Some Results on Lotto Designs

by

Ben Pak Ching Li

A thesis
Submitted to the Faculty of Graduate Studies
in Partial Fulfilment of the Requirements
for the degree of

Doctor of Philosophy

Department of Computer Science
University of Manitoba
Winnipeg, Manitoba, Canada

0-612-45009-0

THE UNIVERSITY OF MANITOBA

FACULTY OF GRADUATE STUDIES
*****
COPYRIGHT PERMISSION PAGE

Some Results on Lotto Designs

BY

Ben Pak Ching Li

A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University of Manitoba in partial fulfillment of the requirements of the degree

of

Doctor of Philosophy

Abstract

Given parameters n, k, p, t, an (n, k, p, t) Lotto design is a collection of k-sets such that any arbitrary p-set, which are chosen from an n-set, intersects at least one k-set in the collection in at least t elements. The number $L(n, k, p, t)$ is size of the minimal (n, k, p, t) Lotto design. We provide constructions and techniques for determining upper and lower bounds for $L(n, k, p, t)$. We also provide computer programs that generate upper bounds for $L(n, k, p, t)$.

Acknowledgements

I would like to begin by thanking my supervisor Dr. John van Rees. He has helped me greatly with the research and the writing of this thesis. Without his support, this thesis would not have been possible.

I would like to thank Dr. John Bate, Dr. Bob Quackenbush and Dr. Wal Wallis for being on my committee and taking the time to read this thesis.

I would also like to thank my wife, Joanne, and my parents for their support.

Contents

List of Tables

Chapter 1

Introduction

The study of Lotto designs is relatively new in combinatorics. The first major study of Lotto designs was performed by Bate [1]. Lotto designs are directly related to lottery wheels such as the Canadian Lotto 6/49. In general, this is how lotteries work :

- A buyer buys tickets and each ticket has k distinct numbers on it chosen by the buyer. The numbers are between 1 and n.
- The government closes the buying and then picks p of the n numbers, randomly.
- If one of the buyer's tickets and the government's pick intersects in t numbers or more, he/she wins a prize.
- The bigger the value of t, the greater the prize. For small t, there is usually no prize.

For Lotto 6/49, $n = 49$, $k = 6$, $p = 6$ and your ticket has to intersect the government's pick in at least $t = 3$ numbers in order to win a prize. In Lotto 6/49, it is clear that you need to buy all $C(49, 6) = 13983816$ tickets in order to be guaranteed a win of the big jackpot where one of your tickets is exactly the same as the government's pick.

A Lotto design with parameters n, k, p, and t can be informally thought of as a collection of tickets, each with k numbers from 1 to n such that any p numbers chosen from 1 to n will intersect one of the tickets in at least t numbers. Consider the following question about Lotto 6/49: What is the minimum number of tickets one needs to buy to be guaranteed matching at least three numbers of the government's pick? This question can be restated as : What is the smallest number of tickets in a Lotto design where $n = 49$, $k = 6$, $p = 6$, and $t = 3$? The answer for has not be determined by it is known to be between 87 and 169. For $t = 2$, the answer is 19.

The fundamental question in studying Lotto designs is : Given n, k, p, t, what is the smallest number of tickets in a Lotto design satisfying the given parameters? The goal of this thesis is to try to answer this question. In some special cases, such as when $p = t$ and $k = t$ the answer is known. However, not much is known about Lotto designs in general.

1.1 Definitions

In this section, we shall define the terms that will be used throughout this thesis. The goal of this section is to make this thesis as self-contained as possible. We begin by defining a Lotto design.

Definition 1.1.1 : *If x is an integer, then B is an x-***set** *if B is a set containing x elements.*

Definition 1.1.2 : *If n is an integer, let $X(n)$ be the set of integers from 1 to n.*

Definition 1.1.3 : *If X is a set, and y is an integer where $y \leq |X|$, then Y is a* y-**subset** *of X if Y is a y-set such that $Y \subseteq X$.*

We now formally defined a Lotto design.

Definition 1.1.4 : *Suppose n, k, p and t are integers and $\mathcal{B}$ is a collection of k-subsets of a set X of n elements (usually X is $X(n)$). We say $\mathcal{B}$ is an (n, k, p, t)* **Lotto design** *if an arbitrary p-subset of $X(n)$ intersects some k-set of $\mathcal{B}$ in at least t elements. The k-sets in $\mathcal{B}$ are known as the* **blocks** *of the Lotto design $\mathcal{B}$. The elements of X are known as the* **varieties** *of the design.*

An (n, k, p, t) Lotto design is usually denoted as $\mathcal{B}$ where $\mathcal{B}$ are the blocks of the design. It is assumed that the elements of the blocks are chosen from $X(n)$. Lotto designs can also be denoted by $(X, \mathcal{B})$ where $\mathcal{B}$ denotes the blocks of the design and X denotes the set from which the elements of the blocks of $\mathcal{B}$ are chosen. From here on , we assume that the elements of the design are from $X(n)$ unless explicitly stated otherwise. We will also assume that $k \leq n, p \leq n, t \leq min\{k, p\}$. These assumptions will guarantee that the parameters are not ambiguous. The following is an example of an $(7, 5, 4, 3)$ Lotto design with 3 blocks.

Example 1.1.1 *The following three blocks form a $(7, 5, 4, 3)$ Lotto design.*

$$\begin{array}{l}
\{1, \ 2, \ 3, \ 4, \ 7\}, \\
\{1, \ 2, \ 5, \ 6, \ 7\}, \\
\{3, \ 4, \ 5, \ 6, \ 7\}.
\end{array}$$

Suppose that $\mathcal{B}$ is a collection of k-sets and P is an arbitrary p-set. If P intersects some block in $\mathcal{B}$ in t elements, we say that P is "t-represented" by a block from $\mathcal{B}$. We now formally define this term.

Definition 1.1.5 : *Suppose $\mathcal{B}$ is a (n, k, p, t) Lotto design. If P is p-set that intersects some block of $\mathcal{B}$ in t or more elements, then P is said to be t-* **represented** *(or simply* **represented***) by a block of $\mathcal{B}$ or represented by $\mathcal{B}$.*

Clearly, $\mathcal{B}$ is an (n, k, p, t) Lotto design if and only if every p-set is represented by $\mathcal{B}$.

It should be clear that if $\mathcal{B}$ is an (n, k, p, t) Lotto design with b blocks, then adding additional k-subsets of $X(n)$ to $\mathcal{B}$ will generate other (n, k, p, t) Lotto designs. Hence Lotto designs in general are not unique. However, for fixed n, k, p and t, there will exist an integer $b > 0$ such that no (n, k, p, t) Lotto design will have fewer than b blocks. This leads to the following definition.

Definition 1.1.6 : *Let $L(n, k, p, t)$ denote the minimum number of blocks of any (n, k, p, t) Lotto design. An (n, k, p, t) Lotto design with $L(n, k, p, t)$ blocks is called an* **optimal** *or* **minimal** (n, k, p, t) **Lotto design**. *We also define $LD^*(n, k, p, t; b)$ to denote an optimal (n, k, p, t) Lotto design with b blocks.*

The main goal of the thesis is to determine the value of $L(n, k, p, t)$. For small values of n, k, p and t, this task is often easy. In general, the determination of $L(n, k, p, t)$ is very difficult.

Given an (n, k, p, t) Lotto design, each element from $X(n)$ appears in the design with a certain frequency. If $i \geq 0$, let f_i denote the number of elements that appear i times in the design. Let f_i^+ denote the number of elements of frequency at least i.

During the construction of Lotto designs, collections of k-sets will be formed which may or may not be Lotto designs. In this case we call them "Potential Lotto designs" or simply "Potential designs". This term will be useful in distinguishing between an actual (n, k, p, t) Lotto design or simply a collection of k-sets that may or may not be a (n, k, p, t) Lotto design.

Another useful concept is the complement of a Lotto design. Suppose $\mathcal{B}$ is an (n, k, p, t) Lotto design. The complement of $\mathcal{B}$ is the collection of $(n-k)$-subsets of $X(n)$ $\{X(n) \setminus B : B \in \mathcal{B}\}$. Bate [1] showed that the complement of an (n, k, p, t) Lotto design is an $(n, n-k, n-p, n-k-p+t)$ Lotto design. We now formally define the complement of a Lotto design.

Definition 1.1.7 : *Suppose $\mathcal{B}$ is a (n, k, p, t) Lotto design. The* **complement** *of $\mathcal{B}$ is the collection of $n-k$-subsets of $X(n)$, $\{X(n) \setminus B : B \in \mathcal{B}\}$.*

In a Lotto design, there may be elements that do not appear together in any of the blocks. In chapter 5, we will be using this concept often.

Definition 1.1.8 : *Let i be an integer such that $i < n$. Consider $\{a_j : j = 1 \text{ to } i\}$ where each a_j is an element of an (n, k, p, t) Lotto design B. These elements are* **disjoint (in the design)** *if $|A \cap B| \leq 1$ for all $B \in \mathcal{B}$.*

Lotto designs are relatively new in the field of combinatorics. They are a generalization of other types of designs. We now discuss several of these designs.

One special class of Lotto designs is *Covering designs*. Covering designs are Lotto designs where $p = t$. Extensive research has gone into Covering designs and much is known. We now give a formal definition of Covering designs.

Definition 1.1.9 : *Let $n \geq k \geq t$ be three integers. An (n, k, t)* **Covering design** *is a pair $(X, \mathcal{B})$ where X is an n-set of elements (points) and $\mathcal{B}$ is a collection of k-subsets (blocks) of X, such that every t-subset of X is represented by $\mathcal{B}$. The covering number $C(n, k, t)$ is used to denote the minimum number of blocks in any (n, k, t) Covering design.*

Combining definitions 1.1.4 and 1.1.9, we see that $L(n, k, t, t) = C(n, k, t)$.

One of the most famous results on Covering designs is the Schöheim bound [20] which states :

$$C(n, k, t) \geq \left\lceil \frac{n}{k} \left\lceil \frac{n-1}{k-1} \cdots \left\lceil \frac{n-t+1}{k-t+1} \right\rceil \cdots \right\rceil \right\rceil . \tag{1.1}$$

The recursive form of this bound is:

$$C(n, k, t) = \left\lceil \frac{n}{k} C(n-1, k-1, t-1) \right\rceil . \tag{1.2}$$

Other designs that are related to both Lotto designs and Covering designs are Turán designs.

Definition 1.1.10 : *Let $n \geq p \geq t$ be three integers. An (n, p, t)* **Turán design** *is a pair $(X, \mathcal{B})$ where X is a n-set of elements (points) and $\mathcal{B}$ is a collection of t-subsets (blocks) of X, such that every p-subset of X contains a block of $\mathcal{B}$. The Turán number $T(n, p, t)$ is used to denote the minimum number of blocks in any (n, p, t) Turán design.*

Combining definitions 1.1.4 and 1.1.10, we see that $L(n, t, p, t) = T(n, p, t)$. Also, it is easy to show that the complement of a Covering design is a Turán design and vice versa.

1.2 Organization of the Thesis

The main goal of this thesis is to develop methods for computing the lower and upper bounds for $L(n, k, p, t)$. Upper bounds are usually obtained by construction techniques while lower bounds often have to be obtained analytically. Another goal of this thesis is to construct tables that contain a wide range of values for $L(n, k, p, t)$ where $5 \leq n \leq 20$, $2 \leq k, p \leq n$ and $2 \leq t \leq min\{k, p\}$.

Chapter 2 gives a brief history of the work that has been done relating to Lotto designs. The results from Bate's thesis [1] and from Bate and van Rees [2] relating to Lotto designs with $t = 2$ are discussed. A lower bound formula for $L(n, k, p, 2)$ appearing in [11] is stated and proved. Also, the computational approach of constructing Lotto designs by Simulated Annealing [19] is also discussed.

Constructions are one of the most common techniques for obtaining upper bounds on Lotto designs. Chapter 3 begins with a discussion of BIBDs and the construction of Lotto designs from BIBDs. Special classes of BIBDs such as symmetric BIBDs and resolvable BIBDs may also be used to construct Lotto designs. We then state the monotonicity theorems that are quite useful for improving our tables. These monotonicity results may be applied to the tables as soon as there is a change in the tables. Our next result is a construction which we term "Semi-direct product". This construction takes several designs and puts them together to form a larger design. The designs constructed using this technique are sometimes quite good. The last major result in this chapter is a theorem on m-ing a design. This theorem give sufficient conditions for $L(mn, mk, p, t) \leq L(n, k, p, t)$ to hold. A few other constructions are also stated which have been generalized from results on Covering designs.

Lower bounds for Lotto designs are hard to compute. The approach of constructing a design cannot help compute lower bounds. Instead lower bounds are usually given as formulas derived analytically. Chapter 4 lists the lower bound formulas that we have developed. The first formula is a generalization of Schöheim's lower bound formula for Covering designs. The rest of the formulas apply to specific groups of Lotto designs that are determined by their parameters.

In Chapter 5, we compute the values of $L(n, k, p, t)$ for certain values of n, k, p and t on an individual basis. One of the main goals of this chapter is to update the tables from Bate's thesis [1].

Chapter 6 describes the computer programs that have been developed to compute upper bounds for $L(n, k, p, t)$. One of our new algorithms is based on an exhaustive search taking into account average frequencies of elements in the design. Other algorithms we discuss are greedy algorithms and random algorithms. Results from simulated annealing are also discussed. The search for $L(n, k, p, t)$ can also be done using integer programming. We used the CPLEX program to compute values of $L(n, k, p, t)$. We also discuss how to improve the performance of our programs in general. Finally, we discuss the organization of the tables used to hold the upper and lower bounds of $L(n, k, p, t)$.

Chapter 7 lists the lower and upper bounds for $L(n, k, p, t)$ that we have collected.

Chapter 8 gives concluding comments for this thesis. We also discuss possible further work that may be done on Lotto designs.

1.3 Frequently Used Notation

Some of the symbols used frequently throughout this thesis are defined as follows :

- $C(n, r)$ – $n!/((n-r)!r!)$
- f_i – The number of elements of frequency i
- f_i^+ – The number of elements of frequency i or more
- $X(n)$ – The set $\{1, 2, 3, ..., n\}$
- $\lceil x \rceil$ – The ceiling of x
- $\lfloor x \rfloor$ – The floor of x

Chapter 2

History

2.1 Introduction

Lotto designs are a generalization of Covering designs. The amount of research that has been done in Lotto designs is less than the amount done in other designs such as Covering designs, BIBDS, t-designs and so forth. In this chapter, we will discuss the major contributions to the study of Lotto designs. We begin by stating the main results from Bate's doctoral thesis which dealt mostly with the study of $L(n, k, p, 2)$. We then discuss a lower bound formula for $L(n, k, p, 2)$ by Füredi, Székely and Zubor. The results of Bate and van Rees are then mentioned where they computed the value of $L(n, 6, 6, 2)$ for $n \leq 54$. Finally we mention the other contributions which did not fit in with the previous three results.

2.2 Bate's Thesis

One of the earliest mathematical studies on Lotto designs was performed by Bate in his Doctoral thesis [1]. However, Bate did not use the term Lotto designs. Instead, he called them "generalized (T, K, L, V) designs" . In his work, the emphasis was in dealing with designs where the value of t was 2. By generalizing a result on covering designs, Bate was able to show the following result which states that the complement of a Lotto design is a Lotto design.

Theorem 2.2.1 : *The complement of an (n, k, p, t) Lotto design is an $(n, n-k, n-p, n-k-p+t)$ Lotto design and hence $L(n, k, p, t) = L(n, n-k, n-p, n-k-p+t)$.*

By employing a graph theoretical approach, Bate showed the following:

Theorem 2.2.2 :

$$L(n, 2, p, 2) = \left\lceil \frac{n^2 - (p-1)n}{2(p-1)} \right\rceil .$$

This particular result was first proved by Turàn [22] in 1941 and since then, additional proofs have appeared.

A formula for $L(n, 3, 3, 2)$ was also given in Bate's thesis. This result was independently derived by Brouwer [6] using Covering designs and stated in terms of Covering numbers. However, we will state Bate's version.

Theorem 2.2.3 :

$$L(n,3,3,2) = \begin{cases} \lceil \frac{n^2-2n}{12} \rceil & \textit{if } n \equiv 2,4,6 \textit{ (mod 12))} \\ \lceil \frac{n^2-2n}{12} \rceil + 1 & \textit{if } n \equiv 0,8,10 \textit{ (mod 12))} \\ \lceil \frac{n^2-n}{12} \rceil & \textit{if } n \equiv 1,3,5,7 \textit{ (mod 12))} \\ \lceil \frac{n^2-n}{12} \rceil + 1 & \textit{if } n \equiv 9,11 \textit{ (mod 12))} \end{cases}$$

For Turàn designs, Bate stated the following old result of Turàn.

Theorem 2.2.4 : $L(3n, 3, 4, 3) \leq 3C(n, 3) + 3nC(n, 2)$.

Proof : Partition the $3n$ elements into three sets A, B and C evenly. Consider all possible 3-sets of the forms $\{x_1, x_2, x_3\}$, $\{y_1, y_2, y_3\}$, $\{z_1, z_2, z_3\}$, $\{x_1, x_2, y_1\}$, $\{y_1, y_2, z_1\}$ and $\{z_1, z_2, x_1\}$ where $x_1, x_2, x_3 \in A$, $y_1, y_2, y_3 \in B$ and $z_1, z_2, z_3 \in C$. There are $3C(n,3) + 3nC(n,2)$ such 3-sets. It is easy to show that these 3-sets form an $(3n, 3, 4, 3)$ Lotto design. □

The following two results are similar to the result stated above in Theorem 2.2.4 and hence are stated without proofs.

Theorem 2.2.5 : $L(3n+1, 3, 4, 3) \leq 2C(n,3) + C(n+1,3) + (2n+1)C(n,2) + nC(n,2)$.

Theorem 2.2.6 : $L(3n+2, 3, 4, 3) \leq C(n,3) + 2C(n+1,3) + (2n+1)C(n+1,2) + (n+1)C(n,2)$.

Another important result from Bate's thesis is a theorem that allows us to consider only minimal designs which have no elements of frequency zero, without loss of generality.

Theorem 2.2.7 : *If $L(n,k,p,t) \geq \frac{n}{k}$ then there exists a minimal design which contains every element.*

Theorem 2.2.8 : *If $L(n,k,p,t) < \frac{n}{k}$ then there exists a minimal design in which blocks are pairwise disjoint.*

Theorems 2.2.7 and 2.2.8 generalize to the following two results, respectively. The proofs for the generalizations are nearly identical to those given by Bate.

Theorem 2.2.9 *: If there exists an $LD(n, k, p, t; b)$ Lotto design where $b \geq \frac{n}{k}$ then there exist an $LD(n, k, p, t; b)$ Lotto design which contains every element.*

Theorem 2.2.10 *: If there exists an $LD(n, k, p, t; b)$ Lotto design where $b < \frac{n}{k}$ then there exist an $LD(n, k, p, t; b)$ Lotto design which consists of distinct elements.*

A computer program included in Bate's thesis described how one can construct, for small parameters, a minimal (n, k, p, t) lottery design. The basic principle of this computer program was to find a minimal (n, k, p, t) lottery design by considering all possible potential designs. The basis of this search program is given by the following algorithm.

Algorithm 2.2.1 *:*

```
/* returns true if design found*/
Bool GenerateDesign()
{
    Find first p-set not covered
    If all are represented then
        Return true meaning a design has been found.
    Else
    {
        If design contains maximum number of blocks allowed then
                Return false to mean no design found
        Else
        {
                For each k-set representing the p-set do
```

```
        {
            Add the k-set to the design.
            Flag every p-set represented by this k-set.
            Call GenerateDesign and store its return value in
            variable named found.
            If found is true then
                Return true to mean a design has been found.
            else
            {
                Unflag the p-sets.
                Remove k-set from design.
            }
        }
        Return false meaning not found.
    }
}
```

With the exception of small values of n, these searches are not feasible because of exponential growth. Therefore, techniques to reduce the size of the search tree were employed. One of these techniques was based on the concept of preclusion. The idea behind preclusion is to determine how many p-sets can be covered by a k-set. If at any point in the search, the number of k-sets remaining to be selected for a design cannot cover the remaining uncovered p-sets, the search returns to the previous level. This simple technique often provided a dramatic improvement in the reduction of the size of the search tree. The other technique that was implemented involved the idea of isomorphism rejection. Bate defined the terms *trivially isomorphic* and *trivially equivalent* as follows :

Definition 2.2.1 : *Two elements are trivially isomorphic if every block in a par-*

tially constructed design contains either both of them or neither of them. Two k-sets are trivially equivalent if one of them can be transformed into the other by replacing elements with trivially isomorphic elements.

Bate's algorithm kept a table of the trivially isomorphic elements in the Lotto design constructed by the program. When a new k-set B is added to the design, this table is first updated, followed by an attempt to complete the design. The table is then restored to its former status when the attempt is complete. Following this, the k-set B and any k-sets which are trivially isomorphic to B are marked and rejected at this level and any lower levels. Once every possible k-set has been attempted, all of the k-sets are unmarked before the program returns to the previous level. According to Bate, this form of isomorphism testing resulted in a dramatic decrease in the size of the search tree with only a moderate increase in overhead. Table 2.1 from [1] illustrates the size of the search tree for several designs.

n	k	l	t	Basic Algorithm Nodes searched	With Preclusion Nodes searched	With Isomorphism Rejection Nodes searched
8	3	2	2	16,663,323	211,205	7,224
9	3	3	2	5,126,742	2,567,524	10,173
7	3	4	3	324,467	80,234	7,786
8	4	2	2	432,061	38,920	5,710
8	3	5	3	958,842	130,275	2,335
8	4	4	3	283,972	88,183	8.676
9	4	5	3	859,644	845,825	1,471
9	5	2	2	747,682	145,192	4,062

Table 2.1: Comparison of Sizes of Search Trees in Exhaustive Search

Even with these techniques, the computer program could handle only small parameters.

Bate also included a collection of tables containing the the lower and upper bound for $L(v, k, p, 2)$ where $v \leq 16$ and $k, p < v$.

2.3 Result of Füredi, Székely and Zubor

Lower bounds are very difficult to compute for Lotto designs. The special case for $t = 2$ was analyzed by Füredi, Székely and Zubor [11] in 1996. Their approach was take a multi-graph representing an $(n, k, p, 2)$ Lotto design, transform it into another multi-graph with the same degree sequence as the original multi-graph which contains $p - 1$ disjoint subgraphs and analyze this new graph to obtain a lower bound for $L(n, k, p, 2)$. Using this approach, Füredi, Székely and Zubor determined the following lower bound for $L(n, k, p, 2)$:

$$L(n,k,p,2) \geq_{(\sum_{i=1}^{p-1} a_i)}^{min} \frac{1}{k}\left(\sum_{i=i}^{p-1} a_i \left\lceil \frac{a_i - 1}{k-1} \right\rceil\right).$$

Before proceeding to prove the above result (since no complete proof was given), we thank John Bate for his help in this section. We begin by stating some definitions.

Definition 2.3.1 : *A* **multi-graph** $\mathcal{G}$ *is a set of vertices and edges between the vertices such that loops and multiple edges between two vertices are possible.*

Definition 2.3.2 : *A multi-graph* $\mathcal{G}$ *is a* **complete graph** *if any two distinct vertices are adjacent.*

Definition 2.3.3 : *An* **independent set** *of size p in a multi-graph is a set of p vertices where no two distinct vertices are adjacent.*

If $\mathcal{G}$ is a multi-graph, then H is a subgraph of $\mathcal{G}$ if the vertices of H are a subset of of the vertices of $\mathcal{G}$ and if x and y are vertices in $\mathcal{H}$ joined by i edges, then x and y are vertices in $\mathcal{G}$ joined by i or more edges. Note that nothing is said about the relationship between H and the rest of $\mathcal{G}$. We now prove a result on multi-graphs that is a slight generalization of a result in Bollobás [4].

Theorem 2.3.1 : *If $\mathcal{G}$ is a multi-graph with vertex set $V = \{x_1, ..., x_n\}$ and $\mathcal{G}$ contains no independent set I with $|I| \geq p$, then there is a graph $\mathcal{H}$ with the same vertex set V such that*

$$deg_{\mathcal{G}}(x_i) = deg_{\mathcal{H}}(x_i)$$

for $i = 1, 2, .., n$ and $\mathcal{H}$ contains $p-1$ subgraphs $A_1, A_2, ..., A_{p-1}$ such that each is a complete graph and $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^{p-1} A_i = V$. It should be noted that there may exist edges between the $p-1$ subgraphs in the graph $\mathcal{H}$.

Proof : We use induction on p. For $p = 2$, we have $\mathcal{G} = \mathcal{H} = K^n$, the complete graph. Hence the theorem is true for $p = 2$. Assume the theorem is true for $p-1$. We now show it is true for p. Let x_m be the vertex connected to the minimum number of other distinct vertices of $\mathcal{G}$. Let W denote the set of vertices not adjacent to x_m and let Y denote the set of vertices adjacent to x_m. The set W cannot contain an independent set of size $p-1$ since if it did, $W \cup \{x_m\}$ would contain an independent set of size p which contradicts our assumptions. By the induction hypothesis, W can be made to consist of $(p-2)$ complete subgraphs $A_1, A_2, ...A_{p-2}$ without changing the degrees of the vertices, and for which the vertex sets of the subgraphs are disjoint by their union is W. Now all vertices in Y may be made adjacent to each other as follows. Suppose $y_1 \in Y$ is not adjacent to $y_2 \in Y$. Then, as all vertices in $\mathcal{G}$ are adjacent to at least $|Y|$ distinct vertices, y_1 and y_2 must be connected to vertices $w_1, w_2 \in W$ where w_1 could equal w_2. Remove the edges y_1, w_1 and y_2, w_2, and add edges between y_1 and y_2 and between w_1 and w_2. Repeat this process until all vertices in Y are adjacent to each other. Clearly, $\{x_m\} \cup Y$ is the $(p-1)^{th}$ complete subgraph of $\mathcal{H}$ and no degree has been altered.□

If there exists an $(n, k, p, 2; b)$ Lotto design B, we can construct a multi-graph $\mathcal{G}$ with $b * C(k, 2)$ edges as follows :

1. Let the vertices of the graph be the n varieties in the design.
2. For every pair (i, j) that appear in the Lotto design, add an edge between i and j. We note that there may be multiple edges between two vertices.

The multi-graph $\mathcal{G}$ has the following properties :

A) The degree at each vertice is a multiple of $k - 1$

B) There is no independent set of size larger than $p - 1$

We now state and prove the main result from Füredi et. al.[11].

Theorem 2.3.2 *:*

$$L(n, k, p, 2) \geq_{(\sum_{i=1}^{p-1} a_i)}^{min} \frac{1}{k} \left(\sum_{i=i}^{p-1} a_i \left\lceil \frac{a_i - 1}{k - 1} \right\rceil \right).$$

Proof : Let $b = L(n, k, p, t)$. Then there exists a multi-graph $\mathcal{G}$ with vertex set $V = \{x_1, ..., x_n\}$ which is the multi-graph for an $(n, k, p, 2; b)$ Lotto design. This multi-graph has $b * C(k, 2)$ edges. By Theorem 2.3.1, there exist a multi-graph $\mathcal{H}$ whose vertex set is V, $deg_{\mathcal{G}}(x_i) = deg_{\mathcal{H}}(x_i)$ for $i = 1$ to n and the vertex set V can be partitioned into $p - 1$ subsets $A_1, A_2, ..., A_{p-1}$ where $|A_i| = a_i$ and A_i is a complete subgraph of $\mathcal{H}$ for $i = 1$ to $p - 1$.

Consider $x \in A_i$, for arbitrary i. Since x has degree a multiple of $k - 1$ and x

is adjacent to at least $a_i - 1$ elements in the multi-graph $\mathcal{H}$, the degree of x is at least $\lceil \frac{a_i-1}{k-1} \rceil (k-1)$. Hence the number of edges in $\mathcal{H}$ is at least

$$\geq \frac{1}{2} \sum_{i=1}^{p-1} a_i \left\lceil \frac{a_i - 1}{k-1} \right\rceil (k-1).$$

Since this is true for any multi-graph $\mathcal{H}$ with the properties :

1. Each vertex has degree a multiple of $k-1$
2. The vertex set of $\mathcal{H}$ can be partition into $p-1$ subsets $A_1, A_2, \ldots, A_{p-1}$ where $|A_i| = a_i$ and A_i is a complete subgraph of $\mathcal{H}$ for $i = 1$ to $p-1$,

we have the number of edges in $\mathcal{G}$ is at least

$$\min_{(\sum_{i=1}^{p-1} a_i)} \frac{1}{2} \sum_{i=1}^{p-1} a_i \left\lceil \frac{a_i - 1}{k-1} \right\rceil (k-1).$$

But the number of edges in G is $b * C(k,2)$. Thus,

$$b * C(k,2) \geq \min_{(\sum_{i=1}^{p-1} a_i)} \frac{1}{2} \sum_{i=1}^{p-1} a_i \left\lceil \frac{a_i - 1}{k-1} \right\rceil (k-1),$$

which gives

$$b \geq \min_{(\sum_{i=1}^{p-1} a_i)} \frac{1}{k} \sum_{i=1}^{p-1} a_i \left\lceil \frac{a_i - 1}{k-1} \right\rceil,$$

as required. □

The lower bound stated above is one of the best lower bound formulas currently known for $L(n, k, p, 2)$.

The following is a standard result for obtaining an upper bound for $L(n, k, p, 2)$ that is stated in Füredi et. al [11].

Theorem 2.3.3 *:*

$$L(n, k, p, 2) \leq \min_{a_1 + \ldots + a_{p-1} = n} \left(L(a_1, k, 2, 2) + \ldots L(a_{p-1}, k, 2, 2) \right).$$

Proof: Let A_i be a partition of the numbers 1 to n, where $|A_i| = a_i$, for $i = 1$ to $p-1$. We can construct a $(n, k, p, 2)$ design by putting $p - 1$ $(a_i, k, 2, 2)$ designs together where $i = 1$ to $p - 1$. To show that it is an $(n, k, p, 2)$ design, let $P = (x_1, x_2, .., x_p)$ be an arbitrary p-set. By the pigeon hole principle, at least two x_i's must be from the same A_j for some j. Thus P must be represented by the $(a_j, k, 2, 2)$ design used to form the $(n, k, p, 2)$ design. □

Füredi et. al [11] gave a non-exhaustive list of cases where the bounds meet. The approach taken by the authors generated good lower bounds for $L(n, k, p, 2)$. However, no known, similar technique can be applied to generate good lower bounds for $L(n, k, p, t)$ where $t \geq 3$.

2.4 Result of Bate and Van Rees

In 1998, Bate and van Rees [2] determined the values for $L(n, 6, 6, 2)$ for $n \leq 54$. Their approach was to examine the frequencies of elements in "nice" designs. By considering only these "nice" designs, they were able to determine the values of $L(n, 6, 6, 2)$ for $n \leq 54$. Before we state the main results in [2], we will give several definitions.

Definition 2.4.1 : *An* **independent set** *in a Lotto design is a set of size* ≥ 2 *of elements, no pair of which occurs together in any block of the design. It is* **maximal** *if the set can not be enlarged. The elements in an independent set are called* **independent elements** *and the blocks in a Lotto design containing independent elements are called* **independent blocks**.

It is be obvious that there may be numerous independent sets in a design and that the independent elements are relative to some fixed independent set.

Definition 2.4.2 : *An element that appears only once in the independent blocks of a design is called a* **single**.

Definition 2.4.3 : *Given an independent element of frequency i, the i blocks containing this independent element is called an* i-**clique**.

Definition 2.4.4 : *A* **maximum independent set** *for a design is an independent set whose size is not less than the size of any independent set for the design.*

It is clear that the maximum size of an independent set in a (n, k, p, t) Lotto design cannot be larger than $p - 1$. The next result states, that, for certain values of n, k and p, there must exist minimal Lotto designs with an independent set of size $p - 1$.

Theorem 2.4.1 : *An $LD^*(n, k, p, 2; b)$ with $n > k(p - 2)$ implies the existence of an $LD^*(n, k, p, 2; b)$ which has an independent set of size $p - 1$.*

The following result states that every element in an $(n, k, p, 2)$ Lotto design must occur in the independent blocks of an independent set with $p - 1$ elements. Clearly, if an independent set of size $p - 1$ exists in an $(n, k, p, 2)$ Lotto design, then it is a maximum (and hence a maximal) independent set.

Theorem 2.4.2 : *In any $LD^*(n, k, p, 2; b)$ with an independent set of size $p - 1$, every element of the design must occur in the independent blocks of the independent set.*

More generally, we have the following result.

Theorem 2.4.3 : *In any (n, k, p, t) Lotto design, every element of the design must occur at least once in the independent blocks of any maximal independent set.*

Corollary 2.4.1 : *The number of independent blocks of any maximal independent set of size m from an (n, k, p, t) Lotto design is at least $\left\lceil \frac{n-m}{k-1} \right\rceil$.*

Definition 2.4.5 : *An* **isolated block** *of a Lotto design is a block that contains only elements of frequency one in the design.*

Under certain conditions, all the elements of frequency one occur only in blocks that contain only frequency one elements. The following result states these conditions.

Theorem 2.4.4 : *If $n \geq k(p-1)$ and there is an $(n, k, p, 2)$ Lotto design with b blocks, then there exists and $(n, k, p, 2)$ Lotto design with b blocks such that there are rk elements of frequency one occurring in r blocks, for $r \geq 0$.*

Definition 2.4.6 : *A* **nice (n,k,p,t;b) design** *is a Lotto design with b blocks wherein each element occurs at least once, the elements of frequency one occur in isolated blocks and there is an independent set of size $p-1$.*

It turns out that if $b \geq \frac{n}{k}$ and $n \geq k(p-1)$ then there exists a nice $(n, k, p, t; b)$ Lotto design.

Theorem 2.4.5 : *If $L(n, k, p, t) = b$, $b \geq \frac{n}{k}$ and $n \geq k(p-1)$ then there exists a nice $LD^*(n, k, p, t; b)$.*

By considering only nice designs and analyzing the frequencies of elements in these designs, Bate and van Rees were able to show the following results.

Theorem 2.4.6 :

n	L(n,6,6,2)
35	9
36	9
37	9
38	11
39	11
40	12
41	13
42	13
43	14
44	15
45	15
46	16
47	17
48	18
49	19
50	19
51	20
52	21
53	22
54	23

Another important result that first appeared in Hartman's thesis [16] and was applied by Bate and Van Rees was the following :

Theorem 2.4.7 : *In an (n, k, p, t) Lotto design, there exist at least*

$$\left\lceil \frac{f_2}{\left\lfloor \frac{3k}{2} \right\rfloor} \right\rceil$$

disjoint elements of frequency two.

2.5 Other Historical Results

Many of the results on Lotto designs were influenced by the study of Covering and Turán designs. The study of Lotto designs is relatively new compared to the study Covering and Turán designs. We now list other important results known about Lotto designs. We first state a result by Hanani, Ornstein and Sós [15] which gives a lower bound for $L(n, k, p, 2)$.

Theorem 2.5.1 : $L(n, k, p, 2) \geq \frac{n(n-p+1)}{k(k-1)(p-1)}$.

The following lower bound for $L(n, k, p, t)$ made be found in [19].

Theorem 2.5.2 :

$$L(n, k, p, t) \geq \frac{C(n, k)}{\sum_{i=t}^{min(k,p)} C(k, i) * C(n-k, p-i)}.$$

De Caen [9] determined a lower bound for Turán designs which states:

Theorem 2.5.3 : $T(n, p, t) \geq \frac{C(n,t)}{C(p-1,t-1)} \frac{n-p+1}{n-t+1}$.

Brouwer and Voorhoeve [7] determined that $L(n, k, p, t) \geq \frac{T(n,p,t)}{C(k,t)}$. This result along with that of de Caen yields the following general lower bound formula for $L(n, k, p, t)$.

Theorem 2.5.4 : $L(n, k, p, t) \geq \frac{C(n,t)}{C(p-1,t-1)C(k,t)} \frac{n-p+1}{n-t+1}$.

Currently, this is the best known general lower bound for $L(n, k, p, t)$. Many upper bound formulas are derived from construction techniques. One of these is the following upper bound construction for $L(n, k, p, t)$ found in The CRC handbook [8].

Theorem 2.5.5 : *If* $n = n_1 + n_2$ *and* $p = p_1 + p_2 - 1$, *then* $L(n, k, p, t) \leq L(n_1, k, p_1, t)$ $+ L(n_2, k, p_2, t)$.

This result was applied to $L(49, 6, 6, 3)$ yielding

$$L(49, 6, 6, 3) \leq L(22, 6, 3, 3) + L(27, 6, 4, 3) \leq 77 + 91 = 168.$$

Thus, in Canada's Lotto 6/49, if you buy a certain 169 tickets, you are guaranteed to match 3 numbers giving you $10 .

Computer programs have also aided in the study of Lotto designs. One such program (named **cover**) is the probabilistic search by Nurmela and Ostergard [19] based on Simulated Annealing. The authors did not use the term" Lotto designs" but instead used "Covering designs". Simulated annealing is an optimization technique, based on the process of physical annealing. Physical annealing is a process where a crystal is cooled down from the liquid phase to the solid phase in a heat bath. If the cooling process is done carefully enough, the energy state of the solid at the end of the cooling is at its minimum. The general idea of simulated annealing, taken from the Metropolis algorithm [18], is stated in the following pseudo-code.

1. Obtain an initial solution S and an initial temperature T
2. While *stop criterion* is not satisfied do the following :
    (a) While *inner loop criterion* not satisfied do the following :
        i. Select a neighbor S' of S.
        ii. Let $\delta = cost(S') - cost(S)$.
        iii. If $\delta \leq 0$, set $S = S'$.
        iv. If $\delta > 0$, set $S = S'$ with probability $e^{-\delta/T}$.
    (b) Reduce temperature T

3. Return S

When using simulated annealing to construct an (n, k, p, t) Lotto design , we have to define the cost function and the neighborhood of a solution. A *solution* may be any potential design. The most natural way of defining the cost function is :

$$cost(S) = \text{Number of } p\text{-sets not represented by the solution S.}$$

The most natural neighborhood structure is given as follows: Suppose we have a solution S. We can make another solution by selecting one of the k-sets K in S and replacing it with another k-set K' not already in S. This is done by replacing a point in K by a point not belonging to K to generate the required k-set K'. We define the neighborhood of S to be the set of all solutions obtained from S by one such change. The *stop criterion* and *inner loop criterion* in steps 2 and 2a may vary. To reduce the temperature in step 2b, the simplest technique is to multiply the current temperature by a constant less than one.

Simulated Annealing has given excellent upper bounds for many values of $L(n, k, p, t)$. However, the memory and time requirements become large as the parameters become large. Nonetheless, simulated annealing is a useful tool for computing upper bounds for $L(n, k, p, t)$ and for constructing Lotto Designs.

2.6 Conclusion

Lotto designs with parameters $(n, k, p, 2)$ have been studied and much is known about them. Many lower and upper bound formulas are available for determining $L(n, k, p, 2)$. However, for general (n, k, p, t) Lotto designs, not much is known. Computer searches do exist for constructing (n, k, p, t) designs, but they require large amounts of resources. In the next chapter, we shall discuss some construction techniques for Lotto designs which give upper bounds for $L(n, k, p, t)$.

Chapter 3

Upper Bound Constructions

3.1 Introduction

The main goal of this thesis is to determine the values of $L(n, k, p, t)$. In this chapter, we describe constructions that give upper bounds for $L(n, k, p, t)$. We begin by considering BIBDs as sources of Lotto designs. Then we study formulas that relate the lower and upper bounds of Lotto designs which we called monotonicity formulas. Next we describe a construction known as the *semi-direct product* construction which entails putting several Lotto designs together to form a new Lotto design. We conclude this chapter with several other upper bound constructions for Lotto designs.

3.2 Lotto Designs from BIBDs

The study of balanced incomplete block designs (BIBDs) was started by Euler and has been continuously studied since the Fisher and Yates paper [10] in 1935. Our focus is on determining which BIBDs are Lotto designs. We begin by stating conditions that a BIBD must satisfy in order for it to be a Lotto design. Then we consider resolvable BIBDs and symmetric BIBDs. We state conditions that must be satisfied in order for them to be Lotto designs. We begin by giving a definition for a BIBD. A good reference on BIBDs can be found in "Combinatorial Theory" by Hall [14].

Definition 3.2.1 : *A (v, b, r, k, λ) BIBD is a collection of b k-sets with elements chosen from $X(v)$ such that each element appears r times and each pair of elements in $X(v)$ appears λ times in the collection of k-sets.*

Theorem 3.2.1 : *If $\mathcal{B}$ is the set of blocks of a (v, b, r, k, λ) BIBD and p, t are positive integers where $\lfloor \frac{pr}{t-1} \rfloor C(t-1, 2) + C(pr - \lfloor \frac{pr}{t-1} \rfloor (t-1), 2) < C(p, 2)\lambda$, then $\mathcal{B}$ is the set of blocks of an (v, k, p, t) Lotto design. Hence $L(v, k, p, t) \leq b$.*

Proof : Let $\mathcal{B}$ denote the blocks of a (v, b, r, k, λ) BIBD. Suppose P is a p-set not represented by any block in $\mathcal{B}$. By the definition of a BIBD, each pair appears λ times in $\mathcal{B}$. Thus, the pairs of P appear $C(p, 2)\lambda$ times in $\mathcal{B}$. On the other hand, since P is not represented in $\mathcal{B}$, at most $t-1$ elements from P can appear together in any block of $\mathcal{B}$. The maximum number of pairs of P that can be forced in $\mathcal{B}$ is

$$\left\lfloor \frac{pr}{t-1} \right\rfloor C(t-1, 2) + C(pr - \left\lfloor \frac{pr}{t-1} \right\rfloor (t-1), 2). \tag{3.1}$$

But $\lfloor \frac{pr}{t-1} \rfloor C(t-1, 2) + C(pr - \lfloor \frac{pr}{t-1} \rfloor (t-1), 2) < C(p, 2)\lambda$ by assumption, which is a contradiction. Hence all p-sets are represented in $\mathcal{B}$ and therefore $\mathcal{B}$ is an (v, k, p, t) Lotto design. □

We now give an example of the application of Theorem 3.2.1 using the existance of a $(17, 34, 16, 8, 7)$ BIBD [8].

Example 3.2.1 : *Since there exists a $(17, 34, 16, 8, 7)$ BIBD that satisfies the conditions of Theorem 3.2.1 with $p = 6$ and $t = 4$, $L(17, 8, 6, 4) \leq 34$.*

The following three results are consequences of Theorem 3.2.1.

Corollary 3.2.1 : *Suppose there exists a (v, b, r, k, λ) BIBD. If $r < 2\lambda$, then $L(v, k, 5, 4) \leq b$.*

Proof : With the current parameters, Equation (3.1) becomes

$$\left\lfloor \frac{5r}{3} \right\rfloor C(3,2) + C(5r - \left\lfloor \frac{5r}{3} \right\rfloor 3, 2).$$

This simplifies to $5r$ if $3|r$ and to $5r - 1$ if $3 \not| r$. In any case , $p \leq 5r < 10\lambda$. So we can apply Theorem 3.2.1 and obtain $L(v, k, 5, 4) \leq b$. □

Corollary 3.2.2 : *Suppose there exists a (v, b, r, k, λ) BIBD. If $r < \frac{5}{2}\lambda$, then $L(v, k, 6, 4) \leq b$.*

Proof : With the current parameters, Equation (3.1) reduces to

$$\left\lfloor \frac{6r}{3} \right\rfloor C(3,2) + C(6r - \left\lfloor \frac{6r}{3} \right\rfloor 3, 2) = 6r.$$

Since $r < \frac{5}{2}\lambda$, then $6r < 15\lambda$. Thus, the conditions of Theorem 3.2.1 are satisfied and $L(v, k, 6, 4) \leq b$. □

Corollary 3.2.3 : *Suppose there exists a (v, b, r, k, λ) BIBD. If $r < 3\lambda$, then $L(v, k, 7, 4) \leq b$.*

Proof : With the current parameters, Equation (3.1) reduces to

$$\left\lfloor \frac{7r}{3} \right\rfloor C(3,2) + C(7r - \left\lfloor \frac{7r}{3} \right\rfloor 3, 2). \tag{3.2}$$

Regardless of whether 3 divides r or not, Equation (3.2) is less than or equal to $7r$. Since $r < 3\lambda$, we have $7r < 21\lambda$. Thus, the conditions of Theorem 3.2.1 are satisfied and $L(n, k, 7, 4) < b$. □

Theorem 3.2.1 is a general result that covers all types of BIBDs. It is possible to determine weaker conditions than those given by Theorem 3.2.1 if the design is resolvable. We begin by stating the definition of a *resolvable BIBD.*

Definition 3.2.2 : *A (v, b, r, k, λ) BIBD is resolvable if the blocks of the BIBD can be partitioned into sets in such a way that every partition contains each element exactly once. A BIBD that is resolvable is called a resolvable BIBD (RBIBD). The partitions are called resolution classes.*

The parameters for a resolvable BIBD must be of the form (kx, rx, r, k, λ) where x is the number of blocks in each resolution class and $r = \frac{\lambda(kx-1)}{k-1}$. Here is an example of a $(9, 12, 4, 3, 1)$ resolvable BIBD with four resolution classes.

Example 3.2.2 : *The following is a* $(9, 12, 4, 3, 1)$ *RBIBD.*

{1, 2, 3} {4, 5, 6} {7, 8, 9}
{1, 4, 7} {2, 5, 8} {3, 6, 9}
{1, 5, 9} {2, 6, 7} {3, 4, 8}

{1, 6, 8}
{2, 4, 9}
{3, 5, 7}

For RBIBDs, a slightly weaker condition may be stated which is sufficient for it to be a Lotto design.

Theorem 3.2.2 : *If there exists a resolvable BIBD with parameters* $(xk, \frac{\lambda x(xk-1)}{k-1}, \frac{\lambda(xk-1)}{k-1}, k, \lambda)$ *and if* p, t *are positive integers where*

$$\left(\lfloor\frac{p}{t-1}\rfloor C(t-1,2) + C(p - \lfloor\frac{p}{t-1}\rfloor(t-1),2))\lambda\frac{(xk-1)}{(k-1)} < C(p,2)\lambda\right)$$

, then $L(xk,k,p,t) \leq b$.

Proof : Let $\mathcal{B}$ denote the blocks of a $(xk, \frac{\lambda x(xk-1)}{k-1}, \frac{\lambda(xk-1)}{k-1}, \frac{\lambda(xk-1)}{k-1}, k, \lambda)$ BIBD. Suppose P is a p-set not represented by any block in $\mathcal{B}$. By the definition of a BIBD, each pair appears λ times in $\mathcal{B}$. Thus, the pairs of P appear $C(p,2)\lambda$ times in $\mathcal{B}$. On the other hand, since P is not represented in $\mathcal{B}$, at most $t-1$ elements from P can appear together in any block of $\mathcal{B}$. In each resolution class, the elements of the set P must appear exactly once. Thus the maximum number of pairs of P in any given resolution class of the design is :

$$\left\lfloor\frac{p}{t-1}\right\rfloor C(t-1,2) + C(p - \left\lfloor\frac{p}{t-1}\right\rfloor(t-1),2).$$

Since there are $\frac{\lambda(xk-1)}{k-1}$ resolution classes in the design, the maximum number of pairs from P in the design is

$$\left[\left\lfloor\frac{p}{t-1}\right\rfloor C(t-1,2) + C(p - \left\lfloor\frac{p}{t-1}\right\rfloor(t-1),2)\right]\frac{\lambda(xk-1)}{k-1}. \tag{3.3}$$

But by assumption,

$$\left[\left\lfloor\frac{p}{t-1}\right\rfloor C(t-1,2) + C(p - \left\lfloor\frac{p}{t-1}\right\rfloor(t-1),2)\right]\frac{\lambda(xk-1)}{k-1} < C(p,2)\lambda.$$

This is a contradiction since the number of pairs of P occurring in $\mathcal{B}$ must be at least $C(p,2)\lambda$. Therefore every p-set is represented by the design. □

Corollary 3.2.4 : *If a* $(2a+2, 4a+2, 2a+1, a+1, a)$ *RBIBD exists, and* $4 < t < 2a+4$, *then* $L(2a+2, a+1, 2t-3, t) \leq 4a+2$.

Proof : Consider a $(2a+2, 4a+2, 2a+1, a+1, a)$ RBIBD $\mathcal{B}$. Using the notation of theorem 3.2.2, $p = 2t-3$, $\lambda = a$, $k = a+1$ and $x = 2$. Equation 3.3 becomes

$$\left[\left\lfloor \frac{2t-3}{t-1} \right\rfloor C(t-1,2) + C(2t-3 - \left\lfloor \frac{2t-3}{t-1} \right\rfloor (t-1), 2)\right] (2a+1).$$

If this is less than $aC(2t-3,2)$, then the conditions of theorem 3.2.2 are satisfied and hence our result holds. Now,

$$\left[\left\lfloor \frac{2t-3}{t-1} \right\rfloor C(t-1,2) + C(2t-3 - \left\lfloor \frac{2t-3}{t-1} \right\rfloor (t-1), 2)\right] (2a+1) < aC(2t-3,2)$$

if and only if $2t^2 - (8+2a)t + 4a + 8 < 0$. This is the case if and only if $4 < t < 2a+4$. But we assumed that this was the case. Hence, $\mathcal{B}$ is a $(2a+2, a+1, 2t-3, t)$ Lotto design with $4a+2$ blocks. □

We now state some examples of resolvable BIBDs that are Lotto designs. Notice that one RBIBD may give several Lotto designs.

Example 3.2.3 : *Consider a* $(12, 22, 11, 6, 5)$ *resolvable BIBD. Since*

$$\left[\left\lfloor \frac{5}{3} \right\rfloor C(3,2) + C(5 - \left\lfloor \frac{5}{3} \right\rfloor (3), 2)\right] 11 < 5C(5,2),$$

then $L(12,6,5,4) \leq 22$ *by Corollary 3.2.4.*

Example 3.2.4 : *Consider an* $(8,14,7,4,3)$ *RBIBD. By Corollary 3.2.4, we have* $L(8,4,3,3) \leq 14$ *and* $L(8,4,5,4) \leq 14$.

Example 3.2.5 : *Consider a* $(12, 22, 11, 6, 5)$ *RBIBD. By Corollary 3.2.4, we have* $L(12, 6, 3, 3) \leq 22$ *and* $L(12, 6, 5, 4) \leq 22$.

Example 3.2.6 : *Consider a* $(16, 30, 15, 8, 7)$ *RBIBD. By Corollary 3.2.4, we have* $L(16, 8, 3, 3) \leq 30$, $L(16, 8, 5, 5) \leq 30$ *and* $L(16, 8, 7, 5) \leq 30$.

Symmetric BIBDs are another special type of BIBD. For a specific type of symmetric BIBD, a sufficient condition may be stated for it to be a Lotto design.

Definition 3.2.3 : *A BIBD is a symmetric BIBD (SBIBD) if* $v = b$.

The following result applies specifically to symmetric designs.

Theorem 3.2.3 : *If there exists a* $(4a+3, 4a+3, 2a+1, 2a+1, a)$ *symmetric BIBD and* $a > t-2$, *then* $L(4a+3, 2a+1, 2t-2, t) \leq 4a+3$.

Proof: Let $\mathcal{B}$ denote the blocks of a $(4a+3, 2a+1, a)$ symmetric BIBD. Suppose P is a $(2t-2)$-set not represented by any block in $\mathcal{B}$. There are $aC(2t-2, 2)$ pairs of P in $\mathcal{B}$. On the other hand, each variety appears $2a+1$ times in $\mathcal{B}$. Since P can intersect a block in $\mathcal{B}$ in at most $t-1$ elements, the maximum number of these pairs of elements of P can be obtained if P intersects blocks in $\mathcal{B}$ only in $t-1$ or zero elements. The number of blocks that P can intersect in $t-1$ elements is $\lfloor \frac{(2a+1)(2t-2)}{t-1} \rfloor = 2(2a+1)$ giving $2(2a+1)C(t-1, 2)$ pairs. Now $2(2a+1)C(t-1, 2) \leq C(2t-2, 2)a$ which simplifies to $a \leq t-2$ which is a contradiction. Hence, every $(2t-2)$-set must be represented by $\mathcal{B}$ and hence $\mathcal{B}$ is an $(4a+3, 2a+1, 2t-2, t)$ Lotto design. □

The following is an example of an SBIBD that is a Lotto design.

Example 3.2.7 : *Consider a* $(15, 15, 7, 7, 3)$ *symmetric design. Since* $a = 3, t = 4$ *and* $a > t - 2$, *then the* 15 *blocks of the BIBD form an* $(15, 7, 6, 4)$ *Lotto design.*

As we have seen, under certain conditions, balanced incomplete block designs may be Lotto designs, and thus may be able to give upper bounds for Lotto Designs. However, these upper bounds are not always very good due to the strict conditions that BIBDs adhere to. In most cases, the upper bounds for $L(n, k, p, t)$ derived from BIBDs were superceded by upper bounds generated by other methods. However, a BIBD was used in determining that $L(13, 4, 5, 3) = 13$.

3.3 Monotonicity Formulas

Monotonicity formulas relate one Lotto design to another in which one or more of the parameters of the designs differ by one. An upper bound of a Lotto design may be used to determine upper bounds for other Lotto designs. Similarly, a lower bound of one Lotto design my be used to determine lower bounds for other Lotto designs. In our computer programs, we applied the monotonicity formulas each time an upper or lower bound changed.

Theorem 3.3.1 : $L(n, k, p, t) \geq L(n, k + 1, p, t)$.

Proof : Let $\mathcal{B}$ be an (n, k, p, t) Lotto design. For each block in $\mathcal{B}$, form a new block by concatenating any element to that block that is not in it. We claim these new blocks form an $(n, k + 1, p, t)$ Lotto design. To see this, let P be a p-set. Then there is some block in $\mathcal{B}$ that meets P in a t-set. The new block formed from this block will also meet P in that t-set. Thus the collection of new blocks formed from $\mathcal{B}$ is an (n, k, p, t) Lotto design, and hence $L(n, k, p, t) \geq L(n, k + 1, p, t)$. □

Theorem 3.3.2 : $L(n, k, p, t) \geq L(n+1, k+1, p, t)$

Proof : Let $\mathcal{B}$ be an (n, k, p, t) Lotto design. For each block in $\mathcal{B}$, form a new block by concatenating the new element $n+1$. This new collection of blocks is clearly an $(n+1, k+1, p, t)$ Lotto design. □

Theorem 3.3.3 : $L(n, k, p, t) \geq L(n, k, p+1, t)$

Proof : Let $\mathcal{B}$ be the blocks of an (n, k, p, t) Lotto design. Let P be a $(p+1)$-set. Since every p-set is represented by some block in $\mathcal{B}$, then P is represented by some block in $\mathcal{B}$. Thus $\mathcal{B}$ forms an $(n, k, p+1, t)$ Lotto design. □

Theorem 3.3.4 : $L(n, k, p, t) \leq L(n, k, p, t+1)$

Proof :Let $\mathcal{B}$ be the blocks of an $(n, k, p, t+1)$ Lotto design. Let P be an arbitrary p-set. Then P intersects some block of $\mathcal{B}$ in at least $t+1$ elements and hence in at least t elements. Thus $\mathcal{B}$ forms an (n, k, p, t) Lotto design. □

Theorem 3.3.5 : $L(n, k, p, t) \leq L(n+1, k, p, t)$

Proof : Consider an $(n+1, k, p, t)$ Lotto design. Select an element x, and delete every occurrence from the design. To the blocks which have been shortened, add any element not already appearing in the block. This is possible since $k < n$. Clearly, this new collection of blocks forms an (n, k, p, t) Lotto design. □

Theorem 3.3.6 : $L(n, k, p, t) \leq L(n, k+1, p, t+1)$

Proof: Let $\mathcal{B}$ be the blocks of an $(n, k+1, p, t+1)$ Lotto design. We will construct a set of k-sets $\mathcal{B}'$ such that $|\mathcal{B}| \geq |\mathcal{B}'|$. Let $\mathcal{B}' = \{B' : B' = B \backslash \{\text{the largest element of } B\}, B \in \mathcal{B}\}$. Clearly $|\mathcal{B}| \geq |\mathcal{B}'|$. Consider an arbitrary p-set P. It intersects some block B in $\mathcal{B}$ in at least $t+1$ elements. Hence it intersects a corresponding B' of $\mathcal{B}'$ in at least t elements. Hence $\mathcal{B}'$ are the blocks of an (n, k, p, t) Lotto design. □.

Theorem 3.3.7 : $L(n, k, p, t) \leq L(n, k, p+1, t+1)$

Proof: Let $\mathcal{B}$ be the blocks of an $(n, k, p+1, t+1)$ Lotto design. Consider an arbitrary p-set P. Adjoin one element to the p-set P. This $(p+1)$-set will intersect some block of $\mathcal{B}$ in at least $t+1$ elements. Hence P will intersect the same block in at least t elements. Thus the blocks of $\mathcal{B}$ form an (n, k, p, t) Lotto design. □

Corollary 3.3.1 : $L(n, k, p, t) \leq L(n+1, k, p+1, t+1)$

Proof: By Theorem 3.3.5, $L(n, k, p, t) \leq L(n+1, k, p, t)$ and by Theorem 3.3.7, $L(n+1, k, p, t) \leq L(n+1, k, p+1, t+1)$. This implies $L(n, k, p, t) \leq L(n+1, k, p+1, t+1)$. □

Corollary 3.3.2 : $L(n, k, p, t) \leq L(n+1, k+1, p+1, t+1)$

Proof: By Theorem 2.2.1, $L(n, k, p, t) = L(n, n-k, n-p, n-k-p+t)$ and $L(n+1, k+1, p+1, t+1) = L(n+1, n-k, n-p, n-k-p+t)$. Thus it suffices to show $L(n, n-k, n-p, n-k-p+t) \leq L(n+1, n-k, n-p, n-k-p+t)$. But this follows immediately from Theorem 3.3.5. □

The following result is a generalization of a result on Covering designs which states: $C(n, k, t) \leq C(n+1, k+1, t+1)$.

Corollary 3.3.3 : $L(n, k, p, t) \leq L(n+1, k+1, p, t+1)$

Proof: By Theorem 3.3.1, $L(n, k, p, t) \leq L(n, k+1, p, t+1)$ and by Theorem 3.3.5, $L(n, k+1, p, t+1) \leq L(n+1, k+1, p, t+1)$. Hence $L(n, k, p, t) \leq L(n+1, k+1, p, t+1)$. □

Corollary 3.3.4 : $L(n, k, p, t) \geq L(n+1, k+1, p+1, t)$

Proof: By Theorem 3.3.2, $L(n, k, p, t) \geq L(n+1, k+1, p, t)$ and by Theorem 3.3.3, $L(n+1, k+1, p, t) \geq L(n+1, k+1, p+1, t)$. Hence, $L(n, k, p, t) \geq L(n+1, k+1, p+1, t)$. □

Corollary 3.3.5 : $L(n, k, p, t) \geq L(n+1, k, p+1, t)$.

Proof : Let $\mathcal{B}$ be the blocks of an (n, k, p, t) Lotto design. Let P be an $(p+1)$-set with elements chosen from $X(n+1)$. If $n+1 \in P$, then $P \setminus \{n+1\}$ is a p-set and hence is represented by some block of $\mathcal{B}$ as $\mathcal{B}$ is an (n, k, p, t) Lotto design. Otherwise, if $n+1 \notin P$, then P must be represented by some block of $\mathcal{B}$ as $\mathcal{B}$ is an (n, k, p, t) Lotto design. In any case, P is represented by a block of $\mathcal{B}$ and hence $\mathcal{B}$ is an $(n+1, k, p+1, t)$ Lotto design. Thus, $L(n, k, p, t) \geq L(n+1, k, p+1, t)$. □

Corollary 3.3.6 : $L(n, k, p, t) \leq L(n+1, k, p, t+1)$

Proof: By Theorem 3.3.5, $L(n, k, p, t) \leq L(n+1, k, p, t)$ and by Theorem 3.3.4, $L(n+1, k, p, t) \leq L(n+1, k, p, t+1)$. Hence, $L(n, k, p, t) \leq L(n+1, k, p, t+1)$. □.

Corollary 3.3.7 : $L(n, k, p, t) \geq L(n, k+1, p+1, t)$.

Proof : By Theorem 3.3.1, $L(n,k,p,t) \geq L(n,k+1,p,t)$ and by Theorem 3.3.3, $L(n,k+1,p,t) \geq L(n,k+1,p+1,t)$. Hence $L(n,k,p,t) \geq L(n,k+1,p+1,t)$.

The main purpose of the monotonicity results in this thesis is to apply them to our tables of lower and upper bounds as soon as a lower bound or an upper bound changes in the table.

3.4 Semi-Direct Product Construction

Constructing Lotto designs based on other Lotto designs is one way to obtain upper bounds. The method described in this section uses this technique. We termed the construction in this section "Semi-Direct Product" construction. The construction technique described may be used to obtain an (n,k,p,t) Lotto design for arbitrary values of n,k,p and t. In general, the larger the value of k, the better the construction will be.

We begin with a useful lemma.

Lemma 3.4.1 : *Suppose $\mathcal{B}$ is an (n,k,p,t) Lotto design. If $m \leq p$ is an integer, then any m-set intersects some block of $\mathcal{B}$ in at least $t-p+m$ elements.*

Proof : Let M be an m-set. Add $p-m$ elements to M to form a p-set P. Since P now has p elements, it is represented by some block B in $\mathcal{B}$. Since at most $p-m$ of the elements added to M to form P can appear in B, then at least $t-p+m$ elements from M appear in B. □

We now state the main result of this section. This result will give us a way to compute upper bounds for Lotto designs.

Theorem 3.4.1 : *Suppose n, k, p, t, n_1, k_1 and r are integers such that $n_1 < n$, $p - r \geq t$, $k_1 \geq t - r - 1$ and $k_1 = k - n + n_1$. Then $L(n, k, p, t) \leq L(n_1, k, p - r, t) + L(n_1, k_1, p - r - 1, t - r - 1)$.*

Proof : We shall proceed to construct an (n, k, p, t) Lotto design using $(n_1, k, p-r, t)$ and $(n_1, k_1, p - r - 1, t - r - 1)$ Lotto designs.

Let $\mathcal{A}$ be an $(n_1, k, p - r, t)$ Lotto design, $\mathcal{B}'$ be an $(n_1, k_1, p - r - 1, t - r - 1)$ Lotto design and let C be the set $\{n_1 + 1, n_1 + 2, ..., n\}$. For each block in $\mathcal{B}'$, adjoin the set C to it. Denote this new collection of k-sets as $\mathcal{B}$. We claim that $\mathcal{A} \cup \mathcal{B}$ is an (n, k, p, t) Lotto design.

Consider a p-set P from $X(n)$. Suppose $p-x$ elements in P come from $\{1, 2, 3, ..., n_1\}$ and the remaining x elements of P come from $\{n_1 + 1, n_1 + 2, ..., n\}$. Since $n - n_1 = k - k_1$, the set $\{n_1+1, n_1+2, ..., n\}$ has exactly $k - k_1$ elements and hence x can be at most $k - k_1$. We need to show that the p-set is represented by $\mathcal{A} \cup \mathcal{B}$. If $x \leq r$ then the p-set would be represented by a block in $\mathcal{A}$ since it is an $(n_1, k, p - r, t)$ Lotto design. If $r + 1 \leq x \leq p$, then by Lemma 3.4.1, there is a block from $\mathcal{B}'$ that intersects P in at least $t - x$ and P intersects C in x elements. Hence P is represented by a block in $\mathcal{B}$.

Since P is always represented by some block in $\mathcal{A} \cup \mathcal{B}$, we conclude that $\mathcal{A} \cup \mathcal{B}$ is an (n, k, p, t) Lotto design. □

If $r \geq n - n_1$ in the previous theorem, then $\mathcal{B}$ is not used. The theorem can be simplified to $L(n, k, p, t) \leq L(n_1, k, p - r, t)$ which is essentially Theorem 3.3.5.

A construction of a $(20, 10, 6, 4)$ Lotto design using Theorem 3.4.1 will now be given.

Example 3.4.1 : Let $n = 20$, $k = 10$, $p = 6$, $t = 4$, $n_1 = 15, r = 1$. Then $k_1 = 5$. Let $\mathcal{A} = \{\{1, 3, 5, 6, 7, 10, 11, 12, 13, 14\}, \{2, 4, 5, 6, 7, 8, 9, 10, 12, 15\}, \{1, 2, 3, 4, 8, 9, 11, 13, 14, 15\}\}$. Let $\mathcal{B}' = \{\{3, 5, 6, 10, 12\}, \{1, 7, 9, 11, 14\}, \{2, 4, 8, 13, 15\}\}$ and let $C = \{16, 17, 18, 19, 20\}$. It can be shown that $\mathcal{A}$ is a $(15, 10, 5, 4)$ Lotto design and $\mathcal{B}'$ is an $(15, 5, 4, 2)$ Lotto design. We construct $\mathcal{B}$ from $\mathcal{B}'$ and C by adjoining C to each block of $\mathcal{B}'$. Thus $\mathcal{B} = \{\{3, 5, 6, 10, 12, 16, 17, 18, 19, 20\}, \{1, 7, 9, 11, 14, 16, 17, 18, 19, 20\}, \{2, 4, 8, 13, 15, 16, 17, 18, 19, 20\}\}$. By Theorem 3.4.1, $\mathcal{A} \cup \mathcal{B}$ is a $(20, 15, 6, 4)$ Lotto design. The blocks of this $(20, 15, 6, 4)$ Lotto design are $\{\{1, 3, 5, 6, 7, 10, 11, 12, 13, 14\}, \{2, 4, 5, 6, 7, 8, 9, 10, 12, 15\}, \{1, 2, 3, 4, 8, 9, 11, 13, 14, 15\}, \{3, 5, 6, 10, 12, 16, 17, 18, 19, 20\}, \{1, 7, 9, 11, 14, 16, 17, 18, 19, 20\}, \{2, 4, 8, 13, 15, 16, 17, 18, 19, 20\}\}$.

By applying Theorem 3.4.1, we can compute uppers bounds for $L(n, k, p, t)$ by varying the value of n_1 (which in turn varies k_1) in the construction of an (n, k, p, t) Lotto design and taking the minimum size of all the designs constructed. This approach can be easily programmed and used to update our tables. Table 3.1 displays upper bounds for $L(19, 10, 6, 4)$ using the semi-direct product construction. We fixed the value of r from Theorem 3.4.1 at 1.

n_1	Sub-designs used	upper bound obtained
18	$(18, 10, 5, 4)$	14
17	$(17, 10, 5, 4)$ and $(17, 8, 4, 2)$	11
16	$(16, 10, 5, 4)$ and $(16, 7, 4, 2)$	8
15	$(15, 10, 5, 4)$ and $(15, 6, 4, 2)$	6
14	$(14, 10, 5, 4)$ and $(14, 5, 4, 2)$	6
13	$(13, 10, 5, 4)$ and $(13, 4, 4, 2)$	8
12	$(12, 10, 5, 4)$ and $(12, 3, 4, 2)$	11
11	$(11, 10, 5, 4)$ and $(11, 2, 4, 2)$	16

Table 3.1: Some upper bounds generated by semi-product construction

We stop computing Table 3.1 at $n_1 = 11$ because for $n_1 < 11$, the construction makes no sense. In general, for fixed r, we would stop when $k_1 < t - r - 1$ for an (n, k, p, t)

design. From Table 3.1, we see that the best upper bound for $L(19, 10, 6, 4)$ determined using the semi-direct product construction is 6.

The semi-direct product construction allowed us to construct an arbitrary (n, k, p, t) Lotto design from smaller Lotto designs. This construction empowered us with another way of obtaining an upper bound for $L(n, k, p, t)$. This construction works best when k is large. This is because a large value of k gives more ways of constructing a Lotto design from smaller ones. This construction does not work well for cases where k is close to p or t. Finally, since this construction depends on other Lotto designs, better upper bounds for these designs would generate a better design.

3.5 Other Upper Bound Constructions

In this section, we will state other upper bound constructions that we have developed.

One way of constructing an $(n+1, k+1, p+1, t+1)$ Lotto design from an (n, k, p, t) Lotto design and an $(n, k+1, p+1, t+1)$ Lotto design is to attach to each row of the (n, k, p, t) design the element $n+1$. The blocks of this new design along with the blocks of the $(n, k+1, p+1, t+1)$ design form an $(n+1, k+1, p+1, t+1)$ Lotto design. This result is a direct generalization of a result on Covering designs that can be found in the CRC handbook [8].

Theorem 3.5.1 : $L(n+1, k+1, p+1, t+1) \leq L(n, k, p, t) + L(n, k+1, p+1, t+1)$.

Proof: Let $\mathcal{A}$ be an (n, k, p, t) Lotto design and $\mathcal{B}$ be and $(n, k+1, p+1, t+1)$ Lotto design. To each block of $\mathcal{A}$, attach the element $n+1$ and denote this new collection by $\mathcal{A}^*$. We claim that $\mathcal{A}^* \cup \mathcal{B}$ is an $(n+1, k+1, p+1, t+1)$ Lotto design. To show this, let P be a $(p+1)$-set. If $n+1 \notin P$ then there is a block in $\mathcal{B}$ that represents P. Otherwise, $n+1 \in P$. Then, $P \setminus \{n+1\}$ is a p-set and hence intersects some block A from $\mathcal{A}$ in t elements. The block $\{n+1\} \cup A$ belongs in $\mathcal{A}^*$ by construction, and intersects P in $t+1$ elements. Hence $\mathcal{A}^* \cup \mathcal{B}$ is an $(n+1, k+1, p+1, t+1)$ Lotto design and $L(n+1, k+1, p+1, t+1) \leq L(n, k, p, t) + L(n, k+1, p+1, t+1)$. $\square$

In the study of Covering designs, it is well known that $C(mn, mk, t) \leq C(n, k, t)$ for any integer $m \geq 1$. For Lotto designs, the obvious generalization is not always true. An extra condition must be added to the theorem.

Theorem 3.5.2 : *Suppose there exists an (n, k, p, t) Lotto design D_1 where every $(t-1)$-set is contained in some block of the design. Then $L(mn, mk, p, t) \leq L(n, k, p, t)$ where $m > 1$ is an integer.*

Proof: Let D_1 be an (n, k, p, t) Lotto design such that every $(t-1)$-set appears in the design and the elements of D_1 are $\{1^1, 2^1, 3^1, ..., n^1\}$. Suppose the blocks of D_1 are ordered using some ordering. Since every $(t-1)$-set appears in the design, so does every u-set, for any integer $u < t$. Create $m-1$ copies $D_2, D_3, ..., D_m$ of D_1 keeping D_1's ordering of the blocks, relabeling the elements $\{1^1, 2^1, ..., n^1\}$ in D_i with $\{1^i, 2^i, ..., n^i\}$, for $i = 2$ to m in the obvious way. For each j from 1 to $|D_1|$, unite the j^{th} blocks of $D_1, D_2, ..., D_m$ to form a mk-set. Let D denote the set of all mk-sets formed. We claim that D is an (nm, km, p, t) Lotto design.

Suppose P is a p-set in D. In order to show that D is a Lotto design, we need to show that P is represented by some block of D. Let P be denoted $\{(x_1)^{i_1}, (x_2)^{i_2}, ..., (x_p)^{i_p}\}$ where $x_j \in \{1, 2, ..., n\}$ and $i_j \in \{1, 2, ..., m\}$ for $j = 1$ to p. There are three possible cases that we need to consider.

Case 1 : Suppose all the x_j's are distinct, then the p-set $\{(x_1)^1, (x_2)^1, ..(x_p)^1\}$ is represented by some block in D_1. Without loss of generality suppose $\{(x_1)^1, (x_2)^1, ..., (x_t)^1\}$ appears in this block of D_1. Then the corresponding block in D will contain the subset $\{(x_1)^{i_1}, (x_2)^{i_2}, ..., (x_t)^{i_t}\}$ of P. Thus P is represented in this case.

Case 2 : Suppose not all the x_j's are distinct and some x_j appears at least t times in the collection $\{x_1, x_2, ...x_p\}$. Without loss of generality, suppose $x_1 = x_2 = ... = x_t$. Since $(x_1)^1$ has to appear in some block of D_1, the corresponding block in D must contain the subset $\{(x_1)^{i_1}, (x_2)^{i_2}, ..., (x_t)^{i_t}\}$ of P. Thus P is represented in this case.

Case 3 : Suppose not all the x_j's are distinct and at most $t-1$ of any of the x_j's are the same, then select t elements from the collection $\{x_1, x_2, ..., x_p\}$ such that at least two of the chosen elements are the same. Without loss of generality, suppose we chose $x_1, x_2, ..x_t$ where $x_1 = x_t$. Now there are at most $t-1$ distinct elements in the collection $\{(x_1)^1, (x_2)^1, ..., (x_t)^1\}$, since $x_1 = x_t$. By hypothesis, the set $\{(x_1)^1, (x_2)^1, ..., (x_t)^1\}$ is contained in some block of D_1. Then the corresponding block in D will contain the subset $\{(x_1)^{i_1}, (x_2)^{i_2}, ..., (x_t)^{i_t}\}$ of P. Thus P is represented in this case.

In each case, the p-set P is represented in D. Hence D is an (nm, km, p, t) Lotto design and $L(mn, km, p, t) \leq L(n, k, p, t)$. □

The construction process above is called "m-ing a design". The following is a special case of a result from Bate [1] which we will need, to prove a later theorem.

Theorem 3.5.3 : *If $L(n, k, p, 2) \geq \frac{n}{k}$ then there exists a $(n, k, p, 2)$ Lotto design* *with $\frac{n}{k}$ blocks such that every element appears in the design.*

The following theorem states that m-ing an $(n,k,p,2)$ Lotto design is always possible if every element appears in the original design.

Theorem 3.5.4 : *If $L(n,k,p,2) = b \geq \frac{n}{k}$ then $L(mn,mk,p,t) \leq b$.*

Proof: By Theorem 3.5.3 there is an $(n,k,p,2)$ Lotto design with b blocks such that every element appears in the design. By Theorem 3.5.2, $L(mn,mk,p,t) \leq L(n,k,p,2) = b$ □

Theorem 3.5.5 : *$L(mn, 2m, zn+2, 2z+2) \leq C(n,2)$ for $z \leq m$, where m is an integer.*

Proof : Let D_1 be the $(n,2,2,2)$ Lotto design which consists of blocks that are all pairs from the n-set. Order the blocks using some ordering. Create $m-1$ copies $D_2, D_3, ..., D_m$ of D_1, each keeping D_1's ordering of the blocks, relabeling the elements $\{1^1, 2^1, ..., n^1\}$ in D_i with $\{1^i, 2^i, ..., n^i\}$, for $i = 2$ to m in the obvious way. For each j from 1 to $|D_1|$, unite the j^{th} blocks of $D_1, D_2, ..., D_m$ to form a mn-set. Let D denote the set of all $2m$-sets formed. We claim that D is an $(mn, 2m, zn+2, 2z+2)$ Lotto design. To show that D is an $(mn, 2m, zn+2, 2z+2)$ Lotto design, let $P = \{(x_1)^{i_1}, (x_2)^{i_2}, ..., (x_{zn+2})^{i_{zn+2}}\}$ be an arbitrary $(zn+2)$-set chosen from the elements of D, where $x_j \in \{1, 2, ..., n\}$ and $i_j \in \{1, 2, ..., m\}$ for $j = 1$ to $zn+2$. We need to show it intersects some block of D in at least $2z+2$ elements. We note that there is at least one element in the collection $\{x_1, x_2, ..., x_{zn+2}\}$ that occurs $z+1$ or more times in the collection, as $\frac{zn+2}{n} > z$. We need to consider the possible cases :

Case 1 : Suppose the collection $\{x_1, x_2, x_3, ..., x_{zn+2}\}$ contains two elements, say x_1 and x_{z+2} such that $x_1 \neq x_{z+2}$ and x_1 is appears at least $z+1$ times in the collection and x_{z+2} appears at least $z+1$ times in the collection also. Then, without loss of generality, assume $x_1 = x_2 = ... = x_{z+1}$ and $x_{z+2} = x_{z+3} = ... = x_{2z+2}$. Since D_1 is an $(n, 2, 2, 2)$ Lotto design the the set $\{(x_1)^1, (x_{z+2})^1\}$ must appear in some block of D_1. Thus the subset $\{(x_1)^{i_1}, (x_2)^{i_2}, ..., (x_{z+1})^{i_{z+1}}, (x_{z+2})^{i_{z+2}}, ..., (x_{2z+2})^{i_{2z+2}}\}$ (which contains $2z+2$ elements of P) must be contained in some block of D. Since P was arbitrarily chosen, D is an $(mn, 2m, zn+2, 2z+2)$ Lotto design.

Case 2 : Suppose the collection $\{x_1, x_2, x_3, ..., x_{zn+2}\}$ contains only one element, say x_1 that appears at least $z+1$ times in the collection. It is easy to see that there exists an element x_2 distinct from x_1 that appears in the collection, and x_1 along with x_2 together appear at least $2z+2$ times in the collection. To see this, suppose that x_1 appears $z+l$ times in the collection where $l \geq 1$ and suppose every element in the collection distinct from x_1 appears fewer than $z-l+2$ times in the collection. Counting the total number of elements in the collection, there are at most $(n-1)(z-l+1)+(z+l) = zn+(n-nl+2l-1)$ elements. We can easily show that $n-nl+2l-1 < 2$ since $\frac{2l-3}{l-1} < 2$. Hence our method of counting showed that there are less than $zn+2$ elements in the collection which is a contradiction. Considering the elements x_1 and x_2 and using almost the same argument as in the previous case, we conclude that P must be contained in some block of D. Since P was arbitrarily chosen, D is an $(mn, 2m, zn+2, 2z+2)$ Lotto design. □

Table 3.2 shows some upper bounds resulting from Theorem 3.5.5.

Gordon et. al [12] stated the following theorem for Covering designs.

mn	k	l	t	# of blocks	n
12	6	6	4	6	4
16	8	6	4	6	4
16	8	10	6	6	4
20	8	12	6	10	5
20	10	6	4	6	4
20	10	14	8	6	4

Table 3.2: Some Upper Bounds generated by m-ing a base design

Theorem 3.5.6 : $C(n+1,k+1,t+1) \leq \lfloor (2-\frac{k}{n})C(n,k,t) \rfloor + C(n-1,k+1,t+1)$.

We shall generalize Theorem 3.5.6 for Lotto designs with the following construction.

Consider an (n,k,p,t) Lotto design D. Let $x \in X(n)$ and choose 2 new points x' and x'' not in $X(n)$. On the blocks of D, perform the following operations :

1. If a block B of D does not contain x, replace B with the two blocks : $B \cup \{x'\}$ and $B \cup \{x''\}$.
2. If a block B of D contains x, replace B with the block $(B \setminus \{x\}) \cup \{x', x''\}$.

Let us denote this new design by D'. Finally, add an $(n-1,k+1,p+1,t)$ Lotto design E on $X(n) \setminus \{x\}$ to the end of D'.

Theorem 3.5.7 : *The above construction gives an* $(n+1,k+1,p+1,t)$ *Lotto design on the elements* $X(n) \setminus \{x\} \cup \{x', x''\}$.

Proof : Consider a $(p+1)$-set $P = \{a_1, a_2, ..., a_{p+1}\}$. We must show it is represented by some block in $D' \cup E$. Consider the three possible cases :
Case 1 : Suppose $x' \notin P$ and $x'' \notin P$. Then P is represented by E.

Case 2 : Suppose $x' \in P$ but $x'' \notin P$. Then $P \setminus \{x'\} \cup \{x\}$ is represented by some block of B in D. If $x \in B$, then P is represented by the block B' constructed from B. Otherwise, if $x \notin B$, then the block $B \cup \{x'\}$ from D' represents P. Symmetrically, the same is true if $x'' \in P$ but $x' \notin P$.

Case 3 : Suppose x' and x'' is in P. Let $P = \{x', x'', a_1, a_2, ..., a_{p-2}\}$. Consider the set $\{x, a_1, a_2, ..., a_{p-2}, b\}$ where b is any other element. This set is represented by some block, B, in the original design. If B contains x, then $B \setminus \{x\}$ intersects P in at least $t-2$ elements. But then, P intersects $B \setminus \{x\} \cup \{x', x''\}$ in t elements. If B does not contain x then it intersects P in at least $t-1$ elements and either $B \cup \{x'\}$ or $B \cup \{x''\}$ can represent P in t elements.

Since in each case, P was represented by the constructed design, the design must be an $(n+1, k+1, p+1, t)$ Lotto design. □

Corollary 3.5.1 : *If $x \in X(n)$ and b_x denotes the number of blocks in a minimal (n, k, p, t) Lotto design that contain x, then $L(n+1, k+1, p+1, t) \leq 2L(n, k, p, t) - b_x + L(n-1, k+1, p+1, t)$.*

Proof : The construction creates $b_x + 2(L(n, k, p, t) - b_x) + L(n-1, k+1, p+1, t)$ blocks. Hence $L(n+1, k+1, p+1, t) \leq 2L(n, k, p, t) - b_x + L(n-1, k+1, p+1, t)$. □

The next result follows immediately from Corollary 3.5.1.

Corollary 3.5.2 : *$L(n+1, k+1, p+1, t) \leq \left\lfloor (2 - \frac{k}{n} L(n, k, p, t) \right\rfloor + L(n-1, k+1, p+1, t)$.*

Proof : For a minimal (n, k, p, t) Lotto design, there exists some $x \in X(n)$ such that

x appears in at least $\left\lceil \frac{k}{n} L(n,k,p,t) \right\rceil$ blocks of the design. Then by Corollary 3.5.1, we have

$$L(n+1,k+1,p+1,t) \le 2L(n,k,p,t) - \left\lceil \frac{k}{n} L(n,k,p,t) \right\rceil + L(n-1,k+1,p+1,t).$$

Hence we are done. □

The following results determine upper bound formulas for fixed values of p and t.

Theorem 3.5.8 : *$L(k+2,k,4,3) \le 3$, for $k \ge 3$.*

Proof: The blocks $B_1 = \{1,2,3,...,k\}$, $B_2 = \{1,2,3,...,k-1,k+1\}$ and $B_3 = \{1,2,3,...,k-3,k,k+1,k+2\}$ form a $(k+2,k,4,3)$ Lotto design. To show this, consider a 4-set P. If P is not represented by B_1 or B_2, then $|P \cap B_1| \le 2$ which implies $\{k+1,k+2\} \subset P$. Similarly, $|P \cap B_2| \le 2$ implying that $\{k,k+2\} \subset P$. Thus, $\{k,k+1,k+2\} \subset P$ which is contained in B_3. Hence, the three blocks B_1, B_2, B_3 form an $(k+2,k,4,3)$ Lotto design. □

Theorem 3.5.9 : *$L(2k+1,k,5,3) \le 5$, for $k \ge 4$.*

Proof : Consider the five blocks : $B_1 = \{1,2,3,...,k-2,k-1,k\}$, $B_2 = \{1,2,3,...,k-2,k+1,2k+1\}$, $B_3 = \{3,...,k-2,k-1,k,k+1,2k+1\}$, $B_4 = \{k+2,...,2k-1,2k,2k+1\}$ and $B_5 = \{k+1,k+2,...,2k-1,2k\}$. We claim these five blocks make up a $(2k+1,k,5,3)$ Lotto design. To show this, divide $X(2k+1)$ into two sets $A = \{1,2,3,...,k\}$ and $B = \{k+1,k+2,...,2k+1\}$. Consider an arbitrary 5-set P. Clearly, if three or more of P's elements come from A, then it will be represented by the block B_1. Similarly, if four or more of P's elements come from B, then it will be represented by the block B_5.

We consider the remaining case where two elements of P are from A and three elements of P are from B. If $2k+1 \notin P$, then $|\{k+1, k+2, ..., 2k\} \cap P| = 3$, which implies that P is represented by B_5. So assume that $2k+1 \in P$. Similarly, if $k+1 \notin P$, then $|\{k+2, ..., 2k, 2k+1\} \cap P| \geq 3$, which implies that P is represented by B_4. Thus we assume $k+1 \in P$. So both $k+1$ and $2k+1 \in P$. Now, consider an element from $P \cap A$ (there are two of these); it must be in B_2 or B_3 or both. In any case, since $k+1$ and $2k+1$ appear in both B_2 and B_3, P is represented in this case.

We have shown that an arbitrary 5-set is represented by these five blocks, and hence, it forms a $(2k+1, k, 5, 3)$ Lotto design. □

Theorem 3.5.10 *: $L(2k+2, k, 8, 4) \leq 4$, for $k \geq 4$.*

Proof : We claim that the blocks $B_1 = \{1, 2, 3, ..., k\}$, $B_2 = \{k+3, k+4, ..., 2k+2\}$, $B_3 = \{k+1, k+2, ..., 2k-2, 2k-1, 2k\}$ and $B_4 = \{k+1, k+2, ..., 2k-2, 2k+1, 2k+2\}$ form a $(2k+2, k, 8, 4)$ Lotto design. To show this, divide $X(2k+2)$ into two sets $A = \{1, 2, 3, ..., k\}$ and $B = \{k+1, k+2, ..., 2k+1, 2k+2\}$. Consider an arbitrary 8-set P. Clearly if four or more of P's elements come from A, then it will be represented by the block B_1. Similarly, if six or more of P's elements come from B, then it will be represented by the blocks B_2 or B_4.

We consider the remaining case where three elements of P are from A and five elements of P are from B. Split the set B into two subsets $C = \{k+1, k+2, ..., 2k-2\}$ and $D = \{2k-1, 2k, 2k+1, 2k+2\}$, and consider the possible subcases. If two or more elements of P appear in C, then P is represented by either B_3 or B_4. If one of P appear in C, then P is represented by B_2. It is not possible for not elements of P to appear in C since $|P| = 5$ and $|D| = 4$. Thus all subcases have been considered.

We have shown that an arbitrary 8-set is represented by these four blocks, and hence it is a $(2k+2, k, 8, 4)$ Lotto design. □

Theorem 3.5.11 : $L(3k+2, k, 8, 3) \leq 5$, *for* $k \geq 4$.

Proof: Let the elements 1,2,3,...,$k+4$ have frequency one and the rest of the elements have frequency two. Put the elements 1,2,..., k in the first block B_1. Let the second block B_2 be $\{k+1, k+2, k+3, k+4, k+5, ..., 2k\}$. Now put the elements $k+5, k+6, ..., 3k+2$ into the last three blocks B_3, B_4 and B_5 in any way that does not place the same element twice in any block. We claim this is a $(3k+2, k, 8, 3)$ Lotto design. To show this, consider an arbitrary 8-set P chosen from the two sets $A = \{1, 2, ..., k+4\}$ and $B = \{k+5, ..., 3k+2\}$. We must show that this 8-set is represented. We denote the number of elements chosen for P from each set by (a, b) where a is the number of elements from A and b is the number of elements from B. If $a \geq 5$, then clearly, B_1 or B_2 contains at least three elements of the 8-set and hence, P is represented in this case. If $a = 4$, then it is easy to see that we may assume that exactly two elements chosen from A appear in B_1 and the remaining two elements chosen from A appears in B_2. Otherwise, either B_1 or B_2 would contain at least three elements of the 8-set, and hence, the P is represented. As B_2 contains two elements from the 8-set, we may assume that no element in $\{k+5, k+6, ..., 2k\}$ may belong to the P, or else B_2 would represent P. So the 4 elements of the P chosen from B must actually be contained in $\{2k+1, 2k+2, ..., 3k+2\}$, all of which have frequency two. This implies that at least one of three blocks B_3, B_4 and B_5 contains 3 elements of the P. Finally, consider the remaining cases where $(a, b) = (4-x, 4+x)$ where $0 < x \leq 4$. If 0 or 1 elements of P are from $\{k+5, k+6, ..., 2k\}$, then there are at least $3+x \geq 4$ elements of P from $\{2k+1, 2k+2, ..., 3k+2\}$ which can be handled by the case where $a = 4$. If 2 elements of P are from $\{k+5, k+6, ..., 2k\}$, then there are at least $2+x \geq 3$ elements of P from $\{2k+1, 2k+2, ..., 3k+2\}$, which implies that the elements of P appear at least 3(2)+2=8 times in the blocks B_3, B_4 and B_5. Clearly, some block must contain at least three of these elements of P and hence P is represented in this case. Finally, if 3 or more elements of P are from $\{k+5, k+6, ..., 2k\}$, then P is represented by the block B_2. So, no matter what 8-set we choose, it is always represented. Thus, the construction design is a $(3k+2, k, 8, 3)$ Lotto Design with five blocks. □

The next three results although specific are useful.

Theorem 3.5.12 : $L(16, 8, 4, 3) \leq 7$

Proof : The blocks $\{1,3,4,6\}, \{1,2,5,7\}, \{3,4,6,7\}, \{2,3,5,8\}, \{2,4,5,8\}, \{1,6,7,8\}$, and $\{2,5,6,8\}$ form an $(8,4,4,3)$ Lotto design on seven blocks where every 2-set appears in the design. Hence by Theorem 3.5.2, $L(16, 8, 4, 3) \leq 7$. □

Theorem 3.5.13 : $L(18, 8, 4, 3) \leq 9$.

Proof : Since we can generate a $(9,4,4,3)$ Lotto design cyclically from the block $\{0,1,2,4\}$ modulo 9, every 2-set appears in the constructed design. Hence by Theorem 3.5.2, we can double this design to get an $(18,8,4,3)$ Lotto design with 9 blocks. Thus, $L(18, 8, 4, 3) \leq 9$. □

Theorem 3.5.14 : $L(18, 9, 4, 3) \leq 6$.

Proof: The blocks $\{1,2,3\}, \{1,3,5\}, \{2,4,5\}, \{1,4,6\}, \{3,4,6\}, \{2,5,6\}$ form an $(6,3,4,3)$ Lotto design in which every 2-set appears. By tripling this design, we get an $(18,9,4,3)$ Lotto design. Hence $L(18, 9, 4, 3) \leq 6$. □

3.6 Conclusion

We have presented several constructions of Lotto designs in this chapter. These constructions give upper bounds for $L(n, k, p, t)$. A few of these results are generalized from the theory of Covering designs. The semi-direct product is one construction that could be generalized further. All the constructions from this chapter have been incorporated into our computer programs and they help generate upper bounds for $L(n, k, p, t)$.

Chapter 4

Lower Bound Formulas

4.1 Introduction

In the previous chapter, we presented several techniques for improving the upper bound for $L(n,k,p,t)$. In this chapter, we discuss techniques for improving the lower bound for $L(n,k,p,t)$. Upper bounds are usually much easier to obtain than lower bounds since upper bounds can be obtained often from constructions. Lower bounds on the other hand cannot be generated from constructions. Lower bounds often must be derived mathematically.

Our first lower bound formula is a generalization of the Schönheim bound for Covering designs. We follow this with lower bound formulas for $L(n,k,p+1,p)$ and $L(n,k,p+2,p)$. After that we give a result that states the value of $L(n,k,p,t)$ for an infinite class of parameters. Finally, we prove several other sporadic lower bound results.

4.2 Generalized Schönheim Formula

The main goal of this section is to derive a lower bound formula for Lotto designs that is a generalization of Schönheim's lower bound formula for Covering designs.

Definition 4.2.1 : *If $(X, \mathcal{B})$ is an (n, k, p, t) Lotto design and S is a subset of X, let $\mathcal{B}^{S*} = \{T \in \mathcal{B} : |S \cap T| \leq k - t\}$ and let $\mathcal{B}^S = \{T^* : \text{if } T \in \mathcal{B}^{S*} \text{ then } T^* = T \setminus S \cup X^T \text{ where } |X^T| = |T \cap S| \text{ and } X^T \subseteq X \setminus S\}$.*

Theorem 4.2.1 : *Let $(X, \mathcal{B})$ be an (n, k, p, t) Lotto design and let S be a subset of X such that $|S| \leq n - p$ and $|S| \leq n - k$. If $\mathcal{B}^S \neq \emptyset$, then $(X \setminus S, \mathcal{B}^S)$ is an $(n - |S|, k, p, t)$ Lotto design.*

Proof : It is easy to see that blocks in $\mathcal{B}^S$ are made up of elements from $X \setminus S$ and since $n \geq k + |S|$, each block will not have repeated elements (although there could be repeated blocks). Let P be a p-set from $X \setminus S$, then since $(X, \mathcal{B})$ is a Lotto design, there exists some $T \in \mathcal{B}$ such that $|T \cap P| \geq t$. Since $P \cap S = \emptyset$, then $k \geq |(P \cap T)| + |(S \cap T)| \geq t + |S \cap T|$. Hence $|S \cap T| \leq k - t$. As $\mathcal{B} \neq \emptyset$, $T \in \mathcal{B}^{S*}$ and we are done. □

It is possible to take (n, k, p, t) Lotto-designs found using techniques such as exhaustive search, hill climbing or Simulated Annealing and try to find an optimal S such that an $(n - |S|, k, p, t)$ Lotto design is generated which has a better upper bound than currently known. The only problem is that the number of choices for S becomes prohibitively large as n gets large. A future goal is to try discover the types of subsets S which will improve upper bounds.

Definition 4.2.2 : *If $(X, \mathcal{B})$ is an (n, k, p, t) Lotto design and S is a subset of X,* *then we define* $deg_{\mathcal{B}}(S) = |\{T \in \mathcal{B} : S \subseteq T\}|$.

Lemma 4.2.1 : *Let* $(X, \mathcal{B})$ *be an* (n, k, p, t) *Lotto design and* $S \subseteq X$ *such that* $|S| = k - t + 1$. *Then* $|\mathcal{B}^S| = |\mathcal{B}| - deg_{\mathcal{B}}(S)$.

Proof : We see that $|\mathcal{B}^S| = |\{T \in \mathcal{B} : |S \cap T| \leq k - t\}|$ and $deg_{\mathcal{B}}(S) = |\{T \in \mathcal{B} : |S \cap T| = k - t + 1\}|$. It is easy to see that any $T \in \mathcal{B}$ must be counted in one and only one of the two stated collections. Thus $|\mathcal{B}| = |\mathcal{B}^S| + deg_{\mathcal{B}}(S)$. That is, $|\mathcal{B}^S| = |\mathcal{B}| - deg_{\mathcal{B}}(S)$. □

Lemma 4.2.2 : *If* $n \geq k - t + 1 + p$ *and* $n \geq 2k - t + 1$ *and* $(X, \mathcal{B})$ *is an* (n, k, p, t) *design, then there cannot exist an* $S \subseteq X$ *where* $|S| = k - t + 1$ *such that* $S \subseteq T$ *for all* $T \in \mathcal{B}$.

Proof: Suppose there is some (n, k, p, t) design $(X, \mathcal{B})$ and some $S \subseteq X$ where $|S| = k - t + 1$ and $S \subseteq T$ for every $T \subseteq \mathcal{B}$. Then consider a p-set Q which has empty intersection with S. Such p-sets exist, since $n \geq k - t + 1 + p$. As $(X, \mathcal{B})$ is a Lotto design, there exists $T \in \mathcal{B}$ such that $|Q \cap T| \geq t$. However, as $Q \cap S = \emptyset$, $S \subseteq T$ and $|S| = k - t + 1$, $|T \cap Q| \leq t - 1$. This is a contradiction. Hence our lemma holds. □

This lemma implies that for a given (n, k, p, t) Lotto design $(X, \mathcal{B})$, any S with $|S| = k - t + 1$, $n \geq 2k - t + 1$ and $n \geq k - t + 1 + p$, will generate an $(n - k + t - 1, k, p, t)$ design. Notice that is not true in general as stated by the formulation of Theorem 4.2.1. More specifically, Theorem 4.2.1 states that $\mathcal{B}^S$ must be non-empty. Lemma 4.2.2 states that this is always true under certain conditions.

Corollary 4.2.1 : *If an (n,k,p,t) Lotto design with x blocks exist with $n \geq p+k-t+1$ and $n \geq 2k-t+1$ then an $(n-k+t-1,k,p,t)$ Lotto design with y blocks exists where $y \leq x$*

Proof : Consider a set $S \subseteq X$ where $|S| = k-t+1$. By Lemma 4.2.2, $\mathcal{B}^S \neq \emptyset$. Then, $|S| \leq n-p$ and $|S| \leq n-k$ and hence, Theorem 4.2.1 holds which immediately give the desired result. □

Example 4.2.1 :Consider the following $(14,5,4,3)$ Lotto Design $(X,\mathcal{B})$ where $X = \{0,1,...,13\}$ which we obtained by simulated annealing :

$\{0, 1, 2, 5, 7\}$
$\{1, 3, 4, 6, 7\}$
$\{2, 3, 5, 8, 10\}$
$\{0, 3, 6, 9, 10\}$
$\{2, 4, 6, 9, 11\}$
$\{1, 2, 4, 10, 11\}$
$\{5, 6, 7, 10, 11\}$
$\{0, 1, 6, 8, 12\}$
$\{4, 5, 6, 8, 12\}$
$\{0, 4, 8, 9, 12\}$
$\{5, 7, 9, 10, 12\}$
$\{1, 2, 3, 11, 12\}$
$\{0, 7, 8, 11, 12\}$
$\{0, 3, 4, 5, 13\}$
$\{2, 7, 8, 9, 13\}$
$\{1, 5, 9, 11, 13\}$
$\{3, 8, 9, 11, 13\}$
$\{1, 8, 10, 11, 13\}$
$\{0, 2, 6, 12, 13\}$
$\{4, 7, 10, 12, 13\}$

Let $S = \{6,8,11,12\}$. By Theorem 4.2.1, the following collection of 5-sets form an $(10,5,4,3)$ Lotto design with 17 blocks whose elements are from $\{0,1,2,3,4,5,7,9,10,13\}$.

{0,	1,	2,	5,	7}
{1,	3,	4,	5,	7}
{2,	3,	5,	9,	10}
{0,	3,	5,	9,	10}
{2,	4,	5,	9,	13}
{1,	2,	4,	10,	13}
{4,	5,	7,	10,	13}
{0,	4,	5,	9,	10}
{5,	7,	9,	10,	13}
{1,	2,	3,	4,	5}
{0,	3,	4,	5,	13}
{2,	5,	7,	9,	13}
{1,	5,	9,	10,	13}
{3,	4,	9,	10,	13}
{1,	2,	9,	10,	13}
{0,	2,	3,	7,	13}
{4,	7,	9,	10,	13}

We now state our main Theorem which is a generalization of the Schönheim's lower bound for Covering designs.

Theorem 4.2.2 : *If $(X, \mathcal{B})$ is an (n, k, p, t) Lotto design and $S \subseteq X$ such that $|S| = k - t + 1$, $n \geq 2k - t + 1$ and $n \geq k - t + 1 + p$ then*

$$|\mathcal{B}| \geq \left\lceil \frac{\binom{n}{k-t+1} L(n-k+t-1, k, p, t)}{\binom{n}{k-t+1} - \binom{k}{k-t+1}} \right\rceil$$

Proof : By Corollary 4.2.1 $|\mathcal{B}^S| \geq L(n - k + t - 1, k, p, t)$ and by Lemma 4.2.1 $|\mathcal{B}| - deg_{\mathcal{B}}(S) \geq L(n - k + t - 1, k, p, t)$. By Lemma 4.2.2, summing over all $S \subseteq X$ such that $|S| = k - t + 1$ yields :

$$\sum |\mathcal{B}| - \sum deg_{\mathcal{B}}(S) \geq \sum L(n - k + t - 1, k, p, t).$$

By simple counting, we get the following :

$$\binom{n}{k-t+1}|\mathcal{B}| - \binom{k}{k-t+1}|\mathcal{B}| \geq \binom{n}{k-t+1} L(n-k+t-1, k, p, t).$$

By re-arranging the terms, we get the desired result. □

The following result follows immediately from Theorem 4.2.2.

Corollary 4.2.2 :

$$L(n,k,p,t) \geq \left\lceil \frac{\binom{n}{k-t+1} L(n-k+t-1,k,p,t)}{\binom{n}{k-t+1} - \binom{k}{k-t+1}} \right\rceil.$$

for $n \geq 2k - t + 1$ *and* $n \geq k - t + 1 + p$.

Corollary 4.2.3 : *If* $n \geq 2k - t + 1$ *and* $n \geq k - t + 1 + p$, *then* $L(n, k, p, t) > L(n - k + t - 1, k, p, t)$

Corollary 4.2.4 : *If* $n \geq 2k - t + 1$ *and* $n \geq k - t + 1 + p$, *then* $L(n, k, p, t) > L(m, k, p, t)$ *for all* $m \leq n - k + t - 1$.

Proof : Use Corollary 4.2.3 along with the fact that $L(n, p, k, t) \geq L(n-1, k, p, t)$. □

We can now derive the Schönheim bound from Theorem 4.2.2.

Corollary 4.2.5 : $L(n', k', t', t') \geq \lceil \frac{n'}{k'} L(n' - 1, k' - 1, t' - 1, t' - 1) \rceil$.

Proof : Put $n = n'$, $k = n' - k'$, $t = n' - k'$ and $p = n' - t'$ in Theorem 4.2.2. We must show that $n \geq 2k - t + 1$ and $n \geq k - t + 1 + p$. We see that $2k - t + 1 = 2(n' - k') - (n' - k') + 1 = n' - k' + 1 \leq n'$, as required. We also have $k - t + 1 + p = (n' - k') - (n' - k') + 1 + p = p + 1$. If $n = p$, then $n' = n' - t'$ and hence $t' = 0$ which is impossible. Thus $n \geq p + 1$ and both conditions of Theorem 4.2.2 are satisfied. Applying Theorem 4.2.2, we have

$$L(n,t,p,t) \geq \left\lceil \frac{n * L(n-1,t,p,t)}{n-t} \right\rceil .$$

Since Turán designs and Covering designs are complements,

$$L(n,n-t,n-p,n-p) \geq \left\lceil \frac{n * L(n-1,n-1-t,n-1-p,n-1-p)}{n-t} \right\rceil .$$

Hence,

$$L(n,k',t',t') \geq \left\lceil \frac{n}{k'} L(n-1,k'-1,t'-1,t'-1) \right\rceil ,$$

which is the Schönheim lower bound for Covering designs. □

4.3 Other Lower Bound Formulas

In this section we state other lower bound formulas that we have discovered. Only formulas that give infinitely many lower bounds will be stated here. Lower bounds determined on an individual basis are discussed in the next chapter.

Theorem 4.3.1 : $L(n,k,t{+}1,t) \geq min\{L(n,k,t{-}1,t{-}1), L(n{-}t{+}1,k{-}t{+}2,2,2)\}$.

Proof : Consider a minimal $(n,k,t+1,t)$ Lotto design $\mathcal{B}$. If every $(t-1)$-set appears in some block of an $\mathcal{B}$, then $\mathcal{B}$ is an $(n,k,t-1,t-1)$ Lotto design and hence, $L(n,k,t{+}1,t) \geq L(n,k,t{-}1,t{-}1)$. Otherwise, suppose the $(t{-}1)$-set $\{1,2,...,t{-}1\}$

does not appear in any block of B; then consider the $(t+1)$-set $\{1, 2, ..., t-1, x, y\}$. It can only be represented by a k-set containing $\{x, y\}$ and exactly $t-2$ elements from $\{1, 2, ..., t-1\}$. Thus all 2-sets chosen from $\{t, t+1, t+2, ..., n\}$ must appear in $\mathcal{B}$. Thus, $L(n, k, t+1, t) \geq L(n-t+1, k-t+2, 2, 2)$. Combining both cases gives us the formula $L(n, k, t+1, t) \geq min\{L(n, k, t-1, t-1), L(n-t+1, k-t+2, 2, 2)\}$, as required. □

Theorem 4.3.2 : $L(n, k, t+2, t) \geq min\{L(n, k, t, t-1), L(n-t, k-t+2, 2, 2)\}$.

Proof: Consider a minimal $(n, k, t+2, t)$ Lotto design $\mathcal{B}$. If every t-set intersects some block of $\mathcal{B}$ in at least $t-1$ elements, then $\mathcal{B}$ is an $(n, k, t, t-1)$ Lotto design and hence, $L(n, k, t+2, t) \geq L(n, k, t, t-1)$. Otherwise, suppose that no $(t-1)$ elements of the t-set $\{1, 2, ..., t-1, t\}$ appear together in $\mathcal{B}$. Consider the $(t+2)$-set $\{1, 2, ..., t-1, t, x, y\}$. It can only be represented by a k-set that must include $\{x, y\}$ and exactly $t-2$ elements from $\{1, 2, ..., t\}$, since at most $t-2$ elements from $\{1, 2, 3, ..., t\}$ may appear together in a block of the design. Hence, every pair $\{x, y\}$ where $x, y \notin \{1, 2, ..., t\}$ must occur at least once in the $\mathcal{B}$. This implies $L(n, k, t+2, t) \geq L(n-t, k-t+2, 2, 2)$. Combining both cases gives us the formula $L(n, k, t+2, t) \geq min\{L(n, k, t, t-1), L(n-t, k-t+2, 2, 2)\}$, as required. □

Theorem 4.3.3 : $L(n, k, 6, 3) \geq min\{L(n, k, 4, 2), L(n-4, k-1, 2, 2)\}$.

Proof : Consider a minimal $(n, k, 6, 3)$ Lotto design $\mathcal{B}$. Either each 4-set intersects some block of $\mathcal{B}$ in at least two elements or some 4-set intersects each block of the design in at most one element. If every 4-set intersects some block of the design in at least 2 elements, then $L(n, k, 6, 3) \geq L(n, k, 4, 2)$. If a 4-set, say $\{1, 2, 3, 4\}$, does not intersect any block of $\mathcal{B}$ in more than one element, then the 6-set $\{1, 2, 3, 4, x, y\}$ where $x, y \in X(n) \setminus \{1, 2, 3, 4\}$ must be represented by a block that contains both x

and y. This implies $L(n, k, 6, 3) \geq L(n-4, k-1, 2, 2)$. Combining both cases, we get the formula : $L(n, k, 6, 3) \geq min\{L(n, k, 4, 2), L(n-4, k-1, 2, 2)\}$. □

The following result is a generalization of a result from Bate [1].

Theorem 4.3.4 : *If $k_1 > k_2$ then*

$$L(n, k_1, p, t) \geq \frac{L(n, k_2, p, t)}{L(k_1, k_2, t, t)}.$$

Proof: Consider a (n, k_1, p, t) Lotto design $\mathcal{B}$ containing b blocks. For each block, construct a (k_1, k_2, t, t) Lotto design. Let $\mathcal{C}$ denote the set of all k_2-sets in the b (k_1, k_2, t, t) Lotto designs. We claim that $\mathcal{C}$ is an (n, k_2, p, t) Lotto design. To show this, consider an arbitrary p-set P. P must intersect some block of $\mathcal{B}$ in t elements as $\mathcal{B}$ is an (n, k_1, p, t) Lotto design. These t elements must be a block in $\mathcal{C}$ by definition. Hence, $\mathcal{C}$ is an (n, k_2, p, t) Lotto design. So $L(n, k_2, p, t) \leq L(n, k_1, p, t) * L(k_1, k_2, t, t)$, or $L(n, k_1, p, t) \geq \frac{L(n, k_2, p, t)}{L(k_1, k_2, t, t)}$. □

The following two results give the value of $L(n, k, p, t)$ for infinite sets of parameters.

Theorem 4.3.5 : *If $n - k \leq p - t$ then $L(n, k, p, t) = 1$.*

Proof : Let B be an arbitrary k-set. There are exactly $n - k$ elements not in B. If P is a p-set, since $n - k \leq p - t$, then $n - k + t \leq p$ which implies at least t elements from p must be from B. Hence the single set B form an (n, k, p, t) Lotto design if $n - k \leq p - t$. □

Theorem 4.3.6 : *If $n - k \geq p - t + 1$ and $\left\lceil \frac{r}{r+1} n \right\rceil \leq k$ where $r = \left\lfloor \frac{p}{p-t+1} \right\rfloor$, then $L(n, k, p, t) = r + 1$.*

Proof : Suppose $n - k \geq p - t + 1$ and $\left\lceil \frac{r}{r+1} n \right\rceil \leq k$ where $r = \left\lfloor \frac{p}{p-t+1} \right\rfloor$. We first need to show that an (n, k, p, t) Lotto design cannot have fewer than $r + 1$ blocks. Suppose there is such a design with r blocks. As $n - k \geq p - t + 1$, pick $p - t + 1$ elements from the complement of each block and denote this set as P. There are at most $r*(p-t+1)$ distinct elements chosen. Since $r = \left\lfloor \frac{p}{p-t+1} \right\rfloor$, then $r*(p-t+1) \leq p$. If $|P| < p$, add any other distinct elements to P until $|P| = p$. The set P intersects the complement of each block in at least $p - t + 1$ elements and hence will intersect each block of the design in at most $t - 1$ elements. This contradicts the assumption that there is an (n, k, p, t) Lotto design with r blocks. Hence we conclude that $L(n, k, p, t) \geq r + 1$. Now we need to show that $L(n, k, p, t) \leq r + 1$. We can do this by constructing such a design. Since $\left\lceil \frac{r}{r+1} n \right\rceil \leq k$, then $(r + 1)(n - k) \leq n$. So pick $r + 1$ $(n - k)$-sets by filling them with distinct elements. This implies that no element in these blocks is repeated. Since $r = \left\lfloor \frac{p}{p-t+1} \right\rfloor$ then $r + 1 > \frac{p}{p-t+1}$ which implies $p < (r + 1)(p - t + 1) \leq (r + 1)(n - k)$. This means that least one of the $(n - k)$-sets intersects any p-set in at most $p - t$ elements. This means that the p-set will intersect the complement of the $(n - k)$-set in at least t elements. The complements of the $(n - k)$-sets are blocks of the (n, k, p, t) Lotto design that we desire. Hence we have shown that $L(n, k, p, t) \leq r + 1$. Since we have inequalities in both directions, we conclude that $L(n, k, p, t) = r + 1$. □

We conclude this chapter with some lower bound formulas for several infinite classes of Lotto designs. We can use these formulas to update our tables via a computer program. These results along with the monotonicity formulas of the previous chapter may be combined to yield lower bounds for other Lotto designs.

Theorem 4.3.7 : *If $k \geq 3$, then $L(2k+1, k, 5, 3) \geq 5$.*

Proof : For $k = 3$, the design in question is a $(7, 3, 5, 3)$ Turan design. By complementing the design, we get a $(7, 4, 2, 2)$ design. From Bate's tables, $L(7, 4, 2, 2) = 5$, hence the result is true for $k = 3$. We will now prove the result for $k \geq 4$. Suppose there is a $(2k+1, k, 5, 3)$ Lotto design with four blocks. If a given 5-set is not represented in this design, then the 5-set will intersect every block of the complement in at least three elements. We will show that we can always construct such a 5-set that intersects every block of the complement in at least three elements. Let B_1, B_2, B_3, B_4 denote the four $(k+1)$-sets in the complement of the design. Suppose that there is an element of frequency zero in the design, say the element 1. Then that element appears in each block of the complement. From this point on, we will be speaking with respect to the complement blocks of the design unless otherwise stated. There are $4k$ spots left to be filled using $2k$ elements. The average frequency is 2.

Case 1 : If there is another element, say 2, of frequency 4, then there are $4k - 4$ spots left to be filled by $2k - 1$ elements. The average frequency of the remaining elements is $1 + \frac{2k-3}{2k-1}$. Since $k \geq 4$, then there exists an element of frequency 2 or higher. For if not, then every element has frequency at most 1 and the total number of spots that can be filled is $8 + 1(2k-1) = 2k + 7 < 4k + 4$, as $k \geq 4$. So suppose that the element 3 has frequency at least 2. Without loss of generality, assume the element 3 belongs to B_1 and B_2. The 5-set $\{1, 2, 3, x, y\}$ where $x \in B_3$ and $y \in B_4$ is a 5-set that intersects each block in at least three elements which implies that this 5-set is not represented by the Lotto design. This is a contradiction and hence it is impossible to have two elements of frequency 4 in the complement.

Case 2 : If there is an element, say 2, of frequency 3, then there are $4k - 3$ spots left to be filled by $2k - 1$ elements. The average frequency is $1 + \frac{2k-2}{2k-1}$. If there is another element of frequency 3, say 3, then the blocks of the complement would look like: $B_1 = \{1, 2, , ...\}, B_2 = \{1, 2, 3, ...\}, B_3 = \{1, 2, 3, ...\}$ and $B_4 = \{1, 3, ...\}$, or else $B_1 = \{1, 2, 3, ...\}$, $B_2 = \{1, 2, 3, ...\}$, $B_3 = \{1, 2, 3, ...\}$ and $B_4 = \{1, ...\}$. In the former case, the 5-set $\{1, 2, 3, x, y\}$ where $x \in B_1$ and $y \in B_4$ intersects each block of the complement in at least 3 elements, which is a contradiction. Similarly if $B_1 = \{1, 2, 3, ...\}$, $B_2 = \{1, 2, 3, ...\}$, $B_3 = \{1, 2, 3, ...\}$ and $B_4 = \{1, ...\}$, then the 5-set $\{1, 2, 3, x, y\}$ where $x \in B_4$ and $y \in B_4$ intersects each block of the complement in at least 3 elements, which is a contradiction. Hence there is only 1 element of frequency 3 in the complements, which in turn implies there is exactly one element of frequency 1 in the original blocks. Let the element 2 occur in B_1, B_2 and B_3. Thus, all other elements must have frequency 2, that is, there are $2k - 2$ elements of frequency 2 in the complements of the blocks. There must be an element of frequency 2 in block B_4, say 3. Since it has frequency 2, it must also appear in one of B_1, B_2 or B_3. Without loss of generality, assume that element 3 occurs in B_3 and B_4. Since B_3 has room for only $k-2$ additional elements, there is an element, say 4, that does not appear in B_3 but does appear in B_4. Without loss of generality, assume that $4 \in B_1$ and $4 \in B_4$. Now the 5-set $\{1, 2, 3, 4, x\}$ where $x \in B_1$ intersects every block of the complement in at least 3 elements. This is a contradiction and hence implies that there are no elements of frequency three in the complement, if there is an element of frequency four in the complement.

Case 3 : In the remaining case, all elements, other than element 1, have frequency 2. Suppose element 2 belongs to both B_1 and B_2. There are $4k - 2$ spots left to be filled by $2k - 1$ elements. Since there are $2k$ spots remaining in blocks B_3 and B_4 together and only $2k - 1$ elements remain, B_3 and B_4 shared a common element, say 3. Now, since B_1 and B_2 must be distinct blocks, suppose the element 4 is in B_1 and not in B_2. Since the element 4 has frequency 2, then suppose it also is in B_3. Finally, there are $2k - 2$ spots left to be filled between B_2 and B_4 using $2k - 3$ elements. Hence, there exists a common element between these two blocks, say element 5. The 5-set $\{1, 2, 3, 4, 5\}$ intersects every block in the complement in at least three elements, which is a contradiction.

If there is an element of frequency zero in the Lotto design, every scenario yields a contradiction. Hence we conclude that there must not be any elements of frequency zero in the Lotto design. Now there must be at least one element of frequency 1 in the Lotto design. For if not, then every element in the Lotto design has frequency at least two, and hence the total in the design is at least $2(2k+1) = 4k+2 > 4k$, which is a contradiction. Thus, we can assume that element 1 has frequency 3 in the complement. Using the same reasoning again, there is another element of frequency 3, say the element 2. There are two cases for the arrangements of these two elements.
Case a : $B_1 = \{1, 2, ...\}$, $B_2 = \{1, 2, ...\}$, $B_3 = \{1, 2, ...\}$ and $B_4 = \{3, 4, 5, ..., k+3\}$. Then there are $k-2$ elements remaining to be used. These $k-2$ elements cannot fill up the $k-1$ spots in any one of the blocks B_1, B_2 or B_3. Hence, there exist three elements, say 3, 4 and 5 such that x is in B_1, y is in B_2 and z is in B_3 where $x, y, z \in \{3, 4, 5\}$. Now the 5-set $\{1, 2, 3, 4, 5\}$ intersects every block of the complement in at least three elements, which is a contradiction.
Case b : $B_1 = \{1, 2, ...\}$, $B_2 = \{1, 2, ...\}$, $B_3 = \{1, ...\}$ and $B_4 = \{2, ...\}$. There are $2k$ spots remaining to be filled between blocks B_3 and B_4 using $2k-1$ elements. So there exists some common element between these two blocks, say the element 3. If there is an element of frequency 3, say x, that belongs to every block except say B_1, then the 5-set $\{1, 2, 3, x, y\}$ where $y \in B_1$ intersects every block of the complement in at least three elements. On the other hand, if no element of frequency three exists, then every remaining element must be of frequency two. Since B_1 and B_2 are distinct, let the element 4 be in B_1 and not in B_2. Then since the element 4 has frequency 2, then it must appear in either B_3 or B_4. Without loss of generality, assume it is in B_3. Now B_2 and B_4 have $2k-2$ spots remaining to be filled between them using $2k-3$ elements. This implies B_2 and B_4 must have a common element, say the element 5. Thus the 5-set $\{1, 2, 3, 4, 5\}$ intersects every block of the complement in at least 3 elements, which is a contradiction.

Since every scenario leads to a contradiction, the assumption that there is a $(2k+1, k, 5, 3)$ Lotto design on 4 blocks must be false, and hence $L(2k+1, k, 5, 3) \geq 5$. $\square$

Theorem 4.3.8 : *If* $k \geq 3$, *then* $(2k+2, k, 6, 3) \geq 4$.

Proof: Assume that $L(2k+2, k, 6, 3) \leq 3$ and consider a $(2k+2, k, 6, 3)$ Lotto design with three blocks. We shall construct a 6-set that is not represented by this design and hence such a design cannot exist.

There are $3k$ total occurrences in the design. We consider the cases for the number of elements of frequency 0.

Case 1 : Suppose $f_0 = 0$. Then, since each element can have frequency at most 3, the number of elements of frequency two or more is at most $k-2$. This is because if there are $k-1$ or more elements of frequency at least two, then there are at most $k+3$ elements of frequency one, which give a total occurrences of at least $(k-1)2 + (k+3) = 2k - 2 + k + 3 = 3k + 1 > 3k$. This implies that there are at least two elements of frequency one in each block. Pick two such elements from each block to form a 6-set that will not be represented.

Case 2 : Suppose $f_0 = 1$. Pick that element x, as part of our 6-set. There are at most $k-1$ elements of frequency two or more since if there were k elements of frequency two or more, then the total occurance of the the elements is at least $2(k)+(k+1) = 3k+1 > 3k$, which is a contradiction. Hence, there exists an element of frequency one in each of the three blocks. Pick three such elements a, b, c to be in our 6-set. It should be noted that there are at least $k+2$ elements of frequency one in the design. Consider the following subcases :

Case 2a : Suppose there exists elements e and f of frequency one in two different blocks such that they are not a, b or c. Then the 6-set $\{a, b, c, e, f, x\}$ is not represented by any block of the design.

Case 2b : Suppose no such frequency one elements e and f exists as stated in case 2a. Then, since there are at least $k+2$ elements of frequency one, there must be exactly $k+2$ elements of frequency one. The $k-1$ elements of frequency one which is not a, b or c must appear with one of these three elements. Without loss of generality, assume that element is a. Then, the block containing element a is comprised entirely of frequency 1 elements. Let d be an element in this block that is not the element a. Now pick some frequency two element e in the design. The 6-set $\{a, b, c, d, e, x\}$ is not represented by any block of the design.

Case 3: Suppose $f_0 = 2$. Pick these two elements x, y as part of our 6-set. It is easy to see that there are at least k elements of frequency one. We consider the following cases based upon where the elements of frequency one lie, in the design.

Case 3a : Suppose all the elements of frequency one appear in one block B_1 in the design. It is easy to see then that the number of elements of frequency one is exactly k, since $1(k) + 2(k) = 3k$. Pick two elements a, b from B_1. Pick two elements e, f from the remaining blocks that do not appear in B_1. The 6-set $\{a, b, e, f, x, y\}$ is not represented by any block of the design.

Case 3b : Suppose the frequency one elements occur in only two blocks B_1 and B_2. Pick $a \in B_1$ and $b \in B_2$ where a, b have frequency one. Note that since $k \geq 3$ and there are at least k elements of frequency one, one of the two blocks B_1, B_2 must have at least two elements of frequency one. If each of these blocks contain two or more elements of frequency two, then picking another element of frequency one from each of B_1 and B_2, say c and d respectively. Then the 6-set $\{a, b, c, d, x, y\}$ is not represented by any block in the design. Otherwise, only one of the two blocks, say B_1 has at least two elements of frequency one and B_2 has exactly one element of frequency one. Let $c \neq a$ be an element of frequency one in B_1. Since $B_1 \neq B_2$, there exists an element $d \in B_2 \setminus B_1$ such that $d \neq a, b, c$. The 6-set $\{a, b, c, d, x, y\}$ is not represented by any block in the design.

Case 3c : Suppose each block contains an element of frequency one. Pick one element of frequency one from each block, say a, b and c. Now pick any other element e that appears in the design. The 6-set $\{a, b, c, e, x, y\}$ is not represented by any block of the design.

The cases where $3 \leq f_0 \leq 6$ trivially lead to contradictions. Hence $L(2k+2, k, 6, 3) \geq 4$.□

Theorem 4.3.9 : *$L(3k+1, k, 7, 3) \geq 6$ for $k \geq 3$.*

Proof : Suppose there is a $(3k+1, k, 7, 3)$ Lotto design with five blocks. Let us denote these blocks as B_1, B_2, B_3, B_4 and B_5. By Theorem 2.2.9, we may assume every element appears in this design. Looking at other frequencies, we see that at least $k+2$ elements have frequency one and at most $2k-1$ elements have frequency two or more. We will proceed to show that these five blocks cannot form a $(3k+1, k, 7, 3)$ Lotto design. We will do this by considering the number of blocks which contain elements of frequency one. We begin by considering the case where the elements of frequency one appear in exactly two blocks.

Case 1 : The elements of frequency one appear in only two blocks, say B_1 and B_2. Let a and b be two elements of frequency one from B_1 and let c and d be two elements of frequency one from B_2. There are $3k+1$ elements in the Lotto design and at most $2k$ of them are in B_1 and B_2. Thus, at least $k+1$ elements must occur only in B_3, B_4 or B_5. It is easy to see that in these $k+1$ elements, there are at least three that have frequency two. Let x, y, z be three such elements. These three elements can occur in B_3, B_4 and B_5 in three different ways.
Case 1a : $B_3 = \{x, y, ...\}$, $B_4 = \{x, z, ...\}$ and $B_5 = \{y, z, ...\}$. Then the 7-set $\{a, b, c, d, x, y, z\}$ is not represented by the design.
Case 1b : $B_3 = \{x, y, z, ...\}$, $B_4 = \{x, y, ...\}$ and $B_5 = \{z, ...\}$. Then there are at least $k+1-3 = k-2$ other elements that occur only in B_3, B_4 or B_5. If they were all frequency three elements, then they would occur in B_3, B_4 and B_5. But then B_3

would contain $k-2+3 = k+1$ elements, which is a contradiction. Since B_3 contains $\{x, y, z\}$, it can contain another $k-3$ elements. But there are at least $k-2$ elements that appears only in B_3, B_4 and B_5 (besides x, y and z). Hence one of the $k-2$ elements, say element w, has frequency two and appears in B_4 and B_5. The 7-set $\{a, b, c, d, y, z, w\}$ is not represented in the design.
Case 1c : $B_3 = \{x, y, z, ...\}$ and $B_4 = \{x, y, z, ...\}$. Then there are at least $k+1-3 = k-2$ other elements that occur only in B_3, B_4 or B_5. Since $B_3 \neq B_4$, one of these $k-2$ elements, say element w, must have frequency two and appear in B_5. It also appears in either B_3 or B_4. In either case, it takes us back to case 1b.

We now consider the cases where the elements of frequency one appear in exactly three blocks.

Case 2 : Suppose B_2 contains exactly one element of frequency one, B_3 contains exactly one element of frequency one and B_1 contains the rest of the elements of frequency one (that is, B_4 and B_5 have no elements of frequency one). Since the number of elements of frequency one is at least $k+2$ and they must appear in B_1, B_2 or B_3, B_1 must be made up entirely of frequency one elements. Let a and b be two elements of frequency one from B_1, and let c and d be the elements of frequency one from B_2 and B_3, respectively. Since there are $2k-1$ elements remaining to be considered and only $2k-2$ spots open between B_2 and B_3, there exists an element x such that $x \in B_4$ and $x \in B_5$ and x is not in any other block. If B_2 and B_3 are disjoint, then we must be able to find two elements $y \in B_2$ and $z \in B_3$ such that they do not appear together in B_4 and B_5. Then, the 7-set $\{a, b, c, d, x, y, z\}$ is not represented, which is a contradiction. On the other hand, if there is an element e such that $e \in B_2 \cap B_3$, then there there exists another element $y \in B_4 \cap B_5$ such that y is not in any other block. Then the 7-set $\{a, b, c, d, e, x, y\}$ is not represented by any block in the design, which is a contradiction.

Case 3 : Suppose the elements of frequency one appear at least twice in each of B_1 and B_2, exactly once in B_3 and do not appear in B_4 or B_5. Let a and b be two elements of frequency one from B_1, c and d be two elements of frequency one from B_2 and e the element of frequency one from B_3. Then there is at least one element x in both B_4 and B_5 such that it is not in any other block because the first three blocks contain at most $3k$ distinct elements. Also there must be an element y in B_3, B_4 or B_5 such that it is not in B_1 or B_2 since the first two blocks contain at most $2k$ distinct elements. The 7-set $\{a, b, c, d, e, x, y\}$ is not represented by any block of the design. This is a contradiction.

Case 4 : Suppose the frequency one elements appear at most $3k$ times and are contained within the blocks B_1, B_2 and B_3. Let a, b be frequency one elements in B_1, c, d be frequency one elements in B_2 and e, f be frequency one elements in B_3. There exists an element x not in B_1, B_2 or B_3 as there are $3k + 1$ elements in the design and the first three blocks may contain at most $3k$ distinct elements. Since x has frequency greater than one but does not appear in B_1, B_2 or B_3, then x must appear in B_4 and B_5. The 7-set $\{a, b, c, d, e, f, x\}$ is not represented by any block in the design. This is a contradiction.

We now deal with the case where the elements of frequency one appear in exactly four blocks.

Case 5 : Suppose B_1 has at least $k - 1$ elements of frequency one and B_2, B_3, B_4 each have exactly one element of frequency one. Then all other elements have frequency two or more except for possibly one additional element of frequency one in B_1. Let a, b be elements of frequency one chosen from B_1 and c, d, e be the elements of frequency one from B_2, B_3 and B_4, respectively. If B_2 and B_3 pairwise disjoint, there are at least $(k-1)(k-2)$ pairs $\{x, y\}$ where $x \in B_2$, $y \in B_3$, $x \notin B_1$, $y \notin B_1$ and both x and y have frequency two or more. The block B_4 may contain at most $C(k-1, 2)$ of these pairs. As $C(k-1, 2) < (k-1)(k-2)$, there exists a pair $\{x', y'\}$ where $x' \in B_2, y' \in B_3, x' \notin B_1$ and $y' \notin B_1$ such that $\{x', y'\} \not\subseteq B_4$. Since B_2 and B_3 are pairwise disjoint, the 7-set $\{a, b, c, d, e, x', y'\}$ is not represented by any block in the design which is a contradiction. On the other hand, if there is an element common between B_2 and B_3, say f, then we see that, not counting the elements, c, d and f, there may be at most $2k-4$ distinct elements of frequency two or more in B_2 and B_3. Thus B_1, B_2 and B_3 may contain at most $3k-1$ distinct elements. This means that some element, say g, occurs in both B_4 and B_5 only. Then the 7-set $\{a, b, c, d, e, f, g\}$ is not represented by any block in the design.

Case 6 : Suppose that B_1 and B_2 each contain at least two elements of frequency one, and B_3 and B_4 each contain exactly one element of frequency one. Let a, b be elements of frequency one from B_1, c, d be elements of frequency one from B_2, e be the element of frequency one from B_3 and f be the element of frequency one from B_4. Since B_1 and B_2 together may only contain $2k$ distinct elements, there is an element x different from e or from f that does not appear in B_1 or B_2. The 7-set $\{a, b, c, d, e, f, x\}$ is not represented by the blocks of the design.

Case 7 : Suppose B_1, B_2, B_3 and B_4 each contain at least one element of frequency one and at least three of these blocks contain two or more elements of frequency one. It is trivial that a 7-set may be formed that is not represented by the design.

We now deal with the cases where the frequency one elements appear in every block.

Case 8 : If exactly one element frequency one appears in each block, then $k = 3$, $f_1 = 5$, $f_2 = 5$ and $f_3^+ = 0$. Let a, b, c, d and e denote the elements of frequency one from each block. Assume B_1 and B_2 contain an element x which has frequency two. Since at most five distinct elements appear in B_1 and B_2, there exists an element y not in B_1 or B_2 with frequency two. The 7-set $\{a, b, c, d, e, x, y\}$ is not represented by the blocks of the design.

Case 9 : Consider the case where B_1 contains at least two elements of frequency one and B_2, B_3, B_4 and B_5 each contain at least one element of frequency one. Let a, b be elements of frequency one chosen from B_1 and let c, d, e, f be elements of frequency one chosen from B_2, B_3, B_4 and B_5, respectively. Clearly, there exists an element x that does not appear in B_1 and is different from a, b, c, d, e and f. The 7-set $\{a, b, c, d, e, f, x\}$ is a 7-set not represented by any block in the design.

We have considered every possible case and each case led to a contradiction. Hence our assumption that a $(3k+1, k, 7, 3)$ Lotto design with five blocks exists cannot be correct, and thus $L(3k+1, k, 7, 3) \geq 6$. $\square$

The above result improves our table entries for $L(13, 4, 7, 3)$, $L(16, 5, 7, 3)$ and $L(19, 6, 7, 3)$, now all have the value 6. This result also allows the monotonicity results to improve the tables further.

Theorem 4.3.10 : *$L(k+2, k, 4, 3) \geq 3$, for $k \geq 3$*

Proof : Suppose there is a $(k+2, k, 4, 3)$ Lotto design with 2 blocks. Since $\frac{k+2}{k} < 2$, for $k >\geq 3$, by Theorem 2.2.9 , we may assume that there is a $(k+2, k, 4, 3)$ Lotto design which contains no element of frequency zero. Consider such a design. We may conclude that there are 4 elements of frequency one and $k-2$ elements of frequency

2. Since there are two blocks, the $k-2$ elements of frequency two must appear once in each block. Each block must contain exactly two elements of frequency one. The elements of frequency one form a 4-set that is not represented. This contradicts the assumption that there is a $(k+2, k, 4, 3)$ Lotto design with 2 blocks. Thus we conclude that no design of size two exists, so $L(k+2, k, 4, 3) \geq 3$. $\square$

Theorem 4.3.11 : *$L(3k+2, k, 8, 3) \geq 5$, for $k \geq 3$.*

Proof : Suppose $\mathcal{B}$ is a $(3k+2, k, 8, 3)$ Lotto design on four blocks. Since $\frac{3k+2}{k} \leq 4$ for $k \geq 2$, by Theorem 2.2.9, we may assume that this design has no elements of frequency zero. By analyzing the frequencies of the design, there are at least $2k+4$ elements of frequency one and at most $k-2$ elements of frequency two or higher. Each block must have at least two elements of frequency one. We can pick eight elements of frequency one, two from each block to form an 8-set that is not represented by the blocks of the design. Thus, there cannot exist a $(3k+2, k, 8, 3)$ Lotto design on 4 blocks. $\square$

Theorem 4.3.12 : *$L(3k+3, k, 9, 3) \geq 4$, for $k \geq 3$.*

Proof : Suppose $\mathcal{B}$ is a $(3k+3, k, 9, 3)$ Lotto design with three blocks. By Theorem 2.2.8, we may assume that the elements in the design of frequency 0 or 1. There are exactly $3k$ elements in the design and hence three elements do not appear in the design. A 9-set consisting of these three elements plus two elements from each of the three blocks is not represented by the design. Hence $L(3k+3, k, 9, 3) \geq 4$. $\square$

Theorem 4.3.13 : *If $k \geq 4$, $L(2k+3, k, 9, 4) \geq 4$.*

Proof : Suppose that $L(2k+3,k,9,4) = 3$; then there is a $(2k+3,k,9,4)$ Lotto design $\mathcal{B}$ with three blocks B_1, B_2 and B_3. Since $\frac{2k+3}{k} \leq 3$, Theorem 2.2.9 implies that $\mathcal{B}$ has no elements of frequency zero. By analyzing the frequencies of elements in $\mathcal{B}$ we see that the number of elements of frequency one is at least $k+6$ and the number of elements of frequency two or higher is at most $k-3$. Since there are at most $k-3$ elements of frequency two or higher, each block must contain at least three elements of frequency one. From each block of the design, select three elements of frequency one. The nine selected elements form a 9-set that is not represented by the design. Hence $L(2k+3,k,9,4) \geq 4$. □

Theorem 4.3.14 : *$L(2k+2,k,8,4) \geq 4$, for $k \geq 4$.*

Proof : By Theorem 4.3.13 $L(2k+3,k,9,4) \geq 4$, for $k \geq 4$. Then by Theorem 3.3.5, $L(2k+2,k,8,4) \geq L(2k+3,k,9,4)$ and hence $L(2k+2,k,8,4) \geq 4$, for $k \geq 4$. □

Theorem 4.3.15 : *$L(3k+2,k,11,4) \geq 5$, for $k \geq 4$.*

Proof : Suppose $\mathcal{B}$ is a $(3k+2,k,11,4)$ Lotto design with four blocks B_1, B_2, B_3 and B_4. Since $\frac{3k+2}{k} \leq 4$, by Theorem 2.2.9, we may assume that this design has no elements of frequency zero. By analyzing the frequencies of the design, there are at least $2k+4$ elements of frequency one and at most $k-2$ elements of frequency two or higher. This implies that each block of the design must contain at least two elements of frequency one. If there are at least three blocks with at least three elements of frequency one, then we may select three elements of frequency one from three of these blocks and two elements of frequency one from the remaining block in the design to form an unrepresented 11-set in the design. Otherwise there must two blocks with two elements of frequency one, say blocks B_3 and B_4. Then the blocks B_1 and B_2

must be made up of strictly frequency one elements since $f_0 \geq 2k+4$. Let x be an element that does not appear in B_1 or B_2. Then the 11-set consisting of three elements of frequency one from each of B_1 and B_2, two elements of frequency one from each of B_3 and B_4 and x form an unrepresented 11-set in the design. Thus, the four blocks B_1, B_2, B_3 and B_4 cannot form a $(3k+2, k, 11, 4)$ Lotto design. This implies that $L(3k+2, k, 11, 4) \geq 5$. □

Theorem 4.3.16 : *If* $k \geq 4$, $L(3k+3, k, 12, 4) \geq 5$.

Proof : Suppose $L(3k+3, k, 12, 4) \leq 4$ and consider a $(3k+3, k, 12, 4)$ Lotto design $\mathcal{B}$ with 4 blocks B_1, B_2, B_3 and B_4. As $3k+3 \leq 4k$ for $k \geq 4$, Theorem 2.2.9 implies that we may assume that our design has no elements of frequency zero. By analyzing the frequencies of the elements in the design, we conclude there are at least $2k+6$ elements of frequency 1. We claim that each block in the design contains at least three elements of frequency 1. For if not, suppose B_4 has at most two elements of frequency 1. The remaining $k-2$ elements in B_4 must have frequency 2 or more and hence must appear again in the other three blocks. Since there are at least $2k+6-2 = 2k+4$ elements of frequency 1 in B_1, B_2 and B_3, these three blocks may contain at most $3k-(2k+4) = k-4$ other elements which is less than the $k-2$ elements they must hold. Hence, each block in the design has at least three elements of frequency 1. By selecting three elements of frequency 1 from each of the 4 blocks, we could form a 12-set that is not represented by any block in the design. Hence, $L(3k+3, k, 12, 4) \geq 5$. □

Theorem 4.3.17 : *If* $k \geq 5$, $L(2k+4, k, 12, 5) \geq 4$.

Proof : Suppose $L(2k+4, k, 12, 5) \leq 3$ and consider a $(2k+4, k, 12, 5)$ Lotto design with 3 blocks B_1, B_2 and B_3. As $2k+4 \leq 3k$ for $k \geq 5$, Theorem 2.2.9 implies that we may assume that our design has no elements of frequency zero. By analyzing the frequencies of the elements in the design, we conclude there are at least $k+8$ elements of frequency 1. We claim that each block in the design contains at least four elements of frequency 1. For if not, suppose B_3 has at most three elements of frequency 1. The remaining $k-3$ elements in B_3 must have frequency 2 or more and hence must appear again in the other two blocks. Since there are at least $k+8-3=k+5$ elements of frequency 1 in B_1 and B_2, these two blocks may contain at most $2k-(k+5)=k-5$ other elements which is less than the $k-3$ elements they must hold. Hence, each block in the design has at least four elements of frequency 1. By selecting four elements of frequency 1 from each of the 3 blocks, we form a 12-set that is not represented by any block in the design. Hence, $L(2k+4,k,12,5)\geq 4$. □

Theorem 4.3.18 : *If* $k \geq 5$, $L(2k+3,k,11,5)\geq 4$.

Proof : By Theorem 4.3.17, $L(2k+4,k,12,5)\geq 4$ for $k\geq 5$. Then by Theorem 3.3.5, $L(2k+3,k,11,5)\geq L(2k+4,k,12,5)$ and hence $L(2k+3,k,11,5)\geq 4$ for $k\geq 5$. □

4.4 Conclusion

A generalization of the Schönheim bound was proved for Lotto designs. We have also stated proved bound formulas for infinite classes of parameters. These lower bound formulas may be used to improve our collection of tables. Other results similar to those given by Theorem 4.3.17 may also be obtained.

Chapter 5

Special Cases

5.1 Introduction

In this chapter, we will improve the lower and upper bounds for $L(n, k, p, t)$ on a case by case basis. Different techniques will be applied to improve these bounds. For the cases where $t = 2$, the results from Bate and van Rees [2] may be applied. The main goal of this Chapter is to complete the gaps in Bate's tables [1] and ascertain some of the smaller values.

5.2 Lower Bounds

The results from Bate and van Rees which are stated in Chapter Two are referenced quite often in this section. Before we begin, we would like to state several results that are useful for determining if a Lotto design has elements of a certain frequency.

Theorem 5.2.1 : *If there exists an element of frequency x in an (n, k, p, t) Lotto design with y blocks , then there exists an $(n-(x(k-1)+1), k, p-1, t)$ Lotto design* *with $y - x$ blocks, where $n \geq max\{(x+1)(k-1)+2, x(k-1)+p\}$.*

Proof : Let the element a have frequency x in the (n, k, p, t) Lotto design. Delete the blocks from the design that contain the element a. In the remaining blocks, replace the elements that appeared in the deleted blocks with arbitrary elements. This is possible since n is large enough. Let R be a $(p-1)$-set chosen from the $n-xk+x-1$ elements. Then $\{a\} \cup R$ is a p-set and so was represented in the original design. But since the elements of R and a did not occur together in any block, $\{a\} \cup R$ must have been represented in one of the $y - x$ blocks not containing a. Hence R would still be represented by one of the new $y - x$ blocks. Hence the $y - x$ blocks form an $(n - (x(k-1)+1), k, p-1, t)$ Lotto design. □

The following result is immediate from Theorem 5.2.1

Corollary 5.2.1 : *If there exists an element of frequency zero in an (n, k, p, t) Lotto design with y blocks, then there exists an $(n-1, k, p-1, t)$ Lotto design on y blocks.*

Corollary 5.2.2 : *If there exists an element of frequency one in an (n, k, p, t) Lotto design with y blocks, then there exists an $(n-k+t-2, k, p-1, t)$ Lotto design with $y-1$ blocks, where $n > 2k-t+1$ and $n > k+p-1$.*

Proof : Suppose the element 1 has frequency one in an (n, k, p, t) Lotto design, and say it occurs in the block $B_1 = \{1, 2, ..., k\}$. Delete B_1 from the design and delete all but $t-2$ elements that are in B_1, say do not delete $2, 3, ..., t-1$ and delete $1, t, t+1, ..., k$. In the remaining blocks, replace $t, t+1, ..., k$ with arbitrary remaining elements. This is possible as n is large enough. Let R be any $(p-1)$-set chosen from $\{2, 3, 4, ..., t-1\} \cup \{k+1, k+2, ..., n\}$. Consider $R \cup \{1\}$. This set is represented in the original design by some block that is not B_1 as B_1 contains at most $t-1$

elements from $R \cup \{1\}$. Thus, this block must still be in the new design with those same elements. Hence, R is represented in the new design. □.

The following result is immediate from Corollary 5.2.1 and Corollary 5.2.2.

Corollary 5.2.3 : *If $L(n,k,p,t) < min\{L(n-1,k,p-1,t), L(n-k+t-2,k,p-1,t)+1\}$, then the elements of any minimal (n,k,p,t) Lotto design must have frequency two or more.*

The next four results may be used to improve lower bounds for $L(n,k,p,t)$ for any given n,k,p,t. These results were helpful in improving several of the lower bounds found in Bate's tables.

Theorem 5.2.2 : *If $L(n,k,p,t) = b < \frac{n}{k}$, then $L(n,k,p,t) \geq L(n-1,k,p-1,t)$.*

Proof : Assume $b < L(n-1,k,p-1,t)$. Then there exists an (n,k,p,t) Lotto design with b blocks whose average occurrence for elements in the design is $\frac{kb}{n} < \frac{k}{n}\frac{n}{k} = 1$. This means that there is an element of frequency zero in the Lotto design. By Theorem 5.2.1, there exists an $(n-1,k,p-1,t)$ Lotto design on b blocks. This is a contradiction as we assumed $b < L(n-1,k,p-1,t)$. □

Theorem 5.2.3 : *If $L(n,k,p,t) = b < \frac{2n}{k}$, then $L(n,k,p,t) \geq 1 + L(n-k+t-2,k,p-1,t)$.*

Proof : Assume $b < 1 + L(n-k+t-2,k,p-1,t)$. Then there exists an (n,k,p,t) Lotto design on b blocks whose average occurrence for elements in the design is

$\frac{kb}{n} < \frac{k}{n}\frac{2n}{k} = 2$. This means that there must be an element of frequency zero or frequency one in the design. If there is an element of frequency zero, then by Theorem 5.2.1, there exists an $(n-1, k, p-1, t)$ Lotto design on $b < 1+L(n-k+t-2, k, p-1, t)$ blocks. But by Corollary 4.2.3, $1 + L(n-k+t-2, k, p-1, t) \leq L(n-1, k, p-1, t)$. This is a contradiction. If there is an element of frequency one, then by Corollary 5.2.2, there exists an $(n-k+t-2, k, p-1, t)$ Lotto design on $b-1 < L(n-k, k, p-1, t)$ blocks, which is again a contradiction. Since in either case we get a contradiction, the theorem is true. □

Theorem 5.2.4 : *If* $L(n, k, p, t) = b < \frac{3n}{k}$, *then* $L(n, k, p, t) \geq min\{\frac{2n}{k}, 1 + L(n-k+t-2, k, p-1, t)\}$.

Proof: Assume $b < min\{\frac{2n}{k}, 1+L(n-k+t-2, k, p-1, t)\}$. There exists an (n, k, p, t) Lotto design with b blocks where the average occurrence for elements is less than three. This means that there must be an element of frequency zero, frequency one or frequency two. If there is an element of frequency zero, then by Theorem 5.2.1, there exists an $(n-1, k, p-1, t)$ Lotto design on $b < 1 + L(n-k+t-2, k, p-1, t) \leq L(n-1, k, p-1, t)$ blocks, which is a contradiction. Similarly, if there is an element of frequency one, then by Theorem 5.2.1, there exists an $(n-k+t-2, k, p-1, t)$ on $b-1 < L(n-k+t-2, k, p-1, t)$ blocks. Again, this is a contradiction. Therefore, every element has frequency at least two and so there are at least $\frac{2n}{k}$ blocks in the design. □

The following result is used often throughout the Chapter. It determines when a pair of elements must appear together in a block of a Lotto design. In the proofs of the next several results, the terms *single*, *clique* and *i-cliques* are used. These terms were first introduced in Section 2.4. Also recall the term *disjoint* from Section 1.1.

Lemma 5.2.1 : *Suppose* $\{a_1, a_2, ..., a_{p-1}\}$ *is a maximum independent set in an* $(n, k, p, 2)$ *Lotto design. If* x, y *are two singles appearing in different blocks with one of the independent elements, say* a_1*, then* x, y *must appear together in some block of the design.*

Proof : Consider the p-set $\{x, y, a_2, a_3, ..., a_{p-1}\}$. It must be represented by some block in the design. Since $a_2, a_3, ...a_{p-1}$ are independent elements in the design, no two of them can appear together in a block of the design. As x, y are singles appearing with a_1, they cannot appear with any of the other independent elements. Hence x, y must appear together in a block of the design. □

We now proceed with determining the values of $L(n, k, p, t)$ on a case by case basis.

Theorem 5.2.5 : $L(14, 3, 4, 2) = 11$.

Proof : From Bate's tables, $10 \leq L(14, 3, 4, 2) \leq 11$. Assume that $L(14, 3, 4, 2) = 10$ and consider a $(14, 3, 4, 2)$ Lotto design on 10 blocks. By Corollary 5.2.3, all elements in the design must have frequency at least 2. There are at least 12 elements of frequency two and at most 2 elements of frequency three or more. By Theorem 2.4.7, there are 3 disjoint elements of frequency two, say 1, 6 and 11, forming a maximal independent set in the design. The 6 independent blocks that contain these 3 elements contain every element in the design, by Theorem 2.4.3. In these 6 blocks, there are exactly 4 elements that occur twice (where three of these elements are the independent elements). Thus there are exactly 10 singles. Therefore there exists some clique that has 4 singles, say $\{1, 2, 3\}, \{1, 4, 5\}$. The element 2 must appear with elements 4 and 5, and the element 3 must appear with elements 4 and 5. To see why the element 2 must appear with the element 4, suppose they did not appear together. Then the 4-set $\{2, 4, 6, 11\}$ is not represented in the design. The argument for why the other pairs $\{2, 5\}$, $\{3, 4\}$ and $\{3, 5\}$ must appear in the design is similar. This implies that the elements 4 and 5 (or 2 and 3) have frequency at least 3. If there is another clique with 4 singles, then using the same argument as given above, another 2 elements would have frequency at least 3, which contradicts the fact that there are at most 2 elements of frequency three or more. So, without loss of generality, the two remaining cliques would look like $\{6, 7, 8\}, \{6, 9, 10\}$ and $\{11, 12, 13\}, \{11, 14, 10\}$ where the element 10 is the non-single. Now the 4-set $\{1, 7, 10, 12\}$ is not represented in the six blocks intersecting the independent set $\{1, 6, 11\}$. Since there are already two elements of frequency at least three, the element 10 cannot occur again. Also, the element 1 cannot occur again in the design. Thus the pair $\{7, 12\}$ must appear in the design. Now the pair $\{7, 9\}$ must appear in the design or else the 4-set $\{1, 7, 9, 11\}$ will not be represented. Similarly, the pair $\{7, 13\}$ must appear in the design. Since the element 7 must appear with elements 9, 12, 13, this implies that element 7 must have frequency at least three, which contradicts the fact that at most two elements may have frequency three. Thus, $L(14, 3, 4, 2) > 10$. Along with the fact that $L(14, 3, 4, 2) \leq 11$, this implies that $L(14, 3, 4, 2) = 11$. □

Theorem 5.2.6 : $L(16, 3, 4, 2) = 14$.

Proof: From Bate's tables, $13 \leq L(16, 3, 4, 2) \leq 14$. Assume $L(16, 3, 4, 2) = 13$. Then there exists a $(16, 3, 4, 2)$ Lotto design with 13 blocks. By Corollary 5.2.3, all elements have frequency two or more. By looking at frequency counts, we see that at least 9 elements must have frequency two and at most 7 elements have frequency three or more. By Theorem 2.4.7, there exist 3 disjoint elements of frequency two forming a maximal independent set in the design. But these 6 blocks cannot contain every element of the design which contradicts Theorem 2.4.2. Thus $L(16, 3, 4, 2) = 14$.□

Theorem 5.2.7 : $L(18, 3, 4, 2) = 17$.

Proof: By Theorem 2.5.5, $L(18,3,4,2) \leq 17$. Suppose there exists a $(17,3,4,2)$ Lotto design with 16 blocks. The minimum number of blocks that must contain three disjoint elements is 8 since 7 blocks can hold only $2(7)+3 = 17 < 18$ elements. By Theorem 2.4.7, there are at two disjoint elements of frequency two, say 1 and 2. There can be as many as 8 other elements occurring with these two elements. Each of the remaining 8 elements must therefore have frequency at least 4. Therefore the total number of occurances of elements in the design is at least $10(2) + 8(4) = 52 > 16(3) = 48$, which is a contradiction. Hence, $L(18,3,4,2) = 17$. □

Theorem 5.2.8 : $L(19,3,4,2) = 18$.

Proof: By Theorem 2.5.5, $L(19,3,4,2) \leq 18$. Suppose there exists a $(19,3,4,2)$ Lotto design with 17 blocks. By arguing exactly as we did in the previous Theorem, we find that the 17 blocks must hold 56 elements, which is a contradiction. Hence, $L(19,3,4,2) = 18$. □

Theorem 5.2.9 : $L(20,3,4,2) = 20$.

Proof : By Theorem 2.5.5, $L(20,3,4,2) \leq 20$. Assume $L(20,3,4,2) \leq 19$ and consider a $(20,3,4,2)$ Lotto design on 19 blocks. By Corollary 5.2.3, all elements have frequency two or more. By looking at frequency counts, we see that at least 3 elements must have frequency two and at most 17 elements have frequency three or more. If there are 5 or more elements of frequency two, Theorem 2.4.7 implies there are two disjoint elements of frequency two. Since it is not possible for all elements to appear with these two elements, then by Theorem 2.4.3, the remaining elements must each have frequency five or more. This implies that the total number of occurances of elements in the design is $2(5) + 2(5) + 5(10) = 70$ which is larger than 57. Hence the number of elements of frequency two must be 3 or 4. Suppose element 1 has frequency two. There are at least 15 elements of frequency three. The elements of frequency two must all appear together in a 2-clique where element 1 is the independent element of the clique. Then there must exist two disjoint elements of frequency three, say 6 and 13 that do not appear in the 2-clique. $\{1, 6, 13\}$ form a maximal independent set of the design. By Theorem 2.4.3, every element must appear with one of these independent elements. This is clearly not possible, since it requires 9 independent blocks to hold all 20 elements. Hence $L(20, 3, 4, 2) = 20$. □

Theorem 5.2.10 : $L(18, 3, 5, 2) = 13$.

Proof : By Theorem 2.5.5, $L(18, 3, 5, 2) \leq 13$. Suppose $L(18, 3, 5, 2) = 12$ and consider an $(18, 3, 5, 2)$ Lotto design on 12 blocks. By Corollary 5.2.3, every element in the design has frequency two or more. By analyzing the elements' frequencies, we see that every element in the design must have frequency two. By Theorem 2.4.7, there are 5 elements (of frequency two) that do no occur together in a block. Hence, these five elements form a 5-set that is not represented by a block in the design. Thus, $L(18, 3, 5, 2) = 13$. □.

Theorem 5.2.11 : $L(19, 3, 5, 2) = 15$.

Proof : By Theorem 2.5.5, $L(19, 3, 5, 2) \leq 15$. Suppose $L(19, 3, 5, 2) \leq 14$ and consider a $(19, 3, 5, 2)$ Lotto design on 14 blocks. By Theorem 5.2.3, all elements in the design have frequency two or more. By analyzing the frequencies of the elements, we see that there are at least 15 elements of frequency two. By Theorem 2.4.7, there are four disjoint elements of frequency two. Since $p = 5$, this forms a maximum independent set, so by Theorem 2.4.3 every element of the design must appear in one of the independent blocks. There are 8(2)=16 places for the remaining elements. So 14 of the elements are singles and one element appears twice in the independent blocks. Now a least 10 of the singles have frequency two, so there is a clique with at least 3 singles of frequency two. If this clique had 4 singles of frequency two, say $\{1,2,3\}$ and $\{1,4,5\}$ where element 1 is the independent element, then the 4 pairs $\{x,y\}$ where $x \in \{2,3\}$ and $y \in \{4,5\}$ must appear in the design or otherwise $\{a,b,c,x,y\}$ would not be represented, where a,b,c are the other independent elements. Since element 2 has frequency two, the block $\{2,4,5\}$ is forced in the design. As element 3 has frequency two, the block $\{3,4,5\}$ is also forced in the design. This contradicts the assumption that elements 4 and 5 had frequency two in the design. If the clique has 4 singles with 3 singles of frequency two, say $\{1,2,3\}$ and $\{1,4,5\}$ where element 2 has frequency greater than 2, then the block $\{3,4,5\}$ would be forced. As element 4 has frequency two, it cannot appear in a block with the element 2 and hence the 5-set $\{a,b,c,2,4\}$ would not be represented by the blocks in the design, where a,b,c are the other independent elements. Hence, there are exactly two cliques containing 3 singles of frequency two and a non-single. These cliques would then force the 3 singles of frequency two to occur in another block. Now the other two cliques would be forced to have 4 singles, of which 2 have frequency two and 2 have frequency three. Consider $\{11,12,13\}$ and $\{11,14,15\}$ where each element is a single and 12 is a single of frequency two. Element 12 occurs once again and must do so with 14 and 15, forcing the block $\{12,14,15\}$. If 14 or 15 have frequency two (say element 14), then the pair $\{13,14\}$ cannot occur and $\{a,b,c,13,14\}$ would not be represented by the blocks in the design. So element 13 must have frequency two and the block $\{13,14,15\}$ is forced. A similar argument may be applied to the remaining cliques. So we have 8+1+1+2+2 = 14 blocks forced. Now consider the 5-set $\{a,b,x,r,s\}$ where a,b are independent elements that do not occur with the non-single x, and r and s are singles that occur in a clique with x but not in a block with x. This requires another block in the design to represent it. But this contradicts the assumption that $L(19,3,5,2) \leq 14$ and therefore $L(19,3,5,2) = 15$. $\square$

Theorem 5.2.12 : $L(20,3,5,2) = 16$.

Proof : By Theorem 2.5.5, $L(20,3,5,2) \leq 16$. Suppose $L(20,3,5,2) \leq 15$ and consider a $(20,3,5,2)$ Lotto design on 15 blocks. By Corollary 5.2.3, every element in the design has frequency two or more. By analyzing the frequencies of the elements, we see that there are at least 15 elements of frequency two. Theorem 2.4.7 implies there are four disjoint elements of frequency two which make up a maximal independent set. The eight blocks containing four such elements must look like : $\{1,5,6\}$, $\{1,7,8\}$, $\{2,9,10\}$, $\{2,11,12\}$, $\{3,13,14\}$, $\{3,15,16\}$, $\{4,17,18\}$, $\{4,19,20\}$ where the elements $1,2,3$ and 4 are the elements of frequency two that form the independent set. Consider the pairs $\{17,19\}$, $\{17,20\}$, $\{18,19\}$, $\{18,20\}$. These pairs must all appear in the remaining seven blocks of the design, because 5-sets of the form $\{1,2,3,17,19\}$, $\{1,2,3,17,20\}$, $\{1,2,3,18,19\}$ and $\{1,2,3,18,20\}$ must be represented. This requires at least two blocks in the design. Similarly, the pairs $\{5,7\}$, $\{5,8\}$, $\{6,7\}$, $\{6,8\}$, $\{9,11\}$, $\{9,12\}$, $\{10,11\}$, $\{10,12\}$, $\{13,15\}$, $\{13,16\}$, $\{14,15\}$ and $\{14,16\}$ must all appear in the design. To have all these pairs in the design requires at least six blocks. Thus we require at least 16 blocks. Hence, $L(20,3,5,2) = 16$. □

Theorem 5.2.13 : $L(13,4,3,2) = 8$.

Proof : From Bate's tables, $7 \leq L(13,4,3,2) \leq 8$. Assume that $L(13,4,3,2) = 7$ and consider a $(13,4,3,2)$ Lotto design on 7 blocks. By Corollary 5.2.3, every element in the design has frequency two or more. By analyzing the frequency counts, at least 11 elements have frequency two and at most 2 elements have frequency three or more. By Theorem 2.4.7, there are at least 2 disjoint elements of frequency two, say

1 and 8, which form a maximal independent set for the design. Since every element must appear with 1 or 8, there is at most one repeated element not belonging to the independent set, giving at least 10 singles. This means that at least one clique has at least 5 singles. Without loss of generality, suppose this clique contain the blocks $\{1,2,3,4\}$ and $\{1,5,6,7\}$. If this clique has 6 singles, then 2 must appear with 5, 6 and 7, otherwise the 3-sets $\{2,5,8\}$, $\{2,6,8\}$ and $\{2,7,8\}$ will not be represented by the design. Similarly, the element 3 must appear with the elements 5, 6, and 7, and the element 4 must appear with the elements 5, 6 and 7. This implies that three of 2, 3, 4, 5, 6, and 7 must have frequency at least three. This is a contradiction since the number of elements of frequency three or greater is at most 2. Thus, the above clique must have exactly 5 singles (and thus both cliques have 5 singles). Without loss of generality, suppose the element 2 is a non-single. Element 3 must appear with 5, 6 and 7 elsewhere in the design and element 4 must also appear with 5, 6, 7 elsewhere in the design. This clique has 3 elements that appear at least three times. The same is true for the other clique. But this gives too many elements with frequency three. Hence, $L(13,4,3,2) = 8$. □

Theorem 5.2.14 : $L(14,4,3,2) = 9$.

Proof: From Bate's tables, $L(14,4,3,2) = 8$ or 9. Suppose $L(14,4,3,2) = 8$ and consider a $(14,4,3,2)$ Lotto design on 7 blocks. By Corollary 5.2.3, every element in the design has frequency two or more. By analyzing the elements' frequencies, we see there are at least ten elements of frequency two and at most four elements of frequency three or greater. Theorem 2.4.7 implies a maximal independent set with two elements of frequency two. Without loss of generality, suppose the four independent blocks look like $\{1,3,4,5\}$, $\{1,6,7,8\}$, $\{2,9,10,11\}$, $\{2,12,13,14\}$. At least eight of $\{3,4,...,14\}$ has frequency two and hence at least one of the cliques have four elements of frequency two (not including the independent element). Suppose this clique contains the independent element 1. Then without loss of generality, at least two of the elements 3, 4 and 5 (say elements 4 and 5) have frequency two. This implies that the blocks $\{3, 6, 7, 8\}$ and $\{4, 6, 7, 8\}$ are forced. Also the element 5 must appear with 6, 7 and 8 elsewhere in the design. This means that elements 5, 6, 7 have frequency at least three. Hence, only one of the elements between 9 and 14 has frequency at least three. So without loss of generality, suppose elements 9, 10, ...,13 have frequency two and element 14 has frequency at least three. But then the blocks $\{9, 12, 13, 14\}$, $\{10, 12, 13, 14\}$ and $\{11, 12, 13, 14\}$ are forced. There are too many elements of frequency at least three, which is a contradiction. Hence $L(14, 4, 3, 2) = 9$. □

Theorem 5.2.15 : $L(15, 4, 3, 2) = 11$.

Proof : From Bate's tables, $L(15, 4, 3, 2) = 10$ or 11. Suppose $L(15, 4, 3, 2) = 10$ and consider a $(15, 4, 3, 2)$ Lotto design with 10 blocks. By Corollary 5.2.3, every element in the design has frequency at least two. By analyzing the frequencies of the elements in the design, we know there are at least five elements of frequency two and at most 10 elements of frequency three or higher. There cannot exist two disjoint elements of frequency two in the design, for if there were, they would form a maximal independent set and by Theorem 2.4.2, all elements must appear with one of these independent elements. This is impossible since there are too many elements. Thus, by Theorem 2.4.7, the number of elements of frequency two is at most six. We consider the value of f_2.

Case 1 : $f_2 = 6$. Without loss of generality, suppose the elements $\{1, 2, ..., 6\}$ have frequency two. Then the blocks containing the elements of frequency two must look like : $\{1, 2, 3, 4\}$, $\{1, 2, 5, 6\}$ and $\{3, 4, 5, 6\}$. The remaining blocks of the design must form a $(9, 4, 2, 2)$ design. This requires 8 blocks. Thus the design has 11 blocks which is too many.

Case 2 : $f_2 = 5$. Suppose that the elements 1, 2, 3, 4, 5 are frequency two elements. Consider the clique $\{1, 2, 3, x\}$, $\{1, 4, y, z\}$ where one of x, y, z must be the element 5. All occurrences of the elements of frequency two must appear in three blocks of the design where two of the blocks are stated above. Within these three blocks, there are at most two elements that are not of frequency two. Every non frequency two element must appear together in the rest of the design, except possibly for one pair that may appear in the three blocks. This requires at least $L(10, 4, 2, 2) - 1 = 8$ blocks, which means the size of the design is at least 11. This is a contradiction.

Since each case led to a contradiction, we conclude that $L(15, 4, 3, 2) = 11$. □

Theorem 5.2.16 : $L(16, 4, 3, 2) = 12$.

Proof : From Bate's thesis, $L(16, 4, 3, 2) = 11$ or 12. Suppose $L(16, 4, 3, 2) \leq 11$ and consider a $(16, 4, 3, 2)$ Lotto design with 11 blocks. By Corollary 5.2.3, all elements in the design have frequency two or more. By analyzing the elements' frequencies, we see there are at least four elements of frequency two. Suppose element 1 has frequency two. Since not all elements can occur with element 1, there must be another element, say 2, that does not occur with 1 in the design. Then elements 1 and 2 form an maximum independent set. If element 2 had frequency two, then not all elements could occur with element 1 or with element 2, contradicting Theorem 2.4.2. Hence all pairs of elements of frequency two must occur together in some block. By Theorem 2.4.7, the maximum number of non-disjoint frequency two elements is 6. If all elements not occurring with element 1 had frequency four or more, then the total number of occurrences in the design is at least $7(2) + 9(4) = 50 > 44$, which is a contradiction. So, we may assume that there is an element that does not occur with element 1 and has frequency three. We consider the possible number of elements of frequency two in the design.

Case 1 : Suppose the number of elements of frequency two in the design is 6, where the elements $\{1, 2, ..., 6\}$ have frequency two. Then the blocks containing the elements of frequency two must look like : $\{1, 2, 3, 4\}$, $\{1, 2, 5, 6\}$ and $\{3, 4, 5, 6\}$. The remaining blocks of the design must form a $(10, 4, 2, 2)$ design. This requires 9 blocks. Thus the design has 12 blocks, which is too many.

Case 2 : Suppose the number of elements of frequency two in the design is 5. Let 1, 2, 3, 4 and 5 be the elements of frequency two. Then the following is forced : $\{1, 2, 3, x\}$, $\{1, 4, y, z\}$ and $\{2, 3, 4, w\}$, where two of x, y, w, z is the element 5. The other two elements have frequency three or more. It is easy to check that all pairs of frequency three or more elements must occur together in some block or there will be some 3-set $\{a, b, c\}$ not represented by the blocks of the design, where a is an element of frequency two, b and c are not x, y, z or w and c and has frequencies three or more. This requires at least $L(11, 4, 2, 2) - 1 = 11$ blocks. The 1 is subtracted because y and z might be frequency three or more and that might mean that one less than $L(11, 4, 2, 2)$ blocks are needed. But $11 + 3 = 14$ blocks is too many, as we assumed that the design has 13 blocks.

Case 3 : Suppose the number of elements of frequency two in the design is 4. Then the number of elements of frequency three is exactly 12 and there are no elements with frequency four or more. There are three non-isomorphic sub-cases to consider.

Case 3a : The independent blocks are $\{1,2,3,4\}$, $\{1,5,6,7\}$, $\{8,9,10,11\}$, $\{8,9,12,13\}$, $\{8,14,15,16\}$. One of the two blocks containing 1 must have at least two more elements of frequency two. Let elements 2 and 3 have frequency two. Since elements 2 through 7 are singles, the blocks $\{2,5,6,7\}$ and $\{3,5,6,7\}$ are forced. Further, elements 4 and 5 must occur in some block which implies element 5 must have frequency three or more, which is too much.

Case 3b : The independent blocks are $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{7,8,9,10\}$, $\{7,11,12,13\}$, $\{7,14,15,16\}$. The first two blocks have 4 pairs that must appear in the design requiring at least one full block. The remaining 10 elements 7, 8,.., 16 must form an $(10,4,2,2)$ Lotto design, which requires at least 9 blocks. Hence, this case requires at least 12 blocks, which is a contradiction.

Case 3c : The independent blocks are $\{1,2,3,4\}$, $\{1,5,6,7\}$, $\{8,2,9,10\}$, $\{8,11,12,13\}$, $\{8,14,15,16\}$. The element 9 must occur with elements 11 through 16 in the design in two full blocks containing no other elements. This is also true for the element 10. Then elements 11 through 16 have occurred a total of 3 times and cannot occur again. But none of the 9 pairs $\{x,y\}$ where $x \in \{11,12,13\}$ and $y \in \{14,15,16\}$ have yet appeared, so some 3-set $\{3,x,y\}$ is not represented by the blocks of the design. This is a contradiction.

Since each case led to a contradiction, we conclude that $L(16,4,3,2) = 12$. □

Theorem 5.2.17 : $L(17,4,3,2) = 14$.

Proof : From Bate's tables, $L(17,4,3,2) = 12$, 13 or 14. Suppose $L(17,4,3,2) \leq 13$ and consider a $(17,4,3,2)$ Lotto design on 13 blocks. By Corollary 5.2.3, every element of the design must have frequency two or more. By analyzing the frequencies of the elements, we see that there are at least nine elements of frequency two. By Theorem 2.4.7, there exist two disjoint elements of frequency two. These two elements form a maximal independent set, as $p = 3$. By Theorem 2.4.3, every other element in the design must appear with these two elements. But this is clearly not possible. Hence, $L(17, 4, 3, 2) = 14$. $\square$

Theorem 5.2.18 : $L(18, 4, 3, 2) = 15$.

Proof : By Theorem 2.5.5, $L(18, 4, 3, 2) \leq 15$. Suppose $L(18, 4, 3, 2) \leq 14$; then consider an $(18, 4, 3, 2)$ Lotto design with 14 blocks. By Corollary 5.2.3, all elements in the design have frequency two or more. If there exists an element of frequency two, then the elements that are disjoint from it must have frequency four or more by Theorem 2.4.2. This implies that the total occurrences of the elements in the design is at least $4(11) + 7(2) = 58$, which is larger that $4(14) = 56$, a contradiction. So there are no elements of frequency two in the design. By analyzing the frequency of elements, we see that there are at least sixteen elements of frequency three and at most two elements of frequency four or more. If element 1 has frequency three, then there exists an element, say 2, of frequency three that does not occur with element 1 in the design. These two elements form a maximum independent set for the design. By Theorem 2.4.3, every element must occur with at least one of the 2 independent elements. We consider the four possible configurations for the independent blocks.

Case 1 : One of the cliques, say the blocks containing 1, consists of the independent element and 9 singles. At least 7 of the singles have frequency three, so one of the blocks consists of 1 and three singles of frequency three. Let B_1 be that block so $B_1 = \{1, 3, 4, 5\}$, $B_2 = \{1, 6, 7, 8\}$ and $B_3 = \{1, 9, 10, 11\}$. Now 3 must appear with elements 6 through 11 in its remaining two blocks. These blocks are full. The same is true for element 4 and element 5. So elements 6 through 11 each have frequency four or more. But this contradicts the assumption that there are at most two elements of frequency four or more in the design.

Case 2 : Each of the two cliques contains a repeated element besides the independent elements. Consider $B_1 = \{1,3,4,5\}$, $B_2 = \{1,3,6,7\}$, $B_3 = \{1,8,9,10\}$, $B_4 = \{2,11,12,13\}$, $B_5 = \{2,11,14,15\}$ and $B_6 = \{2,16,17,18\}$. The elements 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17 and 18 are singles. If element 3 has frequency three, the block $\{3,8,9,10\}$ is forced. The elements 4, 5, 6, 7 must appear with 8 in the design. But this is also true for elements 9 and 10. This contradicts the fact that there are at most two elements of frequency four or more. So element 3 must have frequency four. A symmetric argument will give that element 11 has frequency four. So, all the singles have frequency three. Now, since 8 must occur with 3, 4, 5, 6, and 7 in its last two remaining blocks, it cannot occur with both 9 and 10 again. So let elements 8 and 9 not occur together again in the design. Thus 3, 4, 5, 6 and 7 must occur in the two blocks with element 8 and in two different blocks with element 9. As 3, 4, 5, 6 and 7 have frequency three, they cannot occur again in the design. Now 3, 4, 5, 6, and 7 must also occur with 11. There is room for two occurances of 11 in those four blocks, but then element 11 can occur with a maximum of four of the five elements 3, 4, 5, 6 and 7. This is a contradiction.

Case 3 : One of the cliques, say the blocks containing 1, has a non-single and eight singles in it. Let element 3 be the non-single. The blocks are $B_1 = \{1,3,4,5\}$, $B_2 = \{1,6,7,8\}$ and $B_3 = \{1,9,10,11\}$. Since there are at most two elements of frequency four or more, one of B_2 or B_3 must have at most one of those elements in it. So let elements 9 and 10 have frequency three. If 9 and 10 occur together again, then the blocks $\{9,a,b,c\}$, $\{9,d,e,10\}$ and $\{10,a,b,c\}$ are forced in the design where $\{a,b,c,d,e\} = \{4,5,6,7,8\}$. But element 11 then has not occurred with a, b, c and so those elements must occur at least four times, which is a contradiction. So elements 9 and 10 only occur together in B_3. Thus elements 4, 5, 6, 7 and 8 must occur in two blocks with element 9 and in two different blocks with element 10. So at least four of the elements 4, 5, 6, 7 and 8 do not occur again but do have to occur with 11. Hence, 11 must occur twice in these blocks and once more. Hence, element 11 has frequency four. Using a similar argument, it can be proved that one of the elements 6, 7 or 8, say 8 is an element of frequency 4. So elements 8 and 11 occur together in a block not containing 9 and 10. So the elements 4, 5, 6, 7, 9, 10 all have frequency three. In particular, elements 4 and 5 have frequency three. They must occur with 6, 7, 8, 9, 10, 11 and this happens in four full blocks where 4 and 5 do not occur together. Now element 6 must occur once with element 4 and once with element 5 and does not occur again. So it must occur with 9, 10 and 11 in these two blocks, say as follows : $\{4,6,9,11\}$ and $\{5,6,10,x\}$. This is without loss of generality. Since elements 7 and 8 must occur with 9, the block $\{5,7,8,9\}$ is forced. Since elements 7 and 8 must also occur with element 4, the block $\{4,7,8,10\}$ is also forced and hence x must be the element 11. But this is a contradiction, as the two singles 7 and 11 have not occurred together in any block of the design.

The next case considers the situation where the two cliques have two elements in common. There are three sub cases.

Case 4a : $B_1 = \{1,3,4,5\}$, $B_2 = \{1,6,7,8\}$, $B_3 = \{1,9,10,11\}$, $B_4 = \{2,3,4,12\}$, $B_5 = \{2,13,14,15\}$ and $B_6 = \{2,16,17,18\}$. Pairs of the type $\{x,y\}$ where $x \in \{5\}$ and $y \in \{6,7,8,9,10,11\}$, or $x \in \{6,7,8\}$ and $y \in \{9,10,11\}$ must appear in the design by Lemma 5.2.1. Similarly, pairs of the type $\{x,y\}$ where $x \in \{12\}$ and $y \in \{13,14,15,16,17,18\}$ or $x \in \{13,14,15\}$ and $y \in \{16,17,18\}$ must appear in the design by Lemma 5.2.1. These pairs will be called **type** A pairs. There are 30 type A pairs. Now the following 3-sets $\{3,6,13\}$, $\{3,7,14\}$, $\{3,8,15\}$, $\{3,9,16\}$, $\{3,10,17\}$

and $\{3, 11, 18\}$ must be represented by pairs. We will call such pairs **type** B^1 pairs. Similarly, 3-sets $\{4, 6, 18\}$, $\{4, 7, 13\}$, $\{4, 8, 14\}$, $\{4, 9, 15\}$, $\{4, 10, 16\}$ and $\{4, 11, 17\}$ must be represented by pairs. We will call such pairs **type** B^2 pairs. We will say that a pair is a **type** B pair if it is either a type B^1 or type B^2 pair. There are no common pairs between the type A pairs, type B^1 pairs and the type B^2 pairs. There are at least 6 type B^1 and 6 type B^2 pairs needed to represent the listed 3-sets. The total number of pairs needed is $30 + 6 + 6 = 42$.

We now proceed to show that any block can contain at most x type A pairs, and y type B pairs which represent distinct 3-sets from our list, where $x + y < 6$. Any block in the design can hold $C(4, 2) = 6$ pairs. So we must show that any block has at least one pair that is neither a type A pair nor represents any of the listed 3-sets. Let $M = \{a, b, c, d\}$ be a block in the design where $M \neq B_i$ for $i = 1$ to 6. If M has two elements from the same block B_i for $i = 1, 2$ or 3, then those two elements form the required pair. Also it is clear that element 3 and element 4 can be interchanged in the following argument. There remain six situations under which M may be formed. Let $X = \{5, 6, 7, 8, 9, 10, 11\}$ and $Y = \{12, 13, 14, 15, 16, 17, 18\}$. Consider the situation where $\{a, b, c\} \subset X$ and $d \in Y$. Clearly, element 5 must be in B, and $\{5, d\}$ is not type A and cannot represent one of our listed 3-sets. Now consider the case where $\{a, b\} \subset X$ and $\{c, d\} \subset Y$. The pair $\{a, c\}$ represents $\{3, a, c\}$ or $\{4, a, c\}$. Suppose it represents $\{3, a, c\}$. Then the pair $\{a, d\}$ must represent $\{4, a, d\}$; $\{b, c\}$ represents $\{4, b, c\}$, and $\{b, d\}$ represents $\{3, b, d\}$. But the 3-sets have been set up to preclude this possibility. So the only situation left to investigate is $M = \{3, b_2, b_3, c\}$ where $b_2 \in B_2$, $b_3 \in B_3$ and $c \in Y$. Now $\{3, b_2\}$ represents $\{3, b_2, x\}$, $\{3, b_3\}$ represents $\{3, b_3, y\}$ and $\{3, c\}$ represents $\{3, z, c\}$. The 3-sets must be distinct. Now $\{b_2, c\}$ cannot represent $\{3, b_2, c\} = \{3, b_2, x\}$ as that 3-set would be represented twice and we need 3-sets to be represented once. So $\{b_2, c\}$ represents $\{4, b_2, c\}$. Similarly, $\{b_3, c\}$ represents $\{4, b_3, c\}$. But there is only one 3-set with 4 and c in it, so $b_2 = b_3$, which is a contradiction. The remaining three situations just have X and Y interchanged. So any block has a pair that is neither type A nor a pair that represents a unique 3-set from our list.

So any block contains at most x type A pairs and y type B pairs that represent unique 3-sets from our list where $x + y < 6$. Since there are 42 pairs needed, we need at least $\frac{42}{5}$ or 9 blocks. This gives us a total of 15 blocks, which is a contradiction.

Case 4b : $B_1 = \{1,3,5,6\}$, $B_2 = \{1,4,7,8\}$, $B_3 = \{1,9,10,11\}$, $B_4 = \{2,3,12,13\}$, $B_5 = \{2,4,14,15\}$ and $B_6 = \{2,16,17,18\}$. Pairs of the type $\{x,y\}$ where $x \in \{5,6\}$ and $y \in \{7,8,9,10,11\}$, or $x \in \{7,8\}$ and $y \in \{9,10,11\}$ must appear in the design by Lemma 5.2.1. Similarly, pairs of the type $\{x,y\}$ where $x \in \{12,13\}$ and $y \in \{14,15,16,17,18\}$, or $x \in \{14,15\}$ and $y \in \{16,17,18\}$ must appear in the design by Lemma 5.2.1. These pairs will be called **type** A **pairs**. There are 32 type A pairs. Now the 3-sets $\{3,7,14\}$, $\{3,8,15\}$, $\{3,9,16\}$, $\{3,10,17\}$ and $\{3,11,18\}$ must be represented by pairs. We will call such pairs **type** B^1 pairs. Similarly, 3-sets $\{4,5,12\}$, $\{4,6,13\}$, $\{4,9,18\}$, $\{4,10,16\}$ and $\{4,11,17\}$ must be represented by pairs. We will call such pairs **type** B^2 pairs. We will say that a pair is a **type** B pair if it is either a type B^1 or type B^2 pair. There are no common pairs between the type A pairs, type B^1 pairs and the type B^2 pairs. There are at least five type B^1 and five type B^2 pairs needed to represent the listed 3-sets. The total number of pairs needed is $32 + 5 + 5 = 42$. Using a tedious argument similar to the one in case 4a, it can be proved that any block in the design has at most x type A pairs in it and can represent y distinct 3-sets from the list of 3-sets above, where $x + y < 6$. Then the total number of blocks in the design is at least $6+9 = 15$, which is a contradiction.

Case 4c : $B_1 = \{1,3,4,5\}$, $B_2 = \{1,6,7,8\}$, $B_3 = \{1,9,10,11\}$, $B_4 = \{2,3,12,13\}$, $B_5 = \{2,4,14,15\}$ and $B_6 = \{2,16,17,18\}$. Pairs of the type $\{x, y\}$ where $x \in \{5\}$ and $y \in \{6,7,8,9,10,11\}$, or $x \in \{6,7,8\}$ and $y \in \{9,10,11\}$ must appear in the design by Lemma 5.2.1. Similarly, pairs of the type $\{x,y\}$ where $x \in \{12,13\}$ and $y \in \{14,15,16,17,18\}$, or $x \in \{14,15\}$ and $y \in \{16,17,18\}$ must appear in the design by Lemma 5.2.1. These pairs will be called **type** A pairs. There are 31 type A pairs. Now the 3-sets $\{3,7,14\}$, $\{3,8,15\}$, $\{3,9,16\}$, $\{3,10,17\}$ and $\{3,11,18\}$ must be represented by pairs. We will call such pairs **type** B^1 pairs. Similarly, 3-sets $\{4,7,12\}$, $\{4,8,13\}$, $\{4,9,18\}$, $\{4,10,16\}$ and $\{4,11,17\}$ must be represented by pairs. We will call such pairs **type** B^2 pairs. We will say that a pair is a **type** B pair if it is either a type B^1 or type B^2 pair. There are no common pairs between the type A pairs, type B^1 pairs and the type B^2 pairs. There are at least five type B^1 and five type B^2 pairs needed to represent the listed 3-sets. The total number of pairs needed is $31 + 5 + 5 = 41$. Using an even more tedious argument similar to the one in case 4a, it can be proved that any block in the design has at most x type A pairs in it and can represent y distinct 3-sets from the list of 3-sets above, where $x + y < 6$. Then the total number of blocks in the design is at least $6+9 = 15$, which is a contradiction.

Since each case led to a contradiction, we conclude that $L(18,4,3,2) = 15$. □

Theorem 5.2.19 : $L(19,4,3,2) = 16$.

Proof : By Theorem 2.5.5, $L(19,4,3,2) \leq 16$. Assume that $L(19,4,3,2) \leq 15$ and consider a $(19,4,3,2)$ Lotto design with 15 blocks. By Corollary 5.2.3, all elements in the design have frequency two or more. If there exists an element of frequency two, then by Theorem 2.4.2, there must exist an element of frequency four which make up a maximal independent set. This implies that the total occurrences of the elements in the design is at least $4(12) + 7(2) = 62$ which is larger that $4(15) = 60$. Hence, there are no elements of frequency two in the design. By analyzing the frequency of the elements, we conclude that there are at least sixteen elements of frequency three and at most three elements of frequency four or more. This implies there are two elements of frequency three, $\{1, 2\}$, that form an maximum independent set. Consider the following non-isomorphic configurations of the independent blocks.

Case 1 : Suppose the independent blocks are $B_1 = \{1, 3, 4, 5\}$, $B_2 = \{1, 6, 7, 8\}$, $B_2 = \{1, 9, 10, 11\}$, $C_1 = \{2, 3, 12, 13\}$, $C_2 = \{2, 14, 15, 16\}$, $C_3 = \{2, 17, 18, 19\}$. By Lemma 5.2.1, pairs of the form $\{x, y\}$ where $x \in \{4, 5\}$, $y \in \{6, 7, 8, 9, 10, 11\}$, or $x \in \{6, 7, 8\}$, $y \in \{9, 10, 11\}$ must appear in the design. Similarly, pairs of the form $\{x, y\}$ where $x \in \{12, 13\}$, $y \in \{14, 15, 16, 17, 18, 19\}$, or $x \in \{14, 15, 16\}$, $y \in \{17, 18, 19\}$ must also appear in the design. We will call the pairs **type** A pairs. There are 42 type A pairs. Now 3-sets of the form $\{3, x, y\}$ where $x \in \{6, 7, 8, 9, 10, 11\}$, $y \in \{14, 15, 16, 17, 18, 19\}$, must also be represented. The pairs that represent these 3-sets will be called **type** B pairs. There are no common pairs between the type A and type B pairs. Consider the 3-sets $\{3, 6, 14\}$, $\{3, 7, 15\}$, $\{3, 8, 16\}$, $\{3, 9, 17\}$, $\{3, 10, 18\}$ and $\{3, 11, 19\}$. These 3-sets require at least 6 distinct pairs to represent them. Hence the number of type B pairs that need to appear in the design is 6. Thus the total number of pairs that need to appear in the design is at least 48.

We now proceed to show that given a block, it can contain at most five pairs of type A and type B unless there are two pairs of type A or 1 pair of type A in the block. To show this, consider a block in the design. It can hold at most $C(4, 2) = 6$ pairs. Clearly it cannot hold six pairs of type A and it cannot hold six pairs of type B. If the block contains five pairs of type A, then it must be of the form $\{a, b, c, d\}$, where $a, b \in B_i$, $c \in B_j$ and $d \in B_k$ or $a, b \in C_i$, $c \in C_j$ and $d \in C_k$, where $i, j, k \in \{1, 2, 3\}$, $i \neq j \neq k$, $a \neq 3$, $b \neq 3$, $c \neq 3$ and $d \neq 3$. Then no pairs of type B are in this block and hence five pairs of type A and B appear in this block. If the block contains four pairs of type A, then it must be of the form $\{a, b, c, d\}$, where $a, b \in B_i$, $c, d \in B_j$ or $a, b \in C_i$, $c, d \in C_j$, where $i, j \in \{1, 2, 3\}$, $i \neq j$, $a \neq 3$, $b \neq 3$, $c \neq 3$ and $d \neq 3$. Again, no pairs of type B are in this block and hence four pairs of type A and B appear in this block. If the block contains three pairs of type A, then it must be of the form form $\{a, b, c, d\}$ where $a \in B_i$, $b \in B_j$ and $c \in B_k$ or $a \in C_i$, $b \in C_j$ and $c \in C_k$ where $i, j, k \in \{1, 2, 3\}$, $i \neq j \neq k$, $a \neq 3$, $b \neq 3$ and $c \neq 3$. Now one of a, b, c, say a, must appear with the element 3. Hence, only the pairs $\{c, d\}$, $\{b, d\}$ may be of type B. Thus, there are at most five pairs of type A and B in this case.

We will now show that there cannot be any blocks in the design that contain one or two pairs of type A. If there are $x > 0$ blocks that contain exactly one pair of type A, then x pairs of type A appears in these blocks. Since there are 42 pairs of type A, $42 - x$ pairs of type A must appear in the remaining $15 - 6 - x = 9 - x$ blocks in the design. As each of the remaining blocks may contain at most five pairs of type A, $5(9 - x) \geq 42 - x$, which implies $x \leq \frac{3}{4}$, a contradiction. Hence, there are no blocks that contain one pair of type A. If there are $x > 0$ blocks that contain exactly two pairs of type A, then $2x$ pairs of type A appears in these blocks. Since there are 42 pairs of type A, $42 - 2x$ pairs of type A must appear in the remaining $15 - 6 - x = 9 - x$ blocks in the design. As each of the remaining blocks may contain at most five pairs of type A, $5(9 - x) \geq 42 - 2x$ which implies $x \leq 1$. If $x = 1$, then $9 - 1 = 8$ blocks must each contain five pairs of type A (and hence no pairs of type B). Thus, the block that contains two pairs of type A must contain all six pairs of type B, which is impossible. Hence, there are no blocks that contain two pairs of type A.

We conclude that a block in the design can contain at most five pairs of type A and B. There are 48 pairs that need to be represented, each block holding at most five of these pairs. This requires at least ten blocks making the total number of blocks in the design to be at least 16. This is a contradiction.

Case **2** : Suppose the independent blocks are $B_1 = \{1,3,4,5\}$, $B_2 = \{1,6,7,8\}$, $B_2 = \{1,9,10,11\}$, $C_1 = \{2,12,13,14\}$, $C_2 = \{2,12,15,16\}$, $C_3 = \{2,17,18,19\}$. As there are at most three elements of frequency four or more, we may assume elements 3 and 4 have frequency three. Since element 3 must appear with $\{6,7,8,9,10,11\}$, then the blocks $\{3,x,y,z\}$, $\{3,w,u,v\}$ are forced in the design, where $x,y,z,w,u,v \in \{6,7,8,9,10,11\}$. Similarly element 4 must appear with $\{6,7,8,9,10,11\}$; then the blocks $\{4,x,y,z\}$, $\{4,w,u,v\}$ are forced in the design, where $x,y,z,w,u,v \in \{6,7,8, 9,10,11\}$. Now element 5 is a single and must appear with $\{6,7,8,9,10,11\}$, forcing these elements to appear once more. Hence, $\{6,7,8,9,10,11\}$ must have frequency four or more which contradicts the fact that at most three elements have frequency 4 or more.

Since each case led to a contradiction, we conclude that $L(19,4,3,2) = 16$. □

Theorem 5.2.20 : $L(20,4,3,2) = 18$.

Proof : By Theorem 2.5.5, $L(20,4,3,2) \leq 18$. Assume that $L(20,4,3,2) \leq 17$ and consider a $(20,4,3,2)$ Lotto design with 17 blocks. By Corollary 5.2.3, all elements in the design have frequency two or more. Let us assume there exists an element of frequency two, say the element 1. By Theorem 2.4.2 and Theorem 2.4.1, there must exist an element of frequency four which, together with element 1, make up a maximal independent set. Hence there are at least 13 elements of frequency four or more, some of which must have frequency four. This implies there are at least five elements of frequency two, all of which must occur with the element 1. The independent blocks can now be determined up to isomorphism. The blocks containing the element 1 are $\{1,2,3,4\}$ and $\{1,5,6,7\}$. There are at least two elements of frequency two in one of the blocks, say elements 2 and 3. By Lemma 5.2.1, pairs of the form $\{x,y\}$ where

$x \in \{2, 3, 4\}$ and $y \in \{5, 6, 7\}$, must occur in the design. Thus the blocks $\{2, 5, 6, 7\}$ and $\{3, 5, 6, 7\}$ are forced and elements 5, 6, 7 must occur at least once more with element 4. Hence, elements 5, 6, 7 have frequency four or more and so there are at most three elements of frequency two occurring with element 1, which is a contradiction. This implies there are no elements of frequency two. We know there are at least 12 elements of frequency three and at most eight elements of frequency four or more. Let element 1 have frequency three. Since at most nine elements of frequency three can occur with the element 1, let element 2 have frequency three, and not occur with element 1 in any block of the design. Then without loss of generality, the independent blocks are $B_1 = \{1, 3, 4, 5\}$, $B_2 = \{1, 6, 7, 8\}$, $B_3 = \{1, 9, 10, 11\}$, $B_4 = \{2, 12, 13, 14\}$, $B_5 = \{2, 15, 16, 17\}$, $B_6 = \{2, 18, 19, 20\}$. Since there are at least tweleve elements of frequency three, there must be at least five of them occurring with element 1 or 2, say element 1. There must be at least two of them in B_1, B_2 or B_3. Suppose elements 3 and 4 are two such elements of frequency three. By Lemma 5.2.1, pairs of the form $\{x, y\}$, where $x \in \{3, 4, 5\}$ and $y \in \{6, 7, 8, 9, 10, 11\}$, must occur in the design. This implies element 3 occurs in two more blocks with just enough spaces for elements 6, 7, 8, 9, 10, 11. The same is true for element 4. Now 6, 7, 8, 9, 10, 11 still has to occur with element 5. Hence those six elements have frequency four or more. This implies that there are at most three elements of frequency three occurring with 1. The same is true for element 2. Hence, there are at most eight elements of frequency three in the design, which is a contradiction . Therefore $L(20, 4, 3, 2) = 18$. □

Theorem 5.2.21 : $L(19, 4, 4, 2) = 11$.

Proof: We know that $L(19, 4, 4, 2) = 10$ or 11. Assume that $L(19, 4, 4, 2) = 10$ and consider a $(19, 4, 4, 2)$ Lotto design on 10 blocks. By Corollary 5.2.3, all elements in the design have frequency two or more. Thus there are at least seventeen elements of frequency two and at most two elements of greater frequency. By Theorem 2.4.7, there are three disjoint elements of frequency two, say the elements 1, 8 and 15, forming a maximal independent set of the design. Since every element must appear in one of the six blocks containing the elements 1, 8, 15 and there are at most two elements repeated in the independent blocks other than the independent elements, there are at least fourteen singles. Either one of the cliques has six singles or two of the three cliques have at least five singles. If one of the cliques has six singles, then assume it looks like $\{1, 2, 3, 4\}$, $\{1, 5, 6, 7\}$. Since there are at most two elements of frequency three or more, we may assume elements 2 and 3 have frequency two. Then the blocks $\{2, 5, 6, 7\}$ and $\{3, 5, 6, 7\}$ are forced in the design. Thus, elements 5, 6, 7 have frequency three or more, which is a contradiction. So there cannot be a clique with six singles, and hence there exists two cliques with five singles each. Suppose these two clique look like $\{1, x, 3, 4\}, \{1, 5, 6, 7\}$, $\{8, y, 10, 11\}$ and $\{8, 12, 13, 14\}$, where x, y are not singles. Now the elements 3, 4, 5, 6, 7 must appear in pairs, or else the two elements not appearing in a pair along with 8 and 15 form a 4-set that is not represented. Similarly, elements 10, 11, 12, 13, 14 must appear in pairs. If elements 3 and 4 both have frequency three or more, then elements 10 and 11 must have frequency two, as $f_2 \geq 17$. This forces the blocks $\{10, 12, 13, 14\}$ and $\{11, 12, 13, 14\}$, which gives too many elements of frequency three or more. Hence, at least one of the elements 3 or 4 must have frequency two. If the both have frequency two, then the blocks $\{3, 5, 6, 7\}$ and $\{4, 5, 6, 7\}$ are forced, giving too many elements of frequency three or more. Hence, without loss of generality, assume element 3 has frequency two and element 4 have frequency three or more. As element 3 has frequency two, it forces the block $\{3, 5, 6, 7\}$. As element 4 must appear with 5, 6 and 7, then 5, 6 and 7 must appear again in the design. This implies that 5, 6 and 7 have frequency three or more, which contradicts the fact that the number of elements of frequency three is at most two. Hence, $L(19, 4, 4, 2) = 11$. □

Theorem 5.2.22 : $L(20, 4, 4, 2) = 12$

Proof : By Theorem 5.2.21, $L(20, 4, 4, 2) \geq 11$. By Theorem 2.5.5, $L(20, 4, 4, 2) \leq 12$. Suppose $L(20, 4, 4, 2) = 11$ and consider a $(20, 4, 4, 2)$ Lotto design with 11 blocks. By Corollary 5.2.3, every element in the design has frequency two or more. By analyzing the elements' frequencies, we conclude that there are at least sixteen elements of frequency two in the design. By Theorem 2.4.7, there is a maximal independent set (of size three) consisting only of elements of frequency two. Let these three elements be $\{1, 2, 3\}$. The three cliques look like $\{1, 4, 5, 6\}$, $\{1, 7, 8, 9\}$, $\{2, 10, 11, 12\}$, $\{2, 13, 14, 15\}$, $\{3, 16, 17, 18\}$, $\{3, 19, 20, x\}$ where $x \in \{4, 5, ..., 18\}$. As there are at least 16 elements of frequency two, there is a clique with six singles where at least two of them have frequency two. We may assume that this clique is $\{2, 10, 11, 12\}$, $\{2, 13, 14, 15\}$. Now the two singles of frequency two may be in the same block or in different blocks of the clique. If they are in the same block, say elements 10 and 11, then as element 10 has frequency two and by Lemma 5.2.1, the block $\{10, 13, 14, 15\}$ is forced in the design. Similarly, the block $\{11, 13, 14, 15\}$ is forced in the design. By Lemma 5.2.1, element 12 must also appear with elements 13, 14, 15 in the design. This implies elements 13, 14, 15 must appear at least once more in the design and hence they all have frequency four or more. Thus, the total occurrences of the elements in the design is at least $17(2) + 3(4) = 46$, which is larger than $11(4) = 44$. This is a contradiction, and hence the two singles cannot appear in the same block. Suppose the two singles of frequency two are elements 10 and 13. As element 10 has frequency two and must appear with elements 13, 14, 15, the block $\{10, 13, 14, 15\}$ is forced in the design. Now element 11 must appear with 13, 14, 15. This implies element 13 must appear again, implying it has frequency three or more, which is a contradiction.

As both cases led to a contradiction, we conclude that $L(20, 4, 4, 2) = 12$. □

Theorem 5.2.23 : $L(15, 5, 3, 2) = 7$.

Proof : From Bate's tables, $6 \leq L(15,5,3,2) \leq 7$. Suppose $L(15,5,3,2) = 6$ and consider a $(15, 5, 3, 2)$ Lotto design on six blocks. By Corollary 5.2.3, every element in the design has frequency two or more. There are exactly 15 elements of frequency two. By Theorem 2.4.7, there are at least three disjoint elements of frequency two in the design. These three elements form a 3-set set that cannot be represented in the design. Hence, $L(15,5,3,2) = 7$. □

Theorem 5.2.24 : $L(16, 5, 3, 2) = 8$

Proof : From Bate's tables $L(16,5,3,2) = 7$ or 8. Suppose $L(16,5,3,2) = 7$ and consider a $(16,5,3,2)$ Lotto design on seven blocks. By Corollary 5.2.3, every element in the design has frequency two or more. By analyzing the elements' frequencies, we conclude that there are at least thirteen elements of frequency two in the design. Table 5.2 states all possible frequency distributions of the elements in the design.

Case	f_2	f_3	f_4	f_5	# of disjoint elements of frequency two
A	13	3	0	0	2
B	14	1	1	0	2
C	15	0	0	1	3

Table 5.1: Possible Frequency Distributions for an $LD^*(16, 5, 3, 2; 7)$.

By Theorem 2.4.7, case C implies there are three disjoint elements of frequency two. These three elements form a 3-set that is not represented by the design. Hence case C need not be considered and we only need to consider cases A and B. Thus, the design must have at least one element of frequency three. Let the elements $\{1, 2\}$ be two disjoint elements of frequency two in the design. These two elements form a maximal independent set of the design. Consider the following 7 non-isomorphic ways that the two cliques may look like.

Case 1 : Suppose the two cliques look like $\{1,3,4,5,6\}$, $\{1,7,8,9,10\}$, $\{2,11,12,13,14\}$ and $\{2,3,4,15,16\}$. Suppose one of elements 5 or 6 has frequency two, say element 5. Then, the block $\{5,7,8,9,10\}$ is forced in the design. But element 6 must also appear with elements 7, 8, 9, 10 in the design which implies 7, 8, 9, 10 must appear at least one more time. Hence, the number of elements of frequency three or more is at least four. This is a contradiction. Hence both elements 5 and 6 have frequency at least three. Arguing exactly the same in the other clique, we conclude that elements 15 and 16 must also have frequency at least three. But this implies that there are at least four elements (5, 6, 15, 16) with frequency three or more. This is a contradiction.

Case 2 : Suppose the two cliques look like $\{1,3,4,5,6\}$, $\{1,7,8,9,10\}$, $\{2,3,11,12,13\}$ and $\{2,4,14,15,16\}$. Arguing as in case 1, elements 5 and 6 must have frequency three or more. As the number of elements of frequency three or more is at most three, at least 5 of the elements 11, 12, 13, 14, 15, 16 must have frequency two. Suppose they are 11, 12, 13, 14, 15. Element 11 must appear with elements 14, 15, 16 and hence the block $B = \{11, x, 14, 15, 16\}$ is forced, where x can be some other element. Now as elements 14 and 15 have frequency two and element 12 must appear with them in a block, we see that $x = 12$. Thus $B = \{11, 12, 14, 15, 16\}$. But element 13 must appear with elements 14, 15, 16 which implies elements 14, 15, 16 must appear again in the design. This is a contradiction as we assumed that elements 14 and 15 have frequency two.

Case 3 : Suppose the two cliques look like $\{1,3,4,5,6\}$, $\{1,7,8,9,10\}$, $\{2,3,11,12,13\}$ and $\{2,7,14,15,16\}$. As there are at most three elements of frequency three or more in the design, one of the two cliques will have both its blocks with at most one single with frequency three or more. Then let the elements 4, 5, 8 and 9 have frequency two. As element 4 must appear with elements 8, 9, 10 in the design, the block $B = \{4, x, 8, 9, 10\}$ is forced, where x is some other element. Now as elements 8 and 9 have frequency two and element 5 must appear with them, we see that $x = 5$. But element 6 must also appear with elements 8, 9, 10 in the design. This implies that elements 8, 9, 10 must appear again which is a contradiction as we assumed that elements 8 and 9 had frequency two.

Case 4 : Suppose the two cliques look like $\{1, 3, 4, 5, 6\}$, $\{1, 7, 8, 9, 10\}$, $\{2, 3, 11, 12, 13\}$ and $\{2, 11, 14, 15, 16\}$. As there are at most three elements of frequency three or more in the design, one of the elements 7, 8, 9, 10 must have frequency two, say element 7. As 7 must appear with 4, 5, 6, the block $B = \{4, 5, 6, 7, x\}$ is forced in the design, where x is some other element. As element 4 must appear with 8, 9, 10 in the design, it must have frequency three or more. Similarly, elements 5 and 6 must have frequency three or more. This implies that elements 8, 9 and 10 have frequency two. This forces the blocks $\{4, 5, 6, 7, x\}$ and $\{4, 5, 6, y, z\}$ where $x, y, z \in \{8, 9, 10\}$. The elements in the other clique must all be of frequency two. The block $\{12, 13, 14, 15, 16\}$ is forced. But the 3-set $\{3, 7, 14\}$ needs to be represented. This can be accomplished by having the pair $\{3, 7\}$, $\{3, 14\}$ or $\{7, 14\}$ in the design. But these elements have frequency two and they cannot appear again as they have already appeared twice in the design. So the block $\{3, 7, 14\}$ cannot be represented which is a contradiction.

Case 5 : Suppose the cliques may look like $\{1, 3, 4, 5, 6\}$, $\{1, 3, 7, 8, 9\}$, $\{2, 10, 11, 12, 13\}$, and $\{2, 10, 14, 15, 16\}$. There is a clique that contains a block with three singles of frequency two. Suppose these three singles are the elements 4, 5, 6. As element 4 must appear with elements 7, 8, 9, the block $B = \{4, x, 7, 8, 9\}$ is forced where x is some other element. Now elements 5 and 6 must also appear with elements 7, 8, 9. This implies that elements 7, 8, 9 must appear at least once more in the design. Hence elements 7, 8 and 9 have frequency three or more in the design. As there are at most three elements of frequency three or more in the design, the rest of the elements must have frequency two. As element 11 must appear with elements 14, 15, 16, this forces the block $B = \{11, x, 14, 15, 16\}$. Now as elements 12 and 13 must appear with elements 14, 15 and 16, this causes elements 14, 15 and 16 to appear at least once more. This is a contradiction as we assumed that elements 14, 15 and 16 had frequency two.

Case 6 : Suppose the cliques may look like $\{1,3,4,5,6\}$, $\{1,7,8,9,10\}$, $\{2,11,12,13,14\}$, and $\{2,11,12,15,16\}$. As there are at least 13 elements of frequency two, at least one of the blocks $\{1,3,4,5,6\}$ and $\{1,7,8,9,10\}$ have two or more singles of frequency two. Suppose elements 3 and 4 are singles of frequency two. As element 3 must appear with elements 7, 8, 9 and 10 in the design, the block $\{3,7,8,9,10\}$ is forced. Similarly the block $\{4,7,8,9,10\}$ is forced. Since element 5 must also appear in the design with the elements 7, 8, 9 and 10, these elements must appear once more. Hence, the elements 7, 8, 9 and 10 must have frequency three or more, which is a contradiction.

Case 7 : Suppose the two cliques look like $\{1,3,4,5,6\}$, $\{1,7,8,9,10\}$, $\{2,3,11,12,13\}$ and $\{2,3,14,15,16\}$. As there are at most three elements of frequency three or more in the design, one of the elements 7, 8, 9, 10 must have frequency two, say element 7. As 7 must appear with 4, 5, 6, the block $B = \{4,5,6,7,x\}$ is forced in the design, where x is some other element. As element 4 must appear with 8, 9, 10 in the design, it must have frequency three or more. Similarly, elements 5 and 6 must have frequency three or more. There are at least four elements in the design with frequency three or more, which is a contradiction.

Since each case led to a contradiction, we conclude that there is no $(16, 5, 3, 2)$ Lotto design on seven blocks. Hence, $L(16, 5, 3, 2) = 8$. □

Theorem 5.2.25 : $L(17, 5, 3, 2) = 9$.

Proof : By Theorem 2.5.5, $L(17, 5, 3, 2) \leq 9$. Suppose $L(17, 5, 3, 2) = 8$ and consider a $(17, 5, 3, 2)$ Lotto design with 8 blocks. By Corollary 5.2.3, every element in the design has frequency two or more. By analyzing the elements' frequencies in the design, we conclude there are at least eleven elements of frequency two and at most four elements with frequency three or more. By Theorem 2.4.7, there is a maximal independent set of size two containing two elements of frequency two. Let $\{1, 2\}$ be this maximal independent set. The blocks containing the elements of the independent set are stated in the following two cases :
Case 1 : Let $B_1 = \{1, 3, 4, 5, 6\}$, $B_2 = \{1, 7, 8, 9, 10\}$, $B_3 = \{2, 11, 12, 13, 14\}$ and $B_4 = \{2, 11, 15, 16, 17\}$. There exists an element from the set $\{3, 4, 5, 6, 7, 8, 9, 10\}$, say 3, that has frequency two. Since 3 must appear with the elements 7, 8, 9, 10 in the design, the block $\{3, 7, 8, 9, 10\}$ is forced. Since 4, 5, 6 must appear with 7, 8, 9, 10 in the design, the elements 7, 8, 9, 10 must have frequency three or more. Similarly, an element from the set $\{12, 13, 14, 15, 16, 17\}$, say 12, must have frequency two. Since 12 must appear with the elements 15, 16, 17, the block $\{x, 12, 15, 16, 17\}$ is forced, where x is any other element. Since 13, 14 must also appear with 15, 16, 17 in the design, the elements 15, 16, 17 must have frequency at least three. This is a contradiction as there are too many elements of frequency three or more.

Case 2 : Let $B_1 = \{1, 3, 4, 5, 6\}$, $B_2 = \{1, 7, 8, 9, 10\}$, $B_3 = \{2, 11, 12, 13, 14\}$ and $B_4 = \{2, 3, 15, 16, 17\}$. If the element 3 has frequency two, then the elements 7, 8, 9, 10 must appear with the elements 11, 12, 13, 14 because of 3-sets of the form

$\{3, x, y\}$, where $x \in \{7, 8, 9, 10\}$ and $y \in \{11, 12, 13, 14\}$ must be represented by the design. Also, elements 4, 5, 6 must appear with 7, 8, 9, 10 and elements 11, 12, 13, 14 must appear with 15, 16, 17 in the design. There are $16 + 12 + 12 = 40$ pairs that must appear in the design. Each block may hold at most six of these pairs. Hence these forty pairs require at least an additional seven blocks in the design, which contradicts the assumption that the design has eight blocks. Hence the element 3 must have frequency three or more. As in the previous case, elements 4, 5, 6 must appear with 7, 8, 9, 10 and elements 11, 12, 13, 14 must appear with 15, 16, 17 in the design. There are $12 + 12 = 24$ pairs that must appear in the design. Each block may hold at most six of these pairs. Hence these twenty four pairs require at least an additional four blocks in the design. But since the element 3 must appear again, it requires yet another block. Thus the design contains at least nine blocks which contradicts the that the design has eight blocks.

Since each case led to a contradiction, we conclude that the is no $(17, 5, 3, 2)$ Lotto design on eight blocks. Hence, $L(17, 5, 3, 2) = 9$. □

Theorem 5.2.26 : $L(18, 5, 3, 2) = 10$.

Proof: By Theorem 2.5.5, $L(18, 5, 3, 2) \leq 10$. Assume $L(18, 5, 3, 2) = 9$. Then there is an $(18, 5, 3, 2)$ Lotto design on nine blocks. By Corollary 5.2.3, every element in the design has frequency two or more. By analyzing the elements' frequencies, there are at least nine elements of frequency two and at most nine elements of frequency three or higher. By Theorem 2.4.7, there exists two disjoint elements of frequency two, which form a maximal independent set in the design. By Theorem 2.4.3, every element must appear in the four independent blocks. The four independent blocks look like $\{1, 3, 4, 5, 6\}$, $\{1, 7, 8, 9, 10\}$, $\{2, 11, 12, 13, 14\}$, $\{2, 15, 16, 17, 18\}$. By Theorem 5.2.1, each pair $\{x, y\}$ where $x \in \{3, 4, 5, 6\}$ and $y \in \{7, 8, 9, 10\}$ must appear in the design. Similarly, each pair $\{x, y\}$ where $x \in \{11, 12, 13, 14\}$ and $y \in \{15, 16, 17, 18\}$ must appear in the design. There are 32 pairs listed above that must appear in the design. Each block may hold at most six of these pairs. This mean the design requires at least $\lceil \frac{32}{6} \rceil = 6$ blocks to hold these 32 pairs. But this implies the design must have at least $4 + 6 = 10$ blocks which contradicts our assumption that the design had 9 blocks. Hence, $L(18, 5, 3, 2) = 10$. □

Theorem 5.2.27 : $L(19, 5, 3, 2) = 11$.

Proof : By Theorem 2.5.5, $L(19, 5, 3, 2) \leq 11$. Assume $L(19, 5, 3, 2) = 10$. Then there is a $(19, 5, 3, 2)$ Lotto design on 10 blocks. By Corollary 5.2.3, every element in the design has frequency two or more. By analyzing the elements' frequencies, there are at least seven elements of frequency two and at most twelve elements of frequency three or higher. If there were more than seven elements of frequency two, we would have two disjoint elements of frequency two which would form a maximum independent set, as $p = 3$. By Theorem 2.4.3, every element must appear in the four independent blocks, which is impossible. Hence there are exactly seven elements of frequency two and exactly twelve elements of frequency three. Now the seven elements of frequency two can only appear one non-isomorphic way : $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 6, 7\}$, $\{4, 5, 6, 7, 8\}$ where the element 8 has frequency three and elements 1, 2, ..., 7 have frequency two. So $\{1, 8\}$ forms an maximum independent set. However, the remaining two blocks containing element 8 can contain only eight other elements. Theorem 2.4.3 requires that these two blocks hold all of the remaining 11 elements. Hence, we have a contradiction and $L(19, 5, 3, 2) = 11$. □

Theorem 5.2.28 : $L(20, 5, 3, 2) = 12$.

Proof : Assume $L(20, 5, 3, 2) \leq 11$. Then there exists a $(20, 5, 3, 2)$ Lotto design on 11 blocks. By Corollary 5.2.3, every element in the design has frequency two or more. By analyzing the elements' frequencies, there are at least five elements of frequency two and at most fifteen elements of frequency three or more. If there are more than seven elements of frequency two, then by Theorem 2.4.7, there are two disjoint elements of frequency two. These can occur with at most $4(4) = 16$ elements, contradicting Theorem 2.4.2. So $5 \leq f_2 \leq 7$ and there cannot be any disjoint elements of frequency two. We consider each possible value of f_2.

If $f_2 = 7$, let the elements 1, 2, 3, 4, 5, 6 and 7 be elements of frequency two and suppose element 1 is part of an independent set of the design. Then the three blocks determined by the element 1 (up to isomorphism) are $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 6, 7\}$ and $\{4, 5, 6, 7, x\}$, where x is an element of frequency three. The remaining blocks of the design must form an $(13, 5, 2, 2)$ Lotto design. This requires at least ten blocks for a total of at least thirteen blocks which is a contradiction.

If $f_2 = 6$, then all elements of frequency two must occur together in some 2-clique. But this cannot happen to all of them in the same block. There are at most two non-single elements occurring in any independent set of blocks. By examining possibilities, we see there are three non-isomorphic ways that the elements of frequency two can occur. The number of elements of frequency two in a block (if not 0) is $\{5, 5, 2\}$, $\{5, 4, 3\}$ or $\{3, 3, 3\}$. We label these case a, case b and case c, respectively. Let elements 1 to 6 have frequency two. We consider each case.

Case a : If the number of elements of frequency two in a block is $\{5, 5, 2\}$. Then consider the three blocks $B_1 = \{1, 2, 3, 4, 5\}$, $B_2 = \{1, 6, 7, 8, 9\}$ and $B_3 = \{2, 3, 4, 5, 6\}$ in the design. B_1 and B_3 share 3 non-singles, which a contradiction.

Case b : If the number of elements of frequency two in a block is $\{5, 4, 3\}$, then consider the three blocks $B_1 = \{1, 2, 3, 4, 5\}$, $B_2 = \{1, 2, 3, 6, x_1\}$ and $B_3 = \{4, 5, 6, x_2, x_3\}$ in the design. It may be that $x_3 = x_1$, but $x_1 \neq x_2$. Now $\{1, x_2\}$ forms an independent set and all elements must occur with it, forcing $\{x_2, 9, 10, 11, 12\}$, $\{x_2, 13, 14, 15, 16\}$, $\{x_2, 17, 18, 19, 20\}$. Now elements 9 through 20 are singles and there must be 48 other pairs of singles that must be in the design (for example $\{9, 13\}$, $\{9, 17\}$, $\{10, 20\}$, etc). The most that can occur in one block is eight pairs. So there must be at least $48/8 = 6$ more blocks. Hence, the design must have at least 12 blocks in all, which is a contradiction.

Case c : If the number of elements of frequency two in a block is $\{4, 4, 4\}$, then consider the three blocks $B_1 = \{1, 2, 3, 4, x_1\}$, $B_2 = \{1, 2, 5, 6, x_2\}$ and $B_3 = \{3, 4, 5, 6, x_3\}$ in the design, where x_1, x_2 and x_3 are not necessarily distinct. If $x_1 = x_2 = x_3$, then the remaining 13 elements form a covering which takes at least 10 blocks as $L(13, 5, 2, 2) = 10$. This is a total of 13 blocks which is a contradiction. Suppose $x_1 = x_2 \neq x_3$. Let y be an element of frequency three so that $\{1, y\}$ is an independent set. Since $\{1, x_3, y\}$ must be represented, the pairs $\{x_3, y\}$ must occur in a block. Similarly, for $\{1, x_3, x_4\}$ where $x_4 > 6, x_4 \neq x_1, x_3$ or y, it must be represented be represented by the pair $\{x_3, x_4\}$. So all pairs of x_3, y and the other 11 elements must occur in some block. This must take at least $L(13, 5, 2, 2) = 10$ blocks. So there are a total of at least $10 + 3 = 13$ blocks, which is a contradiction. If x_1, x_2 and x_3 are distinct, we get a similar contradiction.

If $f_2 = 5$, then $f_3 = 15$ and $f_i = 0$ for all other i. Consider the situation where all elements of frequency two occur in one block $\{1, 2, 3, 4, 5\}$. We list the possible non-isomorphic cases.

Case 1a : Suppose two of the blocks in the design are $\{1,2,3,4,5\}$ and $\{1,6,7,8,9\}$ and there are no non-singles in these two blocks. Then the blocks $\{2,6,7,8,9\}$, $\{3,6,7,8,9\}$, $\{4,6,7,8,9\}$ and $\{5,6,7,8,9\}$ are forced in the design. This implies that the elements 6, 7, 8, and 9 occur too frequently, which is a contradiction.

Case 1b : Suppose two of the blocks in the design are $\{1,2,3,4,5\}$ and $\{1,6,7,8,9\}$ and element 9 is the only non-single. Without loss of generality, the blocks $\{2,3,6,7,8\}$ and $\{4,5,6,7,8\}$ are forced. Now the 3-set $\{3,9,i\}$ must be represented by $\{9,i\}$ where $i \geq 10$. But in two more blocks there is room for only eight such pairs.

Case 1c : Suppose two of the blocks in the design are $\{1,2,3,4,5\}$ and $\{1,6,7,8,9\}$ and element 2 is the only non-single. So the blocks $\{3,6,7,8,9\}$, $\{4,6,7,8,9\}$ and $\{5,6,7,8,9\}$ are forced. The elements 6, 7, 8 and 9 occur too frequently, which is a contradiction.

Case 1d : Suppose two of the blocks in the design are $\{1,2,3,4,5\}$ and $\{1,6,7,8,9\}$ where elements 8 and 9 are the only non-singles. Two blocks are forced in two non-isomorphic ways. They are $\{2,3,4,6,7\}$, $\{5,6,7,x_1,x_2\}$ and $\{2,3,6,7,x_1\}$, $\{4,5,6,7,x_2\}$. In the first case, 3-sets of the form $\{2,8,i\}$ must be represented by the pair $\{8,i\}$, where $i \geq 10$. But there is only room for at most eight such pairs in the remaining two blocks containing the element 8, which is a contradiction. In the other case, if $x_1 \neq 8$, 3-sets of the form $\{2,8,i\}$, for $i \geq 10$, must be represented by $\{8,i\}$, except for perhaps when $i = x_1$. There are at least nine of these pairs that must occur in two blocks (as the element 8 has frequency three), which is impossible. If $x_1 = 8$, 3-sets of the form $\{2,9,i\}$, for $i \geq 10$, must be represented by $\{8,i\}$. There are 11 of these pairs that must occur in two blocks that contain the element 9, which is impossible.

Case 1e : Suppose two of the blocks in the design are $\{1,2,3,4,5\}$ and $\{1,6,7,8,9\}$, where elements 5 and 9 are the only non-singles. Without loss of generality, the blocks $\{2,3,6,7,8\}$ and $\{4,6,7,8,x\}$ are forced where x is some element. Now the 3-set $\{2,9,i\}$ is represented by the pair $\{9,i\}$ for $i \geq 10$. But there is not enough room to hold these pairs.

Case 1f : Suppose two of the blocks in the design are $\{1,2,3,4,5\}$ and $\{1,6,7,8,9\}$, where elements 4 and 5 are the only non-singles. Without loss of generality, the blocks $\{2,6,7,8,9\}$ and $\{3,6,7,8,9\}$ are forced. Let the element 10 be an element of frequency three and let 11 be an element of frequency three that does not occur with 4 or 5. But then, the 3-set $\{4,8,11\}$ is not represented, which is a contradiction.

Case 1g : Suppose two of the blocks in the design are $\{1,2,3,4,5\}$ and $\{1,6,7,8,9\}$, where elements 3, 4, 5, 6, 7, 8 are singles. Without loss of generality, the blocks $\{3,4,6,7,8\}$ and $\{5,6,7,8,x\}$ are forced. Then $\{1,9,i\}$ must be represented by $\{9,i\}$ for $i \geq 10$. But there is not enough room.

Case 1h : Suppose two of the blocks in the design are $\{1,2,3,4,5\}$ and $\{1,2,6,7,8\}$, where 8 is the only non-single. There are two non-isomorphic sub cases determined by blocks that are forced by the two blocks above : i) $\{3,4,5,6,7\}$ and ii) $\{3,4,6,7,x_1\}$, $\{5,6,7,x_2,x_3\}$. In the first sub case, 3-sets of the form $\{3,8,i\}$, for $i \geq 9$, must be represented by the pair $\{8,i\}$. But there is no room to hold all such pairs. In the second sub case, let the element 9 belong in $\{x_1,x_2,x_3\}$. Then 3-sets of the form $\{1,9,i\}$ for $i \geq 10$ must be represented by the pair $\{9,i\}$. Again there is not enough room to hold all such pairs.

Case 1i : Suppose two of the blocks in the design are $\{1,2,3,4,5\}$ and $\{1,2,6,7,8\}$ where 5 is the only non-single (other than 2). Then the blocks $\{3,4,6,7,8\}$, $\{5,9,10,11,12\}$, $\{9,13,14,15,16\}$ and $\{9,17,18,19,20\}$ are forced. Then, 3-sets of the form $\{5,8,i\}$ for $i \geq 13$ must be represented by the pair $\{8,i\}$. There is not enough room to hold all such pairs.

Case 1j : Suppose two of the blocks in the design are $\{1,2,3,4,5\}$ and $\{1,2,3,6,7\}$. This forces the block $\{4,5,6,7,x\}$ where x is some element. Without loss of generality, let $x = 8$; then 3-sets of the form $\{1,8,i\}$ for $i \geq 9$ must be represented by the pair $\{8,i\}$. There is not enough room to hold all such pairs.

The remaining two non-isomorphic cases do not have all the frequency two elements in a single block. We list them as case 2 and case 3.

Case 2 : Suppose the blocks of the design include $\{1,2,3,4,6\}$, $\{1,2,3,5,7\}$, $\{4,5,x_1,x_2,x_3\}$, $\{8,9,10,11,12\}$, $\{8,13,14,15,16\}$ and $\{8,17,18,19,20\}$. The elements 17, 18, 19 and 20 must appear two more times, which is a contradiction.

Case 3 : Suppose the blocks of the design include $\{1,2,3,4,6\}$, $\{1,2,5,7,8\}$, $\{9,10,11,12,13\}$, $\{9,14,15,16,17\}$ and $\{9,18,19,20,x\}$. If $x = 6, 7$ or 8, then use an argument like that of case 2 to get a contradiction. If $x = 10$, then element 14 must appear with elements 10, 11, 12, 13, 18, 19 and 20 with one spot left over. Suppose 15 does not occur with 14. It also appears in two blocks with 10, 11, 12,13, 18, 19, 20 and one space left over. Consider one of 16 or 17 that does not occur in both spots. It must appear in another block with four of 10, 11, 12, 13, 18, 19, 20. But those elements appear four times, which is a contradiction.

Since we have considered all possible cases and each gave a contradiction, $L(20,5,3,2) = 12$. □

Theorem 5.2.29 : $L(20,6,3,2) = 8$.

Proof: Since we know that $L(20,6,3,2) = 7$ or 8, it suffices to show that $L(20,6,3,2) > 7$. Assume $L(20,6,3,2) = 7$ and consider a $(20,6,3,2)$ Lotto design with seven blocks. By Corollary 5.2.3, all elements in the design must have frequency two or more. The number of elements of frequency two must be at least eighteen and the number of elements of frequency three or more is at most two. There are two possible frequency distributions :

1. $f_2 = 18$ and $f_3 = 2$,
2. $f_2 = 19$ and $f_4 = 1$.

By Theorem 2.4.7, the latter distribution implies there are three disjoint elements of frequency two which is not represented by the design. So there must be exactly 18 elements of frequency two and two elements of frequency three. By Theorem 2.4.7, there exist two disjoint elements of frequency two, say 1 and 2. These two elements form a maximal independent set of the design. We consider each non-isomorphic configurations of the two cliques.

Case 1 : Let the two cliques look like $B_1 = \{1,3,4,5,6,7\}$, $B_2 = \{1,8,9,10,11,12\}$, $B_3 = \{2,13,14,15,16,17\}$ and $B_4 = \{2,3,4,18,19,20\}$. As the number of elements of frequency two is at least 18, either B_1 or B_4 has two singles of frequency two. Suppose elements 5 and 6 of B_1 are 2 such singles. Then the block $\{5,8,9,10,11,12\}$ and $\{6,8,9,10,11,12\}$ are forced in the design, by Lemma 5.2.1. Hence elements 8, 9, 10, 11, 12 have frequency three or more, which is a contradiction.

Case 2 : Let the two cliques look like $B_1 = \{1,3,4,5,6,7\}$, $B_2 = \{1,8,9,10,11,12\}$, $B_3 = \{2,3,13,14,15,16\}$ and $B_4 = \{2,4,17,18,19,20\}$. As the number of elements of frequency two is at least eighteen, one of elements 5, 6 or 7 must have frequency two. Suppose element 5 has frequency two. Then the block $\{5,8,9,10,11,12\}$ is forced in the design. Since element 6 must also appear with elements 8, 9, 10, 11, 12, we conclude that elements 8, 9, 10, 11, 12 must have frequency three or more, which is a contradiction.

Case 3 : Let the two cliques look like $B_1 = \{1,3,4,5,6,7\}$, $B_2 = \{1,8,9,10,11,12\}$, $B_3 = \{2,3,13,14,15,16\}$ and $B_4 = \{2,8,17,18,19,20\}$. The block B_1 has at least two singles that have frequency two. Suppose elements 5 and 6 are two such singles. Similarly the block B_2 has at least two singles, say elements 9 and 10, that have frequency two. As the element 5 must appear with elements 9, 10, 11, 12, the block $B = \{5,x,9,10,11,12\}$ is forced where x is some other element. As elements 6 and 7 must also appear with elements 9, 10, 11, 12, the elements 9, 10, 11, 12 must appear at least once more in the design. Hence they must have frequency three or more, which is a contradiction.

Case 4 : Let the two cliques look like $B_1 = \{1,3,4,5,6,7\}$, $B_2 = \{1,8,9,10,11,12\}$, $B_3 = \{2,3,13,14,15,16\}$ and $B_4 = \{2,13,17,18,19,20\}$. The argument for case 3 can be used to handle this case.

Case 5 : Let the two cliques look like $\{1,3,4,5,6,7\}$, $\{1,3,8,9,10,11\}$, $\{2,12,13,14,15,16\}$ and $\{2,12,17,18,19,20\}$. Since there are at most two elements of frequency three, every element in one of the sets $\{4,5,6,7\}$, $\{8,9,10,11\}$, $\{13,14,15,16\}$ or $\{17,18,19,20\}$

must have frequency two. Without loss of generality, suppose it is the set $\{4, 5, 6, 7\}$. Thus, 4 ,5 ,6 ,7 must appear with 8, 9, 10, 11. This requires that the elements 8, 9, 10, 11 appear at least two more times in the design. Hence elements 8, 9, 10, 11 have frequency three or more. This is a contradiction since there is suppose to be at most two elements of frequency three in the design.

Case 6 : Let the two cliques look like $B_1 = \{1, 3, 4, 5, 6, 7\}$, $B_2 = \{1, 8, 9, 10, 11, 12\}$, $B_3 = \{2, 13, 14, 15, 16, 17\}$ and $B_4 = \{2, 13, 14, 18, 19, 20\}$. Since there are only two elements of frequency three, at least two elements from $\{3, 4, 5, 6, 7\}$ must have frequency two. Suppose elements 3 and 4 have frequency two. Since they must appear with elements in $\{8, 9, 10, 11, 12\}$, the block $\{3, 8, 9, 10, 11, 12\}$ is forced in the design. Similarly, the block $\{4, 8, 9, 10, 11, 12\}$ is forced. Now element 5 must also appear with the elements in $\{8, 9, 10, 11, 12\}$. This implies the elements in $\{8, 9, 10, 11, 12\}$ have frequency three or more. This contradicts the assumption that there are only two elements of frequency three.

Case 7 : Let the two cliques look like $B_1 = \{1, 3, 4, 5, 6, 7\}$, $B_2 = \{1, 8, 9, 10, 11, 12\}$, $B_3 = \{2, 3, 13, 14, 15, 16\}$ and $B_4 = \{2, 3, 17, 18, 19, 20\}$. The argument for case 3 can be used to handle this case.

Since every case led to a contradiction, we have $L(20, 6, 3, 2) = 8$. □

Theorem 5.2.30 *:* $L(11, 5, 4, 3) \geq 8$.

Proof : From Bate's thesis [1], $L(11, 5, 4, 3) \geq 7$. Suppose that $L(11, 5, 4, 3) = 7$. Let $\mathcal{B}$ be an optimal $(11, 5, 4, 3)$ Lotto design . If a pair, say $\{1, 2\}$, does not appear in the design, there are $C(9, 2) = 36$ 4-sets containing $\{1, 2\}$. A 4-set $\{1, 2, x, y\}$ must be represented by a block containing $\{1, x, y\}$ or $\{2, x, y\}$. In this case $\mathcal{B}$ must also be a $(9, 4, 2, 2)$ Lotto design. But since $L(9, 4, 2, 2) = 8$, then all pairs must appear in $\mathcal{B}$ if $L(11, 5, 4, 3) = 7$. This implies that $f_1 = f_2 = 0$. Hence there are at least 33 occurrences given at least seven blocks. Consider an element, say 1 that occurs three times. We have the following three possible cases pertaining to the blocks containing the element 1:

Case 1 : The blocks containing element 1 look like $\{1, 2, 3, 4, 5\}$, $\{1, 2, 6, 7, 8\}$ and $\{1, 2, 9, 10, 11\}$. Blocks of the form $\{1, x, y, z\}$ where $x \in \{3, 4, 5\}, y \in \{6, 7, 8\}$ and $z \in \{9, 10, 11\}$ must be represented by a block in $\mathcal{B}$ containing $\{x, y, z\}$. There are 3*3*3 = 27 such triples. Since each block can hold at most four such triples, the minimum number of blocks in $\mathcal{B}$ in this case is 8 which is a contradiction.

Case 2 : The blocks containing element 1 look like $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 6, 7\}$ and $\{1, 8, 9, 10, 11\}$. The number of triples that need to be in the design is $2 * 2 * 4 = 16$, which requires at least four blocks. But element 2 must also occur with element 8 in the design, which requires another block. So there are more than 7 blocks in the design, which is a contradiction.

Case 3 : The blocks containing element 1 look like $\{1, 2, 3, 4, 5\}$, $\{1, 2, 6, 7, 8\}$ and $\{1, 3, 9, 10, 11\}$. The number of triples that must be represented by a block of the design is $2 * 3 * 3 = 18$. This requires at least five blocks and hence the design has at least 8 blocks in this case, which is a contradiction.

Since we have shown that no $(11, 5, 4, 3)$ Lotto design can 5, 6 or 7 blocks, then $L(11, 5, 4, 3)$ is at least 8. □

Theorem 5.2.31 : $L(9,4,4,3) = 9$.

Proof : From Bate [1], $L(9,4,4,3) \geq 8$ and by Simulated Annealing, we were able to show that $L(9,4,4,3) \leq 9$. Assume $L(9,4,4,3) = 8$ and $\mathcal{B}$ is an optimal $(9,4,4,3)$ Lotto design. Since $9 * 4 = 36 > 32$, there are elements of frequency three or less. By Theorem 5.2.1, there cannot be a element of frequency zero, since $L(8,4,3,3) = 14$. Let there be an element of frequency one, say the element 1 that occur in block $\{1,2,3,4\}$. Consider 4-sets of the form $\{1,x,y,z\}$, where $x \in \{2,3,4\}$ and $y,z,w \in \{5,6,7,8,9\}$. Such a 4-set must be represented by a block in $\mathcal{B}$ that contains the triple $\{x_1,y_1,y_2\}$ where $x_1 \in \{2,3,4\}$ and $y_1,y_2 \in \{5,6,7,8,9\}$. There are $3 * C(5,2) = 30$ such triples. Also, consider 4-sets of the form $\{1,y,z,w\}$. Such a 4-set must be represented by a block in $\mathcal{B}$ that contains the triple $\{y,z,w\}$. There are at least $30 + 10 = 40$ triples that must appear in the design $\mathcal{B}$. But each block can hold at most four of these triples, and hence $\mathcal{B}$ must have at least ten blocks. This contradicts our assumption that the size of $\mathcal{B}$ was at most 8. Hence there are no elements of frequency one.

If there is an element of frequency two, say the element 1, then consider the three possible cases :

Case 1 : Suppose the blocks containing the element 1 look like $\{1,2,3,4\}$ and $\{1,5,6,7\}$. If $x \in \{2,3,4\}, y \in \{5,6,7\}$ and $z \in \{8,9\}$, then a 4-set of the form $\{1,x,y,z\}$ must be represented by a block in $\mathcal{B}$ that contains $\{x,y,z\}$. There are $3 * 3 * 2 = 18$ such triples that must appear in $\mathcal{B}$. Only two such triples can fit into a single block. So there are at least 11 blocks in $\mathcal{B}$, which is a contradiction.

Case 2 : Suppose the blocks containing the element 1 look like $\{1,2,3,4\}$ and $\{1,2,5,6\}$. If $x \in \{3,4\}, y \in \{5,6\}$ and $z_1,z_2 \in \{7,8,9\}$, then, as in case 1, twelve triples of the form $\{x,y,z_1\}$ must appear in the blocks of $\mathcal{B}$. Similarly, 6 triples of the form $\{y, z_1, z_2\}$ and 6 triples of the form $\{x, z_1, z_2\}$ must appear in the blocks of $\mathcal{B}$. Now consider a 4-set of the form $\{1, 2, z_1, z_2\}$. It must be represented by a block that contains the triple $\{2, z_1, z_2\}$ and there are three of these triples. Hence $\mathcal{B}$ contains at least

$$\left\lceil \frac{12+6+6+3}{4} \right\rceil = 9$$

blocks, which is a contradiction.

Case 3 : Suppose the blocks containing the element 1 look like $\{1, 2, 3, 4\}$ and $\{1, 2, 3, 5\}$. If $x_1, x_2, x_3 \in \{6, 7, 8, 9\}$, the following table summarizes the types and number of triples that must appear in $\mathcal{B}$ in order for the given 4-sets to be represented.

4-sets to be represented	triple required	# of such triples
$\{1,4,5,x_1\}, \{2,4,5,x_1\}, \{3,4,5,x_1\}$	$\{4,5,x_1\}$	4
$\{1,4,x_1,x_2\}$	$\{4,x_1,x_2\}$	6
$\{1,5,x_1,x_2\}$	$\{5,x_1,x_2\}$	6
$\{1,x_1,x_2,x_3\}$	$\{x_1,x_2,x_3\}$	4
$\{1,2,x_1,x_2\}, \{2,3,x_1,x_2\}$	$\{2,x_1,x_2\}$	6
$\{1,3,x_1,x_2\}$	$\{3,x_1,x_2\}$	6

Form the above table, the thirty-two triples stated must appear in the design $\mathcal{B}$. Since each block can hold at most four triples, this requires at least eight blocks. These eight blocks along with the two blocks containing the element 1 makes the size of $\mathcal{B}$ at least ten, which is a contradiction.

Since all three cases give a contradiction, we conclude that there are no elements of frequency two. Thus, every element has frequency three or greater.

Now either $\mathcal{B}$ contains all pairs or some pair of elements is missing. If every pair occurs in $\mathcal{B}$ then, without loss of generality, three of the blocks in $\mathcal{B}$ look like $\{1, 2, 3, 4\}$,

$\{1,2,5,6\}$ and $\{1,7,8,9\}$. where the element 1 has frequency three since every element in the design must appear with the element 1. If $x \in \{3,4\}$, $y \in \{5,6\}$ and $z \in \{7,8,9\}$, then a 4-set of the form $\{1,x,y,z\}$ must be represented by a block in $\mathcal{B}$ that contains the triple $\{x,y,z\}$ and there are twelve such triples. It can be easily seen that a block in $\mathcal{B}$ can contain at most two triples of this form. Hence six blocks in $\mathcal{B}$ are required to contain these triples. These six blocks along with the three blocks containing the element 1 mean that $\mathcal{B}$ has at least nine blocks, which contradicts the assumption that the size of $\mathcal{B}$ is at most eight. Hence, we may assume that there is a missing pair, say $\{1,2\}$ in $\mathcal{B}$.

Since $\{1,2\}$ does not appear in the design, 4-sets of the form $\{1,2,x_1,x_2\}$ where $x_1, x_2 \in \{3,4,5,6,7,8,9\}$ must be represented by a set that contains either $\{1,x_1,x_2\}$ or $\{2,x_1,x_2\}$. There are twenty-one of these triples. These twenty-one triples can be contained in (the unique) seven blocks in the $(7,7,3,3,1)$ BIBD (Fano Plane). In this case, one of x or y occurs only three times in the seven blocks. The seven blocks look like $\{a,3,4,6\}, \{a,4,5,7\}, \{a,5,6,8\}$, $\{a,6,7,9\}, \{a,3,7,8\}$, $\{a,4,8,9\}$ and $\{a,3,5,9\}$, where a may be either 1 or 2. But then the sets $\{3,5,6,7\}, \{3,4,5,8\}$, $\{4,5,6,9\}$ and $\{5,7,8,9\}$ must be represented by the last block of the design, which is impossible. Hence the twenty-one triples must be contained in all eight blocks. But consider a 4-set of the form $\{x,y,z,w\}$, where $x,y,z,w \notin \{1,2\}$. There are thirty-five such 4-sets that must be represented. Each of the eight blocks can represent at most 4 of these 4-sets. Thus, the eight blocks together can represented at most thirty-two of the thirty-five 4-sets. This is a contradiction. Hence, $L(9,4,4,3) = 9$. $\square$

5.3 Conclusion

This Chapter improved lower bounds for $L(n, k, p, t)$ on a case by case basis. We applied the results from Bate and van Rees[2] to deal with designs where $t = 2$. The results obtained were used to complete the gaps in Bate's tables and find values for other small Lotto designs. The techniques shown in this Chapter may be applied to improve lower bounds of other designs not discussed in this Chapter.

Chapter 6

Computer Programs

6.1 Introduction

This chapter will present several algorithms that may be used to generate upper bounds for $L(n, k, p, t)$. Greedy algorithms are first presented. These algorithms generate a Lotto design by selecting blocks to form a Lotto design based on some ordering on the k-sets and p-sets. The orderings that we considered were lexicographical, reverse lexicographical and colexicographical (colex) ordering. Techniques for optimizing these searches are also discussed.

Another approach similar to the greedy approach is to randomly pick blocks to form a Lotto design. Such designs generally do not give a good upper bound. However, random algorithms are fast and if they are executed frequently enough, decent upper bounds may be obtained.

An exhaustive search algorithm incorporating isomorphic rejection was discussed in Chapter 2. By analyzing the frequencies of elements that appear in a Lotto design, we can use this information to generate a search algorithm that is faster than an exhaustive search. We present a search based on this idea.

Finally, we briefly discuss how integer linear programming may be used to compute $L(n, k, p, t)$.

6.2 Greedy Algorithms

Greedy algorithms are algorithms that solve a problem that requires making a sequence of choices. Each choice is made such that it is the best possible at that point, without regard to how it affects future choices that will make up the solution. Greedy algorithms in general do not generate optimal solutions [17]. Greedy algorithms may be applied to many combinatorial problems such as the "Traveling Salesman Problem", computing the minimum spanning tree of a graph and the construction of Covering and Turàn designs. We can use greedy algorithms to generate Lotto designs. Here is the basic algorithm :

Algorithm 6.2.1 *:*

void GreedyConstruction(n,k,p,t)

{

/ input : n,k,p,t */*

/ output : A Lotto design */*

1) Order the k-sets and p-sets using some ordering.

2) Create data structures for use in the algorithm.

3) Initialize $C = \emptyset$ (it will hold the k-sets picked for the design).

4) While there are still p-sets not represented

{

4a) Compute the first k-set not in C that represents the most p-sets not yet represented and add it to C.

4b) Mark all p-sets represented by this k-set as represented.

4c) Update the number of p-sets still not represented.

}

5) C is now a (n, k, p, t) Lotto design. Remove redundant blocks from C.

}

The algorithm stated above is quite simple to understand and implement. Our goal is to make this algorithm run as fast as possible. It is possible to speed up the algorithm by using more memory (storage). Before describing the algorithm, we would like to formally define lexigraphic, colexigraphic and reverse lexicographical orderings.

Definition 6.2.1 : *Let $(x_1, x_2, ..., x_k)$ and $(y_1, y_2, ..., y_k)$ be two k-tuples where $x_1 < x_2 < ... < x_k$, $y_1 < y_2 < ... < y_k$. We say that $(x_1, x_2, ..., x_k)$ is before $(y_1, y_2, ..., y_k)$ in the* **lexicographical ordering** *if for some $0 \leq j \leq k-1$, $(x_1, x_2, ..., x_j) = (y_1, y_2, ..., y_j)$ and $x_{j+1} < y_{j+1}$.*

Definition 6.2.2 *Let $(x_1, x_2, ..., x_k)$ and $(y_1, y_2, ..., y_k)$ be two k-tuples where $x_1 < x_2 < ... < x_k$, $y_1 < y_2 < ... < y_k$. We say that $(x_1, x_2, ..., x_k)$ is before $(y_1, y_2, ..., y_k)$ in the* **colexicographical ordering** *if either 1) for some $1 \leq j \leq k-1$, $(x_{j+1}, x_{j+2}, ..., x_k) = (y_{j+1}, y_{j+2}, ..., y_k)$ and $x_j < y_j$ or 2) $x_k < y_k$.*

Definition 6.2.3 : *Let $(x_1, x_2, ..., x_k)$ and $(y_1, y_2, ..., y_k)$ be two k-tuples where $x_1 < x_2 < ... < x_k$, $y_1 < y_2 < ... < y_k$. We say that $(x_1, x_2, ..., x_k)$ is before $(y_1, y_2, ..., y_k)$ in the* **reverse lexicographical ordering** *if for some $0 \leq j \leq k-1$, $(x_1, x_2, ..., x_j) = (y_1, y_2, ..., y_j)$ and $x_{j+1} > y_{j+1}$.*

We now give a detailed description of the algorithm and the data structures used in the algorithm.

In step 1, we need to define the ordering of the k-sets and the p-sets. The orderings that we considered were lexicographical, reverse lexicographical and colexicographical. k-sets and p-sets are associated with a non-negative integer that denotes the "rank" of the k-set or p-set with respect to the chosen ordering. The ranks are between 0 and $C(n,k) - 1$ for k-sets and between 0 and $C(n,p) - 1$ for p-sets. Throughout the algorithm, it is necessary to compute the rank of a set and vice versa. Thus we need a method for performing this task. Bàte [1] discusses a method for generating rank and sets using lexicographical ordering, while [21] discusses a method for generating rank and sets using co-lexicographical ordering. Tables 6.1, 6.2 and 6.3 show the ordering of 3-subsets of $X(6)$ using the lexicographical , reverse lexicographical and colexicographical orderings respectively.

In step 2, we define some data structures that are required by the algorithm. We require an array of integers indexed by the ranks of the k-sets. The value at a particular index of the array is non-zero if the k-set whose rank is that index is in the design. It has the value zero otherwise. We shall called this array *kSetIndex*. You may think of this array as a Boolean array that determines if a k-set is in the design or not. Similarly we require an array of integers indexed by the ranks of the p-sets. The value at a particular index of the array is non-zero if the p-set whose rank is that index is represented by the k-sets in the design. It is zero otherwise. We shall called this array *pSetIndex*. The actual value in this array represents how many times that p-set has been represented. That is, how many of the k-sets in the constructed design represent that p-set.

The design is initially empty. k-sets will be added to it until it becomes a Lotto design. In step 3, we initialize the set C to the empty set. C will contain the blocks

rank	set
0	$\{1,2,3\}$
1	$\{1,2,4\}$
2	$\{1,2,5\}$
3	$\{1,2,6\}$
4	$\{1,3,4\}$
5	$\{1,3,5\}$
6	$\{1,3,6\}$
7	$\{1,4,5\}$
8	$\{1,4,6\}$
9	$\{1,5,6\}$
10	$\{2,3,4\}$
11	$\{2,3,5\}$
12	$\{2,3,6\}$
13	$\{2,4,5\}$
14	$\{2,4,6\}$
15	$\{2,5,6\}$
16	$\{3,4,5\}$
17	$\{3,4,6\}$
18	$\{3,5,6\}$
19	$\{4,5,6\}$

Table 6.1: Lexicographical ordering of 3-subsets of $X(6)$

rank	set
0	$\{4,5,6\}$
1	$\{3,5,6\}$
2	$\{3,4,6\}$
3	$\{3,4,5\}$
4	$\{2,5,6\}$
5	$\{2,4,6\}$
6	$\{2,4,5\}$
7	$\{2,3,6\}$
8	$\{2,3,5\}$
9	$\{2,3,4\}$
10	$\{1,5,6\}$
11	$\{1,4,6\}$
12	$\{1,4,5\}$
13	$\{1,3,6\}$
14	$\{1,3,5\}$
15	$\{1,3,4\}$
16	$\{1,2,6\}$
17	$\{1,2,5\}$
18	$\{1,2,4\}$
19	$\{1,2,3\}$

Table 6.2: Reverse lexicographical ordering of 3-subsets of $X(6)$

rank	set
0	$\{1,2,3\}$
1	$\{1,2,4\}$
2	$\{1,3,4\}$
3	$\{2,3,4\}$
4	$\{1,2,5\}$
5	$\{1,3,5\}$
6	$\{2,3,5\}$
7	$\{1,4,5\}$
8	$\{2,4,5\}$
9	$\{3,4,5\}$
10	$\{1,2,6\}$
11	$\{1,3,6\}$
12	$\{2,3,6\}$
13	$\{1,4,6\}$
14	$\{2,4,6\}$
15	$\{3,4,6\}$
16	$\{1,5,6\}$
17	$\{2,5,6\}$
18	$\{3,5,6\}$
19	$\{4,5,6\}$

Table 6.3: Colexicographical ordering of 3-subsets of $X(6)$

in the constructed design.

Step 4 is the most time consuming step of the algorithm. This step must execute until all p-sets are represented, that is, until a Lotto design is constructed. We discuss each of the sub-steps in 4 individually, discussing techniques for speeding up each step where possible.

In step 4a, the algorithm computes the k-set with lowest rank that represents the most p-sets not previously represented by the k-sets in C. This requires that almost every k-set be checked as a possible choice to be in C. The obvious thing to do here is consider every k-set. However, if we store the number of p-sets represented by the k-set chosen in the last iteration of step 4a, then it is not possible to find a k-set that represents more than that number of p-sets in the current iteration. Thus it may be possible to avoid considering every k-set as a choice to put into C in the current iteration. For example, suppose the previous iteration of step 4a has chosen a k-set to be added to the design and it represented 10 p-sets that were not previously represented. In the current iteration of step 4a, if you encounter a k-set that represent 10 p-sets that weren't previously represented, then you will choose this k-set to add to your design and it is not necessary to check the rest of the k-sets. The process of determining how many p-sets are represented by a k-set which were not previously represented is very time consuming. This process has to be performed each time we consider a k-set in step 4a. One easy solution to this problem is to check each p-set to see if it has been represented by checking the array *pSetIndex*. However the running time of this is $O(C(n,p))$ which is much too slow, particularly since this task is done often. It is very easy to see that the number of p-sets represented by a k-set is :

$$\sum_{i=t}^{min\{k,p\}} C(k,i) * C(n-k, p-i). \tag{6.1}$$

It is easy to create a routine to generate these p-sets and check to see if they are already represented by checking the array *pSetIndex*. This is much better than checking all the p-sets but we still have to generate these p-sets each time for a given k-set. To speed up step 4a even further, we propose an array named *numPSetsCovers* of length $C(n, k)$ where each entry of the array holds the number of un-represented p-sets that the k-set, whose rank is the array index, can represent. Initially, the value in each array position is given by equation 6.1.

Once a k-set has been chosen in step 4a, step 4b must mark all p-sets represented by the chosen k-set as represented. The can be done by generating the p-sets that the k-set represents and updating the *pSetIndex* array. Given the rank of the p-sets represented by the k-set chosen in step 4a, the *pSetIndex* entries indexed by these ranks are incremented by 1. Note that we do this for all p-sets represented by the k-set, not just the ones that were not previously represented. This fact allows us to track how many blocks represent a particular p-set in the design which is useful in step 5. We also need to update the array *numPSetsCovers* to reflect the fact that these p-sets are now represented. To do this efficiently, for each p-set, we store the ranks of the k-sets that can represent it. For each p-set, the number of k-sets that can represent it is exactly

$$\sum_{i=t}^{min\{k,p\}} C(k,i) * C(n-k, p-i).$$

An array of pointers (named **pSetNeighbors**) of length $C(n,p)$ where each pointer points to an array of length $\sum_{i=t}^{min\{k,p\}} C(k,i) * C(n-k,p-i)$ is sufficient. The i^{th} pointer will point to the array of k-set ranks that represent the p-set with rank i. This array of pointers can be constructed at the beginning of the algorithm. Once this is done, the k-sets (actually their ranks) that represent a p-set can be determined simply by accessing this array of pointers. By keeping track of the number of unrepresented p-sets that each k-set can still represent, the algorithm is sped up dramatically.

Step 4c takes one step since we know exactly how many p-sets were represented by the k-set chosen in step 4a. Simply subtract this number from the previous number of p-sets not represented.

Once step 4 is complete, a Lotto design has been constructed. The ranks of the blocks in the design are the indices of the non-zero entries in the array *kSetIndex*. The design may contain redundant blocks, that is, blocks that may be removed and the remaining blocks still form a Lotto design. In step 5, redundant blocks are removed. Our approach is very simple. For each block, determine whether or not removing that block will give a Lotto design. If so, remove the block from the design. Otherwise check the next block. The blocks are checked in order based on their rank. This process can be made efficient since the array *pSetIndex* not only states that a p-set is represented or not, but also how many blocks represent it. Thus, given a block in the design, generate all the p-sets that it represents and subtract one from the entries in *pSetIndex* corresponding to these p-sets. If these entries are still non-zero, then the block is redundant and may be removed from the design.

The algorithm runs quite efficiently if we implement all the data structures described above. However, these data structures require a lot of memory. To build a $(20, 10, 6, 4)$ Lotto design, the amount of memory required is about 112 megabytes! But removing the data structures discussed steps 4a and 4b would increase the running time drastically. Adding an additional data structure to determine the p-sets represented by a given k-set would make the algorithm even faster but would approximately double the memory required. Tables 6.4 and 6.5 show the memory requirements and time requirements for several values of n, k, p and t using the greedy algorithm. These results were generated on an AMD K2-300 processor with 64 MB of RAM running under the Linux operating system.

It should be noted that, based on our experiments, the difference in the size of the designs generated for given n, k, p and t using the various orderings of the k-sets and the p-sets was very small.

n	k	p	t	Memory required	Time required	Number of blocks
14	5	6	4	2.0 MB	3 seconds	42 blocks
15	7	6	5	5.2 MB	11 seconds	11 blocks
20	5	4	3	10 MB	19 seconds	78 blocks
20	4	6	3	39 MB	90 seconds	59 blocks

Table 6.4: Memory and time required for greedy algorithm implementing all data structures described above

n	k	p	t	Memory required	Time required	Number of blocks
14	5	6	4	0.5 MB	299 seconds	42 blocks
15	7	6	5	0.9 MB	11 seconds	11 blocks
20	5	4	3	0.5 MB	495 seconds	78 blocks
20	4	6	3	1.8 MB	3086 seconds	59 blocks

Table 6.5: Memory and time required for greedy algorithm without the array of pointers *pSetNeighbors* and the *numPSetsCovers* described above

6.3 Random Algorithms

The idea of a random algorithm is to make a sequence of random choices, to form a solution. Random solutions have been used to generate Covering designs. We present a random algorithm to generate Lotto designs. Here is the basic algorithm.

Algorithm 6.3.1 :

void RandomConstruction(n,k,p,t)

{

1) Seed the random number generator

2) Create data structures for use in the algorithm.

3) Initialize $C = \emptyset$ (it will hold the k-sets pick for the design).

4) While there are still p-sets not represented

{

4a)Randomly pick a k-set to add to C

4b) Mark all p-sets represented by this k-set as represented.

4c) Update the number of p-sets still not represented.

}

5) C is now a (n, k, p, t) Lotto design. Remove redundant blocks from C.

}

The algorithm is very similar to Algorithm 6.2.1 for construction of Lotto designs. The main difference is step 4a. In the random case, step 4a is quite fast since it randomly picks a k-set to add into the design. Being able to compute the number of p-sets represented by a k-set and not by C efficiently is not as critical as for the greedy algorithm. Nonetheless, we may still want to keep the data structures introduced in Algorithm 6.2.1. One reason for this is that we would often like to execute the random algorithm many times for a given n, k, p and t. If we have the data structures of Algorithm 6.2.1 in the random algorithm, the initialization of the data structures is done only once. Step 4a and 4b would be able to make use of them and execute more quickly.

Instead of randomly picking a k-set and adding it to C, we could instead randomly pick w k-sets where $w \geq 1$ and add the one that represents the most p-sets not represented by C to C. Experiments have shown that the larger the value of w the better the results. This would make sense since as w increases, the number of k-sets considered increases.

According to our experiments, our random algorithm did not perform as well as Algorithm 6.2.1. However, that is not to say that they may not perform better given enough iterations of the algorithm.

Using the random algorithm program, we were able to determine that $L(16,6,6,5) \leq 358$, $L(17,6,6,5) \leq 510$, $L(18,6,6,5) \leq 727$, $L(19,6,6,5) \leq 1005$ and $L(20,6,6,5) \leq 1363$.

6.4 Heuristic Search

Chapter 2 described a exhaustive search by Bate [1]. This algorithm determined the value of $L(n,k,p,t)$ given n,k,p and t. One of the major disadvantages of this algorithm was the size of the search tree, which is measured by the number of tree nodes visited by the algorithm. This section discusses a technique to reduce the size of the search tree by considering the allowable frequencies that a constructed design must have. This technique, will however, turn the exhaustive search algorithm into an algorithm that generates only upper bounds for $L(n,k,p,t)$.

Suppose you wanted to generate an (n,k,p,t) Lotto design with b blocks. There are n elements that must be placed in each of the $k*b$ spots in the design. Suppose we divide $k*b$ by n and obtain a quotient x and a remainder r. We can see that if we have $n-r$ elements of frequency x and r elements of frequency $x+1$, then this gives exactly $k*b$ occurrences. This idea gives the basis of our heuristic :

Heuristic 6.4.1 : *Suppose $L(n,k,p,t) \leq b$. Then, there is an (n,k,p,t) Lotto design with b blocks such that at most $kb - n\lfloor b*k/n \rfloor$ elements have frequency $\lfloor b*$*

n	k	p	t	Nodes Searched	Starting Design Size	Best Design Constructed
12	4	2	2	1,258,020	12 blocks	12 blocks
13	4	2	2	206	13 blocks	13 blocks
8	3	2	2	5234	12 blocks	11 blocks
9	3	3	2	178	9 blocks	7 blocks
7	3	4	3	1698	13 blocks	12 blocks

Table 6.6: Performance of Heuristic Search Algorithm

$k/n\rfloor + 1$.

By adding this heuristic to the exhaustive search, we are able to decrease the size of the search tree substantially, thus making the program run faster. However, the search is now not guaranteed to compute $L(n, k, p, t)$ since the heuristic is not always true.

Another minor difference between our algorithm and Bate's is that in our algorithm we have to input the starting size for the design to be constructed. The algorithm will produce designs that cannot be larger than this starting size. Table 6.6 illustrates how this heuristic performs on several values of n, k, p and t.

6.5 Results From Simulated Annealing

In chapter 2, we briefly discussed simulated annealing and how it applied to the construction of Lotto designs. We have used the **cover** program by Nurmela and Ostergard. Table 6.7 states some of the results that were generated by this program.

The main problem with simulated annealing is that a large amount of memory is required for it to determine Lotto designs efficiently. It is possible to reduce the amount of memory required at the expense of a much longer running time. Because

n	k	p	t	upper bound generated
9	4	4	3	9
10	5	4	3	7
10	5	5	4	15
11	5	5	4	24
12	4	4	3	26
12	5	4	3	12
12	6	4	3	6
13	4	4	3	35
13	5	4	3	16
13	6	4	3	9
14	4	4	3	46
14	5	4	3	21

Table 6.7: Some upper bounds obtained by simulated annealing

of this dilemma, the **cover** program could not feasibly be used to compute Lotto designs for all values of n, k, p and t.

6.6 Determining $L(n, k, p, t)$ using Integer Programming

Let n, k, p, t be given. Suppose the k-sets are assigned an index between 0 and $C(n, k) - 1$ and the p-sets are assigned an index between 0 and $C(n, p) - 1$. This assignment may be based on orderings such as lexicographical, colexicographical or reverse lexicographical as seen earlier. These indices are used to determine a unique k-set (or p-set). Create an $C(n, p) \times C(n, k)$ matrix $A = (a_{i,j})$ where

$$a_{i,j} = \begin{cases} 1 & \text{if the p}-\text{set with index i is represented by the k}-\text{set with index j} \\ 0 & \text{otherwise} \end{cases}$$

for $i = 0$ to $C(n, p) - 1$ and for $j = 0$ to $C(n, k) - 1$. Then any binary vector $\vec{x} = (x_0, x_1, ..., x_{C(n,k)-1})$ satisfying the inequality

$$A\vec{x} \geq \vec{1}$$

(where $\vec{1} = (1, 1, 1, ..., 1)$) determines an (n, k, p, t) Lotto design where the k-set with index i is part of the Lotto design if and only if $x_i = 1$. To determine $L(n, k, p, t)$ we need to find a solution vector $\vec{x}$ such that

$$\sum_{i=0}^{C(n,k)-1} x_i$$

is minimized. This implies that the determination of $L(n, k, p, t)$ may be considered to be an integer programming problem.

We used the CPLEX linear programming package to compute $L(n, k, p, t)$ using the above approach. For small values of n, k, p, t, the value of $L(n, k, p, t)$ may be determined given sufficient time. However, for larger parameters, this approach was infeasible. We were able to determine that $L(10, 4, 5, 3) = 7$ which improved our tables.

6.7 Conclusion

Computer programs may be used to construct Lotto designs, thus giving upper bounds on $L(n, k, p, t)$. We have seen several types of computer programs that do this in this chapter. Each program constructs Lotto designs in different ways. Other than exhaustively considering the entire search space, a computer program cannot determine lower bounds for $L(n, k, p, t)$ since a construction gives only information about upper bounds and not lower bounds. Nonetheless, these programs are very useful. Another drawback of these programs is that they typically requires vast amounts of memory or vast amount of CPU cycles or both when constructing designs of good quality. Random algorithms are fast and require little memory but the designs generated are not very good, meaning that the size of these designs is often greater than the size of designs generated using other methods. The exhaustive search, on the other hand, will always determine the value of $L(n, k, p, t)$. However, the search is so computationally expensive that it is not performed except for very small values of n, k, p and t. We have discussed several computer programs in this chapter for constructing Lotto designs. Which program to apply to which set of parameters is not an easy question to answer. In general, the performance of two programs could differ dramatically when constructing designs with the same parameters. This means that different methods of constructing Lotto designs are often required.

Chapter 7

Tables

We begin by providing a short description of how the tables for the upper and lower bounds for $L(n, k, p, t)$ are organized. We then list these tables.

7.1 Organization of Tables

The tables are a major part of this thesis. We now describe how the tables are organized and how they are updated.

The tables contain the lower and upper bounds for $L(n, k, p, t)$ for $5 \leq n \leq 20$, $2 \leq p, k \leq n$, and $2 \leq t \leq min\{k, p\}$. The actual data is stored in two binary data files, one for the lower bounds and one for the upper bounds. Each row in the data files holds the values n, k, p, t, the bound and a comment. The comment usually holds the method that was used to compute the bound. The data files are sorted by n, then k, then p and then finally by t. This was done so that we could easily search for a particular bound. Most of the programs that manipulate the tables make use of this fact.

Many of the results in the previous chapters may be written up as computer programs that can be used to update the table data files. Some of the results in the previous chapters can only be applied to the table once, and these results are usually non-recursive. Other results may be used over and over again, and are usually recursive. An Example of a result that may be repeatedly applied is the monotonicity theorem from Chapter 3. Results from other people can also be used to improve our tables. We have made use of results of Gordon [13] via his web site at *http://sdcc12.ucsd.edu/~xm3dg/cover.html* and of Bluskov [3]. Other times, we may rely on computer searches such as our greedy algorithms, random algorithms, exhaustive searchs and simulated annealing to generate upper bounds for designs.

When an entry has been updated (improved), all recursive results must be re-applied to the tables. This is done until there are no more improvements in the tables. These tables are constantly being updated.

7.2 The Tables

In the tables, the lower bound is followed by the upper bound for each n, k, p, t. The superscripts denote how the bound was derived. For each table, the value of p and t are fixed and the values of n and k vary. The column labels represent the value of n and the row labels represent the value of k. Any row which consist entirely of the values NA and 1 are not displayed.

We now list the meanings of the superscripts.

0. The lower bound of 1 and the upper bound of $C(n, k)$

1. Theorem 2.5.4

2. Schönheim's Lower Bound for Coverings

3. Tables from Bate's thesis [1]

4. Gordon et. al.'s tables [13]

7. Theorem 4.3.5

8. Theorem 2.5.5

9. Results from Bluskov [3]

10. Results from BIBD's (Theorems 3.2.1, 3.2.2 and 3.2.3)

11. Theorem 2.2.1

12-25. Monotonicity theorems from Chapter 3

26. Theorem 3.5.1

28. Theorem 4.3.6

29. Corollary 3.5.2

30. Theorem 4.2.2 (Generalized Schöheim Bound)

31. Theorem 2.2.3

33. Theorems 4.3.1, 4.3.2, 4.3.3

34. Theorem 4.3.4

35. Theorem 3.5.2

36. Theorems 4.3.7, 4.3.8, 4.3.9, 4.3.10, 4.3.11, 4.3.12, 4.3.13, 4.3.14, 4.3.15, 4.3.16, 4.3.17 and 4.3.18

37. Lemma 4.1.1 from Bate [1]

38. Theorems 5.2.2, 5.2.3 and 5.2.4

39. Semi Direct Product (Theorem 3.4.1)

40. Greedy algorithms

41. Simulated Annealing

42. Theorem 3.5.5

43. Individual cases from Chapter 5

44. Boyer et. el. [5]

45. Theorems 2.2.4, 2.2.5 and 2.2.6.

46. Linear programming using CPLEX.

t=2, p=2	5	6	7	8	9	10	11	12	13
2	10^1-10^0	15^1-15^0	21^1-21^0	28^1-28^0	36^1-36^0	45^1-45^0	55^1-55^0	66^1-66^0	78^1-78^0
3	4^1-4^4	6^2-6^3	7^1-7^3	11^2-11^3	12^1-12^3	17^2-17^3	19^1-19^3	24^2-24^3	26^1-26^3
4	3^2-3^4	3^1-3^4	5^3-5^3	6^2-6^3	8^3-8^3	9^3-9^3	11^2-11^3	12^2-12^3	13^1-13^3
5	1^0-1^0	3^2-3^4	3^1-3^4	4^2-4^3	5^3-5^3	6^2-6^3	7^2-7^3	9^3-9^3	10^3-10^3
6	NA	1^0-1^0	3^2-3^4	3^2-3^4	3^1-3^3	4^2-4^3	6^3-6^3	6^2-6^3	7^2-7^3
7	NA	NA	1^0-1^0	3^2-3^4	3^2-3^4	3^1-3^3	4^2-4^3	5^3-5^3	6^3-6^3
8	NA	NA	NA	1^0-1^0	3^2-3^4	3^2-3^4	3^2-3^3	3^1-3^3	4^2-4^3
9	NA	NA	NA	NA	1^0-1^0	3^2-3^4	3^2-3^4	3^2-3^3	3^1-3^3
10	NA	NA	NA	NA	NA	1^0-1^0	3^2-3^4	3^2-3^4	3^2-3^3
11	NA	NA	NA	NA	NA	NA	1^0-1^0	3^2-3^4	3^2-3^4
12	NA	NA	NA	NA	NA	NA	NA	1^0-1^0	3^2-3^4

t=2, p=2 (cont.)	14	15	16	17	18	19	20
2	$91^1 - 91^0$	$105^1 - 105^0$	$120^1 - 120^0$	$136^1 - 136^0$	$153^1 - 153^0$	$171^1 - 171^0$	$190^1 - 190^0$
3	$33^2 - 33^3$	$35^1 - 35^3$	$43^2 - 43^3$	$46^1 - 46^4$	$54^2 - 54^4$	$57^1 - 57^4$	$67^2 - 67^4$
4	$18^2 - 18^3$	$19^2 - 19^3$	$20^1 - 20^3$	$26^2 - 26^4$	$27^2 - 27^4$	$31^4 - 31^4$	$35^2 - 35^4$
5	$12^2 - 12^3$	$13^3 - 13^3$	$15^4 - 15^4$	$16^4 - 16^4$	$18^2 - 18^4$	$19^2 - 19^4$	$21^4 - 21^4$
6	$7^1 - 7^3$	$10^3 - 10^3$	$10^3 - 10^3$	$12^2 - 12^4$	$12^2 - 12^4$	$14^4 - 15^4$	$16^4 - 16^4$
7	$6^2 - 6^3$	$7^2 - 7^3$	$8^3 - 8^3$	$9^4 - 9^4$	$10^4 - 10^4$	$11^4 - 11^4$	$12^2 - 12^4$
8	$5^3 - 5^3$	$6^3 - 6^3$	$6^2 - 6^3$	$7^2 - 7^4$	$7^2 - 7^4$	$9^4 - 9^4$	$9^4 - 9^4$
9	$4^2 - 4^3$	$4^2 - 4^3$	$5^3 - 5^3$	$6^4 - 6^4$	$6^2 - 6^4$	$7^2 - 7^4$	$7^2 - 7^4$
10	$3^1 - 3^3$	$3^1 - 3^3$	$4^2 - 4^3$	$5^4 - 5^4$	$5^4 - 5^4$	$6^4 - 6^4$	$6^2 - 6^4$
11	$3^2 - 3^3$	$3^2 - 3^3$	$3^1 - 3^3$	$4^2 - 4^4$	$4^2 - 4^4$	$5^4 - 5^4$	$6^4 - 6^4$
12	$3^2 - 3^4$	$3^2 - 3^3$	$3^2 - 3^3$	$3^1 - 3^4$	$3^1 - 3^4$	$4^2 - 4^4$	$4^2 - 4^4$
13	$3^2 - 3^4$	$3^2 - 3^4$	$3^2 - 3^3$	$3^2 - 3^4$	$3^2 - 3^4$	$3^1 - 3^4$	$4^2 - 4^4$
14	$1^0 - 1^0$	$3^2 - 3^4$	$3^2 - 3^4$	$3^2 - 3^4$	$3^2 - 3^4$	$3^2 - 3^4$	$3^1 - 3^4$
15	NA	$1^0 - 1^0$	$3^2 - 3^4$	$3^2 - 3^4$	$3^2 - 3^4$	$3^2 - 3^4$	$3^2 - 3^4$
16	NA	NA	$1^0 - 1^0$	$3^2 - 3^4$	$3^2 - 3^4$	$3^2 - 3^4$	$3^2 - 3^4$
17	NA	NA	NA	$1^0 - 1^0$	$3^2 - 3^{11}$	$3^2 - 3^{11}$	$3^2 - 3^{11}$
18	NA	NA	NA	NA	$1^0 - 1^0$	$3^2 - 3^{11}$	$3^2 - 3^{11}$
19	NA	NA	NA	NA	NA	$1^0 - 1^0$	$3^2 - 3^{11}$

t=2, p=3	5	6	7	8	9	10	11	12
2	4^1-4^3	6^1-6^3	9^1-9^3	12^1-12^3	16^1-16^3	20^1-20^3	25^1-25^3	30^1-30^3
3	2^1-2^{11}	2^1-2^3	4^3-4^3	5^3-5^3	7^3-7^3	8^3-8^3	10^3-10^3	11^3-11^3
4	1^0-1^7	2^8-2^{11}	2^1-2^{11}	2^1-2^3	4^3-4^3	4^1-4^3	6^3-6^3	6^3-6^3
5	1^0-1^0	1^0-1^7	2^8-2^{11}	2^1-2^{11}	2^1-2^3	2^1-2^3	4^3-4^3	4^3-4^3
6	NA	1^0-1^0	1^0-1^7	2^8-2^{11}	2^1-2^{11}	2^1-2^3	2^1-2^3	2^1-2^3
7	NA	NA	1^0-1^0	1^0-1^7	2^8-2^{11}	$2^{11}-2^{11}$	2^1-2^3	2^1-2^3
8	NA	NA	NA	1^0-1^0	1^0-1^7	$2^{11}-2^{11}$	$2^{11}-2^{11}$	2^1-2^3
9	NA	NA	NA	NA	1^0-1^0	1^0-1^7	$2^{11}-2^{11}$	$2^{11}-2^{11}$
10	NA	NA	NA	NA	NA	1^0-1^0	1^0-1^7	$2^{11}-2^{11}$

t=2, p=3 (cont.)	13	14	15	16	17	18	19	20
2	$36^1 - 36^3$	$42^1 - 42^3$	$49^1 - 49^3$	$56^1 - 56^3$	$64^1 - 64^8$	$72^1 - 72^8$	$81^1 - 81^8$	$90^1 - 90^8$
3	$13^3 - 13^3$	$14^1 - 14^3$	$18^3 - 18^3$	$19^1 - 19^3$	$23^{46} - 23^8$	$24^1 - 24^8$	$29^{46} - 29^8$	$31^{46} - 31^8$
4	$8^{43} - 8^3$	$9^{43} - 9^3$	$11^{43} - 11^3$	$12^{43} - 12^3$	$14^{43} - 14^8$	$15^{43} - 15^8$	$16^{43} - 16^8$	$18^{43} - 18^8$
5	$5^3 - 5^3$	$6^{38} - 6^3$	$7^{43} - 7^3$	$8^{43} - 8^3$	$9^{43} - 9^8$	$10^{43} - 10^8$	$11^{43} - 11^8$	$12^{43} - 12^8$
6	$4^3 - 4^3$	$4^3 - 4^3$	$4^1 - 4^3$	$5^3 - 5^3$	$6^{38} - 6^8$	$6^{38} - 6^8$	$7^{38} - 7^8$	$8^{43} - 8^8$
7	$2^1 - 2^3$	$2^1 - 2^3$	$4^3 - 4^3$	$4^3 - 4^3$	$4^1 - 4^8$	$5^{38} - 5^8$	$6^{38} - 6^8$	$6^{38} - 6^8$
8	$2^1 - 2^3$	$2^1 - 2^3$	$2^1 - 2^3$	$2^1 - 2^3$	$4^{38} - 4^8$	$4^{38} - 4^8$	$4^{38} - 4^8$	$4^1 - 4^8$
9	$2^3 - 2^3$	$2^1 - 2^3$	$2^1 - 2^3$	$2^1 - 2^3$	$2^1 - 2^{11}$	$2^1 - 2^8$	$4^{38} - 4^8$	$4^{38} - 4^8$
10	$2^{11} - 2^{11}$	$2^3 - 2^3$	$2^1 - 2^3$	$2^1 - 2^3$	$2^1 - 2^{11}$	$2^1 - 2^{11}$	$2^1 - 2^{11}$	$2^1 - 2^8$
11	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^3 - 2^3$	$2^1 - 2^3$	$2^1 - 2^{11}$	$2^1 - 2^{11}$	$2^1 - 2^{11}$	$2^1 - 2^{11}$
12	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^3 - 2^3$	$2^{12} - 2^{11}$	$2^1 - 2^{11}$	$2^1 - 2^{11}$	$2^1 - 2^{11}$
13	$1^0 - 1^0$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^1 - 2^{11}$	$2^1 - 2^{11}$
14	NA	$1^0 - 1^0$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
15	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
16	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
17	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
18	NA	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=2, p=4	5	6	7	8	9	10	11	12
2	$2^1 - 2^8$	$3^1 - 3^3$	$5^1 - 5^3$	$7^1 - 7^3$	$9^1 - 9^3$	$12^1 - 12^3$	$15^1 - 15^3$	$18^1 - 18^3$
3	$1^0 - 1^7$	$2^8 - 2^8$	$2^1 - 2^3$	$3^1 - 3^3$	$3^1 - 3^3$	$5^3 - 5^3$	$6^3 - 6^3$	$8^{38} - 8^3$
4	$1^0 - 1^7$	$1^0 - 1^7$	$2^8 - 2^{11}$	$2^1 - 2^3$	$2^1 - 2^3$	$3^3 - 3^3$	$3^1 - 3^3$	$3^1 - 3^3$
5	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$2^8 - 2^{11}$	$2^8 - 2^{11}$	$2^1 - 2^3$	$2^1 - 2^3$	$3^3 - 3^3$
6	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$2^8 - 2^{11}$	$2^3 - 2^3$	$2^3 - 2^3$	$2^1 - 2^3$
7	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^3 - 2^3$
8	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$
9	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=2, p=4 (cont.)	13	14	15	16	17	18	19	20
2	$22^1 - 22^3$	$26^1 - 26^3$	$30^1 - 30^3$	$35^1 - 35^3$	$40^1 - 40^8$	$45^1 - 45^8$	$51^1 - 51^8$	$57^1 - 57^8$
3	$9^3 - 9^3$	$11^{43} - 11^3$	$12^3 - 12^3$	$14^{43} - 14^3$	$15^8 - 15^8$	$17^{43} - 17^8$	$18^{43} - 18^8$	$19^1 - 20^8$
4	$5^3 - 5^3$	$5^1 - 5^3$	$7^{38} - 7^3$	$7^3 - 7^3$	$9^{38} - 9^8$	$9^{38} - 9^8$	$11^{43} - 11^8$	$12^{43} - 12^8$
5	$3^1 - 3^3$	$3^1 - 3^3$	$3^1 - 3^3$	$5^3 - 5^3$	$5^{12} - 5^8$	$6^{38} - 6^8$	$7^{38} - 7^8$	$8^{38} - 8^8$
6	$2^1 - 2^3$	$3^3 - 3^3$	$3^3 - 3^3$	$3^1 - 3^3$	$3^1 - 3^8$	$3^1 - 3^8$	$5^{38} - 5^8$	$5^{38} - 5^8$
7	$2^1 - 2^3$	$2^1 - 2^3$	$2^1 - 2^3$	$3^3 - 3^3$	$3^{12} - 3^{12}$	$3^1 - 3^8$	$3^1 - 3^8$	$3^1 - 3^8$
8	$2^3 - 2^3$	$2^3 - 2^3$	$2^1 - 2^3$	$2^1 - 2^3$	$2^1 - 2^8$	$3^{11} - 3^{14}$	$3^{11} - 3^{12}$	$3^1 - 3^8$
9	$2^{11} - 2^{11}$	$2^3 - 2^3$	$2^3 - 2^3$	$2^3 - 2^3$	$2^1 - 2^{11}$	$2^1 - 2^8$	$2^1 - 2^8$	$3^{11} - 3^{14}$
10	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^3 - 2^3$	$2^3 - 2^3$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^1 - 2^{11}$	$2^1 - 2^8$
11	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^3 - 2^3$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^1 - 2^{11}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
13	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
14	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
15	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
16	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
17	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=2, p=5	6	7	8	9	10	11	12	13
2	$2^1 - 2^8$	$3^1 - 3^3$	$4^1 - 4^3$	$6^1 - 6^3$	$8^1 - 8^3$	$10^1 - 10^3$	$12^1 - 12^3$	$15^1 - 15^3$
3	$1^0 - 1^7$	$2^8 - 2^8$	$2^1 - 2^3$	$3^3 - 3^3$	$3^1 - 3^3$	$4^1 - 4^3$	$4^1 - 4^3$	$6^3 - 6^3$
4	$1^0 - 1^7$	$1^0 - 1^7$	$2^8 - 2^8$	$2^3 - 2^3$	$2^1 - 2^3$	$3^3 - 3^3$	$3^3 - 3^3$	$3^1 - 3^3$
5	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^8 - 2^{11}$	$2^3 - 2^3$	$2^3 - 2^3$	$2^1 - 2^3$	$3^3 - 3^3$
6	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^8 - 2^{11}$	$2^{11} - 2^{11}$	$2^3 - 2^3$	$2^3 - 2^3$
7	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^3 - 2^3$
8	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$
9	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=2, p=5 (cont.)	14	15	16	17	18	19	20
2	$18^1 - 18^3$	$21^1 - 21^3$	$24^1 - 24^3$	$28^1 - 28^8$	$32^1 - 32^8$	$36^1 - 36^8$	$40^1 - 40^8$
3	$7^3 - 7^3$	$9^{38} - 9^3$	$10^{38} - 10^3$	$12^{38} - 12^8$	$12^{38} - 13^8$	$13^{38} - 15^8$	$16^{43} - 16^8$
4	$4^3 - 4^3$	$4^1 - 4^3$	$4^1 - 4^3$	$6^{38} - 6^8$	$6^1 - 6^8$	$8^{38} - 8^8$	$8^{38} - 8^8$
5	$3^3 - 3^3$	$3^1 - 3^3$	$3^1 - 3^3$	$4^{11} - 4^8$	$4^1 - 4^8$	$4^1 - 4^8$	$4^1 - 4^8$
6	$2^1 - 2^3$	$3^3 - 3^3$	$3^3 - 3^3$	$3^{12} - 3^8$	$3^1 - 3^8$	$3^1 - 3^8$	$4^{11} - 4^8$
7	$2^3 - 2^3$	$2^3 - 2^3$	$2^1 - 2^3$	$3^{11} - 3^{12}$	$3^{11} - 3^8$	$3^{11} - 3^8$	$3^{11} - 3^8$
8	$2^3 - 2^3$	$2^3 - 2^3$	$2^3 - 2^3$	$2^{12} - 2^8$	$2^1 - 2^8$	$3^{11} - 3^{12}$	$3^{11} - 3^8$
9	$2^{11} - 2^{11}$	$2^3 - 2^3$	$2^3 - 2^3$	$2^{12} - 2^{11}$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^1 - 2^8$
10	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^3 - 2^3$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
14	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
15	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
16	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=2, p=6	7	8	9	10	11	12	13	14
2	$2^1 - 2^8$	$3^1 - 3^3$	$4^1 - 4^3$	$5^1 - 5^3$	$7^1 - 7^3$	$9^1 - 9^3$	$11^1 - 11^3$	$13^1 - 13^3$
3	$1^0 - 1^7$	$2^8 - 2^8$	$2^1 - 2^3$	$3^3 - 3^3$	$3^1 - 3^3$	$4^3 - 4^3$	$4^1 - 4^3$	$5^1 - 5^3$
4	$1^0 - 1^7$	$1^0 - 1^7$	$2^8 - 2^8$	$2^3 - 2^3$	$2^1 - 2^3$	$3^3 - 3^3$	$3^3 - 3^3$	$3^1 - 3^3$
5	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^8 - 2^8$	$2^3 - 2^3$	$2^3 - 2^3$	$2^1 - 2^3$	$3^3 - 3^3$
6	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^3 - 2^3$	$2^3 - 2^3$	$2^3 - 2^3$
7	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^3 - 2^3$
8	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$
9	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=2, p=6 (cont.)	15	16	17	18	19	20
2	$15^1 - 15^3$	$18^1 - 18^3$	$21^1 - 21^8$	$24^1 - 24^8$	$27^1 - 27^8$	$30^1 - 30^8$
3	$5^1 - 5^3$	$7^3 - 7^3$	$8^{38} - 8^8$	$10^{38} - 10^8$	$11^{38} - 11^8$	$13^{38} - 13^8$
4	$4^3 - 4^3$	$4^3 - 4^3$	$4^1 - 4^8$	$5^{11} - 5^8$	$5^1 - 5^8$	$5^1 - 5^8$
5	$3^3 - 3^3$	$3^3 - 3^3$	$3^1 - 3^8$	$4^{11} - 4^8$	$4^{11} - 4^8$	$4^{11} - 4^8$
6	$2^3 - 2^3$	$3^3 - 3^3$	$3^{12} - 3^8$	$3^{12} - 3^8$	$3^{12} - 3^8$	$3^{12} - 3^8$
7	$2^3 - 2^3$	$2^3 - 2^3$	$2^{12} - 2^8$	$3^{11} - 3^8$	$3^{11} - 3^8$	$3^{11} - 3^8$
8	$2^3 - 2^3$	$2^3 - 2^3$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$
9	$2^{11} - 2^{11}$	$2^3 - 2^3$	$2^{12} - 2^{11}$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
10	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
15	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=2, p=7	8	9	10	11	12	13	14	15
2	$2^1 - 2^8$	$3^1 - 3^3$	$4^1 - 4^3$	$5^1 - 5^3$	$6^1 - 6^3$	$8^1 - 8^3$	$10^1 - 10^3$	$12^1 - 12^3$
3	$1^0 - 1^7$	$2^8 - 2^8$	$2^1 - 2^3$	$3^3 - 3^3$	$3^3 - 3^3$	$4^3 - 4^3$	$4^1 - 4^3$	$5^3 - 5^3$
4	$1^0 - 1^7$	$1^0 - 1^7$	$2^8 - 2^8$	$2^3 - 2^3$	$2^3 - 2^3$	$3^3 - 3^3$	$3^3 - 3^3$	$3^3 - 3^3$
5	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^8 - 2^8$	$2^3 - 2^3$	$2^3 - 2^3$	$2^3 - 2^3$	$3^3 - 3^3$
6	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^3 - 2^3$	$2^3 - 2^3$	$2^3 - 2^3$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^3 - 2^3$	$2^3 - 2^3$
8	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$
9	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=2, p=7 (cont.)	16	17	18	19	20
2	$14^1 - 14^3$	$16^1 - 16^8$	$18^1 - 18^8$	$21^1 - 21^8$	$24^1 - 24^8$
3	$5^1 - 5^3$	$6^1 - 6^8$	$6^1 - 6^8$	$8^{38} - 8^8$	$9^{38} - 9^8$
4	$4^3 - 4^3$	$4^{12} - 4^8$	$4^{12} - 4^8$	$5^{11} - 5^8$	$5^{11} - 5^8$
5	$3^3 - 3^3$	$3^{12} - 3^8$	$3^{12} - 3^8$	$4^{11} - 4^8$	$4^{11} - 4^8$
6	$2^3 - 2^3$	$3^{11} - 3^8$	$3^{11} - 3^8$	$3^{11} - 3^8$	$3^{11} - 3^8$
7	$2^3 - 2^3$	$2^{12} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$	$3^{11} - 3^8$
8	$2^3 - 2^3$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
9	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
10	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=2, p=8	9	10	11	12	13	14	15	16
2	$2^1 - 2^8$	$3^1 - 3^3$	$4^1 - 4^3$	$5^1 - 5^3$	$6^1 - 6^3$	$7^1 - 7^3$	$9^1 - 9^3$	$11^1 - 11^3$
3	$1^0 - 1^7$	$2^8 - 2^8$	$2^1 - 2^3$	$3^3 - 3^3$	$3^3 - 3^3$	$4^3 - 4^3$	$4^3 - 4^3$	$5^3 - 5^3$
4	$1^0 - 1^7$	$1^0 - 1^7$	$2^8 - 2^8$	$2^3 - 2^3$	$2^3 - 2^3$	$3^3 - 3^3$	$3^3 - 3^3$	$3^3 - 3^3$
5	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^3 - 2^3$	$2^3 - 2^3$	$2^3 - 2^3$	$3^3 - 3^3$
6	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^3 - 2^3$	$2^3 - 2^3$	$2^3 - 2^3$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^3 - 2^3$	$2^3 - 2^3$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^3 - 2^3$
9	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=2, p=8 (cont.)	17	18	19	20
2	$13^1 - 13^8$	$15^1 - 15^8$	$17^1 - 17^8$	$19^1 - 19^8$
3	$5^1 - 5^8$	$6^{11} - 6^8$	$6^1 - 6^8$	$7^1 - 7^8$
4	$4^{11} - 4^8$	$4^{11} - 4^8$	$4^{11} - 4^8$	$5^{11} - 5^8$
5	$3^{12} - 3^8$	$3^{12} - 3^8$	$3^{12} - 3^8$	$4^{11} - 4^8$
6	$2^{12} - 2^8$	$3^{11} - 3^8$	$3^{11} - 3^8$	$3^{11} - 3^8$
7	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$
8	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
9	$2^{12} - 2^{11}$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
10	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=2, p=9	10	11	12	13	14	15	16	17
2	$2^1 - 2^8$	$3^1 - 3^3$	$4^3 - 4^3$	$5^1 - 5^3$	$6^1 - 6^3$	$7^1 - 7^3$	$8^1 - 8^3$	$10^1 - 10^4$
3	$1^0 - 1^7$	$2^8 - 2^8$	$2^3 - 2^3$	$3^3 - 3^3$	$3^3 - 3^3$	$4^3 - 4^3$	$4^3 - 4^3$	$5^{11} - 5^8$
4	$1^0 - 1^7$	$1^0 - 1^7$	$2^8 - 2^8$	$2^3 - 2^3$	$2^3 - 2^3$	$3^3 - 3^3$	$3^3 - 3^3$	$3^{12} - 3^8$
5	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^3 - 2^3$	$2^3 - 2^3$	$2^3 - 2^3$	$3^{11} - 3^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^3 - 2^3$	$2^3 - 2^3$	$2^{12} - 2^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^3 - 2^3$	$2^{12} - 2^8$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=2, p=9 (cont.)	18	19	20
2	$12^1 - 12^8$	$14^1 - 14^8$	$16^2 - 16^8$
3	$5^{11} - 5^8$	$6^{11} - 6^8$	$6^{11} - 6^8$
4	$4^{11} - 4^8$	$4^{11} - 4^8$	$4^{11} - 4^8$
5	$3^{11} - 3^8$	$3^{11} - 3^8$	$3^{11} - 3^8$
6	$2^{12} - 2^8$	$3^{11} - 3^8$	$3^{11} - 3^8$
7	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
8	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
9	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
10	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=2, p=10	11	12	13	14	15	16	17	18
2	2^1-2^8	3^3-3^3	4^3-4^3	5^3-5^3	6^3-6^3	7^1-7^3	8^1-8^4	9^1-9^4
3	1^0-1^7	2^8-2^8	2^3-2^3	3^3-3^3	3^3-3^3	4^3-4^3	$4^{12}-4^8$	$5^{11}-5^8$
4	1^0-1^7	1^0-1^7	$2^{11}-2^8$	2^3-2^3	2^3-2^3	3^3-3^3	$3^{12}-3^8$	$3^{12}-3^8$
5	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	2^3-2^3	2^3-2^3	$2^{12}-2^8$	$3^{11}-3^8$
6	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	2^3-2^3	$2^{12}-2^8$	$2^{12}-2^8$
7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$
8	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$
9	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$

t=2, p=10 (cont.)	19	20
2	$11^1 - 11^8$	$13^1 - 13^8$
3	$5^{11} - 5^8$	$6^{11} - 6^8$
4	$4^{11} - 4^8$	$4^{11} - 4^8$
5	$3^{11} - 3^8$	$3^{11} - 3^8$
6	$2^{12} - 2^8$	$3^{11} - 3^8$
7	$2^{12} - 2^8$	$2^{12} - 2^8$
8	$2^{12} - 2^8$	$2^{12} - 2^8$
9	$2^{12} - 2^8$	$2^{12} - 2^8$
10	$2^{11} - 2^{11}$	$2^{12} - 2^8$
11	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=2, p=11	12	13	14	15	16	17	18	19	20
2	2^1-2^8	3^3-3^3	4^3-4^3	5^3-5^3	6^3-6^3	7^4-7^4	8^1-8^4	9^1-9^8	10^1-10^8
3	1^0-1^7	2^8-2^8	2^3-2^3	3^3-3^3	3^3-3^3	$4^{11}-4^8$	$4^{11}-4^8$	$5^{11}-5^8$	$5^{11}-5^8$
4	1^0-1^7	1^0-1^7	$2^{11}-2^8$	2^3-2^3	2^3-2^3	$3^{11}-3^8$	$3^{11}-3^8$	$3^{11}-3^8$	$4^{11}-4^8$
5	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	2^3-2^3	$2^{12}-2^8$	$2^{12}-2^8$	$3^{11}-3^8$	$3^{11}-3^8$
6	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$
7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$
8	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$
9	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$
10	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$

t=2, p=12	13	14	15	16	17	18	19	20
2	$2^1 - 2^8$	$3^3 - 3^3$	$4^3 - 4^3$	$5^3 - 5^3$	$6^4 - 6^4$	$7^4 - 7^4$	$8^2 - 8^8$	$9^1 - 9^8$
3	$1^0 - 1^7$	$2^{11} - 2^8$	$2^3 - 2^3$	$3^3 - 3^3$	$3^{12} - 3^8$	$4^{11} - 4^8$	$4^{11} - 4^8$	$5^{11} - 5^8$
4	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^3 - 2^3$	$2^{12} - 2^8$	$3^{11} - 3^8$	$3^{11} - 3^8$	$3^{11} - 3^8$
5	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=2, p=13	14	15	16	17	18	19	20
2	$2^1 - 2^8$	$3^3 - 3^3$	$4^3 - 4^3$	$5^4 - 5^4$	$6^4 - 6^4$	$7^2 - 7^8$	$8^2 - 8^8$
3	$1^0 - 1^7$	$2^{11} - 2^8$	$2^3 - 2^3$	$3^{11} - 3^8$	$3^{11} - 3^8$	$4^{11} - 4^8$	$4^{11} - 4^8$
4	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$	$3^{11} - 3^8$
5	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=2, p=14	15	16	17	18	19	20
2	$2^1 - 2^8$	$3^3 - 3^3$	$4^4 - 4^4$	$5^4 - 5^4$	$6^2 - 6^8$	$7^2 - 7^8$
3	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$	$3^{11} - 3^8$	$4^{11} - 4^8$
4	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$
5	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=2, p=15	16	17	18	19	20
2	$2^1 - 2^8$	$3^4 - 3^4$	$4^4 - 4^4$	$5^2 - 5^8$	$6^2 - 6^8$
3	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$	$3^{11} - 3^8$
4	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
5	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=2, p=16	17	18	19	20
2	$2^1 - 2^8$	$3^4 - 3^4$	$4^2 - 4^8$	$5^2 - 5^8$
3	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$
4	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
5	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=2, p=17	18	19	20
2	$2^1 - 2^8$	$3^2 - 3^8$	$4^2 - 4^8$
3	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
4	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=2, p=18	19	20
2	$2^1 - 2^8$	$3^2 - 3^8$
3	$1^0 - 1^7$	$2^{11} - 2^8$

t=2, p=19	20
2	$2^1 - 2^8$

t=3, p=3	5	6	7	8	9	10	11	12	13
3	$10^1 - 10^0$	$20^1 - 20^0$	$35^1 - 35^0$	$56^1 - 56^0$	$84^1 - 84^0$	$120^1 - 120^0$	$165^1 - 165^0$	$220^1 - 220^0$	$286^1 - 286^0$
4	$4^2 - 4^4$	$6^2 - 6^4$	$12^4 - 12^4$	$14^1 - 14^3$	$25^2 - 25^4$	$30^1 - 30^4$	$47^2 - 47^4$	$57^2 - 57^4$	$78^2 - 78^4$
5	$1^0 - 1^0$	$4^2 - 4^4$	$5^2 - 5^4$	$8^4 - 8^4$	$12^3 - 12^3$	$17^4 - 17^4$	$20^4 - 20^4$	$27^2 - 29^4$	$32^2 - 34^4$
6	NA	$1^0 - 1^0$	$4^2 - 4^4$	$4^2 - 4^4$	$7^4 - 7^4$	$10^4 - 10^4$	$11^2 - 11^4$	$15^4 - 15^4$	$20^4 - 21^4$
7	NA	NA	$1^0 - 1^0$	$4^2 - 4^4$	$4^2 - 4^4$	$6^4 - 6^4$	$8^4 - 8^4$	$11^4 - 11^4$	$13^4 - 13^4$
8	NA	NA	NA	$1^0 - 1^0$	$4^2 - 4^4$	$4^2 - 4^4$	$5^2 - 5^4$	$6^2 - 6^4$	$10^4 - 10^4$
9	NA	NA	NA	NA	$1^0 - 1^0$	$4^2 - 4^4$	$4^2 - 4^4$	$4^2 - 4^4$	$6^4 - 6^4$
10	NA	NA	NA	NA	NA	$1^0 - 1^0$	$4^2 - 4^4$	$4^2 - 4^4$	$4^2 - 4^4$
11	NA	NA	NA	NA	NA	NA	$1^0 - 1^0$	$4^2 - 4^4$	$4^2 - 4^4$
12	NA	NA	NA	NA	NA	NA	NA	$1^0 - 1^0$	$4^2 - 4^4$

t=3, p=3 (cont.)	14	15	16	17	18	19	20
3	$364^1 - 364^0$	$455^1 - 455^0$	$560^1 - 560^0$	$680^1 - 680^0$	$816^1 - 816^0$	$969^1 - 969^0$	$1140^1 - 1140^0$
4	$91^1 - 91^4$	$124^2 - 124^4$	$140^1 - 140^4$	$183^2 - 183^4$	$207^2 - 207^4$	$257^2 - 258^4$	$285^1 - 285^4$
5	$37^1 - 43^4$	$54^2 - 57^4$	$61^2 - 65^4$	$68^1 - 68^4$	$94^2 - 94^4$	$103^2 - 108^4$	$124^4 - 133^4$
6	$24^4 - 25^4$	$30^2 - 31^4$	$35^4 - 38^4$	$43^4 - 44^4$	$48^4 - 48^4$	$57^2 - 63^4$	$64^2 - 72^4$
7	$15^4 - 15^4$	$15^2 - 15^4$	$23^4 - 24^4$	$25^4 - 28^4$	$31^2 - 33^4$	$33^2 - 35^4$	$40^4 - 45^4$
8	$11^4 - 11^4$	$13^4 - 13^4$	$14^2 - 14^4$	$17^4 - 18^4$	$21^4 - 22^4$	$24^4 - 28^4$	$28^4 - 30^4$
9	$8^4 - 8^4$	$10^4 - 10^4$	$11^4 - 11^4$	$13^4 - 14^4$	$14^2 - 16^4$	$15^2 - 17^4$	$20^4 - 24^4$
10	$5^2 - 5^4$	$7^4 - 7^4$	$8^4 - 8^4$	$9^4 - 11^4$	$11^4 - 12^4$	$13^4 - 14^4$	$14^2 - 15^4$
11	$4^2 - 4^4$	$5^2 - 5^4$	$6^4 - 6^4$	$7^2 - 7^4$	$9^4 - 10^4$	$9^4 - 11^4$	$11^4 - 14^4$
12	$4^2 - 4^4$	$4^2 - 4^4$	$4^2 - 4^4$	$6^4 - 6^4$	$6^2 - 6^4$	$9^4 - 9^4$	$9^4 - 10^4$
13	$4^2 - 4^4$	$4^2 - 4^4$	$4^2 - 4^4$	$4^2 - 4^4$	$5^2 - 5^4$	$6^4 - 6^4$	$8^4 - 8^4$
14	$1^0 - 1^0$	$4^2 - 4^4$	$4^2 - 4^4$	$4^2 - 4^4$	$4^2 - 4^4$	$5^2 - 5^4$	$6^4 - 6^4$
15	NA	$1^0 - 1^0$	$4^2 - 4^4$	$4^2 - 4^4$	$4^2 - 4^4$	$4^2 - 4^4$	$4^2 - 4^4$
16	NA	NA	$1^0 - 1^0$	$4^2 - 4^4$	$4^2 - 4^4$	$4^2 - 4^4$	$4^2 - 4^4$
17	NA	NA	NA	$1^0 - 1^0$	$4^2 - 4^{11}$	$4^2 - 4^{11}$	$4^2 - 4^{11}$
18	NA	NA	NA	NA	$1^0 - 1^0$	$4^2 - 4^{11}$	$4^2 - 4^{11}$
19	NA	NA	NA	NA	NA	$1^0 - 1^0$	$4^2 - 4^{11}$

t=3, p=4	5	6	7	8	9	10	11	12
3	$3^1 - 3^5$	$6^4 - 6^4$	$12^3 - 12^3$	$20^3 - 20^3$	$30^3 - 30^3$	$45^3 - 45^3$	$63^4 - 63^4$	$84^4 - 84^4$
4	$1^0 - 1^7$	$3^8 - 3^{11}$	$4^{11} - 4^{11}$	$6^3 - 6^3$	$9^{43} - 9^{11}$	$12^{34} - 14^{11}$	$16^{34} - 19^{41}$	$21^{34} - 26^{41}$
5	$1^0 - 1^0$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$5^{11} - 5^9$	$6^{33} - 7^9$	$8^{43} - 9^9$	$9^{33} - 12^9$
6	NA	$1^0 - 1^0$	$1^0 - 1^7$	$3^{11} - 3^9$	$3^{11} - 3^9$	$4^{11} - 4^9$	$5^{33} - 5^9$	$6^{33} - 6^9$
7	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$4^{33} - 4^{11}$
8	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$
9	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$
10	NA	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=3, p=4 (cont.)	13	14	15	16	17	18	19	20
3	$110^2 - 112^{45}$	$140^2 - 144^{45}$	$175^2 - 180^{45}$	$216^2 - 225^{45}$	$263^2 - 275^{45}$	$316^2 - 330^{45}$	$376^2 - 396^{45}$	$443^2 - 468^{45}$
4	$28^{34} - 35^{41}$	$35^{34} - 46^{41}$	$44^{34} - 58^{41}$	$54^{34} - 72^{41}$	$66^{34} - 89^{41}$	$79^{34} - 107^{41}$	$94^{34} - 127^{41}$	$111^{34} - 156^{26}$
5	$11^{34} - 16^{9}$	$14^{34} - 20^{9}$	$18^{34} - 25^{9}$	$22^{34} - 32^{9}$	$27^{34} - 40^{9}$	$32^{34} - 49^{9}$	$38^{34} - 57^{9}$	$45^{34} - 71^{9}$
6	$7^{33} - 9^{9}$	$7^{33} - 11^{9}$	$10^{33} - 15^{9}$	$11^{34} - 16^{9}$	$14^{34} - 21^{9}$	$16^{34} - 26^{9}$	$19^{34} - 32^{9}$	$23^{34} - 39^{9}$
7	$6^{33} - 6^{14}$	$6^{33} - 6^{41}$	$7^{33} - 9^{41}$	$7^{33} - 11^{41}$	$9^{33} - 14^{41}$	$10^{33} - 19^{11}$	$11^{33} - 20^{11}$	$13^{34} - 22^{11}$
8	$4^{33} - 4^{11}$	$5^{33} - 5^{11}$	$6^{33} - 6^{41}$	$6^{33} - 7^{41}$	$7^{33} - 9^{12}$	$7^{33} - 9^{42}$	$9^{33} - 14^{12}$	$9^{33} - 14^{42}$
9	$3^{11} - 3^{11}$	$3^{12} - 3^{18}$	$4^{33} - 4^{11}$	$5^{33} - 5^{11}$	$6^{33} - 6^{12}$	$6^{33} - 6^{42}$	$7^{33} - 9^{14}$	$7^{33} - 11^{11}$
10	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$4^{33} - 4^{11}$	$4^{33} - 4^{11}$	$5^{33} - 6^{11}$	$6^{33} - 6^{14}$	$6^{33} - 8^{11}$
11	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{18}$	$4^{33} - 4^{11}$	$5^{33} - 5^{11}$	$5^{33} - 6^{11}$
12	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$4^{33} - 4^{11}$	$4^{33} - 4^{11}$
13	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
14	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
15	NA	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
16	NA	NA	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
17	NA	NA	NA	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
18	NA	NA	NA	NA	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$

t=3, p=5	6	7	8	9	10	11	12	13
3	2^1-2^8	5^4-5^4	8^3-8^3	12^3-12^3	20^3-20^3	29^4-29^4	40^4-40^4	52^4-52^4
4	1^0-1^7	2^8-2^{11}	2^1-2^8	5^3-5^3	$7^{46}-7^8$	$8^{34}-10^8$	$10^{34}-12^8$	$13^{34}-13^{10}$
5	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{11}-2^{11}$	2^1-2^3	$5^{43}-5^8$	$5^{36}-6^8$	$6^{34}-9^8$
6	1^0-1^0	1^0-1^7	1^0-1^7	$2^{11}-2^9$	$2^{11}-2^9$	2^1-2^9	2^1-2^8	$5^{36}-5^8$
7	NA	1^0-1^0	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{11}-2^{11}$	$2^{11}-2^{11}$	2^1-2^{11}
8	NA	NA	1^0-1^0	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{11}-2^{11}$	$2^{11}-2^{11}$
9	NA	NA	NA	1^0-1^0	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{11}-2^{11}$
10	NA	NA	NA	NA	1^0-1^0	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$

t=3, p=5 (cont.)	14	15	16	17	18	19	20
3	$67^2 - 70^8$	$84^2 - 89^{44}$	$104^2 - 112^8$	$127^2 - 140^8$	$153^2 - 168^8$	$182^2 - 204^8$	$215^2 - 240^8$
4	$17^{34} - 20^8$	$21^{34} - 26^8$	$26^{34} - 28^8$	$32^{34} - 39^8$	$39^{34} - 44^8$	$46^{34} - 55^8$	$54^{34} - 60^8$
5	$7^{34} - 10^8$	$9^{34} - 13^8$	$11^{34} - 16^8$	$13^{34} - 20^8$	$16^{34} - 24^8$	$19^{34} - 28^8$	$22^{34} - 32^8$
6	$5^{36} - 5^8$	$5^{36} - 8^8$	$6^{34} - 8^8$	$7^{34} - 11^8$	$8^{34} - 12^9$	$10^{34} - 15^8$	$11^{34} - 18^8$
7	$2^1 - 2^8$	$5^{36} - 5^8$	$5^{36} - 5^8$	$5^{36} - 7^8$	$5^{36} - 8^8$	$6^{33} - 10^8$	$7^{34} - 12^8$
8	$2^{12} - 2^{11}$	$2^1 - 2^{11}$	$2^1 - 2^8$	$5^{36} - 5^8$	$5^{36} - 5^8$	$5^{36} - 6^8$	$5^{36} - 7^8$
9	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^1 - 2^{11}$	$2^1 - 2^8$	$5^{36} - 5^8$	$5^{36} - 5^8$
10	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^1 - 2^{11}$	$2^1 - 2^8$
11	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^1 - 2^{11}$
12	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
14	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
15	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
16	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
17	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=3, p=6	7	8	9	10	11	12	13	14
3	$2^1 - 2^8$	$4^4 - 4^4$	$7^3 - 7^3$	$10^4 - 10^4$	$16^4 - 16^4$	$22^4 - 22^4$	$30^4 - 30^4$	$40^4 - 40^4$
4	$1^0 - 1^7$	$2^8 - 2^8$	$2^1 - 2^8$	$4^3 - 4^3$	$4^{12} - 5^8$	$6^{34} - 7^8$	$8^{34} - 10^8$	$10^{34} - 12^8$
5	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^8$	$2^1 - 2^3$	$4^{36} - 4^8$	$4^{36} - 4^8$	$4^{36} - 6^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^9$	$2^{11} - 2^9$	$2^3 - 2^3$	$2^1 - 2^8$	$4^{36} - 4^9$
7	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
8	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
9	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$
10	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=3, p=6 (cont.)	15	16	17	18	19	20
3	$50^2 - 50^8$	$62^2 - 65^8$	$76^2 - 80^8$	$92^2 - 98^8$	$110^2 - 119^8$	$130^2 - 140^8$
4	$13^{34} - 15^8$	$16^{34} - 20^8$	$19^{34} - 23^8$	$23^{34} - 28^8$	$28^{34} - 33^8$	$33^{34} - 40^8$
5	$6^{24} - 8^8$	$7^{34} - 10^8$	$8^{34} - 12^8$	$10^{34} - 14^8$	$11^{34} - 17^8$	$13^{34} - 20^8$
6	$4^{36} - 4^9$	$4^{36} - 5^9$	$4^{36} - 6^9$	$6^{24} - 7^9$	$6^{34} - 9^9$	$7^{34} - 10^9$
7	$2^{12} - 2^8$	$4^{36} - 4^{14}$	$4^{36} - 4^8$	$4^{36} - 4^8$	$4^{36} - 5^8$	$4^{36} - 7^8$
8	$2^{12} - 2^{11}$	$2^{12} - 2^8$	$2^{12} - 2^8$	$4^{36} - 4^8$	$4^{36} - 4^8$	$4^{36} - 4^8$
9	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$	$2^{12} - 2^8$	$4^{36} - 4^8$
10	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
12	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
15	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
16	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=3, p=7	8	9	10	11	12	13	14	15
3	$2^1 - 2^8$	$3^1 - 3^4$	$6^3 - 6^3$	$9^4 - 9^4$	$12^4 - 12^4$	$16^4 - 16^4$	$22^4 - 22^4$	$30^4 - 30^4$
4	$1^0 - 1^7$	$2^8 - 2^8$	$2^8 - 2^8$	$3^3 - 3^3$	$3^1 - 3^3$	$6^{36} - 6^8$	$6^{36} - 8^8$	$8^{34} - 10^8$
5	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$2^3 - 2^3$	$3^{18} - 3^{14}$	$3^{18} - 3^{12}$	$3^1 - 3^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$3^{18} - 3^{14}$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^8$	$2^{12} - 2^8$
8	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
9	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$
10	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=3, p=7 (cont.)	16	17	18	19	20
3	$37^2 - 39^8$	$45^2 - 49^8$	$54^2 - 60^8$	$65^2 - 72^8$	$77^2 - 87^8$
4	$10^{34} - 12^8$	$12^{34} - 14^8$	$14^{34} - 17^8$	$17^{34} - 19^8$	$20^{34} - 25^8$
5	$6^{36} - 6^8$	$6^{36} - 7^8$	$6^{36} - 10^8$	$7^{34} - 11^8$	$8^{34} - 14^8$
6	$3^{18} - 3^{14}$	$3^{18} - 3^{12}$	$3^1 - 3^8$	$6^{36} - 6^8$	$6^{36} - 6^8$
7	$2^{12} - 2^8$	$3^{18} - 3^{14}$	$3^{18} - 3^{14}$	$3^{18} - 3^{14}$	$3^{18} - 3^8$
8	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^{14}$	$3^{11} - 3^{14}$
9	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
10	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
12	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
15	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=3, p=8	9	10	11	12	13	14	15	16
3	$2^1 - 2^8$	$3^1 - 3^4$	$5^3 - 5^3$	$8^4 - 8^4$	$11^4 - 11^4$	$14^4 - 14^4$	$18^4 - 18^4$	$24^4 - 24^4$
4	$1^0 - 1^7$	$2^8 - 2^8$	$2^8 - 2^8$	$3^3 - 3^3$	$3^3 - 3^3$	$5^{36} - 5^8$	$5^{36} - 6^8$	$7^{38} - 8^8$
5	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$2^3 - 2^3$	$3^{18} - 3^{12}$	$3^{18} - 3^8$	$3^{17} - 3^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$3^{18} - 3^{14}$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^8$
9	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$
10	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=3, p=8 (cont.)	17	18	19	20
3	$30^2 - 32^8$	$36^2 - 40^8$	$43^2 - 49^8$	$51^2 - 59^8$
4	$8^{34} - 11^8$	$9^{34} - 13^8$	$11^{34} - 16^8$	$13^{34} - 17^8$
5	$5^{36} - 5^8$	$5^{36} - 5^8$	$5^{36} - 7^8$	$7^{38} - 9^8$
6	$3^{18} - 3^{12}$	$3^{18} - 3^8$	$3^{18} - 3^8$	$5^{36} - 5^8$
7	$2^{12} - 2^8$	$3^{11} - 3^{14}$	$3^{11} - 3^{14}$	$3^{11} - 3^{14}$
8	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^{14}$
9	$2^{12} - 2^{11}$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
10	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
12	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=3, p=9	10	11	12	13	14	15	16	17
3	2^1-2^8	3^4-3^4	4^1-4^3	7^4-7^4	10^4-10^4	13^4-13^4	16^4-16^4	20^4-20^4
4	1^0-1^7	2^8-2^8	2^8-2^8	3^3-3^3	3^3-3^3	4^3-4^3	4^3-4^3	$6^{38}-7^8$
5	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$	2^3-2^3	$3^{18}-3^8$	$3^{18}-3^8$	$3^{18}-3^8$
6	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$3^{18}-3^{12}$
7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$
8	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{11}-2^8$	$2^{12}-2^8$
9	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{12}-2^{11}$
10	1^0-1^0	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$

t=3, p=9 (cont.)	18	19	20
3	$24^2 - 24^8$	$29^2 - 32^8$	$35^2 - 40^8$
4	$6^{34} - 9^8$	$8^{34} - 11^8$	$9^{34} - 13^8$
5	$4^{11} - 4^{12}$	$4^{11} - 4^{12}$	$4^{11} - 4^8$
6	$3^{18} - 3^8$	$3^{18} - 3^8$	$3^{18} - 3^8$
7	$2^{12} - 2^8$	$3^{11} - 3^{14}$	$3^{11} - 3^{14}$
8	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
9	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
10	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
12	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=3, p=10	11	12	13	14	15	16	17	18
3	$2^1 - 2^8$	$3^4 - 3^4$	$4^3 - 4^3$	$6^3 - 6^3$	$9^4 - 9^4$	$12^4 - 12^4$	$15^4 - 15^4$	$18^4 - 18^4$
4	$1^0 - 1^7$	$2^8 - 2^8$	$2^{11} - 2^8$	$3^3 - 3^3$	$3^3 - 3^3$	$4^3 - 4^3$	$4^{12} - 4^8$	$5^{11} - 6^8$
5	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$2^3 - 2^3$	$3^{18} - 3^8$	$3^{18} - 3^8$	$3^{18} - 3^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^8$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=3, p=10 (cont.)	19	20
3	$22^2 - 22^8$	$26^2 - 28^8$
4	$6^{34} - 7^8$	$7^{34} - 9^8$
5	$4^{11} - 4^{12}$	$4^{11} - 4^8$
6	$3^{11} - 3^8$	$3^{11} - 3^8$
7	$2^{12} - 2^8$	$3^{11} - 3^{14}$
8	$2^{12} - 2^8$	$2^{12} - 2^8$
9	$2^{12} - 2^8$	$2^{12} - 2^8$
10	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
12	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=3, p=11	12	13	14	15	16	17	18	19	20
3	2^2-2^8	3^4-3^4	4^3-4^3	5^3-5^3	8^4-8^4	11^4-11^4	14^4-14^4	17^4-17^4	20^2-20^8
4	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$	3^3-3^3	3^3-3^3	$4^{18}-4^8$	$4^{18}-4^8$	$5^{11}-5^8$	$5^{11}-5^8$
5	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$	2^3-2^3	$3^{18}-3^8$	$3^{18}-3^8$	$3^{18}-3^8$	$4^{11}-4^8$
6	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$3^{11}-3^8$	$3^{11}-3^8$
7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$
8	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$
9	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$
10	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{12}-2^8$
11	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$

t=3, p=12	13	14	15	16	17	18	19	20
3	2^2-2^8	3^4-3^4	4^3-4^3	5^3-5^3	7^4-7^4	10^4-10^4	13^4-13^4	16^2-16^8
4	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$	3^3-3^3	$3^{12}-3^8$	$4^{11}-4^8$	$4^{11}-4^8$	$5^{11}-5^8$
5	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$	$2^{12}-2^8$	$3^{11}-3^8$	$3^{11}-3^8$	$3^{11}-3^8$
6	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$3^{11}-3^8$
7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$
8	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$
9	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$
10	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$

t=3, p=13	14	15	16	17	18	19	20
3	2^2-2^8	3^4-3^4	4^3-4^3	5^4-5^4	6^4-6^4	9^4-9^4	11^2-12^8
4	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$	$3^{18}-3^8$	$3^{18}-3^8$	$4^{11}-4^8$	$4^{11}-4^8$
5	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$3^{11}-3^8$	$3^{11}-3^8$
6	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$
7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$
8	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$
9	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$

t=3, p=14	15	16	17	18	19	20
3	$2^2 - 2^8$	$3^4 - 3^4$	$4^4 - 4^4$	$5^4 - 5^4$	$6^4 - 6^4$	$8^2 - 8^8$
4	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$	$3^{11} - 3^8$	$4^{11} - 4^8$
5	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=3, p=15	16	17	18	19	20
3	$2^2 - 2^8$	$3^4 - 3^4$	$4^4 - 4^4$	$5^4 - 5^4$	$6^2 - 6^8$
4	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$	$3^{11} - 3^8$
5	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=3, p=16	17	18	19	20
3	$2^2 - 2^8$	$3^4 - 3^4$	$4^4 - 4^4$	$5^2 - 5^8$
4	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$
5	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=3, p=17	18	19	20
3	$2^2 - 2^8$	$3^4 - 3^4$	$4^2 - 4^8$
4	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
5	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=3, p=18	19	20
3	$2^2 - 2^8$	$3^2 - 3^8$
4	$1^0 - 1^7$	$2^{11} - 2^8$

t=3, p=19	20
3	$2^2 - 2^8$

t=4, p=4	5	6	7	8	9	10	11	12	13
4	$5^1 - 5^0$	$15^1 - 15^0$	$35^1 - 35^0$	$70^1 - 70^0$	$126^1 - 126^0$	$210^1 - 210^0$	$330^1 - 330^0$	$495^1 - 495^0$	$715^1 - 715^0$
5	$1^0 - 1^0$	$5^2 - 5^4$	$9^2 - 9^4$	$20^4 - 20^4$	$30^4 - 30^4$	$50^2 - 51^4$	$66^1 - 66^4$	$113^2 - 113^4$	$149^2 - 157^4$
6	NA	$1^0 - 1^0$	$5^2 - 5^4$	$7^2 - 7^4$	$12^4 - 12^4$	$20^4 - 20^4$	$32^4 - 32^4$	$40^4 - 41^4$	$59^2 - 66^4$
7	NA	NA	$1^0 - 1^0$	$5^2 - 5^4$	$6^2 - 6^4$	$10^4 - 10^4$	$17^4 - 17^4$	$20^4 - 24^4$	$28^4 - 30^4$
8	NA	NA	NA	$1^0 - 1^0$	$5^2 - 5^4$	$5^2 - 5^4$	$9^4 - 9^4$	$12^4 - 12^4$	$18^4 - 18^4$
9	NA	NA	NA	NA	$1^0 - 1^0$	$5^2 - 5^4$	$5^2 - 5^4$	$8^4 - 8^4$	$10^4 - 10^4$
10	NA	NA	NA	NA	NA	$1^0 - 1^0$	$5^2 - 5^4$	$5^2 - 5^4$	$7^4 - 7^4$
11	NA	NA	NA	NA	NA	NA	$1^0 - 1^0$	$5^2 - 5^4$	$5^2 - 5^4$
12	NA	NA	NA	NA	NA	NA	NA	$1^0 - 1^0$	$5^2 - 5^4$

t=4, p=4 (cont.)	14	15	16	17	18	19	20
4	$1001^1 - 1001^0$	$1365^1 - 1365^0$	$1820^1 - 1820^0$	$2380^1 - 2380^0$	$3060^1 - 3060^0$	$3876^1 - 3876^0$	$4845^1 - 4845^0$
5	$219^2 - 230^4$	$274^4 - 295^4$	$397^2 - 405^4$	$476^1 - 492^4$	$659^2 - 664^4$	$787^2 - 846^4$	$1028^2 - 1083^4$
6	$75^2 - 80^4$	$93^2 - 118^4$	$144^2 - 152^4$	$173^2 - 188^9$	$205^4 - 236^9$	$298^2 - 330^9$	$344^2 - 400^4$
7	$40^4 - 44^4$	$52^4 - 57^4$	$69^2 - 76^4$	$85^4 - 100^4$	$111^4 - 131^4$	$131^4 - 167^4$	$163^2 - 217^4$
8	$23^4 - 24^4$	$29^4 - 30^4$	$30^2 - 30^4$	$49^4 - 54^4$	$57^4 - 68^4$	$74^2 - 94^4$	$83^2 - 116^4$
9	$16^4 - 16^4$	$19^4 - 21^4$	$24^4 - 27^4$	$27^2 - 28^4$	$34^4 - 42^4$	$45^4 - 51^4$	$54^4 - 72^4$
10	$9^4 - 9^4$	$12^4 - 14^4$	$16^4 - 19^4$	$19^4 - 23^4$	$24^4 - 26^4$	$27^2 - 34^4$	$30^2 - 38^4$
11	$6^2 - 6^4$	$8^4 - 8^4$	$12^4 - 12^4$	$13^4 - 16^4$	$15^4 - 20^4$	$19^4 - 23^4$	$24^4 - 29^4$
12	$5^2 - 5^4$	$5^2 - 5^4$	$7^2 - 7^4$	$10^4 - 10^4$	$12^4 - 12^4$	$15^4 - 18^4$	$15^4 - 20^4$
13	$5^2 - 5^4$	$5^2 - 5^4$	$5^2 - 5^4$	$7^4 - 7^4$	$9^4 - 9^4$	$11^4 - 11^4$	$14^4 - 16^4$
14	$1^0 - 1^0$	$5^2 - 5^4$	$5^2 - 5^4$	$5^2 - 5^4$	$6^2 - 6^4$	$9^4 - 9^4$	$10^4 - 10^4$
15	NA	$1^0 - 1^0$	$5^2 - 5^4$	$5^2 - 5^4$	$5^2 - 5^4$	$6^2 - 6^4$	$8^4 - 8^4$
16	NA	NA	$1^0 - 1^0$	$5^2 - 5^4$	$5^2 - 5^4$	$5^2 - 5^4$	$5^2 - 5^4$
17	NA	NA	NA	$1^0 - 1^0$	$5^2 - 5^{11}$	$5^2 - 5^{11}$	$5^2 - 5^{11}$
18	NA	NA	NA	NA	$1^0 - 1^0$	$5^2 - 5^{11}$	$5^2 - 5^{11}$
19	NA	NA	NA	NA	NA	$1^0 - 1^0$	$5^2 - 5^{11}$

t=4, p=5	6	7	8	9	10	11	12	13
4	$3^1 - 3^5$	$7^1 - 7^4$	$14^1 - 14^4$	$30^4 - 30^4$	$50^4 - 50^4$	$81^4 - 84^4$	$122^4 - 126^4$	$177^4 - 185^4$
5	$1^0 - 1^7$	$3^{11} - 3^9$	$5^{11} - 5^9$	$9^{11} - 9^9$	$10^{34} - 14^9$	$17^{34} - 22^9$	$25^{34} - 35^9$	$36^{34} - 50^9$
6	$1^0 - 1^0$	$1^0 - 1^7$	$3^{11} - 3^9$	$3^{11} - 3^9$	$7^{11} - 7^9$	$7^{11} - 10^9$	$10^{11} - 14^9$	$12^{34} - 21^9$
7	NA	$1^0 - 1^0$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$4^{18} - 5^{11}$	$6^{11} - 9^{11}$	$6^{33} - 13^{39}$
8	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$4^{33} - 6^{11}$
9	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$
10	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$
11	NA	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=4, p=5 (cont.)	14	15	16	17	18	19	20
4	$248^2 - 272^{11}$	$339^2 - 386^{11}$	$452^2 - 531^{11}$	$592^2 - 713^{11}$	$762^2 - 937^{11}$	$966^2 - 1210^{11}$	$1208^2 - 1537^{11}$
5	$50^{34} - 72^9$	$68^{34} - 99^9$	$91^{34} - 134^9$	$119^{34} - 193^9$	$153^{34} - 256^9$	$194^{34} - 319^9$	$242^{34} - 400^9$
6	$17^{34} - 31^9$	$23^{34} - 42^9$	$31^{34} - 54^9$	$40^{34} - 71^9$	$51^{34} - 81^9$	$65^{34} - 120^9$	$81^{34} - 149^9$
7	$8^{34} - 18^{39}$	$10^{34} - 26^{11}$	$13^{34} - 35^{39}$	$17^{34} - 43^{39}$	$22^{34} - 58^{39}$	$28^{34} - 75^{39}$	$35^{34} - 95^{39}$
8	$6^{33} - 9^{39}$	$6^{33} - 13^{39}$	$7^{33} - 15^{40}$	$9^{34} - 23^{39}$	$11^1 - 31^{39}$	$14^{34} - 38^{39}$	$18^{34} - 42^{39}$
9	$4^{33} - 5^{11}$	$5^{33} - 8^{11}$	$6^{33} - 11^{39}$	$6^{33} - 14^{39}$	$7^{33} - 18^{39}$	$8^1 - 23^{39}$	$10^1 - 30^{39}$
10	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$4^{33} - 5^{11}$	$5^{33} - 8^{11}$	$6^{33} - 11^{39}$	$6^{33} - 15^{39}$	$7^{33} - 15^{40}$
11	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$4^{33} - 5^{11}$	$4^{33} - 7^{11}$	$5^{33} - 10^{39}$	$6^{33} - 12^{39}$
12	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$4^{33} - 5^{11}$	$5^{33} - 7^{11}$
13	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$4^{33} - 5^{11}$
14	$1^0 - 1^0$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
15	NA	$1^0 - 1^0$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
16	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
17	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
18	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=4, p=6	7	8	9	10	11	12	13	14
4	$3^2 - 3^{11}$	$6^4 - 6^4$	$12^4 - 12^4$	$20^4 - 20^4$	$33^4 - 34^4$	$50^4 - 51^4$	$73^4 - 79^4$	$103^4 - 119^4$
5	$1^0 - 1^7$	$3^{11} - 3^{11}$	$4^{11} - 4^{11}$	$6^{33} - 7^{11}$	$7^{33} - 10^{11}$	$10^{34} - 18^{39}$	$15^{34} - 26^{39}$	$21^{34} - 37^{39}$
6	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^9$	$3^{11} - 3^9$	$5^{33} - 5^9$	$6^{33} - 6^9$	$7^{33} - 10^9$	$7^{33} - 14^9$
7	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$4^{33} - 4^{11}$	$5^{33} - 5^{11}$	$6^{33} - 8^{39}$
8	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{12} - 3^{24}$	$4^{33} - 4^{11}$
9	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{12} - 3^{18}$
10	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$
11	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=4, p=6 (cont.)	15	16	17	18	19	20
4	$141^2 - 175^{11}$	$188^2 - 248^{11}$	$246^2 - 340^{11}$	$317^2 - 455^{11}$	$402^2 - 597^{11}$	$503^2 - 768^{11}$
5	$29^{34} - 53^{39}$	$38^{34} - 72^{39}$	$50^{34} - 98^{39}$	$64^{34} - 129^{39}$	$81^{34} - 169^{39}$	$101^{34} - 220^{40}$
6	$10^{33} - 19^9$	$13^{34} - 26^9$	$17^{34} - 36^9$	$22^{34} - 42^9$	$27^{34} - 54^9$	$34^{34} - 66^9$
7	$7^{33} - 11^{39}$	$7^{33} - 17^{39}$	$9^{33} - 22^{39}$	$10^{33} - 29^{39}$	$12^{34} - 35^{11}$	$15^{34} - 49^{39}$
8	$6^{33} - 6^{26}$	$6^{33} - 6^{35}$	$7^{33} - 11^{39}$	$7^{33} - 14^{39}$	$9^{33} - 20^{39}$	$9^{33} - 22^{39}$
9	$4^{33} - 4^{11}$	$5^{33} - 5^{11}$	$6^{33} - 6^{39}$	$6^{33} - 8^{39}$	$7^{33} - 10^{39}$	$7^{33} - 15^{39}$
10	$3^{12} - 3^{11}$	$3^{11} - 3^{11}$	$4^{33} - 4^{11}$	$5^{33} - 5^{11}$	$6^{33} - 6^{39}$	$6^{33} - 6^{35}$
11	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{14}$	$4^{33} - 4^{11}$	$4^{33} - 4^{11}$	$5^{33} - 5^{40}$
12	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{24}$	$4^{33} - 4^{11}$
13	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{24}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
15	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
16	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
17	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=4, p=7	8	9	10	11	12	13	14	15
4	$2^1 - 2^8$	$5^4 - 5^4$	$10^4 - 10^4$	$17^4 - 17^4$	$26^4 - 26^4$	$38^4 - 40^4$	$54^4 - 58^4$	$74^4 - 81^4$
5	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^1 - 2^8$	$5^{11} - 5^{11}$	$6^{34} - 9^{40}$	$8^{34} - 14^8$	$11^{34} - 18^8$	$15^{34} - 29^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$2^8 - 2^{11}$	$2^{11} - 2^{11}$	$2^1 - 2^8$	$5^{11} - 5^{40}$	$5^{11} - 8^8$	$5^{34} - 10^{35}$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^1 - 2^3$	$4^{18} - 5^{40}$
8	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{11} - 2^{11}$
9	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
10	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$
11	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=4, p=7 (cont.)	16	17	18	19	20
4	$99^2 - 121^{11}$	$130^2 - 173^{11}$	$168^2 - 238^{11}$	$213^2 - 319^{11}$	$267^2 - 419^{11}$
5	$20^{34} - 39^8$	$26^{34} - 50^8$	$34^{34} - 60^8$	$43^{34} - 81^8$	$54^{34} - 96^8$
6	$7^{34} - 14^8$	$9^{34} - 19^8$	$12^{34} - 24^8$	$15^{34} - 32^8$	$18^{34} - 40^8$
7	$4^{13} - 7^8$	$4^{34} - 11^8$	$5^{34} - 12^8$	$7^{34} - 16^8$	$8^{34} - 20^8$
8	$2^{12} - 2^8$	$4^{18} - 5^{40}$	$4^{13} - 6^8$	$4^{34} - 9^{39}$	$4^{34} - 10^8$
9	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$	$4^{18} - 6^8$	$4^{13} - 6^8$
10	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
12	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
13	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
15	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
16	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=4, p=8	9	10	11	12	13	14	15	16
4	2^1-2^8	4^4-4^4	8^4-8^4	12^4-12^4	19^4-19^4	27^4-29^4	37^4-42^4	50^4-59^4
5	1^0-1^7	$2^{11}-2^8$	2^8-2^8	$4^{36}-4^8$	$4^{36}-6^8$	$6^{34}-10^8$	$8^{34}-14^8$	$10^{34}-18^8$
6	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{11}-2^8$	$2^{11}-2^8$	$4^{36}-4^8$	$4^{36}-4^8$	$4^{36}-7^{26}$
7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{11}-2^{11}$	$2^{12}-2^8$	2^3-2^3	$4^{36}-4^8$
8	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{11}-2^{11}$	$2^{12}-2^{11}$	2^3-2^3
9	1^0-1^0	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{11}-2^{11}$	$2^{12}-2^{11}$
10	NA	1^0-1^0	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{11}-2^{11}$
11	NA	NA	1^0-1^0	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$

t=4, p=8 (cont.)	17	18	19	20
4	$66^2 - 85^8$	$85^2 - 119^8$	$108^2 - 154^8$	$135^2 - 196^8$
5	$14^{34} - 23^8$	$17^{34} - 31^8$	$22^{34} - 42^8$	$27^{34} - 52^8$
6	$6^{18} - 10^8$	$6^{34} - 14^8$	$8^{34} - 17^8$	$9^{34} - 21^8$
7	$4^{36} - 4^8$	$4^{36} - 6^8$	$4^{36} - 9^8$	$6^{18} - 11^8$
8	$2^{12} - 2^8$	$4^{36} - 4^8$	$4^{36} - 4^8$	$4^{36} - 4^8$
9	$2^{12} - 2^{11}$	$2^{12} - 2^8$	$2^{12} - 2^8$	$4^{36} - 4^8$
10	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
12	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
13	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
15	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=4, p=9	10	11	12	13	14	15	16	17
4	$2^1 - 2^8$	$4^4 - 4^4$	$6^4 - 6^4$	$10^4 - 10^4$	$14^4 - 14^4$	$21^4 - 21^4$	$28^4 - 28^4$	$37^4 - 42^4$
5	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$4^{36} - 4^8$	$4^{36} - 5^8$	$5^{34} - 8^8$	$6^{34} - 10^8$	$8^{34} - 14^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$2^{12} - 2^8$	$4^{36} - 4^8$	$4^{36} - 4^8$	$4^{36} - 6^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^3 - 2^3$	$4^{36} - 4^8$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^8$	$2^{12} - 2^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
10	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
11	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=4, p=9 (cont.)	18	19	20
4	$48^2 - 60^8$	$61^2 - 80^8$	$77^2 - 100^8$
5	$10^{34} - 18^8$	$13^{34} - 23^8$	$16^{34} - 28^8$
6	$5^{18} - 6^8$	$5^{34} - 10^8$	$6^{34} - 13^8$
7	$4^{36} - 4^8$	$4^{36} - 5^8$	$4^{36} - 6^8$
8	$2^{12} - 2^8$	$4^{36} - 4^8$	$4^{36} - 4^8$
9	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
10	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
12	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
13	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=4, p=10	11	12	13	14	15	16	17	18
4	$2^2 - 2^8$	$3^4 - 3^4$	$6^4 - 6^4$	$9^4 - 9^4$	$13^4 - 13^4$	$19^4 - 19^4$	$25^4 - 26^4$	$33^4 - 34^4$
5	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$3^{18} - 3^{12}$	$3^3 - 3^3$	$5^{24} - 6^8$	$5^{34} - 9^8$	$7^{34} - 12^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$2^{12} - 2^8$	$3^3 - 3^3$	$3^{12} - 3^{12}$	$3^{12} - 3^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$3^{18} - 3^{14}$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
11	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=4, p=10 (cont.)	19	20
4	$42^2 - 48^8$	$53^2 - 64^8$
5	$9^{34} - 15^8$	$11^{34} - 19^8$
6	$5^{24} - 6^8$	$5^{13} - 8^8$
7	$3^{18} - 3^{14}$	$3^{18} - 3^8$
8	$2^{12} - 2^8$	$3^{11} - 3^{14}$
9	$2^{12} - 2^8$	$2^{12} - 2^8$
10	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
12	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
13	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=4, p=11	12	13	14	15	16	17	18	19	20
4	2^2-2^8	3^4-3^4	5^4-5^4	8^4-8^4	12^4-12^4	17^4-17^4	24^4-24^4	31^4-31^4	39^2-40^8
5	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$	$3^{18}-3^8$	3^3-3^3	$5^{36}-5^8$	$5^{36}-7^8$	$7^{34}-10^8$	$8^{34}-14^8$
6	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$	$2^{12}-2^8$	$3^{18}-3^{12}$	$3^{18}-3^8$	$3^{18}-3^8$	$5^{36}-5^8$
7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$3^{11}-3^{14}$	$3^{11}-3^{14}$
8	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$
9	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{12}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$
10	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{12}-2^{11}$	$2^{12}-2^8$
11	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{12}-2^{11}$
12	1^0-1^0	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$

t=4, p=12	13	14	15	16	17	18	19	20
4	2^2-2^8	3^4-3^4	5^4-5^4	7^4-7^4	11^4-11^4	15^4-15^4	19^4-19^4	26^4-26^4
5	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$	$3^{18}-3^8$	$3^{18}-3^8$	$5^{36}-5^8$	$5^{36}-6^8$	$6^{34}-9^8$
6	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$	$2^{12}-2^8$	$3^{18}-3^8$	$3^{18}-3^8$	$3^{18}-3^8$
7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$3^{11}-3^{14}$
8	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$
9	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$
10	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{12}-2^8$
11	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$

t=4, p=13	14	15	16	17	18	19	20
4	$2^2 - 2^8$	$3^4 - 3^4$	$4^4 - 4^4$	$7^4 - 7^4$	$9^4 - 9^4$	$13^4 - 13^4$	$17^4 - 17^4$
5	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$3^{18} - 3^8$	$3^{18} - 3^8$	$4^{11} - 4^{12}$	$4^{11} - 4^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$	$3^{11} - 3^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=4, p=14	15	16	17	18	19	20
4	$2^2 - 2^8$	$3^4 - 3^4$	$4^4 - 4^4$	$6^4 - 6^4$	$9^4 - 9^4$	$12^4 - 12^4$
5	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$3^{18} - 3^8$	$3^{18} - 3^8$	$4^{11} - 4^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=4, p=15	16	17	18	19	20
4	$2^2 - 2^8$	$3^4 - 3^4$	$4^4 - 4^4$	$6^4 - 6^4$	$8^4 - 8^4$
5	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$	$3^{11} - 3^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=4, p=16	17	18	19	20
4	$2^2 - 2^8$	$3^4 - 3^4$	$4^4 - 4^4$	$5^4 - 5^4$
5	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=4, p=17	18	19	20
4	$2^2 - 2^8$	$3^4 - 3^4$	$4^4 - 4^4$
5	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
6	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=4, p=18	19	20
4	$2^2 - 2^8$	$3^4 - 3^4$
5	$1^0 - 1^7$	$2^{11} - 2^8$

t=4, p=19	20
4	$2^2 - 2^8$

t=5, p=5	6	7	8	9	10	11	12	13	14
5	6^1-6^0	21^1-21^0	56^1-56^0	126^1-126^0	252^1-252^0	462^1-462^0	792^1-792^0	1287^1-1287^0	2002^1-2002^0
6	1^0-1^0	6^2-6^4	12^2-12^4	30^4-30^4	50^4-50^4	92^2-100^4	132^1-132^4	245^2-245^4	348^2-371^4
7	NA	1^0-1^0	6^2-6^4	9^2-9^4	20^4-20^4	33^4-34^4	55^4-59^4	75^4-78^4	118^2-138^4
8	NA	NA	1^0-1^0	6^2-6^4	8^2-8^4	16^4-16^4	26^4-26^4	33^4-43^4	49^4-55^4
9	NA	NA	NA	1^0-1^0	6^2-6^4	7^2-7^4	12^4-12^4	19^4-19^4	28^4-33^4
10	NA	NA	NA	NA	1^0-1^0	6^2-6^4	6^2-6^4	11^4-11^4	14^4-14^4
11	NA	NA	NA	NA	NA	1^0-1^0	6^2-6^4	6^2-6^4	10^4-10^4
12	NA	NA	NA	NA	NA	NA	1^0-1^0	6^2-6^4	6^2-6^4
13	NA	NA	NA	NA	NA	NA	NA	1^0-1^0	6^2-6^4

t=5, p=5 (cont.)	15	16	17	18	19	20
5	$3003^1 - 3003^0$	$4368^1 - 4368^0$	$6188^1 - 6188^0$	$8568^1 - 8568^0$	$11628^1 - 11628^0$	$15504^1 - 15504^0$
6	$548^2 - 580^4$	$731^4 - 808^4$	$1125^2 - 1213^4$	$1428^1 - 1547^4$	$2087^2 - 2175^4$	$2624^2 - 2900^4$
7	$161^2 - 190^4$	$213^2 - 285^4$	$350^2 - 405^4$	$445^2 - 589^4$	$557^4 - 706^4$	$852^2 - 1003^4$
8	$75^4 - 89^4$	$104^4 - 117^4$	$147^2 - 190^4$	$192^4 - 269^4$	$264^4 - 381^4$	$328^4 - 520^4$
9	$39^4 - 42^4$	$52^4 - 62^4$	$58^4 - 81^4$	$98^4 - 124^4$	$121^4 - 173^4$	$165^2 - 247^4$
10	$24^4 - 27^4$	$31^4 - 38^4$	$41^4 - 49^4$	$49^2 - 54^4$	$65^4 - 94^4$	$90^4 - 120^4$
11	$13^4 - 13^4$	$18^4 - 22^4$	$25^4 - 32^4$	$32^4 - 43^4$	$42^4 - 52^4$	$50^2 - 85^4$
12	$9^4 - 9^4$	$12^4 - 12^4$	$17^4 - 17^4$	$20^4 - 24^4$	$24^4 - 39^4$	$32^4 - 42^4$
13	$6^2 - 6^4$	$8^4 - 8^4$	$11^4 - 11^4$	$15^4 - 15^4$	$18^4 - 21^4$	$24^4 - 34^4$
14	$6^2 - 6^4$	$6^2 - 6^4$	$7^2 - 7^4$	$9^4 - 9^4$	$14^4 - 14^4$	$16^4 - 18^4$
15	$1^0 - 1^0$	$6^2 - 6^4$	$6^2 - 6^4$	$6^2 - 6^4$	$9^4 - 9^4$	$12^4 - 12^4$
16	NA	$1^0 - 1^0$	$6^2 - 6^4$	$6^2 - 6^4$	$6^2 - 6^4$	$8^2 - 8^4$
17	NA	NA	$1^0 - 1^0$	$6^2 - 6^{11}$	$6^2 - 6^{11}$	$6^2 - 6^{11}$
18	NA	NA	NA	$1^0 - 1^0$	$6^2 - 6^{11}$	$6^2 - 6^{11}$
19	NA	NA	NA	NA	$1^0 - 1^0$	$6^2 - 6^{11}$

t=5, p=6	7	8	9	10	11	12	13	14
5	$4^2 - 4^5$	$11^4 - 11^4$	$25^4 - 25^4$	$50^4 - 51^4$	$92^4 - 100^4$	$158^4 - 177^4$	$257^4 - 297^4$	$400^4 - 471^4$
6	$1^0 - 1^7$	$4^{11} - 4^9$	$7^{11} - 7^9$	$12^{11} - 14^9$	$17^{11} - 22^9$	$27^{34} - 38^9$	$43^{34} - 61^9$	$67^{34} - 100^9$
7	$1^0 - 1^0$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$5^{11} - 5^{11}$	$8^{11} - 10^{11}$	$10^{11} - 18^{39}$	$15^{11} - 31^{29}$	$20^{34} - 43^{40}$
8	NA	$1^0 - 1^0$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$	$6^{11} - 7^{11}$	$8^{11} - 14^{11}$	$9^{11} - 21^{11}$
9	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$	$6^{11} - 6^{11}$	$6^{11} - 10^{11}$
10	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$	$5^{11} - 5^{11}$
11	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$
12	NA	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$4^{11} - 4^{11}$

t=5, p=6 (cont.)	15	16	17	18	19	20
5	$600^{2} - 743^{26}$	$873^{2} - 1129^{26}$	$1237^{2} - 1660^{26}$	$1713^{2} - 2373^{26}$	$2325^{2} - 3308^{11}$	$3100^{2} - 4509^{11}$
6	$100^{34} - 152^{9}$	$146^{34} - 251^{9}$	$207^{34} - 374^{9}$	$286^{34} - 544^{9}$	$388^{34} - 740^{9}$	$517^{34} - 929^{9}$
7	$29^{34} - 72^{40}$	$42^{34} - 105^{11}$	$59^{34} - 148^{40}$	$82^{34} - 206^{40}$	$111^{34} - 277^{40}$	$148^{34} - 369^{40}$
8	$12^{11} - 33^{11}$	$17^{11} - 46^{40}$	$23^{34} - 64^{11}$	$31^{34} - 93^{11}$	$42^{34} - 126^{40}$	$56^{34} - 171^{40}$
9	$7^{11} - 16^{39}$	$9^{11} - 24^{39}$	$12^{11} - 35^{39}$	$14^{34} - 49^{40}$	$19^{34} - 63^{40}$	$25^{34} - 86^{40}$
10	$5^{11} - 8^{11}$	$6^{11} - 12^{39}$	$8^{11} - 18^{39}$	$9^{11} - 25^{39}$	$10^{34} - 34^{39}$	$13^{34} - 49^{40}$
11	$4^{11} - 4^{11}$	$5^{14} - 6^{11}$	$5^{11} - 10^{11}$	$6^{11} - 15^{11}$	$8^{11} - 21^{39}$	$8^{11} - 28^{39}$
12	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$	$5^{11} - 5^{11}$	$5^{12} - 7^{11}$	$5^{11} - 12^{11}$	$6^{11} - 17^{39}$
13	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$	$5^{11} - 5^{11}$	$5^{12} - 7^{11}$	$5^{12} - 10^{11}$
14	$1^{0} - 1^{7}$	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$	$4^{12} - 4^{11}$	$4^{12} - 4^{18}$	$4^{12} - 6^{11}$
15	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$	$4^{12} - 4^{11}$	$4^{12} - 4^{11}$
16	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$	$4^{12} - 4^{11}$
17	NA	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$
18	NA	NA	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$4^{11} - 4^{11}$

t=5, p=7	8	9	10	11	12	13	14	15
5	$3^{2} - 3^{11}$	$8^{4} - 8^{4}$	$17^{4} - 17^{4}$	$32^{4} - 32^{4}$	$55^{4} - 59^{4}$	$90^{4} - 101^{4}$	$140^{4} - 166^{4}$	$210^{4} - 264^{4}$
6	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$	$4^{11} - 4^{11}$	$8^{11} - 9^{11}$	$10^{34} - 14^{11}$	$15^{34} - 31^{11}$	$24^{34} - 48^{11}$	$35^{34} - 74^{39}$
7	$1^{0} - 1^{7}$	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$5^{33} - 6^{11}$	$7^{11} - 10^{11}$	$8^{33} - 18^{39}$	$10^{34} - 29^{40}$
8	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$4^{33} - 4^{11}$	$5^{33} - 8^{11}$	$6^{33} - 13^{39}$
9	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{12} - 3^{18}$	$4^{33} - 4^{11}$
10	NA	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
11	NA	NA	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$
12	NA	NA	NA	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$

t=5, p=7 (cont.)	16	17	18	19	20
5	$306^2 - 422^{11}$	$434^2 - 643^{11}$	$601^2 - 945^{11}$	$816^2 - 1348^{11}$	$1088^2 - 1871^{11}$
6	$51^{34} - 110^{40}$	$73^{34} - 163^{11}$	$101^{34} - 232^{40}$	$136^{34} - 321^{40}$	$182^{34} - 438^{40}$
7	$15^{34} - 43^{40}$	$21^{34} - 61^{40}$	$29^{34} - 84^{40}$	$39^{34} - 114^{40}$	$52^{34} - 156^{40}$
8	$7^{33} - 19^{39}$	$9^{33} - 27^{39}$	$11^{34} - 39^{40}$	$15^{34} - 53^{40}$	$20^{34} - 71^{40}$
9	$6^{33} - 9^{11}$	$6^{33} - 13^{39}$	$7^{33} - 19^{39}$	$7^{33} - 28^{39}$	$10^{33} - 37^{11}$
10	$4^{33} - 4^{11}$	$5^{33} - 6^{11}$	$6^{33} - 9^{39}$	$6^{33} - 14^{39}$	$7^{33} - 19^{39}$
11	$3^{12} - 3^{11}$	$3^{12} - 3^{18}$	$4^{33} - 4^{11}$	$5^{33} - 7^{11}$	$6^{33} - 11^{39}$
12	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$4^{33} - 4^{11}$	$4^{33} - 4^{11}$
13	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
14	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
15	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
16	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
17	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=5, p=8	9	10	11	12	13	14	15	16
5	$3^2 - 3^{11}$	$6^4 - 6^4$	$11^4 - 11^4$	$20^4 - 24^4$	$33^4 - 43^4$	$52^4 - 77^4$	$78^4 - 120^4$	$114^4 - 191^4$
6	$1^0 - 1^7$	$3^{11} - 3^{11}$	$4^{11} - 4^{11}$	$6^{11} - 6^{11}$	$6^{34} - 13^{39}$	$9^{34} - 21^{39}$	$13^{34} - 32^{39}$	$19^{34} - 50^{39}$
7	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$5^{11} - 5^{11}$	$6^{11} - 8^{39}$	$6^{11} - 13^{39}$	$6^{34} - 20^{39}$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$4^{11} - 4^{11}$	$4^{12} - 5^{11}$	$4^{12} - 8^{39}$
9	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{12} - 3^{21}$	$4^{11} - 4^{11}$
10	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{12} - 3^{21}$
11	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$
12	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=5, p=8 (cont.)	17	18	19	20
5	$162^2 - 301^{11}$	$225^2 - 455^{11}$	$306^2 - 662^{11}$	$408^2 - 935^{11}$
6	$27^{34} - 76^{39}$	$38^{34} - 109^{11}$	$51^{34} - 152^{11}$	$68^{34} - 228^{11}$
7	$8^{34} - 29^{11}$	$11^{34} - 41^{11}$	$15^{34} - 55^{11}$	$20^{34} - 77^{11}$
8	$4^{12} - 11^{39}$	$5^{34} - 19^{39}$	$6^{34} - 27^{11}$	$8^{34} - 39^{39}$
9	$4^{12} - 6^{26}$	$4^{12} - 8^{39}$	$4^{12} - 12^{39}$	$4^{34} - 17^{39}$
10	$4^{11} - 4^{11}$	$4^{12} - 5^{11}$	$4^{12} - 7^{26}$	$4^{12} - 9^{39}$
11	$3^{12} - 3^{18}$	$3^{12} - 3^{21}$	$4^{11} - 4^{11}$	$4^{12} - 5^{11}$
12	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{21}$	$4^{11} - 4^{11}$
13	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{21}$
14	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
15	$1^{0} - 1^{7}$	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
16	$1^{0} - 1^{7}$	$1^{0} - 1^{7}$	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$

t=5, p=9	10	11	12	13	14	15	16	17
5	$2^1 - 2^8$	$6^4 - 6^4$	$11^4 - 11^4$	$18^4 - 18^4$	$28^4 - 33^4$	$42^4 - 59^4$	$62^4 - 92^4$	$88^4 - 149^4$
6	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^8$	$6^{11} - 6^{24}$	$6^{11} - 9^{39}$	$7^{34} - 16^{39}$	$11^{34} - 23^{39}$	$15^{34} - 39^{40}$
7	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{11} - 2^8$	$6^{11} - 6^{26}$	$6^{11} - 9^{29}$	$6^{11} - 14^{39}$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$	$4^{18} - 6^{26}$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
10	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
11	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
12	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=5, p=9 (cont.)	18	19	20
5	$122^2 - 232^{11}$	$166^2 - 346^{11}$	$222^2 - 498^{11}$
6	$21^{34} - 56^{11}$	$28^{34} - 80^8$	$37^{34} - 100^8$
7	$7^{30} - 18^8$	$8^{34} - 29^8$	$11^{34} - 40^8$
8	$4^{18} - 7^{40}$	$4^{17} - 13^{39}$	$4^{34} - 16^8$
9	$2^{12} - 2^8$	$4^{18} - 6^{26}$	$4^{18} - 7^{40}$
10	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
12	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
13	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
14	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
15	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=5, p=10	11	12	13	14	15	16	17	18
5	$2^1 - 2^8$	$5^4 - 5^4$	$10^4 - 10^4$	$16^4 - 16^4$	$24^4 - 27^4$	$35^4 - 45^4$	$50^4 - 64^4$	$70^4 - 104^4$
6	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$5^{11} - 5^8$	$5^{11} - 8^8$	$6^{34} - 12^{39}$	$9^{34} - 19^8$	$12^{34} - 26^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^8$	$2^{11} - 2^8$	$5^{11} - 5^8$	$5^{11} - 6^8$	$5^{11} - 10^{40}$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^8$	$2^{12} - 2^8$	$4^{18} - 5^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
11	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
12	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=5, p=10 (cont.)	19	20
5	$95^2 - 156^8$	$127^2 - 226^8$
6	$16^{34} - 34^8$	$22^{34} - 50^8$
7	$6^{30} - 14^8$	$7^{34} - 19^8$
8	$4^{13} - 5^8$	$4^{13} - 8^8$
9	$2^{12} - 2^8$	$4^{18} - 5^8$
10	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
12	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
13	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
14	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=5, p=11	12	13	14	15	16	17	18	19	20
5	$2^2 - 2^8$	$4^4 - 4^4$	$8^4 - 8^4$	$12^4 - 14^4$	$18^4 - 22^4$	$26^4 - 36^4$	$36^4 - 50^4$	$49^4 - 76^4$	$66^2 - 102^8$
6	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$4^{36} - 4^8$	$4^{36} - 5^8$	$5^{34} - 10^8$	$6^{34} - 12^{40}$	$9^{34} - 20^8$	$11^{34} - 26^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$2^{12} - 2^8$	$4^{36} - 4^8$	$4^{36} - 4^8$	$4^{36} - 7^8$	$5^{21} - 10^8$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$4^{36} - 4^8$	$4^{36} - 4^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
12	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
13	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=5, p=12	13	14	15	16	17	18	19	20
5	$2^2 - 2^8$	$4^4 - 4^4$	$7^4 - 7^4$	$12^4 - 12^4$	$17^4 - 17^4$	$24^4 - 28^4$	$33^4 - 42^4$	$44^4 - 57^4$
6	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$4^{36} - 4^8$	$4^{36} - 5^8$	$5^{18} - 7^8$	$6^{34} - 11^8$	$8^{34} - 16^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$2^{12} - 2^8$	$4^{36} - 4^8$	$4^{36} - 4^8$	$4^{36} - 6^8$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$4^{36} - 4^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=5, p=13	14	15	16	17	18	19	20
5	$2^2 - 2^8$	$3^4 - 3^4$	$6^4 - 6^4$	$10^4 - 10^4$	$15^4 - 15^4$	$21^4 - 22^4$	$28^4 - 34^4$
6	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$3^{18} - 3^{12}$	$3^{18} - 3^8$	$5^{18} - 7^8$	$5^{34} - 8^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$3^{18} - 3^{14}$	$3^{18} - 3^8$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^8$
11	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=5, p=14	15	16	17	18	19	20
5	$2^2 - 2^8$	$3^4 - 3^4$	$6^4 - 6^4$	$9^4 - 9^4$	$14^4 - 14^4$	$19^4 - 19^4$
6	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$3^{18} - 3^8$	$3^{18} - 3^8$	$4^{11} - 6^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$3^{11} - 3^{14}$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=5, p=15	16	17	18	19	20
5	$2^2 - 2^8$	$3^4 - 3^4$	$5^4 - 5^4$	$9^4 - 9^4$	$12^4 - 12^4$
6	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$3^{18} - 3^8$	$3^{18} - 3^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=5, p=16	17	18	19	20
5	$2^2 - 2^8$	$3^4 - 3^4$	$5^4 - 5^4$	$8^4 - 8^4$
6	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$3^{11} - 3^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
8	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=5, p=17	18	19	20
5	$2^2 - 2^8$	$3^4 - 3^4$	$4^4 - 4^4$
6	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$
7	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=5, p=18	19	20
5	$2^2 - 2^8$	$3^4 - 3^4$
6	$1^0 - 1^7$	$2^{11} - 2^8$

t=5, p=19	20
5	$2^2 - 2^8$

t=6, p=6	7	8	9	10	11	12	13	14	15
6	$7^1 - 7^0$	$28^1 - 28^0$	$84^1 - 84^0$	$210^1 - 210^0$	$462^1 - 462^0$	$924^1 - 924^0$	$1716^1 - 1716^0$	$3003^1 - 3003^0$	$5005^1 - 5005^0$
7	$1^0 - 1^0$	$7^2 - 7^4$	$16^2 - 16^4$	$45^4 - 45^4$	$81^4 - 84^4$	$158^2 - 177^4$	$251^4 - 264^4$	$490^2 - 508^4$	$746^2 - 825^4$
8	NA	$1^0 - 1^0$	$7^2 - 7^4$	$12^2 - 12^4$	$29^4 - 29^4$	$50^4 - 51^4$	$90^4 - 101^4$	$132^4 - 159^4$	$222^2 - 283^4$
9	NA	NA	$1^0 - 1^0$	$7^2 - 7^4$	$10^2 - 10^4$	$22^4 - 22^4$	$38^4 - 40^4$	$52^4 - 77^4$	$82^4 - 100^4$
10	NA	NA	NA	$1^0 - 1^0$	$7^2 - 7^4$	$9^2 - 9^4$	$16^4 - 16^4$	$27^4 - 29^4$	$42^4 - 59^4$
11	NA	NA	NA	NA	$1^0 - 1^0$	$7^2 - 7^4$	$8^2 - 8^4$	$14^4 - 14^4$	$21^4 - 21^4$
12	NA	NA	NA	NA	NA	$1^0 - 1^0$	$7^2 - 7^4$	$7^2 - 7^4$	$13^4 - 13^4$
13	NA	NA	NA	NA	NA	NA	$1^0 - 1^0$	$7^2 - 7^4$	$7^2 - 7^4$
14	NA	NA	NA	NA	NA	NA	NA	$1^0 - 1^0$	$7^2 - 7^4$

t=6, p=6 (cont.)	16	17	18	19	20
6	$8008^1 - 8008^0$	$12376^1 - 12376^0$	$18564^1 - 18564^0$	$27132^1 - 27132^0$	$38760^1 - 38760^0$
7	$1253^2 - 1329^4$	$1776^4 - 2048^4$	$2893^2 - 3261^4$	$3876^1 - 4608^4$	$5963^2 - 6765^4$
8	$322^2 - 408^4$	$453^2 - 620^4$	$788^2 - 918^4$	$1057^2 - 1507^4$	$1393^4 - 1995^4$
9	$134^4 - 172^4$	$197^4 - 272^4$	$294^2 - 446^4$	$406^4 - 630^4$	$587^4 - 1002^4$
10	$63^4 - 84^4$	$89^4 - 146^4$	$105^4 - 176^4$	$187^4 - 287^4$	$242^4 - 433^4$
11	$35^4 - 45^4$	$48^4 - 63^4$	$68^4 - 102^4$	$85^2 - 137^4$	$119^4 - 224^4$
12	$19^4 - 19^4$	$26^4 - 36^4$	$38^4 - 48^4$	$51^4 - 81^4$	$70^4 - 102^4$
13	$12^4 - 12^4$	$17^4 - 17^4$	$24^4 - 28^4$	$30^4 - 42^4$	$37^4 - 66^4$
14	$7^2 - 7^4$	$11^4 - 11^4$	$15^4 - 15^4$	$21^4 - 22^4$	$26^4 - 32^4$
15	$7^2 - 7^4$	$7^2 - 7^4$	$10^4 - 10^4$	$13^4 - 13^4$	$19^4 - 19^4$
16	$1^0 - 1^0$	$7^2 - 7^4$	$7^2 - 7^4$	$9^4 - 9^4$	$12^4 - 12^4$
17	NA	$1^0 - 1^0$	$7^2 - 7^{11}$	$7^2 - 7^{11}$	$8^2 - 8^{11}$
18	NA	NA	$1^0 - 1^0$	$7^2 - 7^{11}$	$7^2 - 7^{11}$
19	NA	NA	NA	$1^0 - 1^0$	$7^2 - 7^{11}$

t=6, p=7	8	9	10	11	12	13	14	15
6	$4^{1} - 4^{5}$	$12^{4} - 12^{4}$	$30^{4} - 30^{4}$	$66^{4} - 66^{4}$	$132^{1} - 132^{4}$	$251^{1} - 264^{4}$	$445^{1} - 471^{4}$	$751^{1} - 789^{4}$
7	$1^{0} - 1^{7}$	$4^{11} - 4^{11}$	$8^{11} - 8^{11}$	$16^{11} - 19^{11}$	$25^{11} - 35^{11}$	$43^{11} - 61^{11}$	$64^{1} - 122^{26}$	$115^{11} - 219^{29}$
8	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$4^{11} - 4^{11}$	$6^{11} - 6^{11}$	$10^{11} - 12^{11}$	$15^{11} - 26^{39}$	$24^{11} - 48^{40}$	$32^{11} - 81^{40}$
9	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$	$8^{11} - 10^{11}$	$11^{11} - 18^{11}$	$13^{11} - 32^{39}$
10	NA	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$	$6^{11} - 8^{11}$	$8^{11} - 14^{11}$
11	NA	NA	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$	$5^{11} - 6^{11}$
12	NA	NA	NA	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$
13	NA	NA	NA	NA	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$4^{11} - 4^{11}$

t=6, p=7 (cont.)	16	17	18	19	20
6	$1214^{1} - 1377^{11}$	$1891^{1} - 2280^{11}$	$2856^{1} - 3616^{11}$	$4199^{1} - 5535^{11}$	$6030^{1} - 8220^{11}$
7	$174^{1} - 366^{29}$	$271^{1} - 603^{40}$	$408^{1} - 913^{40}$	$600^{1} - 1346^{40}$	$862^{1} - 1985^{40}$
8	$50^{11} - 118^{40}$	$68^{1} - 208^{40}$	$102^{1} - 306^{11}$	$152^{11} - 442^{11}$	$216^{1} - 615^{40}$
9	$19^{11} - 52^{40}$	$29^{11} - 82^{40}$	$40^{11} - 129^{40}$	$56^{11} - 186^{40}$	$74^{11} - 268^{40}$
10	$11^{11} - 23^{39}$	$14^{11} - 37^{39}$	$19^{11} - 59^{40}$	$21^{11} - 85^{40}$	$34^{11} - 124^{40}$
11	$6^{11} - 10^{11}$	$9^{11} - 19^{11}$	$11^{11} - 29^{39}$	$14^{11} - 45^{39}$	$16^{11} - 63^{39}$
12	$4^{11} - 4^{11}$	$5^{11} - 9^{11}$	$6^{11} - 12^{40}$	$8^{11} - 24^{11}$	$10^{11} - 34^{39}$
13	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$	$5^{11} - 7^{11}$	$6^{11} - 11^{11}$	$7^{11} - 16^{11}$
14	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$	$4^{12} - 4^{11}$	$5^{11} - 6^{11}$	$5^{11} - 8^{11}$
15	$1^{0} - 1^{7}$	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$	$4^{12} - 4^{11}$	$4^{12} - 4^{11}$
16	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$	$4^{12} - 4^{11}$
17	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$
18	NA	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$4^{11} - 4^{11}$

t=6, p=8	9	10	11	12	13	14	15	16
6	3^2-3^{11}	9^4-9^4	20^4-20^4	40^4-41^4	75^4-78^4	132^4-159^4	220^4-279^4	352^4-448^4
7	1^0-1^7	$3^{11}-3^{11}$	$6^{11}-6^{11}$	$9^{11}-12^{11}$	$12^{11}-21^{11}$	$20^{11}-43^{11}$	$32^{34}-81^{40}$	$51^{34}-134^{11}$
8	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$	$3^{11}-3^{11}$	$6^{11}-9^{11}$	$7^{11}-14^{11}$	$10^{11}-29^{40}$	$13^{34}-44^{40}$
9	1^0-1^0	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$	$3^{11}-3^{11}$	$4^{33}-6^{11}$	$5^{33}-10^{39}$	$6^{33}-20^{39}$
10	NA	1^0-1^0	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$	$3^{11}-3^{11}$	$3^{11}-3^{11}$	$4^{33}-7^{26}$
11	NA	NA	1^0-1^0	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$	$3^{11}-3^{11}$	$3^{12}-3^{11}$
12	NA	NA	NA	1^0-1^0	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$	$3^{11}-3^{11}$
13	NA	NA	NA	NA	1^0-1^0	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$

t=6, p=8 (cont.)	17	18	19	20
6	$544^2 - 791^{11}$	$816^2 - 1319^{11}$	$1193^2 - 2099^{11}$	$1705^2 - 3217^{11}$
7	$78^{34} - 212^{40}$	$117^{34} - 333^{40}$	$171^{34} - 489^{40}$	$244^{34} - 713^{40}$
8	$20^{34} - 68^{40}$	$30^{34} - 109^{40}$	$43^{34} - 162^{40}$	$61^{34} - 235^{40}$
9	$8^{11} - 31^{11}$	$11^{11} - 47^{40}$	$15^{34} - 65^{11}$	$21^{34} - 118^{39}$
10	$6^{33} - 14^{39}$	$6^{33} - 21^{39}$	$7^{33} - 29^{39}$	$9^{34} - 49^{39}$
11	$4^{33} - 6^{11}$	$5^{33} - 10^{40}$	$6^{33} - 16^{39}$	$6^{33} - 21^{39}$
12	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$4^{33} - 7^{11}$	$5^{33} - 10^{39}$
13	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$4^{33} - 6^{11}$
14	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
15	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
16	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
17	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=6, p=9	10	11	12	13	14	15	16	17
6	$3^2 - 3^{11}$	$7^4 - 7^4$	$15^4 - 15^4$	$28^4 - 30^4$	$49^4 - 55^4$	$82^4 - 100^4$	$132^4 - 186^4$	$204^4 - 333^4$
7	$1^0 - 1^7$	$3^{11} - 3^{11}$	$4^{11} - 4^{11}$	$7^{11} - 9^{11}$	$8^{11} - 18^{39}$	$12^{34} - 33^{40}$	$19^{34} - 52^{40}$	$30^{34} - 86^{11}$
8	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$5^{11} - 5^{11}$	$7^{11} - 11^{39}$	$7^{11} - 19^{39}$	$8^{34} - 31^{40}$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$4^{11} - 4^{11}$	$4^{12} - 7^{11}$	$4^{12} - 11^{39}$
10	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{12} - 3^{24}$	$4^{11} - 4^{11}$
11	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{12} - 3^{18}$
12	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
13	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=6, p=9 (cont.)	18	19	20
6	$306^{2} - 572^{11}$	$448^{2} - 936^{11}$	$640^{2} - 1470^{11}$
7	$44^{34} - 136^{40}$	$64^{34} - 205^{40}$	$92^{34} - 303^{11}$
8	$11^{34} - 48^{40}$	$16^{34} - 69^{40}$	$23^{34} - 101^{11}$
9	$4^{1} - 20^{39}$	$6^{34} - 29^{11}$	$8^{34} - 44^{11}$
10	$4^{12} - 7^{40}$	$4^{12} - 13^{39}$	$4^{34} - 20^{39}$
11	$4^{11} - 4^{11}$	$4^{12} - 5^{11}$	$4^{12} - 10^{39}$
12	$3^{12} - 3^{11}$	$3^{12} - 3^{24}$	$4^{11} - 4^{11}$
13	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{24}$
14	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
15	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
16	$1^{0} - 1^{7}$	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$

t=6, p=10	11	12	13	14	15	16	17	18
6	3^2-3^{11}	6^4-6^4	13^4-13^4	23^4-24^4	39^4-42^4	63^4-84^4	98^4-131^4	147^4-235^4
7	1^0-1^7	$3^{11}-3^{11}$	$4^{11}-4^{11}$	$6^{11}-6^{11}$	$6^{34}-13^{39}$	$9^{34}-24^{39}$	$14^{34}-37^{39}$	$21^{34}-62^{40}$
8	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$	$3^{11}-3^{11}$	$5^{11}-5^{11}$	$6^{11}-6^{35}$	$6^{11}-13^{39}$	$6^{34}-21^{39}$
9	1^0-1^7	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$	$3^{11}-3^{11}$	$4^{11}-4^{24}$	$4^{12}-5^{11}$	$4^{12}-8^{39}$
10	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$	$3^{11}-3^{11}$	$3^{12}-3^{21}$	$4^{11}-4^{11}$
11	1^0-1^0	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$	$3^{12}-3^{11}$	$3^{12}-3^{21}$
12	NA	1^0-1^0	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$	$3^{12}-3^{11}$
13	NA	NA	1^0-1^0	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$

t=6, p=10 (cont.)	19	20
6	$215^2 - 399^{11}$	$308^2 - 644^{11}$
7	$31^{34} - 95^{11}$	$44^{34} - 143^{40}$
8	$8^{34} - 34^{11}$	$11^{34} - 54^{39}$
9	$4^{12} - 13^{39}$	$5^{30} - 20^{39}$
10	$4^{12} - 6^{26}$	$4^{12} - 6^{35}$
11	$4^{11} - 4^{11}$	$4^{12} - 5^{11}$
12	$3^{12} - 3^{21}$	$3^{12} - 3^{24}$
13	$3^{12} - 3^{11}$	$3^{12} - 3^{21}$
14	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
15	$1^{0} - 1^{7}$	$3^{11} - 3^{11}$

t=6, p=11	12	13	14	15	16	17	18	19	20
6	$2^1 - 2^8$	$6^4 - 6^4$	$11^4 - 11^4$	$19^4 - 21^4$	$31^4 - 38^4$	$48^4 - 63^4$	$72^4 - 108^4$	$106^4 - 171^4$	$152^2 - 285^{11}$
7	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^8$	$6^{11} - 6^{11}$	$6^{11} - 11^{39}$	$8^{30} - 18^{39}$	$11^{34} - 29^{39}$	$16^{34} - 48^{40}$	$22^{34} - 73^{40}$
8	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{11} - 2^8$	$6^{11} - 6^{39}$	$6^{11} - 9^{39}$	$6^{11} - 16^{39}$	$7^{30} - 24^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$	$4^{18} - 7^{26}$	$4^{18} - 10^{39}$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
12	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
13	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
14	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=6, p=12	13	14	15	16	17	18	19	20
6	2^2-2^8	5^4-5^4	10^4-10^4	16^4-19^4	25^4-32^4	38^4-48^4	56^4-89^4	80^4-141^4
7	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$	$5^{11}-5^8$	$5^{11}-8^{40}$	$6^{34}-15^8$	$8^{34}-24^8$	$12^{34}-35^8$
8	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{11}-2^8$	$2^{11}-2^8$	$5^{11}-5^8$	$5^{11}-7^8$	$5^{11}-10^{35}$
9	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{12}-2^{11}$	$2^{12}-2^8$	$2^{12}-2^8$	$4^{18}-5^8$
10	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{12}-2^{11}$	$2^{12}-2^{11}$	$2^{12}-2^8$
11	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{12}-2^{11}$	$2^{12}-2^{11}$
12	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$	$2^{12}-2^{11}$
13	1^0-1^0	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$

t=6, p=13	14	15	16	17	18	19	20
6	$2^2 - 2^8$	$4^4 - 4^4$	$8^4 - 8^4$	$13^4 - 16^4$	$20^4 - 24^4$	$30^4 - 42^4$	$43^4 - 60^4$
7	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$4^{11} - 4^8$	$4^{11} - 7^8$	$5^{34} - 12^8$	$7^{34} - 16^8$
8	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$2^{12} - 2^8$	$4^{11} - 4^8$	$4^{11} - 4^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=6, p=14	15	16	17	18	19	20
6	$2^2 - 2^8$	$4^4 - 4^4$	$7^4 - 7^4$	$12^4 - 12^4$	$18^4 - 21^4$	$26^4 - 32^4$
7	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$4^{11} - 4^8$	$4^{11} - 5^8$	$5^{18} - 10^8$
8	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$	$4^{11} - 4^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{12} - 2^8$	$2^{12} - 2^8$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^8$
11	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=6, p=15	16	17	18	19	20
6	2^2-2^8	4^4-4^4	6^4-6^4	11^4-11^4	16^4-18^4
7	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$	$4^{11}-4^8$	$4^{11}-5^8$
8	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$	$2^{12}-2^8$
9	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$
10	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$

t=6, p=16	17	18	19	20
6	2^2-2^8	3^4-3^4	6^4-6^4	10^4-10^4
7	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$	$3^{18}-3^8$
8	1^0-1^7	1^0-1^7	$2^{11}-2^8$	$2^{12}-2^8$
9	1^0-1^7	1^0-1^7	1^0-1^7	$2^{11}-2^8$

t=6, p=17	18	19	20
6	2^2-2^8	3^4-3^4	6^4-6^4
7	1^0-1^7	$2^{11}-2^8$	$2^{11}-2^8$
8	1^0-1^7	1^0-1^7	$2^{11}-2^8$

t=6, p=18	19	20
6	2^2-2^8	3^4-3^4
7	1^0-1^7	$2^{11}-2^8$

t=6, p=19	20
6	2^2-2^8

t=7, p=7	8	9	10	11	12	13	14	15
7	$8^1 - 8^0$	$36^1 - 36^0$	$120^1 - 120^0$	$330^1 - 330^0$	$792^1 - 792^0$	$1716^1 - 1716^0$	$3432^1 - 3432^0$	$6435^1 - 6435^0$
8	$1^0 - 1^0$	$8^2 - 8^4$	$20^2 - 20^4$	$63^4 - 63^4$	$122^4 - 126^4$	$257^2 - 297^4$	$445^4 - 471^4$	$919^2 - 979^4$
9	NA	$1^0 - 1^0$	$8^2 - 8^4$	$15^2 - 15^4$	$40^4 - 40^4$	$73^4 - 79^4$	$140^4 - 166^4$	$220^4 - 279^4$
10	NA	NA	$1^0 - 1^0$	$8^2 - 8^4$	$12^2 - 12^4$	$30^4 - 30^4$	$54^4 - 58^4$	$78^4 - 120^4$
11	NA	NA	NA	$1^0 - 1^0$	$8^2 - 8^4$	$11^2 - 11^4$	$22^4 - 22^4$	$37^4 - 42^4$
12	NA	NA	NA	NA	$1^0 - 1^0$	$8^2 - 8^4$	$10^2 - 10^4$	$18^4 - 18^4$
13	NA	NA	NA	NA	NA	$1^0 - 1^0$	$8^2 - 8^4$	$9^2 - 9^4$
14	NA	NA	NA	NA	NA	NA	$1^0 - 1^0$	$8^2 - 8^4$

t=7, p=7 (cont.)	16	17	18	19	20
7	$11440^1 - 11440^0$	$19448^1 - 19448^0$	$31824^1 - 31824^0$	$50388^1 - 50388^0$	$77520^1 - 77520^0$
8	$1492^2 - 1722^4$	$2663^2 - 3040^4$	$3996^4 - 4690^4$	$6871^2 - 7949^4$	$9690^1 - 12134^4$
9	$395^2 - 543^4$	$609^2 - 860^4$	$906^2 - 1285^4$	$1664^2 - 2185^4$	$2349^2 - 3621^4$
10	$132^4 - 186^4$	$228^4 - 320^4$	$355^4 - 555^4$	$559^2 - 901^4$	$812^4 - 1469^4$
11	$62^4 - 92^4$	$98^4 - 131^4$	$146^4 - 255^4$	$182^4 - 364^4$	$340^4 - 647^4$
12	$28^4 - 28^4$	$50^4 - 64^4$	$72^4 - 108^4$	$108^4 - 184^4$	$142^2 - 284^4$
13	$16^4 - 16^4$	$25^4 - 26^4$	$36^4 - 50^4$	$56^4 - 89^4$	$79^4 - 146^4$
14	$8^2 - 8^4$	$15^4 - 15^4$	$24^4 - 24^4$	$33^4 - 42^4$	$43^4 - 60^4$
15	$8^2 - 8^4$	$8^2 - 8^4$	$14^4 - 14^4$	$19^4 - 19^4$	$28^4 - 34^4$
16	$1^0 - 1^0$	$8^2 - 8^4$	$8^2 - 8^4$	$13^4 - 13^4$	$17^4 - 17^4$
17	NA	$1^0 - 1^0$	$8^2 - 8^{11}$	$8^2 - 8^{11}$	$11^{11} - 12^{11}$
18	NA	NA	$1^0 - 1^0$	$8^2 - 8^{11}$	$8^2 - 8^{11}$
19	NA	NA	NA	$1^0 - 1^0$	$8^2 - 8^{11}$

t=7, p=8	9	10	11	12	13	14	15	16
7	$5^2 - 5^5$	$17^4 - 17^4$	$47^4 - 47^4$	$113^4 - 113^4$	$245^4 - 245^4$	$490^4 - 508^4$	$919^4 - 979^4$	$1634^4 - 1768^4$
8	$1^0 - 1^7$	$5^{11} - 5^{11}$	$10^{11} - 10^{11}$	$21^{11} - 26^{11}$	$36^{11} - 50^{11}$	$67^{11} - 100^{11}$	$115^{34} - 219^{11}$	$205^{34} - 434^{40}$
9	$1^0 - 1^0$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$8^{11} - 8^{11}$	$13^{11} - 13^{11}$	$21^{11} - 37^{39}$	$35^{11} - 74^{11}$	$51^{11} - 134^{40}$
10	NA	$1^0 - 1^0$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$6^{11} - 6^{11}$	$10^{11} - 12^{11}$	$15^{11} - 29^{11}$	$19^{11} - 50^{39}$
11	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$5^{11} - 5^{11}$	$8^{11} - 10^{11}$	$10^{11} - 18^{11}$
12	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$5^{11} - 5^{11}$	$7^{11} - 8^{11}$
13	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$5^{11} - 5^{11}$
14	NA	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$5^{11} - 5^{11}$

t=7, p=8 (cont.)	17	18	19	20
7	$2778^{2} - 3145^{26}$	$4546^{2} - 5425^{26}$	$7198^{2} - 9041^{26}$	$11074^{2} - 14576^{26}$
8	$348^{34} - 739^{11}$	$569^{34} - 1237^{40}$	$900^{34} - 1989^{40}$	$1385^{34} - 3100^{40}$
9	$84^{11} - 231^{11}$	$127^{34} - 379^{40}$	$200^{34} - 597^{11}$	$308^{34} - 924^{11}$
10	$30^{11} - 86^{40}$	$47^{11} - 144^{11}$	$69^{11} - 229^{11}$	$102^{11} - 355^{40}$
11	$15^{11} - 39^{40}$	$21^{11} - 62^{11}$	$29^{11} - 98^{11}$	$34^{34} - 154^{40}$
12	$8^{11} - 14^{11}$	$12^{11} - 26^{11}$	$16^{11} - 48^{11}$	$21^{11} - 73^{11}$
13	$6^{11} - 7^{11}$	$7^{11} - 12^{11}$	$9^{11} - 20^{11}$	$12^{11} - 35^{11}$
14	$5^{12} - 5^{11}$	$5^{12} - 6^{11}$	$7^{11} - 10^{11}$	$8^{11} - 16^{11}$
15	$5^{11} - 5^{11}$	$5^{12} - 5^{11}$	$5^{12} - 5^{11}$	$6^{11} - 9^{11}$
16	$1^{0} - 1^{7}$	$5^{11} - 5^{11}$	$5^{12} - 5^{11}$	$5^{12} - 5^{11}$
17	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$5^{11} - 5^{11}$	$5^{12} - 5^{11}$
18	NA	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$5^{11} - 5^{11}$

t=7, p=9	10	11	12	13	14	15	16	17
7	$4^2 - 4^{11}$	$11^4 - 11^4$	$27^4 - 29^4$	$59^4 - 66^4$	$118^4 - 138^4$	$222^4 - 283^4$	$395^4 - 543^4$	$672^4 - 979^4$
8	$1^0 - 1^7$	$4^{11} - 4^{11}$	$6^{11} - 6^{11}$	$11^{11} - 16^{11}$	$17^{11} - 31^{11}$	$29^{11} - 72^{40}$	$50^{34} - 118^{40}$	$84^{34} - 231^{40}$
9	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$5^{11} - 5^{11}$	$7^{11} - 10^{11}$	$10^{11} - 19^{11}$	$15^{11} - 43^{40}$	$20^{11} - 68^{11}$
10	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$	$6^{14} - 8^{11}$	$7^{11} - 14^{11}$	$8^{11} - 29^{40}$
11	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$	$6^{11} - 6^{11}$	$6^{12} - 10^{11}$
12	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$	$5^{11} - 5^{11}$
13	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$
14	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$

t=7, p=9 (cont.)	18	19	20
7	$1100^2 - 1770^{26}$	$1742^2 - 3089^{26}$	$2680^2 - 5173^{11}$
8	$138^{34} - 383^{11}$	$218^{34} - 618^{11}$	$335^{34} - 971^{40}$
9	$31^{34} - 120^{40}$	$49^{34} - 192^{40}$	$75^{34} - 303^{40}$
10	$11^{11} - 48^{40}$	$17^{11} - 71^{11}$	$23^{34} - 111^{40}$
11	$7^{11} - 18^{11}$	$8^{11} - 34^{40}$	$10^{11} - 52^{40}$
12	$5^{12} - 6^{11}$	$6^{11} - 14^{11}$	$7^{11} - 24^{11}$
13	$4^{12} - 4^{18}$	$5^{14} - 6^{11}$	$5^{16} - 10^{11}$
14	$4^{12} - 4^{11}$	$4^{12} - 4^{18}$	$5^{11} - 5^{11}$
15	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$	$4^{12} - 4^{11}$
16	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$
17	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$

t=7, p=10	11	12	13	14	15	16	17	18
7	$3^2 - 3^{11}$	$9^4 - 9^4$	$20^4 - 21^4$	$40^4 - 44^4$	$75^4 - 89^4$	$134^4 - 172^4$	$228^4 - 320^4$	$374^4 - 610^4$
8	$1^0 - 1^7$	$3^{11} - 3^{11}$	$5^{11} - 5^{11}$	$7^{11} - 11^{11}$	$10^{34} - 26^{40}$	$17^{34} - 46^{40}$	$29^{34} - 82^{11}$	$47^{34} - 144^{40}$
9	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$5^{11} - 8^{11}$	$7^{11} - 17^{39}$	$9^{11} - 27^{39}$	$11^{34} - 47^{11}$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$4^{11} - 5^{11}$	$4^{11} - 11^{11}$	$5^{11} - 19^{39}$
11	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{12} - 3^{18}$	$4^{11} - 6^{11}$
12	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
13	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
14	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=7, p=10 (cont.)	19	20
7	$593^2 - 1149^{11}$	$913^2 - 2028^{11}$
8	$75^{34} - 236^{40}$	$115^{34} - 376^{40}$
9	$17^{34} - 71^{40}$	$26^{34} - 115^{40}$
10	$6^{11} - 29^{40}$	$8^{34} - 46^{40}$
11	$4^{12} - 13^{39}$	$5^{11} - 20^{39}$
12	$4^{11} - 5^{11}$	$4^{12} - 8^{11}$
13	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
14	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
15	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
16	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=7, p=11	12	13	14	15	16	17	18	19	20
7	$3^2 - 3^{11}$	$7^4 - 7^4$	$15^4 - 15^4$	$29^4 - 30^4$	$52^4 - 62^4$	$89^4 - 146^4$	$146^4 - 255^4$	$232^4 - 467^4$	$357^2 - 866^{26}$
8	$1^0 - 1^7$	$3^{11} - 3^{11}$	$4^{11} - 4^{11}$	$7^{11} - 9^{11}$	$7^{34} - 15^{11}$	$12^{34} - 35^{39}$	$19^{34} - 59^{11}$	$29^{34} - 98^{40}$	$45^{34} - 159^{40}$
9	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$5^{11} - 5^{11}$	$7^{11} - 11^{39}$	$7^{11} - 19^{39}$	$7^{34} - 29^{39}$	$10^{34} - 52^{40}$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$4^{11} - 4^{11}$	$4^{12} - 6^{11}$	$4^{12} - 12^{39}$	$4^{11} - 20^{39}$
11	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{21}$	$4^{11} - 4^{11}$	$4^{12} - 7^{40}$
12	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{21}$	$4^{11} - 4^{11}$
13	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{21}$
14	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
15	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=7, p=12	13	14	15	16	17	18	19	20
7	$3^2 - 3^{11}$	$6^4 - 6^4$	$13^4 - 13^4$	$24^4 - 27^4$	$41^4 - 49^4$	$68^4 - 102^4$	$108^4 - 184^4$	$167^4 - 308^4$
8	$1^0 - 1^7$	$3^{11} - 3^{11}$	$4^{11} - 4^{11}$	$6^{11} - 7^{11}$	$6^{34} - 14^{39}$	$9^{34} - 25^{39}$	$14^{34} - 45^{39}$	$21^{34} - 73^{40}$
9	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$5^{11} - 5^{11}$	$6^{11} - 8^{39}$	$6^{11} - 14^{39}$	$6^{11} - 21^{39}$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{14}$	$4^{11} - 4^{11}$	$4^{12} - 6^{11}$	$4^{12} - 9^{39}$
11	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{24}$	$4^{11} - 4^{11}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{24}$
13	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
14	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=7, p=13	14	15	16	17	18	19	20
7	$2^2 - 2^8$	$6^4 - 6^4$	$11^4 - 11^4$	$19^4 - 23^4$	$32^4 - 43^4$	$51^4 - 81^4$	$79^4 - 146^4$
8	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^8$	$6^{11} - 6^{18}$	$6^{11} - 11^{39}$	$8^{30} - 21^{39}$	$10^{34} - 34^{39}$
9	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{11} - 2^8$	$6^{11} - 6^{39}$	$6^{11} - 11^{39}$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^{11}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=7, p=14	15	16	17	18	19	20
7	$2^2 - 2^8$	$5^4 - 5^4$	$9^4 - 11^4$	$15^4 - 20^4$	$24^4 - 39^4$	$37^4 - 66^4$
8	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$5^{11} - 6^8$	$5^{11} - 10^{39}$	$6^{30} - 17^{39}$
9	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^8$	$2^{11} - 2^8$	$5^{11} - 5^{40}$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$	$2^{12} - 2^8$
11	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^{11}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=7, p=15	16	17	18	19	20
7	$2^2 - 2^8$	$5^4 - 5^4$	$9^4 - 10^4$	$15^4 - 18^4$	$24^4 - 34^4$
8	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$5^{11} - 5^8$	$5^{11} - 7^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$2^{11} - 2^8$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{12} - 2^8$
11	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=7, p=16	17	18	19	20
7	$2^2 - 2^8$	$4^4 - 4^4$	$9^4 - 9^4$	$14^4 - 16^4$
8	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$4^{11} - 4^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$
10	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=7, p=17	18	19	20
7	$2^2 - 2^8$	$4^4 - 4^4$	$8^4 - 8^4$
8	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$
9	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=7, p=18	19	20
7	$2^2 - 2^8$	$4^4 - 4^4$
8	$1^0 - 1^7$	$2^{11} - 2^8$

t=7, p=19	20
7	$2^2 - 2^8$

t=8, p=8	9	10	11	12	13	14	15	16
8	$9^1 - 9^0$	$45^1 - 45^0$	$165^1 - 165^0$	$495^1 - 495^0$	$1287^1 - 1287^0$	$3003^1 - 3003^0$	$6435^1 - 6435^0$	$12870^1 - 12870^0$
9	$1^0 - 1^0$	$9^2 - 9^4$	$25^2 - 25^4$	$84^4 - 84^4$	$177^4 - 185^4$	$400^2 - 471^4$	$751^4 - 789^4$	$1634^2 - 1768^4$
10	NA	$1^0 - 1^0$	$9^2 - 9^4$	$18^2 - 18^4$	$52^4 - 52^4$	$103^4 - 119^4$	$210^4 - 264^4$	$352^4 - 448^4$
11	NA	NA	$1^0 - 1^0$	$9^2 - 9^4$	$15^2 - 15^4$	$40^4 - 40^4$	$74^4 - 81^4$	$114^4 - 191^4$
12	NA	NA	NA	$1^0 - 1^0$	$9^2 - 9^4$	$13^2 - 13^4$	$30^4 - 30^4$	$50^4 - 59^4$
13	NA	NA	NA	NA	$1^0 - 1^0$	$9^2 - 9^4$	$12^2 - 12^4$	$24^4 - 24^4$
14	NA	NA	NA	NA	NA	$1^0 - 1^0$	$9^2 - 9^4$	$11^2 - 11^4$
15	NA	NA	NA	NA	NA	NA	$1^0 - 1^0$	$9^2 - 9^4$

t=8, p=8 (cont.)	17	18	19	20
8	$24310^1 - 24310^0$	$43758^1 - 43758^0$	$75582^1 - 75582^0$	$125970^1 - 125970^0$
9	$2819^2 - 3355^4$	$5326^2 - 6098^4$	$8436^4 - 10641^4$	$15269^2 - 18590^4$
10	$672^2 - 979^4$	$1097^2 - 1629^4$	$1722^2 - 2510^4$	$3328^2 - 4525^4$
11	$204^4 - 333^4$	$374^4 - 610^4$	$614^4 - 1090^4$	$1017^2 - 1859^4$
12	$88^4 - 149^4$	$147^4 - 235^4$	$232^4 - 467^4$	$304^4 - 738^4$
13	$37^4 - 42^4$	$70^4 - 104^4$	$106^4 - 171^4$	$167^4 - 308^4$
14	$20^4 - 20^4$	$33^4 - 34^4$	$49^4 - 76^4$	$80^4 - 141^4$
15	$10^2 - 10^4$	$18^4 - 18^4$	$31^4 - 31^4$	$44^4 - 57^4$
16	$9^2 - 9^4$	$9^2 - 9^4$	$17^4 - 17^4$	$26^4 - 26^4$
17	$1^0 - 1^0$	$9^2 - 9^{11}$	$9^2 - 9^{11}$	$16^{11} - 16^{11}$
18	NA	$1^0 - 1^0$	$9^2 - 9^{11}$	$9^2 - 9^{11}$
19	NA	NA	$1^0 - 1^0$	$9^2 - 9^{11}$

t=8, p=9	10	11	12	13	14	15	16	17
8	$5^2 - 5^5$	$19^4 - 19^4$	$57^4 - 57^4$	$149^4 - 157^4$	$348^4 - 371^4$	$746^4 - 825^4$	$1492^4 - 1722^4$	$2819^4 - 3355^4$
9	$1^0 - 1^7$	$5^{11} - 5^{11}$	$11^{11} - 11^{11}$	$28^{11} - 35^{11}$	$50^{11} - 72^{11}$	$100^{11} - 152^{11}$	$174^{11} - 366^{11}$	$348^{11} - 739^{40}$
10	$1^0 - 1^0$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$9^{11} - 9^{11}$	$17^{11} - 20^{11}$	$29^{11} - 53^{39}$	$51^{11} - 110^{40}$	$78^{11} - 212^{11}$
11	NA	$1^0 - 1^0$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$7^{11} - 7^{11}$	$13^{11} - 15^{11}$	$20^{11} - 39^{11}$	$27^{11} - 76^{39}$
12	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$5^{11} - 5^{11}$	$10^{11} - 12^{11}$	$14^{11} - 23^{11}$
13	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$5^{11} - 5^{11}$	$8^{11} - 11^{11}$
14	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$5^{11} - 5^{11}$
15	NA	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$5^{11} - 5^{11}$

t=8, p=9 (cont.)	18	19	20
8	$5075^{2} - 6311^{11}$	$8766^{2} - 11396^{11}$	$14610^{2} - 19811^{11}$
9	$564^{34} - 1359^{40}$	$1012^{11} - 2386^{11}$	$1624^{34} - 4031^{11}$
10	$138^{11} - 383^{40}$	$211^{11} - 670^{40}$	$325^{34} - 1119^{40}$
11	$44^{11} - 136^{40}$	$75^{11} - 236^{40}$	$114^{11} - 391^{11}$
12	$21^{11} - 56^{40}$	$31^{11} - 95^{40}$	$45^{11} - 159^{11}$
13	$10^{11} - 18^{11}$	$16^{11} - 34^{11}$	$22^{11} - 73^{11}$
14	$6^{11} - 9^{11}$	$9^{11} - 15^{11}$	$11^{11} - 26^{11}$
15	$5^{12} - 5^{11}$	$6^{11} - 7^{11}$	$8^{11} - 14^{11}$
16	$5^{11} - 5^{11}$	$5^{12} - 5^{11}$	$5^{12} - 5^{11}$
17	$1^{0} - 1^{7}$	$5^{11} - 5^{11}$	$5^{12} - 5^{11}$
18	$1^{0} - 1^{0}$	$1^{0} - 1^{7}$	$5^{11} - 5^{11}$

t=8, p=10	11	12	13	14	15	16	17	18
8	$4^2 - 4^{11}$	$12^4 - 12^4$	$32^4 - 34^4$	$75^4 - 80^4$	$161^4 - 190^4$	$322^4 - 408^4$	$609^4 - 860^4$	$1097^4 - 1629^4$
9	$1^0 - 1^7$	$4^{11} - 4^{11}$	$8^{11} - 8^{11}$	$14^{11} - 20^{11}$	$23^{11} - 42^{11}$	$42^{11} - 105^{40}$	$68^{11} - 208^{11}$	$127^{11} - 379^{40}$
10	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$5^{11} - 5^{11}$	$9^{11} - 13^{11}$	$13^{11} - 26^{11}$	$21^{11} - 61^{40}$	$30^{11} - 109^{40}$
11	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$	$7^{11} - 10^{11}$	$9^{11} - 19^{11}$	$11^{11} - 41^{40}$
12	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$	$6^{11} - 7^{11}$	$6^{11} - 14^{11}$
13	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$	$5^{11} - 5^{11}$
14	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$
15	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$

t=8, p=10 (cont.)	19	20
8	$1895^{2} - 3144^{11}$	$3159^{2} - 5787^{11}$
9	$211^{34} - 670^{11}$	$351^{34} - 1150^{40}$
10	$49^{11} - 192^{11}$	$71^{34} - 327^{40}$
11	$16^{11} - 69^{11}$	$26^{11} - 115^{11}$
12	$8^{11} - 29^{11}$	$11^{11} - 54^{39}$
13	$5^{11} - 10^{11}$	$7^{11} - 19^{11}$
14	$4^{12} - 4^{18}$	$5^{18} - 8^{11}$
15	$4^{12} - 4^{11}$	$4^{12} - 4^{11}$
16	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$
17	$1^{0} - 1^{7}$	$4^{11} - 4^{11}$

t=8, p=11	12	13	14	15	16	17	18	19	20
8	$3^2 - 3^{11}$	$10^4 - 10^4$	$24^4 - 25^4$	$52^4 - 57^4$	$104^4 - 117^4$	$197^4 - 272^4$	$355^4 - 555^4$	$614^4 - 1090^4$	$1024^2 - 2121^{11}$
9	$1^0 - 1^7$	$3^{11} - 3^{11}$	$6^{11} - 6^{11}$	$10^{11} - 15^{11}$	$13^{11} - 35^{39}$	$23^{11} - 64^{40}$	$40^{34} - 129^{40}$	$69^{34} - 229^{40}$	$114^{34} - 391^{40}$
10	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$6^{11} - 8^{11}$	$9^{11} - 22^{39}$	$11^{11} - 39^{11}$	$15^{11} - 65^{40}$	$23^{34} - 111^{11}$
11	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$4^{11} - 6^{11}$	$5^{11} - 12^{11}$	$6^{11} - 27^{40}$	$8^{11} - 44^{40}$
12	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$4^{11} - 9^{11}$	$4^{11} - 16^{11}$
13	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$	$4^{11} - 6^{11}$
14	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
15	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
16	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=8, p=12	13	14	15	16	17	18	19	20
8	$3^{2}-3^{11}$	$7^{4}-7^{4}$	$15^{4}-15^{4}$	$30^{4}-30^{4}$	$58^{4}-81^{4}$	$105^{4}-176^{4}$	$182^{4}-364^{4}$	$304^{4}-738^{4}$
9	$1^{0}-1^{7}$	$3^{11}-3^{11}$	$4^{11}-4^{11}$	$7^{11}-11^{11}$	$9^{11}-23^{39}$	$14^{11}-49^{40}$	$21^{34}-85^{11}$	$34^{34}-154^{11}$
10	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$3^{11}-3^{11}$	$3^{11}-3^{11}$	$5^{11}-7^{11}$	$7^{11}-14^{39}$	$7^{11}-28^{39}$	$9^{11}-49^{39}$
11	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$3^{11}-3^{11}$	$3^{12}-3^{11}$	$4^{11}-4^{11}$	$4^{11}-9^{39}$	$4^{11}-17^{39}$
12	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$3^{11}-3^{11}$	$3^{12}-3^{11}$	$3^{12}-3^{24}$	$4^{11}-4^{11}$
13	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$3^{11}-3^{11}$	$3^{12}-3^{11}$	$3^{12}-3^{24}$
14	NA	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$3^{11}-3^{11}$	$3^{12}-3^{11}$
15	NA	NA	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$3^{11}-3^{11}$

t=8, p=13	14	15	16	17	18	19	20
8	$3^2 - 3^{11}$	$7^4 - 7^4$	$14^4 - 14^4$	$27^4 - 28^4$	$49^4 - 54^4$	$85^4 - 137^4$	$142^4 - 284^4$
9	$1^0 - 1^7$	$3^{11} - 3^{11}$	$4^{11} - 4^{11}$	$7^{11} - 9^{18}$	$7^{11} - 18^{39}$	$10^{34} - 34^{39}$	$16^{34} - 63^{39}$
10	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{18}$	$5^{11} - 5^{11}$	$7^{11} - 10^{39}$	$7^{11} - 19^{39}$
11	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$4^{11} - 4^{11}$	$4^{12} - 6^{11}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{24}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
14	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=8, p=14	15	16	17	18	19	20
8	$3^2 - 3^{11}$	$6^4 - 6^4$	$13^4 - 14^4$	$24^4 - 26^4$	$42^4 - 52^4$	$70^4 - 102^4$
9	$1^0 - 1^7$	$3^{11} - 3^{11}$	$4^{11} - 4^{11}$	$6^{11} - 6^{11}$	$6^{11} - 15^{39}$	$8^{34} - 28^{39}$
10	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{21}$	$5^{11} - 5^{11}$	$6^{11} - 6^{35}$
11	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{18}$	$4^{11} - 4^{11}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=8, p=15	16	17	18	19	20
8	$2^2 - 2^8$	$6^4 - 6^4$	$11^4 - 12^4$	$19^4 - 23^4$	$32^4 - 42^4$
9	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^8$	$6^{11} - 6^{24}$	$6^{11} - 12^{39}$
10	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$	$2^{11} - 2^8$
11	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^{11}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=8, p=16	17	18	19	20
8	$2^2 - 2^8$	$5^4 - 5^4$	$9^4 - 11^4$	$15^4 - 20^4$
9	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$	$5^{11} - 6^8$
10	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$	$2^{11} - 2^8$
11	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^{11}$

t=8, p=17	18	19	20
8	$2^2 - 2^8$	$5^4 - 5^4$	$9^4 - 10^4$
9	$1^0 - 1^7$	$2^{11} - 2^8$	$2^{11} - 2^8$
10	$1^0 - 1^7$	$1^0 - 1^7$	$2^{11} - 2^8$

t=8, p=18	19	20
8	$2^2 - 2^8$	$4^4 - 4^4$
9	$1^0 - 1^7$	$2^{11} - 2^8$

t=8, p=19	20
8	$2^2 - 2^8$

t=9, p=9	10	11	12	13	14	15	16	17
9	$10^{1}-10^{0}$	$55^{1}-55^{0}$	$220^{1}-220^{0}$	$715^{1}-715^{0}$	$2002^{1}-2002^{0}$	$5005^{1}-5005^{0}$	$11440^{1}-11440^{0}$	$24310^{1}-24310^{0}$
10	$1^{0}-1^{0}$	$10^{2}-10^{11}$	$30^{2}-30^{11}$	$110^{11}-112^{11}$	$248^{11}-272^{29}$	$600^{2}-743^{26}$	$1214^{11}-1377^{29}$	$2778^{2}-3145^{26}$
11	NA	$1^{0}-1^{0}$	$10^{2}-10^{11}$	$22^{2}-22^{11}$	$67^{11}-70^{11}$	$141^{11}-175^{29}$	$306^{11}-422^{29}$	$544^{11}-791^{29}$
12	NA	NA	$1^{0}-1^{0}$	$10^{2}-10^{11}$	$18^{2}-18^{11}$	$50^{11}-50^{11}$	$99^{11}-121^{29}$	$162^{11}-301^{29}$
13	NA	NA	NA	$1^{0}-1^{0}$	$10^{2}-10^{11}$	$15^{2}-15^{11}$	$37^{11}-39^{11}$	$66^{11}-85^{11}$
14	NA	NA	NA	NA	$1^{0}-1^{0}$	$10^{2}-10^{11}$	$14^{2}-14^{11}$	$30^{11}-32^{11}$
15	NA	NA	NA	NA	NA	$1^{0}-1^{0}$	$10^{2}-10^{11}$	$13^{2}-13^{11}$
16	NA	NA	NA	NA	NA	NA	$1^{0}-1^{0}$	$10^{2}-10^{11}$

t=9, p=9 (cont.)	18	19	20
9	$48620^{1} - 48620^{0}$	$92378^{1} - 92378^{0}$	$167960^{1} - 167960^{0}$
10	$5075^{2} - 6311^{29}$	$10120^{2} - 12292^{29}$	$16872^{11} - 22553^{29}$
11	$1100^{2} - 1770^{26}$	$1895^{2} - 3144^{29}$	$3131^{2} - 5469^{29}$
12	$306^{11} - 572^{29}$	$593^{11} - 1149^{29}$	$1024^{11} - 2121^{29}$
13	$122^{11} - 232^{29}$	$215^{11} - 399^{29}$	$357^{11} - 866^{26}$
14	$48^{11} - 60^{11}$	$95^{11} - 156^{11}$	$152^{11} - 285^{29}$
15	$24^{11} - 24^{11}$	$42^{11} - 48^{11}$	$66^{11} - 102^{11}$
16	$12^{2} - 12^{11}$	$22^{11} - 22^{11}$	$39^{11} - 40^{11}$
17	$10^{2} - 10^{11}$	$11^{2} - 11^{11}$	$20^{11} - 20^{11}$
18	$1^{0} - 1^{0}$	$10^{2} - 10^{11}$	$10^{2} - 10^{11}$
19	NA	$1^{0} - 1^{0}$	$10^{2} - 10^{11}$

t=9, p=10	11	12	13	14	15	16	17	18
9	$6^{2}-6^{5}$	$24^{4}-24^{4}$	$78^{4}-78^{4}$	$219^{4}-230^{4}$	$548^{4}-580^{4}$	$1253^{4}-1329^{4}$	$2663^{4}-3040^{4}$	$5326^{4}-6098^{4}$
10	$1^{0}-1^{7}$	$6^{11}-6^{11}$	$13^{11}-13^{11}$	$35^{11}-46^{11}$	$68^{11}-99^{11}$	$146^{11}-251^{11}$	$271^{11}-603^{11}$	$569^{11}-1237^{11}$
11	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$6^{11}-6^{11}$	$11^{11}-11^{11}$	$21^{11}-26^{11}$	$38^{11}-72^{39}$	$73^{11}-163^{40}$	$117^{11}-333^{40}$
12	NA	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$6^{11}-6^{11}$	$9^{11}-9^{11}$	$16^{11}-20^{11}$	$26^{11}-50^{11}$	$38^{11}-109^{40}$
13	NA	NA	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$6^{11}-6^{11}$	$7^{11}-7^{11}$	$12^{11}-14^{11}$	$17^{11}-31^{11}$
14	NA	NA	NA	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$6^{11}-6^{11}$	$6^{11}-6^{11}$	$9^{11}-13^{11}$
15	NA	NA	NA	NA	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$6^{11}-6^{11}$	$6^{12}-6^{11}$
16	NA	NA	NA	NA	NA	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$6^{11}-6^{11}$

t=9, p=10 (cont.)	19	20
9	$10120^{2} - 12292^{11}$	$18400^{2} - 23540^{11}$
10	$1012^{34} - 2386^{40}$	$1840^{34} - 4416^{40}$
11	$218^{11} - 618^{40}$	$351^{11} - 1150^{11}$
12	$64^{11} - 205^{11}$	$115^{11} - 376^{11}$
13	$28^{11} - 80^{11}$	$44^{11} - 143^{11}$
14	$13^{11} - 23^{11}$	$22^{11} - 50^{11}$
15	$8^{11} - 11^{11}$	$11^{11} - 19^{11}$
16	$6^{12} - 6^{11}$	$7^{11} - 9^{11}$
17	$6^{11} - 6^{11}$	$6^{12} - 6^{11}$
18	$1^{0} - 1^{7}$	$6^{11} - 6^{11}$

t=9, p=11	12	13	14	15	16	17	18	19	20
9	$4^{2}-4^{11}$	$13^{4}-13^{4}$	$37^{4}-43^{4}$	$93^{4}-118^{4}$	$213^{4}-285^{4}$	$453^{4}-620^{4}$	$906^{4}-1285^{4}$	$1722^{4}-2510^{4}$	$3131^{2}-5469^{11}$
10	$1^{0}-1^{7}$	$4^{11}-4^{11}$	$9^{11}-9^{11}$	$18^{11}-25^{11}$	$31^{11}-54^{11}$	$59^{11}-148^{40}$	$102^{11}-306^{40}$	$200^{11}-597^{40}$	$325^{11}-1119^{11}$
11	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$4^{11}-4^{11}$	$7^{11}-7^{11}$	$11^{11}-16^{11}$	$17^{11}-36^{11}$	$29^{11}-84^{40}$	$43^{11}-162^{40}$	$75^{11}-303^{11}$
12	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$4^{11}-4^{11}$	$4^{11}-4^{11}$	$8^{11}-12^{11}$	$12^{11}-24^{11}$	$15^{11}-55^{40}$	$23^{11}-101^{40}$
13	NA	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$4^{11}-4^{11}$	$4^{11}-4^{11}$	$6^{11}-10^{11}$	$8^{11}-17^{11}$	$11^{11}-40^{11}$
14	NA	NA	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$4^{11}-4^{11}$	$4^{12}-4^{11}$	$5^{11}-7^{11}$	$6^{11}-13^{11}$
15	NA	NA	NA	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$4^{11}-4^{11}$	$4^{12}-4^{11}$	$4^{12}-4^{11}$
16	NA	NA	NA	NA	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$4^{11}-4^{11}$	$4^{12}-4^{11}$
17	NA	NA	NA	NA	NA	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$4^{11}-4^{11}$

t=9, p=12	13	14	15	16	17	18	19	20
9	$4^{2}-4^{11}$	$12^{4}-12^{4}$	$30^{4}-31^{4}$	$69^{4}-76^{4}$	$147^{4}-190^{4}$	$294^{4}-446^{4}$	$559^{4}-901^{4}$	$1017^{4}-1859^{4}$
10	$1^{0}-1^{7}$	$4^{11}-4^{11}$	$7^{11}-7^{11}$	$11^{11}-16^{11}$	$17^{11}-43^{39}$	$31^{11}-93^{40}$	$56^{34}-186^{11}$	$102^{34}-355^{11}$
11	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$4^{11}-4^{11}$	$5^{11}-5^{11}$	$7^{11}-11^{11}$	$10^{11}-29^{39}$	$15^{11}-53^{11}$	$21^{11}-118^{39}$
12	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$4^{11}-4^{11}$	$4^{12}-4^{11}$	$6^{14}-7^{11}$	$7^{11}-16^{11}$	$8^{11}-39^{39}$
13	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$4^{11}-4^{11}$	$4^{12}-4^{11}$	$6^{11}-6^{11}$	$6^{12}-11^{11}$
14	NA	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$4^{11}-4^{11}$	$4^{12}-4^{11}$	$5^{11}-5^{11}$
15	NA	NA	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$4^{11}-4^{11}$	$4^{12}-4^{11}$
16	NA	NA	NA	$1^{0}-1^{0}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$1^{0}-1^{7}$	$4^{11}-4^{11}$

t=9, p=13	14	15	16	17	18	19	20
9	$3^2 - 3^{11}$	$10^4 - 10^4$	$23^4 - 24^4$	$49^4 - 54^4$	$98^4 - 124^4$	$187^4 - 287^4$	$340^4 - 647^4$
10	$1^0 - 1^7$	$3^{11} - 3^{11}$	$5^{11} - 5^{11}$	$9^{11} - 14^{11}$	$11^{11} - 31^{39}$	$19^{34} - 63^{40}$	$34^{34} - 124^{11}$
11	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$5^{11} - 8^{11}$	$9^{11} - 20^{39}$	$10^{11} - 37^{40}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$4^{11} - 5^{11}$	$4^{11} - 10^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$3^{12} - 3^{11}$
14	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
15	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=9, p=14	15	16	17	18	19	20
9	$3^2 - 3^{11}$	$8^4 - 8^4$	$17^4 - 18^4$	$34^4 - 42^4$	$65^4 - 94^4$	$119^4 - 224^4$
10	$1^0 - 1^7$	$3^{11} - 3^{11}$	$4^{11} - 4^{11}$	$7^{11} - 9^{11}$	$8^{11} - 23^{39}$	$13^{11} - 49^{40}$
11	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$5^{11} - 6^{11}$	$7^{11} - 15^{39}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$	$4^{11} - 4^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=9, p=15	16	17	18	19	20
9	3^2-3^{11}	7^4-7^4	14^4-16^4	27^4-34^4	50^4-85^4
10	1^0-1^7	$3^{11}-3^{11}$	$4^{11}-4^{11}$	$7^{11}-9^{24}$	$7^{11}-15^{11}$
11	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$	$3^{12}-3^{18}$	$5^{11}-5^{11}$
12	1^0-1^7	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$	$3^{12}-3^{11}$
13	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$

t=9, p=16	17	18	19	20
9	3^2-3^{11}	6^4-6^4	13^4-14^4	24^4-29^4
10	1^0-1^7	$3^{11}-3^{11}$	$4^{11}-4^{11}$	$6^{11}-8^{39}$
11	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$	$3^{12}-3^{21}$
12	1^0-1^7	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$

t=9, p=17	18	19	20
9	2^2-2^8	6^4-6^4	11^4-14^4
10	1^0-1^7	$2^{11}-2^{11}$	$2^{11}-2^8$
11	1^0-1^7	1^0-1^7	$2^{11}-2^{11}$

t=9, p=18	19	20
9	2^2-2^8	6^4-6^4
10	1^0-1^7	$2^{11}-2^8$

t=9, p=19	20
9	2^2-2^8

t=10, p=10	11	12	13	14	15	16	17	18
10	$11^1 - 11^0$	$66^1 - 66^0$	$286^1 - 286^0$	$1001^1 - 1001^0$	$3003^1 - 3003^0$	$8008^1 - 8008^0$	$19448^1 - 19448^0$	$43758^1 - 43758^0$
11	$1^0 - 1^0$	$11^2 - 11^{11}$	$36^2 - 36^{11}$	$140^{11} - 144^{11}$	$339^{11} - 386^{29}$	$873^2 - 1129^{26}$	$1891^{11} - 2280^{29}$	$4546^2 - 5425^{26}$
12	NA	$1^0 - 1^0$	$11^2 - 11^{11}$	$26^2 - 26^{11}$	$84^{11} - 89^{11}$	$188^{11} - 248^{29}$	$434^{11} - 643^{29}$	$816^{11} - 1319^{29}$
13	NA	NA	$1^0 - 1^0$	$11^2 - 11^{11}$	$21^2 - 21^{11}$	$62^{11} - 65^{11}$	$130^{11} - 173^{29}$	$225^{11} - 455^{29}$
14	NA	NA	NA	$1^0 - 1^0$	$11^2 - 11^{11}$	$18^2 - 18^{11}$	$45^{11} - 49^{11}$	$85^{11} - 119^{11}$
15	NA	NA	NA	NA	$1^0 - 1^0$	$11^2 - 11^{11}$	$16^2 - 16^{11}$	$36^{11} - 40^{11}$
16	NA	NA	NA	NA	NA	$1^0 - 1^0$	$11^2 - 11^{11}$	$15^2 - 15^{11}$
17	NA	NA	NA	NA	NA	NA	$1^0 - 1^0$	$11^2 - 11^{11}$

t=10, p=10 (cont.)	19	20
10	$92378^{1} - 92378^{0}$	$184756^{1} - 184756^{0}$
11	$8766^{2} - 11396^{29}$	$18400^{2} - 23540^{29}$
12	$1742^{2} - 3089^{26}$	$3159^{2} - 5787^{29}$
13	$448^{11} - 936^{29}$	$913^{11} - 2028^{29}$
14	$166^{11} - 346^{29}$	$308^{11} - 644^{29}$
15	$61^{11} - 80^{11}$	$127^{11} - 226^{11}$
16	$29^{11} - 32^{11}$	$53^{11} - 64^{11}$
17	$14^{2} - 14^{11}$	$26^{11} - 28^{11}$
18	$11^{2} - 11^{11}$	$13^{2} - 13^{11}$
19	$1^{0} - 1^{0}$	$11^{2} - 11^{11}$

t=10, p=11	12	13	14	15	16	17	18	19	20
10	$6^2 - 6^5$	$26^4 - 26^4$	$91^4 - 91^4$	$274^4 - 295^4$	$731^4 - 808^4$	$1776^4 - 2048^4$	$3996^4 - 4690^4$	$8436^4 - 10641^4$	$16872^2 - 22553^{11}$
11	$1^0 - 1^7$	$6^{11} - 6^{11}$	$14^{11} - 14^{11}$	$44^{11} - 58^{11}$	$91^{11} - 134^{11}$	$207^{11} - 374^{11}$	$408^{11} - 913^{11}$	$900^{11} - 1989^{11}$	$1624^{11} - 4031^{40}$
12	$1^0 - 1^0$	$1^0 - 1^7$	$6^{11} - 6^{11}$	$12^{11} - 12^{11}$	$26^{11} - 28^{11}$	$50^{11} - 98^{39}$	$101^{11} - 232^{11}$	$171^{11} - 489^{11}$	$335^{11} - 971^{11}$
13	NA	$1^0 - 1^0$	$1^0 - 1^7$	$6^{11} - 6^{11}$	$10^{11} - 10^{11}$	$19^{11} - 23^{11}$	$34^{11} - 60^{11}$	$51^{11} - 152^{40}$	$92^{11} - 303^{40}$
14	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$6^{11} - 6^{11}$	$8^{11} - 8^{11}$	$14^{11} - 17^{11}$	$22^{11} - 42^{11}$	$37^{11} - 100^{11}$
15	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$6^{11} - 6^{11}$	$6^{11} - 6^{11}$	$11^{11} - 16^{11}$	$16^{11} - 28^{11}$
16	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$6^{11} - 6^{11}$	$6^{11} - 6^{11}$	$9^{11} - 13^{11}$
17	NA	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$6^{11} - 6^{11}$	$6^{12} - 6^{11}$
18	NA	NA	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$6^{11} - 6^{11}$

t=10, p=12	13	14	15	16	17	18	19	20
10	$5^2 - 5^{11}$	$18^4 - 18^4$	$54^4 - 57^4$	$144^4 - 152^4$	$350^4 - 405^4$	$788^4 - 918^4$	$1664^4 - 2185^4$	$3328^4 - 4525^4$
11	$1^0 - 1^7$	$5^{11} - 5^{11}$	$11^{11} - 11^{11}$	$22^{11} - 32^{11}$	$40^{11} - 71^{11}$	$82^{11} - 206^{40}$	$152^{34} - 442^{40}$	$308^{11} - 924^{40}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$7^{11} - 7^{11}$	$13^{11} - 20^{11}$	$22^{11} - 42^{11}$	$39^{11} - 114^{40}$	$61^{11} - 235^{40}$
13	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$6^{11} - 6^{11}$	$10^{11} - 14^{11}$	$15^{11} - 32^{11}$	$20^{11} - 77^{40}$
14	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$5^{12} - 5^{11}$	$7^{11} - 11^{11}$	$9^{11} - 21^{11}$
15	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$5^{12} - 5^{11}$	$7^{11} - 9^{11}$
16	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$5^{12} - 5^{11}$
17	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$

t=10, p=13	14	15	16	17	18	19	20
10	$4^2 - 4^{11}$	$13^4 - 13^4$	$35^4 - 38^4$	$85^4 - 100^4$	$192^4 - 269^4$	$406^4 - 630^4$	$812^4 - 1469^4$
11	$1^0 - 1^7$	$4^{11} - 4^{11}$	$8^{11} - 8^{11}$	$14^{11} - 21^{11}$	$22^{11} - 58^{39}$	$42^{11} - 126^{40}$	$74^{34} - 268^{11}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$5^{18} - 5^{11}$	$8^{11} - 12^{11}$	$12^{11} - 35^{40}$	$20^{11} - 71^{40}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$	$6^{11} - 9^{11}$	$8^{11} - 20^{11}$
14	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$	$6^{11} - 6^{11}$
15	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$
16	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$

t=10, p=14	15	16	17	18	19	20
10	3^2-3^{11}	10^4-10^4	25^4-28^4	57^4-68^4	121^4-173^4	242^4-433^4
11	1^0-1^7	$3^{11}-3^{11}$	$6^{11}-6^{11}$	$10^{11}-19^{39}$	$14^{11}-38^{39}$	$25^{11}-86^{40}$
12	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$	$3^{11}-3^{11}$	$6^{11}-10^{11}$	$9^{11}-22^{39}$
13	1^0-1^7	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$	$3^{11}-3^{11}$	$4^{11}-7^{11}$
14	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$	$3^{12}-3^{11}$
15	1^0-1^0	1^0-1^7	1^0-1^7	1^0-1^7	1^0-1^7	$3^{11}-3^{11}$

t=10, p=15	16	17	18	19	20
10	$3^2 - 3^{11}$	$9^4 - 9^4$	$21^4 - 22^4$	$45^4 - 51^4$	$90^4 - 120^4$
11	$1^0 - 1^7$	$3^{11} - 3^{11}$	$5^{11} - 5^{11}$	$9^{11} - 14^{18}$	$10^{11} - 30^{39}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$	$5^{11} - 7^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{12} - 3^{11}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=10, p=16	17	18	19	20
10	$3^2 - 3^{11}$	$7^4 - 7^4$	$15^4 - 17^4$	$30^4 - 38^4$
11	$1^0 - 1^7$	$3^{11} - 3^{11}$	$4^{11} - 4^{11}$	$7^{11} - 11^{39}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=10, p=17	18	19	20
10	$3^2 - 3^{11}$	$7^4 - 7^4$	$14^4 - 15^4$
11	$1^0 - 1^7$	$3^{11} - 3^{11}$	$4^{11} - 4^{11}$
12	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=10, p=18	19	20
10	$3^2 - 3^{11}$	$6^4 - 6^4$
11	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=10, p=19	20
10	$2^2 - 2^8$

t=11, p=11	12	13	14	15	16	17	18
11	$12^1 - 12^0$	$78^1 - 78^0$	$364^1 - 364^0$	$1365^1 - 1365^0$	$4368^1 - 4368^0$	$12376^1 - 12376^0$	$31824^1 - 31824^0$
12	$1^0 - 1^0$	$12^2 - 12^{11}$	$42^2 - 42^{11}$	$175^{11} - 180^{11}$	$452^{11} - 531^{29}$	$1237^2 - 1660^{26}$	$2856^{11} - 3616^{29}$
13	NA	$1^0 - 1^0$	$12^2 - 12^{11}$	$30^2 - 30^{11}$	$104^{11} - 112^{11}$	$246^{11} - 340^{29}$	$601^{11} - 945^{29}$
14	NA	NA	$1^0 - 1^0$	$12^2 - 12^{11}$	$24^2 - 24^{11}$	$76^{11} - 80^{11}$	$168^{11} - 238^{29}$
15	NA	NA	NA	$1^0 - 1^0$	$12^2 - 12^{11}$	$21^2 - 21^{11}$	$54^{11} - 60^{11}$
16	NA	NA	NA	NA	$1^0 - 1^0$	$12^2 - 12^{11}$	$18^2 - 18^{11}$
17	NA	NA	NA	NA	NA	$1^0 - 1^0$	$12^2 - 12^{11}$

t=11, p=11 (cont.)	19	20
11	$75582^{1} - 75582^{0}$	$167960^{1} - 167960^{0}$
12	$7198^{2} - 9041^{26}$	$14610^{2} - 19811^{29}$
13	$1193^{11} - 2099^{29}$	$2680^{2} - 5173^{29}$
14	$306^{11} - 662^{29}$	$640^{11} - 1470^{29}$
15	$108^{11} - 154^{11}$	$222^{11} - 498^{29}$
16	$43^{11} - 49^{11}$	$77^{11} - 100^{11}$
17	$17^{2} - 17^{11}$	$35^{11} - 40^{11}$
18	$12^{2} - 12^{11}$	$16^{2} - 16^{11}$
19	$1^{0} - 1^{0}$	$12^{2} - 12^{11}$

t=11, p=12	13	14	15	16	17	18	19	20
11	$7^2 - 7^5$	$33^4 - 33^4$	$124^4 - 124^4$	$397^4 - 405^4$	$1125^4 - 1213^4$	$2893^4 - 3261^4$	$6871^4 - 7949^4$	$15269^4 - 18590^4$
12	$1^0 - 1^7$	$7^{11} - 7^{11}$	$18^{11} - 18^{11}$	$54^{11} - 72^{11}$	$119^{11} - 193^{11}$	$286^{11} - 544^{11}$	$600^{11} - 1346^{11}$	$1385^{11} - 3100^{11}$
13	$1^0 - 1^0$	$1^0 - 1^7$	$7^{11} - 7^{11}$	$14^{11} - 14^{11}$	$32^{11} - 39^{11}$	$64^{11} - 129^{39}$	$136^{11} - 321^{11}$	$244^{11} - 713^{11}$
14	NA	$1^0 - 1^0$	$1^0 - 1^7$	$7^{11} - 7^{11}$	$12^{11} - 12^{11}$	$23^{11} - 28^{11}$	$43^{11} - 81^{11}$	$68^{11} - 228^{29}$
15	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$7^{11} - 7^{11}$	$10^{11} - 10^{11}$	$17^{11} - 19^{11}$	$27^{11} - 52^{11}$
16	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$7^{11} - 7^{11}$	$8^{11} - 8^{11}$	$13^{11} - 17^{11}$
17	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$7^{11} - 7^{11}$	$7^{11} - 7^{11}$
18	NA	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$7^{11} - 7^{11}$

t=11, p=13	14	15	16	17	18	19	20
11	$5^2 - 5^{11}$	$19^4 - 19^4$	$61^4 - 65^4$	$173^4 - 188^{11}$	$445^4 - 589^4$	$1057^4 - 1507^4$	$2349^4 - 3621^4$
12	$1^0 - 1^7$	$5^{11} - 5^{11}$	$12^{11} - 12^{11}$	$27^{11} - 40^{11}$	$51^{11} - 81^{11}$	$111^{11} - 277^{11}$	$216^{11} - 615^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$9^{11} - 9^{11}$	$16^{11} - 24^{11}$	$27^{11} - 54^{11}$	$52^{11} - 156^{11}$
14	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$6^{11} - 6^{11}$	$11^{11} - 17^{11}$	$18^{11} - 40^{11}$
15	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$5^{11} - 5^{11}$	$8^{11} - 14^{11}$
16	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$5^{12} - 5^{11}$
17	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$

t=11, p=14	15	16	17	18	19	20
11	$4^2 - 4^{11}$	$15^4 - 15^4$	$43^4 - 44^4$	$111^4 - 131^4$	$264^4 - 381^4$	$587^4 - 1002^4$
12	$1^0 - 1^7$	$4^{11} - 4^{11}$	$9^{11} - 9^{11}$	$16^{11} - 26^{11}$	$28^{11} - 75^{39}$	$56^{11} - 171^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$6^{11} - 6^{11}$	$10^{11} - 15^{11}$	$15^{11} - 49^{39}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$	$7^{11} - 10^{11}$
15	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$
16	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$

t=11, p=15	16	17	18	19	20
11	$4^2 - 4^{11}$	$12^4 - 12^4$	$31^4 - 33^4$	$74^4 - 94^4$	$165^4 - 247^4$
12	$1^0 - 1^7$	$4^{11} - 4^{11}$	$6^{11} - 6^{11}$	$11^{11} - 20^{39}$	$18^{11} - 42^{39}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$5^{11} - 5^{11}$	$7^{11} - 12^{11}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{12} - 4^{11}$
15	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$

t=11, p=16	17	18	19	20
11	$3^2 - 3^{11}$	$10^4 - 10^4$	$24^4 - 28^4$	$54^4 - 72^4$
12	$1^0 - 1^7$	$3^{11} - 3^{11}$	$6^{11} - 6^{11}$	$9^{11} - 14^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$	$3^{11} - 3^{11}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=11, p=17	18	19	20
11	$3^2 - 3^{11}$	$9^4 - 9^4$	$20^4 - 24^4$
12	$1^0 - 1^7$	$3^{11} - 3^{11}$	$4^{11} - 4^{11}$
13	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=11, p=18	19	20
11	$3^2 - 3^{11}$	$7^4 - 7^4$
12	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=11, p=19	20
11	$3^2 - 3^{11}$

t=12, p=12	13	14	15	16	17	18	19	20
12	$13^1 - 13^0$	$91^1 - 91^0$	$455^1 - 455^0$	$1820^1 - 1820^0$	$6188^1 - 6188^0$	$18564^1 - 18564^0$	$50388^1 - 50388^0$	$125970^1 - 125970^0$
13	$1^0 - 1^0$	$13^2 - 13^{11}$	$49^2 - 49^{11}$	$216^{11} - 225^{11}$	$592^{11} - 713^{29}$	$1713^2 - 2373^{26}$	$4199^{11} - 5535^{29}$	$11074^2 - 14576^{26}$
14	NA	$1^0 - 1^0$	$13^2 - 13^{11}$	$35^2 - 35^{11}$	$127^{11} - 140^{11}$	$317^{11} - 455^{29}$	$816^{11} - 1348^{29}$	$1705^{11} - 3217^{29}$
15	NA	NA	$1^0 - 1^0$	$13^2 - 13^{11}$	$28^2 - 28^{11}$	$92^{11} - 98^{11}$	$213^{11} - 319^{29}$	$408^{11} - 935^{29}$
16	NA	NA	NA	$1^0 - 1^0$	$13^2 - 13^{11}$	$24^2 - 24^{11}$	$65^{11} - 72^{11}$	$135^{11} - 196^{11}$
17	NA	NA	NA	NA	$1^0 - 1^0$	$13^2 - 13^{11}$	$21^2 - 21^{11}$	$51^{11} - 59^{11}$
18	NA	NA	NA	NA	NA	$1^0 - 1^0$	$13^2 - 13^{11}$	$19^2 - 19^{11}$
19	NA	NA	NA	NA	NA	NA	$1^0 - 1^0$	$13^2 - 13^{11}$

t=12, p=13	14	15	16	17	18	19	20
12	$7^2 - 7^5$	$35^4 - 35^4$	$140^4 - 140^4$	$476^4 - 492^4$	$1428^4 - 1547^4$	$3876^4 - 4608^4$	$9690^4 - 12134^4$
13	$1^0 - 1^7$	$7^{11} - 7^{11}$	$19^{11} - 19^{11}$	$66^{11} - 89^{11}$	$153^{11} - 256^{11}$	$388^{11} - 740^{11}$	$862^{11} - 1985^{11}$
14	$1^0 - 1^0$	$1^0 - 1^7$	$7^{11} - 7^{11}$	$15^{11} - 15^{11}$	$39^{11} - 44^{11}$	$81^{11} - 169^{39}$	$182^{11} - 438^{11}$
15	NA	$1^0 - 1^0$	$1^0 - 1^7$	$7^{11} - 7^{11}$	$12^{11} - 13^{11}$	$28^{11} - 33^{11}$	$54^{11} - 96^{11}$
16	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$7^{11} - 7^{11}$	$11^{11} - 11^{11}$	$20^{11} - 25^{11}$
17	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$7^{11} - 7^{11}$	$9^{11} - 9^{11}$
18	NA	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$7^{11} - 7^{11}$

t=12, p=14	15	16	17	18	19	20
12	$5^2 - 5^{11}$	$20^4 - 20^4$	$68^4 - 68^4$	$205^4 - 236^{11}$	$557^4 - 706^4$	$1393^4 - 1995^4$
13	$1^0 - 1^7$	$5^{11} - 5^{11}$	$14^{11} - 14^{11}$	$32^{11} - 49^{11}$	$65^{11} - 120^{11}$	$148^{11} - 369^{11}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$9^{11} - 9^{11}$	$19^{11} - 28^{11}$	$34^{11} - 66^{11}$
15	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$8^{11} - 8^{11}$	$13^{11} - 20^{11}$
16	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$5^{11} - 5^{11}$
17	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$

t=12, p=15	16	17	18	19	20
12	$4^2 - 4^{11}$	$16^4 - 16^4$	$48^4 - 48^4$	$131^4 - 167^4$	$328^4 - 520^4$
13	$1^0 - 1^7$	$4^{11} - 4^{11}$	$10^{11} - 10^{11}$	$19^{11} - 32^{11}$	$35^{11} - 95^{39}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$7^{11} - 7^{11}$	$11^{11} - 18^{11}$
15	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$4^{11} - 4^{11}$
16	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$

t=12, p=16	17	18	19	20
12	$4^2 - 4^{11}$	$12^4 - 12^4$	$33^4 - 35^4$	$83^4 - 116^4$
13	$1^0 - 1^7$	$4^{11} - 4^{11}$	$7^{11} - 7^{11}$	$13^{11} - 22^{39}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$	$5^{11} - 5^{11}$
15	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$

t=12, p=17	18	19	20
12	$3^2 - 3^{11}$	$11^4 - 11^4$	$28^4 - 30^4$
13	$1^0 - 1^7$	$3^{11} - 3^{11}$	$6^{11} - 6^{11}$
14	$1^0 - 1^7$	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=12, p=18	19	20
12	$3^2 - 3^{11}$	$9^4 - 9^4$
13	$1^0 - 1^7$	$3^{11} - 3^{11}$

t=12, p=19	20
12	$3^2 - 3^{11}$

t=13, p=13	14	15	16	17	18	19	20
13	$14^1 - 14^0$	$105^1 - 105^0$	$560^1 - 560^0$	$2380^1 - 2380^0$	$8568^1 - 8568^0$	$27132^1 - 27132^0$	$77520^1 - 77520^0$
14	$1^0 - 1^0$	$14^2 - 14^{11}$	$56^2 - 56^{11}$	$263^{11} - 275^{11}$	$762^{11} - 937^{29}$	$2325^2 - 3308^{29}$	$6030^{11} - 8220^{29}$
15	NA	$1^0 - 1^0$	$14^2 - 14^{11}$	$40^2 - 40^{11}$	$153^{11} - 168^{11}$	$402^{11} - 597^{29}$	$1088^{11} - 1871^{29}$
16	NA	NA	$1^0 - 1^0$	$14^2 - 14^{11}$	$32^2 - 32^{11}$	$110^{11} - 119^{11}$	$267^{11} - 419^{29}$
17	NA	NA	NA	$1^0 - 1^0$	$14^2 - 14^{11}$	$27^2 - 27^{11}$	$77^{11} - 87^{11}$
18	NA	NA	NA	NA	$1^0 - 1^0$	$14^2 - 14^{11}$	$24^2 - 24^{11}$
19	NA	NA	NA	NA	NA	$1^0 - 1^0$	$14^2 - 14^{11}$

t=13, p=14	15	16	17	18	19	20
13	$8^2 - 8^5$	$43^4 - 43^4$	$183^4 - 183^4$	$659^4 - 664^4$	$2087^4 - 2175^4$	$5963^4 - 6765^4$
14	$1^0 - 1^7$	$8^{11} - 8^{11}$	$23^{11} - 23^{11}$	$79^{11} - 107^{11}$	$194^{11} - 319^{11}$	$517^{11} - 929^{11}$
15	$1^0 - 1^0$	$1^0 - 1^7$	$8^{11} - 8^{11}$	$17^{11} - 17^{11}$	$46^{11} - 55^{11}$	$101^{11} - 220^{11}$
16	NA	$1^0 - 1^0$	$1^0 - 1^7$	$8^{11} - 8^{11}$	$13^{11} - 15^{11}$	$33^{11} - 40^{11}$
17	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$8^{11} - 8^{11}$	$13^{11} - 13^{11}$
18	NA	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$8^{11} - 8^{11}$

t=13, p=15	16	17	18	19	20
13	$6^2 - 6^{11}$	$26^4 - 26^4$	$94^4 - 94^4$	$298^4 - 330^{11}$	$852^4 - 1003^4$
14	$1^0 - 1^7$	$6^{11} - 6^{11}$	$15^{11} - 15^{11}$	$38^{11} - 57^{11}$	$81^{11} - 149^{11}$
15	$1^0 - 1^7$	$1^0 - 1^7$	$6^{11} - 6^{11}$	$11^{11} - 11^{11}$	$22^{11} - 32^{11}$
16	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$6^{11} - 6^{11}$	$8^{11} - 8^{11}$
17	NA	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$6^{11} - 6^{11}$

t=13, p=16	17	18	19	20
13	$5^2 - 5^{11}$	$18^4 - 18^4$	$57^4 - 63^4$	$163^4 - 217^4$
14	$1^0 - 1^7$	$5^{11} - 5^{11}$	$11^{11} - 11^{11}$	$23^{11} - 39^{11}$
15	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$	$8^{11} - 8^{11}$
16	$1^0 - 1^7$	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$

t=13, p=17	18	19	20
13	$4^2 - 4^{11}$	$14^4 - 15^4$	$40^4 - 45^4$
14	$1^0 - 1^7$	$4^{11} - 4^{11}$	$8^{11} - 8^{11}$
15	$1^0 - 1^7$	$1^0 - 1^7$	$4^{11} - 4^{11}$

t=13, p=18	19	20
13	$4^2 - 4^{11}$	$12^4 - 12^4$
14	$1^0 - 1^7$	$4^{11} - 4^{11}$

t=13, p=19	20
13	$3^2 - 3^{11}$

t=14, p=14	15	16	17	18	19	20
14	$15^1 - 15^0$	$120^1 - 120^0$	$680^1 - 680^0$	$3060^1 - 3060^0$	$11628^1 - 11628^0$	$38760^1 - 38760^0$
15	$1^0 - 1^0$	$15^2 - 15^{11}$	$64^2 - 64^{11}$	$316^{11} - 330^{11}$	$966^{11} - 1210^{29}$	$3100^2 - 4509^{29}$
16	NA	$1^0 - 1^0$	$15^2 - 15^{11}$	$45^2 - 45^{11}$	$182^{11} - 204^{11}$	$503^{11} - 768^{29}$
17	NA	NA	$1^0 - 1^0$	$15^2 - 15^{11}$	$36^2 - 36^{11}$	$130^{11} - 140^{11}$
18	NA	NA	NA	$1^0 - 1^0$	$15^2 - 15^{11}$	$30^2 - 30^{11}$
19	NA	NA	NA	NA	$1^0 - 1^0$	$15^2 - 15^{11}$

t=14, p=15	16	17	18	19	20
14	$8^2 - 8^5$	$46^4 - 46^4$	$207^4 - 207^4$	$787^4 - 846^4$	$2624^4 - 2900^4$
15	$1^0 - 1^7$	$8^{11} - 8^{11}$	$24^{11} - 24^{11}$	$94^{11} - 127^{11}$	$242^{11} - 400^{11}$
16	$1^0 - 1^0$	$1^0 - 1^7$	$8^{11} - 8^{11}$	$18^{11} - 18^{11}$	$54^{11} - 60^{11}$
17	NA	$1^0 - 1^0$	$1^0 - 1^7$	$8^{11} - 8^{11}$	$16^{11} - 16^{11}$
18	NA	NA	$1^0 - 1^0$	$1^0 - 1^7$	$8^{11} - 8^{11}$

t=14, p=16	17	18	19	20
14	$6^2 - 6^{11}$	$27^4 - 27^4$	$103^4 - 108^4$	$344^4 - 400^4$
15	$1^0 - 1^7$	$6^{11} - 6^{11}$	$16^{11} - 16^{11}$	$45^{11} - 71^{11}$
16	$1^0 - 1^7$	$1^0 - 1^7$	$6^{11} - 6^{11}$	$12^{11} - 12^{11}$
17	$1^0 - 1^0$	$1^0 - 1^7$	$1^0 - 1^7$	$6^{11} - 6^{11}$

t=14, p=17	18	19	20
14	$5^2 - 5^{11}$	$19^4 - 19^4$	$64^4 - 72^4$
15	$1^0 - 1^7$	$5^{11} - 5^{11}$	$12^{11} - 12^{11}$
16	$1^0 - 1^7$	$1^0 - 1^7$	$5^{11} - 5^{11}$

t=14, p=18	19	20
14	$4^2 - 4^{11}$	$16^4 - 16^4$
15	$1^0 - 1^7$	$4^{11} - 4^{11}$

t=14, p=19	20
14	$4^2 - 4^{11}$

t=15, p=15	16	17	18	19	20
15	$16^1 - 16^0$	$136^1 - 136^0$	$816^1 - 816^0$	$3876^1 - 3876^0$	$15504^1 - 15504^0$
16	$1^0 - 1^0$	$16^2 - 16^{11}$	$72^2 - 72^{11}$	$376^{11} - 396^{11}$	$1208^{11} - 1537^{29}$
17	NA	$1^0 - 1^0$	$16^2 - 16^{11}$	$51^2 - 51^{11}$	$215^{11} - 240^{11}$
18	NA	NA	$1^0 - 1^0$	$16^2 - 16^{11}$	$40^2 - 40^{11}$
19	NA	NA	NA	$1^0 - 1^0$	$16^2 - 16^{11}$

t=15, p=16	17	18	19	20
15	$9^2 - 9^5$	$54^4 - 54^4$	$257^4 - 258^4$	$1028^4 - 1083^4$
16	$1^0 - 1^7$	$9^{11} - 9^{11}$	$29^{11} - 29^{11}$	$111^{11} - 156^{26}$
17	$1^0 - 1^0$	$1^0 - 1^7$	$9^{11} - 9^{11}$	$19^{11} - 20^{11}$
18	NA	$1^0 - 1^0$	$1^0 - 1^7$	$9^{11} - 9^{11}$

t=15, p=17	18	19	20
15	$6^2 - 6^{11}$	$31^4 - 31^4$	$124^4 - 133^4$
16	$1^0 - 1^7$	$6^{11} - 6^{11}$	$18^{11} - 18^{11}$
17	$1^0 - 1^7$	$1^0 - 1^7$	$6^{11} - 6^{11}$

t=15, p=18	19	20
15	$5^2 - 5^{11}$	$21^4 - 21^4$
16	$1^0 - 1^7$	$5^{11} - 5^{11}$

t=15, p=19	20
15	$4^2 - 4^{11}$

t=16, p=16	17	18	19	20
16	$17^1 - 17^0$	$153^1 - 153^0$	$969^1 - 969^0$	$4845^1 - 4845^0$
17	$1^0 - 1^0$	$17^2 - 17^{11}$	$81^2 - 81^{11}$	$443^{11} - 468^{11}$
18	NA	$1^0 - 1^0$	$17^2 - 17^{11}$	$57^2 - 57^{11}$
19	NA	NA	$1^0 - 1^0$	$17^2 - 17^{11}$

t=16, p=17	18	19	20
16	$9^2 - 9^5$	$57^4 - 57^4$	$285^4 - 285^4$
17	$1^0 - 1^7$	$9^{11} - 9^{11}$	$31^{11} - 31^{11}$
18	$1^0 - 1^0$	$1^0 - 1^7$	$9^{11} - 9^{11}$

t=16, p=18	19	20
16	$7^2 - 7^{11}$	$35^4 - 35^4$
17	$1^0 - 1^7$	$7^{11} - 7^{11}$

t=16, p=19	20
16	$5^2 - 5^{11}$

t=17, p=17	18	19	20
17	$18^1 - 18^0$	$171^1 - 171^0$	$1140^1 - 1140^0$
18	$1^0 - 1^0$	$18^2 - 18^{11}$	$90^2 - 90^{11}$
19	NA	$1^0 - 1^0$	$18^2 - 18^{11}$

t=17, p=18	19	20
17	$10^2 - 10^5$	$67^4 - 67^4$
18	$1^0 - 1^7$	$10^{11} - 10^{11}$

t=17, p=19	20
17	$7^2 - 7^{11}$

t=18, p=18	19	20
18	$19^1 - 19^0$	$190^1 - 190^0$
19	$1^0 - 1^0$	$19^2 - 19^{11}$

t=18, p=19	20
18	$10^2 - 10^5$

t=19, p=19	20
19	$20^1 - 20^0$

Chapter 8

Conclusion

8.1 Summary

The determination of the value of $L(n,k,p,t)$ is an exponential problem. The use of an exhaustive search program can compute the value of $L(n,k,p,t)$ given enough CPU cycles and memory. However, this approach is not feasible is most cases.

In this thesis we approach the problem of determining $L(n,k,p,t)$ by finding upper and lower bounds. We have developed constructions to determine upper bounds. These included using BIBDs, combining designs, and adjoining designs to construct new Lotto designs. Constructions are quite easy to program into a computer which was useful for updating our table of upper bounds.

We have also developed several techniques for determining lower bounds for $L(n,k,p,t)$. Our first formulation was a generalization of the Schöheim bound for Covering designs. We also have formulas for computing lower bounds for infinite classes of parameters. These formulas can all be programmed to automatically update our table of lower bounds.

Another approach to computing the value of $L(n, k, p, t)$ is to compute them one at a time. This approach can be quite tedious since it only handles one case at a time. However, it is quite useful for completing gaps such as those in Bate's tables. We took this approach in Chapter 5 to complete Bate's tables [1].

Computer programs can be used to compute upper bounds by constructing designs. When no formula is available for determining upper bounds or when a formula gives an upper bound that is not very good, we may apply computer programs to generate upper bounds or improve upper bounds for $L(n, k, p, t)$. We have discussed several types of algorithms : greedy algorithms, random algorithms, a heuristic search and linear programming. Each of these programs constructs a design in a different manner. But they can all be used to compute upper bounds for $L(n, k, p, t)$. In general, the random algorithm produced the biggest upper bounds while the heuristic search and the linear programming approach produced the smallest (best) upper bounds of the three algorithms. However, the heuristic search and the linear programming approach were the slowest while the random algorithm was the fastest of the three algorithms.

A major purpose of this thesis is to track the upper and lower bounds for $L(n, k, p, t)$ and update them as better results are obtained. We have listed these tables in Chapter 7.

8.2 Further Work

The work presented in this thesis has barely scratched the surface on Lotto designs. There is much more that can be done on this topic. We will briefly state several ideas we have for future work.

It may be possible to state a lower bound formula similar to Theorem 2.3.2 for the case where $t \geq 3$. Such a formula would be quite helpful since we don't have any good lower bound formulas that apply to cases other than $t = 2$. The ideas of Bate and van Rees [2] may be used to determine the value of $L(n, k, p, 2)$ by considering "nice" designs and the frequency of the elements in these designs. It may be possible to formulate similar ideas that can be used for computing $L(n, k, p, t \geq 3)$. Finally, we could parallelize the exhaustive search algorithm and the heuristic search algorithm to speed up the algorithms. By parallelizing these algorithms, it may be feasible to apply them to larger values of n, k, p and t than can currently be done.

Bibliography

[1] J.A. Bate. *A Generalized Covering Problem*, Ph.D. thesis, University of Manitoba, 1978.

[2] J.A. Bate, G.H.J. van Rees. *Lotto Designs*, The Journal of Combinatorial Mathematics and Combinatorial Computing 28 (1998), 15-39.

[3] Bluskov, Private Communication.

[4] B. Bollobás. *Extremal Graph Theory*, Academic Press, London-New York, 1978.

[5] E.D. Boyer, D.L. Kreher, S. P. Radziszowski, J. Zou, A. Sidorenko. *On* $(n,5,3)$-*Turán systems*, Ars Combinatoria 37 (1994), 13-31.

[6] A.E. Brouwer, *Some Lotto numbers from an extension of Turán's thoerem*, Math. Center (Amsterdam) Report, 1978.

[7] A.E. Brouwer, M. Voorhoeve. *Turán theory and the Lotto problem*, Math. Center Tracts 106 (1979), 99-105.

[8] C. J. Colbourn, J. H. Dinitz. *The CRC Handbook of Combinatorial Designs*, CRC Press (1996).

[9] D. de Caen. *Extension of a theorem of Moon and Moser on complete subgraphs*, Ars Combin. 16 (1983), 5-10.

[10] R. A. Fisher, F. Yates, *Statistical Tables for Biological, Agricultural and Medical Research*, First Edition, Oliver and Boyd Ltd, 1938.

[11] Z. Füredi, G. Székely and Z. Zubor. *On The Lottery Problem*, Journal of Combinatorial Designs, **4** (1996), 5-10.

[12] D. M. Gordon, O. Patashnik, G. Kuperberg. *New Constructions for Covering Designs*, Journal of Combinatorial Designs, **3**(4) (1995), 269-284.

[13] D. M. Gordon. *Web Site of Covering Bounds at http://sdcc12.ucsd.edu/~xm3dg/cover.html* (May 19, 1998).

[14] M. Hall Jr. *Combinatorial Theory*, Blaisdell, Waltham, Mass., 1967.

[15] H. Hanani, D. Ornstein, V.T. Sós. *On the lottery problem*, Magyar Tud. Acad. Mat. Kutató 9 (1964), 155-158.

[16] C. Hartman. *Extremal Problems in Graph Theory*, Ph.D. thesis, Indiana State University, 1977.

[17] H. R. Lewis, L. Denenberg. *Data Structures and Their Algorithms*, Harper Collins Publishers, New York, 1991.

[18] N. Metropolis, A. Resenbluth, M. Roeentbluth., A. Teller, E. Teller. *Equations of stable calculations by fast computing machines*, J. Chem. Physics 21 (1953), 1087-1092.

[19] K.J. Nurmela, P.R.J. Ostergard. *Upper Bounds for Covering Designs by Simulated Annealing*, Congr. Numer. 96 (1993), 93-111.

[20] J. Schönheim. *On Coverings*, Pacific Journal of Mathematics, **14** (1964), 1405-1411.

[21] D. Stanton, D. White. *Constructive Combinatorics*, Springer-Verlag, New York, 1986.

[22] P. Turàn. *On the theory of Graphs*, Coll. Math 3, 1954, 19-30.