THE MATHEMATICAL THEORY OF SYMMETRY IN ARCHITECTURE

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ABSTRACT

The mathematical theory of symmetry is a significant part of the knowledge an architect applies in design.

The mathematical idea of symmetry is defined as the highest possible group of automorphisms mapping any structural configuration onto itself. The nature of mathematics is a prescriptive framework of rules which enhance and externalize a designer's insight. Space is created by the awareness of relations between architectural elements; in which psychological systems of cognition are the most important process. The clear cognition of space, what might be called order, relies upon the presence of an underlying structure which transforms architectural elements into self-regulating wholes. The mathematical theory of symmetry classifies the structure of certain space creating configurations, enhancing and externalizing a designer's insight into order.

The derivation of the mathematical theory of symmetry emphasizes the combination of an underlying Bravais lattice with a point group distributed on that lattice. The one Bravais lattice in one dimension combines with two point groups to produce the seven "Freize" groups of symmetrical configurations. The five Bravais lattices in two dimensions combine with ten point groups to produce the seventeen "Wallpaper" groups of symmetrical configurations. The fourteen Bravais lattices in three dimensions combine with thirty-two point groups to produce the two hundred thirty "Fedorov" groups of symmetrical configurations in space.

The application of the mathematical theory of symmetry in design is as an arousal moderating device to provide order with structural complexity; to reach maximum aesthetic preferences for the resulting work of architecture. An approach to design involves the selective search for an aesthetically appropriate underlying structure through varying the dimensions and angles of Bravais lattices; and varying the elements in the point groups combined with those lattices. The potential for thoughtful creativity with the theory is shown by the unselfconscious use of symmetry structures in a number of diverse sorts of architecture.

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Whoever condemns the supreme certainty of mathematics feeds on confusion, and can never silence the contradictions of the sophistical sciences, which lead to an eternal quackery.

-- Leonardo da Vinci

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INTRODUCTION

Music, mathematics, and architecture are among the highest pursuits of the human mind. This is because composers, mathematicians, and architects have sought to go beyond surface appearances to create deeper patterns. Patterns of harmony and dissonance, of equality and value, of shape and and material. They are patterns in sound, in ideas, and in light; patterns that create time, logic, and space. Patterns, that to which some kind of repeated arrangement may be found[1], are the keys that unlock understanding for people. Pattern finding and pattern making are the essence of most human activity. In all significant products of human energy, in pieces of music, in theories of mathematics, in works of architecture, abstract pattern is manifest.

The architect, whatever else he may be, is a maker of patterns. Frank Lloyd Wright argued that patterns distinguish works of architecture from mere buildings:

> Architecture is abstract. Abstract form is the pattern of the essential. It is, we may see, spirit in objectified forms. Strictly speaking, abstraction has no reality except as it is embodied in materials. Realization of form is always geometrical. That is to say, it is mathematical. We call it pattern. Geometry is the obvious framework upon

> which nature works to keep her scale in designing. She relates things to each other and to the whole, while meantime she gives to your eye most subtle, mysterious, and apparently spontaneous irregularity in effects. So it is through the embodied abstract that any true architect, or any true artist, must work to put his inspiration into ideas of form in the realm of created things. To arrive at expressive form, he too, must work from within, with the geometry of mathematic pattern. Building is itself only architecture when it is essential pattern significant of purpose.[2]

This thesis seeks to explore an architecture of significant pattern making. It is an attempt to bring together the abstract study of patterns in mathematics with the material embodiment of patterns by architects. The overall goal is to link mathematics and architecture as one creative activity of the mind.

The composition of music is also a making of patterns. A piece of music is firstly a composition that provides the patterns underlying the subsequent performance. The design of architecture should be exactly such a composition that provides the patterns underlying the making of the building. Rudolf Arnheim has commented, "that the forces which organize visual shapes and endow them with expression were embodied in the geometry of architecture with a purity found elsewhere only in music."[3] A design should provide a structure which organizes the spaces created in the work of architecture. Like the composition which organizes the piece of music, this structure is a result of the desire of the human mind to understand. It is a desire to find and make patterns. Just as music is only noise without a structure provided by its composition, architecture is only building without a structure provided by its design.

Mathematicians, as well, have created patterns. Theorems of mathematics provide the patterns underlying the use of mathematics. They provide a structure which organizes the ideas of mathematics. G.H. Hardy has revealed the special beauty of mathematical patterns:

> A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas. A painter makes patterns with shapes and colours, a poet with words. ...A mathematician, on the other hand, has no material but ideas, and so his patterns are likely to last longer, since ideas wear less with time than words. The mathematician's patterns, like the painter's or poet's, must be beautiful; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test; there is no permanent place in the world for ugly mathematics.[4]

The mathmematical theory of symmetry is one of the very special results of mathematician's desire to understand the structure of their patterns. It is a theory of structure; and like compositions for pieces of music or designs for works of architecture, it underlies the surface appearance of mathematics.

Ultimately, this thesis is about structure. It is about the investigation of structures to create space. The resource for that investigation is the mathematical theory of symmetry. The investigation is relevant to the practice of design simply because the architect must satisfy the human need to understand a work of architecture. Human needs in architecture are not primarily the material comfort or sound engineering of the building, they are matters of the mind. These needs are not met by good planning and good construction, they require a concern for the aesthetic qualities of space. One of the most significant of those qualities is order. Order does not imply empty rigid geometric forms, but rather a clarity of pattern from an underlying structure. The significance of the investigation lies in its application in the creation of order in design. The order in a work of architecture is the result of the symmetry of its underlying structure.

In the Team 10 Primer the authors noted that, "Each generation feels a new dissatisfaction, and conceives a new idea of order. This is architecture."[5] It also seems to be the case that an emerging generation of architects are coming to an ever-increasing abstraction in the practice of design. Hopefully, this thesis will link the search for a new idea of order with the abstract knowledge of the mathematical theory of symmetry. It is very important that this link should provide a relevant contribution to the changing practice of design in architecture. The arguments of this thesis, therfore, seem to divide into three parts. The first part will be a groundwork of theory from which to make the link between the mathematical theory of symmetry and the design of works of architecture. The second part will be an explanation and illustration of the resources of the mathematical theory of symmetry. The third part will be some speculations about the contribution of the mathematical theory of symmetry in the practice of design.

The first part of the thesis is concerned with a theory forming the basis of the link between the mathematical theory of symmetry and design in architecture. There are four important concepts in that theory. The first is the precise mathematical idea of symmetry itself, which is discussed in section 1.1 within the context of architecture. The second concept is the nature of mathematics, discussed in section 1.2, which shows how mathematical knowledge may be linked with an art activity like the practice of design. The third concept is an understanding of space creation, discussed in section 1.3, which links spaces in mathematics with spaces created in design. The fourth concept is the crucial investigation of structure that emerges from these discussions, in section 1.4, which completes the link within the context of principles for creating order in architecture. The overall goal of the theory part is to establish the thesis that the mathematical theory of symmetry may be a significant part of the knowledge an architect applies in design.

The second part of the thesis is concerned with the explanation of the mathematical theory of symmetry by graphical illustration of the resources of the theory. There are four steps in that explanation. The first step is a non-technical exposition of the derivation of the mathematical theory of symmetry in section 2.1 which leads into the visual presentation following. The next step is a quick illustration of symmetry in one dimension, in section 2.2, with a careful but non-technical commentary. The third step is an exhaustive illustration of symmetry in two dimensions, in section 2.3, with the same sort of commentary. The fourth step is the extension of the illustration to symmetry in three dimensions, in section 2.4, with a complete commentary that should allow an exhaustive visual understanding of symmetry in space. The overall goal of the resources part is to support the thesis that the mathematical theory of symmetry may be a significant part of the knowledge an architect applies in design.

The third part of the thesis is concerned with speculations about the contribution of the mathematical theory of symmetry to the practice of architecture. There are four areas of speculation to be considered. The first to be considered is the application of the mathematical theory of symmetry in design, in section 3.1, which shows it to be a means for dealing with the aesthetics of complexity in architecture. The second area to be considered is an approach to design, discussed in section 3.2, with the mathematical theory of symmetry. The third area to be considered is the potential for application in design that is provided by the mathematical theory of symmetry, discussed in section 3.3, which concludes all the arguments presented. The fourth and final area for speculation is the directions for research, suggested in section 3.4, that follow from this thesis. The overall goal of the speculations part is to evaluate the thesis that the mathematical theory of symmetry may be a significant part of the knowledge an architect applies in design.

An introduction to the technical definitions needed to discuss the mathematical theory of symmetry is covered by a mathematical appendix. The definitions and theorems have been presented separately from the body of the thesis because they require some special knowledge in mathematics. Only a small number of mathematicians have investigated the subltetics of the proofs leading to the mathematical theory of symmetry. For those who wish to pursue these proofs, the elements of mathematics in the appendix are just a beginning. The general importance of the mathematical appendix is the precise definitions for some of the terms used in the thesis.

It will be difficult to judge the success or failure of this thesis. In terms of meeting specific goals, it will be successful if the first part does establish the thesis, the second part does support the thesis, and the third does evaluate the thesis. In a certain sense, it will be successful if it allows the conclusion that the mathematical theory of symmetry may indeed be a significant part of the knowledge that an architect applies in design. Beyond this, the success of the thesis can only be measured in terms of the stimulation that it provides. If the architect is motivated to investigate and experiment with the mathematical theory of symmetry, then the thesis will be quite successful. However, like all theoretical investigations of the art of architecture, the success or failure of the idea of applying the mathematical theory of symmetry in design can only be judged by the evaluation of actual buildings resulting from that application.

The inspiration for this thesis has come from two sources. The first source is the tradition of the significance of basic design in architecture, embodied most recently at the Ulm Hochschule fur Gestaltung. The idea that design is a skill that may be learned through abstract basic exercises is still a relevant idea in architecture. A faith in the ability and responsibility of the designer to integrate these exercises into the practice of architecture is a moral position. William Huff argued in the Ulm Journal that:

> ...the designer's prime concern is his responsibility for the aesthetic culture, in which he must ultimately take a moral position. The designer is the coordinator, the integrator, the unifier of the environment--where he works more in terms of relationships or arrangements, than of objects or elements.[6]

The second source of inspiration is the trend towards the academic study of the practice of architecture, in particular the application of knowledge from disciplines not formerly studied by designers. The application of "modern" mathematics surveyed by Lionel March and Philip Steadman in The Geometry of Environment[7] is most encouraging contribution to that trend. March has said of the education of the designer:

> If the architectural and planning education is to be anything more than the acquisition of a bag of unrelated tricks, the style of the bag being considered more important in this case perhaps than any of the tricks it might contain, then its educators must eschew fashion and popularity for nothing less than the tough discipline of Simon's notion of the sciences of the artificial. Contemporary engineering education is already well developed in this direction: environmental design education should be no exception. When a school of environmental design adopts as its motto "research pays", then we shall know that the much needed transformation in education and professional attitudes has taken place.[8]

This is also a moral position. The imaginative and creative exploitation of mathematical knowledge in design will be an important step into the future discipline of architecture. If this thesis is but a small contribution in that direction, if it does no more than to add to the change in moral positions towards mathematical research in architecture, then it does something of value.

PART ONE: THEORY

1.1 THE IDEA OF SYMMETRY

People associate different ideas with the word symmetry. Certainly, the mathematician means something very different by it than the average architectural critic. For the mathematical theory of symmetry to be applied in design, it is necessary that the architect first understand the idea of symmetry. Symmetry, like so many words used to describe works of architecture, has no precise common definition. The Oxford Concise Dictionary defines symmetry as beauty resulting from "right proportion between the parts of a body of any whole." Webster's Dictionary defines symmetry as the "similarity of form or arrangement on either side of a dividing line; beauty of form or proportion as a result of such correspondence." The average architectural critic probably defines symmetry as the reflection of the parts of a figure about an axis, although neither dictionary definition mentions mirror reflection. The intent of this section is to abandon these vague notions by exploring the mathematical idea of symmetry, within the context of architecture, reaching a precise definition.

Symmetry is not a new word, neither is it new to apply the idea of symmetry in architecture. The origin of the idea and the word is Ancient

Greece, from the root words "sym", meaning together, and "metron", meaning to measure. Literally the idea of symmetry was "to measure together". The application of the idea in architecture as part of the basic design knowledge of the architect was suggested as early as Vitruvius in The Ten Books on Architecture. Vitruvius included "symmetria" as one of the fundamental principles for design in architecture. In Book I, Chapter II, Vitruvius suggested this definition:

> Symmetry is a proper agreement between the members of the work itself, and relations between the different parts and the whole general scheme, in accordance with a certain part selected as standard.[9]

Certainly, Vitruvius is the first authority that may be cited for the application of the idea of symmetry in architecture. What Vitruvius meant by the word symmetry is probably close to its Greek origins, but clearly it may be interpreted as something more than just mirror reflection about an axis. That something more may only be modular coordination, but it may just as well be the mathematical idea of symmetry.

In order to understand the mathematical idea of symmetry, it is necessary to develop a precise definition of symmetry using some mathematical concepts. The clearest method of developing this definition is to follow the model of two major studies of symmetry available in English; Hermann Weyl's Symmetry[10] and Aleksei Shubnikov's Symmetry in Science and Art.[11] The problem is not that symmetry has lost its meaning, but that its meaning has lost its usefulness in activities like architectural design. To regain that usefulness a precise definition of symmetry using mathematical concepts should be built up from the common notion of mirror reflection about an axis. Such a definition should allow the idea of symmetry to be applied in design.

There appears to be a simple reason that the limited notion of mirror reflection about an axis persists as the idea of symmetry in art. It is this symmetry that is found in most mobile higher life forms, including the human body. For this reason, Huff observed:

> Man, professing to have been made in the image of his god, has, in turn, seen the universe replicated in himself. For him the most persistent of symmetries is the one possessed of his own body--bilateral symmetry. His aesthetic preferences are intermingled with his corporal being, and his products often reflect that condition.[12]

In architecture this idea of symmetry has manifested itself most clearly in the Beaux Arts tradition that relies upon axial planning, both in cities and buildings (Fig. 1.101). The reaction in the Modern Movement against the simplistic and overpowering nature of this tradition is part of the reason the idea of symmetry, even in this limited notion, has lost its usefulness in architectural design. One of the more disturbing discussions in recent literature on this idea of symmetry advocates an even more simplistic application of mirror reflection about an axis in the elevation of houses as a metaphor for the human body, or the human face. Charles Moore and Kent Bloomer in <u>Body, Memory, and Architecture</u> have taken this position:

> Front doors and house facades almost always exhibit a measure of symmetry. In traditional architecture this was achieved with porticoes and balanced facades, whereas today the symmetry is more likely to be expressed by special bushes standing guard at each side of the front entrance. This is certainly related to the frontal symmetry of mobilization characteristic of body posture, where the eyes and ears are focused for defence. In houses these symmetries are facial and are usually oriented to the public.[13]

The Winslow House by Frank Lloyd Wright, with its symmetrical front (Fig. 1.102) and asymmetrical rear (Fig. 1.103), is given in evidence to support this interpretation. This seems to be a clear misunderstanding of Wright's stated

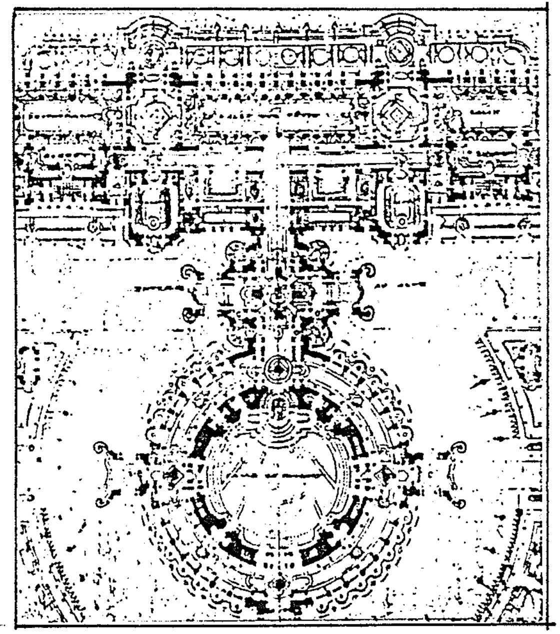

Fig. 1.101

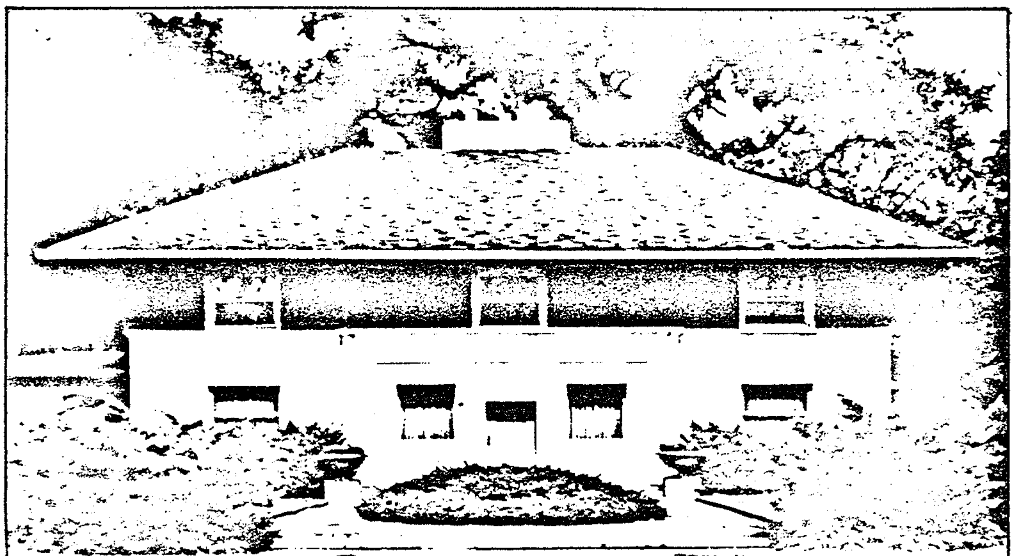

Fig. 1.102

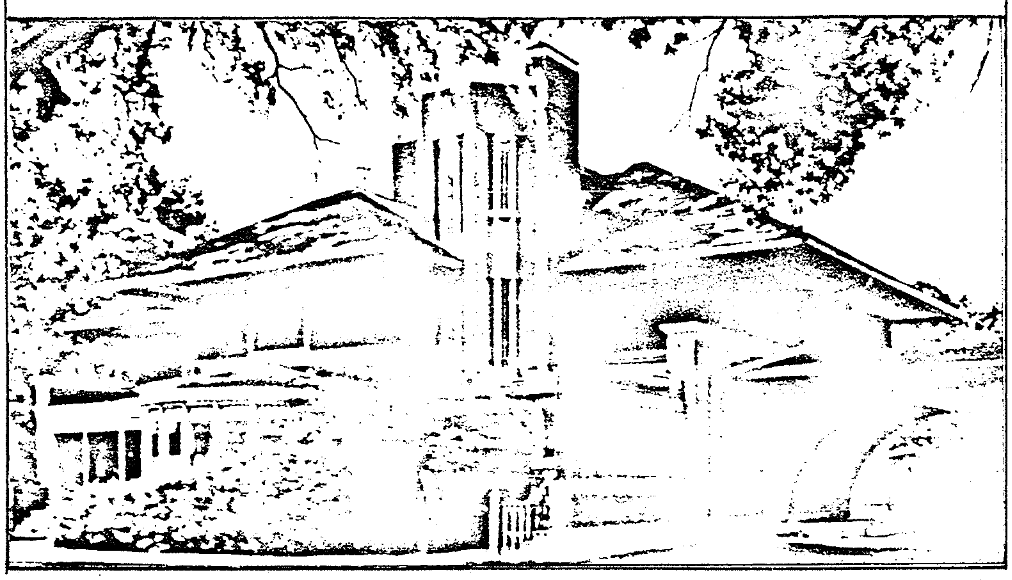

Fig. 1.103

belief in the abstract nature of architecture; and given Wright's extensive use of geometric patterns both in plan and elevation in other works, the symmetry of the front facade of the Winslow House is best seen as an attempt to create order.

The idea of symmetry in architecture should quickly abandon the limited notion of mirror reflection about an axis in either of the examples above. That is because the effect of symmetry is neither due to the monumental power of axial planning, nor due to the imitation of the human body. Shubnikov argued that there is a more subtle reason for effect of mirror reflection about an axis:

> An ink blot is not really beautiful. However, if we fold a piece of paper in two before the ink is dry, we obtain a picture which conveys a pleasing impression. Here the determining factor giving the idea of beauty is the regular mutual disposition of parts of the figure, that is, its symmetry.[14]

The seed of the mathematical idea of symmetry is in this position on the reason for the use of mirror reflection in art.

Bilateral symmetry entails the concept of the parts on either side of the axis being exactly the same, only reversed in sense relative to one another. In some traditional ritual systems even this limited idea of symmetry might be denied because left and right have intrinsic symbolic qualities.[15] However, in modern Western thought, left and right are considered indiscernible. The left side of a symmetrical arrangement has no intrinsic qualities that the right side doesn't, and vice-versa. The condition of being the left or right side of a mirror reflection is called enantiomorphism. The nature of an enantiomorph is that in no sense of super-position can it be made to coincide with its reflection. To use the often given example, there is no way that a left hand glove may be worn on the right hand. Therefore, the idea of left and right enantiomorphs forming a symmetrical configuration involves an operation of the human mind. One is already developing knowledge that is the basis of the mathematical idea of symmetry.

The abstract mathematical concept at the root of the mathematical theory of symmetry is the notion of geometric equality. Clearly the operation of the mind which links enantiomorphs into symmetrical configurations involves noticing their equal size, equal shape, and equal position relative to an axis. Shubnikov has suggested this leads to a basic, yet precise, definition of symmetrical as, "any object which consists of geometrically equal parts appropriately disposed to one another."[16] All mathematics is built upon the manipulation of equalities; the mathematical theory of symmetry is built from the geometric equalities between the parts of symmetrical configurations.

The concept of geometric equality admits many configurations besides just mirror reflections about an axis. Many figures in which the parts are not reversed in sense, yet are still geometrically equal, may be described as symmetrical. Mathematicians call those configurations in which the sense of the parts is not mirrored, direct symmetries. Those in which the sense of the parts is mirrored, are called opposite symmetries.

In order to classify and differentiate symmetrical configurations mathematicians have developed the concept of symmetry operations. The concept is that be a "motion", in the abstract non-kinematic sense, each part of a symmetrical configuration may be made to coincide with another part with which it is geometrically equal. It is a useful idea because there is the parallel in design when an architect speaks of "moving" an element in plan. If the sense of the part is not mirrored, the operation producing direct symmetry, then it is called a proper motion. If the sense of the part is mirrored, the operation producing opposite symmetry, then it is call an improper motion.

It is worthwhile illustrating the concept of a symmetry operation graphically (Fig. 1.104). Architects seem to be the most familiar with two dimensional illustrations from their conventions of drawing plan, section, and elevation. In two dimensions, mathematicians have identified just four symmetry operations, two proper motions of translation and rotation, and two improper motions of reflection and glide relfection. These four operations produce all the possible symmetrical configurations in the plane. In the drawing, the motion of an arbitrary triangle according to each operation has been indicated. Each configuration consisting of two triangles should be recognized as symmetrical.

The motion of a symmetry operation does not alter the lengths, angles, of ratios within the part, respecting the concept of geometric equality. Such an operation is called an isometry in mathematics. Architects are familiar with this term from their graphics, a projection of a plan that, unlike a perspective, does not alter the lenghts, angles, or ratios in the plan is also called an isometric. A symmetry operation is, by definition, an isometric operation. Shubnikov has suggested a further, more precise, definition of symmetrical as, "any finite or infinite figure which may be made to coincide with itself by one or several isometric operations."[17] This definition should be enough to make the mathematical idea of symmetry clear in design.

But, one more level of precision is reached by abstracting two

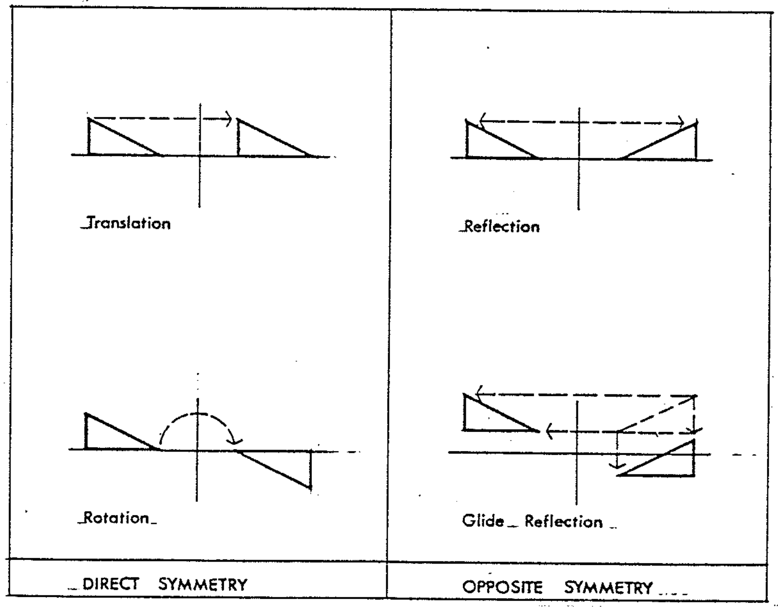

Fig. 1.104

other mathematical concepts contained in this definition. The first concept should be easily understood, enough applications of a symmetry operation will "move" the part back to its original position. The mathematcian calls this complete motion an automorphism. For example, in the illustration of the rotation operation (Fig. 1.104), two motions through 180° make an automorphism for the triangle. Any symmetrical configuration contains an automorphism consisting of two or more symmetry operations.[18] The second conept is more technical, the automorphisms resulting from symmetry operations have the property of forming a group (DEFINITION D:01, Mathematical Appendix). It is sufficient to understand that groups have a particular mathematical structure that may be studied to, in this case, allow the classification of symmetrical configurations by their group structure.

A very precise definition of the mathematical idea of symmetry, based on these two concepts, was developed by Weyl: "Given a spatial configuration, T, those automorphisms of space which leave T unchanged form a group, G, and the group describes exactly the symmetry possessed by T."[19] Shubnikov also developed a similiar precise definition.[20] It has been slightly altered to provide the working definition of symmetry for this thesis. Symmetry is defined as the highest possible group of automorphisms mapping any structural object, consisting of geometrically equal parts, onto itself. The mathematical idea of symmetry contained in this definition should be a clear foundation for the mathematical theory of symmetry in architecture.

Two other ideas which involve the idea of symmetry in their definitions may be useful for the application of the mathematical theory of symmetry in design. The first idea is asymmetry, which is defined only as the abscence of symmetry. However, Weyl suggested that asymmetry refers more accurately to the near presence of symmetry.[21] This should be contrasted with the second idea, which is dissymmetry. Dissymmetry is defined as the purposeful variation from the symmetry any structural object might otherwise have. Deliberate variation from an expected symmetry with elements of dissymmetry may be an important idea in basic design. The subtle distinction between asymmetry and dissymmetry is significant in any application of symmetry in art.

It is interesting to notice the unselfconscious use of the mathematical idea of symmetry in architecture. March and Steadman present several examples of two dimensional symmetry in works of modern architecture, they argued, "It could be said that those who were the most successful innovators of architectural form, in particular Le Corbusier and Frank Lloyd Wright, were those who most understood symmetry as an abstract idea."[22] Examples of Corbusier and Wright plans reveal the creation of a complex variety of spaces within a pattern of different symmetrical configurations. The design of such works appears to be the search for a structure created by symmetry operations applied to the arrangement of architectural elements.

The architecture of Louis Kahn, influenced by his study of Roman architecture, his education in the Beaux Arts tradition, and his own ideas on order, demonstrates an understanding of symmetry operations in design. The mathematical idea of symmetry may be used to study the order in the design of the National Assembly building for Bangladesh (Fig. 1.105). In Kahn's plan for the building it is possible to see the underlying structure resulting from the symmetry of the configuration. The assembly hall itself is created by two elements, a column and a wall, which are reflected about a line, then rotated eight turns about a centre point (Fig. 1.106). The eight lines generated by this operation become the controlling feature of the design. About every second line, the spaces for administration offices is created by the reflection of a square about the line, of course this also creates a four turn rotation about the centre point (Fig. 1.106). About the east-west axis, there are reflections at either end of half-cylinders and rectangles creating space for dining/recreation and ministers' lounges (Fig. 1.107). About the north-south axis, there are four turn rotations at either end creating space for each entry (Fig. 1.107). Between the assembly hall and each of these entries/lounges are stairs or elevators reflected about the axis. But the four turn rotation about the centre point is varied from by the large circular element of the ablution court on the north axis. This is a clear case of dissymmetry. Also, Kahn has turned the mosque main entry, while retaining its rotational symmetry, a few degrees off the line of the axis. This seems to be a case of asymmetry in the sense suggested by Weyl. The application of the idea of symmetry, with the interplay of

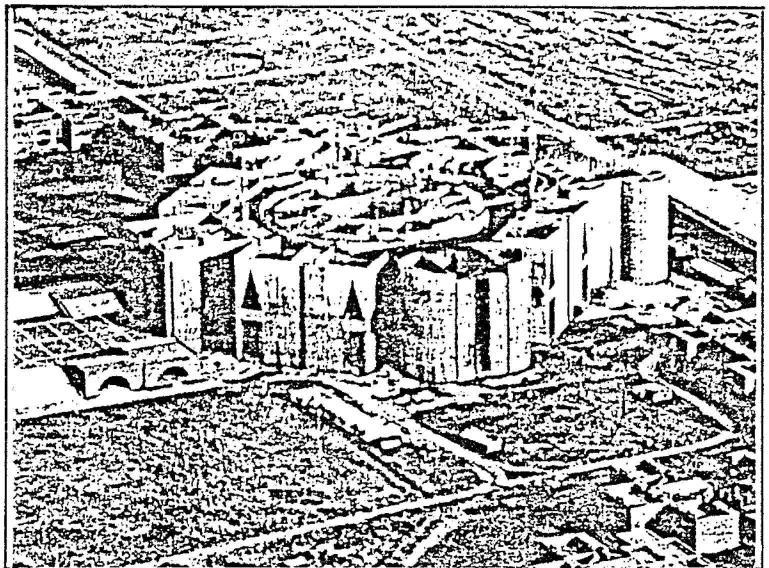

Fig. 1.105

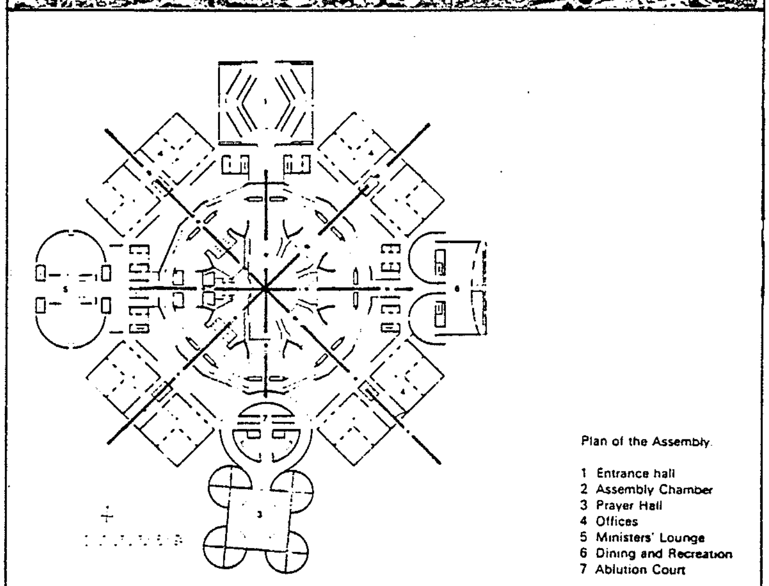

Fig. 1.106

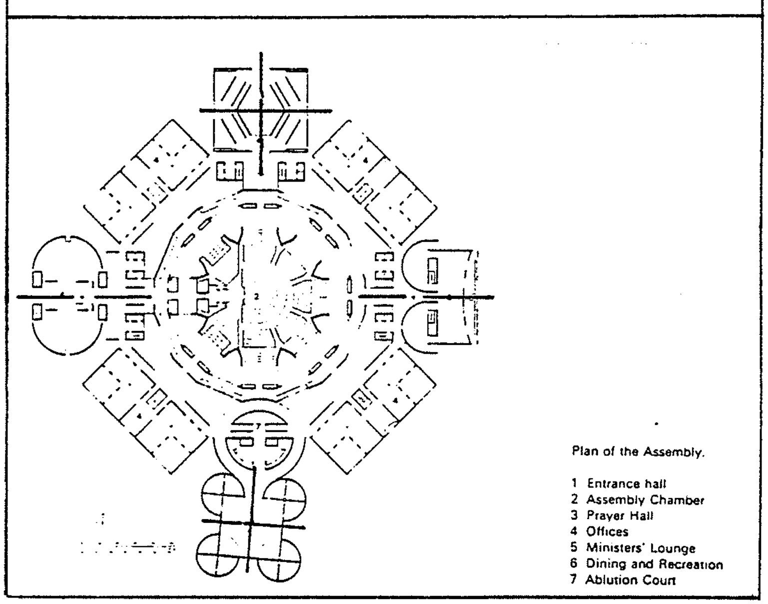

Fig. 1.107

asymmetry and dissymmetry, to create order in design is clearly demonstrated by this and other examples of Kahn's architecture.

The point of this example has been not just to show the presence of the idea of symmetry in a major work of modern architecture, but to show the usefulness of the mathematical theory of symmetry as a way of understanding order in architecture. It is the abstract structure which orders the space creating elements of the building that is the feature to be admired in Kahn's architecture. It is <u>not</u> the forms of the elements, cylinders or cubes or triangles, that make the design so interesting; it is the underlying structure in which those forms are used. Unfortunately, the National Assembly building does exhibit some of the simplistic and overpowering tendencies reminiscent of the Beaux Arts tradition, however it is an example of the subtle appreciation of the idea of symmetry in architecture.

For a knowledge of the mathematical theory of symmetry to be applied in design, the architect should understand the idea of symmetry as a principle for creating order. Kahn's National Assembly building is an example of that principle, not a result to be copied. March and Steadman included a useful quote from Owen Jones' <u>The Grammar of Ornament</u>[23] at this point in their argument:

> The principles discoverable in the works of the past belong to us, not so the results. It is taking the end for the means. No improvement can take place in the art of the present generation until all classes, artists, manufacturers, and the public, are better educated in art, and the existence of general principles is more fully recognized. If the artist, earnest in his search after knowledge, will only lay aside all temptation to indolence, will examine for himself the arts of the past, compare them with works of nature, bend his mind to a thorough appreciation of the principles which reign in each, he cannot fail to be a creator...[24]

Therefore, it is not for the student of architecture, in the practice of design, to imitate the geometric form of the elements in the architecture of Frank Lloyd Wright, Le Corbusier, or Louis Kahn; but to appreciate the symmetry structure present in that architecture.

The development of the premise that the mathematical theory of symmetry may be a significant part of the knowledge an architect applies in design, follows naturally from an appreciation of the idea of symmetry in architecture. An appreciation of the mathematical idea of symmetry in architecture seems to involve three theoretical issues. As defined in this section, the idea of symmetry provides a mathematical method for differentiating and classifying abstract structures underlying spatial configurations. The three issues raised are, therefore, the nature of that method, the creation of those configurations, and the investigation of the abstract structures. The next three sections of the theory part will cover the nature of mathematics, an understanding of space creation, and the investigation of structure. The arguments provided by combining the conepts disscussed in each section should establish the thesis that the mathematical theory of symmetry may be a significant part of the knowledge an architect applies in design.

1.2 THE NATURE OF MATHEMATICS

Whenever the application of mathematics to an art activity, such as design, is proposed there is a theoretical problem to be confronted. Many critics insist that mathematical knowledge, because of the very nature of mathematics, has no place in the judgements creating works of art. They suggest that mathematics is only suitable for the explanations of science. The issue here is not whether architecture is an art or a science, but what is the nature of mathematics. If the mathematical theory of symmetry, or any mathematical knowledge, is to be applied in design, then the nature of that knowledge must allow that application. The intent of this section is to justify the application of mathematics in architecture, by considering the philosophical foundation of mathematics.

As an epigraph, Weyl once used Goethe's remark that, "Mathematicians are like Frenchman; whatever you say to them they translate into their own language, and forthwith it is something entirely different."[25] This reflects the suspicion many artists hold that it is not justified to apply mathematics in art. Equally many architects seem to be mystified by the propositions of mathematics. Indeed, an ignorance of mathematics has attained a certain social status among artists, especially architects. They have no clear concept of mathematical entities, such as numbers or points in space. Nor do they have a clear concept of the source for mathematical truths, such as arithmetic or geometric equalities. These are the two basic problems in the philosophy of mathematics. The resolution of them provides foundation for any application of mathematics in architecture.

The philosophy of mathematics is a relatively recent development, beginning only about the turn of the century. Many answers have been offered by many philosopher to resolve the two problems stated above, but three major trends may be identified: logicism, formalism, and intuitionism. Each trend has several weaknesses, especially when considering the different justifications for applying any mathematics in architecture they provide. A more pertinent approach to the philosophy of mathematics, which does justify the application of mathematics in architecture, is developed from the criticism of these three trends.

Logicism is the label given to the philosophy of mathematics developed by such philosophers as Gottlieb Frege and Bertrand Russell. The general position is very simple, mathematics is logic, and nothing more than logic. It suggests that mathematics may be reduced to logic; this is the programme for Russell and Whitehead's Principia Mathematica.[26] In logicism, mathematical entities are simply defined by the mathematician as nominal entities within a system of logic. The source of mathematical truth, in logicism, is the logical relations between these entities. Thus, the propositions of mathematics are persuasive just because they are logic.

The other two trends in the philosophy of mathematics developed as reactions to the logicist positions. Many philosophers argued that although mathematics proceeds according to the rules of logic, it is not initself logic. The suggestion is that mathematical entities have some kind of real existence, and that the source of mathematical truth is more than just the logic used by mathematicians.

Formalism is the label given to the philosophy of mathematics developed by the mathematician David Hilbert; and later extended by such philosophers as H.B. Curry. The general position is that mathematics is an example of the construction and manipulation of formal systems. The programme was to show that the propositions of mathematics were about, rather that in, the formal system of symbols used by mathematicians. In formalism, mathematical entities are discovered by the mathematican in the symbolic representation for formal systems of mathematics. The source of mathematical truth, in formalism, is the formal relations between these entities. Thus, the propositions of mathematics are persuasive because they are part of the "game" for aformal system.

Intuitionism is the label given to the philosophy of mathematics developed by the philosopher L.E.J. Brouwer; and later extended by Arend Heyting in direct opposition to formalism. The general position, influenced by Kant's synthetic a-priori classification of mathematics, is that mathematics is an activity of intuitive concept formation by the mind. The programme was to show that the propositions of mathematics are the result of the conception and manipulation of certain a-priori mathematical intuitions. In intuitionism, mathematical entities are discovered by the mathematician in the introspection of his intuition. The source of mathematical truth, in intuitionism, is the "self-evident" relations between these entities. Thus, the propositions of mathematics are persuasive because they are constructed through intuitions of the human mind.

These three positions are not as clear cut or schematic as presented here; neither do they characterize every attempt in modern philosophy to account for mathematics. In certain sense they are all plausible; mathematics sometimes seems to be just a system of logic, other times to be a game of manipulating a formal system, and occasionally to be a working of human intuition at a very deep level. Architectural decisions seem remarkably similiar, they often have all three of these characteristics themselves. Various trends in design methods which apply mathematics in architecture, therefore, seem to follow one of these three positions on the nature of mathematics.

Any discussion about the philosophy of mathematics should mention Godel's proof, a brilliant result in mathematical logic, that casts doubt on the whole subject. Nagel and Newman have summarized Godel's achievement:

> Godel's conclusions are two-fold. In the first place he showed that no metamathematical proof is possible for the formal consistency of a system comprehensive enough to contain the whole of arithmetic. Godel's second main conclusion is even more surprising and revolutionary in its import, for it made evident a fundamental limitation in the power of the axiomatic method. Godel showed that Principia, or any other system within which arithmetic can be developed is essentially incomplete.[27]

Godel never argued that any of the philosophical foundations of mathematics were wrong; he simply showed that any programme they might propose would not consistently account for all of arithmetic. A parallel warning should be issued to those who seek to use a mathematical programme to account for all architectural decisions. The application of mathematics in architecture is not such a panacea.

In Remarks on the Foundation of Mathematics[28] Wittgenstein criticized all of the three major positions described above: logicism, formalism, and intuitionism. A review of his criticism will lead to an acceptable view of the nature of mathematics that justifies its application in architecture. Wittgenstein's most fundamental position rejects the idea, which has come down from Plato, that mathematics is a body of knowledge about mathematical entities. Wittgenstein expressed the opinion that, "the mathematicain is an inventor, not a discoverer."[29] This position rejects the idea that the mathematical entities are discovered by people and the idea that mathematical truth is discovered in relations between those entities. Accepting Wittgenstein's position, it is possible to critisize design methods which apply mathematics in architecture following the three positions of logicism, formalism, and intuitionism.

There have been several methods justifying the application of mathematics to design based on an assumption of the logicist position. The early work of Christopher Alexander is typical of an attempt to reduce architectural design to logic by applying mathematics. Alexander discussed the force of logic in the introduction of <u>Notes on the Synthesis of Form</u> and concluded in the epilogue:

> The shapes of mathematics are abstract, of course, and the shapes of architecture concrete and human. But that difference is ineesential. The crucial quality of shape, no matter of what kind, lies in its organization, and when we think of it in this way we call it form. Man's feeling for mathematical form was able to develop only from his feeling for the processes of proof. I believe that our feeling for architectural form can never reach a comparable order of development, until we too have first learned a comparable feeling for the process of design.[30]

Alexanders emphasis on the logic of the process of proof underlying mathematics reveals his belief that the application of mathematics in architecture would be just a comparable reduction of the design process to logic.

An attempt to reduce design to logic through mathematics is questionable because it assumes the position of logicism. Mathematical entities are not discovered in any system of logic to which the architect may appeal in making decisions. Similarly, the source of mathematical truth is not discovered in the logic of the proof of mathematical propositions. The architect applying mathematics does not guarantee the logic of the design process. It is not acceptable for the application of mathematics in architecture to claim that mathematics, initself, makes the design process logical.

Recently, there have been developed several significant methods justifying the application of mathematics to design based on an assumption of the formalist position. The research into applying graph theory using computer-aided design is typical of an attempt to formalize architectural design by applying mathematics. Steadman has discussed the basis for this approach:

> It is, by now, a well established idea that the theory of graphs might find useful application in architectural layout and planning. It is usual to represent a graph with a diagram, showing points joined by the appropriate lines, and to refer to this diagram itself as the graph. Graph points might be used to represent the relation of adjacency between pairs of rooms. It is possible to regard the plan itself as forming yet another different kind of graph. The plan graph and the corresponding adjacency graph bear a special relationship to each other. They are mathematical duals.[31]

Steadman's emphasis on the formal diagram being the graph, and the formal architectural plan a mathematically dual diagram/graph, reveals his belief that the application of mathematics in architecture would formalize the design process.

An attempt to formalize design through mathematics is questionable because it assumes the position of formalism. Mathematical entities are not discovered in the diagrams or representations that the architect may use to make decisions. Similarly, the source of mathematical truth is not discovered in the representations of mathematical propositions. The architect applying mathematics does not make valid the formal plans of the design process. It is not acceptable for the application of mathematics in architecture to claim that mathematics, initself, makes the design process formalized.

There have also been many methods justifying the application of mathematics to design based on an assumption of the intuitionist position. The approach to architecture using geometrical forms of Buckminster Fuller is typical of an attempt to make intuitively reliable architectural design by applying mathematics. Fuller has poetically stated the basis of his intuitive approach:

> Key to humanity's scientific discoveries/Technical inventions/Design conceptioning/And production realizations. That key is the first/ And utterly unpremediated event/Of having come unwittingly upon/An heretofore unknown truth/Of an a-priori universe/An eternal principle. And then moments later/A second intuitive awareness/Regarding what the conceiving individual human/Must do at once/To capture the awareness of/And secure the usefulness of/That eternally reliable generalized principle/For all humanity.[32]

Fuller's emphasis on the intuitive discovery of eternal principles of mathematics and the awareness of their immediate usefulness, reveals his belief that the application of mathematics in architecture would reliably reflect the intuitive design process.

An attempt to reflect intuitive design through mathematics is questionable because it assumes the position of intuitionism. Mathematical entities are not discovered in an a-priori intuition of which the architect is aware in making decisions. Similarly, the source of mathematical truth is not discovered in intuitive introspection of mathematical propositions. The architect applying mathematics does not make reliable the intuitive judgements of the design process. It is not acceptable for the application of mathematics in architecture to claim that mathematics, initself, makes the design process reliable intuition.

Though they misunderstand the nature of mathematics, and consequently are liable to make unjustified claims, each method has resulted some favourable directions for the application of mathematics in architecture. Alexander's work, if nothing else, has produced a significant change in the attitude of architects to the academic study of design process. The research of Steadman and others in graph theory will lead to a very powerful planning tool with the advent of computer aided architecture. The engineering principles of Fuller are already consider landmarks of construction techniques in architecture. While this may vindicate these methods, an alternative philosophical position leading to a justification of the application of mathematics in architecture is required.

A justification for applying mathematics to design is suggested by further readings of Wittgenstein. The intention of the architect applying mathematics should be no different than if he were not applying mathematics. In many instances, mathematics, geometry in particular, has been applied in architecture with the intention of turning otherwise ordinary designs into something special. Wittgenstein opposed this kind of thinking:

> The comparison with alchemy suggests itself. We might speak of a kind of slchemy in mathematics. It is the earmark of this mathematical alchemy that mathematical propositions are regarded as statements about mathematical objects, and so mathematics as the exploration of these objects. In a certain sense it is not

> possible to appeal to the meaning of the signs in mathematics, just because it is only mathematics that gives them their meaning.[33]

Clearly, if the architect's intention in the use of mathematics, like alchemy, is to automatically produce something that another design method would not, then the nature of mathematics has already been misunderstood. The intention in the application of mathematics, just as with any design method, must be to produce good architecture. The evaluation of good or bad architecture cannot be made according to whether or not mathematics was applied in the design.

Once Wittgenstein's position that the mathematician is an inventor is accepted, the two central problems of the philosophy of mathematics may be resolved.[34] A concept of mathematical entities is easily developed; there exist no such things as mathematical entities. There are no such things as a number or a point in space. They are only ideas invented by human beings; the meaning of which has become clear through the use of those ideas, not by their being or representing objects or properties of objects. For example, symmetry is only an idea invented by some person, themeaning of which has become established in mathematics by its use for understanding the structure of certain configurations. Symmetry is not an entity, it is not defined in a system of logic, or discovered in formal representations or intuitions of the world. The application of mathematics to design is not the result of either logical, formal, or intuitive mathematical entities discovered in architecture. Such entities do not exist to be discovered, for they are invented.

Similarly, a concept of mathematical truth is easily developed; mathematical propositions are true by virtue of the conventions for use of mathematical ideas. The source of mathematical truth is the tacit agreement between people about what to accept as truth. Arithmetic or geometric equalities are only truths invented by human beings; the acceptance of which has been the result of conventions about mathematical ideas, not by their being relations between mathematical entities. For example, the isomorphism (DEFINITION D:08, Mathematical Appendix) of certain symmetry groups are propositions invented by some person, the truth of which has been accepted according to the conventions agreed to for use of such ideas. The isomorphism of symmetry groups are not truths discovered in the logical, formal, or intuitive relations between mathematical entities. The application of mathematics to design is not the result of either logical, formal, or intuitive mathematical relations discovered to be true of architecture. Such mathematical truths are not to be discovered, for they are invented.

The nature of mathematics can be best understood as simply a human activity, similar to the activity of language. Mathematics is a set of ideas and conventions about those ideas that has evolved in human culture. A proposition of mathematics is persuasive because we are bound by the rules of mathematics, much as we are bound by the rules of language when we speak. To make or apply propositions of mathematics requires being bound by the meaning of the ideas and agreement to the truth conventions of mathematics; much as to make or apply propostions in languages requires being bound by the meaning of the words and agreement to the grammar of languages. Therefore, mathematics may serve as an alternative way of understanding and communicating design intentions. Mathematical propositions then have the status of rules bounding the design process. The architect is free to creatively apply those rules according to design intentions. However, applying mathematics requires accepting what it presribes.

Mathematics, rather that being a body of knowledge about mathematical entities, is a body of prescriptive rules about mathematical ideas; one of which is the idea of symmetry.

Wittgenstein expressed the status of mathematical propositions in this way:

> The mathematical proposition has the typical (but that doesn't mean simple) role of a rule. If you know a mathematical proposition that's not to say you yet know anything. If there is confusion in our operations, if everyone calculates differently, and each one differently at different times, then there isn't any calculating yet; if we agree, then we have only set our watches, but not yet measured any time. If you know a mathematical proposition, that's not to say you yet know anythin. The mathematical proposition is only supposed to supply a framework for a description.[35]

The mathematical propositions that make up the mathematical theory of symmetry therefore provide a framework for a description, that is, for an understanding and communication, of the structure certain configurations. They, like all mathematics, in no way limit the creativity of the designer, just as language in no way limits the creativity of the poet.

The architect, who understands the nature of mathematics in this way, justifies the application of mathematics in architecture simply because it does give a creative framework of rules within which to design. The mathematical theory of symmetry does give a framework for understanding and communicating the structure of space creating configurations of architectural elements. A knowledge of the theory is, initself, "not to yet know anything", but the application of that knowledge in design may be a powerful creative framework for describing design intentions. The architect whose mind is prejudiced either for or against the application of mathematics in architecture, by assuming some other position on the nature of mathematics, is limited in the creative frameworks that may be applied in design.

There is not only that justification for the application of mathematics in architecture, but also positive value that results from the application. The value in applying mathematics to design lies in increasing the creative capacity of the mind, and the ability to make public that capacity. Pascal observed in the Pensees that:

> Mathematicians who are merely mathematicians therefore reason soundly as long as everything is explained to them by definitions and principles, otherwise they are unsound and intolerable, because they reason only from clearly defined principles.
> And intuitive minds which are merely intuitive lack the patience to go right into the firat principles of speculative and imaginative matters which they have never seen in practice and are quite outside ordinary experience.[36]

It is intrinsically valuable for the designer who has, heretofore, relied upon intuitive insights into architecture to apply mathematics to subjective judgements in design. Equally, subjective judgements should and do temper the application of mathematics in architecture. The capacity of the architect to have creative insights into architecture, and to employ those insights in design, is greatly increased by applying mathematical knowledge with value judgements. The second intrinsic value of mathematics in architecture is, in the words of J. Christopher Jones:

> ...to make public the hitherto private thinking of designers; to externalize the design process. In some cases this is done with words, sometimes in mathematical symbols, and nearly always with a diagram representing parts of the design problem and the relationships between them. Clearly, the underlying aim is to bring designing into the open so that other people can see what is going on and contribute to it information and insights that are outside the designer's knowledge and experience.[37]

More than any other design method proposed in the last decade of research, the application of mathematical knowledge in architecture achieves this externalization. Mathematics increases the ability to externalize the design process, not because it is an objective method of making design logical, or formalizing design, or reflecting design intuitions; but because it is an intersubjective method, like language, for describing design intentions.

The mathematical theory of symmetry in architecture is, therefore, a framework of prescriptive rules, not to replace, but to increase and externalize an architect's insight into design. Specifically, it provides an external method for describing the structure of certain space creating configurations. The application of that knowledge in design is a way of understanding and communicating the intention to create order, by giving a structure to the space created by architectural elements. The architect must have a special understanding of space creation and must investigate structure in a special way for the mathematical knowledge of symmetry to be useful for externalizing that part of the design process. The next two sections of the theory part will cover an understanding of space creation, and the ivestigation of structure. This should further establish the thesis that the mathematical theory of symmetry may be a significant part of the knowledge an architect applies in design.

1.3 AN UNDERSTANDING OF SPACE CREATION

Architecture is commonly defined as an art of space creation, that is, the essence of architectural design, unlike any other art, is the creation of space. This definition has become rather useless in the practice of architecture since few architects explain exactly what they mean by the word space. There is little apparent consensus on a philosophical and psychological concept of space on which to base design methods. The common practice seems to be to distinguish between several concepts of space by prefacing it with another term. Combinations such as virtual space, physical space, personal space, perceptual space and existential space are typical in the literature related to space in architecture. It is no wonder that people are confused by endless, and in most cases inconsequential, distinctions between kinds of space. For this reason many designers seem to avoid confronting the essential issue of space creation. For the mathematical theory of symmetry, which describes the structure of spatial configurations, to be applied in design it is important to develop an understanding of space creation. The intent of this section is to establish an understanding of space creation in architecture that allows

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the application of the mathematical theory of symmetry in design.

The unfortunate effect of the excessive use of jargon in most discussions of space in architecture is that many designer have lapsed into a naive understanding of space creation. Susanne Langer was forced to the conclusion:

> Architecture is so generally regarded as an art of space, meaning actual, practical space, and building is so certainly the making of something that defines and arranges spatial units, that everybody talks about architecture as 'spatial creation' without asking what is created, or how space is involved.[38]

The haphazard consideration of space by architects has lead to an understanding of space creation based on what might be philosophically and psychologically naive concepts. This understanding leaves most important questions about space not only unanswered, but unconsidered.

There are two important questions to be resolved by architects in a concept of space on which to base design methods. The first is the straightforward but difficult question, what is space? This is essentially a philosophical problem to resolve the nature of the existence of space. The second is the more involved question, how do people know space? This is essentially a psychological problem to resolve the relative functions of perception and cognition of space.

Most architects seem uniformed on the answers to these questions. Charles Moore's essay on space in Dimensions reflected an unclear concept of space:

> Space in architecture is a special category of free space, phenomenally created by the architect when he gives a part of free space shape and scale. ... We talk of 'making' a space, and others point out that we have not made a space at all; it was there all along. What we have done, or tried to do, when we cut a piece of space off from the continuum of all space, is to make it recognizable as a domain, responsive to the perceptual dimensions of its inhabitants.[39]

Moore's phrases seem to be based on the philosphical position that space, or the continuum of space, exists as some sort of object that designers divide up, or cut off pieces. They also seem to be based on the psychological position that people parceive space initself as some sort of domain. The suggestion is that designers then control the "dimensions" of space as an object, and thereby control the perception of that object. This leads to an architectural determinism in design, suggesting that space creation is a moulding of some object and people's perception of that object. Such determinism would not allow an application of the mathematical theory of symmetry in design, because symmetry does not concern itself with the dimensions of space but the structure of spatial configurations. Alternative philosophical andpsychological positions are required to allow that application.

The philosophical problem of the nature of the existence of space is as old as the activity of philosophy. It does not seem necessary to discuss the method or grounds on which people know spacexists; in order to discuss the nature of its existence. There are many theories that explain the grounds on which people claim knowledge of objects; which are defined as epistemological positions. Theories that explain the nature of the existence of those objects are defined as ontological positions. The question of what is space that architects should resolve is an ontological problem.

There are three basic ontological positions about space that should be considered. The first considers space to be a substance existing in reality much like ordinary objects. The second considers space to be an awareness of relations between those objects, not as any sort of entity existing initself. The third considers space to be an integral part of human existance being the basis of the relationship between man and reality. The architect should distinguish between these ontological positions, as not all of them suggest or allow that the mathematical theory of symmetry, or similar methods, may be included in an understanding of space creation.

The view of space as a substance existing in reality much like an object of everyday experience, is the most widely held view among non-philosophers. It is this philosophical position to which a concept of space like the one expressed by Moore is most likely to reduce. This position commonly explains space by characterizing it as a recepatacle, container, or arena in which objects exist. The idea is that "empty space" is something which does exist and may be experienced by itself. Similarly, mathematical points and lines as ideas about space may be conceived independent of the experience of objects, through the experience of space itself. An object is thought of as being placed in space, or of having its extension in space. There is very little in the way of justification ever given for this position; except that it is the "natural" concept of space as it is used in ordinary language. However, thisconcept was held by almost every philosopher from Plato to the beginning of this century; and by most scientists under the influence of Newtonian classical physics.

The view of space as an awareness of relations between objects, but not an entity itself, is less obvious to the non-philosopher. It is this philosophical position to which many philsophers have resorted since the experimentts of Mach and the development of the theory of relativity by Einstein. Many philosophers now specializing in space and time[40] suggest that those who believe space to be a substance are beign misled by the appearance of ordinary language. The propenent of this position was Leibniz in the 18th Century, who debated the topic of space in letters to a followar of Newton:

> I hold space to be something merely relative, as time is; that is, I hold it to be an order of coexistences, as time is an order of succesions. For space denotes, in terms of possiblity, an order of things which exist at the same time, considered as existing together, without inquiring into their manner of existing.[41]

not

The idea is that "empty space" is something that does not exist and may not be experienced by itself. Similarly, mathematical points and lines as ideas about space may only be conceived through the experience of objects in the appropriate relations (or at least the possibility of such experience). Space is thought of as itself being conceived through the awareness of relations between objects. The primary justification for this position is that space as a substance cannot be detected the way ordinary objects are; that is, there are no epistemological grounds for space as a substance. The use of space in ordinary language should be understood as not asserting the existence of a substance, but only as attributing actual or possible relations between objects; asserting the existence of those objects only.

The view of space as an integral part of human existence is a founding concept of existential philosophy. Proponents of this view, including Merleau-Ponty and Heidegger, are occasionally cited by architectural theorists such as Christian Norberg-Schulz.[42] This position commonly explains space as an inescapable phenomenum of the relation between man and reality; nothing more, space is neither a physical substance nor a conceptual awareness. Human Existence is characterized as spatial, that space cannot be separated from man's being, it is aphenomenum that cannot be classed as either an external substance or an internal concept. The idea is that questions about the existence of "empty space" do not make sense. Similarly mathematical points and lines as ideas about space may be conceived through the phenomena of man's being in space, they are not associated with any experience of space as a substance or any possible experience of objects in certain relations. The justification for this position seems to be self-fulfilling; it assumes space to be a phenomenum of human existence because human existence is inescapably spatial because space is a phenomenum of human existence. Because of this circular reasoning, it is doubtful whether this position actually provide an ontology for space.

The application of the mathematical theory of symmetry in design implies the use of ideas about space such as rotation about points and reflection about lines. An account of the ontology of space must allow for the usefulness of mathematical ideas such as these. The existentialist position may be easily rejected; not only is it self-fulfilling, but it is basically useless in design. Norberg-Schulz has been typically vague about design method; he indicated only that, "Architectural space, therefore, can be defined as a concretization of man's existential space."[43] Concretization is not really a principle for doing design. The more relevant decision is therefore to be made between the position that space is a substance and the position that space is an awareness of relations. Both of these have clear implications for design methods involing mathematical ideas.

If the architect accepts the position that space is a substance, then the emphasis in the design process would be the creation of elements in space. The direction of design methods would be towards the dimensioning of architectural elements; that is, the form of architecural objects as they have their extension in space. The obvious tendency would think of space in terms of substantive volumes in which objects are to be created. On the other hand, if the architect accepts the position that space is an awareness of relations between objects, then the emphasis in the design process would be the creation of spaces by architectural elements. The direction of design methods would be towards the arranging of architectural elements; that is, the structure of relations between architectural objects as they create an awareness of space. The obvious tendency would think of space in terms of relations created between objects. Clearly, principles for doing design in architecture tend to follow one of these two positions.

The application of the mathematical theory of symmetry in design is based on the architect's acceptance of the position that space is an awareness of relations between objects. If architecture is to be concerned with space creation, it suggests that the designer think in terms of relations that create space, not in terms of three dimensional objects in space. The emphasis of the design process should be the structure of relations between architectural objects, rather than the form of those objects. Architecture does not and should not deal with the invention of three dimensional objects in space; and thus the concept of space as a substance is to be eliminated. Design should deal with the creation of relations between objects (often virtually two dimensional objects such as wall planes); and thus the concept of space as an awareness of relations between objects is to be accepted. An understanding of space creation should be based on the ontological position that space exists only in relations between objects. This understanding best allows the application of the mathematical theory of symmetry because it implies design is based on principles about the relations between objects that create an awareness of space. The mathematical theory of symmetry describes the structure underlying relations between elements of certain spatial configurations. The theory may be a useful principle for doing design if the understanding of space creation is based on the ontological position that space is an awareness of relations between objects.

The architect must still consider the psychological problem of the relative function of perception and cognition of space. A designer should have some concept of the effect of perception and cognition in the processes by which people know space. This is a central activity of environmental psychology, a rather new discipline in contrast to the philosophy of space. The question of how do people know space that architects should resolve is an environmental problem.

The important distinction is between the two psychological processes of perception and cognition. A possible distinction between them is the process/product distinction; in which perception is a process leading to cognition as a product. This seems to simplistic, because cognition clearly seems to be more than just the result of perception. The distinction between perception and cognition may be better interpreted as a particular process, perception; versus a general system, cognition, which involves that process. Cognition then refers to general systems of the mind including the processes of perception, recording into memory, organizing into images, and thinking about things. The boundary may only be a matter of physical size, perception being a particular process in response to the immediate envirionment; with cognition being a general process of awareness of the larger context. Roger Downs and David Stea developed this into useful definitions of the two terms:

> We reserve the term perception for the process that occurs because of the presence of an object, and that results in the immediate apprehension of that object by one or more of the senses. Temporally

> it is closely connected with events in the immediate surroundings and in general is linked with immediate behaviour. ...Cognition need not be linked with immediate behaviour and therefore need not be directly related to any objects or events occuring in the proximate environment. Consequently, it may be conected with what has passed or what is going to happen in the future.[44]

What distinguishes proccesses of perception from systems of cognition is an emphasis on the study of responses to the presence of objects rather than the study of attitudes and dispositions not related to the presence of objects. The study of systems of cognition relies on the effects of objects from the past or in the future, or so large as they cannot be seen at once, or part of an overallcontext.

An understanding of space creation involving the ontological position that space is not a substance, seems to also involve the position that perception is not the most important process in the way people know space. Clearly, the concept of space as a substance would suggest that the perception of that substance is a simple and direct way of knowing space. But the acceptance of the concept of space as an awareness of relations between objects suggests that the cognition of those relations is the actual way people know space; that the process of perception is only involved in the apprehension of those objects. Assuming that design methods should respond to the way people might act in space, and assuming that action is linked with the way people know space, then an understanding of space creation should emphasize systems of cognition. The trend in environmental psychology seems to accept this position; for example, Downs and Stea argued that, "Human spatial behaviour is dependent on the individual's cognitive map of the spatial environment."[45] In The Psychology of Place, David Canter stressed that the essence of the argument is, "that any act is made in relation to the context within which the individual thinks himself to be."[46] Clearly, principles for doing design should emphasize systems of cognition.

The application of the mathematical theory of symmetry in design is based on the architect's acceptance of the position that space is known through systems of cognition. If architecture is to be concerned with space creation, it suggests that the designer think in terms of cognition of the relations that create space, not in terms of perception of three dimensional objects in space. The emphasis of the design process should be the cognition of the structure of relations between architectural objects, rather than the perception of the form of those objects. Thus, the process of the perception of space as a substance is to be eliminated. Design should deal with the creation of clear cognitions of relations between objects; and thus the knowledge of space through systems of cognition is to be accepted. An understanding of space creation should be based on the psychological position that space is known only through the cognition of relations between objects. This understanding best allows the application of the mathematical theory of symmetry because it implies design is based on principles about the relations between objects that are known through systems of cognition. The mathematical theory of symmetry describes the structure underlying the relations between elements that effects the cognition of certain spatial configurations. The theory may be a useful principle for doing design if the understanding of space creation is based on the psychological position that space is known through systems of cognition.

An understanding of space creation for the application of the mathematical theory of symmetry in architecture is based on these two important positions. The first is that, ontologically, space should be postulated only as the awareness of relations between objects. The second is that, psychologically, space should be understood as known through systems of cognition of those relations. In The Dynamics of Architectural Form, Arnheim suggested a practical importance of accepting these positions:

> By way of lofty abstraction we have come accross a fundamental principle of practical importance to the architect. In spite of what spontaneous perception indicates, space is no way given by itself. It is created by a particular constellation of natural and man-made objects to which the architect contributes. In the mind of the creator, user, or beholder, every architectural constellation establishes its own spatial framework. This framework derives form the simplest structure compatible with the physical and psychological situation.[47]

It is desirable that architecture, as an art of space creation, be produced by the capability of the human mind to work at the level of abstract relations between material objects. The design process is rooted in basic study of and reflection upon those abstract relations that create an awareness of space. Architecture based on this understanding of space creation is an abstract art activity on a very high level.

The mathematical theory of symmetry may be a significant part of the knowledge that an architect applies in design becasue it describes the structure underlying the relations that create space in certain configurations. The study of structure in the reflective abstraction of the mathematical theory of symmetry providesa basic knowledge of the creation of space. The application of that knowledge in design is a way of creating a clear cognition of space through the presence of a structure in the relations between architectural elements. The architect must investigate structure in a special way to direct the creation of space through the use of mathematical knowledge of symmetry. The next section of the theory part will cover that investigation of structure. This should finally establish the thesis stated above.

1.4 THE INVESTIGATION OF STRUCTURE

Order seems to be universally recognized as one of the basic concerns of the architect in design. This thesis has stressed the idea that the significance of the mathematical theory of symmetry in architecture is its application for the purpose of giving order. It achieves this by giving a structure to the relations between architectural elements creating space. Indeed, order can be equated in architecture, as inalmost every art activity, with the presence of an underlying and abstract structure in its creation. The intent of this section is to investigate the abstract idea of structure in the context of giving direction to the application of the mathematical theory of symmetry in design.

The role of order in architecture should be understood before the investigation of abstract structures is developed. Arnheim defined order in the essay <u>Entropy and Art</u> as:

> ...a necessary condition for anything the human mind is to understand. Arrangements such as the layout of a city or building, a set of tools, a display of merchandise, the verbal exposition of facts or ideas, or a painting or piece of music are called orderly when an observer or listener can grasp their overall structure and the ramifications of the structure is some detail. Order makes it possible to focus on what is alike and what is different, what belongs together and what is

> segregated. When nothing superfluous is included and nothing indispensible is left out, then one can understand the interrelation of the whole and its parts, as well as the hierarchic scale of importance and power by which some features are dominant, others subordinate.[48]

Order must be present in a work of architecture as a prerequisite for comprehending the space created by the relations between architectural elements. It allows for the clear cognition of the spaces of a building. The design process in architecture must include the creation of spatial order among the goals of its methods. This involves the creation of a structure that facilitates the particular design intentions of the designer. Arnheim's view was that;

> Order must be understood as indispensible to the functioning of any organized system, whether its function be physical or mental. Just as neither an engine nor an orchestra nor a sports team can perform without the integrated cooperation of all its parts, so a work of art or architecture cannot fulfill its function and transmit its message unless it presents an ordered pattern. Order is possible at any level of complexity...but if there is no order, there is no way of telling what the work is trying to say.[49]

Order is necessary for the communication of design intentions about space creation through a work of architecture. The knowledge of abstract structure described by the mathematical theory of symmetry contains just the sort of order in spatial configurations that seems to be necessary in architecture. This is the purpose of a design method involving the application of the mathematical theory of symmetry as an abstract structure.

Unfortunately, structure has become a fashionable word to describe many different ideas. Indeed, in the sense of structuralism, it has lead to a very academic set of jargons and convoluted categories.[50] The investigation of structure which is so essential to the creation of order in architecture and art must avoid these drawbacks associated with structuralism. In fact, the idea of structure must be kept separate from any particular academic exercise of structuralism.

The source of the importance of the idea of structure lies in claims of its association with fundamental proccesses of the human mind, such as the cognition of space. The application of knowledge about abstract structure has been explained by Edmund Leach as based on belief that,

> ...concepts in the mind can be combined and recombined by some deeper level of mental process, a kind of meta-thinking which does not of itself generate conscious thoughts but makes creative originality possible in that it consists in the establishment of relations between relations.[51]

The application of the mathematical theory of symmetry in architecture exactly this kind of meta-thinking that makes thoughtful creativity possible in design.

As suggested above, it is necessary to investigate the "nature of the affirmative ideal that goes with the very idea of structure,"[52] as Piaget did,independent of structuralism. Moreover, the investigation of that ideal suggest directions for the application of structural knowledge, exemplified by the mathematical theory of symmetry, in architecture. Piaget identified three key ideas which together comprise the idea of structure. They are wholeness, transformation, and self-regulation. Each of these ideas provide specific directions for the creative application of structural knowledge in design.

The idea of wholeness is crucial to the investigation of structure. A distinction must be made between structures and aggregates. Structures are whole, while aggregates are built up by the association of separate parts. Structures do, of course, have identifiable elements, but these elements are subordinated to the rules of composition of the whole. It is the framework of rules which relate each element to every other that defines a whole. The whole is not reducible to a one-by-one association of its elements. It is important to note that with a structure it is not necessary or important to say whether the whole precedes the parts, or whether the parts precede the whole; such a question is irrelevant. It is the relations between the parts according to a framework of rules defining the whole that is the important feature of structures.

The design direction implied by accepting the idea of wholeness in architecture is significant. A building should be thought of as a whole, not as an aggregate of individual places through an assocation according to certain program requirements. This requires replacing the popular emphasis in design methods upon the sense of place and accomodation of function with an emphasis on the creation of structures relating places and functions into a clearly defined whole. It would not be important for the architect to understand how elements of a building combine to create particular perceptions of place or accomodate particular activities. Rather, the relations between those combinations of elements throughout the whole of a work of architecture should be important in design. The acceptance of this direction could radically alter the way architects approach design. It suggests that the architect in the creation of a work of architecture emphasize a structure providing a framework for relations between the space creating elements of the building. A direction of the application of the mathematical theory of symmetry in design is to provide a structure which defines the whole in the resulting work of architecture.

The idea of transformation is the most obvious aspect of the investigation of structure. An application of the idea of structure inherently involves the presence of transforming operations. A part or element is transformed or changed through the rules of composition of the structure. A distinction might be made between styles and structures. Styles have rules for the formation of each element of the work; while structures have rules only for the transformation of those elements. Structure do, of course, have elements that have form, but that form issubordinated to the rules for transformation, which relate each element to every other. It is important to not that with a structure it is not necessary that the form of the elements be the same for every work of the same kind; the nature of transformations is that they may operate on any form of elements. It is the relations between the elements according to a framework of rules transforming each element that is the important feature of structures.

The design direction implied by accepting the idea of transformation in architecture is also significant. A building should be thought of as the result of creatively applied trnsformations of space creating elements; not just the invention of those elements. The particular forms of the elements is not as important as their transformation by the rules underlying their arrangement into certain relations. This requires replacing the popular emphasis in design methods upon the invention of formal elements that create particular effects with an emphasis on the creation of structures that transforms these elements into clear relationships. It would not be important for the architect to invent a vocabulary or forms or to rely on any particular formal style. Rather, the relations between those elements of whatever form through the transformation of them into a work of architecture should be important in design. The acceptance of this direction could radically alter the way architects approach design. It suggests that the architect in the creation of a work of architecture emphasize a structure providing a framework for relations transforming the space creating elements of the building. A direction of the application of the mathematical theory of symmetry in design is to provide a structure which transforms each element in the resulting work of architecture.

The idea of self-regulation is quite necessary to the investigation of structure. Self-regulation is defined by two inherent consequences of creating a structure. First, the result of transformations of elements within the whole arealso elements of that whole. Second, no transformations are applied to elements within the whole that violate the framework of rules which define that whole. A distinction is to be made between self-regulation and regularity. Self-regulation conserves the structure that was created; while regularity merely repeats the forms that have been invented. Structures do, of course, repeat formsbut that regularity is subordinated to the self-regulation relating each element to every other. It is important to note that with a structure it is not necessary to regularly repeat certain formal elements as often and whereever possible; the nature of self-regulation is that self-regulation controls the arrangement of elements. It is the relations between the elements controlled by the self-regulation that is the important feature of structures.

The design direction implied by accepting the idea of self-regulation is again significant. A building should be thought of as controlled by the self-regulation of the structure, consistently and completely, underlying the relations between space creating elements. The regular repetition of particular forms of elements is not as important as the consistency and completeness of the self-regulation in relations between those elements. This requires replacing the popular emphasis in design methods upon the repetition of "successful" or preferred forms with an emphasis on the creation of structures that themselves regulate the relations between these elements. It would not be important for the architect just to use certain formal elements as regularly as possible. Rather, the relations between those elements should control the self-regulation of them into a work of architecture. The acceptance of this direction could radically alter the way architects approach design. It suggests that the architect in the creation of a work of architecture emphasize a structure providing a framework for relations that is self-regulating. A direction of the application of the mathematical theory of symmetry in design is to provide a self-regulating structure that controls each element in the resulting work of architecture.

The combination of these three features of the investigation of structure which give direction to the application of the mathematical theory of symmetry in architecture results in a very special attitude to design methods. The cognition of spatial order that might be produced by the underlying presence of an abstract structure is a very deep level of human thought. The application of structural knowledge in design is an entirely human activity, and the results of that activity should produce a very human quality in architecture. In collaboration with the painter Amadee Ozenfant, Le Corbusier concluded that any art activity has but one goal, "...to put the spectator in a state of a mathematical quality, that is, a state of an elevated order."[53] An application of the mathematical knowledge of structure embodied in such things as the theory of symmetry should be understood as a method for creating that quality in architecture.

It is architecture which provides the most appropriate human activity for the application of the mathematical theory of symmetry simply because it, more than any other art[54], is produced by space creating structures. Abstract structure resulting in spatial order seems to be essence of architectural design. Le Corbusier inspired this point of view by argueing that:

> ...in plastic art, the senses should be strongly moved in order to predispose the mind to release into play the subjective reactions without which there is no work of art. But there is no art worth having without this excitement of an intellectual order, of a mathematical order; architecture is the art which up until now has the most strongly induced states of this category. The reason is that everything in architecture is expressed by order...[55]

The investigation of structure directs the application of the mathematical theory of symmetry in design towards anemphasis in design methods on wholeness, transformation, and self-regulation. This imparts the quality of a mathematical order in the resulting works of architecture.

There is danger latent in the application of mathematical knowledge of structure in design. That danger is the tendency to lapse into design methods that are theories of proportion, instead of principles of order. For many good reasons[56] architects have come by and large to reject mathematical proportion as part of their design methods. One exception was Le Corbusier who developed an arithemetic system of proportion in <u>Le Modulor</u>.[57] It, when seen in the context of the quotation above, exemplifies the confusion between proportion and order, which results in mathematics being misdirected away from the investigation of structure towards the investigation of form.

The boundary of theories of proportion have been defined by Scholfield as those studies, "...concerned only with the relationship of the shapes and sizes of objects which please the eye."[58] Theories of proportion seem to contain two components. The first is an emphasis on a particular vocabulary of shapes; and the second is a system, usually based on a series of numerical ratios, for giving dimensions to those shapes. Scholfield explained what happens in design methods relying on theories of proportion, "Once admired shpaes have been selected--and this is where the difficulty lies--architectural proportion becomes a straightforward matter of using them as often as possible."[59]

Theories of proportion may be seen in direct contrast to principles of order by opposing their emphasis on form to an emphasis on structure. The motivation for theories of proportion is generally the desire for a perceptual beauty of form in a work of architecture. The direction of design methods emphasize the individual parts, the formation of those parts and the mere repition of them. On the other hand, the motivation for principles of order is generally the desire for cognitive clarity of structure in a work of architecture. The direction of design methods emphasize the overall whole, the transformation of parts into that whole, and the self-regulation of the whole. Clearly, theories of proportion emphasize form in design; while principles of order emphasize structure in design.

Scholfield argued that, "the object of architectural proportion is the creation of visible order by the repetition of shapes."[60] This reveals confusion, for clearly theories of proportion have a totally different content and direction for design than do principles of order. There is no way that mere proportion may achieve the underlying abstract structure that is the essence of order in architecture. The mathematical theory of symmetry provides a description of structure; and the application of such knowledge in design is a method for creating order. It is not theoretically coherent to apply the mathematical theory of symmetry to proportion architectural elements or volumes. An application of the mathematical theory of symmetry may employ any vocabulary of shapes with any proportion; and transform them according to the three directions suggested by the investigation of structure. The mathematical theory of symmetry provides a description of certain space creating structures; to apply it to the invention of forms is to confuse the distinction between order and proportion.

Some kind of boundary should be drawn between sculpture and architecture. Sculpture seems to emphasize the invention of three-dimensional forms in space. Architecture, on the other hand, should emphasize the creation of three-dimensional structures making space. One of the reasons that the so-called Modern Movement seems to have reached a point where the architecture produced is aesthetically empty seems to be designers assuming the primary role of "form-givers." Such a role suggest that every design problem requires the invention of new formal elements; establishing new forms "following" function. The distinction between sculpture and architecture suggests that designers should assume the primary role of creating structure. This role might allow the production of architecture that communicates aesthetic information about space. Every design problem would require only the creative application of structures relating existing or adapted formal elements; establishing clear cognitions of space. The architect applying the mathematical theory of symmetry should understand that it is an investigation of structures for orderly space creation; not a study for sculpting the forms of architectural elements.

The results of the investigation of structure, an understanding of space creation, and realizing the nature of mathematics seem to involve a change in attitude towards design methods in architecture. An important feature of the art of architecture is the striking parallel between the mental habits of the designer and the character of the resulting architecture. The application of the mathematical theory of symmetry in architecture lies as part of the trend towards the abstract study of human comprehension of the environment; specifically the clear cognition of space. Therefore, the mental habits of the designer, involving an abstract knowledge of structures available to create space, will be reflected in the resulting work of architecture. By adopting the mental habits of mathematicians in the investigation of structure through the mathematical theory of symmetry, the designer should see it paralleled in the clear orderly spaces of the resulting architecture.

The theory part of the thesis may now be concluded by simply argueing back through the ideas of the four sections. Clearly, order in the comprehension of a work of architecture is the result of there being an abstract structure underlying the design. The structure emphasizes the definition of the whole, the transformation of elements, and the self-regulation of the structure in design methods. Those design methods result in the clear cognition by people of the relations between the elements of the work of architecture. The cognition of those relations creates an awareness of space, which is nothing more than such an awareness. Mathematics provides a prescriptive framework of rules for externalizing the designers insight into the creation of these relations. The idea of symmetry is based on the idea of mapping structural configurations, consisting of geometrically equal parts, onto themselves. The mathematical theory of symmetry differentiates and classifies configurations according to the structure of symmetry relations they contain. The structures the theory prescribes will be paralled by an orderly cognition of space in the architecture resulting from an application of the theory. Such an order is one of the essential qualities in works of architecture. Therefore, the thesis that the mathematical theory of symmetry may be a significant part of the knowledge an architect applies in design is established.

PART TWO: RESOURCES

2.1 THE MATHEMATICAL THEORY OF SYMMETRY

Fascination with symmetry has existed for many centuries, but the rigorous mathematical investigation of symmetry structure is a relatively recent investigation. Indeed, the mathematical tools for describing structure and classifying symmetrical configurations were only developed in the 19th Century. The emergence of what may be called a mathematical theory of symmetry has only taken place with modern mathematics. Because such a theory may be a significant part of the knowledge an architect applies in design, the resources of the theory should be integrated into design education. The intent of this whole part is to provide a non-technical, but precise, explanation of the mathematical theory of symmetry. The primary vehicle for that explanation will be the visual illustration of the structures provided by the theory. The intent of this particular section is to outline the derivation of the theory that is illustrated in the following three sections. An exposition of the technical definitions and theorems that might accompany an explanation of the mathematical theory of symmetry has been limited to the Mathematical Appendix. Reference will be made to that appendix where necessary.

There are several slightly different ways authors take to approach an explanation of the mathematical theory of symmetry. The difference between them is of emphasis, not direction. It is generally due to the context in the theory is to be applied. March and Steadman presented in The Geometry of Environment[61] a very coherent explanation of symmetry in one and two dimensions in the context of its presence as an underlying structure in works of architecture. This is the only readily available discussion of the mathematical theory of symmetry in the specific context of architecture. It emphasizes an architect's intuitive understanding of the mathematical ideas of mappings and transformations. The mathematical idea of symmetry operations (see Section 1.1, Fig. 1.104) was used to explain the symmetry structures described by the theory. This is a reasonable procedure to allow the non-mathematician architect to visualize symmetry in two dimensions. But the explanation here will be different. This is because the intention is to make the application useful as a design method in architecture; both in two and three dimensions.

The emphasis in the following illustrations is not on the symmetry operation, but on the whole structure of symmetrical configurations. The level of precision to which the working definition of symmetry was taken in Section 1.1 suggest that the group structure of symmetrical configurations be emphasized. The idea of a group of automorphisms (DEFINITION D:09, Mathematical Appendix) is the basis of the illustrations. The architect's intuitive understanding of the mathematical ideas of lattices and point groups distributed on those lattices will be the key to understanding the illustrations. It is hoped that this will not only allow the non-mathematician to visualize symmetry groups in two and three dimensions; but, because of the architect's experience with grids, to allow the application of symmetry structures in design.

Ultimately, the direction of the presentation here is the explanation of symmetry groups associated with the symmetrical configurations illustrated. A table of those groups will be presented in the appendix (TABLE MA:01, Mathematical Appendix). Group theory is a branch of modern higher mathematics that, among other things, provide the tools for the investigation of structure and the derivation of the mathematical thoery of symmetry. The concept of a group (DEFINITION D:01, Mathematical Appendix) was invented by the French mathematician Galois in 1830. It has proved to be one of the most powerful and significant abstractions in all of mathematics. Newman explained the significance of the concept by suggesting,

> The theory of groups is a branch of mathematics in which one does something to something and then compares the result with the result obtained form doing the same thing to something else, or something else to the same thing. This is a broad definition but it is not trivial. The theory is a supreme example of the art of mathematical abstraction. It is concerned only with the fine filigree of underlying relationships; it is the most powerful instrument yet invented for illuminating structure.[62]

The theory also sounds remarkalby appropriate in design. The idea of group theory is to differentiate and classify the structure of relations in the thing to which it is applied. It is possible to show that there are a finite number of space creating configurations of geometrically equal parts possessing an automorphism. Those automorphisms form a group, and by definition this is the symmetry group associated with those configurations. The mathematical theory of symmetry describes the finite number of structurally different symmetrical configurations in one, two and three dimensions using group theory. The following sections illustrate examples of symmetrical configurations associated with each symmetry group.

The reason that there are only a finite number of symmetry groups is because of the requirement for automorphism of the whole configuration. A group of automorphisms of an entire configuration is the result of the presence of an underlying lattice and the repeated appearance of a point group at each point in that lattice. A symmetrical configuration is always produced simply by combining a point group with a lattice. There are a finite number of symmetry groups just because there are a limited number of structurally different lattices and a limited number of point groups compatible with those lattices.

A lattice may be defined as a collection of points arranged in such a way that each point has the same spatial relationships in the same directions as every other point in the lattice. Intuitively, the architectural equivalent is a grid, in which every point of intersection of parallel grid lines is identical with every other. Because grids are common design devices in architecture, it is appropriate to emphasize the idea of an underlying lattice in symmetrical configurations. The classic study of lattices in mathematics was undertaken by another French mathematician, Bravais, in 1850. Bravais showed that there are only fourteen structurally different lattices in three dimensions, distinguished by their unit cells. The architectural equivalent of a unit cell would be the smallest bay defined in a grid. The fourteen spatial lattices have become known as Bravais lattices. Later, it was shown that there are only five Bravais lattices in two dimensions; and only a single Bravais lattice in one dimension.

A point group may be defined as a group of automorphisms acting about a point which leave that point "invariant", that is, the same for each automorphism. Intuitively, a point group may be equated with an arrangement of architectural elements about a point such that each element is the same distance and at the same angle from the point. The investigation of point groups was origninally undertaken to study the shapes of crystals. Weyl attributed the first listing of the two-dimensional point groups to Leonardo da Vinci[63] although he did not use the mathematics of group theory, obviously. The mathematical treatment of the subject proceeded only after the development of group theory. It has been shown that there are only thirty-two different pointgroups in three dimensions, ten in two dimensions, and just two in one dimension that are compatible with the Bravais lattices at each dimension. This results from a proof of the "cystallographic restriction"; excellent examples of which may be found in Weyl[64] and in March and Steadman.[65]

The mathematical theory of symmetry was first formulated at the very end of the 19th Century by combining these point groups with the Bravais lattices in an exhaustive manner; using group theory to identify and classify all possible symmetry groups. The theory was the product of two major independent works. The Russian crystallographer Fedorov was the first, about 1885, to establish the existence of only two hundred thirty symmetry groups in space. But the German mathematican Schoenflies was the first to publish, about 1891, the mathematical derivation and exhaustive classification of the two hundred thirty groups. Schoenflies did concede Fedorov the credit for establishing the mathematical theory of symmetry.[66] It has also been shown using similar mathematical classification that there are only seventeen symmetry groups in two dimensions, and only seven in one dimension. Despite the fact that the mathematical theory of symmetry did not completely emerge until the 20th Century, all seventeen symmetry structures possible in two dimensions may be found in the decorative tradition of Ancient Egyptian art. Weyl commented:

> One can hardly overestimate the depth of geometric imagination and inventiveness reflected in these patterns. Theor construction is far from mathematically trivial. The art of ornament contains in implicit form the oldest piece of higher mathematics known to us. To be sure the conceptual means for a complete abstract formulation of the underlying problem, namely the mathematical notion of a group of transformations, was not provided before the 19th Century; and only on this basis is one able to prove that the 17 symmetries already implicitly known to the Egyptian crafstman exhaust all possibilities.[67]

Indeed,all decorative patterns, wallpaper patterns, frieze patterns, and similar ornamentation in architecture, as it has been used for centuries in many parts of the world, is based on two-dimensional symmetry.[68]

Because Fedorov is credited with their enumeration, the two hundred thirty space groups describing symmetrical configurations in three dimensions are commonly called the "Fedorov groups". But the origins of the plane groups describing symmetrical configurations in two dimensions are lost in antiquity, hence they are just called the "Wallpaper groups". Similarly, the linear groups describing symmetrical configuration in one dimension are called "Frieze groups". The mathematical theory of symmetry is about the identification and classification of these symmetry groups. The numbers associated with the theory may be summarized in a table (TABLE 2.1:01).

TABLE 2.1:01

Dimension	Bravais lattices	Point Groups	Symmetry Groups
One	1	2	7 (Freize)
Two	5	10	17 (Wallpaper)
Three	14	32	230 (Fedorov)

This table indicates the number of Bravais lattices and the number of point groups compatible with them; they combine to form the number of symmetry groups indicated.

The format for the explanation of the mathematical theory of symmetry in the next three sections is to illustrate examples of all the Bravais lattices and all the point groups compatible with them. In the section on symmetry in one dimension, typical symmetrical configurations associated with all seven Frieze groups are illustrated. Similarly, in the section on symmetry in two dimensions, typical symmetrical configurations associated with all seventeen Wallpaper groups are illustrated. However, in the section on symmetry in three dimensions, it would simply not be practical to illustrate symmetrical configurations associated with all two hundred thirty Fedorov groups. Only one such configuration has been illustrated just as an example. However, with all fourteen Bravais lattices and all thirty two point groups in three dimensions illustrated, it is still possible to use the section as a resource for application in design. This is the advantage of the approach taken here, as opposed to the approach taken by March and Steadman.

The mathematical theory of symmetry is one of those areas in which there are several competing notations, each with its own advantage and disadvantage. In general, to the lower left in the illustrations that follow is the International symbol, sometimes called the Hermann-Maugin notation. It consits of numbers indicating the number of turns of rotation, for example, "3", indicates a three turn rotation in an automorphism. There are also small letters; "m", indicating a mirror reflection, "g", indicating a glide reflection. A number with a bar over it indicates an inversion in space through the point about which there is a rotation of that number of turns. To indicate that state of the underlying lattice in two dimensions, lower case letters, "p" and "c", are used, meaning primitive and centered. Similarly, the state of the underlying lattice in three dimensions is indicated with upper case letters, "P", "C", "F", and "I", meaning primitive, centered, face-centered,and body-centered. The meaning of these states should be clear through the visual information in the illustrations of the Bravais lattices in the following three sections. To deonte a symmetry with the International symbol, one first writes a letter indicating the state of the underlying lattice, then a series of numbers and letters indicating the rotations and reflections present. For example, "p4gm", indicates a symmetry group based on a primitive two dimensional lattice, with four turn rotations, glide reflection, and mirror rflection.

In general, to the lower right in the illustrations that follow is a mathematical notation. These are of two types. The primary notation is the symbols developed by Schoenflies in his pioneering enumeration of the symmetry groups. To indicate the point groups, the symbol uses an upper case letter, "C", "D", "T", and "O", which is associated with the structure of the group in relation to the symmetry of cyclic, dihedral, tetrahedral, and octahedral solids. The letter is followed with a sub-script number which indicates the number of turns of rotation in the point group. For example, "C_4", indicates the four turn cyclic group. However, to indicate the symmetry groups, the Schoenflies symbol is often replaced by second type of notation[69] that is a sort of shorthand. For symmetry groups in one dimension, the upper case letter, "F", for "frieze, is used. For symmetry groups in two dimensions, the upper case letter, "W", for wallpaper, is used. These are then followed by a sub-script number, indicating the number of turns of rotation; and by a super-script number, arbitrarily indicating the different reflections present, if any. So the symmetry group, "p4gm", in the International symbols is given the symbol, "W_4^2" in this mathematical shorthand. However, for the Fedorov symmetry groups in three dimensions, there is no shorthand. For these groups, it is conventional to use the Schoenflies symbol for the point group, again with a super-script number arbitrarily indicating the different instances of all the symmetry groups based on that same point group. For example, "D_2^8", indicates the eighth instance of symmetry groups in three dimensions based on the two turn dihedral point group.

Both the International and Schoenflies symbols are more subtle than the explanation above. They both, also, have certain advantages over the other as systems of notation. The International symbol tells more about the actual physical symmetrical configuration in terms of the underlying lattice and symmetry operations. This is why it is preferred by cyrstallographers. Unfortunately, it tells very little about the mathematical group structure associated with the configuration. The Schoenflies symbol does tell exactly that, it is a "pure" mathematical notation indicating the group structure. It allows the mathematician to investigate and classify symmetry groups according to their mathematical properties. This is why it is preferred by mathematicians. Because they are each better for certain purposes, both the International and Schoenflies symbols have been included in the illustrations following.

There are also conventions for numbering the symmetry groups established in the International Tables of X-Ray Crystallography.[70] These numbers are also shown in the right side of the title box for each illustration of a symmetry group in the following sections. These table might be consulted in any event, as they provide an encyclopedia of the two hundred thirty Fedorov symmetry groups in space.

The next three sections illustrate the resources of the mathematical theory of symmetry as they might be applied in architecture. Perhaps the architect will most appreciate the visual explanation of the theory provided by illustrations of symmetrical configurations, rather than a technical mathematical explanation. Irregardless of notations and emphasis, a designer would certainly develop a deep sense for the mathematical theory of symmetry by actually drawing symmetrical configurations. Similarly, the observation of symmetrical configurations in both the man-made and natural worlds would help develop that sense. The illustration of Bravais lattices and point groups in the following three sections are intended as an educational resource for those exercises. Those exercises, together with the theoretical groundwork provided in Part One, should support the thesis that the mathematical theory of symmetry may be a significant part of the knowledge an architect applies in design.

2.2 SYMMETRY IN ONE DIMENSION

The intent of this section is to illustrate the single Bravais lattice, two point groups, and seven freize groups of symmetry in one dimensions. The presentation here is the same as it will be in the following two sections, a small black asymmetric triangle has been taken as an element to be transformed into symmetrical configurations. The choice of elements is totally arbitrary, although if the form of the element were symmetrical itself (say involving a reflection) it would not be suitable for illustrating all possible symmetrical configurations.

In one dimension, the single Bravais lattice consists of a series of points in a straight line, separated by an arbitrary dimension 'a' (Fig. 2.201). The two point groups that may be combined with that lattice are the cyclic group of one turn rotation, C_1 , and the cyclic "symmetry" group, C_S , which might also have been called the dihedral group of one turn rotation, D_1, (Fig. 2.201). These two point groups combine with the single Bravais lattice to produce the seven fieze groups, F_1 , F_1^1 , F_1^2 , and F_1^3 (Fig. 2.202); F_2 , F_2^1 , F_2^2 (Fig. 2.203). These groups exhaust all the possibilities for symmetrical configurations in one dimension.

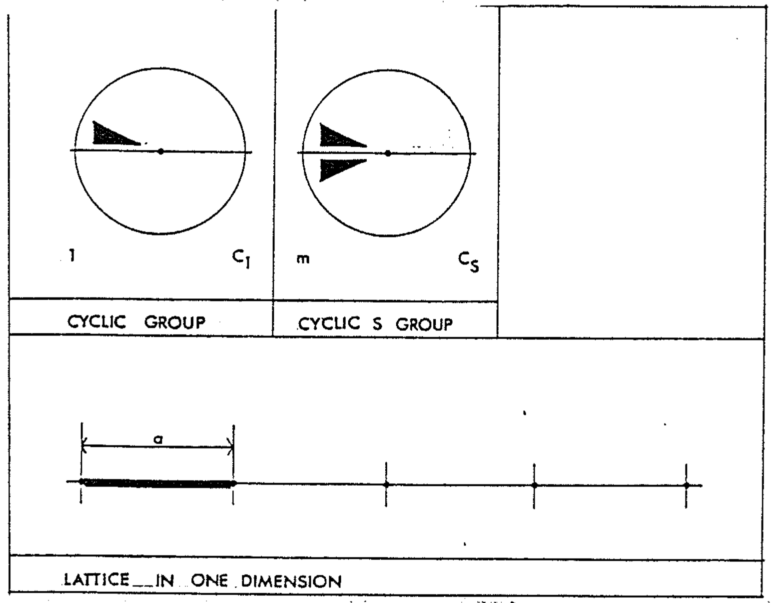


Fig. 2.201

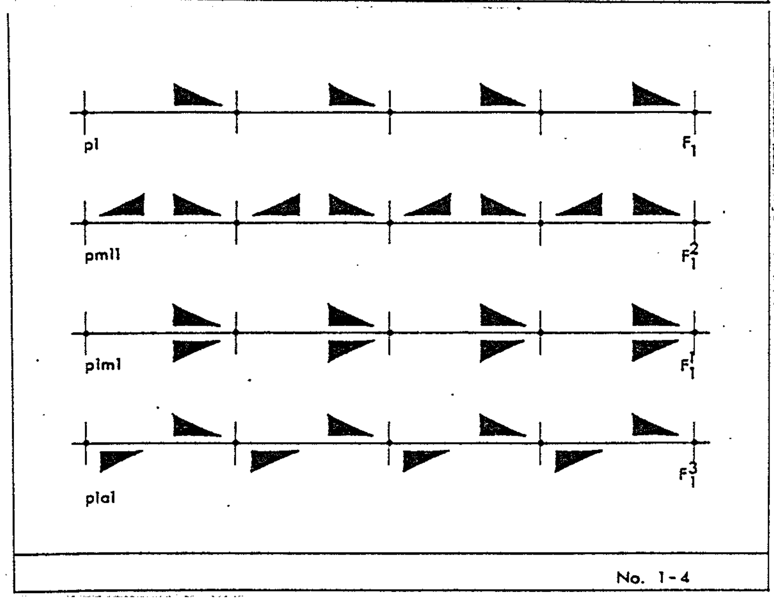


Fig. 2.202

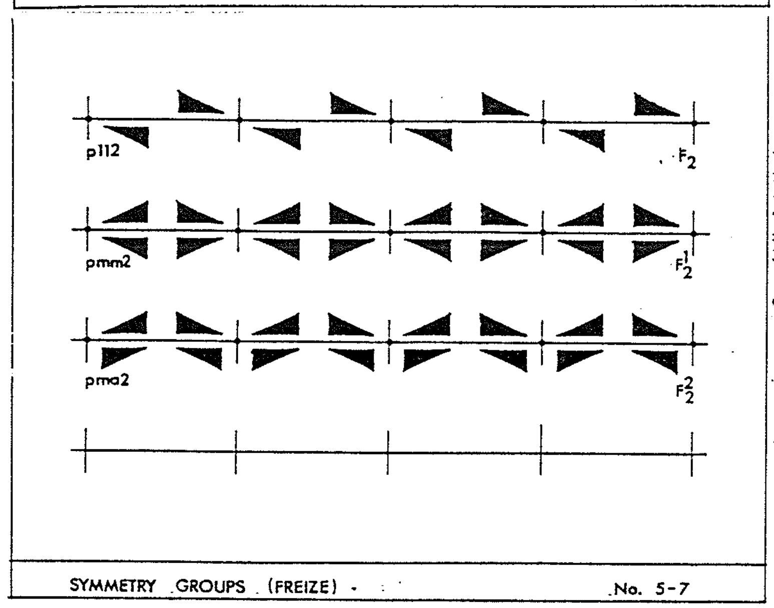


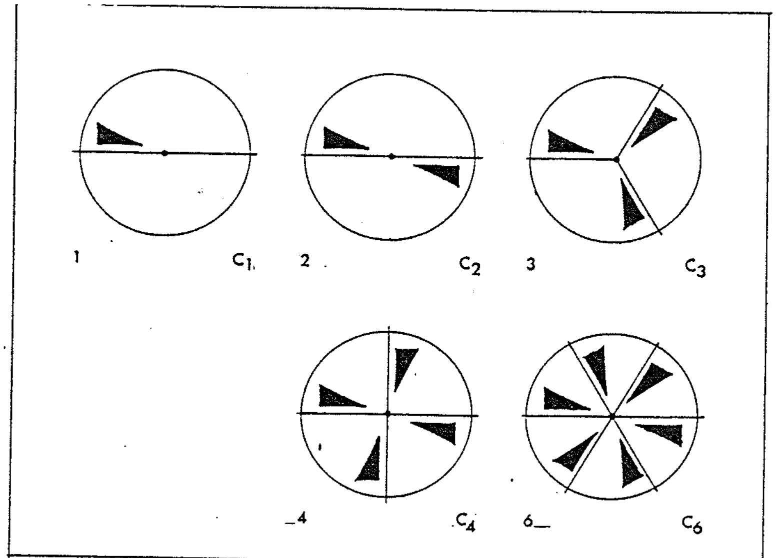
Fig. 2.203

2.3 SYMMETRY IN TWO DIMENSIONS

The intent of this section is to illustrate the five Bravais lattices, ten point groups, and seventeen wallpaper groups of symmetry in two dimensions. The presentation uses the same black asymmetric triangle transformed into symmetrical configurations associated with each group. But, unlike the previous section, this section is divided into five sub-sections, one for each of the Bravais lattices and the symmetry groups based on those lattices. In each of the sub-sections, the unit cell of the Bravais lattice has been indicated in heavy lines; and the conditions which create that cell are indicated.

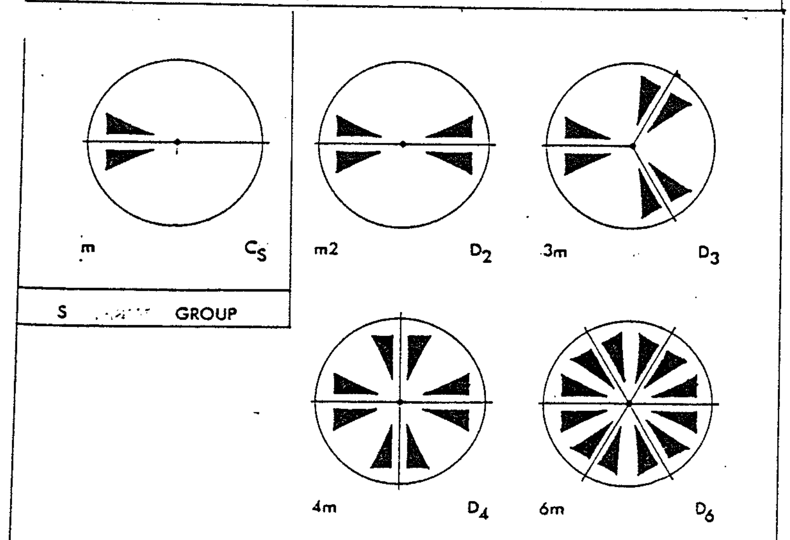
But, first the ten point groups which may be combined with all five lattices are illustrated. The proof of the crystallographic restriction (see reference in section 2.1) establishes that only point groups of one, two, three, four, and six turn rotations are present in symmetrical configurations. Therefore, the ten point groups in two dimensions are C_1 , C_2 , C_3 , C_4 , and C_6 (Fig. 2.301); and C_S (or D_1), D_2 , D_3 , D_4 , and D_6 (Fig. 2.302). Each of these point groups combine with certain of the five Bravais lattices to produce the seventeen wallpaper groups of symmetry in two dimensions.

Fig. 2.301

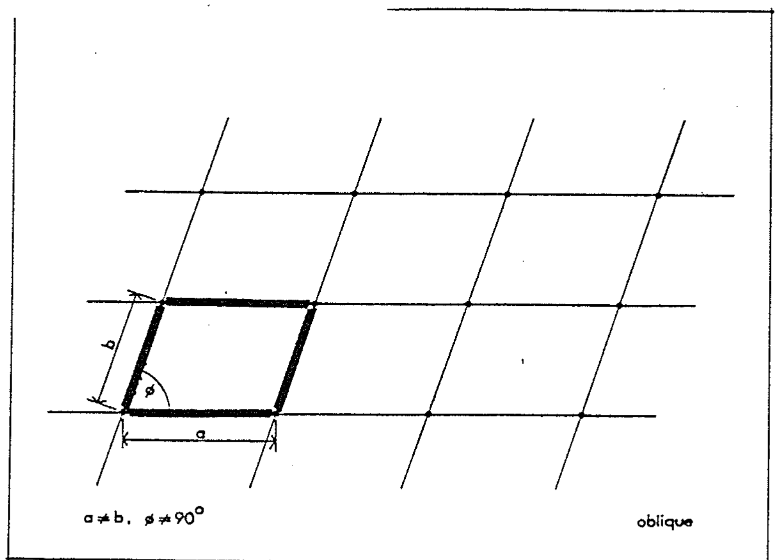


CYCLIC POINT GROUPS

Fig. 2.302



DIHEDRAL POINT GROUPS



LATTICE IN TWO DIMENSIONS

2.3.1 OBLIQUE Fig. 2.303

The first Bravais lattice in two dimensions is the oblique lattice (Fig. 2.303). The unit cell is primitive, in the sense that there are no points inside the cell created by a grid of parallel lines through the points. The unit cell consists in two unequal arbitrary lengths, a and b, with an angle between them, Ø , of anything except 90°. This is the most general and least restricted lattice in two dimensions.

The oblique lattice combines with the cyclic point groups, C_1 and C_2 , to produce the first two symmetrical configurations of the seventeen wallpaper groups of symmetry in two dimensions. They are the groups: W_1 (Fig. 2.304) and W_2 (Fig. 2.305). No other point groups combine with the oblique lattice to produce any symmetry distinct from these two groups.

This is typical of the presentations to follow in each of the sub-sections; it is hoped that a feeling for the approach used will allow the use of the illustrations as resources for design.

Fig. 2.304

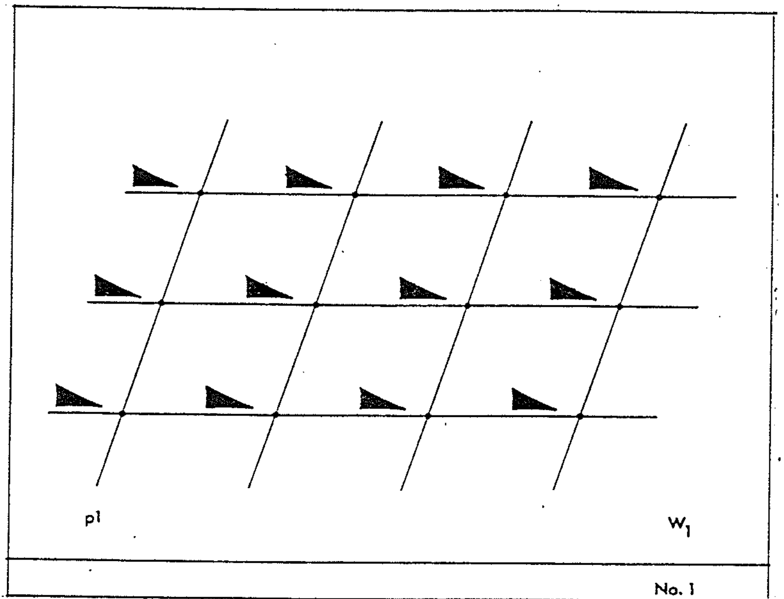
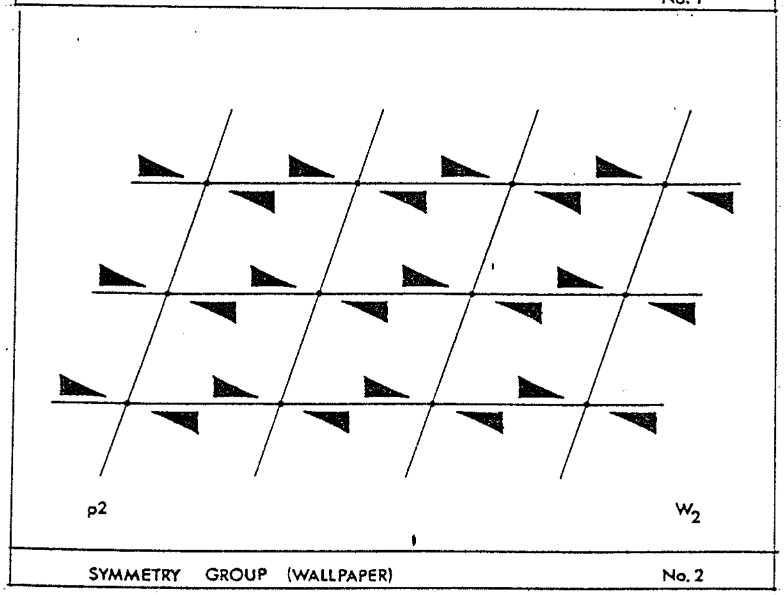
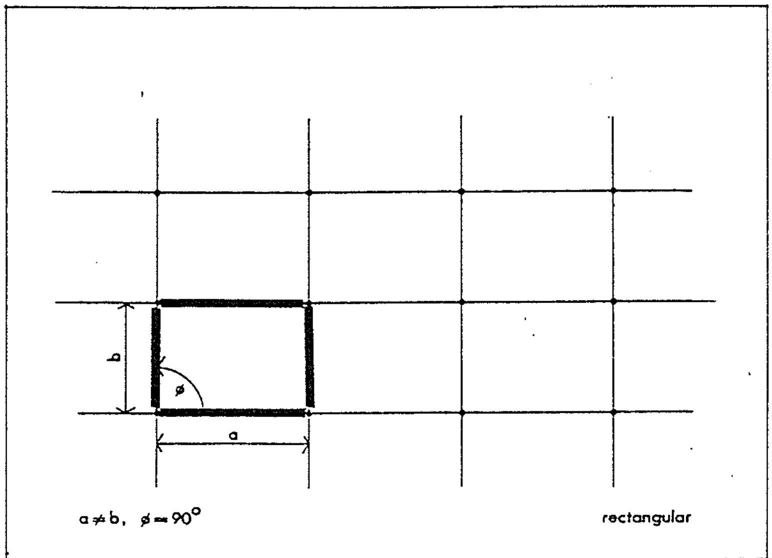


Fig. 2.305





LATTICE IN TWO DIMENSIONS

2.3.2 RECTANGULAR Fig. 2.306

The second Bravais lattice in two dimensions is the rectangular lattice (Fig. 2.306). The unit cell is also primitive, consisting in two unequal arbitrary lengths, a and b, with an angle between them, $\emptyset$, of exactly 90°. The lattice is, obviously, equivalent to the rectangular grids often used by architects. In this sense this is a very important sub-section of symmetry in two dimensions.

The rectangular lattice combines with the point groups, C_1, C_2 , C_S in two ways, and D_2 to produce the next five symmetrical configurations of the seventeen wallpaper groups of symmetry in two dimensions. They are the groups: W_1^2 (Fig. 2.307), W_1^3 (Fig. 2.308), W_2^2 (Fig. 2.309), W_2^3 (Fig. 2.310), and W_2^4 (Fig. 2.311). No other point groups combine with the oblique lattice to produce any symmetry distinct from these five groups. It should be noted that W_2^4 (Fig. 2.311) has been shown with the point group C_2 at an angle bisecting the angle of the lattice. This is necessary if the point group is to occur at every point in the lattice, but this group is often shown with the point group not on the angle at every other point in the lattice.[71]

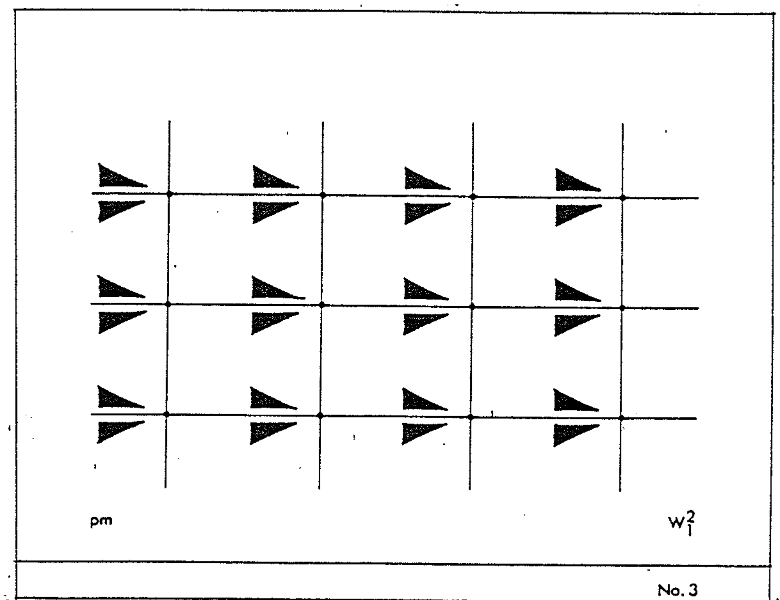


Fig. 2.307

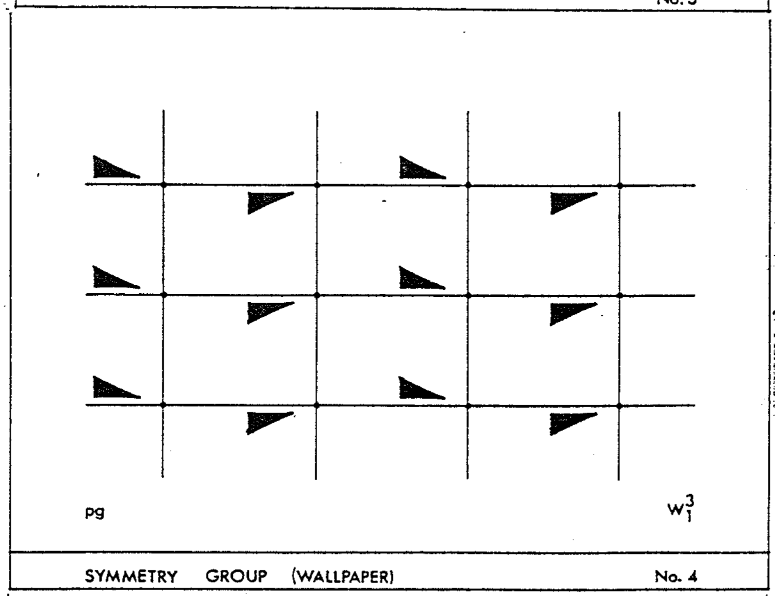


Fig. 2.308

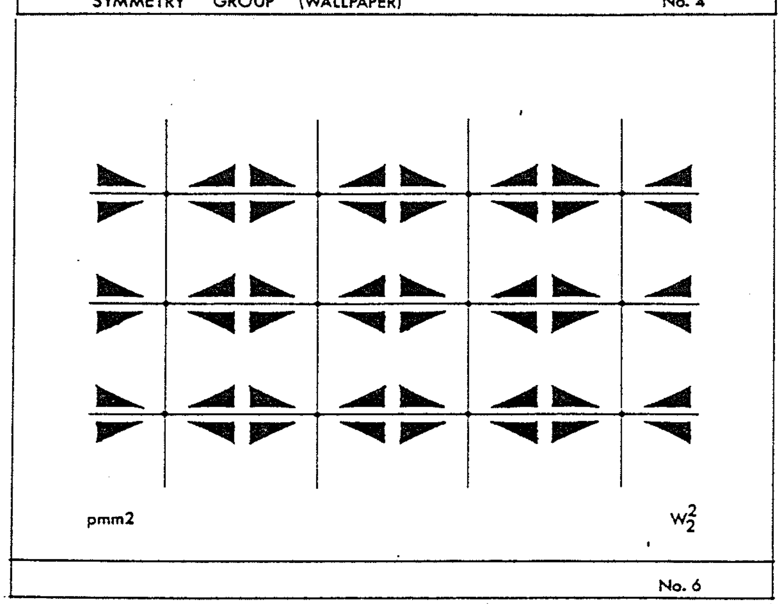


Fig. 2.309

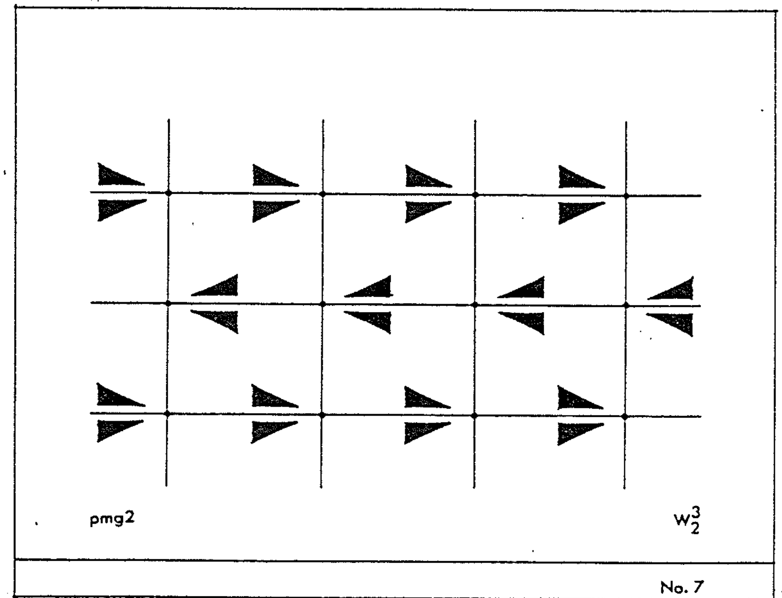


Fig. 2.310

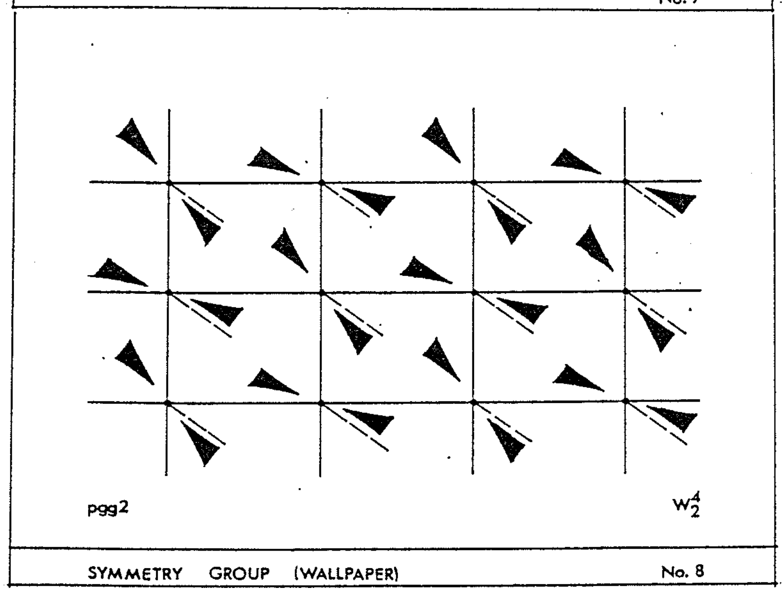
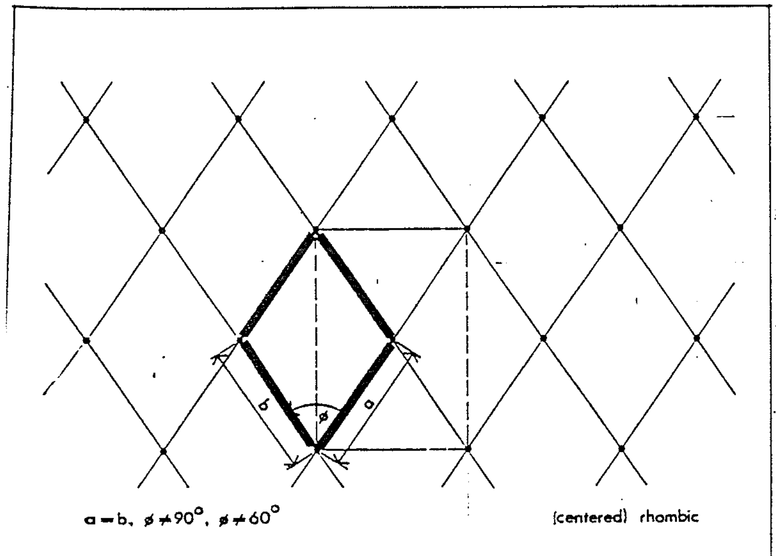


Fig. 2.311



LATTICE IN TWO DIMENSIONS

2.3.3 RHOMBIC Fig. 2.312

The third Bravais lattice in two dimensions is the rhombic lattice (Fig. 2.313). The unit cell is not primitive, it is centered because it is possible to draw a cell created by parallel lines (shown as broken) that has a point exactly at its center. The unit cell consists in two equal arbitrary lengths, a = b, with an angle between them, $\emptyset$, of anything except 60° or 90°.

The rhombic lattice combines with the point groups, C_S and D_2 , to produce two more symmetrical configurations of the seventeen wallpaper groups of symmetry in two dimensions. They are the groups: W_1^1 (Fig. 2.313), and W_2^1 (Fig. 2.314). No other point groups combine with the rhombic lattice to produce any symmetry distinct from these two groups.

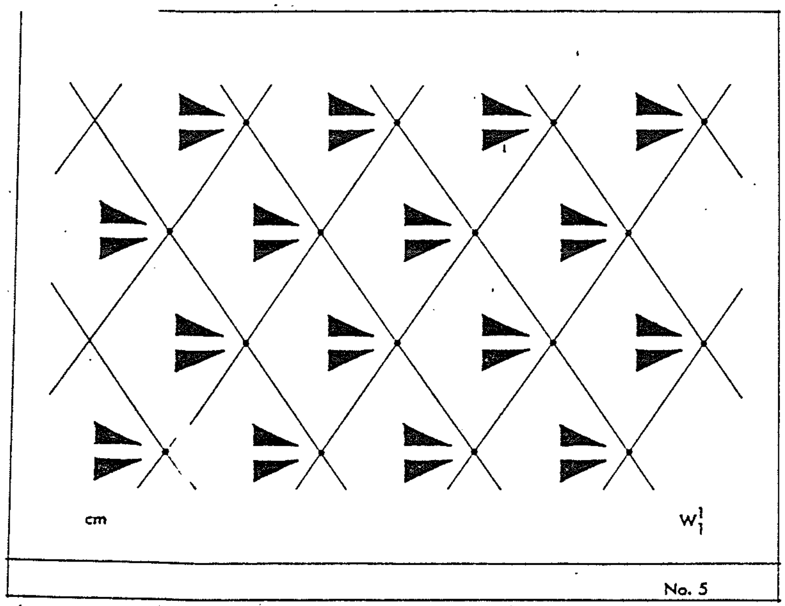


Fig. 2.313

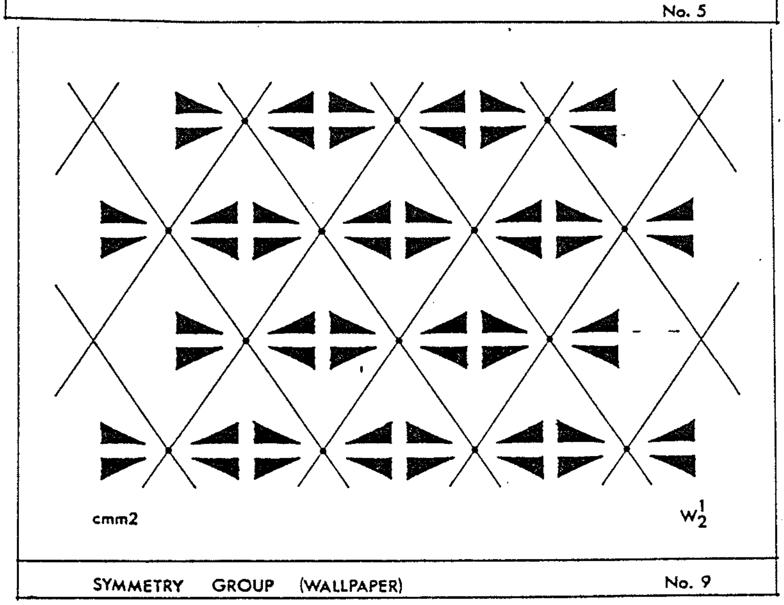
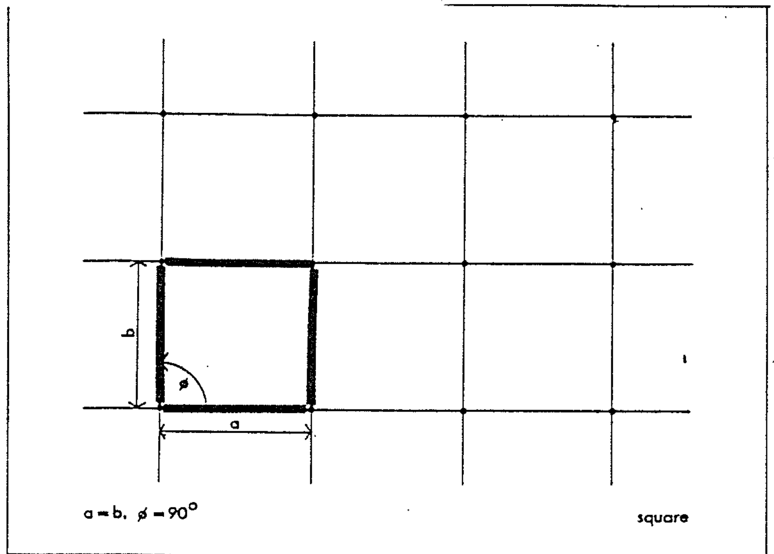


Fig. 2.314



LATTICE IN TWO DIMENSIONS

2.3.4 SQUARE Fig. 2.315

The fourth Bravais lattice in two dimensions is the square lattice (Fig. 2.315). The unit cell is, once again, primitive, consisting in two equal arbitrary lenghts, a = b, and an angle between them, $\emptyset$, of exactly 90^{o}. This is also equivalent to the square grids commonly used as design devices in architecture.

The square lattice combines with the point groups, C_4 in two ways and D_4, to produce three more symmetrical configurations of the seventeen wallpaper groups of symmetry in two dimensions. They are the groups: W_4 (Fig. 2.316), W_4^1 (Fig. 2.317), and W_4^2 (Fig. 2.318). No other point groups combine with the square lattice to produce any symmetry distinct from these three groups. It should be noted that W_4^2 (Fig. 2.318) again involve the point group C_4 at an angle bisecting the angle of the lattice, and again this group is often shown with the point group not on the angle at every other point in the lattice.[72]

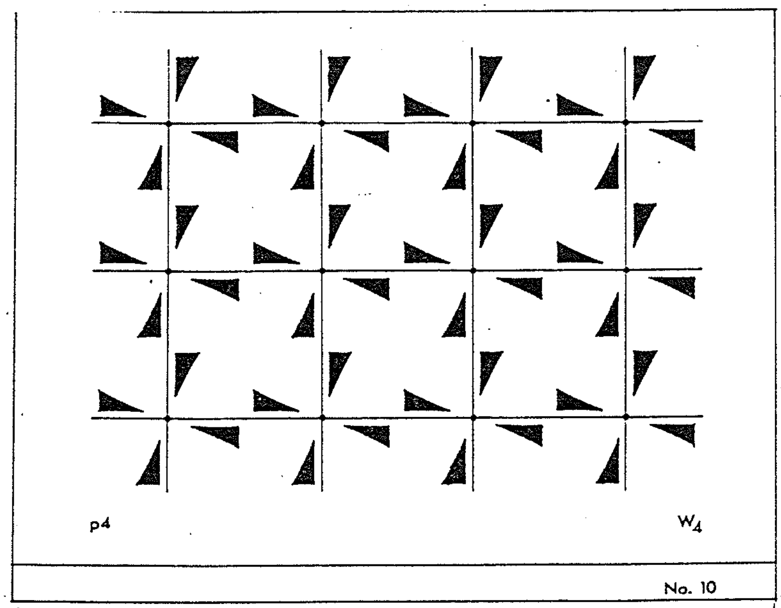


Fig. 2.316

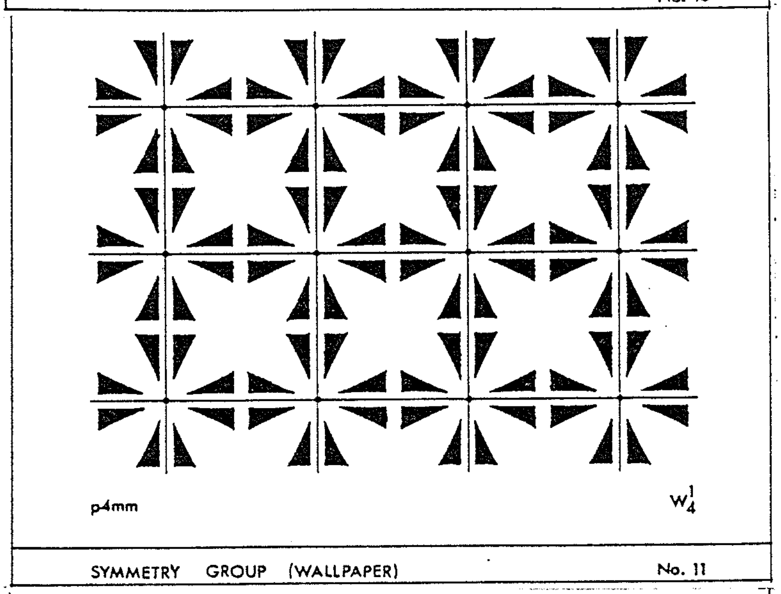


Fig. 2.317

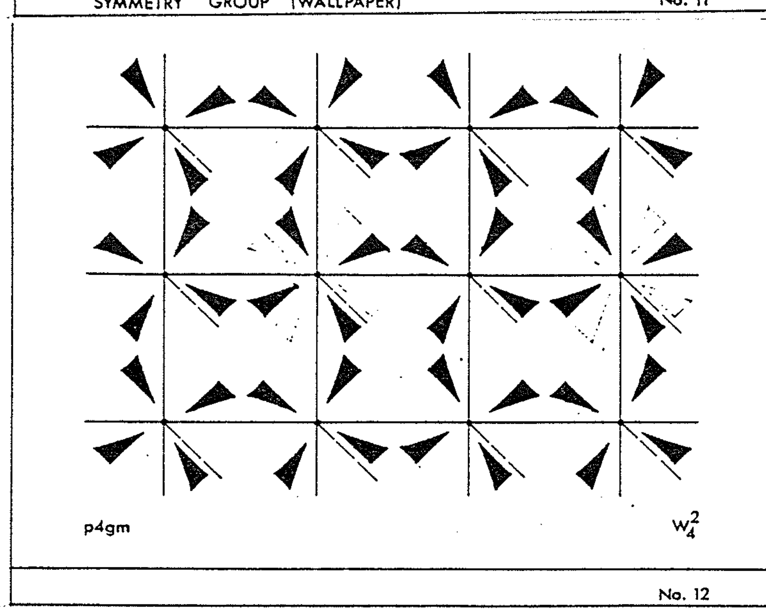
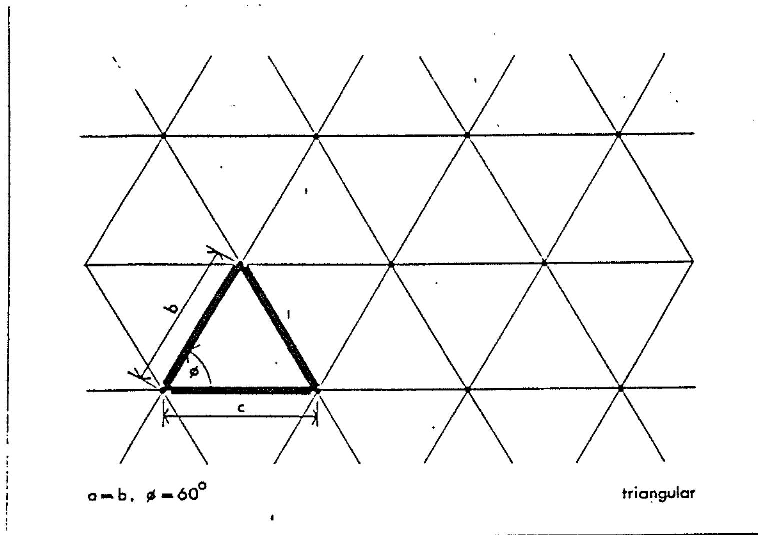


Fig. 2.318



2.3.5 TRIANGULAR Fig. 2.319

LATTICE IN TWO DIMENSIONS

The fifth and final Bravais lattice in two dimensions is the triangular lattice (Fig. 2.319). The unit cell is primitive, consisting in two equal arbitrary dimensions, a = b, with an angle between them, ø , of exactly 60°. The lattice is also not disimiliar to some grids in modern architecture, especially those based on the use of space frames, which often involve 60° geometry.

The triangular lattice combines with the point groups, C_3 , C_6 , D_3 in two ways, and D_6 , to produce the final five symmetrical configurations of the seventeen wallpaper groups of symmetry in two dimensions. They are the groups: W_3 (Fig. 2.320), W_3^1 (Fig. 2.321), W_3^2 (Fig. 2.322), W_6 (Fig. 2.323), and W_6^1 (Fig. 2.324). No other point groups combine with the triangular lattice to produce any symmetry distinct from these five groups.

This completes the illustration of the seventeen wallpaper symmetry groups in two dimensions. These groups exhaust all the possibilities for symmetrical configurations in two dimensions.

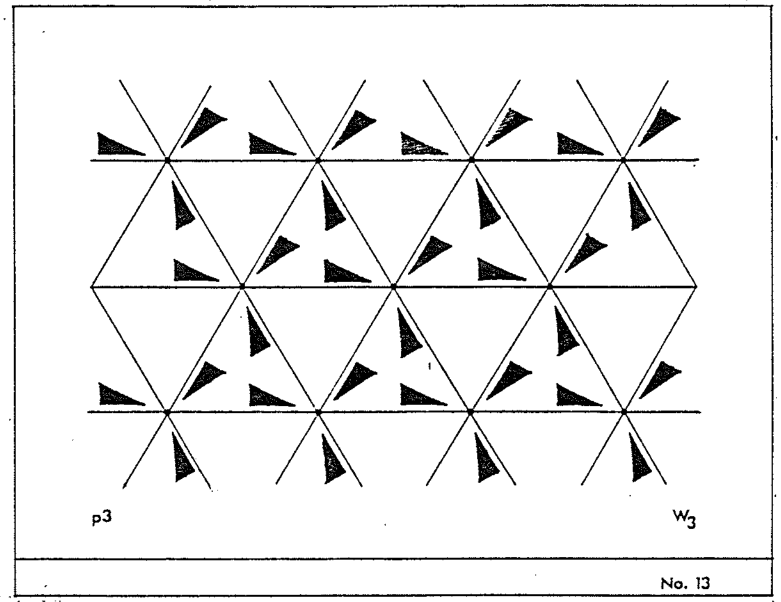


Fig. 2.320

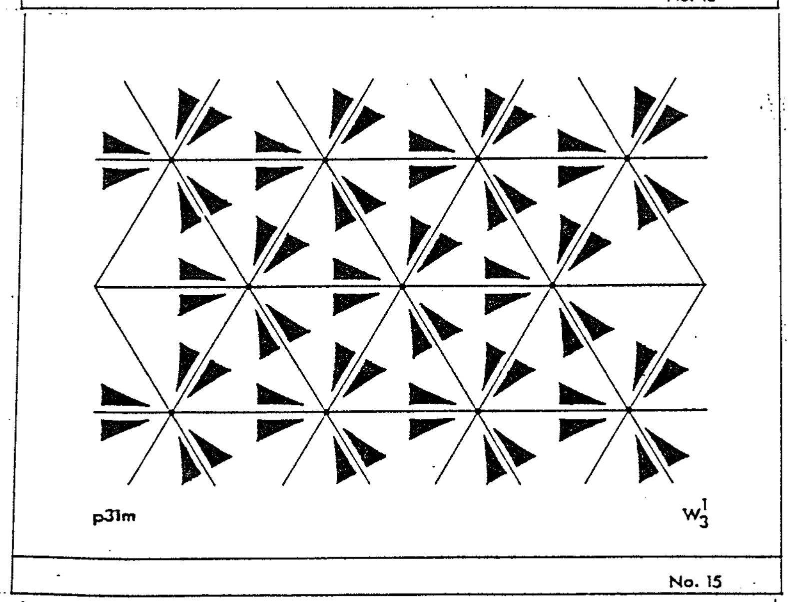


Fig. 2.321

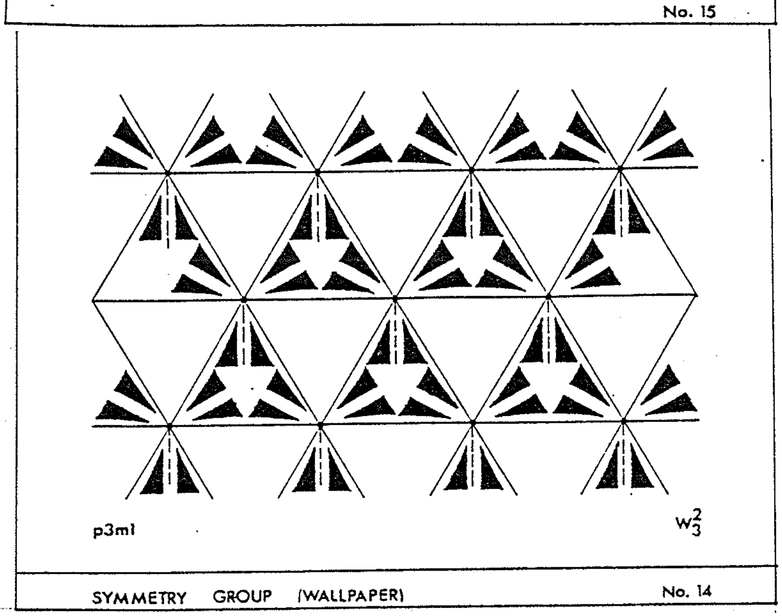


Fig. 2.322

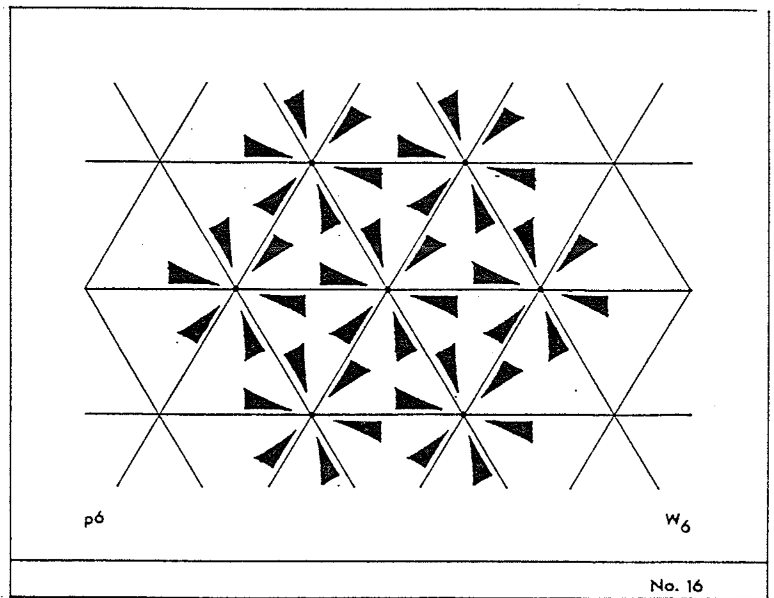


Fig. 2.323

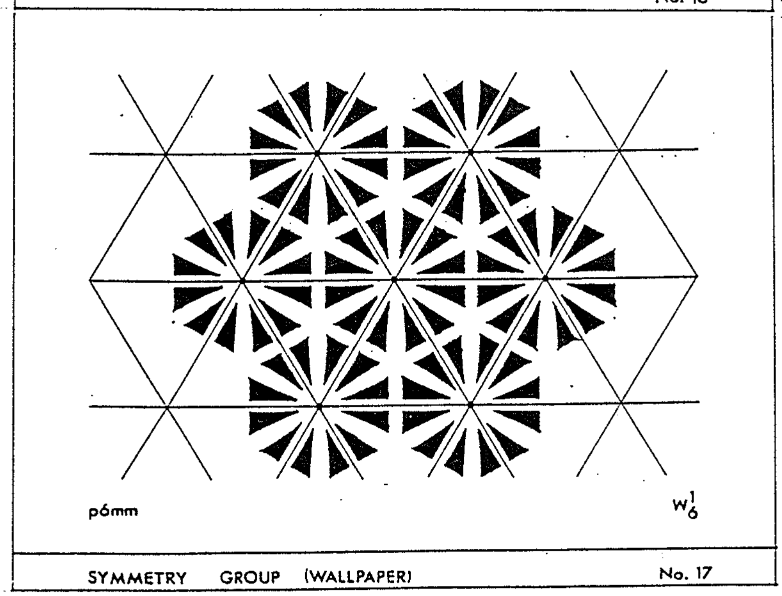
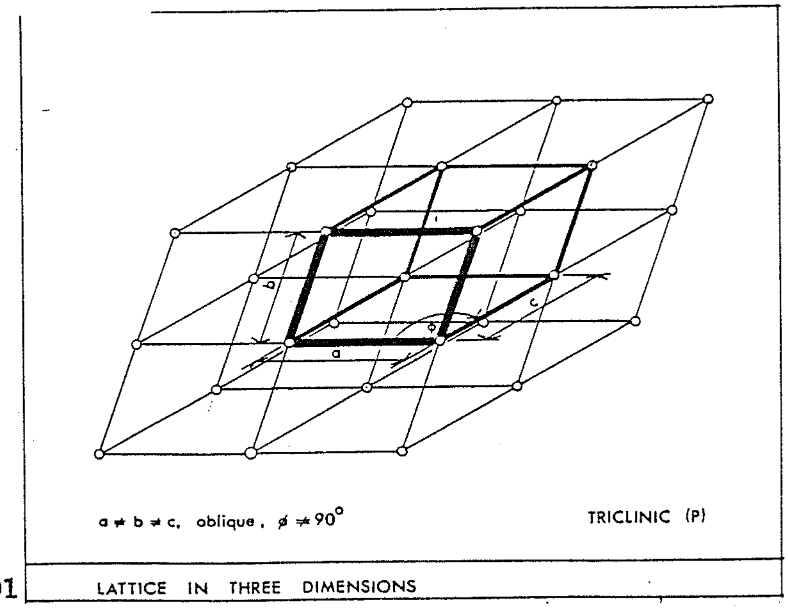


Fig. 2.324

2.4 SYMMETRY IN THREE DIMENSIONS

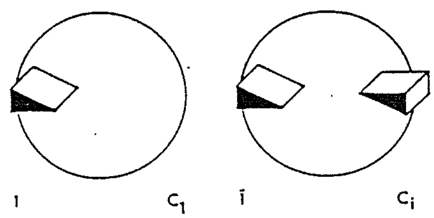
The intent of this section is to illustrate the fourteen Bravais lattices, and thirty two point groups that combine to produce the two hundred thirty Fedorov groups of symmetry in three dimensions. As indicated earlier, it is not practical to illustrate all two hundred thirty symmetry groups in space, but the illustrations provided should be enough to allow the designer to use this section as a resource. Like the previous section, this section is divided into sub-sections. The fourteen Bravais lattices and thirty two point groups may be classified into seven systems; commonly called the seven crystal classes in three dimensions. Like previous sections, a small black asymmetric triangle,projected into a triangular solid, has been taken as the element. Similarly, the Bravais lattices have been shown in projection, with the unit cells indicated in heavy lines, and the conditions which create that cell are indicated with the two dimensional lattice facing the viewer also indicated. Because architecture is an art of space creation (see section 1.3), symmetry in three dimensions may be the most important part of the mathematical theory of symmetry.



2.4.1 TRICLINIC Fig. 2.401

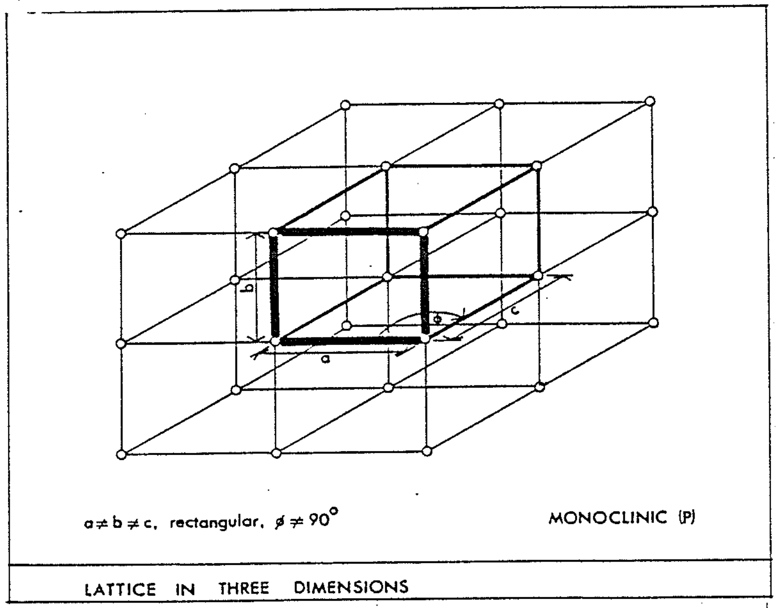
The first crystal class in three dimensions is the triclinic class. In this class there is only one Bravais lattice, the primitive triclinic (Fig. 2.401). The unit cell consists in three unequal arbitrary lengths, a, b, and c; with an oblique two dimensional lattice facing the viewer, projecting into spaceat an angle (to the page), ∅, of anything except 90°. Again, this is the most general and least restricted lattice in three dimensions.

This single lattice combines with two point groups in three dimensions, they are the cyclic point groups, C_1 and C_i (Fig. 2.402). The two triclinic point groups combine with the primitive lattice to generate only two of the two hundred thirty Fedorov groups in space (see TABLE MA:01, Mathematical Appendix). The triclinic crystal class contains just these two symmetry groups in space.



TRICLINIC POINT GROUPS

Fig. 2.402



2.4.2 MONOCLINIC Fig. 2.403

The second crystal class in three dimensions is the monoclinic class. In this class there are two Bravais lattices, the primitive monoclinic (Fig. 2.403) and the centered monoclinic (Fig. 2.404). The primitive unit cell consists in three unequal arbitrary lengths, a, b, and c; with a rectangular lattice facing the viewer, projecting into space at an angle, ϕ , of anything except 90°. The centered unit cell is identical, except there is a point at the center in the side of cell projecting into space.

These two lattices combine with three point groups in three dimensions, they are the cyclic point groups, C_2 , C_S , and C_{2h} (Fig. 2.405). These groups may be drawn in two ways, since they occur in crystals in both ways, they have been called the 1st setting (top row, Fig. 2.405) and the 2nd setting (bottom row, Fig. 2.406). The three monoclinic point groups combine with the primitive lattice to generate eight of the Fedorov groups; and with the centered lattice to gnerate five more (see TABLE MA:01, Mathematical Appendix). In total, the monoclinic crystal class contains thirteen symmetry groups in space.

Fig. 2.404

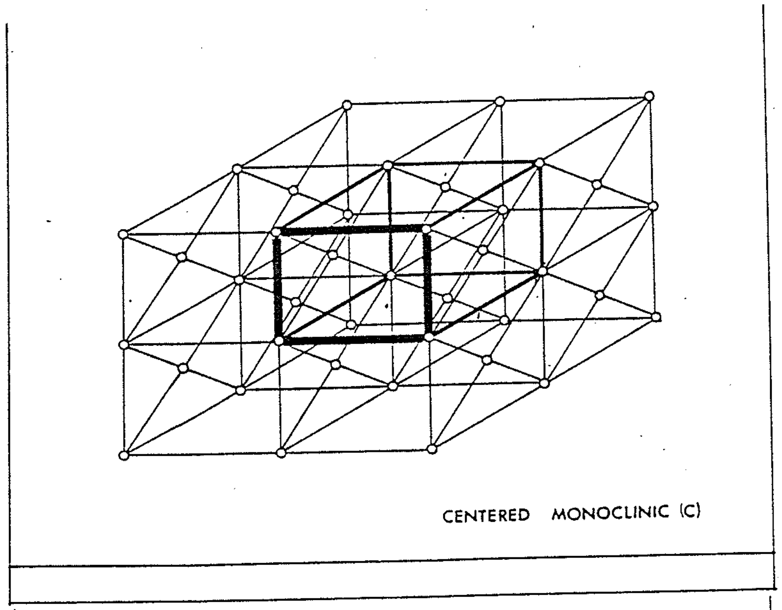
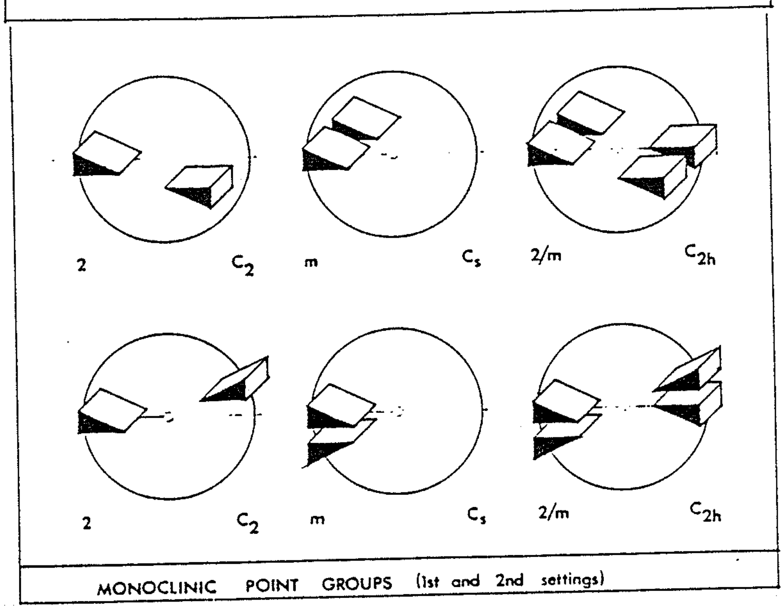
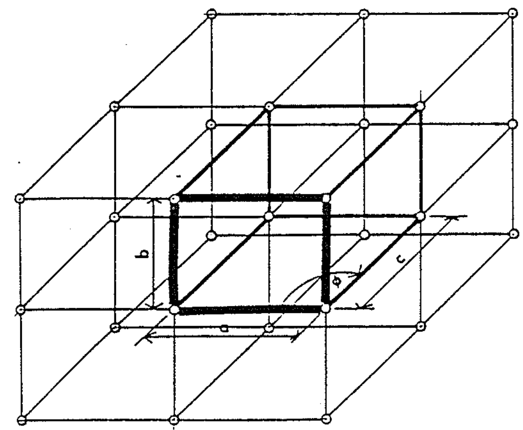


Fig. 2.405





LATTICE IN THREE DIMENSIONS

2.4.3 ORTHORHOMBIC Fig. 2.406

The third crystal class in three dimensions is the orthorhombic class. In this class there are four Bravais lattices, the primitive orthorhombic (Fig. 2.406), the centered orthorhombic (Fig. 2.407), the face-centered orthorhombic (Fig. 2.408), and the body-centered orthorhombic (Fig. 2.409). The primitive unit cell consists in three unequal lengths, a, b, and c; with a rectangular lattice facing the viewer, projecting into space at an angle, Ø , of exactly 90°. The centered unit cell is identical, except there is a point at the center of the side of the cell projecting into space. The face-centered unit cell is also identical, except there is a point at the center of every face of the cell. The body-centered unit cell is also identical to the primitive unit cell, except there is a point at the center of the body of the cell.

These four lattices combine with three point groups in three dimensions, they are the cyclic point group, C_{2v} , and the dihedral point groups, D_2 and D_{2h} (Fig. 2.410). The three orthorhombic point groups combine with the primitive lattice to generate thirty of the Fedorov groups; with the centered lattice to generate fifteen; with the face-centered to generate five; and with

(cont'd. p. 93)

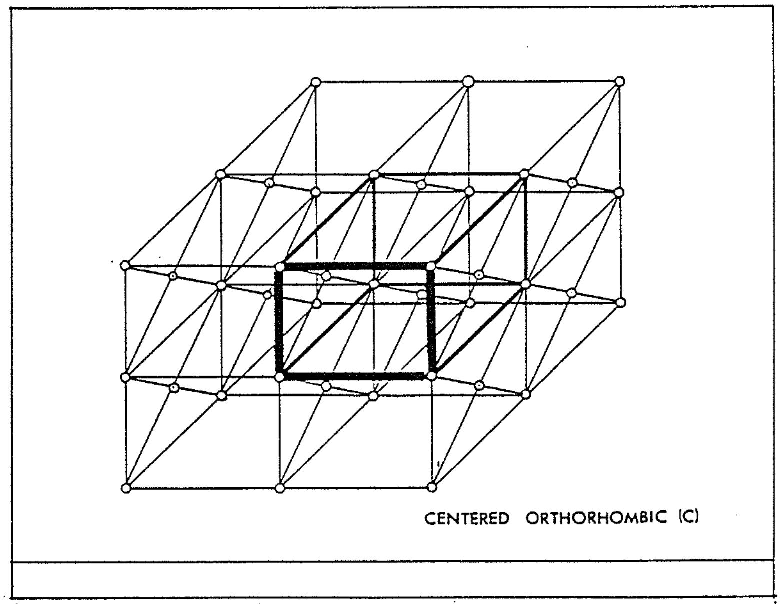


Fig. 2.407

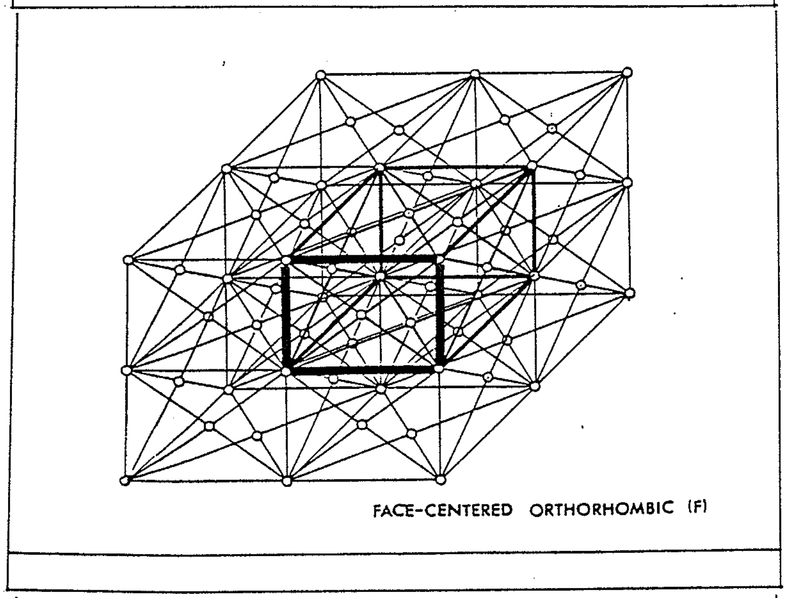


Fig. 2.408

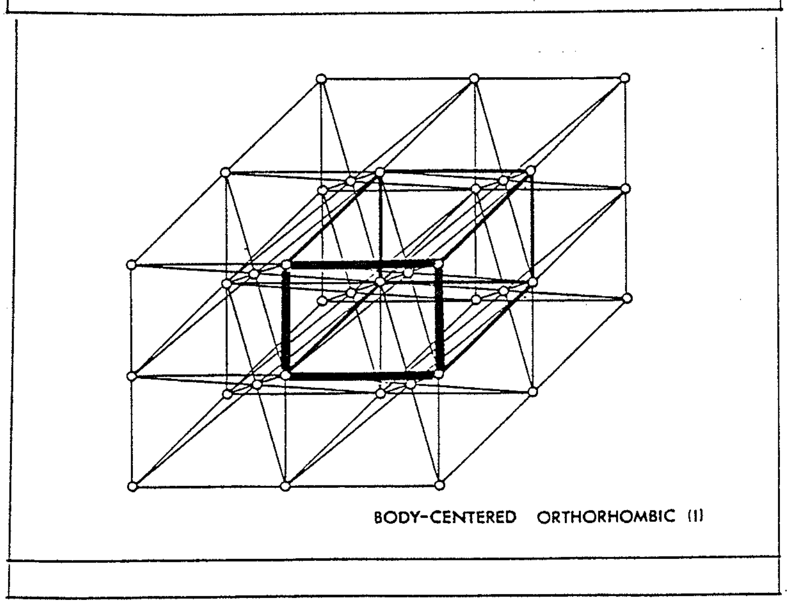
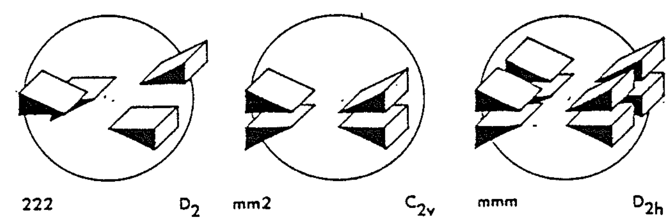


Fig. 2.409

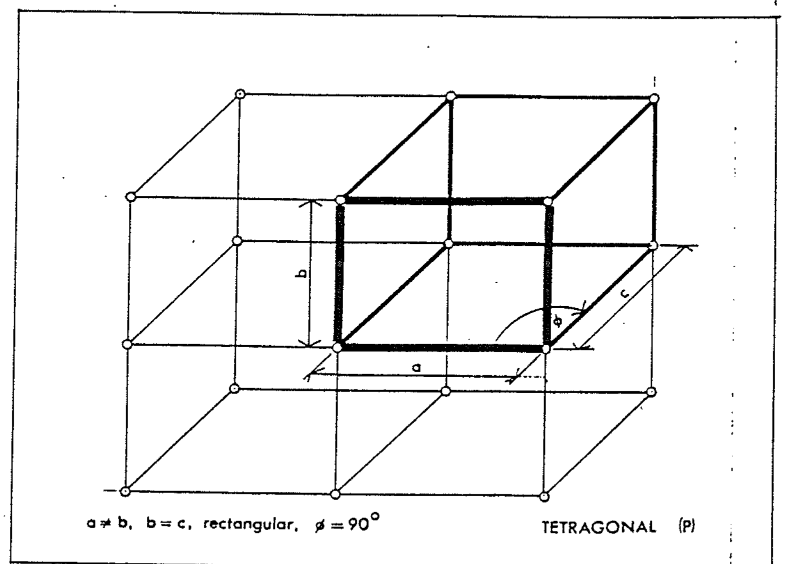


ORTHORHOMBIC POINT GROUPS

Fig. 2.410

(cont'd. from p. 92)

the body-centered to generate another nine of the Fedorov groups (see TABLE MA:01, Mathematical Appendix). In total, the orthorhombic crystal class contains fifty nine symmetry groups in space.



LATTICE IN THREE DIMENSIONS

2.4.4 TETRAGONAL Fig. 2.411

The fourth crystal class in three dimensions is the tetragonal class. In this class there are two Bravais lattices, the primitive tetragonal (Fig. 2.411), and the body-centered tetragonal (Fig. 2.412). The primitive unit cell consists in two unequal arbitrary lengths, a and b, and a third length, c , equal to one of them, b = c ; with a rectangular lattice facing the viewer, projecting into space at an angle, ø , of exactly 90°. The body-centered unit cell is identical, except that there is a point at the center of the body of the cell.

These two lattices combine with seven point point groups in three dimensions, they are the cyclic point groups, C_4 , C_{4h} and the special point group S_4 (Fig. 2.413); as well as the dihedral point groups, D_4 , D_{4h} , D_{2d} and the cyclic point group, C_{4v} (Fig. 2.414). The seven tetragonal point groups combine with the primitive lattice to generate forty nine of the Fedorov groups; and with the body-centered lattice to generate another nineteen. (see TABLE MA:01, Mathematical Appendix). In total, the tetragonal crystal class contains fifty eight symmetry groups in space.

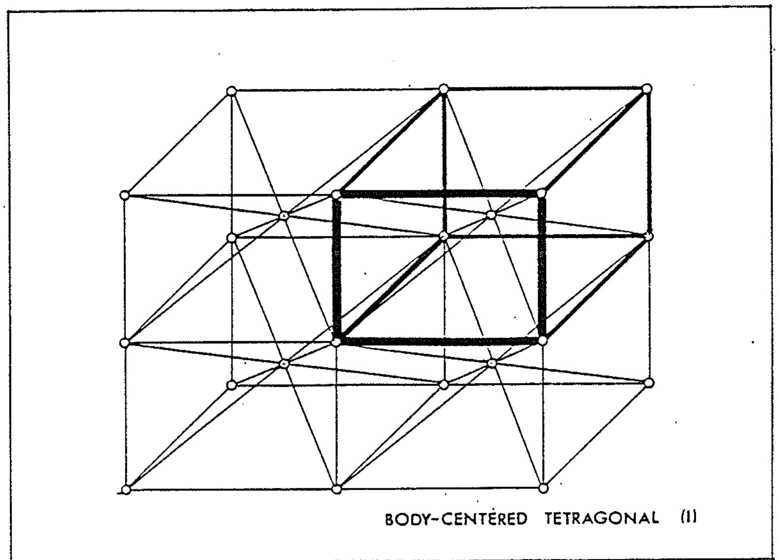


Fig. 2.412

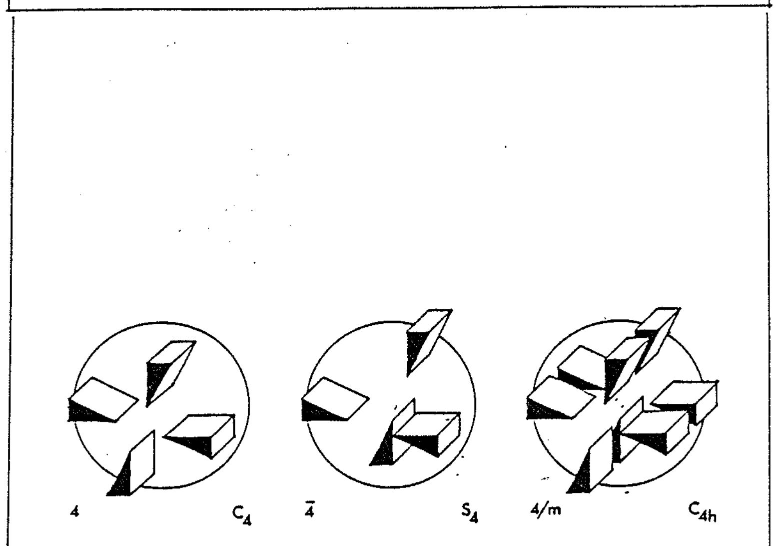
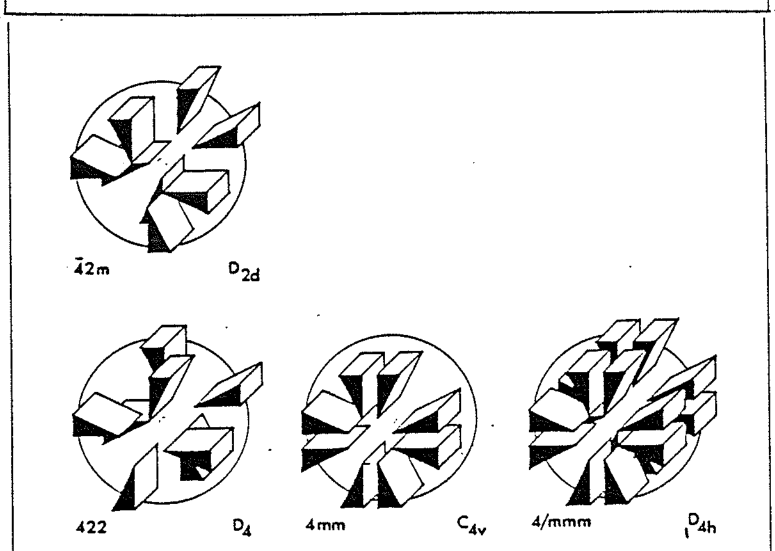
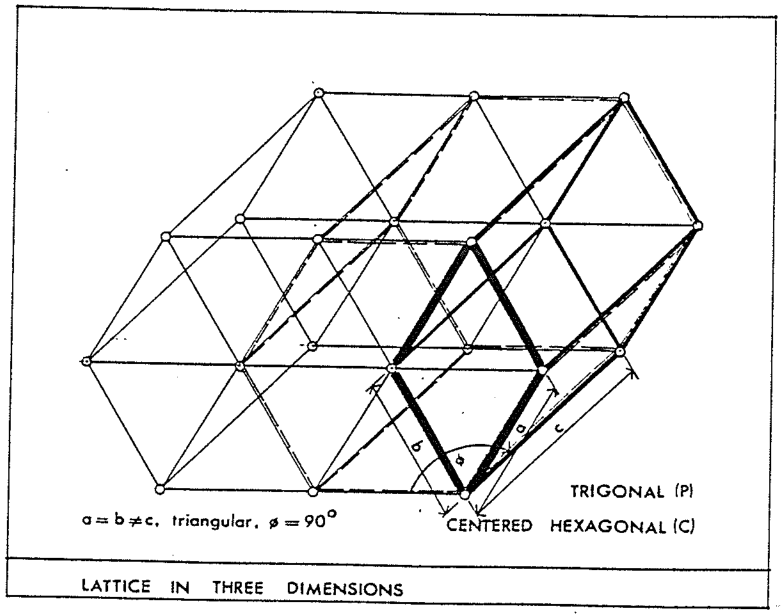


Fig. 2.413



TETRAGONAL POINT GROUPS

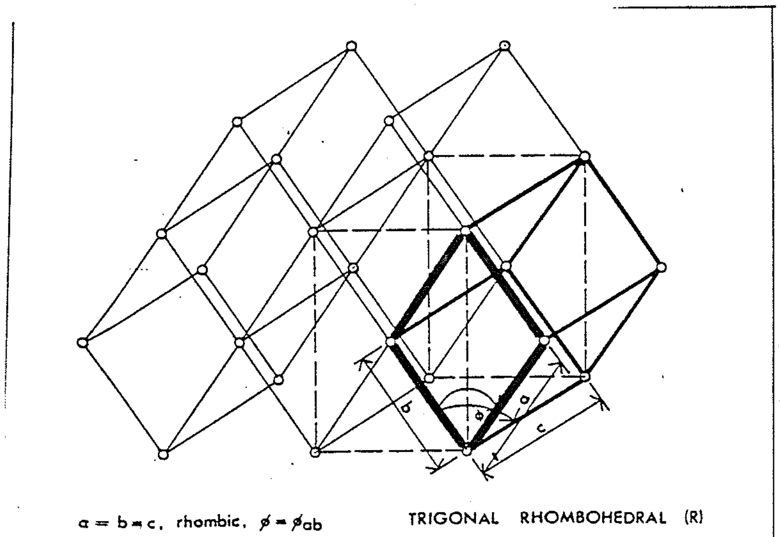
Fig. 2.414



2.4.5 TRIGONAL Fig. 2.415

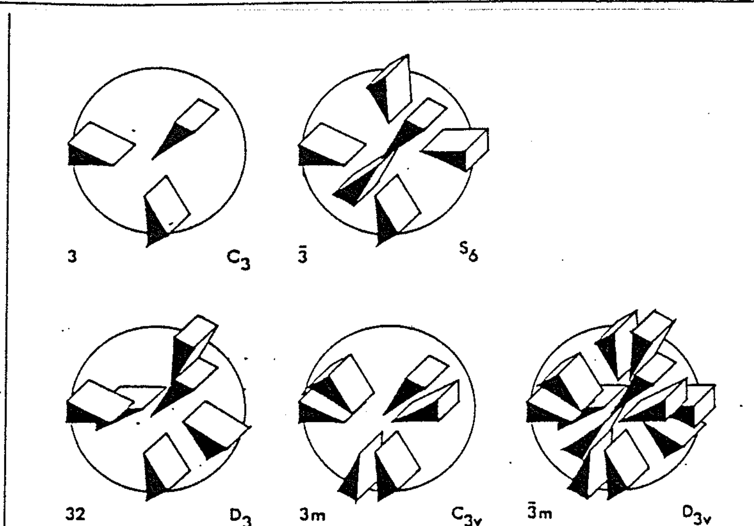
The fifth crystal class in three dimensions is the trigonal class. In this class there are two Bravais lattices, the primitive trigonal (Fig. 2.415) and the trigonal rhombohedral (Fig. 2.416). The primitive unit cell consists in two equal lengths, a = b , and a third length, c , not equal to either a or b; with a tiangular lattice facing the viewer, projecting into space at an angle, $\emptyset$, of exactly 90°. The rhombohedral unit cell consists in three equal dimensions, a = b = c ; with a rhombic lattice facing the viewer, projecting into space at an angle, $\emptyset$, exactly equal to the angle, $\emptyset_{ab}$, in the rhombic lattice.

These two lattices combine with five point groups in three dimensions, they are the cyclic point groups, C_3 , C_{3v} , the special point group, S_6 , and the dihedral point groups, D_3 , D_{3v} (Fig. 2.417). The five trigonal point groups combine with the primitive lattice to generate eighteen of the Fedorov groups, and with the rhombohedral lattice to generate another seven (see TABLE MA:01, Mathematical Appendix). In total, the trigonal crystal class contains twenty five symmetry groups in space.



LATTICE IN THREE DIMENSIONS

Fig. 2.416



TRIGONAL POINT GROUPS

Fig. 2.417

2.4.6 HEXAGONAL

The sixth crystal class in three dimensions is the hexagonal class. In this class there is just one Bravais lattice, the centered hexagonal (Fig. 2.415) which is not distinct from the primitive trigonal. It is not considered a separate member of the fourteen Bravais lattices. The centered hexagonal unit cell, shown in broken line, consists in the same conditions as the trigonal primitive.

This lattice combines with seven point groupsin three dimensions, they are the cyclic point groups, C_6 , C_{6h} , C_{3h} (Fig. 2.418); as well as the cyclic point group, C_{6v} , and the dihedral point groups, D_{3h} , D_6 , D_{6h} (Fig. 2.419). The seven point groups combine with the centered hexagonal lattice to generate twenty seven of the Fedorov groups (see TABLE MA:01, Mathematical Appendix). The hexagonal crystal class contains just these twenty seven symmetry groups in space.

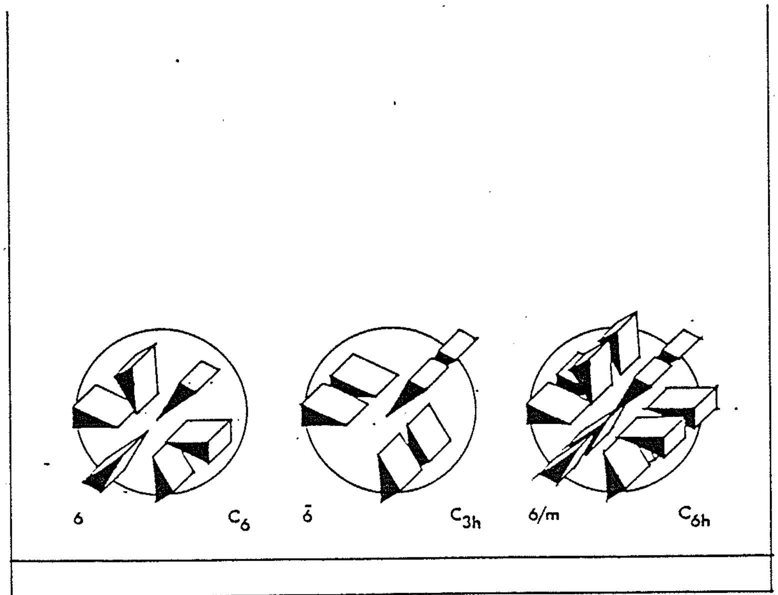
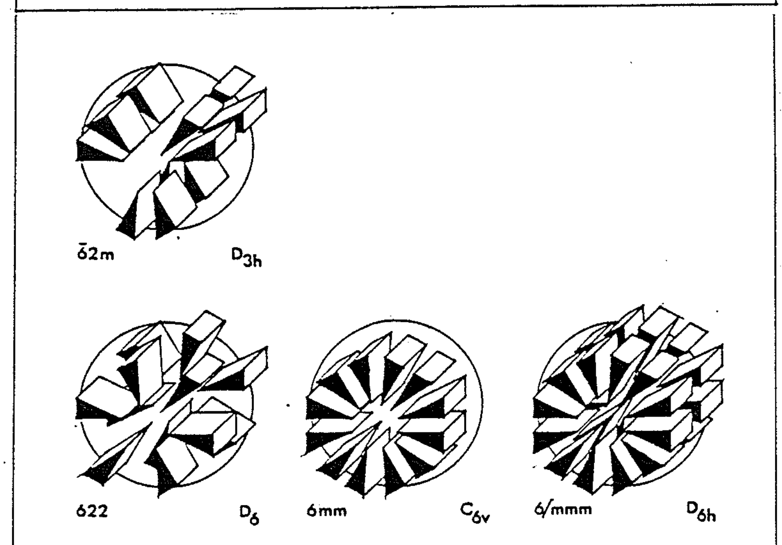
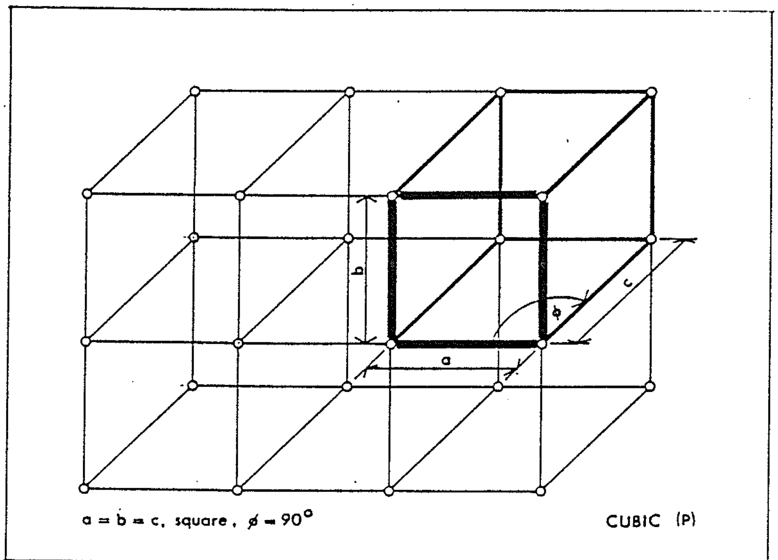


Fig. 2.418



HEXAGONAL POINT GROUPS

Fig. 2.419



2.4.7 CUBIC Fig. 2.420

The seventh and final crystal class in three dimensions is the cubic class. In this class there are three Bravais lattices, the primitive cubic (Fig. 2.420), the face-centered cubic (Fig. 2.421), and the body-centered cubic (Fig. 2.422). The primitive unit cell consists in three equal lengths, a = b = c , with a square lattice facing the viewer, projecting into space at an angle, $\emptyset$, of exactly 90°. The face-centered unit cell is identical, except that there is a point at the center of every face of the cell. The body-centered unit cell is also identical, except that there is a point at the center of the body of the cell.

These three lattices combine with five point groups in three dimensions, they are the tetrahedral point groups, T , T_h (Fig. 2.423); the tetrahedral point group, T_d , and the octahedral point group, O (Fig. 2.424); as well as the octahedral point group, O_h (Fig. 2.425). Each of the illustrations of these point groups shows only half of the elements, there is an identical pattern of elements in the three directions facing away from the viewer. The five cubic point groups combine with the primitive lattice to generate fifteen of the Fedorov groups; with the face centered lattice to generate

(cont'd. p. 101)

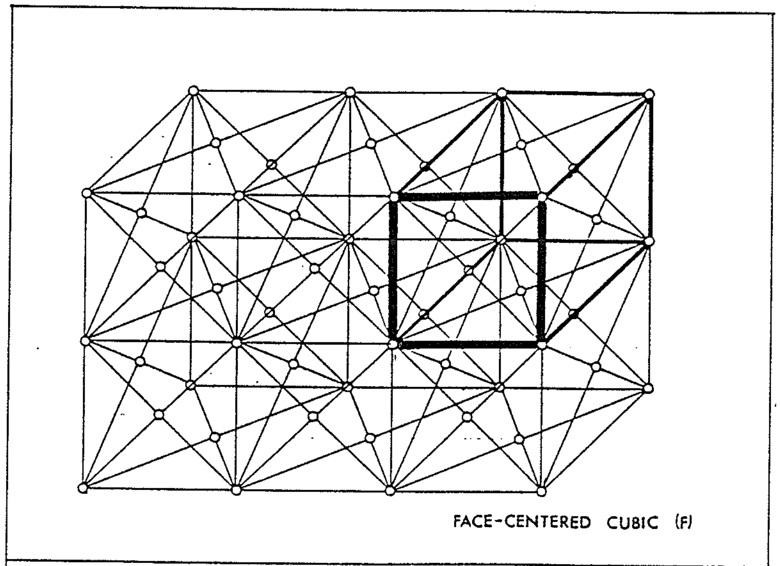


Fig. 2.421

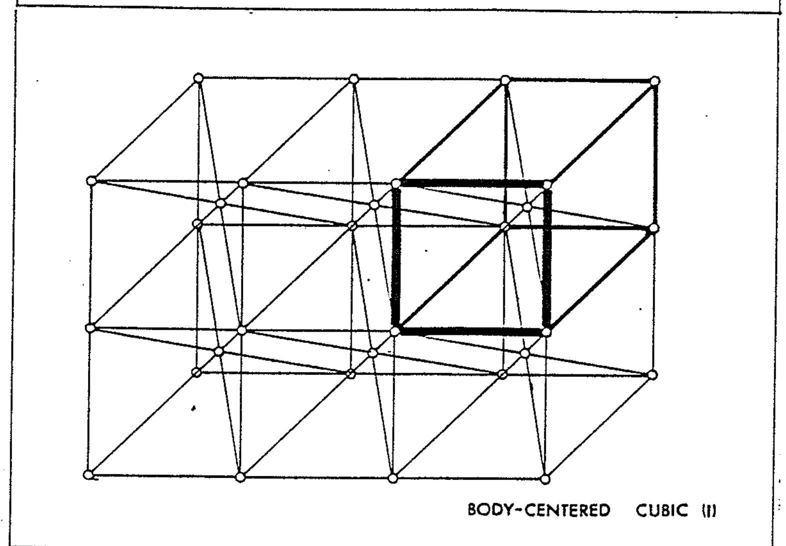


Fig. 2.422

(cont'd. from p. 100)

eleven; and with the body-centered lattice to generate the final ten Fedorov groups (see TABLE MA:01, Mathematical Appendix). In total, the cubic crystal class contains thirty six symmetry groups in space.

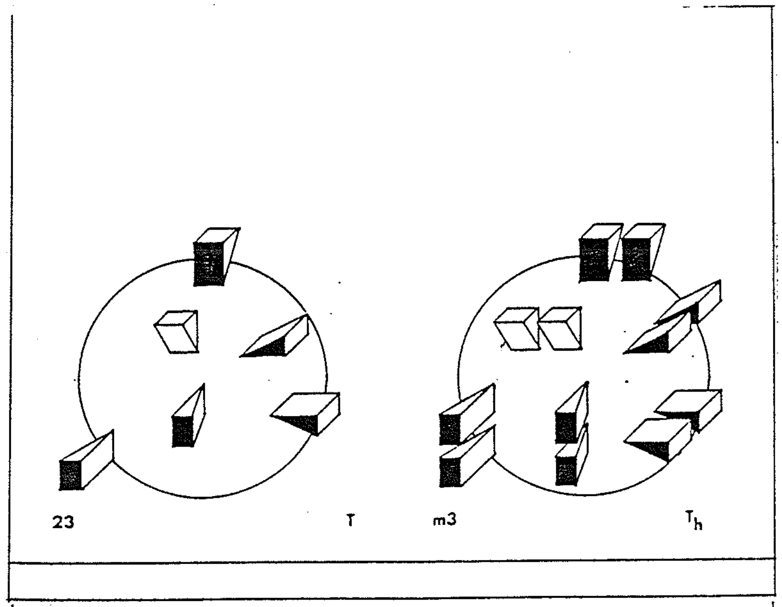


Fig. 2.423

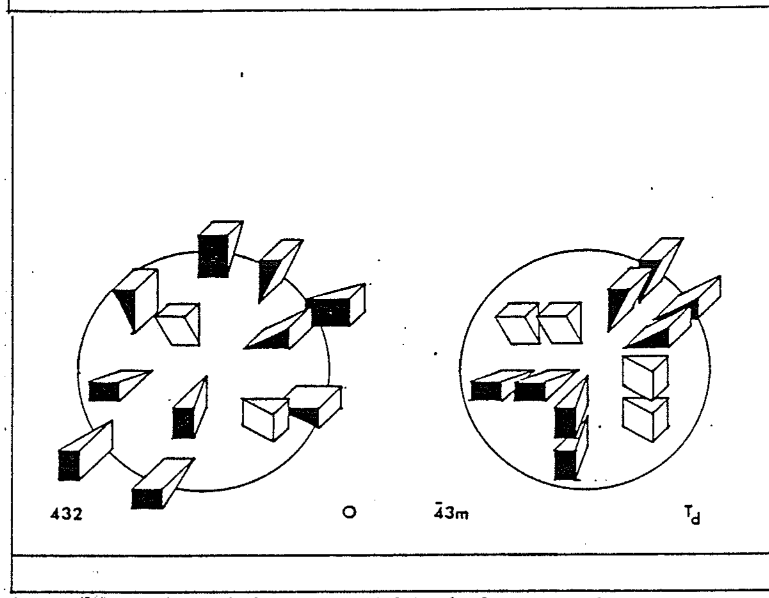


Fig. 2.424

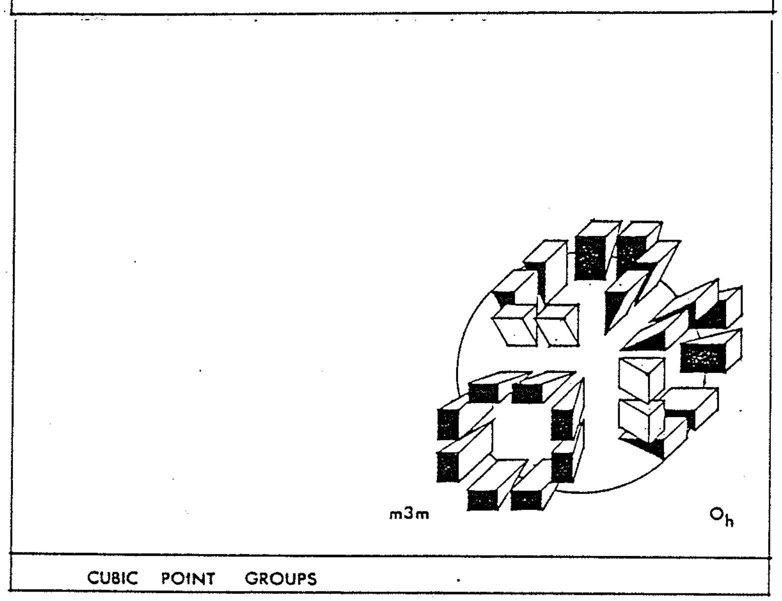


Fig. 2.425

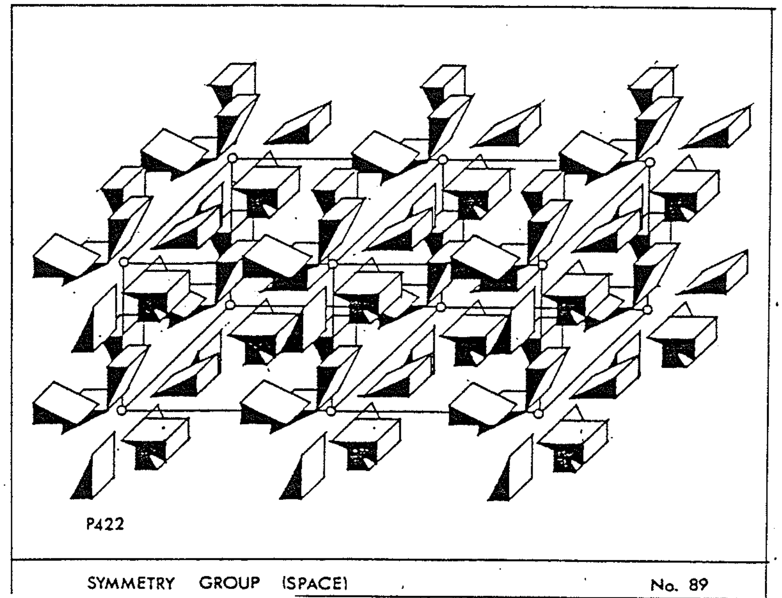


Fig. 2.426

This completes the illustration of the fourteen Bravais lattices and thirty two point groups in three dimensions. To provide an example of the symmetrical configurations that may be generated from these, there is an illustration of the Fedorov group, D_4^1 (Fig. 2.426), that results from the combination of the point group, D_4 , with the primitive tetragonal lattice. All two hundred thirty Fedorov groups in space might be so illustrated, but for practical space limitations they are not shown here.

This also completes the explanation of the resources of the mathematical theory of symmetry in Part Two. With the visual information here, together with the Mathematical Appendix, the thesis that the mathematical theory of symmetry may be a significant part of the knowledge an architect applies in design is supported.

PART THREE: SPECULATIONS

3.1 THE AESTHETICS OF COMPLEXITY

Sir Henry Wotton's The Elements of Architecture, first published in 1624, provided a famous description of the conditions for good architecture, "In architecture, as in all other operative arts, the end must direct the operation. The end is build well. Well building has three conditions: Commodity, Firmness, and Delight."[73] The application of the mathematical theory of symmetry must be directed to one of those conditions in the resulting work of architecture. The intent of this overall part is to make some speculations about the contribution of that application in architecture. The particular intent of this section is to speculate that the mathematical theory of symmetry is a means directed by the end of aesthetic delight and a method for dealing with the aesthetics of complexity in architecture.

A work of art that is only a work of art, that serves no other purpose, has its aesthetic methods as an end in themselves. Nothing more is communicated by the work than to draw attention to the intrinsic values of the aesthetic effects of the work. But, architecture, almost by definition, must serve some other purpose; including the accomodation of human activity, and, more importantly, the communication of a meaningful awareness of space. The work of architecture has the communication of spatial information, to express and induce meaning, as one of its primary ends. The fact that architecture is also an art suggests that the methods of the designer and their effects in the resulting work of architecture are channels for communication. A work of art, such as architecture, that draws attention to extrinsic values of aesthetic effects, such as the awareness of space, allows communication between viewer and creator. The methods applied in these works are channels for that communication.

The mathematical theory of symmetry in architecture is a method for the creation of order in the transmission of spatial information by the work of architecture. The creation of this order is not an important aesthetic effect if the design problem is very well understood, and therefore the spatial information communicated is simple. However, most design problems are not well understood[74] and the information communicated by the architecture is complex. Indeed, the complexity of information communicated is an important aspect of the study of design problems. Herbert Simon has defined complexity in design:

> ...by a complex system I mean one made up of a large number of parts that interact in a nonsimple way. In such systems, the whole is more than the sum of the parts, not in an ultimate, metaphysical sense, but in the improtant pragmatic sense that, given the properties of the parts and the laws of their interaction, it is not a trivial matter to infer the properties of the whole.[75]

Complexity is an important topic in any discussion of methods for creating order, because there is an interaction between the aesthetic effects of order and complexity. The nature of that interaction should guide the application of methods such as the mathematical theory of symmetry in design.

An important point to be made is the recognition of the paradox of studying the methods of the artist and the effects of those methods in works of art as one thing. The persistence of this paradox usually results in aesthetic recipes linking certain methods and effects with certain types of information to be communicated. Those recipes then become the criteria for the evaluation of good or bad works of art; that is, they became laws of beauty. This was the subject of classical aesthetics. That should not be the subject of a discussion aesthetic methods and effects in design. Anton Ehrensweig explained that:

> There was no need for disputing at length their spurious laws of beauty. The passage of time alone saw to that. With the rise modern psychology, the aestheticians changed their aim. Instead of search for objective properties of beauty in the external world they turned inwards to find the source of the aesthetic experience in our own mind.[76]

The study of aesthetic methods and effects in works of art should be seen in the context of understanding psychological processes. The application of the mathematical theory of symmetry as a methodfor creating the effect of order in architecture is not an aesthetic recipe for good architecture (see section 1.1, p. 30). It is only a way of understanding certain psychological processes involved in aesthetic preferences of individuals.

It is appropriate to rely upon experimental psychology to make speculations about the aesthetics of complexity in design to which the mathematical theory of symmetry is applied. D.E. Berlyne was a pioneer in the experimental study of psychological processes in aesthetics; his major concept is "arousal", which is defined as:

> ...the activating or energizing aspects of motivation or emotion. This work has given rise to the psychophysiological concept of 'arousal', which, among many other ares of research that it has affected, seems to have great potential for throwing light on

> aesthetic phenomena. A human being or higher animal can be regarded as possessing, at a particular moment, a particular 'level of arousal' or 'activation'. His position along this dimensions cane be regarded roughly as a measure of how wide awake, alert, or emotionally excited he is.[77]

The concept of arousal is significant because it allows the experimental study of aesthetic mehtods and effects. In the context of works of architecture, it is clear that if the work is to communicate an awareness of space, then it must arouse the people involved in the creation and appreciation of the work. The study of design methods aimed at certain aesthetic effects, such as spatial order, is interested in those qualities that influence levels of arousal. Berlyne suggested:

> ...it will be convenient to refer to all properties of stimulus patterns that ten, on the whole, to raise arousal as the arousal potential. This term will denote something like the psychological strength of a stimulus pattern, the degree to which it can disturb and alert the organism, the ease with which it can take over control and overcome the claims of competing stimuli.[78]

The two concepts of arousal and arousal potential provide a psychological context within which to understand the application of the mathematical theory of symmetry in design as a method for dealing with the aesthetics of complexity.

Berlyne identified three classes of properties that influence the arousal potential of stimulus patterns such as works of art. A work of architecture may be thought of as just such a stimulus pattern. The first class is the psychophysiological properties, which refer to the effects of the intensity and frequency of physical stimuli. In architecture, this includes things like bright lights, loud noises, intense colours, hard surfaces, crowds of people, and similar things. The second class is the ecological properties, which refer to the effects of boilogical and envirionmental conditions. In architecture, this includes things like shelter from wind and rain, warm air, natural sunlight, close contact with the ground, the way energy is consumed, and similar things. The third class is the collative properties, which refer to the effects of relations and lack of relations between stimuli or conditions. In architecture, this includes things like order, complexity, novelty, symbolic associations, cognitive images, and similar things. These latter properties are those to which methods applying knowledge of structure, such as the mathematical theory of symmetry, are directed.

The activity of design must take into account all three types of effects. The work of architecture does provide certain intensity and frequency of physical stimuli, reflecting the architect's concern to accomodate the function of the building. Equally, the work of architecture does provide certain biological and environmental conditions, reflecting the architect's to meet the technological demands of the building. The arousal potential resulting from these two classes of properties relates to the planning and construction of the building; or in Wotton's conditions for well building, the commodity and firmness. But, it is the third class of effects that relates to the aesthetics of the building; the delight of well building. The work of architecture does provide certain relations between stimuli and conditions, reflecting the architect's concern to indicate the relative aesthetic values of elements of the building. It is the structure of these relations that should be central to the discussion of architectural aesthetics. Of course, the planning and construction of the building are important, because poor quality in either one may provide such arousal potential as to detract from the structure of relations that convey the aesthetic effect. But, granted that the planning and construction difficulties of a building may be handled adequately by the training of architects; it is the knowledge of structure applied by architects that conveys the aesthetic effects of a building.

The mathematical theory of symmetry may be a significant part of an investigation of structures in space (see section 1.4). The application of the theory as a method for creating the relations between architectural elements creates one of the basic aesthetic effects in architecture, that is, the quality of order. The application should be directed by the aesthetic concerns of an architect, not either the planning or construction concerns. The important question becomes, what aesthetic concern is dealt with by an application of the mathematical theory of symmetry to create order?

The effects of structures that increase arousal generally depend on the contrast of elements with accompanying elements or previous elements of the same sort. In architecture, devices such as the juxtaposition of bright colours and different shapes produce this kind of arousal. The increased number of forms of elements and wide variation in the awareness of space createvariety increasing the arousal potential. Novelty is one of the most common devices of this sort. The innovation resulting from the invention of new elements or the use of existing elements in new ways provides excitement in the design. A work of architecture that differs in a striking way from previous examples of the same building type, produces a significant raising of arousal. Raising certain expectations for the building, then surprising the viewer with something unexpected or not contiguous with what preceded it, is a basic device for increasing arousal potential. Some degree of unpredictability or ambiguity will also raise the level of arousal in the viewer. Contrasting patterns of stimulation may also be very arousing. Perhaps, all these devices can be collected in the idea of complexity, as defined earlier. In general, structural complexity increases the arousal potential of the work of architecture.

The effects of structures that decrease arousal generally depend on the association of elements with accompanying elements or previous elements of the same sort. In architecture, devices such as the blending of complementary colours and similar shapes produce this kind of lack of arousal. The decreased number of forms of elements and limited variation in the awareness of space create redundancy decreasing the arousal potential. Familiarity is one of the most common devices of this sort. The convention resulting from the adoption of existing elements or the use of new elements in existing ways provides tradition in the design. A work of architecture that differs in no striking way from previous examples of the same building type, produces a significant lowering of arousal. Raising certain expectations for the building, then fulfilling the viewer with the thing expected or contiguous with what preceded it, is a basic device for decreasing arousal potential. Some degree of predictability or clarity will also lower the level of arousal in the viewer. Similar patterns of stimulation may also not be arousing. Perhaps, all these devices can be collected in the idea of order, as defined earlier (see section 1.4). In general, structural order decreases the arousal potential of the work of architecture.

It is important to relate the aesthetic preferences of both the designer and the viewer to the raising and lowering of arousal through structural complexity and order. Clearly, if the work of architecture has the intention to create an aesthetic awareness of space, then the designer should be concerned with maximizing the aesthetic preferences for the work. Aesthetic preference cannot be simply equated with beauty or pleasingness, it refers only to the individual's judgement of the attention that is due the work, both for its methods and its content. Certainly, the application of the mathematical theory of symmetry in design must be part of the methods for creating an aesthetically preferable work of architecture.

There is a need to balance the devices that create structural complexity with the devices that create structural order. But the nature of their interaction is not simple. This is one of the crucial issues in discussions of architectural aesthetics. Berlyne saw common ground in most aesthetic philosophies; he argued:

> Despite the very different terms in which the two components have been specified through the centuries, it is not hard to discern common ground. There is always one factor, whether it be called 'multiplicity', 'variety', or 'complexity' that can be expected to raise arousal. Then, there is the other factor, 'unity', 'order', or 'lawfulness' that can be expected to lower arousal or at least keep arousal within bounds.[79]

It makes sense to suggest that works of architecture must be stimulating; they must include arousal increasing devices of complexity. Generally then, as the complexity of a work increases, so will the arousal of the viewer and this will make the work more aesthetically preferable. But at some point, increases in arousal will become uncomfortable for the viewer and this will make the work less aesthetically preferable. Eventually, at some extreme level of arousal, the work will actually become aversive. Berlyne suggested that the nature of the relationship between aesthetic preference and arousal might be a Wundt curve[80] (Fig. 3.101).

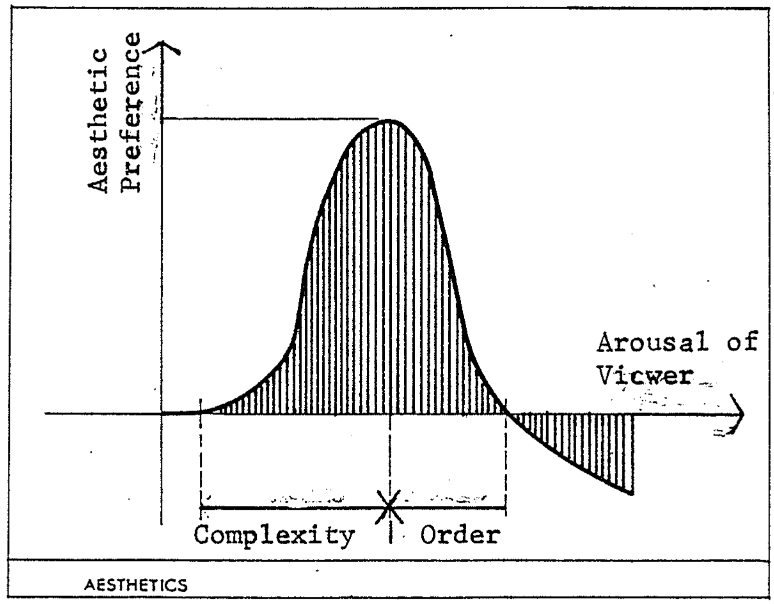


Fig. 3.101

To create a work of architecture that is at the point or in the range of maximum aesthetic preference for the viewer, there is the need to moderate the stimulation of complexity with the arousal decreasing devices of order. In design, the creation of structural complexity is desirable, but only to a point. After that point, any increase in structural complexity should be balanced with an increase in structural order; to allow the level of arousal of the viewer to remain in the range of maximum aesthetic preference. Certainly, every viewer's judgement is individual for each work of architecture; hence, the profile of the curve and exact point of maximum aesthetic preference, is different for each viewer and each building. However, the shape of the curve; the nature of the relationship it explains between aesthetic preference and the interaction of structural complexity with structural order, is significant.

The mathematical theory of symmetry in architecture is a method for creating structural order in the awareness of space. It is a device for moderating the effects of increased complexity. Due to the inherent structural complexity in the creation of a large number or variety of spaces, the contemporary architect will almost certainly be compelled to use some method for creating structural order in design. An application of the mathematical theory of symmetry is a method for dealing with this aesthetic concern to create structural order to balance complexity.

In architecture, as in most art, the state of maximum aesthetic preference may be described as ordered complexity. Venturi argued the case for complexity in modern architecture admirably:

> Architects can no longer afford to be intimidated by the puritanically moral language of orthodox Modern architecture. I like elements which are hybrid rather than pure, compromising rather than clean, distorted rather than straightforward, ambiguous rather than articulated, perverse as well as impersonal, boring as well as interesting, accomodating rather than excluding, redundant and equivocal rather than direct and clear. I am for messy vitality over obvious unity.[81]

Indeed, in terms of creating aesthetically preferable works of architecture through increased structural complexity, "more is not less."[82] But, and this is an important point, the application of devices to increase complexity in design must be restrained at some point by the application of methods for creating structural order. Increasing complexity must be balanced by increasing order in the underlying structure. Arnheim has made a similar commentary on Venturi:

> Order is found at all levels of complexity. The more complex the structure, the greater the need for order and the more admirable its achievement, because it is harder to obtain. Venturi shows many excellent examples of complexity. But he misleads in asserting that those complexities involve contradiction and therefore are disorderly, which in fact most of them are not. The misuse of the term contradiction must not be permitted to justify the existence of chaotic wilfulness, caused in our time by the atomization of society and the breakdown of the sense of form.[83]

Both Venturi and Arnheim have made good arguments. There should be structural complexity in the work of architecture to increase the likely aesthetic preferences of the viewers, but this does not imply the abandonment of structural order.

The speculation here is that the application of the mathematical theory of symmetry is a method for dealing with the aesthetics of complexity in architecture. It allows the designer to moderate the effects of increasing complexity in the number and variety of spaces created by applying the theory to structure the relations between the elements used to make space. This has the effect of creating a balance between the inherent structural complexity of the design and the structural order provided by the theory. This balance is aimed at the aesthetic concerns of the designer to make the work worthy of attention by the viewer, to communicate a meaningful awareness of space. Clearly, the mathematical theory of symmetry must be evaluated as a significant part of the knowledge an architect applies in design.

3.2 AN APPROACH TO DESIGN

Once the architect develops a knowledge of the mathematical theory of symmetry, perhaps through basic design exercises based on the resources of the theory (see Part Two), and is aware of the theory behind its application (see Part One); the most important thing is an approach to design involving that application. The intent of this section is to make speculations about an approach to design, in which the resources of the mathematical theory of symmetry are applied. Clearly, this section must also follow the speculations made about the end, that is, the aesthetics of complexity (see section 3.1), to which the theory is applied. This section is very much an operational conclusion about the application of the mathematical theory of symmetry in design. This conclusion should not be seen as an architectural "how to" or recipe book, but only as speculations about an approach to design with the theory.

It is essential to reiterate the level and role of the mathematical theory of symmetry in architecture. Symmetry is a method for creating underlying structures in works of architecture; that is, self-regulating transformations of architectural elements into space creating wholes.

Symmetry is not a method for inventing forms in works of architecture; that is, particular formations of architectural elements. The proper application of the mathematical theory of symmetry is as principles of order in architecture, not as theories of proportion. The level of an underlying structure and the role of creating order are the basis of any approach to design applying the mathematical thoery of symmetry. These two ideas should be kept in mind at all times.

The effect of applying knowledge of the mathematical theory of symmetry in design is on the aesthetic quality of structural order in the work of architecture. This must also be kept in perspective at all times. Symmetry is an arousal moderating device to balance the arousal stimulating devices of complexity in architecture. Symmetry allows an architect to judge the range of maximum aesthetic preference in complex works of architecture. The creation of a work in that range is an important part of communication with architecture; and is dependent on the knowledge and application of structural devices such as symmetry. But, the whole work of architecture depends also on the planning and construction of the building. The mathematical theory of symmetry does not concern itself with these aspects of design, and should not be applied to them. It ultimately deals with only the aesthetic quality of order in complexity.

Some functional organiztions seem to imply an inherent symmetry, but the forcing of planning problems into symmetrical configurations is still a mistaken idea. The application of symmetry to planning is totally inappropriate in the light of the theoretical basis for that application. One of the dangers to be avoided in an approach to design is the application of symmetry to the planning of the building; that is, to the space allocation and functional organization of the building. Similarly, some technological systems seem to imply an inherent symmetry, but the simplification of construction problems into symmetrical configurations for that reason is still a mistaken idea. The application of symmetry to negineering is totally inappropriate in the light of the theoretical basis for that application. Another danger to be avoided in an approach to design is the application of symmetry to the construction of the building; that is, the building systems and performance specifications of the building. An approach to design should only apply symmetry to the aesthetic problems of space creation.

The mathematical theory of symmetry is only a framework of prescriptive rules to be applied in creating an underlying structure effecting the aesthetic quality of order in the work of architecture. An approach to design must be based on that position. It makes no sense to extend symmetry as methods to resolve planning or construction problems. Therefore, there seems to be two prerequisites for the application of the mathematical theory os symmetry in design. That is, the architect must also resolve the planning and construction difficulties through other design methods. But, these methods may aim at developing space allocations and functional organization that meet the planning requirements of the design; with the idea of applying the mathematical theory of symmetry for the associated aesthetic problems. These methods may also aim at devising building systems and outlining performance specifications that meet the construction requirements of the design; with the idea of applying symmetry for the associated aesthetic problems. For example, the result of planning may be a functional zoning of activities from public to private, served to servant, or one sort to another sort that lends itself to being organized on a Bravais lattice. The result of investigating construction may be a set of industrialized components such as exterior and interior wall panels, columns and beams, roof and floor spans, or windows and doors that lends itself to being organized into point groups. Therefore, an approach to design may involve particular sorts of planning and construction that are directed at being compatible with an application of the mathematical theory of symmetry. The mathematical theory of symmetry should not be applied to force planning or simplify construction, but planning and construction may be made compatible with the theory.

Once the planning and construction concerns of the designer have been resolved to some satisfactory point, there are two approaches to design to go about ordering elements into aesthetically preferable works by applying the mathematical theory of symmetry. The reason there are two approaches is that symmetry groups in space are generated by the combination of a Bravais lattice and a point group on that lattice. A designer may either select and fix the Bravais lattice in accordance with the zoning established by planning methods; then "play" with various point groups to arrange the elements from construction vocabularies on that lattice, to make the spaces of the building. Or, a designer may select and fix the point groups in accordance with the construction vocabulary; then "play" with various Bravais lattices to position the elements in accordance with planning, to make the spaces of the building. In any particular design process, an architect may actually choose to do both, switching back and forth between the two approaches selectively searching[84] for the space creating structure that seems to maximize aesthetic preferences.

In either approach to design, a designer need not play with all thirty two point groups and all 14 Bravais lattices at the same time. Because they are arranged into seven crystal classes, once the selection of an appropriate crystal class has been made, the designer works just with the point groups and lattices in that class. The selection of a crystal class is a judgement that the class will best integrate ideas about the planning, construction, and aesthetic quality of the work of architecture. This judgement is not as critical as it may seem at first, because there is no reason that the selection should be fixed. Anyway, within most crystal classes there is enough variety of symmetry groups and creative possibilities to accomodate most approaches to design. The more critical judgement is about what to fix or what to "play" with when generating symmetrical configurations from Bravais lattices and point groups associated with particular planning and construction.

Creative originality is one of the most desirable by-products of the application of the mathematical theory of symmetry in creating an underlying structure to the work of architecture. Both approaches to design suggested above have this feature of allowing great amounts of creativity. Bravais lattices in space generally have three arbitrary lengths in their unit cells, and in some cases a wide range of angles possible between them. A designer has infinite choices for dimensioning those lengths, as long as the dimensions meet the conditions for the unit cell. A designer, where possible, has infinite choice for setting the angles in the lattice, as long as they meet the conditions for the unit cell. An architect "plays" with a lattice by changing the dimensions and angles selected within the boundaries prescribed by the conditions of the unit cell. Similarly, point groups in space also involve an arbitrary length in the distance of the elements from the point, and an arbitrary orientation to that point. Thus, an architect "plays with a point group by either changing the actual elements in it, or by changing the distance or orientation of elements to the point within the boundaries presrcibed by the structure of the point group. The varying of dimensions and setting of angles are the two basic methods in an approach to design combining Bravais lattices and point groups to generate any number of symmetrical configurations with the same underlying structure; that is, described by the same symmetry group.

There are more creative possibilities in the application of the mathematical theory of symmetry groups in creating an underlying structure. Perhaps, the most important of these is the layering of several symmetry groups together to create a more complex underlying structure responding to greater complexity. It is an endless source of possible structures for an imaginative designer. Layering involves the use of several Bravais lattices, of the same or different class, at the same time producing sophisticated composite lattices; for example, the tartan layering of grids. Similarly, layering may also involve the use of several point groups of the same class on a single Bravais lattice, to provide an interesting interaction between symmetrical configurations. Layering, then involves the creation of several symmetrical configurations at the same time by creating structures that are aggregates of many symmetry groups. An approach to design involving layering of symmetry groups creates complex yet ordered underlying structures making space that may be the most powerful method for dealing with the aesthetics of complexity in architecture.

It is important to suggest that an approach to design should consider the application of asymmetry and dissymmetry in connection with an application of the mathematical theory of symmetry in architecture. This is another source of creative possibilities. Weyl described the application of asymmetry in works of art as not, "...merely the absence of symmetry. Even in asymmetric designs one feels symmetry as the norm from which one deviates under the influence of forces of a non-formal character."[85] In works of architecture, asymmetry may be created by the slight deviation from an underlying structure in the design. It is important to note that the reason for this deviation is usually non-formal; that is, not in response to aesthetic qualities in the work. If for some reason of planning or construction deviation from symmetry makes sense, then the application of asymmetry in that situation is justified with the underlying space creating structure. On the other hand, dissymmetry is the purposeful breaking of a potential symmetry group by varying an element of a point group, or warping a Bravais lattice. It is primarily for the aesthetic emphasis of some special space of the building, not for planning or construction reasons. The contrast of one-of-a-kind dissymmetry with the symmetry of an overall space creating structure may be one of the most important methods of artistic communication. Shubnikov believed it to be an essential component:

> Symmetry, considered as a law of regular composition of structural objects, is similar to harmony. More precisely, symmetry is one of its components, while the other component is dissymmetry. In our opinion the whole aesthetics of scientific and artistic creativity lies in the ability to feel this where others fail to perceive it.[86]

The application of asymmetry and dissymmetry with the mathematical theory of symmetry are matters for individual judgement for designers. They both enhance the opportunity for imaginative application of the mathematical thoery of symmetry in an approach to design.

An approach to design applying the mathematical theory of symmetry, with all the creative possibilities suggested above, leads to some interesting and significant aesthetic effects, through the quality of spatial order, in the works of architecture created. These effects are of three kinds; they parallel closely the properties of crystals in nature as Weyl described:

> The dynamics of the crystal lattice is also responsible for the crystal's physical behaviour, in particular for the manner of its growth, and this in turn determines the particular shape it assumes under the influence of environmental factors. No wonder then that crystals actually occuring in nature display the possible types of symmetry in that abundance of different forms at which Hans Castorp on his Magic Mountain marvelled. The visible characteristics of physical objects usually are the result of constitution and environment.[87]

The three effects on works of architecture of the underlying symmetry structure suggested by this passage are that first, just as with snowflakes, there is no limit to the inventiveness possible in the particular building; second, that the manner of growth and change of the building, like a crystal, is regulated by the underlying structure; and third, that the uniqueness of a particular building, like the particular crystal, is the result of the influence of the environment. It is the combination of "constitution and environment", which may be interpreted in architecture as the general symmetry of the underlying space creating structure and the particular social or environmental context of the building, that determines the final visible product of an approach to design applying the mathematical theory of symmetry. It is the design response within the symmetry of the underlying structure to the social and enviromental contexts that really results in the individual spatial order in every work of architecture. This is the final and perhaps most important speculation about an approach to design with the mathematical theory of symmetry.

3.3 THE POTENTIAL OF SYMMETRY IN ARCHITECTURE

Having taken an approach to design applying the mathematical theory of symmetry (see section 3.2) as a method for creating spatial order, the next specualtion that is important is the resulting potential for that method. The easiest way of seeing that potential is to examine diverse numbers of work of architecture with a variety of aesthetic intentions, to see that each might have applied the mathematical theory of symmetry in their design. The intent of this section is to make specualtions about the widespread potential of symmetry in architecture. The vehicle for that speculation is the presentation and discussion of the underlying space creating structure in a number of designs from several architects.

The first architect to be considered is, appropriately, Louis Kahn, whose Bangldesh National Assembly building (Fig. 1.105) was used to illustrate the mathematical idea of symmetry. Kahn's most famous statement on spatial order was, "Order is.../Design is form-making in order."[88] Clearly, Kahn felt that a designer's insight into order was so intangible and deeply intuitive that he couldn't put any words after, "order is..." But, it is also clear, both in the remainder of that statement plus his entire output of designs, that the quality of order in the spaces created in architecture was essential for Kahn's philosophy.

Kahn's design sketches for the Bryn Mawr Dormitory (Fig. 3.301) may be interpreted as a selective search for a space creating structure. It is a method that orders the arrangement of individual rooms that Kahn is searching for in these sketches. The point to be made here is that the mathematical theory of symmetry in two dimensions might have been applied to describe and classify the structures Kahn was sketching. It might have been much

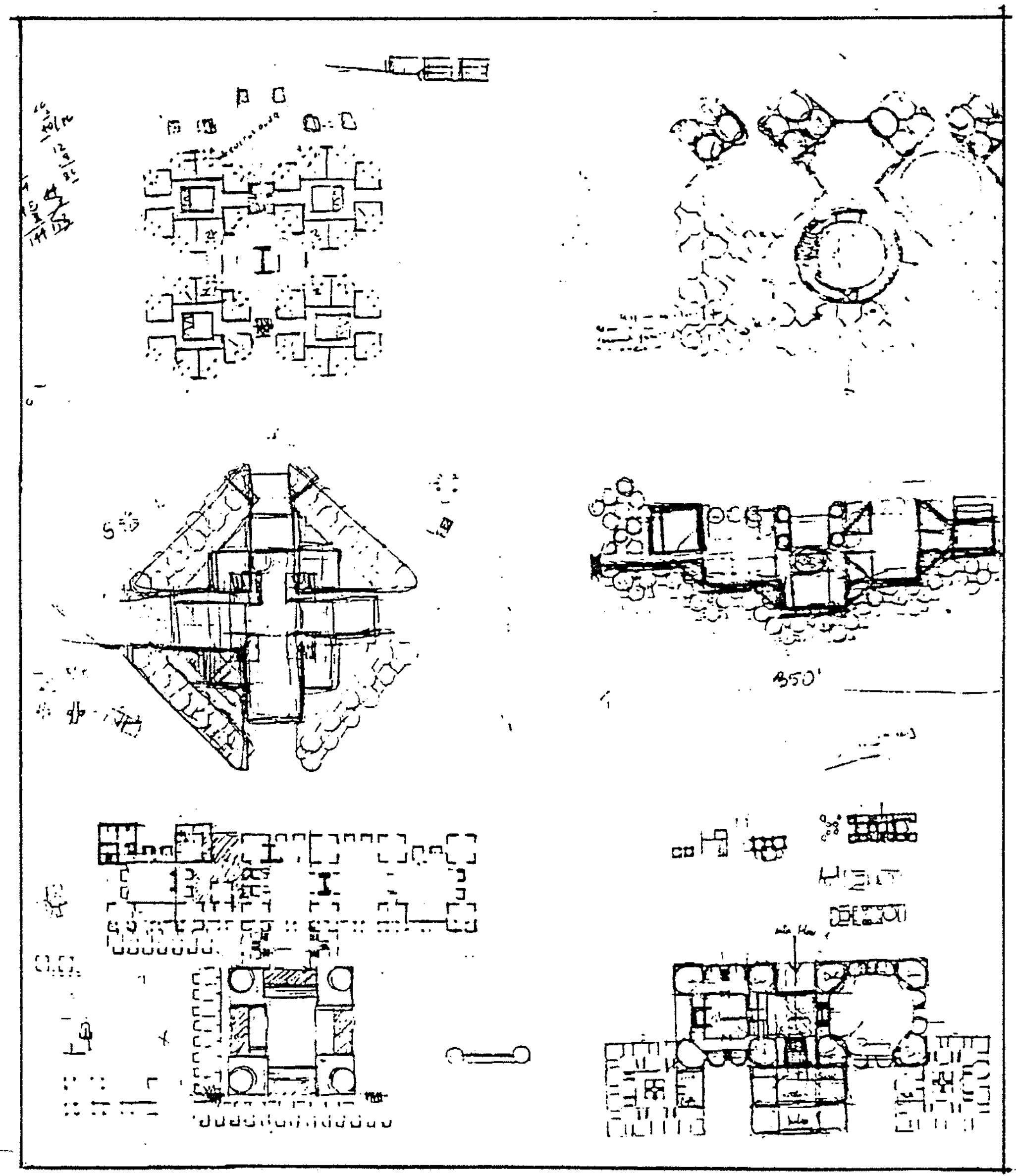


Fig. 3.301

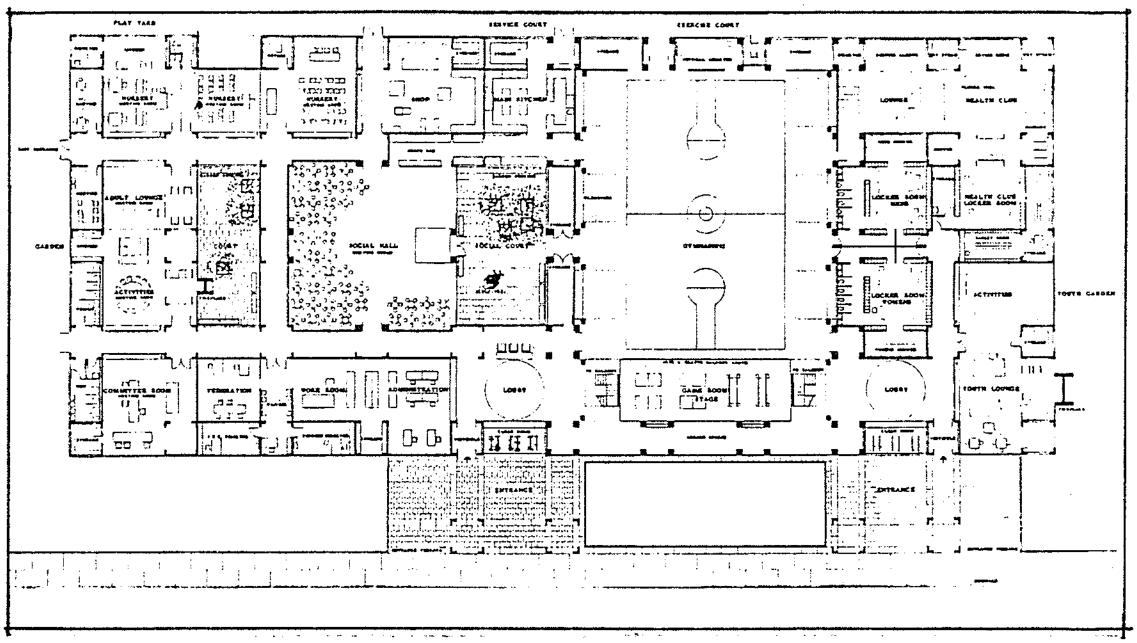


Fig. 3.302

easier to approach this design through the mathematical theory of symmetry. Surely then, Kahn's insight into the order he desired, and the creation of that order in design, is not so deep or mysteriously intuitive that it could not have been externalized through the mathematical theory of symmetry.

The plan for Kahn's Trenton Jewish Community Center (Fig. 3.302) provides support for this speculation. The basic spatial order of this design is created by L-shaped column elements arranged in the point group, C_S , on a square Bravais lattice in two dimensions; combining to generate a symmetrical configuration associated with the wallpaper group, W_4^1 . The structural complexity in the variety of spaces created in the building is balanced by the order of this symmetry. Occasionally this order has been broken, an example of dissymetry, by the omission of these columns to accomodate large spaces such as the gymnasium. Clearly, the mathematical theory of symmetry might have been applied in the design of this building to externalize the underlying space creating structure of the work.

The second architect to be considered is a former employee of Kahn's, Moshe Safdie, whose Habitat housing development (Fig. 3.303) is known and studies widely. The apparent disordered complexity of the Habitat facades, actually depends on two simple repeats of symmetrical clusterings. Safdie's original design for a much larger Habitat (Fig. 3.304); as well as his first post-Habitat design for the Public Housing Authority of Washington, D.C. (Fig. 3.305), both reveal the clarity of three dimensional symmetry underlying the design. The original Habitat appears to be based on a symmetrical configuration from the trigonal crystal class; while the Washington, D.C. design appears to be based on one from the tetragonal class.

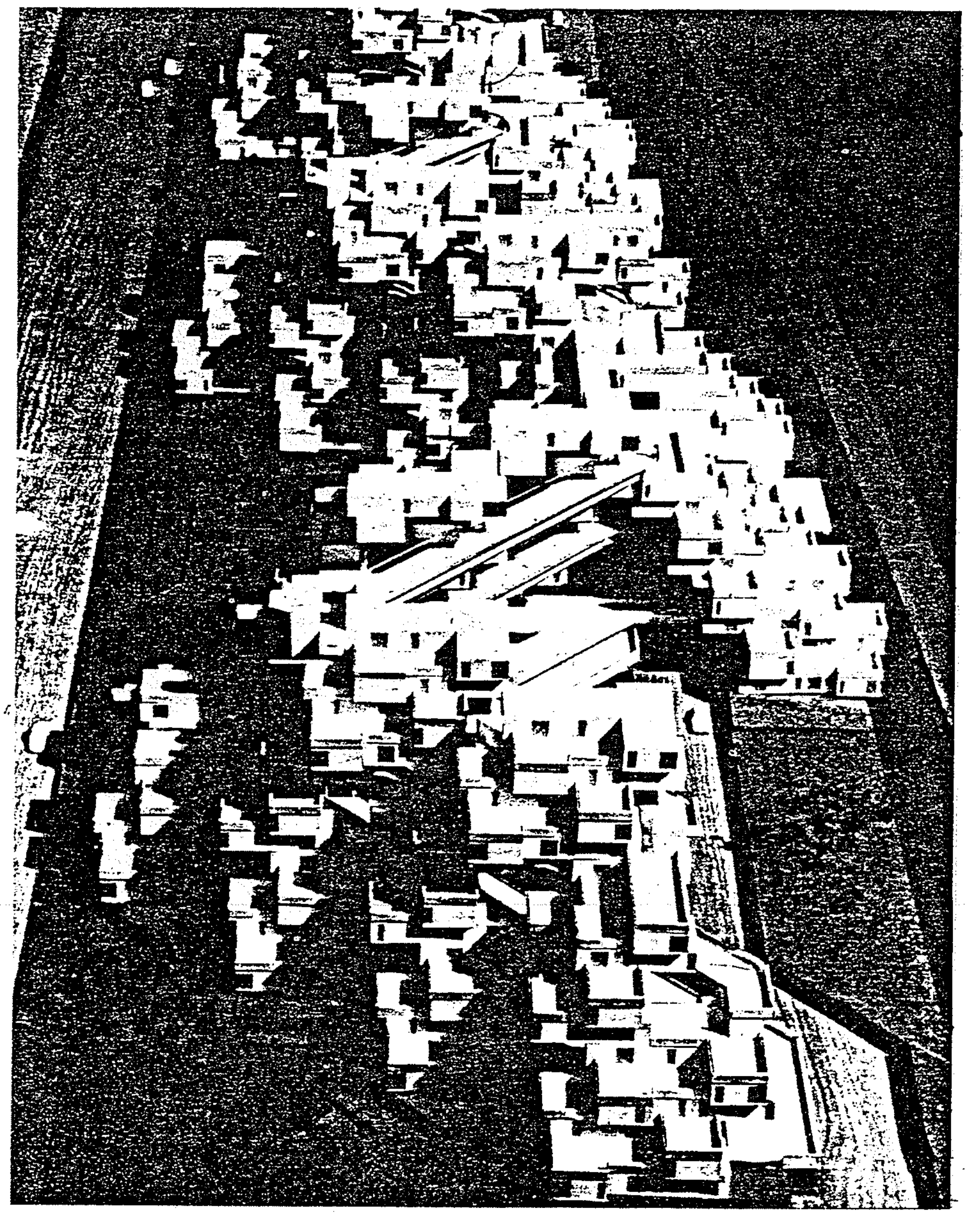
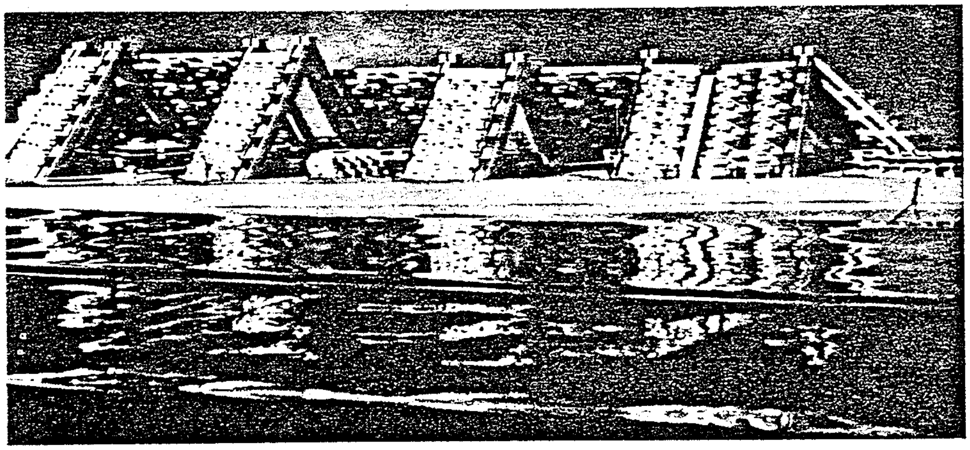


Fig. 3.303



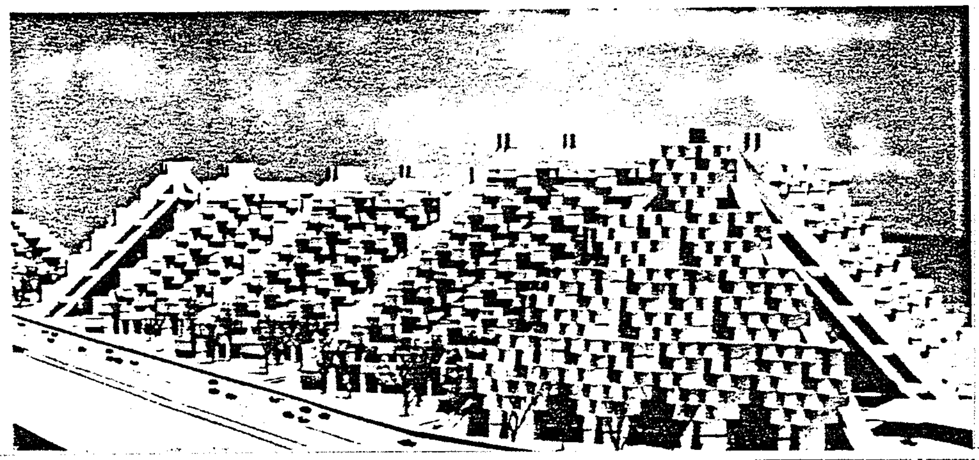


Fig. 3.304

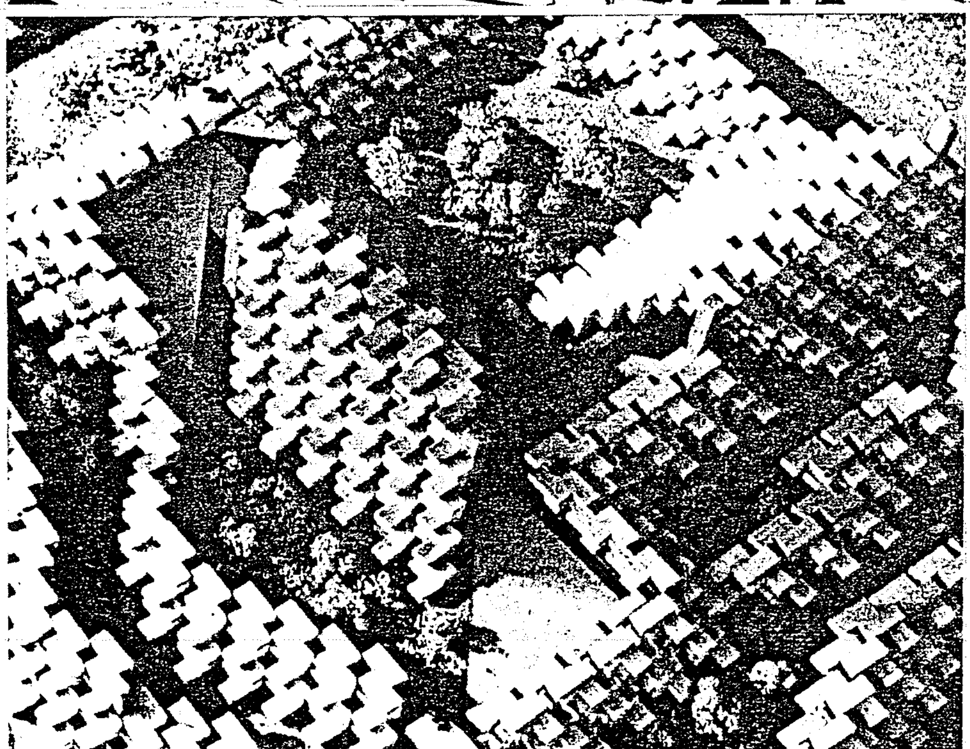


Fig. 3.305

Clearly, Sardie's philosophy that the housing environment may be created through the arrangement of prefabricated units into whole developments relies on the idea that there is an underlying structure in those arrangements. Indeed, Safdie seems to also search for order in complexity; to create aesthetically pleasing designs from the complexity of highly industrialized technologies. He argued about architecture in general that:

> Once the environment is thought of in terms of morphology, then it is easy to see and say that the environment is made up of a multitude of structures and that the understanding of these structures is essential to the understanding of the design process.[89]

Clearly, the theoretical basis for the application of the mathematical theory of symmetry shows it to be part of the basic investigation of space creating structures. The mathematical theory of symmetry might have been applied in these large scale housing developments based on the repition of prefabricated units; to externalize the underlying space creating structure of these complex designs.

The third architect to be considered, also relies on industrialized technology, is the Japanese designer, Kisho Kurokawa, who is a leader of the metabolist movement in architecture.[90] Metabolism, in general, relies on the separation of the physical structure that holds the building up, from the space creating components. This allows the building to grow and change, indefinitely. Kurokawa pioneered the use of industrialized capsules in architecture. The Nakagin Capsule Tower (Fig. 3.306) is an example of Kurokawa's approach to design. It clearly reveals the search for an underlying structure in the arrangement of these capsules; each layer investigates the possible relationships between the capsules. An application of the mathematical theory of symmetry might have provided a clear three dimensional

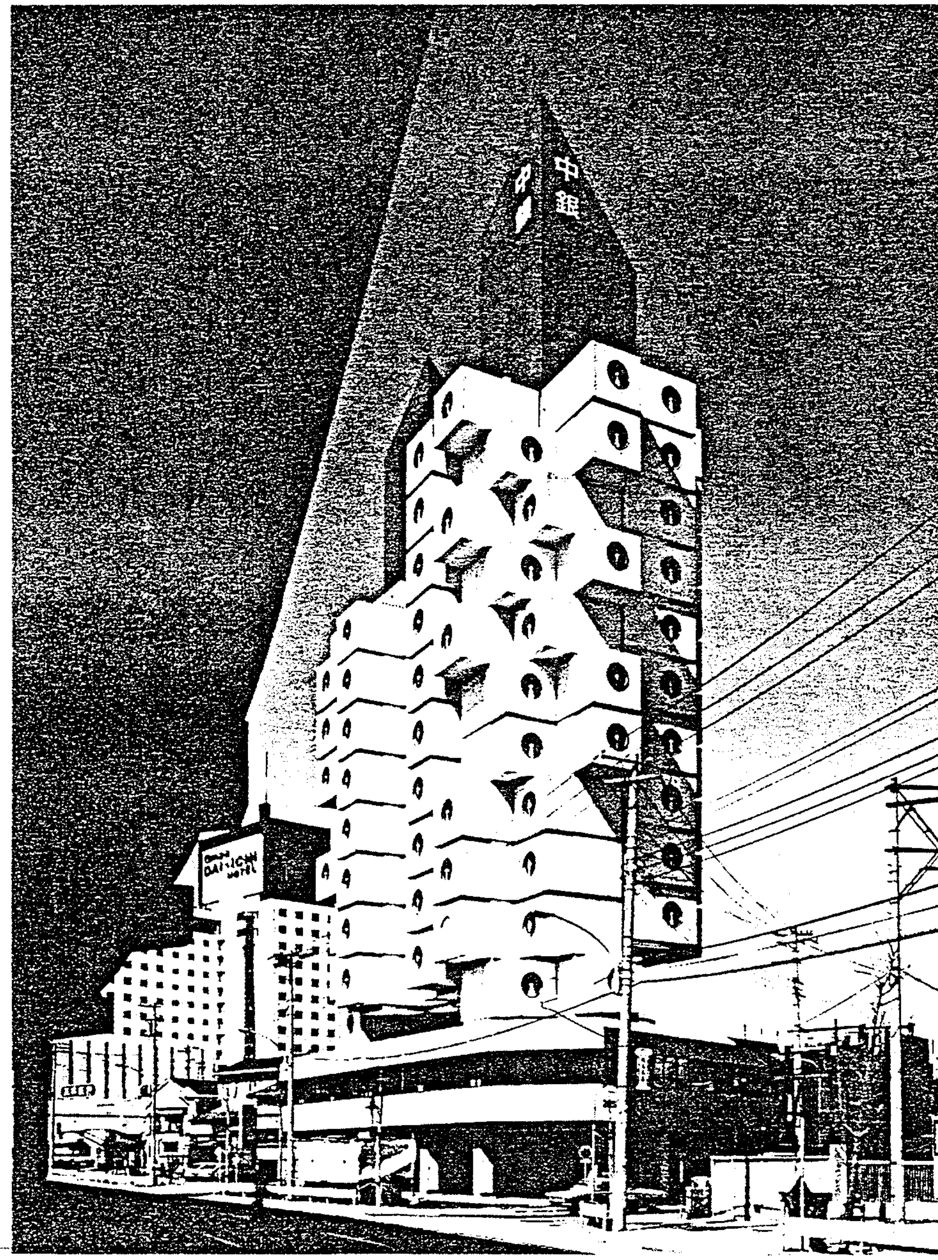


Fig. 3.306

structure in this work that would have added to the visual richness.

Kurokawa's Takara Beautillion (Fig. 3.307) is the prime example of metabolist architecture. The main physical support for the building is provided by twelve right-angled steel tubes welded together to make six arms; this is an example of the three dimensional point group, T_d , from the cubic crystal class. This is combined with the primitive cubic Bravais lattice (see section, Fig. 3.308), to generate a symmetrical configuration associated with the Fedorov group, T_d^1 . Clearly, the mathematical theory of symmetry

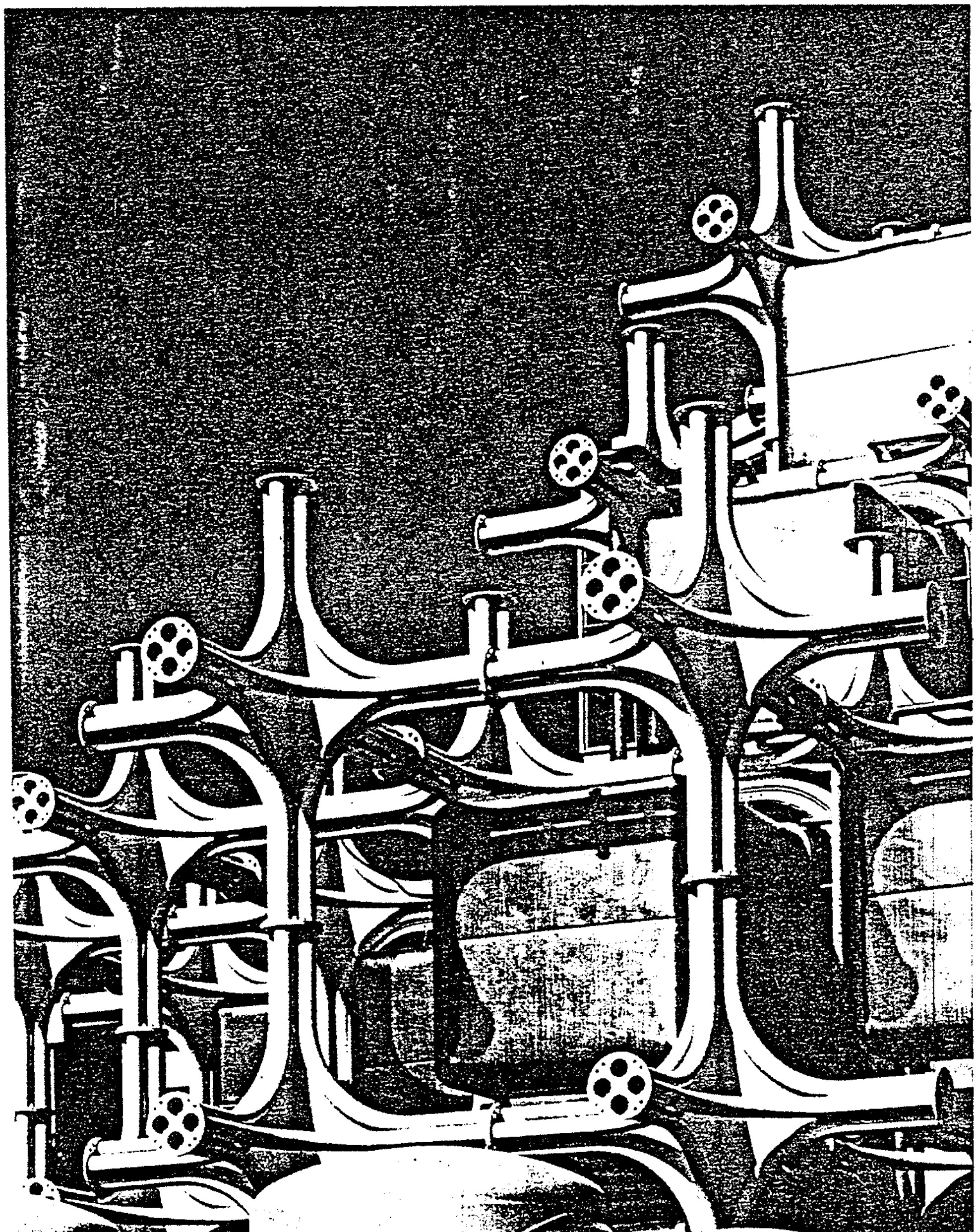


Fig. 3.307

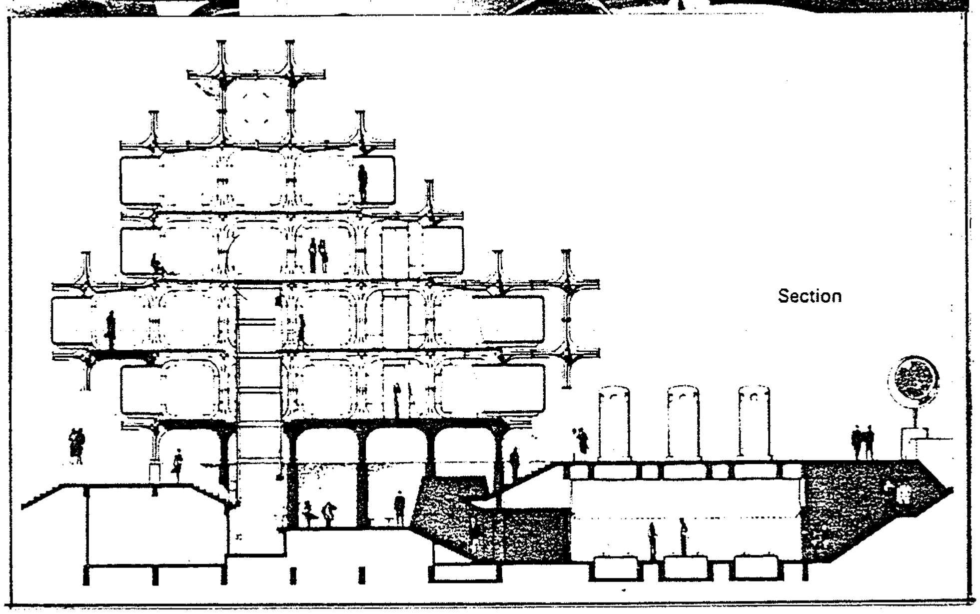


Fig. 3.308

might have been applied in this metabolist design. The fact that the building was assembled in a few days, disassembled equally rapidly; and might have grown or changed in between not only demonstrates the concept of metabolism in architecture, but the power of the mathematical theory of symmetry as a method for creating an underlying structure that responds to the context to determine the final character of the building.

The concept of a megastructure may or may not be, "Urban futures of the recent past,"[91] and a dead issue in the energy conscious design of contemporary architecture. The philosophy behind megastructure, large physical frames carrying essential services, infilled with a variety of space making components, involve an overwhelming complexity. The result was the application of simple symmetry underlying the design of the frames. The Graz-Ragnitz project by Domenig and Huth (Figs. 3.309 and 3.310), which Banham labelled "the ultimate megastructure model,"[92] has a complex three dimensional structure based on a body-centered tetragonal Bravais lattice in three dimensions. The infill appears to be totally wilfull, a sort of uninteresting dissymetry, in contrast to the overwhelming presence of the frame. However, most megastructures might also have been designed with an application of the mathematical theory of symmetry, both to the frame and the repeated elements infilling that frame.

The impression might be at this point that the mathematical theory of symmetry is a method for creating spatial order to balance the complexity of large scale, technologically innovative, highly industrialized projects. But, structural complexity may just be the product of planning difficulties due to the large number of spaces involved. An example of this are the university projects of the next architects to be considered, Candilis, Josic,

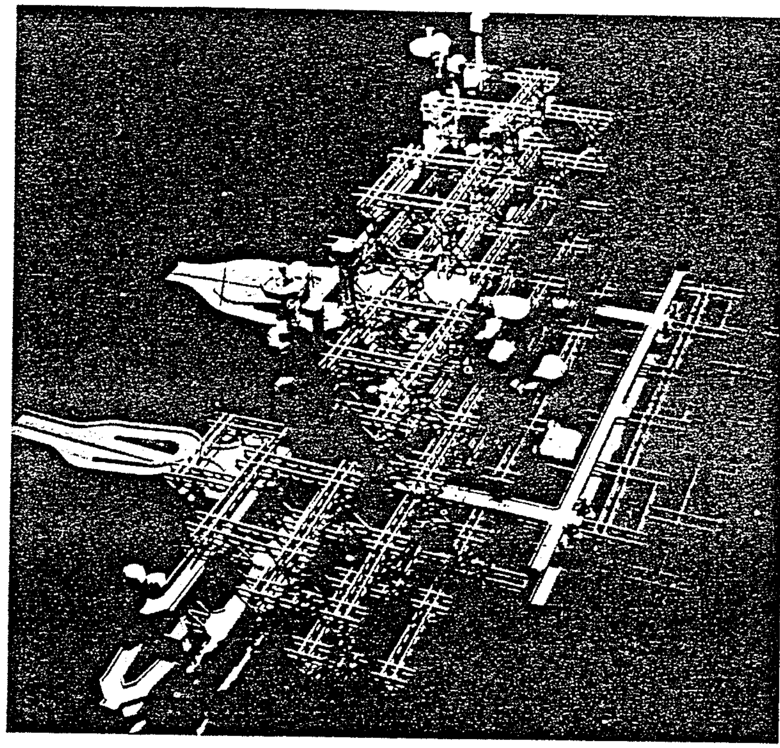


Fig. 3.309

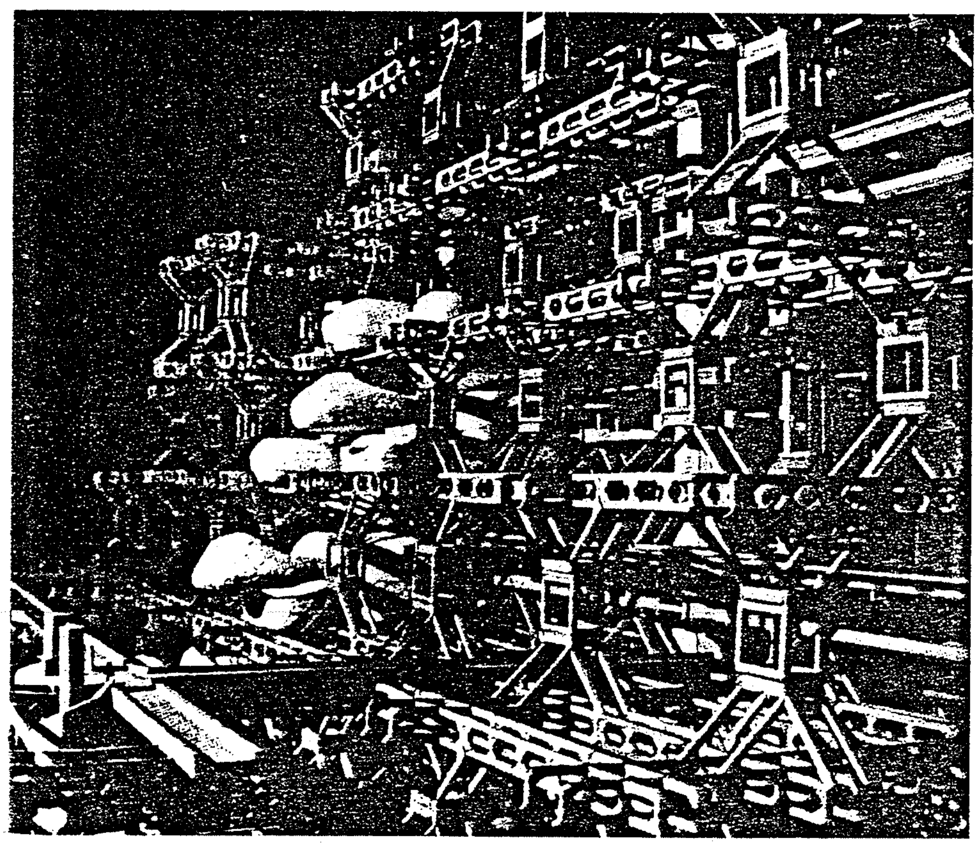


Fig. 3.310

and Woods. The University of Zurich project (Fig. 3.311) demonstrates the organization of complex plans by a relatively simple underlying structure. The basic organizing elements are the stairs, washroom, and service rooms in little plans which have no internal symmetry, they might be considered as point groups, C_1 . These are then arranged on a large square Bravais lattice; generating a symmetrical configuration associated with the wallpaper group, W_1 . It should be noted that Candilis did not take advantage of the fact it was a square lattice, it might just as well have been an oblique lattice. This incredibly simple two dimensional symmetry seems to be enough to order the high structural complexity in the plan. Within each bay, individual underlying structures may be found that might be described by various frieze and wallpaper symmetry groups. Indeed, it seems to be this layering of many structures within the simple overall structure that is the basis of the symmetry in this plan. Not surprisingly, asymmetry and dissymmetry also occur for various planning and emphasis reasons. This project clearly might have applied the mathematical theory of symmetry in its design to externalize Candilis' approach to the order in its formal organization.

The sixth and final architect to be considered is Herman Herzberger. His design for an office building, the Centraal Beheer in Appeldoorn, Holland (Fig. 3.312), has been praised[93] for its attempt to create qualities of space. The underlying structure that creates those spaces that accomodate a variety of different places is one of the more subtle symmetrical configurations in two dimensions. The basic element is the T-shaped column arranged in the point group, C_4 , on a square lattice on the angle bisecting the cells, generating a symmetrical configuration associated with the wallpaper

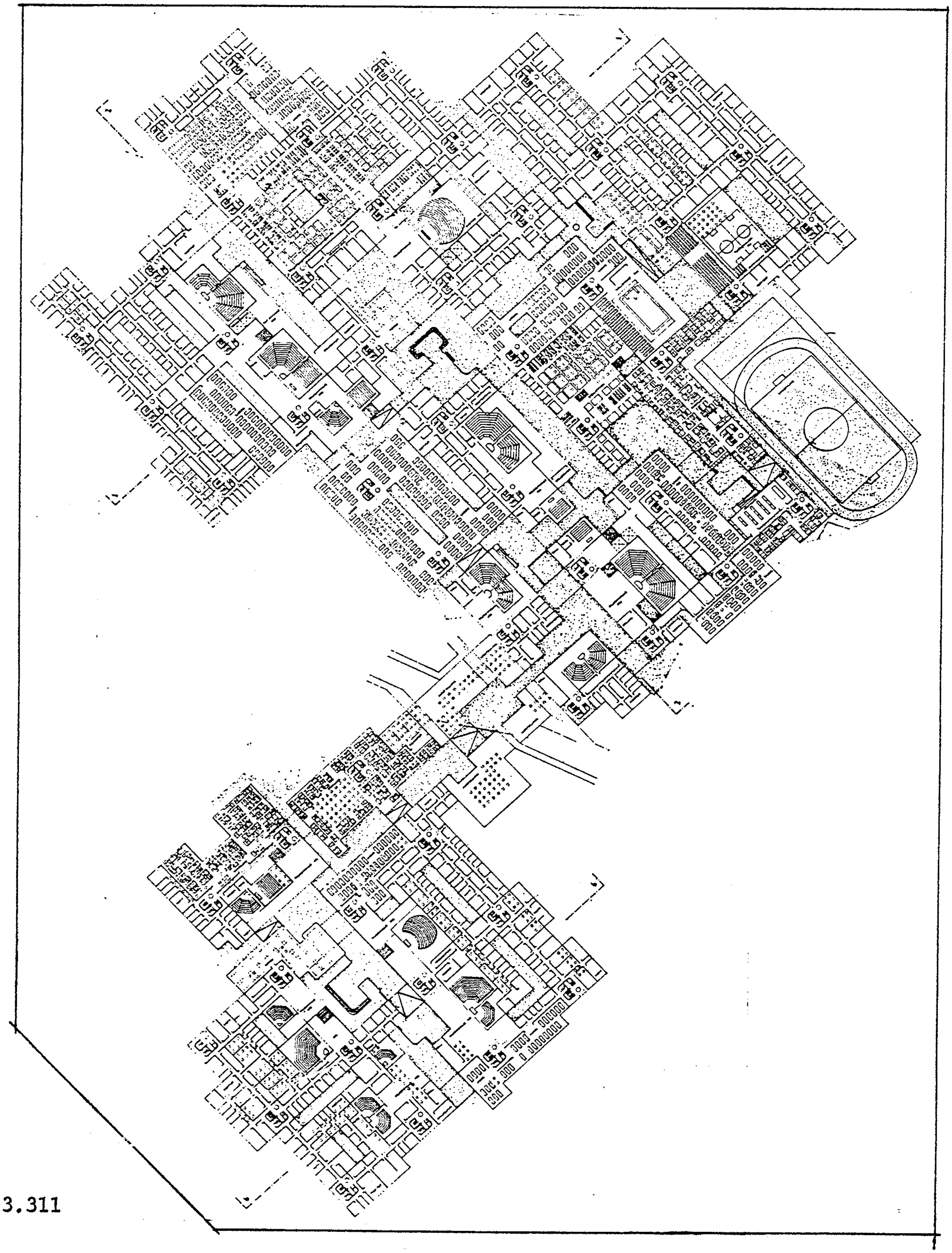


Fig. 3.311

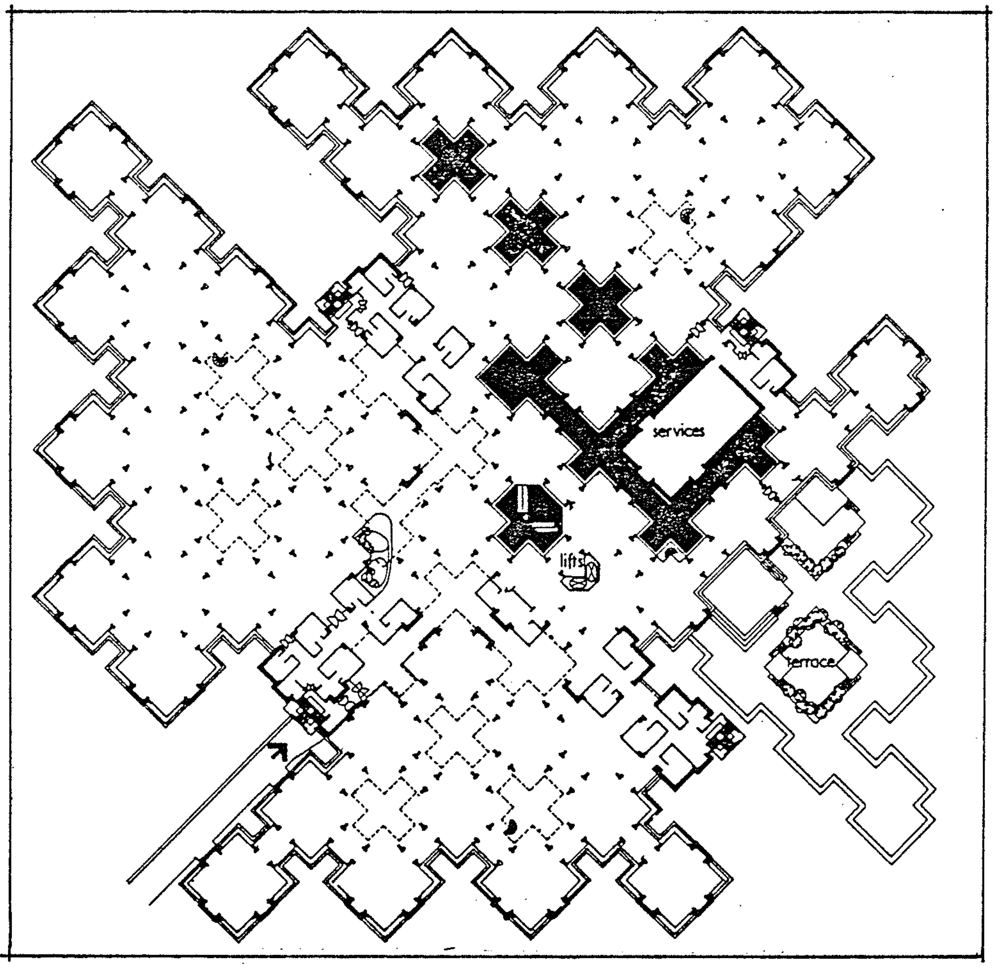


Fig 3.312

group, W_4^2 . The quality of order resulting from this symmetry subtly balances the complexity of the variety of spaces inside the building. It produces an aesthetically preferable design that seems to strike just the right balance between order and structural complexity. Clearly, the mathematical theory of symmetry might have been applied in this design with some careful judgement.

The purpose of providing all these examples of the potential of the mathematical theory of symmetry in symmetry is not to argue that they are all good architecture. The intent is only to show that the mathematical theory of symmetry does have potential as a design method, not only because it is theoretical possible as established in Part One, or because the resources of the theory tend to support that as established in Part Two, but because it is possible to see that potential in a number of diverse sorts of architecture.

This concludes the speculations that attempt to evaluate the thesis that the mathematical theory of symmetry may be a significant part of the knowledge an architect applies in design. It also concludes the arguments of the whole thesis. Weyl concluded his famous lecture on symmetry with this thought:

> Symmetry is a vast subject, significant in art and nature. Mathematics lies at its root, and it would be hard to find a better one on which to demonstrate the working of the mathematical intellect. I hope I have not completely failed in giving you an indication of its many ramifications, and in leading you up the ladder from intuitive concepts to abstract ideas.[94]

Hopefully, the mathematical theory of symmetry may be brought within the everyday activity of design in architecture. There certainly is the pedagogical implication that it should be taught in any course of basic design. Clearly, it is a significant method for the creation of the aesthetic quality of spatial order in architecture; it deserves to be among the methods of the designer. The mathematical theory of symmetry may indeed be a significant part of the knowledge an architect applies in design.

3.4 DIRECTIONS FOR RESEARCH

The only remaining task of the thesis is to suggest some directions for research that follow from the work put into this thesis. The intent of this section is to make some speculations about those directions for research as a post-script to the conclusions reached earlier. There are four distinct areas to be considered as possible spin-offs from this thesis. They are first, the extension of the mathematical theory to include colour symmetry; second, a philosophical investigation of the foundation of mathematics in architecture; third, the development of a semiotic approach to architectural syntax based on symmetry; and fourth, an operationalization of the application of the theory of symmetry within computer aided design.

The first area of research that might be considered is to extend the mathematical theory of symmetry as it has been presented here into colour symmetry. The possibility for describing coloured symmetrical configurations with group theory was developed by Shubnikov, Belov and others in the U.S.S.R. in the 1950's.[95] The idea is based on the concept of treating colour equivalence exactly as the geometric equivalence that is the basis of classical symmetry (see section 1.1). With only two colours, Shubnikov has shown that the two hundred thirty Fedorov groups in space, may be extended exhaustively to one thousand six hundred fifty one groups in space. These so-called Shubnikov groups might be applied to architecture because the one aspect affecting the aesthetic quality of the work, other than the spatial order, is the materials used in the space creating elements. That both aspects might be dealt with through the mathematical theory of symmetry seems to open even more creative potential for the designer. The use of colour and materials is an important aesthetic concern in space creation, and the mathematical theory of symmetry, if extended to colour symmetry, would provide a method for relating colours and materials to the underlying structure in the design.

The second area of research that might be considered is the more academic exercise of a theory establishing the philosohical foundation of mathematics applied in architecture. In trying to clarify the nature of mathematics (see section 1.2), it emerged that the philosophy of mathematics is a fascinating and unduly ignored part of the increasing application of mathematics in design. If the practice of architecture is to become a rigourous discipline like the practice of medicine, then there is a need to understand how mathematical methods fit into that discipline, just as medicine needs to understand how its methods fit into their discipline. Architecture is not a science, in the sense that science aims only at explanation, while architecture aims at change. There is a need then to understand the discipline of architecture itself. Such an understanding of architecture and mathematical methods in design is inherently normative, not descriptive. The ethics in an appraach to architecture applying mathematics should be the starting point for this research. The research in the philosophical foundation of design is among the most significant that should be associated with the application of mathematical methods in architecture.

The third area of research that might be considered is the development of a semiotic approach to architectural syntax based on themmathematical theory of symmetry. The discussion of architectural aesthetics (see section 3.1) assumed the work of architecture to be a channel for communication between designer and user. The study of the semiotics of that communication must assess what provides the syntax for that communication. The orderly cognition of space through a clear underlying structure must surely have some relevance to that syntax. Hence, the mathematical theory of symmetry might provide a method for describing spatial syntax in architecture through the symmetry of the relations between architectual elements. This is however, a highly speculative assumption that requires a much further theoretically basis in architectural semiotics. But, any research in the externalizing of architectural syntax would lead to significant design methodsfor the architect. The mathematical theory of symmetry applied to the relationships between architectural elements may be involved in those methods.

The fourth, and by far the most stimulating, area of research that might be considered is an operationalizing of the application of the mathematical theory of symmetry within computer aided design. The resources of the mathematical theory of symmetry as presented in Part Two are passive knowledge. The next step in research should be activating that knowledge in the design process so that it actually becomes a technique for doing design, not just a method applied in design. This requires mathematically representing the Fedorov groups in space with an active description, perhapse sets of matrices, into which the architect could substitute dimensions and angles;and, of course, a description of the space creating elements. This would suggest an interactive computer program that allows the designer to "play" with Bravais lattices and point groups as suggested in an approach to design applying the mathematical theory of symmetry (see section 3.2). This involves a system for representing architectural elements and a system for manipulating them according to an underlying structure. These might be two directions that computer aided design should investigate. The exciting future of mathematics in architecture almost certainly lies in computer aided design. The interactive situation where a designer could use subjective judgements with an operational application of the mathematical theory of symmetry would allow computer aided design to become involved in aesthetic concerns such as the creation of spatial order in architecture.

This research, like all research, should go beyond research to become practice. As was indicated in the introduction, the ultimate goal of research in architecture is to change the way buildings are designed in the "real world". The only test of that research is. ultimately, the evaluation of buildings designed and built with these methods. The knowledge of the the mathematical theory of symmetry must be involved in more than just research or academic exercises, it should involve a significant architectural practice with that knowledge. This will be the most significant "research" that involves the mathematical theory of symmetry.

MATHEMATICAL APPENDIX

The intent of this appendix is to present an introduction to the elements of group theory involved in the mathematical theory of symmetry. The presentation consists primarily of definitions and well-established theorems from group theory. The discussion is limited, and no rigourous mathematical derivation of symmetry groups is attempted; they must be found in the technical literature on the mathematical theory of symmetry listed in the Bibliography. The obvious starting point is the definition of a group, everything follows from that.

DEFINITION D:01 GROUP A group, G , is a set of elements together with a composition law, called a product, such that:

(1) the product of any two elements, ab , of the group is defined and there is an element, c , in the group, such that $ab = c$;
(2) the product is associative: $a(bc) = (ab)c$, for all $a,b,c \in G$;
(3) there exists a unique identity element, e , in the group: $ea = ae = a$, for all $a \in G$; and
(4) for every element in the group there exists a unique inverse element: for all $a \in G$, there exists $a^{-1} \in G$, such that $aa^{-1} = a^{-1}a = e$.

DEFINITION D:02 ABELIAN GROUP A group, G , is said to be Abelian if it meets all the conditions in D:01, and the extra condition:

(5) the product is commutative: $ab = ba$, for all $a,b \in G$.

DEFINITION D:03 ORDER The number of elements in a group, G , is said to be the order of the group; and is denoted $|G|$.

DEFINITION D:04 GENERATORS A set of elements of a group, G , is said to be a set of generators of the group if any element a G can be written as the product of the powers of the generators and their inverses

DEFINITION D:05 DEFINING RELATIONS A set of relations satisfied by the generators of a group, G , which are sufficient to completely determine every element of the group, is said to be the defining relations of G.

DEFINITION D:06 MAPPING A mapping, $\emptyset$, of a set, S , to a set, T ; denoted $\emptyset: S \rightarrow T$, is a rule which assigns to each element $s \in S$ a unique element $t \in T$.

DEFINITION D:07 HOMOMORPHISM Given two groups, G and H , a mapping, $\emptyset: G \rightarrow H$, which preserves the multiplication, $\emptyset(a)\emptyset(b) = \emptyset(ab)$ for all $a,b \in G$, is said to be an homomorphism.

DEFINITION D:08 ISOMORPHISM A homomorphism, $\emptyset$, of G onto H, $\emptyset:G \rightarrow H$, in which there is a one-to-one correspondence of the elements of G with the elements of H is said to be an isomorphism. Two groups, G and H , are said to isomorphic, denoted $G \cong H$, if there is an isomorphism mapping G onto H, and an inverse isomorphism mapping H onto G.

DEFINITION D:09 AUTOMORPHISM An isomorphism, $\emptyset$, which maps a group, G , onto itself, $\emptyset:G \rightarrow G$, is said to be an automorphism.

THEOREM TH:01 If $\emptyset_1$ and $\emptyset_2$ are two automorphism of a group, G , then the mapping product, $\emptyset_1\emptyset_2$, where the multiplication, $(\emptyset_1\emptyset_2)a = \emptyset_1(\emptyset_2 a)$, for all $a \in G$ is preserved, is also an automorphism. Further the set of all automorphisms of a group, G , is itself a group, of which the composition law is the mapping product just defined.

It is significant here to reiterate the definition of symmetry that was developed in section 1.1. The symmetry of a configuration was defined as the highest possible (of the highest order) of automorphisms mapping any structural configuration onto itself. Two more definitions should allow the classification of symmetry groups. They are:

DEFINITION D:11 SUBGROUP A subset, H , of elements of a group, G , that themselves form a group under the same composition law as in G, is said to a subgroup of G. All groups have at least two subgroups, namely the group itself and the group consisting of the identity element alone; these two groups are called improper subgroups, all others are called proper subgroups.

DEFINITION D:12 OUTER DIRECT PRODUCT Let G be a group, with two proper subgroups, H and K , such that:
(1) if $h \in H$, and $k \in K$, then $hk = kh$;
(2) all $g \in G$ may be expressed in the form, $g = hk$;
(3) the intersection of the set of elements of H and the set of elements of K is the identity element, e , of G, $H \cap K = e$.
Then G is said to be the outer direct product of H and K, and is written, $G = H \times K$.

With these definitions, it is now possible to make a table of the Fedorov groups; classifying them according to crystal class, Bravais lattice, point groups, isomorphisms, and order. The number of Fedorov symmetry groups with these properties is also shown in the table (TABLE MA:01), [illegible]

TABLE MA:01

Crystal Class	State of Lattice	Point Group	Isomorphism	Order	No. of Symmetry Groups
Triclinic	primitive	C_1	—	1	1
		C_i	C_2	2	1
Monoclinic	primitive	C_2	—	2	2
		C_s	C_2	2	2
		C_{2h}	$C_{2h} \cong C_2 \times C_2$	4	4
	centered	C_2	—	2	1
		C_s	C_2	2	2
		C_{2h}	$C_{2h} \cong C_2 \times C_2$	4	2
Orthorhombic	primitive	D_2	$D_2 \cong C_2 \times C_2$	4	4
		C_{2v}	D_2	4	10
		D_{2h}	$D_{2h} \cong D_2 \times C_2$	8	16
	centered	D_2	$D_2 \cong C_2 \times C_2$	4	2
		C_{2v}	D_2	4	7
		D_{2h}	$D_{2h} \cong D_2 \times C_2$	8	6

Crystal Class	State of Lattice	Point Group	Isomorphism	Order	No. of Symmetry Groups
	face-centered	D_2	$D_2 \cong C_2 \times C_2$	4	1
		C_{2v}	D_2	4	2
		D_{2h}	$D_{2h} \cong D_2 \times C_2$	8	2
	body-centered	D_2	$D_2 \cong C_2 \times C_2$	4	2
		C_{2v}	D_2	4	3
		D_{2h}	$D_{2h} \cong D_2 \times C_2$	8	4
Tetragonal	primitive	C_4	—	4	4
		S_4	C_4	4	1
		C_{4h}	$C_{4h} \cong C_4 \times C_2$	8	4
		D_4	—	8	8
		C_{4v}	D_4	8	8
		D_{2d}	D_4	8	8
		D_{4h}	$D_h \cong D_4 \times C_2$	16	16
	body-centered	C_4	—	4	2
		S_4	C_4	4	1
		C_{4h}	$C_{4h} \cong C_4 \times C_2$	8	2
		D_4	—	8	2
		C_{4v}	D_4	8	4
		D_{2d}	D_4	8	4
		D_{4h}	$D_{4h} \cong D_4 \times C_2$	16	4
Trigonal	primitive	C_3	—	3	3
		S_6	C_6	6	1
		D_3	—	6	6
		C_{3v}	D_3	6	4
		D_{3v}	D_6	12	4

Crystal Class	State of Lattice	Point Group	Isomorphism	Order	No. of Symmetry Groups
	rhombohedral	C_3	—	3	1
		S_6	C_6	6	1
		D_3	—	6	1
		C_{3v}	D_3	6	2
		D_{3v}	D_6	12	2
Hexagonal	centered (primitive trigonal)	C_6	$C_6 \cong C_3 \times C_2$	6	6
		C_{3h}	C_6	6	1
		C_{6h}	$C_{6h} \cong C_6 \times C_2$	12	2
		D_6	$D_6 \cong D_3 \times D_2$	12	6
		C_{6v}	D_6	12	4
		D_{3h}	D_6	12	4
		D_{6h}	$D_{6h} \cong D_6 \times C_2$	24	4
Cubic	primitive	T	—	12	2
		T_h	$T_h \cong T \times C_2$	24	3
		O	—	24	4
		T_d	O	24	2
		O_h	$O_h \cong O \times C_2$	48	4
	face-centered	T	—	12	1
		T_h	$T_h \cong T \times C_2$	24	2
		O	—	24	2
		T_d	O	24	2
		O_h	$O_h \cong O \times C_2$	48	4
	body-centered	T	—	12	2
		T_h	$T_h \cong T \times C_2$	24	2
		O	—	24	2
		T_d	O	24	2
		O_h	$O_h \cong O \times C_2$	48	2

NOTES

[1] Note that this definition differs from Alexander's idea of a pattern, see Christopher Alexander, et al, A Pattern Language (New York: Oxford University Press, 1977)

[2] Frank Lloyd Wright, The Future of Architecture (New York: Horizon Press, Inc. 1953), p. 60

[3] Rudolf Arnheim, The Dynamics of Architectural Form (Berkely: University of California Press, 1977), p. 2

[4] G.H. Hardy, A Mathematician's Apology (Cambridge: Cambridge University Press, 1967), p. 84-85

[5] Team 10 Primer, ed. Alison Smithson (Cambridge, Mass.: MIT Press, 1968), p. 82

[6] William Huff, "An Argument for Basic Design" Ulm Journal, 12/13, (1965), p. 27

[7] Lionel March and Philip Steadman, The Geometry of Environment (London: RIBA Publications Ltd, 1971).

[8] Lionel March, Editorial Environment and Planning B, vol. 2 (1975), p. 126

[9] Vitruvius, The Ten Books on Architecture, trans. M.H. Morgan, (New York: Dover Publications, 1960), p. 14

[10] Hermann Weyl, Symmetry (Princeton: Princeton University Press, 1952)

[11] A. V. Shubnikov et al, Symmetry in Science and Art, trans. and ed. David Harker (New York: Plenum Press, 1974)

[12] William Huff, Symmetry: An Appreciation of its Presence in Men's Consciousness (New York: n.p. , n.d.), Part 6, p. 3

13 Kent Bloomer and Charles Moore, Body, Memory, and Architecture (New Haven: Yale University Press, 1977), p. 46

14 Shubnikov, p. 7

15 see, Michael C. Corballis, The Psychology of Left and Right (New York: Halsted Press, 1976); see also Weyl, pp. 22-25.

16 Shubnikov, p. 4

17 Shubnikov, p. 127

18 Clearly, all configurations possess the single automorphism created by rotation of 360°. This is too trivial to qualify it as symmetrical.

19 Weyl, p. 45

20 Shubnikov, p. 308

21 see footnote 85

22 March and Steadman, p. 40

23 Owen Jones, The Grammar of Ornament (London: Day and Son, 1856)

24 March and Steadman, p. 86

25 Hermann Weyl, "The Mathematical Way of Thinking" The World of Mathematics, ed. James R. Newman, (New York: Simon and Schuster, 1956), p. 1832

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