A548

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~~Problems in Elliptic Integrals~~

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A 548

The Ellipse.

The ellipse is the locus of a point which moves so that its distance from a fixed point, called the focus, is in a constant ratio to its distance from a fixed straight line called the directrix. This ratio is denoted by "e", where "e" is less than unity. By analytical geometry the equation of this locus is found to be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $b^2 = a^2(1-e^2)$ or $e^2 = \frac{a^2-b^2}{a^2}$.

Calculus enables us to find and consider several important properties and relations of plane curves, such as; lengths of arcs, areas enclosed, radii of curvature, centres of gravity, equations and lengths of normals, tangents and subtangents, involutes, evolutes, envelopes, asymptotes – parallel and oblique to the axis –, parabolic asymptotes, pedal equations, intrinsic equations, polar equations, tangential polar equations, polar reciprocal curve, points of inflexion and multiple points.

The integrations and differentiations to find the above relations are in most cases simple though a little complicated owing to the numbers of terms involved. In the case of the ellipse all the differentiations can be obtained without any great difficulty, but one of the integrations cannot be obtained directly, namely, the integral to find the length of an arc of the curve.

This geometric problem, namely, to determine the length of an arc of the ellipse between arbitrary given terminal points, led to the integrals which are known as elliptic integrals.

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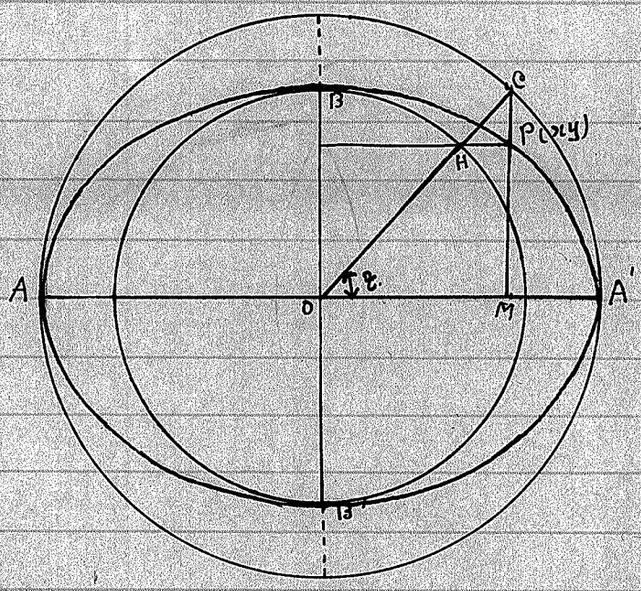
Lemma. Parametric Equation.

Instead of representing points on a

locus by means of two coordinates, one independent and one dependent, it is sometimes more convenient to introduce a third variable and to express x and y or r and θ in terms of this third variable, thus defining the curve by means of two equations instead of one. This is called the parametric equation of the curve.



Construction:-

Describe circles having the major and minor axis of the ellipse as diameters.

Take any point P on the ellipse. Drop a perpendicular from this point on the major axis and produce the line upwards to meet the major auxilliary circle in C. Join OC and through P and draw PK parallel to major axis.

The angle COA is the eccentric angle for the point P and is generally denoted by "t".

Equation:-

$$x = OM = OC \cos t = a \cos t.$$

$$y = PM = KO = OH \sin t = b \sin t.$$

To find the length of an arc of the ellipse using cartesian equation $x^2/a^2 + y^2/b^2 = 1$.

The expression for the element of arc is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2} \qquad \frac{dy}{dx} = \pm \frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}}$$

$$\therefore S = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

$$= \int_{x_0}^{x_1} \sqrt{1 + \frac{b^2}{a^2}\cdot\frac{x^2}{a^2 - x^2}}\, dx.$$

$$= \int_{x_0}^{x_1} \sqrt{\frac{a^4 - (a^2 - b^2)x^2}{a^2(a^2 - x^2)}}\, dx.$$

$$= \int_{x_0}^{x_1} \sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}}\, dx$$

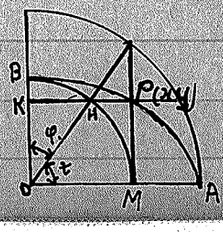
$$= a\int_{x_0}^{x_1} \sqrt{\frac{1 - e^2\left(\frac{x}{a}\right)^2}{1 - \left(\frac{x}{a}\right)^2}}\cdot d\left(\frac{x}{a}\right)$$

$$= a\int_{x_0}^{x_1} \sqrt{\frac{1 - e^2x^2}{1 - x^2}}\, dx$$

$$= a\int_{x_0}^{x_1} \sqrt{\frac{1 - k^2x^2}{1 - x^2}}\, dx \quad \text{where } k^2 = e^2.$$

Therefore the length of an elliptic quadrant depends upon $\int_{x_0}^{x_1} \sqrt{\frac{1 - k^2x^2}{1 - x^2}}\, dx$, which is known as an elliptic integral of the second class. This integral can be simplified by the substitution $x = \sin\phi$, under this transformation it becomes $\int_{\phi_0}^{\phi_1} \sqrt{1 - k^2\sin^2\phi}\, d\phi$.

The eccentricity "e" is always positive and less than unity. Therefore "k" must always be positive and less than unity. The number "k" is called the modulus of the integral while the angle ϕ is called the amplitude. The angle "ϕ" is always the complement of the eccentric angle "z".



To find the length of an arc of the ellipse using the parametric equation $x = a\cos t \quad y = b\sin t$.

The expression for the element of arc is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

$$dx = -a\sin t \cdot dt \qquad dy = b\cos t \cdot dt.$$

$$\therefore s = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

$$= \int_{t_0}^{t_1} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt.$$

$$= \int_{t_0}^{t_1} \sqrt{a^2 + (b^2 - a^2)\cos^2 t}\, dt$$

$$= a\int_{t_0}^{t_1} \sqrt{1 - \frac{a^2 - b^2}{a^2}\cos^2 t}\, dt.$$

$$= a\int_{t_0}^{t_1} \sqrt{1 - e^2\cos^2 t}\, dt.$$

$$= a\int_{\varphi_0}^{\varphi_1} \sqrt{1 - k^2\sin^2\varphi}\, d\varphi \qquad \sin\varphi = \frac{\pi}{2} - t.$$

$\therefore$ Length of a quadrant $S_q = a\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2\varphi}\, d\varphi$.

as was mentioned above, this is an integral of the second class and is denoted by $E(k\varphi)$.

Computation of an elliptic integral of the second class.

The approximate numerical value of an elliptic integral where the modulus and the amplitude are given can readily be computed by Landen's reduction formula.

The particular integral whose value we want to compute is.

$$E(k\varphi) = \int^{\varphi} \sqrt{1 - k^2\sin^2\varphi}\, d\varphi.$$

In applying Landen's transformation we shall apply it to a more general integral

namely,

$$G(k\varphi) = \int^{\varphi} \frac{\alpha + \beta \sin^2\varphi}{\sqrt{1 - k^2\sin^2\varphi}}\, d\varphi,$$ of which $E(k\varphi)$ is a special case when the values 1 and $-k^2$ are assigned to α and β respectively.

$$G(k\varphi) = \int_0^{\varphi} \frac{\alpha + \beta \sin^2\varphi}{\sqrt{1 - k^2\sin^2\varphi}}\, d\varphi.$$

Substitutions. φ is replaced by φ_1 where φ and φ_1 are connected by the relation

$$\tan\varphi = \frac{\sin 2\varphi_1}{k + \cos 2\varphi_1} \quad \text{----------} \quad (1).$$

Then.

(a). $$k\sin\varphi = \sin(2\varphi_1 - \varphi) \quad \text{----------} \quad (2).$$

Proof. $$\frac{\sin\varphi}{\cos\varphi} = \frac{\sin 2\varphi_1}{k + \cos 2\varphi_1}$$

$$(k + \cos 2\varphi_1)\sin\varphi = \sin 2\varphi_1 \cos\varphi.$$

$$k\sin\varphi = \sin 2\varphi_1 \cos\varphi - \cos 2\varphi_1 \sin\varphi$$

$$= \sin(2\varphi_1 - \varphi).$$

(b) $$\tan(\varphi - \varphi_1) = \frac{1-k}{1+k}\tan\varphi_1 \quad \text{----------} \quad (3).$$

Proof:

$$k = \frac{\sin(2\varphi_1 - \varphi)}{\sin\varphi}.$$

$$\frac{1-k}{1+k} = \frac{\sin\varphi - \sin(2\varphi_1 - \varphi)}{\sin\varphi + \sin(2\varphi_1 - \varphi)}.$$

$$= \frac{2\cos\varphi_1 \sin(\varphi - \varphi_1)}{2\sin\varphi_1 \cos(\varphi - \varphi_1)}.$$

$$= \cot\varphi_1 \tan(\varphi - \varphi_1).$$

$$\therefore \tan\varphi_1 \frac{1-k}{1+k} = \tan(\varphi - \varphi_1).$$

(c). $$\sqrt{1 - k^2\sin^2\varphi} = \frac{1 + k\cos 2\varphi_1}{\sqrt{1 + k^2 + 2k\cos 2\varphi_1}} \quad \text{------} \quad (4).$$

Proof.

$$\sin^2\varphi = \frac{1}{\operatorname{cosec}^2\varphi} = \frac{1}{1 + \cot^2\varphi}.$$

$$= \frac{1}{1 + \left(\frac{k + \cos 2\varphi_1}{\sin 2\varphi_1}\right)^2}.$$

$$= \frac{\sin^2 2\varphi_1}{1 + k^2 + 2k\cos 2\varphi_1}$$

$$\therefore 1 - k^2\sin^2\varphi = 1 - \frac{k^2\sin^2 2\varphi_1}{1 + k^2 + 2k\cos 2\varphi_1}$$

$$= \frac{\sin^2 2\varphi_1}{1+k^2+2k\cos 2\varphi_1}$$

$$\therefore 1-k^2\sin^2\varphi = 1 - \frac{k^2\sin^2 2\varphi_1}{1+k^2+2k\cos 2\varphi_1}$$

$$= \frac{(1+k\cos 2\varphi_1)^2}{1+k^2+2k\cos 2\varphi_1}$$

$$\therefore \sqrt{1-k^2\sin^2\varphi} = \frac{1+k\cos 2\varphi_1}{\sqrt{1+k^2+2k\cos 2\varphi_1}}$$

(d). $$d\varphi = \frac{2(1+k\cos 2\varphi_1)}{1+k^2+2k\cos 2\varphi_1} \quad \text{---------- (5).}$$

Proof:

Differentiating (1).

$$\sec^2\varphi\, d\varphi = \frac{2\cos 2\varphi_1\, d\varphi_1}{k+\cos 2\varphi_1} + \sin 2\varphi_1 \frac{2\sin 2\varphi_1}{(k+\cos 2\varphi_1)^2} d\varphi_1$$

$$d\varphi = \frac{2(k\cos 2\varphi_1+1)\cos^2\varphi}{(k+\cos 2\varphi_1)^2}\, d\varphi_1$$

$$\cos^2\varphi = 1-\sin^2\varphi = \frac{(k+\cos 2\varphi_1)^2}{1+k^2+2k\cos 2\varphi_1}$$

$$\therefore d\varphi = \frac{2(k\cos 2\varphi_1+1)}{(k+\cos 2\varphi_1)^2} \cdot \frac{(k+\cos 2\varphi_1)^2}{(1+k^2+2k\cos 2\varphi_1)} d\varphi_1$$

$$= \frac{2(k\cos 2\varphi_1+1)}{1+k^2+2k\cos 2\varphi_1}\, d\varphi_1$$

(e). $$\sqrt{1-k^2\sin^2\varphi} + k\cos\varphi = \sqrt{1+k^2+2k\cos 2\varphi_1} \quad \text{---- (6).}$$

Proof, $$\cos^2\varphi = 1-\sin^2\varphi$$

$$= \frac{(k+\cos 2\varphi_1)^2}{1+k^2+2k\cos 2\varphi_1}$$

$$\therefore k\cos\varphi = \frac{k^2+k\cos 2\varphi_1}{\sqrt{1+k^2+2k\cos 2\varphi_1}}$$

$$\therefore \sqrt{1-k^2\sin^2\varphi} + k\cos\varphi = \frac{1+k^2+2k\cos 2\varphi_1}{\sqrt{1+k^2+2k\cos 2\varphi_1}}$$

$$= \sqrt{1+k^2+2k\cos 2\varphi_1}.$$

$$G(k,\varphi) = \int_0^\varphi \frac{\alpha+\beta\sin^2\varphi}{\sqrt{1-k^2\sin^2\varphi}}\, d\varphi.$$

$$= \int_0^\varphi \frac{\alpha\, d\varphi}{\sqrt{1-k^2\sin^2\varphi}} + \beta/k^2 \int_0^\varphi \frac{1-(1-k^2\sin^2\varphi)}{\sqrt{1-k^2\sin^2\varphi}}\, d\varphi$$

$$= \int_0^\varphi \frac{(\alpha+\beta/k^2)\, d\varphi}{\sqrt{1-k^2\sin^2\varphi}} - \beta/k^2 \int_0^\varphi \sqrt{1-k^2\sin^2\varphi}\, d\varphi.$$

$$G(k\varphi) - \beta/k \sin\varphi = \int_0^{\varphi} \frac{(\alpha + \beta/k^2)\, d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} - \beta/k^2 \int_0^{\varphi} \sqrt{1 - k^2\sin^2\varphi}\, d\varphi - \beta/k \int_0^{\varphi} \cos\varphi\, d\varphi$$

$$= \int_0^{\varphi} \frac{(\alpha + \beta/k^2)\, d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} - \beta/k^2 \int_0^{\varphi} \left(\sqrt{1 - k^2\sin^2\varphi} + k\cos\varphi\right) d\varphi.$$

Substituting φ_1 for φ:

$$= \int_0^{\varphi_1} \frac{2(\alpha + \beta/k^2)\, d\varphi_1}{\sqrt{1 + k^2 + 2k\cos 2\varphi_1}} - \beta/k^2 \int_0^{\varphi} \sqrt{1 + k^2 + 2k\cos 2\varphi_1}\, \frac{2(1 + k\cos 2\varphi_1)\, d\varphi_1}{1 + k^2 + 2k\cos 2\varphi_1}$$

$$= 2\int_0^{\varphi_1} \frac{\alpha - \beta/k\cos 2\varphi_1}{\sqrt{1 + k^2 + 2k\cos 2\varphi_1}}\, d\varphi_1$$

$$= 2\int_0^{\varphi_1} \frac{\alpha - \beta/k + \frac{2\beta}{k}\sin^2\varphi_1}{\sqrt{1 + k^2 + 2k(1 - 2\sin^2\varphi_1)}}\, d\varphi_1$$

$$= \frac{2}{1+k} \int_0^{\varphi} \frac{(\alpha - \beta/k) + \frac{2\beta}{k}\sin^2\varphi_1}{\sqrt{1 - \frac{4k}{(1+k)^2}\sin^2\varphi_1}}\, d\varphi_1$$

$$\therefore G(k\varphi) = \beta/k \sin\varphi + \frac{2}{1+k} G(k_1 \varphi_1).$$

where $k_1 = \frac{2\sqrt{k}}{1+k}$, $\alpha_1 = \alpha - \beta/k$, $\beta_1 = \frac{2\beta}{k}$.

By repeating this same substitution, we see

$$G(k_1\varphi_1) = \frac{\beta_1}{k_1} \sin\varphi_1 + \frac{2}{1+k_1} \cdot G(k_2\varphi_2)$$

$\therefore$ By a repeated use of this same reduction

$$G(k\varphi) = \beta/k \sin\varphi + \frac{\beta_1}{\sqrt{k}} \sin\varphi_1 + \sqrt{\frac{k_1}{k}}\beta_2 \sin\varphi_2 + + + +$$
$$+ + + + + + \sqrt{\frac{k_1 k_2 - - - - k_{n-1}}{k}}\, G(k_n \varphi_n)$$

where $\alpha_p = \alpha - \beta/k\left(1 + \frac{2}{k_1} + \frac{2^2}{k_1 k_2} + - - - - - - - - \frac{2^{p-1}}{k_1 k_2 - k_{p-1}}\right)$.

$$\beta_p = \frac{2\beta}{k\, k_1 - - - - - k_{p-1}}$$

As n is increased k_n rapidly approaches unity and the limiting value of

$$G(k_n\varphi_n) = \int_0^{\Phi} \frac{\alpha_n + \beta_n \sin^2\varphi}{\sqrt{1 - \sin^2\varphi}}\, d\varphi.$$

$$= \int_0^{\Phi} (\alpha_n + \beta_n)\sec\varphi\, d\varphi - \int_0^{\Phi} \beta_n \cos\varphi\, d\varphi.$$

$$= (\alpha_n + \beta_n)\log\tan\left(\tfrac{\pi}{4} + \tfrac{\Phi}{2}\right) - \beta_n \sin\Phi.$$

But $F'(k\varphi) = \sqrt{\frac{k_1 k_2 k_3 \cdots k_{n-1}}{k}} \log \tan\left(\frac{\pi}{4} + \frac{\Phi}{2}\right)$ where $F(k\varphi)$ denotes an elliptic integral of the first class.

$$\therefore G(k\varphi) = (\alpha_n + \beta_n) F(k\varphi) - \beta_n \sin\Phi + \beta_1 k \sin\varphi + \frac{\beta_1}{\sqrt{k}} \sin\varphi_1$$
$$+ + + + + + \sqrt{\frac{k_1 k_2 \cdots k_{n-1}}{k}} \beta_{n-1} \sin\varphi_{n-1}.$$

$$= F(k\varphi)\left[\alpha - \beta_1 k\left(1 + \frac{2}{k_1} + \frac{2^2}{k_1 k_2} + + \frac{2^{n-1}}{k_1 k_2 \cdots k_{n-1}} - \frac{2^n}{k_1 k_2 \cdots k_{n-1}}\right)\right.$$
$$\left. + \beta_1 k\left\{\sin\varphi + \frac{2}{\sqrt{k}} \sin\varphi_1 + + + \frac{2^{n-1}}{\sqrt{k} k_1 \cdots k_{n-2}} \sin\varphi_{n-1} - \frac{2^n}{\sqrt{k} k_1 \cdots k_{n-1}} \sin\Phi\right\}\right.$$

Now by putting $\alpha = 1$ and $\beta = -k^2$

$$G(k\varphi) = E'(k\varphi).$$
$$= F(k\varphi)\left[1 + k\left(1 + \frac{2}{k_1} + \frac{2^2}{k_1 k_2} + + + \right.\right.$$
$$\left.\left. + \frac{2^{n-1} - 2^n}{k_1 k_2 \cdots k_{n-1}}\right)\right] - k\left[\sin\varphi + \frac{2}{\sqrt{k}} \sin\varphi_1 + \frac{2^2}{\sqrt{k} k_1} \sin\varphi_2 + +\right.$$
$$\left. + + + \frac{2^{n-1}}{\sqrt{k} k_1 \cdots k_{n-2}} \sin\varphi_{n-1} - \frac{2^n}{\sqrt{k} k_1 \cdots k_{n-1}} \sin\Phi\right].$$

The expression for the quadrant of an ellipse was $S_q = a \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\varphi}\, d\varphi$.

$$= a\, E(k\varphi).$$
$$= a\left\{F(k\varphi)\left[1 + k\left(1 + \frac{2}{k_1} + \frac{2^2}{k_1 k_2} + + + + -\right)\right]\right.$$
$$\left. - k\left(\sin\varphi + \frac{2}{\sqrt{k}} \sin\varphi_1 + + + + -\right)\right\}.$$

To compute the length of a quadrant of the ellipse $x^2/16 + y^2/8 = 1$. by means of the formula

$$S_q = a\left\{F(k\varphi)\left[1 + k\left(1 + \frac{2}{k_1} + + + -\right)\right] - k\left(\sin\varphi + \frac{2}{\sqrt{k}} \sin\varphi_1 + + -\right)\right\}.$$

here $a = 4$ $\quad k = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

To evaluate $F\left(\frac{\sqrt{2}}{2}\, \frac{\pi}{2}\right)$.

$$\log k = \tfrac{1}{2} \log 2 - \log 2.$$
$$= \cdot 1505150 - \cdot 3010300$$
$$= \bar{1} \cdot 8494850$$
$$k = \text{antilog } \bar{1} \cdot 8494850$$
$$= \cdot 707107.$$

$\frac{1}{2}\log k = \bar{1}.9247425$

$\log 2 = \underline{.3010300}$

$.2257725$

$\log(1+k) = .2322608$

$k_1 = \text{antilog}.2322608$

$= .985171$

$\frac{1}{2}\log k_1 = \bar{1}.9967559$

$\log 2 = \underline{.3010300}.$

$.2977859.$

$\log(1+k_1) = .2977980$

$k_2 = \text{antilog}.2977980$

$= .999972.$

$\frac{1}{2}\log k_2 = \bar{1}.9999939$

$\log 2 = \underline{.3010300}.$

$.3010239.$

$\log(1+k_2) = .3010240.$

$k_3 = \text{antilog}\ \bar{1}.9999999.$

$= 1$ — approximately

$\log k = \bar{1}.8494850.$

$\log \sin \frac{\pi}{2} = 0$

$\log \sin(2\varphi_1 - \frac{\pi}{2}) = \bar{1}.8498850.$

$\therefore 2\varphi_1 - \frac{\pi}{2} = 45^0$

$\varphi_1 = 67^\circ\ 30'.$

$\log k_1 = \bar{1}.9935117.$

$\log \sin \varphi_1 = \underline{\bar{1}.9656153.}$

$\log \sin(2\varphi_2 - \varphi_1) = \bar{1}.9591270$

$2\varphi_2 - 67^\circ 30' = 65^\circ\ 31'\ 48''$

$\varphi_2 = 66^\circ\ 30'\ 54''.$

$\log k_2 = \bar{1}.9999879.$

$\log \sin \varphi_2 = \underline{\bar{1}\ 9624473.}$

$\log \sin (2\varphi_3 - \varphi_2) = \bar{1}.9624352$

$2\varphi_3 - 66°30'54'' = 66°\ 30'\ 41''$

$\varphi_3 = 66°\ 30'\ 47.5''$

$\log k_3 = \bar{1}.9999999$

$\log \sin \varphi_3 = \underline{\bar{1}.9624428}$

$\log (\sin(\varphi_4 - \varphi_3) = \bar{1}.9624427$

$2\varphi_4 - 66°30'47.5'' = 66°\ 30'\ 47''$

$\varphi_4 = 66°\ 30'\ 47.3''$

$F\left(\frac{\sqrt{2}}{2}\ \frac{\pi}{2}\right) = \log_\varepsilon \tan\left(\frac{\pi}{4} + 33°\ 15'\ 23.6''\right) \sqrt{\frac{k_1 k_2}{k}}.$

$= \log_\varepsilon \tan(78°\ 15'\ 23.6'') \sqrt{\frac{k_1 k_2}{k}}.$

$= \frac{.6822473}{\mu} \sqrt{\frac{k_1 k_2}{k}}.$

where μ is the modulus of the common system of logarithms.

$\log F\left(\frac{\sqrt{2}}{2}\ \frac{\pi}{2}\right) = \log .6822473 - \log \mu + \frac{1}{2}\log k_1 + \frac{1}{2}\log k_2 - \frac{1}{2}\log k.$

$= \bar{1}.8339419 - \bar{1}.6377798 + \bar{1}.9967559$

$+ \bar{1}.999939 - \bar{1}.9247425$

$= .2681693$

$\therefore F\left(\frac{\sqrt{2}}{2}\ \frac{\pi}{2}\right) = 1.8542654.$

Now $s_9 = a\left\{F(k\ \tfrac{\pi}{2})\left[1 + k\left(1 + \frac{2^2}{k_1^2} + \frac{2^2}{k_1 k_2} - \frac{2^3}{k_1 k_2}\right)\right] - k(\sin\varphi + + -)\right\}.$

$\log \frac{2}{k_1} = \log 2 - \log k_1 = .3010300 - \bar{1}.9935117.$

$\frac{2}{k_1} = \text{antilog}\ .3075183.$

$= 2.0801104.$

$\log \frac{2^2}{k_1 k_2} = 2\log 2 - \log k_1 - \log k_2$

$= .6020600 - \bar{1}.9935117 - \bar{1}.9999879.$

$\frac{2^2}{k_1 k_2} = \text{antilog}\ .6085604$

$= 4.060925$

$\log \frac{2^3}{k_1 k_2} = 3\log 2 - \log k_1 - \log k_2$

$= .9030900 - \bar{1}.9935117 - \bar{1}.9999879$

$\frac{2^3}{k_1 k_2} = \text{antilog}\ .9095904$

$= 8.120644$

$$1 + k\left(1 + \frac{2}{k_1} + \frac{2^2}{k.k_2} - \frac{2^3}{k.k_1.k_2}\right) = 1 + (1 + 2.030104 + 4.060352 - 8.120644)k.$$

$$= 1 - .728476$$

$$= .271524.$$

$$\sin\varphi = \sin\frac{\pi}{2} = 1.$$

$$\log\frac{2}{\sqrt{k}}\sin\varphi_1 = \log 2 - \tfrac{1}{2}\log k + \log\sin\varphi_1$$

$$= .3010300 - \bar{1}.9247425 + \bar{1}.9656153.$$

$$\frac{2}{\sqrt{k}}\sin\varphi_1 = \text{antilog}.3419028.$$

$$= 2.197367.$$

$$\log\frac{2^2}{\sqrt{k}\,k_1}\sin\varphi_2 = 2\log 2 - \tfrac{1}{2}\log k - \tfrac{1}{2}\log k_1 + \log\sin\varphi_2$$

$$= .6020600 - \bar{1}.9247425$$

$$- \bar{1}.9967559 + \bar{1}.9624473.$$

$$\frac{2^2}{\sqrt{k}\,k_1}\sin\varphi_2 = \text{antilog}.6430089$$

$$= 4.395506.$$

$$\log\frac{2^3}{\sqrt{k}\,k_1.k_2}\sin\varphi_3 = 3\log 2 - \tfrac{1}{2}\log k - \tfrac{1}{2}\log k_1 - \tfrac{1}{2}\log k_2$$

$$+ \log\sin\varphi_3.$$

$$= .9030900 - \bar{1}.9247425$$

$$- \bar{1}.9967559 - \bar{1}.9999939 + .$$

$$\frac{2^3}{\sqrt{k}\,k_1.k_2} = \text{antilog}.9440395$$

$$= 8.791024.$$

$$-k\left(\sin\varphi + \frac{2}{\sqrt{k}}\sin\varphi_1 + \frac{2^2}{\sqrt{k}\,k_1}\sin\varphi_2 - \frac{2^3}{\sqrt{k}\,k_1.k_2}\sin\varphi_3\right)$$

$$= -k(1 + 2.197367 + 4.395506 - 8.791024)$$

$$= k(1.981851) = .8470452.$$

$$\therefore g_q = 4\left\{F\left(\tfrac{\sqrt{2}}{2}\,\tfrac{\pi}{2}\right) \times .271524 + .8470452\right\}.$$

$$= 4\left\{1.8542554 \times .271524 + .8470452\right\}.$$

$$= 4 \times 1.3505\bar{2} = 5.40208$$

In reducing the formula for the computation of the elliptic integral of the second class, the modulus was made to approach unity as its limit. Therefore if the modulus were small

I.

Summary of Computations.

$k = \cdot 707107$	$\log k = \bar{1} \cdot 8494850$
$k_1 = \cdot 985117$	$\log k_1 = \bar{1} \cdot 9935117$
$k_2 = \cdot 999972$	$\log k_2 = \bar{1} \cdot 9999879$
$k_3 = 1 \cdot$	$\log k_3 = \bar{1} \cdot 9999999$

$\frac{1}{2} \log k = \bar{1} \cdot 9247425$

$\frac{1}{2} \log k_1 = \bar{1} \cdot 9967559$

$\frac{1}{2} \log k_2 = \bar{1} \cdot 9999939$

$\frac{1}{2} \log k_3 = \bar{1} \cdot 9999999$

$\varphi_1 = 67^\circ\ 30'$	$\log \sin \varphi_1 = \bar{1} \cdot 9656153$
$\varphi_2 = 66^\circ\ 30'\ 54''$	$\log \sin \varphi_2 = \bar{1} \cdot 9624473$
$\varphi_3 = 66^\circ\ 30'\ 47.5''$	$\log \sin \varphi_3 = \bar{1} \cdot 9624428$
$\varphi_4 = 66^\circ\ 30'\ 47.3''$	$\log \sin \mu = \bar{1} \cdot 6377798$

$F(\frac{\sqrt{2}}{2}, \frac{\pi}{2}) = 1 \cdot 8542354$

$E(\frac{\sqrt{3}}{2}, \frac{\pi}{2}) = 1 \cdot 3505120$

$Sg = 5 \cdot 4020800$

the computation would be laborious, as the number of terms to get a very approximate value would have to be great. Hence a second transformation is introduced. In this transformation the modulus is decreased, while the amplitude is increased, and ultimately is made to approach zero as its limit as n is increased indefinitely

The first transformation is always used if the modulus is large, i.e. greater than one half, but if it is small the latter is generally used. The reduction of this second transformation is similar to the first, the substitutions being the same, except for the fact that the subscripts are interchanged. By a repeated application of these we get

$$E(k\tfrac{\pi}{2}) = F(k\tfrac{\pi}{2})\left[1 - \frac{k^2}{2}\left(1 + \frac{k_1}{2} + \frac{k_1 k_2}{2^2} + + + + \right)\right].$$

where $F(k\frac{\pi}{2}) = \frac{\pi}{2}(1+k_1)(1+k_2)$

and $k_1 = \dfrac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}$.

To compute the length of a quadrant of the ellipse $x^2/81 + y^2/54 = 1$ using the formula

$$Sq = a\left\{F(k\tfrac{\pi}{2})\left[1 - \frac{k^2}{2}\left(1 + \frac{k_1}{2} + \frac{k_1 k_2}{2^2} + + + \right)\right]\right\}.$$

$a = 9 \qquad k = \frac{\sqrt{3}}{3}$.

To evaluate $F(\frac{\sqrt{3}}{3}\frac{\pi}{2})$.

$\log k = \frac{1}{2}\log 3 - \log 3$.

$= .9385607 - .4771213$

$k = \text{antilog } \bar{1}.7614994.$

$= .577350.$

$\sqrt{1-k^2} = .8164964.$

$$k_1 = \frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}} = \frac{.1835036}{1.8164964}$$

$= .1010206.$

$$\sqrt{1-k_1^2} = .994884.$$

$$k_2 = \frac{.005116}{1.994884}$$
$$= .0025^-65^-15^-$$

$$\sqrt{1-k_2^2} = .999997.$$

$$k_3 = \frac{.0000003}{1.9999997}$$
$$= 0 \text{ — approximately.}$$

$$F'\left(\frac{\sqrt{3}}{3}\frac{\pi}{2}\right) = \frac{\pi}{2}(1.1010206)(1.0025^-65^-1).$$
$$= 1.733918.$$

$$\log \frac{k_1}{2} = \log k_1 - \log 2.$$
$$= \bar{1}.0044101 - .3010300$$
$$\frac{k_1}{2} = \text{antilog } \bar{3}.7033801$$
$$= .05^-05^-103.$$

$$\log \frac{k_1 k_2}{2^2} = \log k_1 + \log k_2 - 2\log 2.$$
$$= \bar{1}.0044101 + \bar{3}.4091129 - .6020600$$
$$\frac{k_1 k_2}{2^2} = \text{antilog } \bar{5}.8114630$$
$$= .0000647^-83.$$

$$1 - \frac{k_2^2}{2}\left(1 + \frac{k_1}{2} + \frac{k_1 k_2}{2^2}\right) = 1 - \frac{k_2^2}{2}(1 + .05^-05^-103 + .0000647)$$
$$= 1 - .175^-095^-8$$
$$= .824^-9042.$$

$$\therefore E\left(\frac{\sqrt{3}}{3}\frac{\pi}{2}\right) = 1.733918 \times .8249042$$
$$= 1.433016.$$

$$\therefore S_9 = a\,E\left(\frac{\sqrt{3}}{3}\frac{\pi}{2}\right)$$
$$= 9 \times 1.433016$$
$$= 12.897144.$$

The length of a quadrant of an ellipse is given by $a.E(k\frac{\pi}{2})$. Hence we see that the

Summary of Computations.

$k = \cdot 577350$ $\quad \log k = \bar{1} \cdot 761\,4394$

$k_1 = \cdot 1010206$ $\quad \log k_1 = \bar{1} \cdot 0044101$

$k_2 = \cdot 0025651$ $\quad \log k_2 = \bar{3} \cdot 4091729$

$k_3 = \cdot 0000001$ $\quad \log k_3 = \bar{7} \cdot 1760912$

$F(\sqrt{3}/3 \; \pi/2) = 1 \cdot 733918$

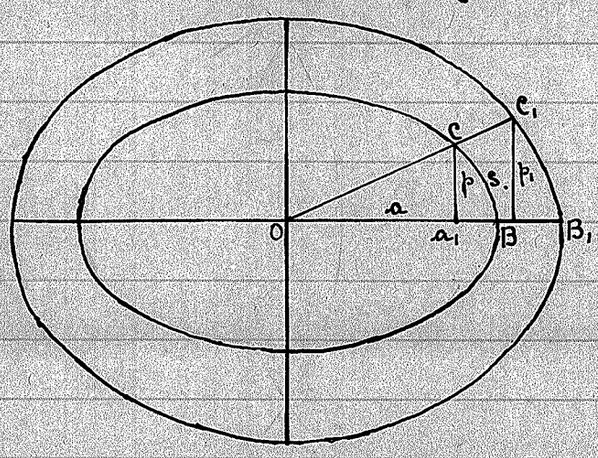
$E(\sqrt{3}/3 \; \pi/2) = 1 \cdot 433016$

$S_q = 12 \cdot 897144.$

length varies directly as the intercept on the major axis, the constant of proportionality being $E(k\frac{\pi}{2})$. Now k is the eccentricity of the curve and in all confocal ellipses the eccentricity is the same. Therefore if the value of $E(k\frac{\pi}{2})$ be computed for some value of k, then we can determine the length of a quadrant of any ellipse whose eccentricity is equal to k. That is, having computed the length of one ellipse, we can determine the length of a quadrant of a whole system of confocal ellipses by multiplying the numerical value of the constant of proportionality by the intercept on the major axis

Since the triangles are similar the sides must be proportional i.e. $\frac{\beta}{\beta_1} = \frac{a}{a_1}$. But if the angle the vector makes with the major axis is indefinitely diminished the arc and perpendicular become nearly equal, and before we come to the limit they are equal since the curve cuts the axis at right angles $\therefore \frac{\beta}{\beta_1} = \frac{s}{s_1}$ and hence $\frac{a}{a_1} = \frac{s}{s_1}$

i.e. $s_1 = \frac{a_1}{a} s$



From data sheets I and II we see if $k = \frac{\sqrt{2}}{2}$ $F(k\frac{\pi}{2}) = 1.8542$ and $E(k\frac{\pi}{2}) = 1.3505$ and if $k = \frac{\sqrt{2}}{3}$ $F(k\frac{\pi}{3}) = 1.7339$ and $E(k\frac{\pi}{3}) = 1.4330$. Therefore by increasing the modulus, it is evident that the numerical value of an integral of the

first class, of amplitude $\frac{\pi}{2}$, is increased, while the numerical value of an integral of the second class, and of the same amplitude is decreased.

By our definition the modulus cannot be negative nor can it exceed unity, hence the limiting values are 0 and 1.

The values of $F(k\frac{\pi}{2})$, when k assumes the values 0 and 1, are found by substituting the values of k in $\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$.

$$\text{if } k=0 \qquad F(k\tfrac{\pi}{2}) = \int_0^{\frac{\pi}{2}} d\varphi = \frac{\pi}{2} \text{ ---------- (1).}$$

$$k=1 \qquad F(k\tfrac{\pi}{2}) = \int_0^{\frac{\pi}{2}} \sec\varphi\, d\varphi = \log\tan\tfrac{\pi}{2} = \infty \text{ ---- (2).}$$

Similarly the values of $E(k\frac{\pi}{2})$ are found by substituting the values of k in $\int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\varphi}\cdot d\varphi$.

$$\text{if } k=0 \qquad E(k\tfrac{\pi}{2}) = \int_0^{\frac{\pi}{2}} d\varphi = \frac{\pi}{2} \text{ ---------- (3).}$$

$$k=1 \qquad E(k\tfrac{\pi}{2}) = \int_0^{\frac{\pi}{2}} \cos\varphi\, d\varphi = 1 \text{ ---------- (4).}$$

As k the modulus decreases and approaches its lower limit the curve approaches a circle and when $k=0$ i.e. the eccentricity is equal to zero, the curve actually becomes a circle of radius "a".

$$\text{Lt}_{k=0}\, g = a\,E(k\tfrac{\pi}{2}) = \frac{\pi a}{2} \quad \text{from (3).}$$

and we also have $g = a\,F(k\frac{\pi}{2})\left\{1-\frac{k^2}{2}\left(1+\frac{k_1}{2}-\frac{k_1 k_2}{2\cdot}\right)\right\}$

$$\text{if } k=0 \qquad = a\cdot\frac{\pi}{2}\{1-0\} = \frac{\pi a}{2}.$$

The quantity $\frac{\pi a}{2}$ we know to be an expression for the length of a quadrant of a circle of radius "a". Therefore the expression to obtain the length of the arc of an ellipse holds even if the modulus be decreased to its lower limiting value.

If k assumes the upper limit 1. the curve under discussion will be a parabola since its eccentricity is unity.

This curve is not a closed curve and consequently we shall repeat the definition for the intercept on the major axis.

The intercept on the major axis is the distance from the centre of the curve to the point of intersection of the curve and the major axis.

The equation of the parabola is of the second degree, therefore the major axis intersects it in two points and as the centre of the curve is at infinity, these intercepts are infinite.

As before $S_q = a\,E(k\frac{\pi}{2})$

If $k=1$ $\quad = \infty . 1 = \infty$

and we also have $S_q = a\left\{F(k\frac{\pi}{2})\left[1+\frac{2}{\pi}+\;+\;-\right\}-k(\sin\varphi\;+\;+\;-)\right\}$

If $k=1$ $\quad = \infty . \infty . = \infty$

If the parabola be symmetrically situated with regard to the axis, the curve will extend to infinity and the length of the arc in each of the two quadrants is infinite. This agrees with the result found by the formula. Hence as a conclusion, we may say the formula to determine the length of an arc of the ellipse is not only true for the values of k less than unity but they are also true for the limits 0 and 1. i.e. the expression holds for every value on the range from zero to one.

Hyperbola

To rectify the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The expression for the element of arc is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\therefore s = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

$$y = \pm b/a \sqrt{x^2 - a^2} \qquad \frac{dy}{dx} = \pm b/a \frac{x}{\sqrt{x^2 - a^2}}.$$

$$\therefore s = \int_0^{x_1} \sqrt{1 + \frac{b^2}{a^2} \cdot \frac{x^2}{x^2 - a^2}}\, dx.$$

$$= \int_{x_0}^{x} \frac{\sqrt{a^2(x^2 - a^2) + b^2 x^2}}{a^2(x^2 - a^2)}\, dx.$$

$$= \int_{x_0}^{x_1} \sqrt{\frac{1 - e^2 x^2}{1 - x^2}}\, dx.$$

When the arc is measured from the vertex to the point whose coordinates are $(x\,y)$.

$$s = \int_a^x \sqrt{\frac{1 - e^2 x^2}{1 - x^2}}\, dx.$$

put $x = \sin\varphi \qquad dx = \cos\varphi\, d\varphi.$

$$= \int_{\varphi_0}^{\varphi_1} \sqrt{1 - e^2 \sin^2\varphi}\, d\varphi.$$

This integral differs from an elliptic integral of the second class, only in that e^2 is greater than unity. A substitution can be found which reduces the integral to an integral of the type form.

Substitution.

$$\text{put } \sin^2\psi = e^2 \sin^2\varphi$$

$$\cos^2\varphi = \frac{1}{e^2}(e^2 - e^2\sin^2\varphi) = \frac{1}{e^2}(e^2 - \sin^2\psi)$$

$$\cos\psi\, d\psi = e\cos\varphi\, d\varphi = \frac{1}{e}\sqrt{1 - \frac{1}{e^2}\sin^2\psi}\, d\psi$$

$$\sqrt{1 - e^2\sin^2\varphi} = \cos\psi.$$

$$\therefore S = l\int_{\psi_0}^{\psi} \frac{\cos^2\psi\, d\psi}{\sqrt{1-\frac{1}{l^2}\sin^2\psi}}$$

$$= 1/k\int_{\psi_0}^{\psi} \frac{(1-\sin^2\psi)\, d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

$$= \frac{1}{k^3}\left[\int_{\psi_0}^{\psi} \frac{1-k^2\sin^2\psi}{\sqrt{1-k^2\sin^2\psi}}\, d\psi + \int_{\psi_0}^{\psi} \frac{k^2-1}{\sqrt{1-k^2\sin^2\psi}}\, d\psi.\right\}.$$

$$= \frac{1}{k^3}\left[\int_0^{\psi} \Delta\psi.\, d\psi - \int_0^{\psi_0} \Delta\psi\, d\psi + (k^2-1)\left\{\int_0^{\psi}\frac{d\psi}{\Delta\psi} - \int_0^{\psi_0}\frac{d\psi}{\Delta\psi}\right\}\right].$$

To rectify the Hyperbola using $ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}.dy$ as the expression for the element of arc.

$\therefore s = \int^{y} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$ where the arc is measured from the vertex to the point whose coordinates are $(x\ y)$.

$$x = \pm a/b\sqrt{y^2+b^2} \qquad \frac{dx}{dy} = \pm\frac{a}{b}\cdot\frac{y}{\sqrt{y^2+b^2}}.$$

$$\therefore s = \int_0^{y} \sqrt{1+\frac{a^2y^2}{b^2(y^2+b^2)}}\, dy.$$

$$= \int_0^{y} \sqrt{\frac{b^2y^2+a^2y^2+b^4}{b^2(y^2+b^2)}}\, dy.$$

$$= \int_0^{y.} \sqrt{\frac{(a^2+b^2)y^2+b^4}{b^2(y^2+b^2}}\, dy.$$

$$= \int_0^{y} \sqrt{\frac{l^2a^2y^2}{b^4}+1}{\frac{y^2}{b^2}+1}\, dy.$$

Substitution :- put $\frac{a\,l\,y}{b^2} = \tan\varphi.$

$$dy = \frac{b^2}{al}\sec\varphi\, d\varphi \qquad y^2/\varphi^2 = \frac{b^2}{a^2l^2}\tan^2\varphi.$$

$$\therefore s = \int_0^{\varphi} \frac{\sqrt{1+\tan^2\varphi}}{\sqrt{1+\frac{b^2\tan^2\varphi}{a^2l^2}}}\,\frac{b^2}{al}\sec^2\varphi.\, d\varphi.$$

$$= \int_0^{\varphi} \frac{\sec^3\varphi.\ b^2/al.\, d\varphi.}{\sqrt{1+\frac{b^2}{a^2l^2}\tan\varphi.}}$$

$$= \frac{b^2}{al}\int_0^{\varphi} \frac{\sec^3\varphi\,.\,d\varphi.}{\sqrt{\sec^2\varphi-\tan^2\varphi+\frac{b^2}{a^2l^2}\tan^2\varphi.}}$$

$$= \frac{b^2}{ae}\int_0^{\varphi} \frac{\sec^2\varphi\, d\varphi}{\sqrt{1+\left(\frac{b^2}{a^2e^2}-1\right)\sin^2\varphi}}$$

$$= \frac{b^2}{ae}\int_0^{\varphi} \frac{\sec^2\varphi\, d\varphi}{\sqrt{1-\frac{1}{e^2}\sin^2\varphi}}$$

$$= \frac{b^2}{ae}\int_0^{\varphi} \frac{\sec^2\varphi\, d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

Transformation.

$$\frac{1}{\sqrt{1-k^2\sin^2\varphi}} = \frac{1}{1-k^2}\cdot\frac{1-k^2}{\sqrt{1-k^2\sin^2\varphi}}$$

$$= \frac{1}{1-k^2}\cdot\frac{1-k^2(\sin^2\varphi+\cos^2\varphi)}{\sqrt{1-k^2\sin^2\varphi}}$$

$$= \frac{1}{1-k^2}\left\{\frac{1-k^2\sin^2\varphi}{\sqrt{1-k^2\sin^2\varphi}} - \frac{k^2\cos^2\varphi}{\sqrt{1-k^2\sin^2\varphi}}\right\}.$$

$$\therefore S = \frac{b^2}{ae(1-k^2)}\left\{\int_0^{\varphi}\sec^2\varphi\left[\sqrt{1-k^2\sin^2\varphi} - \frac{k^2\cos^2\varphi}{\sqrt{1-k^2\sin^2\varphi}}\right]d\varphi\right\}$$

$$= \frac{b^2}{ae(1-k^2)}\left\{\int_0^{\varphi}\sec^2\varphi\sqrt{1-k^2\sin^2\varphi}\,d\varphi - k^2F(k\varphi)\right\}.$$

Reduction of $\int_0^{\varphi}\sec^2\varphi\sqrt{1-k^2\sin^2\varphi}\cdot d\varphi$. Integrating by parts.

$$I = \tan\varphi\sqrt{1-k^2\sin^2\varphi} + \int_0^{\varphi}\frac{k^2\sin^2\varphi}{\sqrt{1-k^2\sin^2\varphi}}\,d\varphi$$

$$= \tan\varphi\sqrt{1-k^2\sin^2\varphi} + \int_0^{\varphi}\Delta\varphi\cdot d\varphi - \int_0^{\varphi}\frac{d\varphi}{\Delta\varphi}.$$

$$= \tan\varphi\sqrt{1-k^2\sin^2\varphi} + F(k\varphi) - E(k\varphi).$$

$$\therefore S = \frac{b^2}{ae(1-k^2)}\left\{\tan\varphi\sqrt{1-k^2\sin^2\varphi} + F(k\varphi) - E(k\varphi) - k^2F(k\varphi)\right\}.$$

$$= \frac{b^2}{ae(1-k^2)}\left\{\tan\varphi\sqrt{1-k^2\sin^2\varphi} + F(k\varphi)(1-k^2) - E(k\varphi)\right\}.$$

Transformation.

$$\frac{b^2}{ae(1-k^2)} = \frac{b^2}{ae}\cdot\frac{1}{1-\frac{1}{e^2}} = \frac{b^2}{ae}\cdot\frac{1}{1-\frac{a^2}{a^2+b^2}}.$$

$$= \frac{b^2}{ae}\cdot\frac{1}{\frac{b^2}{a^2e^2}} = ae.$$

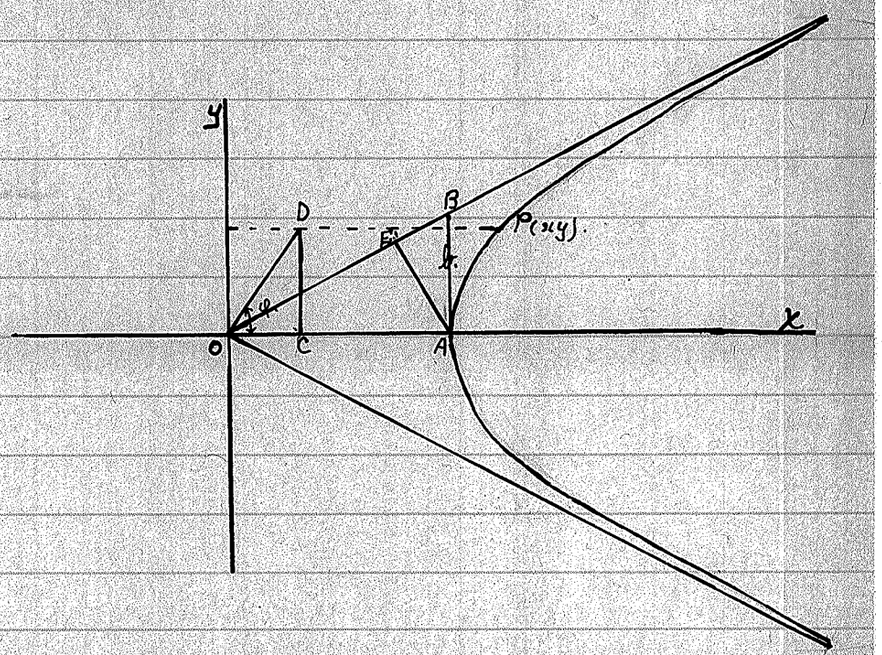
$$\therefore S = ae\tan\varphi\cdot\Delta\varphi + \frac{b^2}{ae}\cdot F(k\varphi) - ae\,E(k\varphi).$$

To determine the amplitude "φ":

$$\tan\varphi = \frac{ae\,y}{b^2}$$

$$\varphi = \tan^{-1}\frac{y}{\frac{b^2}{ae}}.$$

φ is an angle of a right angled triangle in which the perpendicular is y and the side adjacent $\frac{b^2}{ae}$.



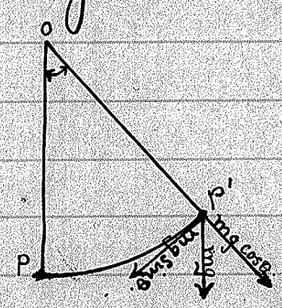
$\frac{b^2}{ae} = \frac{b^2}{a}\cdot\frac{a}{\sqrt{a^2+b^2}} = \frac{b^2}{\sqrt{a^2+b^2}}$ i.e. $\frac{b^2}{ae}$ is equal to the projection of b on the asymptote. Therefore the side adjacent of the right angled triangle is equal to the projection of b on the asymptote.

Deduce the construction.

Project "b" on the asymptote. Lay off a distance on the major axis equal to the projection BF, call it OC. At C erect a perpendicular CD equal to y the ordinate of P. Join OD. Then COD is the required angle for $\tan\varphi = \frac{y}{\frac{b^2}{ae}}$.

Pendulum.

A heavy particle suspended by a perfectly flexible, weightless thread forms what is called a simple pendulum. Although it is impossible to realize a simple pendulum we may closely approach the required conditions if we suspend a small metal sphere by a long and very thin thread. The distance between the point of suspension and the centre of the metal sphere will be the length of the simple pendulum. This length is usually denoted by the letter "l".



Let the pendulum OP move to the position OP'. The force acting on it is "mg" acting vertically downwards. This force may be resolved into two components acting at right angles, one $mg\cos\theta$ acting along OP', producing tension in the string, the other $mg\sin\theta$ acting along the tangent, tending to bring the pendulum back to equilibrium.

Thus the physical forces acting are $mg\cos\theta$ along the normal and $mg\sin\theta$ along the tangent, while the theoretical forces are $ml\left(\frac{d\theta}{dt}\right)^2$ acting along the normal and $ml\frac{d^2\theta}{dt^2}$ acting in the direction of θ increasing.

Setting up the equation of motion by equating the theoretical and physical forces we get.

$$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta \quad \text{---------- (1)}$$

or $$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

If during the motion "θ" remains so small that θ is approximately equal to sin θ, then

$$\frac{d^2\theta}{dt^2} + g/l\,\theta = 0$$

Integrating this differential equation

$$\theta = A\cos\sqrt{g/l}\cdot t + B\sin\sqrt{g/l}\cdot t.$$
$$= a\cos(\sqrt{g/l}\cdot t - \alpha).$$

Where $A = a\cos\alpha$ and $B = a\sin\alpha$.

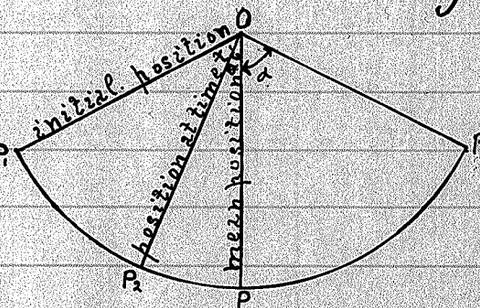
This equation represents harmonic motion of period $2\pi\sqrt{l/g}$ i.e. $T = 2\pi\sqrt{l/g}$.

Again considering ① we have.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

$$\text{or } 2\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2}\cdot dt = -2\frac{g}{l}\sin\theta\, d\theta$$

$$\text{or } \int d\left(\frac{d\theta}{dt}\right)^2 = \int \frac{2g}{l}\cdot d(\cos\theta).$$



Limits.

initial value of $\frac{d\theta}{dt} = 0$

value at any time $t = \frac{d\theta}{dt}$

initial value of $\theta = \alpha$

value at any time $t = \theta$

$$\therefore \int_0^{\frac{d\theta}{dt}} d\left(\frac{d\theta}{dt}\right)^2 = \int_\alpha^\theta \frac{2g}{l}\cdot d(\cos\theta).$$

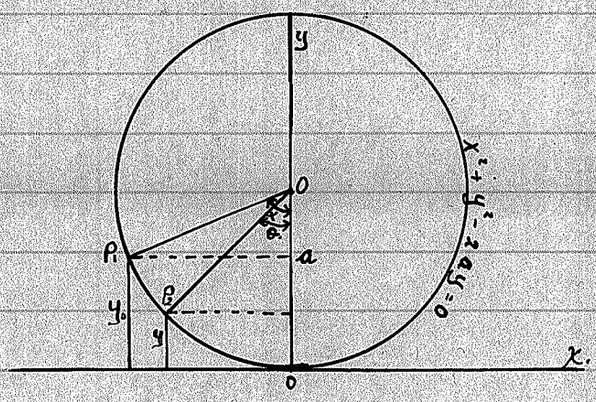
$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l}(\cos\theta - \cos\alpha).$$

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}}\cdot\sqrt{\cos\theta - \cos\alpha}.$$

$$\sqrt{\frac{2g}{l}}\cdot dt = \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

Integrating.

$$\sqrt{\frac{2g}{l}}\cdot t = \int_\alpha^\theta \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}$$



$a = l$.

$\frac{a-y_0}{a} = \cos\alpha$ or $y_0 = a(1-\cos\alpha)$ ——————— ①

$\frac{a-y}{a} = \cos\theta$ ∴ $y = a(1-\cos\theta)$ ——————— ②.

$\frac{x}{a} = \sin\theta$ ∴ $x = a\sin\theta$ ——————— (c).

$dy = a\sin\theta\, d\theta$ ——————— (d)

$dx = a\cos\theta\, d\theta$ ——————— (e).

$\cos^2\theta = (1 - y/a)^2$ ——————— (f).

$\therefore t = \sqrt{\frac{l}{2g}} \cdot \int_0^\theta \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}$. where t is the time required to reach the point whose ordinate is y.

$= \sqrt{\frac{l}{2g}} \cdot \int_0^y \frac{\frac{dy}{l}}{\sqrt{(\frac{2y}{l} - \frac{y^2}{l^2})(\frac{y_0 - y}{l})}}$.

$= \frac{l}{\sqrt{2g}} \int_0^y \frac{dy}{\sqrt{(2ly - y^2)(y_0 - y)}}$.

Substitutions ——————— ③.

put $x^2 = y/y_0$ $dy = 2xy_0\, dx$.

$t = \frac{l}{\sqrt{2g}} \cdot \int_0^x \frac{2y_0 x\, dx}{\sqrt{(y_0 + y_0x^2)(2ly_0x^2 - y_0^2x^4)}}$

$= \frac{l}{\sqrt{2g}} \int_0^x \frac{dx}{\sqrt{(1-x^2)(2l - y_0x^2)}}$.

$= \sqrt{\frac{l}{g}} \int_0^x \frac{dx}{\sqrt{(1-x^2)(1 - \frac{y_0}{2l}x^2)}}$

$= \sqrt{\frac{l}{g}} \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ where $k^2 = \frac{y_0}{2l}$.

k will be less than unity if $y_0 < 2l$, equal to unity if $y_0 = 2l$ and greater than unity if $y_0 > 2l$. In this last case substitution ② fails, so we put $x^2 = y/2l$ $\quad dy = 4ax\,dx$. and the integral reduces to.

$$t = l\sqrt{\frac{2}{gy_0}} \cdot \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \qquad \text{where } k^2 = \frac{2l}{y_0}.$$

The time required by the pendulum to reach its greatest height, i.e. in the first case the time of a half vibration, and in the second case the time of a half revolution, will depend upon $\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ an integral of the first class.

Substitution ——————————— ③

put $x = \sin\varphi$ ——————— (a)

$1 - x^2 = 1 - \sin^2\varphi$ ——————— (b)

$1 - k^2x^2 = 1 - k^2\sin^2\varphi$ ——————— (c)

$dx = \cos\varphi\,d\varphi$ ——————— (d).

Limits —

when $x = 0$ $\quad \sin\varphi = 0$ $\quad \varphi = 0$

$x = 1$ $\quad \sin\varphi = 1$ $\quad \varphi = \frac{\pi}{2}$.

$$\therefore \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}},$$

To determine φ the amplitude.

$k^2 = y/2l$ ——————————— ①.

$\sin\varphi = x = \sqrt{y/y_0}$ ——————————— ②.

From substitution ① — (a) $\quad y_0 = a(1-\cos\alpha)$.

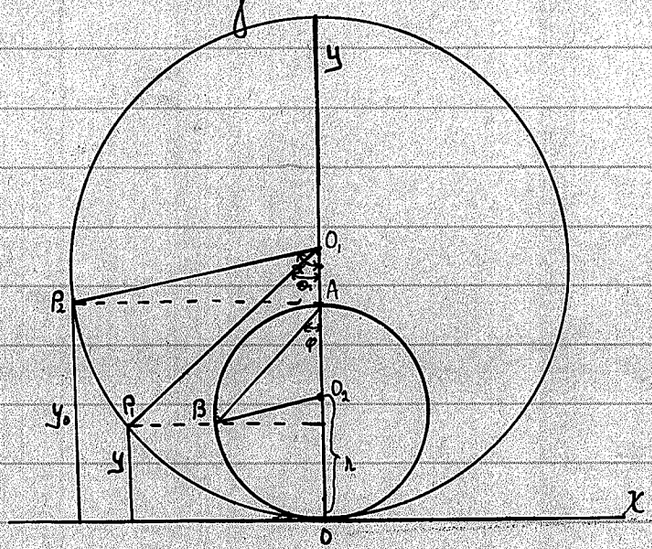
$\therefore y/2l = \frac{1}{2}(1-\cos\alpha) = \sin^2\frac{\alpha}{2}$.

$\therefore k = \sin\frac{\alpha}{2}$. ——————————— ③.

But α is one half the angle through which the pendulum swings, consequently the modulus of the elliptic integral is the sine of one fourth the angle through which the pendulum swings.

$$\text{Sin}\,\varphi = \sqrt{y/y_0} = \sqrt{\frac{y/2l}{y_0/2l}} = \frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}}.$$

Hence we can compute the sine of the amplitude of the integral when the angle through which the pendulum has risen is given.



Construction:- Draw a circle of radius "a". O_1O is the mean position of the pendulum, P_1 position at any time "t", P_2 the highest point the pendulum reaches. Drop a perpendicular from P_2 on O_1O, call the point of intersection A. Draw a circle on OA as diameter. Drop a perpendicular from P_1 on O_1O cutting the circle in B. Join AB and O_2B. Then the angle BAO is the required angle "φ" corresponding to any arc PP_1.

Proof- $\frac{r-y}{r} = \cos BO_2O$

$1 - y/r = 1 - \cos BO_2O$ but $y_0 = 2r$.

$\therefore y/y_0 = \frac{1}{2}(1 - \cos BO_2O)$

$= \frac{1}{2}\left[1 - \left\{1 - 2\sin^2\frac{BO_2O}{2}\right\}\right]$.

$= \sin^2\frac{BO_2O}{2}$

$= \sin^2 BAO$

$\therefore \sqrt{y/y_0} = \sin BAO$

$\therefore \text{Sin}\,\varphi = \text{Sin}\,BAO$

$\angle\varphi = \angle BAO$

Reduction of $\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$ by Landens Transformation

Replace φ by φ_1 where the following relations exist.

$$\tan\varphi = \frac{\sin 2\varphi_1}{k+\cos 2\varphi_1} \quad \text{----------} \quad (1)$$

$$k\sin\varphi = \sin(2\varphi_1 - \varphi) \quad \text{----------} \quad (2)$$

$$\tan(\varphi-\varphi_1) = \frac{1-k}{1+k}\tan\varphi_1 \quad \text{----------} \quad (3)$$

$$d\varphi = \frac{(k+\cos 2\varphi_1)^2}{1+k^2+2k\cos 2\varphi_1}\, d\varphi_1 \quad \text{----} \quad (4)$$

$\therefore F(k\varphi) = \frac{2}{1+k} F(k_1\varphi_1) = \frac{k_1}{k} F(k_1\varphi_1)$, by successive transformations similar to the one above, we get

$F(k\varphi) = k_n \sqrt{\frac{k_1 k_2 \cdots k_{n-1}}{k}}\, F(k_n\varphi_n)$ at the limit $k_n = 1$ and φ_n approaches a limiting value Φ.

$$F(k_n\varphi_n) \longrightarrow F(1\Phi) = \int_0^{\Phi} \frac{d\varphi}{\sqrt{1-\sin^2\varphi}}.$$

$$= \log_\varepsilon \tan\left(\frac{\pi}{4}+\frac{\Phi}{2}\right).$$

$$F(k\varphi) = \log_\varepsilon \tan\left(\frac{\pi}{4}+\frac{\Phi}{2}\right)\sqrt{\frac{k_1 k_2 \cdots k_{n-1}}{k}}$$

$\therefore$ The time of a half vibration is.

$$\sqrt{l/g}\cdot \log_\varepsilon \tan\left(\frac{\pi}{4}+\frac{\Phi}{2}\right)\sqrt{\frac{k_1 k_2 \cdots k_{n-1}}{k}}.$$

Hence the periodic time, or the time of a complete oscillation

$$T = 4\sqrt{l/g}\cdot \log_\varepsilon \tan\left(\frac{\pi}{4}+\frac{\Phi}{2}\right)\sqrt{\frac{k_1 k_2 \cdots k_{n-1}}{k}}.$$

In the reduction of $F(k\frac{\pi}{2})$, we have increased the modulus and decreased the amplitude, but by means of these same substitutions with subscripts interchanged we could have derived the formula $F(k\frac{\pi}{2}) = \frac{\pi}{2}(1+k_1)(1+k_2)\cdots(1+k_{n-1})$. where the modulus has been decreased and approaches zero as its limit while the amplitude has been increased. This latter formula is better adapted for the computation of periodic times of pendulums, as the modulus in these computations is always a very small quantity.

To find the periodic time when arc is 1°.

$k = \sin 16' = .0043633.$

$\log k = \bar{3}.6398160$

$\log k^2 = \bar{5}.2796320$

$k^2 = \text{antilog } \bar{5}.2796320.$

$= .0000019038.$

$$k_1 = \frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}$$

$\log k_1 = \log .0000098 - \log 1.9999902.$

$= \bar{6}.9912261 - .3010283.$

$= \bar{6}.6901978.$

$k_1 = \text{antilog } \bar{6}.6901978$

$= .0000049$

$k_2 = 0.$

$F(\sin 15' \tfrac{\pi}{2}) = \tfrac{\pi}{2}(1+k_1).$

$\log F(\sin 15' \tfrac{\pi}{2}) = \log \pi + \log(1+k_1) - \log 2.$

$= .4971525 + .0000021 - .3010300.$

$F(\sin 15' \tfrac{\pi}{2}) = \text{antilog } .1961230$

$= 1.5708207.$

$T = 4\sqrt{l/g} \cdot F(k\tfrac{\pi}{2}).$

$\log T = 2\log 2 - \tfrac{1}{2}\log g + \log F(k\tfrac{\pi}{2}) + \tfrac{1}{2}\log l.$

$= .6020600 - 1.4936120 + .1961230 + \tfrac{1}{2}\log l.$

$= \bar{1}.3025700 + \tfrac{1}{2}\log l.$

cf $l = 10$ cms.

$\log T = \bar{1}.3025700 + .5.$

$T = \text{antilog } \bar{1}.8025700$

$= .634702$ seconds.

cf $l = 20$ cms.

$\log T = \bar{1}.3025700 + .6505150.$

$T = \text{antilog } \bar{1}.9530850$

$= .897604$ seconds.

cf $l = 30$ cms.

$\log T = \bar{1}.3025700 + .7385607.$

$T = \text{antilog } .0411307$

$= 1.0997335$ seconds.

cf $l = 40$ cms.

$\log T = \bar{1}.3025700 + .8010300$

$T = \text{antilog } .1036000$

$= 1.269403$ seconds.

cf $l = 50$ cms.

$\log T = \bar{1}.3025700 + .8494850$

$T = \text{antilog } .1520550$

$= 1.419236$ seconds.

cf $l = 60$ cms

$\log T = \bar{1}.3025700 + .8890757$

$T = \text{antilog } .1916457$

$= 1.554697$ seconds.

cf $l = 70$ cms.

$\log T = \bar{1}.3025700 + .9225490$

$T = \text{antilog } .2251190$

$= 1.679264$ seconds.

cf $l = 80$ cms

$\log T = \bar{1}.3025700 + .9515740$

$T = \text{antilog } .2541150$

$= 1.795210$ seconds.

cf $l = 90$ cms.

$\log T = \bar{1}.3025700 + .9771213$

$T = \text{antilog } .2796913$

$= 1.904164$ seconds.

cf $l = 100$ cms.

$\log T = \bar{1}.3025700 + 1.$

$T = \text{antilog } .3025700$

$= 2.007104$ seconds.

cf $l = 125$ cms.

$\log T = \bar{1}.3025700 + 1.0484550$

$T = \text{antilog } .3510250$

$= 2.244016$ seconds.

cf $l = 150$ cms.

$\log T = \bar{1}.3025700 + 1.0880457$

$T = \text{antilog } .3906157$

$= 2.458191$ seconds.

Summary of Computations.

Periodic times of pendulums of various lengths and swinging through different arcs.

	Arc	10 cms.	20 cms.	30 cms.	40 cms.	50 cms.
A	small.	·634699	·897602	1·099331	1·269398	1·419230
B.	1°	·634702	·897604	1·099335	1·269403	1·419236
	2°	·634711	·897618	1·099352	1·269423	1·419258
	3°	·634726	·897634	1·099372	1·269446	1·419293
	4°	·634747	·897668	1·099415	1·269494	1·419338
	5°	·634774	·897706	1·099464	1·269549	1·419399
	6°	·634804	·897749	1·099514	1·269609	1·419466
	8°	·634889	·897869	1·099661	1·269779	1·419656
	10°	·634998	·898232	1·099824	1·269996	1·419899
	20°	·635805	·899306	1·101320	1·271811.	1·421928.
C.		Periodic Time using the Formula $T = 2\pi\sqrt{l/g}\cdot(1+k^2/4)$				
	3°	·634726	·897634	1·099371	1·269446	1·419292
	5°	·634774	·897706	1·099463.	1·269549	1·419400

A. using the formula $T = 2\pi\sqrt{l/g}$.

B - using the formula $T = 4\sqrt{l/g}\cdot F(k\,\frac{\pi}{2})$.

C - " " " $T = 2\pi\sqrt{l/g}\,(1+\frac{k^2}{4})$.

Periodic times of pendulums of various lengths swinging through different arcs.

	Arc.	70 cms	80 cms	80 cms	90 cms	100 cms.
A	small	1·554689	1·679255	1·795204	1·904096	2·007094
B.	1°	1·554697	1·679264	1·795210	1·904104	2·007104
	2°	1·554720	1·679289	1·795235	1·904135	2·007132
	3°	1·554747	1·679319	1·795266	1·904168	2·007180
	4°	1·554807	1·679383	1·795336	1·904242	2·007248
	5°	1·554875	1·679455	1·795414	1·904323	2·007333
	6°	1·554947	1·679535	1·795498	1·904414	2·007428
	8°	1·555256	1·679760	1·795739	1·904669	2·007697
	10°	1·555421	1·680047	1·796045	1·904995	2·008041
	20°	1·557644	1·682447	1·798612	1·907716	2·010909
C.						
	3°	1·554748	1·679319	1·795266	1·904170	2·007180
	5°	1·554876	1·679456	1·795416	1·904324	2·007334

IV.
Summary of Computations.

Periodic times of pendulums of various lengths swinging through different arcs.

	Arc.	125 cms	150 cms.	200 cms.
A.	small.	2·244004	2·458179	2·838460
B.	1°	2·244016	2·458191	2·838474
	2°	2·244041	2·458221	2·838517.
	3°	2·244094	2·458283	2·838581
	4°	2·244170	2·458366	2·838675
	5°	2·244266	2·458472	2·838798
	6°	2·244373	2·458587	2·838932
	8°	2·244622	2·458917	2·839313
	10°	2·245058	2·459337	2·839798
	20°	2·248265	2·462851	2·843856
C.				
	3°	2·244095	2·458284	2·838582
	5°	2·244267.	2·458471.	2·838799.

V. Summary of Computations

VI.

Summary of Computations.

$F(\sin 15' \ \frac{\pi}{2}) = 1.570806$

$F(\sin 30' \ \frac{\pi}{2}) = 1.570829$

$F(\sin 45' \ \frac{\pi}{2}) = 1.570858$

$F(\sin 1^\circ \ \frac{\pi}{2}) = 1.570919$

$F(\sin 1^\circ 15' \ \frac{\pi}{2}) = 1.570986$

$F(\sin 1^\circ 30' \ \frac{\pi}{2}) = 1.571061$

$F(\sin 2^\circ \ \frac{\pi}{2}) = 1.571271$

$F(\sin 2^\circ 30' \ \frac{\pi}{2}) = 1.571540$

$F(\sin 5^\circ \ \frac{\pi}{2}) = 1.573785$

$g = 980.$

$\frac{1}{2} \log g = .1969456$

$\frac{1}{2} \log 10 = .5000000$

$\frac{1}{2} \log 20 = .6505150$

$\frac{1}{2} \log 30 = .7385607$

$\frac{1}{2} \log 40 = .8010300$

$\frac{1}{2} \log 50 = .8494850$

$\frac{1}{2} \log 60 = .8907757$

$\frac{1}{2} \log 70 = .9225490$

$\frac{1}{2} \log 80 = .9515450$

$\frac{1}{2} \log 90 = .9771213$

$\frac{1}{2} \log 100 = 1.000000$

$\frac{1}{2} \log 125 = 1.048450$

$\frac{1}{2} \log 150 = 1.088046 7$

$\frac{1}{2} \log 200 = 1.150515 0$

If $l = 200$ cms.

$\log T = 1.3025700 + 1.1505150$

$T = \text{antilog } .4530850$

$= 2.8384474$ seconds.

To determine the periodic time by expanding $(1 - k^2 \sin^2\varphi)^{-1/2}$ and integrating term by term

$$(1 - k^2 \sin^2\varphi)^{-1/2} = 1 + \tfrac{1}{2} k^2 \sin^2\varphi + \frac{1\cdot 3}{2\cdot 4} k^4 \sin^4\varphi + + + +$$

$$\int_0^\varphi \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} = \int_0^\varphi d\varphi + \frac{k^2}{2}\int_0^\varphi \sin^2\varphi\, d\varphi + \frac{1\cdot 3}{2\cdot 4} k^4 \int_0^\varphi \sin^4\varphi \cdot d\varphi + +$$

$$= \varphi + \frac{k^4}{4}(\varphi - \cos\varphi \sin\varphi) - \frac{9}{2\cdot 7} k^4 \sin^3\varphi \cos\varphi + \frac{9}{64} k^4 (\varphi - \sin\varphi\cos\varphi)$$

If $\varphi = \frac{\pi}{2}$. $\quad F(k\tfrac{\pi}{2}) = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}$

$$= \frac{\pi}{2}\left(1 + \frac{k^2}{4} + \frac{9k^4}{64} + + \quad\right).$$

$$\therefore T = 2\pi\sqrt{l/g}\cdot\left(1 + \frac{k^2}{4} + \frac{9k^4}{64} + + + \quad\right).$$

$$= 2\pi\sqrt{l/g}\left(1 + \frac{k^2}{4}\right) \text{ - approximately.}$$

This approximation holds to the 8th decimal place for arcs less than 4° and is correct to the 7th for arcs less than 8°.

From the equation of motion set up by equating the physical and theoretical forces we have $\frac{d^2\theta}{dt^2} = -g/l \sin\theta$.

$$= -g/l\left(\theta - \frac{\theta^3}{16} + - + - + \quad\right).$$

but as θ is a small quantity all terms higher than the fourth may be neglected

Integrating.

$$\int_0^{\frac{d\theta}{dt}} d\left(\frac{d\theta}{dt}\right)^2 = -g/l\int_\alpha^\theta d(\theta^2) + \frac{g}{12l}\int_\alpha^\theta d(\theta^4).$$

$$\left(\frac{d\theta}{dt}\right)^2 = g/l(\alpha^2 - \theta^2) - g/12l(\alpha^4 - \theta^4).$$

$$= g/l\left\{(\alpha^2 - \theta^2)\left(1 - \frac{\alpha^2 + \theta^2}{12}\right)\right\}.$$

$$\frac{d\theta}{dt} = -\sqrt{g/l}\cdot\sqrt{(\alpha^2-\theta^2)\left(1-\frac{\alpha^2+\theta^2}{12}\right)}.$$

Separating variables and integrating.

$$-\sqrt{g/l}\int_0^t dt = \int_\alpha^0 \frac{d\theta}{\sqrt{(\alpha^2-\theta^2)\left(1-\frac{\alpha^2+\theta^2}{12}\right)}}.$$

$$t\sqrt{g/l} = \int_\theta^\alpha \frac{d\theta}{\sqrt{(\alpha^2-\theta^2)\left(1-\frac{\alpha^2+\theta^2}{12}\right)}}.$$

put $\theta = \alpha\sin\varphi$ $\quad d\theta = \alpha\cos\varphi\, d\varphi$.

$\alpha^2-\theta^2 = \alpha^2\cos^2\varphi$ and $T = 4t$ where T is the time of a complete oscillation.

Limits. when $t=0$ θ which is the angle at any time "t" is equal to zero and when $t = \frac{T}{4}$ $\theta = \alpha$ $\therefore$ new limits are 0 and $\frac{\pi}{2}$.

$$\therefore \sqrt{g/l}\,\frac{T}{4} = \int_0^{\frac{\pi}{2}} \frac{\alpha\cos\varphi\, d\varphi}{\alpha\cos\varphi\sqrt{1-\frac{\alpha^2+\alpha^2\sin^2\varphi}{12}}}$$

$$= \int_0^{\frac{\pi}{2}} \left\{(12-\alpha^2)-\alpha^2\sin^2\varphi\right\}^{-1/2} d\varphi.$$

$$= \left(\frac{12-\alpha^2}{12}\right)^{-1/2}\int_0^{\frac{\pi}{2}}\left(1-\frac{\alpha^2\sin^2\varphi}{12-\alpha^2}\right)^{-1/2} d\varphi.$$

$$= \left(1+\frac{\alpha^2}{24}+++\right)\int_0^{\frac{\pi}{2}}\left(1+\frac{\alpha^2\sin^2\varphi}{2(12-\alpha^2)}++\right)d\varphi$$

$$= \left(1+\frac{\alpha^2}{24}\right)\left(\frac{\pi}{2}+\frac{\alpha^2}{2(12-\alpha^2)}\,\frac{\pi}{4}\right).$$

$$= \frac{\pi}{2}\left(1+\frac{\alpha^2}{24}\right)\left\{1+\frac{\alpha^2}{48\left(1-\frac{\alpha^2}{48}\right)}\right\}$$

$$= \frac{\pi}{2}\left(1+\frac{\alpha^2}{24}\right)\left(1+\frac{\alpha^2}{48}\right) \text{ approximately}$$

$$= \frac{\pi}{2}\left(1+\frac{3\alpha^2}{48}+\frac{\alpha^4}{24\cdot 48}\right).$$

$$= \frac{\pi}{2}\left(1+\frac{\alpha^2}{16}\right) \text{ --- approximately}$$

$$\therefore T = 2\pi\sqrt{l/g}\cdot\left(1+\frac{\alpha^2}{16}\right).$$

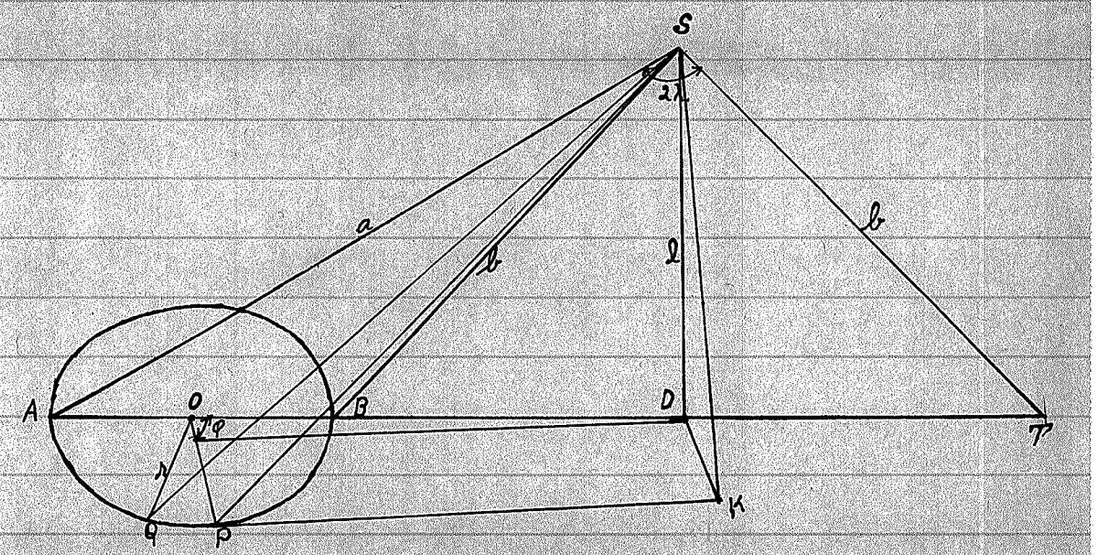
Periodic Times using the formula $T = 2\pi\sqrt{l/g}\cdot\left(1+\frac{\alpha^2}{16}\right)$.

Arc.	10 cms	30 cms	60 cms	100 cms	200 cms.
1°	·634702	1·099335	1·419236	2·007104	2·838474
2°	·634711	1·099351	1·419257	2·007182	2·838515
3°	·634726	1·099371	1·419291	2·007180	2·838582
4°	·634747	1·099414	1·419337	2·007247	2·838676
5°	·634774	1·099464	1·419400	2·007334	2·838799
6°	·634807	1·099519	1·419473	2·007438	2·838946
8°	·634891	1·099665	1·419661	2·007704	2·839321
10°	·635003.	1·099851	1·419910	2·008055	2·839820

VII

Summary of Computations

To find the area of the Surface of the Oblique Cone.



Notations.—

$OB = r$ $\quad AS = a$ $\quad SD = l$

$OD = m$ $\quad BS = b$ $\quad AT = 2m.$

$\angle BOP = \varphi$ $\quad \angle POQ = \Delta\varphi$ $\quad \angle AST = 2\lambda.$

Relations.—

$$l^2 + (m+r)^2 = a^2 \quad \text{---------- (1).}$$

$$l^2 + (m-r)^2 = b^2 \quad \text{---------- (2).}$$

$$l^2 + m^2 + r^2 = \frac{a^2+b^2}{2} \quad \text{---------- (3).}$$

$$4mr = a^2 - b^2 \quad \text{---------- (4).}$$

$$m^2 = \frac{(a^2-b^2)^2}{16r^2} \quad \text{---------- (5).}$$

$$m = r + b\sin\lambda \quad \text{---------- (6).}$$

$$l = b\cos\lambda \quad \text{---------- (7).}$$

$$4m^2 = a^2 + b^2 - 2ab\cos 2\lambda \quad \text{---------- (8).}$$

Step ①.

To find the area of the elemental triangle SPQ.

$$\Delta.SPQ = \tfrac{1}{2}(QP.SK) = \tfrac{1}{2}\, r.\Delta\varphi.SK.$$

$$SK = \sqrt{SD^2 + DK^2}.$$

$$= \sqrt{l^2 + (r - m\cos\varphi)^2}.$$

$\therefore$ Area of triangle $= \frac{r}{2}\cdot\sqrt{l^2 + (r - m\cos\varphi)^2}.\,\Delta\varphi$ ---①.

Step ②.

$\therefore$ Area of Surface of the Oblique cone.

$$S/2 = \frac{r}{2}\int_0^{\pi}\sqrt{l^2 + (r - m\cos\varphi)}\,.\,d\varphi.$$

$$S = r\int_0^{\pi}\sqrt{l^2 + (r - m\cos\varphi)}\;\; d\varphi.$$

Transformation –

put $\cos\varphi = x$. $\quad \sin\varphi = \sqrt{1-x^2}$

$$d\varphi = -\frac{dx}{\sqrt{1-x^2}}.$$

Limits. $\quad \varphi = 0 \qquad x = 1.$

$\qquad\qquad \varphi = \pi \qquad x = -1.$

$$\therefore S = r\int_{-1}^{1}\sqrt{\frac{l^2 + (r - mx)^2}{1-x^2}}\;\; dx.$$

Reduction of $\int_{-1}^{+1}\sqrt{\frac{l^2 + (r-mx)^2}{1-x^2}}\;dx$

Transformation:

Let $x = \frac{p + qy}{1 + by}$ $\qquad dx = \frac{q - p}{(1+y)^2}\,dy.$

$$l^2 + (r - mx)^2 = l^2 + r^2 + 2mr\cdot\left(\frac{p+qy}{1+y}\right) + m^2\left(\frac{p+qy}{1+y}\right)^2.$$

$$= \frac{(l^2 + r^2 + m^2p^2 - 2mrp) + 2y[l^2 + r^2 - mr(p+q) + m^2pq]}{(1+y)^2}$$

$$+ \frac{(l^2 + r^2 - 2mrq + m^2q^2)\,y^2}{(1+y)^2}$$

$$1 - x^2 = 1 - \left(\frac{p+qy}{1+y}\right)^2$$

$$= \frac{(1-p^2) + 2y(1-pq) + (1-q^2)y^2}{(1+y)^2}.$$

To find p and q.

So far no restrictions have been placed on p and q; we can choose them so that the term involving y will vanish

$\therefore l^2 + r^2 - mr(p+q) + m^2 pq = 0$

and $1 - pq = 0$

$$p + q = \frac{l^2 + r^2 + m^2}{mr}$$

$$pq = 1.$$

$$p - q = \sqrt{\left(\frac{l^2+r^2+m^2}{mr}\right)^2 - 4}.$$

$$\therefore 2p = \frac{l^2+r^2+m^2}{mr} + \sqrt{\frac{(l^2+r^2+m^2)^2}{(mr)^2} - 4}.$$

$$p = \frac{1}{4mr}\left[2(l^2+r^2+m^2) + \sqrt{(l^2+r^2+m^2)^2 - 16m^2r^2}\right]$$

from relations ③ and ④.

$$l^2 + m^2 + r^2 = \frac{a^2+b^2}{2}$$

$$4mr = a^2 - b^2.$$

$$\therefore p = \frac{1}{a^2-b^2}\left[a^2+b^2 \pm \sqrt{(a^2+b^2)^2 - (a^2-b^2)^2}\right].$$

$$= \frac{a^2+b^2 \pm 2ab}{a^2-b^2}$$

$$= \frac{a-b}{a+b}.$$

$$2q = \frac{l^2+r^2+m^2}{mr} - \sqrt{\frac{(l^2+r^2+m^2)^2}{m^2r^2} - 4}.$$

$$q = \frac{1}{4mr}\left[2(l^2+r^2+m^2) - \sqrt{4(l^2+r^2+m^2) - 16m^2r^2}\right].$$

$$= \frac{1}{a^2-b^2}\left[a^2+b^2 - \sqrt{(a^2+b^2)^2 - (a^2-b^2)^2}\right].$$

$$= \frac{a^2+b^2 \mp 2ab}{a^2-b^2}$$

$$= \frac{a+b}{a-b}$$

Limits (a). when $x = 1$.

$$1 = \frac{p+qy}{1+pqy}$$

$$y = \frac{p-1}{1-q} = \frac{\frac{a-b}{a+b}-1}{1-\frac{a+b}{a-b}} = \frac{a-b}{a+b}.$$

(b) when $x = -1$.

$$-1 = \frac{p+qy}{1+pqy}$$

$$y = -\frac{p+1}{q+1} = -\frac{\frac{a-b}{a+b}+1}{1+\frac{a-b}{a+b}} = -\frac{a-b}{a+b}.$$

Under the condition that the coefficient of $y = 0$

$$l^2+(x-mc)^2 = \frac{(l^2+r^2+m^2p^2-2mrp)+(l^2+r^2+m^2p^2-2mrq)y^2}{(1+y)^2}$$

Substitutions –

$$l^2+r^2+m^2 = \frac{a^2+b^2}{2}$$

$$l^2+r^2 = \frac{a^2+b^2}{2} - m^2$$

$$4mr = a^2-b^2$$

$$m^2 = \frac{(a^2-b^2)^2}{16r^2}$$

$$p = \frac{a-b}{a+b} \qquad p^2-1 = -\frac{4ab}{(a+b)^2}$$

$$q = \frac{a+b}{a-b} \qquad q^2-1 = \frac{4ab}{(a-b)^2}.$$

$$l^2+(x-mc)^2 = \frac{\frac{a^2+b^2}{2}+m^2(p^2-1)-2mrp+\left\{\frac{a^2+b^2}{2}+m^2(q^2-1)-2mrq\right\}y^2}{(1+y)^2}$$

$$= \frac{\frac{a^2+b^2}{2}+\frac{(a^2-b^2)^2}{16r^2}\cdot\frac{4ab}{(a+b)^2}-\frac{1}{2}(a^2-b^2)\frac{a-b}{a+b}+\left\{\frac{a^2+b^2}{2}-\frac{a^2-b^2}{2}\cdot\frac{a+b}{a-b}+\frac{a^2-b^2}{16r^2}\cdot\frac{4ab}{(a+b)^2}\right\}y^2}{(1+y)^2}$$

$$= \frac{\left\{ab-\frac{ab(a-b)^2}{4r^2}\right\}+\left\{-ab+\frac{ab(a+b)^2}{4r^2}\right\}y^2}{(1+y)^2}.$$

$$= \frac{ab\left\{1-\frac{(a-b)^2}{4r^2}+\left[\frac{(a+b)^2}{4r^2}-1\right]y^2\right\}}{(1+y)^2}.$$

$$1-x^2 = \frac{(1-p^2)+(1-q^2)y^2}{(1+y)^2}$$

$$= \frac{\frac{4ab}{(a+b)^2}-\frac{4ab}{(a-b)^2}y^2}{(1+y)^2}$$

$$= \frac{4ab}{(a+b)^2}\cdot\frac{1-\left(\frac{a+b}{a-b}\right)^2y^2}{(1+y)^2}.$$

Hence $\lambda \int_{-1}^{1} \frac{l^2 + (\lambda - mx)^2}{\sqrt{(1-x^2)\{l^2 + (\lambda - mx)^2\}}}\, dx.$

$$= \lambda \int_{-\frac{a-b}{a+b}}^{\frac{a-b}{a+b}} \frac{ab\left\{1 - \frac{a-b}{4\lambda^2} + \left[\frac{(a+b)^2}{4\lambda^2} - 1\right] y^2\right\} \frac{4ab}{a^2-b^2} \cdot \frac{dy}{(1+y)^2}}{\sqrt{\frac{4a^2b^2}{(a+b)^2} \cdot \left[1 - \left(\frac{a+b}{a-b}\right)^2 y^2\right]\left[1 - \frac{(a-b)^2}{4\lambda^2} + \left\{\frac{(a+b)^2}{4\lambda^2} - 1\right\} y^2\right]}}.$$

$$= \frac{2\lambda ab}{a-b} \int_{-\frac{a-b}{a+b}}^{\frac{a-b}{a+b}} \frac{1 - \frac{(a-b)^2}{4\lambda^2} + \left[\frac{(a+b)^2}{4\lambda^2} - 1\right] y^2}{\sqrt{\left[1 - \left(\frac{a+b}{a-b}\right)^2 y^2\right]\left[1 - \frac{(a-b)^2}{4\lambda^2} + \left\{\frac{(a+b)^2}{4\lambda^2} - 1\right\} y^2\right]}} \cdot \frac{dy}{(1+y)^2}.$$

Substitution – from relations ① and ② we have.

$$4m\lambda = a^2 - b^2$$

$$4m^2 = a^2 - b^2 - 2ab\cos 2\lambda.$$

$$1 - \frac{a-b}{4\lambda^2} = 1 - \frac{4m^2}{(a^2-b^2)^2}(a-b)^2$$

$$= \frac{(a+b)^2 - 4m^2}{(a+b)^2}$$

$$= \frac{(a+b)^2 - (a^2 + b^2 - 2ab\cos 2\lambda)}{(a+b)^2}$$

$$= \frac{2ab + 2ab\cos 2\lambda}{(a+b)^2}$$

$$= \frac{4ab\cdot\cos^2\lambda}{(a+b)^2}.$$

$$\frac{(a+b)^2}{4\lambda^2} - 1 = (a+b)^2 \cdot \frac{4m^2}{(a^2-b^2)^2} - 1.$$

$$= \frac{4m^2 - (a-b)^2}{(a-b)^2}$$

$$= \frac{a^2 + b^2 - 2ab\cos 2\lambda - (a-b)^2}{(a-b)^2}$$

$$= \frac{2ab - 2ab\cos 2\lambda}{(a-b)^2}$$

$$= \frac{4ab\sin^2\lambda}{(a-b)^2}.$$

$$\therefore S = \frac{2\pi ab}{a-b} \int_{-\frac{a-b}{a+b}}^{\frac{a-b}{a+b}} \frac{4ab\left[\frac{\cos^2\lambda}{(a+b)^2} + \frac{\sin^2\lambda}{(a-b)^2}y^2\right]\frac{dy}{(1+y)^2}}{\sqrt{\left[1-\left(\frac{a+b}{a-b}\right)^2 y^2\right]\left[4ab\left\{\frac{\cos^2\lambda}{(a+b)^2} + \frac{\sin^2\lambda}{(a-b)^2}y^2\right\}\right]}}.$$

$$= \frac{4\pi(ab)^{3/2}}{a^2-b^2} \int_{-\frac{a-b}{a+b}}^{\frac{a-b}{a+b}} \frac{1 + \left(\frac{a+b}{a-b}\right)^2 \tan^2\lambda\, y^2}{\sqrt{\left[1-\left(\frac{a+b}{a-b}\right)^2 y^2\right]\left[1+\left(\frac{a+b}{a-b}\right)^2 \tan^2\lambda\, y^2\right]}} \cdot \frac{dy}{(1+y)^2}.$$

Substitution.

put $\frac{a+b}{a-b}\tan\lambda = \rho$ and $\frac{a+b}{a-b} = \sigma$.

$$\therefore S = \frac{4\pi(ab)^{3/2}\cos\lambda}{a^2-b^2} \int_{-\frac{a-b}{a+b}}^{\frac{a-b}{a+b}} \frac{(1+\rho^2y^2)(1-y)^2\,dy}{(1-y^2)^2\sqrt{(1-\sigma^2y^2)(1+\rho^2y^2)}}.$$

$$= \frac{8\pi(ab)^{3/2}\cos\lambda}{a^2-b^2} \int_0^{\frac{a-b}{a+b}} \frac{(1+\rho^2y^2)(1+y^2)}{(1-y^2)^2} \cdot \frac{dy}{\sqrt{(1-\sigma^2y^2)(1+\rho^2y^2)}}.$$

Since $\int_{-\frac{a-b}{a+b}}^{\frac{a-b}{a+b}} -2y(1+\rho^2y^2)\frac{dy}{\sqrt{(1-\sigma^2y^2)(1+\rho^2y^2)}} = 0$

i.e. $\int_{-y_1}^{y_1} y^{n+1}\,dy = 0$

n = positive, even integer.

Transformation from "y" to "z".

put $y^2 = \frac{1-z^2}{\sigma^2}$

$y = \frac{\sqrt{1-z^2}}{\sigma}$ $\quad dy = -\frac{z}{\sigma}\frac{dz}{\sqrt{1-z^2}}$.

$$1-y^2 = 1 - \frac{1-z^2}{\sigma^2} = \frac{\sigma^2-1+z^2}{\sigma^2}.$$

$$1+\rho^2y^2 = 1 + \frac{\rho^2(1-z^2)}{\sigma^2} = \frac{\sigma^2+\rho^2-\rho^2z^2}{\sigma^2}.$$

$$1-\sigma^2y^2 = 1 - \frac{\sigma^2(1-z^2)}{\sigma^2} = z^2.$$

$$z^2 = 1-\sigma^2y^2 = 1-\left(\frac{a+b}{a-b}\right)^2 y^2.$$

Limits

if $y=0$ $\quad z^2 = 1-\left(\frac{a+b}{a-b}\right)^2 y^2 = 1$ $\quad \therefore z = 1$

if $y, \frac{a-b}{a+b}$ $\quad z^2 = 1-\left(\frac{a+b}{a-b}\right)^2\cdot\left(\frac{a-b}{a+b}\right)^2 = 0$ $\quad \therefore z = 0$

$$\therefore S = \frac{8\pi(ab)^{3/2}\cos\lambda}{(\sigma^2+\rho^2)^{1/2}(a^2-b^2)} \int_0^1 \frac{\sigma^4+\sigma^2+\rho^2\sigma^2+\rho^2-t^2(\sigma^2+\sigma^2\rho^2+2\rho^2)+\sigma^2t^4}{(\sigma^2-1+t^2)^2} \frac{dt}{\sqrt{T}}$$

where $\sqrt{T} = (1-t^2)\left(1-\frac{\rho^2}{\rho^2+\sigma^2}\cdot t^2\right)$.

Substitutions.—

$$\frac{\rho^2}{\rho^2+\sigma^2} = \frac{\left(\frac{a+b}{a-b}\right)^2 \tan^2\lambda}{\left(\frac{a+b}{a-b}\right)^2\tan^2\lambda + \left(\frac{a+b}{a-b}\right)^2}$$

$$= \frac{\tan^2\lambda}{\tan^2\lambda+1} = \frac{\tan^2\lambda}{\sec^2\lambda} = \sin^2\lambda.$$

$$(\sigma^2+\rho^2)^{1/2} = \sqrt{\left(\frac{a+b}{a-b}\right)^2 + \left(\frac{a+b}{a-b}\right)^2\tan^2\lambda}$$

$$= \frac{a+b}{a-b}\cdot\sqrt{1+\tan^2\lambda} = \frac{a+b}{a-b}\cdot\sec\lambda.$$

The expression $\frac{\sigma^4+\sigma^2+\rho^2\sigma^2+\rho^2-t^2(\sigma^2+\rho^2\sigma^2+2\rho^2)+\rho^2t^4}{(\sigma^2-1+t^2)}$

can be broken up into the following partial fractions.

$$\rho^2 - \sigma^2(3\rho^2+1)\frac{1}{\sigma^2-1+t^2} + 2\sigma^4(1+\rho^2)\cdot\frac{1}{(\sigma^2-1+t^2)^2}.$$

$$\therefore S = \frac{8\pi(ab)^{3/2}\cos^2\lambda}{(a+b)^2}\left[\int_0^1 \frac{\rho^2\,dt}{\sqrt{(1-t^2)(1-\sin^2\lambda\cdot t^2)}} - \int_0^1 \frac{\sigma^2(3\rho^2+1)\,dt}{(\sigma^2-1+t^2)\sqrt{(1-t^2)(1-\sin^2\lambda\cdot t^2)}}\right.$$

$$\left. + \int_0^1 \frac{2\sigma^4(1+\rho^2)\,dt}{(\sigma^2-1+t^2)^2\sqrt{(1-t^2)(1-\sin^2\lambda\,t^2)}}\right].$$

Substitutions.—

$$\rho^2 = \left(\frac{a+b}{a-b}\right)^2\tan^2\lambda.$$

$$\sigma^2 = \left(\frac{a+b}{a-b}\right)^2.$$

$$\sigma^2-1-t^2 = \left(\frac{a+b}{a-b}\right)^2 - 1 + t^2.$$

$$= \frac{(a+b)^2-(a-b)^2+t^2(a-b)^2}{(a-b)^2}$$

$$= \frac{4ab-(a-b)^2t^2}{(a-b)^2}$$

$$= \frac{4ab}{(a-b)^2}\cdot\left(1-\frac{(a-b)^2}{4ab}\cdot t^2\right).$$

$$= \frac{1}{n}\cdot(1-nt^2)$$

where $\frac{(a-b)^2}{4ab} = n$.

Hence $\vartheta = \frac{8\pi(ab)^{3/2}\sin^2\lambda}{(a-b)^2}\int_0^1 \frac{dt}{\sqrt{T}} - 2\pi(ab)^{1/2}\cos^2\lambda\left\{1+3\left(\frac{a+b}{a-b}\right)^2\sin^2\lambda\right.$

$\left.\int_0^1 \frac{dt}{(1+nt^2)\sqrt{T}}\right\} + \frac{\pi(a+b)^2\cos^2\lambda\cdot\left\{1+\left(\frac{a+b}{a-b}\right)^2\sin^2\lambda\right\}}{(ab)^{1/2}}\int_0^1 \frac{dt}{(1+nt^2)^2\sqrt{T}}$

where $T = (1-t^2)(1-\sin^2\lambda\cdot t^2)$.

Reduction of $z_2 = \int_0^1 \frac{dt}{(1+nt^2)^2\sqrt{T}}$. — Reference

Adamack's Introduction to the Calculus. articles 121 – 126.

This integral can be broken up into one of the form $k_0 y_0 + k_1 y_1 + k_2 z_1 + \psi(t\sqrt{T})_0^1$.

where.

y_0 is a complete integral of first class.

$\therefore y_0 = \int_0^1 \frac{dt}{\sqrt{T}}$

y_1 is a complete integral of second class.

$\therefore y_1 = \int_0^1 \frac{(1-nt^2)\,dt}{\sqrt{T}}$

z_1 is a complete integral of the third class.

$\therefore z_1 = \int_0^1 \frac{dt}{(1+nt^2)\sqrt{T}}$.

For an integral of the third class, when V is positive and equal to 2., we have the reduction formula

$$z_2 = \frac{1}{k_0}\left[\frac{t\sqrt{T}}{1+nt^2}\right\}_0^1 - k_1 z_1 - k_2 z_2 - k_3 z_3\Big\}.$$

$$k_0 = \frac{1}{n^2}\left[n^2(2V-2) + n(2V-2) + n k^2(2V-2) + k^2(2V-2)\right]$$

$$= \frac{2V-2}{n^2}\left[n^2 + n + n k^2 + k^2\right\}.$$

$$= \frac{2V-2}{n^2}\left[(n+1)(n+k^2)\right\}.$$

$$= \frac{2}{n^2}(n+1)(n+\sin^2\lambda).$$

$$\lambda_1 = \frac{3-2\nu}{n^2}\left[n^2 + 2n + 2nk^2 + 3k^2 \right\}.$$

$$= -\frac{2\nu-3}{n^2}\left[n(n+2) + k^2(2n+3) \right\}.$$

$$= -\frac{1}{n^2}\cdot\left[n(n+2) + \sin^2\lambda(2n+3) \right\}.$$

$$\lambda_2 = \frac{2\nu-4}{n^2}\left[(n+k^2)(n+3) \right\}.$$

$$= \frac{0}{n^2}\left[(n+\sin^2\lambda)(n+3) \right\}.$$

$$= 0.$$

$$\lambda_3 = -\frac{(2\nu-3)}{n^2}\cdot k^2.$$

$$= \frac{\sin^2\lambda}{n^2}.$$

$$\therefore Z_2 = \frac{n^2}{2(n+1)(n+\sin^2\lambda)}\left[\frac{n(n+2)+\sin^2\lambda(2n+3)}{n^2} Z_1 - \frac{\sin^2\lambda}{n^2} Z_{-1} \right\}.$$

$$Z_{-1} = \int_0^{\prime} \frac{(1+nt^2)\,dt}{\sqrt{T}} = y_0 + n y_1 \quad \text{i.e. } Z_{-1} \text{ is linearly expressed in terms of } y_0 \text{ and } y_1.$$

$$\therefore Z_2 = \frac{n(n+2)+\sin^2\lambda(2n+3)}{2(n+1)(n+\sin^2\lambda)} Z_1 - \frac{\sin^2\lambda}{2(n+1)(n+\sin^2\lambda)} y_0 - \frac{n\sin^2\lambda}{2(n+1)(n+\sin^2\lambda)} y_1 -$$

By this reduction formula.

$$\frac{n(a+b)^2\cos^2\lambda}{(ab)^{1/2}}\cdot\left[1+\left(\frac{a+b}{a-b}\right)^2\tan^2\lambda \right\}\int_0^{1} \frac{dt\cdot}{(1+nt^2)^2\sqrt{(1-t^2)(1-\sin^2\lambda\cdot t^2)}}.$$

$$= \frac{n(n+2)+\sin^2\lambda(2n+3)}{2(n+1)(n+\sin^2\lambda)}\cdot\frac{n(a+b)^2\cos^2\lambda}{(ab)^{1/2}}\cdot\left[1+\left(\frac{a+b}{a-b}\right)^2\tan^2\lambda \right\} Z_1$$

$$-\frac{\sin^2\lambda}{2(n+1)(n+\sin^2\lambda)}\cdot\frac{n(a+b)^2\cos^2\lambda}{(ab)^{1/2}}\cdot\left[1+\left(\frac{a+b}{a-b}\right)^2\tan^2\lambda \right\} y_0.$$

$$-\frac{n\sin^2\lambda}{2(n+1)(n+\sin^2\lambda)}\cdot\frac{n(a+b)^2\cos^2\lambda}{(ab)^{1/2}}\cdot\left[1+\left(\frac{a+b}{a-b}\right)^2\tan^2\lambda \right\} y_1.$$

Where y_0, y_1, Z_1 are complete integrals of the first, second, and third classes respectively.

$$\therefore \Phi = \left\{ \frac{8r(ab)^{3/2}}{(a-b)^2} \sin^2\lambda - \frac{r(a+b)^2\cos^2\lambda}{(ab)^{1/2}} \left[1 + \left(\frac{a+b}{a-b}\right)^2 \tan^2\lambda \right\} \frac{\sin^2\lambda}{2(n+1)(n+\sin^2\lambda)} \right\} y_0$$

$$- \left\{ \frac{n\sin^2\lambda}{2(n+1)(n+\sin^2\lambda)} \cdot \frac{r(a+b)^2\cos^2\lambda}{(ab)^{1/2}} \left[1 + \left(\frac{a+b}{a-b}\right)^2 \tan^2\lambda \right\} \right\} y_1$$

$$+ \left\{ \frac{n(n+2)+\sin^2\lambda(2n+3)}{2(n+1)(n+\sin^2\lambda)} \cdot \frac{r(a+b)^2\cos^2\lambda}{(ab)^{1/2}} \left[1 + \left(\frac{a+b}{a-b}\right)^2 \tan^2\lambda \right\} - 2r(ab)^{1/2}\cos^2\lambda \left[1 + 3\left(\frac{a+b}{a-b}\right)^2 \tan^2\lambda \right\} \right\} z_1.$$

Special Cases—

Case I. Let m be greater than r and $b = 0$

Then $\lambda = \frac{\pi}{2}$ $\sin\lambda = 1$ $\cos\lambda = 0$ and $a - b = 2r$.

$$\Phi = \left\{ \frac{8r(ab)^{3/2}}{4r^2} - \frac{r(a+b)^2}{(ab)^{1/2}} \left(\frac{a+b}{a-b}\right)^2 \frac{1}{2(n+1)^2} \right\} \int_0^1 \frac{dt}{1-t^2} \cdot$$

$$- \left\{ \frac{n}{2(n+1)^2} \cdot \frac{r(a+b)^2}{(ab)^{1/2}} \cdot \left(\frac{a+b}{a-b}\right)^2 \right\} \int_0^1 \frac{dt}{1-t^2} \cdot$$

$$+ \left\{ \frac{n}{2(n+1)^2} \cdot \frac{r(a+b)^2}{(ab)^{1/2}} \cdot \left(\frac{a+b}{a-b}\right)^2 \right\} \int_0^1 dt \cdot$$

$$+ \left\{ \frac{n(n+2)+(2n+3)}{2(n+1)^2} \cdot \frac{r(a+b)^2}{(ab)^{1/2}} \cdot \left(\frac{a+b}{a-b}\right)^2 - 2r(ab)^{1/2} 3\left(\frac{a+b}{a-b}\right)^2 \right\} \int_0^1 \frac{dt}{(1+nt^2)(1-t^2)}$$

$$= \left\{ \frac{2(ab)^{3/2}}{r} - \frac{2(ab)^{3/2}}{r} \right\} \int_0^1 \frac{dt}{1-t^2} - \frac{n}{2(n+1)^2} \cdot \frac{r(a+b)^2}{(ab)^{1/2}} \cdot \left(\frac{a+b}{a-b}\right)^2 \int_0^1 \frac{dt}{1-t^2} \cdot +$$

$$\frac{n}{2(n+1)^2} \cdot \frac{r(a+b)^2}{(ab)^{1/2}} \left(\frac{a+b}{a-b}\right)^2 \int_0^1 dt + \left\{ \frac{(n+1)+2}{2(n+1)} \cdot \frac{r(a+b)^2}{(ab)^{1/2}} - 6r(ab)^{1/2} \left(\frac{a+b}{a-b}\right)^2 \right\} \int_0^1 \frac{dt}{(1+nt^2)(1-t^2)}$$

$$= \left\{ \frac{n}{2(n+1)^2} r \frac{(a+b)^2}{(ab)^{1/2}} \cdot \left(\frac{a+b}{a-b}\right)^2 \right\} \int_0^1 dt - \left\{ \frac{n}{2(n+1)^2} \cdot \frac{r(a+b)^2}{(ab)^{1/2}} \cdot \frac{(a+b)^2}{(a-b)^2} \right\} \int_0^1 \frac{dt}{1-t^2}$$

$$+ \left\{ \frac{r(a+b)^2}{2(ab)^{1/2}} \cdot \left(\frac{a+b}{a-b}\right)^2 + \frac{r(a+b)^2}{(n+1)(ab)^{1/2}} \left(\frac{a+b}{a-b}\right)^2 - 6r(ab)^{1/2} \left(\frac{a+b}{a-b}\right)^2 \right\} \int_0^1 \frac{dt}{n+1} \left(\frac{n}{1+nt^2} + \frac{1}{1-t^2} \right).$$

$$= 2r(ab)^{1/2} - \left\{ \frac{n}{2(n+1)^2} \cdot \frac{r(a+b)^2}{(ab)^{1/2}} \cdot \left(\frac{a+b}{a-b}\right)^2 \right\} \int_0^1 \frac{dt}{1-t^2} \cdot$$

$$+ \frac{1}{n+1} \left\{ \frac{r(a+b)^2}{2(ab)^{1/2}} \cdot \left(\frac{a+b}{a-b}\right)^2 + \frac{1}{n+1} \frac{r(a+b)^2}{(ab)^{1/2}} \cdot \left(\frac{a+b}{a-b}\right)^2 - 6r(ab)^{1/2} \left(\frac{a+b}{a-b}\right)^2 \right\} \int_0^1 \frac{dt}{1-t^2} \cdot$$

$$+ \frac{n}{n+1} \left\{ \frac{r(a+b)^2}{2(ab)^{1/2}} \left(\frac{a+b}{a-b}\right)^2 + \frac{r(a+b)^2}{(n+1)(ab)^{1/2}} \left(\frac{a+b}{a-b}\right)^2 - 6r(ab)^{1/2} \left(\frac{a+b}{a-b}\right)^2 \right\} \int_0^1 \frac{dt}{1+nt^2} \cdot$$

$$= 2r(ab)^{1/2} + \left\{ \frac{6(ab)^{3/2}}{2} - \frac{6(ab)^{3/2}}{1} \right\} \int_0^1 \frac{dt}{1-t^2}.$$

$$+ \left[\frac{r(a+b)^2}{2(ab)^{1/2}} + \frac{1}{n+1} \frac{r(a+b)^2}{(ab)^{1/2}} \cdot - 6r(ab)^{1/2} \right\} \int_0^1 \frac{dt}{1+nt^2}.$$

$$= 2r(ab)^{1/2} + \left\{ \frac{(a+b)^2}{2} - 2ab \right\} \frac{r}{(ab)^{1/2}} \int_0^1 \frac{dt}{1+nt^2}.$$

$$= 2r(ab)^{1/2} + \left\{ \frac{(a+b)^2}{2} - 2ab \right\} \frac{r}{(ab)^{1/2}} \frac{1}{2n} \cdot \tan^{-1} n.$$

Substitutions —

$$\frac{(a+b)^2}{2} - 2ab = \frac{(a-b)^2}{2}.$$

$$= \frac{4r^2}{2} = 2r^2.$$

$$\frac{r}{2n(ab)^{1/2}} = \frac{a-b}{(ab)^{1/2}} \cdot \frac{(ab)^{1/2}}{a-b} = 1.$$

$$\sqrt{ab} = \sqrt{b(b+2r)}$$

$$= \sqrt{b^2+2br}.$$

$$\therefore S = 2r\sqrt{b^2+2br} + 2r^2 \tan^{-1} \frac{a-b}{(ab)^{1/2}}.$$

$$= 2r\sqrt{b^2+2br} + 2r^2 \left(\frac{\pi}{2} - \cot^{-1} \frac{r}{\sqrt{b^2+2br}} \right).$$

$$= \pi r^2 + 2r \left\{ \sqrt{b^2+2br} - \cos^{-1} \frac{r}{r+b} \right\}.$$

Case II. Let $b = 0$ $\quad m = r$.

$$\therefore S = \pi r^2 + 2r \{ 0 - \cos^{-1} 1 \}.$$

$$= \pi r^2.$$

Case III. Let $l = 0$ $\quad m < r$.

$\lambda = 0$ $\quad \cos\lambda = 1$ $\quad \sin\lambda = \tan\lambda = 0$ $\quad a+b = 2r$.

Substituting these values in the general result we see two of the terms vanish.

$$\therefore S = \left\{ \frac{n+2}{2(n+1)} \cdot \frac{r(a+b)^2}{(ab)^{1/2}} - 2r(ab)^{1/2} \right\} \int_0^1 \frac{dt}{(1+nt^2)\sqrt{1-t^2}}.$$

$$= \left\{ \frac{r(a+b)^2}{2(ab)^{1/2}} + \frac{r(a+b)^2}{2(n+1)(ab)^{1/2}} - 2r(ab)^{1/2} \right\} \int_0^1 \frac{dt}{(1+nt^2)\sqrt{1-t^2}}.$$

$$= \left\{ \tfrac{1}{2} \cdot \frac{r(2r)^2}{(ab)^{1/2}} + \frac{2ab}{(a+b)^2} \cdot \frac{r(a+b)^2}{(ab)^{1/2}} - 2r(ab)^{1/2} \right\} \int_0^1 \frac{dt}{(1+nt^2)\sqrt{1-t^2}}.$$

$$= \frac{2r^3}{(ab)^{1/2}} \cdot \int_0^{\pi} \frac{d\theta}{1+n\sin^2\theta} = \frac{2r^3}{(ab)^{1/2}} \cdot \frac{1}{n+1} \cdot \frac{\pi}{2} = \pi r^2.$$

Lemma.

To Evaluate $\int_0^{\pi} \frac{d\varphi}{1+n\sin^2\varphi}$.

Substitution: put $\tan\varphi = u$

$$d\varphi = \frac{du}{1+u^2}$$

$$\sin^2\varphi = \frac{\tan^2\varphi}{1+\tan^2\varphi} = \frac{u^2}{1+u^2}.$$

Limits

When $\varphi = 0$ $\quad u = 0$

$\varphi = \pi$ $\quad u = -0$.

But as $\tan\varphi$ is discontinuous when $\varphi = \frac{\pi}{2}$, we integrate from 0 to $\frac{\pi}{2}$ and from $\frac{\pi}{2}$ to π.

i.e. $0 \to \infty$ and $\infty \to -0$

ie. $I\}_0^{\infty} + I\}_{\infty}^{-0} = 2\,I\}_0^{\infty}$.

$$\therefore I = 2\int_0^{\infty} \frac{dx}{(1+x^2)\left(1+\frac{nx^2}{1+x^2}\right)}.$$

$$= 2\int_0^{\infty} \frac{dx}{1+(n+1)x^2}.$$

$$= \frac{2}{n+1}\cdot \tan^{-1}(n+1)x^2\Big\}_0^{\infty} = \frac{\pi}{n+1}.$$

Case IV $\quad a = b \quad m = r \quad n = \frac{(a-b)^2}{4ab} = 0$.

$$\frac{(a+b)^2}{4ab} = \frac{(a-b)^2 + 4ab}{4ab} = n+1 = 1.$$

$\lambda = 0$, As the cone becomes more and more nearly right circular λ continuously decreases and ultimately will approach zero.

Substituting these values in the general formula, the first two terms on the right vanish.

$$\therefore S = \left\{\frac{n(n+2)}{2(n+1)n}\cdot\frac{r(a+b)^{1/2}}{(ab)^{1/2}} - 2r(ab)^{1/2}\right\}\Sigma_1.$$

$$= 2ar\int_0^1 \frac{dt}{\sqrt{1-t^2}} \qquad \text{since } n = 0.$$

$$= 2ar\int_0^{\frac{\pi}{2}} d\theta.$$

$$= 2ar\cdot\frac{\pi}{2} = \frac{a}{2}\cdot 2\pi r.$$