} =& [P(z_n, t_{z_n})][P(z_{n-1}, t_{z_n})]^{-1}[P(z_{n-1}, t_{z_{n-1}})][P(z_{n-2}, t_{z_{n-1}})]^{-1} \ldots\ldots \\ & [P(z_{m+2}, t_{z_{m+2}})][P(z_{m+1}, t_{z_{m+2}})]^{-1}[P(z_{m+1}, t_{z_{m+1}})][P(z_m, t_{z_{m+1}})]^{-1} \\ & [\tilde{P}(z_m, t_{z_m})][\tilde{P}(z_{m-1}, t_{z_m})]^{-1}[\tilde{P}(z_{m-1}, t_{z_{m-1}})][\tilde{P}(z_{m-2}, t_{z_{m-1}})]^{-1} \ldots\ldots \\ & [\tilde{P}(z_2, t_{z_2})][\tilde{P}(z_1, t_{z_2})]^{-1}[\tilde{P}(z_1, t_{z_1})][\tilde{P}(z_0, t_{z_1})]^{-1}\{\tilde{R}(z_0)\} \end{aligned} \tag{117}$$

or written in matrix form:

$$\begin{Bmatrix} D_{d_{z_n}} \\ F_{d_{z_n}} \end{Bmatrix} = \begin{bmatrix} S_{d_1} & S_{d_2} \\ S_{d_3} & S_{d_4} \end{bmatrix} \begin{bmatrix} \tilde{S}_{w_1} & \tilde{S}_{w_2} \\ \tilde{S}_{w_3} & \tilde{S}_{w_4} \end{bmatrix} \begin{Bmatrix} \tilde{D}_{w_{z_0}} \\ \tilde{F}_{w_{z_0}} \end{Bmatrix} \tag{118}$$

At the bottom of the tank, all displacement boundary conditions are zero, i.e. $\tilde{D}_{wz0} = 0$ and at the top, all force boundary conditions are zero, namely, $F_{dzn} = 0$. Hence, for such a case *Equation* (118) simplifies to

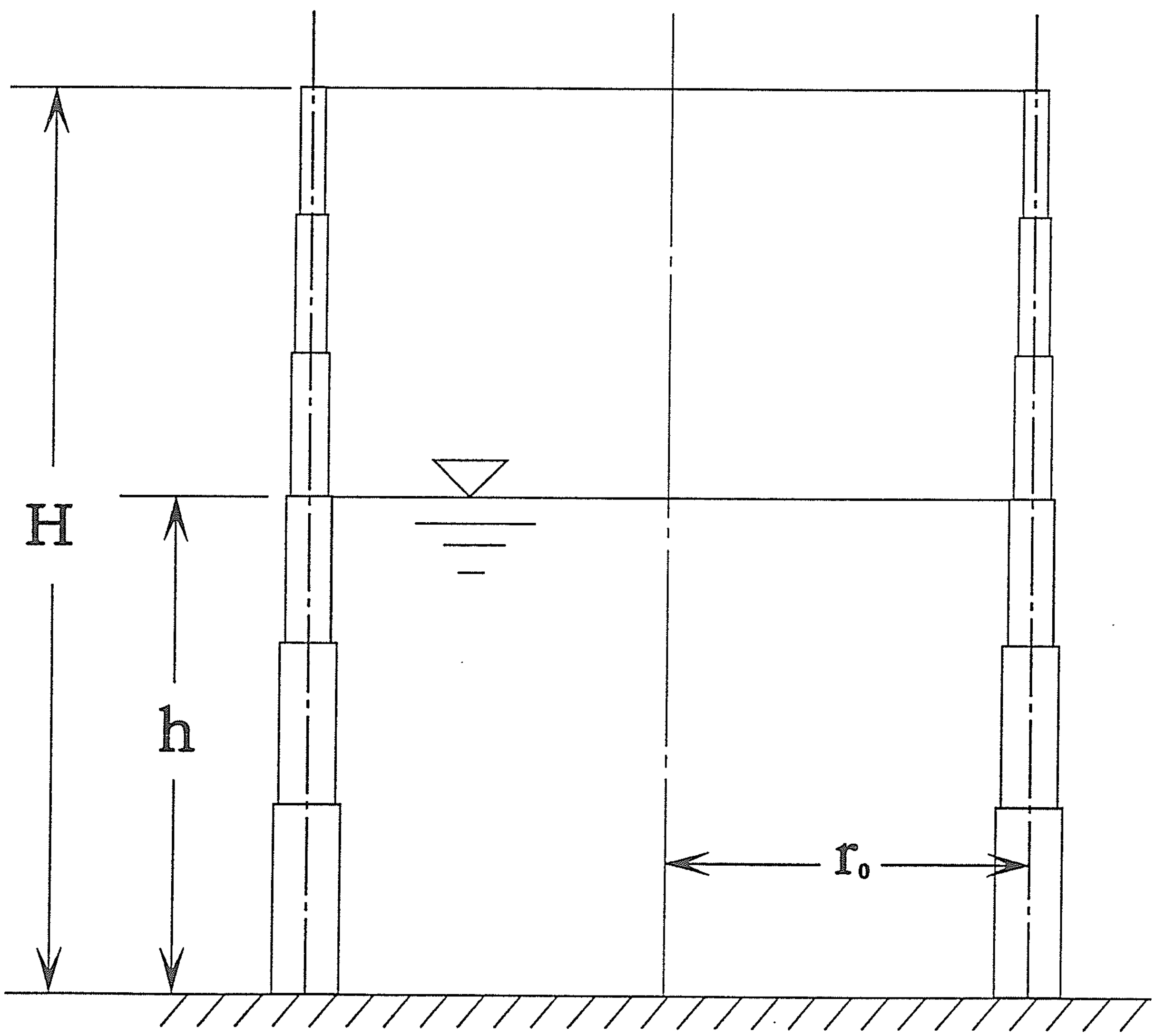


Figure 6 A six-element discretization of a tapered cylindrical tank.

$$\{D_{d_{zn}}\} = [S_{d_1}\tilde{S}_{w_2} + S_{d_2}\tilde{S}_{w_4}]\{\tilde{F}_{w_{z0}}\} \tag{119}$$

$$\{0\} = [S_{d_3}\tilde{S}_{w_2} + S_{d_4}\tilde{S}_{w_4}]\{\tilde{F}_{w_{z0}}\} \tag{120}$$

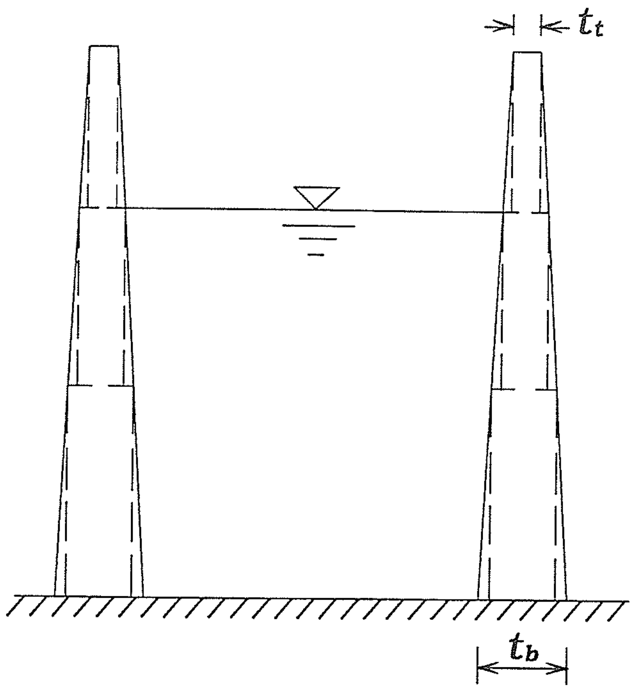
From *Equation* (120), the frequency equation is thus given by,

$$| \, S_{d_3}\tilde{S}_{w_2} + S_{d_4}\tilde{S}_{w_4} \, | = 0 \tag{121}$$

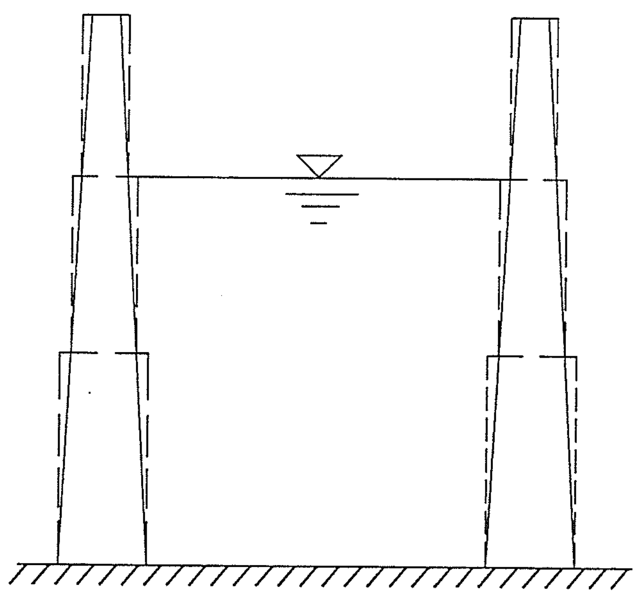
Solving *Equation* (121) yields the natural frequencies, and the associated mode shapes of the coupled fluid-elastic tank system.

3.4 Illustrative Numerical Examples and Parametric Study

Using the procedure formulated for the constant wall tank, the analysis is extended via the transfer matrix approach, to investigate the variable wall tank. Although our proposed technique can handle any types of axial non-uniformity in the wall thickness, we will assume for this simulation, a linear thickness variation. As given in *Table* 1, the thickness varies from t_t = 2.54 cm (1 in) at the top, to a bottom thickness $t_b = 5t_t$. Six elements of constant thickness each, are employed in the discretization of the tank, regardless of whether it is a short or tall tank. The discretization schemes used involve 'inner' and 'outer' elements as illustrated in *Figure* 7. Once again, the three fluid loading conditions corresponding to an empty tank, a completely filled tank and a partially filled tank are analyzed. For the partially filled tank, the fluid depth to tank height ratio h/H assumes values of 0.4, 0.6 and 0.8 in our analysis. As in the case of the constant thickness solutions, convergence is achieved after 20 iterations, with the number of terms used in the summation being 10.



(a) Inside Element Discretization



(b) Outside Element Discretization

Figure 7 Inner and outer element discretization schemes.

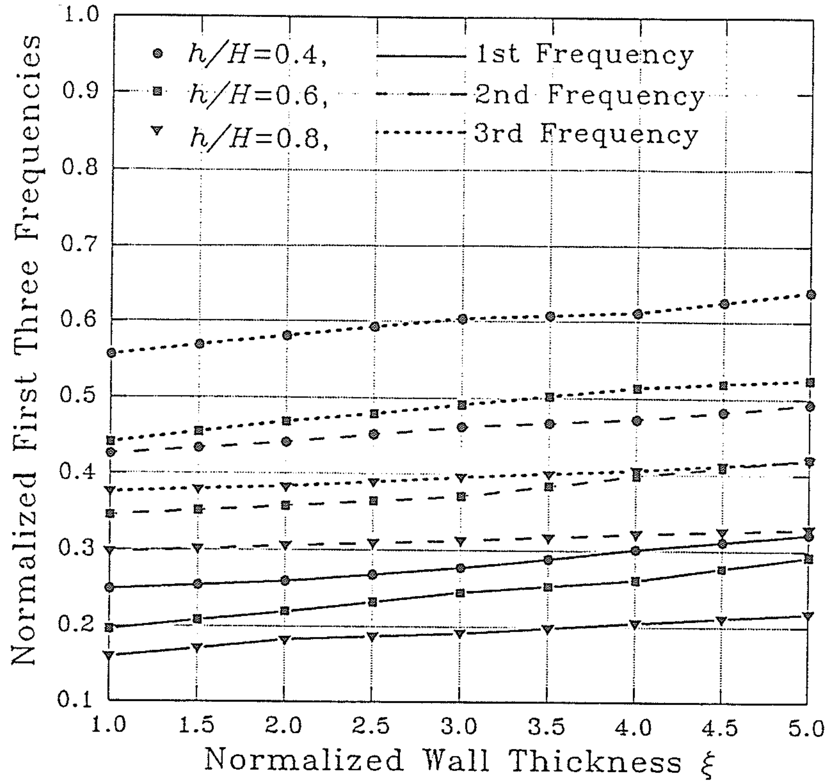
The first three frequencies for short and tall tanks, for varying depths of fluid ($h/H = 0.4, 0.6, 0.8$), are given in *Figure* 8. The frequencies of short tanks are normalized with respect to the fundamental frequency of an empty short tank whereas, for tall tanks, they are normalized with respect to the fundamental frequency of an empty tall tank. As for the wall thickness, it is normalized by defining $\xi = t_b/t_t$. From the figure, it is obvious that as the level of water in the tank increases, the natural frequency decreases. This is understandable since the mass of the fluid-structure system increases with the water level while the structural stiffness remains constant. Note that as the wall thickness increases, the natural frequencies increases and this is consistent with the fact the tank is now becoming more and more stubby.

It would be interesting to compare the prediction of the eigenquantities of a variable thickness tank with those of an equivalent constant thickness tank. By *equivalent*, we mean choosing its wall thickness so that it is equal to the average thickness of the variable thickness tank. In our example, the variable thickness tank has a top thickness $t_t = 2.54$ cm (1 in) which increases linearly to a bottom thickness of $t_b = 2t_t$. Thus, the wall thickness of the equivalent constant thickness tank is 3.81 cm (1.5 in). The frequency results are summarized in *Tables* 5-7 for an empty tank, a completely-filled tank and a partially-filled tank ($h/H = 0.6$) respectively. As shown, the analysis of the variable thickness tank is carried out using both the 'inside' element discretization scheme and the 'outside' element discretization scheme. Since the 'inside' element mesh produces a slightly more slender structure than the 'outside' element mesh, the frequency predictions of the former are slightly lower than the latter. To get a fair comparison with the solutions of the constant wall thickness tank, the mean value of these 'inside' and 'outside' frequency results is computed. From these results, with frequency predictions under 5% difference, it is clear that the equivalent constant thickness tank is a very good approximation of the variable thickness tank for a linearly-varying wall thickness. One could also argue that since tions of the constant thickness tank have already been checked against published results, this excellent agreement in a way, verifies the method formulated for analyzing a variable thickness tank.

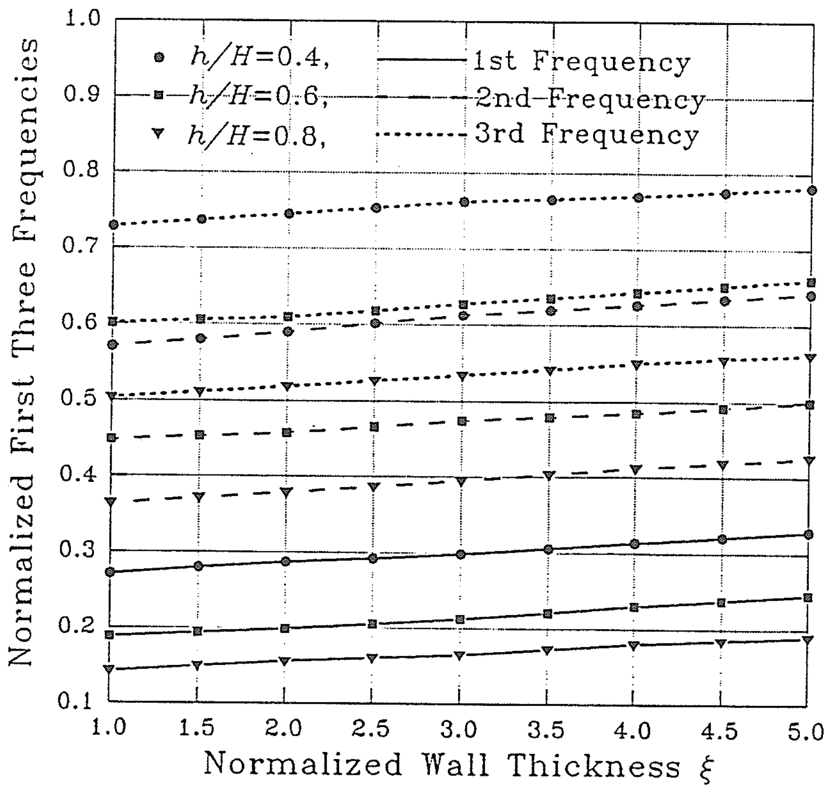
Table 5 *Natural frequencies of empty, variable and constant thickness tanks.*

Tank Type	Mode Number	Natural Frequency (Hz) Inside Element	Outside Element	Average	Constant Thickness
Short	1	45.010	45.438	45.224	45.050
	2	45.350	45.800	45.575	45.360
	3	45.600	46.100	45.850	45.420
Tall	1	58.210	58.724	58.467	58.300
	2	109.011	109.200	109.106	108.990
	3	111.850	112.400	112.125	111.771

The axial and radial displacement mode shapes associated with the fundamental natural frequency for the variable and the equivalent constant thickness tanks mentioned previously are sketched in *Figures* 9-11. These plots correspond respectively, to the three fluid loading conditions of an empty tank, a completely-filled tank and a partially-filled tank ($h/H = 0.6$) for both short and tall tanks. Examination of these figures reveal that in the case of the tall tank, both its axial and radial displacement mode shapes for the empty tank and the completely-filled tank are very similar. This is not true for the short tank where its mode shapes are quite different, particularly the radial displacement mode shape. This observation is true for the variable thickness tank, as well as the constant thickness tank. On the other hand, the free vibration mode shapes of the partially-filled tank ($h/H = 0.6$) as depicted, are different from those of either the empty tank or the completely-filled tank which is expected. Note also, the very good agreement



(a) SHORT TANK



(b) TALL TANK

Figure 8 Normalized natural frequencies partially-filled tanks (h/H = 04, 0.6, 0.8).

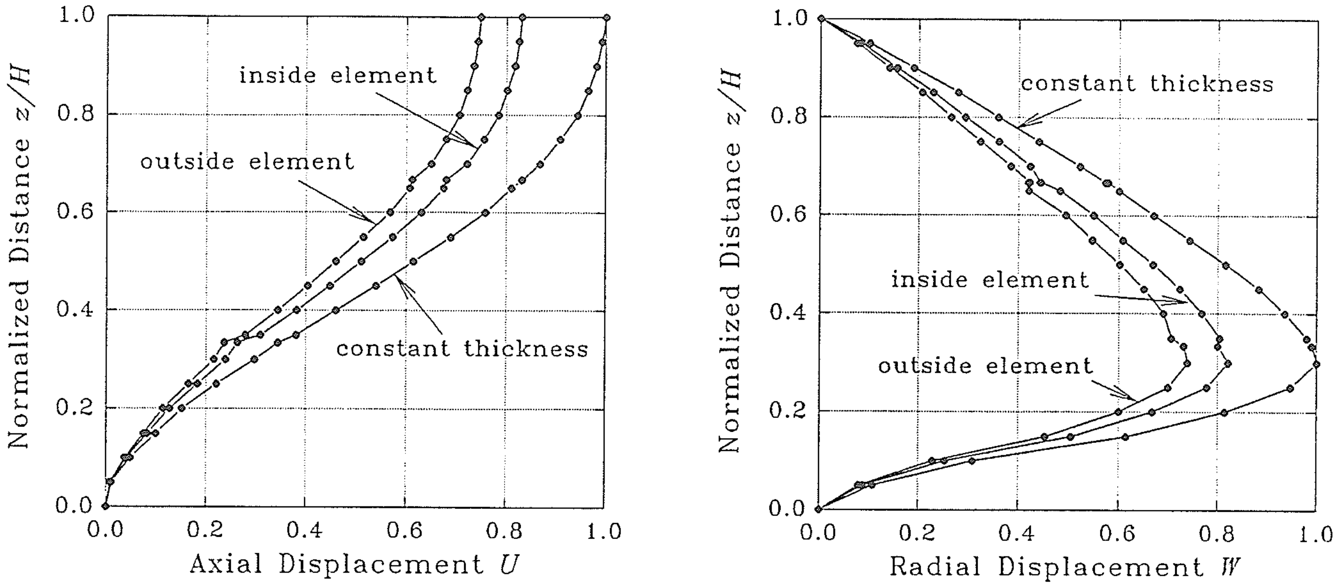
that is obtained in the mode shapes between the variable thickness tank and the equivalent constant thickness tank.

Table 6 Natural frequencies of completely-filled, variable and constant thickness tanks.

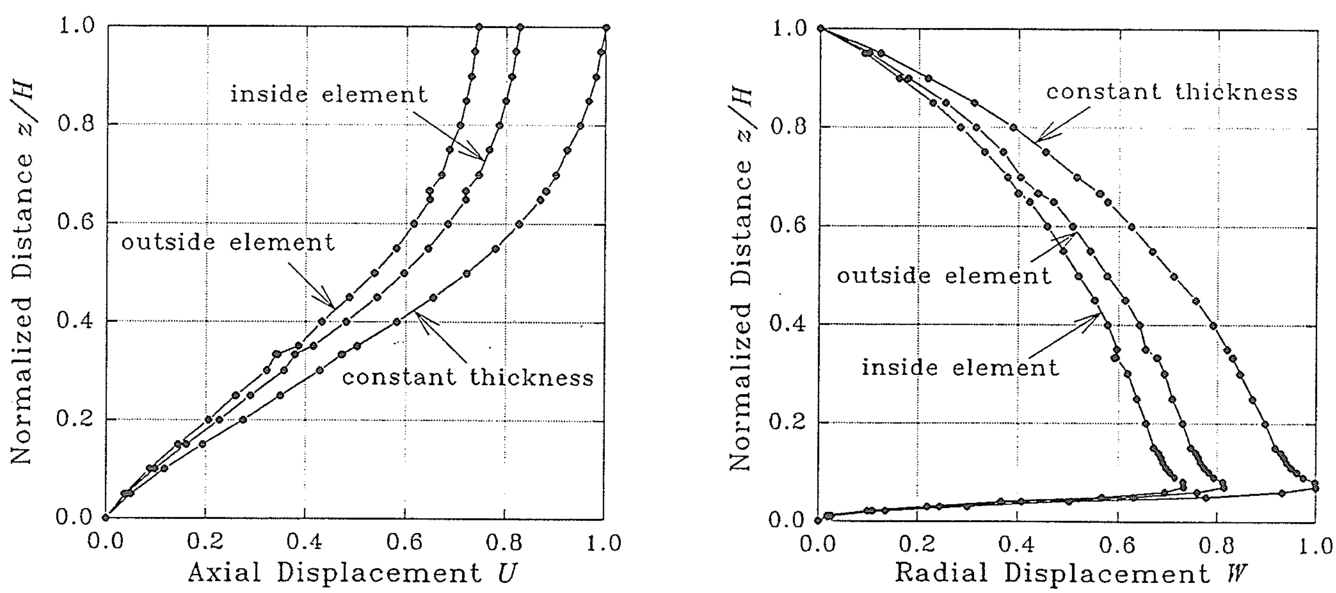
Tank Type	Mode Number	Natural Frequency (Hz)			
		Inside Element	Outside Element	Average	Constant Thickness
Short	1	7.000	7.511	7.455	7.120
	2	11.750	12.746	12.248	11.980
	3	15.695	16.100	15.898	15.540
Tall	1	7.080	7.700	7.390	7.145
	2	18.310	18.650	18.480	18.230
	3	26.340	27.951	27.146	26.635

Table 7 Natural frequencies of partially-filled, variable and constant thickness tanks ($h/H = 0.6$).

Tank Type	Mode Number	Natural Frequency (Hz)			
		Inside Element	Outside Element	Average	Constant Thickness
Short	1	9.419	9.781	9.600	9.210
	2	15.890	16.400	16.145	15.940
	3	20.600	20.858	20.729	20.342
Tall	1	11.040	11.910	11.475	11.250
	2	26.010	26.850	26.430	26.246
	3	34.900	35.499	35.200	34.910



(a) SHORT TANK



(b) TALL TANK

Figure 9 Comparison of mode shapes between an empty tank of variable thickness and constant thickness.

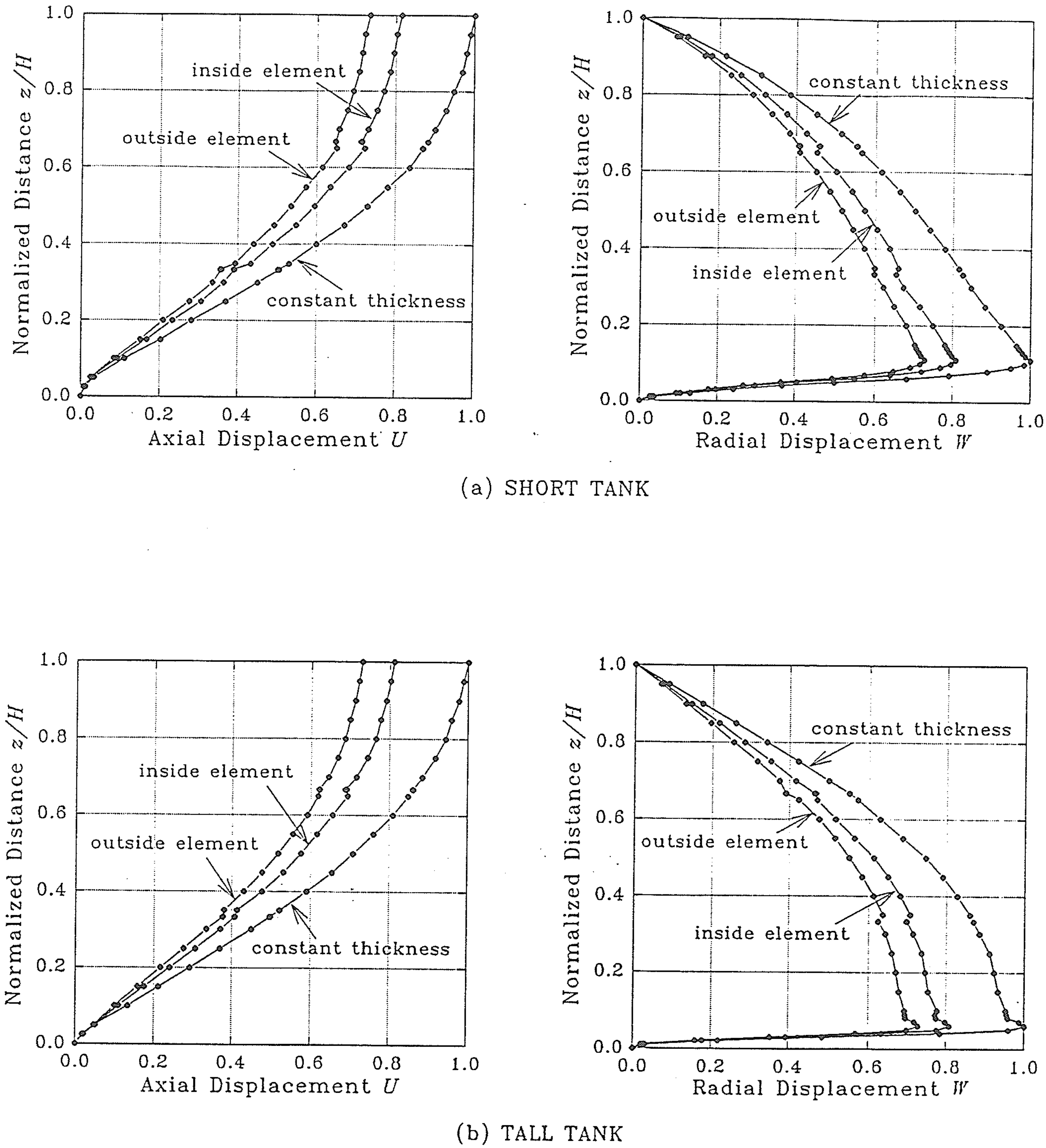
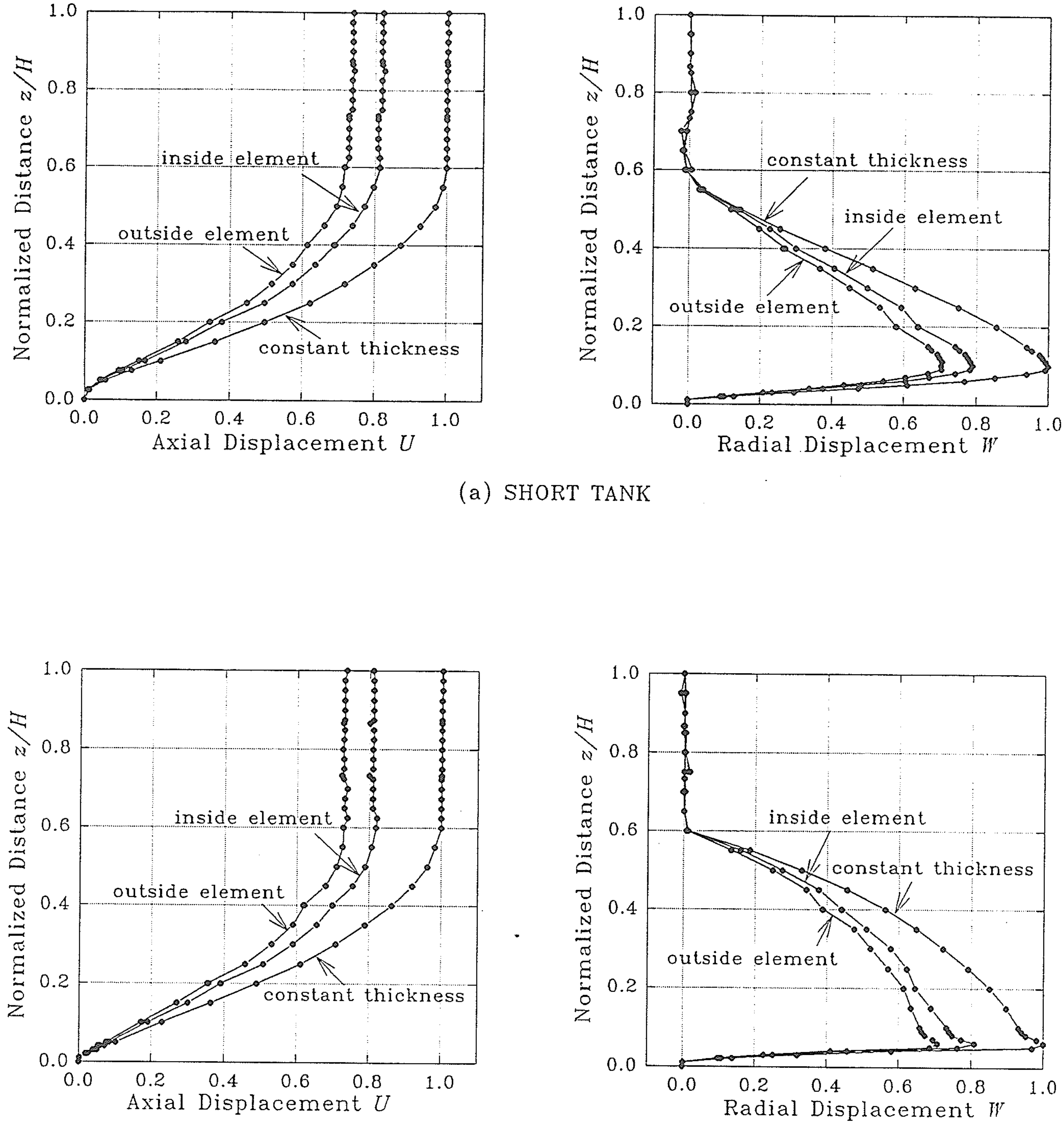


Figure 10 Comparison of mode shapes between a completely-filled tank of variable thickness and constant thickness.



(a) SHORT TANK

(b) TALL TANK

Figure 11 Comparison of mode shapes between a partially-filled tank of variable thickness and constant thickness.

CHAPTER 4

SUMMARY AND CONCLUSIONS

An analytical method for the free vibration analysis of axially non-uniform cylindrical storage tanks, partially filled with fluid is presented. The fluid is modelled as an inviscid, incompressible fluid governed by the potential flow theory while the tank is based on Flugge's thin shell theory. The coupled partial differential equations for this fluid-structure interaction problem, subject to the sixteen boundary and continuity conditions, are solved exactly for the case of a constant wall thickness tank. Using this solution as the basis for analyzing a variable wall thickness tank which is discretized into a series of elements, each of constant thickness, a general procedure via the transfer matrix approach, is suggested. To assess the accuracy of the constant thickness solutions, comparison of the eigenquantities with published results is shown. Parametric studies involving a variable thickness tank whose wall thickness is linearly-varying and an equivalent constant wall thickness tank are carried out. The results demonstrate versatility and accuracy of the proposed technique.

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APPENDIX I

NOMENCLATURE

A_j, B_j, C_j	arbitrary coefficients of the homogeneous solution
a_{ij}	elements of the matrix defined by *Equation* (29)
b_{ij}	elements of the matrix defined by *Equation* (46)
$C_{11}, ..., C_{55}$	coefficients of the charateristic equation given by *Equation* (31)
D	$=Et_z/(1-\nu^2)$
E	Young's modulus
f_m	coefficient defined in *Equation* (50)
G_{1m}, G_{2m}, G_{3m}	coefficients defined in (42)-(44)
H	height of tank
h	liquid depth
I_n	modified Bessel function
K	$=Et_z^3/12(1-\nu^2)$
k_{ij}	elements of the matrix defined by *Equation* (65)
n	number of circumferential waves
P	hydrodynamic pressure
p_{ij}	elements of the matrix defined by *Equation* (86)
$\tilde{p}_{ij}$	elements of the matrix defined by *Equation* (108)
r_0	radius of tank
t_z	shell thickness
U, V, W	shell displacements
α_j, β_j	coefficients defined by *Equation* (34)

σ	characteristic roots defined by *Equation* (31)
λ_m	eigenvalues
ρ_s	mass density of shell
ρ_f	mass density of fluid
ϕ	liquid velocity potential
ω	natural frequency of the system

APPENDIX II

VARIOUS TERMS OF THE CHARACTERISTIC EQUATION

The following terms refer to the characteristic equation given in *Equation* (31):

$$C_{11} = l_5(l_9 - l_3^2)$$

$$C_{22} = l_5 l_{10} + l_9(l_6 + l_1 l_5) + l_7^2 + l_2^2 l_9 + 2l_2 l_3 l_7 - l_3^2 l_6 - 2l_3 l_4 l_5$$

$$C_{33} = l_5 l_{11} + l_{10}(l_6 + l_1 l_5) + l_1 l_6 l_9 + 2l_7 l_8 + l_1 l_7^2 + l_2^2 l_{10} + 2l_2 l_3 l_8 + 2l_2 l_4 l_7 - 2l_3 l_4 l_6 - l_4^2 l_5$$

$$C_{44} = l_{11}(l_6 + l_1 l_5) + l_1 l_6 l_{10} + l_8^2 + 2l_1 l_7 l_8 + l_2^2 l_{11} + 2l_2 l_4 l_8 - l_4^2 l_6$$

$$C_{55} = l_1 l_6 l_{11} + l_1 l_8^2$$

where

$$l_1 = \frac{\rho_s(1-\nu^2)\omega^2}{E} - \frac{(1-\nu)n^2}{2r_0^2}(1 + \frac{t_z^2}{12r_0^2})$$

$$l_2 = \frac{n(1+\nu)}{2r_0}$$

$$l_3 = -\frac{t_z^2}{12r_0}$$

$$l_4 = \frac{\nu}{r_0} - \frac{(1-\nu)t_z^2 n^2}{24r_0^3}$$

$$l_5 = \frac{1-\nu}{2}(1 + \frac{t_z^2}{4r_0^2})$$

$$l_6 = \frac{\rho_s(1-\nu^2)\omega^2}{E} - \frac{n^2}{r_0^2}$$

$$l_7 = \frac{(3-\nu)nt_z^2}{24r_0^2}$$

$$l_8 = -\frac{n}{r_0^2}$$

$$l_9 = \frac{t_z^2}{12}$$

$$l_{10} = -\frac{n^2t_z^2}{6r_0^2}$$

$$l_{11} = \frac{1}{r_0^2} + \frac{t_z^2}{12r_0^4} + \frac{n^4t_z^2}{12r_0^4} - \frac{n^2t_z^2}{6r_0^4} - \frac{\rho_s(1-\nu^2)\omega^2}{E}$$

APPENDIX III

EXPLICIT EXPRESSIONS FOR THE VARIOUS MATRICES

The following terms refer to the matrix defined by *Equation* (29):

$$a_{11} = \sigma^2 + \frac{\rho_s(1-\nu^2)\omega^2}{E} - \frac{(1-\nu)n^2}{2r_0^2}(1 + \frac{t_z^2}{12r_0^2})$$

$$a_{12} = \frac{n(1+\nu)\sigma}{2r_0}$$

$$a_{13} = -\frac{t_z^2\sigma^3}{12r_0} + [\frac{\nu}{r_0} - \frac{(1-\nu)t_z^2n^2}{24r_0^3}]\sigma$$

$$a_{21} = -a_{12}$$

$$a_{22} = \frac{1-\nu}{2}(1 + \frac{t_z^2}{4r_0^2})\sigma^2 + [\frac{\rho_s(1-\nu^2)\omega^2}{E} - \frac{n^2}{r_0^2}]$$

$$a_{23} = \frac{(3-\nu)nt_z^2}{24r_0^2}\sigma^2 - \frac{n}{r_0^2}$$

$$a_{31} = a_{13}$$

$$a_{32} = -a_{23}$$

$$a_{33} = \frac{t_z^2\sigma^4}{12} - \frac{n^2t_z^2\sigma^2}{6r_0^2} + [\frac{1}{r_0^2} + \frac{t_z^2}{12r_0^4} + \frac{n^4t_z^2}{12r_0^4} - \frac{n^2t_z^2}{6r_0^4} - \frac{\rho_s(1-\nu^2)\omega^2}{E}]$$

The following terms refer to the matrix defined by *Equation* (46):

$$b_{11} = \frac{\rho_s(1-\nu^2)\omega^2}{E} - \lambda_m^2 - \frac{(1-\nu)n^2}{2r_0^2}(1 + \frac{t_z^2}{12r_0^2})$$

$$b_{12} = -\frac{n(1+\nu)\lambda_m}{2r_0}$$

$$b_{13} = -\frac{t_z^2\lambda_m^3}{12r_0} + [-\frac{\nu}{r_0} + \frac{(1-\nu)t_z^2n^2}{24r_0^3}]\lambda_m$$

$$b_{21} = b_{12}$$

$$b_{22} = -\frac{1-\nu}{2}(1 + \frac{t_z^2}{4r_0^2})\lambda_m^2 + [\frac{\rho_s(1-\nu^2)\omega^2}{E} - \frac{n^2}{r_0^2}]$$

$$b_{23} = -\frac{(3-\nu)nt_z^2}{24r_0^2}\lambda_m^2 - \frac{n}{r_0^2}$$

$$b_{31} = -b_{13}$$

$$b_{32} = -b_{23}$$

$$b_{33} = \frac{t_z^2\lambda_m^4}{12} + \frac{n^2t_z^2\lambda_m^2}{6r_0^2} + [\frac{1}{r_0^2} + \frac{t_z^2}{12r_0^4} + \frac{n^4t_z^2}{12r_0^4} - \frac{n^2t_z^2}{6r_0^4} - \frac{\rho_s(1-\nu^2)\omega^2}{E}]$$

The following terms refer to the matrix defined by *Equation* (65):

$$k_{11} = \alpha_1 \qquad k_{12} = \alpha_2 \qquad \ldots \qquad k_{18} = \alpha_8$$

$$k_{19} = 0 \qquad k_{110} = 0 \qquad \ldots \qquad k_{116} = 0$$

$$k_{21} = \beta_1 + \sum_{m=1}^{\infty} B_2P_mT_{1m} \qquad \ldots \qquad k_{28} = \beta_8 + \sum_{m=1}^{\infty} B_2P_mT_{8m}$$

$$k_{29} = 0 \qquad k_{210} = 0 \qquad \ldots \qquad k_{216} = 0$$

$$k_{31} = 1 + \sum_{m=1}^{\infty} B_3 P_m T_{1m} \qquad \ldots \qquad k_{38} = 1 + \sum_{m=1}^{\infty} B_3 P_m T_{8m}$$

$$k_{39} = 0 \qquad k_{310} = 0 \qquad \ldots \qquad k_{316} = 0$$

$$k_{41} = \sigma_1 \qquad k_{42} = \sigma_2 \qquad \ldots \qquad k_{48} = \sigma_8$$

$$k_{49} = 0 \qquad k_{410} = 0 \qquad \ldots \qquad k_{416} = 0$$

$$k_{51} = 0 \qquad k_{52} = 0 \qquad \ldots \qquad k_{58} = 0$$

$$k_{59} = (\alpha_1 \sigma_1 + \frac{\nu}{r_0}\beta_1 n + \frac{\nu}{r_0}) \qquad \ldots \qquad k_{516} = (\alpha_8 \sigma_8 + \frac{\nu}{r_0}\beta_8 n + \frac{\nu}{r_0})$$

$$k_{61} = 0 \qquad k_{62} = 0 \qquad \ldots \qquad k_{68} = 0$$

$$k_{69} = (\beta_1 \sigma_1 - \frac{\alpha_1 n}{r_0}) \qquad \ldots \qquad k_{616} = (\beta_8 \sigma_8 - \frac{\alpha_8 n}{r_0})$$

$$k_{71} = 0 \qquad k_{72} = 0 \qquad \ldots \qquad k_{78} = 0$$

$$k_{79} = (\sigma_1^2 - \frac{\nu n^2}{r_0^2}) \qquad \ldots \qquad k_{716} = (\sigma_8^2 - \frac{\nu n^2}{r_0^2})$$

$$k_{81} = 0 \qquad k_{82} = 0 \qquad \ldots \qquad k_{88} = 0$$

$$k_{89} = (\sigma_1^3 - \frac{\nu n^2 \sigma_1}{r_0^2}) \qquad \ldots \qquad k_{816} = (\sigma_8^3 - \frac{\nu n^2 \sigma_8}{r_0^2})$$

$$k_{91} = \alpha_1 e^{\sigma_1 h} + \sum_{m=1}^{\infty} B_1 P_m T_{1m} \sin \lambda_m h \qquad \ldots \qquad k_{98} = \alpha_8 e^{\sigma_8 h} + \sum_{m=1}^{\infty} B_1 P_m T_{8m} \sin \lambda_m h$$

$$k_{99} = -\alpha_1 e^{\sigma_1 h} \qquad \ldots \qquad k_{916} = -\alpha_8 e^{\sigma_8 h}$$

$$k_{101} = \beta_1 e^{\sigma_1 h} + \sum_{m=1}^{\infty} B_2 P_m T_{1m} \cos \lambda_m h \qquad \ldots \qquad k_{108} = \beta_8 e^{\sigma_8 h} + \sum_{m=1}^{\infty} B_2 P_m T_{8m} \cos \lambda_m h$$

$$k_{109} = -\beta_1 e^{\sigma_1 h} \qquad \ldots \qquad k_{1016} = -\beta_8 e^{\sigma_8 h}$$

$$k_{111} = e^{\sigma_1 h} + \sum_{m=1}^{\infty} B_3 P_m T_{1m} \cos \lambda_m h \qquad \ldots \qquad k_{118} = e^{\sigma_8 h} + \sum_{m=1}^{\infty} B_3 P_m T_{8m} \cos \lambda_m h$$

$$k_{119} = -e^{\sigma_1 h} \qquad \ldots \qquad k_{1116} = -e^{\sigma_8 h}$$

$$k_{121} = \sigma_1 e^{\sigma_1 h} - \sum_{m=1}^{\infty} B_3 P_m T_{1m} \lambda_m \sin \lambda_m h \qquad \ldots$$

$$k_{128} = \sigma_8 e^{\sigma_8 h} - \sum_{m=1}^{\infty} B_3 P_m T_{8m} \lambda_m \sin \lambda_m h$$

$$k_{129} = -\sigma_1 e^{\sigma_1 h} \qquad \ldots \qquad k_{1216} = -\sigma_8 e^{\sigma_8 h}$$

$$k_{131} = (\alpha_1 \sigma_1 + \frac{\nu}{r_0} \beta_1 n + \frac{\nu}{r_0}) e^{\sigma_1 h} + \sum_{m=1}^{\infty} (B_1 \lambda_m + \frac{\nu}{r_0} B_2 n + \frac{\nu}{r_0} B_3) P_m T_{1m} \cos \lambda_m h \qquad \ldots$$

$$k_{138} = (\alpha_8 \sigma_8 + \frac{\nu}{r_0} \beta_8 n + \frac{\nu}{r_0}) e^{\sigma_8 h} + \sum_{m=1}^{\infty} (B_1 \lambda_m + \frac{\nu}{r_0} B_2 n + \frac{\nu}{r_0} B_3) P_m T_{8m} \cos \lambda_m h$$

$$k_{139} = -(\alpha_1 \sigma_1 + \frac{\nu}{r_0} \beta_1 n + \frac{\nu}{r_0}) e^{\sigma_1 h} \qquad \ldots \qquad k_{1316} = -(\alpha_8 \sigma_8 + \frac{\nu}{r_0} \beta_8 n + \frac{\nu}{r_0}) e^{\sigma_8 h}$$

$$k_{141} = (\beta_1 \sigma_1 - \frac{\alpha_1 n}{r_0}) e^{\sigma_1 h} - \sum_{m=1}^{\infty} (B_2 \lambda_m + \frac{B_1 n}{r_0}) P_m T_{1m} \sin \lambda_m h \qquad \ldots$$

$$k_{148} = (\beta_8\sigma_8 - \frac{\alpha_8 n}{r_0})e^{\sigma_8 h} - \sum_{m=1}^{\infty}(B_2\lambda_m + \frac{B_1 n}{r_0})P_m T_{8m}\sin\lambda_m h$$

$$k_{149} = -(\beta_1\sigma_1 - \frac{\alpha_1 n}{r_0})e^{\sigma_1 h} \qquad \ldots \qquad k_{1416} = -(\beta_8\sigma_8 - \frac{\alpha_8 n}{r_0})e^{\sigma_8 h}$$

$$k_{151} = (\sigma_1^2 - \frac{\nu n^2}{r_0^2})e^{\sigma_1 h} - \sum_{m=1}^{\infty}(\lambda_m^2 + \frac{n^2\nu}{r_0^2})B_3 P_m T_{1m}\cos\lambda_m h \qquad \ldots$$

$$k_{158} = (\sigma_8^2 - \frac{\nu n^2}{r_0^2})e^{\sigma_8 h} - \sum_{m=1}^{\infty}(\lambda_m^2 + \frac{n^2\nu}{r_0^2})B_3 P_m T_{8m}\cos\lambda_m h$$

$$k_{159} = -(\sigma_1^2 - \frac{\nu n^2}{r_0^2})e^{\sigma_1 h} \qquad \ldots \qquad k_{1516} = -(\sigma_8^2 - \frac{\nu n^2}{r_0^2})e^{\sigma_8 h}$$

$$k_{161} = (\sigma_1^3 - \frac{\nu n^2\sigma_1}{r_0^2})e^{\sigma_1 h} + \sum_{m=1}^{\infty}(\lambda_m^3 + \frac{n^2\nu\lambda_m}{r_0^2})B_3 P_m T_{1m}\sin\lambda_m h \qquad \ldots$$

$$k_{168} = (\sigma_8^3 - \frac{\nu n^2\sigma_8}{r_0^2})e^{\sigma_8 h} + \sum_{m=1}^{\infty}(\lambda_m^3 + \frac{n^2\nu\lambda_m}{r_0^2})B_3 P_m T_{8m}\sin\lambda_m h$$

$$k_{169} = -(\sigma_1^3 - \frac{\nu n^2\sigma_1}{r_0^2})e^{\sigma_1 h} \qquad \ldots \qquad k_{1616} = -(\sigma_8^3 - \frac{\nu n^2\sigma_8}{r_0^2})e^{\sigma_8 h} \qquad (122)$$

The following terms refer to the matrix defined by *Equation* (86):

$$p_{11} = \alpha_1 e^{\sigma_1 z} \qquad \ldots\ldots \qquad p_{18} = \alpha_8 e^{\sigma_8 z}$$

$$p_{21} = \beta_1 e^{\sigma_1 z} \qquad \ldots\ldots \qquad p_{28} = \beta_8 e^{\sigma_8 z}$$

$$p_{31} = e^{\sigma_1 z} \qquad \ldots\ldots \qquad p_{38} = e^{\sigma_8 z}$$

$$p_{41} = \sigma_1 e^{\sigma_1 z} \qquad \ldots\ldots \qquad p_{48} = \sigma_8 e^{\sigma_8 z}$$

$$p_{51} = D(\alpha_1\sigma_1 + \frac{\nu}{r_0}\beta_1 + \frac{\nu}{r_0})e^{\sigma_1 z} \qquad \ldots \qquad p_{58} = D(\alpha_8\sigma_8 + \frac{\nu}{r_0}\beta_8 + \frac{\nu}{r_0})e^{\sigma_8 z}$$

$$p_{61} = \frac{D(1-\nu)}{2}(\beta_1\sigma_1 - \frac{\alpha_1}{r_0})e^{\sigma_1 z} \qquad \ldots \qquad p_{68} = \frac{D(1-\nu)}{2}(\beta_8\sigma_8 - \frac{\alpha_8}{r_0})e^{\sigma_8 z}$$

$$p_{71} = K(\sigma_1^2 - \frac{\nu}{r_0^2})e^{\sigma_1 z} \qquad \ldots \qquad p_{78} = K(\sigma_8^2 - \frac{\nu}{r_0^2})e^{\sigma_8 z}$$

$$p_{81} = K(\sigma_1^3 - \frac{\nu\sigma_1}{r_0^2})e^{\sigma_1 z} \qquad \ldots \qquad p_{88} = K(\sigma_8^3 - \frac{\nu\sigma_8}{r_0^2})e^{\sigma_8 z} \qquad (123)$$

The following terms refer to the matrix defined by *Equation* (108):

$$\tilde{p}_{11} = \alpha_1 e^{\sigma_1 z} + \sum_{m=1}^{\infty} B_1 P_m T_{1m} \sin\lambda_m z \qquad \ldots \qquad \tilde{p}_{18} = \alpha_8 e^{\sigma_8 z} + \sum_{m=1}^{\infty} B_1 P_m T_{8m} \sin\lambda_m z$$

$$\tilde{p}_{21} = \beta_1 e^{\sigma_1 z} + \sum_{m=1}^{\infty} B_2 P_m T_{1m} \cos\lambda_m z \qquad \ldots \qquad \tilde{p}_{28} = \beta_8 e^{\sigma_8 z} + \sum_{m=1}^{\infty} B_2 P_m T_{8m} \cos\lambda_m z$$

$$\tilde{p}_{31} = e^{\sigma_1 z} + \sum_{m=1}^{\infty} B_3 P_m T_{1m} \cos\lambda_m z \qquad \ldots \qquad \tilde{p}_{38} = e^{\sigma_8 z} + \sum_{m=1}^{\infty} B_3 P_m T_{8m} \cos\lambda_m z$$

$$\tilde{p}_{41} = \sigma_1 e^{\sigma_1 z} - \sum_{m=1}^{\infty} B_3 P_m T_{1m} \lambda_m \sin\lambda_m z \qquad \ldots\ldots \qquad \tilde{p}_{48} = \sigma_8 e^{\sigma_8 z} - \sum_{m=1}^{\infty} B_3 P_m T_{8m} \lambda_m \sin\lambda_m z$$

$$\tilde{p}_{51} = D[(\alpha_1\sigma_1 + \frac{\nu}{r_0}\beta_1 + \frac{\nu}{r_0})e^{\sigma_1 z} + \sum_{m=1}^{\infty}(B_1\lambda_m + \frac{\nu}{r_0}B_2 + \frac{\nu}{r_0}B_3)P_m T_{1m} \cos\lambda_m z] \qquad \ldots$$

$$\tilde{p}_{58} = D[(\alpha_8\sigma_8 + \frac{\nu}{r_0}\beta_8 + \frac{\nu}{r_0})e^{\sigma_8 z} + \sum_{m=1}^{\infty}(B_1\lambda_m + \frac{\nu}{r_0}B_2 + \frac{\nu}{r_0}B_3)P_m T_{8m}\cos\lambda_m z]$$

$$\tilde{p}_{61} = \frac{D(1-\nu)}{2}[(\beta_1\sigma_1 - \frac{\alpha_1}{r_0})e^{\sigma_1 z} - \sum_{m=1}^{\infty}(B_2\lambda_m + \frac{B_1}{r_0})P_m T_{1m}\sin\lambda_m z] \quad \ldots$$

$$\tilde{p}_{68} = \frac{D(1-\nu)}{2}[(\beta_8\sigma_8 - \frac{\alpha_8}{r_0})e^{\sigma_8 z} - \sum_{m=1}^{\infty}(B_2\lambda_m + \frac{B_1}{r_0})P_m T_{8m}\sin\lambda_m z]$$

$$\tilde{p}_{71} = K[(\sigma_1^2 - \frac{\nu}{r_0^2})e^{\sigma_1 z} - \sum_{m=1}^{\infty}(\lambda_m^2 + \frac{\nu}{r_0^2})B_3 P_m T_{1m}\cos\lambda_m z] \quad \ldots$$

$$\tilde{p}_{78} = K[(\sigma_8^2 - \frac{\nu}{r_0^2})e^{\sigma_8 z} - \sum_{m=1}^{\infty}(\lambda_m^2 + \frac{\nu}{r_0^2})B_3 P_m T_{8m}\cos\lambda_m z]$$

$$\tilde{p}_{81} = K[(\sigma_1^3 - \frac{\nu\sigma_1}{r_0^2})e^{\sigma_1 z} + \sum_{m=1}^{\infty}(\lambda_m^3 + \frac{\nu\lambda_m}{r_0^2})B_3 P_m T_{1m}\sin\lambda_m z] \quad \ldots$$

$$\tilde{p}_{88} = K[(\sigma_8^3 - \frac{\nu\sigma_8}{r_0^2})e^{\sigma_8 z} + \sum_{m=1}^{\infty}(\lambda_m^3 + \frac{\nu\lambda_m}{r_0^2})B_3 P_m T_{8m}\sin\lambda_m z] \tag{124}$$

APPENDIX IV

THE COMPUTER PROGRAM

PROGRAM NUCYL

Free Vibration Analysis of a Variable Thickness,

Flexible Cylindrical Tank Partially Filled with Fluid

Programed by

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May, 1992

CONSTANT THICKNESS SOLUTION FOR PARTIALLY-FILLED TANK

material property and tanks dimension

	Em = 21091837000	Ds = 7840	Df = 1000 v = 0.3
short tank :	H = 12.19	r0 = 18.29	ts = 0.0254
tall tank :	H = 21.95	r0 = 7.32	ts = 0.0254
water depth :	h = 0.6 H		

```
writeto(cst3fout):

eqn:=w^2*cos(la(m)*h)+9.8*la(m)*sin(la(m)*h):
fsolve(eqn=0,la(m)):
solutions:='':
la(m):=solutions:

L1:=Ds*w^2*(1-v^2)/Em-(1-v)*(1+ts^2/(12*r0^2))/(2*r0^2):
L2:=(1+v)/(2*r0):
L3:=-ts^2/(12*r0):
L4:=(v/r0)-ts^2*(1-v)/(24*r0^3):
L5:=0.5*(1-v)*(1+ts^2/(4*r0^2)):
L6:=Ds*w^2*(1-v^2)/Em-1/r0^2:
L7:=ts^2*(3-v)/(24*r0^2):
L8:=-1/r0^2:
L9:=ts^2/12:
L10:=-ts^2/(6*r0^2):
L11:=(1/r0^2)-Ds*w^2*(1-v^2)/Em:

C1:=simplify(L5*(L9-L3^2)):
C2:=simplify(L5*L10+L9*(L6+L1*L5)+L7^2+L9*L2^2-2*L3*L4*L5-L6*L3^2+2*L2*L3*L7):
C3:=simplify(L5*L11+L10*(L6+L1*L5)+L1*L6*L9+2*L7*L8+L1*L7^2+L10*L2^2-L5*L4^2\
          -2*L3*L4*L6+2*L2*L3*L8+2*L2*L4*L7):
C4:=simplify(L11*(L6+L1*L5)+L1*L6*L10+L8^2+2*L1*L7*L8+L11*L2^2-L6*L4^2\
          +2*L2*L4*L8):
C5:=simplify(L1*L6*L11+L1*L8^2):

eqn:=C1*x^8+C2*x^6+C3*x^4+C4*x^2+C5:
solve(eqn=0,x):
solutions:='':
```

```
x1:=solutions[1]:
x2:=solutions[2]:
x3:=solutions[3]:
x4:=solutions[4]:
x5:=solutions[5]:
x6:=solutions[6]:
x7:=solutions[7]:
x8:=solutions[8]:

a1:=evalc(Re(x1)):
b1:=evalc(Im(x1)):
a2:=evalc(Re(x2)):
b2:=evalc(Im(x2)):
a3:=evalc(Re(x3)):
b3:=evalc(Im(x3)):
a4:=evalc(Re(x4)):
b4:=evalc(Im(x4)):
a5:=evalc(Re(x5)):
b5:=evalc(Im(x5)):
a6:=evalc(Re(x6)):
b6:=evalc(Im(x6)):
a7:=evalc(Re(x7)):
b7:=evalc(Im(x7)):
a8:=evalc(Re(x8)):
b8:=evalc(Im(x8)):

a11(1):=simplify(x1^2+L1):
a11(2):=simplify(x2^2+L1):
a11(3):=simplify(x3^2+L1):
a11(4):=simplify(x4^2+L1):
a11(5):=simplify(x5^2+L1):
a11(6):=simplify(x6^2+L1):
a11(7):=simplify(x7^2+L1):
a11(8):=simplify(x8^2+L1):

a12(1):=simplify(L2*x1):
a12(2):=simplify(L2*x2):
a12(3):=simplify(L2*x3):
a12(4):=simplify(L2*x4):
a12(5):=simplify(L2*x5):
a12(6):=simplify(L2*x6):
a12(7):=simplify(L2*x7):
a12(8):=simplify(L2*x8):
```

```
a21(1):=-a12(1):
a21(2):=-a12(2):
a21(3):=-a12(3):
a21(4):=-a12(4):
a21(5):=-a12(5):
a21(6):=-a12(6):
a21(7):=-a12(7):
a21(8):=-a12(8):

a13(1):=simplify(L3*x1^3+L4*x1):
a13(2):=simplify(L3*x2^3+L4*x2):
a13(3):=simplify(L3*x3^3+L4*x3):
a13(4):=simplify(L3*x4^3+L4*x4):
a13(5):=simplify(L3*x5^3+L4*x5):
a13(6):=simplify(L3*x6^3+L4*x6):
a13(7):=simplify(L3*x7^3+L4*x7):
a13(8):=simplify(L3*x8^3+L4*x8):

a31(1):=a13(1):
a31(2):=a13(2):
a31(3):=a13(3):
a31(4):=a13(4):
a31(5):=a13(5):
a31(6):=a13(6):
a31(7):=a13(7):
a31(8):=a13(8):

a22(1):=simplify(L5*x1^2+L6):
a22(2):=simplify(L5*x2^2+L6):
a22(3):=simplify(L5*x3^2+L6):
a22(4):=simplify(L5*x4^2+L6):
a22(5):=simplify(L5*x5^2+L6):
a22(6):=simplify(L5*x6^2+L6):
a22(7):=simplify(L5*x7^2+L6):
a22(8):=simplify(L5*x8^2+L6):

a23(1):=simplify(L7*x1^2+L8):
a23(2):=simplify(L7*x2^2+L8):
a23(3):=simplify(L7*x3^2+L8):
a23(4):=simplify(L7*x4^2+L8):
a23(5):=simplify(L7*x5^2+L8):
a23(6):=simplify(L7*x6^2+L8):
a23(7):=simplify(L7*x7^2+L8):
a23(8):=simplify(L7*x8^2+L8):
```

```
a32(1):=-a23(1):
a32(2):=-a23(2):
a32(3):=-a23(3):
a32(4):=-a23(4):
a32(5):=-a23(5):
a32(6):=-a23(6):
a32(7):=-a23(7):
a32(8):=-a23(8):

a33(1):=simplify(L9*x1^4+L10*x1^2+L11):
a33(2):=simplify(L9*x2^4+L10*x2^2+L11):
a33(3):=simplify(L9*x3^4+L10*x3^2+L11):
a33(4):=simplify(L9*x4^4+L10*x4^2+L11):
a33(5):=simplify(L9*x5^4+L10*x5^2+L11):
a33(6):=simplify(L9*x6^4+L10*x6^2+L11):
a33(7):=simplify(L9*x7^4+L10*x7^2+L11):
a33(8):=simplify(L9*x8^4+L10*x8^2+L11):

y1:=simplify((a12(1)*a23(1)-a13(1)*a22(1))/(a11(1)*a22(1)-a12(1)*a21(1))):
y2:=simplify((a12(2)*a23(2)-a13(2)*a22(2))/(a11(2)*a22(2)-a12(2)*a21(2))):
y3:=simplify((a12(3)*a23(3)-a13(3)*a22(3))/(a11(3)*a22(3)-a12(3)*a21(3))):
y4:=simplify((a12(4)*a23(4)-a13(4)*a22(4))/(a11(4)*a22(4)-a12(4)*a21(4))):
y5:=simplify((a12(5)*a23(5)-a13(5)*a22(5))/(a11(5)*a22(5)-a12(5)*a21(5))):
y6:=simplify((a12(6)*a23(6)-a13(6)*a22(6))/(a11(6)*a22(6)-a12(6)*a21(6))):
y7:=simplify((a12(7)*a23(7)-a13(7)*a22(7))/(a11(7)*a22(7)-a12(7)*a21(7))):
y8:=simplify((a12(8)*a23(8)-a13(8)*a22(8))/(a11(8)*a22(8)-a12(8)*a21(8))):

y1:=evalc(y1):
y2:=evalc(y2):
y3:=evalc(y3):
y4:=evalc(y4):
y5:=evalc(y5):
y6:=evalc(y6):
y7:=evalc(y7):
y8:=evalc(y8):

z1:=simplify((a13(1)*a21(1)-a11(1)*a23(1))/(a11(1)*a22(1)-a12(1)*a21(1))):
z2:=simplify((a13(2)*a21(2)-a11(2)*a23(2))/(a11(2)*a22(2)-a12(2)*a21(2))):
z3:=simplify((a13(3)*a21(3)-a11(3)*a23(3))/(a11(3)*a22(3)-a12(3)*a21(3))):
z4:=simplify((a13(4)*a21(4)-a11(4)*a23(4))/(a11(4)*a22(4)-a12(4)*a21(4))):
z5:=simplify((a13(5)*a21(5)-a11(5)*a23(5))/(a11(5)*a22(5)-a12(5)*a21(5))):
z6:=simplify((a13(6)*a21(6)-a11(6)*a23(6))/(a11(6)*a22(6)-a12(6)*a21(6))):
z7:=simplify((a13(7)*a21(7)-a11(7)*a23(7))/(a11(7)*a22(7)-a12(7)*a21(7))):
z8:=simplify((a13(8)*a21(8)-a11(8)*a23(8))/(a11(8)*a22(8)-a12(8)*a21(8))):
```

```
z1:=evalc(z1):
z2:=evalc(z2):
z3:=evalc(z3):
z4:=evalc(z4):
z5:=evalc(z5):
z6:=evalc(z6):
z7:=evalc(z7):
z8:=evalc(z8):

b11(m):=simplify(L1-la(m)^2):
b12(m):=simplify(-L2*la(m)):
b13(m):=simplify(L3*la(m)^3-L4*la(m)):
b21(m):=b12(m):
b22(m):=simplify(L6-L5*la(m)^2):
b23(m):=simplify(-L7*la(m)^2+L8):
b31(m):=-b13(m):
b32(m):=-b23(m):
b33(m):=simplify(L9*la(m)^4-L10*la(m)^2+L11):

with(linalg):
b(m):=matrix([[b11(m),b12(m),b13(m)],[b21(m),b22(m),b23(m)],[b31(m),b32(m),\
              b33(m)]]):
D(m):=det(b(m)):

B1(m):=simplify((b12(m)*b23(m)-b13(m)*b22(m))/D(m)):
B2(m):=simplify((b13(m)*b21(m)-b11(m)*b23(m))/D(m)):
B3(m):=simplify((b11(m)*b22(m)-b12(m)*b21(m))/D(m)):

alias(I=BesselI):
I1(m):=I(1,la(m)*r0):
I2(m):=I(0,la(m)*r0)/2+I(2,la(m)*r0)/2:
I(m):=I1(m)/I2(m):

P(m):=simplify(2*Df*(1-v^2)*w^2*I(m)/Em*ts*(2*la(m)^2*h+\
              la(m)*sin(2*la(m)*h))^(1/2)):

T1(m):=simplify(evalf(exp(a1)*(cos(b1*h)+I*sin(b1*h))*(x1*cos(la(m)*h)+\
              la(m)*sin(la(m)*h))-x1)/(x1^2+la(m)^2)):
T1(m):=evalc(T1(m)):
T2(m):=simplify(evalf(exp(a2)*(cos(b2*h)+I*sin(b2*h))*(x2*cos(la(m)*h)+\
              la(m)*sin(la(m)*h))-x2)/(x2^2+la(m)^2)):
T2(m):=evalc(T2(m)):
T3(m):=simplify(evalf(exp(a3)*(cos(b3*h)+I*sin(b3*h))*(x3*cos(la(m)*h)+\
              la(m)*sin(la(m)*h))-x3)/(x3^2+la(m)^2)):
T3(m):=evalc(T3(m)):
```

```
T4(m):=simplify(evalf(exp(a4)*(cos(b4*h)+I*sin(b4*h))*(x4*cos(la(m)*h)+\
              la(m)*sin(la(m)*h))-x4)/(x4^2+la(m)^2)):
T4(m):=evalc(T4(m)):
T5(m):=simplify(evalf(exp(a5)*(cos(b5*h)+I*sin(b5*h))*(x5*cos(la(m)*h)+\
              la(m)*sin(la(m)*h))-x5)/(x5^2+la(m)^2)):
T5(m):=evalc(T5(m)):
T6(m):=simplify(evalf(exp(a6)*(cos(b6*h)+I*sin(b6*h))*(x6*cos(la(m)*h)+\
              la(m)*sin(la(m)*h))-x6)/(x6^2+la(m)^2)):
T6(m):=evalc(T6(m)):
T7(m):=simplify(evalf(exp(a7)*(cos(b7*h)+I*sin(b7*h))*(x7*cos(la(m)*h)+\
              la(m)*sin(la(m)*h))-x7)/(x7^2+la(m)^2)):
T7(m):=evalc(T7(m)):
T8(m):=simplify(evalf(exp(a8)*(cos(b8*h)+I*sin(b8*h))*(x8*cos(la(m)*h)+\
              la(m)*sin(la(m)*h))-x8)/(x8^2+la(m)^2)):
T8(m):=evalc(T8(m)):

S21:=simplify(sum(B2(m)*P(m)*T1(m),m=1..10)):
S22:=simplify(sum(B2(m)*P(m)*T2(m),m=1..10)):
S23:=simplify(sum(B2(m)*P(m)*T3(m),m=1..10)):
S24:=simplify(sum(B2(m)*P(m)*T4(m),m=1..10)):
S25:=simplify(sum(B2(m)*P(m)*T5(m),m=1..10)):
S26:=simplify(sum(B2(m)*P(m)*T6(m),m=1..10)):
S27:=simplify(sum(B2(m)*P(m)*T7(m),m=1..10)):
S28:=simplify(sum(B2(m)*P(m)*T8(m),m=1..10)):

S31:=simplify(sum(B3(m)*P(m)*T1(m),m=1..10)):
S32:=simplify(sum(B3(m)*P(m)*T2(m),m=1..10)):
S33:=simplify(sum(B3(m)*P(m)*T3(m),m=1..10)):
S34:=simplify(sum(B3(m)*P(m)*T4(m),m=1..10)):
S35:=simplify(sum(B3(m)*P(m)*T5(m),m=1..10)):
S36:=simplify(sum(B3(m)*P(m)*T6(m),m=1..10)):
S37:=simplify(sum(B3(m)*P(m)*T7(m),m=1..10)):
S38:=simplify(sum(B3(m)*P(m)*T8(m),m=1..10)):

S91:=simplify(sum(B1(m)*P(m)*T1(m)*sin(la(m)*h),m=1..10)):
S92:=simplify(sum(B1(m)*P(m)*T2(m)*sin(la(m)*h),m=1..10)):
S93:=simplify(sum(B1(m)*P(m)*T3(m)*sin(la(m)*h),m=1..10)):
S94:=simplify(sum(B1(m)*P(m)*T4(m)*sin(la(m)*h),m=1..10)):
S95:=simplify(sum(B1(m)*P(m)*T5(m)*sin(la(m)*h),m=1..10)):
S96:=simplify(sum(B1(m)*P(m)*T6(m)*sin(la(m)*h),m=1..10)):
S97:=simplify(sum(B1(m)*P(m)*T7(m)*sin(la(m)*h),m=1..10)):
S98:=simplify(sum(B1(m)*P(m)*T8(m)*sin(la(m)*h),m=1..10)):

S101:=simplify(sum(B2(m)*P(m)*T1(m)*cos(la(m)*h),m=1..10)):
S102:=simplify(sum(B2(m)*P(m)*T2(m)*cos(la(m)*h),m=1..10)):
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S103:=simplify(sum(B2(m)*P(m)*T3(m)*cos(la(m)*h),m=1..10)):
S104:=simplify(sum(B2(m)*P(m)*T4(m)*cos(la(m)*h),m=1..10)):
S105:=simplify(sum(B2(m)*P(m)*T5(m)*cos(la(m)*h),m=1..10)):
S106:=simplify(sum(B2(m)*P(m)*T6(m)*cos(la(m)*h),m=1..10)):
S107:=simplify(sum(B2(m)*P(m)*T7(m)*cos(la(m)*h),m=1..10)):
S108:=simplify(sum(B2(m)*P(m)*T8(m)*cos(la(m)*h),m=1..10)):

S111:=simplify(sum(B3(m)*P(m)*T1(m)*cos(la(m)*h),m=1..10)):
S112:=simplify(sum(B3(m)*P(m)*T2(m)*cos(la(m)*h),m=1..10)):
S113:=simplify(sum(B3(m)*P(m)*T3(m)*cos(la(m)*h),m=1..10)):
S114:=simplify(sum(B3(m)*P(m)*T4(m)*cos(la(m)*h),m=1..10)):
S115:=simplify(sum(B3(m)*P(m)*T5(m)*cos(la(m)*h),m=1..10)):
S116:=simplify(sum(B3(m)*P(m)*T6(m)*cos(la(m)*h),m=1..10)):
S117:=simplify(sum(B3(m)*P(m)*T7(m)*cos(la(m)*h),m=1..10)):
S118:=simplify(sum(B3(m)*P(m)*T8(m)*cos(la(m)*h),m=1..10)):

S121:=simplify(sum(B3(m)*P(m)*T1(m)*la(m)*sin(la(m)*h),m=1..10)):
S122:=simplify(sum(B3(m)*P(m)*T2(m)*la(m)*sin(la(m)*h),m=1..10)):
S123:=simplify(sum(B3(m)*P(m)*T3(m)*la(m)*sin(la(m)*h),m=1..10)):
S124:=simplify(sum(B3(m)*P(m)*T4(m)*la(m)*sin(la(m)*h),m=1..10)):
S125:=simplify(sum(B3(m)*P(m)*T5(m)*la(m)*sin(la(m)*h),m=1..10)):
S126:=simplify(sum(B3(m)*P(m)*T6(m)*la(m)*sin(la(m)*h),m=1..10)):
S127:=simplify(sum(B3(m)*P(m)*T7(m)*la(m)*sin(la(m)*h),m=1..10)):
S128:=simplify(sum(B3(m)*P(m)*T8(m)*la(m)*sin(la(m)*h),m=1..10)):

S131:=simplify(sum(P(m)*T1(m)*cos(la(m)*h)*(B1(m)*la(m)+\
            v*(B2(m)+B3(m))/r0),m=1..10)):
S132:=simplify(sum(P(m)*T2(m)*cos(la(m)*h)*(B1(m)*la(m)+\
            v*(B2(m)+B3(m))/r0),m=1..10)):
S133:=simplify(sum(P(m)*T3(m)*cos(la(m)*h)*(B1(m)*la(m)+\
            v*(B2(m)+B3(m))/r0),m=1..10)):
S134:=simplify(sum(P(m)*T4(m)*cos(la(m)*h)*(B1(m)*la(m)+\
            v*(B2(m)+B3(m))/r0),m=1..10)):
S135:=simplify(sum(P(m)*T5(m)*cos(la(m)*h)*(B1(m)*la(m)+\
            v*(B2(m)+B3(m))/r0),m=1..10)):
S136:=simplify(sum(P(m)*T6(m)*cos(la(m)*h)*(B1(m)*la(m)+\
            v*(B2(m)+B3(m))/r0),m=1..10)):
S137:=simplify(sum(P(m)*T7(m)*cos(la(m)*h)*(B1(m)*la(m)+\
            v*(B2(m)+B3(m))/r0),m=1..10)):
S138:=simplify(sum(P(m)*T8(m)*cos(la(m)*h)*(B1(m)*la(m)+\
            v*(B2(m)+B3(m))/r0),m=1..10)):

S141:=simplify(sum(P(m)*T1(m)*sin(la(m)*h)*(B2(m)*la(m)+B1(m)/r0),m=1..10)):
S142:=simplify(sum(P(m)*T2(m)*sin(la(m)*h)*(B2(m)*la(m)+B1(m)/r0),m=1..10)):
S143:=simplify(sum(P(m)*T3(m)*sin(la(m)*h)*(B2(m)*la(m)+B1(m)/r0),m=1..10)):
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S144:=simplify(sum(P(m)*T4(m)*sin(la(m)*h)*(B2(m)*la(m)+B1(m)/r0),m=1..10)):
S145:=simplify(sum(P(m)*T5(m)*sin(la(m)*h)*(B2(m)*la(m)+B1(m)/r0),m=1..10)):
S146:=simplify(sum(P(m)*T6(m)*sin(la(m)*h)*(B2(m)*la(m)+B1(m)/r0),m=1..10)):
S147:=simplify(sum(P(m)*T7(m)*sin(la(m)*h)*(B2(m)*la(m)+B1(m)/r0),m=1..10)):
S148:=simplify(sum(P(m)*T8(m)*sin(la(m)*h)*(B2(m)*la(m)+B1(m)/r0),m=1..10)):

S151:=simplify(sum(B3(m)*P(m)*T1(m)*cos(la(m)*h)*(la(m)^2+v/r0^2),m=1..10)):
S152:=simplify(sum(B3(m)*P(m)*T2(m)*cos(la(m)*h)*(la(m)^2+v/r0^2),m=1..10)):
S153:=simplify(sum(B3(m)*P(m)*T3(m)*cos(la(m)*h)*(la(m)^2+v/r0^2),m=1..10)):
S154:=simplify(sum(B3(m)*P(m)*T4(m)*cos(la(m)*h)*(la(m)^2+v/r0^2),m=1..10)):
S155:=simplify(sum(B3(m)*P(m)*T5(m)*cos(la(m)*h)*(la(m)^2+v/r0^2),m=1..10)):
S156:=simplify(sum(B3(m)*P(m)*T6(m)*cos(la(m)*h)*(la(m)^2+v/r0^2),m=1..10)):
S157:=simplify(sum(B3(m)*P(m)*T7(m)*cos(la(m)*h)*(la(m)^2+v/r0^2),m=1..10)):
S158:=simplify(sum(B3(m)*P(m)*T8(m)*cos(la(m)*h)*(la(m)^2+v/r0^2),m=1..10)):

S161:=simplify(sum(B3(m)*P(m)*T1(m)*sin(la(m)*h)*la(m)*(la(m)^2+v/r0^2),\
            m=1..10)):
S162:=simplify(sum(B3(m)*P(m)*T2(m)*sin(la(m)*h)*la(m)*(la(m)^2+v/r0^2),\
            m=1..10)):
S163:=simplify(sum(B3(m)*P(m)*T3(m)*sin(la(m)*h)*la(m)*(la(m)^2+v/r0^2),\
            m=1..10)):
S164:=simplify(sum(B3(m)*P(m)*T4(m)*sin(la(m)*h)*la(m)*(la(m)^2+v/r0^2),\
            m=1..10)):
S165:=simplify(sum(B3(m)*P(m)*T5(m)*sin(la(m)*h)*la(m)*(la(m)^2+v/r0^2),\
            m=1..10)):
S166:=simplify(sum(B3(m)*P(m)*T6(m)*sin(la(m)*h)*la(m)*(la(m)^2+v/r0^2),\
            m=1..10)):
S167:=simplify(sum(B3(m)*P(m)*T7(m)*sin(la(m)*h)*la(m)*(la(m)^2+v/r0^2),\
            m=1..10)):
S168:=simplify(sum(B3(m)*P(m)*T8(m)*sin(la(m)*h)*la(m)*(la(m)^2+v/r0^2),\
            m=1..10)):

k11:=evalf(y1):
k12:=evalf(y2):
k13:=evalf(y3):
k14:=evalf(y4):
k15:=evalf(y5):
k16:=evalf(y6):
k17:=evalf(y7):
k18:=evalf(y8):
k19:=0:
k110:=0:
k111:=0:
k112:=0:
k113:=0:
```

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k114:=0:
k115:=0:
k116:=0:

k21:=simplify(evalf(z1+S21)):
k22:=simplify(evalf(z2+S22)):
k23:=simplify(evalf(z3+S23)):
k24:=simplify(evalf(z4+S24)):
k25:=simplify(evalf(z5+S25)):
k26:=simplify(evalf(z6+S26)):
k27:=simplify(evalf(z7+S27)):
k28:=simplify(evalf(z8+S28)):
k29:=0:
k210:=0:
k211:=0:
k212:=0:
k213:=0:
k214:=0:
k215:=0:
k216:=0:

k31:=simplify(evalf(1+S31)):
k32:=simplify(evalf(1+S32)):
k33:=simplify(evalf(1+S33)):
k34:=simplify(evalf(1+S34)):
k35:=simplify(evalf(1+S35)):
k36:=simplify(evalf(1+S36)):
k37:=simplify(evalf(1+S37)):
k38:=simplify(evalf(1+S38)):
k39:=0:
k310:=0:
k311:=0:
k312:=0:
k313:=0:
k314:=0:
k315:=0:
k316:=0:

k41:=simplify(evalf(x1)):
k42:=simplify(evalf(x2)):
k43:=simplify(evalf(x3)):
k44:=simplify(evalf(x4)):
k45:=simplify(evalf(x5)):
k46:=simplify(evalf(x6)):
k47:=simplify(evalf(x7)):
```

```
k48:=simplify(evalf(x8)):
k49:=0:
k410:=0:
k411:=0:
k412:=0:
k413:=0:
k414:=0:
k415:=0:
k416:=0:

k51:=0:
k52:=0:
k53:=0:
k54:=0:
k55:=0:
k56:=0:
k57:=0:
k58:=0:
k59:=simplify(evalf(y1*x1+v*z1/r0+v/r0)):
k510:=simplify(evalf(y2*x2+v*z2/r0+v/r0)):
k511:=simplify(evalf(y3*x3+v*z3/r0+v/r0)):
k512:=simplify(evalf(y4*x4+v*z4/r0+v/r0)):
k513:=simplify(evalf(y5*x5+v*z5/r0+v/r0)):
k514:=simplify(evalf(y6*x6+v*z6/r0+v/r0)):
k515:=simplify(evalf(y7*x7+v*z7/r0+v/r0)):
k516:=simplify(evalf(y8*x8+v*z8/r0+v/r0)):

k61:=0:
k62:=0:
k63:=0:
k64:=0:
k65:=0:
k66:=0:
k67:=0:
k68:=0:
k69:=simplify(evalf(z1*x1-y1/r0)):
k610:=simplify(evalf(z2*x2-y2/r0)):
k611:=simplify(evalf(z3*x3-y3/r0)):
k612:=simplify(evalf(z4*x4-y4/r0)):
k613:=simplify(evalf(z5*x5-y5/r0)):
k614:=simplify(evalf(z6*x6-y6/r0)):
k615:=simplify(evalf(z7*x7-y7/r0)):
k616:=simplify(evalf(z8*x8-y8/r0)):

k71:=0:
```

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k72:=0:
k73:=0:
k74:=0:
k75:=0:
k76:=0:
k77:=0:
k78:=0:
k79:=simplify(evalf(x1^2-v/r0^2)):
k710:=simplify(evalf(x2^2-v/r0^2)):
k711:=simplify(evalf(x3^2-v/r0^2)):
k712:=simplify(evalf(x4^2-v/r0^2)):
k713:=simplify(evalf(x5^2-v/r0^2)):
k714:=simplify(evalf(x6^2-v/r0^2)):
k715:=simplify(evalf(x7^2-v/r0^2)):
k716:=simplify(evalf(x8^2-v/r0^2)):

k81:=0:
k82:=0:
k83:=0:
k84:=0:
k85:=0:
k86:=0:
k87:=0:
k88:=0:
k89:=simplify(evalf(x1^3-v*x1/r0^2)):
k810:=simplify(evalf(x2^3-v*x2/r0^2)):
k811:=simplify(evalf(x3^3-v*x3/r0^2)):
k812:=simplify(evalf(x4^3-v*x4/r0^2)):
k813:=simplify(evalf(x5^3-v*x5/r0^2)):
k814:=simplify(evalf(x6^3-v*x6/r0^2)):
k815:=simplify(evalf(x7^3-v*x7/r0^2)):
k816:=simplify(evalf(x8^3-v*x8/r0^2)):

k91:=simplify(y1*evalf(exp(a1*h)*(cos(b1*h)+I*sin(b1*h)))+S91):
k92:=simplify(y2*evalf(exp(a2*h)*(cos(b2*h)+I*sin(b2*h)))+S92):
k93:=simplify(y3*evalf(exp(a3*h)*(cos(b3*h)+I*sin(b3*h)))+S93):
k94:=simplify(y4*evalf(exp(a4*h)*(cos(b4*h)+I*sin(b4*h)))+S94):
k95:=simplify(y5*evalf(exp(a5*h)*(cos(b5*h)+I*sin(b5*h)))+S95):
k96:=simplify(y6*evalf(exp(a6*h)*(cos(b6*h)+I*sin(b6*h)))+S96):
k97:=simplify(y7*evalf(exp(a7*h)*(cos(b7*h)+I*sin(b7*h)))+S97):
k98:=simplify(y8*evalf(exp(a8*h)*(cos(b8*h)+I*sin(b8*h)))+S98):
k99:=S91-k91:
k910:=S92-k92:
k911:=S93-k93:
k912:=S94-k94:
```

```
k913:=S95-k95:
k914:=S96-k96:
k915:=S97-k97:
k916:=S98-k98:

k101:=simplify(z1*evalf(exp(a1*h)*(cos(b1*h)+I*sin(b1*h)))+S101):
k102:=simplify(z2*evalf(exp(a2*h)*(cos(b2*h)+I*sin(b2*h)))+S102):
k103:=simplify(z3*evalf(exp(a3*h)*(cos(b3*h)+I*sin(b3*h)))+S103):
k104:=simplify(z4*evalf(exp(a4*h)*(cos(b4*h)+I*sin(b4*h)))+S104):
k105:=simplify(z5*evalf(exp(a5*h)*(cos(b5*h)+I*sin(b5*h)))+S105):
k106:=simplify(z6*evalf(exp(a6*h)*(cos(b6*h)+I*sin(b6*h)))+S106):
k107:=simplify(z7*evalf(exp(a7*h)*(cos(b7*h)+I*sin(b7*h)))+S107):
k108:=simplify(z8*evalf(exp(a8*h)*(cos(b8*h)+I*sin(b8*h)))+S108):
k109:=S101-k101:
k1010:=S102-k102:
k1011:=S103-k103:
k1012:=S104-k104:
k1013:=S105-k105:
k1014:=S106-k106:
k1015:=S107-k107:
k1016:=S108-k108:

k111:=simplify(evalf(exp(a1*h)*(cos(b1*h)+I*sin(b1*h)))+S111):
k112:=simplify(evalf(exp(a2*h)*(cos(b2*h)+I*sin(b2*h)))+S112):
k113:=simplify(evalf(exp(a3*h)*(cos(b3*h)+I*sin(b3*h)))+S113):
k114:=simplify(evalf(exp(a4*h)*(cos(b4*h)+I*sin(b4*h)))+S114):
k115:=simplify(evalf(exp(a5*h)*(cos(b5*h)+I*sin(b5*h)))+S115):
k116:=simplify(evalf(exp(a6*h)*(cos(b6*h)+I*sin(b6*h)))+S116):
k117:=simplify(evalf(exp(a7*h)*(cos(b7*h)+I*sin(b7*h)))+S117):
k118:=simplify(evalf(exp(a8*h)*(cos(b8*h)+I*sin(b8*h)))+S118):
k119:=S111-k111:
k1110:=S112-k112:
k1111:=S113-k113:
k1112:=S114-k114:
k1113:=S115-k115:
k1114:=S116-k116:
k1115:=S117-k117:
k1116:=S118-k118:

k121:=simplify(x1*evalf(exp(a1*h)*(cos(b1*h)+I*sin(b1*h)))-S121):
k122:=simplify(x2*evalf(exp(a2*h)*(cos(b2*h)+I*sin(b2*h)))-S122):
k123:=simplify(x3*evalf(exp(a3*h)*(cos(b3*h)+I*sin(b3*h)))-S123):
k124:=simplify(x4*evalf(exp(a4*h)*(cos(b4*h)+I*sin(b4*h)))-S124):
k125:=simplify(x5*evalf(exp(a5*h)*(cos(b5*h)+I*sin(b5*h)))-S125):
k126:=simplify(x6*evalf(exp(a6*h)*(cos(b6*h)+I*sin(b6*h)))-S126):
```

```
k127:=simplify(x7*evalf(exp(a7*h)*(cos(b7*h)+I*sin(b7*h)))-S127):
k128:=simplify(x8*evalf(exp(a8*h)*(cos(b8*h)+I*sin(b8*h)))-S128):
k129:=-S121-k121:
k1210:=-S122-k122:
k1211:=-S123-k123:
k1212:=-S124-k124:
k1213:=-S125-k125:
k1214:=-S126-k126:
k1215:=-S127-k127:
k1216:=-S128-k128:

k131:=simplify((y1*x1+v*z1/r0+v/r0)*evalf(exp(a1*h)*(cos(b1*h)+\
            I*sin(b1*h)))+S131):
k132:=simplify((y2*x2+v*z2/r0+v/r0)*evalf(exp(a2*h)*(cos(b2*h)+\
            I*sin(b2*h)))+S132):
k133:=simplify((y3*x3+v*z3/r0+v/r0)*evalf(exp(a3*h)*(cos(b3*h)+\
            I*sin(b3*h)))+S133):
k134:=simplify((y4*x4+v*z4/r0+v/r0)*evalf(exp(a4*h)*(cos(b4*h)+\
            I*sin(b4*h)))+S134):
k135:=simplify((y5*x5+v*z5/r0+v/r0)*evalf(exp(a5*h)*(cos(b5*h)+\
            I*sin(b5*h)))+S135):
k136:=simplify((y6*x6+v*z6/r0+v/r0)*evalf(exp(a6*h)*(cos(b6*h)+\
            I*sin(b6*h)))+S136):
k137:=simplify((y7*x7+v*z7/r0+v/r0)*evalf(exp(a7*h)*(cos(b7*h)+\
            I*sin(b7*h)))+S137):
k138:=simplify((y8*x8+v*z8/r0+v/r0)*evalf(exp(a8*h)*(cos(b8*h)+\
            I*sin(b8*h)))+S138):
k139:=S131-k131:
k1310:=S132-k132:
k1311:=S133-k133:
k1312:=S134-k134:
k1313:=S135-k135:
k1314:=S136-k136:
k1315:=S137-k137:
k1316:=S138-k138:

k141:=simplify((z1*x1-y1/r0)*evalf(exp(a1*h)*(cos(b1*h)+\
            I*sin(b1*h)))+S141):
k142:=simplify((z2*x2-y2/r0)*evalf(exp(a2*h)*(cos(b2*h)+\
            I*sin(b2*h)))+S142):
k143:=simplify((z3*x3-y3/r0)*evalf(exp(a3*h)*(cos(b3*h)+\
            I*sin(b3*h)))+S143):
k144:=simplify((z4*x4-y4/r0)*evalf(exp(a4*h)*(cos(b4*h)+\
            I*sin(b4*h)))+S144):
k145:=simplify((z5*x5-y5/r0)*evalf(exp(a5*h)*(cos(b5*h)+\
```

```
                    I*sin(b5*h)))+S145):
k146:=simplify((z6*x6-y6/r0)*evalf(exp(a6*h)*(cos(b6*h)+\
                    I*sin(b6*h)))+S146):
k147:=simplify((z7*x7-y7/r0)*evalf(exp(a7*h)*(cos(b7*h)+\
                    I*sin(b7*h)))+S147):
k148:=simplify((z8*x8-y8/r0)*evalf(exp(a8*h)*(cos(b8*h)+\
                    I*sin(b8*h)))+S148):
k149:=-S141-k141:
k1410:=-S142-k142:
k1411:=-S143-k143:
k1412:=-S144-k144:
k1413:=-S145-k145:
k1414:=-S146-k146:
k1415:=-S147-k147:
k1416:=-S148-k148:

k151:=simplify((x1^2-v/r0^2)*evalf(exp(a1*h)*(cos(b1*h)+\
                    I*sin(b1*h)))+S151):
k152:=simplify((x2^2-v/r0^2)*evalf(exp(a2*h)*(cos(b2*h)+\
                    I*sin(b2*h)))+S152):
k153:=simplify((x3^2-v/r0^2)*evalf(exp(a3*h)*(cos(b3*h)+\
                    I*sin(b3*h)))+S153):
k154:=simplify((x4^2-v/r0^2)*evalf(exp(a4*h)*(cos(b4*h)+\
                    I*sin(b4*h)))+S154):
k155:=simplify((x5^2-v/r0^2)*evalf(exp(a5*h)*(cos(b5*h)+\
                    I*sin(b5*h)))+S155):
k156:=simplify((x6^2-v/r0^2)*evalf(exp(a6*h)*(cos(b6*h)+\
                    I*sin(b6*h)))+S156):
k157:=simplify((x7^2-v/r0^2)*evalf(exp(a7*h)*(cos(b7*h)+\
                    I*sin(b7*h)))+S157):
k158:=simplify((x8^2-v/r0^2)*evalf(exp(a8*h)*(cos(b8*h)+\
                    I*sin(b8*h)))+S158):
k159:=-S151-k151:
k1510:=-S152-k152:
k1511:=-S153-k153:
k1512:=-S154-k154:
k1513:=-S155-k155:
k1514:=-S156-k156:
k1515:=-S157-k157:
k1516:=-S158-k158:

k161:=simplify((x1^3-v*x1/r0^2)*evalf(exp(a1*h)*(cos(b1*h)+\
                    I*sin(b1*h)))+S161):
k162:=simplify((x2^3-v*x2/r0^2)*evalf(exp(a2*h)*(cos(b2*h)+\
                    I*sin(b2*h)))+S162):
```

```
k163:=simplify((x3^3-v*x3/r0^2)*evalf(exp(a3*h)*(cos(b3*h)+\
              I*sin(b3*h)))+S163):
k164:=simplify((x4^3-v*x4/r0^2)*evalf(exp(a4*h)*(cos(b4*h)+\
              I*sin(b4*h)))+S164):
k165:=simplify((x5^3-v*x5/r0^2)*evalf(exp(a5*h)*(cos(b5*h)+\
              I*sin(b5*h)))+S165):
k166:=simplify((x6^3-v*x6/r0^2)*evalf(exp(a6*h)*(cos(b6*h)+\
              I*sin(b6*h)))+S166):
k167:=simplify((x7^3-v*x7/r0^2)*evalf(exp(a7*h)*(cos(b7*h)+\
              I*sin(b7*h)))+S167):
k168:=simplify((x8^3-v*x8/r0^2)*evalf(exp(a8*h)*(cos(b8*h)+\
              I*sin(b8*h)))+S168):
k169:=S161-k161:
k1610:=S162-k162:
k1611:=S163-k163:
k1612:=S164-k164:
k1613:=S165-k165:
k1614:=S166-k166:
k1615:=S167-k167:
k1616:=S168-k168:
```

solve the frequencies w

```
with(linalg):
K:=matrix([[k11,k12,k13,k14,k15,k16,k17,k18,k19,k110,k111,k112,k113,k114,\
           k115,k116],\
          [k21,k22,k23,k24,k25,k26,k27,k28,k29,k210,k211,k212,k213,k214,\
           k215,k216],\
          [k31,k32,k33,k34,k35,k36,k37,k38,k39,k310,k311,k312,k313,k314,\
           k315,k316],\
          [k41,k42,k43,k44,k45,k46,k47,k48,k49,k410,k411,k412,k413,k414,\
           k415,k416],\
          [k51,k52,k53,k54,k55,k56,k57,k58,k59,k510,k511,k512,k513,k514,\
           k515,k516],\
          [k61,k62,k63,k64,k65,k66,k67,k68,k69,k610,k611,k612,k613,k614,\
           k615,k616],\
          [k71,k72,k73,k74,k75,k76,k77,k78,k79,k710,k711,k712,k713,k714,\
           k715,k716],\
          [k81,k82,k83,k84,k85,k86,k87,k88,k89,k810,k811,k812,k813,k814,\
           k815,k816],\
          [k91,k92,k93,k94,k95,k96,k97,k98,k99,k910,k911,k912,k913,k914,\
           k915,k916],\
          [k101,k102,k103,k104,k105,k106,k107,k108,k109,k1010,k1011,k1012,\
           k1013,k1014,k1015,k1016],\
          [k111,k112,k113,k114,k115,k116,k117,k118,k119,k1110,k1111,k1112,\
```

```
 k1113,k1114,k1115,k1116],\
[k121,k122,k123,k124,k125,k126,k127,k128,k129,k1210,k1211,k1212,\
 k1213,k1214,k1215,k1216],\
[k131,k132,k133,k134,k135,k136,k137,k138,k139,k1310,k1311,k1312,\
 k1313,k1314,k1315,k1316],\
[k141,k142,k143,k144,k145,k146,k147,k148,k149,k1410,k1411,k1412,\
 k1413,k1414,k1415,k1416],\
[k151,k152,k153,k154,k155,k156,k157,k158,k159,k1510,k1511,k1512,\
 k1513,k1514,k1515,k1516],\
[k161,k162,k163,k164,k165,k166,k167,k168,k169,k1610,k1611,k1612,\
 k1613,k1614,k1615,k1616]]):

A:=det(K);
```

END OF FILE

CONSTANT THICKNESS SOLUTION FOR PARTLY-FILLED TANK

(mode shape)

natural frequency:

short w = 8.71 tall w = 10.87

```
writeto(ms3out):
```

axial mode shape of partly-filled tank (wet part)

```
U(z):=exp(xi1*z)*(C1*(gam1*cos(eta1*z)-del1*sin(eta1*z))+C2*(del1*\
    cos(eta1*z)+gam1*sin(eta1*z)))+exp(-xi1*z)*(C3*(-gam1*cos(eta1*z)-\
    del1*sin(eta1*z))+C4*(del1*cos(eta1*z)-gam1*sin(eta1*z)))+\
    exp(xi2*z)*(C5*(gam2*cos(eta2*z)-del2*sin(eta2*z))+C6*(del2*\
    cos(eta2*z)+gam2*sin(eta2*z)))+exp(-xi2*z)*(C7*(-gam2*cos(eta2*z)-\
    del2*sin(eta2*z))+C8*(del2*cos(eta2*z)-gam2*sin(eta2*z)))+\
    (C1*Pa11+C2*Pa12+C3*Pa13+C4*Pa14+C5*Pa15+C6*Pa16+C7*Pa17+C8*Pa18):
```

axial mode shape of partly-filled tank (dry part)

```
u(z):=exp(xi1*z)*(c1*(gam1*cos(eta1*z)-del1*sin(eta1*z))+c2*(del1*\
    cos(eta1*z)+gam1*sin(eta1*z)))+exp(-xi1*z)*(c3*(-gam1*cos(eta1*z)-\
    del1*sin(eta1*z))+c4*(del1*cos(eta1*z)-gam1*sin(eta1*z)))+\
    exp(xi2*z)*(c5*(gam2*cos(eta2*z)-del2*sin(eta2*z))+c6*(del2*\
    cos(eta2*z)+gam2*sin(eta2*z)))+exp(-xi2*z)*(c7*(-gam2*cos(eta2*z)-\
    del2*sin(eta2*z))+c8*(del2*cos(eta2*z)-gam2*sin(eta2*z))):
```

radial mode shape of partly-filled tank (wet part)

```
W(z):=exp(xi1*z)*(C1*sin(eta1*z)+C2*cos(eta1*z))+exp(-xi1*z)*(C3*sin(eta1*z)+\
   C4*cos(eta1*z))+exp(xi2*z)*(C5*sin(eta2*z)+C6*cos(eta2*z))+exp(-xi2*z)*\
   (C7*sin(eta2*z)+C8*cos(eta2*z))+\
   (C1*Pa31+C2*Pa32+C3*Pa33+C4*Pa34+C5*Pa35+C6*Pa36+C7*Pa37+C8*Pa38):
```

radial mode shape of partly-filled tank (dry part)

```
w(z):=exp(xi1*z)*(c1*sin(eta1*z)+c2*cos(eta1*z))+exp(-xi1*z)*(c3*sin(eta1*z)+\
   c4*cos(eta1*z))+exp(xi2*z)*(c5*sin(eta2*z)+c6*cos(eta2*z))+exp(-xi2*z)*\
   (c7*sin(eta2*z)+c8*cos(eta2*z)):
```

interface condition

```
U(j):=exp(xi1*j)*(C1*(gam1*cos(eta1*j)-del1*sin(eta1*j))+C2*(del1*\
    cos(eta1*j)+gam1*sin(eta1*j)))+exp(-xi1*j)*(C3*(-gam1*cos(eta1*j)-\
    del1*sin(eta1*j))+C4*(del1*cos(eta1*j)-gam1*sin(eta1*j)))+\
    exp(xi2*j)*(C5*(gam2*cos(eta2*j)-del2*sin(eta2*j))+C6*(del2*\
    cos(eta2*j)+gam2*sin(eta2*j)))+exp(-xi2*j)*(C7*(-gam2*cos(eta2*j)-\
    del2*sin(eta2*j))+C8*(del2*cos(eta2*j)-gam2*sin(eta2*j)))+\
    (C1*Pa11+C2*Pa12+C3*Pa13+C4*Pa14+C5*Pa15+C6*Pa16+C7*Pa17+C8*Pa18):

Uj:=diff(U(j),j):

u(j):=exp(xi1*j)*(c1*(gam1*cos(eta1*j)-del1*sin(eta1*j))+c2*(del1*\
    cos(eta1*j)+gam1*sin(eta1*j)))+exp(-xi1*j)*(c3*(-gam1*cos(eta1*j)-\
    del1*sin(eta1*j))+c4*(del1*cos(eta1*j)-gam1*sin(eta1*j)))+\
    exp(xi2*j)*(c5*(gam2*cos(eta2*j)-del2*sin(eta2*j))+c6*(del2*\
    cos(eta2*j)+gam2*sin(eta2*j)))+exp(-xi2*j)*(c7*(-gam2*cos(eta2*j)-\
    del2*sin(eta2*j))+c8*(del2*cos(eta2*j)-gam2*sin(eta2*j))):

uj:=diff(u(j),j):

V(j):=exp(xi1*j)*(C1*(mu1*cos(eta1*j)-eps1*sin(eta1*j))+C2*(eps1*\
    cos(eta1*j)+mu1*sin(eta1*j)))+exp(-xi1*j)*(C3*(-mu1*cos(eta1*j)-\
    eps1*sin(eta1*j))+C4*(eps1*cos(eta1*j)-mu1*sin(eta1*j)))+\
    exp(xi2*j)*(C5*(mu2*cos(eta2*j)-eps2*sin(eta2*j))+C6*(eps2*\
    cos(eta2*j)+mu2*sin(eta2*j)))+exp(-xi2*j)*(C7*(-mu2*cos(eta2*j)-\
    eps2*sin(eta2*j))+C8*(eps2*cos(eta2*j)-mu2*sin(eta2*j)))+\
    (C1*Pa21+C2*Pa22+C3*Pa23+C4*Pa24+C5*Pa25+C6*Pa26+C7*Pa27+C8*Pa28):

Vj:=diff(V(j),j):

v(j):=exp(xi1*j)*(c1*(mu1*cos(eta1*j)-eps1*sin(eta1*j))+c2*(eps1*\
    cos(eta1*j)+mu1*sin(eta1*j)))+exp(-xi1*j)*(c3*(-mu1*cos(eta1*j)-\
    eps1*sin(eta1*j))+c4*(eps1*cos(eta1*j)-mu1*sin(eta1*j)))+\
    exp(xi2*j)*(c5*(mu2*cos(eta2*j)-eps2*sin(eta2*j))+c6*(eps2*\
    cos(eta2*j)+mu2*sin(eta2*j)))+exp(-xi2*j)*(c7*(-mu2*cos(eta2*j)-\
    eps2*sin(eta2*j))+c8*(eps2*cos(eta2*j)-mu2*sin(eta2*j))):

vj:=diff(v(j),j):

W(j):=exp(xi1*j)*(C1*sin(eta1*j)+C2*cos(eta1*j))+exp(-xi1*j)*(C3*sin(eta1*j)+\
```

```
  C4*cos(eta1*j))+exp(xi2*j)*(C5*sin(eta2*j)+C6*cos(eta2*j))+exp(-xi2*j)*\
  (C7*sin(eta2*j)+C8*cos(eta2*j))+\
  (C1*Pa31+C2*Pa32+C3*Pa33+C4*Pa34+C5*Pa35+C6*Pa36+C7*Pa37+C8*Pa38):

Wj:=diff(W(j),j):
Wjj:=diff(Wj,j):

w(j):=exp(xi1*j)*(c1*sin(eta1*j)+c2*cos(eta1*j))+exp(-xi1*j)*(c3*sin(eta1*j)+\
  c4*cos(eta1*j))+exp(xi2*j)*(c5*sin(eta2*j)+c6*cos(eta2*j))+exp(-xi2*j)*\
  (c7*sin(eta2*j)+c8*cos(eta2*j)):

wj:=diff(w(j),j):
wjj:=diff(wj,j):

Nj:=Uj+(mu/r0)*(V(j)+W(j)):
Njtheta:=Vj-U(j)/r0:
Mj:=Wjj-(mu/r0^2)*W(j):
Mjj:=diff(Mj,j):
Qj:=Mjj:

nj:=uj+(mu/r0)*(v(j)+w(j)):
njtheta:=vj-u(j)/r0:
mj:=wjj-(mu/r0^2)*w(j):
mjj:=diff(mj,j):
qj:=mjj:

j:=h:
U(h):=exp(xi1*j)*(C1*(gam1*cos(eta1*j)-del1*sin(eta1*j))+C2*(del1*\
   cos(eta1*j)+gam1*sin(eta1*j)))+exp(-xi1*j)*(C3*(-gam1*cos(eta1*j)-\
   del1*sin(eta1*j))+C4*(del1*cos(eta1*j)-gam1*sin(eta1*j)))+\
   exp(xi2*j)*(C5*(gam2*cos(eta2*j)-del2*sin(eta2*j))+C6*(del2*\
   cos(eta2*j)+gam2*sin(eta2*j)))+exp(-xi2*j)*(C7*(-gam2*cos(eta2*j)-\
   del2*sin(eta2*j))+C8*(del2*cos(eta2*j)-gam2*sin(eta2*j)))+\
   (C1*Pa11+C2*Pa12+C3*Pa13+C4*Pa14+C5*Pa15+C6*Pa16+C7*Pa17+C8*Pa18):


V(h):=exp(xi1*j)*(C1*(mu1*cos(eta1*j)-eps1*sin(eta1*j))+C2*(eps1*\
   cos(eta1*j)+mu1*sin(eta1*j)))+exp(-xi1*j)*(C3*(-mu1*cos(eta1*j)-\
   eps1*sin(eta1*j))+C4*(eps1*cos(eta1*j)-mu1*sin(eta1*j)))+\
   exp(xi2*j)*(C5*(mu2*cos(eta2*j)-eps2*sin(eta2*j))+C6*(eps2*\
   cos(eta2*j)+mu2*sin(eta2*j)))+exp(-xi2*j)*(C7*(-mu2*cos(eta2*j)-\
   eps2*sin(eta2*j))+C8*(eps2*cos(eta2*j)-mu2*sin(eta2*j)))+\
   (C1*Pa21+C2*Pa22+C3*Pa23+C4*Pa24+C5*Pa25+C6*Pa26+C7*Pa27+C8*Pa28):
```

```
W(h):=exp(xi1*j)*(C1*sin(eta1*j)+C2*cos(eta1*j))+exp(-xi1*j)*(C3*sin(eta1*j)+\
    C4*cos(eta1*j))+exp(xi2*j)*(C5*sin(eta2*j)+C6*cos(eta2*j))+exp(-xi2*j)*\
    (C7*sin(eta2*j)+C8*cos(eta2*j))+\
    (C1*Pa31+C2*Pa32+C3*Pa33+C4*Pa34+C5*Pa35+C6*Pa36+C7*Pa37+C8*Pa38):

j:=h:
Wj:=Wj:
Nj:=Nj:
Njtheta:=Njtheta:
Mj:=Mj:
Qj:=Qj:

j:=h:
u(h):=exp(xi1*j)*(c1*(gam1*cos(eta1*j)-del1*sin(eta1*j))+c2*(del1*\
    cos(eta1*j)+gam1*sin(eta1*j)))+exp(-xi1*j)*(c3*(-gam1*cos(eta1*j)-\
    del1*sin(eta1*j))+c4*(del1*cos(eta1*j)-gam1*sin(eta1*j)))+\
    exp(xi2*j)*(c5*(gam2*cos(eta2*j)-del2*sin(eta2*j))+c6*(del2*\
    cos(eta2*j)+gam2*sin(eta2*j)))+exp(-xi2*j)*(c7*(-gam2*cos(eta2*j)-\
    del2*sin(eta2*j))+c8*(del2*cos(eta2*j)-gam2*sin(eta2*j))):

v(h):=exp(xi1*j)*(c1*(mu1*cos(eta1*j)-eps1*sin(eta1*j))+c2*(eps1*\
    cos(eta1*j)+mu1*sin(eta1*j)))+exp(-xi1*j)*(c3*(-mu1*cos(eta1*j)-\
    eps1*sin(eta1*j))+c4*(eps1*cos(eta1*j)-mu1*sin(eta1*j)))+\
    exp(xi2*j)*(c5*(mu2*cos(eta2*j)-eps2*sin(eta2*j))+c6*(eps2*\
    cos(eta2*j)+mu2*sin(eta2*j)))+exp(-xi2*j)*(c7*(-mu2*cos(eta2*j)-\
    eps2*sin(eta2*j))+c8*(eps2*cos(eta2*j)-mu2*sin(eta2*j))):

w(h):=exp(xi1*j)*(c1*sin(eta1*j)+c2*cos(eta1*j))+exp(-xi1*j)*(c3*sin(eta1*j)+\
   c4*cos(eta1*j))+exp(xi2*j)*(c5*sin(eta2*j)+c6*cos(eta2*j))+exp(-xi2*j)*\
   (c7*sin(eta2*j)+c8*cos(eta2*j)):

j:=h:
wj:=wj:
nj:=nj:
njtheta:=njtheta:
mj:=mj:
qj:=qj:

e9:=U(h)-u(h)=0:
e10:=V(h)-v(h)=0:
e11:=W(h)-w(h)=0:
e12:=Wj-wj=0:
e13:=Nj-nj=0:
e14:=Njtheta-njtheta=0:
e15:=Mj-mj=0:
```

```
SolutionSet:=solve({e1,e2,e3,e4,e5,e6,e7,e8,e9,e10,e11,e12,e13,e14,e15},\
          {C1,C2,C3,C4,C5,C6,C7,C8,c1,c2,c3,c4,c5,c6,c7}):
assign(SolutionSet):
```

END OF FILE

VARIABLE THICKNESS SOLUTION FOR PARTLY-FILLED TANK

material property and tanks dimension

	Em = 21091837000	Ds = 7840 Df = 1000	v = 0.3
short tank:	H = 12.19	r0 = 18.29	ts = 0.0254
tall tank :	H = 21.95	r0 = 7.32	ts = 0.0254
water depth :	h = 0.6 H		

element[1] :		
inside :	ts1 = (9/5)ts	outside : ts1 = 2ts
element[2] :		
inside :	ts2 = (8/5)ts	outside : ts2 = (9/5)ts
element[3] :		
inside :	ts3 = (7/5)ts	outside : ts3 = (8/5)ts
element[4] :		
inside :	ts4 = (19/15)ts	outside : ts4 = (7/5)ts
element[5] :		
inside :	ts5 = (17/15)ts	outside : ts5 = (19/15)ts
element[6] :		
inside :	ts6 = ts	outside : ts6 = (17/15)ts

```
writeto(vst3fout):


         dry part: element[4]


L1:=Ds*wf^2*(1-v^2)/Em-(1-v)*(1+ts4^2/(12*r0^2))/(2*r0^2):
L2:=(1+v)/(2*r0):
L3:=-ts4^2/(12*r0):
L4:=(v/r0)-ts4^2*(1-v)/(24*r0^3):
L5:=0.5*(1-v)*(1+ts4^2/(4*r0^2)):
L6:=Ds*wf^2*(1-v^2)/Em-1/r0^2:
L7:=ts4^2*(3-v)/(24*r0^2):
L8:=-1/r0^2:
L9:=ts4^2/12:
L10:=-ts4^2/(6*r0^2):
L11:=(1/r0^2)-Ds*wf^2*(1-v^2)/Em:

C11:=simplify(L5*(L9-L3^2)):
```

```
C22:=simplify(L5*L10+L9*(L6+L1*L5)+L7^2+L9*L2^2-2*L3*L4*L5-L6*L3^2+\
        2*L2*L3*L7):
C33:=simplify(L5*L11+L10*(L6+L1*L5)+L1*L6*L9+2*L7*L8+L1*L7^2+L10*L2^2-\
        L5*L4^2 -2*L3*L4*L6+2*L2*L3*L8+2*L2*L4*L7):
C44:=simplify(L11*(L6+L1*L5)+L1*L6*L10+L8^2+2*L1*L7*L8+L11*L2^2-L6*L4^2\
        +2*L2*L4*L8):
C55:=simplify(L1*L6*L11+L1*L8^2):

eqn:=C11*x^8+C22*x^6+C33*x^4+C44*x^2+C55:
solve(eqn=0,x):
solutions:=":

x1:=solutions[1]:
x2:=solutions[3]:
x3:=solutions[4]:
x4:=solutions[2]:
x5:=solutions[5]:
x6:=solutions[7]:
x7:=solutions[8]:
x8:=solutions[6]:

xi1:=abs(evalc(Re(x1))):
eta1:=abs(evalc(Im(x1))):
xi2:=abs(evalc(Re(x5))):
eta2:=abs(evalc(Im(x5))):

a11(1):=simplify(x1^2+L1):
a11(2):=simplify(x2^2+L1):
a11(3):=simplify(x3^2+L1):
a11(4):=simplify(x4^2+L1):
a11(5):=simplify(x5^2+L1):
a11(6):=simplify(x6^2+L1):
a11(7):=simplify(x7^2+L1):
a11(8):=simplify(x8^2+L1):

a12(1):=simplify(L2*x1):
a12(2):=simplify(L2*x2):
a12(3):=simplify(L2*x3):
a12(4):=simplify(L2*x4):
a12(5):=simplify(L2*x5):
a12(6):=simplify(L2*x6):
a12(7):=simplify(L2*x7):
a12(8):=simplify(L2*x8):

a21(1):=-a12(1):
```

```
a21(2):=-a12(2):
a21(3):=-a12(3):
a21(4):=-a12(4):
a21(5):=-a12(5):
a21(6):=-a12(6):
a21(7):=-a12(7):
a21(8):=-a12(8):

a13(1):=simplify(L3*x1^3+L4*x1):
a13(2):=simplify(L3*x2^3+L4*x2):
a13(3):=simplify(L3*x3^3+L4*x3):
a13(4):=simplify(L3*x4^3+L4*x4):
a13(5):=simplify(L3*x5^3+L4*x5):
a13(6):=simplify(L3*x6^3+L4*x6):
a13(7):=simplify(L3*x7^3+L4*x7):
a13(8):=simplify(L3*x8^3+L4*x8):

a31(1):=a13(1):
a31(2):=a13(2):
a31(3):=a13(3):
a31(4):=a13(4):
a31(5):=a13(5):
a31(6):=a13(6):
a31(7):=a13(7):
a31(8):=a13(8):

a22(1):=simplify(L5*x1^2+L6):
a22(2):=simplify(L5*x2^2+L6):
a22(3):=simplify(L5*x3^2+L6):
a22(4):=simplify(L5*x4^2+L6):
a22(5):=simplify(L5*x5^2+L6):
a22(6):=simplify(L5*x6^2+L6):
a22(7):=simplify(L5*x7^2+L6):
a22(8):=simplify(L5*x8^2+L6):

a23(1):=simplify(L7*x1^2+L8):
a23(2):=simplify(L7*x2^2+L8):
a23(3):=simplify(L7*x3^2+L8):
a23(4):=simplify(L7*x4^2+L8):
a23(5):=simplify(L7*x5^2+L8):
a23(6):=simplify(L7*x6^2+L8):
a23(7):=simplify(L7*x7^2+L8):
a23(8):=simplify(L7*x8^2+L8):

a32(1):=-a23(1):
```

```
a32(2):=-a23(2):
a32(3):=-a23(3):
a32(4):=-a23(4):
a32(5):=-a23(5):
a32(6):=-a23(6):
a32(7):=-a23(7):
a32(8):=-a23(8):

a33(1):=simplify(L9*x1^4+L10*x1^2+L11):
a33(2):=simplify(L9*x2^4+L10*x2^2+L11):
a33(3):=simplify(L9*x3^4+L10*x3^2+L11):
a33(4):=simplify(L9*x4^4+L10*x4^2+L11):
a33(5):=simplify(L9*x5^4+L10*x5^2+L11):
a33(6):=simplify(L9*x6^4+L10*x6^2+L11):
a33(7):=simplify(L9*x7^4+L10*x7^2+L11):
a33(8):=simplify(L9*x8^4+L10*x8^2+L11):

y(1):=simplify((a12(1)*a23(1)-a13(1)*a22(1))/(a11(1)*a22(1)-a12(1)*a21(1))):
y(2):=simplify((a12(2)*a23(2)-a13(2)*a22(2))/(a11(2)*a22(2)-a12(2)*a21(2))):
y(3):=simplify((a12(3)*a23(3)-a13(3)*a22(3))/(a11(3)*a22(3)-a12(3)*a21(3))):
y(4):=simplify((a12(4)*a23(4)-a13(4)*a22(4))/(a11(4)*a22(4)-a12(4)*a21(4))):
y(5):=simplify((a12(5)*a23(5)-a13(5)*a22(5))/(a11(5)*a22(5)-a12(5)*a21(5))):
y(6):=simplify((a12(6)*a23(6)-a13(6)*a22(6))/(a11(6)*a22(6)-a12(6)*a21(6))):
y(7):=simplify((a12(7)*a23(7)-a13(7)*a22(7))/(a11(7)*a22(7)-a12(7)*a21(7))):
y(8):=simplify((a12(8)*a23(8)-a13(8)*a22(8))/(a11(8)*a22(8)-a12(8)*a21(8))):

gam1:=abs(evalc(Re(y(1)))):
del1:=abs(evalc(Im(y(1)))):
gam2:=abs(evalc(Re(y(5)))):
del2:=abs(evalc(Im(y(5)))):

z(1):=simplify((a13(1)*a21(1)-a11(1)*a23(1))/(a11(1)*a22(1)-a12(1)*a21(1))):
z(2):=simplify((a13(2)*a21(2)-a11(2)*a23(2))/(a11(2)*a22(2)-a12(2)*a21(2))):
z(3):=simplify((a13(3)*a21(3)-a11(3)*a23(3))/(a11(3)*a22(3)-a12(3)*a21(3))):
z(4):=simplify((a13(4)*a21(4)-a11(4)*a23(4))/(a11(4)*a22(4)-a12(4)*a21(4))):
z(5):=simplify((a13(5)*a21(5)-a11(5)*a23(5))/(a11(5)*a22(5)-a12(5)*a21(5))):
z(6):=simplify((a13(6)*a21(6)-a11(6)*a23(6))/(a11(6)*a22(6)-a12(6)*a21(6))):
z(7):=simplify((a13(7)*a21(7)-a11(7)*a23(7))/(a11(7)*a22(7)-a12(7)*a21(7))):
z(8):=simplify((a13(8)*a21(8)-a11(8)*a23(8))/(a11(8)*a22(8)-a12(8)*a21(8))):

mu1:=abs(evalc(Re(z(1)))):
eps1:=abs(evalc(Im(z(1)))):
mu2:=abs(evalc(Re(z(5)))):
eps2:=abs(evalc(Im(z(5)))):
```

```
U(z):=exp(xi1*z)*(C1*(gam1*cos(eta1*z)-del1*sin(eta1*z))+C2*(del1*\
      cos(eta1*z)+gam1*sin(eta1*z)))+exp(-xi1*z)*(C3*(-gam1*cos(eta1*z)-\
      del1*sin(eta1*z))+C4*(del1*cos(eta1*z)-gam1*sin(eta1*z)))+\
      exp(xi2*z)*(C5*(gam2*cos(eta2*z)-del2*sin(eta2*z))+C6*(del2*\
      cos(eta2*z)+gam2*sin(eta2*z)))+exp(-xi2*z)*(C7*(-gam2*cos(eta2*z)-\
      del2*sin(eta2*z))+C8*(del2*cos(eta2*z)-gam2*sin(eta2*z))):

Uz:=diff(U(z),z):

V(z):=exp(xi1*z)*(C1*(mu1*cos(eta1*z)-eps1*sin(eta1*z))+C2*(eps1*\
      cos(eta1*z)+mu1*sin(eta1*z)))+exp(-xi1*z)*(C3*(-mu1*cos(eta1*z)-\
      eps1*sin(eta1*z))+C4*(eps1*cos(eta1*z)-mu1*sin(eta1*z)))+\
      exp(xi2*z)*(C5*(mu2*cos(eta2*z)-eps2*sin(eta2*z))+C6*(eps2*\
      cos(eta2*z)+mu2*sin(eta2*z)))+exp(-xi2*z)*(C7*(-mu2*cos(eta2*z)-\
      eps2*sin(eta2*z))+C8*(eps2*cos(eta2*z)-mu2*sin(eta2*z))):

Vz:=diff(V(z),z):

W(z):=exp(xi1*z)*(C1*sin(eta1*z)+C2*cos(eta1*z))+exp(-xi1*z)*(C3*sin(eta1*z)+\
      C4*cos(eta1*z))+exp(xi2*z)*(C5*sin(eta2*z)+C6*cos(eta2*z))+exp(-xi2*z)*\
      (C7*sin(eta2*z)+C8*cos(eta2*z)):

Wz:=diff(W(z),z):
Wzz:=diff(Wz,z):

Nz:=Uz+(v/r0)*(V(z)+W(z)):
Nztheta:=Vz-U(z)/r0:
Mz:=Wzz-(v/r0^2)*W(z):
Mzz:=diff(Mz,z):
Qz:=Mzz:

e1:=U(z)=0:
e2:=V(z)=0:
e3:=W(z)=0:
e4:=Wz=0:
e5:=Nz=0:
e6:=Nztheta=0:
e7:=Mz=0:
e8:=Qz=0:
z:=(11/15)*H:

with(linalg):
P8:=genmatrix([e1,e2,e3,e4,e5,e6,e7,e8],[C1,C2,C3,C4,C5,C6,C7,C8]);

z:='z':
```

```
e1:=U(z)=0:
e2:=V(z)=0:
e3:=W(z)=0:
e4:=Wz=0:
e5:=Nz=0:
e6:=Nztheta=0:
e7:=Mz=0:
e8:=Qz=0:
z:=(3/5)*H:

with(linalg):
P7:=genmatrix([e1,e2,e3,e4,e5,e6,e7,e8],[C1,C2,C3,C4,C5,C6,C7,C8]):

P7i:=inverse(P7);
```

dry part: element[5]

```
z:='z':
L1:=Ds*wf^2*(1-v^2)/Em-(1-v)*(1+ts5^2/(12*r0^2))/(2*r0^2):
L2:=(1+v)/(2*r0):
L3:=-ts5^2/(12*r0):
L4:=(v/r0)-ts5^2*(1-v)/(24*r0^3):
L5:=0.5*(1-v)*(1+ts5^2/(4*r0^2)):
L6:=Ds*wf^2*(1-v^2)/Em-1/r0^2:
L7:=ts5^2*(3-v)/(24*r0^2):
L8:=-1/r0^2:
L9:=ts5^2/12:
L10:=-ts5^2/(6*r0^2):
L11:=(1/r0^2)-Ds*wf^2*(1-v^2)/Em:

e1:=U(z)=0:
e2:=V(z)=0:
e3:=W(z)=0:
e4:=Wz=0:
e5:=Nz=0:
e6:=Nztheta=0:
e7:=Mz=0:
e8:=Qz=0:
z:=(13/15)*H:

with(linalg):
P10:=genmatrix([e1,e2,e3,e4,e5,e6,e7,e8],[C1,C2,C3,C4,C5,C6,C7,C8]);
```

```
z:='z':
e1:=U(z)=0:
e2:=V(z)=0:
e3:=W(z)=0:
e4:=Wz=0:
e5:=Nz=0:
e6:=Nztheta=0:
e7:=Mz=0:
e8:=Qz=0:
z:=(11/15)*H:

with(linalg):
P9:=genmatrix([e1,e2,e3,e4,e5,e6,e7,e8],[C1,C2,C3,C4,C5,C6,C7,C8]):

P9i:=inverse(P9);
```

dry part: element[6]

```
z:='z':
L1:=Ds*wf^2*(1-v^2)/Em-(1-v)*(1+ts6^2/(12*r0^2))/(2*r0^2):
L2:=(1+v)/(2*r0):
L3:=-ts6^2/(12*r0):
L4:=(v/r0)-ts6^2*(1-v)/(24*r0^3):
L5:=0.5*(1-v)*(1+ts6^2/(4*r0^2)):
L6:=Ds*wf^2*(1-v^2)/Em-1/r0^2:
L7:=ts6^2*(3-v)/(24*r0^2):
L8:=-1/r0^2:
L9:=ts6^2/12:
L10:=-ts6^2/(6*r0^2):
L11:=(1/r0^2)-Ds*wf^2*(1-v^2)/Em:

e1:=U(z)=0:
e2:=V(z)=0:
e3:=W(z)=0:
e4:=Wz=0:
e5:=Nz=0:
e6:=Nztheta=0:
e7:=Mz=0:
e8:=Qz=0:
z:=1*H:

with(linalg):
P12:=genmatrix([e1,e2,e3,e4,e5,e6,e7,e8],[C1,C2,C3,C4,C5,C6,C7,C8]);
```

```
z:='z':
e1:=U(z)=0:
e2:=V(z)=0:
e3:=W(z)=0:
e4:=Wz=0:
e5:=Nz=0:
e6:=Nztheta=0:
e7:=Mz=0:
e8:=Qz=0:
z:=(13/15)*H:

with(linalg):
P11:=genmatrix([e1,e2,e3,e4,e5,e6,e7,e8],[C1,C2,C3,C4,C5,C6,C7,C8]):

P11i:=inverse(P11);
```

wet part: element[1]

```
z:='z':

h1:=0.2*H:
eqn:=wf^2*cos(la(m)*h1)+9.8*la(m)*sin(la(m)*h1):
fsolve(eqn=0,la(m)):
solutions:='':
la(m):=solutions:

L1:=Ds*wf^2*(1-v^2)/Em-(1-v)*(1+ts1^2/(12*r0^2))/(2*r0^2):
L2:=(1+v)/(2*r0):
L3:=-ts1^2/(12*r0):
L4:=(v/r0)-ts1^2*(1-v)/(24*r0^3):
L5:=0.5*(1-v)*(1+ts1^2/(4*r0^2)):
L6:=Ds*wf^2*(1-v^2)/Em-1/r0^2:
L7:=ts1^2*(3-v)/(24*r0^2):
L8:=-1/r0^2:
L9:=ts1^2/12:
L10:=-ts1^2/(6*r0^2):
L11:=(1/r0^2)-Ds*wf^2*(1-v^2)/Em:

C11:=simplify(L5*(L9-L3^2)):
C22:=simplify(L5*L10+L9*(L6+L1*L5)+L7^2+L9*L2^2-2*L3*L4*L5-\
            L6*L3^2+2*L2*L3*L7):
C33:=simplify(L5*L11+L10*(L6+L1*L5)+L1*L6*L9+2*L7*L8+L1*L7^2+L10*L2^2-\
```

```
              L5*L4^2-2*L3*L4*L6+2*L2*L3*L8+2*L2*L4*L7):
C44:=simplify(L11*(L6+L1*L5)+L1*L6*L10+L8^2+2*L1*L7*L8+L11*L2^2-L6*L4^2\
             +2*L2*L4*L8):
C55:=simplify(L1*L6*L11+L1*L8^2):

eqn:=C11*x^8+C22*x^6+C33*x^4+C44*x^2+C55:
solve(eqn=0,x):
solutions:=":

x1:=solutions[1]:
x2:=solutions[3]:
x3:=solutions[4]:
x4:=solutions[2]:
x5:=solutions[5]:
x6:=solutions[7]:
x7:=solutions[8]:
x8:=solutions[6]:

xi1:=abs(evalc(Re(x1))):
eta1:=abs(evalc(Im(x1))):
xi2:=abs(evalc(Re(x5))):
eta2:=abs(evalc(Im(x5))):

a11(1):=simplify(x1^2+L1):
a11(2):=simplify(x2^2+L1):
a11(3):=simplify(x3^2+L1):
a11(4):=simplify(x4^2+L1):
a11(5):=simplify(x5^2+L1):
a11(6):=simplify(x6^2+L1):
a11(7):=simplify(x7^2+L1):
a11(8):=simplify(x8^2+L1):

a12(1):=simplify(L2*x1):
a12(2):=simplify(L2*x2):
a12(3):=simplify(L2*x3):
a12(4):=simplify(L2*x4):
a12(5):=simplify(L2*x5):
a12(6):=simplify(L2*x6):
a12(7):=simplify(L2*x7):
a12(8):=simplify(L2*x8):

a21(1):=-a12(1):
a21(2):=-a12(2):
a21(3):=-a12(3):
a21(4):=-a12(4):
```

```
a21(5):=-a12(5):
a21(6):=-a12(6):
a21(7):=-a12(7):
a21(8):=-a12(8):

a13(1):=simplify(L3*x1^3+L4*x1):
a13(2):=simplify(L3*x2^3+L4*x2):
a13(3):=simplify(L3*x3^3+L4*x3):
a13(4):=simplify(L3*x4^3+L4*x4):
a13(5):=simplify(L3*x5^3+L4*x5):
a13(6):=simplify(L3*x6^3+L4*x6):
a13(7):=simplify(L3*x7^3+L4*x7):
a13(8):=simplify(L3*x8^3+L4*x8):

a31(1):=a13(1):
a31(2):=a13(2):
a31(3):=a13(3):
a31(4):=a13(4):
a31(5):=a13(5):
a31(6):=a13(6):
a31(7):=a13(7):
a31(8):=a13(8):

a22(1):=simplify(L5*x1^2+L6):
a22(2):=simplify(L5*x2^2+L6):
a22(3):=simplify(L5*x3^2+L6):
a22(4):=simplify(L5*x4^2+L6):
a22(5):=simplify(L5*x5^2+L6):
a22(6):=simplify(L5*x6^2+L6):
a22(7):=simplify(L5*x7^2+L6):
a22(8):=simplify(L5*x8^2+L6):

a23(1):=simplify(L7*x1^2+L8):
a23(2):=simplify(L7*x2^2+L8):
a23(3):=simplify(L7*x3^2+L8):
a23(4):=simplify(L7*x4^2+L8):
a23(5):=simplify(L7*x5^2+L8):
a23(6):=simplify(L7*x6^2+L8):
a23(7):=simplify(L7*x7^2+L8):
a23(8):=simplify(L7*x8^2+L8):

a32(1):=-a23(1):
a32(2):=-a23(2):
a32(3):=-a23(3):
a32(4):=-a23(4):
```

```
a32(5):=-a23(5):
a32(6):=-a23(6):
a32(7):=-a23(7):
a32(8):=-a23(8):

a33(1):=simplify(L9*x1^4+L10*x1^2+L11):
a33(2):=simplify(L9*x2^4+L10*x2^2+L11):
a33(3):=simplify(L9*x3^4+L10*x3^2+L11):
a33(4):=simplify(L9*x4^4+L10*x4^2+L11):
a33(5):=simplify(L9*x5^4+L10*x5^2+L11):
a33(6):=simplify(L9*x6^4+L10*x6^2+L11):
a33(7):=simplify(L9*x7^4+L10*x7^2+L11):
a33(8):=simplify(L9*x8^4+L10*x8^2+L11):

y1:=simplify((a12(1)*a23(1)-a13(1)*a22(1))/(a11(1)*a22(1)-a12(1)*a21(1))):
y2:=simplify((a12(2)*a23(2)-a13(2)*a22(2))/(a11(2)*a22(2)-a12(2)*a21(2))):
y3:=simplify((a12(3)*a23(3)-a13(3)*a22(3))/(a11(3)*a22(3)-a12(3)*a21(3))):
y4:=simplify((a12(4)*a23(4)-a13(4)*a22(4))/(a11(4)*a22(4)-a12(4)*a21(4))):
y5:=simplify((a12(5)*a23(5)-a13(5)*a22(5))/(a11(5)*a22(5)-a12(5)*a21(5))):
y6:=simplify((a12(6)*a23(6)-a13(6)*a22(6))/(a11(6)*a22(6)-a12(6)*a21(6))):
y7:=simplify((a12(7)*a23(7)-a13(7)*a22(7))/(a11(7)*a22(7)-a12(7)*a21(7))):
y8:=simplify((a12(8)*a23(8)-a13(8)*a22(8))/(a11(8)*a22(8)-a12(8)*a21(8))):

gam1:=abs(evalc(Re(y1))):
del1:=abs(evalc(Im(y1))):
gam2:=abs(evalc(Re(y5))):
del2:=abs(evalc(Im(y5))):

z1:=simplify((a13(1)*a21(1)-a11(1)*a23(1))/(a11(1)*a22(1)-a12(1)*a21(1))):
z2:=simplify((a13(2)*a21(2)-a11(2)*a23(2))/(a11(2)*a22(2)-a12(2)*a21(2))):
z3:=simplify((a13(3)*a21(3)-a11(3)*a23(3))/(a11(3)*a22(3)-a12(3)*a21(3))):
z4:=simplify((a13(4)*a21(4)-a11(4)*a23(4))/(a11(4)*a22(4)-a12(4)*a21(4))):
z5:=simplify((a13(5)*a21(5)-a11(5)*a23(5))/(a11(5)*a22(5)-a12(5)*a21(5))):
z6:=simplify((a13(6)*a21(6)-a11(6)*a23(6))/(a11(6)*a22(6)-a12(6)*a21(6))):
z7:=simplify((a13(7)*a21(7)-a11(7)*a23(7))/(a11(7)*a22(7)-a12(7)*a21(7))):
z8:=simplify((a13(8)*a21(8)-a11(8)*a23(8))/(a11(8)*a22(8)-a12(8)*a21(8))):

mu1:=abs(evalc(Re(z1))):
eps1:=abs(evalc(Im(z1))):
mu2:=abs(evalc(Re(z5))):
eps2:=abs(evalc(Im(z5))):

b11(m):=simplify(L1-la(m)^2):
b12(m):=simplify(-L2*la(m)):
b13(m):=simplify(L3*la(m)^3-L4*la(m)):
```

```
b21(m):=b12(m):
b22(m):=simplify(L6-L5*la(m)^2):
b23(m):=simplify(-L7*la(m)^2+L8):
b31(m):=-b13(m):
b32(m):=-b23(m):
b33(m):=simplify(L9*la(m)^4-L10*la(m)^2+L11):


with(linalg):
b(m):=matrix([[b11(m),b12(m),b13(m)],[b21(m),b22(m),b23(m)],[b31(m),b32(m),\
            b33(m)]]):
D(m):=det(b(m)):


B1(m):=simplify((b12(m)*b23(m)-b13(m)*b22(m))/D(m)):
B2(m):=simplify((b13(m)*b21(m)-b11(m)*b23(m))/D(m)):
B3(m):=simplify((b11(m)*b22(m)-b12(m)*b21(m))/D(m)):

alias(I=BesselI):
I1(m):=I(1,la(m)*r0):
I2(m):=I(0,la(m)*r0)/2+I(2,la(m)*r0)/2:
I(m):=I1(m)/I2(m):

P(m):=simplify(2*Df*(1-v^2)*wf^2*I(m)/Em*ts1*(2*la(m)^2*h1+\
            la(m)*sin(2*la(m)*h1))^(1/2)):

f1(m):=simplify(int(exp(xi1*z)*sin(eta1*z)*cos(la(m)*z),z=0..h1)):
f2(m):=simplify(int(exp(xi1*z)*cos(eta1*z)*cos(la(m)*z),z=0..h1)):
f3(m):=simplify(int(exp(-xi1*z)*sin(eta1*z)*cos(la(m)*z),z=0..h1)):
f4(m):=simplify(int(exp(-xi1*z)*cos(eta1*z)*cos(la(m)*z),z=0..h1)):
f5(m):=simplify(int(exp(xi2*z)*sin(eta2*z)*cos(la(m)*z),z=0..h1)):
f6(m):=simplify(int(exp(xi2*z)*cos(eta2*z)*cos(la(m)*z),z=0..h1)):
f7(m):=simplify(int(exp(-xi2*z)*sin(eta2*z)*cos(la(m)*z),z=0..h1)):
f8(m):=simplify(int(exp(-xi2*z)*cos(eta2*z)*cos(la(m)*z),z=0..h1)):

Pa11:=simplify(sum(sin(la(m)*z)*B1(m)*P(m)*f1(m),m=1..10))
Pa12:=simplify(sum(sin(la(m)*z)*B1(m)*P(m)*f2(m),m=1..10))
Pa13:=simplify(sum(sin(la(m)*z)*B1(m)*P(m)*f3(m),m=1..10))
Pa14:=simplify(sum(sin(la(m)*z)*B1(m)*P(m)*f4(m),m=1..10))
Pa15:=simplify(sum(sin(la(m)*z)*B1(m)*P(m)*f5(m),m=1..10))
Pa16:=simplify(sum(sin(la(m)*z)*B1(m)*P(m)*f6(m),m=1..10))
Pa17:=simplify(sum(sin(la(m)*z)*B1(m)*P(m)*f7(m),m=1..10))
Pa18:=simplify(sum(sin(la(m)*z)*B1(m)*P(m)*f8(m),m=1..10))

Pa21:=simplify(sum(cos(la(m)*z)*B2(m)*P(m)*f1(m),m=1..10))
```

```
Pa22:=simplify(sum(cos(la(m)*z)*B2(m)*P(m)*f2(m),m=1..10))
Pa23:=simplify(sum(cos(la(m)*z)*B2(m)*P(m)*f3(m),m=1..10))
Pa24:=simplify(sum(cos(la(m)*z)*B2(m)*P(m)*f4(m),m=1..10))
Pa25:=simplify(sum(cos(la(m)*z)*B2(m)*P(m)*f5(m),m=1..10))
Pa26:=simplify(sum(cos(la(m)*z)*B2(m)*P(m)*f6(m),m=1..10))
Pa27:=simplify(sum(cos(la(m)*z)*B2(m)*P(m)*f7(m),m=1..10))
Pa28:=simplify(sum(cos(la(m)*z)*B2(m)*P(m)*f8(m),m=1..10))

Pa31:=simplify(sum(cos(la(m)*z)*B3(m)*P(m)*f1(m),m=1..10))
Pa32:=simplify(sum(cos(la(m)*z)*B3(m)*P(m)*f2(m),m=1..10))
Pa33:=simplify(sum(cos(la(m)*z)*B3(m)*P(m)*f3(m),m=1..10))
Pa34:=simplify(sum(cos(la(m)*z)*B3(m)*P(m)*f4(m),m=1..10))
Pa35:=simplify(sum(cos(la(m)*z)*B3(m)*P(m)*f5(m),m=1..10))
Pa36:=simplify(sum(cos(la(m)*z)*B3(m)*P(m)*f6(m),m=1..10))
Pa37:=simplify(sum(cos(la(m)*z)*B3(m)*P(m)*f7(m),m=1..10))
Pa38:=simplify(sum(cos(la(m)*z)*B3(m)*P(m)*f8(m),m=1..10))

U(z):=exp(xi1*z)*(C1*(gam1*cos(eta1*z)-del1*sin(eta1*z))+C2*(del1*\
    cos(eta1*z)+gam1*sin(eta1*z)))+exp(-xi1*z)*(C3*(-gam1*cos(eta1*z)-\
    del1*sin(eta1*z))+C4*(del1*cos(eta1*z)-gam1*sin(eta1*z)))+\
    exp(xi2*z)*(C5*(gam2*cos(eta2*z)-del2*sin(eta2*z))+C6*(del2*\
    cos(eta2*z)+gam2*sin(eta2*z)))+exp(-xi2*z)*(C7*(-gam2*cos(eta2*z)-\
    del2*sin(eta2*z))+C8*(del2*cos(eta2*z)-gam2*sin(eta2*z)))+\
    (C1*Pa11+C2*Pa12+C3*Pa13+C4*Pa14+C5*Pa15+C6*Pa16+C7*Pa17+C8*Pa18):

Uz:=diff(U(z),z):

V(z):=exp(xi1*z)*(C1*(mu1*cos(eta1*z)-eps1*sin(eta1*z))+C2*(eps1*\
    cos(eta1*z)+mu1*sin(eta1*z)))+exp(-xi1*z)*(C3*(-mu1*cos(eta1*z)-\
    eps1*sin(eta1*z))+C4*(eps1*cos(eta1*z)-mu1*sin(eta1*z)))+\
    exp(xi2*z)*(C5*(mu2*cos(eta2*z)-eps2*sin(eta2*z))+C6*(eps2*\
    cos(eta2*z)+mu2*sin(eta2*z)))+exp(-xi2*z)*(C7*(-mu2*cos(eta2*z)-\
    eps2*sin(eta2*z))+C8*(eps2*cos(eta2*z)-mu2*sin(eta2*z)))+\
    (C1*Pa21+C2*Pa22+C3*Pa23+C4*Pa24+C5*Pa25+C6*Pa26+C7*Pa27+C8*Pa28):

Vz:=diff(V(z),z):

W(z):=exp(xi1*z)*(C1*sin(eta1*z)+C2*cos(eta1*z))+exp(-xi1*z)*(C3*sin(eta1*z)+\
    C4*cos(eta1*z))+exp(xi2*z)*(C5*sin(eta2*z)+C6*cos(eta2*z))+exp(-xi2*z)*\
    (C7*sin(eta2*z)+C8*cos(eta2*z))+\
    (C1*Pa31+C2*Pa32+C3*Pa33+C4*Pa34+C5*Pa35+C6*Pa36+C7*Pa37+C8*Pa38):


Wz:=diff(W(z),z):
Wzz:=diff(Wz,z):
```

```
Nz:=Uz+(v/r0)*(V(z)+W(z)):
Nztheta:=Vz-U(z)/r0:
Mz:=Wzz-(v/r0^2)*W(z):
Mzz:=diff(Mz,z):
Qz:=Mzz:

e1:=U(z)=0:
e2:=V(z)=0:
e3:=W(z)=0:
e4:=Wz=0:
e5:=Nz=0:
e6:=Nztheta=0:
e7:=Mz=0:
e8:=Qz=0:
z:=(1/5)*H:

with(linalg):
P2:=genmatrix([e1,e2,e3,e4,e5,e6,e7,e8],[C1,C2,C3,C4,C5,C6,C7,C8]);

e1:=U(z)=0:
e2:=V(z)=0:
e3:=W(z)=0:
e4:=Wz=0:
e5:=Nz=0:
e6:=Nztheta=0:
e7:=Mz=0:
e8:=Qz=0:
z:=0:

with(linalg):
P1:=genmatrix([e1,e2,e3,e4,e5,e6,e7,e8],[C1,C2,C3,C4,C5,C6,C7,C8]):
P1i:=inverse(P1);
```

wet part: element[2]

```
la(m):='la(m)':
z:='z':
h2:=0.4*H:
eqn:=wf^2*cos(la(m)*h2)+9.8*la(m)*sin(la(m)*h2):
fsolve(eqn=0,la(m)):
solutions:='':
la(m):=solutions:
```

```
L1:=Ds*wf^2*(1-v^2)/Em-(1-v)*(1+ts2^2/(12*r0^2))/(2*r0^2):
L2:=(1+v)/(2*r0):
L3:=-ts2^2/(12*r0):
L4:=(v/r0)-ts2^2*(1-v)/(24*r0^3):
L5:=0.5*(1-v)*(1+ts2^2/(4*r0^2)):
L6:=Ds*wf^2*(1-v^2)/Em-1/r0^2:
L7:=ts2^2*(3-v)/(24*r0^2):
L8:=-1/r0^2:
L9:=ts2^2/12:
L10:=-ts2^2/(6*r0^2):
L11:=(1/r0^2)-Ds*wf^2*(1-v^2)/Em:

e1:=U(z)=0:
e2:=V(z)=0:
e3:=W(z)=0:
e4:=Wz=0:
e5:=Nz=0:
e6:=Nztheta=0:
e7:=Mz=0:
e8:=Qz=0:
z:=(2/5)*H:

with(linalg):
P4:=genmatrix([e1,e2,e3,e4,e5,e6,e7,e8],[C1,C2,C3,C4,C5,C6,C7,C8]);

e1:=U(z)=0:
e2:=V(z)=0:
e3:=W(z)=0:
e4:=Wz=0:
e5:=Nz=0:
e6:=Nztheta=0:
e7:=Mz=0:
e8:=Qz=0:
z:=(1/5)*H:

with(linalg):
P3:=genmatrix([e1,e2,e3,e4,e5,e6,e7,e8],[C1,C2,C3,C4,C5,C6,C7,C8]):
P3i:=inverse(P3);
```

wet part: element[3]

```
la(m):='la(m)':
```

```
z:='z':
h3:=0.6*H:
eqn:=wf^2*cos(la(m)*h3)+9.8*la(m)*sin(la(m)*h3):
fsolve(eqn=0,la(m)):
solutions:='':
la(m):=solutions:

L1:=Ds*wf^2*(1-v^2)/Em-(1-v)*(1+ts3^2/(12*r0^2))/(2*r0^2):
L2:=(1+v)/(2*r0):
L3:=-ts3^2/(12*r0):
L4:=(v/r0)-ts3^2*(1-v)/(24*r0^3):
L5:=0.5*(1-v)*(1+ts3^2/(4*r0^2)):
L6:=Ds*wf^2*(1-v^2)/Em-1/r0^2:
L7:=ts3^2*(3-v)/(24*r0^2):
L8:=-1/r0^2:
L9:=ts3^2/12:
L10:=-ts3^2/(6*r0^2):
L11:=(1/r0^2)-Ds*wf^2*(1-v^2)/Em:

e1:=U(z)=0:
e2:=V(z)=0:
e3:=W(z)=0:
e4:=Wz=0:
e5:=Nz=0:
e6:=Nztheta=0:
e7:=Mz=0:
e8:=Qz=0:
z:=(3/5)*H:

with(linalg):
P6:=genmatrix([e1,e2,e3,e4,e5,e6,e7,e8],[C1,C2,C3,C4,C5,C6,C7,C8]);

e1:=U(z)=0:
e2:=V(z)=0:
e3:=W(z)=0:
e4:=Wz=0:
e5:=Nz=0:
e6:=Nztheta=0:
e7:=Mz=0:
e8:=Qz=0:
z:=(2/5)*H:

with(linalg):
P5:=genmatrix([e1,e2,e3,e4,e5,e6,e7,e8],[C1,C2,C3,C4,C5,C6,C7,C8]):
P5i:=inverse(P5);
```

```
S:=multiply(P12,P11i,P10,P9i,P8,P7i,P6,P5i,P4,P3i,P2,P1i);
K:=submatrix(S,5..8,5..8);

A:=det(K);
```

END OF FILE

VARIABLE THICKNESS SOLUTION FOR PARTLY-FILLED TANK

(MODE SHAPE)

natural frequency:

short tank	w = 9.419	w = 9.781
tall tank	w = 11.040	w = 11.910

```
writeto(vms3out):

axial mode shape for partly-filled tank (dry part)

U(z):=exp(xi1*z)*(C1*(gam1*cos(eta1*z)-del1*sin(eta1*z))+C2*(del1*\
    cos(eta1*z)+gam1*sin(eta1*z)))+exp(-xi1*z)*(C3*(-gam1*cos(eta1*z)-\
    del1*sin(eta1*z))+C4*(del1*cos(eta1*z)-gam1*sin(eta1*z)))+\
    exp(xi2*z)*(C5*(gam2*cos(eta2*z)-del2*sin(eta2*z))+C6*(del2*\
    cos(eta2*z)+gam2*sin(eta2*z)))+exp(-xi2*z)*(C7*(-gam2*cos(eta2*z)-\
    del2*sin(eta2*z))+C8*(del2*cos(eta2*z)-gam2*sin(eta2*z))):

radial mode shape for partly-filled tank (dry part)

W(z):=exp(xi1*z)*(C1*sin(eta1*z)+C2*cos(eta1*z))+exp(-xi1*z)*(C3*sin(eta1*z)+\
    C4*cos(eta1*z))+exp(xi2*z)*(C5*sin(eta2*z)+C6*cos(eta2*z))+exp(-xi2*z)*\
    (C7*sin(eta2*z)+C8*cos(eta2*z)):

axial mode shape for partly-filled tank (wet part)

U(z):=exp(xi1*z)*(C1*(gam1*cos(eta1*z)-del1*sin(eta1*z))+C2*(del1*\
    cos(eta1*z)+gam1*sin(eta1*z)))+exp(-xi1*z)*(C3*(-gam1*cos(eta1*z)-\
    del1*sin(eta1*z))+C4*(del1*cos(eta1*z)-gam1*sin(eta1*z)))+\
    exp(xi2*z)*(C5*(gam2*cos(eta2*z)-del2*sin(eta2*z))+C6*(del2*\
    cos(eta2*z)+gam2*sin(eta2*z)))+exp(-xi2*z)*(C7*(-gam2*cos(eta2*z)-\
    del2*sin(eta2*z))+C8*(del2*cos(eta2*z)-gam2*sin(eta2*z)))+\
    (C1*Pa11+C2*Pa12+C3*Pa13+C4*Pa14+C5*Pa15+C6*Pa16+C7*Pa17+C8*Pa18):

radial mode shape for partly-filled tank (wet part)
```

```
W(z):=exp(xi1*z)*(C1*sin(eta1*z)+C2*cos(eta1*z))+exp(-xi1*z)*(C3*sin(eta1*z)+\
  C4*cos(eta1*z))+exp(xi2*z)*(C5*sin(eta2*z)+C6*cos(eta2*z))+exp(-xi2*z)*\
  (C7*sin(eta2*z)+C8*cos(eta2*z))+\
  (C1*Pa31+C2*Pa32+C3*Pa33+C4*Pa34+C5*Pa35+C6*Pa36+C7*Pa37+C8*Pa38):

s15:=det(submatrix(S,1..1,5..5)):
s16:=det(submatrix(S,1..1,6..6)):
s17:=det(submatrix(S,1..1,7..7)):
s18:=det(submatrix(S,1..1,8..8)):
s25:=det(submatrix(S,2..2,5..5)):
s26:=det(submatrix(S,2..2,6..6)):
s27:=det(submatrix(S,2..2,7..7)):
s28:=det(submatrix(S,2..2,8..8)):
s35:=det(submatrix(S,3..3,5..5)):
s36:=det(submatrix(S,3..3,6..6)):
s37:=det(submatrix(S,3..3,7..7)):
s38:=det(submatrix(S,3..3,8..8)):
s45:=det(submatrix(S,4..4,5..5)):
s46:=det(submatrix(S,4..4,6..6)):
s47:=det(submatrix(S,4..4,7..7)):
s48:=det(submatrix(S,4..4,8..8)):
s55:=det(submatrix(S,5..5,5..5)):
s56:=det(submatrix(S,5..5,6..6)):
s57:=det(submatrix(S,5..5,7..7)):
s58:=det(submatrix(S,5..5,8..8)):
s65:=det(submatrix(S,6..6,5..5)):
s66:=det(submatrix(S,6..6,6..6)):
s67:=det(submatrix(S,6..6,7..7)):
s68:=det(submatrix(S,6..6,8..8)):
s75:=det(submatrix(S,7..7,5..5)):
s76:=det(submatrix(S,7..7,6..6)):
s77:=det(submatrix(S,7..7,7..7)):
s78:=det(submatrix(S,7..7,8..8)):
```

solve forces for node 0

```
z:='z':
Qz0:=1:

e5:=s55*Nz0+s56*Nztheta0+s57*Mz0+s58=0:
e6:=s65*Nz0+s66*Nztheta0+s67*Mz0+s68=0:
e7:=s75*Nz0+s76*Nztheta0+s77*Mz0+s78=0:
```

```
SolutionSet:=solve({e5,e6,e7},{Nz0,Nztheta0,Mz0});
assign(SolutionSet);
```

use the results of node 0 to get the displacements for node 6

```
Uz6:=simplify(s15*Nz0+s16*Nztheta0+s17*Mz0+s18*Qz0);
Vz6:=simplify(s25*Nz0+s26*Nztheta0+s27*Mz0+s28*Qz0);
Wz6:=simplify(s35*Nz0+s36*Nztheta0+s37*Mz0+s38*Qz0);
DWz6:=simplify(s45*Nz0+s46*Nztheta0+s47*Mz0+s48*Qz0);
```

solve the displacements and forces for node 5

```
z:='z':

P65:=multiply(P12,P11i):
R5:=vector([Uz5,Vz5,Wz5,DWz5,Nz5,Nztheta5,Mz5,Qz5]):
Vec5:=multiply(P65,R5):
MA5:=matrix(8,1,Vec5):

e1:=det(submatrix(MA5,1..1,1..1))=Uz6:
e2:=det(submatrix(MA5,2..2,1..1))=Vz6:
e3:=det(submatrix(MA5,3..3,1..1))=Wz6:
e4:=det(submatrix(MA5,4..4,1..1))=DWz6:
e5:=det(submatrix(MA5,5..5,1..1))=0:
e6:=det(submatrix(MA5,6..6,1..1))=0:
e7:=det(submatrix(MA5,7..7,1..1))=0:
e8:=det(submatrix(MA5,8..8,1..1))=0:

SolutionSet:=solve({e1,e2,e3,e4,e5,e6,e7,e8},{Uz5,Vz5,Wz5,DWz5,Nz5,Nztheta5,\
                Mz5,Qz5});
assign(SolutionSet);
```

use the results of node 5 to get the displacements and forces for node 4

```
z:='z':
P54:=multiply(P10,P9i):
R4:=vector([Uz4,Vz4,Wz4,DWz4,Nz4,Nztheta4,Mz4,Qz4]):
Vec4:=multiply(P54,R4):
MA4:=matrix(8,1,Vec4):

e1:=det(submatrix(MA4,1..1,1..1))=Uz5:
e2:=det(submatrix(MA4,2..2,1..1))=Vz5:
e3:=det(submatrix(MA4,3..3,1..1))=Wz5:
e4:=det(submatrix(MA4,4..4,1..1))=DWz5:
e5:=det(submatrix(MA4,5..5,1..1))=Nz5:
```

```
e6:=det(submatrix(MA4,6..6,1..1))=Nztheta5:
e7:=det(submatrix(MA4,7..7,1..1))=Mz5:
e8:=det(submatrix(MA4,8..8,1..1))=Qz5:

SolutionSet:=solve({e1,e2,e3,e4,e5,e6,e7,e8},{Uz4,Vz4,Wz4,DWz4,Nz4,Nztheta4,\
                Mz4,Qz4});
assign(SolutionSet);
```

use the results of node 4 to get the displacements and forces for node 3

```
z:='z':
P43:=multiply(P8,P7i):
R3:=vector([Uz3,Vz3,Wz3,DWz3,Nz3,Nztheta3,Mz3,Qz3]):
Vec3:=multiply(P43,R3):
MA3:=matrix(8,1,Vec3):

e1:=det(submatrix(MA3,1..1,1..1))=Uz4:
e2:=det(submatrix(MA3,2..2,1..1))=Vz4:
e3:=det(submatrix(MA3,3..3,1..1))=Wz4:
e4:=det(submatrix(MA3,4..4,1..1))=DWz4:
e5:=det(submatrix(MA3,5..5,1..1))=Nz4:
e6:=det(submatrix(MA3,6..6,1..1))=Nztheta4:
e7:=det(submatrix(MA3,7..7,1..1))=Mz4:
e8:=det(submatrix(MA3,8..8,1..1))=Qz4:

SolutionSet:=solve({e1,e2,e3,e4,e5,e6,e7,e8},{Uz3,Vz3,Wz3,DWz3,Nz3,Nztheta3,\
                Mz3,Qz3});
assign(SolutionSet);
```

use the results of node 3 to get the displacements and forces for node 2

```
z:='z':
P32:=multiply(P6,P5i):
R2:=vector([Uz2,Vz2,Wz2,DWz2,Nz2,Nztheta2,Mz2,Qz2]):
Vec2:=multiply(P32,R2):
MA2:=matrix(8,1,Vec2):

e1:=det(submatrix(MA2,1..1,1..1))=Uz3:
e2:=det(submatrix(MA2,2..2,1..1))=Vz3:
e3:=det(submatrix(MA2,3..3,1..1))=Wz3:
e4:=det(submatrix(MA2,4..4,1..1))=DWz3:
e5:=det(submatrix(MA2,5..5,1..1))=Nz3:
e6:=det(submatrix(MA2,6..6,1..1))=Nztheta3:
e7:=det(submatrix(MA2,7..7,1..1))=Mz3:
e8:=det(submatrix(MA2,8..8,1..1))=Qz3:
```

```
SolutionSet:=solve({e1,e2,e3,e4,e5,e6,e7,e8},{Uz2,Vz2,Wz2,DWz2,Nz2,Nztheta2,\
                Mz2,Qz2});
assign(SolutionSet);
```

use the results of node 2 to get the displacements and forces for node 1

```
z:='z':
P21:=multiply(P4,P3i):
R1:=vector([Uz1,Vz1,Wz1,DWz1,Nz1,Nztheta1,Mz1,Qz1]):
Vec1:=multiply(P21,R1):
MA1:=matrix(8,1,Vec1):

e1:=det(submatrix(MA1,1..1,1..1))=Uz2:
e2:=det(submatrix(MA1,2..2,1..1))=Vz2:
e3:=det(submatrix(MA1,3..3,1..1))=Wz2:
e4:=det(submatrix(MA1,4..4,1..1))=DWz2:
e5:=det(submatrix(MA1,5..5,1..1))=Nz2:
e6:=det(submatrix(MA1,6..6,1..1))=Nztheta2:
e7:=det(submatrix(MA1,7..7,1..1))=Mz2:
e8:=det(submatrix(MA1,8..8,1..1))=Qz2:

SolutionSet:=solve({e1,e2,e3,e4,e5,e6,e7,e8},{Uz1,Vz1,Wz1,DWz1,Nz1,Nztheta1,\
                Mz1,Qz1});
assign(SolutionSet);
```

solve coefficents of element [1]

```
z:='z':
e1:=U(z)=Uz1:
e2:=V(z)=Vz1:
e3:=W(z)=Wz1:
e4:=Wz=DWz1:
e5:=Nz=Nz1:
e6:=Nztheta=Nztheta1:
e7:=Mz=Mz1:
e8:=Qz=Qz1:
z:=(1/5)*H:

SolutionSet:=solve({e1,e2,e3,e4,e5,e6,e7,e8},{C1,C2,C3,C4,C5,C6,C7,C8});
assign(SolutionSet);
```

solve coefficents of element [2]

```
z:='z':
```

```
C1:='C1':
C2:='C2':
C3:='C3':
C4:='C4':
C5:='C5':
C6:='C6':
C7:='C7':
C8:='C8':

e1:=U(z)=Uz2:
e2:=V(z)=Vz2:
e3:=W(z)=Wz2:
e4:=Wz=DWz2:
e5:=Nz=Nz2:
e6:=Nztheta=Nztheta2:
e7:=Mz=Mz2:
e8:=Qz=Qz2:
z:=(2/5)*H:

SolutionSet:=solve({e1,e2,e3,e4,e5,e6,e7,e8},{C1,C2,C3,C4,C5,C6,C7,C8});
assign(SolutionSet);
```

solve coefficents of element [3]

```
z:='z':
C1:='C1':
C2:='C2':
C3:='C3':
C4:='C4':
C5:='C5':
C6:='C6':
C7:='C7':
C8:='C8':

e1:=U(z)=Uz3:
e2:=V(z)=Vz3:
e3:=W(z)=Wz3:
e4:=Wz=DWz3:
e5:=Nz=Nz3:
e6:=Nztheta=Nztheta3:
e7:=Mz=Mz3:
e8:=Qz=Qz3:
z:=(3/5)*H:

SolutionSet:=solve({e1,e2,e3,e4,e5,e6,e7,e8},{C1,C2,C3,C4,C5,C6,C7,C8});
```

```
assign(SolutionSet);

 solve coefficents of element [4]

z:='z':
C1:='C1':
C2:='C2':
C3:='C3':
C4:='C4':
C5:='C5':
C6:='C6':
C7:='C7':
C8:='C8':

e1:=U(z)=Uz4:
e2:=V(z)=Vz4:
e3:=W(z)=Wz4:
e4:=Wz=DWz4:
e5:=Nz=Nz4:
e6:=Nztheta=Nztheta4:
e7:=Mz=Mz4:
e8:=Qz=Qz4:
z:=(11/15)*H:

SolutionSet:=solve({e1,e2,e3,e4,e5,e6,e7,e8},{C1,C2,C3,C4,C5,C6,C7,C8});
assign(SolutionSet);

 solve coefficents of element [5]

z:='z':
C1:='C1':
C2:='C2':
C3:='C3':
C4:='C4':
C5:='C5':
C6:='C6':
C7:='C7':
C8:='C8':

e1:=U(z)=Uz5:
e2:=V(z)=Vz5:
e3:=W(z)=Wz5:
e4:=Wz=DWz5:
e5:=Nz=Nz5:
e6:=Nztheta=Nztheta5:
```

```
e7:=Mz=Mz5:
e8:=Qz=Qz5:
z:=(13/15)*H:

SolutionSet:=solve({e1,e2,e3,e4,e5,e6,e7,e8},{C1,C2,C3,C4,C5,C6,C7,C8});
assign(SolutionSet);

 solve coefficents of element [6]

z:='z':
C1:='C1':
C2:='C2':
C3:='C3':
C4:='C4':
C5:='C5':
C6:='C6':
C7:='C7':
C8:='C8':

e1:=U(z)=Uz6:
e2:=V(z)=Vz6:
e3:=W(z)=Wz6:
e4:=Wz=DWz6:
e5:=Nz=0:
e6:=Nztheta=0:
e7:=Mz=0:
e8:=Qz=0:
z:=H:

SolutionSet:=solve({e1,e2,e3,e4,e5,e6,e7,e8},{C1,C2,C3,C4,C5,C6,C7,C8});
assign(SolutionSet);
```

END OF FILE