SOME POINTS IN ALGEBRAIC NUMBERS DEFINED BY A QUARTIC EQUATION.

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INTRODUCTION.

The object of the present thesis is to work out some points in Algebraic Numbers defined by a quartic equation.

In the first chapterthe reduction of an algebraic number of the form $f(x)/g(x)$ to the form $F(x)$, where f, g and F are all polynomials, is worked out explicitly.

In the second cha ter a transformation of the cubic is worked out, expressing it as the sum of two expressions, one of which is the form it takes when roots are equal, and the other is linear and contains the discriminant as a factor. This transformation has been found to simplify the corresponding problem in the cubic fields. The same transformation is also worked out for the quartic.

In the third chapter, taking an integer of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ the seven integral combinations known to be rational integers are worked out as an application. Also, the basis is found for the special case $x^4 + s = 0$.

CHAPTER 1.

Sec. 1.

Any algebraic number in the field K(x), where x is of degree "n" is of the form $\frac{f(x)}{g(x)}$ where f(x) and g(x) are polynomials of degree "n -1" at most. The existance of a polynomial r(x), also of degree "n - 1" at most, such that $r(x) = \frac{f(x)}{g(x)}$ has been shown in general, but the method does not give the actual result. The object of this chapter is to determine a formula for finding the coefficients of r(x) for the general cubic and quartic fields given by $x^3 + qx + r = 0$ and $x^4 + qx^2 + rx + s = 0$.

Sec. 2. The Cubic.

For the cubic field $x^3 + qx + r = 0$, any algebraic number in this field is of the form $\frac{ax^2+bx+c}{dx^2+fx+h}$. We wish to determine u, v, and w, as given by $\frac{ax^2+bx+c}{dx^2+fx+h} = ux^2 + vx + w$.

Multiplying we have,

$$ax^2 + bx + c = dux^4 + (uf + vd)x^3 + (hu + fv + wd)x^2 + (vh + wf)x + hw .$$

But we have that, $x^3 + qx + r = 0$ or $x^3 = -(qx + r)$ and $x^4 = -(qx^2 + rx)$. Substituting these values of x^4 and x^3, we have,

$$ax^2 + bx + c = (hu - duq + fv + dw)x^2 + (-dru - qfu + vh - qdv + wf)x + (-ruf - rdv + hw)$$

Sec. 2.

Equating coefficients,

$(h - dq)u + fv + dw = a$

$-(dr + qf)u + (h - qd)v + fw = b$

$-rfu - rdv + hw = c$

$$u = \frac{\begin{vmatrix} a & f & d \\ b & h-qd & f \\ c & -rd & h \end{vmatrix}}{\begin{vmatrix} h-dq & f & d \\ -(dr+qf) & h-qd & f \\ -rf & -rd & h \end{vmatrix}} \qquad v = \frac{\begin{vmatrix} h-dq & a & d \\ -(dr+qf) & b & f \\ -rf & c & h \end{vmatrix}}{\begin{vmatrix} h-dq & f & d \\ -(dr+qf) & h-qd & f \\ -rf & -rd & h \end{vmatrix}}$$

$$w = \frac{\begin{vmatrix} h-dq & f & d \\ -(dr+qf) & h-qd & b \\ -rf & -rd & c \end{vmatrix}}{\begin{vmatrix} h-dq & f & d \\ -(dr+qf) & h-qd & f \\ -rf & -rd & h \end{vmatrix}}$$

In order that u, v, and w have real and finite values it is sufficient toshow that the denominator above does not vanish. Expanding the denominator we have:

$$\begin{vmatrix} h-dq & f & d \\ -(dr+qf) & h-qd & f \\ -rf & -rd & h \end{vmatrix} = d\begin{vmatrix} -(dr+qf) & h-qd \\ -rf & -rd \end{vmatrix} - f\begin{vmatrix} h-dq & f \\ -rf & -rd \end{vmatrix} + h\begin{vmatrix} h-dq & f \\ -(dr+qf) & h-qd \end{vmatrix}$$

$$= r^2d^3 + rd^2qf + 3rfdh - 2d^2rfq - rf^3 + h^3 - 2h^2dq + hd^2q^2 + hqf^2$$

$$= r^2d^3 - rd^2qf + hd^2q^2 + 3rfdh - 2h^2dq - rf^3 + h^3 + hqf^2$$

This expression is, however, identical with the

resultant of the equation $x^3 + qx + r = 0$ and $dx^2 + fx + h = 0$ as obtained by the G.C.D. process. But $x^3 + qx + r = 0$ is an irreducible equation of the third degree, hence it can have no factor in common with another algebraic equation of lower degree. Therefore the resultant of the equations $x^3 + qx + r = 0$ and $dx^2 + fx + h = 0$ cannot be zero, for if it were the two equations would have a common factor contrary to hypothesis. Therefore the denominator in the values for u, v, and w does not vanish and u, v, and w have real and finite values.

Note.

The resultant of the two equations $x^3 + qx + r = 0$ and $dx^2 + fx + h = 0$ as obtained above differs from that obtained by substitution in the resultant obtained by Burnside & Panton Vol.2. Sec. 156 for two cubics in that Burnside & Panton's result gives $d^2qrf - hd^2q^2$ where we have $- rd^2qf + hd^2q^2$.

Sec. 3. The Quartic.

Any algebraic number in the quartic field

$x^4 + qx^3 + rx + s = 0$ is of the form $\frac{ax^3 + bx^2 + cx + d}{ex^3 + fx^2 + hx + k}$.

We wish to find values for u, v, w, t, such that

$$\frac{ax^3 + bx^2 + cx + d}{ex^3 + fx^2 + hx + k} = ux^3 + vx^2 + wx + t.$$

Multiplying we have,

$$ax^3 + bx^2 + cx + d = eux^6 + (fu + ev)x^5 + (hu + fv + ew)x^4$$
$$(ku + hv + fw + et)x^3 + (kv + hw + ft)x^2 + (kw + ht)x + kt.$$

But $x^4 = -(qx^3 + rx + s)$

$x^5 = -x(qx^3 + rx + s)$

$x^6 = -x^2(qx^3 + rx + s) = (q^2x^3 + qrx + qs - rx^3 - sx^2)$

Therefore $ax^3 + bx^2 + cx + d = (-reu + ku - qfu - qev + hv + fw + et)x^3 + (q^2eu - seu - rfu - qhu - qfv - rev + kv - qew + hw + ft)x^2 + (qreu - sfu - rhu - sev - rfv - wew + kw + ht)x + qseu - shu - sfv - sew + kt$

Equating coefficients

$a = (k - re - qf)u + (h - qe)v + fw + et$

$b = (q^2e - se - rf - qh)u + (k - re - qf)v + (h - qe)w + ft$

$c = (qre - sf - rh)u + (-se - rf)v + (k - re)w + ht$

$d = (qse - sh)u - sfv - sew + kt$

$$u = \frac{\begin{vmatrix} a & h-qe & f & e \\ b & k-re-qf & h-qe & f \\ c & -(se+rf) & k-re & h \\ d & -sf & -se & k \end{vmatrix}}{\begin{vmatrix} k-er-qt & h-qe & f & e \\ eq^2-es-rf-qh & k-er-qf & h-qe & f \\ eqr-sf-rh & -(se+rf) & k-re & h \\ qse-sh & -sf & -se & k \end{vmatrix}}$$

Similar expressions can be written down for v, w, and t the denominator being the same in allcases.

If we multiply both numerator and denominator of $\frac{ax^3+bx^2+cx+d}{ex^3+fx^2+hx+d}$ by $ex_1^3+fx_1^2+hx_1+k$ and similar expressions in x_2 and x_3, where x_1, x_2, x_3, are the conjugates of x, then the denominator will be a symmetric function and therefore ex pressible in terms of the coefficients of $x^4+qx^2+rx+s=0$. In the numerator we will have symmetric expressions except for terms involving x. For exam le : $x_1^2+x_2^2+x_3^2$ = the symmetric function minus x^2 = an expression in the coefficients of $x^4+qx^2+rx+s=0$ minus x^2. Hence the numerator will involve the two sets of coefficients and powers of x which may be reduced to the form ux^3+vx^2+wx+t by using $x^4+qx^2+rx+s=0$. These coefficients have unique finite values unless the denominator is zero, that is unless $x^4+qx^2+rx+s=0$ and $ex^3+fx^2+hx+k=0$ have a common root,(since the denominator is the resultant of the two equations) which is impossible. Hence the denominator cannot be zero.

CHAPTER 2.

Sec.1.

If $f(x) = 0$ be the equation generating an algebraic number field, the particular case in which $f(x)$ has repeated factors mod.p, where p^2/Δ the discriminant, is of importance. If $d(x)$ denote the G.C.D. of $f(x)$ and $f'(x)$, then $f(x) - (d(x))^2 . g(x)$ should vanish when two roots of $f(x)$ become equal. We should therefore expect the remainder to be a multiple of Δ, since $\Delta = 0$ is the necessary and sufficient condition for equal roots. By choosing the coefficients of $g(x)$ so as to make as many coefficients of the difference vanish as possible, we obtain an expression of low degree which is of assistance in finding the basis of the algebraic number field generated by $f(x) = 0$. The object of the present chapter is to exhibit these relations for the cubic and quartic. These equations are taken in the form:

$$x^3 + qx + r = 0$$

$$x^4 + qx^2 + rx + s = 0.$$

to which forms they can be reduced by a rational transformation.

2.

The Cubic: General Case.

The cubic $x^3 + qx + r = 0$ reduces to the form with a squared factor when $\Delta = 0$. This suggests that $x^3 + qx + r = 0$ and the form with the squared factor differ by a quantity which is a multiple of Δ.

To find the form with a squared factor, we have that $x^3 + qx + r$ and $3x^2 + q$ have a common factor. The common factor is obtained by going through the G.C.D. process:

$$\begin{array}{l} 3x^2+q) \; x^3 + qx + r \quad | \times 3 \\ \qquad\quad 3x^3 + 3qx + 3r \quad | \; x \\ \hline \qquad\quad 2qx + 3r) \; 3x^2 + q \end{array}$$

Hence $2qx + 3r$ must be the common factor or $x + \frac{3r}{2q}$

The cubic with equal roots therefore takes the form $(x + \frac{3r}{2q})^2 \cdot (x - \frac{3r}{q})$, the latter being determined from the fact that the sum of the roots, that is, the coefficient of x^2 must be zero. We should now expect that, $x^3 + qx + r - (x + \frac{3r}{2q})^2 \cdot (x - \frac{3r}{q}) = M(\Delta)$

or multiplying by $4q^3$ to clear of fractions,

$4q^3(x^3 + qx + r) - (2qx + 3r)^2 \cdot (qx - 3r) = M(\Delta)$

By actual multiplication we have,

$4q^3(x^3 + qx + r) - (4q^2x^2 + 12qrx + 9r^2) \cdot (qx - 3r) =$

$(4q^3 + 27r^2) \cdot (qx + r) = \Delta(qx \; r)$ since the discriminant of the equation $x^3 + qx + r = 0$ is $4q^3 + 27r^2$.

$x^3 + qx + r \equiv 0$ is $4q^3 + 27r^2$.

3. The Cubic: Special Case.

If $q \equiv 0$ in the equation $x^3 + qx + r \equiv 0$ the G.C.D. process fails to give a linear repeated factor. Hence the repeated factor is triple, viz. x^3 and we have the special relation,

$$(x^3 + r) - x^3 \equiv M(\Delta)$$

In this case the discriminant Δ of the equation $x^3 + r \equiv 0$ is $27r^2$, and we thus have,

$$(x^3 + r) - x^3 = r = \frac{\Delta}{27r} .$$

4. The Quartic: General Case.

Before considering the quartic equation $x^4 + Qx^2 + Rx + S \equiv 0$, we will first consider the equation $x^4 + 6qx^2 + 4rx + s \equiv 0$ and obtain our results for this equation and then deduce our results for the former equation by substituting Q for $6q$, R for $4r$ and S for s in the results obtained for the latter. We do this because the value of the discriminant as obtained from Burnside & Panton, is given in the binomial form for the quartic equation, namely; $\Delta \equiv 256(I^3 - 27J^2)$ where

Theory of equations, Page 144.

4. $I = a_0a_4 - 4a_1a_3 + 3a_2^2$

$J = a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3$

Substituting we get

$I = s + 3q^2$

$J = qs - r^2 - q^2$

Therefore as the discriminant of the equation

$x^4 + 6qx^2 + 4rx + s = 0$ we have

$= 256\left\{ (s + 3q^2)^3 - 27\ (qs - r^2 - q^3)^2 \right\}$

$= 256\left\{ 81q^4s - 54q^3r^2 - 18q^2s^2 + 54qr^2s + (s^3 - 27r^4) \right\}$

The quartic $x^4 + 6qx^2 + 4rx + s = 0$ reduces to a form with a squared factor when $\Delta = 0$. This suggests as for the cubic that the difference between the above and the form with a squared factor is a multiple of Δ.

To find the form with a squared factor we have that $x^4 + 6qx^2 + 4rx + s$ and $4x^3 + 12qx + 4r$ must have a common factor. This factor is obtained by the C.C.D. process as follows :

$x^3 + 3qx + r \ ) \ x^4 + 6qx^2 + 4rx + s \ (\ x$

$x^4 + 3qx^2 + rx$

$3qx^2 + 3rx + s$

$3qx^2 + 3rx + s \ ) \ x^3 + 3qx + r \ |$ mult. $3q$

$3qx^3 + 9q^2x + 3qr \ (\ 1$

$3qx^3 + 3rx^2 + sx$

$$-3rx^2 + 9q^2x - sx + 3qr \quad | \quad .q$$

$$-3qrx^2 + (9q^3 - qs)x + 3q^2r \quad (-r$$

$$\underline{-3qrx^2 - 3r^2x - rs}$$

$$(9q^3 - qs + 3r^2)x + r(3q^2 + s)$$

Hence $(9q^3 - qs + 3r^2)x + r(3q^2 + s)$ must be the repeated factor or $x = \frac{r(3q^2+s)}{9q^3 - qs + 3r^2}$

The quartic with equal roots therefore takes the form $(Ax + B)^2.(A^2x^2 - 2ABx + k) = 0$

where $A = 9q^3 - qs + 3r^2$

and $B = r(3q^2 + s)$

and k is an unknown to be determined by equating coefficients. We determine k so that the coefficient of x^2 in the difference below vanishes.

We have as the difference,

$A^4(x^4 + 6qx^2 + 4rx + s) - (Ax + B)^2.(A^2x^2 - 2ABx + k)$

The coefficient of x^2 vanishes in this difference if $A^4.6q = A^2k - 3A^2B^2$ i.e. if $k = 6qA^2 + 3B^2$

Substituting this value of k in the difference above, we have,

$$4A(rA^3 - 3qA^2B - B^3)x + (A^4s - 6qA^2B^2 - 3B^4)$$

We wish to show that this difference is a multiple of Δ.

First, considering the coefficient of x, we have:

4. $4A(rA^3 - 3qA^2B - B^3) \equiv 4Ar\{(9q^3 + 3r^2 - qs)^3 - r^2(3q^2+s)^3 - 3q(9q^3 + 3r^2 - qs)^2(3q^2 + s)\}$

$\equiv 4Ar\{(9q^3 + 3r^2 - qs)^2(9q^3 + 3r^2 - qs - 9q^3 - 3qs) - r^2(3q^2 + s)^3\}$

$\equiv 4Ar\{(9q^3 + 3r^2 - qs)^2(3r^2 - 4qs) - r^2(3q^2 + s)^3\}$

$\equiv 4Ar\{-324q^7s + 216q^6r^2 + 72q^5s^2 - 297q^4r^2s + q^3(162r^4 - 4s^3) + 18q^2r^2s^2 - 54qr^4s + r^2(27r^4 - s^3)\}$

Dividing this expression by Δ we get as the quotient $-\frac{4Ar}{250}(4q^3 + r^2) \equiv -\frac{Ar}{64}(4q^3 + r^2)$. Thus we have that the coefficient of x in the difference is a multiple of Δ.

Now considering the constant term in the difference, we have,

$A^4s - 6qA^2B^2 - 3B^4 \equiv (9q^3 + 3r^2 - qs)^4s - 6q(9q^3 + 3r^2 - qs)^2 \cdot r^2(3q^2 + s)^2 - 3r^4(3q^2 + s)^4$

$\equiv (9q^3 + 3r^2 - qs)^2\{s(9q^3 + 3r^2 - qs)^2 - 6qr^2(3q^2 + s)^2 - 3r^4(3q^2 + s)^4$

$\equiv 81^2sq^{12} - 81.54q^{11}r^2 - 81.36q^{10}s^2 + 81.84q^9r^2s + q^8(486s^3 - 3159r^4) - 52.54q^7r^2s^2 - q^6(36s^4 - 2430r^4s) + q^5(396r^2s^3 - 486r^6) + q^4(s^5 - 1242r^4s^2) - q^3(18r^2s^4 - 648r^8s) + 54q^2r^4s^3 - 162qr^6s^2 - 3r^4s(s^3 - 27r^4)$

Dividing this by Δ we get as the quotient

$\frac{1}{250}(81q^8 - 18q^6s + 18q^5r^2 + q^4s^2 - 18q^3r^2s - 3r^4s)$

Hence we have as the difference between

$x^4 + 6qx^2 + 4rx + s$ and the form with the squared factor,

$$\frac{}{256} - 4Ar(4q^3 + r^2)x + (81q^8 - 18q^6s + 18q^5r^2 + q^4s^2 - 18q^3r^2s - 3r^4s)$$

For the quartic $x^4 + Qx^2 + Rx + S = 0$ we will obtain corresponding results to those obtained for $x^4 + 6qx^2 + 4rx + s = 0$ by substituting Q for 6q, R for 4r and S for s.

$$I = S + \frac{Q^2}{12} \qquad J = \frac{QS}{6} - \frac{R^2}{16} - \frac{Q^3}{216}$$

$$= 16Q^4S - 4Q^3R^2 - 128Q^2S^2 + 144QR^2S + 256S^3 - 27R^4$$

$$A = \frac{Q^3}{6} + \frac{3R^2}{16} - \frac{QS}{6} \qquad B = \frac{R}{4}\left(\frac{Q^2}{12} + S\right)$$

We will have as the difference,

$$A^4(x^4 + Qx^2 + Rx + S) - (Ax + B)^2.(A^2x^2 - 2ABx + QA^2 + 3B^2)$$

$$-\frac{\Delta}{256}\left\{-AR\left(\frac{Q^3}{54} + \frac{R^2}{16}\right)x + \left(\frac{Q^8}{2^7.6^4} - \frac{Q^6S}{2.6^4} + \frac{Q^5R^2}{6^3.32} + \frac{Q^4S^2}{6^4} - \frac{Q^3R^2S}{192} - \frac{3R^4S}{256}\right)\right.$$

For practical purposes the quotient $81q^8 - 18q^6s + 18q^5r^2 + q^4s^2 - 18q^3r^2s - 3r^4s$ may easily be reduced to the form $q^2A^2 - 3rB(r^2 + 4q^3)$ or for the quartic $x^4 + Qx^2 + Rx + S = 0$ this will take the form $\frac{Q^2A^2}{36} - \frac{3R}{4}.B\left(\frac{R^2}{16} + \frac{Q^3}{54}\right)$ where A and B here have the values as given for this quartic equation.

As a check upon these results suppose we take the equation whose roots are 4, 1, -2, -3, that is, the equation $(x - 4)(x - 1)(x + 2)(x + 3) = 0$.

or $x^4 - 15x^2 - 10x + 24 = 0$.

The discriminant of this equation as determined from the product of the squared differences of the roots is 2,286,144 .

As determined from $256(I^3 - 27J^2)$ it is

$$256\left\{\left(\frac{171}{4}\right)^3 - 27\left(\frac{405}{8}\right)^2\right\} = 2,286,144 \text{ as above.}$$

$$A = \frac{(-15)}{24} + \frac{3(-10)^2}{16} - \frac{(-15)24}{6} = -\frac{495}{8}$$

$$B = \frac{-10}{4}\left(\frac{225}{12} + 24\right) = -\frac{855}{8}$$

The coefficient of x in the difference is,

$$AR\left\{A^2\left(\frac{3R^2}{16} - \frac{4QS}{6}\right) - B^2\left(\frac{Q^2}{12} + S\right)\right\}$$

$$= AR\left\{\left(\frac{-495}{8}\right)^2\left(\frac{300}{16} + \frac{4.15.24}{6}\right) - \left(\frac{855}{8}\right)^2\left(\frac{225}{12} + 24\right)\right.$$

$$= AR\left\{\left(\frac{495}{8}\right)^2\left(\frac{1035}{4}\right) - \frac{855^2.171}{256}\right\}$$

$$= -10A\left(\frac{128,595,600}{256}\right)$$

Dividing this by the discriminant we get,

$$-\frac{10A}{256}.56\tfrac{1}{4}$$

Our quotient as obtained above was,

$$-\frac{AR}{256}\left(\frac{Q^3}{54} + \frac{R^2}{16}\right) = \frac{10A}{256}\left(-\frac{15^3}{54} + \frac{100}{16}\right) = \frac{10A}{256}.\left(-\frac{125}{2} + \frac{25}{4}\right)$$

$$= -\frac{10A}{256}.56\tfrac{1}{4} \text{ as above.}$$

The constant coefficient in the difference which is given by $A^4S - QA^2B^2 - 3B^4$

$$= \left(\frac{495}{8}\right)^4.24 + 15\left(\frac{495}{8}\right)^2\left(\frac{171}{4}\right)^2\frac{100}{16} - \frac{3.10^4}{4}\left(\frac{171}{4}\right)^4$$

$$= \frac{1,440,894,015,000}{4^6} \quad \frac{268,679,100,0875}{4^6} \quad \frac{1,603,192,651,875}{4^6}$$

$$= \frac{2,524,372,500}{4^6} = \frac{631,123,093,125}{4^5}$$

4. Dividing by the discriminant we get as the quotient $\frac{1}{256}\cdot(69,616.\frac{29}{256})$

From the formula we get as the quotient,

$$\frac{1}{256}\left(\frac{Q^2A^2}{36} - \frac{3R^3B}{64} - \frac{Q^3RB}{72}\right)$$

$$= \left(\frac{405^2 \cdot 225}{64 \cdot 36} - \frac{9 \cdot 10^3 \cdot 855}{64 \cdot 8} + \frac{15^3 \cdot 10 \cdot 855}{72 \cdot 8}\right)\frac{1}{256}$$

$$= \left(\frac{6,125,025}{256} - \frac{1,282,500}{256} + \frac{12,825,000}{256}\right)\frac{1}{256}$$

$$= \frac{17,668,125}{256} \cdot \frac{1}{256} = \frac{1}{256}(69,616.\frac{29}{256}) \quad \text{as above.}$$

5. The Quartic; General Case; Second Solution

In the quartic $x^4 + Qx^2 + Rx + S = 0$, Q is of weight 2, R of weight 3 and S of weight 4. In the difference between $x^4 + Qx^2 + Rx + S$ and the form with the squared factor the coefficient of x, which is $4A(\frac{R}{8}A^3 - \frac{Q}{2}A^2B - B^3)$, is of weight 37. The discriminant is of weight 12 and as it is a factor of the coefficient, we have a weight of 15 left, and since R alone has an odd weight, it must occur as a factor also.

Now if $A = 0$ the coefficient of x vanishes, hence also A must be a factor of weight 6. This will leave a balance of weight 6 to account for. Therefore the coefficient of x must be equal to $A.R(aQ^3 + bR^2 + cQS)$ since, Q^3, R^2, QS are the only possible combinations of Q, R, and S to form a weight of 6.

Here we have three unknowns, a, b, c, which can

5. be determined by taking three different equations.

By using the equation $x^4 + x = 0$ we get terms involving R only. We get terms involving only Q & R by using the equation $x^4 + x^2 + x = 0$ and terms involving Q, R and S by using $x^4 + x^2 + x + 1 = 0$.

Table of constants for each Equation

Equation	Δ $= 256(I^3 - 27J^2)$	A $= \frac{Q^3}{24} + \frac{3R^2}{16} - \frac{QS}{6}$	B $= \frac{R}{4}\left(\frac{Q^2}{12} + S\right)$	Coeff. of x in Diff. $4A\left(\frac{A^3R}{4} - \frac{QA^2B}{2} - B^3\right)$	Quotient $= \frac{AR}{256}\left(\frac{Q^3}{54} + \frac{R^2}{16}\right)$
(1) $x^4 + x = 0$	-27	$3/16$	0	$\frac{3^4}{16^4}$	$-\frac{3}{16^4}$
(2) $x^4 + x^2 + x = 0$	-31	$11/48$	$\frac{1}{48}$	$\frac{11 \cdot 31 \cdot 35}{48^4}$	$\frac{-11 \cdot 35}{16^2 \cdot 48 \cdot 432}$
(3) $x^4 + x^2 + x + 1 = 0$	257	$1/16$	$+\frac{13}{48}$	$-\frac{8995}{48^3 \cdot 16}$	$-\frac{35}{16^3 \cdot 432}$

From equation (1) we have,

$$-27 \cdot 1 \cdot \frac{3}{16}(a \cdot 0 + b \cdot 1 + c \cdot 0) = \frac{3^4}{16^4}$$

$$\text{or } b = -\frac{3^4}{16^4} \cdot \frac{16}{3^4} = \frac{-1}{16^3}$$

From equation (2) we have

$$-31 \cdot 1 \cdot \frac{11}{48}(a \cdot 1 + b \cdot 1 + c \cdot 0) = \frac{11 \cdot 31 \cdot 35}{48^4}$$

$$a + b = -\frac{35}{48^3}$$

$$a = -\frac{35}{48^3} + \frac{1}{16^3} = -\frac{35}{48^3} + \frac{27}{48^3} = -\frac{8}{48^3} = \frac{-1}{24^3} = \frac{-1}{4^4 \cdot 54}$$

From equation (3) we have

$$a + b + c = -\frac{35}{48^3}$$

$$c = -\frac{35}{48^3} + \frac{35}{48^3} = 0$$

That is, we have no QS term in the quotient. Therefore the coefficient of x in the difference is

$$-\frac{AR}{256}\left(\frac{Q^3}{54} + \frac{R^2}{16}\right)$$

We might have concluded that the coefficient

of QS was zero to begin with because S occurs only to the third power in the difference and also to the third power in Δ .Thus by removing Δ we would remove S.

The constant coefficient in the difference, $A^4S - QA^2B^2 - 3B^4$ is of weight 28,and as Δ divides this we will have a balance of weight left of 16 on dividing by Δ which is of weight 12. Hence the difference must be of the form, $\Delta(aQ^8 + bQ^6S + cQ^5R^2 + dQ^4S^2 + eQ^3R^2S + fQ^2R^4 + gQR^2S^2 + hR^4S + iS^4 + jQ^2S^3)$,as these combinations of Q,R and S are the only ones that make up a weight of 16.But by inspection we see that the highest power of S in the constant coefficient is 5,and as S^3appears in the discriminant powers of S higher than the second cannot appear in our quotient when we divide by the discriminant, hence we can eliminate the terms S^4 and Q^2S^3 in the above form of the quotient. This leaves only eight unknowns to solve for,and these can be solved by substituting from eight given equations.

The equations $x^4 + x^2 + x = 0$

$x^4 + x^2 + 2x = 0$

$x^4 - x^2 + x = 0$ will give the coefficients of R^4Q^2, Q^5R^2, Q^8.

The equation $x^4 + x + 1 = 0$ will give the coefficients of x^4, the equations, $x^4 + x^3 + 1 = 0$

$$x^4 + x^2 - 1 = 0$$

$$x^4 + x^2 + 1 = 0$$

$$x^4 + x - 1 = 0$$

will give the remaining coefficients.

Table of Constants for each Equation

Equation	Δ $256(I^3-27J^2)$	A $\frac{Q^3}{24}+\frac{3R^2}{16}-\frac{QS}{6}$	$\frac{R}{4}\left(\frac{Q^2}{12}+S\right)$	Constant coefficient in difference. $\frac{Q^2A^2}{36}-\frac{3R^3B}{64}-\frac{Q^3RB}{72}$
(1) $x^4+x^2+x=0$	-31	$\frac{11}{48}$	$\frac{1}{48}$	$-\frac{31\cdot 4}{48^4}$
(2) $x^4+x^2+2x=0$	-448	$\frac{19}{24}$	$\frac{1}{24}$	$-\frac{91}{144\times 576}$
(3) $x^4-x^2+x=0$	-23	$\frac{7}{48}$	$\frac{1}{48}$	$\frac{46}{48^4}$
(4) $x^4+x+1=0$	229	$\frac{3}{16}$	$\frac{1}{4}$	$-\frac{687}{16^4}$
(5) $x^4+x^2+1=0$	144	$-\frac{1}{8}$	0	$\frac{1}{8^4}$
(6) $x^4+x^2-1=0$	-400	$\frac{5}{24}$	0	$\frac{-5^4}{24^4}$
(7) $x^4+x^2+x+1=0$	257	$\frac{1}{16}$	$\frac{13}{48}$	$-\frac{87123}{48^4}$
(8) $x^4+x^2+x-1=0$	-575	$\frac{19}{48}$	$-\frac{11}{48}$	$-\frac{379\times 575}{48^4}$

Therefore substituting in $\Delta(aQ^8 + bQ^6S + cQ^5R^2 + dQ^4S^2 + eQ^2R^2S + fQ^2R^2 + gQR^2S^2 + hR^4S)$ for Q,R,S and from these eight equations and equating to the corresponding constant coefficients in the difference we get eight equations involving a,b,c,d,e,f,g,h.

We have,

(1) $-31(a + c + f) = -\frac{31.4}{48^4}$

(2) $a + 4c + 16f = \frac{13}{48^4}$

(3) $a - c + f = -\frac{46}{48^4 . 23}$

(4) $h = -\frac{3}{16^4}$

(5) $a + b + d = \frac{9}{16 . 24^4}$

(6) $a - b + d = \frac{25}{24^4 . 16}$

(7) $a + b + c + d + e + f + g + h = -\frac{339}{48^4}$

(8) $a - b + c + d - e + f + g - h = \frac{379}{48^4}$

Solving we get, $a = \frac{1}{48^4}$, $b = -\frac{1}{2 . 24^4}$ $c = \frac{1}{27 . 16^4}$ $d = \frac{1}{3 . 48^3}$, $e = -\frac{108}{48^4}$, $f = 0$, $g = 0$, $h = -\frac{3}{16^4}$,.

We have therefore the constant coefficient in the difference in the form, $\frac{\Delta}{256}(\frac{Q^8}{12^4} + \frac{Q^6S}{2 . 6^4} + \frac{Q^5R^2}{12^3 . 4} + \frac{Q^4S^2}{6^4} - \frac{Q^3R^2S}{4 . 48} - \frac{3R^4S}{16^2})$ which is identical to the result obtained in the preceding section.

Sec.6. In the case of the quartic $x^4 + 6qx^2 + 4rx + s = 0$, we had as the difference between it and the form with

6. equal roots the expression, $4A(rA^3 - 3qA^2B - B^3)x + (A^4s - 6qA^2B^2 - 3B^4)$ where $A \equiv 9q^3 + 3r^2 - qs$ and $B \equiv r(3q^2 + s)$

We also had the discriminant Δ which was equal to $256(I^3 - 27J^2)$ where $J \equiv qs - r^2 - q^3$

$$I \equiv 3q^2 + s$$

We notice that there is a great similarity between the expressions for A and J; and for B and I and we find that A and B can be expressed quite simply in terms of I and J. We see at once that $B \equiv r.I$, and by adding and subtracting 3J from the expression for A we get $A \equiv 6q^3 + 2qs - 3J \equiv 2qI - 3J$.

By substituting these values for A and B in the expression for the above difference it is evident that our work in showing that the difference is a multiple of the discriminant will be considerably shortened.

Thus substituting in the coefficient of x for A and B, we have,

$$\begin{aligned}
&4(2qI - 3J)\left\{r(2qI - 3J)^3 - 3q(2qI - 3J)^2rI - r^3I^3\right\} \\
&\equiv 4r(2qI - 3J)\left\{(2qI - 3J)^2(2qI - 3J - 3qI) - r^2I^3\right\} \\
&\equiv 4r(2qI - 3J)\left\{(4q^2I^2 - 12qIJ + 9J^2)(-3J - qI) - r^2I^3\right\} \\
&\equiv 4r(2qI - 3J)\left\{-12q^2I^2J + 36qIJ^2 - 27J^3 - 4q^3I^3 + 12q^2I^2J \right. \\
&\qquad\qquad \left. - 9qJ^2I - r^2I^3\right\} \\
&\equiv 4r(2qI - 3J)\left\{-I^3(4q^3 + r^2) + 27J^2(qI - J)\right\} \\
&\equiv -4r(2qI - 3J)\left\{I^3(4q^3 + r^2) - 27J^2(4q^3 + r^2)\right\}
\end{aligned}$$

6. $= -4r(2qI - 3J)(4q^3 + r^2)(I^3 - 27J^2)$

Hence on dividing by $\Delta = 256(I^3 - 27J^2)$

we have as the quotient

$$-\frac{4r}{256}(2qI - 3J)(4q^3 + r^2) = -\frac{4rA}{256}(4q^3 + r^2) \quad \text{as before.}$$

Now taking the constant term in the difference we have on substituting for A and B

$(2qI - 3J)^4 s - 6q(2qI - 3J)^2 r^2 I^2 - 3r^4 I^4$ and putting $s = I - 3q^2$, then on multiplying out and collecting the terms with respect to powers of I we have:

$$16q^4I^5 - 3I^4(32q^3J + 16q^6 + 8q^3r^2 + r^4) + (72I^3Jq^2) \times$$
$$(3J + 4q^3 + r^2) - 54qI^2J^2(4J + 12q^3 + r^3) + 81IJ^3(J + 8q^3)$$
$$- 243J^4q^2$$
$$= 16q^4I^5 - 3I^4\left(32q^3J + (4q^3 + r^2)^2\right) + 72I^3Jq^2(3J + qI - J)$$
$$- 54qI^2J^2(4J + 8q^3 + qI - J) + 81IJ^3(J + 8q^3) - 243J^4q^2$$
$$= 16q^4I^5 - 3I^4\left(32q^3J + (qI - J)^2\right) + 72I^3Jq^2(qI + 2J)$$
$$- 54qI^2J^2(8q^3 + 3J + qI) + 81IJ^3(J + 8q^3) - 243J^4q^2$$
$$= -3I^6q^2 + 2I^5q(8q^3 + 3J) - 3I^4J(8q^3 + J) + 90I^3J^2q^2$$
$$-54I^2qJ^2(8q^3 + 3J) + 81IJ^3(8q^3 + J) - 243J^4q^2$$

Dividing this expression by the discriminant $256(I^3 - 27J^2)$ we get as the quotient

$$\frac{1}{256}\left\{-3q^2I^3 + 2I^2q(8q^3 + 3J) - 3IJ(8q^3 + J) + 9J^2q^2\right\}$$
$$= \frac{1}{256}\left\{-3q^2I^3 + 16I^2q^4 + 6I^2qJ - 24q^3IJ - 3IJ^2 + 9J^2q^2\right\}$$
$$= \frac{1}{256}\left\{q^2(9J^2 - 24qIJ + 16I^2q^2) - 3I(q^2I^2 - 2IqJ + J^2)\right\}$$
$$= \frac{1}{256}\left\{q^2(3J - 4qI)^2 - 3I(qI - J)^2\right\}$$

6. $= \frac{1}{256}\left\{ q^2(2qI - 3J)^2 - 3I(4q^3 + r^2)^2 + 12q^3I(qI - J)\right\}$

$= \frac{1}{256}\left\{ q^2A^2 - 3I(4q^3 + r^2)^2 + 12q^3I(4q^3 + r^2)\right\}$

$= \frac{1}{256}\left\{ q^2A^2 - 3I(4q^3 + r^2)(4q^3 + r^2 - 4q^3)\right\}$

$= \frac{1}{256}\left\{ q^2A^2 - 3Ir^2(4q^3 + r^2)\right\}$

$= \frac{1}{256}\left\{ q^2A^2 - 3rB(4q^3 + r^2)\right\}$

as before

7. The Quartic; Special Case.

If $q = 0$ in the general quartic, we have a quartic equation of the form $x^4 + rx + s = 0$. The condition for equal roots is that $x^4 + rx + s$ and $4x^3 + r$ have a common root. The common root is obtained as before by the G.C.D. process as follows:

$$4x^3 + r \;)\; x^4 + rx + s \;\Big|\; \text{Mult. by } 4$$

$$4x^4 + 4rx + 4s \;(\; x$$

$$\underline{4x^4 + rx}$$

$$3rx + 4s \;)\; 4x^3 + r$$

Hence $3rx + 4s$ must be the common factor, or $x + \frac{4s}{3r}$. The biquartic with equal roots, therefore takes the form $(x + \frac{4s}{3r})^2(x^2 - \frac{8s}{3r}x + \frac{48s^2}{9r^2}) = 0$

The latter part is determined from the fact that the coefficients of x^3 and x^2 are zero.

We wish to show that

$$(x^4 + rx + s) - (x + \tfrac{4s}{3r})^2(x^2 - \tfrac{8s}{3r}x + \tfrac{48s^2}{9r^2}) \equiv M(\Delta)$$

By actual multiplication we have on multiplying by $81r^4$ to clear of fractions

$$81r^4(x^4 + rx + s) - (81x^4r^4 + 768s^3rx + 768s^4)$$

7. $\equiv 3xr(27r^4 - 256s^3) + 3s(27r^4 - 256s^3)$

$\equiv (27r^4 - 256s^3)(3xr + 3s) \equiv 3 \cdot \Delta(rx \quad s)$

Since the discriminant of the quartic

$\equiv 27r^4 - 256s^3$. We may obtain this discriminant by carrying out the G.C.D. process above one step farther .

8. The Quartic; Special Case .

In the quartic if $q \equiv 0$ and $9q^3 + 3r^2 - qs \equiv 0$, that is, $A \equiv 0$, then the repeated factor cannot be linear, as the coefficient of x will vanish, therefore it must be quadratic, that is we will have two pairs of repeated roots. Hence since the sum of the two is zero, they are of the form $\alpha_0\alpha_1) - \alpha_0 r\alpha_1$

Hence $r \equiv 0$ and we will therefore have the equation $x^4 + qx^2 + s \equiv 0$ to deal with. The condition that $x^4 + qx^2 + s \equiv 0$ has equal roots, is that, $x^4 + qx^2 + s$ and $4x^3 + 2qx$ have a root in common. This root or factor is obtained by the G.C.D. process thus,

$4x^3 + 2qx \;)\; x^4 + qx^2 + s \;\big|$ Mult. by 4

$4x^4 + 4qx^2 + 4s \quad (x$

$\underline{4x^4 + 2qx^2}$

$2qx^2 + 4s$

Hence $qx^2 + 2s$ must be the common factor, and the biquadratic with equal roots takes the form $(qx^2 + 2s + k)^2$, where k is so chosen that the coefficient

of x^2 will disappear in the difference,

$q^2(x^4+qx^2+s) - (qx^2+2s+k)^2$

Therefore $q^3 = 4qs+2qk$ and $k = \frac{q^2-4s}{2}$

Substituting for k we have,

$q^2(x^4+qx^2+s) - (qx^2+2s+\frac{q^2-4s}{2})^2$

$= 4q^2(x^4+qx^2+s) - (2qx^2+q^2)^2$

$= q^2(4s-q^2) = \Delta.\ \frac{1^2}{16s(4s-q^2)}$

Since when $r = 0$ we have as the value of

$16s(q^4 - 8q^2s + 16s^2) = 16s(4s-q^2)^2$

The difference obtained in the case of the quartic $x^4+qx^2+s = 0$ is not the same as that obtained from the quartic $x^4+qx^2+rx+s = 0$ by putting $r = 0$, in the difference, but this is due to the fact that when we clear of fractions in the difference we multiply by a different quantity in each case. Also for the quartic $x^4+rx+s = 0$ the difference obtained directly from this equation is a multiple of that obtained from the general case by putting $q = 0$.

Table of all Cases.

case	Equation	Discriminant Δ	Difference between equation and form with squared factor	Difference divided by Δ
(1)	$x^3+qx+s=0$	$4q^3+27r^2$	$q(4q^3+27r^2)x+r(4q^3+27r^2)$	$qx+r$
(2)	$x^3+r=0$	$27r^2$	r	$\frac{1}{27r}$
(3)	$x^4+Qx^2+Rx+S=0$	$256(I^3-27J^2)$	$4A\left(\frac{RA^3}{4}-\frac{QA^2B}{2}-B^3\right)+A^4S-QA^2B-3B^4$	$-AR\left(\frac{Q^3}{54}+\frac{R^2}{16}\right)+\frac{QA^2}{36}-\frac{3RB}{4}\left(\frac{Q^3}{54}+\frac{R^2}{16}\right)$
		$I=S+\frac{Q^2}{12}$	$A=\frac{Q^3}{24}+\frac{3R^2}{16}-\frac{QS}{6}$	
		$J=-\frac{QS}{6}-\frac{R^2}{16}-\frac{Q^3}{216}$	$B=\frac{R}{4}\left(\frac{Q^2}{12}+S\right)$	
(4)	$x^4+rx+s=0$	$27r^4-256s^3$	$3r(27r^4-256s^3)x+3s(27r^4-256s^3)$	$3(r^4+s)$
(5)	$x^4+qx^2+s=0$	$16s(4s-q^2)^2$	$q^2(4s-q^2)$	$\frac{q^2}{16s(4s-q^2)}$

/

CHAPTER 8.

Sec. 1.

In the field $x^4 + qx^2 + rx + s = 0$, let $a_0 + a_1x + a_2x^2 + a_3x^3 = y$ an integer or let,

$(a_0 - y) + a_1x + a_2x^2 + a_3x^3 = 0$ - - - - - - - - - (1)

Multiplying equation (1) by x and substituting $-qx^2 - rx - s$ for x^4, we have,

$-a_3s + (a_0 - y - a_3r)x + (a_1 - a_3q)x^2 + a_2x^3 = 0$ - - (2).

Multiplying equation (2) by x and substituting for x^4 again as above, we have, $-a_2s + (a_2r + a_3s)x + (a_0 - y - a_2q - a_3r)x^2 + (a_1 - a_3q)x^3 = 0$ - - - - -(3)

Multiplying equation (3) by x and substituting for x^4 we have, $(a_3qs - a_1s) + (a_3qr - a_2s - a_1r)x + (a_3q^2 - a_3s - a_2r - a_1q)x^2 + (a_0 - y - a_2q - a_3r)x^3 = 0$ (4

We now have four equations from which $1, x, x^2, x^3$, may be eliminated. The result is an equation of the fourth degree in y, whose coefficients must be rational integers.## vis.

$$\begin{vmatrix} a_0 - y & a_1 & a_2 & a_3 \\ -a_3s & (a_0 - y - a_3r) & (a_1 - a_3q) & a_2 \\ -a_2s & -(a_2r + a_3s) & (a_0 - y - a_2q - a_3r) & a_1 - a_3q \\ a_3qs - a_1s, & (a_3qr - a_2s - a_1r), & (a_3q^2 - a_3s - a_2r - a_1q), & (a_0 - y - a_2q - a_3r \end{vmatrix} = 0.$$

H.R. Wilson, in "Integers and Basis of a Number Field". Theorem 4(a) Page 116.

Trans. American Math. Society, Jan. 1927.

The result of expanding this determinant, as shown in the citation, is an equation of the form

$$y^4 - \frac{1}{\lfloor 3}\frac{\delta^3 H}{\delta a_0^3} y^3 + \frac{1}{\lfloor 2}\frac{\delta^2 H}{\delta a_0^2} y^2 - \frac{\delta H}{\delta a_0} y + H = 0.$$

The object of this section is to evaluate H, and from it determine the remaining coefficients.

H will be found by putting $y = 0$ in the above determinant and expanding. Expanding in terms of the fourth row and its minors we have,

$$H = -(a_3 qs - a_1 s)\begin{vmatrix} a_1 & a_2 & a_3 \\ (a_0 - a_3 r) & a_1 - a_3 q & a_2 \\ -(a_2 r + a_3 s), & (a_0 - a_2 q - a_3 r), & (a_1 - a_3 q) \end{vmatrix}$$

$$+(a_3 qr - a_2 s - a_1 r)\begin{vmatrix} a_0 & a_2 & a_3 \\ -a_3 s & a_1 - a_3 q & a_2 \\ -a_2 s & a_0 - a_2 q - a_3 r & a_1 - a_3 q \end{vmatrix}$$

$$+\left(q(a_1 - a_3 q) + (a_2 r + a_3 s)\right)\begin{vmatrix} a_0 & a_1 & a_3 \\ -a_3 s & a_0 - a_3 r & a_2 \\ -a_2 s & -(a_2 r + a_3 s) & a_1 - a_3 q \end{vmatrix}$$

$$+(a_0 - a_2 q - a_3 r)\begin{vmatrix} a_0 & a_1 & a_2 \\ -a_3 s & a_0 - a_3 r & (a_1 - a_3 q) \\ -a_2 s & -(a_2 r + a_3 s) & (a_0 - a_2 q - a_3 r) \end{vmatrix}$$

Expanding these four third order determinants collecting terms and arranging them in powers of a_0, we have,

$$
\begin{aligned}
E = {} & a_0^4 - a_0^3(2a_2q + 3a_3r) \\
& + a_0^2\Big\{q(a_1 - a_3q)^2 + 3(a_2r + a_3s)(a_1 - a_3q) \\
& + (a_2q + a_3r)(a_2q + 3a_3r) + s(a_1a_3 + 2a_2^2)\Big\} \\
& - a_0\Big\{r(a_1 - a_3q)^3 + (a_1 - a_3q)^2(2a_2s + a_3qr) \\
& + (a_1 - a_3q)\Big\{a_2r(a_2q + 3a_3r) + s(2a_1a_2 + 3a_3^2r)\Big\} \\
& + a_3r(a_3r + a_2q)^2 + 2a_1a_3s(a_2q + a_3r) \\
& - a_2(a_2r + 2a_3s)^2 + a_2^2s(3a_3r + 2a_2q)\Big\} \\
& + \Big\{a_1s(a_1 - a_3q)^3 + a_3s(a_1 - a_3q)^2(a_1q + a_3s) \\
& + (a_1 - a_3q)\Big\{a_1a_2s(a_2q + 3a_3r) + a_3s^2(a_3^2q + a_1a_3 - 3a_2^2)\Big\} \\
& + a_1a_3s(a_2q + a_3r)^2 - 2a_2a_3^2s^2(a_2q + a_3r) \\
& - s(a_2r + a_3s)(a_1a_2^2 - a_3^3s) + a_2^4s^2\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta E}{\delta a_0} = {} & 4a_0^3 - 3a_0^2(2a_2q + 3a_3r) \\
& + 2a_0\Big\{q(a_1 - a_3q)^2 + 3(a_2r + a_3s)(a_1 - a_3q) \\
& + (a_2q + a_3r)(a_2q + 3a_3r) + s(a_1a_3 + 2a_2^2)\Big\} \\
& - \Big\{r(a_1 - a_3q)^3 + (a_1 - a_3q)^2(2a_2s + a_3qr) \\
& + (a_1 - a_3q)\Big\{a_2r(a_2q + 3a_3r) + s(2a_1a_2 + 3a_3^2r)\Big\} \\
& + a_3r(a_3r + a_2q)^2 + 2a_1a_3s(a_2q + a_3r) \\
& - a_2(a_2r + 2a_3s)^2 + a_2^2s(3a_3r + 2a_2q)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\lfloor 2}\frac{\delta^2 E}{\delta a_0^2} = {} & \frac{1}{\lfloor 2}\Big[12a_0^2 - 6a_0(2a_2q + 3a_3r) \\
& + 2\Big\{q(a_1 - a_3q)^2 + 3(a_2r + a_3s)(a_1 - a_3q) \\
& + (a_2q + a_3r)(a_2q + 3a_3r) + s(a_1a_3 + 2a_2^2)\Big\}\Big]
\end{aligned}
$$

$$
\frac{1}{\lfloor 2}\frac{\delta^3 E}{\delta a_0^3} = \frac{1}{\lfloor 3}\Big\{24a_0 - 6(2a_2q + 3a_3r)\Big\}
$$

Sec.2.

As a check upon our results, suppose we consider the field $x^4 - x^2 + x + 1 = 0$ and $y = x^3 - x^2 + x + 2$... (2) an integer in the field. Multiplying equation (a) by x, x^2, and x^3 and substituting from $x^4 - x^2 + x + 1 = 0$, for x^4, x^5, and x^6, we will have four equations. viz.

$$1 - y + x - x^2 + x^3 = 0$$

$$-1 + (1 - y)x + 2x^2 - x^3 = 0$$

$$1 - \qquad -yx^2 + 2x^3 = 0$$

$$-2 - x + 2x^2 - yx^3 = 0$$

from which $1, x, x^2, x^3$, may be eliminated by determinants. We have:

$$\begin{vmatrix} 2-y & 1 & -1 & 1 \\ -1 & 1-y & 2 & -1 \\ 1 & 0 & -y & 2 \\ -2 & -1 & 2 & -y \end{vmatrix} = 0.$$

or expanding in terms of the fourth row and its minors, we have,

$$2\begin{vmatrix} 1 & -1 & 1 \\ 1-y & 2 & -1 \\ 0 & -y & 2 \end{vmatrix} \quad -1\begin{vmatrix} 2-y & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -y & 2 \end{vmatrix}$$

$$-2\begin{vmatrix} 2-y & 1 & 1 \\ -1 & 1-y & -1 \\ 1 & 0 & 2 \end{vmatrix} \quad -y\begin{vmatrix} 2-y & 1 & -1 \\ -1 & 1-y & 2 \\ 1 & 0 & -y \end{vmatrix} = 0$$

Expanding these four third order determinants, we have,

as the result, $y^4 - 3y^3 + y^2 + 4y - 1 = 0$.

If we now substitute in our expression for E as obtained above, the values of q, r, s, a_0, a_1, a_2, a_3, as derived from the equations $x^4 - x^2 + x + 1 = 0$ and $y = x^3 - x^2 + x + 2$, we have,

$$E = -1; \quad \frac{\delta E}{\delta a_0} = -4; \quad \frac{1}{\lfloor 2} \frac{\delta^2 E}{\delta a_0^2} = 1; \quad \frac{1}{\lfloor 3} \frac{\delta^3 E}{\delta a_0^3} = 0.$$

These coefficients give us the fourth degree equation in y, $y^4 - 3y^3 + y^2 + 4y - 1 = 0$, which is identical to that obtained above by eliminating directly, thus checking our above results.

Sec. 3.

It is also shown in the paper already cited, that

$$\frac{1}{\lfloor 2} \frac{\delta^3 E}{\delta a_0^2 \delta a_1} + \frac{1}{\lfloor 2} \frac{\delta^3 E}{\delta a_0^2 \delta a_2}; \quad \frac{1}{\lfloor 2} \frac{\delta^3 E}{\delta a_0^2 \delta a_3}; \text{ are}$$

rational integers.

Obtaining the value of these three expressions from E, we have,

$$\frac{1}{\lfloor 2} \frac{\delta^3 E}{\delta a_0^2 \delta a_1} = 2q(a_1 - a_3 q) + 3a_2 r + 4a_3 s$$

$$\frac{1}{\lfloor 2} \frac{\delta^3 E}{\delta a_0^2 \delta a_2} = -6a_0 q + 3r(a_1 - a_3 q) + 4a_2 s + 2a_2 q + 4a_3 qr.$$

$$\frac{1}{\lfloor 2} \frac{\delta^3 E}{\delta a_0^2 \delta a_3} = -9a_0 r - 2q^2(a_1 - a_3 q) + 4a_1 s + a_2 qr + 6a_3 r^2.$$

Sec. 4.

If in the field $x^4 + s \neq 0$, where s is not divisble by any factor to the fourth power, that is, the equation is in the normal form, $a_0 + a_1x + a_2x^2 + a_3x^3 = y$ is an integer, we can determine the seven rational integers for this field from those determined above by putting q and r equal to zero. We have:

$$(1)\quad E = a_0^4 + 2a_0^2s(2a_1a_3 + a_2^2) - 4a_0a_2s(a_1^2 - a_3^2s) + a_1^4s + a_2^4s^2 + a_3^4s^3 + 2a_1^2a_3^2s^2 - 4a_1a_2^2a_3s^2$$

$$(2)\quad \frac{\delta E}{\delta a_0} = 4a_0^3 + 4a_0^2s(2a_1a_3 + a_2^2) - 4a_2s(a_1^2 - a_3^2s)$$

$$(3)\quad \frac{1}{\lfloor 2}\frac{\delta^2 E}{\delta a_0^2} = 6a_0^2 + 2s(2a_1a_3 + a_2^2)$$

$$(4)\quad \frac{1}{\lfloor 3}\frac{\delta^3 E}{\delta a_0^3} = 4a_0.$$

$$(5)\quad \frac{1}{\lfloor 2}\frac{\delta^3 E}{\delta a_0^2 \delta a_1} = 4a_3s$$

$$(6)\quad \frac{1}{\lfloor 2}\frac{\delta^3 E}{\delta a_0^2 \delta a_2} = 4a_2s$$

$$(7)\quad \frac{1}{\lfloor 2}\frac{\delta^3 E}{\delta a_0 \delta a_3^2} = 4a_1s.$$

It is proved in general that a field contains an integer of the form $\omega_1 = 1$. We wish to determine firstly, the necessary and sufficient conditions that the field $x^4 + s = 0$ contains an integer of the form $\omega_2 = a_0 + a_1x$, where $a_1 \neq$ integer, and secondly, the conditions for an integer of the form $\omega_3 = a_0 + a_1x + a_2x^2$. It is also proved in general that we may take the highest coefficient in the form $1/d$ where d^2/Δ where Δ is the discriminant

of the qquation generating the field and is in this particular case $256s^3$.

To determine the conditions for an integer of the form $\omega_2 = a_0 + a_1 x$, writing $a_1 = \frac{1}{d}$, we have:

(1) $d^2/256s^3$.

(2) $E = a_0^4 + \frac{s}{d^4}$ = a rational integer.

(3) $\frac{\delta E}{\delta a_0} = 4a_0^3$ = ...

(4) $\frac{1}{\lfloor 2} \frac{\delta^2 E}{\delta a_0^2} = 6a_0^2$ = ...

(5) $\frac{1}{\lfloor 3} \frac{\delta^3 E}{\delta a_0^3} = 4a_0$ = ...

(6) $\frac{1}{\lfloor 2} \frac{\delta^3 E}{\delta a_0^2 \delta a_3} = \frac{4s}{d}$ = ...

From (3) a_0 must be a rational integer, and reducing if necessary by multiples of $\omega_1 = 1$ we may take $a_0 \neq 0$. From (2) since s contains no factor in the form of a fourth power, we have that d = 1, and therefore $a_1 = 1$. Therefore, we have an integer of the form, $\omega_2 = 0 + 1.x = x$.

To determine the conditions for an integer of the form $\omega_3 = a_0 + a_1 x + a_2 x^2$, writing $a_2 = \frac{1}{d}$ and substituting ,we have:

(1) $d^2/256s^3$

(2) $E = a_0^4 + \frac{2a_0^2 s}{d^2} - \frac{4a_0 a_1^2 s}{d} + a_1^4 s + \frac{s^2}{d^4}$ a rat. int.

(3) $\frac{\delta E}{\delta a_0} = 4a_0^3 + \frac{4a_0 s}{d^2} - \frac{4a_1^2 s}{d}$ = a rational integer.

(4) $\frac{1}{\lfloor 2} \frac{\delta^2 E}{\delta a_0^2} = 6a_0^2 + \frac{2s}{d^2}$ = a rational integer.

(5) $\frac{1}{\lfloor 3} \frac{\delta^3 E}{\delta a_0^3} = 4a_0$ = a rational integer.

(6) $\frac{1}{12}\frac{\delta^3 H}{\delta a_2^2 \delta a_2} = \frac{4s}{d}$ = a rational integer.

(7) $\frac{1}{12}\frac{\delta^3 H}{\delta a_2^2 \delta a_3} = 4a_1 . s$ = a rational integer.

Of these (1) and (6) are <u>sufficient</u> conditions.

We may consider separately integers where the denominators involve powers of a single prime. The cases where the denominators are powers of 2 are evidently special. Enumerating the different cases we have:

Case (1) a prime p such that p/s but not p^2/s, p not 2

" (2) a prime p " " p^2/s " " p^3/s "

" (3) a prime p " " p^3/s " " p^4/s "

" (4) 2 when s is prime to 2

" (5) 2 when 2/s but not 4/s

" (6) 2 when 4/s but not 8/s

" (7) 2 when 8/s but not 16/s

In cases (1), (2), and (3), the only possible fractional values of a_0 are $\pm\frac{1}{4}$ and $\frac{1}{2}$, and since the denominators cannot contain 2, a_0 must have an integral value which we take as 0.

Case 1.

Case 1. is ruled out at once as by (4), $2a_2^2 s$ is an integer.

Case 2. p^2/s but not p^3/s $p \neq 2$.

From (3) - $\frac{4a_1^2 s}{p}$ = integer. Hence a_1 = integer, taken as 0.

This satisfies (2) for we now have $\frac{s^2}{p^4}$ = integer which is possible since p^2/s , satisfying the sufficient conditions. Hence we have an integer in this case of the form $\frac{x^2}{p}$ as part of ω_3.

Case 3. p^3/s but not p^4/s $\quad p \neq 2$

From (3) $-\frac{4a_1^2 \cdot s}{p}$ = integer.

Hence a_1 = integer taken as 0 pr $\pm\frac{1}{p}$. Taking $a_1 = \pm\frac{1}{p}$, from (2) we have $\frac{s}{p^4} + \frac{s^2}{p^4}$ = rational integer, which is impossible. Taking $a_1 = 0$, from (2) we have $\frac{s^2}{p^4}$ = rational integer, which is satisfied. Hence we have again an integer of the form $\frac{x^2}{p}$. From (4) there is no integer for which $a_2 = \frac{1}{p^2}$.

Case 4. Denominators powers of 2 when 2 is prime to s.

Here $\omega_3 = a_0 + a_1 x + a_2 x^2$ where denominators are powers of 2. From (5), (6),(7), the denominators can be only 2 or 4. In the latter case $2\omega_3$ will have denominators at most 2 and be an integer. From (4), we must have $a_0 = \frac{1}{2}$. From (3), $\frac{s+1}{2} - a_1^2 \cdot s$, is an integer so that $a_1 = 0$. From (2), we have $(s+1)^2/16$ an integer, satisfied if $s + 1 \equiv 0$, mod. 4.

Hence we have the integer $\frac{1}{2}x^2 + \frac{1}{2}$ if and only if $s + 1 \equiv 0$, mod. 4.

Cases 5, 6, 7. Denominators powers of 2 where $2/s$ but not $2^4/s$.

Here, if $a_2 = \frac{1}{2^n}$ where $n > 1$, then $2^{n-1}\omega_3$ will have for leading term $\frac{1}{2}x^2$. From (4), we have $a_0 = 0$ since $2a_2^2 s$, is a rational integer. If $a_1 = \frac{1}{2}$, from (2) $\frac{s}{2^4} + \frac{s^2}{2^4}$ is an integer; impossible unless $2^4/s$. Hence $a_1 = 0$. From (2) again $2^4/s^2$ or $4/s$. Hence we have an integer of the form, $\frac{x^2}{2}$ when $4/s$. This may arise from $2\omega_3$ where $\omega_3 = a_0 + a_1 x + \frac{x^2}{4}$, for, a_0, $a_1 = 0, \frac{1}{2}$. If $a_0 = 0$, from (4), $8/s$. In this case (3) is satisfied and (2) requires that $a_1^4 s + \frac{s^2}{64}$ be an integer. Since $8/s$ the latter term is integral and therefore $a_1 = 0$, as $2^4/s$ is excluded. We have therefore an integer of the form $\frac{x^2}{4}$ when $8/s$. This again may arise from $2\omega_3$, where $\omega_3 = a_0 + a_1 x + \frac{x^2}{8}$, a_0, $a_1 = 0, \frac{1}{2}$. This, however, violates (4). Returning to $\omega_3 = a_0 + a_1 x + \frac{x^2}{4}$ where $a_0 = \frac{1}{2}$, from (4), $2(\frac{1}{4})^2 s$ is of the form $\frac{d}{2}$, where d is odd; so that $4/s$ but not $8/s$. In this case, (3) is satisfied, but writing $s = 4s'$, where s' is odd, (2) gives $\frac{1}{16} + \frac{s'}{8} - 2a_1^2 s'$ as a rational integer. Since $a_1 \neq 0$ or $\frac{1}{2}$, this is impossible. Hence we have:

Case 5. no fractional integer.

Case 6. $= \frac{x^2}{2}$

Case 7. $\neq \frac{x^2}{4}$.