THE UNIVERSITY OF MANITOBA

WHITMAN'S CONDITION

by

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ABSTRACT

A lattice L is said to satisfy Whitman's condition (W) if for all $a,b,c,d \in L$, $a \wedge b \leq c \vee d$ implies $a \leq c \vee d$, $b \leq c \vee d$, $a \wedge b \leq c$, or $a \wedge b \leq d$. This condition was introduced in 1941 as part of P. M. Whitman's solution to the word problem for lattices. In recent years, however, lattices satisfying (W) have received increased attention in their own right. In this thesis we continue this trend. We investigate a number of problems, each of which will see (W) playing a more or less central role.

After a short chapter on preliminaries, in Chapter 1 we prove a characterization of modular lattices satisfying (W) in terms of the exclusion of two finite lattices as sublattices. The chapter ends with an expository discussion of methods of constructing lattices satisfying (W).

Let FC denote the class of all lattices with no infinite chains. For a lattice L in FC we say that the condition $Q_{FC}(L)$ holds if for each $M \in$ FC and every epimorphism $\phi : M \to L$ there exists a homomorphism $\psi : L \to M$ such that $\psi\phi = id_L$. In Chapter 2 (the content of which is also due to B. A. Davey) we prove that a lattice L in FC satisfies (W) if and only if $Q_{FC}(L)$ holds. This has several interesting consequences, among which is an extension of a result of R. Freese.

A lattice L is said to satisfy the bounded epimorphism condition if whenever M is a lattice and $\phi : M \to L$ is a bounded epimorphism there exists a homomorphism $\psi : L \to M$ such that $\psi\phi = id_L$. Each lattice satisfying the bounded epimorphism condition satisfies (W). In Chapter 3 we give an infinite family of finite lattices, each of which satisfies (W) but fails the bounded epimorphism condition. We also exhibit a property sufficient to imply that a finite lattice satisfies the bounded epimorphism condition.

In Chapter 4 we introduce the concept of semidistributivity and with it Jonsson's conjecture: a finite lattice is a sublattice of a free lattice if and only if it is semidistributive and satisfies (W). The history of this conjecture is outlined.

For lattices L and M, Hom(L,M) denotes the set of all homomorphisms of L into M, endowed with the pointwise partial order. A lattice L is called catalytic if Hom(L,M) is a lattice for every lattice M. In Chapter 5 (due also to T. G. Kucera) we prove that the class of finite catalytic lattices lies somewhere between the class of finite sublattices of a free lattice and the class of finite semidistributive lattices satisfying (W).

Finally, we have included two papers (written jointly with I. Rival) that deal with the topic of finite planar sublattices of a free lattice. The first confirms Jonsson's conjecture for finite planar lattices, and the second exhibits a family $\mathfrak{F}$ of finite lattices with the property that a finite lattice is a planar sublattice of a free lattice if and only if it does not contain a sublattice isomorphic to a member of $\mathfrak{F}$.

ACKNOWLEDGEMENTS

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CHAPTER 0. INTRODUCTION.

We begin with a question to which nearly everyone reading this thesis will know the answer.

Let F be a lattice freely generated by a set X, and let p and q be elements of F. How can we determine whether or not $p \leq q$ holds in F? The answer has been supplied by P.M. Whitman [27].

(i) If $p = p_1 \vee p_2$ then $p \leq q$ if and only if $p_1 \leq q$ and $p_2 \leq q$.

(ii) If $q = q_1 \wedge q_2$ then $p \leq q$ if and only if $p \leq q_1$ and $p \leq q_2$.

(iii) If p and q are elements of X then $p \leq q$ if and only if $p = q$ in X.

(iv) If $p \in X$ and $q = q_1 \vee q_2$ then $p \leq q$ if and only if $p \leq q_1$ or $p \leq q_2$.

(v) If $p = p_1 \wedge p_2$ and $q \in X$ then $p \leq q$ if and only if $p_1 \leq q$ or $p_2 \leq q$.

(vi) If $p = p_1 \wedge p_2$ and $q = q_1 \vee q_2$ then $p \leq q$ if and only if $p_1 \leq q$ or $p_2 \leq q$ or $p \leq q_1$ or $p \leq q_2$.

A particular pair p, q of elements of F may be decomposed by repeatedly invoking whichever of the above rules apply, until the truth or falsity of $p \leq q$ is revealed.

As the only property among the above that is both nontrivial and generator-free, (vi) is the most interesting, and leads to our first definition.

DEFINITION 0.1. A lattice L is said to satisfy Whitman's condition (or simply, (W)) if whenever $a, b, c, d \in L$ satisfy $a \wedge b \leq c \vee d$, either $a \leq c \vee d$ or $b \leq c \vee d$ or $a \wedge b \leq c$ or $a \wedge b \leq d$ holds.

In the chapters that follow we shall investigate a number of problems, each of which will see (W) playing a more or less central role. For the most part the lattice-theoretic notation of G. Grätzer [13, 15] will be adopted. Terminology to be used in this thesis will be introduced throughout; however we will devote the rest of this chapter to compiling some of the more basic definitions.

An element a of a lattice L is called <u>join reducible</u> if there are elements x, y of L such that $a = x \vee y$ while $a \neq x$, $a \neq y$. An element is <u>join irreducible</u> if it is not join reducible. By dualizing we obtain the concepts of <u>meet reducible</u> and <u>meet irreducible</u> elements. An element is <u>doubly reducible</u> if it is both join reducible and meet reducible, and <u>doubly irreducible</u> if it is both join irreducible and meet irreducible. If $x, y \in L$ and $x > y$, we will say that x <u>covers</u> y if there is no $z \in L$ such that $x > z > y$. For $x, y \in L$ satisfying $x \geq y$, the <u>interval</u> $[y, x]$ of L is the sublattice $\{z \in L \mid y \leq z \leq x\}$.

If L is a lattice, we will denote the lattice of ideals of L by $\mathcal{I}(L)$; $\mathcal{D}(L)$ will denote the lattice of dual ideals of L, ordered by reverse set inclusion.

Let C be a chain and let $\{L_i \mid i \in C\}$ be a family of disjoint lattices. The <u>linear sum</u> of the lattices $\{L_i \mid i \in C\}$ is a lattice $\langle L, \leq \rangle$ defined by $L = \bigcup_{i \in C} L_i$, where $x \leq y$ in L if and only if one of the following holds:

(i) $x, y \in L_i$ for some $i \in C$ and $x \leq y$ in L_i; or

(ii) $x \in L_i$, $y \in L_j$ and $i < j$ in C.

In the particular case $I = \{1, 2\}$, we denote the linear sum of the lattices L_1 and L_2 by $L_1 \oplus L_2$.

Let $\underset{\sim}{K}$ be a class of lattices. A lattice $L \in \underset{\sim}{K}$ is *projective in* $\underset{\sim}{K}$ if for all lattices $A, B \in \underset{\sim}{K}$, for each epimorphism $\phi : A \twoheadrightarrow B$, and for each homomorphism $\psi : L \longrightarrow B$, there exists a homomorphism $\psi' : L \longrightarrow A$ such that $\psi'\phi = \psi$. The finite, finitely generated, and arbitrary lattices projective in the class $\underset{\sim}{L}$ of all lattices have been characterized respectively by R. McKenzie [23], A. Kostinsky [21], and R. Freese and J.B. Nation [7]. A lattice L is a *retract* of a lattice M if there are homomorphisms $\phi : M \twoheadrightarrow L$ and $\psi : L \longrightarrow M$ such that $\psi\phi = id_L$; in this case it follows that ϕ is an epimorphism and ψ is an embedding. An epimorphism $\phi : M \twoheadrightarrow L$ is *lower bounded* if the inverse image $x\phi^{-1}$ has a least element for each $x \in L$. ϕ is *upper bounded* if $x\phi^{-1}$ has a greatest element for each $x \in L$. ϕ is *bounded* if it is both lower and upper bounded. We will denote the least (greatest) element of $x\phi^{-1}$, if it exists, by 0_x (1_x).

For each integer $n \geq 3$, we denote the $n+2$-element lattice of length two by M_n. In particular the five-element, modular, nondistributive lattice will be denoted M_3. The five-element non-modular lattice will be denoted N_5. The n-element chain, where n is a positive integer, will be denoted by $\underset{\sim}{n}$.

Finally, if L is a lattice violating (W) and a, b, c, d are elements of L satisfying $a \wedge b \leq c \vee d$, $a \nleq c \vee d$, $b \nleq c \vee d$, $a \wedge b \nleq c$, and $a \wedge b \nleq d$, we will abbreviate this situation by saying that the 4-tuple (a, b, c, d) is a *violation of* (W).

CHAPTER 1. THE PRESENCE AND ABSENCE OF WHITMAN'S CONDITION.

This chapter will be mainly expository and somewhat of an extension of Chapter 0. In it we discuss lattices that satisfy, and lattices that violate, Whitman's condition.

Free lattices, of course, satisfy (W), as do many other lattices; a few of the finite lattices with this property are given in Figure 1.1. The significance of some of these lattices will be pointed out below.

In Figure 1.2 are illustrated some of the lattices that violate (W), with the offending elements a, b, c, and d indicated. The lattice D of Figure 1.2 is an example of a lattice with a doubly reducible element (x). Clearly any lattice containing a doubly reducible element will violate (W). This particular kind of violation is of sufficient independent interest to warrant a separate definition.

DEFINITION 1.1. A lattice L is said to satisfy the condition (X) if L contains no doubly reducible element; that is, there do not exist elements $a, b, c, d, x \in L$ such that $a \wedge b = c \vee d = x$ and a, b, c, and d are all distinct from x.

Trivially, we have that L satisfies (X) if and only if L does not contain the lattice D as a sublattice.

It is an easy consequence of the distributive laws that a distributive lattice satisfies (W) exactly if it satisfies (X). In fact, the countable distributive lattices satisfying (W) have been even more explicitly characterized by F. Galvin and B. Jónsson [8].

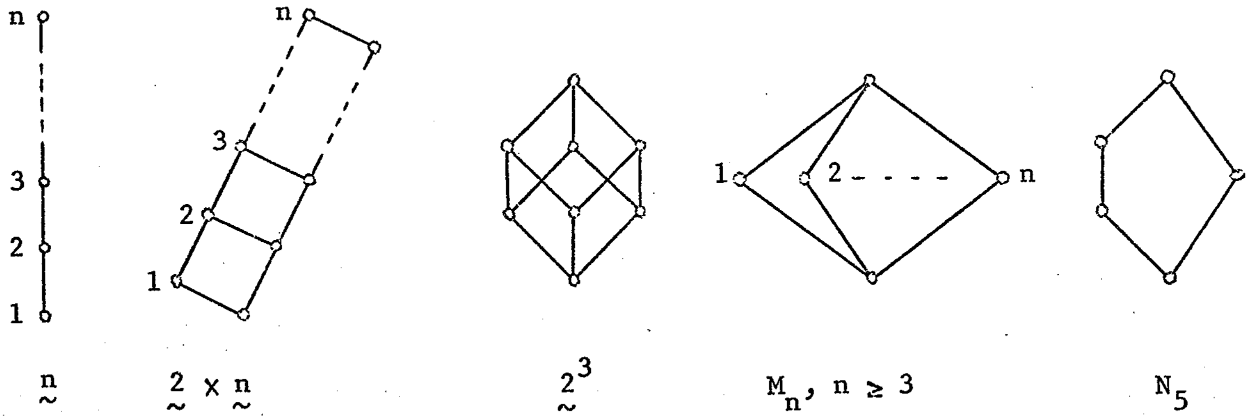


Figure 1.1. Lattices satisfying (W) .

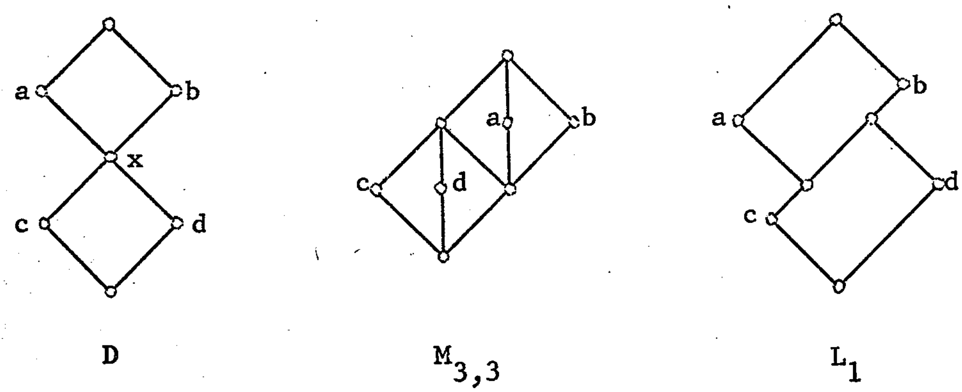


Figure 1.2. Lattices violating (W) .

THEOREM 1.2. <u>A countable distributive lattice</u> L <u>satisfies</u> (W) <u>if and only if</u> L <u>is a linear sum of lattices, each of which is isomorphic to</u> $\underset{\sim}{1}$, $\underset{\sim}{2}^3$, <u>or</u> $\underset{\sim}{2} \times C$, <u>where</u> C <u>is a chain</u>. □

Moreover, Galvin and Jónsson proved that the class of lattices in Theorem 1.2 consists of precisely all distributive sublattices of a free lattice. We return to this topic in Chapter 4.

For what follows, it will be convenient to borrow a definition from [14] .

DEFINITION 1.3. A lattice L is said to satisfy (W′) if whenever $a, b, c, d \in L$ satisfy $c \leq a \wedge b \leq c \vee d$, either $a \leq c \vee d$ or $b \leq c \vee d$ or $a \wedge b \leq c$ or $a \wedge b \leq d$ holds.

Observe that (W′) is weaker than (W) and stronger than (X) . However, for modular lattices (W′) and (X) are equivalent.

LEMMA 1.4. <u>A modular lattice satisfies</u> (W′) <u>if and only if it satisfies</u> (X) .

<u>Proof.</u> Let L be a modular lattice, and assume L satisfies (X) . Let (a, b, c, d) be a violation of (W) in L . If $c \leq a \wedge b$, we have by modularity that $a \wedge b = (c \vee d) \wedge (a \wedge b) = c \vee (d \wedge (a \wedge b))$. Since $a \wedge b \nleq c$ and $a \wedge b \nleq d$, this implies that $a \wedge b$ is doubly reducible, contrary to assumption. Hence $c \nleq a \wedge b$, showing that L satisfies (W′) . The converse is immediate. □

Just as distributive lattices satisfying (W) and modular lattices satisfying (W′) may be characterized by the noncontainment of a particular lattice (namely D) as a sublattice, so may a similar result be proved for modular lattices satisfying (W).

THEOREM 1.5. <u>A modular lattice satisfies</u> (W) <u>if and only if it does not contain either</u> D <u>or</u> $M_{3,3}$ <u>of Figure</u> 1.2 <u>as a sublattice</u>.

<u>Proof.</u> Let L be a modular lattice. It is obvious that if either D or $M_{3,3}$ were a sublattice of L, then L would violate (W). To prove the converse, assume that L does not contain D or $M_{3,3}$ as a sublattice, and suppose that (a', b', c', d') is a violation of (W) in L.

We first show that there exist $c \le c'$, $d \le d'$ such that $(a' \wedge b') \vee c = (a' \wedge b') \vee d = c \vee d$ (in particular the elements a', b', c, d also define a violation of (W)). If both $d' \vee (a' \wedge b') \ge c'$ and $c' \vee (a' \wedge b') \ge d'$ hold, it follows easily that $d' \vee (a' \wedge b') = c' \vee (a' \wedge b') = c' \vee d'$, and we need only set $c = c'$ and $d = d'$. Therefore, without loss of generality, assume $d' \vee (a' \wedge b') \not\ge c'$. Set $d = d'$ and $c = c' \wedge (d \vee (a' \wedge b'))$. We have $a' \wedge b' \le d \vee (a' \wedge b') = (d \vee c') \wedge (d \vee (a' \wedge b')) = d \vee (c' \wedge (d \vee (a' \wedge b'))) = d \vee c$, and since $c \not\le a' \wedge b'$ the elements a', b', c, d define a violation of (W). L satisfies (W$'$) by Lemma 1.4, and hence $c \not\le a' \wedge b'$. Since $(a' \wedge b') \vee c$ is join reducible it cannot be meet reducible. However, $(a' \wedge b') \vee c = (a' \wedge b') \vee (c' \wedge (d \vee (a' \wedge b'))) = ((a' \wedge b') \vee c') \wedge (d \vee (a' \wedge b'))$. Therefore, since $d \vee (a' \wedge b') \not\ge c'$, we are forced to conclude that $(a' \wedge b') \vee c = d \vee (a' \wedge b') = d \vee c$, as desired.

By dualizing the above argument we find elements $a \ge a'$, $b \ge b'$ such that $(c \vee d) \wedge a = (c \vee d) \wedge b = a \wedge b$. Furthermore, since $a' \wedge b' \le a \wedge b \le c \vee d$ it is also true that $(a \wedge b) \vee c = (a \wedge b) \vee d = c \vee d$.

Observe that the elements a, b, c, d still define a violation of (W).

Next we show that $c \vee d$ covers $a \wedge b$. Choose $x \in L$ such that $a \wedge b < x < c \vee d$. By modularity $x = x \vee (a \wedge (c \vee d)) = (x \vee a) \wedge (c \vee d)$, and it follows that x is meet reducible. But a dual argument shows that x is also join reducible, which is impossible.

Since L satisfies (W′), we know that $c \vee (a \wedge b \wedge d) \nleq a \wedge b$; thus we may assume (by replacing c with $c \vee (a \wedge b \wedge d)$) that $c > a \wedge b \wedge d$. Hence $a \wedge b \wedge c = (a \wedge b \wedge d) \vee (a \wedge b \wedge c) = (a \wedge b) \wedge (d \vee (a \wedge b \wedge c))$, and so (by replacing d with $d \vee (a \wedge b \wedge c)$) we may assume $a \wedge b \wedge c = a \wedge b \wedge d$. A dual argument shows that $a \vee c \vee d = b \vee c \vee d$, and since $c \vee d$ covers $a \wedge b$ and L is modular it follows that the elements $\{c, d, a \wedge b, c \vee d, a, b, c \wedge d, a \vee b\}$ form a sublattice (in fact a cover-preserving sublattice) of L isomorphic to $M_{3,3}$, contradicting the hypothesis. Thus L satisfies (W), as claimed. □

The lattice of Figure 1.3 violates (W), but has no finite sublattice violating (W). In particular, arbitrary lattices satisfying (W) cannot be characterized by the exclusion of finitely many finite lattices. The situation with respect to arbitrary finite lattices satisfying (W) is still unclear.

We should mention a related result due to R. Antonius and I. Rival [1]: <u>a semidistributive lattice with no infinite chains satisfies</u> (W) <u>if and only if it does not contain a sublattice isomorphic to</u> D <u>or</u> L_1 <u>of Figure</u> 1.2. The concept of semidistributivity will be introduced in Chapter 4.

The rest of this chapter will deal with the construction of lattices satisfying (W). It is obvious that a sublattice of a lattice satisfying (W) must again satisfy (W) (in fact this

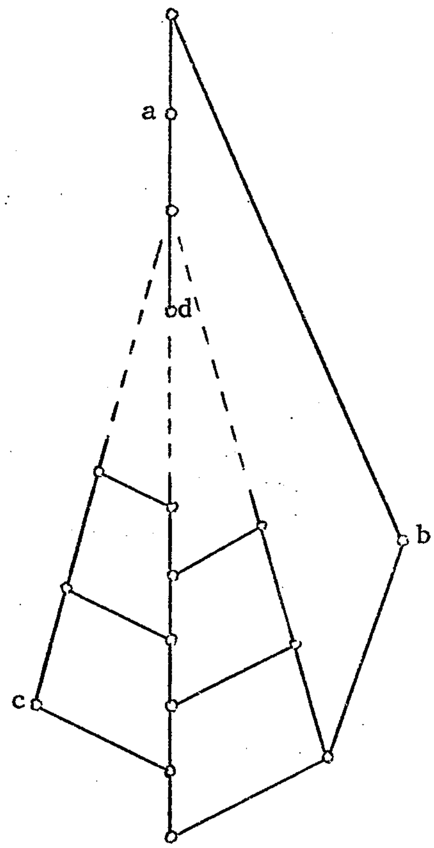
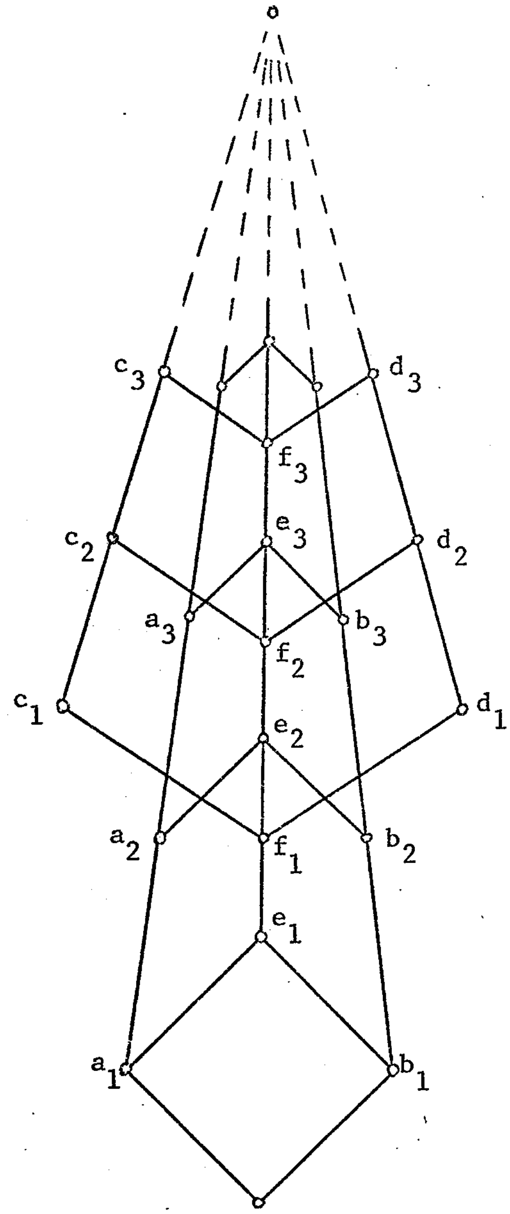


Figure 1.3.



L

Figure 1.4. L satisfies (X) but $\mathcal{I}(L)$ violates (X) .

Letting $\langle S \rangle$ denote the ideal generated by a subset S of L , we see that $\langle \{a_1, a_2, \cdots\} \rangle \vee \langle \{b_1, b_2, \cdots\} \rangle = \langle \{e_1, f_1, e_2, f_2, \cdots\} \rangle$

$$= \langle \{c_1, c_2, \cdots\} \rangle \wedge \langle \{d_1, d_2, \cdots\} \rangle .$$

Thus $\mathcal{I}(L)$ violates (X) .

property has already been tacitly assumed in Theorem 1.5). On the other hand the reader will experience no difficulty in determining that condition (W) is not in general preserved by homomorphic images or direct products. K. Baker and A. Hales [2] have made the observation that the ideal lattice of a lattice satisfying (W) again satisfies (W). In contrast, condition (X) is not preserved in general by ideal lattices (see Figure 1.4 for a counterexample); in fact, G. Grätzer [14] has proved that <u>for every lattice</u> L <u>there exists a lattice</u> M <u>satisfying</u> (X) <u>such that</u> L <u>is isomorphic to a sublattice of</u> $\mathcal{J}(M)$.

Another approach to the construction of lattices satisfying (W) is suggested by Figure 1.5. In each case a violation of (W) has been "repaired" by "splitting" the interval containing the violation. In the examples illustrated, this procedure always yields a lattice satisfying (W); the reader, however, is invited to attempt the same thing with the lattice of Figure 1.6. The procedure, due to A. Day [4], is known as the "interval" or "splitting" construction, and will now be defined in a general setting.

DEFINITION 1.6. Let L be a lattice and let I be an interval of L. The lattice L[I] is defined by: $L[I] = (L \setminus I) \cup (I \times \underset{\sim}{2})$, where $x \leq y$ in L[I] if and only if one of the following holds.

(a) $x, y \in L \setminus I$ with $x \leq y$ in L;

(b) $x = (s, i)$, $y \in L \setminus I$ with $s \leq y$ in L;

(c) $x \in L \setminus I$, $y = (t, j)$ with $x \leq t$ in L;

(d) $x = (s, i)$, $y = (t, j)$ with $s \leq t$ in L and $i \leq j$ in $\underset{\sim}{2}$.

Observe that there is a natural homomorphism ϕ of L[I] onto L defined by

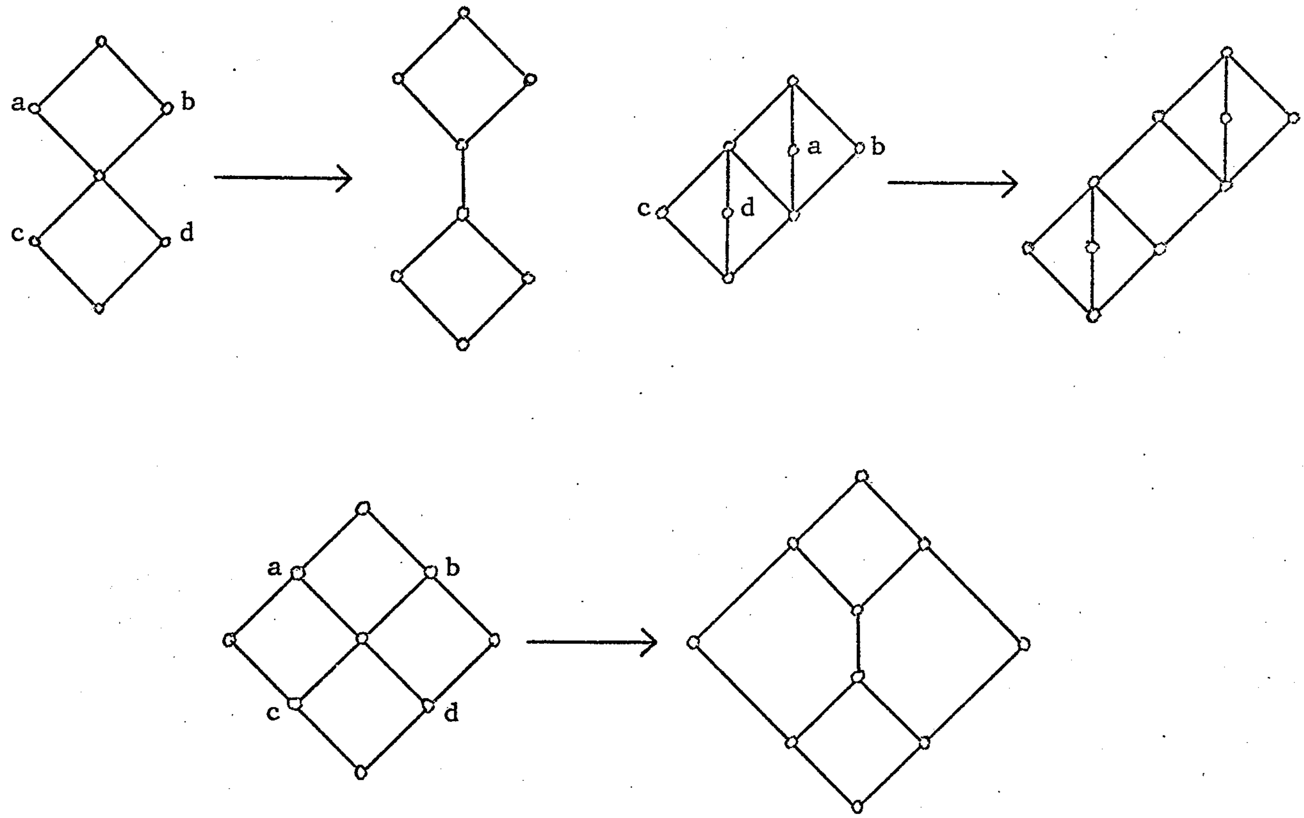
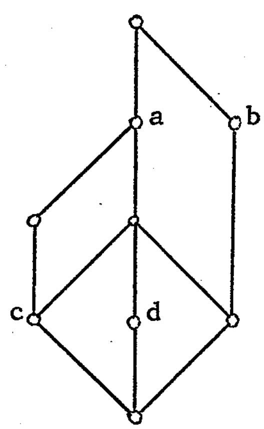


Figure 1.5. Repairing violations of (W) .



L

Figure 1.6.

$$x\phi = \begin{cases} x & \text{if } x \in L \setminus I \\ s & \text{if } x = (s, i) \end{cases} .$$

Now if L is a lattice, and (a, b, c, d) is a violation of (W) in L, we set $I = [a \wedge b, c \vee d]$. It is clear that the violation of (W) defined by a, b, c, and d has been repaired in $L[I]$. By an inverse limit construction, based on repeated applications of the interval construction, A. Day [5] has proved that *for every lattice* L *there exists a lattice* M *satisfying* (W) *and a bounded homomorphism of* M *onto* L. R. Freese and J.B. Nation have shown that for finite L, there need not exist a finite lattice M with this property. In fact, observe that when the interval construction is applied to the lattice L of Figure 1.6, a new violation of (W) is created. It follows that infinitely many applications of the interval construction must be made, and so every lattice satisfying (W) of which L is a homomorphic image must be infinite.

CHAPTER 2. MAXIMAL JOIN-PRESERVING MAPS[1].

For lattices L and M, we endow the set of maps of L into M with the pointwise order. That is, if f and g are maps of L into M, we set $f \leq g$ if and only if $xf \leq xg$ for all $x \in L$. Let $\mu : L \longrightarrow M$ be meet-preserving and consider the set

$$J_\mu = \{\psi : L \longrightarrow M \mid \psi \leq \mu,\ \psi \text{ is join-preserving}\}.$$

Clearly a maximal element μ^∞ of J_μ (if such exists) is a maximum element and hence must be given by

$$x\mu^\infty = \bigvee\{x\psi \mid \psi \in J_\mu\}.$$

LEMMA 2.1. Let $\mu : L \longrightarrow M$ be meet-preserving and assume that μ^∞ exists. If a, b are elements of L such that $(a \wedge b)\mu^\infty < a\mu^\infty \wedge b\mu^\infty$, then there exist elements c and d in L such that (a, b, c, d) is a violation of (W).

Proof. Assume that $(a \wedge b)\mu^\infty < a\mu^\infty \wedge b\mu^\infty$ and define $\eta : L \longrightarrow M$ by

$$x\eta = \begin{cases} x\mu^\infty & \text{if } a \wedge b \nleq x, \\ x\mu^\infty \vee (a\mu^\infty \wedge b\mu^\infty) & \text{if } a \wedge b \leq x. \end{cases}$$

Since $\eta > \mu^\infty$ we have $\eta \notin J_\mu$; but since $\mu^\infty \leq \mu$ and μ is meet-preserving it follows easily that $\eta \leq \mu$. Thus there must exist

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1 Most of the content of this chapter is due jointly to B.A. Davey and the author (see [3]).

$c, d \in L$ such that $(c \vee d)\eta \neq c\eta \vee d\eta$. It follows directly from the definition of η that $a \wedge b \leq c \vee d$, $a \wedge b \nleq c$, and $a \wedge b \nleq d$. Let us suppose that $a \leq c \vee d$; then

$$a\mu^{\infty} \wedge b\mu^{\infty} \leq a\mu^{\infty} \leq (c \vee d)\mu^{\infty},$$

so that

$$\begin{aligned}(c \vee d)\eta &= (c \vee d)\mu^{\infty} \vee (a\mu^{\infty} \wedge b\mu^{\infty}) \\ &= (c \vee d)\mu^{\infty} \\ &= c\mu^{\infty} \vee d\mu^{\infty} \\ &= c\eta \vee d\eta,\end{aligned}$$

a contradiction. Hence we have $a \nleq c \vee d$, and, by symmetry, $b \nleq c \vee d$. □

It follows at once that if L satisfies (W) and μ^{∞} exists, then μ^{∞} is a homomorphism. In particular, if M is complete then the constant map $\omega : L \to \{0_M\}$ is in J_{μ}, and thus $J_{\mu} \neq \emptyset$; hence if L satisfies (W) and M is complete, μ^{∞} exists and is a homomorphism.

Let $\underset{\sim}{K}$ be a class of lattices and let $L \in \underset{\sim}{K}$. We shall say that condition $Q_{\underset{\sim}{K}}(L)$ holds if for each $M \in \underset{\sim}{K}$ and every epimorphism $\phi : M \to L$ there exists a homomorphism $\psi : L \to M$ such that $\psi\phi = id_L$. A simple universal-algebraic argument establishes the next lemma.

LEMMA 2.2. <u>Let $\underset{\sim}{K}$ be an equational class of lattices</u>. <u>Then, for all</u> $L \in \underset{\sim}{K}$, <u>the following are equivalent</u>:

(i) L <u>is a retract of a $\underset{\sim}{K}$-free lattice</u>;

(ii) L <u>is projective in $\underset{\sim}{K}$</u>;

(iii) $Q_{\underset{\sim}{K}}(L)$ <u>holds</u>. □

Denote the class of all lattices with no infinite chains by $\mathbf{FC}$. We are interested in proving a result analogous to Lemma 2.2 for the class $\mathbf{FC}$; naturally, condition (W) will be prominent in this result (Theorem 2.4).

A lattice L is *join-complete* if $\bigvee S$ exists in L for every nonempty subset S of L; hence a join-complete lattice with a least element is complete. If A and B are lattices and $\phi : A \longrightarrow B$ is a bounded epimorphism, we define $\underline{\phi} : B \rightarrow A$ and $\overline{\phi} : B \rightarrow A$ by $b\underline{\phi} = 0_b$ and $b\overline{\phi} = 1_b$; it is easily seen that $\underline{\phi}$ is join-preserving and $\overline{\phi}$ is meet-preserving.

LEMMA 2.3. *Let* A, B, *and* L *be lattices with* A *join-complete and* L *satisfying* (W). *Let* $\phi : A \rightarrow B$ *be a bounded epimorphism, and let* $\psi : L \rightarrow B$ *be a homomorphism. Then there exists a homomorphism* $\psi' : L \longrightarrow A$ *such that* $\psi'\phi = \psi$.

Proof. Define $\mu : L \longrightarrow A$ by $\mu = \psi\overline{\phi}$ and $\hat{\mu} : L \longrightarrow A$ by $\hat{\mu} = \psi\underline{\phi}$; then μ is meet-preserving and $\hat{\mu}$ is join-preserving, whence $\hat{\mu} \in J_{\mu}$. Since A is join-complete, it follows easily that μ^{∞} exists. Since L satisfies (W), Lemma 2.1 implies that μ^{∞} is a homomorphism, and clearly $\hat{\mu} \leq \mu^{\infty} \leq \mu$. It follows immediately that $\psi' = \mu^{\infty}$ is the required homomorphism. □

THEOREM 2.4. *For* $L \in \mathbf{FC}$, *the following are equivalent:*

(i) L *satisfies* (W);

(ii) L *is projective in* $\mathbf{FC}$;

(iii) $Q_{\mathbf{FC}}(L)$ *holds*.

Proof. In $\underset{\sim}{F}\underset{\sim}{C}$ every lattice is complete and every epimorphism is bounded, and hence a lattice $L \in \underset{\sim}{F}\underset{\sim}{C}$ which satisfies (W) is projective in $\underset{\sim}{F}\underset{\sim}{C}$ by Lemma 2.3. It is trivial that if L is projective in $\underset{\sim}{F}\underset{\sim}{C}$ then $Q_{\underset{\sim}{F}\underset{\sim}{C}}(L)$ holds, and so it remains to show only that (iii) implies (i). This is an immediate consequence of Day's interval construction. Let (a, b, c, d) be a violation of (W) in L, and let $I = [a \wedge b, c \vee d]$. Then $M = L[I]$ is in $\underset{\sim}{F}\underset{\sim}{C}$, and there is a natural epimorphism $\phi : M \to L$, but it is easy to see that no homomorphism $\psi : L \to M$ satisfying $\psi\phi = id_L$ exists. □

Remark. Theorem 2.4 may also be proved by a slight modification of the methods used by R. McKenzie in [23].

Since (W) is inherited by sublattices we have the following rather surprising corollary.

COROLLARY 2.5. If L is projective in $\underset{\sim}{F}\underset{\sim}{C}$, then every sublattice of L is projective in $\underset{\sim}{F}\underset{\sim}{C}$. □

Let $\underset{\sim}{Equ}(\underset{\sim}{K})$ denote the equational class generated by a class $\underset{\sim}{K}$ of lattices, and denote the equational class generated by a single lattice L by $\underset{\sim}{Equ}(L)$.

COROLLARY 2.6.[1] Let $\underset{\sim}{K}$ be a class of lattices of length less than some fixed natural number, and let L be a finite subdirectly irreducible

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1 Corollary 2.6 as stated is a generalization due to H.P. Gumm of the authors' original result.

lattice which satisfies (W) . *Then* $\mathrm{Equ}(L) \subseteq \mathrm{Equ}(K)$ *if and only if* L *is isomorphic to a sublattice of a lattice in* K .

Proof. Assume $\mathrm{Equ}(L) \subseteq \mathrm{Equ}(K)$. Since L is subdirectly irreducible it follows, by Jónsson's lemma (see [18, Corollary 3.2, p. 115]), that there is an epimorphism $\phi : M \to L$ where M is a sublattice of an ultraproduct P of lattices in K . A first-order sentence holds in an ultraproduct if and only if it holds in "almost all" the factors of the ultraproduct. As "*length less than some fixed natural number*" is a first-order property it follows that P (and hence M) is of finite length and so has no infinite chains. Since $Q_{FC}(L)$ holds, L is isomorphic to a sublattice of M and hence of P . For any finite lattice L , the statement "L *is isomorphic to a sublattice of*" is first-order, whence L is isomorphic to a sublattice of at least one of the factors of P . Thus L is isomorphic to a sublattice of a lattice in K . The converse is trivial. □

We note in passing that both M_3 and N_5 are subdirectly irreducible and satisfy (W) .

An equational class K is *locally finite* if every finitely generated lattice in K is finite; for example, if L is a finite lattice, then $\mathrm{Equ}(L)$ is locally finite.

COROLLARY 2.7. *Let* K *be a locally finite equational class of lattices. Then every finite lattice* $L \in K$ *which satisfies* (W) *is projective in* K .

<u>Proof.</u> Every finite lattice L in $\underset{\sim}{K}$ is a homomorphic image of a finitely generated $\underset{\sim}{K}$-free lattice $\mathfrak{F}_{\underset{\sim}{K}}(n)$. Since $\underset{\sim}{K}$ is locally finite it follows that $\mathfrak{F}_{\underset{\sim}{K}}(n)$ is finite. If L satisfies (W), then, by Theorem 2.4, L is a retract of $\mathfrak{F}_{\underset{\sim}{K}}(n)$ and so, by Lemma 2.2, is projective in $\underset{\sim}{K}$. □

It follows from this corollary that both M_3 and N_5 are projective in the equational classes they generate.

The next theorem is an especially interesting consequence of Lemma 2.3. We will require the following result.

LEMMA 2.8. <u>Let</u> L <u>and</u> M <u>be lattices and let</u> $\phi : M \to L$ <u>be a homomorphism. Define a mapping</u> $\hat{\phi} : \mathcal{J}(M) \to \mathcal{J}(L)$ <u>by</u> $I\hat{\phi} = (I\phi]$, <u>the ideal generated by</u> $I\phi \subseteq L$, <u>for each</u> $I \in \mathcal{J}(M)$. <u>Then</u> $\hat{\phi}$ <u>is a homomorphism; furthermore, if</u> ϕ <u>is onto then</u> $\hat{\phi}$ <u>is onto and upper bounded</u>.

<u>Proof.</u> That $\hat{\phi}$ is a homomorphism is straightforward and is left to the reader. To prove the last part of the lemma, let ϕ be onto, and observe that for $I \in \mathcal{J}(L)$, $I\phi^{-1}$ is an ideal of M; in fact, $I\phi^{-1}$ is clearly the largest ideal of M whose image under $\hat{\phi}$ is I. Hence $\hat{\phi}$ is onto and upper bounded. □

THEOREM 2.9. <u>Let</u> ϕ <u>be a lower bounded homomorphism of a lattice</u> M <u>onto a lattice</u> L <u>satisfying</u> (W). <u>Then</u> $\mathcal{J}(L)$ <u>is a retract of</u> $\mathcal{J}(M)$.

Proof. Let $\hat{\phi} : \mathcal{J}(M) \longrightarrow \mathcal{J}(L)$ be as in Lemma 2.8 and let $I \in \mathcal{J}(L)$. Since ϕ is lower bounded, we may consider $\{0_x \mid x \in I\} \subseteq M$. Observe that the ideal generated by $\{0_x \mid x \in I\}$ is clearly the smallest ideal of M whose image under $\hat{\phi}$ is I. Hence $\hat{\phi}$ is lower bounded and from Lemma 2.8 is bounded. Also $\mathcal{J}(M)$ is join-complete, and since L satisfies (W) so does $\mathcal{J}(L)$. We may therefore apply Lemma 2.3 to complete the proof. □

The next corollary generalizes a result of R. Freese [6].

COROLLARY 2.10. Let L be a lattice satisfying (W). Then L is isomorphic to a sublattice of the lattice of ideals of the lattice of dual ideals of a free lattice. If in addition L has no infinite chains then L is a retract of the lattice of ideals of the lattice of dual ideals of a free lattice.

Proof. Let ϕ' be a homomorphism from a free lattice F onto L. Dualizing Lemma 2.8, there is a lower bounded homomorphism of $\mathcal{D}(F)$ onto $\mathcal{D}(L)$. Since L satisfies (W), so does $\mathcal{D}(L)$; hence from Theorem 2.9 $\mathcal{J}(\mathcal{D}(L))$ is a retract of $\mathcal{J}(\mathcal{D}(F))$. Since L is embedded in $\mathcal{J}(\mathcal{D}(L))$, L is isomorphic to a sublattice of $\mathcal{J}(\mathcal{D}(F))$. If L has no infinite chains then $L \cong \mathcal{J}(\mathcal{D}(L))$ and hence is a retract of $\mathcal{J}(\mathcal{D}(F))$. □

Clearly in this corollary F could have been replaced with any lattice of which L is a homomorphic image.

Remark. Lemma 2.8 and Corollary 2.10 have also been proved in a similar manner by B. Jónsson and J.B. Nation [20].

We present one more interesting consequence of Lemma 2.3. For lattices L and M, $\mathrm{Hom}(L, M)$ denotes the set of all homomorphisms of L into M, endowed with the pointwise partial order.

THEOREM 2.11. <u>A lattice</u> L <u>satisfies</u> (W) <u>if and only if</u> $\mathrm{Hom}(L, M)$ <u>is a lattice for every complete lattice</u> M.

<u>Proof.</u> Assume that L satisfies (W). Let M be a complete lattice, and for $\alpha, \beta \in \mathrm{Hom}(L, M)$, define $\mu : L \to M$ by $x\mu = x\alpha \wedge x\beta$. Then μ is meet-preserving, and, from Lemma 2.3, μ^{∞} exists and is an element of $\mathrm{Hom}(L, M)$; in fact, μ^{∞} is clearly the greatest lower bound of α and β. The least upper bound of α and β is found dually.

Conversely, let (a, b, c, d) be a violation of (W) in L; we shall construct a complete lattice M such that $\mathrm{Hom}(L, M)$ is not a lattice. Let $\bar{L}$ be a complete lattice such that there is an embedding η of L into $\bar{L}$. Set $\bar{x} = x\eta$ for $x \in L$. Next let $K = \underset{\sim}{2}^2 \oplus \underset{\sim}{1}$ and let x_1 and x_2 be the two atoms, y the co-atom, and v the unit of K. Consider $\bar{L} \times K$. The elements $(\bar{a}, y)$, $(\bar{b}, y)$, $(\bar{c}, y)$, and $(\bar{d}, y)$ of $\bar{L} \times K$ define a violation of (W); apply Day's interval construction to obtain $M = (\bar{L} \times K)[(\bar{a}, y) \wedge (\bar{b}, y), (\bar{c}, y) \vee (\bar{d}, y)]$, a complete lattice. Define $\alpha_1, \alpha_2, \beta$, and $\gamma \in \mathrm{Hom}(L, M)$ by

$$x\alpha_i = (\bar{x}, x_i), \quad i = 1, 2,$$

$$x\beta = (\bar{x}, v),$$

$$x\gamma = (1_{\bar{L}}', y), \text{ where } 1_{\bar{L}} \text{ is the unit of } \bar{L}.$$

Suppose α_1 and α_2 have a least upper bound $\alpha \in \mathrm{Hom}(L, M)$; since β and γ are upper bounds for both α_1 and α_2, we get $\alpha_1 \vee \alpha_2 \leq \alpha \leq \beta \wedge \gamma$. It is easy to show that $x\alpha = (\bar{x}, y)$ for $x \in \{a, b, c, d\}$, and from the construction of M we can now deduce that $(a \wedge b)\alpha \nleq (c \vee d)\alpha$. Since $a \wedge b \leq c \vee d$, we have a contradiction. Hence the least upper bound of α_1 and α_2 cannot exist, and we are done. □

Theorem 2.4 suggests several generalizations; we conclude this chapter with counterexamples disposing of two of these. (Chapter 3 will deal with a third.)

First, the assumption of no infinite chains cannot be relaxed to the Ascending Chain Condition. The lattice of Figure 2.1 satisfies the A.C.C., has M_3 as a homomorphic image, but has no sublattice isomorphic to M_3.

We now present a finite lattice L for which $Q_{\underset{\sim}{F}\underset{\sim}{C}}(L)$ fails although L can be embedded into each lattice $M \in \underset{\sim}{F}\underset{\sim}{C}$ of which it is a homomorphic image. The lattice L is shown in Figure 2.2; note that the only violation of (W) is the one indicated by $a, b, c,$ and d.

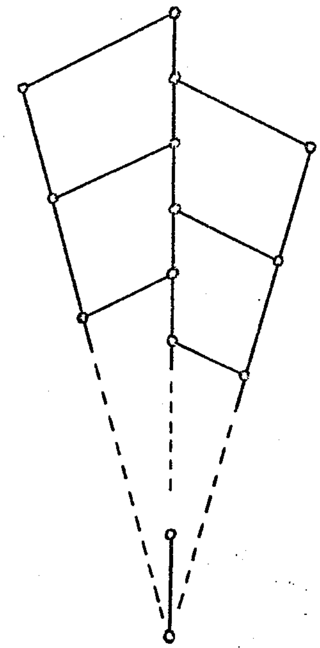

Figure 2.1.

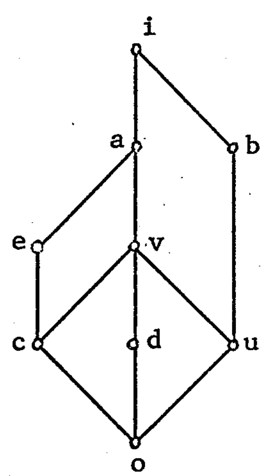

Figure 2.2.

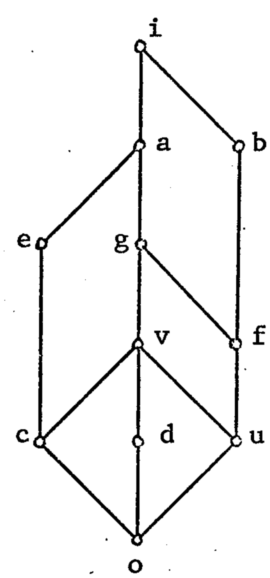

Figure 2.3.

Assume that $M \in \underset{\sim}{F}\underset{\sim}{C}$ and that $\phi : M \to L$ is an epimorphism. Since ϕ is bounded we may define $\mu : L \to M$ by $x\mu = 1_x$ and $\hat{\mu} : L \to M$ by $x\hat{\mu} = 0_x$. Then, as in the proof of Lemma 2.3, $\psi = \mu^{\infty}$ is a join-preserving map which satisfies $\psi\phi = \mathrm{id}_L$. If ψ preserves meets, then it is an embedding and we are through. Otherwise, by Lemma 2.1, ψ preserves all meets except the meet of a and b, and thus $a\psi \wedge b\psi > u\psi$. For all $x \in L$ let $x' = x\psi$. Define $f' = a' \wedge b'$ and $g' = v' \vee f'$; since

$$g'\phi = (v\psi \vee (a\psi \wedge b\psi))\phi = v \vee (a \wedge b) = v ,$$

we have $g' < a'$. Let $\bar{L}$ be the partial lattice shown in Figure 2.3; all meets and joins in $\bar{L}$ are as indicated by the diagram except that the meet of e and g is undefined; then the map $\psi' : \bar{L} \to M$, defined by $x\psi' = x'$, is a partial-lattice monomorphism. Thus the set $\mathcal{P}$ of partial-lattice monomorphisms of $\bar{L}$ into M is nonempty. Consider $\mathcal{J} = \{i\mu \mid \mu \in \mathcal{P}\}$; since M has no infinite chains, $\mathcal{J}$ has at least one minimal element. Let η be an element of $\mathcal{P}$ such that $i\eta$ is minimal in $\mathcal{J}$. For notational convenience we shall identify $\bar{L}$ with its image $\bar{L}\eta$ in M. If $e \wedge g = c$, then

$$\{a, e, g, v, f, c, d, u, 0\}$$

is a sublattice of M isomorphic to L, as required. If $e \wedge g > c$, then $v \vee (e \wedge g) = g$; for otherwise

$$\{a, e, g, v \vee (e \wedge g), e \wedge g, v, f, c, d, u, 0\} ,$$

as a relative sublattice of M, is a monomorphic image of $\bar{L}$ whose unit, a , is strictly less than i , contradicting the choice of η . It is now easily seen that

$$\{i, a, b, e, g, e \wedge g, d, f, 0\}$$

is a sublattice of M isomorphic to L , and we are through.

This example leads us to pose the following problem.

Characterize those lattices in $\underset{\sim}{F}\,\underset{\sim}{C}$ which can be embedded into each lattice M in $\underset{\sim}{F}\,\underset{\sim}{C}$ of which they are a homomorphic image.

CHAPTER 3. THE BOUNDED EPIMORPHISM CONDITION.

One of the results in R. McKenzie's well-known 1972 paper [23] is that a finite lattice is a sublattice of a free lattice if and only if it is a bounded homomorphic image of a free lattice and satisfies (W). In fact he proves the following, which we reformulate for our purposes.

<u>Let</u> ϕ <u>be a bounded homomorphism from a free lattice</u> F <u>onto a finite lattice</u> L <u>satisfying</u> (W). <u>Then there is a homomorphism</u> $\psi : L \longrightarrow F$ <u>such that</u> $\psi\phi = id_L$.

This was soon extended to arbitrary L by A. Kostinsky [21]. In the previous chapter, we proved a result in a similar vein: <u>Let</u> M <u>be a lattice with no infinite chains, and let</u> ϕ <u>be a homomorphism of</u> M <u>onto a lattice</u> L <u>satisfying</u> (W); <u>then there is a homomorphism</u> $\psi : L \longrightarrow M$ <u>such that</u> $\psi\phi = id_L$. Of course ϕ will be bounded. We are led to the following definition.

DEFINITION 3.1. A lattice L satisfies the <u>bounded epimorphism condition</u> if whenever ϕ is a bounded epimorphism from a lattice M onto L, there exists a homomorphism $\psi : L \longrightarrow M$ such that $\psi\phi = id_L$.

The problem of characterizing all lattices satisfying the bounded epimorphism condition was posed in [26], and is still open. It is an immediate consequence of the interval construction that any lattice satisfying the bounded epimorphism condition must satisfy (W). From the preceding discussion one might wonder whether, for finite lattices at least, the converse holds as well. We will first show that the answer to this question is no. Secondly, we will exhibit a property

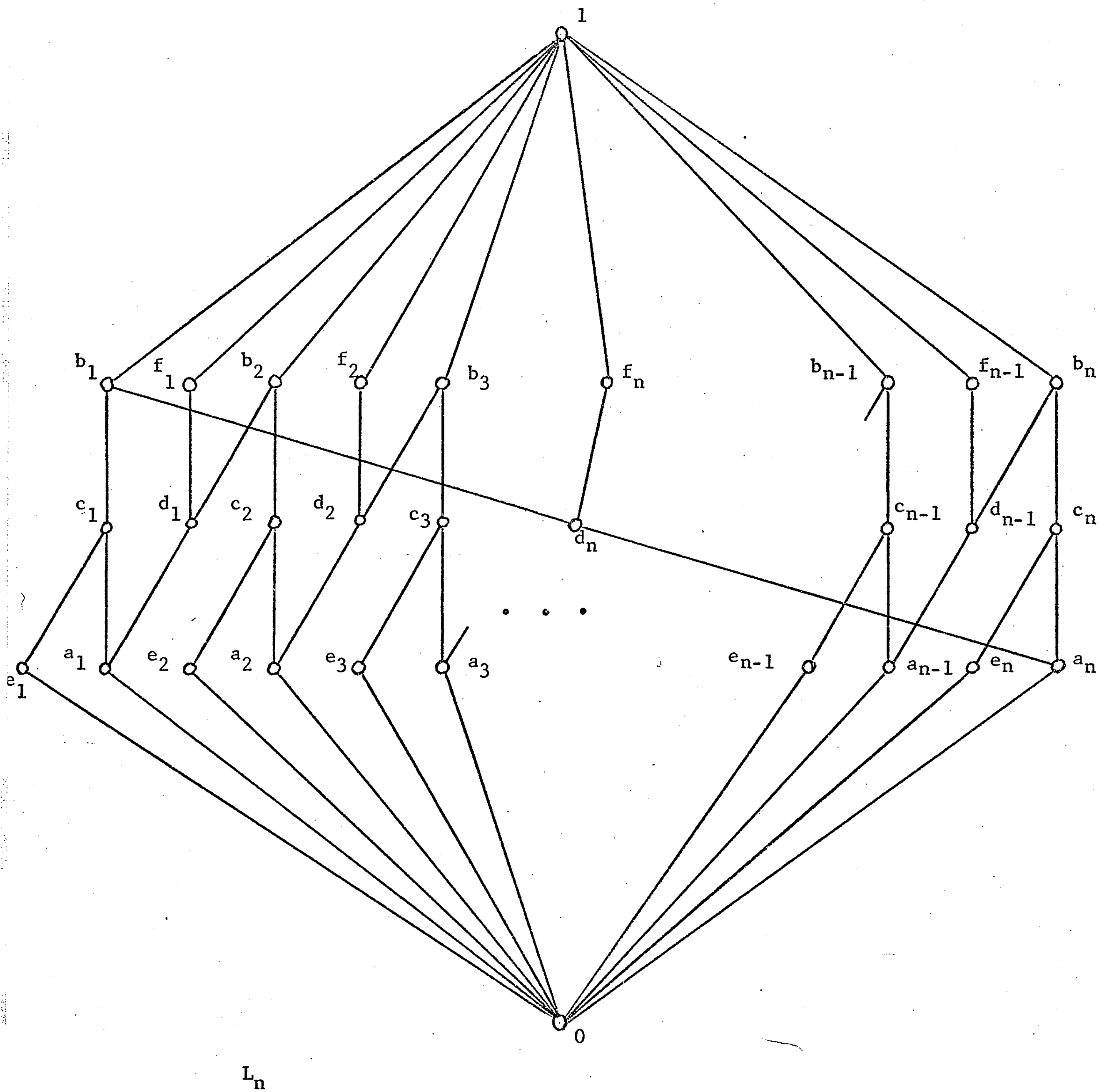
1
b_1
f_1
b_2
f_2
b_3
f_n
b_{n-1}
f_{n-1}
b_n
c_1
d_1
c_2
d_2
c_3
d_n
c_{n-1}
d_{n-1}
c_n
e_1
a_1
e_2
a_2
e_3
a_3
e_{n-1}
a_{n-1}
e_n
a_n
0
L_n

Figure 3.1.

sufficient to imply that a finite lattice satisfies the bounded epimorphism condition.

Let L_n be the lattice of Figure 3.1, where n is any integer ≥ 3 ; it can easily be checked that L_n satisfies (W) . We will construct a lattice M and a bounded epimorphism $\phi : M \longrightarrow L_n$ such that there cannot exist a homomorphism $\psi : L_n \longrightarrow M$ satisfying $\psi\phi = id_{L_n}$. Let $\omega = \{1, 2, 3, \cdots\}$ be the chain of natural numbers with the usual ordering, and let $\omega^* = \{n^* : n \in \omega\}$ denote its dual ; note that $1^* > 2^* > 3^* > \cdots$ holds in ω^* . Define a subposet M_ℓ of $L_n \times (\omega \oplus \omega^*)$ by

$$\begin{aligned} M_\ell = &\{(x, t) \mid x = a_i, c_i, d_i, \text{ or } e_i \text{ for some } i ; \ t \in \omega\} \\ &\cup \{(b_i, nt + i) \mid i = 1, \cdots, n ; \ t \in \omega\} \\ &\cup \{(f_i, nt + i) \mid i = 1, \cdots, n ; \ t \in \omega\} \\ &\cup \{(0, 1), (1, 1^*)\} . \end{aligned}$$

M_ℓ is not a sublattice of $L_n \times (\omega \oplus \omega^*)$; nevertheless, we claim that M_ℓ is a lattice. The following observations are easily verified.

(i) If $x, y \in L_n$ and $t \in \omega$ are such that $x \neq 1$, $y \neq 0$, $(x, t) \in M_\ell$, and $y < x$, then $(y, t) \in M_\ell$.

(ii) Given $x \in L_n \setminus \{0, 1\}$, $t \in \omega$, there is a smallest element $\lambda(x, t)$ of ω such that $\lambda(x, t) \geq t$ and $(x, \lambda(x, t)) \in M_\ell$.

Now the reader may check that for $(x_1, t_1), (x_2, t_2) \in M_\ell$, their meet and join exist in M_ℓ and are defined as follows:

$$(x_1, t_1) \wedge (x_2, t_2) = \begin{cases} (0, 1) & \text{if } x_1 \wedge x_2 = 0 \\ (x_1 \wedge x_2, t_1 \wedge t_2) & \text{otherwise} \end{cases}$$

$$(x_1, t_1) \vee (x_2, t_2) = \begin{cases} (0, 1) & \text{if } (x_1, t_1) = (x_2, t_2) = (0, 1) \\ (1, 1^*) & \text{if } x_1 \vee x_2 = 1 \\ (x_1 \vee x_2, \lambda(x_1 \vee x_2, t_1 \vee t_2)) & \text{otherwise} \end{cases} .$$

Next, L_n is self-dual ; let α be some anti-automorphism of L_n. Define $M_u \subseteq L_n \times (\omega \oplus \omega^*)$ by

$$M_u = \{(x\alpha, t^*) \mid (x, t) \in M_\ell \setminus \{(1, 1^*)\}\} \cup \{(0, 1)\} .$$

By dualizing the preceding argument we see that M_u is a lattice. Let $M = M_\ell \cup M_u$. It can now be readily verified that M is a lattice with joins defined by

$$(x_1, t_1) \vee (x_2, t_2) = \begin{cases} (1, 1^*) & \text{if } x_1 \vee x_2 = 1 \\ (x_1 \vee x_2, t_1 \vee t_2) & \text{if } (x_1, t_1) \text{ or } (x_2, t_2) \text{ is in } M_u \text{ and } x_1 \vee x_2 \neq 1 \\ (x_1 \vee x_2, \lambda(x_1 \vee x_2, t_1 \vee t_2)) & \text{otherwise} \end{cases}$$

and meets defined dually.

Let $\phi : M \to L_n$ be the first projection; that is, $(x, t)\phi = x$ for all $(x, t) \in M$. From the above it is evident that ϕ is a bounded homomorphism of M onto L_n. Suppose $\psi : L_n \to M$ were a homomorphism such that $\psi\phi = id_{L_n}$. For each $x \in L_n$,

$x\psi = (x, t(x))$ for some $t(x) \in \omega \oplus \omega^*$; we will call $x \in L_n$ <u>lower</u> (resp. <u>upper</u>) if $t(x) \in \omega$ (resp. ω^*) . If a_1 and a_2 are lower, so is $b_2 = a_1 \vee a_2$; hence it is impossible that both all a_i's are lower and all b_i's are upper. By symmetry we may suppose that at least one b_i is lower. Let b_k be lower such that $t(b_k)$ is minimal among all $t(b_i)$'s . Then from the construction of M_ℓ , $t(b_k) \equiv k \mod n$. Since $b_k \wedge b_{k-1} = f_{k-1} \wedge b_{k-1} = a_{k-1}$ (where the indices should be read modulo n if necessary), we have that $b_k \psi \wedge b_{k-1} \psi = f_{k-1} \psi \wedge b_{k-1} \psi$; since b_k is lower, one of f_{k-1} and b_{k-1} must be lower, so these meets are defined pointwise. Thus $t(f_{k-1}) \wedge t(b_{k-1}) = t(b_k) \wedge t(b_{k-1}) = t(b_k)$ by the choice of b_k . Since $\omega \oplus \omega^*$ is a chain, $t(f_{k-1})$ and $t(b_{k-1})$ are comparable. Assume $t(f_{k-1}) \le t(b_{k-1})$; then f_{k-1} is lower, and hence $t(f_{k-1}) \equiv k-1 \mod n$ by the construction of M_ℓ . Since $t(f_{k-1}) = t(f_{k-1}) \wedge t(b_{k-1}) = t(b_k) \equiv k \mod n$, this is a contradiction. A similar contradiction arises if $t(b_{k-1}) \le t(f_{k-1})$; hence the homomorphism ψ cannot exist.

The next theorem introduces a property defined for finite lattices which is sufficient to imply that the bounded epimorphism condition holds. In what follows, MR denotes the set of meet reducible elements and JR the set of join reducible elements of the lattice L , and we set $R = MR \cup JR$.

THEOREM 3.2. *Let* L *be a finite lattice satisfying the condition*

(*) *there exists a linear ordering* $r_1, r_2, \cdots, r_n$ *of the elements of* R *such that*

(i) $r_i = a \vee b \in JR$, $r_j \in MR$, $r_i \geq r_j$, $a \not\geq r_j$, $b \not\geq r_j$ *implies* $i < j$,

and dually

(ii) $r_i \in JR$, $r_j = a \wedge b \in MR$, $r_i \geq r_j$, $r_i \not\geq a$, $r_i \not\geq b$ *implies* $i > j$.

Then L *satisfies the bounded epimorphism condition*.

Proof. We shall first observe that L must satisfy (W). Suppose there exist $a, b, c, d \in L$ such that $a \wedge b \leq c \vee d$, $a \wedge b \not\leq c$, $a \wedge b \not\leq d$, $a \not\leq c \vee d$, and $b \not\leq c \vee d$. Letting $r_i = c \vee d$ and $r_j = a \wedge b$ and applying both (i) and (ii), we conclude that both $i < j$ and $i > j$ hold, which is impossible. Thus L satisfies (W).

Let M be a lattice and let $\phi : M \longrightarrow L$ be a bounded epimorphism. If $x \in L$, we will denote the least and greatest element of $x\phi^{-1}$ by 0_x and 1_x respectively; note that $0_x \vee 0_y = 0_{x \vee y}$ and $1_x \wedge 1_y = 1_{x \wedge y}$ hold in M. Define $\psi_1 : L \rightarrow M$ by :

if $r_1 \in JR$, $x\psi_1 = 0_x$ for each $x \in L$

if $r_1 \in MR$, $x\psi_1 = 1_x$ for each $x \in L$

and inductively define $\psi_k : L \longrightarrow M$, $2 \le k \le n$, by:

if $r_k \in JR$,

$$x\psi_k = \bigvee\{r_i\ \psi_{k-1} \mid x \ge r_i \text{ with } i \le k-1\} \vee 0_x \text{ for each } x \in L$$

if $r_k \in MR$,

$$x\psi_k = \bigwedge\{r_i\ \psi_{k-1} \mid x \le r_i \text{ with } i \le k-1\} \wedge 1_x \text{ for each } x \in L .$$

Notice that ψ_k is well-defined for each k , since L satisfies (W) and hence has no doubly reducible elements. We claim that $\psi = \psi_n$ is the homomorphism from L to M that we seek. The proof of this consists of a series of statements, most of which are easily verified. As an aid to the reader we remark that the first few statements, ending with $\underset{\sim}{F}$, show that the maps ψ_k , as elements of the partially ordered set of homomorphisms of L into M , are "nested" .

$\underset{\sim}{A}$ $\psi_k \phi = id_L$ <u>for each</u> k .

$\underset{\sim}{B}$ ψ_k <u>is order preserving for each</u> k .

These two statements follow easily from the definition of ψ_k .

$\underset{\sim}{C}$ $r_k\ \psi_{\ell_1} = r_k\ \psi_{\ell_2}$ <u>for all</u> $\ell_1, \ell_2 \ge k$.

We need only show that $r_k\ \psi_{\ell+1} = r_k\ \psi_\ell$ for each $\ell \ge k$. This is clear, since r_k is one of the r_i's in the definition of $r_k\ \psi_{\ell+1}$.

$\underset{\sim}{D}$ (i) $r_k, r_\ell \in JR$ <u>with</u> $k \le \ell \Rightarrow \psi_k \le \psi_\ell$.

(ii) $r_k, r_\ell \in MR$ <u>with</u> $k \le \ell \Rightarrow \psi_k \ge \psi_\ell$.

Considering (i), we have for each $x \in L$,

$$\begin{aligned} x\psi_k &= \bigvee\{r_i\ \psi_{\ell-1} \mid x \ge r_i \text{ with } i \le k-1\} \vee 0_x \text{ from } \underset{\sim}{C} \\ &\le \bigvee\{r_i\ \psi_{\ell-1} \mid x \ge r_i \text{ with } i \le \ell-1\} \vee 0_x \\ &= x\psi_\ell . \end{aligned}$$

(ii) is dual.

$\underset{\sim}{E}$ (i) $r_k \in JR \Rightarrow \psi_k \leq \psi_{k-1}$.

(ii) $r_k \in MR \Rightarrow \psi_k \geq \psi_{k-1}$.

This follows easily from $\underset{\sim}{A}$, $\underset{\sim}{B}$, and the definition of ψ_k .

$\underset{\sim}{F}$ (i) $r_k \in JR \Rightarrow \psi_k \leq \psi_\ell$ <u>for all</u> $\ell \geq k$.

(ii) $r_k \in MR \Rightarrow \psi_k \geq \psi_\ell$ <u>for all</u> $\ell \geq k$.

Consider (i). If $r_\ell \in JR$ we are done by $\underset{\sim}{D}$(i) . Proceeding by induction, let $\psi_{\ell-1} \geq \psi_k$, and assume $r_\ell \in MR$. Then $\psi_\ell \geq \psi_{\ell-1}$ from $\underset{\sim}{E}$(ii), and hence $\psi_\ell \geq \psi_k$. (ii) is dual.

$\underset{\sim}{G}$ (i) <u>If</u> $x \vee y = r_k \in JR$ <u>then</u> $(x \vee y)\psi_k = x\psi_k \vee y\psi_k$.

(ii) <u>If</u> $x \wedge y = r_k \in MR$ <u>then</u> $(x \wedge y)\psi_k = x\psi_k \wedge y\psi_k$.

Consider (i) . We claim that for all $\ell \leq k-1$, if $r_\ell \leq x \vee y$ then $r_\ell \psi_{k-1} \leq x\psi_k \vee y\psi_k$. First, if $r_\ell \leq x$ then $r_\ell\psi_{k-1} \leq \bigvee\{r_j \psi_{k-1} \mid x \geq r_j \text{ with } j \leq k-1\} \vee 0_x = x\psi_k \leq x\psi_k \vee y\psi_k$, so if $r_\ell \leq x$ or similarly $r_\ell \leq y$ the claim is true. Hence we can assume $r_\ell \leq x \vee y = r_k$, $r_\ell \nleq x$, and $r_\ell \nleq y$, where $\ell \leq k-1$. Since L satisfies (*) , this implies that $r_\ell \notin MR$, and thus $r_\ell \in JR$. We now proceed to prove the claim by induction.
If $\ell = 1$ and we assume $r_1 \in JR$, we have by $\underset{\sim}{C}$ and $\underset{\sim}{A}$ that

$$r_1 \psi_{k-1} = r_1 \psi_1 = 0_{r_1} \leq 0_{x \vee y} = 0_x \vee 0_y \leq x\psi_k \vee y\psi_k ,$$

as desired. Assume that the claim is true for all r_j, $j < \ell$, and let $r_\ell \in JR$. Then from $\underset{\sim}{C}$, $r_\ell \psi_{k-1} = \bigvee\{r_j \psi_{\ell-1} \mid r_\ell \geq r_j \text{ with } j \leq \ell-1\} \vee 0_{r_\ell}$;

for each r_j occurring in this join we have $r_j \le r_\ell \le x \vee y$ and $j < \ell$, and thus $r_j \psi_{\ell-1} = r_j \psi_{k-1} \le x\psi_k \vee y\psi_k$ by $\underset{\sim}{C}$ and the induction hypothesis. Since $0_{r_\ell} \le 0_{x \vee y} = 0_x \vee 0_y \le x\psi_k \vee y\psi_k$, we get $r_\ell \psi_{k-1} \le x\psi_k \vee y\psi_k$, and the claim is proved. Now $(x \vee y)\psi_k = \bigvee \{r_\ell \psi_{k-1} \mid x \vee y \ge r_\ell$ with $\ell \le k-1\} \vee 0_{x \vee y} \le x\psi_k \vee y\psi_k$, and from $\underset{\sim}{B}$ we have equality, proving (i). (ii) is dual.

$\underset{\sim}{H}$ (i) <u>If</u> $x \vee y = r_k \in JR$ <u>then</u> $(x \vee y)\psi_\ell = x\psi_\ell \vee y\psi_\ell$ <u>for all</u> $\ell \ge k$.

(ii) <u>If</u> $x \wedge y = r_k \in MR$ <u>then</u> $(x \wedge y)\psi_\ell = x\psi_\ell \wedge y\psi_\ell$ <u>for all</u> $\ell \ge k$.

For (i), we have

$$\begin{aligned} (x \vee y)\psi_\ell &= (x \vee y)\psi_k && \text{from } \underset{\sim}{C} \\ &= x\psi_k \vee y\psi_k && \text{from } \underset{\sim}{G} \text{ (i)} \\ &\le x\psi_\ell \vee y\psi_\ell && \text{from } \underset{\sim}{F} \text{ (i)} \end{aligned}$$

and so from $\underset{\sim}{B}$ we have equality. (ii) is dual.

Let $\psi = \psi_n$. The theorem follows from $\underset{\sim}{A}$ and $\underset{\sim}{H}$, since ψ_n will preserve all joins and meets. □

<u>Remark.</u> From the previous results it is apparent that the lattice L_n of Figure 3.1 must violate (*) ; this can be demonstrated directly by considering the reducible elements $a_i, b_i, 1 \le i \le n$, of L_n and the fact that $a_{i-1} \vee e_i = b_i > a_i$ and $b_i \wedge f_i = a_i < b_{i+1}$ hold for each i.

CHAPTER 4. WHITMAN'S CONDITION AND SEMIDISTRIBUTIVITY.

We now introduce two lattice properties, known as the semidistributive laws, that are often associated with (W).

($SD_\vee$) for all a, b, c, $a \vee b = a \vee c$ implies $a \vee b = a \vee (b \wedge c)$

($SD_\wedge$) for all a, b, c, $a \wedge b = a \wedge c$ implies $a \wedge b = a \wedge (b \vee c)$

B. Jónsson [17] originated ($SD_\vee$) and ($SD_\wedge$) and observed that they, like (W), hold in sublattices of a free lattice. This brings us to one of the most celebrated conjectures in lattice theory.

CONJECTURE (B. Jónsson-J. Kiefer [19]): A finite lattice is a sublattice of a free lattice if and only if it satisfies ($SD_\vee$), ($SD_\wedge$), and (W).

We wish to outline some of the advances made in the study of finite sublattices of a free lattice. Several new definitions will be required; we present them en masse.

Let L be a lattice and let $X, Y \subseteq L$. We will write that $X << Y$ if for every $x \in X$ there exists $y \in Y$ such that $x \leq y$. The pair $\langle p, J\rangle$, where $p \in L$ and J is a finite subset of L, is called a minimal pair if the following three conditions hold:

(i) $p \notin J$;

(ii) $p \leq \bigvee J$;

(iii) if J' is a finite subset of L such that $p \leq \bigvee J'$ and $J' << J$, then $J' \supseteq J$.

A finite lattice L is said to satisfy the condition ($T_\vee$) if there is a linear ordering $\{x_1, \cdots, x_n\}$ of all the elements of L such that if $\langle x_i, J\rangle$ is a minimal pair and $x_j \in J$, then $i < j$. L is said to satisfy $(T_\wedge)$ if the dual of L satisfies $(T_\vee)$.

A lattice L is <u>transferable</u> if whenever M is a lattice and η' is an embedding of L into $\mathcal{J}(M)$, there is an embedding η of L into M; L is <u>sharply transferable</u> if η can always be chosen such that $x \leq y \Leftrightarrow x\eta \in y\eta'$ for all $x, y \in L$.

The <u>breadth</u> $b(L)$ of the finite lattice L is the smallest integer b such that every join $\bigvee_{i=1}^{b+1} x_i$ of elements of L is equal to a join of b of the x_i's. For instance it is clear that a finite lattice has breadth one if and only if it is a chain. Finally, it is known that the class of all lattices of breadth at most two contains all <u>planar</u> lattices, i.e. all finite lattices that can be drawn on the plane without crossings.

Galvin and Jónsson [8] had already handled the first special case of the Jónsson-Kiefer conjecture in 1961 by showing that it holds for finite distributive lattices; what is more, they explicitly listed all finite distributive sublattices of a free lattice (this result has already been mentioned in Chapter 1).

A lattice L is <u>semidistributive</u> if both $(SD_\vee)$ and $(SD_\wedge)$ hold in L. As the name implies, every distributive lattice is semidistributive. It is also trivial to check that a modular semidistributive lattice is distributive; hence Galvin and Jónsson had really settled the conjecture for all finite modular lattices.

McKenzie showed in 1972 [23] that a finite lattice is a sublattice of a free lattice if and only if it is a bounded homomorphic image of a free lattice and satisfies (W). The work of H.S. Gaskill [9, 10] and later papers by Gaskill, Grätzer, and Platt [11] and Gaskill and Platt [12] established the following characterization.

THEOREM 4.1. The following are equivalent for a finite lattice L:

(i) L is a sublattice of a free lattice;

(ii) L satisfies $(T_\vee)$, $(T_\wedge)$, and (W);

(iii) L is sharply transferable.

In their 1962 paper [19], Jónsson and Kiefer showed that a finite sublattice of a free lattice has breadth at most four. They further proved that if the conjecture holds for all finite lattices of breadth at most three, then it holds for all finite lattices. Using (i) and (ii) of Theorem 4.1 above, I. Rival and the author [24] have proved that the conjecture holds for all finite planar lattices. Furthermore many (but not all) of their techniques are applicable more generally to lattices of breadth two; it is therefore hoped that the conjecture may be settled for breadth two lattices in the near future. In a companion paper [25], I. Rival and the author have also constructed a family $\mathfrak{F}$ of finite lattices with the following property: a finite lattice is a planar sublattice of a free lattice if and only if it does not have a sublattice isomorphic to a member of $\mathfrak{F}$. These two papers have been included here as Appendices 1 and 2.

In closing we remark that a current paper of Jónsson and Nation [20] deals extensively with the topic of sublattices of a free lattice, including a survey of recent work on the subject. In particular they exhibit a finitely generated (infinite) lattice, found by Freese and Nation, that satisfies $(SD_\vee)$, $(SD_\wedge)$, and (W), but is not a sublattice of a free lattice.

CHAPTER 5. LATTICES OF LATTICE HOMOMORPHISMS[1].

Let L and M be lattices, and let Hom(L, M) denote the set of all homomorphisms of L into M , endowed with the pointwise partial order. We introduce the following concept:

DEFINITION 5.1. A lattice L is called *catalytic* if Hom(L, M) is a lattice for every lattice M .

In this chapter we will demonstrate that somewhat close ties exist between catalytic lattices and free lattices. In particular we will prove that retracts of free lattices are catalytic, and that every catalytic lattice is semidistributive and satisfies (W) . R. McKenzie has shown [23] that a finite lattice is a retract of a free lattice if and only if it is a sublattice of a free lattice; hence the class of finite catalytic lattices lies somewhere between the class of finite sublattices of a free lattice and the class of finite semi-distributive lattices satisfying (W) .

If P is a partially ordered set, CF(P) denotes the lattice completely freely generated by P .

THEOREM 5.2. CF(P) *is catalytic for every partially ordered set* P .

Proof. Let M be a lattice, and let f, g : CF(P) $\longrightarrow$ M be homomorphisms. Define $h_0 : P \rightarrow M$ by $xh_0 = xf \vee xg$, and extend

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[1] The content of this chapter is due jointly to T.G. Kucera and the author (see [22]).

h_0 to a homomorphism $h : CF(P) \to M$; it is easy to see that h is the least upper bound of f and g in $\mathrm{Hom}(CF(P), M)$. The greatest lower bound is found dually. □

THEOREM 5.3. A retract of a catalytic lattice is catalytic.

Proof. Let L be a retract of the catalytic lattice K and let $\phi : K \longrightarrow L$ and $\psi : L \longrightarrow K$ be homomorphisms such that $\psi\phi = id_L$. Let M be a lattice and let $f, g \in \mathrm{Hom}(L, M)$. Then ϕf and ϕg are in $\mathrm{Hom}(K, M)$ and so have a join h in $\mathrm{Hom}(K, M)$. We claim that $\psi h : L \longrightarrow M$ is the least upper bound of f and g in $\mathrm{Hom}(L, M)$. Since $h \geq \phi f$, we get that $x\psi h \geq x\psi\phi f = xf$ for all $x \in L$; hence $\psi h \geq f$ (and similarly $\psi h \geq g$). Let $h' \in \mathrm{Hom}(L, M)$ be such that $h' \geq f,g$. Then $\phi h' \geq \phi f, \phi g$, so $\phi h' \geq h$. Thus $h' = \psi\phi h' \geq \psi h$, and our claim is proved. The greatest lower bound of f and g is defined dually. □

COROLLARY 5.4. A retract of a free lattice is catalytic. □

The next result is somewhat incidental, but will be needed in what follows.

PROPOSITION 5.5. A linear sum of catalytic lattices is catalytic.

Proof. Let L be the linear sum of the catalytic lattices $\{L_i \mid i \in C\}$, where C is a chain. Let M be a lattice and let $f, g \in \mathrm{Hom}(L, M)$. For each $i \in C$, set $S_i = \{u \in M \mid u \leq xf \vee xg \text{ for some } x \in L_i\}$. It is easy to see that S_i is an ideal of M for each i.

For each $i \in C$, let $f_i = f|L_i$ and $g_i = g|L_i$. Since $xf_i \leq xf_i \vee xg_i = xf \vee xg \in S_i$, f_i and similarly g_i can be considered as elements of $\mathrm{Hom}(L_i, S_i)$. Let h_i be the join of f_i and g_i in $\mathrm{Hom}(L_i, S_i)$; h_i exists since L_i is catalytic. We claim that if $x \in L_i$, $z \in L_j$, and $i < j$ in C, then $xh_i \leq zh_j$. Indeed, since $xh_i \in S_i$, there exists $y \in L_i$ such that $xh_i \leq yf \vee yg$, and since $i < j$ we have $y < z$. Thus $xh_i \leq yf \vee yg \leq zf \vee zg = zf_j \vee zg_j \leq zh_j$.

Let $h : L \to M$ be defined by: $xh = xh_i$ if $x \in L_i$. From the preceding paragraph it is clear that $h \in \mathrm{Hom}(L, M)$ and that h is the least upper bound of f and g. The greatest lower bound of f and g is defined dually. □

Recall the following result (Theorem 2.11) from Chapter 2: _a lattice_ L _satisfies_ (W) _if and only if_ $\mathrm{Hom}(L, M)$ _is a lattice for every complete lattice_ M. Hence we have immediately that catalytic lattices satisfy (W).

THEOREM 5.6. _Catalytic lattices are semidistributive._

Before proceeding with the proof of this result, we establish a simple lemma concerning lattices violating $(SD_\wedge)$.

LEMMA 5.7. _Let_ L _be a lattice violating_ $(SD_\wedge)$. _Then there exist_ $a, b, c \in L$ _such that_ $a = a \wedge (b \vee c) \neq a \wedge b = a \wedge c$. _Furthermore, letting_ $A_0 = [a)$ _be the principal dual ideal of_ L _generated by_ a, _there are order ideals_ B_0 _and_ C_0 _of_ L _such that_

i) $L = A_0 \cup B_0 \cup C_0$

ii) $A_0 \cap (B_0 \cup C_0) = \emptyset$

iii) $b \in B_0 \setminus C_0$ and $c \in C_0 \setminus B_0$.

Proof. The situation is illustrated in Figure 5.1. We know that there exist $a', b, c \in L$ such that $a' \wedge b = a' \wedge c$ but $a' \wedge b \neq a' \wedge (b \vee c)$. Let $a = a' \wedge (b \vee c)$; the first part of the lemma follows immediately. Now set $B_0 = \{x \in L \mid x \ngeq a, x \ngeq c\}$ and $C_0 = \{x \in L \mid x \ngeq a, x \ngeq b\}$. It is obvious that B_0 and C_0 are order ideals. Let $x \in L \setminus (B_0 \cup C_0)$; then either $x \geq a$ or $x \geq b$, $x \geq c$. But if the latter applies we have $x \geq b \vee c \geq a$, so $x \in [a)$ follows in either case, proving i) . If $x \in B_0$ or $x \in C_0$ then necessarily $x \notin [a)$, proving ii) . Finally, since it is easy to see that a, b, and c must form an antichain, iii) follows. □

Proof of Theorem 5.6. Since the dual of a catalytic lattice is catalytic, we need only prove that catalytic lattices satisfy $(SD_\wedge)$. Let L be a catalytic lattice violating $(SD_\wedge)$. Without loss of generality L has a least element 0 ; for if not, by Proposition 5.5 we can add the one-element lattice $\underset{\sim}{1}$ at the bottom of L , and the result will again be a catalytic lattice violating $(SD_\wedge)$. We will produce a lattice M such that Hom(L, M) is not a lattice, thereby contradicting the fact that L is catalytic.

Let $a, b, c \in L$ and $A_0, B_0, C_0 \subseteq L$ be as in Lemma 5.7 and let K be the lattice of Figure 5.2. Construct the following

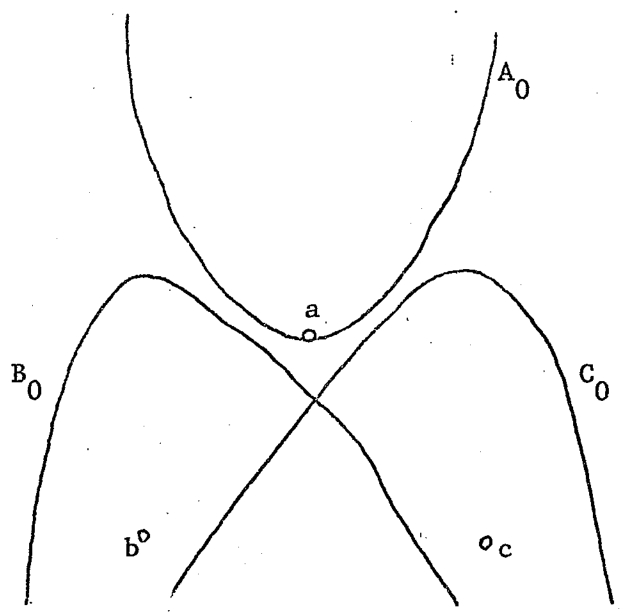

Figure 5.1.

Figure 5.2.

subsets of $L \times K$:

$$B = B_0 \times \{s_1, s_3, s_5, s_7, \cdots\}, \quad C = C_0 \times \{s_2, s_4, s_6, s_8, \cdots\},$$

$$D = (B_0 \cup C_0) \times \{t_1, t_2, t_3, t_4, \cdots\}, \quad E = L \times \{0, 1\}.$$

Set $M = B \cup C \cup D \cup E$.

M is not a sublattice of $L \times K$; nevertheless we claim that, as a subposet of $L \times K$, M is a lattice. Let $X = (x, m)$ and $Y = (y, n)$ be elements of M. We leave it to the reader to check that

$$X \wedge Y = (x \wedge y, m \wedge n)$$

and that the following hold:

i) If $(u, t) \in M$ and $v < u$ then $(v, t) \in M$;

ii) For every $u \in L$ and every $t \in K$ there is a smallest element $\lambda(u, t)$ of K such that $\lambda(u, t) \geq t$ and $(u, \lambda(u, t)) \in M$.

Consequently we have that

$$X \vee Y = (x \vee y, \lambda(x \vee y, m \vee n)).$$

For each $n < \omega$, let $M_n = ((B_0 \cup C_0) \times \{t_n\}) \cup (A_0 \times \{1\})$. From the above, it is easy to see that M_n is a sublattice of M isomorphic to L; let $g_n : L \rightarrow M_n \subseteq M$ be the isomorphism. Let f_1, f_2 be the homomorphisms of L into M given by $f_1(x) = (x, 0)$ and $f_2(x) = (0, s_1)$ for all $x \in L$. Clearly $f_1 < g_n$ and $f_2 < g_n$ for each n.

Since L is catalytic, $f_1 \vee f_2 = h$ exists, and $h \le g_n$ for each n. Thus $ah \le ag_n = (a, 1)$, $bh \le bg_n = (b, t_n)$, and $ch \le cg_n = (c, t_n)$ for all $n < \omega$. But $ah \ge af_1 \vee af_2 = (a, 0) \vee (0, s_1) = (a, 1)$, and similarly $bh \ge (b, s_1)$, $ch \ge (c, s_2)$. Hence $ah = (a, 1)$, and $bh = (b, s_i)$, $ch = (c, s_j)$ for some $i, j < \omega$. Since $b \in B_0 \setminus C_0$ and $c \in C_0 \setminus B_0$, i must be odd and j must be even. Now $a \wedge b = a \wedge c$ implies that $(a \wedge b, s_i) = (a, 1) \wedge (b, s_i) = ah \wedge bh = (a \wedge b)h = (a \wedge c)h = ah \wedge ch = (a, 1) \wedge (c, s_j) = (a \wedge c, s_j) = (a \wedge b, s_j)$. We conclude that $s_i = s_j$, which is impossible. The proof is thereby complete. □

There do exist lattices which are semidistributive and satisfy (W) but are not catalytic. Let L be the lattice of Figure 5.3, let $f \in \mathrm{Hom}(L, L)$ be the identity map, and let $g \in \mathrm{Hom}(L, L)$ be defined by $g(0) = 0$, $g(a_i) = a_{i+1}$, $g(b_i) = b_{i+1}$ for each $i \in \mathbb{Z}$. We leave to the reader the task of verifying that L is semidistributive and satisfies (W). Notice also that the only homomorphic images of L that are isomorphic to chains are $\underset{\sim}{2}$ and $\underset{\sim}{1}$. It is therefore obvious that no element of $\mathrm{Hom}(L, L)$ is greater than both f and g, and hence L is not catalytic.

The construction used in the proof of Theorem 5.7 can be modified to give a proof of the following result: <u>every sharply transferable lattice satisfies</u> $(SD_\wedge)$. For let L be a lattice violating $(SD_\wedge)$, let A_0, K, B, and C be as in the proof of Theorem 5.7, and set $M' = B \cup C \cup (A_0 \times \{1\}) \subseteq L \times K$. Then M' is a

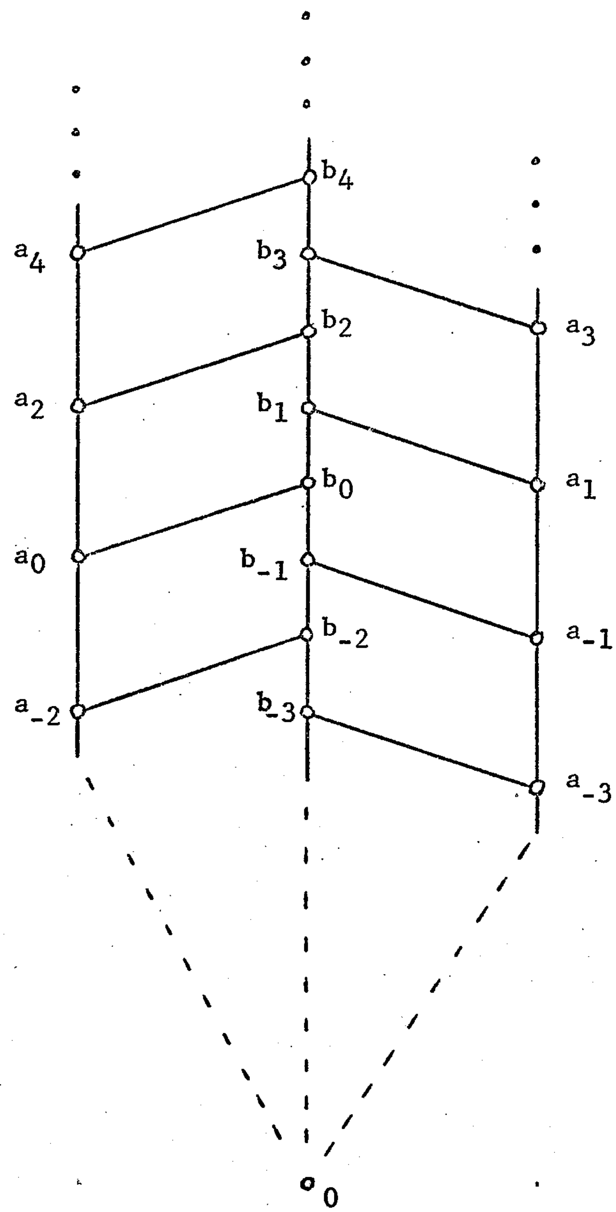

Figure 5.3.

lattice (in fact a sublattice of the lattice M constructed in the proof of Theorem 5.7); furthermore, there is a natural homomorphism ϕ of M' onto L defined by $(x, i)\phi = x$. Define $\psi' : L \to \mathcal{J}(M')$ by

$$x\psi = \text{the ideal generated by } x\phi^{-1}$$

for each $x \in L$. Then ψ is an embedding of L into $\mathcal{J}(M')$, but by arguments similar to those in the proof of Theorem 5.7 it follows that there is no embedding $\psi' : L \to M'$ such that $x \leq y \Leftrightarrow x\psi' \in y\psi$ for all $x, y \in L$.

Remark. This result is an immediate consequence of the recent characterization of sharply transferable lattices by G. Grätzer and C.R. Platt [16].

We conclude with a reference to the concept discussed in Chapter 3.

THEOREM 5.8. Every catalytic lattice satisfies the bounded epimorphism condition.

Proof. Let L be catalytic and let $\phi : M \to L$ be a bounded epimorphism. For each $x \in L$ we denote the least and greatest elements of $x\phi^{-1}$ by 0_x and 1_x respectively. Let A be the lattice of Figure 5.4, and define $K \subseteq M \times A$ by

$$K = \{(0_x, t) \mid x \in L, t < c\} \cup \{(1_x, t) \mid x \in L, t > c\} \cup (M \times \{c\}) .$$

K may not be a sublattice of $M \times A$, but as a poset it is a lattice.

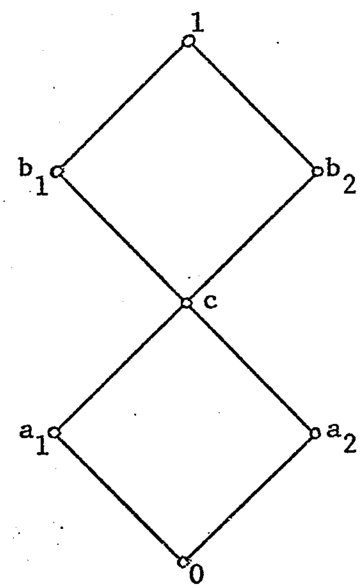

Figure 5.4.

Joins in K are given by

$$(u, s) \vee (v, t) = \begin{cases} (u \vee v,\ s \vee t) & \text{if } s \vee t \leq c \\ \\ (1_{(u \vee v)\phi},\ s \vee t) & \text{if } s \vee t > c \end{cases}$$

and meets are given dually.

Let $f_i, g_i : L \to K$, $i = 1, 2$, be defined by $xf_i = (0_x, a_i)$ and $xg_i = (1_x, b_i)$. It is clear that f_i and g_i are homomorphisms. Since L is catalytic, $\psi = f_1 \vee f_2$ exists, and since $f_1, f_2 \leq g_1, g_2$ we have that $\psi \leq g_1, g_2$. It follows that for each $x \in L$, $(0_x, c) = (0_x, a_1) \vee (0_x, a_2) = xf_1 \vee xf_2 \leq x\psi \leq xg_1 \wedge xg_2 = (1_x, b_1) \wedge (1_x, b_2) = (1_x, c)$. Therefore $x\psi \in M \times \{c\} \cong M$, and so ψ may be regarded as a homomorphism of L into M that satisfies $\psi\phi = \mathrm{id}_L$, as desired. $\square$

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PLANAR SUBLATTICES OF A FREE LATTICE I

by

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Planar sublattices of a free lattice. I

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§1. Introduction.

There are three lattice-theoretic properties that are generally used to open a discussion on sublattices of a free lattice:

(W) *for all* a,b,c,d, $a \wedge b \leq c \vee d$ *implies* $a \wedge b \leq c$, $a \wedge b \leq d$, $a \leq c \vee d$, *or* $b \leq c \vee d$

($SD_\vee$) *for all* a,b,c, $a \vee b = a \vee c$ *implies* $a \vee b = a \vee (b \wedge c)$

($SD_\wedge$) *for all* a,b,c, $a \wedge b = a \wedge c$ *implies* $a \wedge b = a \wedge (b \vee c)$

(W) is one of the conditions present in P. M. Whitman's solution [17] of the word problem for lattices while ($SD_\vee$) and ($SD_\wedge$) were originated by B. Jónsson [8] (cf. R. A. Dean [2]). It is well known that each of these conditions holds in every sublattice of a free lattice.

In the late 1950's Jónsson posed a conjecture (cf. B. Jónsson and J. E. Kiefer [9]) that has in the intervening period attracted considerable attention.

Conjecture. *A finite lattice is a sublattice of a free lattice if and only if it satisfies* ($SD_\vee$), ($SD_\wedge$) *and* (W).

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What is known?

F. Galvin and B. Jónsson [3] considered a special case of the conjecture in 1961: they showed that any finite distributive lattice that satisfies $(\mathbf{SD}_\vee)$, $(\mathbf{SD}_\wedge)$, and $(\mathbf{W})$ is a sublattice of a free lattice. What is more, they could essentially display all finite distributive sublattices of a free lattice: *a finite distributive lattice is a sublattice of a free lattice if and only if it is a linear sum of lattices, each of which is isomorphic to* $\mathbf{1}$, $\mathbf{2}^3$, *or* $\mathbf{2} \times \mathbf{n}$ *for some* n. (For a positive integer m, $\mathbf{m}$ denotes the m-element chain.)

Call a lattice *semidistributive* if it satisfies $(\mathbf{SD}_\vee)$ and $(\mathbf{SD}_\wedge)$. Of course, every distributive lattice is semidistributive while a modular semidistributive lattice is distributive. Hence, Galvin and Jónsson had really settled the conjecture for all finite modular lattices.

In 1962, B. Jónsson and J. E. Kiefer [9] proved that finite sublattices of a free lattice have breadth at most *four*; moreover, they showed that the conjecture will hold for all finite lattices if it holds for finite lattices of breadth at most *three*. Still, all efforts to settle even the breadth *two* case have been unsuccessful.

The purpose of this paper is to settle the conjecture in the affirmative for a rather extensive class of breadth two lattices; namely, *planar* lattices, that is, finite lattices with planar (Hasse) diagrams.

THEOREM. *A finite planar lattice is a sublattice of a free lattice if and only if it satisfies* $(\mathbf{SD}_\vee)$, $(\mathbf{SD}_\wedge)$, *and* $(\mathbf{W})$.

In a companion paper [15] we construct a minimum list $\mathcal{L}$ of lattices such that a finite lattice is a planar sublattice of a free lattice if and only if it contains no sublattice isomorphic to a lattice in $\mathcal{L}$.

Some words about the plan of this paper are in order.

In §2 we shall elaborate on some basic facts concerning breadth, especially in the context of semidistributive lattices of breadth at most two.

One of the most interesting by-products of our investigations stems from the importance of the "finiteness" condition in the conjecture. R. Freese and J. B. Nation (cf. B. Jónsson and J. B. Nation [10]) have already exhibited a finitely generated (infinite) lattice that satisfies $(\underset{\sim}{SD}_\vee)$, $(\underset{\sim}{SD}_\wedge)$, and $(\underset{\sim}{W})$ yet is not a sublattice of a free lattice. In §3 we shall construct a small family of finite partially ordered sets none of which can appear as a subset in a finite breadth two lattice satisfying $(\underset{\sim}{SD}_\vee)$, $(\underset{\sim}{SD}_\wedge)$, and $(\underset{\sim}{W})$. This result contrasts sharply with the problem of characterizing those finite partially ordered sets that generate a finite free lattice (cf. Yu. I. Sorkin [16] and R. Wille [18]). The prospect of enumerating, or at least characterizing, those finite partially ordered sets that cannot appear as subsets in a finite sublattice of a free lattice, may be as illuminating about the structure of free lattices as the solution to the current conjecture itself.

In §4 we shall recall some elementary facts about planarity for lattices.

Our basic approach to the problem originates in the thesis of H. S. Gaskill [4] (cf. [5]) where the conditions $(\underset{\sim}{T}_{\vee})$ and $(\underset{\sim}{T}_{\wedge})$ were first formulated. A paper of H. S. Gaskill, G. Grätzer, and C. R. Platt [6] continued the study of $(\underset{\sim}{T}_{\vee})$ and $(\underset{\sim}{T}_{\wedge})$ along the lines initiated by Gaskill. R. McKenzie had shown in [13] that a finite lattice is a sublattice of a free lattice if and only if it is a bounded homomorphic image of a free lattice and satisfies $(\underset{\sim}{W})$. Gaskill and Platt [7] combined these results to prove that <u>a finite lattice is a sublattice of a free lattice if and only if it satisfies $(\underset{\sim}{T}_{\vee})$, $(\underset{\sim}{T}_{\wedge})$, and $(\underset{\sim}{W})$</u>. This criterion is likely the most practical characterization known for actually determining whether or not a particular finite lattice is a sublattice of a free lattice; it is the one that we shall use. The central ideas in this approach are "minimal pairs" and "cycles". These ideas will be developed in §5 and §6.

The main body of the proof occurs in §7 and §8.

Finally, some retrospective remarks about our proof which for an obvious reason we prefer to include in the introduction. The proof of the main theorem of this paper is long and arduous. In some respects the proof amounts to an accumulation of techniques which in concert enable us to carry a straightforward approach through to the end. We maintain a guarded optimism that this same approach may yet be extended to settle the complete breadth two case of the conjecture. In any case, several of the techniques in our repertoire seem to be of quite independent interest.

§2. Breadth.

The breadth $b(L)$ of a lattice L is the smallest integer b such that every join $\bigvee_{i=1}^{b+1} x_i$ of elements of L is equal to a join of b of the x_i's. Of course, a lattice has breadth one if and only if it is a chain.

What would happen if we were to use meets instead of joins in the definition of $b(L)$? Notice that this definition does not appear to be self-dual: actually, it is.

LEMMA 2.1. *Let L be a lattice with $b(L)=b$, and let b' be the smallest integer such that every meet $\bigwedge_{i=1}^{b'+1} x_i$ of elements of L is equal to a meet of b' of the x_i's. Then $b'=b$.*

Proof. Suppose $b' > b$. It follows that there are elements x_i, $1 \le i \le b+1$, of L such that $\bigwedge_{i=1}^{b+1} x_i$ is not equal to the meet of any b of the x_i's. Let $y_j = \bigwedge\{x_i \mid 1 \le i \le b+1, i \ne j\}$ for each $j \in \{1,\ldots,b+1\}$; we have that $y_j \not\le x_j$, for each j. Now let $z_k = \bigvee\{y_j \mid 1 \le j \le b+1, j \ne k\}$ for each $k \in \{1,\ldots,b+1\}$. Note that $z_k \le x_k$ for each k but that $\bigvee_{j=1}^{b+1} y_j \not\le x_k$ for any k; thus, $z_k \ne \bigvee_{j=1}^{b+1} y_j$ for any k, contradicting $b(L)=b$. Hence $b' \le b$, and a dual argument shows equality. □

For any integer $n \ge 3$, a crown [1] of order $2n$ is a subset

$$C = \{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$$

of a partially ordered set in which $x_1 < y_1, y_1 > x_2, x_2 < y_2, \ldots, x_n < y_n$, and $y_n > x_1$ are the only comparability relations that hold in C (see Figure 1). The next lemma illustrates the usefulness of crowns of order six in a discussion of breadth two lattices.

LEMMA 2.2. <u>A lattice has breadth at most two if and only if it contains no crown of order six.</u> □

Call a finite lattice L <u>dismantlable</u> if we can write $L = \{x_1, x_2, \ldots, x_n\}$ such that $\{x_1, x_2, \ldots, x_i\}$ is a sublattice of L for each $i = 1, 2, \ldots, n$. The notion of dismantlability was first introduced by I. Rival in [14] and it has since played a major role particularly in the study of planar lattices. Dismantlable lattices themselves have been characterized by D. Kelly and I. Rival [11].

LEMMA 2.3. <u>A finite lattice is dismantlable if and only if it contains no crown.</u> □

As a consequence, every dismantlable lattice has breadth at most two. It turns out that, in the presence of $(\underset{\sim}{SD}_\vee)$, the converse is true as well.

LEMMA 2.4. <u>Let</u> L <u>be a finite lattice satisfying</u> $(\underset{\sim}{SD}_\vee)$. <u>Then</u> L <u>is dismantlable if and only if</u> $b(L) \le 2$.

<u>Proof</u>. Let $b(L) \le 2$, and suppose that L is not dismantlable. By Lemma 2.3 L contains a crown ; let

$$\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$$

be a crown of minimum order in L . We may assume that $x_i \vee x_{i+1} = y_i$ and $y_i \wedge y_{i+1} = x_{i+1}, i = 1, \ldots, n-1$, and $x_n \vee x_1 = y_n$, $y_n \wedge y_1 = x_1$ hold in L . Furthermore, since $b(L) \le 2$, Lemma 2.2 implies that $n \ge 4$. Now, if $x_1 \vee x_3 \not> x_2$, $\{x_1, y_1, x_2, y_2, x_3, x_1 \vee x_3\}$ is a crown of order 6, which is impossible ; hence $x_1 \vee x_3 > x_2$ and by duality

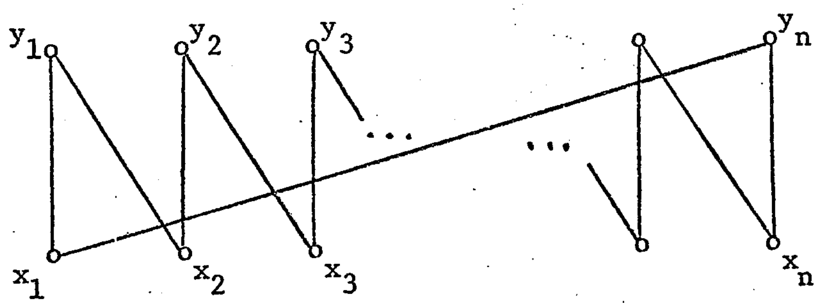

Figure 1. A crown of order 2n.

$y_1 \wedge y_3 < y_2$. Also, if $x_1 \vee x_3 \not> x_4$, it is easy to see that $\{x_1, x_1 \vee x_3, x_3, y_3, x_4, y_4, \ldots, x_n, y_n\}$ will contain a crown of order $< 2n$, which is a contradiction. Therefore $x_1 \vee x_3 > x_4$, and we have by symmetry that $x_1 \vee x_3 \geq x_2 \vee x_4 \geq x_3 \vee x_5 \geq \ldots \geq x_n \vee x_2 \geq x_1 \vee x_3$. Now $y_1 \vee y_2 = x_1 \vee x_2 \vee x_3 = x_1 \vee x_3 = x_2 \vee x_4 = x_2 \vee x_3 \vee x_4 = y_2 \vee y_3$, and hence $(\underset{\sim}{SD}_{\vee})$ implies that $y_1 \vee y_2 = y_2 \vee (y_1 \wedge y_3) = y_2$, a contradiction. Thus L is dismantlable, and the lemma is established. □

In this paper we shall be primarily interested in finite lattices of breadth at most two. Still, it seems worthwhile to prove the next two results in the framework of finite lattices of arbitrary breadth.

Recall that for elements x and y of a lattice L, with $x > y$, x *covers* y (or y *is covered by* x, or x is an *upper cover* of y, or y is a *lower cover* of x) if $x > z \geq y$ implies $z = y$. We write $x \succ y$ or $y \prec x$.

LEMMA 2.5. *Let* n_* *be the maximum number of elements covered by any element of the finite lattice* L. *Then* $n_* \geq b(L)$.

Proof. Let $b = b(L)$. There are elements $x_1, \ldots, x_b$ of L such that $y_j = \bigvee\{x_i \mid 1 \leq i \leq b, i \neq j\} < \bigvee_{i=1}^{b} x_i$ for each $j \in \{1, \ldots, b\}$. Hence for each j there exists $w_j \prec \bigvee_{i=1}^{b} x_i$ such that $y_j \leq w_j$. Moreover $y_j \vee y_k = \bigvee_{i=1}^{b} x_i$ if $j \neq k$; thus all the w_j's are distinct, showing that $\bigvee_{i=1}^{b} x_i$ has at least b lower covers. □

The self-dual nature of breadth has already been demonstrated in Lemma 2.1. Letting n^* be the maximum number of elements covering any element of L, we have $n^* \geq b(L)$ by duality.

LEMMA 2.6. _Let_ L _be a finite lattice satisfying_ $(\underset{\sim}{SD}_\wedge)$ _and let_ n^* _be the maximum number of elements covering any element of_ L. _Then_ $n^* = b(L)$.

Proof. Let $a \in L$ and let $x_1,\ldots,x_{n^*}$ be distinct elements covering a. Since $x_i \wedge x_j = a$ whenever $i \neq j$, successive applications of $(\underset{\sim}{SD}_\wedge)$ show that $x_i \wedge \bigvee\{x_j : 1 \leq j \leq n^*, j \neq i\} = a$ for each $i \in \{1,\ldots,n^*\}$; that is, $x_i \nleq \bigvee\{x_j : 1 \leq j \leq n^*, j \neq i\}$ for each i. It follows that $\bigvee\{x_j : 1 \leq j \leq n^*, j \neq i\} < \bigvee_{j=1}^{n^*} x_j$ for each i, and so $b(L) \geq n^*$. From Lemma 2.5 we conclude $b(L) = n^*$. □

One final result concerning breadth two lattices and we conclude this section.

LEMMA 2.7. _Let_ L _be a finite lattice satisfying_ $(\underset{\sim}{SD}_\wedge)$ _such that_ $b(L) \leq 2$. _Then_

(i) _each element of_ L _has at most two upper covers;_

(ii) _if_ x, y, z _are distinct elements of_ L _and_ $u = x \wedge y = x \wedge z = y \wedge z$ _then_ $u \in \{x,y,z\}$, _that is,_ $\{x,y,z\}$ _is not an antichain._

Proof. (i) This is immediate from Lemma 2.6.

(ii) Let $u = x \wedge y = x \wedge z = y \wedge z$, and suppose $\{x,y,z\}$ is an antichain. Then x,y, and z are all greater than u, and we may choose upper covers x_1,y_1,z_1 of u such that $x \geq x_1$, $y \geq y_1$, and $z \geq z_1$; since $x \wedge y = x \wedge z = y \wedge z = u$, x_1,y_1, and z_1 are distinct, contradicting (i). □

§3. Finiteness.

The purpose of this section is to establish a result that provides us with a powerful technique in applying the finiteness assumption of the conjecture. The result, which is due to I. Rival, also seems to be of considerable independent interest.

For noncomparable elements a and b of a partially ordered set we write $a \parallel b$. An element a of a lattice L, $a \neq 0$, is join irreducible if $a = b \vee c$ implies $a = b$ or $a = c$; $a \neq 1$ is meet irreducible if $a = b \wedge c$ implies $a = b$ or $a = c$. $J(L)$, respectively $M(L)$, shall denote the subset of all join irreducible elements, respectively meet irreducible elements, of L.

If P and Q are partially ordered sets, recall that a one-to-one map $\eta : P \to Q$ is a weak embedding if both η and η^{-1} are order preserving. Consider the partially ordered set $M0$ of Figure 2, and construct partially ordered sets $M1$ and $M2$ (Figure 2) as follows :

$M1 = M0 \cup \{h\}$, where $h < f$, $h \parallel b,c,d,e$, and g, and h may or may not be comparable with a;

$M2 = M1 \cup \{i\}$, where $i < g$, $i \parallel a,b,c,d,f$, and h, and i may or may not be comparable with e.

PROPOSITION 3.1. Let L be a semidistributive lattice satisfying (W) such that $b(L) \leq 2$. Suppose that

(i) there is a weak embedding $\eta : M0 \to L$ such that $f\eta$ and $g\eta$ are join irreducible elements of L,

or

(ii) <u>there is a weak embedding</u> $\eta : M1 \to L$ <u>such that</u> $g\eta$ <u>is a join irreducible element of</u> L ,

<u>or</u>

(iii) <u>there is a weak embedding</u> $\eta : M2 \to L$.

<u>Then</u> L <u>is infinite</u>.

<u>Proof</u>. We will give the proof, assuming that (ii) holds ; it will be easy to see that similar proofs exist in the other two cases.

Suppose (ii), and consider the set

$$C = \{ c\eta \mid \eta : M1 \to L \text{ is a weak embedding, } g\eta \in J(L) \} \subseteq L .$$

By assumption C is nonempty. Suppose that L is finite ; then we can choose η satisfying (ii) such that $c\eta$ is a maximal element of C . Furthermore, letting $x\eta = x'$ for $x \in M1$, we may assume that $a' \vee c' = b'$, $c' \vee e' = d'$, $b' \wedge d' = c'$, and $h' \vee b' = f'$.

Let $c_* < c'$. If $c_* \vee a' < b'$ and $c_* \vee e' < d'$, we have that $c_* \vee a' \ngeq c'$ and $c_* \vee e' \ngeq c'$; hence $(c_* \vee a') \wedge c' = c_* = (c_* \vee e') \wedge c'$. Also, $c_* \leq (c_* \vee a') \wedge (c_* \vee e') \leq b' \wedge d' = c'$, and since $c_* \vee a' \ngeq c'$ we get $(c_* \vee a') \wedge (c_* \vee e') = c_*$. But this is impossible by Lemma 2.7(ii). Thus either $c_* \vee a' = b'$ or $c_* \vee e' = d'$.

Suppose $c_* \vee a' = b'$. By $\underset{\sim}{(W)}$ we must have $f' \wedge d' \nleq b'$. If $(f' \wedge d') \vee b' < f'$ then

$$\{a', (f' \wedge d') \vee b', f' \wedge d', d', e', f', g', h'\} \cong M1 ,$$

and $f' \wedge d' > c'$, contradicting the maximality of $c\eta = c'$. So $(f' \wedge d') \vee b' \nless f'$, which implies that $(f' \wedge d') \vee b' = f'$. Since

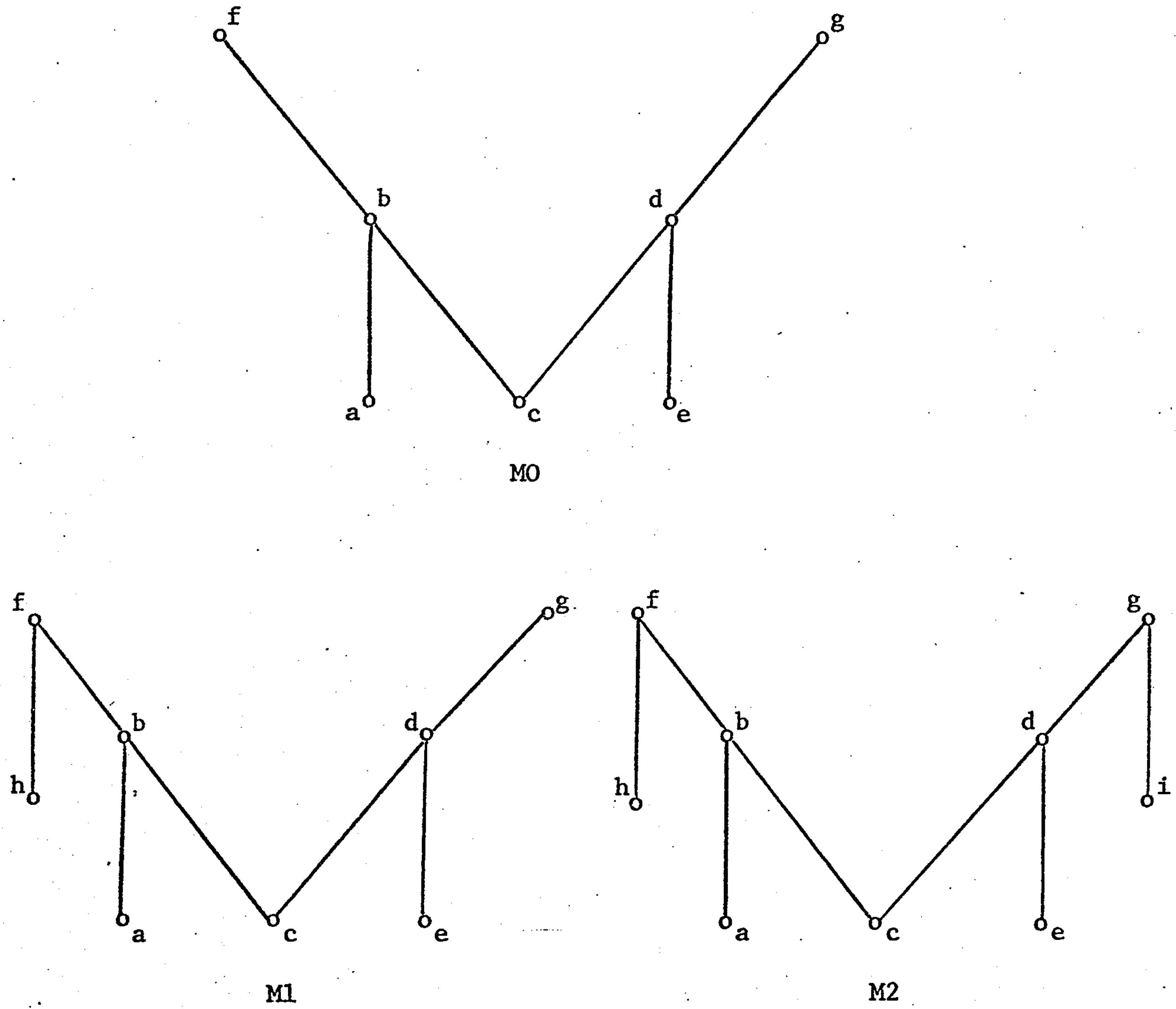

Figure 2

$h' \vee b' = f'$ and $\{h', b', f' \wedge d'\}$ is an antichain, the dual of Lemma 2.7(ii) implies that $(f' \wedge d') \vee h' < f'$. But now $(f' \wedge d') \vee h' \not\geq b'$, and hence

$$\{h', (f' \wedge d') \vee h', f' \wedge d', d', e', f', g', b'\} \cong M1 ,$$

and $f' \wedge d' > c'$, again contradicting the maximality of c'.

Hence $c_* \vee e' = d'$, and by $\underset{\sim}{(W)}$ we must have $b' \wedge g' \not\leq d'$. Now $(b' \wedge g') \vee d' \leq g'$, and since $g' \in J(L)$ we in fact have $(b' \wedge g') \vee d' < g'$. Thus

$$\{a', b', b' \wedge g', (b' \wedge g') \vee d', e', f', g', h'\} \cong M1 ,$$

and $b' \wedge g' > c'$, contradicting the maximality of c'. □

For readers of a statistical bent, we remark that Proposition 3.1 will be applied no less than 18 times in the proof of the main theorem.

§4. Planarity.

Planar lattices were investigated and characterized by D. Kelly and I. Rival [12]. Their characterization, while not needed here, will play a central role in a companion paper [15]. We shall assume familiarity with some of the more transparent concepts concerning planar lattices.

Let $e(L)$ be a planar embedding of the planar lattice L, and let $x \in L$. It is intuitively obvious that the lower covers of x are linearly ordered from left to right. Hence we can define a relation λ on the elements of L as follows: $x \lambda y$ if and only if $x \parallel y$ and there are lower covers x' and y' of $x \vee y$ such that $x \leq x'$, $y \leq y'$, and x' is to the left of y' (with respect to $e(L)$).

For the remainder of this section we let L be a planar (finite) lattice, and we suppose that λ has been defined with respect to some planar embedding of L. The first lemma, due to J. Zilber, sets out the basic properties of λ (for a proof, see [12]).

LEMMA 4.1. λ <u>is a strict partial order on</u> L. <u>Moreover, if</u> $x \parallel y$, <u>then</u> $x \lambda y$ <u>or</u> $y \lambda x$. □

It follows that for $x,y \in L$, exactly one of the following holds: $x = y$, $x < y$, $x > y$, $x \lambda y$, $y \lambda x$. Consequently, the expression "$x \lambda y$" can be read, and thought of, as "x is to the left of y".

LEMMA 4.2 (i) <u>If</u> $x \lambda y$ <u>and</u> $y < z$, <u>then</u> $x \lambda z$ <u>or</u> $x < z$.

(ii) <u>If</u> $x \lambda y$ <u>and</u> $x < z$, <u>then</u> $z \lambda y$ <u>or</u> $z > y$.

<u>Proof</u>. (i) If $x \geq z$ then $x > y$, contradicting $x \lambda y$. If $z \lambda x$ then by transitivity $z \lambda y$, contradicting $y < z$. The conclusion follows from Lemma 4.1. The proof of (ii) is similar. □

<u>Remark</u>. Clearly there is a kind of duality at work here. Corresponding to λ, we may define a partial order λ' on L by: $x \lambda' y$ if and only if $y \lambda x$ (the expression "$x \lambda' y$" of course could stand for "x is to the right of y"). Then given any statement S valid for a particular planar embedding of a planar lattice L, we may replace "left" with "right" and λ with λ' in S without affecting its validity. The resulting statement will be called the <u>reflection</u> of S. With this terminology, Lemma 4.2(ii) is the reflection of Lemma 4.2(i).

While we are on the subject, we may as well point out that the next lemma is "self-reflective".

LEMMA 4.3. *If* $x \lambda y \lambda z$ *then* $x \wedge z < y < x \vee z$.

Proof. Suppose $y \not< x \vee z$. Since $y \lambda z$ and $z < x \vee z$, Lemma 4.2 implies that $y \lambda x \vee z$. But since $x \lambda y$ and $x \vee z > x$, Lemma 4.2 also implies that $x \vee z \lambda y$, a contradiction. Hence $y < x \vee z$ and, dually, $x \wedge z < y$. □

COROLLARY 4.4. *If* $x \lambda y \lambda z$ *then* $x \vee y \vee z = x \vee z$. □

It follows, of course, that a planar lattice has breadth at most two.

For a join irreducible element a of L we denote its unique lower cover by a_*. Similarly, if $a \in M(L)$ we denote its unique upper cover by a^*. (Covering elements shall be separated in the figures by double lines when we wish to emphasize the fact.)

LEMMA 4.5. *Let* L *satisfy* $(\underset{\sim}{\mathrm{SD}}_{\vee})$ *and let* x,y,z *be elements of* L *such that* $x \lambda y \lambda z$, $y \in J(L)$, *and* $y_* < z$. *Then* $x \lambda y \vee z$.

Proof. If $x < y \vee z$ then by Corollary 4.4 $x \vee z = y \vee z$. By $(\underset{\sim}{\mathrm{SD}}_{\vee})$, $y \vee z = (x \wedge y) \vee z$; but since $y \in J(L)$ and $x \| y$, $x \wedge y \leq y_* < z$ and hence $y \vee z = z$, a contradiction. Therefore $x \not< y \vee z$, and by Lemma 4.2 (i) we conclude that $x \lambda y \vee z$. □

Consider the right boundary B of L (of course, with respect to the planar embedding $e(L)$). B certainly contains at least one element of $M(L)$, if $|L| > 1$; for instance B will contain a dual atom of L. Therefore we can speak of the minimal meet irreducible element of B. The first part of the following lemma is due to K. A. Baker, P. C. Fishburn, and F. S. Roberts [1], and the second part (actually, its reflection) is Proposition 2.6 of [12].

LEMMA 4.6. <u>Let</u> B <u>be the right boundary of</u> L <u>and let</u> a <u>be the minimal meet irreducible element of</u> B. <u>Then</u>

(i) a <u>is doubly irreducible</u> ;

(ii) <u>if</u> $x \in L \setminus B$ <u>there exists</u> $y \in B$ <u>such that</u> y <u>is doubly irreducible and</u> $x \lambda y$;

(iii) $x < a$ <u>implies that</u> $x \in B$.

<u>Proof of</u> (iii). If $x \notin B$, choose y as in (ii). From Lemma 4.2, either $a \lambda y$ or $a > y$; since $a \in B$, we have $a > y$. But $y \in M(L)$, contradicting the choice of a. □

This last result will reemerge to play an important role in §6 and beyond.

LEMMA 4.7. <u>Let</u> L <u>be a planar lattice satisfying</u> $(\underset{\sim}{SD}_\wedge)$, <u>and consider a fixed planar embedding of</u> L, <u>with its associated relation</u> λ.

(i) <u>If</u> x,y,y_1,y_2 <u>are elements of</u> L <u>such that</u> $x \lambda y$, $y_1 \succ y$, $y_2 > y$, <u>and</u> $y_1 \lambda y_2$, <u>then</u> $x \vee y = x \vee y_1$.

(ii) <u>If</u> $x \lambda y \lambda z$ <u>then</u> $x \wedge y \neq y \wedge z$.

<u>Proof</u>. (i) It is clear that $x \vee y > y$ implies $x \vee y \geq u$ for some $u \succ y$; from Lemma 2.7(i), $u = y_1$ or $u \leq y_2$. In the first case, $x \vee y \geq x \vee u = x \vee y_1 \geq x \vee y$, so that $x \vee y = x \vee y_1$ as desired. If $u \leq y_2$ then $y_1 \lambda u$ from Lemma 4.2, and so Lemma 4.3 implies that $x \vee y \geq x \vee u \geq x \vee y_1 \geq x \vee y$, showing that $x \vee y = x \vee y_1$ in either case.

(ii) If $x \wedge y = y \wedge z$ then $x \wedge y = y \wedge z = x \wedge z$ from the dual of Corollary 4.4. But this is impossible by Lemma 2.7(ii). □

§5. Minimal Pairs.

Let L be a lattice and let $X,Y \subseteq L$. We shall write $X << Y$ if, for every $x \in X$, there exists $y \in Y$ such that $x \leq y$.

Let $p \in L$ and let J be a finite subset of L. The pair $\langle p,J \rangle$ is called a minimal pair if the following three conditions hold:

(i) $p \not\leq J$;

(ii) $p \leq \bigvee J$;

(iii) if J' is a finite subset of L such that $p \leq \bigvee J'$ and $J' << J$, then $J' \supseteq J$.

We begin this section with some well-known properties of minimal pairs.

LEMMA 5.1. Let $\langle p,J \rangle$ be a minimal pair. Then

(i) J is an antichain;

(ii) $p \not\leq x$ for any $x \in J$;

(iii) $J \subseteq J(L)$;

(iv) $x \in J$ implies that $x_* \not\leq M(L)$;

(v) if $\langle p,J \rangle$ is a minimal pair and $p < p' \leq \bigvee J$ for $p' \in L$, $\langle p',J \rangle$ is also a minimal pair ;

(vi) $b(L) = n$ implies $|J| \leq n$.

Proof. (i) and (vi) are true because otherwise $\bigvee J = \bigvee J'$ for some proper subset J' of J; since $J' << J$, this is a contradiction.

(ii) if $p < x$ for $x \in J$ then, letting $x_* \in L$ be such that $p \leq x_* < x$ and setting $J' = (J \setminus \{x\}) \cup \{x_*\}$, we have that $J' << J$, $p \leq \bigvee J'$, and $J' \not\supseteq J$, contradicting the assumption that $\langle p,J \rangle$ is a minimal pair.

(iii) Let $x = y \vee z \in J$, and set $J' = (J \setminus \{x\}) \cup \{y,z\}$. Then $p \leq \bigvee J'$ and $J' << J$, so we must have that $J \subseteq J'$. This implies that $x = y$ or $x = z$, as desired.

(iv) Since $|J| \geq 2$ from (ii), we can find $x' \in J$ with $x' \neq x$. If $x_* \in M(L)$, then $x' \vee x_* = x' \vee x$; hence, letting $J' = (J \setminus \{x\}) \cup \{x_*\}$, we have that $J' << J$, $\bigvee J' = \bigvee J$, and $J' \supsetneq J$, contradicting the assumption that $\langle p,J \rangle$ is a minimal pair.

(v) This is obvious. □

As a consequence, there is an alternate description of minimal pairs in a lattice of breadth two.

LEMMA 5.2. *Let* $p \in L$ *and* $J \subseteq L$ *with* J *finite, and assume* $b(L) = 2$. *Then* $\langle p,J \rangle$ *is a minimal pair if and only if the following conditions hold:*

(i) $J = \{a,b\}$ *for some* $a,b \in L$;

(ii) $p \leq a \vee b$ *while* $p \nleq a$, $p \nleq b$;

(iii) *if* $a_1 \leq a$, $b_1 \leq b$ *are such that* $p \leq a_1 \vee b_1$, *then* $a_1 = a$ *and* $b_1 = b$. □

We shall require some further techniques to assist us in our investigation of minimal pairs.

LEMMA 5.3. *Let* $b(L) = 2$, *let* $x \in J(L)$, *and let* y *and* p *be elements of* L *such that* $x \parallel y$, $y < p \leq x \vee y$, *and* $p \nleq x_* \vee y$. *Then there exists* $y' \leq y$ *such that* $\langle p, \{x,y'\} \rangle$ *is a minimal pair.*

Proof. Let $Y = \{u \in L \mid u \leq y,\ p \leq x \vee u\}$. Since $y \in Y$, Y is nonempty, and hence we can find a minimal element y' of Y. If $x_1 \leq x$ and $y_1 \leq y'$ are such that $p \leq x_1 \vee y_1$, then $x_1 = x$; for otherwise $x_1 \leq x_*$ and so $x_1 \vee y_1 \leq x_* \vee y < p$. But now $y_1 = y'$ by the minimality of y', hence by Lemma 5.2 we are done. □

LEMMA 5.4. *Let* L *be planar, and let* $\langle p, \{x,y\}\rangle$ *be a minimal pair with* $x < p$ *and* $p \lambda y$. *Let* $p' \in L$ *be such that* $x < p' < x \vee y$ *and* $p' \lambda p$. *Then* $\langle p', \{x,y\}\rangle$ *is a minimal pair.*

Proof. We need only check that (iii) of Lemma 5.2 holds. Let $x' \le x$ and $y' \le y$ with $p' \le x' \vee y'$. If $y' < p$ then $p' \le x' \vee y' \le p$, which is impossible. Since $p \lambda y$, the dual of Lemma 4.2(i) implies that $p' \lambda p \lambda y'$. Now Lemma 4.3 implies $p < y' \vee p' \le x' \vee y'$, and we conclude that $x' = x$ and $y' = y$. □

LEMMA 5.5. *Let* L *be planar and let* $\langle p_1, \{x_1,y_1\}\rangle$, $\langle p_2, \{x_2,y_2\}\rangle$ *be minimal pairs with* $x_1 \lambda y_1$, $y_1 < p_1$, *and* $p_1 = y_2$. *Then either* $x_2 < x_1$ *or* $x_1 \lambda x_2$.

Proof. Let $x_2 \lambda x_1$ or $x_2 \ge x_1$. Then $x_2 \vee y_2 = x_2 \vee y_1$ although $\langle p_2, \{x_2,y_2\}\rangle$ is a minimal pair. □

Now, let L have breadth two and satisfy $(\underset{\sim}{SD}_{\vee})$ as well. We proceed to show that there are, in this case, just two kinds of minimal pairs, as illustrated in Figure 3. Let $\langle p, \{x,y\}\rangle$ be a minimal pair. If $x < p$, this is Figure 3(a) (except that p may equal $x \vee y$). If $y < p$ a similar diagram results. Suppose, then, that x, y, and p form an antichain. Since $x \vee y \ge p$ we have $x \vee y = x \vee y \vee p$. Suppose that $x \vee p \ge y$; then $x \vee p = x \vee y \vee p = x \vee y$, so by $(\underset{\sim}{SD}_{\vee})$ $x \vee (p \wedge y) = x \vee y \ge p$. But since $p \wedge y < y$ this contradicts the assumption that $\langle p, \{x,y\}\rangle$ is a minimal pair. Hence $x \vee p \not\ge y$ and by symmetry $y \vee p \not\ge x$. Thus we arrive at Figure 3(b) (of course, p need not equal the meet of $x \vee p$ and $y \vee p$). Minimal pairs corresponding to Figures 3(a) and 3(b) will be said to be of type (a) and type (b), respectively. It is important to note that the minimal pairs constructed in Lemmas 5.3 and 5.4 are of type (a).

We have shown that in a lattice of breadth two and satisfying $(\underset{\sim}{\mathrm{SD}}_\vee)$, every minimal pair is of type (a) or type (b). Indeed, even minimal pairs of type (b), as we shall shortly see, may be "replaced" by minimal pairs of type (a).

LEMMA 5.6. (i) *Let* L *be a finite breadth two lattice, and let* $\{a,b,c\}$ *be a three-element antichain in* L *with* $a \vee b \parallel a \vee c$. *Then* $a < b \vee c$.

(ii) *If in addition* L *satisfies* $(\underset{\sim}{\mathrm{SD}}_\wedge)$ *then, letting* a' *be a lower cover of* a, *either* $a' \vee b > a$ *or* $a' \vee c > a$.

Proof. (i) If $a \nless b \vee c$ then $\{b, a \vee b, a, a \vee c, c, b \vee c\}$ is a crown of order six, contradicting Lemma 2.2.

(ii) Let us suppose that $a' \vee b \ngtr a$ and $a' \vee c \ngtr a$. Then $a \wedge (a' \vee b) = a' = a \wedge (a' \vee c)$ whence, by $(\underset{\sim}{\mathrm{SD}}_\wedge)$ $a' = a \wedge (a' \vee b \vee c)$. From (i), $b \vee c > a$ so that $a' = a$, which is impossible. □

PROPOSITION 5.7. *Let* L *be a finite semidistributive lattice of breadth two and satisfying* $(\underset{\sim}{\mathrm{W}})$. *Let* $\langle p,\{x,y\}\rangle$ *be a minimal pair of type* (b) *with* $p \in J(L)$. *Then either*

(i) *there is* $y' \in L$ *such that* $y' < p$ *and* $\langle p,\{x,y'\}\rangle$ *is a minimal pair of type* (a),

or

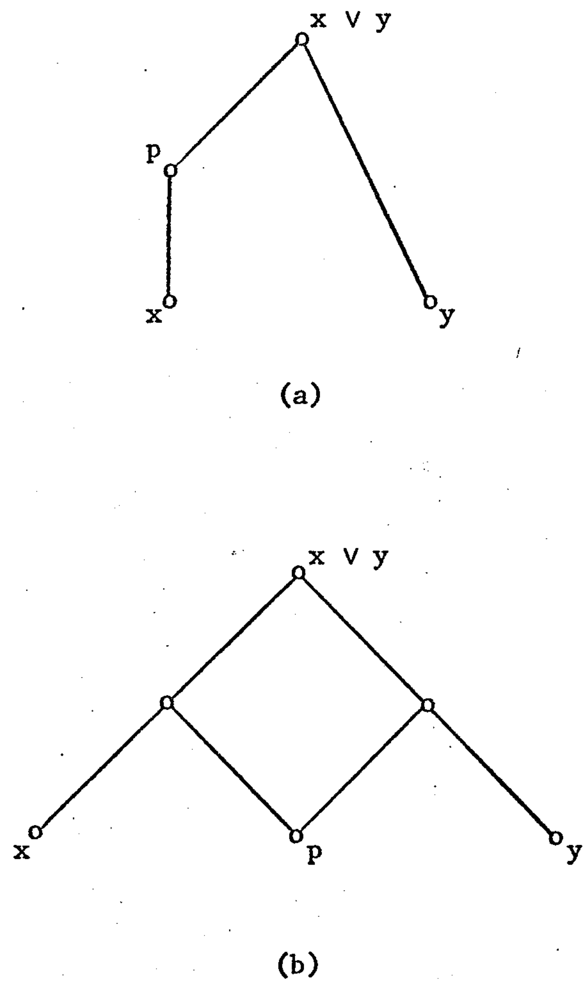

Figure 3. Two kinds of minimal pairs.

(ii) <u>there are elements</u> x', x'' <u>of</u> L <u>such that</u> $x'' < x' < p$, <u>and</u> $\langle p,\{x',y\}\rangle$, $\langle x',\{x,x''\}\rangle$ <u>are minimal pairs of type</u> (a).

<u>Remark</u>. In this proposition the pair $\{x,y\}$ is to be read as an <u>ordered</u> pair, so that the minimal pairs $\langle p,\{x,y\}\rangle$ and $\langle p,\{y,x\}\rangle$ are different. For example, the lattice of Figure 4, with the minimal pair $\langle p,\{x,y\}\rangle$ as indicated, satisfies (ii) but not (i).

<u>Proof</u>. Let $x_* \prec x$ and let us suppose that $p \not> x_*$. If $x_* \not< p \vee y$ then $p \le x_* \vee p \vee y = x_* \vee y$ by Lemma 5.6(i); hence, $x_* < p \vee y$. Now, let $a = (x \vee p) \wedge (x_* \vee y)$ and $b = (x \vee p) \wedge (y \vee p)$. Then $x_* \le a \le b$. As $\langle p,\{x,y\}\rangle$ is a minimal pair and $p \not\geq x_*$, it follows that $p \parallel a$. In addition, $x \vee a \le x \vee b = x \vee p$. If $x \vee a = x \vee p$ then, according to $(\underset{\sim}{SD}_\vee)$, $x \vee p = x \vee (a \wedge p)$. Now choose y' minimal so that $y' \le a \wedge p$ and $x \vee y' > p$. Then $x_* \vee y' \le a$ whence, $\langle p,\{x,y'\}\rangle$ is a minimal pair of type (a). On the other hand, if $x \vee a < x \vee p$ then $x \vee a \not\geq p$. Furthermore, $x \wedge b = x_* = x \wedge (x_* \vee y)$ so that from Lemma 2.7(ii) $a = b \wedge (x_* \vee y) > x_*$ and $a \parallel x$. Applying Lemma 5.6(ii) we have that either $a_* \vee x > a$ or $a_* \vee p > a$, where $a_* \prec a$. Either case is a violation of $(\underset{\sim}{W})$. We conclude that $p > x_*$.

Let p_* be the unique lower cover of p. Then $p_* \ge x_*$. By Lemma 5.6(ii) either $x \vee p_* > p$ or $y \vee p_* > p$. If $x \vee p_* > p$ choose y' minimal so that $y' \le p_*$ and $x \vee y' > p$. Then $\langle p,\{x,y'\}\rangle$ is a minimal pair of type (a). Otherwise, $x \vee p_* \not\geq p$ while $y \vee p_* > p$. Now choose x' minimal so that $x' \le p_*$ and

$x' \vee y > p$. We claim that $\langle p, \{x', y\}\rangle$ is a minimal pair (of type (a)). Let $y' \leq y$ satisfy $x' \vee y' \geq p$. Now, $y' \nleq x \vee p_*$ since $p \leq x' \vee y'$ while $x' \vee x \leq x \vee p_* \ngeq p$. Moreover, $y \vee p_* > p$ implies that $p_* > x_*$ so that $p_* \parallel x$. By Lemma 5.6(i), $x \vee y' \geq p_* \vee y' \geq x' \vee y' \geq p$. As $\langle p, \{x, y\}\rangle$ is a minimal pair it follows that $y' = y$ so that $\langle p, \{x', y\}\rangle$ is a minimal pair of type (a).

Let $x' \parallel x_*$. Set $c = p \wedge (x_* \vee y)$; then $x_* \leq c < p_*$. Since $x \wedge p = x_* = x \wedge (x_* \vee y)$ we have that $c > x_*$; that is, $c \parallel x$. Also, $c \parallel x'$ since $x' \vee y > p$. As $x \vee x' \leq x \vee p_*$, $\underset{\sim}{(W)}$ implies that $x \vee x' \ngeq c$. Let $d = (x \vee x') \wedge (x_* \vee y)$; then, by the dual of Lemma 5.6(i), $d < p$ and so $d < c$. We know that $x \vee d \leq x \vee x'$. If $x \vee d < x \vee x'$ then $x \vee d \ngeq x'$. Since $(x \vee d) \wedge (x_* \vee y) = d = (x' \vee d) \wedge (x_* \vee y)$ we have that $e = (x \vee d) \wedge (x' \vee d) > d$ and $e \nleq x_* \vee y$. Since $d \vee x' \leq p_*$ we have $e \vee y \leq p_* \vee y = x' \vee y$. If $e \vee y = x' \vee y$ then $x' \vee y = (e \wedge x') \vee y$ and $e \wedge x' < x'$, which contradicts the minimality of x'. Thus, $e \vee y < x' \vee y$ and $e \vee y \ngeq x'$. It follows that

$$\{x', x' \vee d, e, e \vee c, c, x \vee x', e \vee y, x, y\} \cong M2.$$

By virtue of Proposition 3.1 L must be infinite which is a contradiction. It follows that $x \vee d = x \vee x'$ whence, by $(\underset{\sim}{SD}_\vee)$ $x \vee (x' \wedge d) > x'$. Choose x'' minimal so that $x'' \leq x' \wedge d$ and $x \vee x'' > x'$. Since $x_* \vee x'' \leq d$, $x_* \vee x'' \ngeq x'$ so that $\langle x', \{x, x''\}\rangle$ is a minimal pair of type (a).

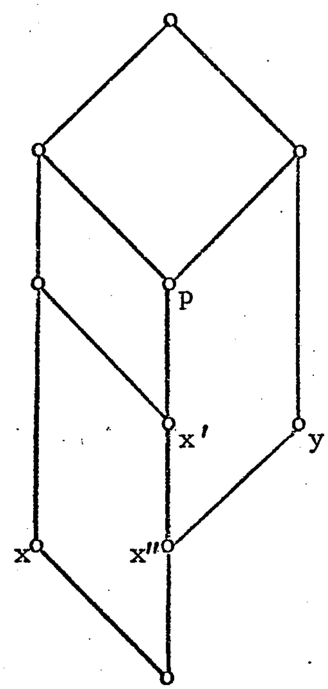

Figure 4

Finally, let $x' > x_*$. In view of the minimality of x', $x_*' \vee y \not\geq x'$, where $x_*' \prec x'$. Now, Lemma 5.6(ii) implies that $x_*' \vee x > x'$. Choose x'' minimal so that $x'' \leq x_*'$ and $x'' \vee x > x'$. Then $x_* \vee x'' < x'$; whence $\langle x', \{x, x''\}\rangle$ is a minimal pair of type (a). □

§6. Cycles.

A finite lattice L is said to satisfy $(T_\vee)$ if there is a linear ordering $\{x_1, \ldots, x_n\}$ of all the elements of L such that if $\langle x_i, J\rangle$ is a minimal pair and $x_j \in J$, then $i < j$. L is said to satisfy $(T_\wedge)$ if the dual of L satisfies $(T_\vee)$.

For us, the significance of these concepts lies in the following result.

THEOREM 6.1 (H. Gaskill and C. R. Platt [7]). *A finite lattice is a sublattice of a free lattice if and only if it satisfies* $(T_\vee)$, $(T_\wedge)$, *and* (W). □

It is clear that a finite lattice L will fail to satisfy $(T_\vee)$ if and only if there is an integer $n > 1$ and minimal pairs $\langle p_i, J_i\rangle$, $i = 1, 2, \ldots, n$, such that $p_{i+1} \in J_i$ for $1 \leq i < n$ and $p_1 \in J_n$; in this case we call $C = \{\langle p_1, J_1\rangle, \langle p_2, J_2\rangle, \ldots, \langle p_n, J_n\rangle\}$ a *cycle* of length n in L, and we write $\ell(C) = n$.

The next result is essentially a corollary of Proposition 5.7.

PROPOSITION 6.2. *Let* L *be a semidistributive lattice of breadth two, and let* L *contain a cycle of minimal pairs. Then* L *contains a cycle, all of whose minimal pairs are of type* (a).

Proof. For each cycle $C = \{\langle p_i, J_i\rangle : i = 1,\ldots,n\}$ in L, define

$$\beta(C) = |\{i : 1 \le i \le n,\ \langle p_i, J_i\rangle \text{ is of type (b)}\}|,$$

and let $\beta_0 = \min\{\beta(C) : C \text{ is a cycle in } L\}$. Choose a cycle $C = \{\langle p_i, J_i\rangle : i = 1,\ldots,n\}$ such that $\beta(C) = \beta_0$. If $\beta_0 = 0$, we have nothing to do. Suppose $\beta_0 > 0$, and, without loss of generality, assume that $\langle p_1, J_1\rangle$ is a minimal pair of type (b). Letting $J_1 = \{x_1, y_1\}$ and $p_2 = x_1$, we apply Proposition 5.7 with $x = x_1$, $y = y_1$, and $p = p_1$. If (i) holds, then

$$C_1 = \{\langle p_1, \{x_1, y'\}\rangle, \langle p_2, J_2\rangle, \ldots, \langle p_n, J_n\rangle\}$$

is a cycle in L with $\beta(C_1) < \beta(C) = \beta_0$, contrary to assumption. Similarly, if (ii) holds, then

$$C_2 = \{\langle p_1, \{x', y_1\}\rangle, \langle x', \{x_1, x''\}\rangle, \langle p_2, J_2\rangle, \ldots, \langle p_n, J_n\rangle\}$$

is a cycle in L with $\beta(C_2) < \beta(C) = \beta_0$, again contrary to assumption. □

Remark. Proposition 6.2 has also been proved by B. Jónsson and J. B. Nation [10, Corollary 6.4], their approach being quite different from ours. Immediate use will be made of this result, but we will eventually require the full force of Proposition 5.7.

We are ready to state our main theorem and begin its proof.

THEOREM 6.3. *A planar lattice L is a sublattice of a free lattice if and only if L is semidistributive and satisfies* (W).

The "only if" direction, of course, is well-known.

The proof of the converse will be by induction on the order of L. Let L be a planar semidistributive lattice satisfying (W);

by Theorem 6.1 it is enough to show L satisfies $(\underset{\sim}{T}_\vee)$ and $(\underset{\sim}{T}_\wedge)$. We assume we are given a fixed planar embedding of L, and any ensuing references to λ or other concepts associated with planarity will be with respect to this planar embedding.

Let a be the minimal meet irreducible on the right boundary of L. By Lemma 4.6, a is doubly irreducible ; hence $L \setminus \{a\}$ is a sublattice of L, and thus is a planar semidistributive lattice satisfying $(\underset{\sim}{W})$ whose order is smaller than that of L. By the induction hypothesis, $L \setminus \{a\}$ satisfies Theorem 6.3.

Suppose L violates $(\underset{\sim}{T}_\vee)$ or $(\underset{\sim}{T}_\wedge)$; by dualizing if necessary, we may assume that L violates $(\underset{\sim}{T}_\vee)$. Thus L contains a cycle $\{\langle p_i, J_i\rangle : i = 1,2,\ldots,n\}$ of minimal pairs, and from Proposition 6.2 we may assume that every $\langle p_i, J_i\rangle$ is of type (a). Clearly $a \in \bigcup_{i=1}^{n} J_i$, for otherwise $\{\langle p_i, J_i\rangle : i=1,2,\ldots,n\}$ will be a cycle in $L \setminus \{a\}$, contradicting the induction hypothesis.

<u>Aside</u>. By definition, $p_{i+1} \in J_i$ for $1 \le i \le n-1$, and $p_1 \in J_n$. For convenience, we will simply write that $p_{i+1} \in J_i$ for $1 \le i \le n$; that is, the subscripts are to be read modulo n. A similar convention will be adopted, usually without comment, throughout this paper.

For each $i \in \{1,2,\ldots,n\}$, define

$$p_i' = \begin{cases} p_i & \text{if } p_i \neq a \\ a^* & \text{if } p_i = a \end{cases}$$

and

$$J_i' = \begin{cases} J_i & \text{if } a \notin J_i \\ (J_i \setminus \{a\}) \cup \{a^*\} & \text{if } a \in J_i \end{cases}$$

We first claim that $p'_{i+1} \in J'_i$ for each $i \in \{1,2,\ldots,n\}$. If $p_{i+1} = a$, then since $p_{i+1} \in J_i$ we have $p'_{i+1} = a^* \in J'_i$. Suppose $p_{i+1} \neq a$; then clearly $p_{i+1} \notin J_i \setminus J_i'$, and since $p'_{i+1} = p_{i+1} \in J_i$ we conclude $p'_{i+1} \in J'_i$, as claimed.

Notice also that $\bigcup_{i=1}^{n} J'_i \subseteq L \setminus \{a\}$; therefore, if each $\langle p'_i, J'_i \rangle$ were a minimal pair in $L \setminus \{a\}$, $\{\langle p'_i, J'_i \rangle : i = 1,2,\ldots,n\}$ would be a cycle of minimal pairs in $L \setminus \{a\}$, contrary to the induction hypothesis. Consequently we choose k such that $\langle p'_k, J'_k \rangle$ is not a minimal pair in $L \setminus \{a\}$. In particular, $\langle p'_k, J'_k \rangle \neq \langle p_k, J_k \rangle$, and hence $a = p_k$ or $a \in J_k$. If $a = p_k$, then $a \notin J_k$, and $a \leq \bigvee J_k$; but since a is doubly irreducible and $a < a^* = p'_k$, it follows that $p'_k \leq \bigvee J_k = \bigvee J'_k$, and from Lemma 5.1 (v) $\langle p'_k, J'_k \rangle$ is a minimal pair in $L \setminus \{a\}$, contrary to assumption. Therefore let $a \in J_k$, and let $J_k = \{x_k, y_k\}$ where $y_k < p_k$.

Suppose $a = y_k$. Then $p_k \geq a^*$, and since $p_k \in J_{k-1}$, Lemma 5.1(iv) implies that $p_k > a^*$. We have $x_k \vee a^* = x_k \vee y_k > p_k$; also, if $b \leq x_k$ is such that $b \vee a^* > p_k$, then since $b \vee a^* = b \vee y_k$ and $\langle p_k, J_k \rangle$ is a minimal pair, $b = x_k$. Since $\langle p'_k, J'_k \rangle = \langle p_k, \{a^*, x_k\} \rangle$ is not a minimal pair in $L \setminus \{a\}$, we can find $c < a^*$, $c \neq a$, such that $x_k \vee c > p_k$. But now $x_k \vee c = x_k \vee a^* = x_k \vee a$, and by $(\underset{\sim}{SD}_\vee)$ we have $x_k \vee a = x_k \vee (c \wedge a)$, contradicting the fact that $\langle p_k, J_k \rangle$ is a minimal pair. Hence $a = x_k$.

Let $\mathcal{C}$ denote the set of all cycles in L whose minimal pairs are all of type (a). For each $C \in \mathcal{C}$ we have seen that we may choose a minimal pair $\langle p_C, J_C \rangle$ of C such that $\langle p'_C, J'_C \rangle$ (defined above) is not a minimal pair in $L \setminus \{a\}$; furthermore, $a \in J_C$ and $a \parallel p_C$. For $C = \{\langle p_i, J_i \rangle : i = 1,2,\ldots,n\} \in \mathcal{C}$ we define

$$\alpha(C) = |\{ i \mid a \in J_i ,\ 1 \le i \le n \}|,$$

and we set $\alpha_0 = \min \{\alpha(C) \mid C \in \mathcal{C}\}$.

We now show that there exists a cycle $\bar{C} \in \mathcal{C}$ and a choice for $\langle p_{\bar{C}}, J_{\bar{C}} \rangle$ such that $\alpha(\bar{C}) = \alpha_0$ and $a^* > p_{\bar{C}}$. Choose $C = \{\langle p_i, J_i \rangle : i = 1,2,\ldots,n\} \in \mathcal{C}$ (where $\langle p_C, J_C \rangle = \langle p_k, J_k \rangle$, say) such that $\alpha(C) = \alpha_0$, and assume that $a^* \not> p_k$. Let $J_k = \{x_k, y_k\}$ where $y_k < p_k$ and $x_k = a$. If $u \le y_k$ is such that $u \vee a^* > p_k$, then since $u \vee a^* = u \vee x_k$ and $\langle p_k, J_k \rangle$ is a minimal pair, $u = y_k$. Since $\langle p_k', J_k' \rangle = \langle p_k, \{y_k, a^*\}\rangle$ is not a minimal pair in $L \setminus \{a\}$, by Lemma 5.3 we can find $x' < a^*$, $x' \ne a$, such that $\langle p_k, \{y_k, x'\}\rangle$ is a minimal pair of type (a). If $x' \mathrel{\lambda} a_*$ then $y_k \vee a_* \ge y_k \vee x' > p_k$, contradicting the fact that $\langle p_k, J_k \rangle$ is a minimal pair; since $x' \mathrel{\lambda} a$ it follows that $x' > a_*$. Let $x_*' \prec x'$. By the choice of x', $y_k \vee x_*' \not\ge p_k$, implying $y_k \vee x_*' \not\ge x'$, and so $(y_k \vee x_*') \wedge x' = x_*'$. If $x_*' \vee a \not\ge x'$, then $(x_*' \vee a) \wedge x' = x_*' = (y_k \vee x_*') \wedge x'$, and also $y_k \vee x_*' \mathrel{\lambda} x' \mathrel{\lambda} x_*' \vee a$, contradicting Lemma 4.7(ii). Thus $x_*' \vee a > x'$, and by Lemma 5.3 there exists $x'' \le x_*'$ such that $\langle x', \{x'', a\}\rangle$ is a minimal pair of type (a). Hence

$$\bar{C} = \{\langle p_1, J_1\rangle, \ldots, \langle p_{k-1}, J_{k-1}\rangle, \langle p_k, \{y_k, x'\}\rangle, \langle x', \{x'', x_k\}\rangle, \langle p_{k+1}, J_{k+1}\rangle, \ldots, \langle p_n, J_n\rangle\}$$

is a cycle in $\mathcal{C}$ with $\alpha(\bar{C}) = \alpha_0$, $\langle p_{\bar{C}}, J_{\bar{C}} \rangle = \langle x', \{x'', x_k\}\rangle$, and $a^* > p_{\bar{C}}$, as desired. For future reference, we summarize this result in a lemma.

LEMMA 6.4. <u>If</u> $C \in \mathcal{C}$ <u>and</u> $\langle p_C, J_C \rangle$ <u>are such that</u> $\alpha(C) = \alpha_0$ <u>and</u> $a^* \not> p_C$, <u>then there exists</u> $\bar{C} \in \mathcal{C}$ <u>and</u> $\langle p_{\bar{C}}, J_{\bar{C}} \rangle$ <u>such that</u> $\alpha(\bar{C}) = \alpha_0$, $\ell(\bar{C}) = \ell(C) + 1$, <u>and</u> $a^* > p_{\bar{C}}$. □

We are now able to choose a specific cycle in L to be studied throughout the remainder of the proof. Let

$$\bar{\mathcal{C}} = \{C \in \mathcal{C} \mid \alpha(C) = \alpha_0 \text{ and there is a choice of } \langle p_C , J_C \rangle \text{ such that } a^* > p_C \}$$

and set $\ell_0 = \min\{\ell(C) \mid C \in \bar{\mathcal{C}}\}$. Choose $C_0 = \{\langle p_i , J_i \rangle \mid i = 1,2,\ldots,\ell_0\} \in \bar{\mathcal{C}}$, where $\ell(C_0) = \ell_0$, and suppose that $\langle p_{C_0}, J_{C_0}\rangle = \langle p_k, J_k\rangle$. Let $J_i = \{x_i, y_i\}$ and $z_i = x_i \vee y_i$ for $i = 1,2,\ldots,\ell_0$. Then we have that $y_k < p_k < z_k = a^*$ and $x_k = a$. Furthermore, let $y_{k*} \prec y_k$; then if $y_{k*} \parallel x_k$, we see that $y_{k*} \vee x_k = a^* > p_k$, contradicting the fact that $\langle p_k, J_k\rangle$ is a minimal pair. Therefore $y_{k*} < x_k = a$, which implies $y_{k*} \leq a_*$. Figure 5 illustrates the minimal pair $\langle p_k, J_k\rangle$, with the understanding that y_{k*} may equal a_*.

We continue our explorations by locating p_{k+1} which, by assumption, is in $J_k = \{x_k, y_k\}$. Our goal now is to prove that $p_{k+1} = x_k$.

Suppose $p_{k+1} = y_k$. Letting $x_{k+1} < p_{k+1}$, we have $x_{k+1} \leq y_{k*}$ and hence $y_{k+1} \lambda x_{k+1}$, since x_{k+1} is on the right boundary by Lemma 4.6. If $z_{k+1} = x_{k+1} \vee y_{k+1} \geq p_k$, then by Lemma 5.1(v) $\langle p_k, J_{k+1}\rangle$ is a minimal pair of type (a), and

$$C = \{\langle p_1, J_1\rangle, \ldots, \langle p_{k-1}, J_{k-1}\rangle, \langle p_k, J_{k+1}\rangle, \langle p_{k+2}, J_{k+2}\rangle, \ldots, \langle p_{\ell_0}, J_{\ell_0}\rangle\}$$

is a cycle in $\mathcal{C}$ for which $\alpha(C) < \alpha(C_0) = \alpha_0$, an impossibility. Also, since $z_k = a^*$, Lemma 4.5 shows that $y_{k+1} \not\leq a^*$. Hence $z_{k+1} \parallel p_k$ and $z_{k+1} \parallel a^*$. Since $y_{k+1} \lambda x_{k+1}$, we deduce from

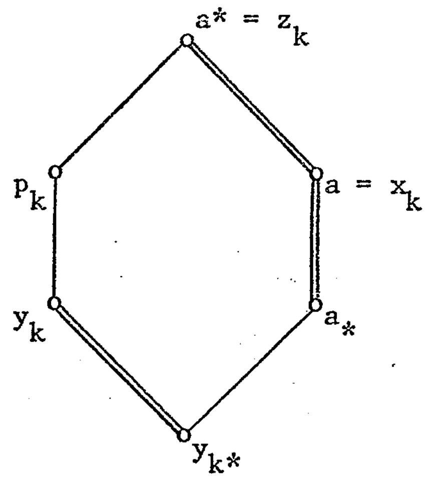

Figure 5

Lemma 4.2 that $y_{k+1} \lambda p_k$ and hence $z_{k+1} \lambda p_k$ and $z_{k+1} \lambda a^*$. Consequently $z_{k+1} < y_{k+1} \vee a$.

We claim that $x_{k+1} = y_{k*}$. Otherwise, $x_{k+1} < y_{k*}$, and $\underset{\sim}{(W)}$ implies that $z_{k+1} = x_{k+1} \vee y_{k+1} \not\geq p_k \wedge a$. Also, x_{k+1} is meet reducible by the choice of a, and hence there exists $u \succ x_{k+1}$ such that $u \lambda a_*$. Since $y_k \in J(L)$ we have $u \lambda y_k$, and by Lemma 4.5 $u \parallel a^*$. However, $y_{k+1} \lambda x_{k+1}$ implies $y_{k+1} \lambda u \lambda y_k$, and so by Lemma 4.7(i) $y_{k+1} \vee u = z_{k+1} = y_{k+1} \vee y_k$. By Lemma 2.7(ii), $u \vee y_k < z_{k+1}$. Therefore, as in Figure 6,

$$\{u,\ u \vee y_k,\ y_k,\ p_k,\ p_k \wedge a,\ z_{k+1},\ a^*,\ y_{k+1},\ a\}$$

is a subset of L isomorphic to M2. By Proposition 3.1 this is a contradiction, and hence $x_{k+1} = y_{k*}$.

Now let $y_{k+1*} \prec y_{k+1}$, and suppose that $y_{k+1*} \lambda x_{k+1}$. From Lemma 4.7(i), $y_{k+1*} \vee x_{k+1} \geq p_{k+1}$, contradicting the fact that $\langle p_{k+1}, J_{k+1}\rangle$ is a minimal pair. Thus $y_{k+1*} < x_{k+1}$.

We use the same arguments to investigate $\langle p_{k+2}, J_{k+2}\rangle$. Suppose $x_{k+1} = p_{k+2}$. If $z_{k+2} \geq p_{k+1}$, then by Lemma 5.1(v) $\langle p_{k+1}, J_{k+2}\rangle$ is a minimal pair of type (a), and

$$C = \{\langle p_1, J_1\rangle, \ldots, \langle p_k, J_k\rangle, \langle p_{k+1}, J_{k+2}\rangle, \langle p_{k+3}, J_{k+3}\rangle, \ldots, \langle p_{\ell_0}, J_{\ell_0}\rangle\}$$

is a cycle in $\bar{\mathcal{C}}$ for which $\ell(C) < \ell(C_0) = \ell_0$, an impossibility. Therefore $z_{k+2} \not\geq p_{k+1}$, and hence $z_{k+2} \not\geq a$. But now $z_{k+2} = x_{k+2} \vee y_{k+2} \geq p_{k+2} = x_{k+1} = p_{k+1} \wedge a$, which is a violation of $\underset{\sim}{(W)}$. Consequently we must have $p_{k+2} = y_{k+1}$, and we may assume

$x_{k+2} \leq y_{k+1*}$ and $y_{k+2} \,\lambda\, y_{k+1}$. By Lemma 4.5, $y_{k+2} \parallel y_{k+1} \vee a$, and $z_{k+2} \not\geq p_{k+1}$ as before. Suppose that $x_{k+2} < y_{k+1*}$; then there exists $u' \succ x_{k+2}$ such that $u' \,\lambda\, a_*$. It follows that $y_{k+2} \,\lambda\, u' \,\lambda\, y_{k+1}$ and $u' \vee y_{k+1} < z_{k+2}$. Also $u' \not\leq y_{k+1} \vee a$, and therefore

$$\{u', u' \vee y_{k+1}, y_{k+1}, z_{k+1}, x_{k+1}, z_{k+2}, y_{k+1} \vee a, y_{k+2}, a\}$$

is a subset of L isomorphic to M2, which is impossible. Hence $x_{k+2} = y_{k+1*}$. Letting $y_{k+2*} \prec y_{k+2}$, we see that $y_{k+2*} < x_{k+2}$, since $\langle p_{k+2}, J_{k+2}\rangle$ is a minimal pair.

If $p_{k+3} = x_{k+2}$ then, as before, $z_{k+3} \not\geq p_{k+2}$ and $z_{k+3} \not\geq p_{k+1}$. But $z_{k+3} = x_{k+3} \vee y_{k+3} \geq p_{k+3} = x_{k+2} = p_{k+2} \wedge p_{k+1}$ is a violation of (W); hence $p_{k+3} = y_{k+2}$. By Lemma 4.7(i), $y_{k+2} \vee x_{k+1} > p_{k+1}$. By Lemma 5.3, there exists $x' \leq x_{k+1}$ such that $\langle p_{k+1}, \{y_{k+2}, x'\}\rangle$ is a minimal pair of type (a). Thus

$$C = \{\langle p_1, J_1\rangle, \ldots, \langle p_k, J_k\rangle, \langle p_{k+1}, \{y_{k+2}, x'\}\rangle, \langle p_{k+3}, J_{k+3}\rangle, \ldots, \langle p_{\ell_0}, J_{\ell_0}\rangle\}$$

is a cycle in $\bar{\mathfrak{C}}$ with $\ell(C) < \ell(C_0) = \ell_0$, which is impossible.

We conclude that $p_{k+1} = x_k = a$, as claimed. An important consequence is:

LEMMA 6.5. (i) $a \in J_i$ implies that $i = k$; in other words, $\alpha_0 = 1$.
(ii) y_k does not equal p_t for any $t \in \{1, \ldots, \ell_0\}$.

Proof. (i) Suppose $a \in J_t$, $t \neq k$; then

$$C = \{\langle p_{k+1}, J_{k+1}\rangle, \ldots, \langle p_t, J_t\rangle\}$$

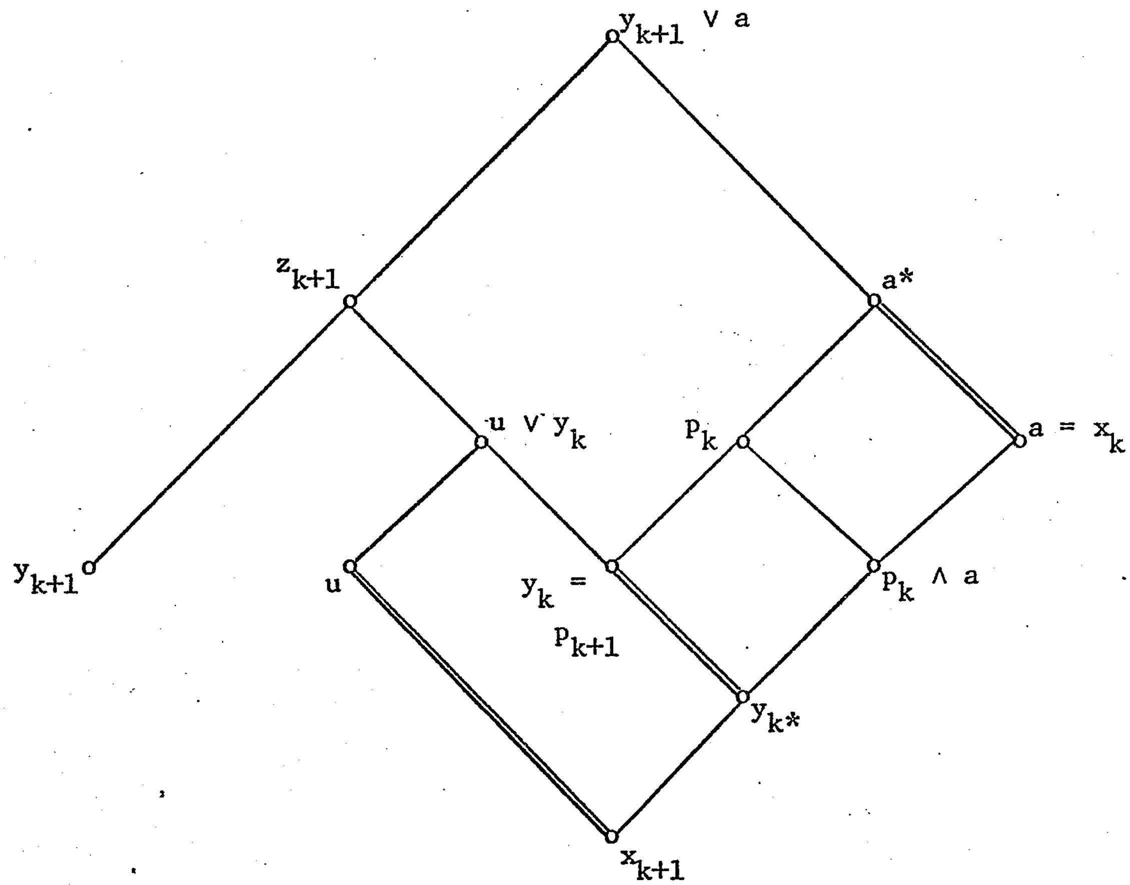

Figure 6

is a cycle in $\mathcal{C}$ with $\alpha(C) < \alpha(C_0) = \alpha_0$, which is impossible.

(ii) Suppose that $y_k = p_t$ for some t ; then $t \neq k + 1$, and so

$$C = \{\langle p_t, J_t\rangle, \ldots, \langle p_k, J_k\rangle\}$$

is a cycle in $\bar{\mathcal{C}}$ with $\ell(C) < \ell_0$, which is impossible. □

The following two lemmas will be useful in asserting the nonexistence of certain minimal pairs, and will be applied many times in conjunction with Lemma 5.3.

LEMMA 6.6. *Let* $1 \leq t \leq \ell_0$ *and let* $x' \in L$ *be such that* $\langle p_t, \{a, x'\}\rangle$ *is a minimal pair of type* (a). *Then* $t = k$ *or* $t = k - 1$ (*modulo* ℓ_0).

Proof. Suppose $t \neq k - 1$ or k and let

$$C = \{\langle p_{k+1}, J_{k+1}\rangle, \ldots, \langle p_{t-1}, J_{t-1}\rangle, \langle p_t, \{a, x'\}\rangle\} .$$

Then $C \in \mathcal{C}$ with $\alpha(C) = \alpha_0 = 1$, $\ell(C) < \ell_0 - 1$. It follows that $C \notin \bar{\mathcal{C}}$; that is, $a^* \not> p_t$. But from Lemma 6.4 we can construct $\bar{C} \in \bar{\mathcal{C}}$ such that $\ell(\bar{C}) = \ell(C) + 1 < \ell_0$; since this is impossible, we are done. □

LEMMA 6.7. *Let* $1 \leq t \leq \ell_0$ *and let* $z \in L$ *be such that* $\langle p_t, \{z, a\}\rangle$ *is a minimal pair. Then* $t = k$, $k - 1$, or $k - 2$ (*modulo* ℓ_0).

Proof. If $\langle p_t, \{z, a\}\rangle$ is of type (a), we are done by Lemma 6.6. Therefore let $\langle p_t, \{z, a\}\rangle$ be of type (b), and let

$$C = \{\langle p_t, \{z, a\}\rangle, \langle p_{k+1}, J_{k+1}\rangle, \ldots, \langle p_{t-1}, J_{t-1}\rangle\}.$$

Assume that $t \neq k$, $k-1$, or $k-2$, and apply Proposition 5.7 with $x = a$, $y = z$, and $p = p_t$. If (i) holds, then there exists $y' \in L$ such that $\langle p_t, \{a, y'\}\rangle$ is a minimal pair of type (a), and

$$C_1 = \{\langle p_t, \{a, y'\}\rangle, \langle p_{k+1}, J_{k+1}\rangle, \ldots, \langle p_{t-1}, J_{t-1}\rangle\}$$

is a cycle in $\mathfrak{C}$ such that $\alpha(C_1) = 1$ and $\ell(C_1) < \ell_0 - 2$. On the other hand, if (ii) holds then there exist $x', x'' \in L$ such that $\langle p_t, \{x', z\}\rangle$ and $\langle x', \{a, x''\}\rangle$ are minimal pairs of type (a), and

$$C_2 = \{\langle p_t, \{x', z\}\rangle, \langle x', \{a, x''\}\rangle, \langle p_{k+1}, J_{k+1}\rangle, \ldots, \langle p_{t-1}, J_{t-1}\rangle\}$$

is a cycle in $\mathfrak{C}$ such that $\alpha(C_2) = 1$ and $\ell(C_2) < \ell_0 - 1$. In either case, Lemma 6.4 again implies that there exists $\overline{C} \in \overline{\mathfrak{C}}$ such that $\ell(\overline{C}) < \ell_0$, which is impossible. □

7. The case $a_* \nleq p_k$.

Before proceeding to the substance of this section, we prove a simple lemma.

LEMMA 7.1. *Let* K *be a lattice satisfying* $(SD_\wedge)$, *and let* u, v, a, b, *and* c *be elements of* K *such that* $u \prec a < v$, $u < b \prec v$, $a \parallel b$, *and* $u < c < v$. *Then either* $c \leq b$ *or* $c \geq a$.

Proof. If $c \nleq b$ and $c \ngeq a$, then $c \vee b = v$ and $c \wedge a = u = b \wedge a$. By $(SD_\wedge)$, $u = (c \vee b) \wedge a = v \wedge a = a$, a contradiction. □

Without loss of generality, we may let $p_k = y_{k-1}$. Assume that $a_* \nleq p_k$; we will eventually arrive at a contradiction. For convenience, this will be accomplished in two parts.

A. Claim. $y_{k-1} \lambda x_{k-1}$.

Suppose that $x_{k-1} \lambda y_{k-1}$; then $x_{k-1} \lambda y_k$ or $x_{k-1} > y_k$. If $x_{k-1} < a^*$, Lemma 4.5 implies that $x_{k-1} > y_k$; but by Lemma 5.1(v) and Lemma 5.4 this would mean that $\langle p_{k-1}, J_k \rangle$ is a minimal pair of type (a), and we can construct a cycle $C \in \bar{\mathcal{C}}$ with $\ell(C) < \ell_0$. This is a contradiction, and therefore $x_{k-1} \lambda a^*$. If $z_{k-1} > a_*$, then by Lemma 4.3 $z_{k-1} = x_{k-1} \vee y_{k-1} = x_{k-1} \vee a_*$, and by $(SD_\vee)$ $z_{k-1} = x_{k-1} \vee (y_{k-1} \wedge a_*)$, contradicting the fact that $\langle p_{k-1}, J_{k-1} \rangle$ is a minimal pair. Hence $z_{k-1} \lambda a_*$ and $z_{k-1} \lambda a^*$ both hold. As a result, $z_{k-1} < x_{k-1} \vee a$ (see Figure 7).

Let $y_{k-1*} \prec y_{k-1}$ and set $u = x_{k-1} \vee y_{k-1*}$. Since $\langle p_{k-1}, J_{k-1} \rangle$ is a minimal pair, $u \ngeq y_{k-1}$ and hence $u \lambda y_{k-1}$. Since $\langle p_k, J_k \rangle$ is a minimal pair, $y_k \vee a_* \ngeq p_k = y_{k-1}$ and hence $y_{k-1} \lambda y_k \vee a_* \lambda a$. Now since $y_{k-1} \vee a = a^* = (y_k \vee a_*) \vee a$, the dual of Lemma 2.7(ii) implies that $y_{k-1} \vee a_* = y_{k-1} \vee (y_k \vee a_*) < a^*$, and so $y_{k-1} \vee a_* \ngeq a$. Moreover, from Lemma 4.7 (ii) $y_{k-1} \wedge (y_k \vee a_*) \neq y_{k-1*}$. It follows that $y_k < y_{k-1*}$. Now, letting $v = z_{k-1} \wedge a^*$, we can choose z such that $v \prec z \leq z_{k-1}$. Of course $z \nleq a^*$. If $z < z_{k-1}$, then by (W) we deduce $z \wedge u \nleq a^* = y_k \vee a$; hence

$$\{z \wedge u,\ z,\ y_{k-1},\ y_{k-1} \vee a_*,\ a_*,\ z_{k-1},\ a^*,\ x_{k-1},\ a\} \cong M2,$$

which is impossible. Therefore $v \prec z_{k-1}$.

Either $p_{k-1} > x_{k-1}$ or $p_{k-1} > y_{k-1}$. If $y_{k-1} < p_{k-1} < z_{k-1}$ then $y_k < p_{k-1} < a^* = z_k$; but then $\langle p_{k-1}, J_k \rangle$ is a minimal pair of type (a), and we can construct a cycle $C \in \bar{\mathcal{C}}$ with $\ell(C) < \ell_0$. Hence we must have $x_{k-1} < p_{k-1} < z_{k-1}$. Assume that $x_{k-1} \neq u$. Since $\langle p_{k-1}, J_{k-1} \rangle$ is a minimal pair, $p_{k-1} \nleq u$. If $p_{k-1} \parallel u$ then $p_{k-1} \ngeq y_{k-1*}$, and

$$\{x_{k-1}, u, y_{k-1*}, y_{k-1} \vee a_*, a_*, p_{k-1} \vee u, a^*, p_{k-1}, a\} \cong M2,$$

which is impossible. On the other hand, if $p_{k-1} > u$ then (recalling that $p_{k-1} \in J(L)$) we have

$$\{x_{k-1}, u, y_{k-1*}, y_{k-1} \vee a_*, a_*, p_{k-1}, a^*, a\} \cong M1,$$

also an impossibility. Thus $x_{k-1} = u$, that is, $x_{k-1} > y_{k-1*}$.

Let $x_{k-1*} \prec x_{k-1}$; either $x_{k-1*} \;\lambda\; v$ or $x_{k-1*} < v$. If $x_{k-1*} \;\lambda\; v$ then, since $\langle p_{k-1}, J_{k-1} \rangle$ is a minimal pair, Lemma 4.2 implies that $x_{k-1} \;\lambda\; x_{k-1*} \vee y_{k-1} \;\lambda\; v$; but now $(x_{k-1*} \vee y_{k-1}) \vee v = z_{k-1} = x_{k-1} \vee (x_{k-1*} \vee y_{k-1})$, contradicting the dual of Lemma 4.7(ii). Hence $x_{k-1*} < v$.

Since $p_{k-1} \in J_{k-2}$, we may let $p_{k-1} = y_{k-2}$. Suppose that $x_{k-2} \;\lambda\; y_{k-2}$; we proceed to show this is impossible.

First assume that $x_{k-2} < z_{k-1}$. Either $x_{k-2} \;\lambda\; x_{k-1}$ or $x_{k-2} > x_{k-1}$; Lemma 4.5 shows that $x_{k-2} > x_{k-1}$. But now $\langle p_{k-2}, J_{k-1} \rangle$ is a minimal pair of type (a), and we can construct a cycle $C \in \bar{\mathcal{C}}$ with $\ell(C) < \ell_0$, which is impossible. Therefore $x_{k-2} \;\lambda\; z_{k-1}$. Since $\langle p_{k-2}, J_{k-2} \rangle$ is a minimal pair, $x_{k-2} \vee x_{k-1} \ngeq y_{k-2}$, and so $x_{k-2} \vee x_{k-1} \ngeq y_{k-1}$. Hence $y_{k-1} \wedge y_{k-2} = y_{k-1*} = y_{k-1} \wedge (x_{k-2} \vee x_{k-1})$, and by $(SD_\wedge)$ we get that $z_{k-2} = (x_{k-2} \vee x_{k-1}) \vee y_{k-2} \nleq y_{k-1}$.

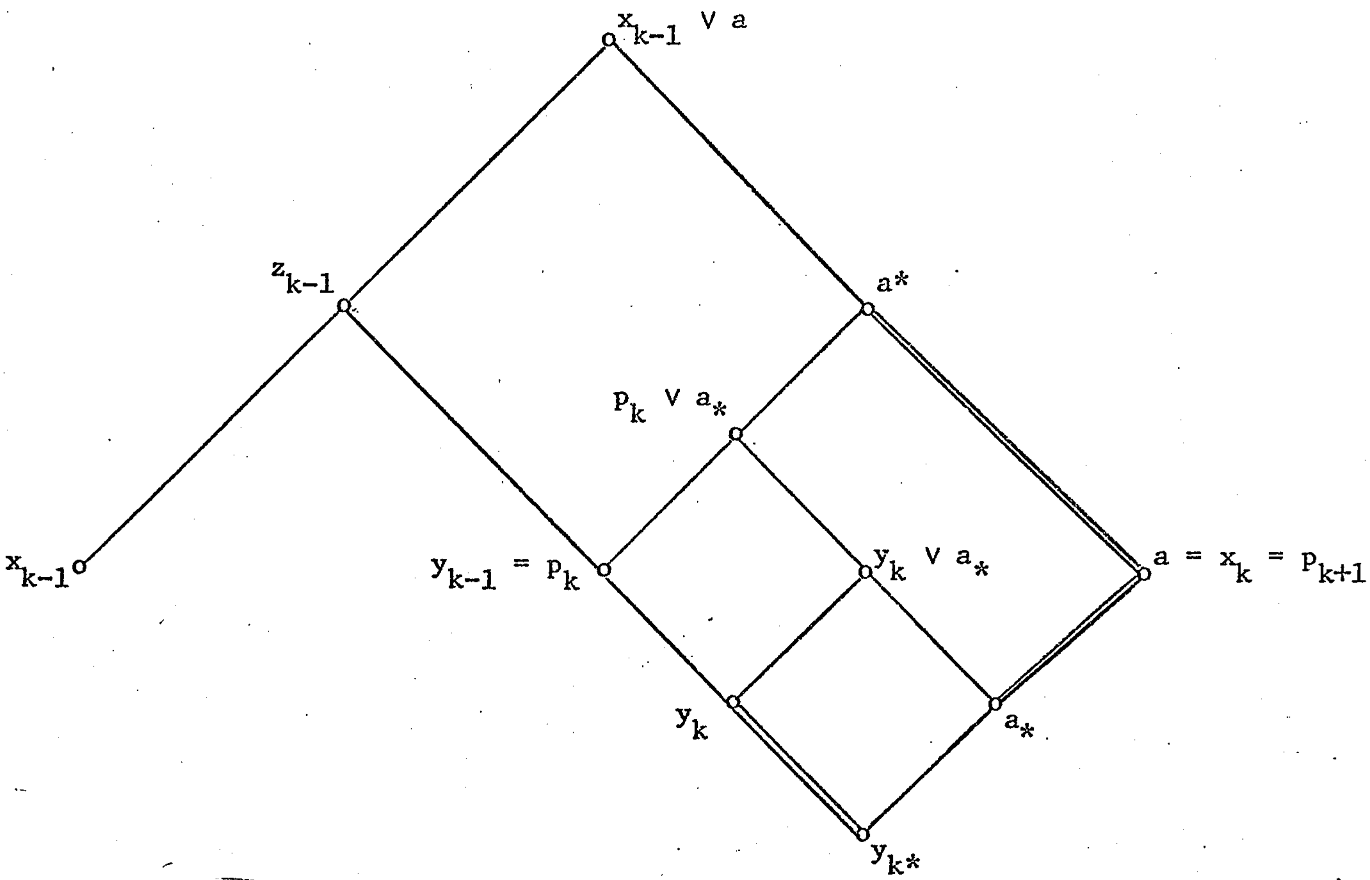

Figure 7

Now since $y_{k-1} \vee a_* < a^*$, $\underset{\sim}{(W)}$ implies that $y_{k-1*} \vee a_* \not\geq v = z_{k-1} \wedge a^*$. Also, since $x_{k-1} \lambda y_{k-1}$ and $y_{k-1*} \lambda a_*$, the reflection of Lemma 4.7(i) shows that $y_{k-1*} \vee a_* = y_{k-1} \vee a_*$. Consequently, $y_{k-1} < v$, and

$$\{z_{k-2}, y_{k-2}, z_{k-1}, v, a^*, x_{k-1}, y_{k-1}, x_{k-2} \vee x_{k-1}, y_{k-1} \vee a_*\} \cong (M2)^d.$$

Thus we conclude that $y_{k-2} \lambda x_{k-2}$.

Since $y_{k-1*} < x_{k-1} < y_{k-2}$, it is clear that $x_{k-2} \not\leq y_{k-1}$. From Lemma 5.5 we have that $x_{k-2} \lambda y_{k-1}$, and so $z_{k-2} \leq z_{k-1}$ by Lemma 4.3. If $x_{k-2} > x_{k-1}$, then $\langle p_{k-2}, J_{k-1}\rangle$ is a minimal pair of type (a), and we can construct a cycle $C \in \bar{\mathcal{C}}$ with $\ell(C) < \ell_0$, which is impossible. Therefore, since $y_{k-2} \lambda x_{k-2}$, we have $x_{k-1} \lambda x_{k-2} \lambda y_{k-1}$. If $x_{k-2} \lambda v$, then by Lemma 4.3 $x_{k-1*} = x_{k-1} \wedge v < x_{k-2} < x_{k-1} \vee v$, contradicting Lemma 7.1. Thus $x_{k-2} < v$. Also, $y_{k-2} \lambda x_{k-1} \vee x_{k-2}$ since $\langle p_{k-2}, J_{k-2}\rangle$ is a minimal pair, and $(x_{k-1} \vee x_{k-2}) \vee v = z_{k-1} = y_{k-2} \vee v$; Lemma 2.7(ii) implies that $z_{k-2} = (x_{k-1} \vee x_{k-2}) \vee y_{k-2} < z_{k-1}$. We now have

$$\{x_{k-1}, x_{k-1} \vee x_{k-2}, x_{k-2}, v, y_{k-1}, z_{k-2}, a^*, y_{k-2}, a\} \cong M2.$$

This contradiction establishes that $y_{k-1} \lambda x_{k-1}$, as advertised.

$\underset{\sim}{B}$. Claim. For all $j < k$, $y_{k*} < p_j < y_k \vee a_*$ and $y_{k*} < x_j$, $y_j < p_k \vee a_*$.

First notice that $y_{k*} < x_{k-1} < a^*$ and $y_{k*} < p_{k-1} < z_{k-1} \leq a^*$, since y_{k*} and a^* are both on the right boundary. Also, it is easy to see that neither x_{k-1} nor p_{k-1} can equal a. By Lemma 7.1, either $p_{k-1} > y_k$ or $p_{k-1} \leq a_*$. If $p_{k-1} \not\leq y_k \vee a_*$, then

$\langle p_{k-1}, J_k \rangle$ is a minimal pair of type (a), and we can construct a cycle $C \in \bar{\mathcal{C}}$ with $\ell(C) < \ell_0$, which is impossible. Hence $p_{k-1} < y_k \vee a_*$, and it follows that $x_{k-1} < p_{k-1}$. We have shown that the claim is true for $j = k - 1$.

Proceeding by induction, we assume that $y_{k*} < p_j < y_k \vee a_*$ and $y_{k*} < x_j, y_j < p_k \vee a_*$ for all j such that $i \leq j < k$, and we consider $\langle p_{i-1}, J_{i-1} \rangle$. Without loss of generality, let $x_i < p_i = y_{i-1}$ (this means that we get $y_{k*} < y_{i-1} < p_k \vee a_*$ gratis). Notice that, since x_i, y_i, and p_i are all distinct from a, Lemma 7.1 implies that each of x_i, y_i and p_i are either less than or equal to a_*, or greater than or equal to y_k.

Assume that $x_{i-1} < p_k \vee a_*$. Certainly $x_{i-1} \not\leq y_{k*}$, since $y_{k*} < y_{i-1}$. If $x_{i-1} \lambda y_{k*}$ then $x_{i-1} \lambda y_k$, and from Lemma 4.5 this is impossible. y_{k*} is on the right boundary, so $x_{i-1} > y_{k*}$. Now p_{i-1} is greater than one of x_{i-1} and y_{i-1}, and in either case $y_{k*} \leq p_{i-1} < z_{i-1} \leq p_k \vee a_*$ and $p_{i-1} \neq y_k$. If $p_{i-1} \not< y_k \vee a_*$, then $p_{i-1} \not\leq a$, and by Lemma 7.1 we have that $p_{i-1} > y_k$. Thus $\langle p_{i-1}, J_k \rangle$ is a minimal pair of type (a), and neither y_{i-1} nor x_{i-1} equals $a = x_k$, showing that $i - 1 \neq k$ (modulo ℓ_0). Hence

$$C = \{\langle p_{i-1}, J_k \rangle, \langle p_{k+1}, J_{k+1} \rangle, \ldots, \langle p_{i-2}, J_{i-2} \rangle\}$$

is a cycle in $\bar{\mathcal{C}}$ with $\ell(C) < \ell_0$, which is impossible. Therefore $p_{i-1} < y_k \vee a_*$, establishing the claim for $j = i - 1$.

We now assume that $x_{i-1} \not< p_k \vee a_*$. Since $y_{i-1} < y_k \vee a_* < p_k \vee a_*$, we have $x_{i-1} \not\leq p_k \vee a_*$. If $p_k \vee a_* \lambda x_{i-1}$, then Lemma 4.3 and $x_{i-1} \lambda a$ imply $y_{k*} < a_* = (p_k \vee a_*) \wedge a < x_{i-1} < (p_k \vee a_*) \vee a = a^*$. Hence from Lemma 7.1 we infer $x_{i-1} \geq y_k$, so $x_{i-1} \geq y_k \vee a_* > y_{i-1}$, a contradiction. Consequently we let $x_{i-1} \lambda p_k \vee a_*$. It follows that $x_{i-1} \lambda y_{i-1} = p_i > x_i$, that either $x_{i-1} \lambda x_i$ or $x_{i-1} > x_i$, and that either $x_{i-1} \lambda y_i$ or $x_{i-1} > y_i$. From Lemma 5.5 we infer $x_i \lambda y_i$, and so $x_{i-1} \lambda y_{i-1} \lambda y_i$. Also $y_i < y_k \vee a_*$, for otherwise $y_k \vee a_* \lambda y_i$ which implies $(y_k \vee a_*) \wedge y_i = a_* = y_i \wedge a$, contradicting Lemma 4.7(ii). If $x_i \leq a_*$ then $y_i \leq a_*$ also, contradicting the fact that $x_i \parallel y_i$. Hence by Lemma 7.1, $y_k \leq x_i < p_i = y_{i-1}$. If $z_{i-1} > y_i$, by Corollary 4.4 $z_{i-1} = x_{i-1} \vee y_{i-1} = x_{i-1} \vee y_i$; by $(\underset{\sim}{SD}_{\vee})$, $z_{i-1} = x_{i-1} \vee (y_{i-1} \wedge y_i)$, contradicting the fact that $\langle p_{i-1}, J_{i-1} \rangle$ is a minimal pair. Thus $z_{i-1} \lambda y_i$ and $z_{i-1} \lambda a_*$.

Let $y_{i-1*} \prec y_{i-1}$, and set $b = x_{i-1} \vee y_{i-1*}$. Since $\langle p_{i-1}, J_{i-1} \rangle$ is a minimal pair, $b \not\geq y_{i-1}$ and so $b \lambda y_{i-1}$. We claim that $y_{i-1*} > y_k$. If $y_{i-1*} = y_k$ then $x_i = y_k$, and so from Lemma 6.5(ii) $y_i = p_{i+1} \leq a_*$. Let $u \succ y_{k*}$ such that $u \leq a_*$. From the reflection of Lemma 4.7(i), $x_i \vee u \geq y_{i-1} = p_i$; since $\langle p_i, J_i \rangle$ is a minimal pair, $u = y_i = p_{i+1}$. Assuming $x_{i+1} < p_{i+1}$, we have $x_{i+1} \leq y_{k*}$. As i is not $k - 1$ (modulo ℓ_0), we have a contradiction to the induction hypothesis. Thus $y_{i-1*} > y_k$, as claimed.

Now let $c = z_{i-1} \wedge (p_k \vee a_*)$; by $(\underset{\sim}{W})$, $c \not\leq y_k \vee a_*$. Also by $(\underset{\sim}{W})$, $c \wedge b \not\leq y_k \vee a_*$. Suppose that $y_i \neq a_*$. By Lemma 7.1, $a_* \prec y_k \vee a_*$, so $y_i \not\geq a_*$. By $(\underset{\sim}{W})$ again, $z_i = x_i \vee y_i \not\geq a_* = (p_k \vee a_*) \wedge a$, while $z_i > p_i = y_{i-1}$. Hence

$$\{b \wedge c,\ c,\ y_{i-1},\ z_i,\ y_i,\ z_{i-1},\ y_k \vee a_*,\ x_{i-1},\ a_*\} \cong M2.$$

Therefore we must have $y_i = a_*$. Since $\langle p_i, J_i\rangle$ is a minimal pair and $z_i = y_k \vee a_*$, we have $x_i = y_k$. Let $v \prec a_*$; then $y_{i-1} \lambda v$ or $y_{i-1} > v$. If $y_{i-1} \lambda v$, then $x_i \vee v \not\geq p_i$ since $\langle p_i, J_i\rangle$ is a minimal pair; hence $p_i \lambda x_i \vee v \lambda a_*$. Now $p_i \vee a_* = y_k \vee a_* = (x_i \vee v) \vee a_*$, and from Lemma 2.7(ii) $p_i \vee v = p_i \vee (x_i \vee v) < y_k \vee a_*$. Also, $v < z_{i-1}$ would imply $z_{i-1} = x_{i-1} \vee y_{i-1} = x_{i-1} \vee v$ by Lemma 4.3, and hence $z_{i-1} = x_{i-1} \vee (y_{i-1} \wedge v)$ by $(SD_\vee)$, contradicting the fact that $\langle p_{i-1}, J_{i-1}\rangle$ is a minimal pair. Hence $z_{i-1} \not\geq v$, and

$$\{b \wedge c,\ c,\ y_{i-1},\ p_i \vee v,\ v,\ z_{i-1},\ y_k \vee a_*,\ x_{i-1},\ a_*\} \cong M2.$$

We conclude that $v < y_{i-1}$. From Lemma 6.5(ii), $x_i = y_k \neq p_{i+1}$, and so $a_* = y_i = p_{i+1}$. Letting $x_{i+1} < p_{i+1}$, we get $x_{i+1} < y_{i-1} < c$; also $y_{i+1} \lambda x_{i+1}$, since x_{i+1} is on the right boundary. Certainly $i + 1 \neq k$ (modulo ℓ_0), so by the induction hypothesis $y_{k*} < y_{i+1} < p_k \vee a_*$. It follows that $y_{i+1} < z_{i-1}$, implying $y_{i+1} \leq c$. But this means $a_* = p_{i+1} < z_{i+1} \leq c$, a contradiction. We have shown that the claim is true for $j = i - 1$, and the proof of $\underset{\sim}{B}$ is complete.

Now, by going completely around the cycle C_0 , we conclude from $\underset{\sim}{B}$ that $y_{k*} < p_k < y_k \vee a_*$. This contradiction completes the case $a_* \not\leq p_k$, and this section.

§8. The case $a_* \leq p_k$.

Gamely continuing with the proof of Theorem 6.3, we may now assume that $a_* \leq p_k$ in L . First we rapidly show that $x_{k-1} \lambda y_{k-1}$, where $y_{k-1} = p_k$. Suppose $y_{k-1} \lambda x_{k-1}$. Since a_* and a^* are on the right boundary, we know that $a_* < x_{k-1} < a^*$. But $x_{k-1} \neq a$, and hence $x_{k-1} \geq y_k$ by Lemma 7.1. Thus $y_k \vee a_* < x_{k-1} < a^*$, and so $y_k \vee a_* < p_{k-1} < z_{k-1} \leq a^* = z_k$. It follows that $\langle p_{k-1}, J_k \rangle$ is a minimal pair of type (a), and we can construct a cycle $C \in \vec{\mathcal{C}}$ such that $\ell(C) < \ell_0$, an impossibility. Therefore $x_{k-1} \lambda y_{k-1}$.

If $x_{k-1} < a^*$, then $x_{k-1} \geq y_k$ from Lemma 4.5, and $\langle p_{k-1}, J_k \rangle$ is a minimal pair of type (a). This is impossible, so $x_{k-1} \lambda a^*$. If $z_{k-1} \geq a^*$, then $z_{k-1} = x_{k-1} \vee y_{k-1} = x_{k-1} \vee a$ from Lemma 4.3, and hence by $(\underset{\sim}{SD}_\vee)$ $z_{k-1} = x_{k-1} \vee (y_{k-1} \wedge a) = x_{k-1} \vee a_*$, contradicting the fact that $\langle p_{k-1}, J_{k-1} \rangle$ is a minimal pair. Thus $z_{k-1} \lambda a^*$.

Set $w = x_{k-1} \vee a_*$. Since $\langle p_{k-1}, J_{k-1} \rangle$ is a minimal pair, $w \ngeq p_{k-1}$ and hence $w \ngeq y_{k-1}$. However, we claim that $w > y_k$. If $y_k \ngeq a_*$ then $y_k \lambda a_*$, and since $x_{k-1} \lambda y_k$ or $x_{k-1} > y_k$ we have $w = x_{k-1} \vee a_* > y_k$ by Lemma 4.3. On the other hand, if $y_k > a_*$ then $y_{k*} = a_*$, and Lemma 4.7(i) implies that $w = x_{k-1} \vee a_* > y_k$.

$\underset{\sim}{A}$. In this part we assume that $p_{k-1} > x_{k-1}$ (later we will deal with the other case, $p_{k-1} > y_{k-1}$). If $w \lambda p_{k-1}$ then $a_* < p_{k-1}$, implying that $w = x_{k-1} \vee a_* \leq p_{k-1}$. It follows that either $p_{k-1} \lambda w$ or $p_{k-1} > w$. In either case, $p_{k-1} \vee y_{k-1} = z_{k-1} = w \vee y_{k-1}$, so $p_{k-1} \vee w < z_{k-1}$ by the dual of Lemma 2.7(ii). Since $p_{k-1} \in J_{k-2}$, we may assume that $p_{k-1} = y_{k-2}$.

<u>Case</u> (i). Let $y_{k-2} \lambda x_{k-2}$.

From Lemma 5.5 we know that $x_{k-2} \lambda y_{k-1}$ or $x_{k-2} < y_{k-1}$. In either case $z_{k-2} = x_{k-2} \vee y_{k-2} \leq z_{k-1}$ follows. Since $\langle p_{k-2}, J_{k-2} \rangle$ is a minimal pair, $x_{k-1} \vee x_{k-2} < z_{k-2}$ and $y_{k-2} \lambda x_{k-1} \vee x_{k-2} \lambda y_{k-1}$. Since $y_{k-2} \vee y_{k-1} = z_{k-1} = (x_{k-1} \vee x_{k-2}) \vee y_{k-1}$, the dual of Lemma 2.7(ii) implies that $z_{k-2} = y_{k-2} \vee (x_{k-1} \vee x_{k-2}) < z_{k-1}$.

We claim that $z_{k-2} > a_*$. Otherwise $z_{k-2} \not> a_*$, from which we get $y_{k-2} \lambda w$ and $z_{k-2} \lambda w$. Hence either $p_{k-2} \lambda w$ or $p_{k-2} < w$. If $p_{k-2} \lambda w$ then from Lemma 4.3 $p_{k-2} > y_{k-2} \wedge w \geq x_{k-1}$, whence $x_{k-1} < p_{k-2} < x_{k-1} \vee a$ and $p_{k-2} \nleq w = x_{k-1} \vee a_*$. Thus from Lemma 5.3 there exists $x' \leq x_{k-1}$ such that $\langle p_{k-2}, \{x', a\} \rangle$ is a minimal pair of type (a), contrary to Lemma 6.6. Therefore $p_{k-2} < w$. Since $p_{k-2} \vee x_{k-1} \nleq y_{k-2}$ we have

$$\{z_{k-2}, p_{k-2} \vee x_{k-1}, w, y_k \vee a_*, y_{k-1}, x_{k-1}, a_*, y_{k-2}, a\} \cong (M2)^d.$$

Hence $z_{k-2} > a_*$.

Now either $x_{k-2} \geq a_*$ or $y_{k-2} > a_*$, since otherwise $x_{k-2} \vee y_{k-2} = z_{k-2} > a_* = z_{k-1} \wedge a$ is a violation of $(\underset{\sim}{W})$. Since $y_{k-2} \lambda x_{k-2}$, we have $x_{k-2} \geq a_*$ in any case, and so $x_{k-1} \vee x_{k-2} \geq w$. If $p_{k-2} < w$ then $x_{k-1} \vee x_{k-2} > p_{k-2}$, contradicting the fact that $\langle p_{k-2}, J_{k-2} \rangle$ is a minimal pair; thus $p_{k-2} \nleq w$. If $p_{k-2} > x_{k-1}$ then by Lemma 5.3 there exists $x' \leq x_{k-1}$ such that $\langle p_{k-2}, \{x', a\} \rangle$ is a minimal pair of type (a), contrary to Lemma 6.6. Therefore $p_{k-2} \not> x_{k-1}$, and so $x_{k-1} \lambda p_{k-2}$, $w \lambda p_{k-2}$, and $p_{k-2} > x_{k-2}$. Since $\langle p_{k-2}, J_{k-2} \rangle$ is a minimal pair, $y_{k-2} \lambda x_{k-1} \vee x_{k-2} \lambda p_{k-2}$. Since $y_{k-2} \vee (x_{k-1} \vee x_{k-2}) = z_{k-2} = y_{k-2} \vee p_{k-2}$, the dual of

Lemma 2.6(ii) implies that $x_{k-1} \vee p_{k-2} = (x_{k-1} \vee x_{k-2}) \vee p_{k-2} < z_{k-2}$, implying $x_{k-1} \vee p_{k-2} \not\geq y_{k-2}$.

Next we claim that $p_{k-2} \vee a \not\geq x_{k-1}$. Otherwise, $p_{k-2} \vee a = x_{k-1} \vee a$, and by $(\underset{\sim}{SD}_{\vee})$ $p_{k-2} \vee a = (x_{k-1} \wedge p_{k-2}) \vee a$. However, $(x_{k-1} \wedge p_{k-2}) \vee a_{*} \leq w$ which means $p_{k-2} \not\leq (x_{k-1} \wedge p_{k-2}) \vee a_{*}$. Thus by Lemma 5.3 there exists $x' \leq x_{k-1} \wedge p_{k-2}$ such that $\langle p_{k-2}, \{x', a\}\rangle$ is a minimal pair of type (a), contradicting Lemma 6.6.

Therefore $p_{k-2} \vee a \not\geq x_{k-1}$ and so

$$\{x_{k-1}, x_{k-1} \vee p_{k-2}, p_{k-2}, p_{k-2} \vee y_{k-1}, y_{k-1}, z_{k-2}, p_{k-2} \vee a, y_{k-2}, a\} \cong M2.$$

We conclude that Case (i) is impossible.

For the remaining two cases, recall that either $y_{k-2} > w$ or $y_{k-2} \,\lambda\, w$.

<u>Case</u> (ii). Let $x_{k-2} \,\lambda\, y_{k-2}$ and $y_{k-2} > w$.

Since $x_{k-2} \,\lambda\, y_{k-2} \,\lambda\, a$, we have $p_{k-2} < z_{k-2} \leq x_{k-2} \vee a$. Let $v = x_{k-2} \vee w$; then $p_{k-2} \not\leq v$ and $y_{k-2} \not\leq v$, since $\langle p_{k-2}, J_{k-2}\rangle$ is a minimal pair. If $p_{k-2} > x_{k-2}$, it follows from Lemma 5.3 that there exists $x' \leq x_{k-2}$ such that $\langle p_{k-2}, \{x', a\}\rangle$ is a minimal pair of type (a). This contradicts Lemma 6.6, and hence $p_{k-2} > y_{k-2}$.

Suppose that $p_{k-2} \leq x_{k-1} \vee a$. Then, since $x_{k-1} \vee a_{*} = w < p_{k-1}$, we may apply Lemma 5.3 again to find $x'' \leq x_{k-1}$ such that $\langle p_{k-2}, \{x'', a\}\rangle$ is a minimal pair of type (a), which is another contradiction of Lemma 6.6. Therefore $p_{k-2} \,\lambda\, x_{k-1} \vee a$.

Next, suppose that $z_{k-2} \geq z_{k-1}$. Since $x_{k-2} \lambda y_{k-2} \lambda y_{k-1}$ we have $z_{k-2} = x_{k-2} \vee y_{k-2} = x_{k-2} \vee y_{k-1}$ by Lemma 4.3 ; but now $z_{k-2} = x_{k-2} \vee (y_{k-2} \wedge y_{k-1})$ by $(\underset{\sim}{\mathrm{SD}}_{\vee})$, and $y_{k-2} \wedge y_{k-1} < y_{k-2}$, contradicting the fact that $\langle p_{k-2}, J_{k-2} \rangle$ is a minimal pair. Thus we have shown that $z_{k-2} \lambda x_{k-1} \vee a$ and $z_{k-2} \lambda z_{k-1}$.

If $v \wedge p_{k-2} > x_{k-1}$, then $v \wedge p_{k-2} \lesseqgtr x_{k-1} \vee a$ by $(\underset{\sim}{\mathrm{W}})$, and

$$\{v \wedge p_{k-2}, p_{k-2}, y_{k-2}, z_{k-1}, y_{k-1}, z_{k-2}, x_{k-1} \vee a, x_{k-2}, a^*\} \cong M2.$$

Therefore $v \wedge p_{k-2} = x_{k-1}$, implying that $x_{k-1} = w > a_*$. Also, we now have that $v \wedge p_{k-2} = v \wedge (x_{k-1} \vee a)$, and so $p_{k-2} \vee a = p_{k-2} \vee (x_{k-1} \vee a) \gtreqless v$ by $(\underset{\sim}{\mathrm{SD}}_{\wedge})$.

We may let $p_{k-2} = y_{k-3}$. By Lemma 5.5, either $x_{k-3} < x_{k-2}$ or $x_{k-2} \lambda x_{k-3}$. Since $a_* < y_{k-3}$ and $\langle p_{k-3}, J_{k-3} \rangle$ is a minimal pair, $x_{k-3} \vee a_* \gtreqless p_{k-3}$. If $x_{k-3} < x_{k-2}$ or $x_{k-2} \lambda x_{k-3} \lambda y_{k-3}$, then $p_{k-3} < z_{k-3} < x_{k-3} \vee a$; therefore from Lemma 5.3 there exists $x' \leq x_{k-3}$ such that $\langle p_{k-3}, \{x', a\} \rangle$ is a minimal pair. This contradicts Lemma 6.7, and we conclude that $y_{k-3} \lambda x_{k-3}$. It follows that $v \lambda x_{k-3}$, and hence $x_{k-2} \vee a_* \gtreqless p_{k-3}$. Now, if $x_{k-3} < x_{k-2} \vee a$ then $p_{k-3} < z_{k-3} \leq x_{k-2} \vee a$; from Lemma 5.3 there exists $x' \leq x_{k-2}$ such that $\langle p_{k-3}, \{x', a\} \rangle$ is a minimal pair, contradicting Lemma 6.7 once more. Thus it must be that $x_{k-2} \vee a \lambda x_{k-3}$, and so $x_{k-3} > a^*$. Finally, if $x_{k-2} \vee y_{k-1} = x_{k-2} \vee a$, then by $(\underset{\sim}{\mathrm{SD}}_{\vee})$ we have $y_{k-1} < x_{k-2} \vee y_{k-1} = x_{k-2} \vee (y_{k-1} \wedge a) = x_{k-2} \vee a_* \leq v$, a contradiction; thus $x_{k-2} \vee y_{k-1} \gtreqless a$. Now if $x_{k-3} > y_{k-2}$ then

$$\{y_{k-3},\ y_{k-3} \vee y_{k-1},\ z_{k-1},\ y_{k-2} \vee a,\ a,\ x_{k-2} \vee y_{k-1},\ x_{k-3},\ v\} \cong M1,$$

while if $x_{k-3} \not\geq y_{k-2}$ then

$$\{y_{k-2},\ z_{k-1},\ y_{k-1},\ a^*,\ a,\ x_{k-2} \vee y_{k-1},\ x_{k-3},\ z_{k-2}\} \cong M1,$$

showing that Case (ii) is impossible.

Case (iii). Let $x_{k-2} \ \lambda\ y_{k-2} \ \lambda\ w$.

Suppose that $x_{k-2} < z_{k-1}$. Then $z_{k-1} = x_{k-1} \vee y_{k-1} = x_{k-2} \vee y_{k-1}$, and $(SD_\vee)$ implies that $z_{k-1} = (x_{k-1} \wedge x_{k-2}) \vee y_{k-1}$. Since $\langle p_{k-1}, J_{k-1}\rangle$ is a minimal pair, $x_{k-2} > x_{k-1}$. Now, $x_{k-1} < p_{k-2} < z_{k-2} \leq z_{k-1}$, and $p_{k-2} \ \lambda\ p_{k-1}$ or $p_{k-2} > p_{k-1}$; hence from Lemma 5.4 and Lemma 5.1(v) $\langle p_{k-2}, J_{k-1}\rangle$ is a minimal pair of type (a), which is impossible. Therefore $x_{k-2} \ \lambda\ z_{k-1}$. If $z_{k-2} > w$ then $z_{k-2} = x_{k-2} \vee y_{k-2} = x_{k-2} \vee w$, and by $(SD_\vee)$ $z_{k-2} = x_{k-2} \vee (y_{k-2} \wedge w)$. Since $\langle p_{k-2}, J_{k-2}\rangle$ is a minimal pair, this is a contradiction; thus $z_{k-2} \ \lambda\ w$. Since $z_{k-2} > x_{k-1}$ and $w = x_{k-1} \vee a_*$, it follows that $z_{k-2} \ \lambda\ a_*$. Finally we have that $x_{k-2} \vee x_{k-1} \ \lambda\ y_{k-2}$ since $\langle p_{k-2}, J_{k-2}\rangle$ is a minimal pair. Therefore

$$\{z_{k-2},\ y_{k-2},\ z_{k-1},\ y_{k-1},\ a^*,\ x_{k-1},\ a_*,\ x_{k-2} \vee x_{k-1},\ a\} \cong (M2)^d.$$

With this contradiction we have succeeded in showing that $p_{k-1} > x_{k-1}$ is impossible.

B. Finally we assume $p_{k-1} > y_{k-1}$. If $p_{k-1} \leq a^*$ then $\langle p_{k-1}, J_k\rangle$ is a minimal pair of type (a), which is impossible. Hence $p_{k-1} \ \lambda\ a^*$.

Recall that $p_{k-1} = y_{k-2}$, and first assume that $x_{k-2} \lambda y_{k-2}$. By Lemma 5.5, $x_{k-2} < x_{k-1}$ or $x_{k-1} \lambda x_{k-2}$, and so $z_{k-2} \leq x_{k-1} \vee y_{k-2} = z_{k-1}$. If $p_{k-2} > y_{k-2}$, then $\langle p_{k-2}, J_{k-1} \rangle$ is a minimal pair of type (a), which is impossible; therefore $p_{k-2} > x_{k-2}$. Also, $x_{k-2} \lambda y_{k-2} \lambda a$ implies $p_{k-2} < x_{k-2} \vee a$, but since $\langle p_{k-2}, J_{k-2} \rangle$ is a minimal pair, $p_{k-2} \nleq x_{k-2} \vee a_*$. Thus by Lemma 5.3 there exists $x' \leq x_{k-2}$ such that $\langle p_{k-2}, \{x', a\} \rangle$ is a minimal pair of type (a), contradicting Lemma 6.6. We conclude that $y_{k-2} \lambda x_{k-2}$, and hence $x_{k-2} > a_*$.

We turn our attention briefly to J_{k+1}. Without loss of generality, let $x_{k+1} \leq a_* < p_{k+1}$; then x_{k+1} is on the right boundary, so $y_{k+1} \lambda x_{k+1}$. If either $x_{k-1} \geq y_{k+1}$ or $x_{k-1} \lambda y_{k+1}$, then $a = p_{k+1} < z_{k+1} \leq x_{k+1} \vee x_{k-1} \leq z_{k-1}$; a contradiction. Hence $y_{k+1} > x_{k-1}$ or $y_{k+1} \lambda x_{k-1}$, and it follows that $z_{k+1} = y_{k+1} \vee p_{k+1} \geq x_{k-1} \vee a > p_{k-1}$. If $\langle p_{k-1}, J_{k+1} \rangle$ is a minimal pair, then we can construct a cycle $C \in \mathcal{C}$ such that $\alpha(C) < \alpha_0$, which is impossible ; hence, letting $x_{k+1*} < x_{k+1}$ and $y_{k+1*} < y_{k+1}$, we conclude that either $y_{k+1} \vee x_{k+1*} > p_{k-1}$ or $y_{k+1*} \vee x_{k+1} > p_{k-1}$. Suppose the former ; then, since $\langle p_{k+1}, J_{k+1} \rangle$ is a minimal pair, $z_{k+1} > y_{k+1} \vee x_{k+1*} = y_{k+1} \vee x_{k+1*} \vee p_{k-1} \geq z_{k+1}$, a contradiction. Therefore $y_{k+1*} \vee x_{k+1} > p_{k-1}$, and so $y_{k+1*} \vee x_{k+1} \geq x_{k-1} \vee p_{k-1} = z_{k-1}$. Clearly $y_{k+1*} \lambda w$, and we now have $y_{k+1*} \vee x_{k+1} \leq y_{k+1*} \vee w \leq y_{k+1*} \vee p_{k-1} = y_{k+1*} \vee x_{k+1}$, implying $y_{k+1*} \vee w = y_{k+1*} \vee p_{k-1}$. From the dual of Lemma 2.7(ii) we get that $z_{k-1} = w \vee p_{k-1} < y_{k+1*} \vee x_{k+1}$, whence $y_{k+1*} \lambda z_{k-1}$. Since $\langle p_{k+1}, J_{k+1} \rangle$ is a minimal pair, we also have $y_{k+1*} \vee x_{k+1} \ngeq p_{k+1}$.

Suppose that $x_{k-2} > a^*$; then

$\{x_{k-1}, z_{k-1}, y_{k-1}, a^*, a, y_{k+1*} \vee x_{k+1}, x_{k-2}, y_{k+1*}\} \cong M1$.

Hence $x_{k-2} \not\geq a^*$, and it follows by Lemma 4.3 that $x_{k-2} < y_{k-2} \vee a \leq x_{k-1} \vee a$. Furthermore $z_{k-2} < y_{k-2} \vee a$, for otherwise $z_{k-2} = x_{k-2} \vee y_{k-2} = y_{k-2} \vee a$, and by $(SD_\vee)$ $z_{k-2} = y_{k-2} \vee (x_{k-2} \wedge a)$, contradicting the fact that $\langle p_{k-2}, J_{k-2}\rangle$ is a minimal pair; hence $z_{k-2} \lambda a$. If $p_{k-2} > y_{k-2}$ then by Lemma 5.3 there exists $x' \leq y_{k-2}$ such that $\langle p_{k-2}, \{x', a\}\rangle$ is a minimal pair of type (a), which is a contradiction to Lemma 6.6. Hence $p_{k-2} > x_{k-2}$.

Let $p_{k-2} = y_{k-3}$. If $x_{k-3} \lambda y_{k-3}$ then by Lemma 4.3 $p_{k-3} < z_{k-3} \leq x_{k-3} \vee a$, while $p_{k-3} \not\leq x_{k-3} \vee a_*$ since $\langle p_{k-3}, J_{k-3}\rangle$ is a minimal pair; by Lemma 5.3 there exists $x' \leq x_{k-3}$ such that $\langle p_{k-3}, \{x', a\}\rangle$ is a minimal pair, contradicting Lemma 6.7. Thus $y_{k-3} \lambda x_{k-3}$. If $x_{k-3} > a^*$ then

$$\{x_{k-1}, z_{k-1}, y_{k-1}, a^*, a, y_{k+1*} \vee x_{k+1}, x_{k-3}, y_{k+1*}\} \cong M1;$$

hence $x_{k-3} \not\geq a^*$, and so $x_{k-3} < x_{k-1} \vee a$. Now $p_{k-3} < x_{k-1} \vee a$ and $w \lambda p_{k-3}$, and by Lemma 5.3 there exists $x'' \leq x_{k-1}$ such that $\langle p_{k-3}, \{x'', a\}\rangle$ is a minimal pair. This contradicts Lemma 6.7, and the proof of Theorem 6.3 is complete.

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PLANAR SUBLATTICES OF A FREE LATTICE II

by

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Alberta,
Canada

Planar sublattices of a free lattice. II

by

Ivan Rival and Bill Sands[†]

In Planar sublattices of a free lattice. I [8] we verify Jónsson's conjecture for finite planar lattices; in particular we obtain a characterization of finite planar sublattices of a free lattice among all finite lattices. In the present paper we use arguments of a quite different flavour to obtain another characterization. Let

$$\mathfrak{F} = \{C_2^3\} \cup \{S_n \mid n \geq 0\} \cup \{L_1, L_2, L_2^d, L_3, L_3^d, L_4\} \cup \{L_5, L_6\}$$

be the family of lattices illustrated in Figures 1,2,3, and 4. Our goal is to prove the following theorem: *a finite lattice is a planar sublattice of a free lattice if and only if it does not have a member of $\mathfrak{F}$ as a sublattice.*

§1. Introduction, and plan of the proof.

A lattice L is *semidistributive* if it satisfies the two conditions

$$(\mathrm{SD}_\vee) \qquad a \vee b = a \vee c \ \text{ implies } \ a \vee b = a \vee (b \wedge c)$$

and

$$(\mathrm{SD}_\wedge) \qquad a \wedge b = a \wedge c \ \text{ implies } \ a \wedge b = a \wedge (b \vee c)\,.$$

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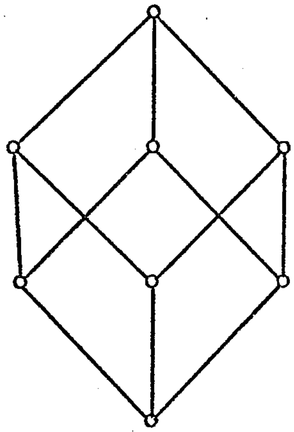
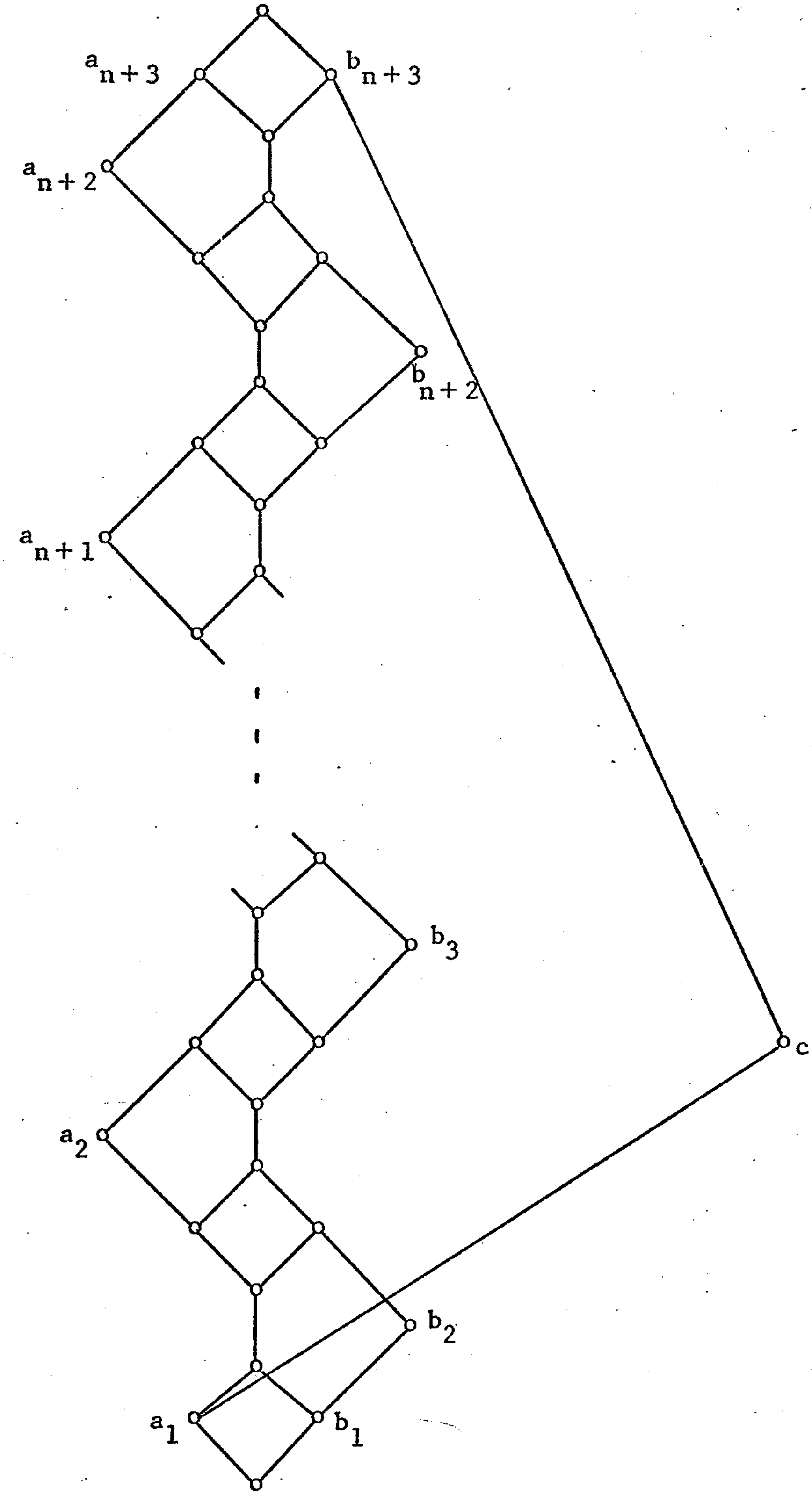


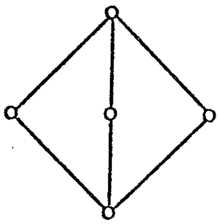
$C_2{}^3$

Figure 1

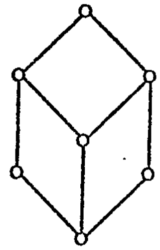


S_n

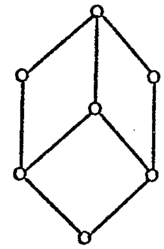
Figure 2



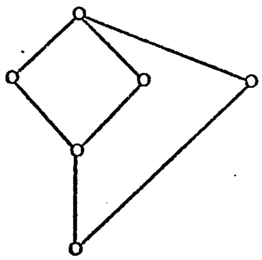
L_1



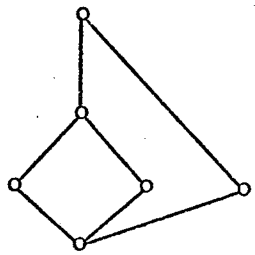
L_2



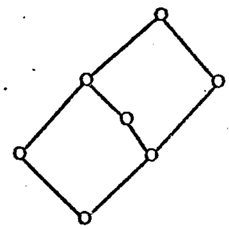
L_2^d



L_3

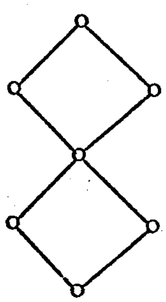


L_3^d

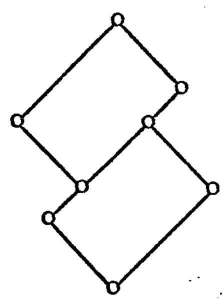


L_4

Figure 3



L_5



L_6

Figure 4

Jónsson [3] has demonstrated that sublattices of a free lattice are semidistributive. Some years earlier, Whitman [9] showed that sublattices of a free lattice satisfy the condition

$$\text{(W)} \quad a \wedge b \leq c \vee d \ \underline{\text{implies}} \ a \wedge b \leq c, a \wedge b \leq d, \ a \leq c \vee d, \ \underline{\text{or}} \ b \leq c \vee d.$$

The celebrated conjecture of Jónsson (see [4]), alluded to at the beginning of this paper, asserts that a finite lattice is a sublattice of a free lattice if and only if it is semidistributive and satisfies (W). For the history of this conjecture we refer the reader to [8].

Let A_3 be the partially ordered set of Figure 5 , let $\mathcal{R} = \{R_n \mid n \geq 0\}$ be the family of partially ordered sets illustrated in Figure 6, and let $\mathcal{S} = \{S_n \mid n \geq 0\}$ be the family of lattices illustrated in Figure 2. Most of the rest of this paper is devoted to the proof of the following two results.

THEOREM 1.1. _A finite semidistributive lattice is planar if and only if it does not contain a member of_ $\{A_3\} \cup \mathcal{R}$ _as a subset_.

THEOREM 1.2. _Let_ L _be a finite semidistributive lattice satisfying_ (W). _Then_ L _contains a member of_ $\{A_3\} \cup \mathcal{R}$ _as a subset if and only if_ L _contains a member of_ $\{C_2^3\} \cup \mathcal{S}$ _as a sublattice_.

We recall two theorems in the spirit of Theorem 1.6 below. The first is due to B. Davey, W. Poguntke, and I. Rival [2], and the second to R. Antonius and I. Rival [1].

THEOREM 1.3. _A finite lattice is semidistributive if and only if it does not contain one of the lattices of Figure 3 as a sublattice_.

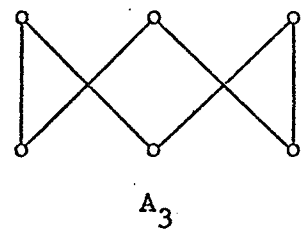


Figure 5

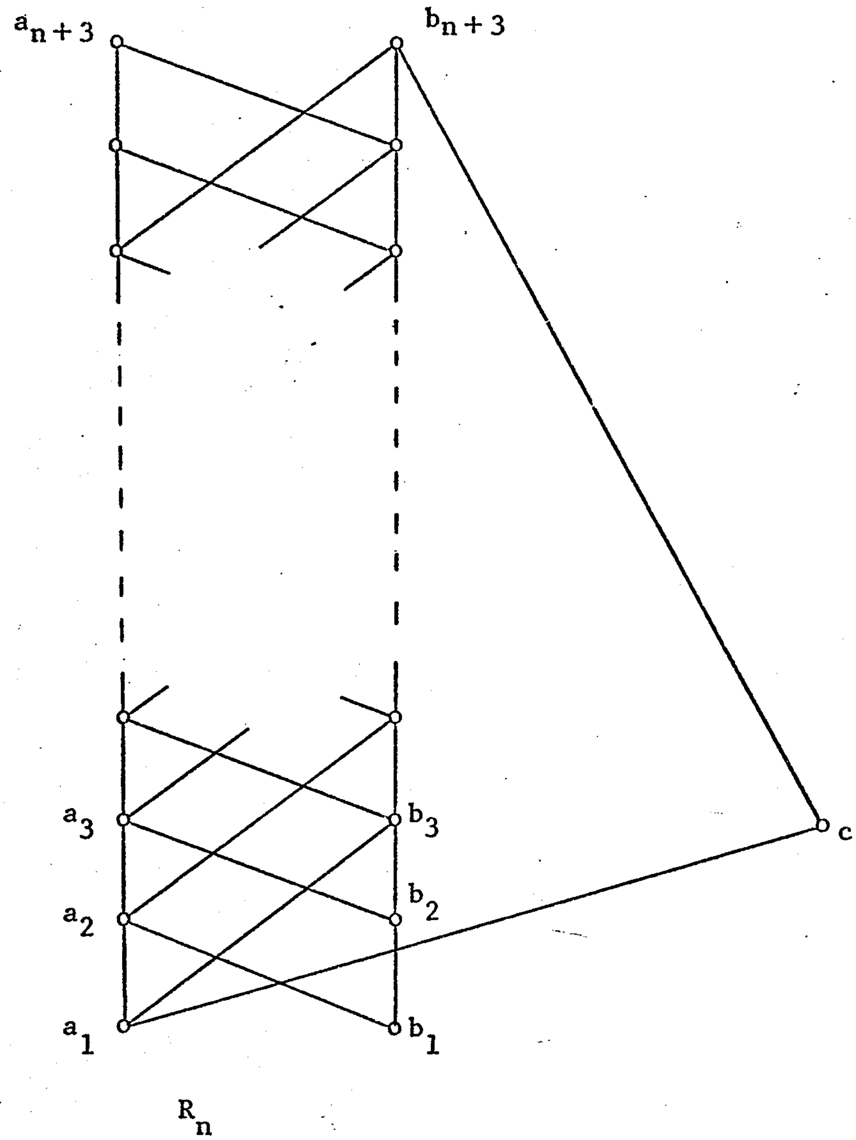


Figure 6

THEOREM 1.4. *A finite semidistributive lattice satisfies* $(\underset{\sim}{W})$ *if and only if it does not contain one of the lattices of Figure 4 as a sublattice.*

Finally we quote the main result from [8].

THEOREM 1.5. *A finite planar lattice is a sublattice of a free lattice if and only if it is semidistributive and satisfies* $(\underset{\sim}{W})$.

Combining the preceding five theorems yields the promised characterization of finite planar sublattices of a free lattice.

THEOREM 1.6. *A finite lattice is a planar sublattice of a free lattice if and only if it does not contain a member of* $\mathfrak{F}$ *as a sublattice.*

Observe that no member of $\mathfrak{F}$ is a sublattice of another member of $\mathfrak{F}$. It follows that Theorem 1.6 is best possible, in the sense that no lattice in $\mathfrak{F}$ may be omitted. Also, while it is true that the lattices S_n are all sublattices of a free lattice, this observation is not essential either to the statement or the proof of the theorem.

§2. Preliminaries.

The *breadth* $b(L)$ of a finite lattice L is the smallest integer b such that every join $\bigvee_{i=1}^{b+1} x_i$ of elements of L is equal to a join of b of the x_i's. For any integer $n \geq 3$, a *crown* of order 2n (Figure 7) is a partially ordered set $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ in which

$$x_1 < y_1,\ y_1 > x_2,\ x_2 < y_2,\ y_2 > x_3,\ \ldots, y_{n-1} > x_n,\ x_n < y_n,\ \text{and}\ y_n > x_1$$

are the only comparability relations.

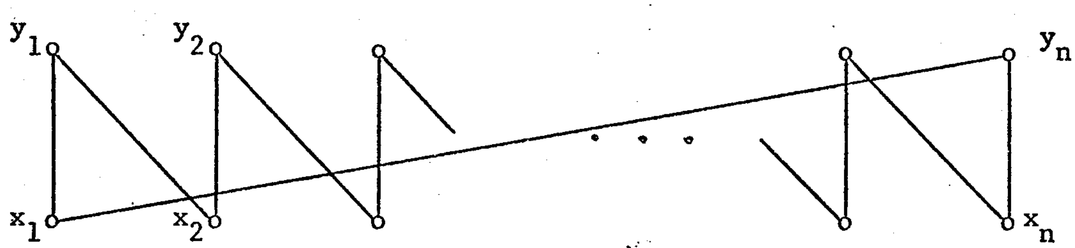


Figure 7. A crown of order 2n

LEMMA 2.1. *Let* L *be a finite semidistributive lattice. The following are equivalent:*

(i) L *contains no crown of order six*;

(ii) $b(L) \leq 2$;

(iii) L *contains no crown.*

Proof. (i) $\Leftrightarrow$ (ii) is Lemma 3.4 of [5], while (ii) $\Leftrightarrow$ (iii) is Lemma 2.4 of [8] together with Theorem 3.1 of [5]. □

LEMMA 2.2. *Let* L *be a finite semidistributive lattice of breadth at most two, and let* $a,b,c \in L$.

(i) *If* $a \vee b = a \vee c = b \vee c$, *then* $\{a,b,c\}$ *is not an antichain.*

(ii) *Either* $a \vee b \geq c$ *or* $a \vee c \geq b$ *or* $b \vee c \geq a$; *in particular,* $\{a \vee b, a \vee c, b \vee c\}$ *is not an antichain.*

Proof. (i) is the dual of Lemma 2.7(ii) of [8]. To prove (ii), suppose that $a \vee b \ngeq c$, $a \vee c \ngeq b$, and $b \vee c \ngeq a$. Then $\{a \vee b, a \vee c, b \vee c\}$ is an antichain, and the join of any pair equals $a \vee b \vee c$, contradicting (i). □

Let n be a positive integer. A <u>down-down fence</u> [6] of length $2n+1$ is a partially ordered set $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_{n+1}\}$ in which

$$x_1 < y_1, y_1 > x_2, x_2 < y_2, y_2 > x_3, \ldots, x_n < y_n, y_n > x_{n+1}$$

are the only comparability relations (see Figure 8).

LEMMA 2.3. *Let L be a finite semidistributive lattice of breadth at most two. Let n be an integer ≥ 2 and let $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_{n+1}\}$ be a down-down fence in L. Then*

(i) $x_1 \vee x_{n+1} > x_i$ *for each* $i \in \{1, \ldots, n+1\}$; *and*

(ii) *there exists* $i \in \{2, \ldots, n\}$ *such that* $x_1 \vee x_i < x_1 \vee x_{n+1}$ *and* $x_i \vee x_{n+1} < x_1 \vee x_{n+1}$.

Proof. (i) First let $n = 2$. Since $x_1 \vee x_2 \leq y_1$ and $x_2 \vee x_3 \leq y_2$, we have that $x_1 \vee x_2 \ngeq x_3$ and $x_2 \vee x_3 \ngeq x_1$, and therefore $x_1 \vee x_3 > x_2$ by Lemma 2.2(ii). Proceeding by induction, assume the result is true for all integers k such that $2 \leq k \leq n-1$. We certainly have that $x_1 \vee x_{n+1} \ngeq y_i$ for any $i \in \{1, \ldots, n\}$. Therefore, if $x_1 \vee x_{n+1} \ngeq x_i$ for any $i \in \{2, \ldots, n\}$, the subset $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_{n+1}, x_1 \vee x_{n+1}\}$ of L is a crown, contradicting Lemma 2.1. Hence we can find $i \in \{2, \ldots, n\}$ such that $x_1 \vee x_{n+1} > x_i$. Now by induction we have $x_1 \vee x_{n+1} \geq x_1 \vee x_i > x_j$ for all $j \in \{1, \ldots, i\}$, and $x_1 \vee x_{n+1} \geq x_i \vee x_{n+1} > x_k$ for all $k \in \{i, \ldots, n+1\}$, as claimed.

Figure 8. A down-down fence of length $2n+1$

(ii) When $n = 2$, $x_1 \vee x_2 \leq x_1 \vee x_3$ and $x_2 \vee x_3 \leq x_1 \vee x_3$ follow from (i). Also, since $x_1 \vee x_2 \leq y_1$ and $x_2 \vee x_3 \leq y_2$ we have $x_1 \vee x_2 < x_1 \vee x_3$ and $x_2 \vee x_3 < x_1 \vee x_3$, as desired. Therefore let $n > 2$. As above, $x_n \vee x_{n+1} < x_1 \vee x_{n+1}$; it follows that if $x_1 \vee x_n < x_1 \vee x_{n+1}$ we are done. Hence, since $x_n < x_1 \vee x_{n+1}$ by part (i), we assume $x_1 \vee x_n = x_1 \vee x_{n+1}$. By induction we choose $j \in \{2,\ldots,n-1\}$ such that $x_1 \vee x_j < x_1 \vee x_n = x_1 \vee x_{n+1}$ and $x_j \vee x_n < x_1 \vee x_{n+1}$. If $x_n \vee x_{n+1} \leq x_j \vee x_n$, then $x_j \vee x_n = x_j \vee x_{n+1}$ by (i), establishing (ii). Therefore let $x_n \vee x_{n+1}$ be noncomparable to $x_j \vee x_n$. We now have that $\{x_1 \vee x_j, x_j \vee x_n, x_n \vee x_{n+1}\}$ is an antichain, and $(x_1 \vee x_j) \vee (x_j \vee x_n) = x_1 \vee x_{n+1} = (x_1 \vee x_j) \vee (x_n \vee x_{n+1})$; from Lemma 2.2(i), $x_j \vee x_{n+1} = (x_j \vee x_n) \vee (x_n \vee x_{n+1}) < x_1 \vee x_{n+1}$, and (ii) follows. □

§ 3. The proof of Theorem 1.1.

By the completion $\underset{\sim}{L}(P)$ of a partially ordered set P to a lattice we shall mean the construction known variously as the " normal completion ", " completion by cuts ", or " MacNeille completion "; recall that a partially ordered set P is a subset of a lattice L exactly when $\underset{\sim}{L}(P)$ is a subset of L . In [6], D. Kelly and I. Rival defined a family $\mathfrak{L}$ of lattices with the property that a finite lattice L is planar if and only if L does not contain a member of $\mathfrak{L}$ as a subset. The family

$$\mathcal{P} = \{A_n \mid n \geq 0\} \cup \{B, B^d, C, C^d, D, D^d\} \cup \{E_n, E_n^d, F_n, G_n, H_n \mid n \geq 0\}$$

of partially ordered sets, which (up to duality) is illustrated in Figure 9, satisfies $\{\underset{\sim}{L}(P) \mid P \in \mathcal{P}\} = \mathcal{L}$ (see [7]). Hence the following is an alternate formulation of the Kelly-Rival result.

THEOREM 3.1. <u>A finite lattice is planar if and only if it does not contain a member of</u> $\mathcal{P}$ <u>as a subset.</u>

To begin the proof of Theorem 1.1, we first observe that if L is a finite planar lattice, L cannot contain a member of $\{A_3\} \cup \mathcal{R}$ as a subset. Certainly $A_3 \nsubseteq L$ by Theorem 3.1. If $R_0 = \{a_1, a_2, a_3, b_1, b_2, b_3, c\}$ is contained in L, then so is $\{a_2, a_3, b_2, b_3, c, a_1\}$, which is isomorphic to the partially ordered set B, contrary to Theorem 3.1. Finally if some $R_n = \{a_1, a_2, \dots, a_{n+3}, b_1, b_2, \dots, b_{n+3}, c\}$, $n \geq 1$, is contained in L, then so is $\{a_1, a_2, \dots, a_{n+2}, b_2, b_3, \dots, b_{n+3}, c\}$, which is isomorphic to G_{n-1}, again a contradiction. We have proven the " only if " direction of Theorem 1.1.

The converse is a little more complicated, and will be established gradually.

THEOREM 3.2. <u>Let</u> L <u>be a finite semidistributive lattice of breadth at most two.</u>

(a) <u>If</u> L <u>contains a member of</u> $\{C, C^d, D, D^d\} \cup \{E_n, E_n^d, F_n \mid n \geq 0\}$ <u>as a subset then</u> L <u>contains</u> B <u>or</u> B^d <u>as a subset.</u>

(b) <u>If</u> L <u>contains</u> H_n <u>as a subset then</u> L <u>contains</u> B, B^d, <u>or</u> G_m <u>as a subset for some</u> $m \leq n$.

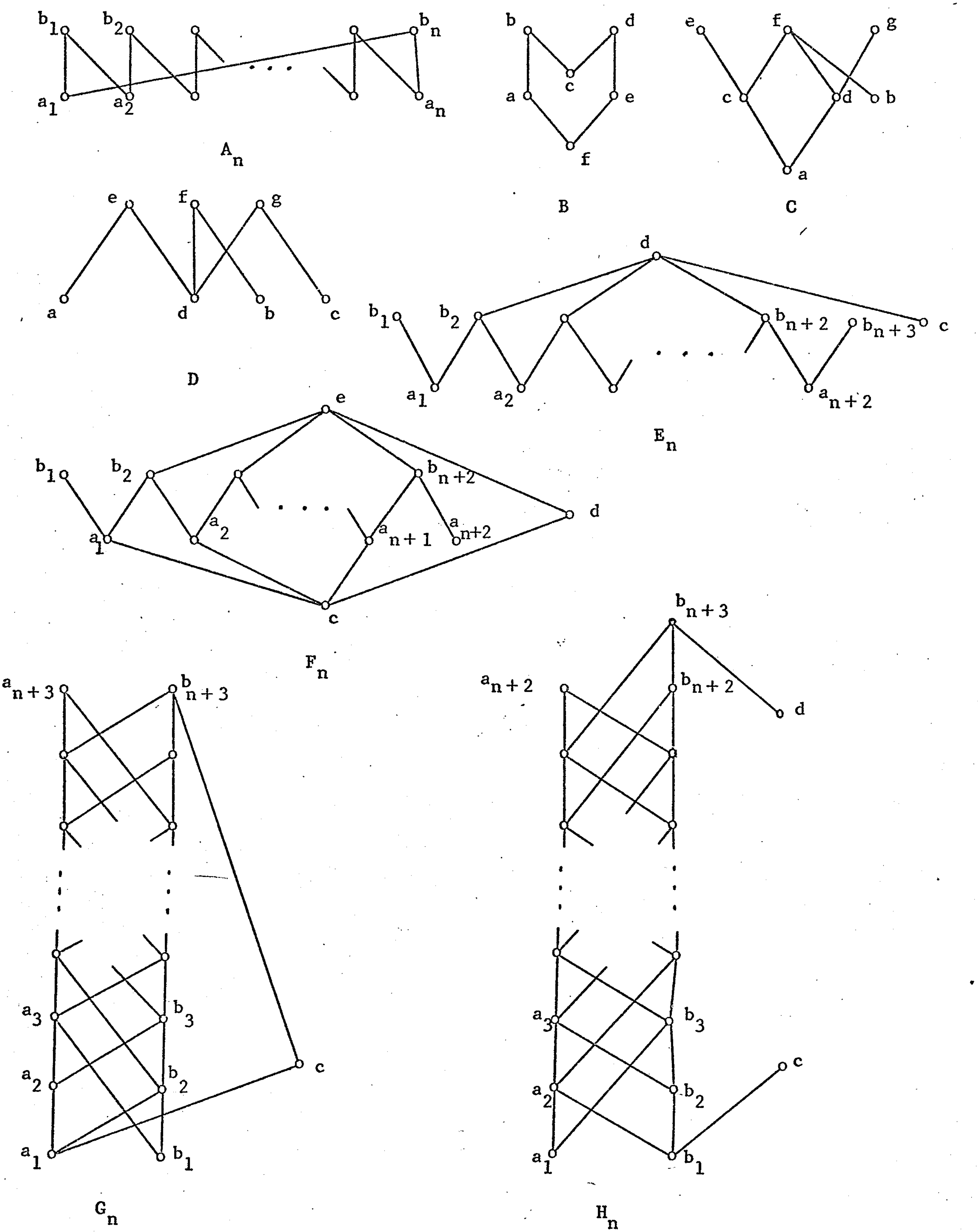


Figure 9

Proof. (a) We proceed through the list of partially ordered sets in (a) in the order given; at each stage we will establish the existence of B or B^d in L, or (what is sufficient) we will exhibit in L a partially ordered set already considered. A similar strategy will be adopted elsewhere in this paper.

Case (i): C.

Choose $C = \{a,b,c,d,e,f,g\} \subseteq L$; observe that we may assume $e \wedge f = c$, $f \wedge g = d$, and $c \wedge d = a$. By the dual of Lemma 2.3 (i) , $e \wedge g = e \wedge f \wedge g = c \wedge d = a$. Next, if $b \wedge c \nleq d$ then $\{b, b \wedge c, e, a, d, f\}$ is a subset of L isomorphic to B^d , as desired ; hence we now let $b \wedge c \leq d$ and similarly $b \wedge d \leq c$, which implies $b \wedge c = b \wedge d$. Thus $b \wedge g = b \wedge f \wedge g = b \wedge d = b \wedge c = b \wedge f \wedge e = b \wedge e$, and by $(\underset{\sim}{SD}_\wedge)$ $b \wedge g = b \wedge (e \vee g)$. It follows that $e \vee g \ngeq b$, and so $e \vee g \ngeq f$. Hence $\{e,c,f,d,g,e \vee g\} \cong B^d$. Of course, a dual argument handles C^d .

Case (ii): D .

This one is easy. Let $D = \{a,b,c,d,e,f,g\} \subseteq L$; by Lemma 2.2(i) and the symmetry of D we may assume that $e \vee f < e \vee f \vee g$, that is, $e \vee f \ngeq g$. Hence $\{e \vee f, g, e, f, a, d, b\} \cong C^d$, and by the previous case we are done.

Case (iii): $\{E_n \mid n \geq 0\}$.

Let $n \geq 0$ be minimal such that there is a subset of L isomorphic to either E_n or E_n^d . We first consider the case $n = 0$.

Without loss of generality, let $E_0 = \{a_1,a_2,b_1,b_2,b_3,c,d\} \subseteq L$. We may assume that $b_1 \wedge b_2 = a_1$, $b_2 \wedge b_3 = a_2$, $a_1 \vee a_2 = b_2$, $b_2 \vee c = d$, and (from the dual of Lemma 2.3(i)) $b_1 \wedge b_3 = a_1 \wedge a_2$. If $a_1 \wedge a_2 \nleq c$ then $\{a_1 \wedge a_2,c,a_1,a_2,b_1,d,b_3\} \cong C$; hence we let $a_1 \wedge a_2 < c$. If $a_1 \vee c > a_2$ and $a_2 \vee c > a_1$ then $a_1 \vee c = a_2 \vee c$, and by $(\underset{\sim}{SD}_\vee)$ $a_1 \vee c = (a_1 \wedge a_2) \vee c = c$, a contradiction. By symmetry we may let $a_1 \nleq a_2 \vee c$. If $a_1 \wedge c \nleq a_2$ then $\{a_1,b_2,a_2,a_2 \vee c,c,a_1 \wedge c\} \cong B$; hence we assume $a_1 \wedge c < a_2$ whereupon $a_1 \wedge c = a_1 \wedge a_2 = a_1 \wedge b_2 \wedge b_3 = a_1 \wedge b_3$. By $(\underset{\sim}{SD}_\wedge)$ $a_1 \wedge c = a_1 \wedge (c \vee b_3)$, and it follows that $a_1 \nleq c \vee b_3$. Hence $b_2 \nleq c \vee b_3$, and we have $\{b_2,d,c,c \vee b_3,b_3,a_2\} \cong B$.

Next assume $n = 1$, and let $E_1 = \{a_1,a_2,a_3,b_1,b_2,b_3,b_4,c,d\} \subseteq L$. We know immediately that $c < b_2 \vee b_3$, for otherwise $\{a_1,a_3,b_1,b_2 \vee b_3,b_4,c,d\} \cong E_0$, contradicting the choice of n . By Lemma 2.2(i) and the symmetry of E_1 , we may assume that $b_3 \vee c < b_2 \vee b_3$, and hence $b_3 \vee c \nleq b_2$. But now $\{a_2,a_3,b_2,b_3,b_4,c,b_3 \vee c\} \cong E_0$, again contradicting the choice of n .

Finally suppose that $n > 1$, and let $\{a_1,\ldots,a_{n+2},b_1,\ldots,b_{n+3},c,d\}$ be a subset of L isomorphic to E_n . We may assume that $a_j \vee a_{j+1} = b_{j+1}$ for each $j \in \{1,\ldots,n+1\}$. Since $\{a_1,b_2,a_2,b_3,\ldots,b_{n+2},a_{n+2}\}$ is a down-down fence, by Lemma 2.3 (ii) we may choose $i \in \{2,\ldots,n+1\}$ such that $a_1 \vee a_i < a_1 \vee a_{n+2}$ and $a_i \vee a_{n+2} < a_1 \vee a_{n+2}$. If $c \nleq a_1 \vee a_i$ and $c \nleq a_i \vee a_{n+2}$, then $\{a_1,a_i,a_{n+2},b_1,a_1 \vee a_i,a_i \vee a_{n+2},b_{n+3},c,d\} \cong E_1$, which is a contradiction ; therefore by symmetry let $c < a_1 \vee a_i$. Since $a_1 \vee a_2 = b_2$, in particular we have $i > 2$. Now set $k = \max\{j \mid 2 \le j \le n+1, a_1 \vee a_j < a_1 \vee a_{n+2}, a_j \vee a_{n+2} < a_1 \vee a_{n+2}\}$. By Lemma 2.3 (i), $a_j < a_1 \vee a_k$ for all $j \in \{1,\ldots,k\}$, which implies that $b_j = a_{j-1} \vee a_j < a_1 \vee a_k$ for each $j \in \{2,\ldots,k\}$; also, since $i \le k$

we have $c < a_1 \vee a_i \leq a_1 \vee a_k$. If $b_{k+1} < a_1 \vee a_k$, then $a_1 \vee a_{k+1} \leq a_1 \vee b_{k+1} \leq a_1 \vee a_k < a_1 \vee a_{n+2}$, and $a_{k+1} \vee a_{n+2} \leq a_k \vee a_{n+2} < a_1 \vee a_{n+2}$ by Lemma 2.3 (i), contradicting the maximality of k . Hence $b_{k+1} \nleq a_1 \vee a_k$, and so $\{a_1,\ldots,a_k,b_1,\ldots,b_{k+1},c,a_1 \vee a_k\} \cong E_{k-2}$. Since $k - 2 < n$, this contradicts the choice of n .

<u>Case</u> (iv) : $\{ F_n \mid n \geq 0\}$.

Let $n \geq 0$ be minimal such that there is a subset of L isomorphic to F_n .

First suppose $n = 0$, and let $F_0 = \{a_1,a_2,b_1,b_2,c,d,e\} \subseteq L$. We may assume that $a_1 \wedge d = c, b_2 \vee d = e, a_1 \vee a_2 = b_2$, and $b_1 \wedge b_2 = a_1$. If $b_1 \vee a_2 \ngtr d$ then $\{b_1,b_1 \vee a_2,a_2,e,d,c\} \cong B$; hence let $b_1 \vee a_2 > d$, and dually $b_1 \wedge a_2 < d$. If $b_1 \vee d \ngtr b_2$ then $\{b_1,b_1 \vee d,d,e,b_2,a_1\} \cong B$; hence we may let $b_1 \vee d > b_2 \vee d = e$, and dually $a_2 \wedge d < c$. But now $b_1 \vee d = b_1 \vee d \vee a_2 = b_1 \vee a_2$, and by $(SD_{\vee})$ we have $b_1 \vee d = b_1 \vee (a_2 \wedge d) = b_1$, a contradiction.

Next suppose $n = 1$, and let $F_1 = \{a_1,a_2,a_3,b_1,b_2,b_3,c,d,e\} \subseteq L$. We may assume that $b_1 \wedge b_2 = a_1$, $a_1 \vee a_2 = b_2$, $b_2 \wedge b_3 = a_2$, and $a_2 \vee a_3 = b_3$. If $b_2 \vee d \ngtr b_3$ then $\{a_1,a_2,b_1,b_2,b_3,d,b_2 \vee d\} \cong E_0$; hence we let $b_2 \vee d > b_3$. If $b_3 \vee d \ngtr b_2$ then $\{a_2,a_3,b_2,b_3,c,d,b_3 \vee d\} \cong F_0$, contradicting the choice of n ; hence $b_3 \vee d > b_2$, and so $b_2 \vee d = b_3 \vee d$. From Lemma 2.2(i), $b_2 \vee b_3 \ngtr d$. But now $\{a_1,a_3,b_1,b_2 \vee b_3,c,d,e\} \cong F_0$, contradicting the choice of n .

Finally suppose $n > 1$, and let $\{a_1,\ldots,a_{n+2},b_1,\ldots,b_{n+2},c,d,e\}$ be a subset of L isomorphic to F_n . We may assume that $a_j \vee a_{j+1} = b_{j+1}$ for each $j \in \{1,\ldots,n + 1\}$. Since $\{a_1,b_2,a_2,b_3,\ldots,b_{n+2},a_{n+2}\}$

is a down-down fence, by Lemma 2.3 (ii) we may choose $i \in \{2,\ldots,n+1\}$ such that $a_1 \vee a_i < a_1 \vee a_{n+2}$ and $a_i \vee a_{n+2} < a_1 \vee a_{n+2}$. If $d \nleq a_1 \vee a_i$ and $d \nleq a_i \vee a_{n+2}$, then $\{a_1,a_i,a_{n+2},b_1,a_1 \vee a_i,a_i \vee a_{n+2}, c,d,e\} \cong F_1$, which is a contradiction ; therefore either $d < a_1 \vee a_i$ or $d < a_i \vee a_{n+2}$. Suppose that $d < a_1 \vee a_i$. Set $k = \max\{j \mid 2 \le j \le n+1, a_1 \vee a_j < a_1 \vee a_{n+2}, a_j \vee a_{n+2} < a_1 \vee a_{n+2}\}$; as in Case (iii), $b_j < a_1 \vee a_k$ for each $j \in \{2,\ldots,k\}$, and $i \le k$ implies that $d < a_1 \vee a_i \le a_1 \vee a_k$. By the maximality of k, we again conclude $b_{k+1} \nleq a_1 \vee a_k$, and so $\{a_1,\ldots,a_k,b_1,\ldots,b_{k+1},d,a_1 \vee a_k\} \cong E_{k-2}$. We now suppose $d < a_i \vee a_{n+2}$. Set $k' = \min\{j \mid 2 \le j \le n+1, a_1 \vee a_j < a_1 \vee a_{n+2}, a_j \vee a_{n+2} < a_1 \vee a_{n+2}\}$. As before, $b_j < a_{k'} \vee a_{n+2}$ for $j \in \{k'+1,\ldots,n+2\}$, and since $k' \le i$ we have $d < a_i \vee a_{n+2} \le a_{k'} \vee a_{n+2}$. From the minimality of k', it follows that $b_{k'} \nleq a_{k'} \vee a_{n+2}$, and hence $\{a_{k'},\ldots,a_{n+2}, b_{k'},\ldots,b_{n+2},c,d,a_{k'} \vee a_{n+2}\} \cong F_{n+1-k'}$. Since $n+1-k' < n$, this contradicts the choice of n.

(b) Let $n \ge 0$ be minimal such that there is a subset of L isomorphic to H_n.

First assume $n = 0$, and let $H_0 = \{a_1,a_2,b_1,b_2,b_3,c,d\} \subseteq L$. If $c \wedge a_2 \nleq b_3$ then $\{c \wedge a_2,a_2,a_1,b_3,b_2,b_1\} \cong B$; hence we let $c \wedge a_2 < b_3$. Since $a_2 \wedge b_3 \ge a_1$, we have $a_2 \wedge b_3 \nleq c$. If $c \wedge b_3 \nleq a_2$ then $\{b_1,d,a_2 \wedge b_3,c \wedge b_3,a_2,b_3,c\} \cong C$; hence assume $c \wedge b_3 < a_2$. If $b_2 \nleq c \vee a_1$ then $\{c,c \vee a_1,a_1,b_3,b_2,b_1\} \cong B$; hence assume $b_2 < c \vee a_1 \le c \vee a_2$ and dually $b_2 > d \wedge a_2$. If $c > a_2 \wedge b_2$ then $c \wedge b_2 = (c \wedge b_3) \wedge b_2 \le a_2 \wedge b_2 \le c \wedge b_2$, implying that $c \wedge b_2 = a_2 \wedge b_2$. Since $c \vee a_2 > b_2$ this is a violation of $(\underset{\sim}{SD}_{\wedge})$, and so $c \not\geq a_2 \wedge b_2$. If $c \vee d \not\geq b_2$ then $\{c, c \vee d, d, b_3, b_2, b_1\} \cong B$; hence assume $c \vee d > b_2$. From $b_2 \vee d \leq b_3$ it follows that $b_2 \vee d \not\geq c$ and $b_2 \vee d \not\geq a_2$. If $c \vee d \not\geq a_2$ then $\{a_2 \wedge b_2, d, a_2, b_2 \vee d, b_1, c, c \vee d\} \cong F_0$; hence assume $c \vee d > a_2$. If $c \vee a_2 \not\geq b_3$ then $\{b_1, b_2, b_3, a_1, a_2, c \vee a_2, c\} \cong G_0$ as desired; hence assume $c \vee a_2 > b_3$. Now $c \vee a_2 \geq c \vee b_3 \geq c \vee d \geq c \vee a_2$, implying $c \vee a_2 = c \vee d$. By $(\underset{\sim}{SD}_{\vee})$ and the above results, $d < c \vee d = c \vee (d \wedge a_2) \leq c \vee b_2 \leq c \vee a_1$. But a dual argument shows that $d \wedge a_2 < c$, and hence $c \vee d = c \vee (d \wedge a_2) = c$, a contradiction.

Therefore $n > 0$. Let $H_n = \{a_1, \ldots, a_{n+2}, b_1, \ldots, b_{n+3}, c, d\} \subseteq L$. If $c \vee d \not> a_{n+1}$ then $\{c, c \vee d, d, b_{n+3}, a_{n+1}, b_1\} \cong B$; hence we let $c \vee d > a_{n+1}$. If $c \vee d \not> b_{n+2}$ then $\{c, c \vee d, d, b_{n+3}, b_{n+2}, b_1\} \cong B$; hence we let $c \vee d \geq d \vee b_{n+2}$. If $d \vee b_{n+2} \not\geq a_{n+1}$ then $\{a_1, \ldots, a_{n+1}, b_1, \ldots, b_{n+1}, d \vee b_{n+2}, c, d\} \cong H_{n-1}$, contradicting the choice of n ; hence $d \vee b_{n+2} > a_{n+1}$. Next, we may assume $c \vee a_{n+2} > b_{n+2}$, for otherwise $\{b_1, \ldots, b_{n+2}, a_1, \ldots, a_{n+1}, c \vee a_{n+2}, c\} \cong G_{n-1}$, as desired. Further, we assume $c \vee a_{n+2} > b_{n+3}$, for otherwise $\{b_1, \ldots, b_{n+3}, a_1, \ldots, a_{n+2}, c \vee a_{n+2}, c\} \cong G_n$; thus we have that $c \vee a_{n+2} \geq c \vee b_{n+3} \geq c \vee d$. We may assume $b_{n+3} \vee c > a_{n+2}$, for otherwise $\{a_2, d, a_{n+2}, b_{n+3}, b_1, c, b_{n+3} \vee c\} \cong F_0$. It follows that $b_{n+3} \vee c = a_{n+2} \vee c$, and by $(\underset{\sim}{SD}_{\vee})$ $b_{n+3} \vee c = (b_{n+3} \wedge a_{n+2}) \vee c$. We may assume $b_{n+3} \wedge a_{n+2} < c \vee d$, for otherwise $\{b_{n+3} \wedge a_{n+2}, b_{n+3}, d, c \vee d, c, b_1\} \cong B$. Thus $c \vee (b_{n+3} \wedge a_{n+2}) = c \vee d$, and by $(\underset{\sim}{SD}_{\vee})$ $c \vee d = c \vee (b_{n+3} \wedge a_{n+2} \wedge d) = c \vee (a_{n+2} \wedge d)$, implying $c \not\geq a_{n+2} \wedge d$. However, $d \wedge a_{n+2} < b_{n+3}$; therefore, letting k be minimal such that $d \wedge a_{n+2} < b_k$, we have $2 \leq k \leq n+3$. If $k = n+3$, then $d \wedge a_{n+2} \not< b_{n+2}$, and $\{b_{n+2}, b_{n+1}, a_{n+2}, d \wedge a_{n+2}, d, b_{n+3}\} \cong B^d$; hence we assume $k < n+3$. If $d \wedge a_{n+2} < a_{k-1}$, then $\{d \wedge a_{n+2}, a_{k-1}, a_k, \ldots, a_{n+2}, b_{k-1}, b_k, \ldots, b_{n+3}, d\} \cong G_{n-k+2}$; hence we assume $d \wedge a_{n+2} \not< a_{k-1}$. If $k = 2$ we have $\{b_1, d \wedge a_{n+2}, c, b_2, d, a_1, b_3\} \cong E_0$; thus we let $k > 2$. But now $\{a_1, \ldots, a_{k-1}, b_1, \ldots, b_k, c, d \wedge a_{n+2}\} \cong H_{k-3}$ where $0 \leq k-3 < n$, contradicting the choice of n. □

As a corollary, we obtain an improvement of Theorem 3.1 for finite semidistributive lattices.

COROLLARY 3.3. <u>A finite semidistributive lattice</u> L <u>is planar if and only if it does not contain</u> A_3, B, B^d, <u>or</u> G_n, $n \geq 0$, <u>as a subset.</u>

<u>Proof.</u> We need only prove the "if" direction. By Lemma 2.1, L contains A_n for some $n \geq 3$ if and only if it contains A_3. By Theorems 3.1 and 3.2 the corollary follows. □

THEOREM 3.4. <u>Let</u> L <u>be a finite semidistributive lattice of breadth at most two.</u>

(a) <u>If</u> L <u>contains</u> B <u>or</u> B^d <u>as a subset then</u> L <u>contains</u> R_0 <u>as a subset.</u>

(b) <u>If</u> L <u>contains</u> G_n <u>as a subset for some</u> $n \geq 0$ <u>then</u> L <u>contains</u> R_m <u>as a subset for some</u> $m \leq n+1$.

<u>Proof.</u> (a) Assume $B = \{a,b,c,d,e,f\} \subseteq L$. We first observe that we must have $a \vee e > c$, for otherwise $\{a \vee e, a \vee c, c \vee e\}$ is an antichain, contrary to Lemma 2.2(ii). If $c \wedge a \not\leq e$, then $\{f, a, b, c \wedge a, c, d, e\}$ is a subset of L isomorphic to R_0, as desired.

Thus we assume $c \wedge a \leq e$ and similarly $c \wedge e \leq a$, which implies $c \wedge a = c \wedge e$. But since $c < a \vee e$ this is a violation of $(SD_\wedge)$. Of course, a dual argument handles B^d.

(b) Let $n \geq 0$ be minimal such that there is a subset $\{a_1, \ldots, a_{n+3}, b_1, \ldots, b_{n+3}, c\}$ of L isomorphic to G_n. Since $b_{n+2} \wedge c \geq a_1$, we have $b_{n+2} \wedge c \nleq b_1$. Choose k minimal such that $b_{n+2} \wedge c < b_k$; then $2 \leq k \leq n+2$. First we assume $k > 2$. If $b_{n+2} \wedge c \nless a_k$, then $\{a_1, \ldots, a_k, b_1, \ldots, b_k, b_{n+2} \wedge c\} \cong G_{k-3}$ where $0 \leq k-3 < n$; on the other hand, if $b_{n+2} \wedge c < a_k$ then $\{b_{n+2} \wedge c, a_k, a_{k+1}, \ldots, a_{n+3}, b_{k-1}, b_k, \ldots, b_{n+3}, c\} \cong G_{n+2-k}$ where $0 \leq n+2-k < n$. In either case we have a contradiction to the choice of n. Hence $k = 2$; that is, $b_{n+2} \wedge c < b_2$, which implies $b_{n+2} \wedge c = b_2 \wedge c$.

We first consider the case $b_2 \wedge c \nless a_2$. Assume that $b_2 \wedge c \nleq a_2 \vee b_1$; then $\{a_2, a_2 \vee b_1, b_1, b_2, b_2 \wedge c, a_1\} \cong B$, and we are done by part (a). Hence $b_2 \wedge c \leq a_2 \vee b_1$. Now, if $a_2 \wedge b_2 \nleq c$ we have $\{a_1, b_1, a_2 \wedge b_2, b_2 \wedge c, a_2, b_2, c\} \cong C$, while if $a_2 \wedge b_2 < c$ we have $\{b_2 \wedge c, b_1, c, (b_2 \wedge c) \vee b_1, a_2 \wedge b_2, a_2, a_2 \vee b_1\} \cong F_0$. In either case we are done by Theorem 3.2 and part (a).

Therefore $b_{n+2} \wedge c = b_2 \wedge c < a_2$, and by duality $a_2 \vee c > b_{n+2}$. If $a_2 \wedge b_1 \leq c$ then $a_2 \wedge b_1 \leq c \wedge b_1 = c \wedge b_2 \wedge b_1 \leq a_2 \wedge b_1$, implying $a_2 \wedge b_1 = c \wedge b_1$, and by $(SD_\wedge)$ we have $a_2 \wedge b_1 = (a_2 \vee c) \wedge b_1 = b_1$, a contradiction. Thus $a_2 \wedge b_1 \nleq c$ and by duality $a_{n+3} \vee b_{n+2} \ngeq c$. Now $\{a_1, \ldots, a_{n+3}, a_{n+3} \vee b_{n+2}, a_2 \wedge b_1, b_1, \ldots, b_{n+3}, c\} \cong R_{n+1}$, and Theorem 3.4 is established. □

The assumption that L has breadth at most two is necessary. For example, the lattice of Figure 10 has breadth three, is semidistributive,

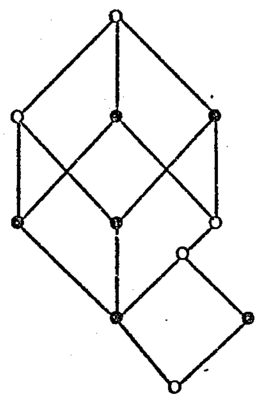


Figure 10

and contains B as a subset (the shaded elements), but does not contain R_0 as a subset.

COROLLARY (THEOREM 1.1). <u>A finite semidistributive lattice is planar if and only if it does not contain</u> A_3 <u>or</u> R_n, $n \geq 0$, <u>as a subset.</u>

<u>Proof</u>. Immediate from Corollary 3.3 and Theorem 3.4. □

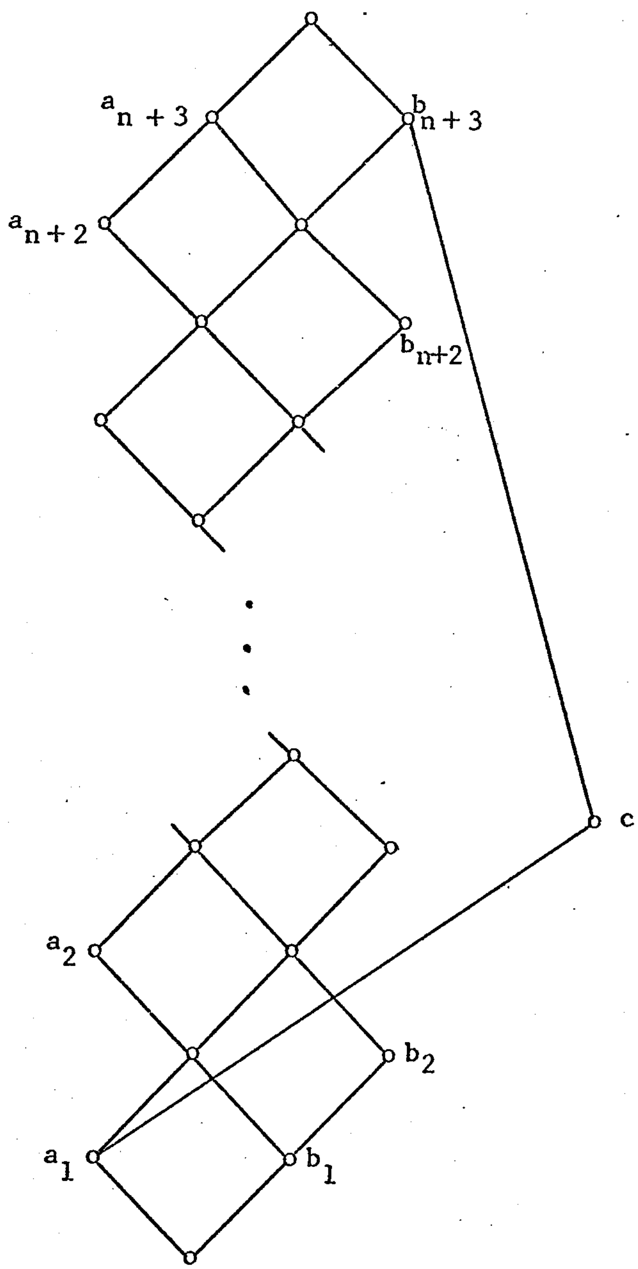
§ 4. <u>The proof of Theorem</u> 1.2.

It is well-known and easy to prove that if L is an arbitrary lattice, A_3 is a subset of L if and only if C_2^3 is a sublattice of L. Also, it is evident from Figures 2 and 6 that R_n is a subset of S_n for each $n \geq 0$; thus if S_n is a sublattice of a lattice L, certainly R_n is a subset of L. This completes the " if " direction of Theorem 1.2.

For each $n \geq 0$, let $P_n = \underset{\sim}{L}(R_n)$, the completion of R_n (see Figure 11) ; recall that if R_n is a subset of a lattice L, so is P_n. We require one more lemma.

LEMMA 4.1. <u>Let</u> L <u>be a lattice satisfying</u> $(\underset{\sim}{W})$, <u>and let</u> $R_n \setminus \{c\}$ <u>be a subset of</u> L <u>for some</u> $n \geq 0$. <u>Then</u> $S_n \setminus \{c\}$ <u>is a sublattice of</u> L.

<u>Proof</u>. Let $R_n \setminus \{c\} = \{a_1,\ldots,a_{n+3},b_1,\ldots,b_{n+3}\} \subseteq L$. Since $\underset{\sim}{L}(R_n \setminus \{c\}) = P_n \setminus \{c\}$, $P_n \setminus \{c\}$ is a subset of L, as indicated in Figure 11. Moreover we claim that the elements $\{a_1,\ldots,a_{n+2},b_2,\ldots,b_{n+3}\}$ of $P_n \setminus \{c\}$ generate a sublattice of L isomorphic to $S_n \setminus \{c\}$. For



P_n

Figure 11

simplicity, we will give the construction only in the case $n = 0$; an induction based on similar arguments will handle the general case. If $n = 0$, the required sublattice of L isomorphic to $S_0 \setminus \{c\}$ is given in Figure 12. Notice that $a_1 \vee (a_2 \wedge b_2) < a_2 \wedge (a_1 \vee b_2)$, $(a_2 \wedge b_3) \vee b_2 < (a_2 \vee b_2) \wedge b_3$, and $a_2 \wedge b_3 \nleq a_1 \vee b_2$ hold by virtue of $(\underset{\sim}{W})$. □

Now let L be a finite semidistributive lattice satisfying $(\underset{\sim}{W})$. We may assume that A_3 is not a subset of L, which implies that $b(L) \leq 2$ by Lemma 2.1. Let $n \geq 0$ be minimal such that there exists a subset of L isomorphic to R_n. Choose $R_n = \{a_1,\ldots,a_{n+3},b_1,\ldots,b_{n+3},c\} \subseteq [a_1 \wedge b_1, a_{n+3} \vee b_{n+3}] \subseteq L$ such that there does not exist a subset of L isomorphic to R_n in any proper subinterval of $[a_1 \wedge b_1, a_{n+3} \vee b_{n+3}]$. Then we have seen that we can find $S_n \subseteq L$, generated by $\{a_1,\ldots,a_{n+2}, b_2,\ldots,b_{n+3},c\}$, such that $S_n \setminus \{c\}$ is a sublattice of L. Observe that we may assume that $a_2 \wedge b_2 = b_1$ and $a_{n+2} \vee b_{n+2} = a_{n+3}$. To complete the proof of Theorem 1.2 we need only show that $a_{n+2} \vee c = a_{n+3} \vee b_{n+3}$ and $b_1 \vee c = b_{n+3}$ (a dual argument handles the corresponding meets).

First, we may assume that $b_{n+3} = (a_{n+3} \wedge b_{n+3}) \vee c$. Hence, since $\{a_{n+2}, a_{n+3}, a_{n+3} \wedge b_{n+3}, b_{n+3}, c\}$ is a down-down fence, Lemma 2.3 (i) implies that $a_{n+2} \vee c = a_{n+2} \vee (a_{n+3} \wedge b_{n+3}) \vee c = a_{n+3} \vee b_{n+3}$, as desired.

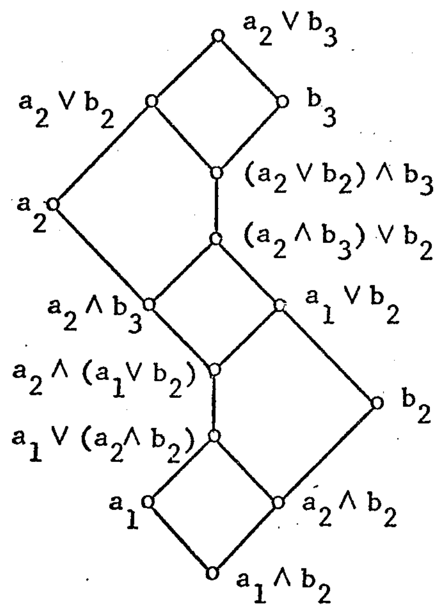

$a_2 \vee b_3$
$a_2 \vee b_2$
b_3
$(a_2 \vee b_2) \wedge b_3$
a_2
$(a_2 \wedge b_3) \vee b_2$
$a_2 \wedge b_3$
$a_1 \vee b_2$
$a_2 \wedge (a_1 \vee b_2)$
b_2
$a_1 \vee (a_2 \wedge b_2)$
a_1
$a_2 \wedge b_2$
$a_1 \wedge b_2$

Figure 12

Now suppose $n = 0$. If $a_2 \wedge b_3 \nleq b_2 \dot{\vee} c$ then $\{a_2 \wedge b_3, a_3 \wedge b_3, b_2, b_2 \dot{\vee} c, c, a_1\}$ is a subset of L isomorphic to B, and is contained in $[a_1 \wedge b_2, b_3]$. By Theorem 3.4 there is a subset of L isomorphic to R_0 which is contained in $[a_1 \wedge b_2, b_3]$, a proper subinterval of $[a_1 \wedge b_1, a_3 \dot{\vee} b_3]$, contrary to assumption. Therefore $a_2 \wedge b_3 < b_2 \dot{\vee} c$, and by (W) we are forced to conclude that $b_3 = b_2 \dot{\vee} c$. If $(a_2 \wedge b_3) \dot{\vee} c \ngeq b_2$ then $\{a_2, a_3, b_2, b_3, (a_2 \wedge b_3) \dot{\vee} c, a_2 \wedge b_3\}$ is a subset of L isomorphic to B, and is contained in $[b_1, a_3 \dot{\vee} b_3]$, a proper subset of $[a_1 \wedge b_1, a_3 \dot{\vee} b_3]$. By Theorem 3.4 we again have a contradiction. Thus $(a_2 \wedge b_3) \dot{\vee} c \geq b_2 \dot{\vee} c$ which implies $(a_2 \wedge b_3) \dot{\vee} c = b_3 = b_2 \dot{\vee} c$. By $(SD_{\vee})$ we conclude that $b_3 = (a_2 \wedge b_3 \wedge b_2) \dot{\vee} c = b_1 \dot{\vee} c$, completing the case $n = 0$.

We now assume $n > 0$. As in the case $n = 0$, our first goal will be to prove that $b_2 \dot{\vee} c = b_{n+3}$. Choose k maximal such that $b_2 \dot{\vee} c > a_k$; it is clear that $1 \leq k \leq n+1$. If $k = 1$, then $\{a_2, a_3, b_2, b_2 \dot{\vee} c, c, a_1\} \cong B$, and by Theorem 3.4 L must contain a subset isomorphic to R_0, contrary to the choice of n. Assume $2 \leq k \leq n$. If $b_2 \dot{\vee} c > b_{k+1}$, then $\{a_1, \ldots, a_{k+2}, b_1, \ldots, b_{k+1}, b_2 \dot{\vee} c, c\} \cong R_{k-1}$, and $k - 1 < n$, contradicting the choice of n. On the other hand, if $b_2 \dot{\vee} c \ngeq b_{k+1}$ then $\{a_k, \ldots, a_{n+2}, b_{k+1}, \ldots, b_{n+3}, b_2 \dot{\vee} c\} \cong G_{n-k}$; by Theorem 3.4 L must contain a subset isomorphic to R_m for some $m \leq n-k+1$, and since $1 \leq n-k+1 \leq n-1$ this again contradicts the choice of n. Therefore $k = n+1$, and so $b_2 \dot{\vee} c > a_{n+1}$. If $b_2 \dot{\vee} c \ngeq b_{n+2}$, then $\{a_{n+2}, a_{n+3}, b_{n+2}, b_{n+3}, b_2 \vee c, a_{n+1}\} \cong B$, which is a contradiction; hence $b_2 \vee c > b_{n+2}$. If $b_2 \vee c \ngeq a_{n+2} \wedge b_{n+3}$ then $\{a_1, \ldots, a_{n+1}, a_{n+2} \wedge b_{n+3}, b_2, \ldots, b_{n+2}, b_2 \vee c, c\}$ is a subset of L isomorphic to G_{n-1}, and is contained in $[a_1 \wedge b_2, b_{n+3}]$, a proper subinterval of $[a_1 \wedge b_1, a_{n+3} \vee b_{n+3}]$. By Theorem 3.4 $[a_1 \wedge b_2, b_{n+3}]$ contains a subset isomorphic to R_m for some $m \leq n$. By the choice of n we must have $m = n$; but this contradicts the minimality of R_n . Thus $b_2 \vee c \geq a_{n+2} \wedge b_{n+3}$, and by (W) we conclude $b_2 \vee c = b_{n+3}$.

Now if $a_2 \vee c \ngeq b_2$, we have $\{a_2, \ldots, a_{n+3}, b_2, \ldots, b_{n+3}, a_2 \vee c\} \cong R_{n-1}$ contradicting the choice of n . Hence $a_2 \vee c \geq b_2$, and so $a_2 \vee c = b_{n+3} = b_2 \vee c$. By $(SD_\vee)$, $b_{n+3} = (a_2 \wedge b_2) \vee c = b_1 \vee c$, and the proof of Theorem 1.2 is complete.

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