A STUDY OF DESCRIBING FUNCTION TECHNIQUES

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Master of Science

by

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ABSTRACT

The conventional describing function method is developed by a procedure that clearly indicates under what conditions it is permissible to use the conventional describing function method to determine the general characteristics of the transient response of a nonlinear system.

The first approximation method of Kryloff and Bogoliuboff is developed, and the results of this method are used to obtain the method of equivalent linearization.

The relationship between the techniques of Kryloff and Bogoliuboff and the conventional describing function method is studied.

A new describing function, called the elliptic describing function, is developed. For most practical systems in which the non-linearity can be represented reasonably closely by an odd cubic approximation, the elliptic describing function is more accurate than the conventional describing function. This describing function is restricted to odd-symmetrical, single-valued, frequency - independent non-linearities.

ACKNOWLEDGEMENTS

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

The design of many types of practical control systems requires the consideration of nonlinear phenomena, and as a result the study of nonlinear systems has been greatly stimulated in recent years. However, as yet, no general method of solving nonlinear problems has been found. The conventional describing function method, which is based on an assumed form of solution similar to that used to develop the methods of Kryloff and Bogoliuboff ., is the most widely used design technique for nonlinear control systems.

1.1 THE PROBLEM

The purpose of this thesis is (1) to study the methods of Kryloff and Bogoliuboff, and to determine their relationship with the conventional describing function; and (2) to attempt to develop a describing function which is more accurate than the conventional describing function.

This study will be restricted to self-excited systems containing one nonlinearity*.

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* However, where the results obtained are applicable to systems with forcing functions, this fact will be stated.

Only nonlinearities with no time varying characteristics will be considered in this study. Examples of the application of the conventional describing function method and the methods of Kryloff and Bogoliuboff are readily found in the literature (1,23,25,29,30), and thus will not be included in this thesis☆. The systems to be studied will be assumed to be in the form shown in Figure 1.1, since by suitable block diagram manipulation, practically all types of control systems containing one nonlinearity can be represented by this form (11, pp. 390 - 402; 33).

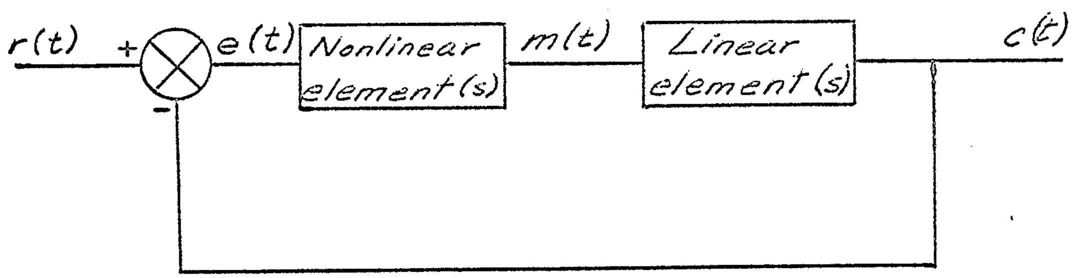

Figure 1.1 General block diagram of a control system incorporating one nonlinear element.

Following this introductory chapter, the conventional describing function method is briefly described. In chapter three, the methods of Kryloff and Bogoliuboff are developed,

☆ Numerals in parentheses refer to the bibliography at the end of the thesis.

and the relationship between the conventional describing function method and the methods of Kryloff and Bogoliuboff are clarified. A new type of describing function, called the elliptic describing function, is introduced in chapter four. The final chapter reviews the findings of this study and states some of the questions which remain unanswered.

1.2 TERMINOLOGY

For the purpose of this study a describing function will be defined as a function which relates a characteristic (or characteristics) of the input to a device to a characteristic (or characteristics) of the output from a device. In some cases a describing function may be defined only for a certain class of inputs.

The transfer function concept of linear analysis is an example of a describing function since it relates the Laplace transforms of the output and input of that device. In nonlinear design and analysis, the most common type of describing function, which we shall denote as the conventional describing function, is defined as the ratio of the phasor of the fundamental component of the output to the phasor of the input sinusoid.

The methods are stated in terms of control systems terminology, although the results can be applied equally well to other physical systems. For applications to circuit theory, the method, developed by Stout, of converting a nonlinear circuit to an equivalent block diagram is particularly useful (26,27,28).

CHAPTER 2

THE CONVENTIONAL DESCRIBING FUNCTION METHOD

The stability of nonlinear systems can be studied through the application of the conventional describing function; and if sustained oscillations exist, their approximate amplitude and frequency can be determined. In some special cases the method can also be used to indicate the general characteristics of the transient response of a nonlinear system. The basic principles involved in the conventional describing function technique will be briefly outlined in this chapter.*

2.1 THE CONVENTIONAL DESCRIBING FUNCTION

If the input to a nonlinear element is sinusoidal, say

$$e = a \sin \omega t, \tag{2.1}$$

then the output can be represented by a Fourier series

$$m = \sum_{n=1}^{\infty} m_n \sin n\omega t. \tag{2.2}$$

The conventional describing function for the nonlinear element is defined by

$$N(a,\omega) = \frac{m_1}{a}. \tag{2.3}$$

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* Although the development has been rearranged somewhat, the method outlined in this chapter is essentially the same as that of Kochenburger (23). The rearrangement of the material was thought desirable in order to clarify the use of the conventional describing method to indicate the general characteristics of the transient response, and also to illustrate more clearly the relationship between the conventional describing function method and the methods of Kryloff and Bogoliuboff.

Constant average components in signals can be handled by considering the nonlinear characteristic to be shifted or displaced by the amount of the constant component. The conventional describing function is then found as described previously by assuming a sinusoidal input to the modified characteristic.

2.2 SYSTEM ANALYSIS

The conventional describing function method is based on the assumption that, insofar as system performance is concerned, the input to the nonlinear element of a system can be adequately approximated by a sinusoid.

The significance of this assumption will be illustrated by Figure 2.1, showing the block diagram of a system with a nonlinear element N and linear elements with a transfer function G(s).

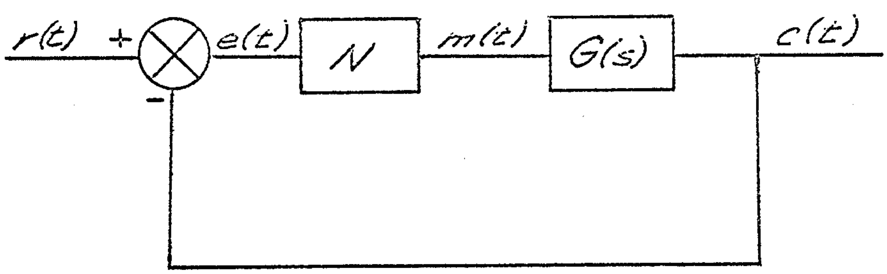

Figure 2.1 General block diagram of a control system incorporating one nonlinear element.

If the input to N is assumed to be sinusoidal, the resulting output will be periodic. The Fourier series of this periodic output will contain components at the fundamental frequency (the frequency of the input) and, in general, all higher harmonic frequencies. The output from the nonlinear device passes through the linear elements and is subtracted from the system input, having the Laplace transform R(s), to give the input signal to the nonlinear device. If the assumption stated is to be valid for this system, the fundamental component must be the only significant component of the feedback signal.

The assumption isusually justified for two reasons: first, the harmonics of the output from the nonlinear device are ordinarily of smaller amplitude than the fundamental; and, second, in most control systems the gain of the linear element decreases as the frequency increases, with the result that in transmission through the linear elements the higher harmonics are attenuated compared with the fundamental.

The conventional describing function is calculated on the assumption of a steady state sinusoidal input. However, it seems reasonable to assume that the describing function will be applicable when the input to the nonlinear element is nearly sinusoidal, with amplitude and frequency changing slowly. This assumption is the basis for the use of the conventional describing function to determine the general characteristics of the transient response; and also for the extension of the method to random input signals.

Let us consider the describing function of the nonlinear element as a transfer function and use Laplace transforms to find the response of the system, keeping in mind that the results obtained are valid only for solutions which are approximately sinusoidal in form.[*]

Consider the system of Figure 2.1 in which the nonlinear element has a describing function $N(a,\omega)$. If it is assumed the system has zero initial conditions, the following equations are readily obtained:

$$C(s) = E(s)\,N(a,\omega)\,G(s), \tag{2.4}$$

$$E(s) = R(s) - C(s), \tag{2.5}$$

and

$$\frac{C(s)}{R(s)} = \frac{G(s)\,N(a,\omega)}{1 + G(s)\,N(a,\omega)}. \tag{2.6}$$ [#]

Equation 2.6 can be thought of in terms of the poles and zeros of the transfer function, where the poles and zeros move slowly around the complex plane. Thus we have used the describing function representation to substitute for an equation which may contain rapidly varying relationships (such as the relations describing the action of a relay) one with slowly varying parameters.

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[*] It should be noted that the use of the Laplace transform to obtain a solution with slowly varying amplitude and frequency involves an additional approximation to those made previously since the Laplace transform method is valid only for differential equations with constant coefficients. However, provided the amplitude and frequency vary only slowly, the Laplace transform method will give an approximate solution. {transfer function

[#] If the initial conditions are not zero the appropriate terms can easily be inserted in equation 2.6 (10, pp. 60-61).

The stability of a linear control system is most easily determined by the use of the Nyquist criterion (10, pp. 118-154). Using this method the stability of a system, with an open loop transfer function $NG(s)$, is determined by finding the number of encirclements the $NG(j\omega)$ plot makes of the -1 point. It is obviously equivalent to find the number of encirclements the $G(j\omega)$ plot makes of the $-1/N$ (or critical) point. This criterion can also be applied to a nonlinear system by using the approximate equation 2.6, except in this case the $-1/N$ (or critical) point is dependent on the amplitude and frequency of the response.

For frequency - independent nonlinearities the position of the critical point can be found by plotting the $-1/N(a)$ locus which is scaled in the amplitude of the input to the nonlinearity. An intersection of the $-1/N(a)$ locus and $G(j\omega)$ locus, say at a_c and ω_c, corresponds to steady state oscillations of amplitude a_c and frequency ω_c. The point a_c, ω_c in Figure 2.2 is an example of such an intersection.

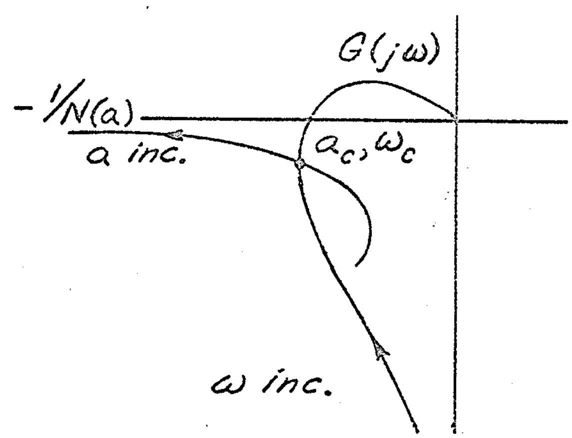

Figure 2.2. System with stable oscillations at a_c, ω_c.

For frequency - dependent nonlinearities a series of $-1/N(a, \omega_n)$ loci, with frequency as a constant parameter for each locus, must be plotted. An intersection of the $-1/N(a, \omega_c)$

locus, say at a_c, with the $G(j\omega)$ plot at ω_c corresponds to a steady state oscillation of amplitude a_c and frequency ω_c. The point a_c, ω_c in Figure 2.3 is an example of such an intersection.

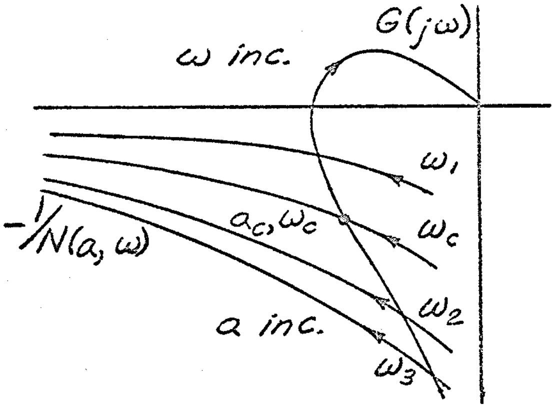

Figure 2.3 System with stable oscillations at a_c, ω_c.

The behaviour of a system when operating in the region near the intersection representing steady state oscillations is important since it determines whether the oscillations can exist in an actual system. For slight disturbances from the point of intersection the response will be nearly sinusoidal, with slowly varying amplitude and frequency, and thus the describing function approach will be valid.

First, consider the system represented by Figure 2.2.* If the system is operating at the point a_c, ω_c, and a slight disturbance increases the amplitude of oscillations, the system enters a stable region and the amplitude of the oscillations will decay to a_c. In the event that the oscillations decrease in amplitude, the system enters an unstable region and the amplitude of the oscillations increases to a_c. Thus, the system will have stable

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* To simplify the discussion all systems are assumed to be open-loop stable and minimum phase.

oscillations with amplitude a_c and frequency ω_c. The same conditions are also found to exist for the system of Figure 2.3.

Another condition, which may exist, is shown in Figure 2.4. If operation at a_c, ω_c ever occurred, the slightest further increase in amplitude would cause the output to grow indefinitely.

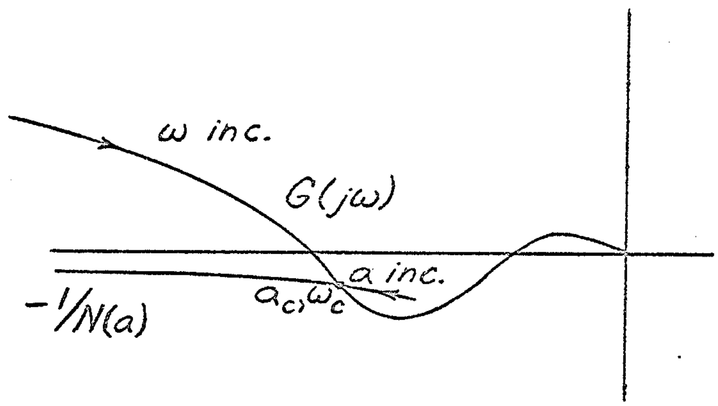

Figure 2.4 System with stability boundary at a_c, ω_c.

Likewise, the result of the slightest decrease would be to cause the output to decrease to zero. Thus stable oscillations cannot exist at a_c, ω_c in a practical system since some disturbance (noise) is always present. The intersection a_c, ω_c in this case is a stability boundary.

A final example is shown in Figure 2.5. Steady state oscillations cannot occur at a_{c1}, ω_{c1} in an actual system, but sustained oscillations can exist at a_{c2}, ω_{c2}.

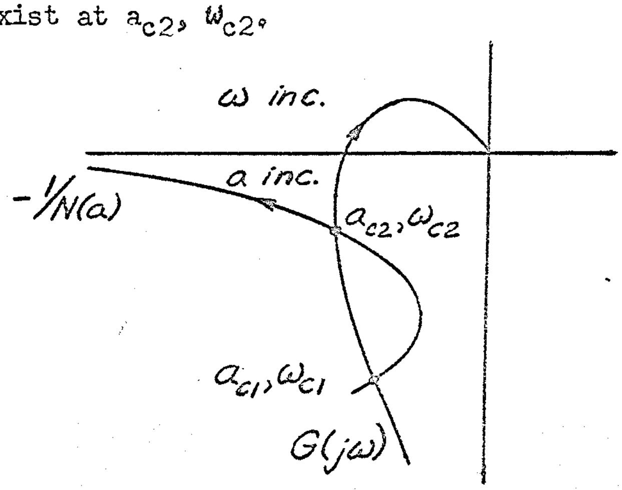

Figure 2.5 System with stability boundary a_{c1}, ω_{c1} and stable oscillations at a_{c2}, ω_{c2}.

Small disturbances which do not excite the system beyond a_{c1}, ω_{c1} will die out completely, but disturbances large enough to excite the system beyond a_{c1}, ω_{c1} will cause sustained oscillations to exist at a_{c2}, ω_{c2}.

These examples by no means exhaust the possible situations which exist, but the same reasoning may be applied to any other cases which arise.[*]

Providing the basic assumptions and limitations are kept well in mind, the method may be of very great value in determining, at least qualitatively, the transient response of a system.[#][†] Therefore, many of the useful design techniques for linear systems, such as the M_{peak} specification, may be applied to nonlinear systems. The describing function may also be used to obtain root-locus plots, again providing the necessary conditions for the validity of the method are observed.

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[*] It is interesting to note that Loeb has developed an alternative method for determining whether the point of intersection corresponds to stable sustained oscillations (11, pp. 419-421).

[#] It is difficult to calculate the transient response precisely because of the difficulty of determining the appropriate variation of sinusoidal amplitude to correspond to a given transient input to the nonlinear device.

[†] For example, in most practical control systems, with one integration or more in the open loop transfer function, the error response to a step input may be considered approximately sinusoidal with varying amplitude and frequency.

2.3 CONCLUDING COMMENTS

The principle of equivalent energy balance, developed by Kryloff and Bogoliuboff in connection with the method of equivalent linearization, can be used as a basis for the calculation of the conventional describing function.* Thus if the output from the nonlinear device is assumed to represent a force and the system output to represent a displacement, then the conventional describing function method can be given the following physical interpretation: the nonlinear force is replaced by an equivalent force such that the active and reactive work per cycle of the actual nonlinear force and equivalent force are approximately equal.#

If information regarding system performance for sinusoidal input signals is desired, then the dual input describing function method should be used since the conventional describing function method will give incorrect results in many cases (12, pp. 133-156; 31; 32). Unfortunately, the development of the dual input describing function method is too lengthy to include in this study.

The main objection to the conventional describing function method is the uncertainty of the accuracy of the results. Johnson has developed a series solution for a nonlinear differential equation, such that the first term of the series is the result of a

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* See Appendix A

The terms active and reactive work per cycle have been defined for mechanical systems in exactly the same way as they are defined for electrical systems. The definition of these terms is explained more fully in Appendix A.

conventional describing function analysis (25). Thus, the higher order terms can be considered as correction terms for the conventional describing function. However, the necessary calculations are complicated and the method is of little practical use.

Despite its limitations, the conventional describing function is widely used because: (1) in many cases there is no practical alternative method; and (2) in most practical cases the results are sufficiently accurate for design purposes.

CHAPTER 3

THE SINUSOIDAL TECHNIQUES OF KRYLOFF AND BOGOLIUBOFF

The first approximation method of Kryloff and Bogoliuboff provides a technique for obtaining an approximate solution of the following type of quasi-linear differential equation:

$$\ddot{x} + \omega^2 x + \mu f(x,\dot{x}) = 0, \qquad (3.1)$$

where $\mu f(x,\dot{x})$ is a single-valued function (1, pp. 181-208).#☆

The method of equivalent linearization of Kryloff and Bogoliuboff consists of obtaining an "equivalent" linear differential equation which has approximately the same solution as the original nonlinear differential equation (1, pp. 231-245).

The methods of Kryloff and Bogoliuboff and the conventional describing function method are similar since the procedure in each case is based on the assumption of a sinusoidal form of solution. In this chapter, the relationship between the methods of Kryloff and Bogoliuboff and the conventional describing function method will be examined.

\# An equation is called quasi-linear if the solution of the equation does not vary appreciably from the solution of the linear equation obtained when $\mu = 0$.

☆ In this thesis, the following notation is adopted to denote time derivatives:

$$\frac{dx}{dt} = \dot{x},\ \frac{d^2x}{dt^2} = \ddot{x},\ -----,\ \frac{d^nx}{dt^n} = x^{(n)}.$$

3.1 THE FIRST APPROXIMATION METHOD OF KRYLOFF AND BOGOLIUBOFF

For $\mu = 0$, equation 3.1 reduces to a linear differential equation with a solution and its first derivative given by the following equations:

$$x = a \sin (\omega t + \phi), \tag{3.2}$$

$$\dot{x} = a\omega \cos (\omega t + \phi). \tag{3.3}$$

Since we are dealing with quasi-linear equations, it appears logical to assume the solution and its first derivative to be of the form of equations 3.2 and 3.3, where a and ϕ are functions of time to be determined.

Differentiating equation 3.2, and for simplicity writing $\theta = \omega t + \phi$, we get

$$\dot{x} = \dot{a} \sin \theta + a\omega \cos \theta + a \dot{\phi} \cos \theta, \tag{3.4}$$

which, by means of equation 3.3, reduces to the following:

$$\dot{a} \sin \theta + a \dot{\phi} \cos \theta = 0. \tag{3.5}$$

Differentiating equation 3.3, we get

$$\ddot{x} = \dot{a}\omega \cos \theta - a\omega^2 \sin \theta - a\omega \dot{\phi} \sin \theta. \tag{3.6}$$

When the values for x, $\dot{x}$, and $\ddot{x}$ from equations 3.2, 3.3, and 3.6 respectively are substituted into equation 3.1, and the proper simplifications made, one obtains

$$\dot{a}\omega \cos \theta - a\omega \dot{\phi} \sin \theta = -\mu f \,(a \sin \theta,\ a\omega \cos \theta). \tag{3.7}$$

Equations 3.5 and 3.7, being linear functions of $\dot{a}$ and $\dot{\phi}$, can now be solved readily for these quantities. We thus get

$$\dot{a} = -\frac{\mu}{\omega} f \,(a \sin \theta,\ a\omega \cos \theta) \cos \theta, \tag{3.8}$$

$$\dot{\phi} = \frac{\mu}{a\omega} f \,(a \sin \theta,\ a\omega \cos \theta) \sin \theta. \tag{3.9}$$

Since a and ϕ are assumed to be slowly varying functions of time, we may, as a first approximation, consider them as being approximately constant during one period of the trigonometric functions involved.

Thus, making use of this approximation, we can expand $f(a \sin \theta, a\omega \cos \theta) \cos \theta$ and $f(a \sin \theta, a\omega \cos \theta) \sin \theta$ into the following Fourier series:

$$f(a \sin \theta, a\omega \cos \theta) \cos \theta = \frac{1}{2} A_o + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta), \quad (3.10)$$

$$f(a \sin \theta, a\omega \cos \theta) \sin \theta = \frac{1}{2} A_o' + \sum_{n=1}^{\infty} (A_n' \cos n\theta + B_n' \sin n\theta), \quad (3.11)$$

where the Fourier coefficients are given by the usual integrals, that is:

$$A_n = \frac{1}{\pi}\int_o^{2\pi} f(a \sin \theta, a\omega \cos \theta) \cos \theta \cos n\theta \, d\theta, \qquad (3.12)$$

$$B_n = \frac{1}{\pi}\int_o^{2\pi} f(a \sin \theta, a\omega \cos \theta) \cos \theta \sin n\theta \, d\theta, \qquad (3.13)$$

$$A_n' = \frac{1}{\pi}\int_o^{2\pi} f(a \sin \theta, a\omega \cos \theta) \sin \theta \cos n\theta \, d\theta, \qquad (3.14)$$

$$B_n' = \frac{1}{\pi}\int_o^{2\pi} f(a \sin \theta, a\omega \cos \theta) \sin \theta \sin n\theta \, d\theta. \qquad (3.15)$$

Using equations 3.10 and 3.11, we can write equations 3.8 and 3.9 in the following form:

$$\mathring{a} = -\frac{\mu A_o}{2\omega} - \frac{\mu}{\omega} \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta), \qquad (3.16)$$

$$\mathring{\phi} = \frac{\mu A_o'}{2a\omega} - \frac{\mu}{a\omega} \sum_{n=1}^{\infty} (A_n' \cos n\theta + B_n' \sin n\theta), \qquad (3.17)$$

Integrating these equations between t and t + T, where T is a period of sin θ and cos θ, and considering a and ϕ as remaining approximately constant in this interval, we get

$$\frac{a(t + T) - a(t)}{T} = -\frac{\mu A_o}{2\omega}, \tag{3.18}$$

$$\frac{\phi(t + T) - \phi(t)}{T} = \frac{\mu A_o'}{2a\omega}. \tag{3.19}$$

Since, by assumption, a and ϕ vary only slowly during one period, we can obtain from equations 3.18 and 3.19 the following approximate equations:

$$\dot{a} = -\frac{\mu A_o}{2\omega}, \tag{3.20}$$

$$\dot{\phi} = \frac{\mu A_o'}{2a\omega}. \tag{3.21}$$

Letting the total phase $\omega t + \phi$ be θ, we have $\dot{\theta} = \omega + \dot{\phi}$. Using the relationship of equation 3.21, we obtain

$$\dot{\theta} = \omega + \frac{\mu A_o'}{2a\omega}. \tag{3.22}$$

The first approximation to the solution of equation 3.1 will then be $x = a \sin\theta$, where the amplitude a and the phase θ are obtained from equations 3.20 and 3.22 respectively.

If, for an amplitude a_1, the following condition exists:

$$\dot{a} = -\frac{\mu A_o}{2\omega} = 0, \tag{3.23}$$

then the first approximation method predicts sustained oscillations with an amplitude a_1.

The question of whether these oscillations can exist in a practical system can be determined by considering the sign of $\dot{a}$ for small departures of a from a_1.

If for a slightly less than a_1, we have $\mathring{a}>0$; and if for a slightly greater than a_1, we have $\mathring{a}<0$, then, for small departures from a_1, the amplitude will tend to return to a_1 (the departure must be small enough that the sign condition for $\mathring{a}$ remains satisfied). This condition indicates that sustained oscillations with an amplitude a_1 can exist in the system.

If $\mathring{a}<0$ for a slightly less than a_1, and if $\mathring{a}>0$ for a slightly greater than a_1, sustained oscillations cannot exist at a_1 in an actual physical system since any disturbance (noise) will cause the amplitude to depart further from a_1. In this case, a stability boundary exists at the amplitude a_1.

If $\mathring{a} = 0$ for a slightly changed from a_1, sustained oscillations will exist at the amplitude to which the oscillation is disturbed.

3.2 THE METHOD OF EQUIVALENT LINEARIZATION OF KRYLOFF AND BOGOLIUBOFF

If the coefficients λ and K of the linear second order differential equation

$$\ddot{x} + \lambda\dot{x} + (\omega^2 + K)\, x = 0 \tag{3.24}$$

are chosen such that its solution, a sin θ, is approximately equal to the first approximation method solution of equation 3.1, the "equivalent" linear equation 3.24 can then be solved to find an approximate solution to equation 3.1.

If the amplitudes of the solutions of the linear equation 3.24 and the nonlinear equation 3.1 are to be equal, the following condition must be satisfied:

$$a = e^{-\frac{\lambda t}{2}}. \qquad (3.25)$$

Differentiating equation 3.25, we obtain

$$\dot{a} = -\frac{\lambda}{2}\ e^{-\frac{\lambda t}{2}}. \qquad (3.26)$$

Substituting equation 3.25 into the above equation gives

$$\dot{a} = -\frac{\lambda}{2}\, a. \qquad (3.27)$$

Using the equation for $\dot{a}$ obtained from the first approximation method (equation 3.20), and solving for λ, we have

$$\lambda = -\frac{\mu A_0}{2a\omega}. \qquad (3.28)$$

The total phase of the solutions of the two equations is equal providing the following condition is satisfied:

$$\theta = \sqrt{K + \omega^2}\ \left[1 - \left(\frac{\lambda}{2\sqrt{K + \omega^2}}\right)^2\right]^{\frac{1}{2}} t + \phi_0. \qquad (3.29)$$

Differentiating equation 3.29, we have

$$\dot{\theta} = \sqrt{K + \omega^2}\ \left[1 - \left(\frac{\lambda}{2\sqrt{K + \omega^2}}\right)^2\right]^{\frac{1}{2}}. \qquad (3.30)$$

Since we are dealing with quasi-linear differential equations of the type (3.1), the damping ratio $\dfrac{\lambda}{2\sqrt{K + \omega^2}}$ is small, and, as an approximation, equation 3.30 becomes

$$\dot{\theta} = \sqrt{K + \omega^2}. \qquad (3.31)$$

Using the equation for $\dot{\theta}$ obtained from the first approximation method (equation 3.22), we obtain

$$\omega + \frac{\mu A_0'}{2a\omega} = \sqrt{K + \omega^2}. \qquad (3.32)$$

Squaring this equation gives

$$K = 2w \left\{\frac{\mu A_o'}{2a\omega}\right\} + \left\{\frac{\mu A_o'}{2a\omega}\right\}^2. \qquad (3.33)$$

For quasi-linear equations, the frequency correction term $\frac{\mu A_o'}{2a\omega}$ will be small compared with ω. Neglecting the second order frequency correction term in equation 3.33, we obtain the following expression for K:

$$K = \frac{\mu A_o'}{a} \qquad (3.34)$$

Thus an "equivalent" linear differential equation, which has approximately the same solution as equation 3.1, is obtained by replacing the term $\mu f(x,\dot{x})$ by an equivalent term $\lambda\dot{x} + Kx$, where λ and K are given by equations 3.28 and 3.34 respectively.

Any of the techniques of linear theory may now be used to study this equivalent linear equation.

Since the solution of the equivalent linear equation 3.24 is

$$x = e^{-\frac{\lambda t}{2}} \sin\left\{\sqrt{K + \omega^2}\left[1 - \left(\frac{\lambda}{2\sqrt{K + \omega^2}}\right)^2\right]^{\frac{1}{2}} t + \phi_o\right\}, \qquad (3.35)$$

or approximately

$$x \doteq e^{-\frac{\lambda t}{2}} \sin\left\{\sqrt{K + \omega^2}\, t + \phi_o\right\}, \qquad (3.36)$$

the solution of the nonlinear equation cannot be found unless a is known so that λ and K can be calculated.

However, by following the procedure used to develop equations 3.25 - 3.28, the following equation for a can be obtained:

$$\dot{a} \doteq -a\frac{\lambda}{2} \qquad (3.37)$$

Thus, the amplitude a, as found from equation 3.37, can be substituted for $e^{\frac{-\lambda}{2}t}$ in equation 3.36, and also used to calculate the parameter K.

It should be noted that the method of equivalent linearization provides no new information, but is merely a different procedure for applying the first approximation method.

3.3 RELATIONSHIP BETWEEN THE CONVENTIONAL DESCRIBING FUNCTION METHOD AND THE TECHNIQUES OF KRYLOFF AND BOGOLIUBOFF.

First, let us see how the describing function method can be applied to a differential equation of the form:

$$\ddot{x} + \omega^2 x + \mu f(x,\dot{x}) = 0. \tag{3.38}$$

An equation of this type can be considered as representing the zero-input error response of a single-loop feedback system (Figure 3.1) with linear

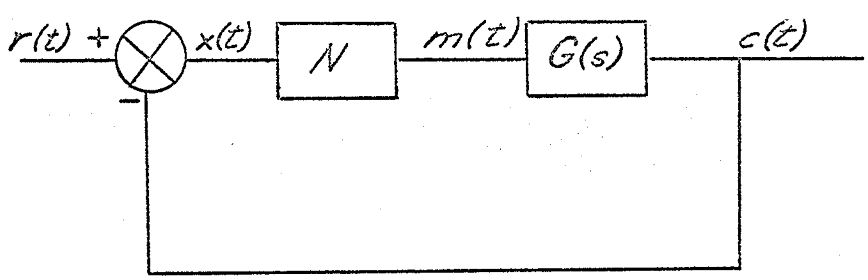

Figure 3.1 Single-loop feedback system

elements having a transfer function

$$G(s) = \frac{1}{s^2 + \omega^2}, \tag{3.39}$$

and, for an error signal x, an output m from the nonlinear element given by the relationship

$$m = \mu f\,(x, \dot{x}). \tag{3.40}$$

If $\mu f\,(x,\dot{x})$ contains a linear term dependent on $\dot{x}$, say $a\dot{x}$, it can be removed from the nonlinear block by making the transfer function of the linear block

$$G(s) = \frac{1}{s^2 + as + \omega^2}\,. \tag{3.41}$$

Now observe that the describing function method can be applied to the system of Figure 3.1 in the normal manner.

For a second order system with a frequency-independent nonlinearity, the procedure used to solve the equivalent linear equation resulting from the Method of Equivalent Linearization can also be applied to effect a solution to the equivalent linear equation obtained from the describing function method.*

However, for frequency-dependent nonlinearities, the solution of the equivalent linear equation resulting from the describing function method is more difficult to obtain since the equivalent parameters are dependent on two unknown functions, that is, the amplitude and frequency (the equivalent parameters are dependent only on the amplitude in the Method of Equivalent Linearization). For this reason, two simultaneous equations involving the amplitude and frequency are obtained. In all but a very few cases, these equations must be solved by a graphical technique.

Since the relationship between the first approximation

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* Supra p.20

method and equivalent linearization has been developed, the conventional describing function method will be compared with only the method of equivalent linearization.

3.3a The Approximant

Although both the methods of Kryloff and Bogoliuboff and the conventional describing function method are based on the assumption of a sinusoidal form of solution, the approximant is not the same for both.

In the conventional describing function approach, the assumed values of the derivatives of the solution are exactly equal to the derivatives of the assumed solution.

In the methods of Kryloff and Bogoliuboff, a frequency of oscillation is assumed, and then a phase correction term ϕ is calculated. However, this correction term is not taken into account in the coefficient of the trigonometric function of the assumed value of the first derivative, and thus the assumed value of the first derivative is not equal to the first derivative of the assumed solution (except when the correction term is zero).

These observations seem to indicate that, providing other approximations are equally accurate for both methods, the describing function method will give more accurate results than the methods of Kryloff and Bogoliuboff.

However, for frequency-dependent nonlinearities, the describing function method requires more calculations since both the amplitude and frequency of oscillations occur in the parameters of the equivalent linear term; whereas the equivalent parameters are dependent only on the amplitude in the equivalent linearization method.

3.3b Nature of the Approximation

The equivalent gain of the nonlinear device for the describing function method is chosen to minimize the mean square error between the actual output and the sinusoidal approximation of the output for a sinusoidal input*.

For the equivalent linearization method of Kryloff and Bogoliuboff, the equivalent gain is chosen such that the rate of change of the amplitude and phase of the solution of the actual nonlinear equation and the equivalent linear equation are approximately equal.

3.3c Results Obtained

If we apply the conventional describing function method to equation 3.1 for the case of a frequency-independent nonlinearity, the nonlinear relationship $\mu f(x)$ is replaced by the function

$$\frac{\mu x}{2a\pi}\int_0^{2\pi} f\,(a \sin \theta) \sin \theta \, d\theta.$$

Therefore, since this is precisely the equivalent term obtained by the method of equivalent linearization, the two methods are equivalent for frequency-independent nonlinearities.

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* The output from the nonlinear device is approximated by the first term of a series, the Fourier series, formed from an orthogonal set of functions, and so the mean square error between the actual and approximating function is minimized (6, pp. 156-158).

It is not possible to obtain the exact relationship between the two methods for the case of frequency-dependent nonlinearities unless the exact form of the nonlinear relationship is known. However, in general, the two methods will yield different results because the approximants are different.

In general, for frequency-dependent nonlinearities, the conventional describing function method gives more accurate results than the method of equivalent linearization since an additional approximation is involved in the latter.

3.4 CONCLUDING COMMENTS

Since all the methods discussed yield approximately the same information, it is important to determine whether one of the methods is preferable for a particular application.

Obviously the describing function method is of much greater applicability for two reasons: first, the nonlinearity need not be single valued as required for the methods of Kryloff and Bogoliuboff; and, second, there is no restriction on the order of the differential equation.

As pointed out previously, the methods of Kryloff and Bogoliuboff involve an additional approximation to those made in the conventional describing function method, and therefore are usually not as accurate.

However, as a result of this additional approximation, the method of equivalent linearization gives an equivalent linear equation which is easier to solve than the equivalent linear equation obtained from the conventional describing function technique. However, the same approximation may be applied to the conventional describing function method, that is, the frequency in the equation for the equivalent linear parameters can be assumed to be ω. If this approximation is made, the equivalent linear equation obtained by the equivalent linearization method and the conventional describing function method become identical. The validity of this type of approximation can be checked by determining whether the frequency correction term is small.

From the material of this chapter, it can be concluded that the conventional describing function method can completely replace the techniques of Kryloff and Bogoliuboff.

If the nonlinear term $\mu f(x,\dot{x})$ is assumed to represent a force and x to represent displacement, the method of equivalent linearization can be given either of the following physical interpretations:*

(1) The nonlinear force $\mu f(x,\dot{x})$ is replaced by an equivalent linear force $K + \lambda\dot{x}$, where K and λ are chosen such that the equivalent force produces the same active and reactive work per cycle as the nonlinear force.

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* See Appendix A

(2) The nonlinear force is replaced by an equivalent linear force $Kx + \lambda\dot{x}$ such that the equivalent linear force is equal to the first harmonic of the nonlinear force.

CHAPTER 4

THE ELLIPTIC DESCRIBING FUNCTION

In this chapter, a new describing function, which is more accurate than the conventional describing function for most practical cases, is defined. This new describing function is restricted to odd-symmetrical, single-valued, frequency-independent nonlinearities.

4.1 A RÉSUMÉ OF PREVIOUS ATTEMPTS TO OBTAIN A MORE ACCURATE DESCRIBING FUNCTION

Klotter (9), Prince (35), and Gibson and Prasanna-Kumar (36) have defined describing functions based on a sinusoidal input to the nonlinear element and an equivalent sinusoidal output which is chosen according to some characteristic of the actual output. However, since these describing functions are difficult to evaluate, and in general are no more accurate than the conventional describing function, they have not become widely used.

West and others (12, pp. 133-156; 31; 32) have defined a describing function based on an input signal composed of two sinusoidal signals of different amplitudes and frequencies, and an equivalent output signal composed of two sinusoidal signals with the same frequencies as the input signals. If the two signals are harmonically related (such as the fundamental and third harmonic), the dual-input describing function can be used to study system oscillations. Unfortunately, the dual-input describing function is difficult to calculate, and therefore it is not normally used to study oscillations in self-excited systems.

Hamel (11, pp. 446-483), Tsypkin (11, pp. 455-483), and Manabe (15) have developed exact methods for finding the steady state response of nonlinear systems, but their methods are applicable only to systems in which the wave shape of the oscillations is known *a priori* at one point in the system.

4.2 CHOICE OF A DESCRIBING FUNCTION

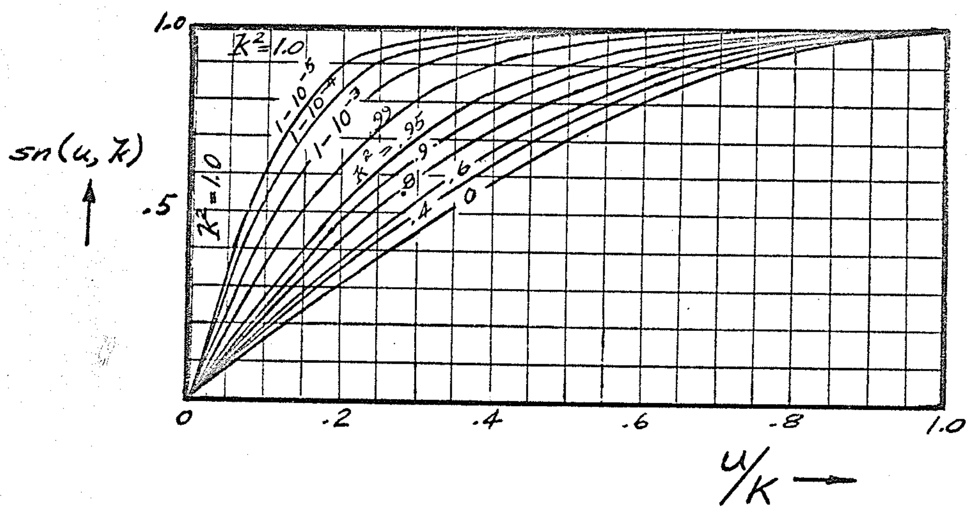
With the exception of the dual-input describing function and the methods developed by Hamel, Tsypkin and Manabe, all describing functions for nonlinear systems have been derived on the assumption of a sinusoidal input signal and a sinusoidal approximation of the output.

If the filtering of the linear elements is not perfect for the harmonics of the output from the nonlinear device, these harmonics will be present in the system output and the input to the nonlinear device. A describing function based on a sinusoidal approximation of the input and output of a nonlinear device can never account for the presence of harmonics in the system.

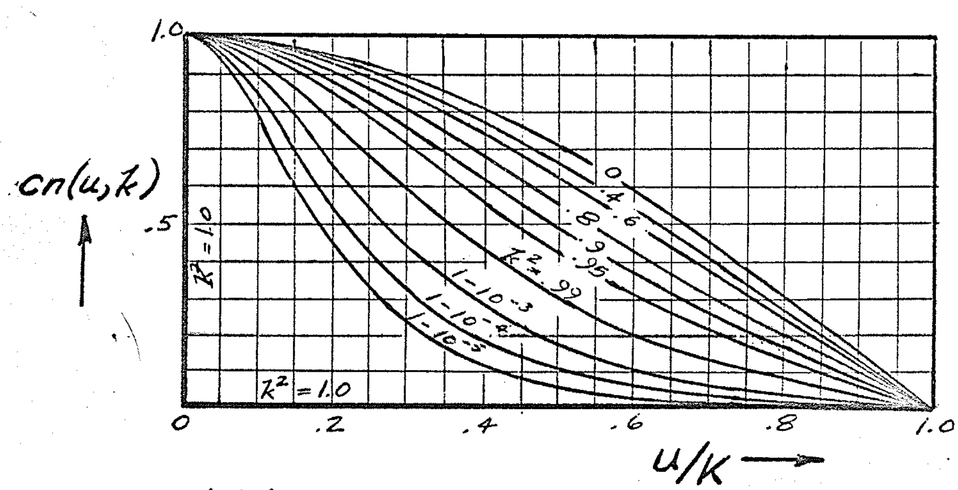
Therefore, it was proposed to define a describing function which would give a more accurate representation of the actual wave shape of system oscillations than is possible with a sinusoidal approximation.

After many unsuccessful attempts had been made to obtain a describing function which would account for the presence of harmonics in the system, it was decided that a restriction on the generality of the problem was necessary if any progress was to be made. Therefore the method will be restricted to odd- symmetrical, single-valued, frequency-independent nonlinearities.

As can be seen from Figure 4.1, it is possible, by varying the modulus k, to obtain a variety of wave shapes with the Jacobian elliptic functions sn (u,k) and cn (u,k).* Therefore, it seems probable that by approximating the input and output



4.1a Sn(u,k) as a function of u/K.



4.1b Cn(u,k) as a function of u/K.
Figure 4.1 Jacobian elliptic functions.

*For definitions see (17) pp. 72-73.

of a device with elliptic functions, a representation of the wave shape of system oscillations, which is more accurate than a sinusoidal approximation, could be obtained.

The following criteria will be applied to obtain the Jacobian elliptic function approximation of a periodic function: the actual function will be approximated by the Jacobian elliptic function which has the same amplitude and period as the actual function, and the minimum mean square error between the actual function and the approximating function for a complete period of the function will be obtained by varying the modulus k.

The choice of a criterion was made on the basis of obtaining a reasonably accurate approximation with a minimum of mathematical difficulties.

Let us denote the instantaneous relationship between the input and output of a nonlinear device by

$$m = f(e), \tag{4.1}$$

where e is the input and m is the output.

The notation

$$e = x_i \operatorname{sn}(u_i, k_i) \tag{4.2}$$

or

$$e = x_i \operatorname{cn}(u_i - K_i, k_i) \tag{4.3}$$

will be used to denote the input signal to the nonlinear device (this choice of reference for the cn(u,k) function considerably simplifies the calculations).

If the radian frequency of the input signal is ω, the relationship between the argument u_i and time t is

$$u_i = \frac{2K_i}{\pi} \omega t, \tag{4.4}$$

where the first complete integral K_i has a modulus k_i.

The output of the nonlinear device will be approximated by the elliptic function

$$m = x_o sn(u_o,k_o) \tag{4.5}$$

or $$m = x_o cn(u_o-K_o,k_o), \tag{4.6}$$

where x_o, u_o, k_o and the elliptic function $\left[sn\ (u_o,k_o)\right.$ or $\left.cn(u_o-K_o,\ k_o)\right]$ to be used are determined on the following basis:

(1) The amplitude x_o is found from the relationship

$$x_o = f(x_i). \tag{4.7}$$

(2) The argument u_o is chosen so that the period of the elliptic function is equal to the period of the actual function, that is,

$$u_o = \frac{2K_o}{\pi}\omega t = \frac{u_i\ K_o}{K_i}, \tag{4.8}$$

where the first complete elliptic integral K_o has a modulus k_o.

(3) The modulus k_o and the elliptic function to be used are chosen so that the mean square error between the actual function and the approximating function for one period of the function is a minimum.

The necessary calculations to obtain the elliptic function approximation are described in detail in appendix B for an input

$$e = x_i sn(u_i,k_i) \tag{4.9}$$

and an assumed form of output

$$m = x_o sn(u_o,k_o) \tag{4.10}$$

(this form of output will be used as the approximation only if it gives less mean square error than the $x_o cn(u_o-K_o,k_o)$ form of output). The calculations for other combinations of input and output functions are similar, and therefore only the results of the calculations will be stated.

As is shown in appendix B (equation B.44), the modulus k_o is to be found from the following equation:

$$\frac{2D}{k_o} - \frac{2E}{k_o(k_o')^2} + \frac{2k_oF}{(k_o')^2} = \frac{x_o}{3k_o^3(k_o')^2}\left\{ K_o\left[4k_o^2 - 4 + (k_o')^2 + 3(k_o')^4\right] + 3E(K_o,k_o)\left[k_o^2 - 1 + (k_o')^2\right]\right\}, \qquad (4.11)$$

where $D = \int_0^{K_o} u_o \,\mathrm{cn}(u_o,k_o)\,\mathrm{dn}(u_o,k_o)\,f\left[x_i\,\mathrm{sn}(u_i,k_i)\right] du_o,$

$$E = \int_0^{K_o} E(u_o)\,\mathrm{cn}(u_o,k_o)\,\mathrm{dn}(u_o,k_o)\,f\left[x_i\,\mathrm{sn}(u_i,k_i)\right] du_o,$$

$$F = \int_0^{K_o} \mathrm{sn}(u_o,k_o)\,\mathrm{cn}^2(u_o,k_o)\,f\left[x_i\,\mathrm{sn}(u_i,k_i)\right] du_o.$$

It is evident from this equation that the necessary calculations to obtain an expression for k_o would be extremely difficult to perform. One of the main difficulties is that the modulus k_o occurs as a parameter in transcendental functions, and therefore the equation must be solved by numerical methods. The calculations would obviously be greatly simplified if the need for a numerical solution was eliminated.

One possible method of avoiding the numerical solution would be to express the nonlinear relationship in such a manner that if k_o and k_i were assumed known the equation could easily be solved for the parameters of the nonlinear relationship. By calculating the parameters of the nonlinear relationship for the full range of k_o and k_i, a graph involving k_o, k_i and the parameters of the nonlinear relationship could be obtained. Then, if k_i and the parameters of the nonlinear relationship were known, the modulus k_o could be obtained from such a graph.

In an attempt to simplify the calculations, a number of methods of approximating the nonlinear relationship were examined. The only suitable method found was an odd cubic polynomial approximation. If the nonlinear relationship is approximated by a cubic

$$m = f(e) \sim b(e + ae^3), \qquad (4.12)$$

only the parameter a is required to specify the nonlinear relationship since the constant b may be considered as part of the gain of the linear elements. If the nonlinear relationship is approximated by an odd cubic polynomial, equation 4.11 (and the equations for other combinations of input and output elliptic functions) can easily be solved for ax_i^2 if k_o and k_i are assumed known, thus making it possible to obtain a graph involving ax_i^2, k_o, and k_i. The required formula is calculated in appendix B and the resulting graph is shown in Figure 4.2, page 35 . This graph also includes the relationship between the parameters for an input $x_i cn(u_i - K_i)$ and an output $x_o cn(u_o - K_o)$. The relationship for the other combinations of input and output functions was not plotted because of their limited usefulness and the large amount of time required for the calculations*.

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* Infra p. 41

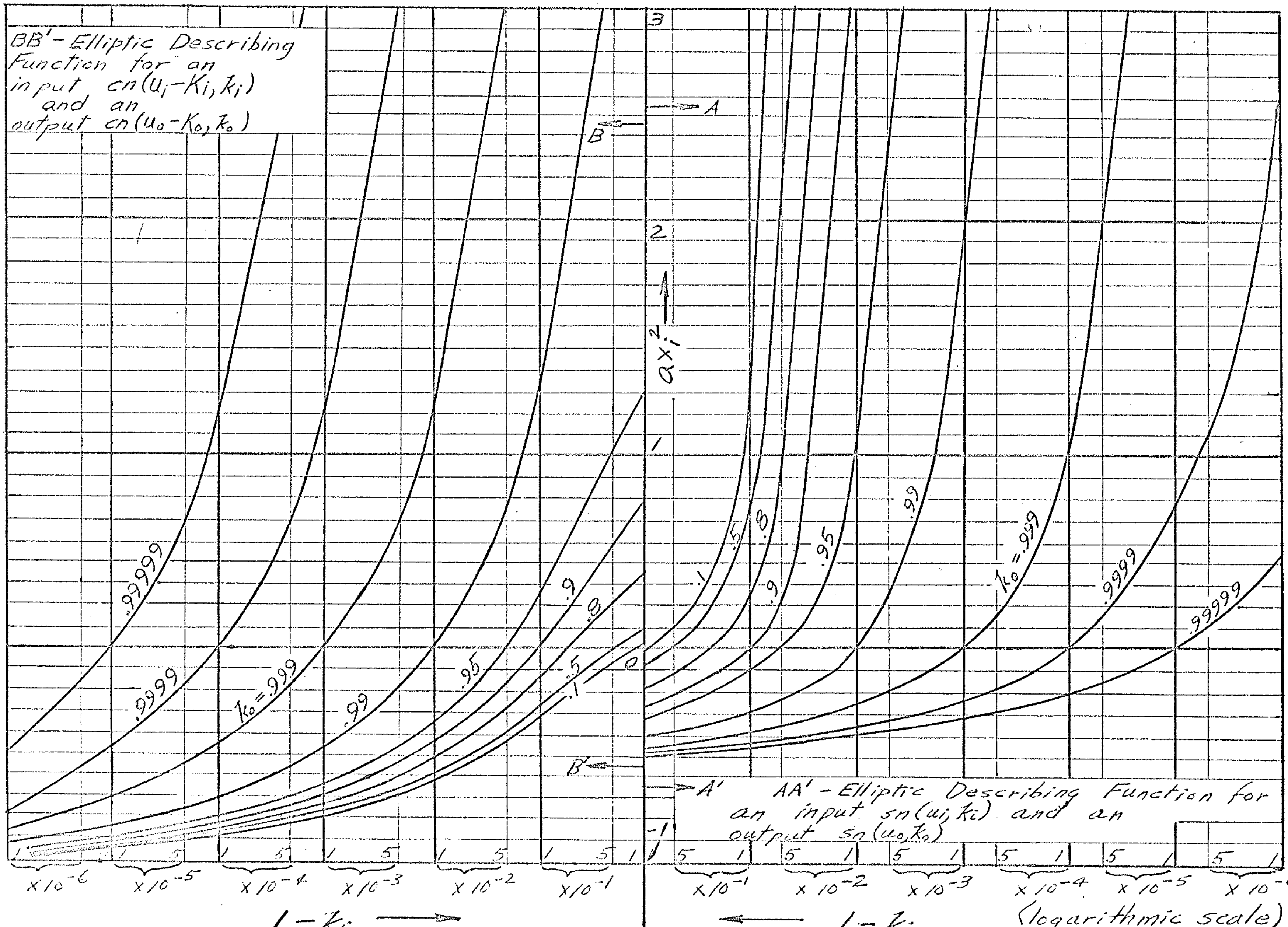

Figure 4.2 Elliptic Describing Function Relationship

Although a cubic approximation may not give a "good fit" for some nonlinearities, the approximation of the nonlinear relationship by a cubic is justified for two reasons:

(1) Most nonlinearities for which the exact methods of Hamel, Tsypkin and Manabe are not applicable may be adequately approximated by a cubic (saturation and dead zone are the most common examples). Therefore, the method will be applicable to most of the systems for which there is no method of obtaining an exact solution.

(2) Soudack (20) has obtained very accurate solutions for a certain class of nonlinear differential equations by approximating the nonlinear relationship by a cubic polynomial. It seems reasonable to assume that a cubic approximation will give accurate results for a wider class of differential equations than those treated by Soudack. It will be assumed that a cubic approximation gives accurate results, and then the validity of this assumption will be tested by comparing the results obtained with the correct solutions.

Since it has not been possible to justify any particular method of approximating the nonlinear relationship with a cubic polynomial, the method will be applied using both the Legendre and Chebyshev approximations to determine whether one of these approximations will give more accurate results than the other.

If equation 4.12 is obtained using a Legendre or Chebyshev approximation the range of the approximation is $-1 \leq e \leq 1$. Thus to obtain the most accurate result, a change of dependent variable should be made in the equation representing system performance such that x_i in the approximate solution of the resulting equation is equal to or only slightly less than one.*

4.3 SYSTEM ANALYSIS

Consider the application of the elliptic describing function to the single-loop system of Figure 4.3 for the condition of zero input signal.# If the nonlinear input-output relationship of the

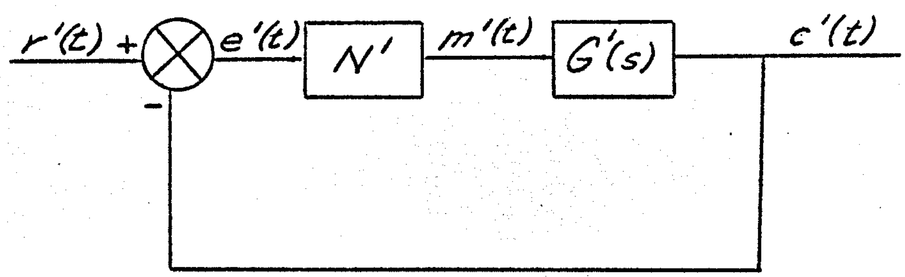

Figure 4.3 General block diagram of control system incorporating one nonlinear element

element denoted by N' in Figure 4.3 is approximated by a cubic polynomial, the necessary elliptic function input $x_i \mathrm{sn}(u_i,k_i)$ to produce an approximate elliptic function output $x_o \mathrm{sn}(u_o,k_o)$ can be determined from equation 4.7 and Figure 4.2. Let us denote this input-output relationship by

$$x_i \mathrm{sn}(u_i,k_i) = f_1 \cdot \left[x_o \mathrm{sn}(u_o,k_o) \right] . \qquad (4.13)$$

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* It is also possible to change the range of the polynomial approximation by a suitable change of variable; however this procedure would require more calculations than the one outlined above.

It is assumed that the proper change of dependent-variable has been made in the system equations by performing a suitable transformation of system parameters. The best procedure for making this transformation is given on pages 46-47.

All possible combinations of sn(u,k) and cn(u,k) functions are necessary to completely specify the input-output relationship, but only the calculations for the sn(u,k) function approximation of input and output are given since the calculations are similar for other cases.

If the output from the linear elements or, equivalently, the negative of the input to the nonlinear element for an elliptic function input $x_o \operatorname{sn}(u_o, k_o)$ to the linear elements is approximated by an elliptic function according to the relationship

$$c = -x_i \operatorname{sn}(u_i, k_i) = f_2 \left[x_o \operatorname{sn}(u_o, k_o) \right] , \qquad (4.14)$$

the condition for sustained oscillations in the system of Figure 4.3 is

$$f_1 \left[x_o \operatorname{sn}(u_o, k_o) \right] = -f_2 \left[x_o \operatorname{sn}(u_o, k_o) \right] . \qquad (4.15)$$

Therefore, to employ the elliptic describing function in system analysis, it is necessary that the response of the linear elements to an elliptic function input be approximated by an elliptic function.

All attempts to calculate the output from the linear elements directly in terms of elliptic functions ended in failure. Because of the ease with which the response of a linear device to sinusoidal input signals can be determined, it was decided to treat the input as a Fourier series, and then attempt to synthesize an approximate elliptic function from the output Fourier series.

The Fourier series expansions of the elliptic functions are given by the following equations (4,pp.167-169):

$$\mathrm{sn}(u,k) = \frac{\pi}{Kk} \sum_{n=0}^{\infty} \frac{\sin\,(2n+1)v}{\sinh\left[\left(n+\frac{1}{2}\right)\pi\frac{K'}{K}\right]}, \tag{4.16}$$

$$\mathrm{cn}(u-K,k) = \frac{\pi}{Kk} \sum_{n=0}^{\infty} \frac{\sin\left[(2n+1)v + n\,180\right]}{\cosh\left[\left(n+\frac{1}{2}\right)\pi\frac{K'}{K}\right]}, \tag{4.17}$$

where v is defined by the following relationship

$$v = \frac{\pi}{2K}\,u. \tag{4.18}$$

Substituting equation 4.4 into equation 4.18 gives

$$v = \omega t,$$

and thus the input to the linear device may be represented by the following expansion:

$$x_o \mathrm{sn}(u_o,k_o) = \frac{X_o\,\pi}{K_o k_o} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\,\omega t}{\sinh\left[\left(n+\frac{1}{2}\right)\pi\frac{K_o'}{K_o}\right]}. \tag{4.19}$$

If the linear elements are characterized by their steady state transfer function

$$G'(j\omega) = A'(\omega)\,e^{j\phi'(\omega)}, \tag{4.20}$$

the response of the linear device to the elliptic function input is

$$c = \frac{x_o\,\pi}{K_o k_o} \sum_{n=0}^{\infty} \frac{A'\left[(2n+1)\omega\right]\,\sin\left\{(2n+1)\omega t + \phi'\left[(2n+1)\omega\right]\right\}}{\sinh\left[\left(n+\frac{1}{2}\right)\pi\frac{K_o'}{K_o}\right]} \tag{4.21}$$

or, equivalently,

$$c = \frac{x_o\,\pi}{K_o k_o} \sum_{n=0}^{\infty} \frac{A'\left[(2n+1)\omega\right]\cos\left\{\phi'(2n+1)\omega + \phi'(\omega)\right\}\,\sin\left\{(2n+1)\omega t + \phi'(\omega)\right\}}{\sinh\left[\left(n+\frac{1}{2}\right)\pi\frac{K_o'}{K_o}\right]}$$

$$+ \frac{A'\left[(2n+1)\omega\right]\sin\left\{\phi'\left[(2n+1)\omega\right] + \phi'(\omega)\right\}\cos\left\{(2n+1)\omega t + \phi'(\omega)\right\}}{\sinh\left[\left(n+\frac{1}{2}\right)\pi\frac{K_o'}{K_o}\right]}. \tag{4.22}$$

It was decided to approximate the series by the elliptic function which gives the least mean square error between the actual function and the approximating function.

Many difficulties were encountered in synthesizing an approximate elliptic function from the Fourier series, and therefore it was found necessary to introduce some further approximations.

From equations 4.16 and 4.17, it is seen that all higher harmonics of the Fourier series of the elliptic function have either zero or 180 degrees phase shift from the fundamental component. Providing the first term of the Fourier series in equation 4.22 is large compared with the higher harmonic terms (which is usually the case for control systems), the best mean square error approximation to equation 4.22 by a sinusoidal series containing terms with either zero or 180 degrees phase shift is obtained by neglecting the cosine terms in equation 4.22.

Therefore, the approximate system output is

$$c = \frac{x_o \pi}{K_o k_o} \sum_{n=0}^{\infty} \frac{A'[(2n+1)\omega] \cos\{\phi'[(2n+1)\omega] - \phi'(\omega)\} \sin\{(2n+1)\omega t + \phi'(\omega)\}}{\sinh\left[\left(n+\frac{1}{2}\right)\frac{\pi K_o'}{K_o}\right]} \qquad (4.23)$$

This Fourier series will be approximated by the elliptic function which has the same first and third harmonic terms in its Fourier expansion. This specification fixes the type of elliptic function to be used, the modulus k, and the amplitude.

Let us write equation 4.23 in the following manner:

$$c = \frac{x_o \pi}{K_o k_o} \left\{ \frac{A'(\omega) \sinh\left(\frac{\pi K'}{2K}\right)}{\sinh\left(\frac{\pi K_o'}{2K_o}\right)} \sum_{n=0}^{1} \frac{\sin[(2n+1)\omega t + \phi'(\omega)]}{\sinh\left[\frac{\pi K'}{K}\left(n+\frac{1}{2}\right)\right]} \right.$$

$$\left. + \sum_{n=2}^{\infty} \frac{A'[(2n+1)\omega] \cos\{\phi'[(2n+1)\omega] - \phi'(\omega)\} \sin[(2n+1)\omega t + \phi'(\omega)]}{\sinh\left[\left(n+\frac{1}{2}\right)\frac{\pi K_o'}{K_o}\right]} \right\}, \qquad (4.24)$$

where k is chosen such that the following equation is satisfied:

$$\frac{A'(3\omega)\cos\left\{\phi'(3\omega)-\phi'(\omega)\right\}\sinh\left(\frac{\pi K_o'}{2K_o}\right)}{A'(\omega)\sinh\left(\frac{3\pi K_o'}{2K_o}\right)} = \frac{\sinh\left(\frac{\pi K'}{2K}\right)}{\sinh\left(\frac{3\pi K'}{K}\right)}.^{*} \qquad (4.25)$$

The series of equation 4.24 is to be approximated by the elliptic function which has the same first two terms in its Fourier series. That is, equation 4.24 is to be approximated by the function

$$c = \frac{kK\, A'(\omega)\sinh\left(\frac{\pi K'}{2K}\right)}{k_oK_o\sinh\left(\frac{\pi K_o'}{2K_o}\right)}\, sn\left(u + \frac{2K\phi'(\omega)}{\pi}, k\right). \qquad (4.26)$$

Since the approximate output elliptic function must equal the negative of the elliptic function input to the nonlinear device (equation 4.14), equation 4.26 can be written as follows:

$$c = -x_i\, sn(u_i,k_i) = f_2\left[x_o\, sn(u_o,k_o)\right]$$

$$= \frac{k_iK_i\, A'(\omega)\sinh\left(\frac{\pi K_i'}{2K_i}\right)}{k_oK_o\sinh\left(\frac{\pi K_o'}{2K_o}\right)}\, sn\left(u + \frac{2K_i\phi'(\omega)}{\pi}, k_i\right). \qquad (4.27)$$

This approximation can now be used to solve equation 4.15 for the parameters of the elliptic function approximation of system oscillations.

---

* This equation can be satisfied only if the phase shift between the first and third harmonic gain is in the range $-\frac{\pi}{2} \leq \phi'(3\omega) - \phi'(\omega) \leq \pi/2$. If the phase shift is outside this range, equation 4.24 should be written with terms of the form $\sin\left[(2n+1)v + n\,180\right]$, and thus would be approximated by a cn(u-K,k) function. Since the phase shift is usually in the range $-\pi/2 \leq \phi'(3\omega) - \phi'(\omega) \leq \pi/2$, the elliptic describing function calculations will not be carried out for phase shifts outside this range.

First, note that since the method has been restricted to frequency-independent nonlinearities there is no "phase shift" between the elliptic function input and output of the nonlinear device, that is, $f_1 \left[x_o sn(u_o, k_o) \right]$ has no phase shift from $x_o sn(u_o, k_o)$. Therefore, using the relationship

$$sn(u + 2K) = -sn(u), \tag{4.28}$$

it is evident $f_2 \left[x_o sn(u_o, k_o) \right]$ must have a "phase shift" of $2 K_i$, from $x_o sn(u_o, k_o)$ if equation 4.15 is to be satisfied. Thus, using equation 4.27, we have

$$2 K_i = \frac{2K_i}{\pi} \phi'(\omega), \tag{4.29}$$

and consequently

$$\phi'(\omega) = \pi. \tag{4.30}$$

Therefore, the frequency of system oscillations is the frequency at which the Nyquist plot of $G'(s)$ crosses the negative real axis. Once the frequency of oscillations is known, $A'(\omega)$ and $A'(3\omega) \cos \left[\phi'(3\omega) - \phi'(\omega) \right]$ can be easily found from the Nyquist plot.

Knowing $A'(\omega)$ and $A'(3\omega) \cos \left[\phi'(3\omega) - \phi'(\omega) \right]$, equation 4.25 may be solved for the k_i, k_o relationship of the linear elements. To calculate this equation for every example would be extremely difficult so it was decided to obtain a plot of the relationship for several ratios of $A'(3\omega) \cos \left[\phi'(3\omega) - \phi'(\omega) \right]$ to $A'(\omega)$. To reduce the number of diagrams required, the relationship of equation 4.25 is plotted on the same Figure as the relationship given by equation B.48 (and given graphically by Figure 4.2). The resulting graph is given in Figure 4.4, page 43 and, to an expanded scale, in Figure 4.5, page 44.

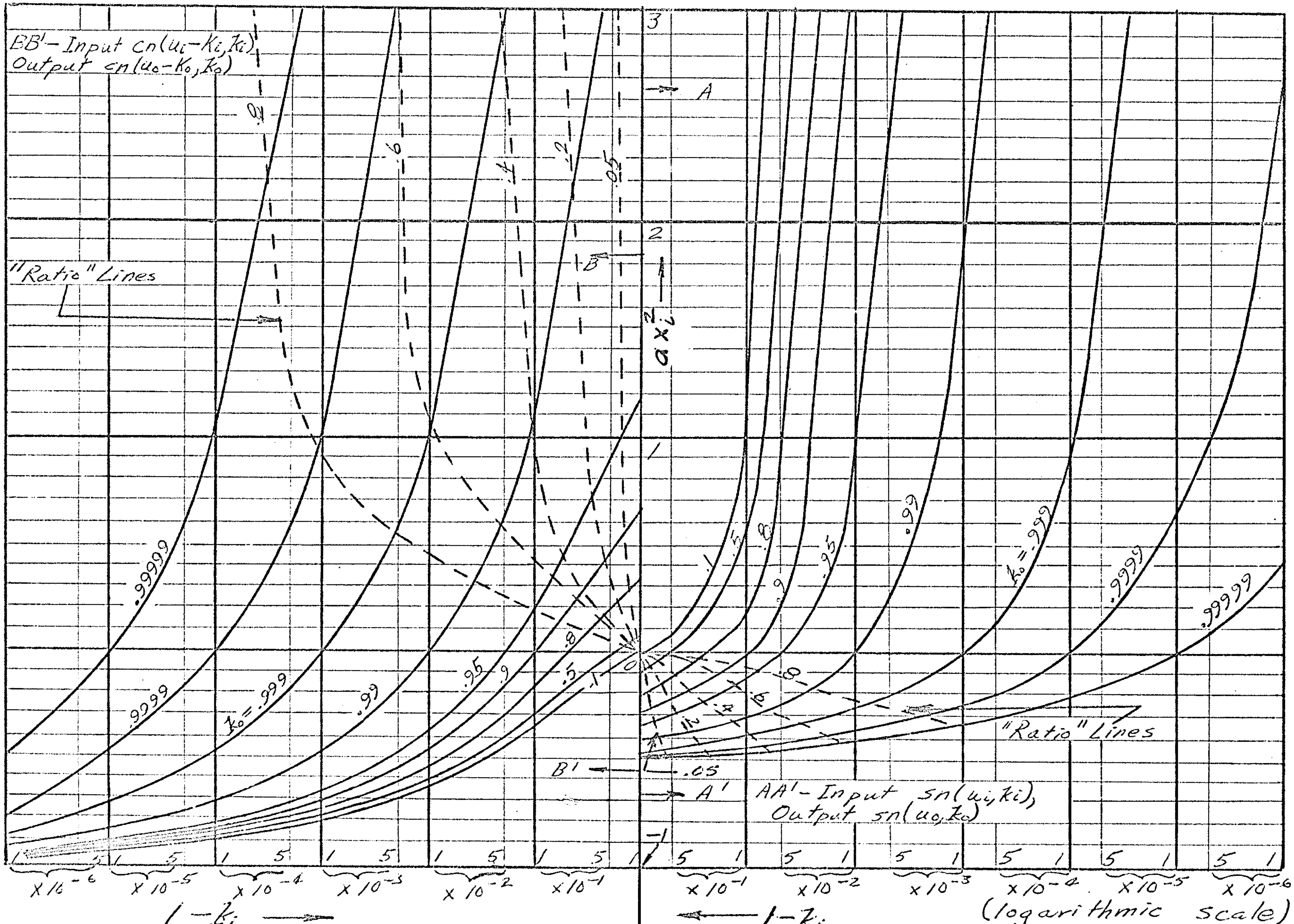

Figure 4.4 Composite Plot of the Elliptic Describing Function Relationship and the $A'(3\omega)\cos[\phi'(3\omega)-\phi'(\omega)]$ to $A'(\omega)$ Relationship

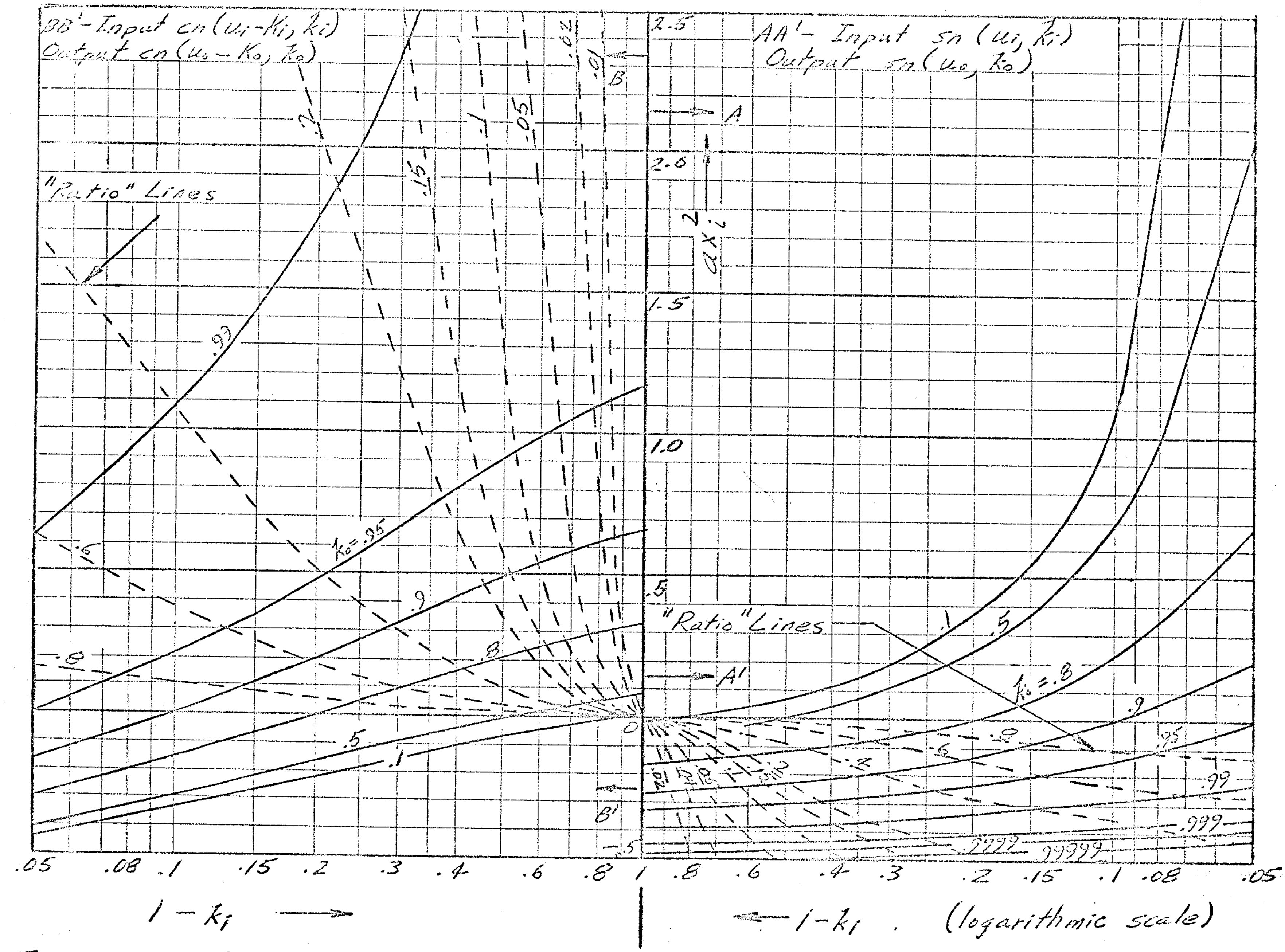

Figure 4.5 Composite Plot of the Elliptic Describing Function Relationship and the $A'(3\omega)\cos[\phi'(3\omega)-\phi'(\omega)]$ to $A'(\omega)$ Relationship (Expanded Scale)

Perhaps, in order to prevent confusion, it would be desirable to examine more closely exactly what Figures 4.4 and 4.5 represent. From a three dimensional plot of the relationship between k_i, k_o, and ax_i^2 given by equation B.48, Figure 4.2 is obtained by plotting several k_i, ax_i^2 curves for constant values of k_o. By plotting the k_i, k_o relationship given by equation 4.25 onto Figure 4.2 (and thus obtaining Figures 4.4 and 4.5), vertical projections from the k_i, k_o plane are being plotted onto the surface representing equation B.48.

Once the ratio $A'(3\omega)\cos\left[\phi'(3\omega) - \phi'(\omega)\right]$ to $A'(\omega)$ has been found, the relationship between k_o and k_i for the system is known to be a point on the "ratio" line of Figures 4.4 and 4.5. The point on this line which corresponds to steady state oscillations can be obtained from the magnitude conditions. Using equation 4.27, the magnitude condition necessary to satisfy equation 4.12 is

$$x_i = (x_i + ax_i^3)b\,\frac{k_iK_i\,A'(\omega)\sinh\left(\frac{\pi K_i'}{2K_i}\right)}{k_oK_o\sinh\left(\frac{\pi K_o'}{2K_o}\right)} \qquad (4.31)$$

Therefore, we have

$$ax_i^2 = \frac{k_oK_o\sinh\left(\frac{\pi K_o'}{2K_o}\right)}{A'(\omega)b\,k_iK_i\sinh\left(\frac{\pi K_i'}{2K_i}\right)} - 1. \qquad (4.32)$$

By plotting this equation for the k_i, k_o locus found from the ratio conditions, an intersection with the surface representing equation B.48 can be obtained. This intersection corresponds to the conditions necessary to satisfy equation 4.15, and thus represents the parameters of the elliptic function approximation of system oscillations.

To facilitate the calculation of equation 4.32, the function $\dfrac{k_o K_o \sinh \left(\frac{\pi K_o'}{2K_o}\right)}{k_i K_i \sinh \left(\frac{\pi K_i'}{2K_i}\right)}$ as a function of k_i is plotted for different values of k_o in Figure 4.6, page 47 and, to an expanded scale, in Figure 4.5, page 48.

As mentioned at the end of the previous section, it is necessary to change the independent variable in the equation describing system performance such that x_i in the approximate solution of the resulting equation is equal to, or only slightly less than, one. For the standard feedback system represented by the block diagram of Figure 1.1, the following relation may be written (for self-excited systems):

$$\frac{\mathcal{L}\{c\}}{\mathcal{L}\{f(-c)\}} = G(s), \tag{4.33}$$

The necessary change in variable is most easily made by setting $c = c'n$ (where n is a constant) in equation 4.33 to obtain

$$\frac{\mathcal{L}\{c'\}}{\mathcal{L}\{f(-c'n)\}} = \frac{G(s)}{n} = G'(s). \tag{4.34}$$

Thus, the change of variable is made by dividing the gain of the linear elements by the normalizing factor n and setting $c = c'n$ in the nonlinear relationship.

However, the value of n required to make x_i of the required magnitude is unknown. Fortunately, a suitable value is usually arrived at by noting the gain of the linear elements and the form of the nonlinearity and in any case the equation need not be completely solved to determine if a satisfactory n has been chosen since if $x_i \leqslant 1$, then $ax_i^2 \leqslant a$, and this condition can be checked when equation 4.32 is being plotted.

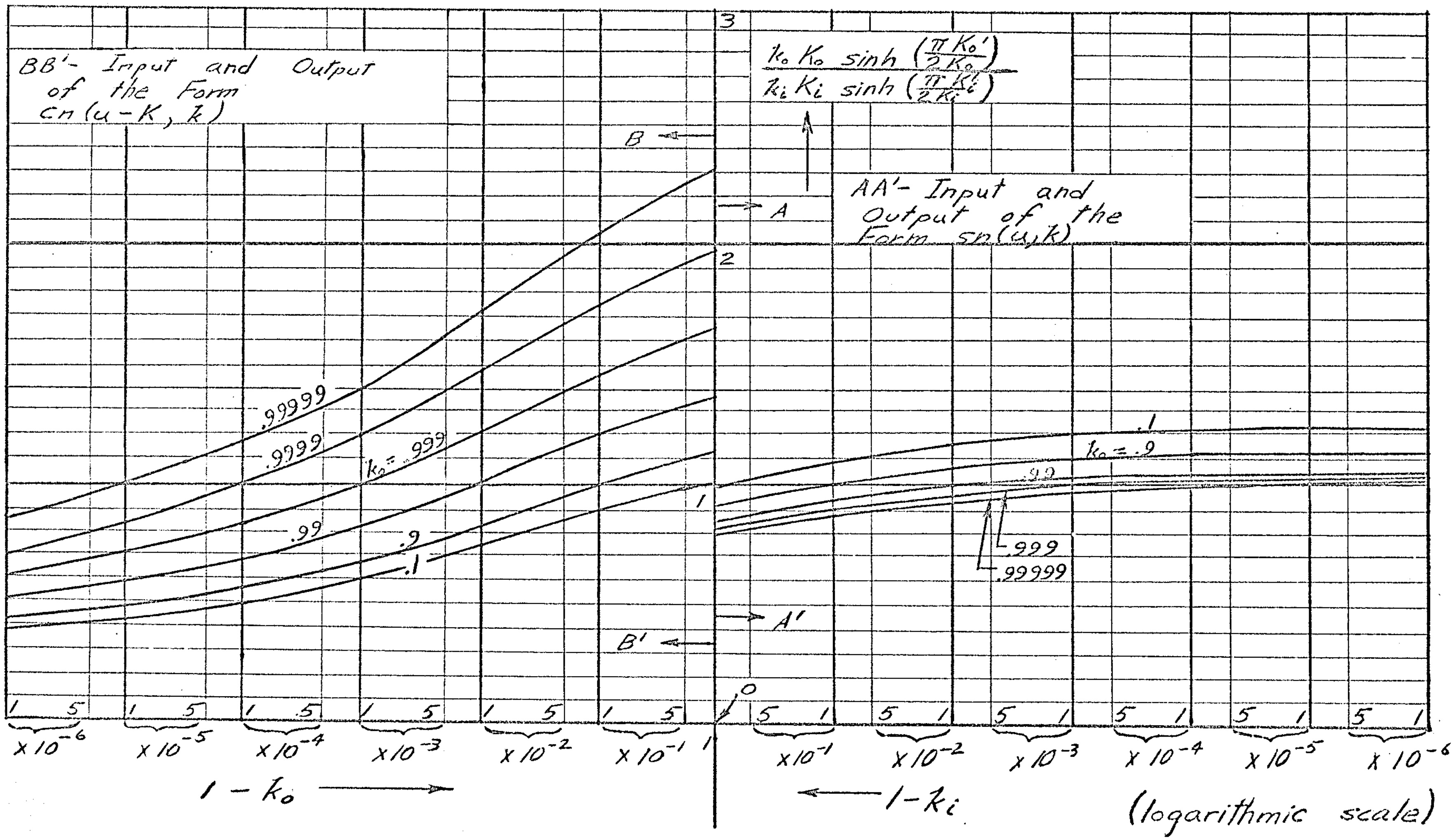

Figure 4.6 The Relationship Between k_i, k_o and $\dfrac{k_o K_o \sinh\left(\frac{\pi K_o'}{2K_o}\right)}{k_i K_i \sinh\left(\frac{\pi K_i'}{2K_i}\right)}$

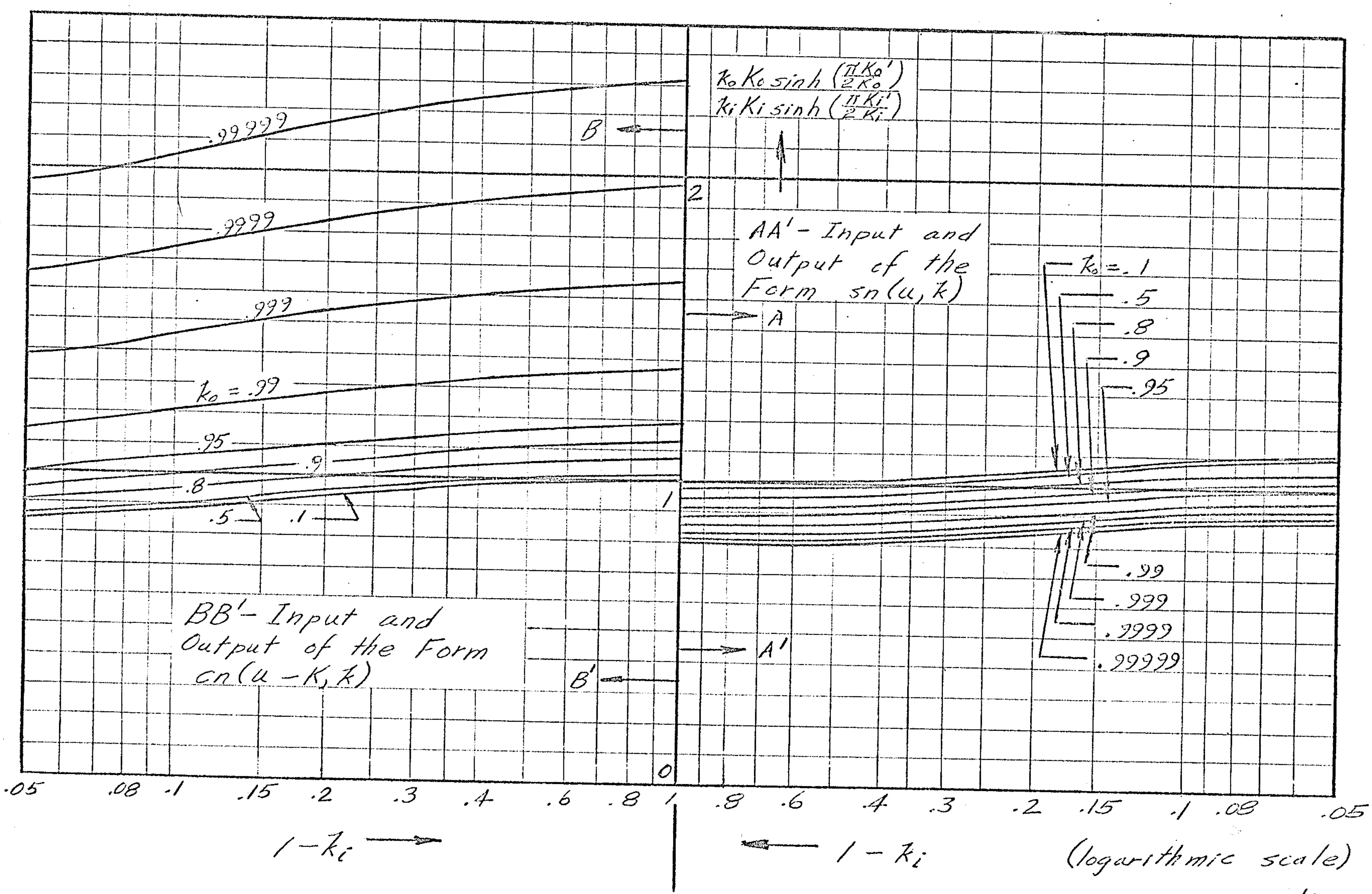

Figure 4.7 The Relationship Between k_i, k_o and $\frac{k_o K_o \sinh\left(\frac{\pi K_o'}{2K_o}\right)}{k_i K_i \sinh\left(\frac{\pi K_i'}{2K_i}\right)}$

(Expanded Scale)

Since knowing the gain of the linear elements to the first harmonic of the system oscillations may be very useful in selecting a normalizing factor, it is usually preferable to calculate the Nyquist plot of the unnormalized system, and then normalize the values from this plot. The $A(3\omega) \cos[\phi(3\omega) - \phi(\omega)]$ to $A(\omega)$ ratio and the phase of the Nyquist plot are unchanged by normalization, and the normalized gain of the first harmonic becomes

$$A'(\omega) = \frac{A(\omega)}{n} . \tag{4.35}$$

The procedure for applying the elliptic describing function to system analysis may be summarized as follows:

(1) The approximate frequency of oscillations is the frequency at which the Nyquist plot of the transfer function of the linear elements crosses the negative real axis.

(2) The ratio $A(3\omega) \cos[\phi(3\omega) - \phi(\omega)]$ to $A(\omega)$ is found from the transfer function of the linear elements, and the corresponding "ratio line" is located in Figure 4.4 (or 4.5).

(3) Approximate the nonlinear relationship by a cubic polynomial (noting the necessary change of variable), and with the aid of Figure 4.6 (or 4.7) plot ax_i^2 versus k_i from equation 4.32 onto Figure4.4 (or 4.5) for the values of k_i, k_o on the ratio line found in step 2.

(4) Find ax_i^2 and k_i from the intersection of the line plotted in step 3 and the surface representing equation B.45, that is, the ratio line found in step 2.

(5) Now knowing k_i, ax_i^2, and a, the elliptic function approximation of the system output can be determined.

Some examples of the application of the elliptic describing function are given in appendix D.

4.4 CONCLUDING COMMENTS

The examples which have been studied (appendix D) seem to indicate that a Legendre polynomial approximation of the nonlinearity gives more accurate results than a Chebyshev approximation, but more examples would have to be studied before a definite conclusion could be reached.

While it is difficult to state precisely on what range the amplitude of the normalized system output must be to give accurate results, the examples studied indicate that accurate results are obtained when x_i is in the range $.6 < x_i < 1$.

Further calculations are necessary to obtain the elliptic describing function relationship for values of the modulus greater than .99999 and for systems in which the phase shift between the first and third harmonic is in the range $\pi/2 \leq \phi(3\omega) - \phi(\omega) \leq -\pi/2$.

The accuracy of the method could be improved if the figures were expanded.

Despite these shortcomings, it has been found that the elliptic describing function, as developed here, gives more accurate results than the conventional describing function. Before the usefulness and the accuracy of the method can be determined, it would be necessary to apply the method to a wider range of examples.

CHAPTER 5

FINAL THOUGHTS

A short résumé of the topics studied and the conclusions reached will be given in this chapter.

5.1 THE CONVENTIONAL DESCRIBING FUNCTION METHOD AND THE SINUSOIDAL TECHNIQUES OF KRYLOFF AND BOGOLIUBOFF

The conventional describing function method was developed in such a manner as to clearly indicate under what conditions the method can be used to determine the characteristics of the transient behaviour of a systems.

The first approximation method of Kryloff and Bogoliuboff was developed, and the results of this method were used to obtain the method of equivalent linearization.

It was found that the conventional describing function and equivalent linearization give exactly the same results for frequency-independent nonlinearities, and, in general, the conventional describing function is more accurate for systems with frequency-dependent nonlinearities.

5.2 THE ELLIPTIC DESCRIBING FUNCTION

A new describing function, the elliptic describing function, was developed. This describing function provides a more accurate representation of the input-output relationship of a nonlinear device than is possible with a sinusoidal approximation.

The main objection to the elliptic describing function, as with the conventional describing function, is that the method gives no sure criteria regarding the existence of sustained oscillations.

Another weakness of the method is the need to change the independent variable in the system equations. Although this has not presented any difficulties in the examples studied, it is possible that someone who was unfamilar with the method might have difficulty in making a satisfactory change in variable in the system equations. In the examples studied, accurate results were obtained when the amplitude of the oscillations in the normalized system was in the range $.6 < x_i < 1$.

For all the examples studied, it was found that the elliptic describing function gave more accurate results than the conventional describing function.

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APPENDIX A

PHYSICAL SIGNIFICANCE OF THE CONVENTIONAL DESCRIBING FUNCTION AND EQUIVALENT LINEARIZATION

Providing the output of the nonlinear device is assumed to represent a force and the system output, say x, a displacement, the conventional describing function and equivalent linearization can be given the following physical interpretations:

(1) The nonlinear force is replaced by an equivalent linear force $Kx + \lambda\dot{x}$, where K and λ are chosen such that the equivalent force produces the same active and reactive work per cycle as the nonlinear force.

(2) The nonlinear force is replaced by an equivalent linear force $Kx + \lambda\dot{x}$ such that the equivalent force is equal to the first harmonic of the nonlinear force.

A.1 PRINCIPLE OF EQUIVALENT BALANCE OF ENERGY

Before a physical significance based on energy relationships can be developed, it is necessary to define a reactive power term for mechanical systems similar to the term "wattless" or reactive power commonly used in alternating current theory.

From alternating current theory, the real (or active) component of power P_a and its "wattless" (or reactive) component P_r are defined as follows:

$$P_a = \frac{1}{T}\int_0^T e\, i \cos \theta \, dt, \qquad (A.1)$$

$$P_r = \frac{1}{T}\int_0^T e\ i\ \sin\Theta\ dt, \qquad (A.2)$$

where e, i, Θ and T are voltage, current, phase angle and period of the function respectively.

If the force is F (t), the active component of power in a mechanical system is

$$P_a = \frac{1}{T}\int_0^T F\ (t)\ \dot{x}\ (t)\ dt, \qquad (A.3)$$

and by analogy with equations A.1 and A.2 the reactive component of power in a mechanical system will be defined as

$$P_r = \frac{1}{T}\int_0^T F\ (t)\ \dot{x}\ (t - T/4)\ dt. \qquad (A.4)$$

The active and reactive work per cycle is obtained by multiplying these expressions by T.

If the nonlinear force is $\mu f\ (x, \dot{x}, ---, x^{(n)})$, the active and reactive work per cycle of the nonlinear force is given by the following equations:

Active work of nonlinear force $\equiv W_a\ (NL) = \int_0^T \mu f(x, \dot{x}, ---, x^{(n)})\dot{x}\ dt,$ (A.5)

Reactive work of nonlinear force $\equiv W_r\ (NL)$

$$= \int_0^T \mu f(x, \dot{x}, ---, x^{(n)})\ \dot{x}\ (t - T/4)\ dt. \qquad (A.6)$$

The active and reactive work per cycle of the equivalent linear force is given by the following expressions:

Active work of equivalent linear force $\equiv W_a\ (L)$

$$= K\int_0^T \dot{x}\ x\ dt + \lambda\int_0^T \dot{x}^2\ dt, \qquad (A.7)$$

Reactive work of equivalent linear force $\equiv W_r\ (L)$

$$= K\int_0^T \dot{x}\ (t - T/4)\ x\ dt + \lambda\int_0^T \dot{x}\ (t - T/4)\ \dot{x}\ dt. \qquad (A.8)$$

A.1a The Conventional Describing Function

Let the generating solution for the nonlinear differential equation be

$$x = a \sin \omega t, \qquad (A.9)$$

where a and ω are constants to be determined. All derivatives of the solution are assumed to be exact derivatives of equation A.9.

Substituting the generating solution into equations A.5 and A.7, we have

$$W_a\ (NL) = \int_0^{2\pi} \mu f\left[a \sin \omega t,\ a\omega \cos \omega t, \text{-----}, (a \sin \omega t)^{(n)}\right]$$

$$a \cos \omega t\ d(\omega t), \qquad (A.10)$$

$$W_a\ (L) = K\int_0^{2\pi} a^2 \cos\ \omega t \sin\ \omega t\ d(\omega t) + \lambda\int_0^{2\pi} a^2 \omega \cos^2 \omega t\ d(\omega t)$$

$$= \lambda a^2 \omega \pi. \qquad (A.11)$$

Equating the expressions for the active work of the nonlinear force and the equivalent linear force, and solving for λ, we have

$$\lambda = \frac{\mu}{\omega a \pi}\int_0^{2\pi} f\left[a \sin \omega t,\ a\omega \cos\ \omega t, \text{---}, (a \sin \omega t)^{(n)}\right]$$

$$\cos \omega t\ d(\omega t). \qquad (A.12)$$

Similarly, the parameter K may be found by equating the reactive work per cycle of the nonlinear and equivalent linear forces (equations A.6 and A.8) for the generating solution given by equation A.9. Whence, the parameter K is given by the following equation:

$$K = \frac{\mu}{\pi a}\int_0^{2\pi} f\left[a \sin \omega t,\ a\omega \cos \omega t, \text{----}, (a \sin \omega t)^{(n)}\right] \sin \omega t\ d(\omega t). \qquad (A.13)$$

Thus, with the generating solution of equation A.9, the nonlinear force is replaced by the equivalent linear force

$$\left[\frac{\mu}{\pi}\int_0^{2\pi} f\left[a \sin \omega t,\ a\omega \cos \omega t, ---, (a \sin \omega t)^{(n)}\right] \sin \omega t \ d(\omega t)\right] \sin \omega t$$
$$+ \left[\frac{\mu}{\pi}\int_0^{2\pi} f\left[a \sin \omega t,\ a\omega \cos \omega t, ---, (a \sin \omega t)^{(n)}\right] \cos \omega t \ d(\omega t)\right] \cos \omega t.$$

This is precisely the equivalent force that is obtained by applying the conventional describing function method to a system with a non-linear relationship

$\mu f\,(x, \dot{x}, ---, x^{(n)})$.

A.1b Equivalent Linearization

The same principle may be applied to obtain a physical interpretation for equivalent linearization.

The generating solution is of the following form:

$$x = a \sin (\omega t + \phi), \qquad (A.14)$$

$$\dot{x} = a\,\omega \cos (\omega t + \phi), \qquad (A.15)$$

where ω is the frequency of the solution for $\mu = 0$, and a and ϕ are functions of time to be determined.

Substituting the generating solution into equations A.5 and A.7, we have

$$W_a\,(NL) = \mu \int_0^{2\pi} f\left[a \sin (\omega t + \phi),\ a\,\omega \cos (\omega t + \phi)\right] a \cos (\omega t + \phi)$$
$$d\,(\omega t), \qquad (A.16)$$

$$W_a(L) = K \int_0^{2\pi} a^2 \sin (\omega t + \phi) \cos (\omega t + \phi)\ d\,(\omega t)$$
$$+ \lambda \int_0^{2\pi} a^2 \omega \cos^2 (\omega t + \phi)\ d\,(\omega t). \qquad (A.17)$$

Since ϕ is assumed to vary slowly over one period, we have

$$d\,(\omega t) \simeq d\,(\omega t + \phi). \tag{A.18}$$

Using the approximation of equation A.18, and for simplicity writing $\theta = \omega t + \phi$, equations A.16 and A.17 become

$$W_a(NL) = \mu \int_0^{2\pi} f(a \sin \theta,\ a\omega \cos \theta)\ a \sin \theta\ d\theta, \tag{A.19}$$

$$W_a(L) = K \int_0^{2\pi} a^2 \sin \theta \cos \theta\ d\,\theta + \lambda \int_0^{2} a^2 \omega \cos^2 \theta\ d\theta. \tag{A.20}$$

If we assume a to be approximately constant over one period, equations A.19 and A.20 become

$$W_a(NL) = \mu a \int_0^{2\pi} f(a \sin \theta,\ a\omega \cos \theta) \sin \theta\ d\theta, \tag{A.21}$$

$$W_a(L) = \lambda a^2 \omega \pi. \tag{A.22}$$

Setting the active work per cycle of the nonlinear force and the equivalent linear force equal (approximately), and solving for λ, we get

$$\lambda = \frac{\mu}{a\pi\omega} \int_0^{2\pi} f(a \sin \theta,\ a\omega \cos \theta) \cos \theta\ d\theta. \tag{A.23}$$

The same approximations may be applied to the reactive work per cycle to obtain the following equation for the parameter K:

$$K = \frac{\mu}{a\pi} \int_0^{2\pi} f(a \sin \theta,\ a\omega \cos \theta) \sin \theta\ d\theta. \tag{A.24}$$

The equivalent linear parameters given by equations A.23 and A.24 are the same as those obtained using the method of equivalent linearization.

A.2 PRINCIPLE OF HARMONIC BALANCE

Another physical interpretation may be obtained if the nonlinear force is replaced by a linear force which is equal to the first harmonic of the nonlinear force.

A.2a The Conventional Describing Function

Again the generating solution is of the form

$$x = a \sin \omega t, \tag{A.25}$$

where a and ω are constants to be determined. All derivatives of the solution are assumed to be exact derivatives of equation A.25.

The equivalent linear term which is equal to the first harmonic of the non-linear force, is

$$\left[\frac{\mu}{\pi}\int_0^{2\pi} f\left[a \sin \omega t,\ a\omega\cos\omega t, ---, (a \sin\omega t)^{(n)}\right]\sin\omega t\ d(\omega t)\right]\sin\omega t$$
$$+\left[\frac{\mu}{\pi}\int_0^{2\pi} f\left[a \sin\omega t,\ a\omega\cos\omega t, ---, (a \sin\omega t)^{(n)}\right]\cos\omega t\ d(\omega t)\right]\cos\omega t.$$

This is precisely the equivalent linear term which would be obtained if the conventional describing function method were used. In fact, the procedure outlined here follows directly from the definition of the conventional describing function.

A.2b Equivalent Linearization

Again the generating solution is of the following form

$$x = a \sin (\omega t + \phi), \tag{A.26}$$

$$\dot{x} = a\,\omega \cos (\omega t + \phi), \tag{A.27}$$

where ω is the frequency of the solution for $\mu = 0$, and a and ϕ are slowly varying functions of time.

Since a and ϕ are assumed to be slowly varying functions of time, the nonlinear force may be expanded as a Fourier series by considering a and ϕ as remaining approximately constant during one period. The first harmonic F of the resulting Fourier series is

$$F = \left[\frac{\mu}{\pi}\int_0^{2\pi} f\,(a \sin \theta,\; a\,\omega \cos \theta)\, \sin \theta \; d\theta\right] \sin\,(\omega t + \phi)$$

$$+ \left[\frac{\mu}{\pi}\int_0^{2\pi} f\,(a \sin \theta,\; a\,\omega \cos \theta)\, \cos \theta \; d\theta\right] \cos\,(\omega t + \phi). \qquad (A.28)$$

The equivalent linear term F_L for the generating solution of equations A.26 and A.27 is

$$F_L = K\, a \sin\,(\omega t + \phi) + \lambda\, a\,\omega \cos\,(\omega t + \phi). \qquad (A.29)$$

By equating the coefficients of $\cos\,(\omega t + \phi)$ and $\sin\,(\omega t + \phi)$ in equations A.28 and A.29, we obtain the following parameters from the principle of harmonic balance:

$$\lambda = \frac{\mu}{a\pi\omega}\int_0^{2\pi} f(a \sin \theta,\; a\,\omega \cos \theta)\, \cos \theta \; d\theta, \qquad (A.30)$$

$$K = \frac{\mu}{a\pi}\int_0^{2\pi} f\,(a \sin \theta,\; a\,\omega \cos \theta)\, \sin \theta \; d\theta. \qquad (A.31)$$

These are precisely the equivalent parameters obtained from the method of equivalent linearization and the principle of equivalent energy balance.[*]

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[*] It should be noted that the last paragraph on page 238 of Minorsky (11) is incorrect. A correct wording would be as follows: It is seen that both principles, that of Equivalent Balance of Energy and that of Harmonic Balance, are equivalent, because the work of the higher harmonics per cycle of the fundamental frequency has been neglected in the equivalent balance of energy concept. Obviously Minorsky's statement, "the work of the higher harmonics per cycle of the fundamental frequency is zero," is incorrect.

APPENDIX B

CALCULATION OF THE ELLIPTIC DESCRIBING FUNCTION

B.1 FIRST DERIVATIVE OF THE ELLIPTIC FUNCTIONS WITH RESPECT TO THEIR MODULUS

The first elliptic integral is defined by the expression

$$F(\emptyset, k) = u = \int_0^{\emptyset} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}, \quad k^2 < 1. \tag{B.1}$$

If we consider u as being constant, and take the derivative of both sides of equation B.1, we obtain the following equation:

$$\int_0^{\emptyset} \frac{k \sin^2 \theta \, d\theta}{(\sqrt{1-k^2 \sin^2\theta})^3} + \frac{1}{\sqrt{1-k^2\sin^2\emptyset}} \frac{d\emptyset}{dk} = 0. \tag{B.2}$$

After performing the integration in equation B.2 (17,p.46), we have

$$k \left[\frac{F-D}{(k')^2} - \frac{\sin \emptyset \cos \emptyset}{(k')^2\sqrt{1-k^2\sin^2\emptyset}} \right] = \frac{-1}{\sqrt{1 - k^2\sin^2\emptyset}} \frac{d\emptyset}{dk}. \tag{B.3}$$

Solving for $d\emptyset/dk$, and observing the relations (17,p.43)

$$D = \frac{F-E}{k^2}, \tag{B.4}$$

$$(k')^2 = 1-k^2, \tag{B.5}$$

we have

$$\frac{d\emptyset}{dk} = -k\sqrt{1-k^2\sin^2\emptyset}\left[\frac{E-(k')^2F}{(k')^2k^2} - \frac{\sin \emptyset \cos \emptyset}{(k')^2\sqrt{1-k^2\sin^2\emptyset}}\right]. \tag{B.6}$$

Using equation B.6, we can write $\frac{d(\sin\emptyset)}{dk}$ in the following form:

$$\frac{d(\sin\emptyset)}{dk} = -k \cos \emptyset\sqrt{1-k^2\sin^2\emptyset}\left[\frac{E-(k')^2F}{(k')^2 k^2} - \frac{\sin\emptyset \cos\emptyset}{(k')^2\sqrt{1-k^2\sin^2\emptyset}}\right]. \tag{B.7}$$

From the definitions of the elliptic functions, we have

$$\mathrm{sn}\,(u,k) = \sin\phi,$$

$$\mathrm{cn}\,(u,k) = \cos\phi,$$

$$\mathrm{dn}\,(u,k) = \sqrt{1 - k^2\sin^2\phi}; \qquad \text{(B.8)}$$

and thus equation B.7 may be written as follows:

$$\frac{d[\mathrm{sn}\,(u,k)]}{dk} = -k\,\mathrm{cn}\,(u,k)\,\mathrm{dn}(u,k)\left[\frac{E-(k')^2F}{(k')^2k^2} - \frac{\mathrm{sn}(u,k)\mathrm{cn}(u,k)}{(k')^2\mathrm{dn}(u,k)}\right] \qquad \text{(B.9)}$$

Similarly, the derivatives of cn(u,k) and dn(u,k) with respect to k are given by the following equations:

$$\frac{d[\mathrm{cn}(u,k)]}{dk} = k\,\mathrm{sn}(u,k)\,\mathrm{dn}(u,k)\left[\frac{E-(k')^2F}{(k')^2(k)^2} - \frac{\mathrm{sn}(u,k)\,\mathrm{cn}(u,k)}{(k')^2\,\mathrm{dn}(u,k)}\right], \qquad \text{(B.10)}$$

$$\frac{d[\mathrm{dn}(u,k)]}{dk} = \frac{-k\,\mathrm{sn}^2(u,k)}{\mathrm{dn}\,(u,k)} + k^3\,\mathrm{cn}(u,k)\,\mathrm{sn}(u,k)\left[\frac{E-(k')^2F}{(k')^2(k)^2} - \frac{\mathrm{sn}(u,k)\,\mathrm{cn}(u,k)}{(k')^2\mathrm{dn}(u,k)}\right] \qquad \text{(B.11)}$$

B.2 APPROXIMATION OF NONLINEAR INPUT-OUTPUT RELATIONSHIPS WITH ELLIPTIC FUNCTIONS

The necessary calculations to obtain the elliptic function approximation are described in detail for an input

$$e = x_i\,\mathrm{sn}(u_i,k_i)$$

and an assumed form of output

$$m = x_o\,\mathrm{sn}(u_o,k_o)$$

(this form of output will be used as the approximation only if it gives less mean square error than the $x_o\mathrm{cn}(u_o-K_o,k_o)$ form of output). The calculations for other combinations of input and output functions are similar, and therefore only the results of the calculations will be stated.

The instantaneous error between the actual function and the Jacobian elliptic function approximant is given by the following equation:

$$\epsilon = f\left[x_i\ sn(u_i,k_i)\right] - x_o\ sn(u_o,k_o). \tag{B.12}$$

It is desired to minimize the mean square value of this expression over one period of u_o, but because of the relations $sn(u,k) = sn(2K-u,k) = -sn(u+2K,k) = -sn(4K-u,k)$, the same value of k_o will be obtained if we minimize the mean square error over one quarter period of the function.

To find the value of k_o which minimizes the mean square error, we must solve the following expression for k_o:

$$\frac{d}{dk_o}\int_0^{K_o}\left\{f\left[x_i sn(u_i,k_i)\right] - x_o sn(u_o,k_o)\right\}^2 du_o = 0. \tag{B.13}$$

Interchanging differentiation and integration, we have

$$\int_0^{K_o}\frac{d}{dk_o}\left\{x_o^2 sn^2(u_o,k_o)\right\} du_o$$

$$= 2\int_0^{K_o}\frac{d}{dk_o}\left[f\left[x_i sn(u_i,k_i)\right] x_o sn(u_o,k_o)\right] du_o. \tag{B.14}$$

Using equation B.9 to perform the differentiation in equation B.14, we have

$$\frac{2x_o^2 A}{k_o} - \frac{2x_o^2 B}{k_o(k_o')^2} + \frac{2x_o^2 k_o C}{(k_o')^2}$$

$$= \frac{2x_o D}{k_o} - \frac{2x_o E}{k_o(k_o')^2} + \frac{2x_o k_o F}{(k_o')^2}, \tag{B.15}$$

where

$$A = \int_0^{K_o} u_o\ sn(u_o,k_o)cn(u_o,k_o)dn(u_o,k_o)\ du_o, \tag{B.16}$$

$$B = \int_0^{K_o} E(u_o)sn(u_o,k_o)cn(u_o,k_o)dn(u_o,k_o)\ du_o, \tag{B.17}$$

$$C = \int_0^{K_o} cn^2(u_o,k_o)sn^2(u_o,k_o)\ du_o, \tag{B.18}$$

$$D = \int_0^{K_o} u_o cn(u_o,k_o)dn(u_o,k_o)\ f\left[x_i sn(u_i,k_i)\right] du_o, \qquad (B.19)$$

$$E = \int_0^{K_o} E(u_o)cn(u_o,k_o)dn(u_o,k_o)\ f\left[x_i sn(u_i,k_i)\right] du_o, \qquad (B.20)$$

$$F = \int_0^{K_o} sn(u_o,k_o)cn^2(u_o,k_o)\ f\left[x_i sn(u_i,k_i)\right] du_o. \qquad (B.21)$$

The integral in equation, B.16 can be integrated by parts if we let

$$u = u_o, \qquad (B.22)$$

$$dv = sn(u_o,k_o)cn(u_o,k_o)\ dn(u_o,k_o), \qquad (B.23)$$

whence

$$du = du_o, \qquad (B.24)$$

$$v = \int sn(u_o,k_o)cn(u_o,k_o)\ dn(u_o,k_o).du_o. \qquad (B.25)$$

If we substitute the equation

$$\frac{d}{du_o}\left[dn(u_o,k_o)\right] = -k_o^2\ sn(u_o,k_o)cn(u_o,k_o) \qquad (B.26)$$

into the integral of equation B.25, we have the following equation:

$$\begin{aligned} v &= \frac{-1}{k_o^2}\int dn(u_o,k_o)\ \frac{d}{du_o}\left[dn(u_o,k_o)\right] du_o \\ &= \frac{-1}{k_o^2}\int dn(u_o,k_o)\ d\left[dn(u_o,k_o)\right] \\ &= \frac{-1}{2k_o^2}\ dn^2(u_o,k_o) \qquad (B.27) \end{aligned}$$

Now performing the integration by parts, we have

$$\begin{aligned} A &= \int udv = uv - \int v\ du \\ &= \frac{-u_o}{2k_o^2}\ dn^2(u_o,k_o)\Big|_0^{K_o} + \frac{1}{2k_o^2}\int_0^{K_o} dn^2(u_o,k_o)du_o. \qquad (B.28) \end{aligned}$$

Using the equation (17,p.78)

$$\int dn^2(u_o,k_o)du_o = E(u_o,k_o), \qquad (B.29)$$

equation B.28 becomes

$$A = \frac{1}{2k_o^2}\left[E(u_o,k_o) - u_o dn^2(u_o,k_o)\right]\Big|_o^{K_o}$$

$$= \frac{1}{2k_o^2}\left[E(K_o,k_o) - K_o(k_o')^2\right]. \qquad (B.30)$$

The integral in equation B.17 can also be integrated by parts if we let

$$u = E(u_o), \qquad (B.31)$$

$$dv = sn(u_o,k_o)cn(u_o k_o)dn(u_o,k_o)\ du_o, \qquad (B.32)$$

whence

$$du = d\,E(u_o), \qquad (B.33)$$

$$v = \int sn(u_o,k_o)cn(u_o,k_o)dn(u_o,k_o)\ du_o. \qquad (B.34)$$

By using the identity (17,p.78)

$$E(u) = \int_o^u dn^2 u\ du, \qquad (B.35)$$

equation B.33 may be written in the form

$$du = dn^2(u_o,k_o)du_o. \qquad (B.36)$$

As found previously (equation B.27), the evaluation of the integral in equation B.34 yields the following result:

$$v = -\frac{1}{2k_o^2}\ dn^2(u_o,k_o). \qquad (B.37)$$

Using equations B.36 and B.37 to perform the integration by parts, we have

$$B = \int_o^{K_o} udv = uv - \int vdu$$

$$= \frac{-E(u_o)}{2k_o^2}\ dn^2(u_o,k_o)\Big|_o^{K_o} + \frac{1}{2k_o^2}\int_o^{K_o} dn^4(u_o,k_o)\ du_o. \qquad (B.38)$$

If we substitute the identities

$$dn^2(u,k) + k^2\ sn^2(u,k) = 1$$

and $sn^2(u,k) + cn^2(u,k) = 1$

into equation B.38, we obtain

$$B = \frac{-E(u_o)}{2k_o^2} \; dn^2(u_o,k_o) \Big|_o^{K_o} + \frac{1}{2k_o^2}\int_o^{K_o} (1-k_o^2)dn^2(u_o,k_o)du_o$$

$$+ \frac{1}{2}\int_o^{K_o} dn^2(u_o,k_o)cn^2(u_o,k_o) \; du_o . \qquad (B.39)$$

Evaluating the integrals (18,pp. 17-18) in equation B.39, we get

$$B = \frac{-E(u_o)}{2k_o^2} \; dn^2(u_o,k_o) + \frac{(1-k_o)^2}{2k_o^2} \; E(u_o,k_o)$$

$$+ \frac{(1+k_o^2)E(u_o,k_o)}{6k_o^2} - \frac{(1-k_o^2)u_o}{6k_o^2}$$

$$+ \frac{k_o^2 sn(u_o,k_o) \; cn(u_o,k_o) \; dn(u_o,k_o)}{6} \Bigg|_o^{K_o}$$

$$= -(k_o')^2\frac{E(K_o,k_o)}{2k_o^2} + \frac{(1-k_o)^2}{2k_o^2} \; E(K_o,k_o)$$

$$+ \frac{(1+k_o^2) \; E(K_o,k_o)}{6k_o^2} - \frac{(1-k_o^2)K_o}{6k_o^2} . \qquad (B.40)$$

Using the identity

$$(k')^2 = 1-k^2,$$

equation B.40 can be reduced to the following expression

$$B = \frac{1}{6k_o^2}\left[(1+ k_o^2) \; E(K_o,k_o)-(k_o')^2K_o\right]. \qquad (B.41)$$

Using the identity

$$sn^2(u,k) + cn^2(u,k) = 1,$$

equation B.18 may be written in the following form:

$$C = \int_o^{K_o} \left[sn^2(u_o,k_o) - sn^4(u_o,k_o)\right]du_o . \qquad (B.42)$$

After evaluating the integrals (18,pp.17-18), we have

$$C = \frac{u_o - E(u_o,k_o)}{k_o^2} \; \frac{-(2 + k_o^2)u_o}{3k_o^4} + \frac{2(1+k_o^2)E(u_o,k_o)}{3k_o^4}$$

$$\left. \frac{-k_o^2 sn(u_o,k_o)cn(u_o,k_o)dn(u_o,k_o)}{3k_o^4} \right|_o^{K_o}$$

$$= \frac{K_o - E(K_o,k_o)}{k_o^2} - \frac{(2+k_o^2)K_o}{3k_o^4}$$

$$+ \frac{2(1 + k_o^2)E(K_o,k_o)}{3k_o^4} . \tag{B.43}$$

Using the results of equations B.30, B.41, and B.43, equation B.15 can be expressed as follows:

$$\frac{2D}{k_o} - \frac{2E}{k_o(k_o')^2} + \frac{2k_o}{(k_o')^2} F = x_o G, \tag{B.44}$$

where D, E, and F are given by equations B.19, B.20, and B.21 respectively, and G is given by the following expression:

$$G = \frac{1}{k_o^3(k_o')^2 3} \left\{ K_o \left[4k_o^2 - 4 + (k_o')^2 + 3\,(k_o')^4 \right] \right.$$

$$\left. + 3E(K_o,k_o) \left[k_o^2 - 1 + (k_o')^2 \right] \right\} . \tag{B.45}$$

If the nonlinear relationship is

$$f(e) = e + ae^3, \tag{B.46}$$

then from equation 4.7, we have

$$x_o = x_i + ax_i^3 . \tag{B.47}$$

Using these relations and the expressions given by equations B.19, B.20, and B.21, the following expression for ax_i^2 can be obtained from equation B.44:

$$ax_i^2 = \frac{\left\{\begin{aligned}&\frac{2}{k_o}\int_0^{K_o}\left[u_o-\frac{E(u_o)}{(k_o')^2}\right]cn(u_o,k_o)dn(u_o,k_o)sn(u_i,k_i)du_o\\&+\frac{2k_o}{(k_o')^2}\int_0^{K_o} sn(u_o,k_o)cn^2(u_o,k_o)sn(u_i,k_i)du_o - G\end{aligned}\right\}}{\left\{\begin{aligned}&G - \frac{2}{k_o}\int_0^{K_o}\left[u_o-\frac{E(u_o)}{(k_o')^2}\right]cn(u_o,k_o)dn(u_o,k_o)sn^3(u_i,k_i)\,du_o\\&+\frac{2k_o}{(k_o')^2}\int_0^{K_o} sn(u_o,k_o)cn^2(u_o,k_o)sn^3(u_i,k_i)\,du_o\end{aligned}\right\}}, \quad (B.48)$$

where G is given by equation B.45, and the relationship between u_o and u_i is given by equation 4.8.

Similar calculations may be performed for the other combinations of input and output functions. The resulting equations are given in table B.1, pages 71 and 72.

It was not possible to reduce equation B.48 further, and thus the integrals in this expression were evaluated by numerical integration.

The elliptic functions were calculated using Landen's scale of increasing amplitudes.*

All numerical calculations were done on a Bendix G-15D digital computer.

A plot of equation B.48 (and the equation for an input $x_i cn(u_i-K_i,k_i)$ and an output $x_o cn(u_o-K_o,k_o)$ is given in figure 4.2.

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* See Appendix C

TABLE B.1

EQUATIONS FOR THE CALCULATION OF THE ELLIPTIC DESCRIBING FUNCTION

Input	Output	Equations for ax_i^2
$x_i sn(u_i,k_i)$	$x_o sn(u_o,k_o)$	$ax_i^2 = \dfrac{\left\{ \dfrac{2}{k_o}\displaystyle\int_0^{K_o}\left[u_o-\dfrac{E(u_o)}{(k_o')^2}\right]cn(u_o,k_o)dn(u_o,k_o)sn(u_i,k_i)du_o + \dfrac{2k_o}{(k_o')^2}\displaystyle\int_0^{K_o} sn(u_o,k_o)cn^2(u_o,k_o)sn(u_i,k_i)du_o - G \right\}}{\left\{ G - \dfrac{2}{k_o}\displaystyle\int_0^{K_o}\left[u_o-\dfrac{E(u_o)}{(k_o')^2}\right]cn(u_o,k_o)dn(u_o,k_o)sn^3(u_i,k_i)du_o + \dfrac{2k_o}{(k_o')^2}\displaystyle\int_0^{K_o} sn(u_o,k_o)cn^2(u_o,k_o)sn^3(u_i,k_i)du_o \right\}}$
$x_i cn(u_i-K_i,k_i)$	$x_o sn(u_o,k_o)$	$ax_i^2 = \dfrac{\left\{ \dfrac{2}{k_o}\displaystyle\int_0^{K_o}\left[u_o-\dfrac{E(u_o)}{(k_o')^2}\right]cn(u_o,k_o)dn(u_o,k_o)cn(u_i-K_i,k_i)du_o + \dfrac{2k_o}{(k_o')^2}\displaystyle\int_0^{K_o} sn(u_o,k_o)cn^2(u_o,k_o)cn(u_i-K_i,k_i)du_o - G \right\}}{\left\{ G - \dfrac{2}{k_o}\displaystyle\int_0^{K_o}\left[u_o-\dfrac{E(u_o)}{(k_o')^2}\right]cn(u_o,k_o)dn(u_o,k_o)cn^3(u_i-K_i,k_i)du_o + \dfrac{2k_o}{(k_o')^2}\displaystyle\int_0^{K_o} sn(u_o,k_o)cn^2(u_o,k_o)cn^3(u_i-K_i,k_i)du_o \right\}}$

TABLE B.1 (CONTINUED)

EQUATIONS FOR THE CALCULATION OF THE ELLIPTIC DESCRIBING FUNCTION

Input	Output	Equations for ax_i^2
$x_i \mathrm{sn}(u_i,k_i)$	$x_o \mathrm{cn}(u_o-K_o,k_o)$	$ax_i^2 = \dfrac{\left\{ -\dfrac{2}{k_o}\displaystyle\int_0^{K_o} \left[u_o-\dfrac{E(u_o)}{(k_o')^2}\right] \mathrm{sn}(u_o-K_o,k_o)\mathrm{dn}(u_o-K_o,k_o)\mathrm{sn}(u_i,k_i)\,du_o -\dfrac{2k_o}{(k_o')^2}\displaystyle\int_0^{K_o} \mathrm{cn}(u_o-K_o,k_o)\mathrm{sn}^2(u_o-K_o,k_o)\mathrm{sn}(u_i,k_i)\,du_o + G \right\}}{\left\{ -G + \dfrac{2}{k_o}\displaystyle\int_0^{K_o} \left[u_o-\dfrac{E(u_o)}{(k_o')^2}\right] \mathrm{sn}(u_o-K_o,k_o)\mathrm{dn}(u_o-K_o,k_o)\mathrm{sn}^3(u_i,k_i)\,du_o - \dfrac{2k_o}{(k_o')^2}\displaystyle\int_0^{K_o} \mathrm{cn}(u_o-K_o,k_o)\mathrm{sn}^2(u_o-K_o,k_o)\mathrm{sn}^3(u_i,k_i)\,du_o \right\}}$
$x_i \mathrm{cn}(u_i-K_i,k_i)$	$x_o \mathrm{cn}(u_o-K_o,k_o)$	$ax_i^2 = \dfrac{\left\{ -\dfrac{2}{k_o}\displaystyle\int_0^{K_o} \left[u_o-\dfrac{E(u_o)}{(k_o')^2}\right] \mathrm{sn}(u_o-K_o,k_o)\mathrm{dn}(u_o-K_o,k_o)\mathrm{cn}(u_i-K_i,k_i)\,du_o -\dfrac{2k_o}{(k_o')^2}\displaystyle\int_0^{K_o} \mathrm{cn}(u_o-K_o,k_o)\mathrm{sn}^2(u_o-K_o,k_o)\mathrm{cn}(u_i-K_i,k_i)\,du_o + G \right\}}{\left\{ -G + \dfrac{2}{k_o}\displaystyle\int_0^{K_o} \left[u_o-\dfrac{E(u_o)}{(k_o')^2}\right] \mathrm{sn}(u_o-K_o,k_o)\mathrm{dn}(u_o-K_o,k_o)\mathrm{cn}^3(u_i-K_i,k_i)\,du_o -\dfrac{2k_o}{(k_o')^2}\displaystyle\int_0^{K_o} \mathrm{cn}(u_o-K_o,k_o)\mathrm{sn}^2(u_o-K_o,k_o)\mathrm{cn}^3(u_i-K_i,k_i)\,du_o \right\}}$

APPENDIX C

NUMERICAL CALCULATION OF ELLIPTIC FUNCTIONS USING LANDEN'S SCALE OF INCREASING AMPLITUDES*

The numerical values of elliptic functions are most easily calculated by using a method of successive transformations which reduce the elliptic integral to an elementary integral. The transformation was first discovered in a geometrical form by Landen in 1775, and since that time many possible variations of the method have been developed. Only the method based on Landen's scale of increasing amplitudes, which is the most useful method for the calculations of this thesis, is developed in this section.

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* Most of the material in this section was obtained from Hancock (15,pp 73-82) and King (16). However in certain places it was found desirable to expand upon the development given in these books.

C.1 INTEGRAL TRANSFORMATIONS

For convenience we introduce the following notation:

$$F(a,b,\phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}, \tag{C.1}$$

$$E(a,b,\phi) = \int_0^{\phi} \sqrt{a^2\cos^2\phi + b^2\sin^2\phi}\, d\phi . \tag{C.2}$$

In order to clarify the relationship between equations C.1 and C.2 and the elliptic integrals, it is convenient to write

$$k^2 = 1-(b^2/a^2), \tag{C.3}$$

$$k' = \frac{b}{a}, \tag{C.4}$$

in which case we have:

$$\sqrt{a^2\cos^2\phi + b^2\sin^2\phi} = a\sqrt{1-k^2\sin^2\phi}. \tag{C.5}$$

The functions C.1 and C.2 are consequently $\frac{1}{a}F(k,\phi)$ and $aE(k,\phi)$, where $F(k,\phi)$ and $E(k,\phi)$ are the usual elliptic integrals (17,p.43).

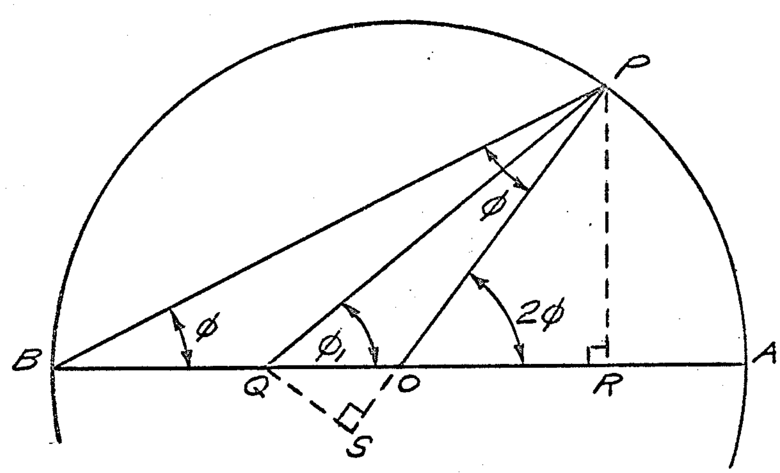


Figure C.1 Geometrical figure for the numerical calculation of elliptic functions

Referring to Figure C.1, let P be a point on a circle with centre O, and let Q be any point on the diameter AB. Further, let us write

$$QA = a \qquad , QB = b,$$

$$\angle AQP = \phi_1 \qquad , \angle AOP = 2\phi \tag{C.6}$$

then obviously

$$\angle ABP = \phi. \tag{C.7}$$

Now, let us define the following functions:

$$a_1 = \tfrac{1}{2}(a+b),$$

$$b_1 = \sqrt{ab},$$

$$c_1 = \tfrac{1}{2}(a-b). \tag{C.8}$$

From these relationships and Figure C.1, we get

$$OA = OB = OP = a_1,$$

$$OQ = a_1 - b = \tfrac{1}{2}(a-b) = c_1. \tag{C.9}$$

Considering triangles QPR and OPR, and using the relationships of C.6, C.7 and C.9, we get

$$PR = QP \sin\phi_1 = a_1 \sin 2\phi, \tag{C.10}$$

$$QR = QP \cos \phi_1 = c_1 + a_1 \cos 2\phi. \tag{C.11}$$

If we square equations C.10 and C.11, and add the resulting equations, we obtain the following:

$$(QP)^2 = a_1^{\,2} \sin^2 2\phi + c_1^{\,2} + a_1^{\,2} \cos^2 2\phi + 2c_1\, a_1 \cos 2\phi,$$

$$= c_1^{\,2} + 2c_1\, a_1 \cos 2\phi + a_1^{\,2},$$

$$= \tfrac{1}{2}(a^2+b^2)(\cos^2\phi + \sin^2\phi) + \tfrac{1}{2}(a^2-b^2)(\cos^2\phi - \sin^2\phi),$$

$$= a^2\cos^2\phi + b^2\sin^2\phi. \tag{C.12}$$

Substituting this result into equations C.10 and C.11 gives the following equations:

$$\sin \phi_1 = \frac{a_1 \sin 2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}\,, \tag{C.13}$$

$$\cos \phi_1 = \frac{c_1 + a_1 \cos 2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}\,. \tag{C.14}$$

Multiplying equations C.13 and C.14 by b_1 and a_1 respectively, squaring the resulting equations, and then adding the squared equations, gives, after considerable manipulation, the following equation:

$$a_1{}^2 \cos^2 \phi_1 + b_1{}^2 \sin^2 \phi_1 = \frac{a_1{}^2 \left(a \cos^2\phi + b \sin^2\phi\right)^2}{a^2 \cos^2\phi + b^2\sin^2\phi}\,. \tag{C.15}$$

Applying the sine law to triangle QOP in Figure C.1 gives

$$\sin (2\phi-\phi_1) = \frac{c_1 \sin(180-2\phi)}{PQ}\,. \tag{C.16}$$

After substituting equations C.8 and C.12 into equation C.16, we obtain

$$\sin (2\phi-\phi_1) = \frac{\frac{1}{2}(a-b)\sin 2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}\,. \tag{C.17}$$

Substituting equation C.13 into equation C.16 gives the following alternative form:

$$\sin(2\phi-\phi_1) = \frac{c_1}{a_1} \sin \phi_1. \tag{C.18}$$

From the geometry of triangle QPS in Figure C.1, we obtain

$$PS = c_1 \cos 2\phi + a_1\,. \tag{C.19}$$

Substituting equation C.8 into equation C.19, and performing suitable manipulations, we obtain

$$PS = a \cos^2\phi + b\sin^2\phi. \qquad (C.20)$$

From triangle QPS, and equations C.12 and C.20, we have

$$\cos(2\phi - \phi_1) = \frac{PS}{PQ} = \frac{a\cos^2\phi + b\sin^2\phi.}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} \qquad (C.21)$$

Substituting equation C.15 into C.21 gives the following result:

$$\cos(2\phi - \phi_1) = \frac{1}{a_1}\sqrt{a_1{}^2\cos^2\phi_1 + b_1{}^2\sin^2\phi_1} \qquad (C.22)$$

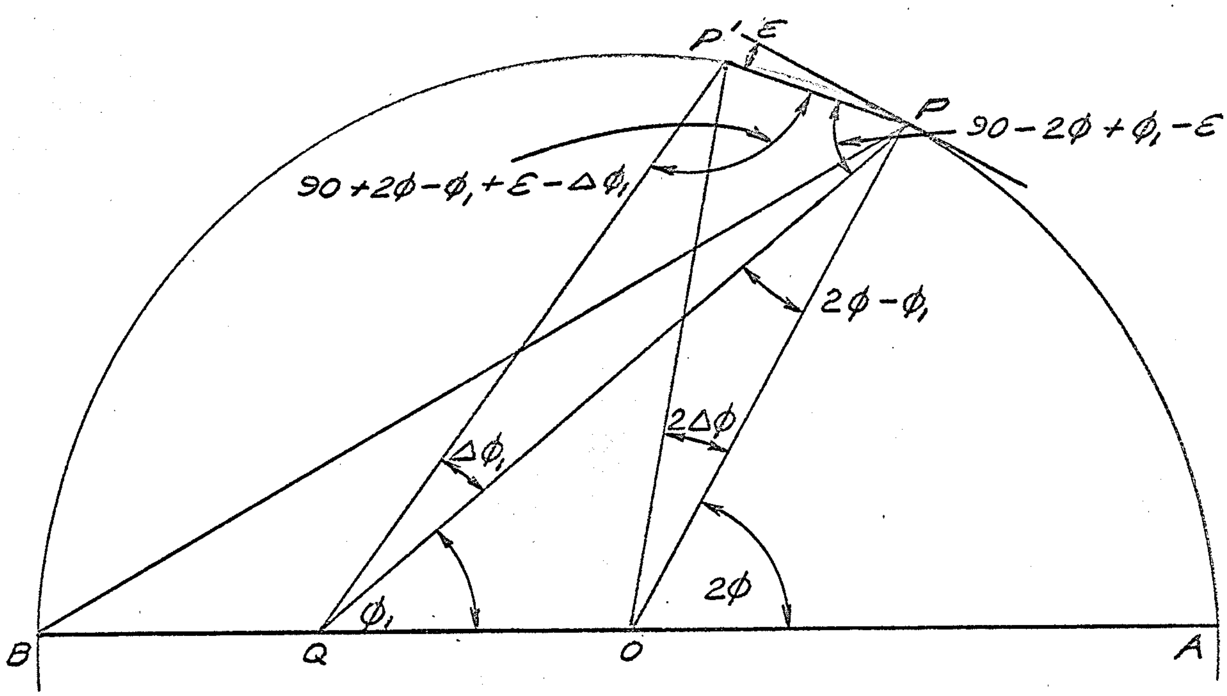


Figure C.2. Geometrical form of Landen's transformation

In figure C.2 a point P′ is chosen on the circle an incremental distance from P. If the small angle between PP′ and the tangent to the circle at P is ε, and if the incremental change in the angles ϕ and ϕ_1, is denoted by $\Delta\phi$ and $\Delta\phi_1$

respectively, then the angles shown in Figure C.2 follow directly from the geometry of the figure.

Applying the sine law to triangle $P'PQ$, we have

$$\frac{\sin \Delta\phi_1}{PP'} = \frac{\sin PP'Q}{PQ},$$

$$\frac{\sin \Delta\phi_1}{PP'} = \frac{\sin(90 + 2\phi - \phi_1 - \Delta\phi_1 + \varepsilon)}{PQ},$$

$$= \frac{\cos\,(2\phi - \phi_1 - \Delta\phi_1 + \varepsilon)}{PQ}. \qquad (C.23)$$

If we let P' approach P, then in the limit we have the following relationships:

$$PP' = 2a_1 d\phi,$$

$$\sin \Delta\phi_1 = d\phi_1,$$

$$2\phi - \phi_1 - \Delta\phi_1 + \varepsilon = 2\phi - \phi_1,$$

and equation C.23 becomes

$$\frac{d\phi_1}{2a_1 d\phi} = \frac{\cos(2\phi - \phi_1)}{PQ}. \qquad (C.24)$$

After substituting equations C.12 and C.22 into equation C.24, and performing suitable manipulations, we have

$$\frac{2d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} = \frac{d\phi_1}{\sqrt{a_1{}^2\cos^2\phi_1 + b_1{}^2\sin^2\phi_1}} \qquad (C.25)$$

Integrating, this expression becomes

$$F(a,b,\phi)\ \tfrac{1}{2}F(a_1,b_1,\phi_1). \qquad (C.26)$$

Applying elementary trigonometric relationships to equation C.18 gives

$$\sin 2\phi \cos\phi_1 - \cos 2\phi \sin\phi_1 = \frac{c_1}{a_1}\sin \phi_1. \qquad (C.27)$$

Multiplying equation C.27 by $\sin \phi_1$, and using elementary trigonometric identities, gives

$$\cos 2\phi = -\frac{c}{a} \sin^2\phi_1 + \cos \phi_1 \left(\sin \phi_1 \sin 2\phi + \cos \phi_1 \cos 2\phi\right). \quad (C.28)$$

Replacing $\sin \phi_1$ and $\cos \phi_1$ in the above equation by the relationships given in equations C.13 and C.14, and again using trigonometric relations, we get

$$\cos 2\phi = \frac{-c_1}{a_1} \sin^2\phi_1 + \cos \phi_1 \left\{\frac{a_1 + c_1 \cos 2\phi}{a^2\cos^2\phi + b^2\sin^2\phi}\right\}. \quad (C.29)$$

By using trigonometric identities, and substituting equations C.8 and C.15 into equation C.29, we obtain the following result:

$$\cos 2\phi = \frac{-c_1}{a_1} \sin^2\phi_1 + \frac{\cos \phi_1}{a_1} \sqrt{a_1^{\,2} \cos^2 \phi_1 + b_1^{\,2} \sin^2 \phi_1} \quad (C.30)$$

Obviously equation C.30 may be written in either of the following forms:

$$2 \cos^2 \phi = 1 - \frac{c_1}{a_1} \sin^2 \phi_1 + \frac{\cos\phi_1}{a_1} \sqrt{a_1^{\,2}\cos^2 \phi_1 + b_1^{\,2} \sin^2 \phi_1}, \quad (C.31)$$

$$2 \sin^2 \phi = 1 - \frac{c_1}{a_1} \sin^2\phi_1 - \frac{\cos \phi_1}{a_1} \sqrt{a_1^{\,2} \cos^2\phi_1 + b_1^{\,2} \sin^2\phi_1}. \quad (C.32)$$

Multiplying equations C.31 and C.32 by a^2 and b^2 respectively, and adding the resulting equations, gives

$$2 \left(a^2 \cos^2 \phi + b^2 \sin \phi\right) = (a^2 + b^2)$$
$$-(a^2-b^2)\frac{c_1}{a_1} \sin^2\phi_1 + (a^2-b^2)\frac{\cos\phi_1}{a_1} \sqrt{a_1^{\,2} \cos^2 \phi_1 + b_1^{\,2} \sin^2\phi_1}. \quad (C.33)$$

Equation C.33 can also be written in the form

$$2(a^2\cos^2\phi + b^2\sin^2\phi) = (a^2-b^2)(\cos^2\phi + \sin^2\phi)$$
$$-(a^2-b^2)\frac{c_1}{a_1} \sin^2\phi_1 + (a^2-b^2)\frac{\cos\phi_1}{a_1} \sqrt{a_1^{\,2}\cos^2\phi_1 + b_1^{\,2}\sin^2\phi_1}. \quad (C.34)$$

Using equations C.8 and trigonometric identities, we can reduce equation C.34 to the following form:

$$2\left(a^2\cos^2\phi + b^2\sin^2\phi\right) + 2b_1^{\,2}$$
$$= 4\left(a_1^{\,2}\cos^2\phi_1 + b_1^{\,2}\sin^2\phi_1\right) + 4c_1\cos\phi_1\sqrt{a_1^{\,2}\cos^2\phi_1 + b_1^{\,2}\sin^2\phi_1}. \qquad (C.35)$$

Multiplying this expression by the differential relation given in equation C.25, and using equations C.8, we have

$$\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}\;d\phi - \sqrt{a_1^{\,2}\cos^2\phi_1 + b_1^{\,2}\sin^2\phi_1}\;d\phi_1$$
$$= c_1\cos\phi_1 d\phi_1 - \frac{ab\,d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}. \qquad (C.36)$$

By using equation C.25 and the relations of equations C.8, we can obtain the following identity:

$$\frac{a^2 d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} - \frac{a_1^{\,2}d\phi_1}{\sqrt{a_1^{\,2}\cos^2\phi_1 + b_1^{\,2}\sin^2\phi_1}}$$
$$= \frac{\left[a^2 - \frac{1}{2}(a-b)^2\right]d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}$$
$$= \frac{c_1^{\,2}d\phi}{2\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} - \frac{ab\,d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}. \qquad (C.37)$$

Subtracting equation C.37 from equation C.36 gives

$$\left[\sqrt{a^2\cos^2\phi + b^2\sin^2\phi} - \frac{a^2}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}\right]d\phi$$
$$= \left[\sqrt{a_1^{\,2}\cos^2\phi_1 + b_1^{\,2}\sin^2\phi_1} - \frac{a_1^{\,2}}{\sqrt{a_1^{\,2}\cos^2\phi_1 + b_1^{\,2}\sin^2\phi_1}}\right]d\phi_1$$
$$+ c_1\cos\phi_1 d\phi_1 - \frac{1}{2}\frac{c^2 d\phi}{\sqrt{a_1^{\,2}\cos^2\phi_1 + b_1^{\,2}\sin^2\phi_1}}. \qquad (C.38)$$

Integrating, this expression becomes

$$E(a,b,\phi) - a^2F(a,b,\phi) = E(a_1,b_1,\phi_1) - a_1^{\,2}F(a_1,b_1,\phi_1) + c_1\sin\phi_1$$
$$-\tfrac{1}{2}c^2F(a,b,\phi). \qquad (C.39)$$

C.2 SUCCESSIVE TRANSFORMATIONS

By the same method as a_1, b_1, c_1 were derived from a, b, we may derive a_2, b_2, c_2 from a_1, b_1, etc., and thus form the following array of numbers:

$$
\begin{array}{lll}
a_1 = (\tfrac{1}{2})(a+b) & , \; b_1 = \sqrt{ab} \; , & c_1 = \tfrac{1}{2}(a-b) \\
a_2 = (\tfrac{1}{2})(a_1+b_1), & b_2 = \sqrt{a_1 b_1}, & c_2 = \tfrac{1}{2}(a_1-b_1) \\
- - - - - & , \; - - - - \; , & - - - - - \\
a_n = (\tfrac{1}{2})(a_{n-1}+b_{n-1}), & b_n = \sqrt{a_{n-1}b_{n-1}}, & c_n = \tfrac{1}{2}(a_{n-1}-b_{n-1}) \\
- - - - - - - - - - & , - - - - - - - - \; , & - - - - - - - - \; .
\end{array} \tag{C.40}
$$

From the following relations:

$$a_1 - b_1 = \frac{(\sqrt{a} - \sqrt{b})^2}{2},$$

$$a_2 - b_2 = \frac{a_1 + b_1}{2} - \sqrt{a_1 b_1} = \frac{a_1 - b_1}{2} - \left[\sqrt{a_1} - \sqrt{b_1}\right]\sqrt{b_1},$$

we have

$$a_2 - b_2 < \frac{a_1 - b_1}{2}.$$

In general, this becomes

$$a_n - b_n < \frac{a_{n-1} - b_{n-1}}{2} \;;$$

and thus, we have

$$\operatorname{Lim}_{n \to \infty} (a_n - b_n) = 0.$$

This limit is approached very rapidly, even when a and b are of very different magnitudes.

When $a_n = b_n$ then equations C.1 and C.2 become

$$F(a_n b_n, \emptyset) = \frac{\emptyset}{a_n}, \tag{C.41}$$

$$E(a_n, b_n, \emptyset) = a_n \emptyset. \tag{C.42}$$

Setting a = 1, b = k', and applying equation C.26 successively for increasing values of n, we have

$$F(k,\emptyset)\ (\tfrac{1}{2})F(a_1,b_1,\emptyset_1) = (\tfrac{1}{2})^2 F(a_2,B_2,\emptyset_2)$$

$$= \cdot\ \cdot\ - - -\ (\tfrac{1}{2})^n F(a_n,b_n,\emptyset_n) = \frac{\emptyset_n}{2^n a_n}, \tag{C.43}$$

where the $\emptyset$'s are calculated from the formula

$$\sin\emptyset_1 = \frac{a_1 \sin 2\emptyset}{\sqrt{a^2\cos^2\emptyset + b^2\sin^2\emptyset}},$$

$$\sin\emptyset_2 = \frac{a_2 \sin 2\emptyset}{\sqrt{a_1{}^2\cos^2\emptyset_1 + b_1{}^2\sin^2\emptyset_1}}, - -. \tag{C.44}$$

If we begin with $\emptyset = \pi/2$, we have

$$\emptyset_1 = \frac{\pi}{2},\ \emptyset_2 = 2\pi,\ \emptyset_3 = 4\pi,\ - - -, \emptyset_n = \frac{\pi}{2}(2)^n; \tag{C.45}$$

and thus obtain

$$F(k,\frac{\pi}{2}) = K = \frac{\emptyset_n \pi}{2}. \tag{C.46}$$

Starting with equation C.39, and then reapplying the equation successively to the $E(a_n,b_n,\emptyset_n) - a_n{}^2\ F(a_n,b_n,\emptyset_n)$ terms, which occur on the right hand side of the equation, we have (note that as n increases $E(a_n,b_n,\emptyset_n) - a_n{}^2\ F(a_n,b_n,\emptyset_n)$ approaches zero)

$$E(a,b,\emptyset) - a^2F(a,b,\emptyset) = c_1\sin\emptyset_1 + c_2\sin\emptyset_2 + -- + c_n\sin\emptyset_n + - - -$$

$$-(\tfrac{1}{2})c^2F(a,b,\emptyset) - c_1{}^2F(a_1,b_1,\emptyset_1) - 2c_2{}^2\ F(a_2,b_2,\emptyset_2)$$

$$-2^{n-1}c_nF(a_n,b_n,\emptyset_n) - - - -. \tag{C.47}$$

where the ϕ's are calculated from equation C.44.

Setting $a = 1$, $b = k'$, and applying equation C.26, equation C.47 becomes

$$E(k,\phi)-F(k,\phi) = c_1\sin\phi_1 + c_2\sin\phi_2 + \cdots + c_n\sin\phi_n + \cdots - \tfrac{1}{2}(c_0^2+2c_1^2+4c_2^2+\cdots+2^n c_n^2+\cdots)F(\phi,k). \tag{C.48}$$

If we begin with $\phi = \frac{\pi}{2}$, and apply equation C.45, we get

$$\frac{K-E(k,\frac{\pi}{2})}{K} = \tfrac{1}{2}(c_0^2+2c_1^2+4c_2^2+\cdots+2^n c_n^2+\cdots). \tag{C.49}$$

By applying relations C.43, C.46, C.48, and C.49 we can calculate the first elliptic integral, the first complete elliptic integral, the second elliptic integral, and the second complete elliptic integral.

C.3 CALCULATION OF THE ANGLE ϕ AND THE JACOBIAN ELLIPTIC FUNCTIONS SN(u,k), DN(u,k) IN TERMS OF THE ARGUMENT u AND THE MODULUS k.

If the array of numbers C.40 is calculated for $a_0 = 1$, $b = k'$, to such a value of n that c_n is less than the allowable error in the calculated values, then ϕ_n for a certain u and k can be found from equation C.43, that is,

$$\phi_n = 2^n a_n u. \tag{C.50}$$

From equation C.18 we can obtain a recurrence formula

$$\sin(2\phi_{n-1} - \phi_n) = \frac{c_n}{a_n}\sin\phi_n, \tag{C.51}$$

which enables us to calculate successively the angles $\phi_{n-1}, \phi_{n-2}, \ldots, \phi$.

Then directly from the definition of the Jacobian elliptic functions, we have

$$\mathrm{sn}(u,k) = \sin \phi,$$

$$\mathrm{cn}(u,k) = \cos \phi,$$

$$\mathrm{dn}(u,k) = \sqrt{1-k^2\sin^2\phi}. \tag{C.52}$$

APPENDIX D

EXAMPLES OF THE APPLICATION OF THE ELLIPTIC DESCRIBING FUNCTION

The steady state response for several systems was calculated using the elliptic describing function, and the results obtained were compared with the results obtained from an analog computer study. An approximation of system oscillations was also obtained using the conventional describing function method.

As an example of the application of the elliptic describing function, consider the nonlinear feedback system of figure D.1, where the nonlinear

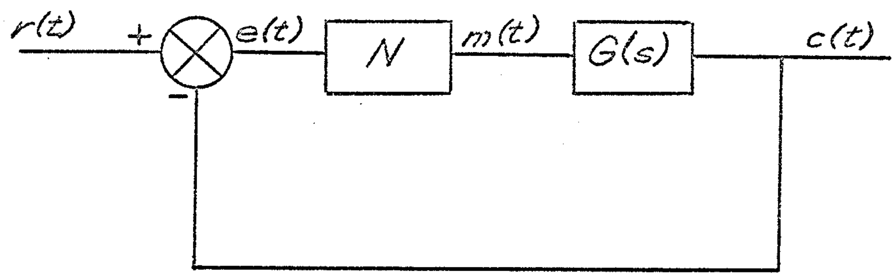


Figure D.1 Nonlinear feedback system

element N is an ideal relay with an output of plus or minus one, and the linear elements have a transfer function

$$G(s)= \frac{.5(s+7)^2}{s(s+1)^2}. \tag{D.1}$$

The Nyquist plot of this transfer function is shown in Figure D.2. This Nyquist plot crosses the negative real axis at two points:

(1) The point a_1 corresponds to an unstable limit cycle.

(2) The point a_2 corresponds to a stable limit cycle with a frequency of 1.58 radians per second.

We will be concerned only with the stable limit cycle.

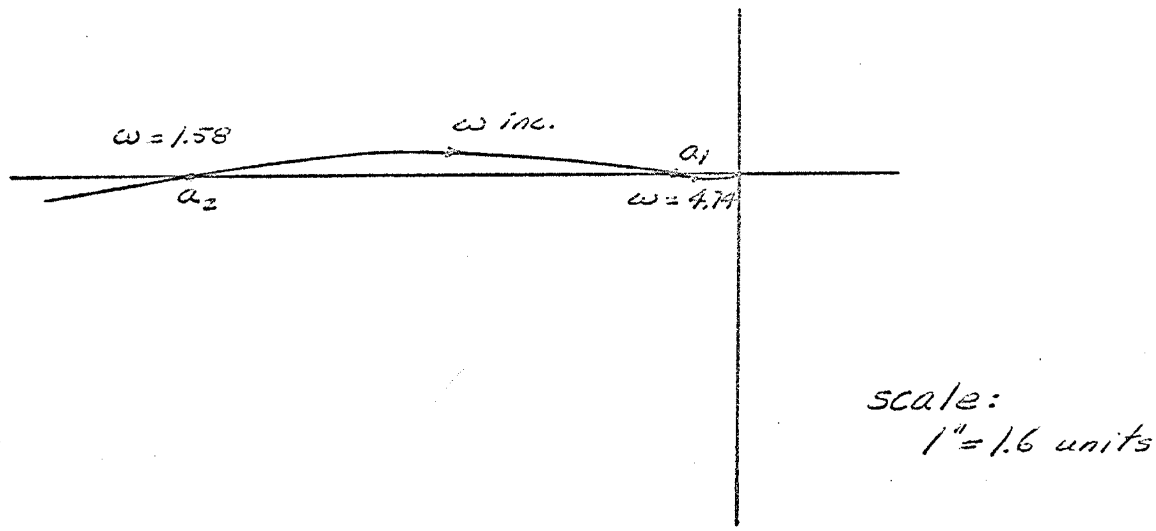


Figure D.2 Nyquist plot

From the Nyquist plot, we find

$$\frac{A(3\omega)\cos\left[\phi(3\omega)-\phi(\omega)\right]}{A(\omega)} = \frac{.316\cos 2}{4.70} = .067. \qquad \text{(D.2)}$$

A "ratio" line for the ratio .067 is sketched on Figure 4.5 (see Figure D.3).

Since the gain of the linear elements to the first harmonic is 4.70, the "gain" of the nonlinear elements to the first harmonic will be approximately 1/4.70 or .213. As a crude approximation, assume the "gain" of the nonlinear device to be the ratio of the amplitude of the output to the amplitude of the input. Using this approximation, and knowing the amplitude of the output from the relay to be one, the "gain" of the nonlinear element will be approximately .213 when the amplitude of the input is .470.

To ensure that the amplitude of the normalized solution is less than one (but not too much less), let us normalize the system output according to the relationship

$$c' = c/10. \qquad \text{(D.3)}$$

Using this relationship, the normalized nonlinear relationship may be approximated by the following polynomials:

Legendre approximation - $m' = 2.81\left[e' - .780(e')^3\right]$, (D.4)

Chebyshev approximation - $m' = 2.55\left[e' - .667(e')^3\right]$. (D.5)

Using a normalizing factor of ten, we have

$$A'(\omega) = \frac{A(\omega)}{10} = .470. \qquad (D.6)$$

First, let us use the elliptic describing function for the Legendre approximation of the nonlinearity.

For the values of k_i, k_o on the "ratio" line, the following values of ax_i^2 are obtained from equation 4.32 (using Figure 4.7 to simplify the calculations):

k_o	k_i	ax_i^2
.99999	.54	.394
.9999	.50	.379
.999	.48	.363
.99	.43	.341
.95	.36	.318
.9	.31	.295
.8	.23	.283
.5	.15	.258.

These values of ax_i^2 and k_i are plotted on Figure D.3 to obtain a line which intersects the .067 "ratio" line.*

---

*If equation 4.32 were plotted for k_i, k_o values in the second quadrant of Figure D.3, another intersection of the "ratio" line would occur. However, since ax_i^2 would be positive and a is negative, the solution would have an imaginary amplitude. Obviously this is not a physically realizable solution.

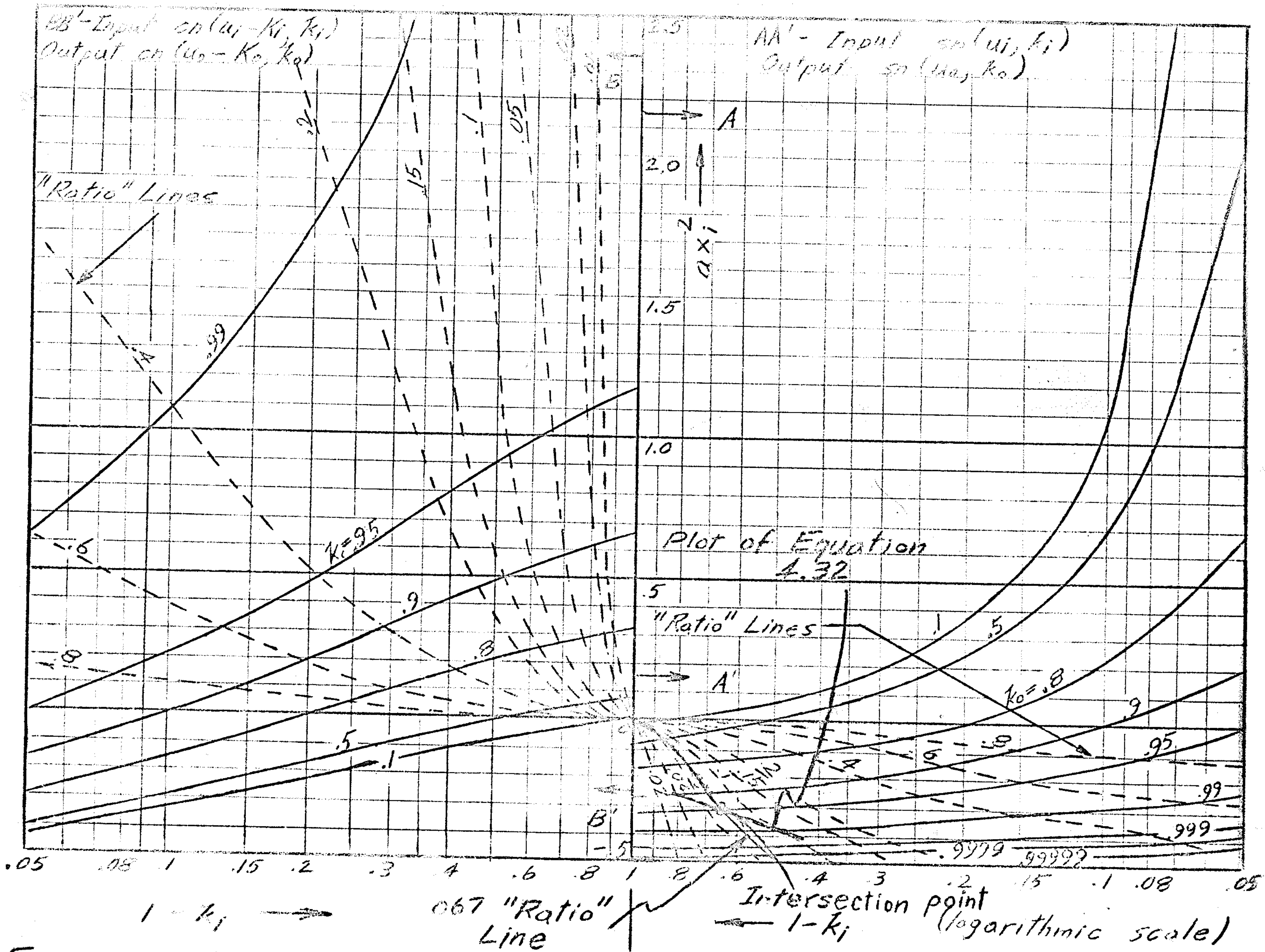


Figure D.3 Composite Plot of the Elliptic Describing Function Relationship and the $A'(3\omega)\cos[\phi'(3\omega) - \phi'(\omega)]$ to $A'(\omega)$ Relationship (Expanded Scale)

This intersection gives us the following parameters of the elliptic function approximation of system oscillations:

$k_i = .38,$

$ax_i^2 = .320.$

Thus, we have

$$x_i = \sqrt{\frac{ax_i^2}{a}} = \sqrt{\frac{.320}{.78}} = .640.$$

This is the amplitude of oscillations for the normalized system. The amplitude for the unnormalized system is 6.40.

Therefore the elliptic function approximation of system oscillations is

$$c = 6.40 \text{ sn}(1.64t, .38).$$

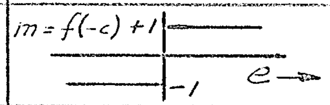
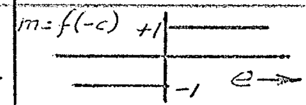
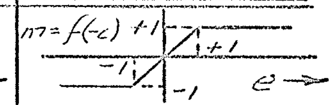
Similarly, the Chebyshev approximation yields the following result:

$$c = 6.48 \text{ sn}(1.72t, .54).$$

The pertinent values for this example and two other examples for which the elliptic describing function has been applied are given in Table D.1.

TABLE D.1

EXAMPLES OF THE APPLICATION OF THE ELLIPTIC DESCRIBING FUNCTION METHOD

System Parameter	System No.1	System No.2	System No.3
Transfer Function G(s)*	$\frac{.5(s+7)^2}{s(s+1)^2}$	$\frac{10}{s(s+1)^2}$	$\frac{10}{s(s+1)^2}$
Nonlinearity			
Analog Computer Simulation #			
Amplitude of Oscillations	6.85	6.89	6.70
Frequency of Oscillations (rps.)	1.55	6.38	6.38
Conventional Describing Function Analysis			
Amplitude of Oscillations	5.98	6.35	6.30
Frequency of Oscillations (rps.)	1.58	6.28	6.28
Percentage Error in Amplitude	12.7	7.85	5.97
Elliptic Describing Function with Legendre Approximation of Nonlinearity			
Normalizing Factor	10	10	10
Legendre Approximation of Normalized Nonlinearity:	$m' = 2.81e' - 2.91(e')^3$	$m' = 2.81e' - 2.91(e')^3$	$m' = 2.51e' - 1.65(e')^3$
Amplitude of Oscillations	6.40	6.97	6.86
Frequency of Oscillations (rps.)	1.58	6.28	6.28
Percentage Error in Amplitude	5.83	1.16	2.39

* All systems are of the type shown in Figure D.1.

\# All analog computer solutions were obtained on a Pace TR 10

TABLE D.1 (CONTINUED)

EXAMPLES OF THE APPLICATION OF THE ELLIPTIC DESCRIBING FUNCTION METHOD

System Parameter	System No.1	System No.2	System No.3
Elliptic Describing Function With Chebyshev Approximation of Nonlinearity			
Normalizing Factor	8	10	10
Chebyshev Approximation of Normalized Nonlinearity:	$m' = 2.55e' - 1.70(e')^3$	$m' = 2.55e' - 1.70(e')^3$	$m' = 2.51e' - 1.65(e')^3$
Amplitude of Oscillations	6.48	6.50	6.47
Frequency of Oscillations (rps.)	1.58	6.28	6.28
Percentage Error in Amplitude	5.41	5.66	3.44