ISOMORPHISMS AND INVOLUTIONS ON THE BANACH ALGEBRAS ASSOCIATED WITH LOCALLY COMPACT GROUPS

BY

HAMID-REZA FARHADI

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H.R. Farhadi

TABLE OF CONTENTS

ABSTRACT

Let G be a locally compact group and $L^1(G)$ be its group algebra. The following results have been established in this thesis:

1. If every continuous automorphism of the algebra $L^1(G)^{**}$ leaves $L^1(G)$ invariant, then G is either compact or abelian.

2. If G is compact and H is a locally compact group and if θ is an algebra isomorphism from $L^1(G)^{**}$ onto $L^1(H)^{**}$, then H is compact and $\theta\big(L^1(G)\big) = L^1(H)$.

3. If G_1 and G_2 are locally compact groups, then any bipositve algebra isomorphism between $L^1(G_1)^{**}$ and $L^1(G_2)^{**}$ is an isometry. This combined with a known result implies that the order structure of $L^1(G)^{**}$ identifies the topological group G in the sense that any bipositive isomorphism between $L^1(G_1)^{**}$ and $L^1(G_2)^{**}$ results in a topological isomorphism between G_1 and G_2. This result also holds when $L^1(G_i)^{**}$ is replaced by a left introverted subspace of $C(G_i)$ containing $C_0(G_i)$, $i = 1, 2$, in which $C(G_i)$ denotes bounded continuous functions on G_i and $C_0(G_i)$ is the subspace of $C(G_i)$ consisting of the functions vanishing at infinity.

4. If G is amenable and there exists an involution on $L^1(G)^{**}$ extending the natural involution of $L^1(G)$, then G is finite. If $L^1(G)^{**}$ is replaced by $LUC(G)^*$ in this assertion, then finiteness is replaced by compactness, where $LUC(G)$ is the space of left uniformly continuous functions on G. However, for every locally compact group G, $WAP(G)^*$ has an involution that extends the involution of $L^1(G)$, in which $WAP(G)$ is the space of weakly almost periodic functions on G.

INTRODUCTION AND HISTORY

The study of the second dual Banach algebras was initiated by R. Arens in [4] and [5]. He showed that to every bounded bilinear map $f : X \times Y \to Z$, between normed spaces X, Y and Z, there are associated two bounded bilinear maps $F, G : X^{**} \times Y^{**} \to Z^{**}$ extending f, where X^{**} denotes the second dual of X, and so on. In fact, if $m \in X^{**}$ and $n \in Y^{**}$, then in the weak-star topology of Z^{**} we have $F(m,n) = \lim_{\alpha} \lim_{\beta} f(x_\alpha, y_\beta)$ and $G(m,n) = \lim_{\beta} \lim_{\alpha} f(x_\alpha, y_\beta)$, where (x_α) and (y_β) are nets in X and Y such that $x_\alpha \to m$ and $y_\beta \to n$ in the weak-star topologies. The concept of regularity of f was also introduced in [5]; f is said to be *regular* if F and G coincide.

When X=Y=Z and f is an algebra multiplication to which X is a Banach algebra, then the two extensions of f, corresponding to F and G respectively, define Banach algebra multiplications on X^{**}, known as the *first* and *second Arens multiplications*.

P. Civin and B. Yood were the first authors who made in [12] an extensive study of $L^1(G)^{**}$, the second dual algebra of the group algebra of a locally compact group, where they also proved some results on the second dual algebras of Banach algebras in general. However some of the results obtained in [12] were for abelian groups. This work was continued by Civin in [11] who studied the maximal left ideals of

$L^1(G)^{**}$ for the case of an abelian locally compact group G.

E.E. Granirer showed in [29] that if G is non-discrete or an infinite amenable discrete group, then the radical of $L^1(G)^{**}$ is non-separable.

In [61], N.J. Young proved that $L^1(G)$ is Arens regular if and only if G is a finite group. That result had been proven in [12] for abelian groups. Another proof of Young's result was later given by A. Ülger in [54].

S. Watanabe proved that $L^1(G)$ is an ideal of $L^1(G)^{**}$ if and only if G is compact [56],[57]. Later, M. Grosser proved this result in [30] using functional analytic methods.

The results on the second dual algebras obtained prior to the year 1979 were surveyed in the article by J. Duncan and S.A.R. Hosseiniun [18].

Renewed interest in the second dual algebra of the group algebra $L^1(G)$ emerged by publication of [36] by N. Isik, A. Ülger and J. Pym, where among other things, it was shown that if G is a compact group, then the topological centre of $L^1(G)^{**}$ is precisely $L^1(G)$.

A.T. Lau and V. Losert showed that $L^1(G)$ is the topological centre of $L^1(G)^{**}$, for any locally compact group G, and as a result, $L^1(G)$ is the algebraic centre of $L^1(G)^{**}$ when G is abelian [42]. In that article, they conjectured that if G and H are locally compact groups and $\theta : L^1(G)^{**} \to L^1(H)^{**}$ is an isometric algebra isomorphism, then G and H are topologically isomorphic; a conjecture which was answered affirmatively afterwards. As they pointed out, the conjecture is fulfilled for the case of abelian groups G and H, based on the fact that $L^1(G)$ is the algebraic centre of $L^1(G)^{**}$ and a result by J.G. Wendel that the group algebra $L^1(G)$ identifies the topological group G [59; Theorem 5]. The conjecture was answered in full by F. Ghahramani and Lau in [23]. In their proof, they used a result by Lau and K. McKennon [43] that if X_i, $i = 1, 2$, is a left introverted subspace of $C(G_i)$ containing $C_0(G_i)$, then any isometric isomorphism between the Banach algebras X_1^* and X_2^* results in a topological isomorphism between the groups G_1 and G_2 $\big(C(G)$ is the space of bounded continuous complex-valued functions on G and $C_0(G)$ stands for those elements of $C(G)$ which vanish at infinity$\big)$. This result of Lau and McKennon generalizes a result due to B.E. Johnson [37]. In [23] it was also shown that if G is a compact group and θ is any isometric algebra isomorphism from $L^1(G)^{**}$ onto $L^1(H)^{**}$, then $\theta\big(L^1(G)\big) = L^1(H)$. This result was later improved by Ghahramani, Lau and Losert in [26] by showing that the compactness assumption on G is not necessary. As a special case of a more general result, Ghahramani and Lau showed that if G and H are compact groups and θ is any continuous algebra isomorphism from $L^1(G)^{**}$ onto $L^1(H)^{**}$, then $\theta\big(L^1(G)\big) = L^1(H)$ [24]. In that paper, they raised the question of whether this result holds without assuming that G and H be compact or abelian (recall that the algebraic centre of $L^1(G)^{**}$ is $L^1(G)$ when G is abelian; therefore the result holds if we assume that G and H are abelian). The main objective of chapter 2 of this thesis is to answer this question by showing that if every continuous isomorphism from $L^1(G)^{**}$ onto itself maps $L^1(G)$ onto $L^1(G)$, then G is compact or abelian. In chapter 2, we also show that for compact groups G and H, any algebra isomorphism from $L^1(G)^{**}$ onto $L^1(H)^{**}$ maps $L^1(G)$ onto $L^1(H)$. This drops the assumption of continuity from the result by Ghahramani and Lau which was described above. Chapter 1 of this thesis is devoted to preliminaries.

In conjunction with the result of [23; Theorem], we prove in chapter 3 that the order structure of $L^1(G)^{**}$ identifies the topological group G; this work has been inspired by the work of Kawada [39] indicating that the order structure of $L^1(G)$ identifies the topological group G. This result also holds for X^*, where X is a C^*-subalgebra and left introverted subspace of $C(G)$ containing $C_0(G)$ (see Theorem 3.4.8). The method of the proof in fact shows that if S and T are compact topological semigroups, if $M(S)$ and $M(T)$ are the corresponding measure algebras and θ is a bipositive isomorphism from $M(S)$ onto $M(T)$, then θ is an isometric isomorphism. Finally in chapter 4, in part we answer a question raised in [18] on the existence of involutions on $L^1(G)^{**}$. More precisely, we show that the existence of such involutions that extend the natural involution of $L^1(G)$ in the case of an amenable group G is necessary (and sufficient) to have G finite. A similar result holds when replacing $L^1(G)^{**}$ by $LUC(G)^*$ and finiteness of G by compactness. It is natural to ask whether this result holds if we consider any left introverted subspace X of $C(G)$ that contains $C_0(G)$ and for which X^* contains a left invariant mean, rather than $LUC(G)$. As we will see, $WAP(G)$–the space of weakly almost periodic functions on G–provides an example of such X for which $WAP(G)^*$ has an involution that extends the natural involution of $L^1(G)$, keeping in mind that $WAP(G)^*$ contains a left invariant mean for every locally compact group G.

CHAPTER I

PRELIMINARIES

This chapter is intended to discuss some basic materials needed in the subsequent chapters, where we deal with the main objectives of this thesis. Materials in this chapter can be found in the literature and include topics from algebras, second duals of Banach algebras, group algebras, Banach spaces and algebras related to locally compact groups, and some classes of groups including amenable groups, Lie groups and etc. We will provide the proofs of some of the theorems for the sake of completeness.

1.1 Radical of an algebra In this manuscript, $\mathbb{F}$ stands for the field of either real or complex numbers, and the field of complex numbers is denoted by $\mathbb{C}$. Let A be an algebra over $\mathbb{F}$ and I be a left ideal of A.

(1) I is a *modular left ideal* of A if there exists $u \in A$ such that the set $A(1-u) = \{x - xu : x \in A\}$ is contained in I. Such an element u is a *right modular identity* for I. A modular left ideal I is *maximal* if there is no modular left ideal J with $I \subsetneqq J \subsetneqq A$

(2) An element $x \in A$ is *left* (respectively right) *quasi-invertible* if there exits $y \in A$ such that $y+x = yx$ (resp. $xy = x+y$); y is a left (right) quasi-inverse of x. An element $x \in A$ is two-sided quasi-invertible if it has a two-sided quasi-inverse y.

A *representation* of an algebra A over X is an algebra homomorphism $\pi : A \to L(X)$, where X is a vector space over $\mathbb{F}$ and $L(X)$ is the space of linear mappings from X into X. A subspace M of X is *invariant* (under π) if $\pi(a)(M) \subseteq M$ holds for all $a \in A$. The representation π is *irreducible* if $\pi \neq 0$ and $\{0\}$ and X are the only invariant subspaces of X.

The proof of the following Proposition can be found in [8; section 24, Proposition 14]

1.1.1 Proposition. *For any algebra A the following sets are identical:*

(a) *the intersection of the kernels of the irreducible representations of A;*

(b) *the intersection of the maximal modular left ideals of A.*

1.1.2 Definition. For an algebra A, the (*Jacobson*) *radical* of A is defined to be the set in Proposition 1.1.1, and it is denoted by $Rad(A)$. An algebra A is *semisimple* if $Rad(A) = \{0\}$; it is a *radical algebra* if $A = Rad(A)$, which actually means that A has no irreducible representation.

The radical $Rad(A)$ is a two sided ideal of A as it is the intersection of the kernels of the irreducible representations. Amongst the semisimple algebras we can list the space of bounded operators on a Banach space [6; Theorem 3.1.4], C^*-algebras, and the group algebras of locally compact groups; for the last one there will be given a proof subsequently.

If A is an algebra over $\mathbb{F}$, the *unitization* of A is defined to be the algebra $A \times \mathbb{F} = \{(a, \alpha) : a \in A, \alpha \in \mathbb{F}\}$ with addition, algebra multiplication, and scalar multiplication defined by

$$(a\,,\,\alpha)+(b\,,\,\beta)=(a+b\,,\,\alpha+\beta),$$

$$(a\,,\,\alpha)(b\,,\,\beta)=(ab+\beta a+\alpha b\,,\,\alpha\beta),$$

and

$$\beta(a\,,\,\alpha)=(\beta a\,,\,\beta\alpha).$$

Note that A can be regarded as an ideal of $A\times\mathbb{F}$ via the map $a\mapsto(a\,,0)$. When A is a Banach algebra, then $A\times\mathbb{F}$ is a Banach algebra with norm defined by $\|(a\,,\,\alpha)\|=\|a\|+|\alpha|$, in which $\|a\|$ is the norm of a in A. Instead of $(a\,,\,\alpha)$, we may use the formal sum $a+\alpha$ or $\alpha+a$, when there is no confusion. If an algebra A has an identity element, we let $\tilde{A}=A$, otherwise we let $\tilde{A}=A\times\mathbb{F}$. The *spectrum* of an element a in a complex algebra A is defined to be $\mathrm{sp}(a)=\{\lambda\in\mathbb{C}:$ $\lambda-a$ is not invertible in $\tilde{A}\}$. When A has no identity element, one can show that $\mathrm{sp}(a)=\{0\}\cup\{\lambda\in\mathbb{C}:\lambda\neq 0$ and $\frac{1}{\lambda}a$ is not quasi-invertible$\}$. The *spectral radius* of $a\in A$ is $r(a)=\sup\{|\lambda|:\ \lambda\in\mathrm{sp}(a)\}$, which is actually equal to $\lim\limits_{n\to\infty}\|a^n\|^{\frac{1}{n}}$ when A is a complex Banach algebra with norm $\|\cdot\|$, [8; section 5, Theorem 8].

From now on, all the vector spaces and algebras are assumed to be on the field of complex numbers unless otherwise stated.

We have the following characterization of the radicals in Banach algebras:

1.1.3 Proposition. *Let A be a Banach algebra. Then $a\in Rad(A)$ if and only if $r(ba)=0$ for all $b\in A$ (equivalently $r(ab)=0$ for all $b\in A$).*

A proof of this can be found in [8; section 25, Proposition 1].

The following theorem is due to B.E. Johnson.

1.1.4 Theorem. *Any two Banach algebra norms on a semisimple algebra are equivalent.*

See [8; section 25, Theorem 9] for a proof.

1.2 Banach A-modules and Cohen's Factorization Theorem

1.2.1 Definition. Let A be a Banach algebra. A Banach space X is called a *Banach left A-module* if a map $(a, x) \longrightarrow ax$ from $A \times X$ into X is so defined that

(i) the mapping $a \longrightarrow ax$ is linear for any fixed $x \in X$,

(ii) the mapping $x \longrightarrow ax$ is linear for every fixed $a \in A$,

(iii) for all $a, b \in A$ and $x \in X$, we have $(ab)x = a(bx)$,

(iv) there exists a constant M such that $\|ax\| \leqslant M\|a\|\|x\|$, for all $a \in A$ and $x \in X$.

Right Banach A-modules are defined if the mapping $(a, x) \longrightarrow ax$ in the above definition is replaced by a mapping $(a, x) \longrightarrow xa$ and properties (i), (ii), (iii), and (iv) are replaced correspondingly $\big($part (iii) is replaced by $x(ab) = (xa)b\big)$. If X is both Banach left and right A-module and for all $a, b \in A$ and $x \in X$, the relation $(ax)b = a(xb)$ holds, then we call X a *Banach A-bimodule* (or Banach two-sided A-module). For instance, every Banach algebra A is a Banach A-bimodule.

1.2.2 Definition. Let A be a Banach algebra and X be a Banach left, right, or two-sided A-module.

(1) A bounded net (a_γ) in A is called a *bounded approximate identity* for X if respectively $\|a_\gamma x - x\| \longrightarrow 0$, $\|xa_\gamma - x\| \longrightarrow 0$, or $\|a_\gamma x - x\| + \|xa_\gamma - x\| \longrightarrow 0$, for all $x \in X$.

(2) We say that A has a bounded left, right, or two sided approximate identity (for A) if it has a bounded approximate identity when regarded as a Banach left, right, or a two sided A-module.

The following theorem is due to P.J. Cohen [13] when $A = X$.

1.2.3 Theorem. *Let A be a Banach algebra and X be a Banach left A-module. If A has a bounded approximate identity for X, then for any $x \in X$ and $\epsilon > 0$ there are $y \in X$ and $a \in A$ such that $x = ay$ and $\|x - y\| < \epsilon$.*

See [8; section 11, Theorem 10] for a proof.

1.3 Second duals of Banach algebras

From now on, the dual of a Banach space X is denoted by X^*.

Let A be a Banach algebra. For any $f \in A^*$ and $a \in A$ we may define a linear functional fa on A by $\langle fa\,, b\rangle = \langle f\,, ab\rangle$, $(b \in A)$. One can check that $fa \in A^*$ and $\|fa\| \leqslant \|f\|\|a\|$. Now for $n \in A^{**}$ we may define $nf \in A^*$ by $\langle nf\,, a\rangle = \langle n\,, fa\rangle$; clearly we have $\|nf\| \leqslant \|n\|\|f\|$. Next for $m \in A^{**}$, define $mn \in A^{**}$ by $\langle mn\,, f\rangle = \langle m\,, nf\rangle$. We have $\|mn\| \leqslant \|m\|\|n\|$, and A^{**} becomes a Banach algebra with the multiplication mn, just defined, referred to as the first (or left) *Arens multiplication*, versus another multiplication on A^{**} called the second Arens multiplication, which is denoted by $m{\cdot}n$ and is defined successively as follows: $\langle m{\cdot}n\,, f\rangle = \langle n\,, f{\cdot}m\rangle$, in which $\langle f{\cdot}m\,, a\rangle = \langle m\,, a{\cdot}f\rangle$, where $\langle a{\cdot}f\,, b\rangle = \langle f\,, ba\rangle$, herein m, n, f, a, and b are taken as above. A Banach algebra A is called *Arens regular* if $mn = m{\cdot}n$ for all $m, n \in A^{**}$. Let us recall that a linear map T from a Banach space X into a Banach space Y is called *weakly compact* if the weak-closure of $T(B)$ in Y is weakly compact, where B denotes the unit ball of X. We have the following theorem in connection with Arens regularity:

1.3.1 Theorem. *For any Banach algebra A, the following statements are equivalent:*

(1) *A is Arens regular;*

(2) *for each $m \in A^{**}$, the map $n \mapsto mn$, from A^{**} into itself, is continuous with respect to weak-star topology;*

(3) *for each $m \in A^{**}$, the map $n \mapsto n{\cdot}m$, from A^{**} into itself, is continuous in weak-star topology;*

(4) *for each $f \in A^*$, the map $a \mapsto fa$, from A into A^*, is weakly compact;*

(5) *for each $f \in A^*$, the map $a \mapsto a{\cdot}f$, from A into A^*, is weakly compact.*

See [18; Theorem 1] for a proof. The equivalence of (1) and (4) will be proven in Proposition 4.6 (see chapter 4) as part of a more general result. This equivalence is what we essentially need in applications, in chapter 3.

One can easily show that the natural embedding i_A from A into A^{**} is an algebra homomorphism when A^{**} is equipped with either Arens multiplication.

From now on, unless otherwise stated, we **always** assume that the second dual of a Banach algebra is equipped with the first Arens multiplication. Note that from the definition of Arens multiplication, A^* has a Banach A-module structure. For $a \in A$ and $f \in A^*$ we will often use af instead of $i_A(a)f$ (that comes from the definition of Arens multiplication); the reader observes easily that in fact af is nothing but $a{\cdot}f$ that was defined as part of the second Arens multiplication.

In the first Arens multiplication, the maps $m \mapsto mn$ and $m \mapsto am$ are weak-star continuous in A^{**}, when $n \in A^{**}$ and $a \in A$ are taken fixed. But, in general, it is not true that the map $m \mapsto nm$ is weak-star continuous for each $n \in A^{**}$. In fact, this is equivalent to A being Arens regular, by Theorem 1.3.1. Amongst the non-regular Banach algebras are the group algebras of infinite locally compact groups (as an example take a non-discrete abelian locally compact group and compare Proposition 1.3.2, Proposition 1.3.3, Proposition 1.5.1, and Proposition 4.1). For a Banach algebra A, the *topological centre* of A^{**} is defined to be the set of those $n \in A^{**}$ for which the maps $m \mapsto nm$ $(m \in A^{**})$ are weak-star continuous in A^{**}. If m, $n \in A^{**}$ and (x_α) and (y_β) are nets in A such that $x_\alpha \to m$ and $y_\beta \to n$ in the weak-star topology, then $mn = \lim_\alpha \lim_\beta x_\alpha y_\beta$ and $m \cdot n = \lim_\beta \lim_\alpha x_\alpha y_\beta$. This establishes the following result, which appeared in [18]; we need this result in Chapter 3.

1.3.2 Proposition. *Let A be commutative. Then A^{**} is commutative under either Arens multiplication if and only if A is Arens regular.*

A net (a_γ) in a Banach algebra A is called a *weak right approximate identity* if for each $a \in A$, $aa_\gamma \to a$ in the weak topology of A. The weak left approximate identities can be defined similarly.

From now on we agree that $\xrightarrow{w^*}$ stands for convergence in the weak-star topology.

1.3.3 Proposition. *Let A be a Banach algebra. Then A^{**} has a right identity E with $\|E\| \leqslant K$, for some $K > 0$, if and only if there exists a weak right approximate* *identity* (a_γ) *in* A *which is bounded by* K *and* $a_\gamma \xrightarrow{w^*} E$ *in* A^{**}.

See [8; section 28, Proposition 7] for a proof. The reader may consult the survey article [18] for further study on this subject.

1.4 Topological groups and related Banach spaces and algebras

1.4.1 Definition. A *topological group* is a group G together with a topology with respect to which the map $(x, y) \mapsto x^{-1}y$ from the topological space $G \times G$ into G is continuous. A *locally compact group* is a topological group in which the underlying topology is locally compact and Hausdorff.

The identity element of a group G is denoted by e_G. If N is a normal subgroup of G, the quotient group of G by N is denoted by G/N and it is the collection of left cosets $\{xN : \ x \in G\}$ with the group operation $(xN)(yN) = xyN$.

1.4.2 Theorem. *Let* G *be a locally compact group and* N *be a closed normal subgroup of* G. *Then* G/N *endowed with the quotient topology induced by the quotient map* $\pi : G \to G/N$, $\pi(x) = xN$, $(x \in G)$, *is a locally compact group. With this topology,* π *is an open map, and furthermore,* G *is compact if and only if* N *and* G/N *are both compact.*

For a proof see Theorem 5.17, Theorem 5.21, Theorem 5.22, Theorem 5.25, Theorem 5.26 of [33].

1.4.3 Definition. For a topological group G we have the following function spaces:

(a) The space of all bounded continuous complex-valued functions on G is denoted by $C(G)$. Unless otherwise stated, we always assume that $C(G)$

is equipped with the topology induced by the norm $\|f\|_u = \sup_{x \in G} |f(x)|$, $(f \in C(G))$. For any $f \in C(G)$ and $x \in G$ we define the left and right translations l_x and r_x by $(l_x f)(y) = f(xy)$ and $(r_x f)(y) = f(yx)$, where $y \in G$. The space $C(G)$ is an algebra with pointwise multiplication. Given $x \in G$, the linear functional $f \mapsto f(x)$ on a subspace X of $C(G)$ is denoted by δ_x and called a point mass.

(b) The space containing the elements of $C(G)$ with compact supports is denoted by $C_{00}(G)$, where the support of an $f \in C(G)$ is defined to be the closure in G of the set $\{x \in G : \ f(x) \neq 0\}$.

(c) $C_0(G)$ denotes the set of those elements of $C(G)$ that vanish at infinity. Note that $C_0(G)$ is a closed subalgebra of $C(G)$, and $C_{00}(G)$ is dense in $C_0(G)$.

(d) $LUC(G)$ is defined to be the set of all elements f of $C(G)$ for which the map $x \mapsto l_x f$, from G into $C(G)$, is continuous. The elements of $LUC(G)$ are called *bounded left uniformly continuous* functions on G.

(e) $RUC(G)$ is defined to be the set of all elements $f \in C(G)$ for which the map $x \mapsto r_x f$, from G into $C(G)$, is continuous. The elements of $RUC(G)$ are called *bounded right uniformly continuous* functions on G. Our definition of spaces $LUC(G)$ and $RUC(G)$ is different from that of [33], where the first one is called the space of right uniformly continuous functions and the second one the space of left uniformly continuous functions. One can see that $C_0(G) \subset LUC(G) \cap RUC(G)$, [33; Theorem 15.4].

(f) A function f in $C(G)$ is called *almost periodic* if the set $\mathcal{O}(f) =$

$\{l_x f : \ x \in G\}$ has compact closure in $C(G)$. The set of all (continuous) almost periodic functions on G is denoted by $AP(G)$. Note that $AP(G)$ can be defined by means of right translations, and indeed $f \in AP(G)$ if and only if $\{r_x f \ : \ x \in G\}$ is relatively compact in $C(G)$. See [33; Theorem 18.1] in this regard.

(g) A function $f \in C(G)$ is called *weakly almost periodic* if the set $\mathcal{O}(f) = \{l_x f : \ x \in G\}$ has compact closure in the weak topology of $C(G)$; the set consisting of these functions is denoted by $WAP(G)$. Right translations, instead of left translations, can be used to define $WAP(G)$. We clearly have $AP(G) \subseteq WAP(G)$.

(h) A closed subspace X of $C(G)$ is called *left introverted* if it is left invariant (that is $l_x X \subseteq X$, for all $x \in X$) and the function nf defined by $(nf)(t) = \langle n\,,\, l_t f\rangle$, $t \in G$, is in X for all $f \in X$ and $n \in X^*$. In this case, if $m, n \in X^*$, one may define the product $mn \in X^*$ by letting $\langle mn\,,\, f\rangle = \langle m\,,\, nf\rangle$, where $f \in X$. From the definition, it is clear that $\|nf\|_u \leqslant \|n\|\|f\|_u$, so that $|\langle mn\,,\, f\rangle| \leqslant |\langle m\,,\, nf\rangle| \leqslant \|m\|\|n\|\|f\|_u$, which in turn shows $\|mn\| \leqslant \|m\|\|n\|$, therefore X^* is a Banach algebra.

The following result is known; a proof for the case of a locally compact group can be found in [9; Corollary 1.26, Theorem 2.21, Corollary 3.7] or in [33; 17.19 part (c) and Theorem 18.8]. The proof we present here is quite elementary and may have not appeared before.

1.4.4 Proposition. *If G is non-compact, then $AP(G) \cap C_0(G) = \{0\}$.*

Proof. Let $f \in AP(G) \cap C_0(G)$. Let $\{K\}$ be the collection of compact subsets of G, ordered by upward inclusion. For each compact set $K \subset G$, let $x_K \in G \setminus K$. Choose a subnet $\{x_L\}$ of $\{x_K\}$ such that $l_{x_L} f \to g$ in $C(G)$ for some $g \in C(G)$. Given $\epsilon > 0$, choose a compact set K such that $|f(t)| < \epsilon$ for $t \notin K$. Then for any $s \in G$, we may select a compact set L enough large such that $x_L \notin Ks^{-1}$ and

$$(1) \qquad \|g - l_{x_L} f\|_u < \epsilon,$$

and hence $|g(s)| \leqslant |g(s) - f(x_L s)| + |f(x_L s)| < 2\epsilon$, which implies that $g = 0$; whence $f = 0$, from (1). $\square$

The elements of the following theorem are known.

1.4.5 Theorem. *The spaces $LUC(G)$, $RUC(G)$, $AP(G)$ and $WAP(G)$ are left and right translation invariant closed subalgebras of $C(G)$. The spaces $LUC(G)$, $AP(G)$ and $WAP(G)$ are left introverted. If we assume that G is locally compact, then $C_0(G)$ is also left introverted and $C_0(G) \subseteq WAP(G) \subseteq LUC(G)$.*

Proof. We only prove the last part of the assertion; the other parts can be found in [7; Chapter 3, Lemmas 5.4, 8.3 and 9.3] and [33; Lemma 19.5]. Assume that G is locally compact. To show that $C_0(G) \subseteq WAP(G)$, let (x_α) be a net in G, $f \in C_0(G)$ and $m \in C(G)^*$. If from a point on each x_α lies in a compact subset of G, then a subnet of (x_α), for example (x_β), and an $x \in G$ can be chosen such that $x_\beta \to x$. Otherwise, by considering the upward direction of compact subsets of G, we can find a subnet (x_β) that approaches infinity. Let n be the restriction of m to $C_0(G)$. Then in the first case let $g = l_x f$ and note that $\langle m, l_{x_\beta} f\rangle = \langle n, l_{x_\beta} f\rangle = (nf)(x_\beta) \to (nf)(x) = \langle m, g\rangle$, while in the second case $\langle m, l_{x_\beta} f\rangle = \langle n, l_{x_\beta} f\rangle = (nf)(x_\beta) \to 0$, because $nf \in C_0(G)$, from the left introvertedness of $C_0(G)$. So $f \in WAP(G)$, as required. For a different approach to $C_0(G) \subseteq WAP(G)$, see [9; Corollary 3.7]. Finally note that from [51; Lemma 5] and [48; Theorem 7], $LUC(G)$ is the largest left introverted subspace of $C(G)$ in the sense that it contains every left introverted subspace of $C(G)$; therefore we have $WAP(G) \subseteq LUC(G)$. $\square$

To every locally compact group G there is associated a positive measure λ, called *left Haar measure*, which has the following properties:

(a) For every open $U \subset G$, we have $\lambda(U) > 0$.

(b) If F is compact, then $\lambda(F) < \infty$.

(c) λ is left invariant, that is $\lambda(xA) = \lambda(A)$ for every $A \subset G$ and any $x \in G$.

(d) λ is regular on the σ-algebra of λ-measurable sets (in the sense of [33; Definition 11.34]).

The positive measure λ with four properties, just cited, is unique up to a positive multiple.

See [33, page 194] for details.

For each real number $p > 0$, we let $L^p(G)$ denote the space of all complex-valued functions f on G measurable with respect to λ and satisfying $\int_G |f|^p \, d\lambda < \infty$; two functions in $L^p(G)$ are taken identical if they differ only on a set of measure zero. By defining $\|f\|_p = \left(\int |f|^p d\lambda\right)^{1/p}$, the space $L^p(G)$, $1 \leqslant p < \infty$, becomes a Banach space with norm $\| \ \|_p$, [33; Theorem 12.8]. All of these spaces contain $C_{00}(G)$ as a dense subspace [33; Theorem 12.10]. A measurable set A is called *locally null* if $\lambda(A \cap F) = 0$, for every compact subset F of G. The space consisting of all bounded measurable functions is denoted by $L^\infty(G)$. Two functions in $L^\infty(G)$ are assumed identical if they differ only on a locally null set. For $f \in L^\infty(G)$, we may define a norm $\|f\|_\infty$ in $L^\infty(G)$ to be the infimum of all $\alpha \geqslant 0$ for which the set $\{x \in G : \ |f(x)| > \alpha\}$ is locally null. With respect to $\|\cdot\|_\infty$, $L^\infty(G)$ is a Banach space. The dual Banach space of $L^1(G)$ is equal to $L^\infty(G)$, where in fact each element of $L^\infty(G)$ can be identified with a bounded linear functional on $L^1(G)$ via the map $f \mapsto \Phi_f \ \big(f \in L^\infty(G)\big)$, in which $\Phi_f(h) = \int_G hf\, d\lambda$, for all $h \in L^1(G)$. See [33; Theorem 12.18] for details. It is customary to write $\int f\, dt$ for $\int f\, d\lambda$, for every measurable function f.

1.5 Some Banach algebras related to locally compact groups

For a locally compact group G, the dual space $C_0(G)^*$ is identified with $M(G)$, the space of regular complex Borel measures (of bounded variation) on the σ-algebra of Borel sets; moreover $M(G)$ becomes a Banach algebra if the product $\mu * \nu$ of any two elements $\mu, \nu \in M(G)$ is defined by

$$\langle \mu * \nu\, ,\, f\rangle = \int_G \int_G f(xy) d\mu(x) d\nu(y), \quad \big(f \in C_0(G)\big).$$

$L^1(G)$ may be identified with a closed ideal of $M(G)$ [33; Theorem 19.18]. For $f, g \in L^1(G)$ and $t \in G$, we have $(f * g)(t) = \int f(s) g(s^{-1}t) d\lambda(s)$ for almost all $t \in G$, where $*$ denotes the multiplication in $L^1(G)$ induced from the multiplication of $M(G)$ [33, Theorem 20.10].

In the sequel, we always assume that G is a locally compact group, unless otherwise stated.

1.5.1 Proposition. *The Banach algebra $L^1(G)$ has a bounded approximate identity.*

Sketch of the Proof. Let $\mathcal{P}$ be the collection of all compact neighbourhoods of e_G, ordered by inclusion, and for each $K \in \mathcal{P}$ let φ_K be the charactristic function of K divided by $\lambda(K)$, where λ is the left Haar measure. Then $\{\varphi_K\}_{K\in\mathcal{P}}$ is an approximate identity for $L^1(G)$ that is bounded by 1. See [33; Theorem 20.15] for details. $\square$

For $\mu \in M(G)$ and $h \in L^p(G), 1 \leqslant p < \infty$, we may define $\lambda_\mu(h) \in L^p(G)$ by

$$\lambda_\mu(h)(t) = \int_G h(s^{-1}t)d\mu(s),$$

[33; Theorem 20.12]; and by a similar argument to the one in [33; Theorem 22.11], the map $\lambda : \mu \mapsto \lambda_\mu$ is a continuous algebra embedding from $M(G)$ into $B\big(L^p(G)\big)$, the space of bounded operators on $L^p(G)$.

1.5.2 Proposition. *The algebras $M(G)$ and $L^1(G)$ are semisimple.*

Proof. Let $H = L^2(G)$ and let $\lambda : M(G) \to B(H)$ be defined as above. To any $\mu \in M(G)$, there is associated a measure $\mu^* \in M(G)$ defined by $\langle \mu^*, \varphi \rangle = \overline{\int \varphi(t^{-1})d\mu(t)}$, and it is routine to see that $\lambda(\mu^*) = \lambda(\mu)^*$, where $\lambda(\mu)^*$ is the adjoint of $\lambda(\mu)$ in $B(H)$. For $\mu \in Rad(M(G))$, we have $r(\mu\mu^*) = 0$ and so $r(\lambda(\mu\mu^*)) = 0$, since an algebra morphism reduces the spectral radii. On the other hand $\lambda(\mu\mu^*) = \lambda(\mu)\lambda(\mu)^*$ is a normal operator, accordingly $\|\lambda(\mu)\|^2 = \|\lambda(\mu\mu^*)\| = r(\lambda(\mu\mu^*)) = 0$, since the spectral radius of a normal operator is equal to its norm (see section 3.2). From $\lambda(\mu) = 0$ we get $\mu = 0$ and so $M(G)$ is semisimple. A similar argument can be furnished to prove the semisimplicity of $L^1(G)$, since $\mu^* \in L^1(G)$ if $\mu \in L^1(G)$. $\square$

Recall from section 1.3, that $L^\infty(G)$ is an $L^1(G)$-bimodule.

1.5.3 Theorem. *In the $L^1(G)$-module structure of $L^\infty(G)$, we have the following factorizations:*

(a) $LUC(G) = L^\infty(G)L^1(G)$;

(b) $RUC(G) = L^1(G)L^\infty(G)$;

(c) $C_0(G) = C_0(G)L^1(G) = L^1(G)C_0(G)$.

See [34; 32.44] and [34; 32.45] for a Proof, or consult Theorem 1.2.3.

Let us see what the representations of the functions $\varphi f \in RUC(G)$, for $f \in L^\infty(G), \varphi \in L^1(G)$ are: For any $\psi \in L^1(G)$, we have that

$$\begin{aligned}\int (\varphi f)(x)\psi(x)\,dx = \langle \varphi f\,,\,\psi\rangle = \langle f\,,\,\psi * \varphi\rangle = \int f(t)(\psi * \varphi)(t)\,dt \\ = \iint f(t)\psi(x)\varphi(x^{-1}t)\,dx\,dt = \iint f(xt)\varphi(t)\psi(x)\,dt\,dx,\end{aligned}$$

which shows that $(\varphi f)(x) = \int f(xt)\varphi(t)\,dt$, for all $x \in G$. One may similarly show that $(f\varphi)(x) = \int f(tx)\varphi(t)\,dt$, for $x \in G$.

Part (a) of the following theorem is perhaps unknown in its generality; it is known for the case $X = LUC(G)$. Part (b) was established by F. Ghahramani and A.T. Lau [23].

1.5.4 Theorem. *Let X be a left introverted subspace of $C(G)$ and E be a right identity of $L^1(G)^{**}$ with $\|E\| = 1$.*

(a) *The linear map $\pi : L^\infty(G)^* \to X^*$, induced by restriction, is an algebra homomorphism.*

(b) *If $X = LUC(G)$, the kernel of the homomorphism in part* (a) *is the space of right annihilators of the algebra $L^1(G)^{**}$. This homomorphism is an isometric isomorphism from $EL^1(G)^{**}$ onto $LUC(G)^*$.*

Proof. **(a)** It follows from [51; Lemma 5] and [48; Theorem 7] that for the locally compact G, $LUC(G)$ is the unique maximal left introverted subspace of $C(G)$; so that $X \subseteq LUC(G)$. For the moment, let us denote by $\pi_\circ$ the linear map $L^\infty(G)^* \to LUC(G)^*$, which is induced by restriction. For $m, n \in L^\infty(G)^*$ and $f \in X$, it is routine to see that $\pi(n)f = \pi_\circ(n)f$. On the other hand, from [40; Lemma 3], we have $nf = \pi_\circ(n)f$, where the left hand side comes from the definition of the Arens multiplication. So $\langle mn\,,\, f\rangle = \langle m\,,\, nf\rangle = \langle m\,,\, \pi(n)f\rangle = \langle \pi(m)\,,\, \pi(n)f\rangle = \langle \pi(m)\pi(n)\,,\, f\rangle$, which indicates that π is an algebra homomorphism.

(b) An element $m \in L^1(G)^{**}$ is a right annihilator if and only if $\varphi m = 0$ for all $\varphi \in L^1(G)$, since the Arens multiplication is weak-star continuous in the first variable. But this is equivalent to having $0 = \langle \varphi m\,,\, f\rangle = \langle m\,,\, f\varphi\rangle$, for all $f \in L^\infty(G)$, and this is equivalent to having m an annihilator of $LUC(G)$, by an application of Theorem 1.5.3. For $m \in L^1(G)^{**}$, $Em - m$ is a right annihilator; so that $\pi(Em - m) = 0$ which shows that $\pi(E)$ is a left identity for the image of π. On the other hand the map $\pi : EL^1(G)^{**} \to LUC(G)^*$ is onto, for if $n \in LUC(G)^*$ and if m is any Hahn-Banach extension of n, then $\pi(Em) = \pi(m) = n$. Now we show that the map $\pi : EL^1(G)^{**} \to LUC(G)^*$ is an isometry: From Proposition 1.3.3, there exists a weak right approximate identity (φ_γ) in $L^1(G)$ such that $\|\varphi_\gamma\| \leqslant 1$, $\varphi_\gamma \xrightarrow{w^*} E$ in $L^1(G)^{**}$. Then for $h \in L^\infty(G)$, $\langle Em\,,\, h\rangle = \lim_\gamma \langle m\,,\, h\varphi_\gamma\rangle = \lim \langle \pi(m)\,,\, h\varphi_\gamma\rangle = \lim \langle \pi(Em)\,,\, h\varphi_\gamma\rangle$ and hence $\|Em\| \leqslant \|\pi(Em)\|$, therefore $\|Em\| = \|\pi(Em)\|$, since $\pi(Em)$ is the restriction of Em. The map $\pi : EL^1(G)^{**} \to LUC(G)^*$ is onto, for if $n \in LUC(G)^*$

and m is any Hahn-Banach extension of n, then $\pi(Em) = \pi(m) = n$, as desired. $\square$

For a left introverted subspace X of $C(G)$ containing $C_0(G)$ we let $C_0(G)^\perp = \{m \in X^* : m(f) = 0 \text{ for all } f \in C_0(G)\}$. Note that for such X, the space $M(G)$ is isometrically and algebraically isomorphic to a subspace of X^* via the map $\mu \mapsto \hat{\mu}$, where $\langle \hat{\mu}, f \rangle = \int f(x)\, d\mu(x)$, for all $f \in X$; therefore we regard $M(G)$ as a closed subalgebra of X^*. The following theorem is due to F. Ghahramani, A.T. Lau and V. Losert [26] for the case $X = LUC(G)$. The proof of that article can be carried out for any X, as verified below:

1.5.5 Theorem. *Let X be a left introverted subspace of $C(G)$ containing $C_0(G)$. Then $C_0(G)^\perp$ is a closed ideal of X^*. If $\mu \in M(G)$ and $m \in C_0(G)^\perp$, then $\|\mu + m\| = \|\mu\| + \|m\|$. Moreover $X^* = M(G) \oplus C_0(G)^\perp$ (direct sum).*

Proof. If $m \in C_0(G)^\perp$, $f \in C_0(G)$ and $\mu \in M(G)$, then for $x \in G$ we have $(mf)(x) = \langle m, l_x f \rangle = 0$, since $l_x f \in C_0(G)$. Therefore $mf = 0$ and $C_0(G)^\perp$ is a left ideal. By Theorem 1.4.5, $C_0(G)$ is left introverted, hence $m\mu \in C_0(G)^\perp$, because $\mu f \in C_0(G)$. If $n \in X^*$ is a Hahn-Banach extension of μ, then $mn = m\mu + m(n - \mu) \in C_0(G)^\perp$, since $n - \mu \in C_0(G)^\perp$ and $C_0(G)$ is a left ideal. So we have $C_0(G)^\perp$ a two-sided ideal. Let $\epsilon > 0$ and $h \in C_{00}(G)$ with $\|h\| < 1$ such that $\|\mu\| \leqslant \mu(h) + \epsilon$, and then choose $g \in C_{00}(G)$ with $0 \leqslant g \leqslant 1$, which takes the value 1 on the support of h. Pick $k \in X$ and $\alpha \in \mathbb{C}$ such that $\|k\|_u \leqslant 1$, $\|m\| \leqslant m(k) + \epsilon$, $|\alpha| = 1$, and

$$(*) \qquad |\mu(\alpha(k - gk) + h)| = |\mu(k - gk)| + |\mu(h)|.$$

Then from $\|\alpha(k-gk)+h\| \leqslant 1$ and $(*)$ we have that $|\mu(k-gk)| \leqslant \epsilon$. If $k' = k-gk+h$, then $\|k'\| \leqslant 1$, and so $\|m\| + \|\mu\| - 3\epsilon \leqslant m(k') + \mu(h) - |\mu(k - gk)| \leqslant |(m + \mu)(k')|$, therefore $\|m\| + \|\mu\| \leqslant \|m + \mu\|$, as desired. Finally note that $C_0(G)^{\perp} \cap M(G) = \{0\}$, and for $m \in X^*$ we have the decomposition $m = \mu + (m - \mu)$ where μ is the restriction of m to $C_0(G)$ and $m - \mu \in C_0(G)^{\perp}$. This completes the proof. $\square$

1.6 Some classes of groups.

Let G be a topological group and let $B\big(AP(G)\big)$ be the space of bounded operators on $AP(G)$. Let $B\big(AP(G)\big)$ be equipped with the strong operator topology, in which, by definition, $T_\alpha \to T$ if $T_\alpha(f) \to T(f)$, in norm, for all $f \in AP(G)$. For $x \in G$, let $L_x \in B\big(AP(G)\big)$ be the left translation by x, that is, $L_x(f) = l_x f$. Then the closure of $\{L_x \ : \ x \in G\}$ in $B\big(AP(G)\big)$, denoted by G^a, is called the *almost periodic compactification* of G, and it can be shown that G^a is a compact topological group and the map $\eta \ : \ x \to L_{x^{-1}}$ from G into G^a is a continuous homomorphism [34; page 311]. The homomorphism η has a relationship with the continuous irreducible unitary representations of G. By a representation of this kind, we mean a (norm) continuous group homomorphism π from G into the group of unitary operators of some $L(H)$, H a Hilbert space, such that the only closed subspaces of H invariant under all $\pi(x)$, $x \in G$, are $\{0\}$ and H. It can be shown that $AP(G)$ is the closure of the linear span of the functions of the form $x \mapsto \langle \pi(x)\xi_1 \, , \, \xi_2 \rangle$, where $\pi \ : \ G \to L(H)$ runs through continuous irreducible finite-dimensional ($\dim H < \infty$) unitary representations of G, and $\xi_1, \xi_2 \in H$ [34; page 312]. So it is immediate that the kernel of the morphism $x \to L_{x^{-1}}$ from G into G^a is equal to the intersection of the kernels of the unitary representations just described. Together with [16; 16.4.1], we have:

1.6.1 Proposition. *The following subgroups of a topological group G are equal:*

(1) *The kernel of the morphism $x \rightarrow L_{x^{-1}}$ from G into G^a;*

(2) *The intersection of the kernels of the continuous morphisms of G into all compact groups;*

(3) *The intersection of the kernels of the continuous irreducible finite dimensional unitary representations of G.*

1.6.2 Definition. A topological group G is called *maximally almost periodic* if $\{e_G\}$ is the subgroup described in Proposition 1.6.1.

Note that G is maximally almost periodic if and only if $AP(G)$ separates the points of G, since $L_{x^{-1}} = L_{y^{-1}}$ in $B\big(AP(G)\big)$ is equivalent to having $f(x^{-1}) = f(y^{-1})$ for all $f \in AP(G)$, by the fact that $AP(G)$ is right-translation invariant.

For a topological group G, we let $\mathfrak{A}(G)$ denote the group of all topological group automorphisms of G. The group of inner automorphisms of G will be denoted by $\Im(G)$.

1.6.3 Definition. If $\mathfrak{B}$ is a subgroup of $\mathfrak{A}(G)$, then G belongs to $[SIN]_{\mathfrak{B}}$ if there exists a neighbourhood base $\mathcal{N}$ for e_G such that for any $\tau \in \mathfrak{B}$ and $V \in \mathcal{N}$, $\tau(V) = V$ (the elements of $\mathcal{N}$ are invariant under the elements of $\mathfrak{B}$). If $\mathfrak{B} = \Im(G)$, instead of $G \in [SIN]_{\mathfrak{B}}$ we say that G is a SIN group (SIN is an abbreviation for "small invariant neighbourhood").

Note that if G is a SIN group and N a closed normal subgroup of G, then G/N is a SIN group as well, since the quotient map from G onto G/N is open and it sends a neighbourhood base at e_G invariant under the inner automorphisms onto a neighbourhood base at $e_{G/N}$ invariant under the inner automorphisms. The following theorem is due to P. Milnes [47].

1.6.4 Theorem. *A locally compact group G is SIN if and only if $LUC(G) = RUC(G)$.*

The following theorem provides a link between maximally almost periodic groups and SIN groups. Note that the connected component of identity in any group is a closed normal subgroup; we denote it by G_0.

1.6.5 Theorem. *Let G be a locally compact group and G_0 be the (connected) component of the identity in G. If G/G_0 is compact, the following statements are equivalent:*

(a) *G is maximally almost periodic;*

(b) *G is a SIN group.*

For a proof see [35; Corollary XII.1, page 55] or [31; Theorem 2.9].

We recall that if G is a Lie group, then G is a locally compact group in which every point x has an open neighbourhood homeomorphic to an open set in an Euclidean space $\mathbb{R}^n$ (for the precise definition see [55]). Furthermore G_0 is an open subgroup of G, since every $x \in G$ has a connected neighbourhood.

We have the following theorem in connection between SIN groups and Lie groups.

1.6.6 Theorem. *Let $\mathfrak{B}$ be a subgroup of $\mathfrak{A}(G)$ containing $\Im(G)$. If $G \in [SIN]_{\mathfrak{B}}$, then every neighbourhood of e_G contains a compact $\mathfrak{B}$-invariant subgroup N such that G/N is a Lie group.*

For a proof see [31; Theorem 2.11].

Let χ be a character of G (a group homomorphism from G into the unit circle) and let X be a left invariant subspace of $L^\infty(G)$. An element $m \in X^*$ is said to be left χ-invariant if $m(l_x f) = \chi(x)m(f)$ for all $f \in X$ and $x \in G$; if χ is the constant function 1 in this definition, m is called left invariant. First we quote the following proposition from [22; pages 567 and 570].

1.6.7 Proposition.

(a) *If* $0 \neq m \in LUC(G)^*$ *and* $m(l_x f) = \chi(x)m(f)$, ($f \in LUC(G)$ *and* $x \in G$) *for some complex-valued function* χ *on* G, *then* χ *is a continuous character of* G.

(b) *If* χ *is a character of* G *and* X *is a left invariant subspace of* $L^\infty(G)$ *with* $\bar{\chi}X = X$, *then an element* $m \in X^*$ *is left invariant if and only if the element* $m_\chi \in X^*$ *is left* χ*-invariant, where* $m_\chi(f) = m(\bar{\chi}f)$, *for* $f \in X$.

Proof. **(a)** The equality $m(l_x f) = \chi(x)m(f)$ shows that $\delta_x m = \chi(x)m$, $x \in G$. So $|\chi(x)|\,\|m\| = \|\delta_x m\| = \|m\|$, and χ maps into the unit circle. On the other hand, $\chi(xy)m = \delta_{xy}m = \delta_x\delta_y m = \chi(x)\chi(y)m$, showing that χ is a character. Now let $f \in LUC(G)$ and $m(f) \neq 0$. If $x_\alpha \to x$ in G, then $l_{x_\alpha}f \to l_x f$ in $LUC(G)$ and so $\chi(x_\alpha)m(f) = m(l_{x_\alpha}f) \to m(l_x f) = \chi(x)m(f)$, which shows that $\chi(x_\alpha) \to \chi(x)$ and so χ is continuous.

(b) For $y \in G$ we have $m_\chi(l_y f) = m\big(\bar{\chi}(l_y f)\big) = \chi(y)m\big(l_y(\bar{\chi}f)\big)$; therefore $m_\chi(l_y f) = \chi(y)m\big(l_y(\bar{\chi}f)\big)$, from which it is easily verified that m is left invariant if and only if m_χ is left χ-invariant. $\square$

Let X be a subspace of $L^\infty(G)$ containing the constant function $\mathbf{1}$ and closed under left translation and complex conjugation. An $m \in X^*$ is called a *left invariant mean* on X, if it is left invariant and $\|m\| = m(\mathbf{1}) = 1$.

1.6.8 Definition. A locally compact group G is *amenable* if there exists a left invariant mean on $L^\infty(G)$.

Note that by [50; Theorem 4.19], the amenability of G is equivalent to the existence of left invariant means on $LUC(G)$.

If G is a compact group, then $LUC(G) = C(G)$, so the normalized left Haar measure m is a left invariant mean on $LUC(G)$ and in fact it is the only left invariant mean. On the other hand, we have the following result in the other direction.

1.6.9 Theorem. *Let G be an amenable locally compact group. If $LUC(G)$ has a unique left invariant mean, then G is compact.*

See [41] for a proof.

CHAPTER II

THE ROLE OF COMPACT GROUPS AND LOCALLY COMPACT ABELIAN GROUPS IN ISOMORPHISMS BETWEEN THE SECOND DUALS OF GROUP ALGEBRAS

This chapter provides a characterization for compact groups and locally compact abelian groups in terms of the continuous (algebra) isomorphisms of the second duals of group algebras.

2.1 Some properties arising from compact groups and abelian locally compact groups

Let A be a Banach algebra. For any $a \in A$, let $\rho_a : A \to A$ be the map $b \mapsto ba$, and let ρ_a^{**} be the second adjoint of ρ_a. Then for $b \in A$, $f \in A^*$ and $m \in A^{**}$ we have $\langle \rho_a^*(f), b\rangle = \langle f, ba\rangle = \langle af, b\rangle$, which shows that $\rho_a^*(f) = af$; therefore $\langle \rho_a^{**}(m), f\rangle = \langle m, af\rangle = \langle ma, f\rangle$, which in turn implies $\rho_a^{**}(m) = ma$. From [19; section VI.4, Theorem 2], if X and Y are Banach spaces and $T : X \to Y$ is a bounded linear operator, then T is weakly compact if and only if $T^{**}(X^{**}) \subseteq Y$. This implies that A is a left ideal of A^{**} if and only if ρ_a is weakly compact, for all $a \in A$. Similarly, A is a right ideal if, and only if, the operator $\lambda_a : b \mapsto ab$, is weakly compact, for all $a \in A$. For a locally compact group G, it is known by [56],[57], and alternatively by [30] that $L^1(G)$ is an ideal of $L^1(G)^{**}$ if and only if G is compact. We include a proof of it.

2.1.1 Proposition. *The following statements are equivalent*

(1) *G is compact.*

(2) *$L^1(G)$ is a left ideal of $L^1(G)^{**}$.*

(3) *$L^1(G)$ is a right ideal of $L^1(G)^{**}$.*

Proof. First, let G be compact and let $\{\psi_n\}$ be a bounded net in $L^1(G)$. To prove that $L^1(G)$ is a right ideal of $L^1(G)^{**}$, it suffices to show that for any φ in $L^1(G)$, the net $\{\varphi\psi_n\}$ has a weakly convergent subnet, as explained earlier. Since the norm bounded sets in dual Banach spaces are relatively weak-star compact, there exists a subnet $\{\psi_{n_k}\}$ of $\{\psi_n\}$ which weak-star converges to some $\mu \in M(G) = C(G)^*$. Then for any $f \in L^\infty(G)$, we have $f\varphi \in C(G)$, and so $\lim_k \langle f\,,\, \varphi\psi_{n_k}\rangle = \lim_k \langle f\varphi\,,\, \psi_{n_k}\rangle = \langle \mu\,,\, f\varphi\rangle = \langle f\,,\, \varphi\mu\rangle$, which shows that $\varphi\psi_{n_k} \to \varphi\mu$ weakly in $L^1(G)$. This proves the claim; a similar argument shows that $L^1(G)$ is a left ideal.

For the converse, assume that $L^1(G)$ is a right ideal. Let $m \in L^1(G)^{**}$ be an annihilator of $C_0(G)$, E be a right identity for $L^1(G)^{**}$ with $\|E\| = 1$, and $\{\varphi_\alpha\}$ be a net in $L^1(G)$ such that $\varphi_\alpha \xrightarrow{w^*} E$. From the assumption, $\varphi_\alpha m \in L^1(G)$. If $f \in C_0(G)$, then $f\varphi_\alpha \in C_0(G)$, from Theorem 1.5.3; so that $0 = \langle m\,,\, f\varphi_\alpha\rangle = \langle \varphi_\alpha m\,,\, f\rangle$. Therefore $\varphi_\alpha m$ is in $L^1(G)$ and it annihilates $C_0(G)$; whence $\varphi_\alpha m = 0$. Then $Em = \lim_\alpha \varphi_\alpha m = 0$. By Theorem 1.5.4, the restriction of m to $LUC(G)$ is also zero. We have proved that any annihilator of $C_0(G)$ is an annihilator of $LUC(G)$; therefore by Hahn-Banach Theorem we have $C_0(G) = LUC(G)$, and from there G must be compact, since $LUC(G)$ contains the constant function 1. $\square$

In section 1.3, for a given Banach algebra A, we defined the topological centre of A^{**}.

The following theorem is due to A.T. Lau and V. Losert and it characterizes the topological centre of $L^1(G)^{**}$. As usual, the second Arens product of $m,n \in L^1(G)^{**}$ is denoted by $m{\cdot}n$.

2.1.2 Theorem. *Let G be a locally compact group. The following are equivalent:*

(a) *m is in the topological centre of $L^1(G)^{**}$.*

(b) *The map $n \mapsto mn$ is weak-star continuous on the norm bounded subsets of $L^1(G)^{**}$.*

(c) *For all $n \in L^1(G)^{**}$, we have $mn = m{\cdot}n$.*

(d) *$m \in L^1(G)$.*

For a proof see [42; Theorem 1]. The following corollary was also announced in [42].

2.1.3 Corollary. *Let G be a locally compact group. Then*

(a) *The algebraic centres of $L^1(G)$ and $L^1(G)^{**}$ coincide.*

(b) *If G is abelian, the algebraic centre of $L^1(G)^{**}$ is equal to $L^1(G)$.*

Proof. (a) Let φ be in the algebraic centre of $L^1(G)$. Then the mappings $\rho_\varphi, \lambda_\varphi : A \to A$ defined by $\rho_\varphi(\psi) = \psi\varphi$ and $\lambda_\varphi(\psi) = \varphi\psi$ are identical; so are their second adjoints. As we have shown already, $\rho_\varphi^{**}(m) = m\varphi$ and similarly $\lambda_\varphi^{**}(m) = \varphi m$, for all $m \in L^1(G)^{**}$. Hence φ is in the algebraic centre of $L^1(G)^{**}$. Now conversely, if n is in the algebraic centre of $L^1(G)^{**}$ and $m_\alpha \to m$ in the weak-star topology of $L^1(G)^{**}$, then $nm_\alpha = m_\alpha n \to mn = nm$; so that n is in the topological centre of $L^1(G)^{**}$, namely in $L^1(G)$. Then, n is in the algebraic centre of $L^1(G)$ too, as required.

(b) If G is abelian, then from [33; Theorem 20.24] $L^1(G)$ is abelian. By part (a), then the algebraic centre of $L^1(G)^{**}$ is equal to $L^1(G)$. □

2.1.4 Corollary. *If G and H are locally compact abelian groups and θ is any algebra isomorphism from $L^1(G)^{**}$ onto $L^1(H)^{**}$, then $\theta(L^1(G)) = L^1(H)$.*

Proof. This follows from part (b) of the above corollary.

An analogue of the above corollary was proved by F. Ghahramani and A.T. Lau, in which G and H are compact groups and θ is assumed to be continuous [24; Theorem 2.5]. In fact, in a more general setting, they proved:

2.1.5 Proposition. *Let A and B be Banach algebras that are two-sided ideals in their second duals and each has a bounded approximate identity. Then any continuous algebra isomorphism from A^{**} onto B^{**} maps A onto B.*

See [24; Lemma 2.2].

We can drop the assumption of continuity from [24; Theorem 2.5], when dealing with compact groups. First we need two results the first of which due to C.A. Akemann [1; Theorem 4].

2.1.6 Theorem. *Let G be compact and let $T : L^1(G) \to L^1(G)$ be a bounded linear operator which commutes with right translations. Then the following are equivalent:*

(1) *T is a compact operator;*

(2) *T is a weakly compact operator;*

(3) *There exists $f \in L^1(G)$ such that for any $\varphi \in L^1(G)$, $T(\varphi) = f * \varphi$.*

2.1.7 Proposition. *Let G and H be compact groups and θ be any algebra isomorphism from $M(G)$ onto $M(H)$. Then $\theta\big(L^1(G)\big) = L^1(H)$.*

Proof. By Theorem 1.1.4 and Proposition 1.5.2, θ is continuous. If $\varphi, \psi \in L^1(G)$ and$\{\mu_n\}$ is a bounded sequence in $M(G)$, then $\{\psi * \mu_n\}$ is a bounded sequence in $L^1(G)$, accordingly $\{\varphi*\psi*\mu_n\}$ has a convergent subsequence by the above Theorem. This shows that the operator $\rho : \mu \longrightarrow \varphi * \psi * \mu$, on $M(G)$, is compact. But then the operator $\nu \mapsto \theta(\varphi * \psi) * \nu$ on $M(H)$ which is the composition of θ, ρ and θ^{-1}, is compact. The restriction of this map to $L^1(H)$ is compact and we are led to existence of an $f \in L^1(H)$ with $\theta(\varphi * \psi) * \eta = f * \eta$, for all $\eta \in L^1(H)$, by the above Theorem 2.1.6. If η runs through a bounded approximate identity in $L^1(H)$, then we get $\theta(\varphi * \psi) = f \in L^1(H)$. So $\theta\big(L^1(G) * L^1(G)\big) \subseteq L^1(H)$, and so $\theta\big(L^1(G)\big) \subseteq L^1(H)$ by Theorem 1.2.3, and eventually $\theta\big(L^1(G)\big) = L^1(H)$, by symmetry. □

Based on the results of [22] we have:

2.1.8 Proposition. *A locally compact group K is amenable if and only if $LUC(K)^*$ has a left ideal of dimension* 1.

Proof. If K is amenable and n is a left invariant mean on $LUC(K)$, then for any $m \in LUC(K)^*$ we have that $mn = \langle m\,,\, \mathbf{1}\rangle n$, where $\mathbf{1}$ is the constant function 1. In fact, for any $f \in LUC(K)$ and $x \in K$, $(nf)(x) = \langle n\,,\, l_x f\rangle = \langle n\,,\, f\rangle$; so that $\langle mn\,,\, f\rangle = \langle m\,,\, nf\rangle = \langle n\,,\, f\rangle\langle m\,,\, \mathbf{1}\rangle$, which implies that $mn = \langle m\,,\, \mathbf{1}\rangle n$. This shows that the one-dimensional subspace generated by n is a left ideal.

Conversely, if J is a one-dimensional left ideal generated by n, then for every

$x \in K$ there exists a complex number $\chi(x)$ such that $\delta_x n = \chi(x)n$. Part (a) of Proposition 1.6.7 shows that χ is a continuous character of K. The equality $\delta_x n = \chi(x)n$ indicates that n is χ-invariant (see section 1.6). and an application of Proposition 1.6.7 provides a left invariant mean on $LUC(K)$.

2.1.9 Proposition. *Let G, H be locally compact groups and let θ be an algebra isomorphism from $L^1(G)^{**}$ onto $L^1(H)^{**}$.*

(a) *If G is amenable, then H is amenable.*

(b) *If G is compact, then H is compact and $\theta\big(L^1(G)\big) = L^1(H)$.*

(c) *If G is abelian, the H is amenable and SIN.*

Proof. Before proceeding with the proofs of (a) and (b), let E and F be right identities of $L^1(G)^{**}$ and $L^1(H)^{**}$, respectively, with $\|E\| = \|F\| = 1$. By Theorem 1.5.4, there exists an isometric algebra isomorphism from $EL^1(G)^{**}$ onto $LUC(G)^*$ induced by restriction. Since $\theta(E)$ and F are right identities of $L^1(H)^{**}$, the algebras $\theta(E)L^1(H)^{**}$ and $FL^1(H)^{**}$ are isomorphic via the map $\theta(E)m \overset{\gamma}{\mapsto} Fm$ $(m \in L^1(H)^{**})$. Combining these isomorphisms with the isomorphism between $FL^1(H)^{**}$ and $LUC(H)^*$, we have an algebra isomorphism Γ from $LUC(G)^*$ onto $LUC(H)^*$.

(a) If G is amenable, then $LUC(G)^*$ has a left ideal I of dimension 1, by Proposition 2.1.8. Then $\Gamma(I)$ is a left ideal of $LUC(H)^*$ of dimension 1, so H is amenable.

(b) If G is compact, then $LUC(G)^* = M(G)$ is semisimple by Proposition 1.5.2. Hence the isomorphic image of $M(G)$ under Γ, namely $LUC(H)^*$, is semisimple. Since G is compact, the left Haar measure λ on G belongs to $M(G)$, and furthermore, it is left invariant; therefore G is amenable. By part (a), H is also amenable. We show that there exists only one left invariant mean on $LUC(H)$, which indicates that H is compact, by Theorem 1.6.9. So let n_1 and n_2 be two arbitrary left invariant means on $LUC(H)$. Then for any $m \in LUC(H)^*$, $m(n_1-n_2) = \langle m\,,\,\mathbf{1}\rangle(n_1-n_2)$ by the left invariance of $n_1 - n_2$. Especially, $(n_1 - n_2)^2 = 0$; hence the spectral radius of $n_1 - n_2$ is zero. From $m(n_1 - n_2) = \langle m\,,\,\mathbf{1}\rangle(n_1 - n_2)$, the spectral radius of $m(n_1 - n_2)$ is zero as well; accordingly, $n_1 - n_2$ is in the radical of $LUC(H)^*$, by Theorem 1.1.3; so that $n_1 = n_2$, as desired.

Now we show that $\theta\big(L^1(G)\big) = L^1(H)$: In Theorem 1.5.4, let π_G be the epimorphism from $L^1(G)^{**}$ onto $M(G)$. The isomorphism θ maps $\ker \pi_G$, the space of right annihilators of $L^1(G)^{**}$, onto $\ker \pi_H$. Consider the induced map $\tilde{\theta}$: $L^1(G)^{**}/\ker \pi_G \to L^1(H)^{**}/\ker \pi_H$. Therefore, $\tilde{\pi}_H\circ\tilde{\theta}\circ\tilde{\pi}_G^{-1}$ is an isomorphism from $M(G)$ onto $M(H)$, in which $\tilde{\pi}_G$ is the induced isomorphism $\tilde{\pi}_G$: $L^1(G)^{**}/\ker \pi_G \to M(G)$. By Proposition 2.1.7, $\big(\tilde{\pi}_H \circ \tilde{\theta} \circ \tilde{\pi}_G^{-1}\big)\big(L^1(G)\big) = L^1(H)$. For any $\phi_1, \phi_2 \in L^1(G)$, we let $\big(\tilde{\pi}_H\circ\tilde{\theta}\circ\tilde{\pi}_G^{-1}\big)(\phi_i) = \psi_i \in L^1(H)$, $i = 1, 2$. Then the restriction of $\theta(\phi_i)$ to $C(H)$ is ψ_i. Now we apply a method used in the proof of [24; Lemma 2.2]: The difference $\theta(\phi_i) - \psi_i$ annihilates $C(H)$, so $\theta(\phi_i) = \psi_i + r_i$ for some right annihilator r_i in the algebra $L^1(H)^{**}$. But for any $f \in L^\infty(H)$, both $f\psi_1$ and $\psi_2 f$ belong to $C(H)$, so that $\langle r_2\,,\,f\psi_1\rangle = 0 = \langle r_1\,,\,\psi_2 f\rangle$, which implies that $\psi_1 r_2 = r_1\psi_2 = 0$, and from there, $\theta(\phi_1 * \phi_2) = \psi_1 * \psi_2 \in L^1(H)$. Now an application of Theorem 1.2.3 finishes the proof.

(c) H is amenable because of part (a) and the fact that any abelian group is amenable [50; Proposition 12.2]. To complete the proof we need to show that

$LUC(H) = RUC(H)$, by Theorem 1.6.4. If for example we had $LUC(H) \not\subseteq RUC(H)$, then we could find an annihilator m of $RUC(H)$ in $L^1(H)^{**}$ and $f \in LUC(H)$ such that $\langle m, f\rangle \neq 0$, by Hahn-Banach Theorem. Write $f = g\psi$ for some $g \in L^\infty(H)$ and $\psi \in L^1(H)$. Then $\langle \psi m, g\rangle = \langle m, g\psi\rangle \neq 0$ while for any $\varphi \in L^1(H)$ and $h \in L^\infty(H)$, $\langle m\varphi, h\rangle = \langle m, \varphi h\rangle = 0$, because $\varphi h \in RUC(H)$. So $\psi m \neq 0$ and

$$(*) \qquad m\varphi = 0 \text{ for all } \varphi \in L^1(H).$$

Let $n = \theta^{-1}(m)$. Then for any $\xi \in L^1(G)$, we have $\theta(\xi) \in L^1(H)$, by an application of Corollary 2.1.3. Then from $(*)$ we have $m\theta(\xi) = 0$; so that $\xi n = n\xi = \theta^{-1}\big(m\theta(\xi)\big) = 0$. This indicates that n is a right annihilator of the algebra $L^1(G)^{**}$, consequently m is a right annihilator of $L^1(H)^{**}$, a result that contradicts $\psi m \neq 0$. Therefore H is a SIN group.

Following the results of Corollary 2.1.4 and [24; Theorem 2.5] , F. Ghahramani and A.T. Lau have raised the question of whether for any locally compact groups G and H and any continuous algebra isomorphism $\theta : L^1(G)^{**} \overset{\text{onto}}{\to} L^1(H)^{**}$ we would necessarily have $\theta\big(L^1(G)\big) = L^1(H)$ (see [24]). The main objective of this chapter is to provide a negative answer to this question (see Theorem 2.2.7 below).

For brevity, let us denote by $\mathfrak{A}$ the class of all locally compact groups G for which every continuous algebra automorphism $T : L^1(G)^{**} \to L^1(G)^{**}$ maps $L^1(G)$ onto $L^1(G)$.

2.2 Characterization of the class $\mathfrak{A}$

By Corollary 2.1.4 and Proposition 2.1.9 part (b) , it is evident that every compact group and every locally compact abelian group belongs to $\mathfrak{A}$. In this section we show that any element of $\mathfrak{A}$ is either abelian or compact. We first prove some lemmas.

2.2.1 Lemma. *Let G and H be locally compact groups and T be an algebra isomorphism from $LUC(G)^*$ onto $LUC(H)^*$. If $T\big(L^1(G)\big) = L^1(H)$, then $T\big(M(G)\big) = M(H)$.*

Proof. Assume that $T\big(L^1(G)\big) = L^1(H)$. Then if $\mu \in M(G)$, we have $T(\mu) = \nu + m$, for some $\nu \in M(H)$ and $m \in C_0(H)^\perp$, by Theorem 1.5.5. From the assumption, for any $\varphi \in L^1(H)$, $T^{-1}(\varphi) \in L^1(G)$, so that $T^{-1}(\varphi) * \mu \in L^1(G)$. Therefore, $\varphi * \nu + \varphi m = \varphi T(\mu) = T\big(T^{-1}(\varphi) * \mu\big) \in L^1(H)$. Hence φm is in both $M(H)$ and $C_0(H)^\perp$, and $\varphi m = 0$. Now let $\{\varphi_\alpha\}$ be a net in $L^1(H)$ such that $\varphi_\alpha \to \delta_e$ in the weak-star topology of $LUC(H)^*$, where e is the identity element of H. Then $m = \delta_e m = \lim_\alpha \varphi_\alpha m = 0$, showing that $T\big(M(G)\big) \subseteq M(H)$, and the proof is complete by symmetry. $\square$

Note that for a locally compact group G and elements $a \in G$, $f \in LUC(G)$ and $m \in LUC(G)^*$, we have that for any $x \in G$, $(\delta_a f)(x) = \langle \delta_a\,, l_x f\rangle = (l_x f)(a) = f(xa) = (r_a f)(x)$, which indicates that $\delta_a f = r_a f$, and furthermore, $\langle \delta_a m\,, f\rangle = \langle \delta_a\,, mf\rangle = (mf)(a) = \langle m\,, l_a f\rangle$. So

$$\langle \delta_a m\,, f\rangle = \langle m\,, l_a f\rangle,$$

and

$$\langle m\delta_a\,, f\rangle = \langle m\,, r_a f\rangle.$$

We use these facts as we go on.

2.2.2 Lemma. *Suppose that there are $f \in LUC(G)$ and $a \in G$ such that $l_a f - r_a f \notin C_0(G)$. Then $G \notin \mathfrak{A}$.*

Proof. Choose $m \in LUC(G)^*$ such that $m \in C_0(G)^\perp$, $\|m\| < 1$, and $\langle m, l_a f - r_a f \rangle \neq 0$, by Hahn-Banach Theorem. Then $\langle \delta_a m - m\delta_a , f \rangle = \langle m, l_a f - r_a f \rangle \neq 0$, which shows that $\delta_a m \neq m\delta_a$. Let E be a right identity for $L^1(G)^{**}$ with $\|E\| = 1$, and let π be the isometric isomorphism from $EL^1(G)^{**}$ onto $LUC(G)^*$, as declared in Theorem 1.5.4. Choose $n \in L^1(G)^{**}$ with $\pi(En) = m$. Then $\|En\| < 1$. Now adjoin an identity 1 to the Banach algebra $L^1(G)^{**}$, as explained in the preliminary chapter, and call the resulting space B. Define $T : B \to B$ by $T(F) = (1 - En)^{-1}F(1 - En)$. Then T maps $L^1(G)^{**}$ onto itself, since $L^1(G)^{**}$ is an ideal of B. Note that $T(E)$ is a right identity for $L^1(G)^{**}$, hence from $(1 - En)T(E) = E(1 - En)$ we will have $T(E) = E$. Accordingly, T restricted to $EL^1(G)^{**}$ is an automorphism of $EL^1(G)^{**}$. If we assume that $T(L^1(G)) = L^1(G)$, then from Lemma 2.2.1 we have $(\pi \circ T \circ \pi^{-1})(\delta_a) = \mu$, for some $\mu \in M(G)$, and therefore $\mu - m\mu = \delta_a - \delta_a m$. Then $\mu = \delta_a$, since $M(G) \cap C_0(G)^\perp = (0)$, and we are led to $m\delta_a = \delta_a m$, which is contradictory. Whence $T(L^1(G)) \neq L^1(G)$. $\square$

2.2.3 Lemma. *Let G be a non-compact non-abelian discrete group. Then $G \notin \mathfrak{A}$.*

Proof. Let $Z(G)$ denote the centre of the group G. Choose $a \in G$ but not in $Z(G)$. Let $H = \{x \in G : xa = ax\}$. Then H is a subgroup of G such that $H \neq G$, because of $a \notin Z(G)$. Since G is infinite, either every coset of H is infinite or there are infinitely many cosets of H in G. This implies that $G \setminus H$, the complement of H in G, is infinite, since $G \neq H$. Therefore, for any finite subset K of G, we have that $G \neq K \cup H$.

Now let $x_1 \in G \setminus H$, and by induction suppose that $x_1, \dots, x_n$ have been chosen such that $\{ax_1, \dots, ax_n\} \cap \{x_1 a, \dots, x_n a\} = \emptyset$. Now let K be the finite set

$$\{x_1, \dots, x_n\} \cup \{ax_1 a^{-1}, \dots, ax_n a^{-1}\} \cup \{a^{-1}x_1 a, \dots, a^{-1}x_n a\}.$$

Then we pick $x_{n+1} \in G \setminus (K \cup H)$. So $x_1, \dots, x_n, x_{n+1}$ are distinct and furthermore $\{ax_n, \dots, ax_{n+1}\} \cap \{x_1 a, \dots, x_{n+1} a\} = \emptyset$. Thus, we have found a sequence $\{x_1, x_2, \dots\}$ such that

$$\{ax_1, ax_2, \dots\} \cap \{x_1 a, x_2 a, \dots\} = \emptyset.$$

Now let f be the characteristic function of the set $D = \{ax_1, ax_2, \dots\}$. Then $l_a f - r_a f \notin C_0(G)$, since for any finite set $L \subseteq G$ there exists an index n with $x_n \notin L$ and $(l_a f - r_a f)(x_n) = 1$. Now by Lemma 2.2.2 the proof is complete by noting that since G is discrete, any bounded complex valued function, especially f, belongs to $LUC(G)$. $\square$

2.2.4 Lemma. *Let N be a closed normal subgroup of a given locally compact group G, $\tilde{G} = G/N$, and let $\pi : G \to \tilde{G}$ be the quotient map. If $F \in LUC(\tilde{G})$, $\pi(a), \pi(b) \in \tilde{G}$ are such that $l_{\pi(a)}F - r_{\pi(b)}F \notin C_0(\tilde{G})$, then there exists $f \in LUC(G)$ with $l_a f - r_b f \notin C_0(G)$.*

Proof. Let f be the composition $F \circ \pi$. Then $f \in C(G)$. If (x_α) is a net in G which tends to the identity element of G, then $\pi(x_\alpha)$ tends to the identity of

$\tilde{G}$ and so $\|l_{x_\alpha} f - f\|_u = \|l_{\pi(x_\alpha)} F - F\|_u \to 0$, which shows that $f \in LUC(G)$. Since $l_{\pi(a)}F - r_{\pi(b)}F \notin C_0(\tilde{G})$, there exists $\epsilon > 0$ such that for any compact set $\tilde{K} \subseteq \tilde{G}$, we can find $\pi(x) \notin \tilde{K}$ with $\left|\left(l_{\pi(a)}F - r_{\pi(b)}F\right)(\pi(x))\right| \geqslant \epsilon$. Now for any compact set $K \subseteq G$, we choose a point $\pi(x) \notin \pi(K)$ as above. Then $x \notin K$ and $|(l_a f - r_b f)(x)| \geqslant \epsilon$, which shows that $l_a f - r_b f \notin C_0(G)$. $\square$

2.2.5 Proposition. *Let H be a non-compact non-abelian SIN group. Then $H \notin \mathfrak{A}$.*

Proof. Choose $a, b \in H$ such that $ab \neq ba$ and let U be a neighbourhood of e_H with $aba^{-1}b^{-1} \notin U$. In Theorem 1.6.6, let $\mathfrak{B} = \Im(G)$ (the group of inner automorphisms of G), and accordingly choose a compact normal subgroup N of H such that $N \subseteq U$ and H/N is a Lie group. Let G be the group H/N. Then G is a SIN group since a continuous homomorphic image of a SIN group is SIN. By Theorem 1.4.2, G is locally compact but not a compact group. Finally, since $aba^{-1}b^{-1} \notin N$, G is non-abelian. Now we may consider the following cases (recall that G_0 is the connected component of G):

Case I. *G/G_0 is compact:*

Then $AP(G)$ separates the points of G, by Theorem 1.6.5 (also see the details after Definition 1.6.2). So, pick $c, d \in G$ with $cd \neq dc$ and $g \in AP(G)$ with $g(cd) \neq g(dc)$. Since G is non-compact, we have $AP(G) \cap C_0(G) = \{0\}$ (Proposition 1.4.4), therefore $l_c g - r_c g \notin C_0(G)$, as $0 \neq l_c g - r_c g \in AP(G)$ by Theorem 1.4.5. The same theorem implies that $g \in LUC(G)$, and by Lemma 2.2.4, there exist $f \in LUC(H)$ and $a \in H$ such that $l_a f - r_a f \notin C_0(H)$; and the assertion is proved in this case, by Lemma

2.2.2.

Case II. *G/G_0 is abelian but not compact:*

We show that there exist $n \in C_0(H)^{\perp}$ and $\alpha \in H$ with $\delta_\alpha n \neq n\delta_\alpha$, and then Lemma 2.2.2 will complete the proof in this case: There exist $r, s \in G$ with $rsr^{-1}s^{-1} \in G_0$ and $rsr^{-1}s^{-1} \neq e$, since G/G_0 is abelian but G is not, where e denotes the identity of G. Choose $h \in C_0(G_0)$ with $h(e) \neq h(rsr^{-1}s^{-1})$, and let $\{G_0x : x \in \omega\}$ be the collection of all disjoint cosets of G_0 in G. Any coset of G_0 is uniquely in the form G_0x, for some $x \in \omega$, and hence we can define a function k on G by $k(tx) = h(t)$, for all $t \in G_0$ and $x \in \omega$. Let (y_α) be any net in G converging to $y \in G$, and choose a coset G_0x, $x \in \omega$, containing y. Since G_0x is open, there exists α_0 such that $\alpha \geqslant \alpha_0$ implies $y_\alpha \in G_0x$. So $k(y_\alpha) = h(y_\alpha x^{-1}) \to h(yx^{-1}) = k(y)$, showing that k is continuous on G. For any compact subset K of G, choose $x_1, \ldots, x_n \in \omega$ such that $K \subseteq \bigcup_{i=1}^{n} G_0x_i$, and let $\epsilon = |h(rsr^{-1}s^{-1}) - h(e)|$. Then for $x \in \omega \setminus \{x_1, \ldots, x_n\}$ we have $x \notin K$ and

$$\left|\left(l_{rsr^{-1}s^{-1}}k - k\right)(x)\right| = |h(rsr^{-1}s^{-1}) - h(e)| = \epsilon,$$

so that $l_{rsr^{-1}s^{-1}}k - k \notin C_0(G)$. Next we show that $k \in LUC(G)$: Since $h \in C_0(G_0)$, there exists a neighbourhood $V \subset G_0$ of the identity such that $|h(xt) - h(t)| < \epsilon$, for $x \in V$ and $t \in G_0$. But then for $t \in G_0$, $x \in V$ and $y \in \omega$, we have that $xt \in VG_0 \subseteq G_0$ and $|k(xty) - k(ty)| = |h(xt) - h(t)| < \epsilon$. Hence $k \in LUC(G)$, as we wanted. From Lemma 2.2.4, there exist α, $\beta \in H$ and $f \in LUC(H)$ such that $l_{\alpha\beta\alpha^{-1}\beta^{-1}}f - f \notin C_0(H)$. Therefore, we may take $m \in C_0(H)^{\perp}$ such that $0 \neq \langle m, l_{\alpha\beta\alpha^{-1}\beta^{-1}}f - f\rangle = \langle \delta_{\alpha\beta\alpha^{-1}\beta^{-1}}m - m, f\rangle$. Hence $\delta_{\alpha\beta\alpha^{-1}\beta^{-1}}m \neq m$, and furthermore if $\delta_{\alpha^{-1}}m \neq m\delta_{\alpha^{-1}}$, then Lemma 2.2.2 would imply that $H \notin \mathfrak{A}$. So we may assume that $\delta_{\alpha^{-1}}m = m\delta_{\alpha^{-1}}$. Then from $\delta_{\alpha\beta\alpha^{-1}\beta^{-1}}m \neq m$ we get

$$\delta_{\alpha^{-1}}(\delta_{\beta^{-1}}m) = \delta_{\alpha^{-1}\beta^{-1}}m \neq \delta_{\beta^{-1}\alpha^{-1}}m = \delta_{\beta^{-1}}\delta_{\alpha^{-1}}m = (\delta_{\beta^{-1}}m)\delta_{\alpha^{-1}}.$$

By letting $n = \delta_{\beta^{-1}}m$, we have $\delta_{\alpha^{-1}}n \neq n\delta_{\alpha^{-1}}$ and $n \in C_0(H)^{\perp}$. Now an application of Lemma 2.2.2 reveals that $H \notin \mathfrak{A}$.

Case III. *G/G_0 is non-abelian and non-compact:*

Since G_0 is open (recall that G is a Lie group), G/G_0 is a discrete group. The proof of Lemma 2.2.3 provides $g \in LUC(G/G_0)$ and $c \in G/G_0$ with $l_c g - r_c g \notin C_0(G/G_0)$. Two applications of Lemma 2.2.4 will then result in the existence of $f \in LUC(H)$ and $a \in H$ with $l_a f - r_a f \notin C_0(H)$, and the proof is complete. $\square$

2.2.6 Lemma. *If $G \in \mathfrak{A}$, then G is a SIN group.*

Proof. Let $G \in \mathfrak{A}$. As in the proof of Proposition 2.1.9 (c), we just need to show that any annihilator m of $RUC(G)$ in $L^1(G)^{**}$ is an annihilator of $LUC(G)$ as well. We may assume that $\|m\| < 1$. Let B be the Banach algebra obtained by adjoining an identity 1 to $L^1(G)^{**}$. Define an automorphism $T : B \to B$ by $T(n) = (1+m)^{-1}n(1+m)$. Since $L^1(G)^{**}$ is an ideal of B, T maps $L^1(G)^{**}$ onto itself. From $G \in \mathfrak{A}$, we have $T\big(L^1(G)\big) = L^1(G)$. Let $\varphi \in L^1(G)$ and $f \in L^\infty(G)$ be arbitrary. Then $T(\varphi) = \psi$ for some $\psi \in L^1(G)$. Since m is an annihilator of $RUC(G)$, we have $m\psi = 0$ (see the proof of Proposition 2.1.9 part (c)). From $T(\varphi) = \psi$, we have $\varphi(1+m) = (1+m)\psi$ and hence $\varphi m = \psi - \varphi \in L^1(G)$. For any $h \in C_0(G)$ we have $h\varphi \in C_0(G) \subseteq RUC(G)$, by Theorem 1.5.3, and therefore $0 = \langle m\,,\, h\varphi\rangle = \langle \varphi m\,,\, h\rangle$, implying that $\varphi m = 0$. Hence $\langle m\,,\, f\varphi\rangle = \langle \varphi m\,,\, f\rangle = 0$;

so that m is an annihilator of $LUC(G)$, since $f\varphi$ is a typical element of $LUC(G)$ by the Theorem 1.5.3. $\square$

2.2.7 Theorem. *A locally compact group G is in $\mathfrak{A}$ if and only if G is abelian or compact.*

Proof. If $G \in \mathfrak{A}$, then Proposition 2.2.5 and Lemma 2.2.6 show that G is either compact or abelian. We have also remarked earlier that compact groups and abelian locally compact groups are in $\mathfrak{A}$. $\square$

CHAPTER III

BIPOSITIVE ISOMORPHISMS BETWEEN THE SECOND DUALS OF GROUP ALGEBRAS

3.1 Discussion Hereafter, by continuity of a linear operator we mean continuity with respect to norm topology unless we specify otherwise. We also assume that isomorphisms are algebraic and surjective.

There have been a number of results on group algebras $L^1(G)$ and related Banach algebras, which tell what structural or topological properties of these algebras identify the locally compact group G. Amongst these results, which will be revealed in more detail later, are a theorem by Y. Kawada [39] on the bipositive algebra isomorphisms of the group algebras $L^1(G)$, indicating that the group G is identified via the order structure of $L^1(G)$. Following this result, J.G. Wendel proved in [59] that a norm-decreasing algebra isomorphism from $L^1(G_1)$ onto $L^1(G_2)$ results in a topological isomorphism between G_1 and G_2. The Wendel's result was independently proved for abelian groups G_1 and G_2 by H. Helson in [32], where he also proved that this assertion holds if $T : L^1(G_1) \to L^1(G_2)$ is an isomorphism of norm less than 2 for abelian groups G_1 and G_2 and if one of the dual groups $\hat{G}_1$ or $\hat{G}_2$ is connected. Later, A.T. Lau and K. McKennon proved in [43] that the Wendel's result still holds if $L^1(G_i)$ is replaced by X_i^*, for left introverted subspaces X_i of $C(G_i)$ containing $C_0(G_i)$, $i = 1, 2$. Based on this result, F. Ghahramani and A.T. Lau proved in [23] that X_i^* can be replaced by $L^1(G_i)^{**}$, $i = 1, 2$. The main objective of this chapter is to prove that the order structure of $L^1(G)^{**}$ identifies the topological group G. (See section 3.4 ahead).

3.2 C^*-algebras An involution on a complex algebra A is a mapping $a \mapsto a^*$ from A into itself with the following properties;

$$(\alpha a + b)^* = \bar{\alpha} a^* + b^*,$$

$$(ab)^* = b^* a^*,$$

$$(a^*)^* = a,$$

for all $a, b \in A$ and complex number α, where $\bar{\alpha}$ denotes the complex conjugate of α. A Banach algebra A is called a *C^*-algebra* if there exists an involution $*$ in A such that $\|aa^*\| = \|a\|^2$, for all $a \in A$. An algebra homomorphism ϕ from a C^*-algebra A with involution $*$ into a C^*-algebra B with involution $\natural$ is called a $*$-homomorphism if $\phi(a^*) = \phi(a)^\natural$, for all $a \in A$. Such a map ϕ is called a $*$-isomorphism if it is one-to-one and onto, and in this case A and B are said to be $*$-isomorphic. Examples of C^*-algebras are $L(H)$, the space of bounded linear operators on a Hilbert space H with adjoint operator taken as the involution, and $C(X)$, the space of continuous complex-valued functions on a compact Hausdorff space X, in which the involution is the complex conjugation.

3.2.1 Proposition. *If A is an commutative C^*-algebra with identity, then there exists a compact Hausdorff space X such that the C^*-algebras A and $C(X)$ are $*$-isomorphic.*

The reader may consult [53; Theorem 1.2.1] for a proof. The space X in this theorem is in fact the maximal ideal space of A, consisting of all non-zero multiplicative linear functionals on A topologized by the restriction of the weak-star topology of A^*.

An element a of a C^*-algebra A is called *normal* if $aa^* = a^*a$. For such an element, we have $\|a^2\|^2 = \|a^2(a^*)^2\| = \|(aa^*)^2\| = \|(aa^*)(aa^*)^*\| = \|aa^*\|^2 = \|a\|^4$, indicating that $\|a^2\| = \|a\|^2$. Then one may show by induction that $\|a^{2^n}\| = \|a\|^{2^n}$, and from there, by taking limit, the spectral radius of a equals $\|a\|$. This was used in the proof of Proposition 1.5.2.

An element a of a C^*-algebra A is called positive, and we write $a \geqslant 0$, if $\mathrm{sp}(a) \subseteq [0, +\infty)$ and $a^* = a$. A (complex) linear functional f on A is said to be positive if $f(a) \geqslant 0$ for all positive $a \in A$.

3.2.2 Proposition. *Let A be a C^*-algebra with identity $\mathbf{1}$ and let f be a linear functional on A.*

(a) *If f is bounded and $\|f\| = f(\mathbf{1})$, then f is positive.*

(b) *If f is positive, then f is bounded and $\|f\| = f(\mathbf{1})$.*

For a proof see [16; Propositions 2.1.4 and 2.1.9].

A C^*-algebra A is called a *von Neumann algebra* if it is the dual space of a Banach space X. It can be shown that X is unique; it is called the predual of A (see [53; Corollary 1.13.3]).

3.2.3 Proposition. *Let X be the predual of a von Neumann algebra. If B is a C^*-algebra, then any bounded linear map from B into X is weakly compact.*

For a proof see [2; Corollary II.9].

3.2.4 Theorem. *Let A be a C^*-algebra. Then*

(a) *A^{**} is a C^*-algebra, and A is Arens regular.*

(b) *Furthermore, if A is commutative, then A^{**} is commutative.*

Proof. (a) A proof of the fact that A is Arens regular can be found in [12, Theorem 7.1]. We can apply Proposition 3.2.3 to prove this part. The proof is probably known. One can even show that A is Arens regular in any algebra multiplication $\circledast$ that makes the norm of A a Banach algebra norm: For any $f \in A^*$ and $a \in A$, let $f \odot a$ denote the element of A^* defined by $\langle f \odot a\,,\, b\rangle = \langle f\,,\, a \circledast b\rangle$, for all $b \in A$. Then $a \mapsto f \odot a$ from A into A^* is a bounded linear map, therefore it is weakly compact by Proposition 3.2.3; Theorem 1.3.1 now shows that A is Arens regular with respect to $\circledast$.

(b) This part follows from part (a) and Proposition 1.3.2. $\square$

Theorem 3.2.4 can be used to find a radical Banach algebra which is at the same time Arens regular: Let A be $C[0\,,\,1]$, the space of complex-valued continuous functions on the interval $[0\,,\,1]$. With pointwise multiplication and the sup-norm $\|\,\|_u$, A is a C^*-algebra. Define a new product $*$ on A by

$$(f * g)(x) = \int_0^x f(x-t)g(t)dt \qquad (f, g \in A,\ x \in [0\,,\,1]).$$

One can easily show that $\|\,\|_u$ is also a Banach algebra norm with respect to $*$. For any $f \in A$, we have

$$\|\overbrace{f * \cdots * f}^{n \text{ times}}\|_u \leqslant \frac{1}{(n-1)!}\|f\|_u^n\,,$$

showing that the spectral radius of f is zero. Since $(A\,,\,*)$ is a commutative algebra, f is in the radical of A, in other words, A is a radical Banach algebra. By part (a) of the above theorem, A provides an example of a radical Banach algebra which is Arens regular. The above is a non-trivial example; of course we could let the multiplication $*$ be zero.

A one-to-one mapping from a C^*-algebra A onto a C^*-algebra B is said to be bipositive if $a \geqslant 0$ whenever $\phi(a) \geqslant 0$, and vice versa.

3.2.5 Theorem. *Let X and Y be compact Hausdorff spaces and let $\phi : C(X) \longrightarrow C(Y)$ be a bipositive linear isomorphism. If ϕ maps the algebra identity of $C(X)$ onto that of $C(Y)$, then ϕ is an algebra isomorphism and moreover there exists a homeomorphism $\tau : Y \to X$ such that $\phi(f) = f \circ \tau$, for all $f \in C(X)$; so that ϕ is an isometric isomorphism.*

For a proof see [38; Corollary 3.4.8].

3.3 Isometric Isomorphisms In this section we quote in detail some of the results mentioned earlier in section 3.1 to have everything ready when we prove the main results of this chapter in the next section.

3.3.1 Definition. Two topological groups G and H are said to be *topologically isomorphic* if there exists a group isomorphism $\tau : G \to H$ which is at the same time a homeomorphism between the topological spaces G and H.

3.3.2 Theorem. *Let X be a left introverted subspace of $C(G)$ containing $C_0(G)$, where G is a locally compact group. If $m \in X^*$ then the following are equivalent:*

(a) *m is invertible and $\|m\| = \|m^{-1}\| = 1$.*

(b) *There exist $x \in G$ and $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $m = \alpha\delta_x$.*

Proof. Assume (a). Using Theorem 1.5.5, write $m = \mu + r$ and $m^{-1} = \nu + s$, where μ, $\nu \in M(G)$ and r, $s \in C_0(G)^{\perp}$. Since $C_0(G)^{\perp}$ is an ideal in X^* and $\delta_e = \mu * \nu + (\mu s + r\nu + rs)$, we have that $\delta_e = \mu * \nu$, and similarly $\delta_e = \nu * \mu$; so that ν is the inverse of μ. But then $1 = \|\mu * \nu\| \leqslant \|\mu\|\|\nu\| \leqslant \|m\|\|m^{-1}\| = 1$, and therefore $\|\mu\| = \|\mu^{-1}\| = 1$, which shows that $r = 0$, since $\|m\| = \|\mu\| + \|r\|$. Now we show that μ is an extreme point of the closed unit ball in $M(G)$, and this completes the proof, since by a known result [14; chapter V, Theorem 8.4] the extreme points are of the form $\alpha\delta_x$, $|\alpha| = 1, x \in G$. To this end, let $\mu = a\mu_1 + (1-a)\mu_2$, where $0 \leqslant a \leqslant 1$, μ_1, $\mu_2 \in M(G)$, and $\|\mu_1\|$, $\|\mu_2\| \leqslant 1$. Then $\delta_e = a\mu^{-1}\mu_1 + (1-a)\mu^{-1}\mu_2$, and since δ_e is an extreme point, we have that $\mu^{-1}\mu_2 = \mu^{-1}\mu_1 = \delta_e$, which implies that $\mu = \mu_1 = \mu_2$, as required.

The proof of (b)$\Longrightarrow$(a) is obvious. $\square$

Let X be a left introverted subspace of $C(G)$, G a locally compact group. A net (m_α) in X^* is said to converge *strictly* to $m \in X^*$ if $\|m_\alpha\varphi - m\varphi\| \to 0$ for all $\varphi \in L^1(G)$. We must explain something here that is needed in the proof of the next lemma: Let X be a left introverted subspace of $C(G)$ containing $C_0(G)$, $\varphi \in L^1(G)$ and $g \in X$. The notation φg can have two meanings; one coming from the left action of $L^1(G)$ on $L^\infty(G)$ as defined in section 1.3, the other coming from the left action of X^* on X as defined in Definition 1.4.3 part (h). It is easily verified that these two interpretations are identical, by using the concrete representation of the elements of $L^1(G)L^\infty(G)$ which was described following Theorem 1.5.3.

3.3.3 Lemma. *Let X_i be a left introverted subspace of $C(G_i)$ containing $C_0(G_i)$,*

$i = 1, 2$. Let $\mu \in M(G_1)$, $\|\mu\| = 1$, and let $\{\mu_\alpha\}$ be a net in $M(G_1)$, bounded by 1, which converges strictly to $\mu \in X_1^*$. If $T : X_1^* \to X_2^*$ is an isometric isomorphism, then $T(\mu_\alpha) \to T(\mu)$ in the weak star topology of ${X_2}^*$.

Proof. Let n be a cluster point of the net $\{T(\mu_\alpha)\}$ in the weak-star topology of ${X_2}^*$. We must show that $n = T(\mu)$. Without loss of generality, we may assume that $T(\mu_\alpha) \xrightarrow{w^*} n$. If $\varphi \in L^1(G)$, then since $\mu_\alpha\varphi \to \mu\varphi$ and T is bounded,

$$T(\mu_\alpha)T(\varphi) \to T(\mu)T(\varphi). \tag{1}$$

From $T(\mu_\alpha) \xrightarrow{w^*} n$ and the fact that the product in ${X_2}^*$ is weak-star continuous in the left variable, we have

$$T(\mu_\alpha)T(\varphi) \xrightarrow{w^*} nT(\varphi). \tag{2}$$

From (1) and (2) we get $\big(T^{-1}(n)\big)\varphi = \mu\varphi$. Any $f \in C_0(G_1)$ can be written as φg for some $\varphi \in L^1(G_1)$ and $g \in C_0(G_1)$; thus $\langle T^{-1}(n)\,,\, f\rangle = \langle\big(T^{-1}(n)\big)\varphi\,,\, g\rangle = \langle \mu\varphi\,,\, g\rangle = \langle \mu\,,\, f\rangle$, indicating that the restriction of $T^{-1}(n)$ to $C_0(G_1)$ equals μ. Then we have $1 = \|\mu\| \leqslant \|T^{-1}(n)\| = \|n\|$. Since $T(\mu_\alpha) \xrightarrow{w^*} n$ and $\|T(\mu_\alpha)\| \leqslant 1$, $\|n\| \leqslant 1$; so $\|T^{-1}(n)\| = \|\mu\| = 1$. By decomposing $T^{-1}(n)$, using Theorem 1.5.5 and applying the norm property in that theorem combined with the equality just found, we must have $T^{-1}(n) = \mu$; equivalently $n = T(\mu)$, as we wished. □

3.3.4 Lemma. *Let L be a locally convex topological vector space and let $\{x_\alpha\}_{\alpha \in A}$ be a net in L converging weakly to some $x \in L$. Then there exists a net $\{y_\beta\}_{\beta \in B} \subseteq L$ with the following properties:*

(a) *$y_\beta \to x$ in the original topology of L.*

(b) *Each* y_β *is a convex combination of the elements in* $\{x_\alpha\}$.

(c) *For each* $\alpha_\circ \in A$ *there exists* $\beta_\circ \in B$ *such that for all* $\beta \geqslant \beta_\circ$, y_β *is a convex combination of the elements in* $\{x_\alpha\}_{\alpha \geqslant \alpha_\circ}$.

For a proof see [15; page 523]. In that article the net $\{y_\beta\}$ having the properties (b) and (c) of Lemma 3.3.4 is said to be *far out* in $\{x_\alpha\}$.

Remarks. **(a)** Let G_i and X_i be as in Lemma 3.3.3, $i = 1, 2$. Let $\tau : G_1 \to G_2$ be a topological isomorphism and ω be a continuous character of G_1. Then for any $\mu \in M(G_1)$ the map $f \mapsto \int_{G_1} \omega(x) f(\tau(x)) d\mu(x)$ defines an elment of $X_2{}^*$. This element of X_2^* is denoted by $U(\mu)$ in Lemma 3.3.5 ahead. In this way, $U : \mu \mapsto U(\mu)$ is a linear mapping from $M(G_1)$ into $X_2{}^*$.

(b) In the next lemma we need the known fact that if G is a locally compact group (or even a locally compact topological space) and $\mu \in M(G)$ is such that $\|\mu\| = 1$ and $\mu \geqslant 0$, then there exists a net $\{\mu_\alpha\}$, each μ_α a convex combination of point masses, such that $\mu_\alpha \to \mu$ in the weak-star topology of $C(G)^*$. If the assertion were not true, then from [52; page 30, corollary 1], there would exist a real-valued $f \in C(G)$ such that $\langle \mu , f \rangle$ does not belong to the closure of the set $\{tf(x) + (1-t)f(y) \ : \ 0 \leqslant t \leqslant 1 \text{ and } x, y \in G\}$. But this closure is the closed interval $[-\|f\|_u , \|f\|_u]$ which necessarily contains $\langle \mu , f \rangle$.

3.3.5 Lemma. *Let* G_i, X_i, $i = 1, 2$, *and* T *be as in* Lemma 3.3.3. *If* T *agrees with* U *(defined above) on point masses, then* $T\big(M(G_1)\big) \subseteq M(G_2)$.

Proof. In order to prove this lemma, it suffices to show that for each $\mu \in M(G_1)$ with $\mu \geqslant 0$ and $\|\mu\| = 1$, $T(\mu) \in M(G_2)$. For such a measure μ we may choose a net $\{\mu_\alpha\}$, each μ_α a convex combination of point masses, for which $\mu_\alpha \xrightarrow{w^*} \mu$ in $C(G_1)^*$, as explained above. Then by the fact that $L^1(G_1)L^\infty(G_1) \subseteq C(G_1)$, we have $\mu_\alpha\varphi \rightarrow \mu\varphi$ weakly in $L^1(G_1)$, for each fixed $\varphi \in L^1(G)$. So by Lemma 3.3.4, there exists a net $\{\nu_\beta\}$ in $M(G_1)$ far out in $\{\mu_\alpha\}$, such that

$$(1) \qquad \nu_\beta\varphi \longrightarrow \mu\varphi$$

in norm. For each $\epsilon > 0$ and $f \in C(G_1)$, there exists $\alpha_\circ$ such that $|\langle \mu_\alpha - \mu\,,\, f\rangle| < \epsilon$, for $\alpha \geqslant \alpha_\circ$. Choose $\beta_\circ$ corresponding to the net $\{\nu_\beta\}$ in Lemma 3.3.4. Then $|\langle \nu_\beta - \mu\,,\, f\rangle| < \epsilon$, for $\beta \geqslant \beta_\circ$, since for such β the measure ν_β is a convex combination of the elements in $\{\mu_\alpha\}_{\alpha \geqslant \alpha_\circ}$. So $\nu_\beta \xrightarrow{w^*} \mu$ in $C(G_1)^*$. Let n be a cluster point of $\{T(\mu_\alpha)\}$ with respect to the weak-star topology of $X_2{}^*$. Without loss of generality, we may assume $T(\mu_\alpha) \xrightarrow{w^*} n$. Then also

$$(2) \qquad T(\nu_\beta) \xrightarrow{w^*} n \quad \text{in } X_2{}^*.$$

Now if we repeat the argument in Lemma 3.3.3 by considering (1) and (2), we will have $\mu\varphi = T^{-1}(n)\varphi$. Then by the proof of Lemma 3.3.3, we come up with $T(\mu) = n$. Next we show that $n = U(\mu)$: From $\nu_\beta \xrightarrow{w^*} \mu$, we easily have

$$(3) \qquad U(\nu_\beta) \xrightarrow{w^*} U(\mu) \quad \text{in } X_2{}^*.$$

Since each ν_β is a linear combination of point masses, the equality $T(\nu_\beta) = U(\nu_\beta)$ holds, and this together with (2) and (3) imply that $n = U(\mu)$ on X_2. Let σ be the restriction of $U(\mu)$ to $C_0(G_2)$. Then from the definition of $U(\mu)$ it is clear that

$$(4) \qquad \|\sigma\| = 1.$$

On the other hand we have

$$(5) \qquad \|n\| = 1,$$

from $T(\mu) = n$, and then the norm property in Theorem 1.5.5 together with (4) and (5) result in the equality $n = \sigma \in M(G_2)$, as required. $\square$

Before we proceed with Theorem 3.3.7, we need the following known result; in a part of the proof we use the Mackey-Arens theorem on dual pairs, a method which is perhaps new in this application. Another way of proving part (c)$\Longrightarrow$(a) is explained in [27; pages 136 and 137].

3.3.6 Lemma. *Let G be a locally compact group and $\mu \in M(G)$. Then the following statements are equivalent:*

(a) $\mu \in L^1(G)$.

(b) *The map $x \mapsto \delta_x * \mu$ (equivalently, the map $x \mapsto \mu * \delta_x$) from G into $M(G)$ is norm continuous.*

(c) *The map $x \mapsto \delta_x * \mu$ (equivalently, the map $x \mapsto \mu * \delta_x$) from G into $M(G)$ is weakly continuous.*

Proof. (a)$\Longrightarrow$(b): Let $\mu \in L^1(G)$. In order to show that $x \mapsto \delta_x * \mu$ is continuous, we may assume $\mu \in C_{00}(G)$, since $C_{00}(G)$ is norm dense in $L^1(G)$. So let $x_\alpha \to x \in G$ and choose a compact neighbourhood V of x containing x_α, $\alpha \geqslant \alpha_\circ$. Let K be the support of the function $\mu \in C_{00}(G)$ and λ be the left Haar measure. Then for $s \in V$, $\mu(s^{-1}t)$ is non-zero only if $t \in VK$, which is a compact set (and has finite measure). So

$$\|\delta_{x_\alpha} * \mu - \delta_x * \mu\| = \|l_{x_\alpha^{-1}}\mu - l_{x^{-1}}\mu\| \leqslant \lambda(VK)\|l_{x_\alpha^{-1}}\mu - l_{x^{-1}}\mu\|_u,$$

for all $\alpha \geqslant \alpha_\circ$, where $\| \, \|_u$ denotes the sup-norm. But μ is uniformly continuous; so $\|\delta_{x_\alpha} * \mu - \delta_x * \mu\| \xrightarrow{\alpha} 0$.

(c)$\Longrightarrow$(a): Let (c) hold. Let $\mathcal{P}$ be the set of all compact neighbourhoods of the identity element e_G in G ordered by inclusion, and for each $W \in \mathcal{P}$ let φ_W be the characteristic function of W divided by $\lambda(W)$ (λ is the Haar measure). For each $m \in M(G)^*$, there exists a compact neighbourhood $\tilde{V}$ of e_G such that $|\langle m\,,\, \delta_x * \mu - \mu\rangle| < \epsilon$, for all $x \in \tilde{V}$. The set $\{\delta_x * \mu \,:\, x \in \tilde{V}\}$ is weakly compact in $M(G)$, since $x \mapsto \delta_x * \mu$ is weakly continuous. Let f_α be a net in $C_0(G)$ such that $f_\alpha \xrightarrow{w^*} m$ in $M(G)^*$. Then by Mackey-Arens theorem on dual pairs [52, Chapter 3, Theorem 7], we have $\langle f_\alpha\,,\, \delta_x * \mu\rangle \xrightarrow{\alpha} \langle m\,,\, \delta_x * \mu\rangle$ uniformly for $x \in \tilde{V}$, and so there exists $\alpha_\circ$ such that $|\langle \mu\,,\, l_t f_\alpha - f_\alpha\rangle| = |\langle \delta_t * \mu - \mu\,,\, f_\alpha\rangle| < 2\epsilon$, for all $t \in \tilde{V}$ and $\alpha \geqslant \alpha_\circ$. Then for $\alpha \geqslant \alpha_\circ$ and $\varphi = \varphi_W$ with $W \subseteq \tilde{V}$ we have

$$|\langle \varphi * \mu - \mu\,,\, f_\alpha\rangle| = \Big| \iint f_\alpha(tx)\varphi(t)d\mu(x)dt - \langle \mu\,,\, f_\alpha\rangle\Big| =$$

$$\Big|\int \langle \mu\,,\, l_t f_\alpha\rangle \varphi(t)dt - \langle \mu\,,\, f_\alpha\rangle\Big| = \Big|\int \langle \mu\,,\, l_t f_\alpha - f_\alpha\rangle\varphi(t)dt\Big| < 2\epsilon,$$

and by taking limit on α, $|\langle m\,,\, \varphi_W * \mu - \mu\rangle| \leqslant 2\epsilon$, when $W \subseteq \tilde{V}$. Hence $\varphi_W * \mu \to \mu$ weakly in $M(G)$, and by Lemma 3.3.4, μ is the norm limit of a convex combinations of the elements in $\{\varphi_W * \mu\}$. Then $\mu \in L^1(G)$, since $L^1(G)$ is a closed ideal of $M(G)$.

3.3.7 Theorem. *Let X_i be a left introverted subspace of $C(G_i)$ containing $C_0(G_i)$, $i = 1, 2$. If $T : {X_1}^* \to {X_2}^*$ is an isometric algebra isomorphism, then*

(a) *G_1 and G_2 are topologically isomorphic.*

(b) $T\big(M(G_1)\big) = M(G_2)$.

(c) $T\big(L^1(G_1)\big) = L^1(G_2)$.

Proof. **(a)** For each $x \in G_1$, both $T(\delta_x)$ and $T(\delta_{x^{-1}})$ have norm one and they are the inverses of one another. So by Theorem 3.3.2, $T(\delta_x) = \omega(x)\delta_{\tau(x)}$ for some $\tau(x) \in G_2$ and $\omega(x) \in \mathbb{C}$ with $|\omega(x)| = 1$. Then $x, y \in G_1$ imply that $\omega(x)\omega(y)\delta_{\tau(x)\tau(y)} = T(\delta_x)T(\delta_y) = T(\delta_{xy}) = \omega(xy)\delta_{\tau(xy)}$, from which $\omega(x)\omega(y) = \omega(xy)$ and $\tau(x)\tau(y) = \tau(xy)$, and therefore ω is a character of G_1 and τ is a group homomorphism from G_1 into G_2. Similarly, we get a group homomorphism $\tilde{\tau} : G_2 \to G_1$ and a character $\tilde{\omega}$ of G_2 with $T^{-1}(\delta_t) = \tilde{\omega}(t)\delta_{\tilde{\tau}(t)}$, for all $t \in G_2$. Combining T and T^{-1}, one may then observe that τ is one-to-one and onto, and furthermore $\tilde{\tau}$ is τ^{-1}. Next we show that τ is continuous: Let (x_α) be a net in G_1 such that $x_\alpha \to x \in G_1$. By Lemma 3.3.6, we have $\delta_{x_\alpha} * \varphi \to \delta_x * \varphi$ for all $\varphi \in L^1(G_1)$. Then by Lemma 3.3.3, $T(\delta_{x_\alpha})$ converges to $T(\delta_x)$ in the weak-star topology of X_2^*. Now let V be any open set containing x and let $f \in C_0(G_2)$ with $f(\tau(x)) = 1$ and $f(t) = 0$ for all $t \notin V$. Then one would have $|\langle T(\delta_{x_\alpha}), f\rangle| \to |\langle T(\delta_x), f\rangle|$, or equivalently, $|f(\tau(x_\alpha))| \to f(\tau(x)) = 1$, from there we must have $f(\tau(x_\alpha)) \neq 0$ from a point on . This indicates that $\tau(x_\alpha) \in V$ and since V was taken to be arbitrary, we have $\tau(x_\alpha) \to \tau(x)$ and τ is continuous. Similarly, τ^{-1} is continuous, yielding that G_1 and G_2 are topologically isomorphic. Further, note that the relations $\omega(x_\alpha)f(\tau(x_\alpha)) \to \omega(x)f(\tau(x))$, $\tau(x_\alpha) \to \tau(x)$ and $f(\tau(x)) \neq 0$ imply that $\omega(x_\alpha) \to \omega(x)$ and ω is continuous.

(b) Let U be the map from $M(G_1)$ into $X_2{}^*$ be defined by $\langle U(\mu), f\rangle = \int \omega(x)f(\tau(x))d\mu(x)$. Note that, by part (a), T and U agree on the point masses. Then by Lemma 3.3.5, we must have $T\big(M(G_1)\big) \subseteq M(G_2)$; whence $T\big(M(G_1)\big) = M(G_2)$ by symmetry.

(c) Let (x_α) be a net in G_1 with $x_\alpha \to x \in G_1$. For any $\psi \in L^1(G_2)$ we have $\delta_{\tau(x_\alpha)} * \psi \to \delta_{\tau(x)} * \psi$ by Lemma 3.3.6. So $T^{-1}(\delta_{\tau(x_\alpha)} * \psi) \to T^{-1}(\delta_{\tau(x)} * \psi)$ and $\delta_{x_\alpha} * T^{-1}\psi \to \delta_x * T^{-1}\psi$. Another application of Lemma 3.3.6 implies that $T^{-1}\psi \in L^1(G_1)$. So $T^{-1}\big(L^1(G_2)\big) \subseteq L^1(G_2)$, and the proof is complete by symmetry. $\square$

Note that this theorem fails to be true if we drop the assumption that X_i contain $C_0(G_i)$, $i = 1, 2$, [43]: In the preliminary chapter, we defined the almost periodic compactification G^a of a topological group G (see section 1.6 for definition and symbols used below). Now let G be a non-compact locally compact group. The fact that $\{L_x : x \in G\}$ is dense in G^a gives rise to an isometric linear isomorphism $\theta : C(G^a) \to AP(G)$, where to each $f \in C(G^a)$ we assign $g = \theta(f) \in AP(G)$ defined by $g(t) = f\big(L_{t^{-1}}\big)$, for all $t \in G$. Then the adjoint $\theta^* : AP(G)^* \to C(G^a)^*$ is an isometric linear isomorphism. But θ^* is in fact an algebra isomorphism: If $m, n \in AP(G)^*$ and f, g are as above, then for $t, s \in G$ we have $(l_t g)(s) = g(ts) = f(L_{(ts)^{-1}}) = f(L_{t^{-1}} L_{s^{-1}}) = l_{L_{t^{-1}}} f(L_{s^{-1}})$, indicating that $\theta\big(l_{L_{t^{-1}}} f\big) = l_t g$. Then $(ng)(t) = \langle n\,,\, l_t g\rangle = \langle n\,,\, \theta\big(l_{L_{t^{-1}}} f\big)\rangle = \langle \theta^*(n)\,,\, l_{L_{t^{-1}}} f\rangle = \big((\theta^*(n))f\big)(L_{t^{-1}})$, and so $ng = \theta\big((\theta^*(n))f\big)$, from which

$$\langle \theta^*(mn)\,,\, f\rangle = \langle mn\,,\, g\rangle = \langle m\,,\, ng\rangle = \langle m\,,\, \theta\big((\theta^*(n))f\big)\rangle = \langle \theta^*(m)\theta^*(n)\,,\, f\rangle,$$

so that

$$\theta^*(mn) = \theta^*(m)\theta^*(n),$$

as desired. Note that since G^a is compact, $C(G^a) = LUC(G^a)$ is left introverted.

3.3.8 Theorem. *In any of the following cases G and H are topologically isomorphic:*

(a) *There exists an isometric epimorphism $T : L^1(G) \to L^1(H)$.*

(b) *There exists an isometric epimorphism $T : L^1(G)^{**} \to L^1(H)^{**}$.*

Proof. In the case of part (a) we may consider the second adjoint T^{**} and that will be an isometric isomorphism from $L^1(G)^{**}$ onto $L^1(H)^{**}$. So it suffices to prove the assertion with the assumption in (b). Let E be a right identity of $L^1(G)^{**}$ with $\|E\| = 1$. Then $F = T(E)$ is a right identity for $L^1(H)^{**}$ and has norm one. Let $\pi_1 : EL^1(G)^{**} \to LUC(G)^*$ and $\pi_2 : FL^1(H)^{**} \to LUC(H)^*$ be the isometric isomorphisms induced by restriction. Then $\pi_2 \circ T \circ \pi_1^{-1}$ defines an isometric isomorphism from $LUC(G)^*$ onto $LUC(H)^*$ and an application of Theorem 3.3.7 part (a) ends the proof. □

The following theorem states that the Banach algebra $L^1(G)^{**}$ identifies the group algebra $L^1(G)$. It was announced in [26].

3.3.9 Theorem. *Let G and H be locally compact groups and T be an isometric algebra isomorphism from $L^1(G)^{**}$ onto $L^1(H)^{**}$. Then $T\big(L^1(G)\big) = L^1(H)$.*

Some Remarks Part (a) of Theorem 3.3.8 is due to J.G. Wendel [59], where he actually proves the existence of a continuous character ω on G, a topological isomorphism $\tau : G \to H$ and a positive constant c such that for all $\varphi \in L^1(G)$ and $t \in G$, $\big(T\varphi\big)\big(\tau(t)\big) = c\,\omega(t)\varphi(t)$.

In the proof of Lemma 3.3.5 we made a slight change to its original proof in [26]. Part (a) of Theorem 3.3.7 was proven by A.T. Lau and M. McKennon in [43]. Their proof was based on a result given in that article indicating that any $\mu \in M(G)$ has a unique norm preserving extension to a bounded linear functional on $C(G)$, a result which was used in [26] often. The implication $(b) \Longrightarrow (c)$ in Theorem 3.3.7 is due to B.E. Johnson [37]. Theorem 3.3.2 appeared in [26] for $LUC(G)$ instead of X, but their proof simply works for X as well; however we made a slightly different approach in that proof. Part (b) of Theorem 3.3.8 is due to F. Ghahramani and A.T. Lau [23].

3.4 Bipositive isomorphisms

3.4.1 Definition.

(a) A real linear space E is called an *ordered vector space* if it has an order relation $\leqslant$ (reflexive, antisymmetric, and transitive) such that for each pair $x, y \in E$ the following two properties hold:

(a1) $x \leqslant y$ implies $x + z \leqslant y + z$, for all $z \in E$.

(a2) $x \leqslant y$ implies $\alpha x \leqslant \alpha y$, for all $\alpha \geqslant 0$.

(b) An ordered vector space E is said to be a *Riesz space* if for each $x, y \in E$ the supremum and infimum of the set $\{x, y\}$ exist in E. The supremum and infimum of the set $\{x, y\}$ are respectively denoted by $x \vee y$ and $x \wedge y$.

(c) A Riesz space E is said to be a *normed Riesz space* if there exists a norm $\|\ \|$ on E such that $\|x\| \leqslant \|y\|$ holds for all $x, y \in E$ with $|x| \leqslant |y|$, in which $|x|$ denotes the element $x \vee 0 + (-x) \vee 0$.

(d) If the norm in part (c) is complete, then E is said to be a *Banach lattice.*

As an example of a Banach lattice we can mention the space of the real measures in $M(X)$, where X is a locally compact Hausdorff topological space. The supremum and infimum of any two real measures μ and ν are simply $\frac{1}{2}(\mu+\nu+|\mu-\nu|)$ and $\frac{1}{2}(\mu+\nu-|\mu-\nu|)$, where $|\cdot|$ denotes the total variation. Another example of a Banach lattice is the space of real-valued functions in $L^1(G)$, where G is a locally compact group. The supremum and infimum are the pointwise supremum and infimum.

Note that if L is any real linear space, then L can be embedded into the complex vector space $\tilde{L} = L \times L$, via $x \mapsto (x\,,\,0)$; the scalar product in $\tilde{L}$ is defined by

$$(\alpha + i\beta)(x\,,\,y) = (\alpha x - \beta y\,,\,\alpha y + \beta x),$$

for real numbers α and β. The space $\tilde{L}$ is called the complexification of L.

3.4.2 Definition. Let E and F be ordered vector spaces and $\tilde{E}, \tilde{F}$ be their complexifications, respectively.

(a) A (real) linear map $T : E \to F$ is called positive whenever $Tx \geqslant 0$ for all $x \in E$ with $x \geqslant 0$.

(b) A (complex) linear map $T : \tilde{E} \to \tilde{F}$ is said to be positive if the restriction of T to E maps into F and it is positive. If $T : \tilde{E} \to \tilde{F}$ is a linear isomorphism, then T is said to be bipositive if both T and T^{-1} are positive.

The following result is due to Y. Kawada [39].

3.4.3 Theorem. *Let $T : L^1(G) \to L^1(H)$ be a bipositive algebra isomorphism. Then G and H are topologically isomorphic.*

Following this result, J.G. Wendel showed that Kawada's result is equivalent to part (a) of Theorem 3.3.8, when dealing with the field of real numbers. More precisely, he showed in [58; Theorem 2 and Theorem 3] that if $P : L^1(G) \to L^1(H)$ is a bipositive isomorphism, then P is an isometry on the real-valued functions, and conversely if $T : L^1(G) \to L^1(H)$ is an isometric isomorphism, then there exists a character χ on H such that the map $P : L^1(G) \to L^1(H)$ defined by $(P\varphi)(t) = \chi(t)(T\varphi)(t)$ $\big(t \in H$ and $\varphi \in L^1(G)\big)$ is a bipositive isomorphism.

Before proving the next theorem, we need the following known lemma:

3.4.4 Lemma. *Let A and B be Banach algebras and $\pi : A \to A^{**}$ be the canonical embedding. Then*

(a) *If $T : A \to B$ is a continuous algebra homomorphism, then $T^{**} : A^{**} \to B^{**}$ is an algebra homomorphism.*

(b) *If A has an identity 1, then $\pi(1)$ is the identity of A^{**}.*

Proof. The proof of (a) can be found in [12; Theorem 6.1].

(b): From section 1.3, we know that any element of A is in the topological centre of A^{**}. If $m \in A^{**}$ and (x_α) is a net in A such that $\pi(x_\alpha) \xrightarrow{w^*} m$ in A^{**}, then $\pi(1)m = \lim_\alpha \pi(1)\pi(x_\alpha) = \lim_\alpha \pi(x_\alpha) = m$. Similarly, one can show that $m\pi(1) = m$.

3.4.5 Lemma. *Let G be a locally compact group. Then the constant function* **1** *in $L^\infty(G)$ is multiplicative on $L^1(G)$.*

Proof. Let f stand for **1**. For $\varphi, \psi \in L^1(G)$, we have

$$\langle f\,,\, \varphi * \psi\rangle = \iint \varphi(s)\psi(s^{-1}t)f(t)\,dt\,ds = \int \varphi(s)\,ds \int \psi(t)\,dt = \langle f\,,\,\varphi\rangle\langle f\,,\,\psi\rangle,$$

which shows that f is multiplicative. □

Our main goal in this chapter is to prove the following result which does not seem to be known already:

3.4.6 Theorem. *Let $T : L^1(G)^{**} \to L^1(H)^{**}$ be a bipositive isomorphism. Then T is an isometry and G and H are topologically isomorphic.*

Proof. We divide the proof into several steps:

Step 1. *T is continuous*:

By [16; Corollary 2.6.4], any $m \in L^1(G)^{**}$, as a linear functional on the C^*-algebra $L^\infty(G)$, has a decomposition $m = m_1 - m_2 + i(m_3 - m_4)$ with $m_i \geqslant 0$ and $\|m_i\| \leqslant \|m\|$. So we just need to show that the set $A = \{\|T(m)\| \,:\, m \geqslant 0,\ \|m\| \leqslant 1\}$ is bounded in the real line. If A were unbounded, then there would exist a sequence $\{m_i\}_{i=1}^\infty$ in $L^1(G)^{**}$ such that $m_i \geqslant 0$, $\|m_i\| \leqslant 1$, and $\|Tm_i\| \to \infty$; we could assume $\|Tm_i\| > i$. Let t_i be a number with $\frac{1}{i\|Tm_i\|} < t_i < \frac{1}{i^2}$. Then $\sum_{i=1}^\infty \|t_i m_i\| \leqslant \sum_{i=1}^\infty \frac{1}{i^2}$, so $m = \sum_{i=1}^\infty t_i m_i$ is a positive element of $L^1(G)^{**}$. If $\mathbf{1}$ denotes the constant function 1, then for each k we have $\sum_{i=1}^k t_i \|Tm_i\| = \sum_{i=1}^k t_i \langle Tm_i\,,\, \mathbf{1}\rangle = \langle T(\sum_{i=1}^k t_i m_i)\,,\, \mathbf{1}\rangle \leqslant \langle Tm\,,\, \mathbf{1}\rangle = \|Tm\|$. By letting $k \to \infty$, we come up with $\sum_{i=1}^\infty t_i\|Tm_i\| < \infty$ which is absurd by $\sum t_i\|Tm_i\| \geqslant \sum \frac{1}{i} = \infty$.

By step 1, the adjoint operator $T^* : L^1(H)^{***} \to L^1(G)^{***}$ can be defined. By Theorem 3.2.4 and Lemma 3.4.4 (b), the second dual of the commutative C^*-algebra $L^\infty(G)$ is a commutative C^*-algebra with identity and there are compact Hausdorff spaces Y and $\hat{Y}$ which yield the identifications $L^\infty(G) = C(Y)$ and $L^\infty(G)^{**} = C(\hat{Y})$, by Proposition 3.2.1. Similarly $L^\infty(H) = C(X)$ and $L^\infty(H)^{**} = C(\hat{X})$ for some compact Hausdorff spaces X and $\hat{X}$. For each $\mu \in M(Y)$ we denote by $\hat{\mu}$ the image of μ under the canonical embedding $M(Y) \to M(Y)^{**} = M(\hat{Y})$. A similar notation will be used for the case of X.

Step 2. *If $\mu \in M(Y)$ and $\mu \geqslant 0$, then $\hat{\mu} \geqslant 0$ in $M(\hat{Y})$:*

An application of Proposition 3.2.2 reveals that $\|\hat{\mu}\| = \|\mu\| = \langle \mu , \mathbf{1} \rangle = \langle \hat{\mu} , \mathbf{1} \rangle$, where $\mathbf{1}$ is the identity of $C(Y)$ (which is the identity of $C(Y)^{**}$ as well). So $\|\hat{\mu}\| = \langle \hat{\mu} , \mathbf{1} \rangle$ and $\hat{\mu} \geqslant 0$, by Proposition 3.2.2.

Step 3. *If $\nu \geqslant 0$ in $M(\hat{Y})$, then there exists a net (μ_i) in $M(Y)$ with $\mu_i \geqslant 0$ and $\hat{\mu}_i \to \nu$ in the weak-star topology of $M(\hat{Y})$. Furthermore we can arrange $\|\mu_i\| \leqslant \|\nu\|$:*

For any $f \in C(\hat{Y})$, choose a net $\{g_s\} \subseteq C(Y)$ with $\hat{g}_s \xrightarrow{w^*} f$ ($\hat{g}_s$ denotes the image of g_s under the canonical mapping $C(Y) \to C(\hat{Y})$). Then for any $\mu \geqslant 0$ in $M(Y)$, $\langle \hat{\bar{g}}_s , \mu \rangle = \langle \mu , \bar{g}_s \rangle = \overline{\langle \mu , g_s \rangle} = \overline{\langle \hat{g}_s , \mu \rangle} \to \overline{\langle f , \mu \rangle} = \overline{\langle \hat{\mu} , f \rangle} = \langle \hat{\mu} , \bar{f} \rangle = \langle \bar{f} , \mu \rangle$, in which bar denotes the complex conjugation, and we have made use of step 2. So $\hat{\bar{g}}_s \xrightarrow{w^*} \bar{f}$. Choose a net $\{\mu_i\}$ in $M(Y)$ with $\|\mu_i\| \leqslant \|\nu\|$ and $\hat{\mu}_i \xrightarrow{w^*} \nu$. For each $\mu \in M(Y)$ let $\mu^* \in M(Y)$ be defined by $\langle \mu^* , h \rangle = \overline{\int \overline{h(x)}\, d\mu(x)}$, for all $h \in C(Y)$. Then we get $\langle \widehat{\mu_i^*} , f \rangle = \langle f , \mu_i^* \rangle = \lim_s \langle \hat{g}_s , \mu_i^* \rangle = \lim_s \langle \mu_i^* , g_s \rangle = \lim_s \overline{\langle \mu_i , \bar{g}_s \rangle} = \lim_s \overline{\langle \hat{\bar{g}}_s , \mu_i \rangle} = \overline{\langle \bar{f} , \mu_i \rangle} = \overline{\langle \hat{\mu}_i , \bar{f} \rangle}$; so that $\widehat{\mu_i^*} = (\hat{\mu}_i)^*$. Since the map $\mu \mapsto \mu^*$ is clearly weak-star continuous on $M(\hat{Y})$, we find that $\frac{1}{2}\widehat{(\mu_i + \mu_i^*)}(f) = \frac{1}{2}\{\langle \hat{\mu}_i , f \rangle + \langle (\hat{\mu}_i)^* , f \rangle\} \to \frac{1}{2}\{\langle \nu , f \rangle + \langle \nu^* , f \rangle\} = \langle \nu , f \rangle$. By replacing μ_i by $\frac{1}{2}(\mu_i + \mu_i^*)$ we can assume that μ_i's are real measures. Now if $\mu_i = \mu_i^+ - \mu_i^-$ is the Jordan decomposition of μ_i, then $\|\mu_i\| = \|\mu_i^+\| + \|\mu_i^-\|$ and $\|\nu\| = \langle \nu , \mathbf{1} \rangle = \lim \langle \hat{\mu}_i , \mathbf{1} \rangle = \lim \langle \mu_i , \mathbf{1} \rangle = \lim(\langle \mu_i^+ , \mathbf{1} \rangle - \langle \mu_i^- , \mathbf{1} \rangle) \leqslant \overline{\lim} \langle \mu_i^+ , \mathbf{1} \rangle \leqslant \overline{\lim}\, \|\mu_i^+\| \leqslant \|\nu\|$, therefore $\langle \mu_i^- , \mathbf{1} \rangle \to 0$, $\|\mu_i^-\| \to 0$, and so $\widehat{\mu_i^+} \xrightarrow{w^*} \nu$, as desired.

Step 4. *If $f \geqslant 0$ in $C(\hat{X})$, then $T^*(f) \geqslant 0$ in $C(\hat{Y})$:*

For any $y \in \hat{Y}$, $\delta_y \geqslant 0$ in $M(\hat{Y})$, and there exists a net $\mu_i \geqslant 0$ in $M(Y)$ with $\hat{\mu}_i \xrightarrow{w^*} \delta_y$, by step 3. So $T^*(f)(y) = \langle \delta_y , T^*(f) \rangle = \lim \langle \hat{\mu}_i , T^*(f) \rangle = \lim \langle T^*(f) , \mu_i \rangle = \lim \langle \widehat{T(\mu_i)} , f \rangle \geqslant 0$, by step 2. Hence $T^*(f)(y) \geqslant 0$ for all $y \in \hat{Y}$, that is $T^*(f) \geqslant 0$.

Step 5. *If $\mu \geqslant 0$ in $M(\hat{Y})$, then $T^{**}(\mu) \geqslant 0$ in $M(\hat{X})$:*

For $f \geqslant 0$ in $C(\hat{X})$, we have $\langle T^{**}(\mu) , f \rangle = \langle \mu , T^*(f) \rangle \geqslant 0$, by step 4; therefore $T^{**}(\mu) \geqslant 0$.

Step 6. *If $1_{\hat{X}}$ denotes the constant function 1 on $\hat{X}$, then $T^*(1_{\hat{X}})$ is positive everywhere as a function in $C(\hat{Y})$.* Moreover, *$T^*(1_{\hat{X}})$ is a multiplicative linear functional on $L^1(G)^{**}$:*

Indeed for all $y \in \hat{Y}$, by step 5, we have $T^*(1_{\hat{X}})(y) = \langle \delta_y , T^*(1_{\hat{X}}) \rangle = \langle T^{**}(\delta_y) , 1_{\hat{X}} \rangle = \|T^{**}(\delta_y)\| > 0$, by Proposition 3.2.2 and the fact that T^{**} is one-to-one. To show that $T^*(1_{\hat{X}})$ is multiplicative, first note that by Lemma 3.4.4., $1_{\hat{X}}$ is nothing but the image of the identity function of $C(X) = L^\infty(H)$ under the canonical mapping $C(X) \to C(\hat{X})$; so it is multiplicative on $L^1(H)^{**}$, from Lemma 3.4.4; and $\langle T^*(1_{\hat{X}}) , mn \rangle = \langle 1_{\hat{X}} , T(mn) \rangle = \langle 1_{\hat{X}} , T(m)T(n) \rangle = \langle 1_{\hat{X}} , T(m) \rangle \langle 1_{\hat{X}} , T(n) \rangle = \langle T^*(1_{\hat{X}}) , m \rangle \langle T^*(1_{\hat{X}}) , n \rangle$; so that $T^*(1_{\hat{X}})$ is multiplicative.

Having done these steps, we may define a function $\theta : C(\hat{X}) \to C(\hat{Y})$ by $\theta(f) = \frac{T^*(f)}{T^*(1_{\hat{X}})}$. One may notice that for all $g \in C(\hat{Y})$, $\theta^{-1}(g) = (T^{-1})^*(T^*(1_{\hat{X}})g)$. A similar assertion to the one in step 4 shows that $(T^{-1})^*$ sends non-negative functions onto non-negative functions, and therefore θ is indeed a bipositive *linear* isomorphism such that $\theta(1_{\hat{X}}) = 1_{\hat{Y}}$. Then by Theorem 3.2.5, θ is an algebraic isomorphism and there exists a homeomorphism $\tau : \hat{Y} \to \hat{X}$ such that $\theta(f)(y) = f(\tau(y))$ for all

$y \in \hat{Y}$ and all $f \in C(\hat{X})$, and equivalently,

$$(*) \qquad T^*(f)(y) = T^*(1_{\hat{X}})(y) f(\tau(y)).$$

We saw in step 6 that $T^*(1_{\hat{X}})$ is a multiplicative linear functional on $L^1(G)^{**}$, and hence $T^*(1_{\hat{X}})$ has norm at most equal to 1. Then $T^*(1_{\hat{X}})$ has the sup-norm at most 1 when regarded as an element of $C(\hat{Y})$. Hence in $(*)$ we have $|T^*(f)(y)| \leqslant |f(\tau(y))| \leqslant \|f\|_u$, showing that T^* is norm decreasing. But then T will be norm decreasing, since $\|T^*\| = \|T\|$, always. By symmetry T is norm-preserving, as required. By Theorem 3.3.8, G and H are topologically isomorphic. $\square$

In step 1 of the above theorem we proved that T is continuous. This fact also follows as a special case of the following theorem:

3.4.7 Theorem. *Any positive operator from a Banach lattice into a normed Riesz space is continuous.*

For a proof see [3, Theorem 12.3].

Using Theorem 3.3.7, we can derive some other closely related results: Let S be a semigroup endowed with a Hausdorff topology in which the semigroup operation $S \times S \to S$ is separately continuous. In this case, we call S a *topological semigroup*, and furthermore if the topology of S is locally compact, then S is called a locally compact semigroup. Note that when the semigroup multiplication makes S a group, then by Ellis's theorem [20] the multiplication is jointly continuous and we are in the situation of a locally compact group. For a topological semigroup S, we can define a left introverted subspace X of $C(S)$ the same way that we defined it for a topological group (see Definition 1.4.3 (h)); for such X, X^* is a Banach algebra.

If $m \in C(S)^*$, then the function mf must be in X and hence in $C(S)$, where $(mf)(x) = \langle m\,,\, l_x f \rangle$, for all $x \in S$. This means that the map $x \mapsto l_x f$, from S into $C(S)$ is weakly continuous. Any function with this property is called *weakly left uniformly continuous*; and the set of all such functions is denoted by $WLUC(S)$. So we have $X \subseteq WLUC(S)$ and it can be verified that $WLUC(S)$ is a left introverted subspace of $C(S)$, and so it is the maximal one. This fact was first observed in [51, Theorem 1].

3.4.8 Theorem. *Let S_i be a topological semigroup and X_i be a left introverted subspace of $C(S_i)$ which is closed under conjugation containing the constant functions, $i = 1, 2$. Let $T : X_1^* \to X_2^*$ be a bipositive isomorphism. Then T is an isometric isomorphism.*

Proof. Under these assumptions X_i is a C^*-algebra and the proof of Theorem 3.4.8 can be carried out the same way as that of Theorem 3.4.6. The only difference occurs in step 6, where we used the fact that $1_{\mathcal{X}}$ is multiplicative on $L^1(H)$. To complete the proof for Theorem 3.4.8, we need to show that if $\mathbf{1}$ is the constant function one on S_2, then $\mathbf{1}$ is multiplicative when considered as a linear functional on X_2^*. Indeed for $m, n \in X_2^*$, and $t \in S_2$ we have $(n\mathbf{1})(t) = \langle n\,,\, l_t \mathbf{1} \rangle = \langle n\,,\, \mathbf{1} \rangle$ and $n\mathbf{1}$ is the constant function $\langle n\,,\, 1 \rangle \mathbf{1}$, from which $\langle mn\,,\, \mathbf{1} \rangle = \langle m\,,\, n\mathbf{1} \rangle = \langle m\,,\, \mathbf{1} \rangle \langle n\,,\, \mathbf{1} \rangle$, as we wished. $\square$

CHAPTER IV

INVOLUTIONS ON THE SUBSPACES OF $L^1(G)^{**}$

If G is a locally compact group, the algebra $L^1(G)$ has a norm-preserving involution. In fact, for $f \in L^1(G)$, let f^* be the function on G defined by $f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1})$, where Δ is the modular function and "bar" stands for the complex conjugation. Then $*$ is a norm-preserving involution on $L^1(G)$ (see [33; Theorem 20.2] for details), which is called the *natural involution.* In [18; page 323], J. Duncan and S.A.R. Hosseiniun ask whether there exists an involution on $L^1(G)^{**}$ which extends the natural involution of $L^1(G)$.

We will answer this question for the class of amenable groups, in the sequel. But first, the following known result:

4.1 Proposition. *For a locally compact group G, the following statements are equivalent:*

(a) *G is discrete.*

(b) $M(G) = L^1(G)$.

(c) $L^\infty(G) = LUC(G)$.

(d) *$L^1(G)$ has an identity.*

(e) *$L^1(G)^{**}$ has an identity.*

Proof. See [33; Theorem 19.15] for the equivalence of (a)$\Longleftrightarrow$(b). $M(G)$ has an identity, so (b)$\Longrightarrow$(d) follows.

The implication (d)$\Longrightarrow$(e) follows from the fact that for $\varphi \in L^1(G)$ the map $n \to \varphi n$ $\left(n \in L^1(G)^{**}\right)$ is weak-star to weak-star continuous and the fact that $L^1(G)$ is weak-star dense in $L^1(G)^{**}$.

(e)$\Longrightarrow$(c) follows from the argument of Lemma 4.2 bellow, where we essentially prove that if $L^1(G)^{**}$ has a left identity, then $L^\infty(G) = LUC(G)$.

To prove (c)$\Longrightarrow$(a), let χ be the characteristic function of the set $\{e_G\}$. Then $\chi \in L^\infty(G) = LUC(G)$, so χ is a continuous function and $\{e_G\} = \{x \in G : \ \chi(x) \neq 0\}$ is an open subset of G; so G is discrete. $\square$

4.2 Lemma. *If there exists an involution on $L^1(G)^{**}$, then G is discrete.*

Proof. Let E be a right identity for $L^1(G)^{**}$, and $*$ be an involution on $L^1(G)^{**}$. Then for any $m \in L^1(G)^{**}$, $E^*m = (m^*E)^* = (m^*)^* = m$, indicating that E^* is a left identity. From the preceding proposition we need to show that $L^\infty(G) = LUC(G)$. If we had $L^\infty(G) \neq LUC(G)$, then by an application of the Hahn-Banach theorem, there would exists a non-zero $m \in L^1(G)^{**}$ annihilating $LUC(G)$. From Theorem 1.5.4, m is a right annihilator. So $0 = E^*m = m$, which is a contradiction. $\square$

When G is a discrete group, then any point mass δ_x belongs to $L^1(G)$, from the previous proposition. Thus for any $f \in L^\infty(G)$, we have $\langle f\,,\,(\delta_x)^*\rangle = \int f(t)(\delta_x)^*(t)\,dt = \int f(t)\overline{(\delta_x)(t^{-1})}\,dt = f(x^{-1}) = \langle f\,,\,\delta_{x^{-1}}\rangle$, indicating that $(\delta_x)^* = \delta_{x^{-1}}$, where $*$ stands for the natural involution on $L^1(G)$.

Also note that if K is any amenable locally compact group and n is any left invariant element of $LUC(K)^*$, then for any $m \in LUC(K)^*$, we have that $mn = \langle m\,,\,\mathbf{1}\rangle n$, by the proof of Proposition 2.1.8, where $\mathbf{1}$ denotes the constant function 1. Moreover, it is known that $LUC(K)^*$ has a two-sided invariant mean. To see this, let $m_1 \in LUC(K)^*$ be a left invariant mean and define $m_2 \in LUC(K)^*$ by $\langle m_2\,,\,f\rangle = \langle m_1\,,\,\check{f}\rangle$, where $\check{f}(x) = f(x^{-1})$, for $x \in K$. Then m_2 is right translation invariant. The left invariance of m_1 and the right invariance of m_2 are equivalent to having $\delta_x m_1 = m_1$ and $m_2\delta_x = m_2$ for all $x \in G$. If we let $m = m_1 m_2$, then $\delta_x m = m\delta_x = m$; so that m is both left and right invariant. From the definition of multiplication, it is easily verified that $\langle m\,,\,\mathbf{1}\rangle = 1$ and $m_2 f \geqslant 0$, for all $f \in LUC(K)$; then m is a positive linear functional on $LUC(K)$. From this, m is a two-sided invariant mean.

In Theorem 1.5.4, we saw that for a locally compact group G, the restriction map induces an isometric algebra isomorphism π_G from $EL^1(G)^{**}$ onto $LUC(G)^*$, where E is a right identity with $\|E\| = 1$. This shows that the original multiplication on $LUC(G)^*$ (as defined in part (h) of Definition 1.4.3) is the same as the Arens multiplication on $LUC(G)^*$, when $LUC(G)^*$ is regarded as a subalgebra of $L^1(G)^{**}$ via the map π_G. This fact was first observed in [40; Lemma 3]. Now we can proceed with the main result of this section.

4.3 Proposition. *Let G be an amenable locally compact group. Then there exists an involution on $L^1(G)^{**}$ extending the natural involution of $L^1(G)$ if and only if G is finite.*

Proof. If G is finite, then $L^1(G)$ is finite dimensional; hence $L^1(G)^{**}$ is finite dimensional. The sufficiency of the assertion is then clear.

Now conversely, let $*$ be an involution on $L^1(G)^{**}$ that extends the involution of $L^1(G)$. From Lemma 4.2 and Proposition 4.1, we have $LUC(G)^* = L^1(G)^{**}$ and the Arens multiplication on $LUC(G)^*$ is the same as the original multiplication, as explained earlier. Choose a two-sided invariant mean m in $LUC(G)^*$. Since G is discrete, we have $(\delta_x)^* = \delta_{x^{-1}}$, for all $x \in G$. Hence $m^*\delta_{x^{-1}} = (\delta_x m)^* = m^*$, and from there, m^* is right invariant. Similarly, m^* is left invariant, and moreover $m^* \neq 0$, since $m \neq 0$. From the left invariance of m, $m^2 = \langle m\,,\, \mathbf{1}\rangle m = m$; therefore $(m^*)^2 = m^*$. On the other hand, since m^* is left invariant, $(m^*)^2 = \langle m^*\,,\, \mathbf{1}\rangle m^*$. So we get $m^* = \langle m^*\,,\, \mathbf{1}\rangle m^*$, and $\langle m^*\,,\, \mathbf{1}\rangle = 1$, since $m^* \neq 0$. Now let n_1 and n_2 be any two left invariant means on $LUC(G)$ and let $n = n_1 - n_2$. Then $m^*n = \langle m^*\,,\, \mathbf{1}\rangle n = n$; so $n = (n^*m)^* = \big(\langle n^*\,,\, \mathbf{1}\rangle m\big)^* = tm^*$, where $t = \overline{\langle n^*\,,\, \mathbf{1}\rangle}$. Since $\langle n_1\,,\, \mathbf{1}\rangle = \langle n_2\,,\, \mathbf{1}\rangle$, we have $0 = \langle n\,,\, \mathbf{1}\rangle = t\langle m^*\,,\, \mathbf{1}\rangle = t$. So $n = 0$, from $n = tm^*$. We have shown that $LUC(G)^*$ has a unique left invariant mean; therefore G is compact by Theorem 1.6.9. Then G is finite, since it is discrete. $\square$

Before proceeding with the next proposition, we point out here that $M(G)$ has an involution extending the natural involution of $L^1(G)$, for every locally compact group G: Indeed, for any $\mu \in M(G)$, define $\mu^\sim \in M(G)$ by $\langle \mu^\sim\,,\, f\rangle = \overline{\langle \mu\,,\, \tilde{f}\rangle}$, where $\tilde{f}(x) = \overline{f(x^{-1})}$, for $f \in C_0(G)$ and $x \in G$; particularly, $(\delta_x)^\sim = \delta_{x^{-1}}$. Then $\sim$ defines an involution with the property required; see [33; Theorem 20.22] for details. A part of the next proposition shows that this involution is actually the unique involution on $M(G)$ which extends the involution of $L^1(G)$.

4.4 Proposition. *Let G be an amenable group. Then $LUC(G)^*$ has an* *involution extending the natural involution of $L^1(G)$ if an only if G is compact.*

Proof. The sufficiency is clear from the above, since $LUC(G)^* = M(G)$ for compact G. Conversely, assume that $LUC(G)^*$ has an involution $*$ as described in the assertion. Let $\{\varphi_\alpha\}$ be a bounded-by-1 approximate identity for $L^1(G)$. Since the involution on $L^1(G)$ is norm-preserving, $\{\varphi_\alpha^*\}$ is also a bounded-by-1 approximate identity for $L^1(G)$. For any $m \in LUC(G)^*$, we have $\varphi_\alpha^* m \xrightarrow{w^*} m$ in $LUC(G)^*$, from Theorem 1.5.3. In particular, for $\mu \in M(G)$ and $f \in C_0(G)$, $\langle \mu^*, f\rangle = \lim_\alpha \langle \varphi_\alpha^* \mu^*, f\rangle = \lim_\alpha \langle (\mu\varphi_\alpha)^*, f\rangle = \lim_\alpha \overline{\langle \mu\varphi_\alpha, \tilde{f}\rangle} = \lim_\alpha \overline{\langle \mu, \varphi_\alpha \tilde{f}\rangle} = \overline{\langle \mu, \tilde{f}\rangle} = \langle \mu^\sim, f\rangle$, in which $\tilde{f}$ and $\mu^\sim$ are taken as above. This shows that the restriction of μ^* to $C_0(G)$ equals $\mu^\sim$; therefore $\|\mu^*\| \geqslant \|\mu^\sim\|$. On the other hand, $(\mu\varphi_\alpha)^* = \varphi_\alpha^* \mu^* \xrightarrow{w^*} \mu^*$; therefore $\|\mu^*\| \leqslant \|\mu\|$, since $\|(\mu\varphi_\alpha)^*\| = \|\mu\varphi_\alpha\| \leqslant \|\mu\|$. From Theorem 1.5.5, we have $\mu^* = \mu^\sim$ and particularly, $(\delta_x)^* = \delta_{x^{-1}}$. Now one can repeat the argument of Proposition 4.3 word for word. □

The result of the preceding proposition may suggest that G is compact whenever there is a left introverted subspace X containing $C_0(G)$ such that X has a left invariant mean and that X^* has an involution extending the natural involution of $L^1(G)$. We will show that this is not the case for $WAP(G)^*$.

4.5 Definition. Let A be a Banach algebra. A linear functional $f \in A^*$ is called *weakly almost periodic* on A if the map $a \mapsto fa$ from A into A^* is weakly compact.

For two elements $m, n \in A^{**}$, we denote the second Arens product of m and n by $m \cdot n$, as usual. Then we have the following known result:

4.6 Proposition. *Let A be a Banach algebra and $f \in A^*$. Then f is weakly* *almost periodic on A if and only if $\langle mn\,,\,f\rangle = \langle m\cdot n\,,\,f\rangle$, for all $m,n\in A^{**}$.*

Proof. First let $f\in A^*$ be weakly almost periodic and $m,n\in A^{**}$. We can assume $\|m\|,\|n\|\leqslant 1$. Then choose nets (a_α) and (b_β) in A, both bounded by 1, such that $a_\alpha \xrightarrow{w^*} m$, $b_\beta \xrightarrow{w^*} n$ and $\{fa_\alpha\}$ weakly converges to some $g\in A^*$, as f is weakly almost periodic. By using Lemma 3.3.4, we can choose a net (c_γ) far out in (a_α) for which $fc_\gamma \longrightarrow g$ in norm. We certainly have $c_\gamma \xrightarrow{w^*} m$. Given $\epsilon>0$, there exists $\gamma_\circ$ such that $\|fc_\gamma - g\|<\epsilon$, for all $\gamma\geqslant\gamma_\circ$. So $|\langle b_\beta\,,\,fc_\gamma - g\rangle|<\epsilon$, i.e.,

$$(1)\qquad |\langle c_\gamma b_\beta\,,\,f\rangle - \langle b_\beta\,,\,g\rangle| < \epsilon,$$

for all $\gamma\geqslant\gamma_\circ$ and all β. Then (1) reveals that

$$(2)\qquad |\langle mn\,,\,f\rangle - \langle n\,,\,g\rangle| = \lim_{\gamma}\lim_{\beta} |\langle c_\gamma b_\beta\,,\,f\rangle - \langle b_\beta\,,\,g\rangle| \leqslant \epsilon.$$

Another time,

$$(3)\qquad |\langle m\cdot n\,,\,f\rangle - \langle n\,,\,g\rangle| = \lim_{\beta}\lim_{\gamma} |\langle c_\gamma b_\beta\,,\,f\rangle - \langle b_\beta\,,\,g\rangle| \leqslant \epsilon,$$

by referring to the fact that $m\cdot b_\beta = mb_\beta$ for $b_\beta\in A$. By comparing (2) and (3), we have $\langle m\cdot n\,,\,f\rangle = \langle mn\,,\,f\rangle$. This proves half of the assertion.

Conversely, let $\langle m\cdot n\,,\,f\rangle = \langle mn\,,\,f\rangle$ hold for all $m,n\in A^{**}$. Denote the map $a\mapsto fa$, from A into A^*, by Γ. Then one can show that $\Gamma^*\,:\,A^{**}\longrightarrow A^*$ has the form $\Gamma^*\,:\,n\mapsto nf$. So for $m\in A^{**}$, $\langle\Gamma^{**}(m)\,,\,n\rangle = \langle m\,,\,nf\rangle = \langle mn\,,\,f\rangle = \langle m\cdot n\,,\,f\rangle = \langle n\,,\,f\cdot m\rangle$; therefore $\Gamma^{**}(m) = f\cdot m\in A^*$. Hence Γ is a weakly compact, by [19; Theorem 2, section VI.4]. $\square$

We recall the definition of weakly almost periodic functions on a locally compact group G. The space of these functions is denoted by $WAP(G)$, as usual. The following theorem is implicit from [60; Lemma 6.3]. See also [54].

4.7 Theorem. *Let $f \in C(G)$. Then f is weakly almost periodic as a functional on $L^1(G)$ if and only if $f \in WAP(G)$.*

4.8 Lemma. *Suppose that A is a Banach algebra with a continuous involution $\sim$. Then $\sim$ has an extension to a one-to-one mapping of A^{**} onto itself, denoted by the same symbol, such that for all m, $n \in A^{**}$, and $\alpha \in \mathbb{C}$, the following hold:*

(a) $m^{\sim\sim} = m$;

(b) $(mn)^{\sim} = n^{\sim} \cdot m^{\sim}$;

(c) $(\alpha m + n)^{\sim} = \bar{\alpha} m^{\sim} + n^{\sim}$;

(d) *The extension of $\sim$ is isometric, whenever $\sim$ is isometric.*

Proof. Let $m, n \in A^{**}$, $f \in A^*$ and $a, b \in A$. Define $\tilde{f} \in A^*$ and $m^{\sim} \in A^{**}$ successively by $\langle \tilde{f}, a \rangle = \overline{\langle f, a^{\sim} \rangle}$ and $\langle m^{\sim}, f \rangle = \overline{\langle m, \tilde{f} \rangle}$.

(a) Clearly $\tilde{\tilde{f}} = f$, therefore $\langle m^{\sim\sim}, f \rangle = \overline{\langle m^{\sim}, \tilde{f} \rangle} = \overline{\overline{\langle m, \tilde{\tilde{f}} \rangle}} = \langle m, f \rangle$, which shows $m^{\sim\sim} = m$.

(b) We have $\langle \tilde{f} a, b \rangle = \langle \tilde{f}, ab \rangle = \overline{\langle f, (ab)^{\sim} \rangle} = \overline{\langle f, b^{\sim} a^{\sim} \rangle} = \overline{\langle a^{\sim} \cdot f, b^{\sim} \rangle} = \langle (a^{\sim} \cdot f)^{\sim}, b \rangle$, indicating that $\tilde{f} a = (a^{\sim} \cdot f)^{\sim}$. Then $\langle n\tilde{f}, a \rangle = \langle n, (a^{\sim} \cdot f)^{\sim} \rangle = \overline{\langle n^{\sim}, a^{\sim} \cdot f \rangle} = \overline{\langle f \cdot n^{\sim}, a^{\sim} \rangle} = \langle (f \cdot n^{\sim})^{\sim}, a \rangle$; so that $n\tilde{f} = (f \cdot n^{\sim})^{\sim}$. Hence $\langle (mn)^{\sim}, f \rangle = \overline{\langle mn, \tilde{f} \rangle} = \overline{\langle m, (f \cdot n^{\sim})^{\sim} \rangle} = \langle m^{\sim}, f \cdot n^{\sim} \rangle = \langle n^{\sim} \cdot m^{\sim}, f \rangle$; so $(mn)^{\sim} = n^{\sim} \cdot m^{\sim}$. Both (c) and (d) are clear from the definition of $\sim$. $\square$

4.9 Corollary. *If A is Arens regular and $\sim$ is a continuous involution of A, then $\sim$ extends to an involution of A^{**}.*

4.10 Remark. Proposition 4.8 shows that when G is an infinite locally compact group, an attempt of using "adjoints"to define an involution on $L^1(G)^{**}$ is futile, since in this case $L^1(G)$ is not Arens regular.

Now we prove our claim:

4.11 Proposition. *For every locally compact group G, the Banach algebra $WAP(G)^*$ has a norm-preserving involution that extends the involution of $L^1(G)$.*

Proof. We assume $f \in WAP(G)$, $m_1, m_2 \in WAP(G)^*$, and that for $i = 1, 2$, n_i is a Hahn-Banach extension of m_i to $L^\infty(G)$. One can verify that $\tilde{f} \in WAP(G)$ whenever $f \in WAP(G)$, in which $\tilde{f}(x) = \overline{f(x^{-1})}$. Note that this $\tilde{f}$ coincides with the $\tilde{f}$ defined in Lemma 4.8 if we let $A = L^1(G)$. Note that from Theorem 1.5.5, we have $\tilde{f} \in WAP(G)$. For any $m \in WAP(G)^*$, define $\hat{m} \in WAP(G)^*$ by $\langle \hat{m}, f \rangle = \overline{\langle m, \tilde{f} \rangle}$. From this definition it is clear that $\hat{\hat{m}} = m$ and $\|\hat{m}\| = \|m\|$, since $\|\tilde{f}\| = \|f\|$. Note that if n is any Hahn-Banach extension of m to $L^\infty(G)$, then $\langle \hat{m}, f \rangle = \langle n^\sim, f \rangle$, where $n^\sim$ is the image of n under the map defined in Lemma 4.8, in which $A = L^1(G)$ is equipped with the natural involution. By Theorem 1.5.4, $n_1 n_2$ is an extension of $m_1 m_2$. Therefore

$$\begin{aligned}
\langle (m_1 m_2)\hat{}\,, f \rangle &= \langle (n_1 n_2)^\sim, f \rangle \\
&= \langle \tilde{n}_2 \cdot \tilde{n}_1, f \rangle \\
&= \langle \tilde{n}_2 \tilde{n}_1, f \rangle \qquad \text{[by Proposition 4.6 and Theorem 4.7]} \\
&= \langle (m_2)\hat{}\,(m_1)\hat{}\,, f \rangle \qquad \text{[by Theorem 1.5.4]}.
\end{aligned}$$

So $(m_1 m_2)\hat{} = (m_2)\hat{}\,(m_1)\hat{}$. It is also clear that $(\alpha m_1 + m_2)\hat{} = \bar{\alpha} \hat{m}_1 + \hat{m}_2$, for any complex number α. The claim in now fulfilled. □

We recall from Proposition 1.5.2 that the left regular representation $\lambda : L^1(G) \to B(L^2(G))$ is a norm-decreasing $*$-homomorphism.

4.12 Proposition. *Let $\lambda : L^1(G) \to B\big(L^2(G)\big)$ be the left regular representation.*

(a) *The image of λ^{**} is a self-adjoint subalgebra of the C^*-algebra $B(\,L^2(G)\,)^{**}$.*

(b) *The kernel of λ^{**} contains the radical of $L^1(G)^{**}$.*

Proof. Let $m \in L^1(G)^{**}$ and (φ_α) be a net in $L^1(G)$ such that $\|\varphi_\alpha\| \leqslant \|m\|$ and $\varphi_\alpha \xrightarrow{w^*} m$. Let φ_α^* be the image of φ_α under the natural involution. Then $\|\varphi_\alpha^*\| \leqslant \|m\|$, so that a subnet of (φ_α^*) converges to some $n \in L^1(G)^{**}$. We can assume $\varphi_\alpha^* \xrightarrow{w^*} n$. By the weak-star continuity of λ^{**} we have that

$$\lim_\alpha \lambda^{**}(\varphi_\alpha) = \lambda^{**}(m)$$

and

$$\lim_\alpha \lambda^{**}(\varphi_\alpha^*) = \lambda^{**}(n).$$

On the other hand, [53; Theorem 1.7.8] indicates that the involution in a von Neumann algebra is continuous in the weak-star topology; so that $\lambda^{**}(n) = \lim_\alpha \lambda^{**}(\varphi_\alpha^*)$ $= \lim_\alpha \lambda(\varphi_\alpha^*) = \lim_\alpha \lambda(\varphi_\alpha)^* = \lim_\alpha \big(\lambda^{**}(\varphi_\alpha)\big)^* = \big(\lambda^{**}(m)\big)^*$, indicating that $\big(\lambda^{**}(m)\big)^* = \lambda^{**}(n)$; hence $\lambda^{**}\big(L^1(G)^{**}\big)$ is self-adjoint, furthermore it is a subalgebra of $B(L^2(G))^{**}$ from Lemma 3.4.4.

Now if we assume that $m \in Rad\big(L^1(G)^{**}\big)$, then the spectral radius of mn is zero, that is $r(mn) = 0$. An algebra homomorphism reduces the spectral radii, so $r\big(\lambda^{**}(m)(\lambda^{**}(m))^*\big) = r(mn) = 0$. Since $\lambda^{**}(m)\lambda^{**}(m)^*$ is a self-adjoint element of $B(L^2(G))^{**}$, we must have $\lambda^{**}(m) = 0$, and the proof is complete. □

4.13 Corollary. *If* $\ker\lambda^{**}$ *denotes the kernel of λ^{**}, then $L^1(G)^{**}/\ker\lambda^{**}$ admits an involution.*

4.14 Corollary. *If G is non-discrete or amenable discrete, then λ^{**} is not one-to-one.*

If it turns out that for every discrete non-amenable group G the map λ^{**} is one-to-one, then $L^1(G)^{**}$ will be semisimple for these groups and an old question regrading to the radical of $L^1(G)^{**}$ [12] will be answered.

In relation to part (b) of Proposition 4.12, one might ask:

Question: Is $\ker\lambda^{**}$ larger than the radical of $L^1(G)^{**}$?.

It would be interesting to know whether the problem of existence of involutions on $L^1(G)^{**}$ for discrete group G is related to existence of a non-zero radical, so we ask:

Question: If G is discrete and $L^1(G)^{**}$ has an involution extending the natural involution of $L^1(G)$, then does it follow that $Rad\big(L^1(G)^{**}\big) = (0)$?

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