

Rigidity Properties of Operator Systems and Partial Order  
Relations in the State Space of  $C^*$ -algebras

by

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## Abstract

Arveson's hyperrigidity conjecture concerns the unique extension property of  $*$ -representations of a  $C^*$ -algebra with respect to a generating operator system. The maximal states in the dilation order fully encapsulate the cyclic representations of a  $C^*$ -algebra with the unique extension property. A reformulation of the conjecture by Davidson and Kennedy raises the question whether the maximal measures in the dilation order are concentrated on a particular set [28]. In this thesis, we address this question for general  $C^*$ -algebras. We show the existence of a projection such that the dilation maximal states are precisely those states which are concentrated on the projection. We also reformulate the conjecture in terms of the non-commutative topological properties of this projection.

Choquet order is a partial order defined on the set of regular Borel probability measures on a compact convex set. With the help of two equivalent characterizations of Choquet order, we define strong dilation relation and sub-division relation on the state space of a  $C^*$ -algebra. The equivalence of the two relations is not known in general. We show that the strong dilation relation is stronger than the sub-division relation. Moreover, we show the equivalence of the strong dilation relation with a *non-commutative* sub-division relation. We also demonstrate that these relations can serve as valuable tools for investigating certain rigidity properties of a generating operator system of a  $C^*$ -algebra.

# Contributions of Authors

Chapter 3 is based on the joint work with Raphaël Clouâtre which is submitted to a peer reviewed journal [20]. Chapter 4 is based on an ongoing solo project of mine intended to be published in a peer reviewed journal.

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*To,*

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# 1

## Introduction

In this thesis, we delve into the preorders defined on state spaces of  $C^*$ -algebras, driven by the profound implications of the hyperrigidity conjecture. The hyperrigidity conjecture, which concerns the unique extension property of a  $*$ -representation of a  $C^*$ -algebra, serves as a central motivation for our exploration. By investigating various preorders, such as the Choquet order, strong dilation order, and sub-division order, we aim to uncover new insights and establish deeper connections within the realm of operator algebras.

To understand the complex interplay between operator systems, the  $C^*$ -algebra they generate, and the completely positive maps on them, we begin with a classical result in approximation theory and see how it motivated another approximation problem in the realm of operator algebras. Korovkin's theorem highlights a fascinating insight: the asymptotic behaviour of certain positive maps on  $C([0, 1])$  can be determined by their action on the set of quadratic polynomials (1953) [43]. To be more precise, let for each  $n$ ,  $\Phi_n : C([0, 1]) \longrightarrow C([0, 1])$  be a positive map such that

$$\lim \|\Phi_n(f) - f\| = 0.$$

for all  $f \in \{1, x, x^2\}$ . Then the convergence holds for every function, i.e. for all

$f \in C([0, 1])$  we get,

$$\lim \|\Phi_n(f) - f\| = 0.$$

This can be proven using the fact that the Choquet boundary of the operator system of the quadratic polynomials coincides with  $[0, 1]$ . Šaškin generalized this theorem to arbitrary metrizable compact Hausdorff spaces [57] in 1967, and Davidson and Kennedy extended this theorem to the non-metrizable case in 2019.

**Theorem 1.0.1** (Šaškin, Korovkin, Davidson, Kennedy [57] [27]). *Let  $X$  be a compact Hausdorff topological space. Let  $G \subset C(X)$  and  $\Phi_\lambda : C(X) \rightarrow C(X)$  be a net of positive maps. Then the following are equivalent:*

(i)  $\lim \|\Phi_\lambda(f) - f\| = 0, \forall f \in G \implies \lim \|\Phi_\lambda(f) - f\| = 0 \forall f \in C(X).$

(ii) *The Choquet boundary of the operator system generated by  $G$  coincides with  $X$ .*

This has inspired many Korovkin-type approximation results [4]. Arveson initiated the study of the Choquet boundary of general operator systems in 1969 in his landmark paper and showed that the Choquet boundary serves as an invariant for the operator systems [5],[6]. Arveson's notion of non-commutative Choquet boundary generalizes the classical notion of Choquet boundary of operator systems contained in a commutative  $C^*$ -algebra.

An **operator system** is a unital, self-adjoint vector subspace of a  $C^*$ -algebra. It is well known that every  $C^*$ -algebra can be realized as a  $C^*$ -subalgebra of  $B(\mathcal{H})$ , where  $B(\mathcal{H})$  denotes the set of all bounded operators on a Hilbert space  $\mathcal{H}$ . Hence, an operator system can also be defined as a unital, self-adjoint subspace of bounded operators on a Hilbert space. For a fixed  $n \in \mathbb{N}$ , let  $T = [t_{i,j}]_{1 \leq i,j \leq n}$ , where  $t_{i,j} \in B(\mathcal{H})$ . Then  $T$  can be viewed as an operator in the Hilbert space  $\mathcal{H}^{(n)}$ ; where  $\mathcal{H}^{(n)}$  is the direct sum of  $n$  copies of  $\mathcal{H}$ . A bounded linear map  $\Phi : S \rightarrow B(\mathcal{H})$  is said

to be **completely positive** if  $[\Phi(s_{i,j})] \geq 0$  in  $B((\mathcal{H})^{(n)})$  for all  $n$  and for all  $n \times n$  matrices  $[s_{i,j}] \geq 0$  with  $s_{i,j} \in S$ .

Suppose  $S$  is an operator system contained in the  $C^*$ -algebra  $B$ . Then a  $*$ -representation is said to have the **unique extension property** or **UEP** with respect to  $S$  if there is no other unital completely positive map on  $B$  that agrees with  $\pi$  on  $S$ . A **boundary representation** of  $S$  is an irreducible  $*$ -representation with the unique extension property relative to  $S$ , and the **Choquet boundary** of an operator system is the collection of all boundary representations of the operator system. The existence of sufficiently many boundary representations is helpful in constructing the *minimal*  $C^*$ -cover of an operator system [30], [7], [26].

Drawing motivation from Korovkin's result, Arveson wondered if a similar rigidity property of a generating operator system in a  $C^*$ -algebra can be characterized by the boundary representations in the non-commutative setup. An operator system  $S$  is said to be **hypperrigid** in  $B = C^*(S)$  if for every isometric  $*$ -representation  $\pi : B \rightarrow B(\mathcal{H})$  and any sequence of unital completely positive maps  $\Phi_n : B(\mathcal{H}) \rightarrow B(\mathcal{H})$  such that

$$\lim \|\Phi_n(\pi(s)) - \pi(s)\| = 0, \quad \forall s \in S$$

then

$$\lim \|\Phi_n(\pi(x)) - \pi(x)\| = 0, \quad \forall x \in B.$$

It is important to mention that Arveson's notion of hyperrigidity is a priori stronger than Korovkin's classical one because, unlike the classical case, Arveson's definition allows the range of the maps  $\Phi_n$  to be entire  $B(\mathcal{H})$  [8]. Arveson proved that a generating operator system is hyperrigid in the  $C^*$ -algebra if and only if every  $*$ -representation of the  $C^*$ -algebra has the unique extension property relative to

the operator system. However, Arveson conjectured that if every irreducible  $*$ -representation has the unique extension property, then that is enough to ensure the hyperrigidity of the generating operator system.

**Arveson's Hyperrigidity conjecture.** The operator system  $S$  is hyperrigid in  $B$  whenever all irreducible  $*$ -representations of  $B$  are boundary representations of  $S$ .

In other words, Arveson conjectured that if every irreducible  $*$ -representation has the unique extension property, then every  $*$ -representation has the unique extension property. Arveson was able to produce many pieces of evidence supporting his conjecture [8]. For instance, the conjecture is true when the  $C^*$ -algebra  $B$  has a countable spectrum. Secondly, let  $B$  be a  $C^*$ -algebra generated by a finite set of unitaries, and  $S$  be the operator system generated by these unitaries. Then  $S$  is hyperrigid in  $B$ . Moreover, Arveson proved a local version of the hyperrigidity conjecture for the commutative case [8]. This was later generalized by Clouâtre [17]. The hyperrigidity conjecture has attracted attention for more than a decade and has been observed to be valid in many cases [42], [38], [18], [17], [19], [37], [40], [29], [52], [34], [28], [23], [20], [55]. Recently, a counter-example of a type-I  $C^*$ -algebra was given to show that the conjecture is not true in general [9]. However, the story of hyperrigidity remains incomplete and interesting. For instance, in some special cases, the hyperrigidity conjecture is equivalent to the essential normality conjecture [38], and the validity of the hyperrigidity conjecture is still not known in those cases.

A recurring theme in this thesis is that the hyperrigidity conjecture can be successfully analyzed by means of the states of a  $C^*$ -algebra. Indeed, cyclic  $*$ -representations with the unique extension property can be used to identify all such representations because this property is preserved under the direct sum of  $*$ -representations. Moreover, a *state* can be associated with every cyclic representation via the GNS

correspondence. So, it is natural to wonder if the unique extension property of the cyclic representation can be translated into a nice property of the corresponding state. Davidson and Kennedy gave an affirmative answer to this question. They showed that a cyclic representation of  $B = C^*(S)$  has the unique extension property relative to  $S$  precisely when the corresponding state is maximal in the *dilation order*, where the dilation order is a partial order defined on the state space of  $B$  [28]. As the name suggests, the dilation order is defined using the idea of the dilation of completely positive maps, and it depends on both the  $C^*$ -algebra  $B$  and the generating operator system  $S$ . Throughout,  $\mathcal{D}(S, B)$  will denote the dilation order determined by the operator system  $S$  and the  $C^*$ -algebra  $B$ . Since a partial order is in particular a relation; we view  $\mathcal{D}(S, B)$  as a subset of  $\mathcal{E}(B) \times \mathcal{E}(B)$ , where  $\mathcal{E}(B)$  denotes the state space of the  $C^*$ -algebra  $B$ .

Davidson and Kennedy carried out a deep investigation of the conjecture in the commutative setup [28]. An operator system contained in a commutative  $C^*$ -algebra is said to be a function system. A function system can be defined axiomatically without mentioning the  $C^*$ -algebra. A detailed account of this approach of function systems can be found in [46]. Let  $S$  be a function system contained in the  $C^*$ -algebra  $C(X)$  where  $X$  is a compact metrizable Hausdorff space. If we consider the set  $K$  to be the state space of  $S$ , then it is a compact convex subset of  $S^*$  in the weak- $*$  topology. It is worth noting that the category of function systems is dual to the category of compact convex subsets of locally convex topological vector spaces. In fact,  $S$  is *complete isometrically isomorphic* to the operator system of all continuous affine functions on  $K$ , denoted by  $A(K)$ . This association imports the theory of convexity to the study of unique extension property of representations of commutative  $C^*$ -algebras. Kadison proved that  $C(K)$  is the *maximal commutative*  $C^*$ -cover of the function system  $S$  [36]. In fact, one can consider the pair  $(A(K), C(K))$  as a prototype to investigate many interactions between the function system and its generating

$C^*$ -algebra. For instance, Davidson and Kennedy reformulated the hyperrigidity conjecture completely in terms of the probability measures on a compact convex set  $K$  and the dilation order determined by the pair  $(A(K), C(K))$ . They compared the dilation order with classical *Choquet order*, which is very handy in determining the boundary measures on compact convex sets. A regular Borel probability measure  $\mu$  on a compact convex set is called a boundary measure if  $\mu(U) = 0$  for all open subsets  $U$  of  $K$  disjoint from the extreme boundary of  $K$ . When  $K$  is assumed to be metrizable, then  $\mu$  is a boundary measure precisely when  $\mu(\partial_e K) = 1$ ; where  $\partial_e K$  denotes the extreme boundary of  $K$ . Davidson and Kennedy showed that the Choquet order is *strictly stronger* than the dilation order, but the hyperrigidity conjecture holds for function systems if and only if the Choquet order and the dilation order share the same set of maximal measures. In other words, the hyperrigidity conjecture holds in the commutative separable case if and only if, for every compact metrizable convex set  $K$ , a regular Borel probability measure  $\mu$  is  $\mathcal{D}(A(K), C(K))$ -maximal precisely when  $\mu(\partial_e K) = 1$ .

This reformulation provides the guiding question for this work. The question raised here is:

**Question 1.** Can dilation maximal states be characterized as those states which are concentrated on a *boundary set*?

We investigate this question without assuming commutativity.

For a unital  $C^*$ -algebra  $B$ , we let  $\mathcal{E}(B)$  denote its state space, and  $\mathcal{E}_p(B)$  denote the pure states. Given a state  $\varphi$  on  $B$ , we let  $\mathcal{R}_\varphi$  denote the set of Borel probability measures  $\mu$  on  $\mathcal{E}_p(B)$  satisfying  $\varphi = \int \omega d\mu(\omega)$ . Such measures always exist, at least in the separable setting [10, Theorem 4.2].

Given a Borel measurable subset  $X \subset \mathcal{E}_p(B)$ , we let  $\Sigma_X$  denote the set of those states  $\varphi$  for which there exists  $\mu \in \mathcal{R}_\varphi$  concentrated on  $X$ . Furthermore, we let  $\Sigma^X$  denote the set of those states  $\varphi$  for which every  $\mu \in \mathcal{R}_\varphi$  is concentrated on  $X$ .

Let  $\Delta \subset \mathcal{E}(B) \times \mathcal{E}(B)$  be partial order, and denote its maximal elements by  $\max(\Delta)$ . A Borel measurable subset  $X \subset \mathcal{E}_p(B)$  will be called a  $\Delta$ -boundary if

$$\max(\Delta) = \Sigma^X = \Sigma_X.$$

We say that the order  $\Delta$  is *hyperrigid* if  $\Sigma_\Omega \subset \max(\Delta)$ , where  $\Omega = \mathcal{E}_p(B) \cap \max(\Delta)$  is the set of pure  $\Delta$ -maximal states. In other words,  $\Delta$  is hyperrigid precisely when states of the form  $\int \omega d\mu(\omega)$  are  $\Delta$ -maximal, where  $\mu$  is a Borel probability measure concentrated on the pure  $\Delta$ -maximal states.

In Section 3.1, the relationships between the above notions are studied. In Corollary 3.1.2, we show that hyperrigidity of  $\Delta$  is equivalent to the existence of a  $\Delta$ -boundary, at least when  $B$  is separable and  $\Delta$  is weak-\* closed and convex.

In Section 3.2, we define the dilation order  $\mathcal{D}(S, B) \subset \mathcal{E}(B) \times \mathcal{E}(B)$  relative to an operator system  $S$  generating  $B$ . Using the terminology introduced above, in Corollary 3.2.1 we show that Arveson's conjecture can be reformulated as saying that  $\mathcal{D}(S, B)$  is hyperrigid whenever all pure states are  $\mathcal{D}(S, B)$ -maximal. In turn, this is equivalent to the existence of a  $\mathcal{D}(S, B)$ -boundary.

Classically, it is known that certain partial orders on the regular Borel probability measures on a compact Hausdorff space always admit boundaries. Let  $Y$  be a compact Hausdorff space and  $\mathcal{F}$  be a cone of real-valued continuous functions on  $Y$ . Let  $\text{Order}(\mathcal{F})$  be the set of pairs of measures  $(\mu, \nu)$  such that

$$\int f d\mu \leq \int f d\nu \quad \forall f \in \mathcal{F}.$$

If  $\mathcal{F}$  separates the points of  $Y$ , then  $\text{Order}(\mathcal{F})$  is a partial order on the regular Borel probability measures on  $Y$ . We say that the cone  $\mathcal{F}$  is *max-stable* if whenever  $f, g \in \mathcal{F}$ , then  $f \vee g \in \mathcal{F}$  where  $f \vee g(x) = \max\{f(x), g(x)\}$  for all  $x \in Y$ . The max-stability of the cone  $\mathcal{F}$  is a sufficient condition for  $\text{Order}(\mathcal{F})$  to have a boundary

[3, Corollary I.5.18]. We thus examine the dilation order from this perspective, so as to determine whether the aforementioned machinery can be employed to produce the boundary that we seek. To do this, we rely rather heavily on the developments in non-commutative Choquet theory and function theory obtained recently by Davidson and Kennedy [27].

When  $B$  is chosen to be the so-called maximal  $C^*$ -cover of  $S$ , we demonstrate in Proposition 3.2.3 that the dilation order is indeed induced by a certain cone  $\Xi$  (Corollary 3.2.2). For a general representation of  $S$ , these findings still enable us to identify the dilation maximal states as the maximal elements of a cone order (Theorem 3.2.3). However, even when  $S$  can be represented in a commutative  $C^*$ -algebra, the corresponding cone is not known to be stable under maxima (Proposition 3.2.2), which means that the aforementioned result from [3] cannot be utilized to show the existence of a boundary for dilation order.

In the context of non-commutative topology, the projections play the role of sets. In other words, the projections can be viewed as non-commutative sets. After observing the difficulty in producing a boundary set for the dilation order, we now ask the question:

**Question 2.** Can we characterize the dilation maximal states as those states which are concentrated on some non-commutative set?

In Section 3.3 we observe that the  $\mathcal{D}(S, B)$  maximality is preserved under absolute continuity (see Theorem 3.3.2), which enables us to use some results of [24] on non-commutative measure theory to produce an affirmative answer to the question raised above (see Theorem 3.3.3).

**Theorem A.** *Let  $B$  be a unital  $C^*$ -algebra and let  $S \subset B$  be an operator system such that  $B = C^*(S)$ . Then, there exists a projection  $\mathfrak{d} \in B^{**}$  with the property that a state  $\phi$  on  $B$  is  $\mathcal{D}(S, B)$ -maximal precisely when  $\phi(\mathfrak{d}) = 1$ .*

We call the projection  $\mathfrak{d}$  mentioned above as the **boundary projection**. In

Section 3.3.1 we examine the non-commutative topological properties of  $\mathfrak{d}$ , in the sense of Akemann [1],[2]. As an application of this, we prove our second main result of this thesis that reformulates the hyperrigidity conjecture in terms of non-commutative regularity property of  $\mathfrak{d}$  (see Corollary 3.3.2).

**Theorem B.** *Assume that  $B$  is separable and that every pure state on  $B$  is  $\mathcal{D}(S, B)$ -maximal. Then, the following statements are equivalent.*

- (i) *The operator system  $S$  is hyperrigid in  $B$ .*
- (ii) *The boundary projection  $\mathfrak{d}$  is closed.*
- (iii) *The boundary projection  $\mathfrak{d}$  is the infimum of a collection of open projections in  $B^{**}$ .*

We have already mentioned that the Choquet order is a partial order defined on the regular Borel probability measures on a compact convex set. Let  $K$  be a compact convex subset of a locally convex topological vector space, and  $P(K)$  denote the set of all regular Borel probability measures on  $K$ . The Choquet order is the set consisting of all pairs  $(\mu, \nu) \in P(K) \times P(K)$  such that

$$\int f d\mu \leq \int f d\nu$$

for all convex functions  $f \in C(K)$ . In other words, the Choquet order is simply  $\text{Order}(\mathcal{F})$  where  $\mathcal{F}$  is the cone of all continuous convex functions on  $K$ . The cone of continuous convex functions can be viewed as the max stable cone generated by the real-valued affine functions on  $K$ . This partial order can be generalized to obtain an abstract Choquet order on the regular Borel probability measures on an arbitrary compact Hausdorff set  $X$ , where  $X$  is not restricted to be convex [3, Chapter I.5]. If  $S$  is an operator system contained in a commutative  $C^*$ -algebra  $B$ , then a natural generalization of the Choquet order to this setup would be the cone order determined

by the max stable cone generated by the self-adjoint part of  $S$ . But constructing such a cone in the non-commutative setup is not straightforward. Thus, lifting the definition of Choquet order to the non-commutative setup is not straightforward. We examine this problem in Chapter 4. We have defined two relations on the state space of a general  $C^*$ -algebra that can be viewed as a natural generalizations of the Choquet order. By analogy with the classical commutative case, we hope that these new relations can eventually serve as meaningful tools in studying the structure of general operator systems.

Davidson and Kennedy provided a dilation-theoretic interpretation of the Choquet order by showing that the Choquet order is equivalent to the strong dilation order. Let  $K$  be a compact convex subset of a locally convex topological vector space, and let  $\mu, \nu$  be in  $P(K)$ . Let  $(\pi_\mu, L^2(\mu), \xi_\mu)$  be the GNS representation of  $\mu$ . Then  $\mu$  is said to be dominated by  $\nu$  in the **strong dilation order** if there exists a ucp map  $\Psi : C(K) \rightarrow L^\infty(\mu)$  such that  $\Psi(a) = \pi_\mu(a)$  for all continuous affine functions  $a$  on  $K$  and  $\int f d\nu = \langle \Psi(f)\xi_\mu, \xi_\mu \rangle$  for all  $f \in C(K)$ . It is well known that  $L^\infty(\mu) = \pi_\mu(C(K))''$ . This observation helps us in defining a strong dilation relation in the state space of a general  $C^*$ -algebra. Let  $S$  be an operator system that generates a  $C^*$ -algebra  $B$ . Let  $\varphi$  and  $\psi$  be states on  $B$  and  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  be the GNS representation of  $\varphi$ . We say that  $\psi$  is a strong dilation of  $\varphi$  if there exists a ucp map  $\Psi : B \rightarrow \pi_\varphi(B)''$  such that  $\pi_\varphi(s) = \Psi(s)$  for all  $s \in S$  and  $\psi(b) = \langle \Psi(b)\xi_\varphi, \xi_\varphi \rangle$  for all  $b \in B$ . Let  $\text{StD}(S, B)$  be the set of all pairs of states  $(\varphi, \psi)$  such that  $\psi$  is a strong dilation of  $\varphi$ . This defines a relation on the state space of  $B$ , and we call it the **strong dilation relation** (see Definition 4.0.1).

On the other hand, it is a classical fact that a pair of regular Borel probability measures  $(\mu, \nu)$  on a compact convex set  $K$  is in the Choquet order if and only if they satisfy a sub-division property. Let  $\mu \in P(K)$ , then a finite subset  $\{\mu_1, \mu_2, \dots, \mu_n\}$  of  $P(K)$  is said to be a subdivision of  $\mu$  if  $\mu$  is a convex combination of  $\mu_1, \mu_2, \dots, \mu_n$ .

[49, Proposition 15.1] shows that  $\mu$  is dominated by  $\nu$  in the Choquet order if and only if for every sub-division  $\{\mu_1, \dots, \mu_n\}$  with convex combination  $\sum_{i=1}^n \lambda_i \mu_i = \mu$ , there exists a corresponding sub-division  $\{\nu_1, \dots, \nu_n\}$  such that  $\sum_{i=1}^n \lambda_i \nu_i = \nu$  and  $\int a d\mu_i = \int a d\nu_i$  for all  $i$  and for all continuous affine functions  $a$  on  $K$ .

Drawing inspiration from above, we have defined a sub-division relation on the state space of a general C\*-algebra in Chapter 4. Let  $S$  be an operator system contained in the C\*-algebra  $B$  and  $\varphi, \psi \in \mathcal{E}(B)$ . Let  $\text{SubD}(S, B) \subset \mathcal{E}(B) \times \mathcal{E}(B)$  be the set containing all pairs of states  $(\varphi, \psi)$  such that whenever  $\{\varphi_1, \dots, \varphi_n\}$  is a sub-division of  $\varphi$  of the form:

$$\varphi = \sum_{i=1}^n \lambda_i \varphi_i$$

then there exists a corresponding sub-division  $\{\psi_1, \dots, \psi_n\}$  of  $\psi$  such that  $(\varphi_i, \psi_i) \in \mathcal{D}(S, B)$  for all  $i$  and

$$\psi = \sum_{i=1}^n \lambda_i \psi_i.$$

$\text{SubD}(S, B)$  defines a relation on the state space of  $B$  and we call it the **sub-division relation** on  $B$  (see Definition 4.0.2). It is not straightforward to see why this is a natural generalization of the sub-division relation in the commutative case. However, we show that this indeed generalizes the sub-division of the classical measures in the non-commutative setup (see Corollary 4.0.1).

In the previous paragraph, we mentioned that the sub-division relation recovers the classical sub-division order when restricted to the regular Borel probability measures on compact convex sets. So, in that case, both sub-division relation and the strong dilation relation are equivalent to the classical Choquet order. This inspires us to view the strong dilation relation and the sub-division relation as two generalizations of the Choquet order in the general setup. We then ask the next obvious question:

**Question 3.** Is the strong dilation relation equivalent to the sub-division rela-

tion?

In Theorem 4.0.1, we show that the strong dilation relation is stronger than the sub-division relation.

**Theorem C.** *Let  $S$  be an operator system contained in a  $C^*$ -algebra  $B$  such that  $C^*(S) = B$ . Then  $\text{StD}(S, B) \subset \text{SubD}(S, B)$ .*

The converse of the containment is not known so far. However, we establish that for a pair of states  $\varphi$  and  $\psi$  of  $B$ ,  $(\varphi, \psi) \in \text{StD}(S, B)$  if and only if the pair of states satisfies some *non-commutative* sub-division property (see Theorem 4.0.3).

It is worth mentioning that the strong dilation relation had previously been defined and studied in the literature only for the very specific case of compact convex subsets of topological vector spaces but not even for general commutative  $C^*$ -algebras. Nevertheless, we show that the strong dilation relation is equivalent to the sub-division relation when the  $C^*$ -algebra is commutative (see Theorem 4.0.2).

The recent counterexample by Bilich and Dor-on [9] shows that the hyperrigidity conjecture is not true in general, indicating that the maximality of the Choquet boundary does not always guarantee the unique extension property for all  $*$ -representations. A natural question arises: Can we weaken the unique extension property to make the hyperrigidity conjecture valid? This question is addressed in a recent work by Clouâtre and Thompson [22], where they propose a weakening of the unique extension property. Let  $S$  be an operator system contained in a  $C^*$ -algebra  $B$  such that  $C^*(S) = B$ . A  $*$ -representation  $\pi : B \rightarrow B(\mathcal{H})$  is said to have the **unique tight extension property** if whenever  $\Psi : B \rightarrow \pi(B)''$  is a unital completely positive map with  $\Psi(s) = \pi(s)$  for all  $s \in S$ , then  $\Psi = \pi$ . Clouâtre and Thompson have shown that Arveson's hyperrigidity conjecture is valid in the separable case if one replaces the unique extension property by the unique tight extension property. More precisely, If  $S$  is a generating operator system contained in a separable  $C^*$ -algebra  $B$  then every irreducible  $*$ -representation has the unique extension property if and

only if every  $*$ -representation has unique tight extension property. In Proposition 4.0.1, we provide a connection between the strong dilation relation on the states of a  $C^*$ -algebra and cyclic representations with the unique tight extension property. This indicates that the strong dilation relation may be a useful tool to investigate the unique tight extension property of  $*$ -representations.

## 2

# Background and preliminaries

## 2.1 Operator system, completely positive maps, unique extension property

Let  $\mathcal{H}$  and  $\mathcal{K}$  be Hilbert spaces. Throughout  $B(\mathcal{H}, \mathcal{K})$  will denote the set of all bounded linear operators  $T : \mathcal{H} \rightarrow \mathcal{K}$ . Moreover, we write  $B(\mathcal{H}) = B(\mathcal{H}, \mathcal{H})$ .

Let  $\mathcal{H}$  be a Hilbert space with the inner product  $\langle \cdot, \cdot \rangle$ . For a cardinal number  $n$ , define  $\mathcal{H}^{(n)}$  as

$$\mathcal{H}^{(n)} = \left\{ (h_i)_{i \leq n} : h_i \in \mathcal{H}, \sum_{i \leq n} \|h_i\|^2 < \infty \right\}.$$

Equipped with the inner product

$$\langle (h_i)_{i \leq n}, (k_i)_{i \leq n} \rangle' = \sum_{i \leq n} \langle h_i, k_i \rangle,$$

$\mathcal{H}^{(n)}$  forms a Hilbert space.

For  $n \in \mathbb{N}$ , let  $M_n(B(\mathcal{H}))$  be the set of all  $n \times n$  matrices with entries from  $B(\mathcal{H})$ . Then every element  $[t_{i,j}]_{i,j \leq n} \in M_n(B(\mathcal{H}))$  can be naturally identified with an element of  $B(\mathcal{H}^{(n)})$ . Conversely, every element  $T \in B(\mathcal{H}^{(n)})$  can be identified with a  $n \times n$  matrix  $[t_{i,j}]_{1 \leq i,j \leq n}$  with  $t_{i,j} \in B(\mathcal{H})$ . This identification behaves well with the

product and involution in  $B(\mathcal{H}^{(n)})$ . More precisely, whenever  $[a_{ij}], [b_{jk}] \in B(\mathcal{H}^{(n)})$ , then

$$[a_{ij}][b_{ij}] = \left[ \sum_{k=1}^n a_{ik} b_{kj} \right] \quad (2.1.1)$$

and,

$$[a_{ij}]^* = [a_{ji}^*]. \quad (2.1.2)$$

Therefore  $M_n(B(\mathcal{H}))$  inherits a  $C^*$ -algebra structure for all  $n \in \mathbb{N}$  through this identification. We refer to [45, Chapter 1] for additional details.

**Definition 2.1.1.** An operator system  $S$  is a vector subspace of a unital  $C^*$ -algebra  $B$  such that  $1_B \in S$  and  $s^* \in S$ , whenever  $s \in S$ .

Let  $B$  be a  $C^*$ -algebra. Then, from the basic theory of  $C^*$ -algebras, we may assume that  $B$  is contained in  $B(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ . With the identification mentioned above,  $M_n(B) = \{[b_{ij}] : b_{ij} \in B, 1 \leq i, j \leq n\}$  forms a  $C^*$ -algebra sitting inside  $M_n(B(\mathcal{H}))$  for each  $n \in \mathbb{N}$ . Furthermore, if  $S$  is an operator system in the  $C^*$ -algebra  $B$ , then  $M_n(S) = \{[s_{ij}] : s_{ij} \in S, 1 \leq i, j \leq n\}$  is an operator system contained in the  $C^*$ -algebra  $M_n(B)$  for each  $n \in \mathbb{N}$ .

Let  $\phi : S_1 \longrightarrow S_2$  be a linear map between two operator systems  $S_1$  and  $S_2$ . Then the  $n$ -th **ampliation** of  $\phi$ , denoted by  $\phi_n : M_n(S_1) \longrightarrow M_n(S_2)$  is defined as,

$$\phi_n([s_{ij}]) = [\phi(s_{ij})], \quad \forall [s_{ij}] \in M_n(S_1).$$

**Definition 2.1.2.** Let  $S_1$  and  $S_2$  be operator systems and  $\phi : S_1 \longrightarrow S_2$  be a linear map. Then,

1.  $\phi$  is said to be **positive** if  $\phi(s) \geq 0$  whenever  $s \in S_1$  and  $s \geq 0$ .

2.  $\phi$  is said to be **completely positive**(cp) if  $\phi_n$  is positive for each  $n \in \mathbb{N}$ .
3.  $\phi$  is said to be **unital completely positive**(ucp) if  $\phi$  is completely positive and  $\phi(1_{S_1}) = 1_{S_2}$ .
4.  $\phi$  is said to be a **complete isometry** if  $\phi_n$  is an isometry for all  $n \in \mathbb{N}$ .
5.  $\phi$  is said to be a **complete contraction** if  $\phi_n$  is a contraction for all  $n \in \mathbb{N}$ .
6.  $\phi$  is said to be **completely bounded** if  $\sup_{n \in \mathbb{N}} \|\phi_n\| < \infty$ .

A unital completely positive linear functional  $\varphi : S \rightarrow \mathbb{C}$  on an operator system  $S$  is said to be a **state**. The set of all states on  $S$  is called the **state space** of  $S$ , denoted by  $\mathcal{E}(S)$ . Let  $S^*$  denote the dual of  $S$ . Then  $\mathcal{E}(S)$  is a compact convex subset of  $S^*$  in the weak-\* topology and we let  $\mathcal{E}_p(S) \subset \mathcal{E}(S)$  denote the set of all extreme points of  $\mathcal{E}(S)$ . Elements of  $\mathcal{E}_p(S)$  are called the **pure states** of  $S$ .

Let  $B$  be a  $C^*$ -algebra. The GNS construction assures that for every state  $\varphi$  of  $B$ , there exists a triple  $(\pi, \mathcal{H}, \xi)$  consisting of a  $*$ -representation  $\pi$  of  $B$  on a Hilbert space  $\mathcal{H}$  and a unit vector  $\xi \in \mathcal{H}$  such that,

$$\varphi(b) = \langle \pi(b)\xi, \xi \rangle \quad \forall b \in B. \quad (2.1.3)$$

A triple  $(\pi, \mathcal{H}, \xi)$  satisfying 2.1.3 is said to be a **representation** of  $\varphi$ . If the vector  $\xi$  is cyclic for the representation  $\pi$ , then  $\pi$  is called the GNS representation of  $\varphi$ . This special construction of the GNS representation can be derived from a more general result by Stinespring. He proved that any unital completely positive map is an isometric *compression* of a  $*$ -representation [45].

**Theorem 2.1.1** (Stinespring's dilation theorem). *Let  $B$  be a unital  $C^*$ -algebra and  $\phi : B \rightarrow B(\mathcal{H})$  be a completely positive map. Then there exists a Hilbert space  $\mathcal{K}$ , a unital  $*$ -representation  $\pi : B \rightarrow B(\mathcal{K})$  and a bounded operator  $V : \mathcal{H} \rightarrow \mathcal{K}$ , with*

$\phi(1) = \|V\|^2$  such that

$$\phi(b) = V^*\pi(b)V \quad \forall b \in B.$$

It is noteworthy that when  $\phi$  is unital and completely positive, the operator  $V$  is an isometry. The representation  $\pi$  is said to be **minimal Stinespring dilation** if the space  $\{\pi(b)Vh : b \in B, h \in \mathcal{H}\}$  is dense in  $\mathcal{K}$ . The minimal Stinespring dilation is unique up to unitary equivalence. In particular, the GNS representation  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  of a state  $\varphi$  is defined to be the minimal Stinespring dilation of  $\varphi$ . If  $(\sigma, \mathcal{K}, \eta)$  is a representation of  $\varphi$ , then the closed cyclic subspace  $\mathcal{H}$  generated by the vector  $\eta$  is a reducing subspace of  $\sigma$ , and  $\sigma$  restricted to  $\mathcal{H}$  is unitarily equivalent to the GNS representation of  $\varphi$ . More precisely, in that case  $\sigma$  is unitarily equivalent to  $\pi_\varphi \oplus \pi'$  where  $\pi'$  is some  $*$ -representation of  $B$ .

The restriction of completely positive maps on an operator subsystem remains completely positive. Thus, a natural question arises: Can a completely positive map on an operator system be extended to a completely positive map on a larger operator system? Arveson answered this question affirmatively.

**Theorem 2.1.2** ([45, Theorem 7.5]). *Let  $S$  be an operator system contained in the  $C^*$ -algebra  $B$  and  $\phi : S \rightarrow B(\mathcal{H})$  a completely positive map. Then there exists a completely positive map  $\psi : B \rightarrow B(\mathcal{H})$  such that  $\psi|_S = \phi$ .*

The extension of a completely positive map is not unique in general. A  $*$ -representation  $\pi : B \rightarrow B(\mathcal{H})$  of a  $C^*$ -algebra is said to have the **unique extension property** relative to an operator system  $S \subset B$  if  $\pi$  is the only unital completely positive extension of  $\pi|_S$ . An irreducible  $*$ -representation of  $B$  with the unique extension property relative to  $S$  is said to be a **boundary representation** of  $S$ . The following result is going to be a useful tool for the later part of the thesis.

**Theorem 2.1.3** ([30], [7]). *Let  $S$  be an operator system contained in the  $C^*$ -algebra  $B$  such that  $C^*(S) = B$  and let  $\phi : B \rightarrow B(\mathcal{H})$  be a ucp map. Then there exists a  $*$ -representation  $\sigma : B \rightarrow B(\mathcal{K})$  with the unique extension property relative to  $S$  and an isometry  $V : \mathcal{H} \rightarrow \mathcal{K}$  such that  $\phi(s) = V^*\sigma(s)V$  for all  $s \in S$ .*

The next result shows that the unique extension property is preserved under direct sum.

**Theorem 2.1.4.** *Let  $S$  be an operator system contained in a  $C^*$ -algebra  $B$  such that  $C^*(S) = B$ . Let  $\pi_i : B \rightarrow \mathcal{H}_i$  be a family of  $*$ -representations and let  $\pi = \bigoplus_i \pi_i$ . Then  $\pi$  has the unique extension property relative to  $S$  if and only if  $\pi_i$  has the unique extension property relative to  $S$  for all  $i$ .*

*Proof.* Suppose  $\pi$  has the unique extension property relative to  $S$ . For all  $i$ , we view  $\mathcal{H}_i$  as a subspace of  $\mathcal{H}$ . Then  $\mathcal{H}_i$  is a reducing subspace of  $\pi$ , and  $\pi_i(b) = \pi(b)|_{\mathcal{H}_i}$ . So applying [21, Lemma 2.8] we have that  $\pi_i$  has the unique extension property relative to  $S$  for all  $i$ .

For the proof of the converse we refer to [8, Proposition 4.4]. □

The next theorem is an wellknown result in the representation theory of  $C^*$ -algebras. This will be useful in the later parts of the thesis. A  $*$ -representation  $\pi : B \rightarrow B(\mathcal{H})$  is said to be **non-degenerate** if for  $h \in \mathcal{H}$ , whenever  $\pi(b)h = 0$  for all  $b \in B$ , then  $h = 0$ . Throughout this thesis, all  $*$ -representations are assumed to be non-degenerate.

**Theorem 2.1.5** ([44, Theorem 5.1.3]). *Let  $B$  be a  $C^*$ -algebra and  $\pi : B \rightarrow B(\mathcal{H})$  be a non-degenerate  $*$ -representation. Then  $\pi$  is a direct sum of cyclic  $*$ -representations.*

The next result says that the unique extension property serves as an invariant for an operator system.

**Theorem 2.1.6** ([5, Theorem 2.1.2]). *Let  $S_1$  and  $S_2$  be two operator systems contained in  $C^*$ -algebras  $B_1$  and  $B_2$  respectively such that  $B_i = C^*(S_i)$  for  $i = 1, 2$ . Let  $\phi : S_1 \rightarrow S_2$  be a complete isometry. Then for every  $*$ -representation  $\sigma : B_1 \rightarrow B(\mathcal{H})$  with the unique extension property, there exists a  $*$ -representation with the unique extension property  $\pi : B_2 \rightarrow B(\mathcal{H})$  such that  $\pi \circ \phi = \sigma$ .*

**Definition 2.1.3.** Let  $S$  be an operator system in a  $C^*$ -algebra  $B$  such that  $C^*(S) = B$ . Then, the operator system  $S$  is said to be **hyperrigid** in  $B$  if every  $*$ -representation of  $B$  has the unique extension property relative to  $S$ .

Theorem 2.1.4 and Theorem 2.1.5 together imply that an operator system  $S$  is hyperrigid in the  $C^*$ -algebra  $B$  precisely when every cyclic representation of  $B$  has the unique extension property relative to  $S$ . The following well known theorem says that one can associate a state to every cyclic  $*$ -representation (see [44, Chapter 3] for details).

**Theorem 2.1.7.** *Let  $B$  be a  $C^*$  algebra and  $\pi : B \rightarrow B(\mathcal{H})$  be a cyclic  $*$ -representation of  $B$  with a cyclic unit vector  $\xi \in \mathcal{H}$ . Then  $(\pi, \mathcal{H}, \xi)$  is unitarily equivalent to the GNS representation of the state  $x \mapsto \langle \pi(x)\xi, \xi \rangle$ .*

This theorem implies that every cyclic  $*$  representation of  $B$  can be identified with the GNS representation of some state of  $B$  (upto unitary equivalence). So a natural question is if we could translate the UEP of a cyclic  $*$ -representation to a nice property of the corresponding state. The answer is provided in Theorem 3.2.1. Moreover, irreducible representations play a crucial role in the context of hyperrigidity conjecture. The following theorem is about irreducible representations and their corresponding state.

**Theorem 2.1.8.** *Let  $B$  be a  $C^*$ -algebra. Let  $\varphi$  be a state on  $B$  and  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  be the GNS representation of  $\varphi$ . Then the following are equivalent:*

1.  $\varphi$  is a pure state.

2.  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  is a irreducible representation.

## 2.2 Non-commutative convexity

### Infinite matrices over an operator space

Let  $\mathcal{H}$  be a Hilbert space and  $m$  and  $n$  be cardinal numbers. Then every bounded linear operator  $T : \mathcal{H}^{(n)} \longrightarrow \mathcal{H}^{(m)}$  can be identified with a  $m \times n$  matrix  $[t_{i,j}]_{i \leq m, j \leq n}$  where  $t_{i,j} \in B(\mathcal{H})$  such that  $[t_{i,j}]_{i \leq m, j \leq n}$  has uniformly bounded finite sub-matrices, i.e.

$$\sup \{ \|[t_{i,j}]_{i \in I, j \in J}\| \} < \infty,$$

where the supremum is taken over all finite subsets  $I$  and  $J$  of  $\{r \leq m\}$  and  $\{r \leq n\}$  respectively. For each finite subsets  $I$  and  $J$  of  $\{r \leq m\}$  and  $\{r \leq n\}$  respectively, we view  $[t_{i,j}]_{i \in I, j \in J}$  as a bounded operator from  $\bigoplus_{j \in J} \mathcal{H}$  to  $\bigoplus_{i \in I} \mathcal{H}$ . Let  $M_{m \times n}(B(\mathcal{H}))$  denote the set of all  $m \times n$  matrices with entries from  $B(\mathcal{H})$  having uniformly bounded sub-matrices. Then,  $M_{m \times n}(B(\mathcal{H}))$  is a normed vector space with the norm defined as

$$\|[t_{i,j}]_{i \leq m, j \leq n}\| = \sup \{ \|[t_{i,j}]_{i \in I, j \in J}\| \}, \quad (2.2.1)$$

where the supremum is taken over all finite subsets  $I$  and  $J$  of  $\{r \leq m\}$  and  $\{r \leq n\}$  respectively. Then  $M_{m,n}(B(\mathcal{H}))$  is isomorphic to  $B(\mathcal{H}^{(n)}, \mathcal{H}^{(m)})$  via the natural identification. For a detailed discussion on infinite matrices with entries from  $B(\mathcal{H})$ , we refer to [13, Section 1.2.25]. Given any cardinal number  $m$ , fix a Hilbert space  $\mathcal{H}_m$  of dimension  $m$ . We then denote  $M_{m,n} = B(\mathcal{H}_n, \mathcal{H}_m)$ .

Let  $\alpha = [\alpha_{i,j}]_{i \leq m, j \leq n} \in M_{m,n}$ . Then the matrix  $[\alpha_{i,j}I]_{i \leq m, j \leq n}$  is in  $M_{m,n}(B(\mathcal{H}))$ , where  $I$  is the identity operator on  $\mathcal{H}$ . If moreover  $x = [x_{i,j}]_{i,j \leq n} \in M_n(B(\mathcal{H}))$  then we define  $\alpha x$  to be the composition of  $[\alpha_{i,j}I]_{i \leq m, j \leq n}$  and  $[x_{i,j}]_{i,j \leq n}$  as Hilbert space operators. So  $\alpha x \in M_{m,n}(B(\mathcal{H}))$ . For  $\beta \in M_{n,p}$  we define  $x\beta \in M_{n,p}(B(\mathcal{H}))$

similarly.

For a cardinal number  $m$ , let  $\{x_i : i \leq m\}$  be a bounded family of matrices over  $B(\mathcal{H})$  with  $x_i \in M_{n_i}(B(\mathcal{H}))$ . Then we define  $\bigoplus_i x_i$  to be the direct sum of  $x_i$ s as Hilbert space operators. Thus  $\bigoplus_i x_i \in M_n(B(\mathcal{H}))$  where  $n = \sum_{i \leq m} n_i$ .

Let  $x = [x_{i,j}]_{i,j \leq n} \in M_n(B(\mathcal{H}))$ , then  $x^*$  is defined as

$$x^* = [x_{j,i}^*]_{i,j \leq n}.$$

## Non-commutative convex set

An **operator space** is a vector space  $E \subset B(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ . Let  $E$  be a fixed operator space in  $B(\mathcal{H})$ . Then, for all cardinal numbers  $n$  we set,

$$M_n(E) = \{[t_{i,j}]_{i,j \leq n} \in M_n(B(\mathcal{H})) : t_{i,j} \in E \ \forall i, j \leq n\}.$$

Then  $M_n(E)$  is again an operator space in  $B(\mathcal{H}^n)$ .

Let  $E$  be a dual operator space with the predual  $E_*$ . Then for each cardinal number  $n$ , there is a natural operator space isomorphism

$$M_n(E) \cong CB(E_*, M_n),$$

where  $CB(E_*, M_n)$  denotes the space of all completely bounded maps from  $E_*$  to  $M_n$ . With this identification, we equip  $M_n(E)$  with the point weak-\* topology, i.e., a net  $(\Phi_\lambda)$  in  $M_n(E)$  converges to  $\Phi$  if  $\Phi_\lambda(x)$  converges to  $\Phi(x)$  in the weak-\* topology of  $M_n$ .

We let  $\mathbb{M}(E) = \coprod_{n \leq \kappa} M_n(E)$ , where the disjoint union is taken over all cardinal numbers  $n \leq \kappa$ , where  $\kappa$  is a sufficiently large infinite cardinal number. We fix the notation  $\mathbb{M} = \mathbb{M}(\mathbb{C})$ . One needs to clearly define the upper bound  $\kappa$  to claim that  $\mathbb{M}(E)$  is indeed a set. We set  $\kappa$  to be the dimension of the minimal infinite

dimensional Hilbert space  $\mathcal{H}$  such that  $E$  can be embedded completely isometrically on  $B(\mathcal{H})$ . For instance, if  $E$  is separable, then we take  $\kappa = \aleph_0$ .

**Definition 2.2.1.** A **non-commutative convex set** or an **nc convex set** over an operator space  $E$  is a set  $K = \coprod_{n \leq \kappa} K_n$  such that  $K_n \subset M_n(E)$  which is closed under direct sum and isometric compression, i.e.,

- (i)  $\bigoplus_i x_i \in K$  for every bounded family  $\{x_i \in K_{n_i}\}$ ,
- (ii)  $\alpha^* x \alpha \in K_n$  for all  $x \in K_m$  and for all isometries  $\alpha \in M_{m,n}$ .

Let  $K$  be an nc convex set over a dual operator space  $E$ . Then  $K$  is said to be **closed** if  $K_n$  is closed for all  $n$  in the point weak-\* topology of  $M_n(E)$ . Similarly,  $K$  is said to be **compact** if  $K_n$  is compact in the point weak-\* topology for all  $n$ .

**Example 1.** Let  $S$  be an operator system, and  $S^*$  denotes its dual space. The dual of an operator space is again an operator space [14], [31]. Moreover, there is a natural operator space isomorphism

$$M_n(S^*) \cong CB(S, M_n)$$

where  $CB(S, M_n)$  is the set of all completely bounded linear maps from  $S$  to  $M_n$ . We endow  $M_n(S^*)$  with the point weak-\* topology. i.e., a net  $\{\Phi_\lambda\}$  in  $M_n(S^*)$  converges to  $\Phi$  if and only if for all  $s \in S$ ,  $\Phi_\lambda(s)$  converges to  $\Phi(s)$  in the weak-\* topology of  $M_n$ .

We set  $K_n = UCP(S, M_n)$  where  $UCP(S, M_n)$  is the set of all unital completely positive maps from  $S$  to  $M_n$ . Then, with the above identification,  $K_n$  is a compact subset of  $M_n(S^*)$ . Moreover, the direct sum and compression of ucp maps are again ucp. Hence,  $K = \coprod_{n \leq \kappa} K_n$  is a compact non-commutative convex set over  $S^*$ . We often refer to  $K$  as the **non-commutative state space** or the **nc state space** of  $S$ .

For additional details on non-commutative convex sets we refer to [27], [39].

## 2.2.1 Non-commutative functions

**Definition 2.2.2.** Let  $K = \coprod_{n \leq \kappa} K_n$  be a compact nc convex set over an operator space  $E$ , and let  $F : K \rightarrow \mathbb{M}(B(\mathcal{H}))$  be a function. We say that  $F$  is a **non-commutative function** or an **nc function** if  $F$  is graded, respects direct sums and is unitarily equivariant, meaning that,

- (i)  $F(K_n) \subset M_n(B(\mathcal{H}))$  for all  $n \leq \kappa$ ,
- (ii)  $F(\bigoplus_i x_i) = \bigoplus_i F(x_i)$  for every bounded family  $\{x_i \in K_{n_i}\}$ ,
- (iii)  $F(u^*xu) = u^*F(x)u$  for all  $x \in K_n$  and for all unitaries  $u \in M_n$ .

Let  $F : K \rightarrow \mathbb{M}(B(\mathcal{H}))$  be an nc function. We set  $F_n = F|_{K_n}$ . Moreover, we define,

$$\|F\|_\infty = \sup_{x \in K} \|F(x)\|.$$

We say that  $F$  is a **bounded nc function** if  $\|F\|_\infty < \infty$ . Let  $B_{\text{nc}}(K)$  denote the set of all bounded nc functions from  $K$  to  $\mathbb{M}$ .

A function  $F$  is said to be an **affine nc function** if  $F$  is an nc function and in addition,  $F$  respects the isometric compression, meaning that

- (iv)  $F(\alpha^*x\alpha) = \alpha^*F(x)\alpha$  for all  $x \in K_n$  and for all isometries  $\alpha \in M_{n,m}$ .

Let  $E$  be a dual operator space, and let  $K$  be an nc convex set over  $E$ . An nc function  $F : K \rightarrow \mathbb{M}$  is said to be **continuous** if each  $F_n : K_n \rightarrow M_n$  is continuous where  $M_n$  is endowed with the  $\sigma$ -strong- $*$  topology. Let  $A_{\text{nc}}(K)$  denote the set of all continuous affine nc functions from  $K$  to  $\mathbb{M}$ . By virtue of [27, Proposition 2.5.3], we have,  $A_{\text{nc}}(K) \subset B_{\text{nc}}(K)$ .

For a nc function  $F : K \rightarrow \mathbb{M}(B(\mathcal{H}))$ , we define the adjoint of  $F$  as  $F^*(x) = F(x)^*$ . Then  $B_{\text{nc}}(K)$  is a  $C^*$ -algebra when endowed with the involution defined

above, pointwise multiplication and the uniform norm  $\|\cdot\|_\infty$ . The function  $I : K \rightarrow \mathbb{M}$  defined by  $I(x) = I_n$  for all  $n$  and  $x \in K_n$  serves as the identity of  $B_{\text{nc}}(K)$ . Clearly  $I \in A_{\text{nc}}(K)$  and  $a^* \in A_{\text{nc}}(K)$  whenever  $a \in A_{\text{nc}}(K)$ . So  $A_{\text{nc}}(K)$  is an operator system contained in the  $C^*$ -algebra  $B_{\text{nc}}(K)$ . Let  $\mathcal{A}$  be the  $C^*$ -algebra generated by  $A_{\text{nc}}(K)$  in  $B_{\text{nc}}(K)$  i.e.,  $\mathcal{A} = C^*(A_{\text{nc}}(K)) \subset B_{\text{nc}}(K)$ . Then by [27, Theorem 4.4.3], the elements of  $\mathcal{A}$  are precisely the continuous nc functions from  $K$  to  $\mathbb{M}$ . We give a detailed account of  $\mathcal{A}$  in Section 2.3.

Let  $m$  be a cardinal number and  $F = [F^{(i,j)}]_{i,j \leq m} \in M_m(\mathcal{A})$ , where each  $F^{(i,j)}$  is a continuous nc function from  $K$  to  $\mathbb{M}$ . The following technical discussion will enable us to view  $F$  as a continuous nc function from  $K$  to  $\mathbb{M}(M_m)$ .

For cardinal numbers  $m, n$ , let  $A \in M_m(M_n)$ . Then we write  $A = [A^{i,j}]_{i,j \leq m}$  where  $A^{i,j} = [A_{k,l}^{i,j}]_{k,l \leq n} \in M_n$  for all  $i, j \leq m$ . Fix a cardinal number  $m$ . For any cardinal number  $n$ , define  $i_n : M_m(M_n) \rightarrow M_n(M_m)$  by:

$$i_n(A)_{i,j}^{k,l} = A_{k,l}^{i,j} \quad \forall k, l \leq n, \forall i, j \leq m.$$

**Lemma 2.2.1.** *Let  $A = [A^{i,j}]_{i,j \leq m} \in M_m(M_n)$  with  $A^{i,j} \in M_n$  and let  $\alpha \in M_n$ . Then,*

$$i_n([\alpha A^{i,j}]_{i,j \leq m}) = \alpha i_n(A). \quad (2.2.2)$$

**Remark 2.2.1.** Observe that in 2.2.2, each  $A^{i,j} \in M_n$  and  $\alpha \in M_n$ . So  $\alpha A^{i,j}$  is again in  $M_n$ . On the other hand,  $i_n(A) \in M_n(M_m) = M_n(B(\mathcal{H}_m))$ , so  $\alpha i_n(A)$  is the operator composition of  $[\alpha_{l,k} I]_{l,k \leq n}$  with  $A$ .

Following the same proof, one can get that if  $\beta \in M_n$ , then  $i_n([A^{i,j} \beta]_{i,j \leq m}) = i_n(A)\beta$ .

*Proof of Lemma 2.2.1.* We let  $[\alpha A^{i,j}]_{i,j \leq m} = X$  with  $X^{i,j} = \alpha A^{i,j}$ . Then

$$X_{k,l}^{i,j} = \sum_{p \leq n} \alpha_{k,p} X_{p,l}^{i,j}.$$

Next we see that  $(\alpha i_n(A))^{k,l} = \sum_{p \leq n} \alpha_{k,p} I i_n(A)^{p,l} = \sum_{p \leq n} \alpha_{k,p} i_n(A)^{p,l} \in M_m$ . Hence,

$$\begin{aligned}
(\alpha i_n(A))_{i,j}^{k,l} &= \left( \sum_{p \leq n} \alpha_{k,p} i_n(A)^{p,l} \right)_{i,j} \\
&= \sum_{p \leq n} \alpha_{k,p} i_n(A)_{i,j}^{p,l} \\
&= \sum_{p \leq n} \alpha_{k,p} A_{p,l}^{i,j} \\
&= X_{k,l}^{i,j}
\end{aligned}$$

Hence  $\alpha i_n(A) = X = [\alpha A_{i,j}]_{i,j \leq m}$ . □

Let  $F = [F^{(i,j)}]_{i,j \leq m} \in M_m(\mathcal{A})$ , where  $F^{(i,j)} \in \mathcal{A}$ . Hence  $F_n^{(i,j)} : K_n \rightarrow M_n$  is a continuous function for all  $n \leq \kappa$ . Therefore  $F_n : K_n \rightarrow M_m(M_n)$  is a continuous function for all  $n \leq \kappa$ . With the identification  $i_n : M_m(M_n) \rightarrow M_n(M_m)$  we see that  $i_n \circ F_n : K_n \rightarrow M_n(M_m)$  for all  $n \leq \kappa$ . Set  $G : K \rightarrow \mathbb{M}(M_m)$  defined as  $G_n = i_n \circ F_n$  for all  $n \leq \kappa$ . Then we claim that  $G$  is an nc function from  $K$  to  $\mathbb{M}(M_m)$ . To see this, let  $x \in K_n$  for some  $n \leq \kappa$  and let  $u \in M_n$  be an unitary. Then we observe that,

$$\begin{aligned}
G(u^* x u) &= G_n(u^* x u) \\
&= i_n F_n(u^* x u) \\
&= i_n \circ \left( F_n^{(i,j)}(u^* x u) \right) \\
&= i_n \circ \left( u^* F_n^{(i,j)}(x) u \right) && \text{[ since each } F^{(i,j)} \text{ is an nc function.]} \\
&= u^* i_n \circ F_n(x) u && \text{[ from Lemma 2.2.1]} \\
&= u^* G(x) u.
\end{aligned}$$

Hence,  $G$  is unitarily equivariant. Let  $n_1$  and  $n_2$  are two cardinal numbers and

$x \in K_{n_1}$  and  $y \in K_{n_2}$ . Let  $n = n_1 + n_2$ . Then,

$$\begin{aligned}
G(x \oplus y)_{i,j}^{k,l} &= G_n(x \oplus y)_{i,j}^{k,l} \\
&= i_n \circ F_n(x \oplus y)_{i,j}^{k,l} \\
&= F_n(x \oplus y)_{k,l}^{i,j} \\
&= \begin{cases} \left(F^{(i,j)}(x)\right)_{k,l} & \text{if } k, l \leq n_1, \\ \left(F^{(i,j)}(y)\right)_{a,b} & \text{if } k = n_1 + a, l = n_2 + b \text{ for some } a, b \leq n_2. \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Similarly  $G(x) \oplus G(y) \in M_n(M_m)$  and,

$$\begin{aligned}
(G(x) \oplus G(y))_{i,j}^{k,l} &= \begin{cases} G(x)_{i,j}^{k,l} & \text{if } k, l \leq n_1 \\ G(y)_{i,j}^{a,b} & \text{if } k = n_1 + a, l = n_2 + b \text{ for some } a, b \leq n_2 \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} F(x)_{k,l}^{i,j} & \text{if } k, l \leq n_1 \\ F(y)_{a,b}^{i,j} & \text{if } k = n_1 + a, l = n_2 + b \text{ for some } a, b \leq n_2 \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} \left(F^{(i,j)}(x)\right)_{k,l} & \text{if } k, l \leq n_1, \\ \left(F^{(i,j)}(y)\right)_{a,b} & \text{if } k = n_1 + a, l = n_2 + b \text{ for some } a, b \leq n_2. \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

This proves that  $G(x \oplus y) = G(x) \oplus G(y)$ . The same proof can be adapted to show that  $G(\bigoplus_i x_i) = \bigoplus_i G(x_i)$  where  $\{x_i : i \in K\}$  is bounded family of elements in  $K$ . Hence,  $G : K \rightarrow \mathbb{M}(M_m)$  is a continuous nc function.

By applying the identification  $i_n : M_m(M_n) \rightarrow M_n(M_m)$ , we have constructed

an nc function  $G$  corresponding to every element  $F = [F^{(i,j)}]_{i,j \leq m}$ . Identifying  $F$  with  $G$ , we consider  $F \in M_m(\mathcal{A})$  as an nc function from  $K$  to  $\mathbb{M}(M_m)$ .

**Definition 2.2.3.** Let  $K$  be a compact nc convex set. An nc function  $F : K \rightarrow \mathbb{M}(M_m)$  is said to be **nc convex** if for all cardinals  $n$ , for all  $x, y \in K_n$  and for all  $t \in [0, 1]$ ,  $F(x)^* = F(x)$  and

$$F(tx + (1 - t)y) \leq tF(x) + (1 - t)F(y) \quad \text{in } M_n(M_m).$$

For a more detailed discussion on convex functions, we refer to [27].

## 2.3 C\*-cover of an operator system

Operator systems can be defined abstractly with no mention of an ambient C\*-algebra or concrete representation on Hilbert space by means of the Choi–Effros theorem [45, Theorem 13.1].

**Definition 2.3.1.** Let  $S$  be an operator system. A C\*-cover of  $S$  is a pair  $(B, \theta)$  consisting of a unital C\*-algebra  $B$  and a unital completely isometric map  $\theta : S \rightarrow B$  such that  $B = C^*(\theta(S))$ .

Note that C\*-covers of an operator system are not unique.

**Definition 2.3.2.** Let  $S$  be an operator system. A C\*-cover  $(A, j)$  is said to be the **maximal C\*-cover** of  $S$  if, given any other C\*-cover  $(B, \theta)$  of  $S$ , there exists a surjective \*-homomorphism  $\pi : A \rightarrow B$  such that  $\pi \circ j = \theta$ .

The maximal C\* cover of an operator system always exists [41] and it is unique up to a \*-isomorphism. Let  $(C_{\max}^*(S), j)$  denote the maximal C\*-cover of  $S$ .

**Lemma 2.3.1.** *Let  $S$  be an operator system and  $(C_{\max}^*(S), j)$  be the maximal C\*-cover of  $S$ . Let  $\phi : S \rightarrow B(H)$  be a unital completely contractive map. Then, there is a unique unital \*-representation  $\hat{\phi} : C_{\max}^*(S) \rightarrow B(H)$  such that  $\hat{\phi} \circ j = \phi$ .*

*Proof.* Consider the map  $j \oplus \phi : S \rightarrow C_{\max}^*(S) \oplus B(H)$ , which is clearly unital and completely isometric. Define also the unital  $*$ -homomorphism  $\pi_0 : C_{\max}^*(S) \oplus B(H) \rightarrow B(H)$  as the projection onto the second component. Now, there is a unital  $*$ -homomorphism  $\sigma : C_{\max}^*(S) \rightarrow C^*((j \oplus \phi)(S))$  such that  $\sigma \circ j = j \oplus \phi$ . It thus suffices to put  $\widehat{\phi} = \pi_0 \circ \sigma$ .

Since  $C^*(j(S)) = C_{\max}^*(S)$ ,  $\widehat{\phi}$  is the unique  $*$ -representation satisfying  $\widehat{\phi} \circ j = \phi$ . □

The next result will help us in constructing a  $C^*$ -cover for an operator system.

**Theorem 2.3.1** ([27, Theorem 3.2.3]). *Let  $S$  be an operator system and  $K$  be the nc state space of  $S$ . Let  $\theta : S \rightarrow B_{nc}(K)$  be the map defined as,*

$$\theta(s)(\phi) = \phi(s) \quad \forall s \in S, \forall \phi \in K.$$

*Then  $\theta(s) \in A_{nc}(K)$  for all  $s \in S$  and moreover,  $S$  is complete isometrically isomorphic to  $A_{nc}(K)$  via  $\theta$ .*

We have already mentioned that  $\mathcal{A} = C^*(A_{nc}(K))$ . Hence  $(\mathcal{A}, \theta)$  is a  $C^*$ -cover of  $S$ . Moreover, [27, Theorem 4.4.3] shows that  $(\mathcal{A}, \theta)$  has the universal property mentioned in Definition 2.3.2.

If  $\phi \in K$ , then  $\phi$  is a ucp map from  $S$  to  $B(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ . In particular,  $\phi$  is a unital complete contraction. Hence by Lemma 2.3.1, there exists a unique  $*$ -homomorphism  $\widehat{\phi} : C_{\max}^*(S) \rightarrow B(\mathcal{H})$  such that  $\widehat{\phi} \circ j = \phi$ .

**Lemma 2.3.2.** *Let  $\Phi : C_{\max}^*(S) \rightarrow \mathcal{A}$  defined as  $\Phi(b)(\phi) = \widehat{\phi}(b)$  for all  $b \in C_{\max}^*(S)$  and for all  $\phi \in K$ . Then  $\Phi$  is a  $*$ -isomorphism such that  $\Phi \circ j = \theta$ .*

*Proof.* From the universal property of  $(C_{\max}^*(S), j)$  mentioned in Definition 2.3.2, there exists a unique surjective  $*$ -homomorphism  $\pi : C_{\max}^*(S) \rightarrow \mathcal{A}$  such that  $\pi \circ j = \theta$ . Since  $(\mathcal{A}, \theta)$  also possesses the same universal property, there exists a

unique surjective  $*$ -homomorphism  $\sigma : \mathcal{A} \longrightarrow C_{\max}^*(S)$  such that  $\sigma \circ \theta = j$ . Observe  $\sigma \circ \pi : C_{\max}^*(S) \longrightarrow C_{\max}^*(S)$  is a  $*$ -homomorphism such that  $\sigma \circ \pi|_{j(S)}$  is the identity map. Since  $C^*(j(S)) = C_{\max}^*(S)$ , we have  $\sigma \circ \pi = \text{id}_{C_{\max}^*(S)}$ . Similarly we can show that  $\pi \circ \sigma = \text{id}_{\mathcal{A}}$ . Hence  $\pi$  is an  $*$ -isomorphism. Now, it is enough to show that  $\pi = \Phi$ . It is straightforward to see that  $\Phi : C_{\max}^*(S) \longrightarrow \mathcal{A}$  is a  $*$ -homomorphism. Moreover, for all  $s \in S$ ,

$$\Phi(j(s))(\phi) = \phi(s) = \theta(s)(\phi) = \pi(j(s))(\phi).$$

Hence  $\pi|_{j(S)} = \Phi|_{j(S)}$  and since  $C_{\max}^*(j(S)) = C_{\max}^*(S)$ , we conclude that  $\Phi = \pi$ .  $\square$

## 2.4 Bidual of a $C^*$ -algebra and absolute continuity of positive linear functionals

In this part, we will discuss some preliminaries regarding von Neumann algebras which will be handy in discussing our results later. It is a fact that every von Neumann algebra attains a unique predual and hence von Neumann algebras can be endowed with the weak- $*$  topology.

Let  $B$  be a  $C^*$ -algebra and  $B^{**}$  be the bidual of  $B$  i.e.,  $B^{**} = (B^*)^*$ . Then  $B^{**}$  is a von Neumann algebra and there is a canonical embedding of  $B$  in the von Neumann algebra  $B^{**}$ . Without loss of generality, we will assume that  $B \subset B^{**}$ . Moreover,  $B$  is weak- $*$  dense in  $B^{**}$ . For more elaborate discussion about this topic we refer to [54, Section III.2] and [51, Section 1.17]. For every bounded functional  $\varphi$  on  $B$ , there exists a unique weak- $*$  continuous functional  $\hat{\varphi}$  on  $B^{**}$  that extends  $\varphi$ . Moreover, if  $\varphi$  is positive, then  $\hat{\varphi}$  is also positive.

Let  $M$  be a von Neumann algebra, then a weak- $*$  continuous state  $\varphi$  on  $M$  is said to be a **normal state** on  $M$ . There are several equivalent characterizations of

normal state. However, the following is the one we are interested in [15, Theorem 2.4.11].

**Theorem 2.4.1.** *Let  $M \subset B(\mathcal{H})$  be a von Neumann algebra. Then the following are equivalent:*

1.  $\varphi$  is a normal state on  $M$ .
2. There exists a positive trace class operator  $T \in B(\mathcal{H})$  such that  $\varphi(x) = \text{tr}(xT)$  for all  $x \in M$ .

Let  $M$  be a von Neumann algebra. If  $\mathcal{I} \subset M$  is a weak-\* closed left ideal of  $M$ , then there exists a projection  $p \in M$  such that  $\mathcal{I} = B^{**}(1 - p)$  [51, Proposition 1.10.1]. Similarly for a right weak-\* closed ideal  $\mathcal{J}$  of  $M$ , there exists a projection  $q \in M$  such that  $\mathcal{J} = M(1 - q)$ . Moreover, if  $\mathcal{I}$  is both left and right ideal which is also weak-\* closed, then there exists a projection  $p \in M$  that commutes with every element of  $M$ , such that  $\mathcal{I} = M(1 - p)$ .

For a positive linear functional  $\varphi$ , let  $L_\varphi = \{x \in B^{**} : \hat{\varphi}(x^*x) = 0\}$ . Then  $L_\varphi$  is a closed left ideal in  $B^{**}$  [53, Lemma 9.6]. Thus, there exists a projection  $\mathfrak{s}_\varphi \in B^{**}$  such that

$$L_\varphi = \{x \in B^{**} : \hat{\varphi}(x^*x) = 0\} = B^{**}(1 - \mathfrak{s}_\varphi).$$

**Definition 2.4.1** ([24]). Let  $\varphi$  and  $\psi$  be positive linear functionals on the  $C^*$ -algebra  $B$ . Then we say that  $\varphi$  is **absolutely continuous** with respect to  $\psi$  and write  $\varphi \ll \psi$  if, given  $x \in B^{**}$  the following holds:

$$\hat{\psi}(x^*x) = 0 \text{ implies } \hat{\varphi}(x^*x) = 0.$$

i.e.  $\varphi \ll \psi$  if and only if  $L_\psi \subset L_\varphi$ .

**Remark 2.4.1.** Let  $Y$  be a compact Hausdorff space. Then by the Riesz representation theorem, the state space of  $C(Y)$  can be identified with  $P(Y)$ , where  $P(Y)$  denotes the set of regular Borel probability measures on  $Y$ . For  $\mu \in P(Y)$ , we view  $\mu$  as a state on  $C(Y)$  in the following manner

$$\mu(f) = \int_Y f d\mu, \quad \forall f \in C(Y).$$

If  $\mu, \nu \in P(Y)$ , then  $\mu$  is absolutely continuous with respect to  $\nu$  in the measure-theoretic sense if and only if  $\mu \ll \nu$  as linear functionals in the  $C^*$ -algebra  $C(Y)$  as defined above [24].

Let  $\Delta$  be a norm closed convex subset of the state space of a  $C^*$ -algebra  $B$ , and  $AC(\Delta)$  be the set of all bounded linear functionals on  $B$  which are absolutely continuous with respect to some state in  $\Delta$ . Then the functionals in  $AC(\Delta)$  are completely characterized by a projection in the bidual  $B^{**}$  in the following sense.

**Theorem 2.4.2** ([24]). *Let  $\Delta$  be a norm closed, convex subset of the state space of a  $C^*$  algebra  $B$  and  $\varphi \in \mathcal{E}(B)$ . Then there exists a projection  $\mathfrak{d} \in B^{**}$  such that  $\varphi \in AC(\Delta)$  if and only if  $\hat{\varphi}(b) = \hat{\varphi}(\mathfrak{d}b)$  for all  $b \in B$ .*

**Remark 2.4.2.** Let  $\Delta$  be a norm closed, convex subset of the state space of a  $C^*$ -algebra  $B$ . Then as discussed above, for every  $\varphi \in \Delta$ , there exists a projection  $\mathfrak{s}_\varphi \in B^{**}$  such that  $L_\varphi = B^{**}(1 - \mathfrak{s}_\varphi)$ . Then, with the notation continuing from the previous theorem,

$$\mathfrak{d} = \bigvee_{\varphi \in \Delta} \mathfrak{s}_\varphi.$$

This means the characterizing projection  $\mathfrak{d}$  is obtained by taking the supremum of support projections of all elements of  $\Delta$ . We refer to [24] for further details.

## 2.5 Non-commutative topology

Let  $Y$  be a compact Hausdorff topological space, and let  $C(Y)$  be the  $C^*$ -algebra of all continuous functions on  $Y$ . Let  $Z$  be a subset of  $Y$ , and let  $\chi_Z$  be the characteristic function of  $Z$ . Then the set  $Z$  is open precisely when there exists an increasing net of functions  $f_\lambda$  in  $C(Y)$  such that,

- (i)  $f_\lambda \geq 0$  for all  $\lambda$ ,
- (ii)  $f_\lambda$  increases to  $\chi_Z$  pointwise on  $Y$ .

This can be seen from Urysohn's Lemma.

Drawing motivation from this, next we define non-commutative topology.

**Definition 2.5.1** ([1],[2]). Let  $B$  be a unital  $C^*$ -algebra. A projection  $p \in B^{**}$  is said to be **open** if there exists an increasing net of positive elements  $\{a_\lambda\}$  in  $B$  such that  $a_\lambda$  converges to  $p$  in the weak\* topology of  $B^{**}$ . A projection  $p$  is said to be **closed** if  $1 - p$  is open.

# 3

## A non commutative boundary for the dilation order

### 3.1 Boundaries and hyperrigidity for pre-orders

Let  $B$  be a unital  $C^*$ -algebra, with  $\mathcal{E}(B)$  representing its state space and  $\mathcal{E}_p(B)$  representing the pure states. In this section, we aim to explore the structure of the maximal elements in certain pre-orders defined on  $\mathcal{E}(B)$ . In the following section, we will apply our findings to a specific partial order, but for now, we will proceed with a more general approach.

**Lemma 3.1.1.** *Let  $B$  be a separable unital  $C^*$ -algebra, and let  $\Delta \subset \mathcal{E}(B) \times \mathcal{E}(B)$  be a weak-\* closed pre-order. Let  $\{a_n : n \geq 1\}$  be a countable dense subset of the self-adjoint part of  $B$ . For each pair of integers  $m, n \geq 1$ , let  $K_{n,m} \subset \mathcal{E}(B)$  consist of those states  $\varphi$  for which there is another state  $\psi$  such that  $(\varphi, \psi) \in \Delta$  and  $\psi(a_n) - \varphi(a_n) \geq 1/m$ . Then, each  $K_{n,m}$  is weak-\* closed, and  $\bigcup_{n,m=1}^{\infty} K_{n,m}$  is the set of states on  $B$  that are not  $\Delta$ -maximal. In particular, the set of  $\Delta$ -maximal states is Borel measurable.*

*Proof.* Let  $\varphi \in K_{n,m}$ . Then there exists a state  $\psi$  such that  $(\varphi, \psi) \in \Delta$  and  $\psi(a_n) -$

$\varphi(a_n) \geq 1/m$ . This implies that  $(\varphi, \psi) \in \Delta$  and  $\psi \neq \varphi$ . Hence  $\varphi$  is not maximal. So,  $\bigcup_{n,m=1}^{\infty} K_{n,m}$  is contained in the set of states on  $B$  that are not  $\Delta$ -maximal. Conversely, if  $\varphi$  is a state on  $B$  which is not  $\Delta$ -maximal, then there is another state  $\psi \neq \varphi$  such that  $(\varphi, \psi) \in \Delta$ . This implies that there must be a self-adjoint element  $b \in B$  such that  $\varphi(b) \neq \psi(b)$ . Upon replacing  $b$  with  $-b$  if necessary, we may assume that  $\psi(b) - \varphi(b) > 0$ . The density of the set  $\{a_n\}$  then easily implies that  $\varphi \in \bigcup_{n,m=1}^{\infty} K_{n,m}$ . We thus conclude that  $\bigcup_{n,m=1}^{\infty} K_{n,m}$  is the set of states on  $B$  that are not  $\Delta$ -maximal.

Fix integers  $m, n \geq 1$ , and let  $(\varphi_i)$  be a net in  $K_{n,m}$  converging to some state  $\varphi \in \mathcal{E}(B)$  in the weak-\* topology. By definition, this means that there is another net of states  $(\psi_i)$  such that  $(\varphi_i, \psi_i) \in \Delta$  and  $\psi_i(a_n) - \varphi_i(a_n) \geq 1/m$ . Upon passing to a cofinal subnet, we may assume that  $(\psi_i)$  also converges to some state  $\psi \in \mathcal{E}(B)$  in the weak-\* topology. Clearly, we then have  $\psi(a_n) - \varphi(a_n) \geq 1/m$ , while  $(\varphi, \psi) \in \Delta$  since  $\Delta$  is assumed to be weak-\* closed. This shows that  $\varphi \in K_{n,m}$ , so indeed  $K_{n,m}$  is closed in the weak-\* topology.

Finally, the previous paragraph implies that the  $\Delta$ -maximal states form a  $G_\delta$ -set, and hence a Borel measurable set.  $\square$

Recall that given a state  $\varphi$  on  $B$ , we let  $\mathcal{R}_\varphi$  denote the set of Borel probability measures  $\mu$  on  $\mathcal{E}(B)$  concentrated on  $\mathcal{E}_\varphi(B)$  and satisfying

$$\varphi(b) = \int \omega(b) d\mu(\omega), \quad b \in B.$$

When  $B$  is separable, such measures always exist [10, Theorem 4.2]. The following is inspired by the proof of [10, Corollary 3.3], and it generalizes the separable version of [27, Proposition 9.2.5] to a large class of pre-orders.

**Theorem 3.1.1.** *Let  $B$  be a separable unital  $C^*$ -algebra. Let  $\Delta \subset \mathcal{E}(B) \times \mathcal{E}(B)$  be a weak-\* closed, convex pre-order. Let  $\varphi$  be a  $\Delta$ -maximal state on  $B$  and let  $\mu$  be a*

measure in  $\mathcal{R}_\varphi$ . Then,  $\mu$  is concentrated on the pure  $\Delta$ -maximal states.

*Proof.* First note that because  $B$  is separable, the set  $\mathcal{E}_p(B)$  is Borel measurable [10, Corollary 3.3 and Lemma 4.1]. Let  $N \subset \mathcal{E}_p(B)$  denote the set of pure states that are not  $\Delta$ -maximal. Using the notation from the previous result we have,  $N = \mathcal{E}_p(B) \cap \left(\bigcup_{n,m=1}^{\infty} K_{n,m}\right)$ . Hence,  $N$  is Borel measurable by Lemma 3.1.1. Our goal is to show that  $\mu(N) = 0$ .

Assume for the sake of contradiction that  $\mu(N) > 0$ . By Lemma 3.1.1, there is a self-adjoint element  $a \in B$  and  $\varepsilon > 0$  such that  $\mu(K) > 0$ , where  $K$  is the set of pure states  $\omega$  on  $B$  for which there is a state  $\psi$  with  $(\omega, \psi) \in \Delta$  and  $\omega(a) - \psi(a) \geq \varepsilon$ . Define a non-zero positive linear functional  $\varphi'$  on  $B$  as

$$\varphi'(b) = \int_K \omega(b) d\mu(\omega), \quad b \in B.$$

Then  $\frac{1}{\mu(K)}\varphi'$  is a state on  $B$ . Let  $\text{co}(K)$  denote the convex hull of  $K$  and  $\overline{\text{co}}(K)$  denote the closure of  $\text{co}(K)$  in the weak-\* topology. We claim that  $\frac{1}{\mu(K)}\varphi' \in \overline{\text{co}}(K)$ . If this is not true, then by convex separation theorem, there exists a self adjoint element  $x \in B$  such that,

$$\gamma(x) \leq 0 < \frac{1}{\mu(K)}\varphi'(x) \quad \forall \gamma \in \overline{\text{co}}(K). \quad (3.1.1)$$

In particular,  $\omega(x) \leq 0$  for all  $\omega \in K$ . Thus,

$$\frac{1}{\mu(K)}\varphi'(x) = \frac{1}{\mu(K)} \int_K \omega(x) d\mu(\omega) \leq 0,$$

which is a contradiction to 3.1.1. Hence  $\frac{1}{\mu(K)}\varphi' \in \overline{\text{co}}(K)$ . This guarantees that there exists a net  $(\beta_i)$  in  $\text{co}(K)$  that converges to  $\frac{1}{\mu(K)}\varphi'$  in the weak-\* topology. For a fixed  $i$ , let

$$\beta_i = \sum_{j=1}^m \lambda_j \omega_j,$$

where  $\omega_j \in K$  and  $\lambda_j \in [0, 1]$  with  $\sum_{j=1}^m \lambda_j = 1$ . By the definition of  $K$ , for each  $\omega_j$ , there exists a state  $\rho_j$  such that  $(\omega_j, \rho_j) \in \Delta$  and  $\omega_j(a) - \rho_j(a) \geq \varepsilon$ . Let  $\psi_i = \sum_{j=1}^m \lambda_j \rho_j$ . Then  $\psi_i \in \mathcal{E}(B)$  and

$$(\beta_i, \psi_i) = \sum_{j=1}^m \lambda_j (\omega_j, \rho_j).$$

Hence by convexity of  $\Delta$ , we have  $(\beta_i, \psi_i) \in \Delta$ . Moreover,

$$\beta_i(a) - \psi_i(a) = \sum_{j=1}^m \lambda_j (\omega_j(a) - \rho_j(a)) \geq \varepsilon.$$

This shows that, for each  $\beta_i$ , there exists a state  $\psi_i$  such that  $(\beta_i, \psi_i) \in \Delta$  and  $\beta_i(a) - \psi_i(a) \geq \varepsilon$ .

Let  $\psi \in \mathcal{E}(B)$  be a weak-\* cluster point of  $(\psi_i)$ . Using that  $\Delta$  is weak-\* closed, taking the weak-\* limit of a cofinal subnet of  $(\beta_i, \psi_i)$ , we find  $(\frac{1}{\mu(K)}\varphi', \psi) \in \Delta$  and

$$\psi(a) - \frac{1}{\mu(K)}\varphi'(a) \geq \varepsilon.$$

In particular,  $\psi \neq \frac{1}{\mu(K)}\varphi'$ . Since  $\varphi$  is assumed to be  $\Delta$ -maximal,  $\varphi \neq \frac{1}{\mu(K)}\varphi'$  and thus  $0 < \mu(K) < 1$ .

Finally, set  $\theta = (\varphi - \varphi') + \mu(K)\psi$ . Then,  $\theta$  is a convex combination of the state  $\psi$  and the state  $\varphi'' = \frac{1}{1-\mu(K)}(\varphi - \varphi')$ , and hence is state itself. We note that

$$(\varphi, \theta) = (1 - \mu(K))(\varphi'', \varphi'') + \mu(K) \left( \frac{1}{\mu(K)}\varphi', \psi \right)$$

so that  $(\varphi, \theta) \in \Delta$  by convexity. Observe that

$$\theta(a) - \varphi(a) = \mu(K) \left( \psi(a) - \frac{1}{\mu(K)}\varphi'(a) \right) \geq \mu(K)\varepsilon$$

so that  $\varphi \neq \theta$ , which contradicts the fact that  $\varphi$  is  $\Delta$ -maximal. Consequently,  $\mu(N) = 0$  as desired.  $\square$

Next, we examine a converse to Theorem 3.1.1. For a Borel measurable subset  $X \subseteq \mathcal{E}_p(B)$ , we define  $\Sigma_X$  as the set of states  $\varphi$  on  $B$  for which there exists a measure  $\mu \in \mathcal{R}_\varphi$  that is concentrated on  $X$ . Additionally, we define  $\Sigma^X$  as the set of states  $\varphi$  on  $B$  for which every measure  $\mu \in \mathcal{R}_\varphi$  is concentrated on  $X$ . Clearly, we have  $\Sigma^X \subseteq \Sigma_X$  as long as  $\mathcal{R}_\varphi$  is non-empty. It follows directly from [10, Lemma 4.1] that if  $\omega$  is a pure state on  $B$ , then  $\mathcal{R}_\omega$  contains only the point mass at  $\omega$ . Therefore,

$$\Sigma^X \cap \mathcal{E}_p(B) = \Sigma_X \cap \mathcal{E}_p(B) = X. \quad (3.1.2)$$

Consider a pre-order  $\Delta \subset \mathcal{E}(B) \times \mathcal{E}(B)$ , with its maximal elements denoted by  $\max(\Delta)$ . Under certain natural conditions, it can be demonstrated that the maximal elements are generally abundant, particularly for partial orders. For a partial order  $\Delta \subset \mathcal{E}(B) \times \mathcal{E}(B)$  and a state  $\varphi$ , we set  $\Delta_\varphi = \{\psi \in \mathcal{E}(B) : (\varphi, \psi) \in \Delta\}$ .

**Proposition 3.1.1.** *Let  $B$  be a unital  $C^*$ -algebra and  $\Delta \subset \mathcal{E}(B) \times \mathcal{E}(B)$  be a partial order. Assume that  $\Delta_\varphi$  is closed in the weak-\* topology for all  $\varphi \in \mathcal{E}(B)$ . Then for all  $\varphi \in \mathcal{E}(B)$ , there exists a  $\Delta$ -maximal state  $\theta$  such that  $(\varphi, \theta) \in \Delta$ .*

*Proof.* Fix a  $\varphi \in \mathcal{E}(B)$ . Let  $C \subset \Delta_\varphi$  be a chain with respect to the partial order  $\Delta$ . There is a cofinal subnet  $\Lambda$  of  $C$  that converges to some state  $\tau$  in the weak-\* topology. Since  $\Delta_\varphi$  is assumed to be weak-\* closed, we see that  $\tau \in \Delta_\varphi$ , whence  $(\varphi, \tau) \in \Delta$ .

Next, fix  $\psi \in C$ . Then,  $\Lambda_\psi = \{\lambda \in \Lambda : (\psi, \lambda) \in \Delta\}$  is a cofinal subnet of  $\Lambda$ . Hence  $\Lambda_\psi$  also converges to  $\tau$ . Using once again that  $\Delta_\psi$  is weak-\* closed, we find that  $\tau \in \Delta_\psi$  and hence  $(\psi, \tau) \in \Delta$ . We conclude that  $\tau$  is an upper bound for  $C$ . By Zorn's lemma,  $\Delta_\varphi$  has a maximal element  $\theta$ .

We claim that  $\theta$  is in fact  $\Delta$ -maximal. To see this, assume that  $\gamma$  is a state on  $B$  such that  $(\theta, \gamma) \in \Delta$ . Then,  $(\varphi, \gamma) \in \Delta$  so that  $\gamma \in \Delta_\varphi$ . Maximality of  $\theta$  in  $\Delta_\varphi$  then forces  $\theta = \gamma$ , thereby completing the proof.  $\square$

**Corollary 3.1.1.** *Let  $B$  be a unital  $C^*$ -algebra. Let  $\Delta \subset \mathcal{E}(B) \times \mathcal{E}(B)$  be a weak-\* closed partial order. For every state  $\varphi$  on  $B$ , there exists a  $\Delta$ -maximal state  $\theta$  such that  $(\varphi, \theta) \in \Delta$ .*

*Proof.* We will show that for all  $\varphi \in \mathcal{E}(B)$ ,  $\Delta_\varphi$  is weak-\* closed. Then the result follows from Proposition 3.1.1.

Let  $\{\psi_\lambda\}$  be a net in  $\Delta_\varphi$  that converges to a state  $\psi$  in the weak-\* topology. Then  $\{(\varphi, \psi_\lambda)\}$  is a net in  $\Delta$  that converges to  $(\varphi, \psi)$  in the weak-\* topology. Since  $\Delta$  is assumed to be closed in the weak-\* topology, we have  $(\varphi, \psi) \in \Delta$ . Hence  $\psi \in \Delta_\varphi$  and this proves that  $\Delta_\varphi$  is weak-\* closed.  $\square$

A Borel measurable subset  $X \subset \mathcal{E}_p(B)$  will be called a  $\Delta$ -boundary if

$$\max(\Delta) = \Sigma^X = \Sigma_X.$$

We also define  $\Omega = \max(\Delta) \cap \mathcal{E}_p(B)$ , meaning that  $\Omega$  is the set of pure  $\Delta$ -maximal states. According to [10, Corollary 3.3 and Lemma 4.1] and Lemma 3.1.1,  $\Omega$  is Borel measurable when  $B$  is separable. The order  $\Delta$  is termed **hyperrigid** if  $\Sigma_\Omega \subset \max(\Delta)$ . This means that a state of the form  $\int \omega d\mu(\omega)$  is  $\Delta$ -maximal if  $\mu$  is a Borel probability measure concentrated on  $\Omega$ . This condition is trivially satisfied if  $\Delta$  has no maximal elements.

The following is the main result of this section, and it shows that hyperrigidity and boundaries are closely related.

**Corollary 3.1.2.** *Let  $B$  be a separable unital  $C^*$ -algebra. Let  $\Delta \subset \mathcal{E}(B) \times \mathcal{E}(B)$  be a weak-\* closed, convex pre-order. Then, the following statements are equivalent.*

- (i) *There is a Borel measurable subset  $X \subset \mathcal{E}_p(B)$  such that  $\max(\Delta) = \Sigma_X$ .*
- (ii) *The set  $\Omega$  of pure  $\Delta$ -maximal states is a  $\Delta$ -boundary.*
- (iii) *The order  $\Delta$  is hyperrigid.*

*Proof.* (i)  $\Rightarrow$  (ii): Using the assumption along with (3.1.2), we find

$$\Omega = \max(\Delta) \cap \mathcal{E}_p(B) = \Sigma_X \cap \mathcal{E}_p(B) = X.$$

Therefore,  $\max(\Delta) = \Sigma_\Omega$ . By virtue of Theorem 3.1.1, we then find

$$\Sigma^\Omega \subset \Sigma_\Omega = \max(\Delta) \subset \Sigma^\Omega$$

which implies that  $\Omega$  is a  $\Delta$ -boundary.

(ii)  $\Rightarrow$  (i) + (iii): This is trivial.

(iii)  $\Rightarrow$  (ii): Assume that  $\Delta$  is hyperrigid, that is,  $\Sigma_\Omega \subset \max(\Delta)$ . Invoking Theorem 3.1.1, we see that  $\max(\Delta) \subset \Sigma^\Omega$ , so that  $\max(\Delta) = \Sigma^\Omega = \Sigma_\Omega$  and  $\Omega$  is a  $\Delta$ -boundary.  $\square$

## 3.2 The dilation order

### 3.2.1 Definition and properties

The main objective of this section is to give a precise definition and some of the properties of the dilation order. Moreover, we will apply the results found in Section 3.1 to relate the hyperrigidity of the dilation order to the hyperrigidity of operator systems.

Let  $B$  be a unital  $C^*$ -algebra and let  $S \subset B$  be an operator system such that  $C^*(S) = B$ . Let  $\varphi$  be a state on  $B$ . Recall that by a **representation** of  $\varphi$  we mean a triple  $(\pi, \mathcal{H}, \xi)$  consisting of a Hilbert space  $\mathcal{H}$ , a unital  $*$ -representation  $\pi : B \rightarrow B(\mathcal{H})$  and a unit vector  $\xi \in \mathcal{H}$  that satisfies

$$\varphi(b) = \langle \pi(b)\xi, \xi \rangle, \quad b \in B.$$

When  $\xi$  happens to be a cyclic vector for  $\pi$ , then the representation  $(\pi, \mathcal{H}, \xi)$  is unitarily equivalent to the GNS representation of  $\varphi$ . In general, we will need to consider non-cyclic representations as well.

**Definition 3.2.1.** Let  $S$  be an operator system contained in a  $C^*$ -algebra  $B$  such that  $C^*(S) = B$ . Let  $\varphi$  and  $\psi$  be states on  $B$  and  $(\pi, \mathcal{H}, \xi)$  and  $(\sigma, \mathcal{K}, \eta)$  be some representations of  $\varphi$  and  $\psi$  respectively. Then we say that  $(\pi, \mathcal{H}, \xi)$  is **dilated** by  $(\sigma, \mathcal{K}, \eta)$  if there exists an isometry  $V : \mathcal{H} \rightarrow \mathcal{K}$  such that  $V\xi = \eta$  and

$$\pi(s) = V^*\sigma(s)V \quad \forall s \in S.$$

We define the subset  $\mathcal{D}(S, B) \subset \mathcal{E}(B) \times \mathcal{E}(B)$  consisting of all pairs of states  $(\varphi, \psi)$  such that a representation  $(\pi, \mathcal{H}, \xi)$  of  $\varphi$  is dilated by some representation  $(\sigma, \mathcal{K}, \eta)$  of  $\psi$  via an isometry  $V : \mathcal{H} \rightarrow \mathcal{K}$ . Following [27] and [28], we call  $\mathcal{D}(S, B)$  the **dilation order relative to  $S$  and  $B$** .

Davidson and Kennedy introduced this order, and they showed that it is a partial order on  $\mathcal{E}(B)$ . The following theorem provides some equivalent characterizations of the dilation order.

**Proposition 3.2.1.** *Let  $S$  be an operator system in a  $C^*$ -algebra  $B$  such that  $C^*(S) = B$  and  $\varphi, \psi \in \mathcal{E}(B)$ . Then the following are equivalent:*

- (i)  $(\varphi, \psi) \in \mathcal{D}(S, B)$ .
- (ii) *The GNS representation  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  of  $\varphi$  is dilated by a representation  $(\sigma, \mathcal{K}, \eta)$  of  $\psi$ .*
- (iii) *Any representation of  $\varphi$  is dilated by some representation of  $\psi$ .*
- (iv) *For any representation  $(\pi, \mathcal{H}, \xi)$  of  $\varphi$ , there exists a ucp map  $\Psi : B \rightarrow B(\mathcal{H})$  such that,*

- (a)  $\Psi(s) = \pi(s)$  for all  $s \in S$ ,
- (b)  $\psi(x) = \langle \Psi(x)\xi, \xi \rangle$  for all  $x \in B$ .

*Proof.* (i)  $\implies$  (ii): Let  $(\pi, \mathcal{H}, \xi)$  be a representation of  $\varphi$  which is dilated by a representation  $(\sigma, \mathcal{K}, \eta)$  of  $\psi$  via an isometry  $V : \mathcal{H} \longrightarrow \mathcal{K}$ . The  $*$ -representation  $\pi$  restricted to the cyclic reducing subspace of  $\mathcal{H}$  generated by  $\xi$  is unitarily equivalent to the GNS representation of  $\varphi$ . Hence, we can write  $\mathcal{H} = \mathcal{H}_\varphi \oplus \mathcal{H}'$  for some Hilbert space  $\mathcal{H}'$  and  $\pi = \pi_\varphi \oplus \pi'$  for some  $*$ -representation  $\pi' : B \longrightarrow B(\mathcal{H}')$ . Let  $W : \mathcal{H}_\varphi \longrightarrow \mathcal{K}$  be defined as  $W = V|_{\mathcal{H}_\varphi}$ . Hence  $W$  is an isometry and  $W(\xi) = \eta$ . For  $s \in S$  and  $h, k \in \mathcal{H}_\varphi$ , we have

$$\begin{aligned}
\langle W^* \sigma(s) W h, k \rangle &= \langle \sigma(s) W h, W k \rangle \\
&= \langle \sigma(s) V h, V k \rangle \\
&= \langle V^* \sigma(s) V h, k \rangle \\
&= \langle \pi(s) h, k \rangle.
\end{aligned}$$

But  $h, k$  are both in  $\mathcal{H}_\varphi$  which is a reducing subspace of  $\pi$ . Hence,  $\langle \pi(s) h, k \rangle = \langle \pi_\varphi(s) h, k \rangle$ . This implies that  $\langle W^* \sigma(s) W h, k \rangle = \langle \pi_\varphi(s) h, k \rangle$  for all  $s \in S$  and for all  $h, k \in \mathcal{H}_\varphi$ . Hence  $W^* \sigma(s) W = \pi_\varphi(s)$  for all  $s \in S$ . This proves that the GNS representation  $(\pi_\varphi, \mathcal{H}_\varphi, \xi)$  of  $\varphi$  is dilated by the representation  $(\sigma, \mathcal{K}, \eta)$  of  $\psi$  via the isometry  $W : \mathcal{H}_\varphi \longrightarrow \mathcal{K}$ .

(ii)  $\implies$  (iii): Let the GNS representation  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  of  $\varphi$  be dilated by a representation  $(\sigma, \mathcal{K}, \eta)$  via the isometry  $V : \mathcal{H}_\varphi \longrightarrow \mathcal{K}$ . Let  $(\pi, \mathcal{H}, \xi)$  be a representation of  $\varphi$ . We write  $\mathcal{H} = \mathcal{H}_\varphi \oplus \mathcal{H}'$  where  $\mathcal{H}_\varphi$  is the cyclic subspace of  $\mathcal{H}$  generated by  $\xi$ ,  $\mathcal{H}'$  is a subspace of  $\mathcal{H}$  and  $\pi = \pi_\varphi \oplus \pi'$ , where  $\pi_\varphi$  and  $\pi'$  are restrictions of  $\pi$  to  $\mathcal{H}_\varphi$  and  $\mathcal{H}'$  respectively. Now let  $\mathcal{K}_0$  be the Hilbert space  $\mathcal{K} \oplus \mathcal{H}'$ . Let  $\sigma_0 : B \longrightarrow B(\mathcal{K}_0)$  be defined as  $\sigma_0 = \sigma \oplus \pi'$ . Then  $\sigma_0$  is a  $*$ -representation. Let  $\eta_0 = \eta \oplus 0 \in \mathcal{K}_0$ . Then

for all  $b \in B$ , we have

$$\langle \sigma_0(b)\eta_0, \eta_0 \rangle = \langle \sigma(b)\eta, \eta \rangle = \psi(b).$$

Hence  $(\sigma_0, \mathcal{K}_0, \eta_0)$  is a representation of  $\psi$ . Let  $W : \mathcal{H} \longrightarrow \mathcal{K}_0$  be defined as

$$W(h \oplus k) = V(h) \oplus k$$

for all  $h \in \mathcal{H}_\varphi$  and for all  $k \in \mathcal{H}'$ . Clearly  $W$  is an isometry. Moreover  $W(\xi) = \eta_0$ .

For all  $s \in S$  and  $h_1, h_2 \in \mathcal{H}_\varphi$  and  $k_1, k_2 \in \mathcal{H}'$  we have

$$\begin{aligned} \langle W^*\sigma_0(s)W(h_1 \oplus k_1), h_1 \oplus k_1 \rangle &= \langle \sigma_0(s)W(h_1 \oplus k_1), W(h_1 \oplus k_1) \rangle \\ &= \langle \sigma_0(s)(V(h_1) \oplus k_1), V(h_2) \oplus k_2 \rangle \\ &= \langle \sigma(s)Vh_1 \oplus \pi'(s)k_1, Vh_2 \oplus k_2 \rangle \\ &= \langle \sigma(s)Vh_1, Vh_2 \rangle + \langle \pi'(s)k_1, k_2 \rangle \\ &= \langle V^*\sigma(s)Vh_1, h_2 \rangle + \langle \pi'(s)k_1, k_2 \rangle \\ &= \langle \pi_\varphi(s)h_1, h_2 \rangle + \langle \pi'(s)k_1, k_2 \rangle \\ &= \langle \pi_\varphi \oplus \pi'(s)(h_1 \oplus k_1), h_2 \oplus k_2 \rangle \\ &= \langle \pi(s)(h_1 \oplus k_1), h_2 \oplus k_2 \rangle. \end{aligned}$$

This proves that  $W^*\sigma(s)W = \pi(s)$  for all  $s \in S$ . Hence we conclude that the representation  $(\pi, \mathcal{H}, \xi)$  of  $\varphi$  is dilated by a representation  $(\sigma_0, \mathcal{K}_0, \eta_0)$  of  $\psi$ .

(iii)  $\implies$  (iv): If  $(\pi, \mathcal{H}, \xi)$  is a representation of  $\varphi$ , then by (iii), there exists a representation  $(\sigma, \mathcal{K}, \eta)$  of  $\psi$  and an isometry  $V : \mathcal{H} \longrightarrow \mathcal{K}$  such that  $V\xi = \eta$  and  $V^*\sigma(s)V = \pi(s)$  for all  $s \in S$ . Let  $\Psi : B \longrightarrow B(\mathcal{H})$  defined as  $\Psi(b) = V^*\sigma(s)V$  for all  $b \in B$ . Then clearly  $\Psi$  is a ucp map. For all  $s \in S$ ,  $\Psi(s) = V^*\sigma(s)V = \pi(s)$ .

Moreover, for all  $b \in B$  we have

$$\begin{aligned}
\langle \Psi(b)\xi, \xi \rangle &= \langle V^* \sigma(b) V \xi, \xi \rangle \\
&= \langle \sigma(b) V \xi, V \xi \rangle \\
&= \langle \sigma(b) \eta, \eta \rangle \\
&= \psi(b).
\end{aligned}$$

This shows the existence of a ucp map  $\Psi : B \rightarrow B(\mathcal{H})$  such that  $\Psi(s) = \pi(s)$  for all  $s \in S$  and  $\psi(b) = \langle \Psi(b)\xi, \xi \rangle$  for all  $b \in B$ .

(iv)  $\implies$  (iii): Let  $(\pi, \mathcal{H}, \xi)$  be a representation of  $\varphi$ . Then by assumption, there exists a ucp map  $\Psi : B \rightarrow B(\mathcal{H})$  such that  $\Psi(s) = \pi(s)$  for all  $s \in S$  and  $\psi(b) = \langle \Psi(b)\xi, \xi \rangle$  for all  $b \in B$ . By Stinespring's dilation theorem (Theorem 2.1.1), there exists a  $*$ -representation  $\sigma : B \rightarrow B(\mathcal{K})$  along with an isometry  $V : \mathcal{H} \rightarrow \mathcal{K}$  such that  $\Psi(b) = V^* \sigma(b) V$ . Let  $\eta = V\xi \in \mathcal{K}$ . Then  $(\sigma, \mathcal{K}, \eta)$  is a representation of  $\psi$  because

$$\langle \sigma(b)\eta, \eta \rangle = \langle \sigma(b)V\xi, V\xi \rangle = \langle V^* \sigma(b) V \xi, \xi \rangle = \langle \Psi(b)\xi, \xi \rangle = \psi(b).$$

Observe that for all  $s \in S$ ,  $V^* \sigma(s) V = \Psi(s) = \pi(s)$ . Hence  $(\pi, \mathcal{H}, \xi)$  is dilated by  $(\sigma, \mathcal{K}, \eta)$  via the isometry  $V : \mathcal{H} \rightarrow \mathcal{K}$ .

(iii)  $\implies$  (i) is trivial. □

The following theorem establishes a nice connection between the dilation order and the unique extension property of cyclic  $*$ -representations.

**Theorem 3.2.1.** *Let  $S$  be an operator system contained in the  $C^*$ -algebra  $B$  such that  $C^*(S) = B$ . Let  $\varphi \in \mathcal{E}(B)$  and  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  be the GNS representation of  $\varphi$ . Then the following are equivalent:*

(i)  $\varphi$  is maximal in the dilation order.

(ii)  $\pi_\varphi$  has the unique extension property relative to  $S$ .

*Proof.* Let  $\pi_\varphi$  have the unique extension property relative to  $S$  and let  $\psi \in \mathcal{E}(B)$  such that  $(\varphi, \psi) \in \mathcal{D}(S, B)$ . Then by Proposition 3.2.1, there exists a ucp map  $\Psi : B \rightarrow B(\mathcal{H}_\varphi)$  such that  $\pi_\varphi(s) = \Psi(s)$  for all  $s \in S$  and  $\psi(b) = \langle \Psi(b)\xi_\varphi, \xi_\varphi \rangle$  for all  $b \in B$ . From the unique extension property of  $\pi_\varphi$ , we have that  $\Psi(b) = \pi_\varphi(b)$  for all  $b \in B$ . Hence  $\psi(b) = \varphi(b)$  for all  $b \in B$ . Hence  $\psi = \varphi$ , so that  $\varphi$  is  $\mathcal{D}(S, B)$ -maximal.

Conversely, let  $\varphi$  be  $\mathcal{D}(S, B)$ -maximal. Let  $\Psi : B \rightarrow B(\mathcal{H}_\varphi)$  be a ucp map such that  $\pi_\varphi(s) = \Psi(s)$ . By Theorem 2.1.3, there exists a  $*$ -representation  $\sigma : B \rightarrow B(\mathcal{K})$  with the unique extension property relative to  $S$  and an isometry  $V : \mathcal{H} \rightarrow \mathcal{K}$  such that  $\Psi(s) = V^*\sigma(s)V$  for all  $s \in S$ . Let  $\eta = V\xi_\varphi$  and  $\psi$  be the state defined as,

$$\psi(b) = \langle \sigma(b)\eta, \eta \rangle \quad \forall b \in B.$$

Then  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  is dilated by  $(\sigma, \mathcal{K}, \eta)$  via the isometry  $V : \mathcal{H} \rightarrow \mathcal{K}$ . Hence  $(\varphi, \psi) \in \mathcal{D}(S, B)$ . From the maximality of  $\varphi$ , we have  $\psi = \varphi$ . Since  $(\sigma, \mathcal{K}, \eta)$  is a representation of  $\varphi$ , we have,  $\sigma = \pi_\varphi \oplus \sigma'$  for some  $*$ -representation  $\sigma'$  of  $B$ . This implies that  $\pi_\varphi$  has the unique extension property relative to  $S$  by Theorem 2.1.4.  $\square$

**Remark 3.2.1.** The unique extension property of  $*$ -representations is preserved under direct sum (Theorem 2.1.4), and every  $*$ -representation can be decomposed into a direct sum of cyclic representations [44, Theorem 5.1.3]. This implies that a  $*$ -representation  $\pi$  has the unique extension property if and only if every cyclic sub-representation of  $\pi$  has the unique extension property. On the other hand, every cyclic representation is unitarily equivalent to the GNS representation of a state on the  $C^*$ -algebra. So, Theorem 3.2.1 ensures that the dilation maximal states are helpful in detecting representations with the unique extension property.

**Remark 3.2.2.** With the help of Theorem 3.2.1, we can translate the statement of

hyperrigidity conjecture to a statement about the states.

**Arveson's hyperrigidity conjecture.** If every pure state is dilation maximal then every state is dilation maximal.

So, in many places throughout the thesis we will naturally assume that every pure state is maximal in the dilation order.

The next result shows that there is an abundance of maximal states in the dilation order.

**Theorem 3.2.2.** *Let  $S$  be an operator system contained in the  $C^*$ -algebra  $B$  such that  $C^*(S) = B$ . Then for all  $\varphi \in \mathcal{E}(B)$ , there exists a state  $\psi \in \mathcal{E}(B)$  such that  $(\varphi, \psi) \in \mathcal{D}(S, B)$  and  $\psi$  is maximal in  $\mathcal{D}(S, B)$ .*

*Proof.* For  $\varphi \in \mathcal{E}(B)$ , let  $\mathcal{D}_\varphi = \{\psi \in \mathcal{E}(B) : (\varphi, \psi) \in \mathcal{D}(S, B)\}$ . In the light of Proposition 3.1.1, it is enough to show that  $\mathcal{D}_\varphi$  is weak-\* closed for all  $\varphi \in \mathcal{E}(B)$ . Let  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  be the GNS representation of  $\varphi$  and  $E_\varphi$  be the collection of all ucp maps  $\Psi : B \rightarrow B(\mathcal{H}_\varphi)$  such that  $\pi_{\varphi|_S} = \Psi|_S$ . Let  $(\Psi_\lambda)$  be a net in  $E_\varphi$  that converges to  $\Psi \in UCP(B, B(\mathcal{H}))$  in the point weak-\* topology. So for all  $b \in B$ ,  $\Psi_\lambda(b)$  converges to  $\Psi(b)$  in the weak-\* topology of  $B$ . In particular,  $\Psi_\lambda(b)$  converges to  $\Psi(b)$  in the weak operator topology, as the weak operator topology is weaker than the weak-\* topology on  $B(\mathcal{H})$  [44, Section 4.2]. Hence, for all  $s \in S$  and for all  $h, k \in \mathcal{H}$ , we have

$$\langle \Psi(s)h, k \rangle = \lim_\lambda \langle \Psi_\lambda(s)h, k \rangle = \langle \pi_\varphi(s)h, k \rangle.$$

Thus,  $\Psi|_S = \pi_{\varphi|_S}$ . This implies that  $E_\varphi$  is a closed subset of  $UCP(B, B(\mathcal{H}))$ . Hence  $E_\varphi$  is compact, being a closed subset of the compact set  $UCP(B, B(\mathcal{H}))$  in the point weak-\* topology. Now define a map  $\alpha : E_\varphi \rightarrow \mathcal{E}(B)$  by

$$\alpha(\Psi)(b) = \langle \Psi(b)\xi_\varphi, \xi_\varphi \rangle \quad \forall b \in B.$$

Applying a similar argument as above, we get that  $\alpha$  is a continuous map where  $E_\varphi$  and  $\mathcal{E}(B)$  are endowed with point weak-\* and weak-\* topologies respectively. From Theorem 3.2.1 it is clear that a state  $\psi \in \mathcal{D}_\varphi$  if and only if  $\psi = \alpha(\Psi)$  for some  $\Psi \in E_\varphi$ . This implies that  $\mathcal{D}_\varphi$  is the image of  $\alpha$ . Since  $\alpha$  is a continuous map and  $E_\varphi$  is compact,  $\mathcal{D}_\varphi$  is also compact, being the image of a compact set under the continuous map  $\alpha$ . Hence  $\mathcal{D}_\varphi$  is a closed set.  $\square$

Using the terminology from Section 3.1, we can now reformulate the notion of hyperrigidity for  $S$ .

**Corollary 3.2.1.** *Let  $B$  be a separable unital  $C^*$ -algebra and let  $S \subset B$  be an operator system such that  $C^*(S) = B$ . Assume that every pure state on  $B$  is  $\mathcal{D}(S, B)$ -maximal. Then, the following statements are equivalent.*

- (i) *The operator system  $S$  is hyperrigid in  $B$ .*
- (ii) *The pre-order  $\mathcal{D}(S, B)$  is hyperrigid.*
- (iii) *There exists a  $\mathcal{D}(S, B)$ -boundary.*

*Proof.* (i)  $\implies$  (ii): Let  $\varphi$  be a state. The GNS representation  $\pi_\varphi$  has the unique extension property relative to  $S$ , as  $S$  is hyperrigid in  $B$ . Hence by Theorem 3.2.1  $\varphi$  is  $\mathcal{D}(S, B)$  maximal. This implies that every state on  $B$  is  $\mathcal{D}(S, B)$  maximal, so trivially  $\mathcal{D}(S, B)$  is hyperrigid.

(ii)  $\implies$  (i): By assumption, the set  $\Omega$  of pure  $\mathcal{D}(S, B)$ -maximal states coincides with the set of pure states on  $B$ . Thus,  $\Sigma_\Omega$  is the entire state space by [10, Theorem 4.2]. Assuming that  $\mathcal{D}(S, B)$  is hyperrigid, we thus conclude that every state on  $B$  is  $\mathcal{D}(S, B)$ -maximal. Hence all cyclic representations have the unique extension property relative to  $S$ . Since every \*-representation can be written as a direct sum of cyclic representations and the unique extension property is preserved under direct sum (by Theorem 2.1.4), we can say that every \*-representation of  $B$  has the unique extension property relative to  $S$ . Hence  $S$  is hyperrigid in  $B$ .

(ii)  $\implies$  (iii): As  $\mathcal{D}(S, B)$  is hyperrigid, we have  $\Sigma^\Omega \subset \Sigma_\Omega \subset \max(\mathcal{D}(S, B))$ . On the other hand by our assumption

$$\Omega = \mathcal{E}_p(B) \cap \max(\mathcal{D}(S, B)) = \mathcal{E}_p(B).$$

Hence we have

$$\mathcal{E}(B) = \Sigma^\Omega \subset \Sigma_\Omega \subset \max(\mathcal{D}(S, B)).$$

Hence  $\Omega$  is a  $\mathcal{D}(S, B)$  boundary.

(iii)  $\implies$  (ii) is trivial from the definitions. □

### 3.2.2 The cone $\Xi$

Let  $S$  be an operator system and  $K = \coprod_{n \leq \kappa} K_n$  be the nc state space of  $S$ . Recall from Section 2.2 that  $\mathcal{A} = C^*(A_{\text{nc}}(K))$  is the  $C^*$ -algebra of all continuous nc functions on  $K$  and  $\Phi : C_{\max}^*(S) \longrightarrow \mathcal{A}$  is the  $*$ -isomorphism determined by

$$\Phi(b)(\phi) = \widehat{\phi}(b), \quad \forall b \in C_{\max}^*(S), \forall \phi \in K$$

With  $\widehat{\phi}$  as defined in Lemma 2.3.1. Let  $\kappa_0$  be an infinite cardinal number greater than the linear dimension of  $S$ . We define  $\Xi \subset C_{\max}^*(S)$  as the set of elements of the form:

$$\sum_{i,j \in I} c_j \overline{c_i} \Phi^{-1}(F^{(i,j)})$$

where  $F = [F^{(i,j)}]_{i,j \leq m} \in M_m(\mathcal{A})$  is a convex nc function for some cardinal  $m \leq \kappa_0$ ,  $I$  is a finite subset of  $\{n \leq m\}$ , and  $\{c_i : i \in I\}$  is a finite subset of complex numbers.

**Lemma 3.2.1.** *The set  $\Xi$  is a cone in  $C_{\max}^*(S)$ .*

*Proof.* Let  $m$  and  $n$  be two cardinal numbers,  $F = [F^{(i,j)}]_{i,j \leq m} \in M_m(\mathcal{A})$  and  $G = [G^{(k,l)}]_{k,l \leq n} \in M_n(\mathcal{A})$  be convex. Then it is easy to verify that  $H = F \oplus G \in M_{m+n}(\mathcal{A})$

is also convex. Let  $I \subset \{r \leq m\}$  and  $J \subset \{r \leq n\}$  be two finite subsets, and  $\{c_i : i \in I\}$  and  $\{d_j : j \in J\}$  be two finite subsets of complex numbers. To prove that  $\Xi$  is a cone, it is enough to show that  $\Xi$  is closed under positive linear combination, i.e. for all  $s, t \in \mathbb{R}^+$

$$s \left( \sum_{i,i' \in I} c_{i'} \bar{c}_i \Phi^{-1}(F^{(i,i')}) \right) + t \left( \sum_{j,j' \in J} d_{j'} \bar{d}_j \Phi^{-1}(G^{(j,j')}) \right) \in \Xi.$$

Let  $J_0 = \{m+j : j \in J\}$ . Then  $I$  and  $J_0$  are two disjoint finite subsets of  $\{r \leq m+n\}$ . Let  $\Lambda = I \cup J_0$ . Then  $\Lambda$  is again a finite subset of  $\{r \leq m+n\}$ . For all  $\lambda \in \Lambda$  let

$$\alpha_\lambda = \begin{cases} c_i \sqrt{s} & \text{if } \lambda \in I, \\ d_j \sqrt{t} & \text{if } \lambda = m+j \text{ for some } j \in J. \end{cases}$$

Hence  $\{\alpha_\lambda : \lambda \in \Lambda\}$  is a finite set of complex numbers.

Observe that

$$H^{(\lambda,\lambda')} = \begin{cases} F^{(\lambda,\lambda')} & \text{if } \lambda, \lambda' \in I, \\ G^{(j,j')} & \text{if } \lambda = m+j, \lambda' = m+j' \text{ for some } j, j' \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} \sum_{\lambda,\lambda' \in \Lambda} \alpha_{\lambda'} \bar{\alpha}_\lambda \Phi^{-1}(H^{(\lambda,\lambda')}) &= \sum_{\lambda,\lambda' \in I} \alpha_{\lambda'} \bar{\alpha}_\lambda \Phi^{-1}(H^{(\lambda,\lambda')}) + \sum_{\lambda,\lambda' \in J_0} \alpha_{\lambda'} \bar{\alpha}_\lambda \Phi^{-1}(H^{(\lambda,\lambda')}) \\ &= \sum_{i,i' \in I} s c_{i'} \bar{c}_i \Phi^{-1}(F^{(i,i')}) + \sum_{j,j' \in J} t d_{j'} \bar{d}_j \Phi^{-1}(G^{(j,j')}) \\ &= s \left( \sum_{i,i' \in I} c_{i'} \bar{c}_i \Phi^{-1}(F^{(i,i')}) \right) + t \left( \sum_{j,j' \in J} d_{j'} \bar{d}_j \Phi^{-1}(G^{(j,j')}) \right). \end{aligned}$$

Since  $H \in \mathcal{M}_{m+n}(\mathcal{A})$  is convex,  $\sum_{\lambda,\lambda' \in \Lambda} \alpha_{\lambda'} \bar{\alpha}_\lambda \Phi^{-1}(H^{(\lambda,\lambda')}) \in \Xi$ . This implies that  $\Xi$

is closed under positive linear combination.

□

Next, to gain a more concrete understanding of the elements of  $\Xi$ , we will examine the restrictions of nc functions on  $K$  to the first level,  $K_1$ , which corresponds to the state space of  $S$ . Recall that if  $F : \mathcal{F} \rightarrow \mathbb{M}(B(\mathcal{H}))$  is an nc function, then  $F_n = F|_{K_n} : K_n \rightarrow M_n(B(\mathcal{H}))$  for all  $n \leq \kappa$ .

Consider the unital  $*$ -homomorphism  $\rho : \mathcal{A} \rightarrow C(K_1)$  defined by

$$[\rho(F)](\varphi) = F_1(\varphi)$$

for each continuous nc function  $F : K \rightarrow \mathbb{M}$  and each  $\varphi \in K_1$ . Additionally, define the natural evaluation map  $\varepsilon : S \rightarrow C(K_1)$  as

$$[\varepsilon(s)](\varphi) = \varphi(s)$$

for each  $s \in S$  and  $\varphi \in K_1$ . This is a unital completely contractive map, so by Lemma 2.3.1, there exists a unique unital surjective  $*$ -homomorphism  $q : C_{\max}^*(S) \rightarrow C(K_1)$  such that  $q \circ j = \varepsilon$  on  $S$ . For  $s \in S$  and  $\varphi \in K_1$ , using Lemma 2.3.2, we have

$$[(\rho \circ \Phi \circ j)(s)](\varphi) = [\Phi(j(s))]_1(\varphi) = \widehat{\varphi}(j(s)) = \varphi(s) = [\varepsilon(s)](\varphi),$$

so that  $\rho \circ \Phi \circ j = \varepsilon = q \circ j$  on  $S$ . This immediately implies that

$$\rho \circ \Phi = q. \tag{3.2.1}$$

We can now provide a more concrete description of the restrictions of the elements in  $\Xi$  to  $K_1$ .

**Proposition 3.2.2.** *Let  $S$  be an operator system with state space  $L$ . Let  $\Gamma \subset C(L)$*

denote the closed cone of continuous convex functions on  $L$ . Let  $\varepsilon : S \rightarrow C(L)$  be the evaluation map, and let  $q : C_{\max}^*(S) \rightarrow C(L)$  be the surjective unital  $*$ -homomorphism satisfying  $q \circ j = \varepsilon$  on  $S$ . Then,  $q(\Xi) \subset \Gamma$  and  $q(\Xi)$  contains all restrictions to  $L$  of affine weak- $*$  continuous functions on  $S^*$ . Furthermore,  $\Gamma$  is the smallest closed cone in  $C(L)$  that is stable under maxima and contains  $q(\Xi)$ .

*Proof.* Recall that for each cardinal  $n \leq \kappa$ ,  $K_n$  is the set of all completely positive maps from  $S$  into  $B(H_n)$ . Hence,  $K_1 = L$ .

Fix  $s \in S$ . The evaluation function  $\hat{s} : K \rightarrow \mathbb{C}$  defines a continuous nc affine function as ( $\hat{s} = \theta(s)$ , from Theorem 2.3.1). By Lemma 2.3.2, we have  $\Phi^{-1}(\hat{s}) = \Phi^{-1}(\theta(s)) = j(s)$ , so that  $j(s) \in \Xi$ . Thus,  $q(\Xi)$  contains  $q(j(s)) = \varepsilon(s)$  for every  $s \in S$ . Note that  $\varepsilon(s)$  is the restriction of the linear functional on  $S^*$  defined by evaluation at  $s$ . This means  $q(\Xi)$  contains restriction to  $L$  of all weak- $*$  continuous linear functionals on  $S^*$ . Moreover  $q(\Xi)$  contains all the constants trivially. Since every affine function on a topological vector space  $V$  can be written as a  $f(\cdot) + c$  where  $f \in V^*$  and  $c$  is a constant [53, Section III.6 ],  $q(\Xi)$  contains restriction of all affine functions on  $S^*$ .

Next, let  $\xi \in \Xi$ . By definition, this means there is a cardinal  $m \leq \kappa$ , a convex continuous nc function  $F = (F^{(i,j)})_{i,j < m} : K \rightarrow \mathbb{M}(B(\mathcal{H}_m))$ , a finite subset  $I \subset \{n < m\}$ , and a set of complex numbers  $\{c_i : i \in I\}$  such that

$$\xi = \sum_{i,j \in I} c_j \bar{c}_i \Phi^{-1}(F^{(i,j)}).$$

Let  $h = (h_n)$  be the vector in  $\bigoplus_{n < m} \mathbb{C}$  such that  $h_n = c_n$  if  $n \in I$ , and  $h_n = 0$  otherwise. Then, using the convexity of  $F$  and applying (3.2.1), we obtain for each  $\varphi, \psi \in K_1$  and  $0 \leq t \leq 1$ , that

$$[q(\xi)](t\varphi + (1-t)\psi) = \sum_{i,j \in I} c_j \bar{c}_i F_1^{(i,j)}(t\varphi + (1-t)\psi)$$

$$\begin{aligned}
&= \langle F_1(t\varphi + (1-t)\psi)h, h \rangle_{H_m} \\
&\leq \langle (tF_1(\varphi) + (1-t)F_1(\psi))h, h \rangle_{H_m} \\
&= t \left( \sum_{i,j \in I} c_j \bar{c}_i F_1^{(i,j)}(\varphi) \right) + (1-t) \left( \sum_{i,j \in I} c_j \bar{c}_i F_1^{(i,j)}(\psi) \right) \\
&= t[q(\xi)](\varphi) + (1-t)[q(\xi)](\varphi).
\end{aligned}$$

We infer that  $q(\xi)$  is convex on  $L$ .

We have thus shown that  $q(\Xi)$  contains all restrictions to  $L$  of affine weak-\* continuous functions on  $S^*$ , and it is contained in  $\Gamma$ . The second conclusion follows directly from this, in light of [3, Corollary I.1.3].  $\square$

### 3.2.3 The pre-order induced by $\Xi$

The motivation for introducing  $\Xi$  is to define a new pre-order. We define

$$\text{Order}(\Xi) \subset \mathcal{E}(C_{\max}^*(S)) \times \mathcal{E}(C_{\max}^*(S))$$

to be the pre-order consisting of those pairs of states  $(\varphi, \psi)$  satisfying  $\varphi(\xi) \leq \psi(\xi)$  for every  $\xi \in \Xi$ .

**Proposition 3.2.3.** *Let  $S$  be an operator system with maximal  $C^*$ -cover  $(C_{\max}^*(S), j)$ . Then,  $\mathcal{D}(j(S), C_{\max}^*(S)) = \text{Order}(\Xi)$ . In particular,  $\mathcal{D}(j(S), C_{\max}^*(S))$  is a convex, weak-\* closed partial order on the state space of  $C_{\max}^*(S)$ .*

*Proof.* According to [27, Theorem 8.5.1], a pair  $(\varphi, \psi) \in \mathcal{D}(j(S), C_{\max}^*(S))$  if and only if

$$[\varphi(\Phi^{-1}(F^{(i,j)}))]_{i,j < m} \leq [\psi(\Phi^{-1}(F^{(i,j)}))]_{i,j < m} \quad \text{in } M_m$$

for every convex nc continuous function  $F = [F^{(i,j)}]_{i,j < m} : K \rightarrow \mathbb{M}(B(\mathcal{H}_m))$  and for

any cardinal number  $m \leq \kappa$ . For such a function  $F$ , this inequality is equivalent to

$$\langle [\varphi(\Phi^{-1}(F^{(i,j)}))]h, h \rangle \leq \langle [\psi(\Phi^{-1}(F^{(i,j)}))]h, h \rangle$$

for every finitely supported vector  $h$  in  $\bigoplus_{n < m} \mathbb{C}$ . Given such a finitely supported vector  $h$ , there is a finite subset  $I \subset \{n < m\}$  and complex numbers  $\{c_i : i \in I\}$  such that

$$\langle [\varphi(\Phi^{-1}(F^{(i,j)}))]h, h \rangle = \varphi \left( \sum_{i,j \in I} c_j \bar{c}_i \Phi^{-1}(F^{(i,j)}) \right)$$

and

$$\langle [\psi(\Phi^{-1}(F^{(i,j)}))]h, h \rangle = \psi \left( \sum_{i,j \in I} c_j \bar{c}_i \Phi^{-1}(F^{(i,j)}) \right).$$

Thus,  $(\varphi, \psi) \in \mathcal{D}(j(S), C_{\max}^*(S))$  is equivalent to

$$\varphi(\xi) \leq \psi(\xi), \quad \xi \in \Xi.$$

In other words,  $\mathcal{D}(j(S), C_{\max}^*(S)) = \text{Order}(\Xi)$ . It is straightforward to verify that this implies  $\mathcal{D}(j(S), C_{\max}^*(S))$  is convex and weak-\* closed.  $\square$

The previous result applies exclusively to operator systems represented in their maximal  $C^*$ -covers. Our next goal is to demonstrate that Proposition 3.2.3 also provides relevant information about dilation maximal states for any representation of  $S$ . To achieve this, we need the following lemma.

**Lemma 3.2.2.** *Let  $B$  be a unital  $C^*$ -algebra generated by an operator system  $S \subset B$ . Let  $q : C_{\max}^*(S) \rightarrow B$  be the surjective unital  $*$ -homomorphism such that  $q \circ j = \text{id}$  on  $S$ . Let  $\varphi$  be a state on  $B$ . Then,  $\varphi$  is  $\mathcal{D}(S, B)$ -maximal if and only if  $\varphi \circ q$  is  $\mathcal{D}(j(S), C_{\max}^*(S))$ -maximal.*

*Proof.* Let  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  be the GNS representation of  $\varphi$ . Then  $\pi_\varphi \circ q : C_{\max}^*(S) \rightarrow$

$B(\mathcal{H}_\varphi)$  is a  $*$ -representation and for all  $x \in C_{\max}^*(S)$ ,

$$\langle \pi_\varphi \circ q(x)\xi_\varphi, \xi_\varphi \rangle = \varphi \circ q(x).$$

Hence  $(\pi_\varphi \circ q, \mathcal{H}_\varphi, \xi_\varphi)$  is a representation of  $\varphi \circ q$ . Surjectivity of  $q$  implies that  $\pi_\varphi \circ q$  is a cyclic  $*$ -representation. Hence  $(\pi_\varphi \circ q, \mathcal{H}_\varphi, \xi_\varphi)$  is unitarily equivalent to the GNS representation of  $\varphi \circ q$ . Since  $q$  is completely isometric on  $S$ , it follows from Theorem 2.1.6 that  $\pi_\varphi$  has the unique extension property relative to  $S$  if and only if  $\pi_\varphi \circ q$  has the unique extension property relative to  $j(S)$ . So if  $\varphi$  is  $\mathcal{D}(S, B)$ -maximal then  $\varphi \circ q$  is  $\mathcal{D}(j(S), C_{\max}^*(S))$ -maximal.

Let  $\psi \in \mathcal{E}(B)$  be such that  $(\varphi, \psi) \in \mathcal{D}(S, B)$ . Then by Proposition 3.2.1 there exists a ucp map  $\Psi : B \rightarrow B(\mathcal{H}_\varphi)$  such that  $\Psi(s) = \pi_\varphi(s)$  for all  $s \in S$  and  $\psi(b) = \langle \Psi(b)\xi_\varphi, \xi_\varphi \rangle$  for all  $b \in B$ .  $\Psi \circ q : C_{\max}^*(S) \rightarrow B(\mathcal{H}_\varphi)$  is a ucp map such that,

$$\Psi \circ q(j(s)) = \Psi(s) = \pi_\varphi(s) = \pi_\varphi \circ q(j(s)),$$

and

$$\langle \Psi \circ q(x)\xi_\varphi, \xi_\varphi \rangle = \psi \circ q(x) \quad \forall x \in C_{\max}^*(S).$$

Hence by Theorem 3.2.1, we get that  $(\varphi \circ q, \psi \circ q) \in \mathcal{D}(j(S), C_{\max}^*(S))$ . If  $\varphi \circ q$  is  $\mathcal{D}(j(S), C_{\max}^*(S))$ -maximal, then  $\varphi \circ q = \psi \circ q$ . Surjectivity of  $q$  implies that  $\varphi = \psi$ . This shows that  $\varphi$  is  $\mathcal{D}(S, B)$ -maximal, when  $\varphi \circ q$  is  $\mathcal{D}(j(S), C_{\max}^*(S))$ -maximal.  $\square$

Building on the notation established above, we now present the main result of this section.

**Theorem 3.2.3.** *Let  $B$  be a unital  $C^*$ -algebra generated by an operator system  $S \subset B$ . Let  $q : C_{\max}^*(S) \rightarrow B$  be the surjective unital  $*$ -homomorphism such that  $q \circ j = \text{id}$  on  $S$ . Then,  $\max(\mathcal{D}(S, B)) = \max(\text{Order}(q(\Xi)))$ .*

*Proof.* Let  $\varphi$  be a state on  $B$ . Suppose  $\varphi$  is  $\mathcal{D}(S, B)$ -maximal. Let  $\psi$  be another

state on  $B$  such that  $(\varphi, \psi) \in \text{Order}(q(\Xi))$ . Then,  $(\varphi \circ q, \psi \circ q) \in \text{Order}(\Xi)$ , and by Proposition 3.2.3,  $(\varphi \circ q, \psi \circ q) \in \mathcal{D}(j(S), C_{\max}^*(S))$ . Furthermore, by Lemma 3.2.2,  $\varphi \circ q$  is  $\mathcal{D}(j(S), C_{\max}^*(S))$ -maximal, which implies  $\varphi \circ q = \psi \circ q$  and hence  $\varphi = \psi$ . This establishes that  $\varphi$  is  $\text{Order}(q(\Xi))$ -maximal.

Conversely, assume  $\varphi$  is  $\text{Order}(q(\Xi))$ -maximal. Let  $\psi$  be another state on  $B$  such that  $(\varphi, \psi) \in \mathcal{D}(S, B)$ . There exist representations  $(\pi, \mathcal{H}, \xi)$  and  $(\sigma, \mathcal{K}, \eta)$  of  $\varphi$  and  $\psi$  respectively, along with an isometry  $V : \mathcal{H} \rightarrow \mathcal{K}$  such that  $V\xi = \eta$  and

$$\pi(s) = V^*\sigma(s)V, \quad s \in S.$$

Arguing as in the previous result, we get that  $(\pi \circ q, \mathcal{H}, \xi)$  and  $(\sigma \circ q, \mathcal{K}, \eta)$  are representations of  $\varphi \circ q$  and  $\psi \circ q$  respectively, and it follows that  $(\varphi \circ q, \psi \circ q) \in \mathcal{D}(j(S), C_{\max}^*(S))$ . Therefore, by another application of Proposition 3.2.3, we have  $(\varphi \circ q, \psi \circ q) \in \text{Order}(\Xi)$ , which means  $(\varphi, \psi) \in \text{Order}(q(\Xi))$ . The maximality of  $\varphi$  in  $\text{Order}(q(\Xi))$  then implies  $\varphi = \psi$ , confirming that  $\varphi$  is  $\mathcal{D}(S, B)$ -maximal.  $\square$

Let us explore some of the ramifications of the previous result in relation to Arveson's hyperrigidity conjecture in the commutative setting.

Let  $B$  be a separable commutative unital  $C^*$ -algebra and let  $S \subset B$  be an operator system with  $C^*(S) = B$ . Let  $L$  denote the state space of  $S$ . Because  $B$  is commutative, the evaluation map  $\varepsilon : S \rightarrow C(L)$  is completely isometric [28, Theorem 2.2]. Let  $q : C_{\max}^*(S) \rightarrow C(L)$  denote the unique surjective unital  $*$ -homomorphism such that  $q \circ j = \varepsilon$  on  $S$ .

Under the assumption that all pure states on  $C(L)$  are  $\mathcal{D}(\varepsilon(S), C(L))$ -maximal, to establish the conjecture, we need to prove that  $\mathcal{D}(\varepsilon(S), C(L))$  is hyperrigid (see Corollary 3.2.1). On the other hand, the property of being hyperrigid only depends on the set of maximal elements, so we may replace the dilation order by any pre-order with the same maximal elements. In light of Theorem 3.2.3, this means that

we can just as well try to show that  $\text{Order}(q(\Xi))$  is hyperrigid.

In turn, by Corollary 3.1.2, this is equivalent to the existence of a boundary for  $\text{Order}(q(\Xi))$ . In this context, we may thus hope to apply the classical machinery of [3, Corollary I.5.18] to construct such a boundary: If a partial order is determined by a max stable cone of functions then the boundary exists for the partial order. This strategy essentially reduces to the one employed in [28]. Indeed, in order for [3, Corollary I.5.18] to be applicable, the cone  $q(\Xi)$  would need to be stable under taking maxima. If this were the case, then by virtue of Proposition 3.2.2, we would know that the closure of  $q(\Xi)$  coincides with the cone of all continuous convex functions on  $L$ . In turn, Theorem 3.2.3 would then imply that the dilation maximal elements coincide with so-called Choquet maximal elements.

### 3.2.4 Uniqueness of $\Xi$

It is natural now to wonder whether  $\Xi$  is the unique cone in  $C_{\max}^*(S)$  that satisfies Proposition 3.2.3. Before we can address this question, we introduce some notation and terminology.

Let  $B$  be a unital  $C^*$ -algebra. Given a pre-order  $\Delta \subset \mathcal{E}(B) \times \mathcal{E}(B)$ , we define the *induced cone* of  $\Delta$  to be the set  $\text{Cone}(\Delta) \subset B$  of self-adjoint elements  $b$  with the property that  $\varphi(b) \leq \psi(b)$  whenever  $(\varphi, \psi) \in \Delta$ . If, conversely, we are given a cone  $\Gamma \subset B$  of self-adjoint elements, we define the *induced order* of  $\Gamma$  to be the pre-order  $\text{Order}(\Gamma) \subset \mathcal{E}(B) \times \mathcal{E}(B)$  consisting of those pairs of states  $(\varphi, \psi)$  satisfying

$$\varphi(\gamma) \leq \psi(\gamma), \quad \gamma \in \Gamma.$$

There exists a certain duality between these objects, as we show next.

**Theorem 3.2.4.** *Let  $B$  be a unital  $C^*$ -algebra and let  $\Gamma \subset B$  be a cone of self-adjoint elements containing both 1 and  $-1$ . Then,  $\text{Cone}(\text{Order}(\Gamma))$  is the norm closure of  $\Gamma$ .*

*Proof.* By continuity, it follows from the definitions that the norm closure of  $\Gamma$  is contained in  $\text{Cone}(\text{Order}(\Gamma))$ . Assume that there is a self-adjoint element  $b \in \text{Cone}(\text{Order}(\Gamma))$  outside the norm closure of  $\Gamma$ . By the convex separation theorem, we can find a bounded linear functional  $\theta$  on  $B$  such that

$$\sup_{c \in \Gamma} (\text{Re } \theta)(c) < (\text{Re } \theta)(b).$$

Here, we let  $\text{Re } \theta = (\theta + \theta^*)/2$ ; this is a self-adjoint bounded linear functional on  $B$ .

Next, let  $c \in \Gamma$ . Then  $tc \in \Gamma$  for every  $t > 0$ , so that

$$t(\text{Re } \theta)(c) < (\text{Re } \theta)(b),$$

which forces  $(\text{Re } \theta)(c) \leq 0$ . Since both 1 and  $-1$  belong to  $\Gamma$ , we find  $(\text{Re } \theta)(1) = 0$ . Therefore,

$$\sup_{c \in \Gamma} (\text{Re } \theta)(c) = 0. \tag{3.2.2}$$

Next, by the Jordan decomposition [47, Lemma 3.2.2], there are positive linear functionals  $\varphi_0, \psi_0$  on  $B$  with the property that  $\text{Re } \theta = \varphi_0 - \psi_0$ . Using that  $(\text{Re } \theta)(1) = 0$ , we infer that there is strictly positive number  $r$  such that  $r = \|\varphi_0\| = \varphi_0(1) = \psi_0(1) = \|\psi_0\|$ . Define  $\varphi = \frac{1}{r}\varphi_0$  and  $\psi = \frac{1}{r}\psi_0$ , which are then states on  $B$  satisfying

$$\sup_{c \in \Gamma} (\varphi(c) - \psi(c)) = 0 < \varphi(b) - \psi(b)$$

by virtue of (3.2.2). In particular,  $(\varphi, \psi) \in \text{Order}(\Gamma)$ . In turn, because  $b$  lies in  $\text{Cone}(\text{Order}(\Gamma))$ , this means that  $\varphi(b) \leq \psi(b)$ , contradicting the previous inequality.  $\square$

One may wonder if the “duality” uncovered above between cones and pre-orders goes in the other direction, namely whether  $\text{Order}(\text{Cone}(\Delta)) = \Delta$  for any pre-order

$\Delta$  on the state space of  $B$ . It is readily seen that  $\Delta$  must be convex and weak-\* closed for this to hold, but at the time of this writing we do not know if these necessary conditions are also sufficient.

We can now address the uniqueness question raised earlier.

**Corollary 3.2.2.** *Let  $S$  be an operator system. Then, the norm closure of  $\Xi$  is the unique closed cone  $\Gamma$  of self-adjoint elements in  $C_{\max}^*(S)$  containing 1 and  $-1$ , and satisfying*

$$\mathcal{D}(j(S), C_{\max}^*(S)) = \text{Order}(\Gamma).$$

*Proof.* First, it is clear that  $\text{Order}(\Xi) = \text{Order}(\overline{\Xi})$ , where  $\overline{\Xi} \subset B$  denotes the norm closure of  $\Xi$ . Hence, we can apply Proposition 3.2.3 to see that

$$\mathcal{D}(j(S), C_{\max}^*(S)) = \text{Order}(\overline{\Xi}).$$

Assume that  $\Gamma \subset C_{\max}^*(S)$  is a closed cone of self-adjoint elements, containing 1 and  $-1$ , and satisfying

$$\mathcal{D}(j(S), C_{\max}^*(S)) = \text{Order}(\Gamma).$$

By virtue of Theorem 3.2.4, we see that  $\overline{\Xi} = \text{Cone}(\mathcal{D}(j(S), C_{\max}^*(S))) = \Gamma$ . □

### 3.3 Detecting hyperrigidity with the boundary projection

We have seen that the existence of a boundary for the dilation order cannot be inferred from the classical Choquet theory. In this section, we will demonstrate the existence of a "non-classical boundary" on which the dilation maximal states are concentrated. Using the techniques of non-commutative absolute continuity, we will show the existence of a projection that characterizes the dilation maximal states.

Moreover, we will give a reformulation of the hyperrigidity conjecture in terms of this projection.

Let  $Y$  be compact and Hausdorff. Then by Riesz representation theorem,  $\mathcal{E}(C(Y))$  can be identified as  $P(Y)$  where  $P(Y)$  is the set of all regular Borel probability measures on  $Y$ . We view each  $\mu \in P(Y)$  as a state on  $C(Y)$  acting as

$$\mu(f) = \int f d\mu.$$

Moreover for each  $\mu \in P(Y)$ , let  $(\pi_\mu, L^2(\mu), \xi_\mu)$  be the GNS representation of  $\mu$ . Then it is well known that

$$\pi_\mu(f)g = fg \quad \forall f \in C(Y), \forall g \in L^2(\mu).$$

The following result is an algebraic version of the classical Radon-Nikodym theorem. Later in this section, we will provide a partial non-commutative version of this result.

**Theorem 3.3.1.** *Let  $\mu$  and  $\nu$  be regular Borel probability measures on a compact Hausdorff space  $Y$  and  $(\pi_\mu, L^2(\mu), \xi_\mu)$  be the GNS representation of  $\mu$ . Then  $\nu \ll \mu$  if and only if there exists  $r \in L^2(\mu)$  such that  $r \geq 0$  ( $\mu$ -a.e.), and  $\nu(f) = \langle \pi_\mu(f)r, r \rangle$  for all  $f \in C(Y)$ .*

*Proof.* If  $\nu \ll \mu$ , then by the Radon-Nikodym theorem, there exists  $g \in L^1(\mu)$  such that  $d\nu = g d\mu$ . Also,  $g \geq 0$  ( $\mu$ -a.e.) as  $\nu$  is a probability measure. Let  $r = g^{\frac{1}{2}}$ . Then,  $|r|^2$  is  $\mu$  integrable as  $|r|^2 = g$  is in  $L^1(\mu)$ . Hence,  $r \in L^2(\mu)$ . Now for  $f \in C(Y)$  we have:

$$\langle \pi_\mu(f)r, r \rangle = \langle fr, r \rangle = \int fr^2 d\mu = \int fg d\mu = \int f d\nu = \nu(f).$$

Conversely, let there exists  $r \in L^2(\mu)$  such that  $r \geq 0$ ,  $\mu$  almost everywhere and  $\nu(f) = \langle \pi_\mu(f)r, r \rangle$  for all  $f \in C(Y)$ . Let  $g = r^2$ , then  $g \in L^1(\mu)$  and for all

$f \in C(Y)$ :

$$\nu(f) = \langle \pi_\mu(f)r, r \rangle = \int fr^2 d\mu = \int fg d\mu.$$

This implies that there exists  $g \in L^1(\mu)$  such that  $d\nu = gd\mu$ . Hence  $\nu \ll \mu$ .  $\square$

Let  $B$  be a unital  $C^*$ -algebra. The bidual  $B^{**}$  is then a von Neumann algebra. If  $\pi : B \rightarrow B(\mathcal{H})$  is a  $*$ -representation, then it admits a unique weak- $*$  continuous extension  $\tilde{\pi} : B^{**} \rightarrow B(\mathcal{H})$ , which is also a  $*$ -representation [53, Section III.2]. Recall from Section 2.4 that for a state  $\varphi$  on  $B$ , its left kernel is

$$L_\varphi = \{x \in B^{**} : \hat{\varphi}(x^*x) = 0\}.$$

Let  $\psi$  be another state on  $B$ , then  $\varphi$  is **absolutely continuous** with respect to  $\psi$  if and only if  $L_\psi \subset L_\varphi$ .

Unlike in the classical setting, however, the existence of some form of a Radon-Nikodym theorem in the general case is a rather subtle issue, and no perfect analogue exists as far as we know; see [50],[48],[32],[56],[33] and the references therein. Fortunately, this difficulty can be circumvented via the following fact.

**Lemma 3.3.1.** *Let  $B$  be a unital  $C^*$ -algebra. Let  $\phi, \psi$  be states on  $B$  with respective GNS representations  $(\pi, \mathcal{H}, \xi)$  and  $(\sigma, \mathcal{K}, \eta)$ . Assume that  $\varphi$  is absolutely continuous with respect to  $\psi$ . Then, the following statements hold.*

1. *There is a unique weak- $*$  continuous  $*$ -representation  $\rho : \tilde{\sigma}(B^{**}) \rightarrow \tilde{\pi}(B^{**})$  such that  $\rho \circ \tilde{\sigma} = \tilde{\pi}$ .*
2. *There is a normal state  $\tau$  on  $B(\mathcal{K})$  such that  $\varphi = \tau \circ \tilde{\sigma}$ .*

*Proof.* (i) The vector  $\xi$  is cyclic for  $\pi$ , so that an element  $x \in B^{**}$  belongs to  $\ker \tilde{\pi}$  if and only if

$$\varphi((xb)^*(xb)) = \|\tilde{\pi}(x)\pi(b)\xi\|^2 = 0$$

for every  $b \in B$ . In other words,

$$\ker \tilde{\pi} = \{x \in B^{**} : xB \subset L_\varphi\}.$$

Similarly,

$$\ker \tilde{\sigma} = \{x \in B^{**} : xB \subset L_\psi\}.$$

Now,  $\varphi$  is absolutely continuous with respect to  $\psi$ , so  $L_\psi \subset L_\varphi$ . We thus infer that  $\ker \tilde{\sigma} \subset \ker \tilde{\pi}$ .

Next, by [51, Proposition 1.10.4] there are central projection  $\mathfrak{s}_\pi, \mathfrak{s}_\sigma \in B^{**}$  satisfying

$$\ker \tilde{\pi} = B^{**}(I - \mathfrak{s}_\pi) \quad \text{and} \quad \ker \tilde{\sigma} = B^{**}(I - \mathfrak{s}_\sigma).$$

By the previous paragraph, we see that  $\mathfrak{s}_\pi \leq \mathfrak{s}_\sigma$ . Hence, the weak-\* continuous \*-homomorphism  $\rho_0 : B^{**}\mathfrak{s}_\sigma \rightarrow B^{**}\mathfrak{s}_\pi$  of multiplication by  $\mathfrak{s}_\pi$  is surjective. Furthermore, there are weak-\* homomomorphic \*-isomorphisms  $\theta_\pi : \tilde{\pi}(B^{**}) \rightarrow B^{**}\mathfrak{s}_\pi$  and  $\theta_\sigma : \tilde{\sigma}(B^{**}) \rightarrow B^{**}\mathfrak{s}_\sigma$  defined as

$$\theta_\pi(\tilde{\pi}(x)) = x\mathfrak{s}_\pi \quad \text{and} \quad \theta_\sigma(\tilde{\sigma}(x)) = x\mathfrak{s}_\sigma$$

for every  $x \in B^{**}$ . The desired map is then  $\rho = \theta_\pi^{-1} \circ \rho_0 \circ \theta_\sigma$ , and it is clearly unique.

(ii) Define a weak-\* continuous state  $\tau_0 : B(\mathcal{H}) \rightarrow \mathbb{C}$  as

$$\tau_0(t) = \langle t\xi, \xi \rangle, \quad t \in B(\mathcal{H}).$$

By [51, Proposition 1.24.5], there is a weak-\* continuous state  $\tau$  on  $B(\mathcal{K})$  extending  $\tau_0 \circ \rho$ , and this state has the desired properties.  $\square$

This fact will be exploited in the following fashion to establish a partial non-commutative version of the Radon-Nikodym theorem.

**Proposition 3.3.1.** *Let  $B$  be a unital  $C^*$ -algebra. Let  $\varphi, \psi$  be states on  $B$  such that  $\varphi$  is absolutely continuous with respect to  $\psi$ . Let  $\sigma : B \rightarrow B(\mathcal{K})$  be the GNS representation of  $\psi$ . Then, the following statements hold.*

- (i) *There is a countable orthonormal set of vectors  $\{e_n\}$  in  $\mathcal{K}$  along with positive numbers  $\{t_n\}$  such that  $\sum_{n=1}^{\infty} t_n = 1$  and*

$$\varphi(b) = \sum_{n=1}^{\infty} t_n \langle \sigma(b)e_n, e_n \rangle, \quad b \in B.$$

- (ii) *The GNS representation of  $\varphi$  is unitarily equivalent to a subrepresentation of  $\sigma^{(\infty)} = \bigoplus_{i \in \mathbb{N}} \sigma$ .*

*Proof.* (i) By Lemma 3.3.1, there is a normal state  $\tau$  on  $B(\mathcal{K})$  such that  $\varphi = \tau \circ \tilde{\sigma}$ . Hence by virtue of [15, Theorem 2.4.41], there is a positive trace class operator  $T$  on  $\mathcal{K}$  with  $\text{tr}(T) = 1$  such that  $\tau(A) = \text{tr}(AT)$  for all  $T \in B(\mathcal{K})$ . Hence,

$$\varphi(b) = \text{tr}(\sigma(b)T), \quad b \in B.$$

Applying the spectral theorem to  $T$ , there is a countable orthonormal set of vectors  $\{e_n\}$  in  $\mathcal{K}$  along with positive numbers  $\{t_n\}$  such that  $\sum_{n=1}^{\infty} t_n = 1$  and

$$\varphi(b) = \sum_{n=1}^{\infty} t_n \langle \sigma(b)e_n, e_n \rangle, \quad b \in B.$$

- (ii) We see that  $\xi = (t_n^{1/2}e_n)$  is a unit vector in  $\mathcal{K}^{(\infty)} = \bigoplus_{i \in \mathbb{N}} \mathcal{K}$  and satisfies

$$\varphi(b) = \langle \sigma^{(\infty)}(b)\xi, \xi \rangle, \quad b \in B.$$

It follows that the GNS representation of  $\varphi$  is unitarily equivalent to the restriction of  $\sigma^{(\infty)}$  to the cyclic subspace generated by  $\xi$ . □

The following consequence of Proposition 3.3.1 is the crucial technical step in achieving our goal in this section.

**Theorem 3.3.2.** *Let  $B$  be a unital  $C^*$ -algebra and let  $S \subset B$  be an operator system such that  $B = C^*(S)$ . Let  $\varphi, \psi$  be states on  $B$  such that  $\varphi$  is absolutely continuous with respect to  $\psi$ . If  $\psi$  is  $\mathcal{D}(S, B)$ -maximal, then so is  $\varphi$ .*

*Proof.* Let  $\sigma : B \rightarrow B(\mathcal{K})$  be the GNS representation of  $\psi$ . Since  $\psi$  is  $\mathcal{D}(S, B)$ -maximal, we infer from Theorem 3.2.1 that  $\sigma$  has the unique extension property relative to  $S$ . In turn, by virtue of Theorem 2.1.4, we see that any subrepresentation of  $\sigma^{(\infty)}$  also has the unique extension property relative to  $S$ . In particular, this is the case for the GNS representation of  $\varphi$  by Proposition 3.3.1(ii). Another application of Theorem 3.2.1 implies that  $\varphi$  is also  $\mathcal{D}(S, B)$  maximal.  $\square$

Let  $C$  be a compact convex set in a locally convex topological vector space. Then a subset  $F \subset C$  is said to be a **face** of  $C$  if whenever  $x, y \in C$  and  $t \in (0, 1)$  are such that  $tx + (1 - t)y \in F$ , then  $x, y \in F$ . Now we can prove one of the main result of the thesis.

**Theorem 3.3.3.** *Let  $B$  be a unital  $C^*$ -algebra and let  $S \subset B$  be an operator system such that  $B = C^*(S)$ . Then, the following statements hold.*

- (i) *The set of  $\mathcal{D}(S, B)$ -maximal elements is a norm-closed face of  $\mathcal{E}(B)$ .*
- (ii) *There exists a projection  $\mathfrak{d} \in B^{**}$  with the property that a state  $\varphi$  on  $B$  is  $\mathcal{D}(S, B)$ -maximal precisely when  $\varphi(\mathfrak{d}) = 1$ .*

*Proof.* (i) Let  $\theta$  be a  $\mathcal{D}(S, B)$ -maximal state on  $B$ . Assume that there is  $0 < t < 1$  and states  $\varphi, \psi$  on  $B$  such that  $\theta = t\varphi + (1 - t)\psi$ . Let  $\beta \in B^{**}$  be such that  $\theta(\beta^*\beta) = 0$ . This implies that

$$t\varphi(\beta^*\beta) + (1 - t)\psi(\beta^*\beta) = 0.$$

Since  $\varphi$  and  $\psi$  are positive linear functionals and  $\beta^*\beta$  is positive, we have  $\varphi(\beta^*\beta) = \psi(\beta^*\beta) = 0$ . This shows that both  $\varphi$  and  $\psi$  are absolutely continuous with respect to  $\theta$ . So  $\varphi, \psi$  are also  $\mathcal{D}(S, B)$ -maximal by Theorem 3.3.2. Therefore the  $\mathcal{D}(S, B)$ -maximal elements form a face of the state space of  $B$ .

Next, let  $(\varphi_n)$  be a sequence of  $\mathcal{D}(S, B)$ -maximal states converging in norm to some state  $\varphi$  on  $B$ . For each  $n$ , let  $(\pi_{\varphi_n}, \mathcal{H}_{\varphi_n}, \xi_{\varphi_n})$  be the GNS representation of  $\varphi_n$ , which has the unique extension property with respect to  $S$  by Theorem 3.2.1. Let  $\tau = \sum_{n=1}^{\infty} \frac{1}{2^n} \varphi_n$ . Hence  $\tau$  is a state on  $B$ . We set  $\mathcal{H} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}_{\varphi_n}$  and  $\xi = \bigoplus_{n \in \mathbb{N}} 2^{-n/2} \xi_{\varphi_n} \in \mathcal{H}$ . Then  $\xi$  is a unit vector and  $\pi = \bigoplus_{n \in \mathbb{N}} \pi_{\varphi_n}$  is a  $*$ -representation on  $\mathcal{H}$ . Moreover,  $(\pi, \mathcal{H}, \xi)$  is a representation of the state  $\tau$ , because for all  $b \in B$ ,

$$\begin{aligned} \langle \pi(b)\xi, \xi \rangle &= \left\langle \bigoplus_{n \in \mathbb{N}} 2^{-n/2} \pi_{\varphi_n}(b) \xi_{\varphi_n}, \bigoplus_{n \in \mathbb{N}} 2^{-n/2} \xi_{\varphi_n} \right\rangle \\ &= \sum_{n \in \mathbb{N}} \frac{1}{2^n} \langle \pi_{\varphi_n}(b) \xi_{\varphi_n}, \xi_{\varphi_n} \rangle \\ &= \sum_{n \in \mathbb{N}} \frac{1}{2^n} \varphi_n(b) \\ &= \tau(b). \end{aligned}$$

Hence, the GNS representation of  $\tau$  is a subrepresentation  $\pi$ . Next note that  $\pi$  has the unique extension property, because of Theorem 2.1.4 and the fact that  $\pi$  is a direct sum of  $*$ -representations  $\pi_{\varphi_n}$  with the unique extension property relative to  $S$ . Hence the GNS representation of  $\tau$  also has the unique extension property, being a subrepresentation of  $\pi$ . So by Theorem 3.2.1  $\tau$  is  $\mathcal{D}(S, B)$ -maximal. Let  $\beta \in B^{**}$  such that  $\tau(\beta^*\beta) = 0$ . Then,

$$\sum_{n \in \mathbb{N}} \frac{1}{2^n} \varphi_n(\beta^*\beta) = 0.$$

Then from the positivity of  $\varphi_n$  and  $\beta^*\beta$  we have  $\varphi_n(\beta^*\beta) = 0$  for all  $n \in \mathbb{N}$ . Hence  $\varphi(\beta^*\beta) = 0$ . This implies that  $\varphi$  is absolutely continuous with respect to  $\tau$ . Hence

by Theorem 3.3.2,  $\varphi$  is  $\mathcal{D}(S, B)$  maximal.

(ii) By (i), we may apply [24, Theorem 3.5] to find a projection  $\mathfrak{d} \in B^{**}$  with the property that a state  $\varphi$  on  $B$  is absolutely continuous with respect to some  $\mathcal{D}(S, B)$ -maximal state precisely when

$$\varphi(b) = \varphi(\mathfrak{d}b), \quad b \in B. \quad (3.3.1)$$

If the above equality holds, then clearly  $\varphi(\mathfrak{d}) = 1$ . Conversely if  $\varphi(\mathfrak{d}) = 1$ , then

$$\varphi(\mathfrak{d}^*\mathfrak{d}) = \varphi(\mathfrak{d}) = 1 = \varphi(\mathfrak{d})^*\varphi(\mathfrak{d}).$$

Then by applying [45, Theorem 3.18], we have that  $\mathfrak{d}$  is in the multiplicative domain of  $\varphi$ . So  $\varphi(\mathfrak{d}b) = \varphi(\mathfrak{d})\varphi(b) = \varphi(b)$  for all  $b \in B$ . In other words 3.3.1 is equivalent to  $\varphi(\mathfrak{d}) = 1$ . Since  $\mathcal{D}(S, B)$  is stable under absolute continuity by Theorem 3.3.2, hence we conclude that a state  $\varphi$  is  $\mathcal{D}(S, B)$ -maximal if and only if  $\varphi(\mathfrak{d}) = 1$ .  $\square$

The projection  $\mathfrak{d}$  in the previous result completely determines the set of  $\mathcal{D}(S, B)$ -maximal elements: these are the states that are “concentrated” on  $\mathfrak{d}$ . This phenomenon is reminiscent of the notion of a  $\mathcal{D}(S, B)$ -boundary from Section 3.1. For this reason, we call  $\mathfrak{d}$  the **boundary projection** of the dilation order.

### 3.3.1 Hyperrigidity and non-commutative topology

As demonstrated in Corollary 3.2.1, the existence of a  $\mathcal{D}(S, B)$ -boundary would provide a positive resolution to Arveson’s hyperrigidity conjecture. In the rest of the paper, we explore whether there is a comparable relationship between the boundary projection  $\mathfrak{d}$  and the conjecture.

According to Theorem 3.2.3, there exists a weak-\* closed convex pre-order on  $\mathcal{E}(B)$  that shares the same maximal elements as  $\mathcal{D}(S, B)$ . When  $B$  is separable,

the set  $\Omega$  of pure  $\mathcal{D}(S, B)$ -maximal states is Borel measurable, as shown in Lemma 3.1.1. Recall that  $\Sigma_\Omega$  denotes the set of states  $\varphi$  on  $B$  for which there exists a Borel probability measure concentrated on  $\Omega$  such that  $\varphi = \int_\Omega \omega d\mu(\omega)$ .

We now give a generalization of [10, Theorem 3.2].

**Corollary 3.3.1.** *Let  $B$  be a separable unital  $C^*$ -algebra and let  $S \subset B$  be an operator system such that  $B = C^*(S)$ . Any state  $\varphi \in \Sigma_\Omega$  satisfies  $\varphi(q) = 0$  if  $q \in B^{**}$  is a closed projection orthogonal to the boundary projection  $\mathfrak{d}$ .*

*Proof.* Fix a closed projection  $q \in B^{**}$  orthogonal to  $\mathfrak{d}$ . Hence  $J = B^{**}(I - q)$  is a weak-\* closed left ideal in  $B^{**}$  and moreover,  $J$  is the weak-\* closure of  $J \cap B$  [12, Theorem 3.3]. If  $(a_\lambda)$  is any approximate unit of the left ideal  $J \cap B$ , then by virtue of [13, Proposition 2.5.8],  $(I - q)$  is the weak-\* limit of  $(a_\lambda)$ . Since  $B$  is separable, one can choose a countable approximate unit  $(a_n)$  in  $J \cap B$ . So  $(I - q)$  is the increasing weak-\* limit of  $(a_n)$ . Let for all  $n \in \mathbb{N}$ ,  $b_n = I - a_n$ . Then each  $a_n$  is contractive, as  $0 \leq a_n \leq I$ . Moreover  $(b_n)$  is a decreasing sequence that converges to  $q$  in the weak-\* topology. This concludes that there is a decreasing contractive sequence  $(b_n)$  converging to  $q$  in the weak-\* topology of  $B^{**}$ .

Let  $\varphi$  be a state in  $\Sigma_\Omega$ . We can find a Borel probability measure  $\mu$  concentrated on the set  $\Omega$  such that

$$\varphi(b) = \int_\Omega \omega(b) d\mu(\omega), \quad b \in B.$$

Next, note that  $\omega(q) \leq \omega(I - \mathfrak{d}) = 0$  for every  $\omega \in \Omega$  by Theorem 3.3.3. By the dominated convergence theorem, we find

$$\varphi(q) = \lim_{n \rightarrow \infty} \int_\Omega \omega(b_n) d\mu(\omega) = \int_\Omega \omega(q) d\mu(\omega) = 0$$

as desired. □

**Remark 3.3.1.** As noted in Corollary 3.2.1, we aim to determine whether the dilation order is hyperrigid. Using the boundary projection from Theorem 3.3.3, dilation order is hyperrigid precisely when  $\varphi(\mathfrak{d}) = 1$  for any  $\varphi \in \Sigma_\Omega$ . Corollary 3.3.1 provides a similar, though slightly weaker, statement. In essence, it implies that each state  $\varphi \in \Sigma_\Omega$  is *concentrated* on the boundary projection  $\mathfrak{d}$ . This conclusion serves as a non-commutative counterpart to the classical concept of a measure being concentrated on a potentially non-measurable set; see [10, Theorem 3.4] and [27, Section 9].

Finally, as an application of Corollary 3.3.1, we show how hyperrigidity can be reformulated in non-commutative topological terms using the boundary projection.

**Corollary 3.3.2.** *Let  $B$  be a separable unital  $C^*$ -algebra and let  $S \subset B$  be an operator system such that  $B = C^*(S)$ . Assume that every pure state on  $B$  is  $\mathcal{D}(S, B)$ -maximal. Then, the following statements are equivalent.*

- (i) *The operator system  $S$  is hyperrigid in  $B$ .*
- (ii) *The boundary projection  $\mathfrak{d}$  is closed.*
- (iii) *The boundary projection  $\mathfrak{d}$  is the infimum of a collection of open projections in  $B^{**}$ .*

*Proof.* (i)  $\Rightarrow$  (ii): In this case, all states on  $B$  are  $\mathcal{D}(S, B)$ -maximal, so that  $\mathfrak{d} = I$  by Theorem 3.3.3, which is then trivially closed.

(ii)  $\Rightarrow$  (iii): This follows from [35, Proposition 2.3].

(iii)  $\Rightarrow$  (i): Assume that there is a collection  $\mathcal{U} \subset B^{**}$  of open projections such that  $\mathfrak{d} = \wedge\{u : u \in \mathcal{U}\}$ . Fix  $u \in \mathcal{U}$ . Then,  $I - u$  is a closed projection orthogonal to  $\mathfrak{d}$ . Since every pure state is  $\mathcal{D}(S, B)$ -maximal, every state on  $B$  lies in  $\Sigma_\Omega$  by [10, Theorem 4.2]. Hence, applying Corollary 3.3.1, we see that  $\varphi(I - u) = 0$  for every state  $\varphi$  on  $B$ , and consequently  $u = I$ . It follows that  $\mathfrak{d} = I$ .  $\square$

# 4

## Strong dilation and Sub-division relation

So far, we have explored the role of dilation order in detecting the unique extension property for an operator system in a  $C^*$ -algebra. In [27], Davidson and Kennedy made a connection between the dilation order and the classical Choquet order defined on the regular Borel probability measures on a compact convex set. Furthermore, they provided an alternate dilation-type description of the Choquet order, which enabled them to extend Cartier's theorem to the non-metrizable case.

The classical Choquet order is equivalent to the strong dilation order and the sub-division order defined on the regular Borel probability measures on a compact convex set. We can naturally extend the definitions of the latter two to define relations on the state space of more general  $C^*$ -algebras. In this section, we define the strong dilation relation and sub-division relations; we study the connection between the two and analyze certain properties of these two relations. By analogy with the classical commutative case, we hope that these new relations can eventually serve as meaningful tools in studying the structure of general operator systems. For instance, in Theorem 4.0.1 we have established a connection between the strong

dilation relation and the *tight extension property* of cyclic  $*$ -representations, which is at the heart of the new amended version of Arveson's hyperrigidity principle.

For a subset  $A$  of  $B(\mathcal{H})$ ,  $A'$  denotes the set of all elements of  $B(\mathcal{H})$  which commute with all elements of  $A$ . Moreover,  $A'' = (A')'$ . From the double commutant theorem it is known that if  $A$  is a  $C^*$ -algebra, then  $A$  is a weak- $*$  dense subset of  $A''$  [44].

**Definition 4.0.1.** Let  $S$  be an operator system contained in the  $C^*$ -algebra  $B$ . Let  $\varphi, \psi \in \mathcal{E}(B)$  and let  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  be the GNS representation of  $\varphi$ . We say that  $\psi$  is a **strong dilation** of  $\varphi$  if there exists a ucp map  $\Psi : B \rightarrow \pi_\varphi(B)''$  such that

- (i)  $\Psi(s) = \pi_\varphi(s)$  for all  $s \in S$ ,
- (i)  $\psi(b) = \langle \Psi(b)\xi_\varphi, \xi_\varphi \rangle$  for all  $b \in B$ .

Let  $\text{StD}(S, B)$  be the subset of  $\mathcal{E}(B) \times \mathcal{E}(B)$  containing all pairs of states  $(\varphi, \psi)$  such that  $\psi$  is a strong dilation of  $\varphi$ .

The reader should compare this definition with [28, Definition 3.1].

**Remark 4.0.1.** Let  $X$  be a compact convex set and  $\mu$  be a regular Borel probability measure on  $X$ . Let  $(\pi_\mu, L^2(\mu), \xi_\mu)$  be the GNS representation of  $\mu$ . Then  $\pi_\mu(C(X))''$  can be naturally identified with  $L^\infty(\mu)$  [25]. A measure  $\mu$  is dominated by  $\nu$  in the strong dilation order (in the definition of Davidson and Kennedy[28]) if there exists a ucp map  $\Psi : C(X) \rightarrow L^\infty(\mu)$  such that  $\pi_\mu(a) = \Psi(a)$  for all  $a \in A(X)$  and  $\nu(f) = \langle \Psi(f)\xi_\mu, \xi_\mu \rangle$  for all  $f \in C(X)$ . Hence when we restrict our attention to this classical set up, the above definition of strong dilation relation coincides with strong dilation order on the regular Borel probability measures on  $X$  as defined in [28].

Let  $\varphi$  be a state of  $B$ . A finite subset  $\{\varphi_1, \dots, \varphi_n\}$  of  $\mathcal{E}(B)$  is said to be a **sub-division** of  $\varphi$  if  $\varphi$  is a convex combination of  $\{\varphi_1, \dots, \varphi_n\}$ , i.e. if there exist scalars  $\lambda_i \in [0, 1]$  such that  $\sum_{i=1}^n \lambda_i = 1$  and  $\sum_{i=1}^n \lambda_i \varphi_i = \varphi$ .

**Definition 4.0.2.** Let  $S$  be an operator system contained in the  $C^*$ -algebra  $B$  and  $\varphi, \psi \in \mathcal{E}(B)$ . Let  $\text{SubD}(S, B) \subset \mathcal{E}(B) \times \mathcal{E}(B)$  be the set of all pairs of states  $(\varphi, \psi)$  such that whenever  $\{\varphi_1, \dots, \varphi_n\}$  is a sub-division of  $\varphi$  of the form:

$$\varphi = \sum_{i=1}^n \lambda_i \varphi_i$$

then there exists a corresponding sub-division  $\{\psi_1, \dots, \psi_n\}$  of  $\psi$  such that  $(\varphi_i, \psi_i) \in \mathcal{D}(S, B)$  for all  $i$  and

$$\psi = \sum_{i=1}^n \lambda_i \psi_i.$$

**Remark 4.0.2.** Note that the previous definition is not a straightforward generalization of the sub-division order defined on the regular Borel probability measures on a compact convex set  $X$  (as defined in [49, Chapter 15]). Classically the notion of the subdivision order was introduced by Cartier-Fell-Meyer which appeared in [16]. Indeed, in the present definition of the sub-division relation, for each  $i$ , one requires  $\nu_i$  such that  $(\mu_i, \nu_i) \in \mathcal{D}(S, B)$ , which is not the same as  $\mu_i \sim \nu_i$ , where we write  $\mu \sim \nu$  if  $\mu(a) = \nu(a)$  for all continuous affine functions on  $X$ . However, we will give arguments to show that, when restricted to the classical setup, this new sub-division relation is equivalent to the classical sub-division relation.

**Remark 4.0.3.** It can be seen easily that both the strong dilation relation and the sub-division relation are antisymmetric, i.e, if  $(\varphi, \psi), (\psi, \varphi) \in \text{StD}(S, B)$  then  $\varphi = \psi$  and if  $(\varphi, \psi), (\psi, \varphi) \in \text{SubD}(S, B)$  then  $\varphi = \psi$ . We do not know at present whether the strong dilation relation is transitive or not. However, the sub-division relation is transitive. Let  $(\varphi, \psi)$  and  $(\psi, \gamma)$  be elements of  $\text{SubD}(S, B)$ . Let  $\{\varphi_1, \dots, \varphi_n\}$  be a sub-division of  $\varphi$ . Then there exists a corresponding sub-division  $\{\psi_1, \dots, \psi_n\}$  of  $\psi$  such that  $(\varphi_i, \psi_i) \in \mathcal{D}(S, B)$  for all  $i$ . Since  $(\psi, \gamma) \in \text{SubD}(S, B)$ , there exists a sub-division  $\{\gamma_1, \dots, \gamma_n\}$  such that  $(\psi_i, \gamma_i) \in \mathcal{D}(S, B)$  for all  $i$ . From transitivity of  $\mathcal{D}(S, B)$ , we have  $(\varphi_i, \gamma_i) \in \mathcal{D}(S, B)$ . Hence,  $(\varphi, \gamma) \in \text{SubD}(S, B)$  and the sub-

division relation is transitive.

The following is the first main result of this section.

**Theorem 4.0.1.** *Let  $S$  be an operator system contained in a  $C^*$ -algebra  $B$  such that  $C^*(S) = B$ . Then  $\text{StD}(S, B) \subset \text{SubD}(S, B)$ .*

*Proof.* Let  $\varphi$  and  $\psi$  be two states on  $B$  such that  $(\varphi, \psi) \in \text{StD}(S, B)$ . Let  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  be the GNS representation of  $\varphi$ . Then there exists a ucp map  $\Psi : B \rightarrow \pi_\varphi(B)''$  such that  $\Psi(s) = \pi_\varphi(s)$  for all  $s \in S$  and  $\psi(b) = \langle \Psi(b)\xi_\varphi, \xi_\varphi \rangle$  for all  $b \in B$ . Let  $\{\varphi_1, \dots, \varphi_n\}$  be a sub-division of  $\varphi$  such that

$$\varphi = \sum_{i=1}^n \lambda_i \varphi_i.$$

Then in particular each  $\varphi_i$  is absolutely continuous with respect to  $\varphi$ , i.e.  $\varphi_i \ll \varphi$ . So from Proposition 3.3.1, for each  $i$  there exists a sequence of unit vectors  $\{\eta_{i,n} : n \in \mathbb{N}\} \subset \mathcal{H}_\varphi$  and a sequence of positive scalars  $\{t_{i,n} : n \in \mathbb{N}\}$  such that  $\sum_{n \in \mathbb{N}} t_{i,n} = 1$  and,

$$\varphi_i(b) = \sum_{n \in \mathbb{N}} t_{i,n} \langle \pi_\varphi(b)\eta_{i,n}, \eta_{i,n} \rangle \quad \forall b \in B. \quad (4.0.1)$$

For each  $i$ , let  $\gamma_i : B(\mathcal{H}_\varphi) \rightarrow \mathbb{C}$  be defined as

$$\gamma_i(T) = \sum_{n \in \mathbb{N}} t_{i,n} \langle T\eta_{i,n}, \eta_{i,n} \rangle \quad \forall T \in B(\mathcal{H}_\varphi).$$

Then  $\gamma_i$  is a norm-continuous unital positive linear functional on  $B(\mathcal{H}_\varphi)$ . Note that

$$\gamma_i(\pi_\varphi(b)) = \varphi_i(b) \quad \forall b \in B, \quad (4.0.2)$$

and for all  $i$ . Hence, for all  $b \in B$ , we have

$$\sum_{i=1}^n \lambda_i \gamma_i(\pi_\varphi(b)) = \sum_{i=1}^n \lambda_i \varphi_i(b) = \varphi(b) = \langle \pi_\varphi(b)\xi_\varphi, \xi_\varphi \rangle.$$

This implies that  $\sum_{i=1}^n \lambda_i \gamma_i(T) = \langle T \xi_\varphi, \xi_\varphi \rangle$  for all  $T \in \pi_\varphi(B)$ . But from the bicommutant theorem  $\pi_\varphi(B)$  is a dense subset of  $\pi_\varphi(B)''$  in the weak operator topology. Moreover each  $\gamma_i$  is continuous in weak operator topology by [11, Theorem 2.1.4]. Hence

$$\sum_{i=1}^n \lambda_i \gamma_i(T) = \langle T \xi_\varphi, \xi_\varphi \rangle, \quad \forall T \in \pi_\varphi(B)'' . \quad (4.0.3)$$

For all  $i$ , define  $\psi_i : B \rightarrow \mathbb{C}$  by

$$\psi_i(b) = \gamma_i(\Psi(b)), \quad \forall b \in B.$$

Then  $\psi_i$  is a state on  $B$  for all  $i$ . Since  $\Psi(b) \in \pi_\varphi(B)''$  for all  $b \in B$ , using Equation 4.0.3 we have:

$$\sum_{i=1}^n \lambda_i \psi_i(b) = \sum_{i=1}^n \lambda_i \gamma_i(\Psi(b)) = \langle \Psi(b) \xi_\varphi, \xi_\varphi \rangle = \psi(b), \quad \forall b \in B.$$

Hence,  $\sum_{i=1}^n \lambda_i \psi_i = \psi$ . It thus only remains to show that  $(\phi_i, \psi_i) \in \mathcal{D}(S, B)$ .

Let  $\pi_\varphi^{(\infty)} : B \rightarrow B(\mathcal{H}_\varphi^{(\infty)})$  be defined as  $\pi_\varphi^{(\infty)} = \bigoplus_{n \in \mathbb{N}} \pi_\varphi$  where  $\mathcal{H}_\varphi^{(\infty)}$  is the countable direct sum of copies of  $\mathcal{H}_\varphi$ . For each  $i$ , set  $\xi_i = \bigoplus_{n \in \mathbb{N}} \sqrt{t_{i,n}} \eta_{i,n}$ . Then  $\xi_i$  is a unit vector in  $\mathcal{H}_\varphi^{(\infty)}$ . From (4.0.1) we have that  $(\pi_\varphi^{(\infty)}, \mathcal{H}_\varphi^{(\infty)}, \xi_i)$  is a representation of  $\varphi_i$  for each  $i$ . For each  $i$ , define  $\Psi^{(\infty)} : B \rightarrow B(\mathcal{H}_\varphi^{(\infty)})$  by  $\Psi^{(\infty)} = \bigoplus_{n \in \mathbb{N}} \Psi$ . Then observe that:

$$\langle \Psi^{(\infty)}(b) \xi_i, \xi_i \rangle = \sum_{n \in \mathbb{N}} t_{i,n} \langle \Psi(b) \eta_{i,n}, \eta_{i,n} \rangle = \gamma_i(\Psi(b)) = \psi_i(b).$$

To summarize:  $(\pi_\varphi^{(\infty)}, \mathcal{H}_\varphi^{(\infty)}, \xi_i)$  is a representation of  $\varphi_i$  and there exists a UCP map  $\Psi^{(\infty)}$  such that,

- (i)  $\pi_\varphi^{(\infty)}(s) = \Psi^{(\infty)}(s)$  for all  $s \in S$ ,
- (ii)  $\langle \Psi^{(\infty)}(b) \xi_i, \xi_i \rangle = \psi_i(b)$  for all  $b \in B$ .

It follows from Theorem 3.2.1 that  $(\varphi_i, \psi_i) \in \mathcal{D}(S, B)$ . Hence, we have a corresponding sub-division of  $\psi$  such that the required conditions of Definition 4.0.2 are satisfied. Hence  $(\varphi, \psi) \in \text{SubD}(S, B)$  and  $\text{StD}(S, B) \subset \text{SubD}(S, B)$ .  $\square$

Note that the proof doesn't require commutativity of  $B$ . Next we will show that the other containment also holds when the  $C^*$ -algebra  $B$  is commutative.

**Lemma 4.0.1.** *Let  $S$  be an operator system contained in a  $C^*$ -algebra  $B$  such that  $B = C^*(S)$  and  $\varphi, \psi \in \mathcal{E}(B)$ . Let  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  be the GNS representation of  $\varphi$  and  $\mathcal{F}$  be a finite set of commuting projections in  $\pi_\varphi(B)'$ . If  $(\varphi, \psi) \in \text{SubD}(S, B)$ , then there exists a ucp map  $\Psi_{\mathcal{F}} : B \rightarrow B(\mathcal{H}_\varphi)$  such that*

- (i)  $\Psi_{\mathcal{F}}(s) = \pi_\varphi(s)$  for all  $s \in S$ ,
- (ii)  $\psi(b) = \langle \Psi_{\mathcal{F}}(b)\xi_\varphi, \xi_\varphi \rangle$  for all  $b \in B$ ,
- (iii)  $\Psi_{\mathcal{F}}(b)P = P\Psi_{\mathcal{F}}(b)$  for all  $b \in B$  and for all  $P \in \mathcal{F}$ .

*Proof.* Let  $\mathcal{F} = \{P_1, \dots, P_n\}$  and let  $\mathcal{G}$  be the subset of  $\pi_\varphi(B)'$  containing all nonzero elements of the form  $X_1 X_2 \dots X_n$  where for all  $1 \leq i \leq n$ ,  $X_i$  is either  $P_i$  or  $1 - P_i$ . Note that every element of  $\mathcal{G}$  is again a nonzero projection. Moreover,  $\mathcal{G}$  is a finite subset of  $\pi_\varphi(B)'$  and we write  $\mathcal{G} = \{Q_1, \dots, Q_k\}$  for some finite  $k$ . It is easy to see that whenever  $h, g \in \mathcal{H}_\varphi$  and  $l \neq r$  then  $\langle Q_l h, Q_r g \rangle = 0$  and  $\sum_{1 \leq j \leq k} Q_j = I$ .

For all  $1 \leq j \leq k$ , and for all  $b \in B$ ,

$$\pi_\varphi(b)Q_j\xi_\varphi = Q_j\pi_\varphi(b)\xi_\varphi.$$

Hence  $\text{range}(Q_j)$  is a reducing subspace of  $\pi_\varphi$  and  $\{Q_j\pi_\varphi(b)\xi_\varphi : b \in B\}$  is dense in the range of  $Q_j$ , so that  $Q_j\xi_\varphi$  is a non zero vector. Let  $\pi_j$  be the restriction of  $\pi_\varphi$  on the reducing subspace  $\mathcal{H}_j = \text{range}(Q_j)$ . For each  $1 \leq j \leq k$ , let  $\lambda_j = \|Q_j\xi_\varphi\|^2$  and

$\varphi_j : B \longrightarrow \mathbb{C}$  be defined as,

$$\varphi_j(b) = \frac{1}{\lambda_j} \langle \pi_j(b) Q_j \xi_\varphi, Q_j \xi_\varphi \rangle \quad \forall b \in B.$$

Then it is clear that  $\varphi_j$  is a state on  $B$  for each  $j$  and  $(\pi_j, \mathcal{H}_j, \frac{1}{\sqrt{\lambda_j}} Q_j \xi_\varphi)$  is a representation of  $\varphi_j$ . Note that each  $\lambda_j \in [0, 1]$  and for all  $b$ ,

$$\begin{aligned} \sum_{1 \leq j \leq k} \lambda_j \varphi_j(b) &= \sum_{1 \leq j \leq k} \langle \pi_j(b) Q_j \xi_\varphi, Q_j \xi_\varphi \rangle \\ &= \sum_{1 \leq j \leq k} \langle \pi_\varphi(b) Q_j \xi_\varphi, Q_j \xi_\varphi \rangle \\ &= \langle \pi_\varphi(b) \sum_{1 \leq j \leq k} Q_j \xi_\varphi, \sum_{1 \leq j \leq k} Q_j \xi_\varphi \rangle \\ &= \langle \pi_\varphi(b) \xi_\varphi, \xi_\varphi \rangle \\ &= \varphi(b). \end{aligned}$$

Hence  $\sum_{1 \leq j \leq k} \lambda_j \varphi_j = \varphi$  and this means  $\{\varphi_j : 1 \leq j \leq k\}$  is a sub-division of  $\varphi$ . Since  $(\varphi, \psi) \in \text{SubD}(S, B)$ , there exists a corresponding sub-division  $\{\psi_j : 1 \leq j \leq k\}$  of  $\psi$  such that  $\sum_{1 \leq j \leq k} \lambda_j \psi_j = \psi$  and  $(\varphi_j, \psi_j) \in \mathcal{D}(S, B)$  for each  $j$ . Since,  $(\pi_j, \mathcal{H}_j, \frac{1}{\sqrt{\lambda_j}} Q_j \xi_\varphi)$  is a representation of  $\varphi_j$  for all  $j$ , by Theorem 3.2.1, there exists a ucp map  $\Psi_j : B \longrightarrow B(\mathcal{H}_j)$  such that  $\pi_j(s) = \Psi_j(s)$  for all  $s \in S$ , and  $\psi_j(b) = \frac{1}{\lambda_j} \langle \Psi_j(b) Q_j \xi_\varphi, Q_j \xi_\varphi \rangle$  for all  $b \in B$ . Now define a ucp map  $\Psi_{\mathcal{F}} : B \longrightarrow B(\mathcal{H}_\varphi)$  as  $\Psi_{\mathcal{F}}(b) = \bigoplus_{j=1}^k \Psi_j(b)$ .  $\Psi_{\mathcal{F}}$  satisfies (i), because

$$\Psi(s) = \bigoplus_{j=1}^k \Psi_j(s) = \bigoplus_{j=1}^k \pi_j(s) = \pi_\varphi(s) \quad \forall s \in S.$$

$\Psi_{\mathcal{F}}$  satisfies (ii), because

$$\langle \Psi_{\mathcal{F}}(b) \xi_\varphi, \xi_\varphi \rangle = \sum_{j=1}^k \langle \Psi_j(b) Q_j \xi_\varphi, Q_j \xi_\varphi \rangle$$

$$\begin{aligned}
&= \sum_{j=1}^k \lambda_j \psi_j(b) \\
&= \psi(b).
\end{aligned}$$

For  $P_i \in \mathcal{F}$ , let  $\mathcal{G}_i$  be the subset of  $\mathcal{G}$  containing all elements of  $\mathcal{G}$  of the form  $X_1 X_2 \dots X_n$  such that  $X_i = P_i$ . Then, note that  $P_j = \sum_{Q \in \mathcal{G}_j} Q$ . Since the elements of  $\mathcal{G}$  are mutually orthogonal projections and the range of each  $Q \in \mathcal{G}$  is a reducing subspace of  $\Psi_{\mathcal{F}}$ , we can conclude that  $\text{range}(P_j)$  is a reducing subspace of  $\Psi_{\mathcal{F}}$ . Hence  $\Psi_{\mathcal{F}}(b)P = P\Psi_{\mathcal{F}}(b)$  for all  $b \in B$  and for all  $P \in \mathcal{F}$ . This shows that  $\Psi_{\mathcal{F}}$  satisfies (iii).  $\square$

Using this lemma, we now prove the converse of Theorem 4.0.1 in the commutative setting.

**Theorem 4.0.2.** *Let  $S$  be an operator system contained in a commutative  $C^*$ -algebra  $B$  such that  $C^*(S) = B$ . Then  $\text{StD}(S, B) = \text{SubD}(S, B)$ .*

*Proof.* We have shown that the containment  $\text{StD}(S, B) \subset \text{SubD}(S, B)$  does not require commutativity. The proof will be completed if we can show the  $\text{SubD}(S, B) \subset \text{StD}(S, B)$ .

Let  $\varphi$  and  $\psi$  be states on  $B$  such that  $(\varphi, \psi) \in \text{SubD}(S, B)$ . Since  $B$  is assumed to be commutative,  $\pi_{\varphi}(B)' = \pi_{\varphi}(B)''$  [44, Example 4.1.2]. Hence  $\pi(B)'$  is also commutative. Let  $\Lambda$  be the collection of all finite subsets of the projections in  $\pi_{\varphi}(B)'$  along with an order  $\leq$  defined by containment, i.e. for  $\mathcal{F}_1, \mathcal{F}_2 \in \Lambda$ ,  $\mathcal{F}_1 \leq \mathcal{F}_2$  if and only if  $\mathcal{F}_1 \subset \mathcal{F}_2$ . Therefore  $(\Lambda, \leq)$  is a directed set.

By the previous lemma, for each  $\mathcal{F} \in \Lambda$ , there exists a ucp map  $\Psi_{\mathcal{F}} : B \rightarrow B(\mathcal{H}_{\varphi})$  such that

- (i)  $\Psi_{\mathcal{F}}(s) = \pi_{\varphi}(s)$  for all  $s \in S$ ,
- (ii)  $\psi(b) = \langle \Psi_{\mathcal{F}}(b)\xi_{\varphi}, \xi_{\varphi} \rangle$  for all  $b \in B$ ,

(iii)  $\Psi_{\mathcal{F}}(b)P = P\Psi_{\mathcal{F}}(b)$  for all  $b \in B$  and for all  $P \in \mathcal{F}$ .

Now consider the net  $\{\Psi_{\mathcal{F}}\}_{\mathcal{F} \in \Lambda}$  in  $\text{UCP}(B, B(\mathcal{H}_{\varphi}))$ . Since  $\text{UCP}(B, B(\mathcal{H}_{\varphi}))$  is compact in the point-weak-\* topology [45, Theorem 7.4], there exists a cofinal subnet  $\{\Psi_{\mathcal{F}}\}_{\mathcal{F} \in \Lambda_0}$  converging to a ucp map  $\Psi \in \text{UCP}(B, B(\mathcal{H}_{\varphi}))$ . It is easy to verify that since  $\Psi$  is the point-weak-\* limit of  $\{\Psi_{\mathcal{F}}\}_{\mathcal{F} \in \Lambda_0}$ ,  $\Psi$  satisfies (i) and (ii).

Let  $P$  be any projection in  $\pi_{\varphi}(B)'$  and take  $\Lambda_1 = \{\mathcal{F} \in \Lambda_0 : \mathcal{F} \geq \{P\}\}$ . Then  $\Lambda_1$  is again a cofinal subnet of  $\Lambda_0$ . For each  $b \in B$  and  $v, w \in \mathcal{H}_{\varphi}$  we get

$$\begin{aligned} \langle \Psi(b)Pv, w \rangle &= \lim_{\mathcal{F} \in \Lambda_1} \langle \Psi_{\mathcal{F}}(b)Pv, w \rangle \\ &= \lim_{\mathcal{F} \in \Lambda_1} \langle P\Psi_{\mathcal{F}}(b)v, w \rangle \\ &= \lim_{\mathcal{F} \in \Lambda_1} \langle \Psi_{\mathcal{F}}(b)v, Pw \rangle \\ &= \langle \Psi(b)v, Pw \rangle \\ &= \langle P\Psi(b)v, w \rangle \end{aligned}$$

This implies that  $P\Psi(b) = \Psi(b)P$  for all  $b \in B$  and for all projections  $P$  in  $\pi_{\varphi}(B)'$ . Note that  $\pi_{\varphi}(B)'$  is a \*-subalgebra of  $B(\mathcal{H}_{\varphi})$  which is closed in the weak operator topology and hence a von Neumann algebra. Since a von Neumann algebra is generated by its projections [44, Theorem 4.1.11], we infer that  $\pi_{\varphi}(B)'$  is generated by its projections. Therefore  $\Psi(b) \in (\pi_{\varphi}(B)')'$ . In other words  $\Psi : B \rightarrow \pi_{\varphi}(B)''$ . We have already seen that  $\Psi$  satisfies (i) and (ii). Hence we have  $(\varphi, \psi) \in \text{StD}(S, B)$ .  $\square$

The next result shows that the strong dilation relation and the sub-division relation are same as the Choquet order when restricted to the classical commutative case.

**Corollary 4.0.1.** *Let  $X$  be a compact convex subset of a locally convex topological vector space and let  $\mu, \nu \in P(X)$ . Then the following are equivalent.*

(i)  $\mu(f) \leq \nu(f)$  for all convex function  $f \in C(X)$ .

(ii)  $(\mu, \nu) \in \text{StD}(A(X), C(X))$ .

(iii)  $(\mu, \nu) \in \text{SubD}(A(X), C(X))$ .

*Proof.* (i)  $\iff$  (ii) because of [28, Theorem 3.7] and (ii)  $\iff$  (iii) because of Theorem 4.0.2.  $\square$

Note that we have used the fact that the projections of  $\pi_\varphi(B)'$  commute with each other to prove Theorem 4.0.2. The equality of these two relations is unknown for the non-commutative setup. However, we will now see that the strong dilation relation is equivalent to a different version of the sub-division relation, and we call it the non-commutative sub-division relation or the nc sub-division relation.

**Definition 4.0.3.** Let  $S$  be an operator system contained in the  $C^*$ -algebra  $B$  such that  $C^*(S) = B$ . Let  $\pi : B \rightarrow B(\mathcal{H})$  and  $\sigma : B \rightarrow B(\mathcal{K})$  be two  $*$ -representations. Then we say that  $\sigma$  is a **strong dilation** of  $\pi$  via the isometry  $V : \mathcal{H} \rightarrow \mathcal{K}$  if whenever  $\pi = \pi_1 \oplus \pi_2$ , where  $\pi_i : B \rightarrow B(\mathcal{H}_i)$  with  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ ; there exists  $\mathcal{K}_i \subset \mathcal{K}$  and  $\sigma_i : B \rightarrow B(\mathcal{K}_i)$  such that:

(i)  $V(\mathcal{H}_i) \subset \mathcal{K}_i$  for  $i = 1, 2$ ,

(ii)  $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$  and  $\sigma = \sigma_1 \oplus \sigma_2$ ,

(iii)  $\pi_i$  is dilated by  $\sigma_i$  for  $i = 1, 2$  via the isometry  $V|_{\mathcal{H}_i} : \mathcal{H}_i \rightarrow \mathcal{K}_i$ .

We define the subset  $\text{ncSubD}(S, B) \subset \mathcal{E}(B) \times \mathcal{E}(B)$  consisting of all pairs of states  $(\varphi, \psi)$  such that there exists a representation of  $\psi$  which is a strong dilation of the GNS representation of  $\varphi$ .

**Lemma 4.0.2.** *Let  $B$  be a unital  $C^*$ -algebra and  $\pi : B \rightarrow B(\mathcal{K})$  be a unital  $*$ -representation of  $B$ . Let  $V : \mathcal{H} \rightarrow \mathcal{K}$  be an isometry and  $\Psi : B \rightarrow B(\mathcal{H})$  be defined as  $\Psi(b) = V^*\pi(b)V$  for all  $b \in B$ . Then for all projections  $P \in \Psi(B)'$ , there exists a projection  $Q \in \pi(B)'$  such that  $QV = VP$ .*

*Proof.* First, we assume that the space  $\{\pi(b)Vh : b \in B, h \in \mathcal{H}\}$  is dense in  $\mathcal{K}$ . Note that  $\pi(B)$  is a  $C^*$ -subalgebra of  $B(\mathcal{K})$  and  $\Psi(B) = V^*\pi(B)V$ . Hence, by applying Arveson's commutant lifting theorem [45, Theorem 12.7], we have that for every projection  $P \in \Psi(B)'$ , there exists a projection  $Q \in \pi(B)'$  such that  $QV = VP$ .

When  $\{\pi(b)Vh : b \in B, h \in \mathcal{H}\}$  is not dense in  $\mathcal{K}$ , then let  $\mathcal{K}_1$  be the closure of  $\{\pi(b)Vh : b \in B, h \in \mathcal{H}\}$ . Then  $\mathcal{K}_1$  is a reducing subspace of  $\pi$  and let  $\pi_1$  be the restriction of  $\pi$  to  $\mathcal{K}_1$ . Then  $V^*\pi_1(b)V = \Psi(b)$  for all  $b$ . By the previous paragraph, there exists  $Q \in \pi_1(B)' \subset B(\mathcal{K}_1)$  such that  $QV = VP$ . Now viewing  $Q$  as a projection in  $B(\mathcal{K})$  by setting  $Q(k) = 0$  for all  $k \in \mathcal{K}_1^\perp$ , we have the result.  $\square$

The following is the second main result of this section.

**Theorem 4.0.3.** *Let  $S$  be an operator system contained in a  $C^*$ -algebra  $B$  such that  $C^*(S) = B$ . Then  $\text{StD}(S, B) = \text{ncSubD}(S, B)$ .*

*Proof.* Let  $\varphi, \psi$  be states on  $B$  such that  $(\varphi, \psi) \in \text{StD}(S, B)$ . Therefore, there exists a ucp map  $\Psi : B \rightarrow \pi_\varphi(B)''$  such that (i) and (ii) of Definition 4.0.1 are satisfied. Let  $\sigma : B \rightarrow B(\mathcal{K})$  be the minimal Stinespring dilation of  $\Psi$  along with the isometry  $V : \mathcal{H}_\varphi \rightarrow \mathcal{K}$  such that

$$\Psi(b) = V^*\sigma(b)V \quad \forall b \in B.$$

Let  $\pi_\varphi = \pi_1 \oplus \pi_2$  where the direct sum is taken corresponding to some reducing subspaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$  such that  $\mathcal{H}_\varphi = \mathcal{H}_1 \oplus \mathcal{H}_2$ . Let  $P$  be the projection onto  $\mathcal{H}_1$ . Thus  $P \in \pi_\varphi(B)'$ . By Lemma 4.0.2, there exists projection  $Q$  in  $\sigma(B)'$  such that  $QV = VP$ . Let  $\mathcal{K}_1 = \text{range}(Q)$  and  $\mathcal{K}_2 = \mathcal{K}_1^\perp$ . Thus  $\mathcal{K}_i$  is a reducing subspace of  $\sigma$ . Define  $\sigma_1$  as the restriction of  $\sigma$  on  $\mathcal{K}_1$  and  $\sigma_2$  as restriction of  $\sigma$  on  $\mathcal{K}_2$ . Note that  $VP_h = QVh$  for all  $h \in \mathcal{H}_\varphi$ , hence  $V(\mathcal{H}_1) \subset \mathcal{K}_1$ . Moreover,

$$V(I - P) = V - VP = V - QV = (I - Q)V.$$

Therefore  $V(\mathcal{H}_2) \subset \mathcal{K}_2$ . Note that  $V_i = V_{|\mathcal{H}_i} : \mathcal{H}_i \rightarrow \mathcal{K}_i$  is an isometry and it is routine to check that  $V_i^* \sigma_i(s) V_i = \pi_i(s)$  for all  $s \in S$ . Hence  $\sigma$  is a strong dilation of  $\pi$  via the isometry  $V : \mathcal{H} \rightarrow \mathcal{K}$  and thus  $(\varphi, \psi) \in \text{ncSubD}(S, B)$ .

Conversely, let  $(\varphi, \psi) \in \text{ncSubD}(S, B)$ . Let  $\sigma$  be a representation of  $\psi$  that is a strong dilation of  $\pi_\varphi$ . Define  $\Psi : B \rightarrow B(\mathcal{H}_\varphi)$  as  $\Psi(b) = V^* \sigma(b) V$  for all  $b \in B$ . Let  $P \in \pi(B)'$ . Hence  $\pi_\varphi$  reduces as  $\pi_\varphi = \pi_1 \oplus \pi_2$  with respect to the reducing subspaces  $\mathcal{H}_1 = \text{range}(P)$  and  $\mathcal{H}_2 = \text{range}(I - P)$ . Since  $\sigma$  is a strong dilation of  $\pi_\varphi$  via  $V$  there exist subspaces  $\mathcal{K}_1$  and  $\mathcal{K}_2$  and  $*$ -representations  $\sigma_i : B \rightarrow B(\mathcal{K}_i)$  such that

- (i)  $V(\mathcal{H}_i) \subset \mathcal{K}_i$  for  $i = 1, 2$ ,
- (ii)  $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$  and  $\sigma = \sigma_1 \oplus \sigma_2$ ,
- (iii)  $\pi_i$  is dilated by  $\sigma_i$  for  $i = 1, 2$  via the isometry  $V_{|\mathcal{H}_i} : \mathcal{H}_i \rightarrow \mathcal{K}_i$ .

Let  $h \in \mathcal{H}_\varphi$ , then:

$$\Psi(b)Ph = V^* \sigma(b) VPh.$$

Note that  $VPh \in \mathcal{K}_1$  and hence  $\sigma(b)VPh \in \mathcal{K}_1$  for all  $b \in B$ , because  $\mathcal{K}_1$  is a reducing subspace of  $\sigma$ . For all  $b \in B$  and  $k \in \mathcal{H}_1^\perp$ ,

$$\langle V^* \sigma(b) VPh, k \rangle = \langle \sigma(b) VPh, Vk \rangle = 0.$$

Hence  $V^* \sigma(b) VPh \in \mathcal{H}_1$ . In other words,  $\mathcal{H}_1$  is a reducing subspace of  $\Psi$ , so that  $\Psi(b)$  commutes with every projection  $P \in \pi_\varphi(B)'$ . But  $\pi_\varphi(B)'$  is a  $*$ -subalgebra of  $B(\mathcal{H}_\varphi)$  which is closed in the weak operator topology, and hence a von Neuman algebra. In the light of [44, Theorem 4.1.11],  $\pi_\varphi(B)'$  is generated by its projections. Hence  $\Psi(b) \in \pi_\varphi(B)''$  for all  $b \in B$ , and we have  $\Psi : B \rightarrow \pi_\varphi(B)''$ . This proves that  $(\varphi, \psi) \in \text{StD}(S, B)$ . □

## Strong dilation relation and the unique tight extension property

The recent counter example by Bilich and Dor-on [9] shows that the hyperrigidity conjecture is not true in general. This means that the maximality of the Choquet boundary does not always guarantee the unique extension property of all  $*$ -representations. This also suggests that the unique extension property of a  $*$ -representation is too strong to be determined from the boundary representations. So a natural question that can be asked is: Can we weaken the unique extension property to make the hyperrigidity conjecture work? This question is addressed in a recent work of Clouâtre and Thompson [22]. The following is a weakening of the unique extension property.

**Definition 4.0.4.** Let  $S$  be an operator system contained in a  $C^*$ -algebra  $B$  such that  $C^*(S) = B$ . Let  $\pi : B \rightarrow B(\mathcal{H})$  be a  $*$ -representation. A unital completely positive map  $\Psi : B \rightarrow B(\mathcal{H})$  is said to be a **tight extension** of  $\pi$  with respect to  $S$  if  $\Psi(s) = \pi(s)$  for all  $s \in S$  and  $\Psi(B) \subset \pi(B)''$ . Moreover,  $\pi$  is said to have the **unique tight extension property** with respect to  $S$  if whenever  $\Psi$  is a tight extension of  $\pi$  with respect to  $S$ , then  $\Psi = \pi$ .

Clouâtre and Thompson have shown that Arveson's hyperrigidity conjecture is valid for the separable case if one replaces the unique extension property by the unique tight extension property.

**Theorem 4.0.4** ([22, Theorem 2.4]). *Let  $S$  be an operator system contained in a unital separable  $C^*$ -algebra  $B$  such that  $C^*(S) = B$ . Then the following are equivalent.*

- (i) *Every irreducible  $*$ -representation of  $B$  is a boundary representation of  $S$ .*
- (ii) *Every unital  $*$ -representation of  $B$  has the unique tight extension property with respect to  $S$ .*

The next result shows that the unique tight extension property is preserved under direct sum.

**Lemma 4.0.3.** *Let  $S$  be an operator system contained in a  $C^*$ -algebra  $B$  such that  $C^*(S) = B$ . Let  $\pi_i : B \longrightarrow B(\mathcal{H}_i)$  be a family of  $*$ -representations. Let  $\mathcal{H} = \bigoplus_i \mathcal{H}_i$  and  $\pi$  be the  $*$ -representation of  $B$  on  $\mathcal{H}$  defined as  $\pi = \bigoplus_i \pi_i$ . If  $\pi_i$  has the unique tight extension property with respect to  $S$  for all  $i$ , then  $\pi$  has the unique tight extension property with respect to  $S$ .*

*Proof.* It is a standard algebraic fact that  $\pi(B)'' \subset \bigoplus_i \pi_i(B)''$ . Let  $\Psi : B \longrightarrow B(\mathcal{H})$  be a unital completely map such that  $\Psi(s) = \pi(s)$  for all  $s \in S$  and  $\Psi(B) \subset \pi(B)''$ . Hence, there exist ucp maps  $\Psi_i : B \longrightarrow \pi_i(B)''$  such that  $\Psi = \bigoplus_i \Psi_i$ . Moreover  $\Psi_i(s) = \pi_i(s)$  for all  $i$  and for all  $s \in S$ . Hence  $\Psi_i = \pi_i$  for all  $i$ , as each  $\pi_i$  has the unique tight extension property with respect to  $S$ . So  $\Psi = \pi$ . This proves that  $\pi$  has the unique tight extension property with respect to  $S$ .  $\square$

Theorem 2.1.5 ensures that every  $*$ -representation  $\pi : B \longrightarrow B(\mathcal{H})$  can be written as  $\pi = \bigoplus_i \pi_i$  where each  $\pi_i : B \longrightarrow B(\mathcal{H}_i)$  is a cyclic representation. If each of these cyclic representations has the unique tight extension property relative to  $S$ , then by the previous theorem  $\pi$  also has the unique tight extension property relative to  $S$ . Moreover, every cyclic representation of a  $C^*$ -algebra is unitarily equivalent to the GNS representation of a state. The next result gives a necessary condition for a cyclic representation to have the unique tight extension property in terms of its corresponding state.

**Proposition 4.0.1.** *Let  $S$  be an operator system contained in a  $C^*$ -algebra  $B$  such that  $C^*(S) = B$  and  $\varphi \in \mathcal{E}(B)$ . Let  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  be the GNS representation of  $\varphi$ . Let  $\pi_\varphi$  have the unique tight extension property relative to  $S$ . Then, for any  $\psi \in \mathcal{E}(B)$  such that  $(\varphi, \psi) \in \text{StD}(S, B)$ , it follows that  $\psi = \varphi$ .*

*Proof.* Let  $\psi \in \mathcal{E}(B)$  such that  $(\varphi, \psi) \in \text{StD}(S, B)$ . Then there exists a ucp map  $\Psi : B \rightarrow \pi_\varphi(B)''$  such that  $\Psi(s) = \pi_\varphi(s)$  and  $\psi(b) = \langle \Psi(b)\xi_\varphi, \xi_\varphi \rangle$ .  $\Psi = \pi$  as  $\pi_\varphi$  has the unique tight extension property relative to  $S$ . Hence  $\psi = \varphi$ .  $\square$

Note that the validity Theorem 4.0.4 without assuming separability is still unknown. Proposition 4.0.1 provides an indication that the techniques of strong dilation relation and the sub-division relation may be helpful in investigating the non-separable case. We do not know the validity of the converse of 4.0.1 at present.

# 5

## Conclusion

In this thesis, we have investigated the complexities of Arveson's hyperrigidity conjecture, offering new insights and advancements. In this section, we will summarize our work, the key findings, and the future directions.

Recall that the maximal elements in the dilation order are crucial for determining the  $*$ -representations with the unique extension property. The primary motivation for this thesis is to understand the structure and properties of these special states. Davidson and Kennedy's equivalent reformulation of the hyperrigidity conjecture in the classical commutative case motivated the question: Does there exist a *boundary set* where the dilation maximal measures are concentrated? We addressed this question without assuming commutativity.

Initially, we explored the structure and properties of more abstract partial orders on the state space of a  $C^*$ -algebra. This investigation yielded many insightful previously unknown results. For instance, when we imposed some mild topological conditions on a partial order, we found that:

- (i) There is an abundance of maximal elements in the partial order (see Proposition 3.1.1).
- (ii) The maximal elements of the partial order always form a Borel measurable set

(see Lemma 3.1.1).

- (iii) The maximal elements lie in the closed convex hull of the pure maximal states 3.1.1.

These results generalize many known classical results about regular Borel probability measures on compact sets to the state space of non-commutative  $C^*$ -algebras. Various partial orders naturally occur in classical Choquet theory (the study of compact convex sets). Therefore, extending these results to the non-commutative setup will be valuable in studying non-commutative Choquet theory. Later we have applied these results for abstract partial orders on our favourite dilation order and shown that the hyperrigidity is equivalent to the maximal elements admitting a boundary.

It is known that if a partial order on the set of regular Borel probability measures is defined by a cone, then the order attains its maximum boundary [3]. In other words, if the cone is nice, then the maximal measures in the partial order determined by the cone are completely characterized by their support. Motivated by Davidson-Kennedy's reformulation of the hyperrigidity conjecture, we asked the question if the dilation order is also determined by a cone of functions in the commutative case. We do not have an answer to this question. However we have found a result that serves this purpose. In this direction we have found that

- (i) There exists a cone of self adjoint elements of a  $C^*$ -algebra such that the partial order determined by the cone has the same set of maximal elements as the dilation order (see 3.2.3).
- (ii) One cannot use the classical techniques of abstract cone orders to produce a boundary set for the dilation order, as we have proved that the cone determining the maximal states is not max-stable.

These results rely heavily on the techniques of non commutative Choquet theory.

Next, we investigated the topological properties of the set of all dilation maximal states. We have found the following important results in this regard:

- (i) the set of all dilation maximal elements is stable under absolute continuity (see Theorem 3.3.2).
- (ii) The set of all dilation maximal states is a norm-closed subset of the state space ( see Theorem 3.3.3).
- (iii) The set of all dilation maximal states is a convex face in the state space (see Theorem 3.3.3).

Next, we gave an answer to the question we began with: Does there exist a subset on which the dilation maximal states are concentrated? We have given an affirmative answer to this question, but with a twist. In non-commutative topology, projections play the role of sets. Using the structure of the dilation maximal states, we have shown that there exists a projection  $\mathfrak{d}$  that completely characterizes the dilation maximal states, and in fact, a state is maximal in the dilation order if and only if it is concentrated on the boundary projection  $\mathfrak{d}$ . Using this characterization of the dilation maximal states, we have proved a result which is slightly weaker than the hyperrigidity conjecture. Here, we summarize the results on the boundary projection, which is a central theme of the thesis. Let  $B$  be a  $C^*$ -algebra generated by an operator system  $S$ . Then

- (i) There exists a projection  $\mathfrak{d}$  such that a state  $\varphi$  of  $B$  is maximal in the dilation order if and only if  $\varphi(\mathfrak{d}) = 1$  (see Theorem 3.3.3).
- (ii) The hyperrigidity conjecture holds if and only if certain non-commutative topological regularities hold for the boundary projection  $\mathfrak{d}$  (see Corollary 3.3.2).

The boundary projection  $\mathfrak{d}$  characterizing the dilation maximal states is very abstract in nature. There is still a lot to investigate about this projection. We have seen that

the hyperrigidity conjecture is equivalent to the boundary projection being closed under certain assumptions. We are tempted to ask the following question:

**Question 5.0.1.** Let  $S$  be an operator system contained in the  $C^*$ -algebra  $B$ . Let  $\mathfrak{d}$  be the boundary projection for the dilation order  $\mathcal{D}(S, B)$ . What are some necessary or sufficient conditions for the projection  $\mathfrak{d}$  to be closed?

As a first step, we are trying to calculate the boundary projection for the case of the recent counter-example of the hyperrigidity conjecture [9]. It will give us some insight to address the problem. Moreover, an answer to this question will help us generate more examples in which the hyperrigidity conjecture holds or fails.

The results so far discussed in this section are vastly inspired by the connection of the dilation order and the Choquet order on the regular Borel probability measures on compact convex sets, which was established by Davidson and Kennedy. In Chapter 4 we have attempted to generalize the Choquet order defined for regular Borel probability measures on compact convex sets. In the classical commutative case, the Choquet order has been a useful tool for studying function spaces. So using two different interpretations of the Choquet order, we have defined strong dilation relation and sub-division relation. We summarize the main result of the section here:

- (i) In general, the dilation relation is stronger than the sub-division relation (see Theorem 4.0.1).
- (ii) For commutative  $C^*$ -algebras, the strong dilation relation and the sub-division order are the same (see Theorem 4.0.2).
- (iii) The strong dilation relation is equivalent to the non-commutative subdivision relation (see Theorem 4.0.3).

The converse of the Theorem 4.0.1 is not known for general  $C^*$ -algebras. So we are motivated to ask:

**Question 5.0.2.** For operator system  $S$  contained in a  $C^*$ -algebra  $B$  with  $C^*(S) = B$ , is it true that  $\text{SubD}(S, B) \subset \text{StD}(S, B)$ ?

The counterexample provided by Bilich and Dor-on [9] tells us that the hyper-rigidity conjecture is not true in general. In other words, the maximality of the boundary representation is not enough to ensure that all  $*$ -representations have the unique extension property. Recently, Clouâtre and Thompson have proved that the hyperrigidity conjecture is true if one replaces the unique extension property with the so-called unique tight extension property. More precisely, if every irreducible  $*$ -representation has the unique extension property, then all  $*$ -representations have the unique tight extension property. We have shown a connection between the unique tight extension property and the strong dilation relation (see Proposition 4.0.1). This indicates that the techniques of the strong dilation relation and the sub-division relation may be useful in studying rigidity properties of operator systems. The converse of the Proposition 4.0.1 is not known at present, which we find to be an intriguing question in this context.

**Question 5.0.3.** Let  $S$  be an operator system contained in the  $C^*$ -algebra  $B$  such that  $C^*(S) = B$ . Let  $\varphi \in \mathcal{E}(B)$  and  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  be the GNS representation of  $\varphi$ . Let  $\varphi$  have the property that whenever there is  $\psi \in \mathcal{E}(B)$  such that  $(\varphi, \psi) \in \text{StD}(S, B)$ , then  $\psi = \varphi$ . Does this imply that  $\pi_\varphi$  has the unique tight extension property relative to  $S$ ?

Let  $X$  be a compact metrizable subset of a locally convex topological vector space and let  $\text{Ch}$  denote the set of all pairs of measures  $(\mu, \nu)$  such that  $\mu$  is dominated by  $\nu$  in the Choquet order. Classically it was known that a pair of measures  $(\mu, \nu) \in \text{Ch}$  precisely when for every sub-division  $\{\mu_1, \mu_2, \dots, \mu_n\}$  of  $\mu$ , there exists a corresponding sub-division  $\{\nu_1, \nu_2, \dots, \nu_n\}$  such that  $\mu_i \sim \nu_i$  for all  $i$ . Recall that we say  $\mu \sim \nu$  if  $\mu(a) = \nu(a)$  for all continuous affine functions on  $X$ . Our result (Corollary 4.0.1)

has shown that  $\text{Ch}$  is precisely the subset of all pairs  $(\mu, \nu) \in \mathcal{D}(S, B)$  such that, whenever there is a sub-division  $\{\mu_1, \mu_2, \dots, \mu_n\}$  of  $\mu$ , there exists a corresponding sub-division  $\{\nu_1, \nu_2, \dots, \nu_n\}$  of  $\nu$  such that  $(\mu_i, \nu_i) \in \mathcal{D}(S, B)$  for all  $i$ . This is a new reformulation of the Choquet order in terms of the sub-division relation.

For clarity, it is worth mentioning that the Choquet order is classically defined only for the Borel measures on compact convex sets, in which case, we have shown in Corollary 4.0.1 that the Choquet order is equivalent to the strong dilation relation and sub-division relation. In [3], an abstract Choquet order is studied, which is defined on regular Borel probability measures on a compact Hausdorff space (without assuming convexity). So far, we do not know if this abstract Choquet order is equivalent to the strong dilation relation and the sub-division relation. This is an intriguing question in this context. Let us frame this question more precisely. Recall that, for a compact convex set  $X$ , the Choquet order is  $\text{Ch} = \text{Order}(\mathcal{F})$  where  $\mathcal{F}$  is the set of all continuous convex functions on  $X$ . An equivalent interpretation of  $\mathcal{F}$  is that:  $\mathcal{F}$  is the smallest norm-closed, max-stable cone generated by the function system of all continuous affine functions on  $X$ . Now let  $Y$  be a compact Hausdorff space and  $S$  be a function system in  $C(Y)$  such that  $S$  separates points of  $Y$ . Let  $\mathcal{G}$  be the norm closure of the set  $\{f_1 \vee f_2 \vee \dots \vee f_n : f_i \in S\}$ . Let  $\text{Order}(\mathcal{G}) = \{(\mu, \nu) : \mu(f) \leq \nu(f) \ \forall f \in \mathcal{G}\}$ . Then  $\text{Order}(\mathcal{G})$  is a partial order.

**Question 5.0.4.** Is  $\text{Order}(\mathcal{G})$  equivalent to the strong dilation relation and the sub-division relation?

A positive answer to this question will enrich the understanding of the abstract Choquet order.

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