AN INVESTIGATION OF THE SINGULAR OPTIMAL CONTROL OF SINGLE-INPUT LINEAR TIME-INVARIANT SYSTEMS WITH NUMERATOR DYNAMICS

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ABSTRACT

The problem of singular optimal control for single-input, linear, time-invariant systems with numerator dynamics is investigated. A review of the transformation which eliminates derivatives of the control function from the system differential equation is made. Pontryagin's Minimum Principle is then given, although not in its most general form. Conditions necessary for the singularity of both the time-optimal and fuel-optimal problems are found. Finally, simple singular problems are solved to illustrate some of the special techniques which must be used in their solution.

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CHAPTER I

INTRODUCTION

This thesis investigates the problem of singular optimal control for single-input, linear, time-invariant systems whose transfer functions contain zeroes, or numerator dynamics. The two most common optimization problems for which singular controls can exist are the time-optimal and the fuel-optimal problems. Because of the form of the general optimal control, the energy-optimal problem cannot have any singular solutions.[1] In general, singular optimal controls can exist for any problem whose Hamiltonian is either a linear function of the control ($\underline{u}$) or a linear function of the control and its absolute value, i.e. $H[\underline{u}, |\underline{u}|]$.

Firstly, a problem inherent in any attempt to deal with systems whose transfer functions contain zeroes will be reviewed. That is the problem of finding a state representation of the system in which derivatives of the control function do not appear.

Secondly, the necessary conditions for both time-optimal and fuel-optimal problems in general will be generated, using Pontryagin's Minimum (Maximum) Principle, and the general form of the optimal control will be given. A singular control will be defined from this and conditions necessary for the existence of a singular control will be derived. Comments will also be made on the more important effects these conditions have on the system.

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[1] M. Athans and P. Falb, Optimal Control: An Introduction to the Theory and its Applications (McGraw-Hill, 1966), p.479.

Finally, simple examples of both types of optimal control problems will be solved for a second-order, single-zero system to illustrate some of the precautions which must be taken in finding optimal controls for problems in which singular optimal controls may exist.

CHAPTER II

TRANSFORMATION OF SYSTEMS WITH NUMERATOR DYNAMICS[1]

The problem in dealing with systems whose transfer functions have zeroes is, of course, the occurrence of derivatives of the control function in the system differential equation. These may cause undesirable discontinuities, especially in the case of optimal control.

Probably the best way to eliminate this problem is to find a state representation of the system which eliminates the derivative of the control. Consider an $n^{\underline{th}}$ order single-input, single-output system represented by the following transfer function with constant coefficients,

$$\frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \dots + b_o}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_o}, \quad \dots\dots\dots\dots \quad 1$$

where y(t) is the output and u(t) is the input. It should be noted that this equation represents the case where the number of zeroes is equal to the number of poles. This, however, is the most general case for all physical systems, since by choosing the numerator coefficients as follows:

$$b_n = b_{n-1} = \dots = b_{m+1} = 0, \quad b_m \neq 0,$$

the case of m zeroes and n poles may be represented. The differential equation representing the above system is

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1 Cf.-M.Athans and P.Falb, Optimal Control: An Introduction to the Theory and its Applications (McGraw-Hill, 1966), pp.182-190.

$$[D^n + a_{n-1}D^{n-1} + \dots + a_1 D + a_o]y(t) = [b_n D^n + b_{n-1}D^{n-1} + \dots$$
$$\dots + b_1 D + b_o]u(t). \qquad \dots\dots 2$$

In order to obtain the exact solution of this differential equation it is necessary to know the 2n initial conditions, $y(o)$, $\dot{y}(o)$, $\ddot{y}(o)$, ..., $y^{(n-1)}(o)$, $u(o)$, $\dot{u}(o)$, ..., $u^{(n-1)}(o)$. With this in mind, the following method will be used to try to determine the state vector.

The state vector $\underline{z}(t)$ will be constructed with components $z_1(t)$, $z_2(t)$, ..., $z_n(t)$ as follows:

$$\begin{aligned}
z_1(t) &= y(t) - h_o u(t) \\
z_2(t) &= \dot{y}(t) - h_o\dot{u}(t) - h_1 u(t) \\
z_3(t) &= \ddot{y}(t) - h_o\ddot{u}(t) - h_1\dot{u}(t) - h_2 u(t) \qquad \dots\dots 3 \\
&\;\;\vdots \\
z_n(t) &= y^{(n-1)}(t) - h_o u^{(n-1)}(t) - h_1 u^{(n-2)}(t) - \dots \\
&\qquad \dots - h_{n-1}u(t)
\end{aligned}$$

or in concise form,

$$z_i(t) = y^{(i-1)}(t) - \sum_{k=0}^{i-1} u^{(k)}(t)h_{i-k-1},$$

where the h_i are n constants whose values must be determined.

The differential equation which the $z_i(t)$ satisfy must now be determined. It can be seen from equations 3 that, for $i = 1, 2, \ldots, n-1$,

$$\dot{z}_i(t) = y^{(i)}(t) - \sum_{k=0}^{i-1} u^{(k+1)}(t)h_{i-k-1}$$

$$= z_{i+1}(t) + h_i u(t) \qquad \ldots\ldots\ldots\ldots\ldots \quad 4$$

since $$z_{i+1}(t) = y^{(i)}(t) - \sum_{k=0}^{i} u^{(k)}(t)h_{i-k}. \qquad \ldots\ldots\ldots\ldots \quad 5$$

The final relation, for $i = n$, is

$$\dot{z}_n(t) = y^{(n)}(t) - \sum_{k=1}^{n} u^{(k)}(t)h_{n-k}. \qquad \ldots\ldots\ldots\ldots \quad 6$$

The system differential equation (2) gives the following relation:

$$y^{(n)}(t) = -\sum_{i=0}^{n-1} a_i y^{(i)}(t) + b_n u^{(n)}(t) + \sum_{k=1}^{n-1} b_k u^{(k)}(t) + b_o u(t). \quad \ldots \quad 7$$

Rewriting equation 5 as

$$y^{(i)}(t) = z_{i+1}(t) + \sum_{k=1}^{i} u^{(k)}(t)h_{i-k} + h_i u(t),$$

successively multiplying by a_i, $i=0, 1, \ldots, n-1$ and adding, yields:

$$\sum_{i=0}^{n-1} a_i y^{(i)}(t) = \sum_{i=0}^{n-1} a_i z_{i+1}(t) + \sum_{i=0}^{n-1} a_i \Big[\sum_{k=1}^{i} u^{(k)}(t)h_{i-k}\Big] + u(t) \sum_{i=0}^{n-1} a_i h_i. \quad 8$$

Substituting this relation into equation 7 gives the relation

$$y^{(n)}(t) \;=\; -\sum_{i=0}^{n-1} a_i z_{i+1}(t) - \sum_{i=0}^{n-1} a_i \Big[\sum_{k=1}^{i} u^{(k)}(t) h_{i-k}\Big] - u(t)\sum_{i=0}^{n-1} a_i h_i . \quad \ldots\; 9$$

Finally the substitution of this relation into equation 6 yields the equation

$$\dot{z}_n(t) \;=\; -\sum_{i=0}^{n-1} a_i z_{i+1}(t) + b_n u^{(n)}(t) + \sum_{k=0}^{n-1} b_k u^{(k)}(t) + b_o u(t) - \sum_{k=1}^{n} u^{(k)}(t) h_{n-k}$$

$$-\sum_{i=0}^{n-1} a_i \Big[\sum_{k=1}^{i} u^{(k)}(t) h_{i-k}\Big] - u(t)\sum_{i=0}^{n-1} a_i h_i . \qquad \ldots\ldots\ldots\ldots\ldots \; 10$$

The relation,

$$-\sum_{i=0}^{n-1} a_i \sum_{k=1}^{i} u^{(k)}(t) h_{i-k} \;=\; -\sum_{k=1}^{n-1} u^{(k)}(t) \sum_{i=0}^{h-k-1} h_i a_{i+k}, \qquad \ldots\ldots \; 11$$

can be verified by direct computation, and substitution of this relation into equation 10 gives

$$\dot{z}_n(t) \;=\; -\sum_{i=0}^{n-1} a_i z_{i+1}(t) + (b_n - h_o) u^{(n)}(t) + u(t)\Big[b_o - \sum_{i=0}^{n-1} a_i h_i\Big]$$

$$+\sum_{k=1}^{n-1} u^{(k)}(t)\Big[b_k - h_{n-k} - \sum_{i=0}^{n-k-1} h_i a_{i+k}\Big]. \qquad \ldots\ldots\ldots\ldots\ldots \; 12$$

This relation makes it possible to determine the values of the h_i. In order to make equation 12 independent of the derivatives of the control u(t), it is obvious that the following relations must be satisfied:

$$h_o = b_n$$

and 13

$$h_{n-k} = b_k - \sum_{i=0}^{n-k-1} h_i a_{i+k} \text{ for } k = 0, 1, 2, \ldots, n-1.$$

Substitution of the above relations into equation 12 gives the following differential equation:

$$\dot{z}_n(t) = - \sum_{i=0}^{n-1} a_i z_{i+1}(t) + h_n u(t). \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad 14$$

Thus, the set of differential equations determined by the $z_i(t)$ has been found and can be written concisely as

$$\underline{\dot{z}}(t) = A\underline{z}(t) + \underline{h}u(t), \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 15$$

where $\underline{h}$ is the column matrix

$$\begin{bmatrix} h_1 \\ h_2 \\ \cdot \\ \cdot \\ \cdot \\ h_n \end{bmatrix}$$

and A is the system matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & . & 0 \\ 0 & 0 & 0 & 1 & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & . & . & 1 \\ -a_o & -a_1 & -a_2 & . & . & . & . & . & -a_{n-1} \end{bmatrix}.$$

The computation of the constants h_i, $i = 0, 1, 2, \ldots, n$, may be done quite easily by successive substitution into equations 13.

The solution of equation 15 is

$$\underline{z}(t) = \{\underline{z}(t_o) + \int_{t_o}^{t} e^{-A(s-t_o)}\underline{h}u(s)ds\}e^{A(t-t_o)} \qquad \ldots\ldots 16$$

and from this it is clear that knowledge of $\underline{z}(t_o)$ and $u(s)$, $t_o \leq s \leq t$, will uniquely determine $\underline{z}(t)$. Moreover, since $y(t) = z_1(t) + h_o u(t)$, the output $y(t)$ is also uniquely determined. The state equations for the system are, therefore,

$$\dot{\underline{z}}(t) = A\underline{z}(t) + \underline{h}u(t)$$

and $\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ 17

$$y(t) = z_1(t) + h_o u(t).$$

These equations show that the zeroes of a transfer function do not affect the modes of the response, i.e. the eigenvalues of the system matrix A are the poles of the transfer function. The zeroes affect only the control term, since the h_i are determined by the coefficients of the numerator of the transfer function, as well as the coefficients of the denominator.

The occurrence of the control terms in equations 3 means that care must be taken in transforming boundary conditions on the output and its derivatives to equivalent conditions on the state variables. The initial state of the output usually presents no special problem because $u(t_o^-) = 0$ for a linear, time-invariant system. The control terms, however, must be taken into account when finding the equivalent state representation of the target set.[2] The equations 3 relating the state variables and the output and its derivatives provide, of course, the desired relations. For high order systems with many zeroes, the determination of the equivalent target set will become a very tedious task, if not impossible. If, however, the system has only one or two zeroes, the task is quite simple. Since the b_i, $i = 2, 3, \ldots, n$ are all zero for systems with a single zero, the h_i, $i = 0, 1, 2, \ldots, n-2$, are also zero and equation 3 reduces to

$$\begin{aligned}
z_1(t) &= y(t) \\
z_2(t) &= \dot{y}(t) \\
&\;\;\vdots \\
z_{n-1}(t) &= y^{(n-2)}(t) \\
z_n(t) &= y^{(n-1)}(t) - h_{n-1}u(t)
\end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots \quad 18$$

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2 see pp.11-12 for definition.

which make for a simple transformation. As will be seen in the examples, special types of problems and boundary conditions may also affect the form of the target set. These, however, restrict the target set even further and therefore equations 18 and more generally 3 are most useful because they determine the maximum extent of the target set, within which all special target sets must lie.

CHAPTER III

FORMULATION OF THE MINIMUM PRINCIPLE

The problem for which Pontryagin's Minimum Principle will be given is not the most general possible, but one which is fairly common and covers the problems to be considered in this thesis. The optimal control problem to be considered is that of driving the linear, time-invariant system, described by the vector differential equation

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 1$$

where $|u_i(t)| \leq 1$, $i = 1, 2, \ldots, r$, from some initial state $\underline{x}(o)$ to some specified, possibly time-varying, target set S while at the same time minimizing the cost functional

$$J(\underline{u}) = \int_o^T L\,[\underline{x}(t), \underline{u}(t)]dt. \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad 2$$

The terminal time T is free, the integrand $L[\underline{x}(t), \underline{u}(t)]$ dt is not an explicit function of time and $L[\underline{x}(t), \underline{u}(t)]$ and $\partial L[\underline{x}(t), u(t)]/\partial \underline{x}$ are continuous. The target set S may be specified in many ways, but the ones which will be of use in the sequel are as follows:

a) S may be a given fixed element of the state space, i.e. $\underline{x}_1$, a fixed end point.

b) S may be a moving point in the state space i.e. $\underline{g}(t)$.

c) S may be a fixed or time-varying target set specified by the following equations:

$$g_i[\underline{x}(t), t] = 0 \qquad i=1, 2, \ldots, n-k \qquad \ldots\ldots\ldots\ldots \quad 3$$
$$1 \leq k \leq n-1$$

or in vector form

$$\underline{g}[\underline{x}(t), t] = \underline{0} \qquad \text{.....................} \qquad 4$$

where $\underline{g}$ is an n-k vector with components g_i. To ensure the suitable behaviour of this target set $g_i[\underline{x}(t), t]$, $\partial g_i[\underline{x}(t),t]/\partial\underline{x}$ and $\partial g_i[\underline{x}(t),t]/\partial t$ must be continuous in $\underline{x}(t)$ and t. Also, the gradient vectors $\partial g_i[\underline{x}(t),t]/\partial\underline{x}$ must be linearly independent for all $(\underline{x},t)$ in S.

The Hamiltonian function for the posed problem is given by the relation

$$H[\underline{x}(t),\underline{p}(t),\underline{u}(t)] = p_o L[\underline{x}(t),\underline{u}(t)] + \left\langle \underline{p}(t), A\underline{x}(t) + B\underline{u}(t) \right\rangle$$

$$= p_o L[\underline{x}(t), \underline{u}(t)] + \left\langle A\underline{x}(t),\underline{p}(t) \right\rangle + \left\langle \underline{u}(t), B'\underline{p}(t) \right\rangle, \qquad \text{.....} \qquad 5$$

where $\underline{p}(t)$ is the costate vector, to be defined later, p_o is a nonnegative constant and $\langle \underline{x},\underline{p} \rangle$ denotes the scalar product of the vectors $\underline{x}$ and $\underline{p}$.

The canonical system of differential equations associated with the problem is

$$\dot{\underline{x}}(t) = \frac{\partial H[\underline{x}(t),\underline{p}(t),\underline{u}(t)]}{\partial \underline{p}(t)} = A\underline{x}(t) + B\underline{u}(t) \qquad \text{...............} \qquad 6$$

$$\dot{\underline{p}}(t) = \frac{-\partial H[\underline{x}(t),\underline{p}(t),\underline{u}(t)]}{\partial \underline{x}(t)} = - p_o \frac{\partial L[\underline{x}(t),\underline{u}(t)]}{\partial \underline{x}} - A'\underline{p}(t). \qquad \text{.....} \qquad 7$$

Any solution, $\hat{\underline{p}}(t)$, of

$$\underline{\dot{p}}(t) = -p_o \frac{\partial L[\underline{\hat{x}}(t),\underline{\hat{u}}(t)]}{\partial \underline{x}} - A'\underline{p}(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots \; 8$$

is said to correspond to $\underline{\hat{u}}(t)$ and $\underline{\hat{x}}(t)$ for all admissible $\underline{\hat{u}}(t)$ and each trajectory $\underline{\hat{x}}(t)$ corresponding to $\underline{\hat{u}}(t)$.

The Minimum Principle states that if an admissible control $\underline{u}^*(t)$ which transfers $\underline{x}(o)$ to S is to be optimal, then it is necessary that there exist a nonnegative constant p_o^* and a function $\underline{p}^*(t)$ such that:

i) $\underline{p}^*(t)$ and $\underline{x}^*(t)$ are a solution of the canonical system equations 6 and 7, satisfying the boundary conditions $\underline{x}^*(o) = \underline{x}(o)$ and $x^*(T)\varepsilon S$,

ii) the Hamiltonian function $H[\underline{x}^*(t), \underline{p}^*(t), \underline{u}(t), p_o^*]$ has an absolute minimum, as a function of $\underline{u}(t)$, over all admissible controls at $\underline{u}(t) = \underline{u}^*(t)$, which implies

$$H[\underline{x}^*(t),\underline{p}^*(t),\underline{u}^*(t),p_o^*] \le H[\underline{x}^*(t),\underline{p}^*(t),\underline{u}(t),p_o^*] \quad \ldots. \; 9$$

for t in [0,T],

iii) the function $H[\underline{x}^*(t),\underline{p}^*(t),\underline{u}^*(t),p_o^*]$ is a constant for t in [0,T],

i.e. $H[\underline{x}^*(t),\underline{p}^*(t),\underline{u}^*(t),p_o^*] = C,$ 10

where C is equal to zero for any fixed target (set or point), is equal to

$$\sum_{i=1}^{n-1} r_i \left.\frac{\partial g_i[\underline{x}^*(t),t]}{\partial t}\right|_T \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 11$$

for a moving target set and is equal to

$$\left\langle \underline{p}^*(T), \left. \frac{d\underline{g}(t)}{dt} \right|_T \right\rangle$$

for the moving point target,

iv) there are no conditions on the final value of the optimum costate [$\underline{p}^*(T)$] if the target set S is a single point (fixed or moving) of the state space. If the target set S is a set of points i.e. target set (c), the final value of the optimal costate must be normal to the target set at $\underline{x}^*(T)$, i.e.

$$\underline{p}^*(T) = \sum_{i=1}^{n-k} \alpha_i \left. \frac{\partial g_i[\underline{x}^*(t),t]}{\partial \underline{x}} \right|_{\underline{x}^*(T),\, T.} \qquad \text{......... 12}$$

The proof of the Minimum Principle is beyond the scope of this thesis. It may be found in numerous books and articles, done with varying degrees of rigour. Some of these works may be found in the bibliography at the end of this thesis.[1]

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[1] M. Athans and P. Falb, Optimal Control: An Introduction to the Theory and its Applications (McGraw-Hill, 1966), pp.284-351; L. S. Pontryagin et al., The Mathematical Theory of Optimal Processes (Interscience Publishers, Inc., New York, 1962); and L. I. Rozonoer, "L. S. Pontryagin Maximum Principle in the Theory of Optimum Systems", Automation and Remote Control, vol. 20, part 2, 1960.

The Minimum Principle is local in nature and thus there may exist several controls which satisfy the conditions necessary for optimality. These controls are commonly called extremal controls. The evaluation of the cost function for each of the extremal controls will usually determine which of them may be optimal. This does not always hold, however, because the existence of extremal controls does not, in general, guarantee the existence of optimal controls. Problems of this type will not be considered, as they do not occur for linear, time-invariant systems. The problem for which there is no solution, optimal or otherwise, is of concern and will be mentioned later.

Now that the necessary conditions for an optimal control have been outlined in general, the next step is to apply them to more specific problems, namely the time-optimal and the fuel-optimal problems.

CHAPTER IV

THE TIME-OPTIMAL PROBLEM

The general formulation of Pontryagin's Minimum Principle will now be used to obtain the conditions necessary for a time-optimal control. The condition necessary for the time-optimal problem to be singular will also be found. These conditions will then be used to obtain the time-optimal solution for a system satisfying the necessary condition for singularity.

I. NECESSARY CONDITIONS FOR THE SINGULAR TIME-OPTIMAL PROBLEM

The time-optimal, or minimum-time problem is very simply the problem of transferring the system from its initial state to the target set as quickly as possible. To achieve this, the measure of performance must be the transition time and therefore the integrand of the cost functional is $L[\underline{x}(t),\underline{u}(t)] = 1$.

The Hamiltonian for the time-optimal problem is given by

$$H[\underline{x}(t),\ \underline{p}(t),\ \underline{u}(t),\ p_o] = p_o + \langle A\underline{x}(t),\ \underline{p}(t)\rangle + \langle \underline{u}(t),\ B'\underline{p}(t)\rangle \quad \ldots\ldots \quad 1$$

and the canonical equations by

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 2$$

$$|u_i(t) \leq 1|,\ i = 1,\ 2,\ \ldots.,\ r$$

$$\dot{\underline{p}}(t) = -A'\underline{p}(t). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 3$$

Another most important relationship can be derived from necessary condition ii) of the Minimum Principle,

$$H[\underline{x}^*(t),\underline{p}^*(t),\underline{u}^*(t),p_o^*] \leq H[\underline{x}^*(t),\underline{p}^*(t),\underline{u}(t),p_o^*]. \quad \ldots \quad 4$$

The substitution of equation 1 into equation 4 yields,

$$p_o^* + \langle A\underline{x}^*(t),\underline{p}^*(t)\rangle + \langle \underline{u}^*(t),B'\underline{p}^*(t)\rangle \leq p_o^* + \langle A\underline{x}^*(t),\underline{p}^*(t)\rangle + \langle \underline{u}(t),B'\underline{p}^*(t)\rangle,$$

which reduces to

$$\langle \underline{u}^*(t),\ B'\underline{p}^*(t)\rangle \leq \langle \underline{u}(t),\ B'\underline{p}^*(t)\rangle, \qquad \ldots\ldots\ldots\ldots\ 5$$

where the equality holds if $\underline{u}(t) = \underline{u}^*(t)$ for all $t \leq T^*$, T^* being the minimum time of operation. Equation 5 may be rewritten as

$$\sum_{j=1}^{r} u_j^*(t)\ q_j^*(t) \leq \sum_{j=1}^{r} u_j(t)\ q_j^*(t) \qquad \ldots\ldots\ldots\ldots\ldots\ 6$$

by defining a vector function

$$\underline{q}^*(t) = B'\underline{p}^*(t)$$

which becomes, in component form

$$q_j^*(t) = \sum_{i=1}^{n} b_{ij}\ p_i^*(t), \quad j = 1, 2, \ldots, r. \qquad \ldots\ldots\ 7$$

The time-optimal control is, of course, the one which minimizes the right hand sides of equations 5 and 6. It is obvious that the control which does this and yet does not violate the magnitude constraints is defined by the relations:

$$u_j^*(t) = +1 \quad \text{if} \quad q_j^*(t) < 0$$

$$u_j^*(t) = -1 \quad \text{if} \quad q_j^*(t) > 0$$

$$|u_j^*(t)| \leq 1 \quad \text{if} \quad q_j^*(t) = 0.$$

A concise expression for the optimal control,

$$u_j^*(t) = - \text{sgn}\,[q_j^*(t)], \qquad j = 1, 2, \ldots, r \qquad \text{.......................} \quad 8$$

or in vector notation,

$$\underline{u}^*(t) = - \underline{\text{Sgn}}\,[\underline{q}^*(t)],$$

may be obtained by defining a function, sgn [....], as

$$\text{sgn}[a] = +1 \quad \text{if } a > 0$$

$$\text{sgn}[a] = -1 \quad \text{if } a < 0$$

$$|\text{sgn}[a]| \leq 1 \quad \text{if } a = 0.$$

These relations now lead to a definition of normal and singular time-optimal problems.

A time-optimal problem is said to be normal if, in the time interval, [0, T*], there is a countable set of times,

$$t_{aj} \in [0, T^*] \qquad j = 1, 2, \ldots, r$$
$$a = 1, 2, \ldots, m$$

such that $q_j^*(t) = 0$ holds if and only if $t = t_{aj}$ and $q_j^*(t) \neq 0$ holds for all other t, for all j = 1, 2, ..., r. In other words, the function

$q_j^*(t)$ is zero only at a finite number of isolated instants of time, yielding a piecewise constant time-optimal control.

A time-optimal problem is said to be singular if, in the time interval [0, T*], there is at least one subinterval $[T_1, T_2]_j \subset [0, T^*]$ such that $q_j^*(t) = 0$ holds for all $t \in [T_1, T_2]_j$. Each such interval $[T_1, T_2]$ is called a singularity interval. During such intervals the Hamiltonian does not lead to a well-defined relationship among $\underline{u}^*(t)$, $\underline{x}^*(t)$ and $\underline{p}^*(t)$, therefore necessitating the use of some other means to determine a relationship among these terms. These means usually involve use of the canonical equations and the Hamiltonian function to obtain a set of necessary conditions which may lead to a relation among $\underline{u}^*(t)$, $\underline{x}^*(t)$ and $\underline{p}^*(t)$. Examples illustrating this procedure will be done in section II.

In order for the time-optimal problem given at the beginning of this section to be singular, it is necessary that, for some j, j = 1, 2, ..., r, the matrix,

$$G_j = [\underline{b}_j \vdots A\underline{b}_j \vdots A^2\underline{b}_j \vdots \cdots \vdots A^{n-1}\underline{b}_j], \qquad \ldots\ldots\ldots\ldots \quad 9$$

be singular, where $\underline{b}j$ is the $j^{\underline{th}}$ column of B, the control matrix. This statement is easily proven. The optimal control is given by equation 8, $u_j^*(t) = -\operatorname{sgn}[q_j^*(t)]$, where $q_j^*(t) = \sum_{i=1}^{m} b_{ij}\, p_i^*(t)$ is given by equation 6. This latter equation may be rewritten as

$$q_j^*(t) = \langle \underline{b}_j, \underline{p}^*(t) \rangle. \qquad \ldots\ldots\ldots\ldots \quad 10$$

The definition of a singular control leads to the requirement that $q_j^*(t)$ must be equal to zero for all $t \varepsilon [T_1, T_2]_j$. Therefore, the relation $\dot{q}_j^*(t) = \ddot{q}_j^*(t) = \ldots. = q_j^{\overset{*}{(n-1)}}(t) = 0$ must also hold for all $t \varepsilon [T_1, T_2]_j$. This and equation 10 yield the following relations, valid for all $t \varepsilon [T_1, T_2]_j$:

$$
\begin{aligned}
q_j^*(t) &= \left\langle \underline{b}_j, \underline{p}^*(t) \right\rangle = 0 \\
\dot{q}_j^*(t) &= \left\langle \underline{b}_j, \dot{\underline{p}}^*(t) \right\rangle = 0 \\
\ddot{q}_j^*(t) &= \left\langle \underline{b}_j, \ddot{p}^*(t) \right\rangle = 0 \\
&\;\;\vdots \\
q_j^{(n-1)}(t) &= \left\langle \underline{b}_j, \underline{p}^{\overset{*}{(n-1)}}(t) \right\rangle = 0.
\end{aligned}
\qquad \ldots\ldots\ldots\ldots \quad 11
$$

Equation 3, however, yields the relation, $\dot{\underline{p}}^*(t) = -A'\underline{p}^*(t)$, which when substituted into equation 11 gives the relations

$$
\begin{aligned}
q_j^*(t) &= \left\langle \underline{b}_j, \underline{p}^*(t) \right\rangle = 0 \\
\dot{q}_j^*(t) &= \left\langle \underline{b}_j, -A'\underline{p}^*(t) \right\rangle = 0 \\
\ddot{q}_j^*(t) &= \left\langle \underline{b}_j, A'^2\underline{p}^*(t) \right\rangle = 0 \\
&\;\;\vdots
\end{aligned}
\qquad \ldots\ldots\ldots \quad 12
$$

$$\vdots \qquad \qquad \qquad \qquad \ldots\ldots \quad 12$$

$$q_j^{(n-1)*}(t) = \left\langle \underline{b}_j,\ (-1)^{n-1}\ A'^{\,n-1}\underline{p}^*(t)\right\rangle = 0.$$

Since $\langle \underline{x},\ A\underline{y}\rangle = \langle A'\underline{x},\ \underline{y}\rangle$ is true for all $\underline{x}$ and $\underline{y}$ and since $\langle -A\underline{x},\ \underline{y}\rangle = -\langle A\underline{x},\ \underline{y}\rangle$ is also true for all $\underline{x}$ and $\underline{y}$, equations 12 may be rewritten as

$$q_j^*(t) = \left\langle \underline{b}_j,\ \underline{p}^*(t)\right\rangle = 0$$

$$\dot{q}_j^*(t) = \left\langle A\underline{b}_j,\ \underline{p}^*(t)\right\rangle = 0$$

$$\ddot{q}_j^*(t) = \left\langle A^2\underline{b}_j,\ \underline{p}^*(t)\right\rangle = 0 \qquad \ldots\ldots\ldots\ldots \quad 13$$

$$\vdots$$

$$q_j^{(n-1)*}(t) = \left\langle A^{n-1}\underline{b}_j,\ \underline{p}^*(t)\right\rangle = 0.$$

The following concise relationship,

$$G_j'\underline{p}^*(t) = \underline{0}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 14$$

may now be written, where G_j is as defined in equation 9. Now, $\underline{p}^*(t)$ is not the zero vector, since this would violate a hypothesis of the Minimum Principle. Consequently, the matrix G_j must be singular, i.e. det $G_j = 0$.

One important thing to notice about this necessary condition for singularity is that it is very similar to the criterion which must be satisfied in order for a system to be completely controllable. This criterion states that in order for a system to be completely controllable, i.e. the system may be driven to the origin from anywhere in the state space, it is necessary and sufficient that the rank of $G = [B \vdots AB \vdots A^2B \vdots .. \vdots A^{n-1}B]$ be at least equal to n, the order of the system. In fact, a system not satisfying the necessary condition for singularity, i.e. det $G_j \neq 0$ for each $j = 1, 2, ..., r$, is completely controllable by each of the control components, $u_j(t)$. If this occurs it is a much stronger condition than complete controllability and is called normality. The two criteria become identical for single-input systems and, in addition to the possibility of a singular solution, the problem of non-existence of solutions becomes much more pronounced. For example, a solution to the problem of driving a completely controllable single-input system from any point in the state space to the origin is guaranteed. This, of course, is not true if the system is not completely controllable. As a result of this, additional constraints must be placed on the initial states of the system, as will be shown in an example done in a following section.

Moreover, the necessary condition for singularity also gives the reason for this investigation of those systems whose transfer functions contain zeroes. The reason is that only systems of this type are able to satisfy the condition necessary for singularity. The existence of a transfer function without zeroes automatically assures that a single-input system is completely controllable and hence non-singular. This is easily shown, since the relationship between input and output is given by

$$Y(s) = H(s)U(s) = \sum_{i=1}^{n} \frac{\rho_i}{s - \lambda_i} U(s) \quad \ldots\ldots\ldots\ldots \quad 15$$

where ρ_i is the residue of the transfer function H(s), at λ_i. For a system with distinct eigenvalues, the expression for ρ_i is

$$\rho_i = \frac{b_o}{(\lambda_i-\lambda_1)(\lambda_i-\lambda_2)\ldots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\ldots(\lambda_i-\lambda_n)} .$$

$$\ldots\ldots\ldots \quad 16$$

Since each of the terms of equation 15 represents a mode of the response, the only way the system can be uncontrollable and hence satisfy the necessary condition for singularity, is for one of the ρ_i to be zero. This, however, can only happen if b_o is zero, a trivial case.

A system with a zero in its transfer function offers another, more reasonable, possibility. The residue in this case is

$$\rho_i = \frac{b_m(\lambda_i-\sigma_1)(\lambda_i-\sigma_2)\ldots(\lambda_i-\sigma_m)}{(\lambda_i-\lambda_1)(\lambda_i-\lambda_2)\ldots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\ldots(\lambda_i-\lambda_n)}$$

$$\ldots\ldots\ldots \quad 17$$

where m is less than n and the σ_j, $j = 1, 2, \ldots, m$ are the zeroes of the transfer function. Here the possibility of ρ_i being equal to zero is not trivial. The only requirement is that, for some $j = 1, 2, \ldots, m$ $\lambda_i = \sigma_j$ holds. A similar argument reaching the same conclusion can be made for systems whose eigenvalues are not distinct. This conclusion can also be reached by looking at an analog computer simulation of such a system.[1]

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[1] M. Athans and P. Falb, Optimal Control: An Introduction to the Theory and its Applications (McGraw-Hill, 1966), p.188.

II. EXAMPLES FOR THE TIME-OPTIMAL PROBLEM

The examples presented here are simple, but still illustrate some of the special techniques necessary to solve singular problems. The problem to be considered is the one of driving the single-input system, represented by the transfer function

$$\frac{Y(s)}{U(s)} = \frac{s + a}{s(s + b)}, \qquad \text{.............................} \quad 1$$

from some initial state $\underline{y}(o)$ to the origin $\underline{y}(T) = \underline{0}$ in the least possible time with the magnitude of the control not greater than one at any time.

The differential equation represented by equation 1 is

$$\ddot{y}(t) + b\dot{y}(t) = \dot{u}(t) + a\,u(t). \qquad \text{..........................} \quad 2$$

As has been shown in Chapter II, suitable state variables are

$$z_1(t) = y(t) - h_o u(t)$$

and $\qquad \text{.....................} \quad 3$

$$z_2(t) = \dot{y}(t) - h_o \dot{u}(t) - h_1 u(t)$$

where, from equations 13 of Chapter II, the following values of h_o, h_1 and h_2 result: $h_o = 0$, $h_1 = 1$ and $h_2 = a - b$. The state equations are, therefore,

$$z_1(t) = y(t)$$

$$\text{.............................} \quad 4$$

$$z_2(t) = \dot{y}(t) - u(t).$$

Equation 2 yields the system matrix,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -b \end{bmatrix}$$

and equation 15 of Chapter II gives the state differential equation,

$$\begin{aligned} \dot{\underline{z}}(t) &= A\underline{z}(t) + \underline{h}u(t) \\ &= \begin{bmatrix} 0 & 1 \\ 0 & -b \end{bmatrix} \underline{z}(t) + \begin{bmatrix} 1 \\ a-b \end{bmatrix} u(t). \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad 5 \end{aligned}$$

The Hamiltonian for this problem is

$$\begin{aligned} H &= p_o + \langle A\underline{z}(t), \underline{p}(t) \rangle + \langle \underline{b}'\underline{p}(t), u(t) \rangle \\ &= p_o + p_1(t)z_2(t) - p_2(t)bz_2(t) \\ &\quad + [p_1(t) + (a-b)p_2(t)]u(t), \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad 6 \end{aligned}$$

where $\dot{\underline{p}}(t) = -A'\underline{p}(t).$ 7

To find the condition on the system necessary to make the problem singular, the matrix G must be found. The matrix G is, from the preceeding section,

$$\begin{aligned} G &= [\underline{h} \quad A\underline{h}] \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad 8 \\ &= \begin{bmatrix} 1 & a-b \\ a-b & -b(a-b) \end{bmatrix}. \end{aligned}$$

This matrix is singular if a is equal to either b, or zero, but only the case where a and b are both equal to one will be examined.

Equations 5, 6 and 7 become, respectively

$$\dot{\underline{z}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \underline{z}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \quad \ldots\ldots\ldots\ldots \quad 9$$

$$H = p_o + p_1(t)z_2(t) - p_2(t)z_2(t) + p_1(t)u(t) \quad .. \quad 10$$

and

$$\dot{\underline{p}}(t) = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \underline{p}(t). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 11$$

Solving for $p_1(t)$ and $p_2(t)$ from equation 11 yields the fact that

$\dot{p}_1(t) = 0$ and therefore $p_1(t) = p_1(o)$, ... 12

Moreover, $\dot{p}_2(t) = -p_1(t) + p_2(t)$ gives the solution

$$p_2(t) = p_1(o) + [p_2(o) - p_1(o)]\, e^t. \quad \ldots\ldots\ldots\ldots \quad 13$$

The system differential equations are, from equation 9,

$$\dot{z}_1(t) = z_2(t) + u(t)$$

and

$$\dot{z}_2(t) = -z_2(t),$$

which yield, upon solution,

$$z_2(t) = z_2(o)e^{-t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 14$$

and

$$z_1(t) = tu(t) + z_1(o) + z_2(o)[1-e^{-t}] \quad \ldots\ldots\ldots\ldots \quad 15$$

where $u(t)$ is piecewise constant.

These last two equations may be used to find the system trajectories in the state space. Equation 14 yields the following expression for time:

$$t = \ln\left[\frac{z_2(o)}{z_2}\right].$$

Substitution of this into equation 15 results in the following equation:

$$z_1 = u \ln\left[\frac{z_2(o)}{z_2}\right] -z_2 + z_1(o) + z_2(o). \quad \ldots\ldots\ldots \quad 16$$

The slopes of the trajectories may be obtained from equation 9 and are given by

$$\frac{dz_2}{dz_1} = -\left(\frac{z_2}{z_2 + u}\right). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 17$$

These equations permit the graphical construction of the trajectories in the state space, for various values of the control variable, as has been done in Figures 1 and 2.

The target set in the state space must now be found. Looking at equations 4 immediately gives the result that $z_1(T) = y(T) = 0$, where T is the terminal time. Now, $\dot{y}(T)$ must also be zero so that the second

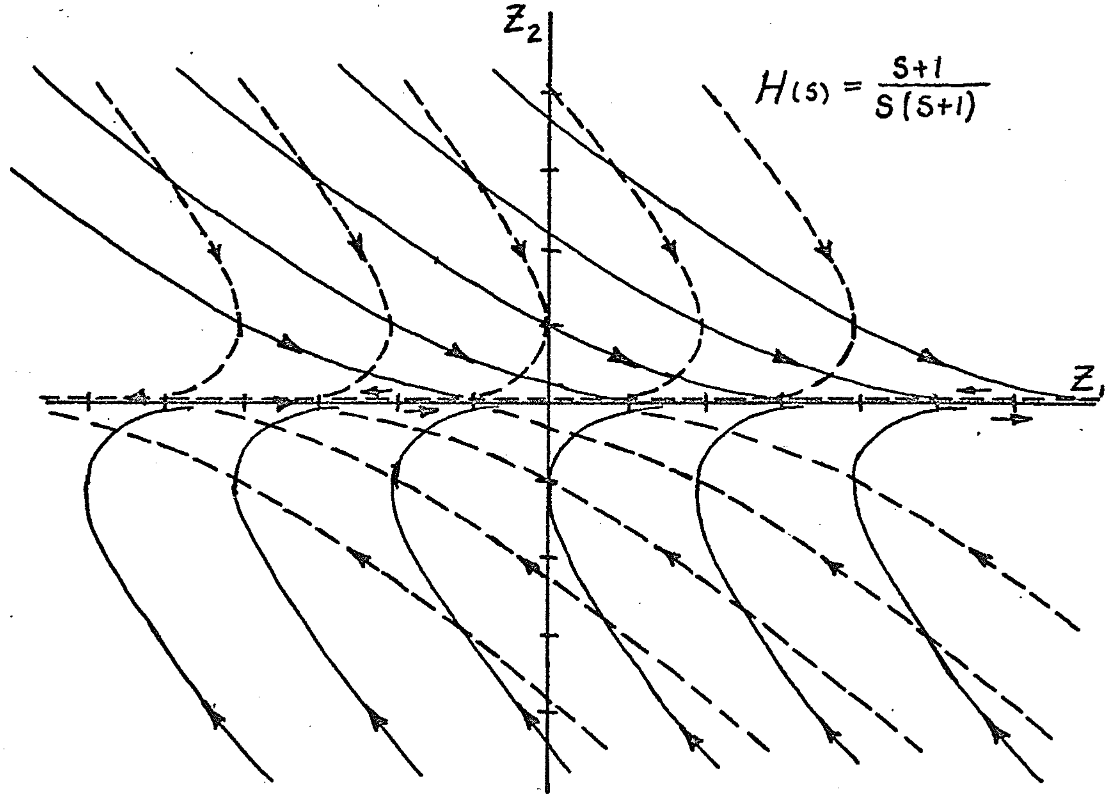

Figure 1. Forced system trajectories, solid lines for u = +1, dashed lines for u = -1.

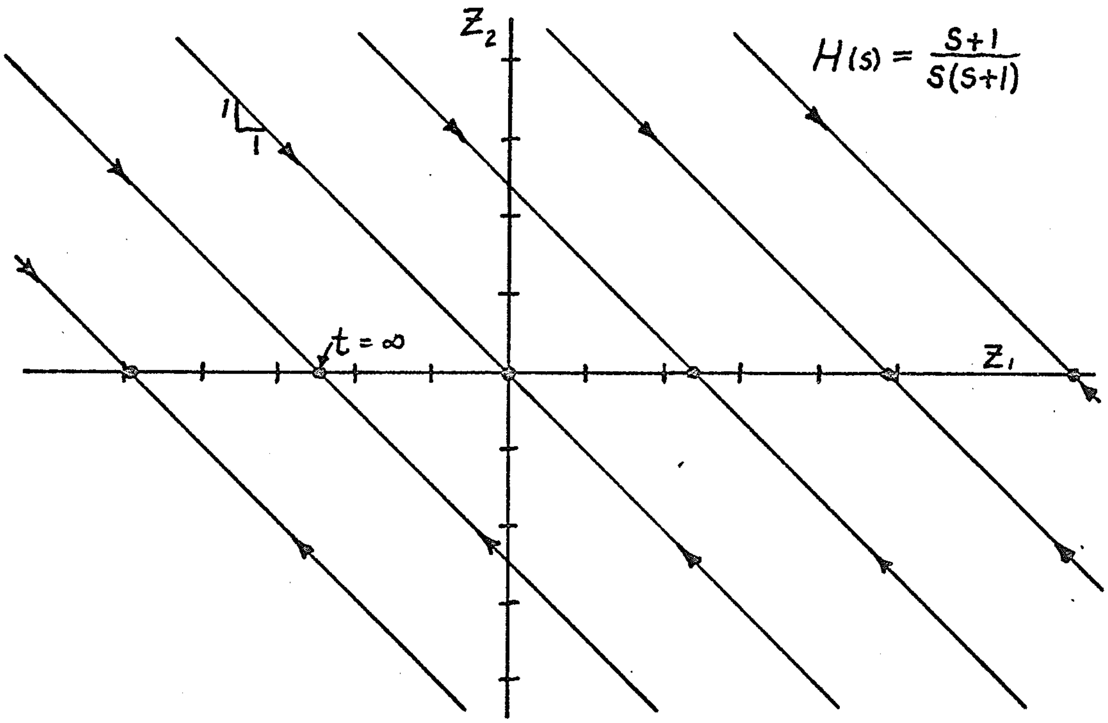

Figure 2. Unforced system trajectories, u = 0.

of equations 4 yields

$$z_2(T) = - u(T). \qquad \text{.........................} \qquad 18$$

The final value of the control, $u(T)$, may take on any value between minus one and plus one. This means that $z_2(T)$ may also range between minus one and plus one. The target set is, therefore, the line segment $|z_2| \leq 1$ and $z_1 = 0$.

Substituting equations 12 to 15 inclusive into equation 10 and letting $p_o = 1$ yields

$$H = 1 + p_1(o)z_2(o)e^{-t} - p_1(o)z_2(o)e^{-t}$$

$$- [p_2(o) - p_1(o)]z_2(o) + p_1(o)\ u(t)$$

$$= 1 - [p_2(o) - p_1(o)]z_2(o) + p_1(o)u(t) = 0. \qquad 19$$

From this equation, the time-optimal control, $u^*(t)$, is given by

$$u^*(t) = - \operatorname{sgn}[p_1(o)]. \qquad \text{.....................} \qquad 20$$

Since $p_1(o)$ is a constant, $u^*(t)$ will also be a constant and therefore the time-optimal control will be

$$u^*(t) = +1 \quad \text{if} \quad p_1(o) < 0,$$

$$u^*(t) \quad = \quad -1 \quad \text{if} \quad p_1(o) > 0,$$

and

$$|u^*(t)| \leq 1 \quad \text{if} \quad p_1(o) \quad = \quad 0.$$

At this point something further should be said about the target set. Several variations of this problem exist each of which may correspond to some physical situation. One is that the output remain at zero for all $t_1 \geq T^*$ either with $u^*(t_1) = 0$ or $u^*(t_1) \neq 0$. Another is the case where the output for $t_1 > T^*$ is immaterial as long as $y(T^*)$ and $\dot{y}(T^*)$ are equal to zero. Each of these conditions will have its effect on the form of the optimal control and the target set. Since these variations of the main problem differ only at and after the minimum response time, the optimal control from any controllable initial state will be identical for each variation except at the terminal time T*. The differences will occur, quite naturally, in the form of the control at and beyond the terminal time and more importantly, in the form of the target set. To illustrate this, each of the three variations of the basic problem will be solved.

Example 1

The first variation will be the most restrictive, that of driving the output and its derivative to zero in the least possible time and maintaining the output at zero with $u(t_1) = 0$ for $t \geq T^*$. This is commonly called the time-optimal regulator problem.

Equation 18 indicates that if $u(t_1) = 0$ holds for $t_1 \geq T^*$ then $z_2(t_1)$ must also equal zero. The target set for this problem is

$z_1 = z_2 = 0$, as obtained from equations 4. The set of initial states is very neatly restricted by equation 14, to all those for which $z_2(o)$ is zero since e^{-t} can never equal zero. Because there is no restriction on $z_1(o)$, the allowable set of initial states is the $z_2 = 0$ axis as shown in Figure 3 below.

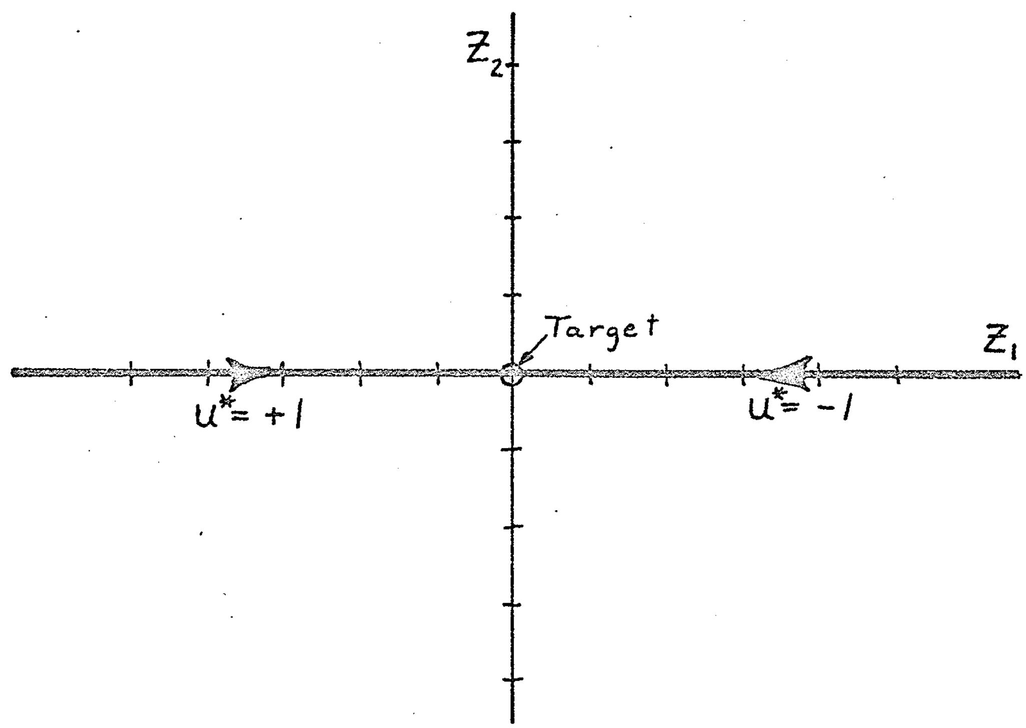

Figure 3. A plot of the allowable initial states and optimal trajectories.

The optimal control and a set of restrictions on the initial conditions of the output are obtained from equations 2 and 4. Since $z_2(o) = 0$ and $|u| \leq 1$ must hold, equation 4 implies that the magnitude of $\dot{y}(o)$ must be less than or equal to one. In fact, the same equation yields the initial value of the optimal control,

$$u^*(o) = \dot{y}(o). \qquad \text{21}$$

Moreover, since $z_2(t)$ is zero for the entire period of operation, $\dot{y}(t)$ must equal $u^*(t)$ for the same period of time. Equation 2 indicates that this is possible for a piecewise constant control if and only if $\ddot{y}(o)$ is zero. If these restrictions are all satisfied, the system can be driven to the origin in minimum time by the following control:

$$u^*(o) = \dot{y}(o)$$

$$u^*(t) = -\operatorname{sgn}[y(o)], \quad 0 < t < T^*$$

$$u^*(T^*) = 0.$$

Because p_1 is a constant, the optimal control does not switch, as can also be seen from the equations above.

Inspection of equation 19 reveals that this example cannot be singular. Since $z_2(o) = 0$, equation 19 reduces to

$$H = 1 + p_1(o)\, u(t) = 0, \qquad \text{22}$$

which implies that $p_1(o)$ is non-zero and that the problem is not singular.

The restrictions on the initial state of the system are a result of the system's being not completely controllable. Strictly speaking, the fact that $\ddot{y}(o)$ must be zero implies, from equation 2, that the system has been under the influence of some piecewise constant control for all $t < 0$, such that $u(o^-)$ is equal to $\dot{y}(o)$. Moreover, once $\ddot{y}$ assumes some value other than zero, it becomes impossible to drive the system to the origin and have it remain there with no control force because of the exponential factor.

The fact that an output may not be exactly zero is usually of little concern in an actual physical system. If this were the case and $\ddot{y}(o)$ were not zero, the best solution would be to drive the system to the $u = 0$ trajectory which goes to the origin of the state space, and remove the control. This allows the system to approach the origin with an exponential decay. Figure 4 illustrates some of the possible trajectories.

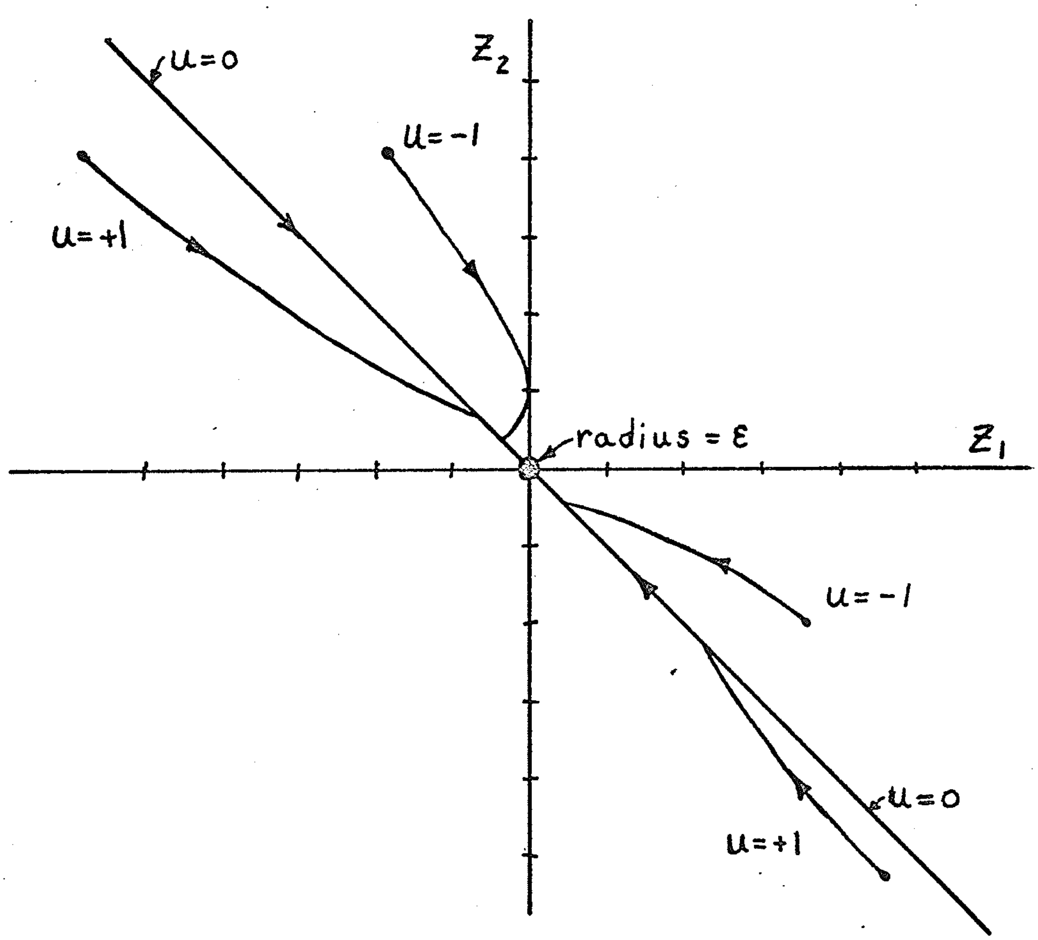

Figure 4. Best trajectories for an asymptotic approach to the origin from various points in the state space.

These trajectories provide the best response because of the uncontrollability of $z_2(t)$, i.e. $z_2(t) = z_2(o)e^{-t}$. In other words, the time taken to come within some distance ε of the $z_2 = 0$ axis is independent of $z_1(o)$ and the control applied. Therefore, the only thing that can be done is to assure that $z_1(t)$ and $z_2(t)$ both decay to zero at the same time. The trajectory which does this is the unforced ($u = 0$) trajectory which approaches the origin.

Example 2

The second variation will be that of driving the system so that $y(T^*) = \dot{y}(T^*) = 0$ becomes true in the least possible time, with no attention being paid to the output or its derivative for $t > T^*$. The target set in this case is, from equations 4,

$$\begin{aligned} z_1 &= 0 \\ |z_2| &\leq 1. \end{aligned} \qquad \text{.................................} \quad 23$$

The optimal controls may be easily obtained from the trajectories of Figure 1 and the condition imposed by equation 18, i.e. $u(T^*) = -z_2(T^*)$. The output and its derivative will of course not remain at zero unless a specific control is maintained for $t > T^*$. This will be shown in example 3.

Figure 5 shows the target set, the optimal control for various regions, some of the corresponding optimal trajectories and the singular regions of the state space. The optimal trajectories can be seen to verify equation 20, which implies that the optimal control does not switch. For this reason, there are no switching curves. The trajectories of interest are those which form the boundaries of the singular regions. These are, as indicated in Figure 5, the trajectories for u = +1 and u = -1 which pass through the points $z_1 = 0$, $z_2 = +1$ and $z_1 = 0$, $z_2 = -1$.

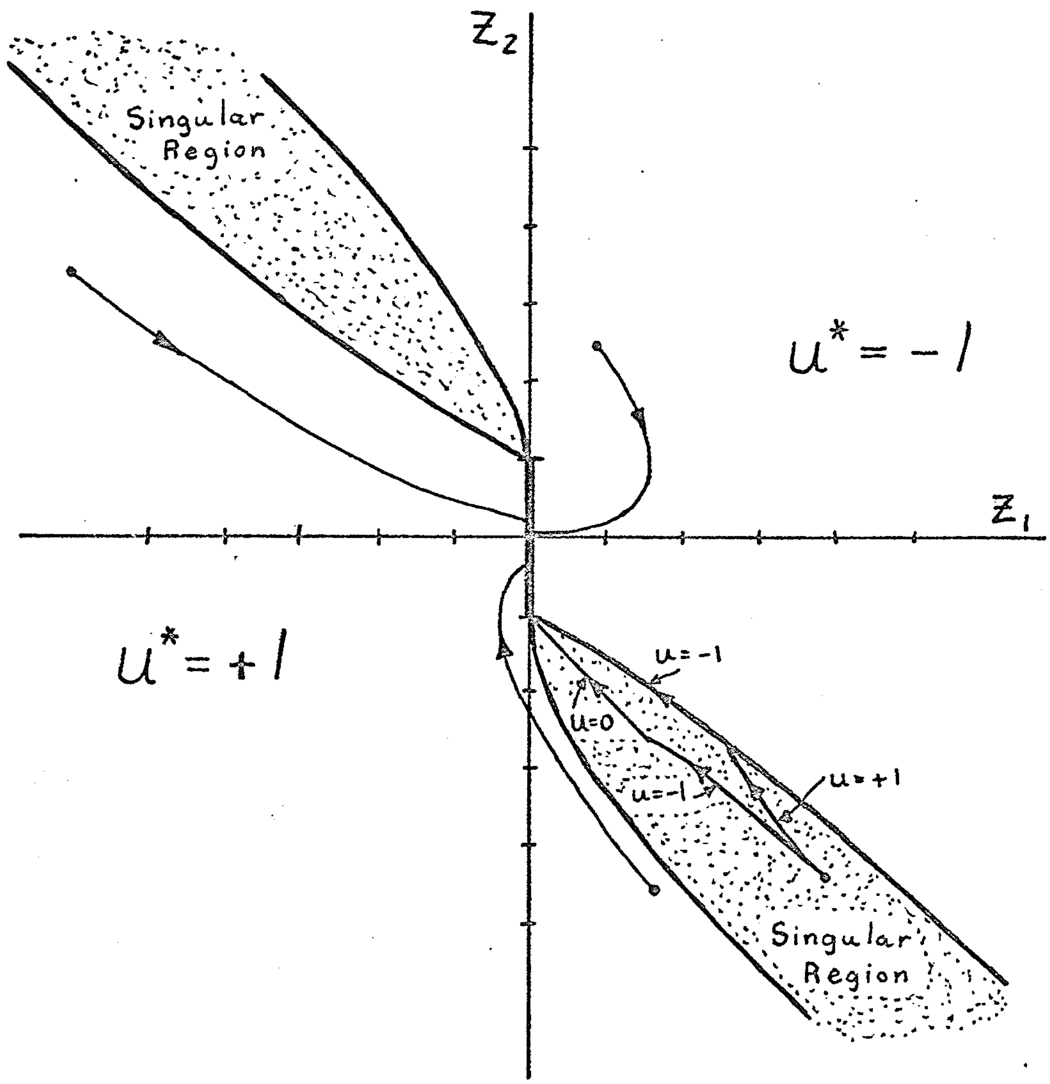

Figure 5. Time-optimal trajectories and singular regions for example 2.

Singular optimal trajectories to any other point of the target cannot exist because of the transversality condition which states that the final value of the costate vector must be normal to the target set. Thus the time-optimal control for any trajectory reaching the interior of the straight line segment target set cannot be singular because $p_1(o) = p_1(T^*) \neq 0$ must hold. The requirement for a singular time-optimal control was, of course, that $p_1(o)$ be zero. Therefore, the only possibility for a singular time-optimal control is one which drives the system to one of the end points of the target set. Interesting results connected with the shape of the reachable region in the state space can be found in the reference cited below.[1]

To show that the time-optimal control from any point inside the indicated regions is singular is not difficult. The time taken to reach $z_1 = 0$, $z_2 = \pm 1$ from any point inside the singular region is independent of the control used. Since $z_2(t) = z_2(o)e^{-t}$ is independent of the control variable, the time taken to reach $z_2 = \pm 1$ depends only on $z_2(o)$. Thus, any control, and there are infinitely many, which drives the system from $z_1(o)$, $z_2(o)$, inside the singular region, to $z_1 = 0$, $z_2 = \pm 1$ is time-optimal.

This non-uniqueness of the time-optimal control in the singular region may be put to good use in many cases. In these cases some other cost functional could be optimized, such as fuel consumed, or some other

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[1] Eliezer Kreindler, "Contributions to the Theory of Time-Optimal Control", Journal of The Franklin Institute, vol.275, no. 4, April 1963, pp.314-344.

constraints[2] could be imposed which might prove useful in certain situations.

Example 3

The solution for this variation of the time-optimal problem, which is to maintain $y(t) = \dot{y}(t) = 0$ with a control not necessarily zero for $t > T^*$, is identical to the previous problem for $t \leq T^*$. The time-optimal control is, therefore, the same as found in example 2.

In order to maintain $y(t) = \dot{y}(t) = 0$ for $t > T^*$, the following equations must be satisfied:

$$\begin{aligned} z_1(t) &= 0 \\ z_2(t) &= -u(t) \\ \dot{z}_1(t) &= z_2(t) + u(t) = 0 \\ \dot{z}_2(t) &= -z_2(t). \end{aligned} \qquad \text{.................. } 24$$

The last of these equations yields the solution

$$z_2(t) = z_2(T^*)e^{-t}.$$

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[2] Frederick E. Thau, "Optimal Time Control of Non-Normal Linear Systems," International Journal of Control, vol.1, no. 4, 1965, pp.363-73.

The second of these gives the control which will satisfy all the required conditions and maintain the output at zero. This control is given by

$$u(t) = - z_2(T^*)e^{-t}.$$

CHAPTER V

THE FUEL-OPTIMAL PROBLEM

Similar to the previous chapter, the general formulation of Pontryagin's Minimum Principle will now be used to obtain the conditions necessary for a control to be fuel-optimal and singular. These conditions will then be used to obtain the fuel-optimal solution for systems satisfying the necessary condition for singularity.

I. NECESSARY CONDITIONS FOR THE SINGULAR FUEL-OPTIMAL PROBLEM

Fuel-optimal, or minimum-fuel problems are in general more complicated than time-optimal problems. Both the theoretical and actual design portions of this type of problem exhibit additional complexity. One of the main reasons for this is the fact that other factors, such as response time, must be considered in most fuel-optimal problems. Probably some of the more common factors which are used are: an upper bound on the response time, a fixed response time, a minimum time solution with a bound on the fuel, or a minimization of some linear combination of fuel and response time. In any event, the physical requirements must naturally determine the problem formulation which best fits the given physical situation.

The fuel-optimal problem will be formulated with the condition of having an upper bound on the response time, in order to eliminate trivial solutions which may occur for stable systems being driven to the origin. Naturally, in order for a solution to exist this upper bound, T_u must be greater than the minimum response time.

The necessary conditions for the fuel-optimal problem are quite similar to those of the time-optimal problem. Again the problem of driving the system $\dot{\underline{z}}(t) = A\underline{z}(t) + B\underline{u}(t)$ to some target set S will be considered. This time, of course, the cost functional,

$$J(\underline{u}) = \int_{t_o}^{T_f} \left\langle \underline{c}, \ |\underline{u}(t)| \right\rangle dt, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 1$$

is different. The vector $\underline{c}$ represents the constants of proportionality between the fuel consumed and the magnitude of the control components $|u_j(t)|$. The canonical equations,

$$\dot{\underline{z}}^*(t) = \frac{\partial H}{\partial \underline{p}^*(t)} = A\underline{z}^*(t) + B\underline{u}^*(t)$$

and $\ldots\ldots\ldots\ldots\ldots\ldots$ 2

$$\dot{\underline{p}}^*(t) = \frac{-\partial H}{\partial \underline{z}^*(t)} = -A'\underline{p}^*(t),$$

are identical to those of the time-optimal problem, since the only difference between the Hamiltonians of the two problems is a function of $\underline{u}^*(t)$ alone. Substituting into equation 5 Chapter III yields

$$H[\underline{z}^*(t),\underline{p}^*(t),\underline{u}^*(t)] = p_o^*\left\langle \underline{c}, |\underline{u}^*(t)| \right\rangle + \left\langle \underline{p}^*(t), A\underline{z}^*(t) \right\rangle + \left\langle \underline{u}^*(t), B'\underline{p}^*(t) \right\rangle. \quad \ldots\ldots \quad 3$$

The necessary condition

$$H[\underline{z}^*(t),p^*(t),\underline{u}^*(t)] \leq H[\underline{z}^*(t),\underline{p}^*(t),\underline{u}(t)]$$

yields, upon reduction, the following relation:

$$p_o^*\left\langle \underline{c}, \ |\underline{u}^*(t)| \right\rangle + \left\langle \underline{u}^*(t), B'\underline{p}^*(t) \right\rangle \leq p_o^*\left\langle \underline{c}, \ |\underline{u}(t)| \right\rangle + \left\langle \underline{u}(t), B'\underline{p}^*(t) \right\rangle. \quad \ldots\ldots \quad 4$$

The difference between this relation and the corresponding time-optimal relation is the term $p_o^*\left\langle \underline{c}, \ |\underline{u}(t)| \right\rangle$. The final value of the costate is independent of the cost functional and therefore the same for both time- and fuel-optimal problems. Expressions for this are given in Chapter III.

As in the time-optimal case, equation 4 gives an expression for $\underline{u}^*(t)$ in terms of $\underline{p}^*(t)$. Minimizing the right hand side of this relation yields the following expression for $\underline{u}^*(t)$:

$$u_j^*(t) = 0 \quad \text{if} \quad \left| \frac{\langle \underline{p}^*(t), \underline{b}_j \rangle}{c_j} \right| < 1$$

$$u_j^*(t) = +1 \quad \text{if} \quad \frac{\langle \underline{p}^*(t), \underline{b}_j \rangle}{c_j} < -1$$

$$u_j^*(t) = -1 \quad \text{if} \quad \frac{\langle \underline{p}^*(t), \underline{b}_j \rangle}{c_j} > +1 \qquad \cdots\cdots\cdots\cdot \quad 5$$

$$0 \leq u_j^*(t) \leq +1 \text{ if } \frac{\langle \underline{p}^*(t), \underline{b}_j \rangle}{c_j} = -1$$

$$-1 \leq u_j^*(t) \leq 0 \text{ if } \frac{\langle \underline{p}^*(t), \underline{b}_j \rangle}{c_j} = +1.$$

A function called the dead-zone function is defined by the above relations, and the fuel-optimal control may be written

$$u_j^*(t) = - \operatorname{dez} \left[\frac{\langle \underline{p}^*(t), \underline{b}_j \rangle}{c_j}\right]. \qquad \cdots\cdots\cdots\cdots\cdots\cdot \quad 6$$

Figure 6, a plot of all possible values of the right hand side of equation 4 vs $\langle \underline{p}^*(t), \underline{b}_j \rangle / c_j$ for all allowable values of $u_j(t)$, shows that the dead-zone function does indeed give the minimum possible right hand side of equation 4.

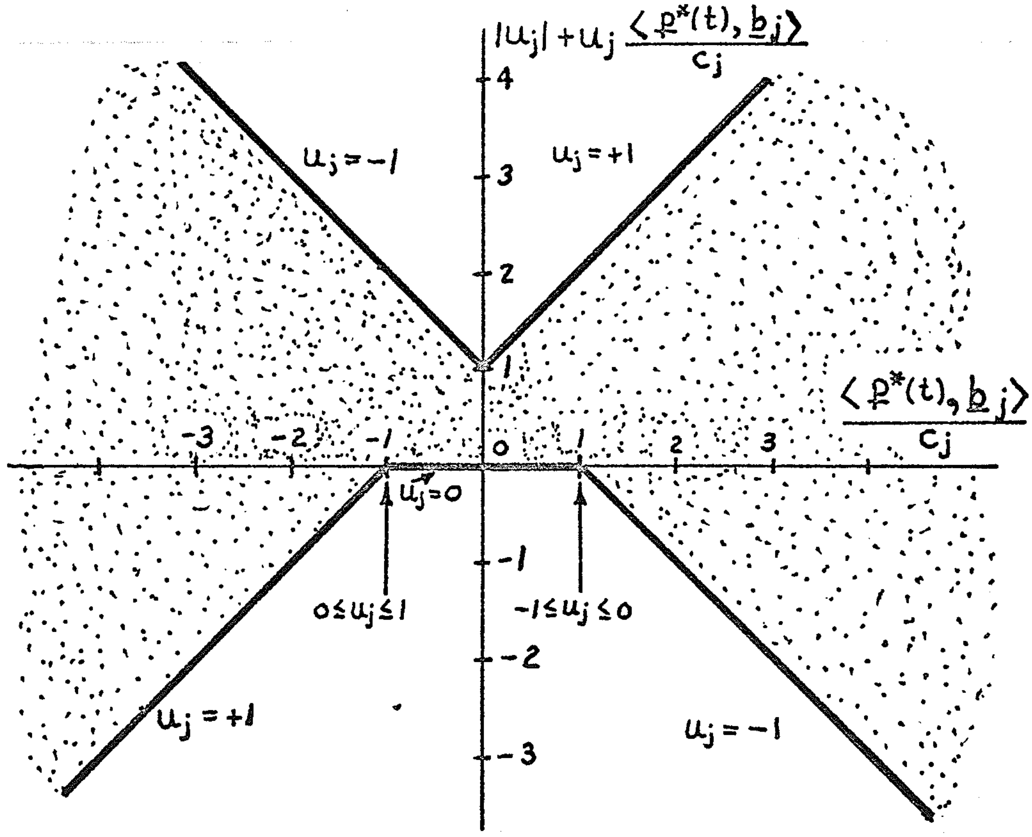

Figure 6. A plot of $|u_j| + u_j \dfrac{\langle \underline{p}^*(t), \underline{b}_j \rangle}{c_j}$ vs. $\dfrac{\langle \underline{p}^*(t), \underline{b}_j \rangle}{c_j}$ for $|u_j| \le 1$.

The singular problem occurs when

$$\langle \underline{p}^*(t), \underline{b}_j \rangle / c_j = \pm 1 \qquad \text{.....................} \quad 7$$

is true for a non-zero period of time. In this case, the sign of the optimal control is known, but its magnitude is not. For a fuel-optimal control to be singular it is necessary that there exists at least one non-zero interval of time during the control interval $[0, T_f]$ over which equation 7 holds. During a singular period these two relations must hold,

$$\dot{\underline{p}}^*(t) = -A'\underline{p}^*(t)$$

and

$$\langle \underline{p}^*(t), \underline{b}_j \rangle = \pm c_j.$$

Differentiating equation 7 repeatedly and substituting these last two relations gives this set of equations:

$$\begin{aligned} \langle A\underline{b}_j, \underline{p}^*(t) \rangle &= 0 \\ \langle A^2\underline{b}_j, \underline{p}^*(t) \rangle &= 0 \\ &\vdots \\ \langle A^n\underline{b}_j, \underline{p}^*(t) \rangle &= 0. \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots \; 8$$

Equations 8 may now be written concisely as

$$G_j' \, A'\underline{p}^*(t) = 0, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 9$$

where, as for the time optimal case G_j is defined as

$$G_j = [\underline{b}_j \,\vdots\, A\underline{b}_j \,\vdots\, \cdots\cdots \,\vdots\, A^{(n-1)}\underline{b}_j].$$

Equation 7 implies that $\underline{p}^*(t)$ is not zero. Equation 9 therefore implies that det $[G_j'A']$ is zero and consequently G_j and/or A are singular. A necessary, but not sufficient condition for a singular fuel-optimal problem is, therefore, that either or both G_j and A be singular. As before, the singularity of G_j implies that the system is not normal and that a single-input system is not completely controllable. This implies, as in the time-optimal problem, the possibility of nonexistence of solutions from certain initial states. This factor compounded with the possibility of the optimal control being singular serves to complicate any solution to this problem.

The other formulations of the fuel-optimal control problem, mentioned at the beginning of this section, are very similar to the bounded response time problem just discussed. The canonical equations and certain transversality conditions are the same for all problems. The Hamiltonian for each formulation differs only by the appearance of different constants while the functional dependence of the Hamiltonian on $\underline{z}(t)$ and $\underline{p}(t)$ does not change. For this reason, the form of the optimal control remains unchanged for each problem. The main differences in each problem formulation are taken care of by the transversality conditions. For this reason, nothing more will be said about them and the only problem which will be dealt with in the subsequent examples will be the fuel-optimal problem with an upper bound on the response time.

If the necessary condition for singularity is satisfied only because A is singular (i.e. the system has at least one zero eigenvalue) the system is completely controllable. In this case, however, if the optimal control is singular, it is singular over the entire control interval. In other words, if the problem is singular for all $t \in [t_1, t_2]$ then $t_1 = 0$ and $t_2 = T_f$ must hold. To prove this, let t_3 be any time such that $0 \leq t_3 \leq t_1$ holds. The solution of the adjoint equation is, for every $t \in [t_1, t_2]$,

$$\hat{\underline{p}}(t) = e^{-A'(t-t_3)}\hat{\underline{p}}(t_3). \qquad 10$$

Since G_j is non-singular, equation 9 yields

$$A'\hat{\underline{p}}(t) = A'e^{-A'(t-t_3)}\hat{\underline{p}}(t_3) = 0. \qquad 11$$

The matrices A' and $e^{-A'(t-t_3)}$ commute and since $e^{-A'(t-t_3)}$ is non-singular, the following relation may be obtained from equation 11:

$$A'\hat{\underline{p}}(t_3) = 0, \qquad 12$$

which means that the problem is singular for all $t_3 \varepsilon [0, t_1]$. Similarly, for all $t_4 \varepsilon [t_2, T_f]$, $A'\hat{\underline{p}}(t_4) = 0$ holds and the problem is singular for the entire interval.[1]

Before the examples are presented, it should be mentioned that the conditions given for the singularity of a problem are necessary only; and even if they are satisfied there will exist, in general, initial states from which the optimal control is normal for the controllable fuel-optimal problem.[2]

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[1] Michael Athans, "On the Uniqueness of the Extremal Controls for a Class of Minimum Fuel Problems", IEEE Trans. Aut. Control, vol.AC-11, no. 4, Oct. 1966, p.664.

[2] Ibid., pp.664-65.

II. EXAMPLES FOR THE FUEL-OPTIMAL PROBLEM

These examples will deal with the problem of finding an admissible control which drives the system given by the transfer function

$$\frac{Y(s)}{U(s)} = \frac{s + a}{s(s + b)} \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \qquad 1$$

from some initial state $\underline{y}(o)$ to the origin $\underline{y}(T) = \underline{0}$ in some time less than or equal to T_f. At the same time the control must minimize the fuel consumed. The cost functional for this problem is therefore given by

$$J(u) = \int_o^T c|u(t)|\, dt, \qquad \dots\dots\dots\dots\dots\dots\dots \qquad 2$$

where T is less than or equal to T_f. The canonical equations are the same as those of the time-optimal problem:

$$\underline{\dot{z}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -b \end{bmatrix} \underline{z}(t) + \begin{bmatrix} 1 \\ a - b \end{bmatrix} u(t), \qquad \dots\dots \qquad 3$$

$$\underline{\dot{p}}(t) = \begin{bmatrix} 0 & 0 \\ -1 & b \end{bmatrix} \underline{p}(t), \qquad \dots\dots\dots\dots\dots\dots\dots \qquad 4$$

$$z_1(t) = y(t)$$

and

$$z_2(t) = \dot{y}(t) - u(t).$$

The target set is also the same, i.e.

$$z_1 = 0$$

$$|z_2| \leq 1.$$

For simplicity, the problem of maintaining the system output at the origin will not be considered in the following examples. The procedures used to determine any special target sets or controls have already been illustrated for the time-optimal problem and they apply to the fuel-optimal problem as well. The Hamiltonian,

$$H = p_o c|u(t)| + \langle A\underline{z}(t), \underline{p}(t) \rangle + \langle \underline{h}'\underline{p}(t), u(t) \rangle \quad .. \qquad 5$$

is different because of the term involving the magnitude of the control. Also different is the condition necessary for the problem to be singular. In order for a fuel-optimal solution to be singular it is necessary that the matrix

$$G = [\underline{h} \ \vdots \ A\underline{h}]$$

$$= \begin{bmatrix} 1 & a - b \\ a - b & -b(a-b) \end{bmatrix} \quad \dots\dots\dots\dots\dots\dots\dots \qquad 6$$

and/or the system matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -b \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad 7$$

be singular. The system matrix just given is obviously singular because of the pole at s = 0. Two cases of the fuel-optimal problem will therefore be examined. The matrix G will be non-singular in the first example and singular in the second.

Example 1

With c = 1, the Hamiltonian for this example becomes

$$H = p_o|u(t)| + p_1(t)z_2(t) - bp_2(t)z_2(t) + p_1(t)u(t)$$

$$+ (a - b)p_2(t)u(t). \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad 8$$

Expansion of equation 4 yields the following differential equations for the adjoint variables,

$$\dot{p}_1(t) = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \qquad 9$$

$$\dot{p}_2(t) = - p_1(t) + bp_2(t)$$

which become upon solution,

$$p_1(t) = p_1(o) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad 10$$

and

$$p_2(t) = \frac{p_1(o)}{b} + [p_2(o) - \frac{p_1(o)}{b}]e^{bt}. \qquad \ldots\ldots\ldots \qquad 11$$

The state differential equations as given by equation 3 are

$$\dot{z}_1(t) = z_2(t) + u(t) \qquad \text{.......................} \qquad 12$$

and

$$\dot{z}_2(t) = -bz_2(t) + (a-b)u(t). \qquad \text{.................} \qquad 13$$

The Laplace transform of equation 13 is, for u(t) piecewise constant,

$$sZ_2 - z_2(o) + bZ_2 = \frac{(a-b)u}{s} .$$

Solving for Z_2 gives

$$Z_2 = \frac{(a-b)u}{s(s+b)} + \frac{z_2(o)}{s+b}$$

$$= \frac{(a-b)\frac{u}{b}}{s} + \frac{z_2(o) - (a-b)\frac{u}{b}}{s+b} .$$

The inverse transformation of this equation yields the desired solution,

$$z_2(t) = z_2(o)e^{-bt} + \frac{a-b}{b}(1-e^{-bt})u. \qquad \text{.............} \qquad 14$$

Substituting equation 14 into equation 12 gives

$$\dot{z}_1(t) = z_2(o)e^{-bt} + \frac{a-b}{b}(1-e^{-bt})u + u.$$

Transforming this equation results in the following equation

$$sZ_1 - z_1(o) = \frac{z_2(o)}{s + b} + \frac{(a-b)\,\frac{u}{b}}{s} - \frac{(a-b)\,\frac{u}{b}}{s + b} + \frac{u}{s},$$

which reduces to

$$Z_1 = \frac{z_1(o)}{s} + \frac{z_2(o)}{s(s+b)} + \frac{\frac{a}{b}\,u}{s^2} - \frac{(a-b)\,\frac{u}{b}}{s(s+b)}$$

$$= \frac{z_1(o)}{s} + \frac{\frac{a}{b}\,u}{s^2} + \frac{1}{b}[z_2(o)-(a-b)\frac{u}{b}]\,[\frac{1}{s} - \frac{1}{s+b}].$$

The inverse transformation of this equation yields

$$z_1(t) = z_1(o) + t\,\frac{a}{b}\,u + \frac{1}{b}\,[z_2(o) - (a-b)\,\frac{u}{b}][1-e^{-bt}]. \qquad \ldots\ldots 15$$

Now that the canonical equations have been solved they can be substituted into equation 8 to yield, upon reduction,

$$H = p_o|u| - bp_2(t)z_2(t) + p_1(o)z_2(t)$$

$$+ u[\frac{a}{b}\,p_1(o) + \{(a-b)p_2(o) - (\frac{a}{b} - 1)p_1(o)\}e^{bt}]. \qquad \ldots\ldots 16$$

The fuel-optimal control can now be written from equation 16 as

$$u^*(t) = -\,\mathrm{dez}\left[\frac{\frac{a}{b}\,p_1(o) + \{(a-b)p_2(o) - (\frac{a}{b} - 1)p_1(o)\}e^{bt}}{p_o}\right]. \qquad \text{.....} \quad 17$$

The possible non-singular fuel-optimal controls are therefore limited to the following sequences:

$[\pm 1, 0, \mp 1]$,

$[\pm 1, 0]$, $[0, \pm 1]$,

$[0]$ or $[\pm 1]$.

The condition for a singular fuel-optimal control is that

$$\left|\frac{\frac{a}{b}\,p_1(o) + \{(a-b)p_2(o) - (\frac{a}{b} - 1)p_1(o)\}e^{bt}}{p_o}\right| = 1$$

hold for some non-zero period of time. This condition requires that the following conditions hold true:

$$(\frac{a}{b} - 1)p_1(o) = (a - b)p_2(o)$$

or

$$p_1(o) = bp_2(o) \qquad \text{..............} \quad 18$$

and

$$p_1(o) = \pm \frac{b}{a} p_o. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad 19$$

It can be seen from these equations that the singular control interval must be the entire period of operation, as was proven in the preceeding section.

If the interior of the target set is reached by a fuel-optimal control, the final value of the adjoint variable must be normal to the target set. The following conditions result:

$$p_1(T) \neq 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad 20$$

$$p_2(T) = 0. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad 21$$

Equation 21 leads to this relation:

$$p_2(T) = \frac{p_1(o)}{b} + [p_2(o) - \frac{p_1(o)}{b}]e^{bT} = 0.$$

In order for the relation just given to hold, it is necessary for

$$p_1(o) = \frac{bp_2(o)}{1-e^{-bT}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad 22$$

to be true. Equations 22 and 18 cannot hold simultaneously, except for the trivial solution where T is zero. Therefore, a singular fuel-optimal control to the interior of the target set cannot exist.

The previously mentioned final condition on $\underline{p}(T)$ does not yield any useful information when the end-points of the target set are reached. Moreover substitution of equation 18 into equation 16 automatically means that the Hamiltonian is identically zero. Therefore, singular fuel-optimal control to the end-points still remains possible. Equations 18 and 19 limit quite severely the initial states from which a singular fuel-optimal control is possible. There are exactly two points in the costate space which satisfy these equations, since in the regular case of the fuel-optimal problem p_o can be made equal to one with no loss of generality. The irregular case occurs when the fuel-optimal solution is also the time-optimal solution. For this case p_o is zero.

The system trajectories may be found from equations 12 to 15 inclusive. Equations 12 and 13 yield the slope of the trajectories,

$$\frac{dz_2}{dz_1} = \frac{b[-\ z_2 + (\frac{a}{b} - 1)u]}{z_2 + u} \; . \qquad \ldots\ldots\ldots\ldots \qquad 23$$

Graphically constructed trajectories for various values of u may be seen in Figures 7 and 8. As might have been expected, they look very much like the trajectories of the system for which the time-optimal solution was obtained.

Before the fuel-optimal control is obtained, one point should be mentioned. Because of the upper limit on the time of operation, there will exist regions in the state space from which solutions will not be possible. These will be regions from which the time-optimal response of the system is greater than T_f.

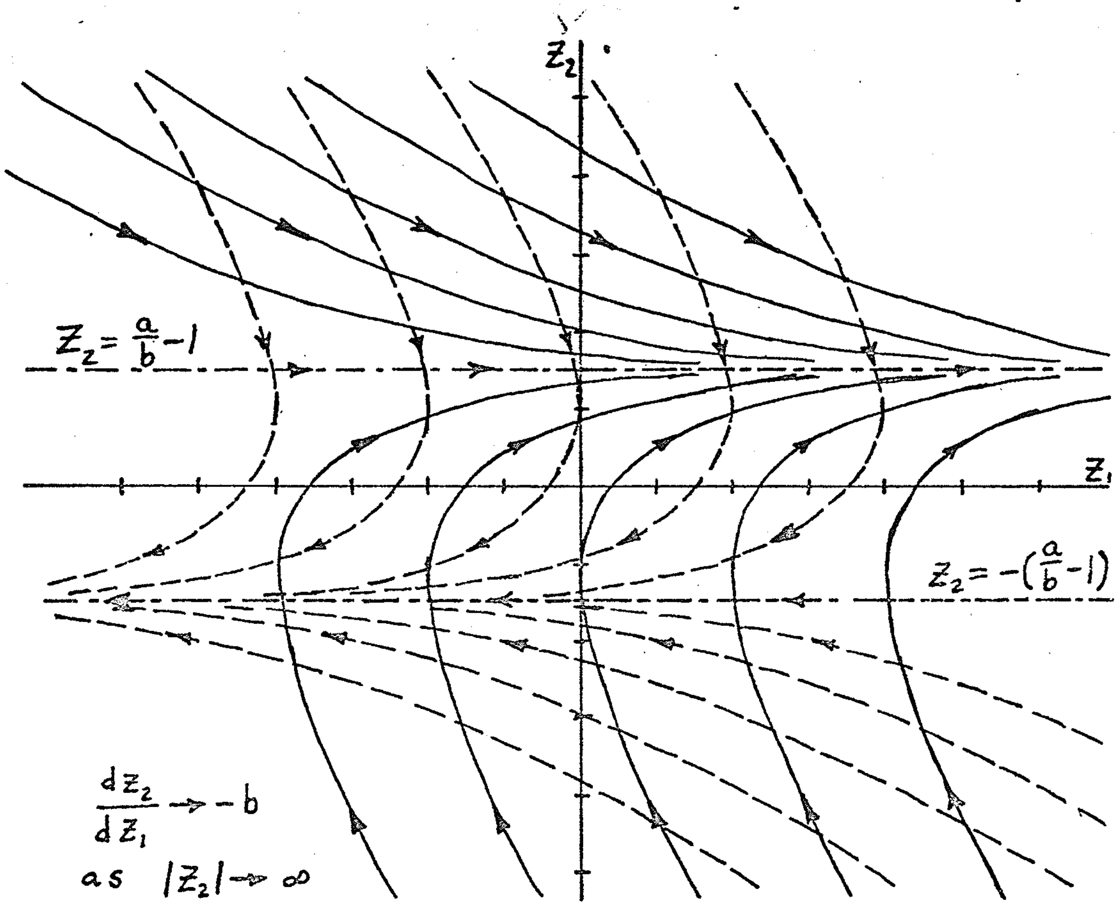

Figure 7. Forced system $[H(s) = \frac{s + a}{s(s+b)}]$ trajectories, solid lines for $u = +1$, dashed lines for $u = -1$.

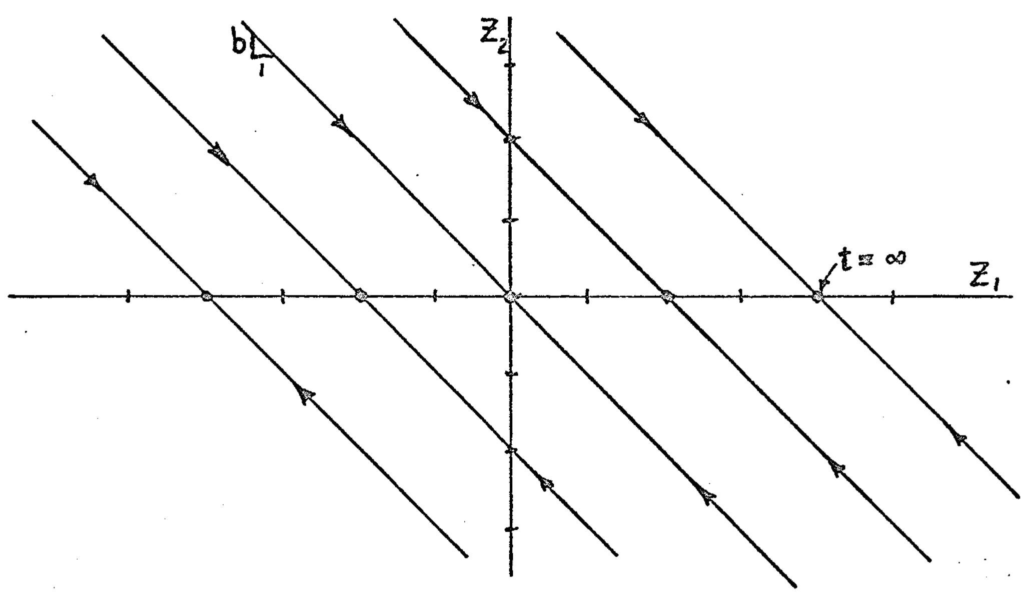

Figure 8. Unforced system $[H(s) = \frac{s + a}{s(s+b)}]$ trajectories, $u = 0$.

Some help in limiting the number of possible fuel-optimal control sequences, which drive the system to the interior of the target set, can be obtained by substituting equation 22 into equation 17. This substitution yields, upon simplification, the relation,

$$u^*(t) = - \operatorname{dez} \left[\frac{p_2(o)\{a - (a - b)e^{b(t-T)}\}}{1 - e^{-bT}} \right]. \quad \ldots \quad 25$$

For given values of a and b, certain control sequences may be eliminated from the list of all possible sequences. If a is greater than b, which is the case illustrated in Figure 7, the allowable fuel-optimal sequences are:

$$[\pm 1, 0], \quad [0], \quad \text{or} \quad [\pm 1].$$

If a is less than b, the allowable control sequences to the interior of the target set become:

$$[0, \pm 1], \quad [0], \quad \text{or} \quad [\pm 1].$$

These restrictions are of immense help in the determination of the fuel-optimal controls for various regions in the state space. The case in which a equals b was not mentioned because it will be done fully in the following example.

Because it is almost impossible to solve this problem in general terms, a specific example will be done, which will illustrate the shapes of regions in the state space and the corresponding fuel-optimal control sequences. For the sake of simplicity, fuel-optimal control sequences will be given over half the state space only. Because of symmetry, the only difference in the corresponding regions of the other half of the state space is that the signs of the respective controls will be reversed.

The parameters to be used in the example problem are as follows:

$$a = 2.5, \quad b = 1.0$$

and

$$T_f = 2.0 \text{ secs.}$$

Figure 9 is a plot of certain regions in the state space and the corresponding fuel-optimal control sequences. The following paragraphs give details about the regions and their fuel-optimal controls.

Region I is that region for which the fuel-optimal control is $u^* = 0$ throughout the period of operation, except of course when the trajectory meets the target set and $u^*(T)$ must equal $-z_2^*(T)$. This control is obviously the only fuel-optimal control throughout this region and on its boundaries. The extent of the region is determined by the upper bound on the response time, T_f and the $u = 0$ trajectory which runs to $z_1 = 0$, $z_2 = -1$. The time-restricted boundary of this region may be found from the following expressions derived from equations 14 and 15,

$$z_2 = z_2(T_f)e^{bT_f} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 26$$

and

$$z_1 = \frac{z_2(T_f)}{b}[1 - e^{bT_f}] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 27$$

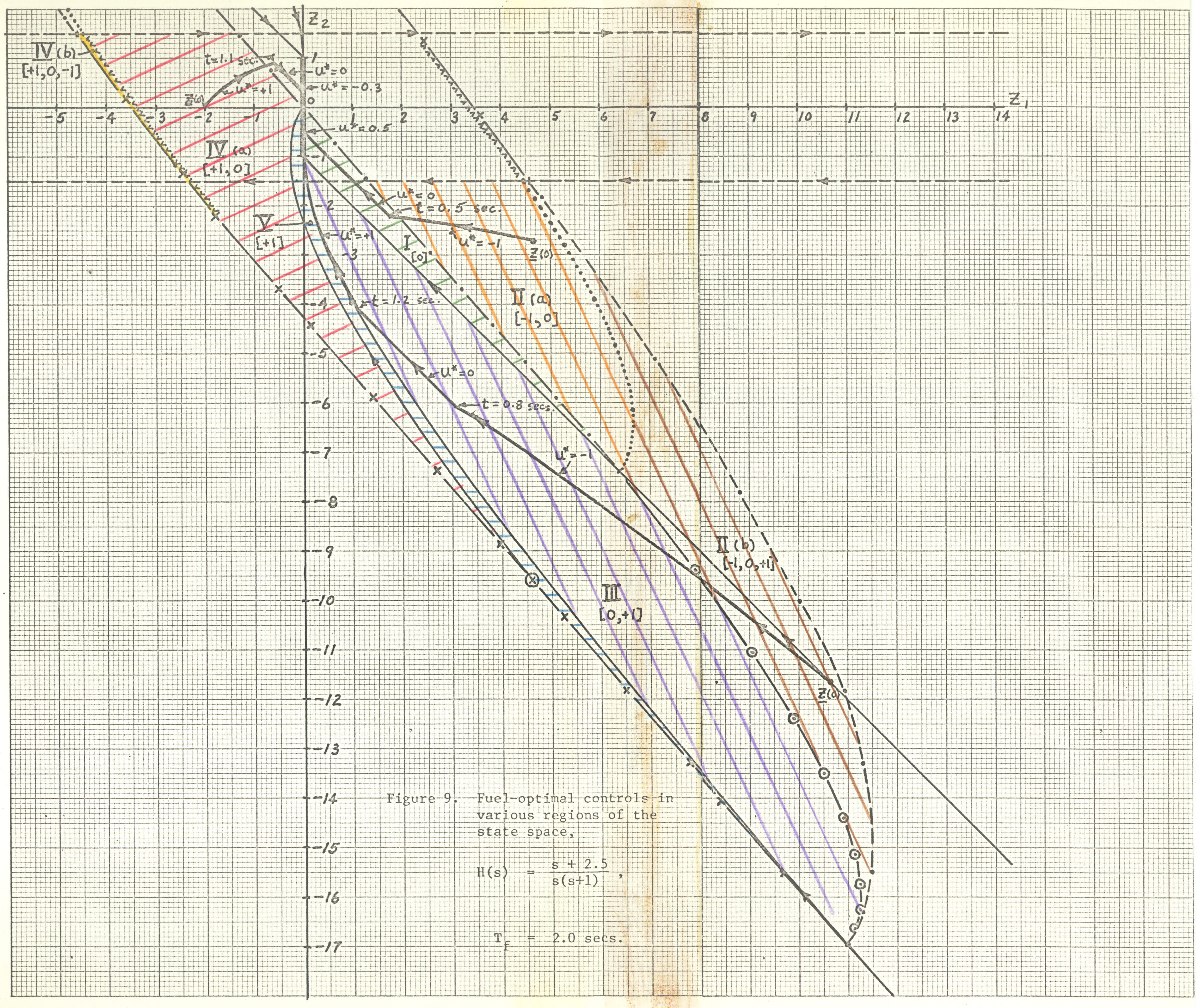

Figure 9. Fuel-optimal controls in various regions of the state space,

$$H(s) = \frac{s + 2.5}{s(s+1)},$$

$$T_f = 2.0 \text{ secs.}$$

where z_1 and z_2 are points on the time-restricted boundary and $z_2(T_f)$ is the point in the target set to which the system will run with $u = 0$, from z_1 and z_2. The response time T in this region will be less than T_f except when $\underline{z}(o)$ is on the time-restricted portion of the boundary.

The fuel-optimal control sequences in regions II(a) and II(b) are $u^* = [-1, 0]$[3] and $u^* = [-1, 0, +1]$ respectively. The optimal control sequence $u^* = [-1, 0]$ always switches to zero from minus one inside or on the non-time-restricted boundary of region I. The switch to the zero control takes place at the earliest possible time, so that the system can operate without consuming fuel for the longest possible time. The final time T is therefore always equal to T_f.

Inside region II(b), the fuel-optimal control sequence is $u^* = [-1, 0, +1]$. This time the switching from minus one to zero takes place inside region III. As for region II(a) the switch takes place at the earliest possible time, so that T equals T_f. The switch from zero to plus one takes place when the state reaches the trajectory for $u = +1$ which runs to $z_1 = 0$, $z_2 = -1$. The fuel-optimal control sequences along the asymptote $z_2 = -1.5$ are $u^* = [-1, 0]$ and $u^* = [-1, 0, +1]$ for initial states on those portions forming part of the boundary for regions II(a) and II(b) respectively.

Finally, on the time-restricted boundary of region II, the problem becomes irregular and the fuel-optimal and time-optimal solutions become identical. The optimal control sequence is, of course, $u^* = [-1, +1]$ where the switch to plus one takes place when the state reaches the trajectory for $u = +1$ which runs to $z_1 = 0$, $z_2 = -1$.

---

[3] this sequence reaches the interior of the target set, cf. p.55.

The final condition on all these fuel-optimal controls is $u^*(T_f) = - z_2^*(T_f)$. The boundary dividing regions II(a) and II(b) and the time-restricted boundary were calculated from these equations derived from equations 14 and 15:

$$z_2 = z_2(T)e^T + 1.5[1 - e^T]u \quad \ldots\ldots\ldots\ldots\ldots \quad 28$$

$$z_1 = z_1(T) - 2.5(T)u + [1.5u - z_2][1 - e^{-T}], \quad \ldots \quad 29$$

by a reverse time procedure.

The fuel-optimal control sequence inside region III is $u^* = [0, +1]$ where the switch to plus one occurs when the state reaches the trajectory for $u = +1$ which passes through $z_1 = 0$, $z_2 = -1$. The terminal time may be less than or equal to T_f. The equality to T_f will hold when $\underline{z}(o)$ is on the time-restricted boundary of region III, which was constructed using equations 28 and 29 as before. The fuel-optimal control sequence $u^*(t) = [0, +1]$ holds throughout the interior of region III and on the entire time-restricted portion of the boundary except the two end points.

Regions IV(a) and IV(b) are analogous to regions II(a) and II(b) respectively. The only difference is that the signs on the fuel-optimal control sequences are reversed, i.e. $u^* = [-1, 0, +1]$ for region II(b) becomes $u^* = [+1, 0, -1]$ for region IV(b). The same comments apply to this region as to region II. In addition, however, it should be noted that region IV(a) is bounded in part by the overall time-restricted boundary. From this portion of the boundary the irregular fuel-optimal control is $u^* = [+1]$.

The fuel-optimal control for the interior and boundaries of region V is $u^* = [+1]$, $u^*(T) = -z_2^*(T)$ since the only other possible control from this region, $u = [0, +1]$, is not fuel-optimal to the interior of the target set. The only end-point reachable from region V with the control sequence $[0, +1]$ is $z_1 = 0$, $z_2 = +1$. Such a trajectory from region V cannot be fuel-optimal, even with no upper bound on the response time. In the third quadrant of the state space, the unforced system trajectories take the system away from the target set into regions from which more fuel is required to reach the target set. The final time may be less than or equal to T_f in this region.

No solution for the stated problem exists outside these regions, except of course for their mirror images, because the upper bound on the response time is less than the minimum time of operation, as previously mentioned.

With all regions and their boundaries having readily identifiable non-singular fuel-optimal controls, no singular fuel-optimal controls could be found. Because of the severe restrictions on $\underline{p}(o)$ and the fact that $\underline{p}$ must be constant in order to satisfy the condition necessary for a singular fuel-optimal control, it seems likely that the singular case involves a trivial initial condition.

Example 2

The matrix G will be made singular for this example by setting a and b equal to one. For purposes of comparison with example 1, the upper bound on the response time, T_f, will be 2.0 seconds. With these parameters, equations 14 to 17 inclusive become,

$$z_2(t) = z_2(o)e^{-t}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad 30$$

$$z_1(t) = z_1(o) + tu + z_2(o)[1 - e^{-t}], \qquad \ldots\ldots\ldots \qquad 31$$

$$H = p_o|u| + [p_1(o) - p_2(t)]z_2(t) + up_1(o) \;.. \qquad 32$$

and

$$u^*(t) = - \text{dez}\ [\frac{p_1(o)}{p_o}]. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad 33$$

Since the argument of the dead-zone function is constant, no switching of the non-singular fuel-optimal control can occur. These controls are therefore limited to the following:

$$u^* = [\pm 1]$$

and

$$u^* = [0].$$

The key to the solution of this particular problem lies in the solution of the state differential equation,

$$\dot{z}_1 = z_2(o)e^{-t} + u.$$

The solution of this equation is given by

$$z_1(\tau) = z_1(o) + z_2(o)[1-e^{-\tau}] + \int_o^{\tau} u dt. \quad .... \quad 34$$

The magnitude of the last term of equation 34 represents the fuel consumed in the time interval $[0, \tau]$ as long as u does not change sign. This is no real restriction, however, since equations 5 show that the polarity of a singular fuel-optimal control cannot change. Thus, the following expressions can be obtained,

$$\left| \int_o^{\tau} u dt \right| = \left| z_1(\tau) - z_1(o) - z_2(o)[1 - e^{-\tau}] \right|,$$

or

$$\left| \int_o^{\tau} u dt \right| = \left| z_1(o) - z_1(\tau) + z_2(o)[1 - e^{-\tau}] \right|. \quad .... \quad 35$$

These equations will prove very useful in finding the fuel-optimal controls in the various regions of the state space shown in Figure 10.

If the terminal time T is less than T_f, the Hamiltonian must be zero throughout the period of operation. This fact in conjunction with equation 32 means that in order for a fuel-optimal control to the interior of the target set, in time T less than T_f, to be singular, either or both of the following equations must hold,

$$z_2(t) = 0 \quad .... \quad 36$$

$$p_1(o) = p_2(t). \quad .... \quad 37$$

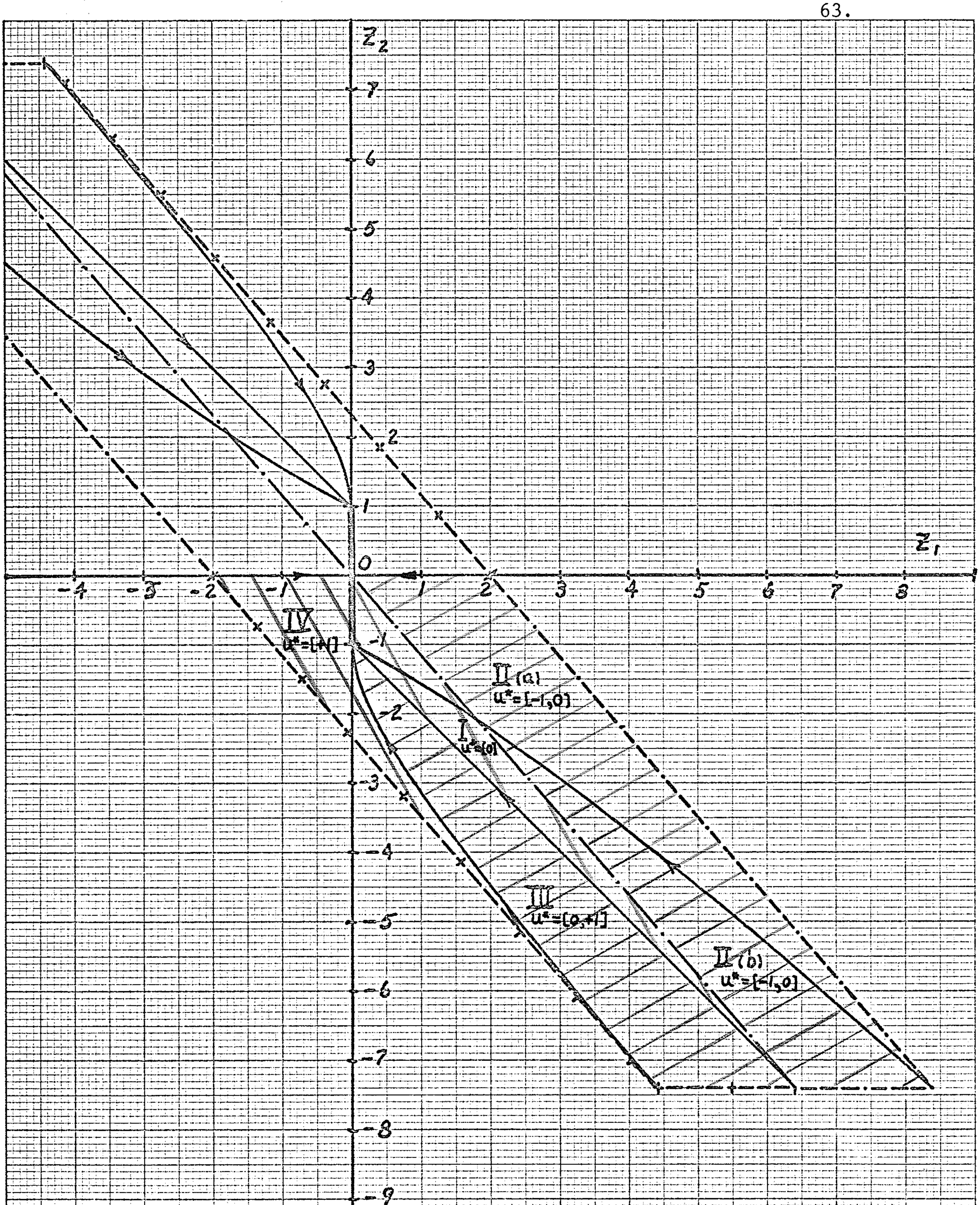

Figure 10. Fuel-optimal controls in various regions of the state space, $H(s) = \frac{s+1}{s(s+1)}$, $T_f = 2.0$ secs.

Because of the transversality condition, $p_1(o) \neq 0$ and $p_2(T) = 0$, equation 37 cannot hold. Looking at equations 34 and 36 leads to the conclusion that singular fuel-optimal control along the $z_2 = 0$ axis is possible. The fuel consumed in driving the system along the $z_2 = 0$ axis depends only on $z_1(o)$. Thus, any control which drives the system to the target in time less than or equal to T_f and does not change sign is acceptable. The equality to T_f holds in this case because $z_2(t) = 0$ means $H = 0$ holds regardless of the terminal time. As was mentioned for the singular region of the time-optimal problem, some other parameter could be minimized along this trajectory. Singular trajectories to the interior of the target set in time T equal to T_f and to the end-points with T less than or equal to T_f are still a possibility.

The effect that the uncontrollability of z_2 has on the regions for which any solutions exist can readily be seen by comparing Figures 9 and 10. The time-restricted boundaries were found in the same way that those of example 1 were found.

Region I remains unchanged from example 1. Nothing further can be said about it since the fuel-optimal control, $u^* = [0]$ in this region, is obviously the only fuel-optimal control.

The fuel-optimal control from a point inside regions II(a) and II(b) and on the boundary separating them may be obtained after consideration of equation 35. The fuel consumed depends on the time of operation of the system under a non-zero control and the difference between the initial state of z_1 and the state of z_1 at the time u becomes zero. Therefore the fuel-optimal control strategy is to minimize the time of operation with a non-zero control and also the difference, $|z_1(o)-z_1(\tau)|$.

The time of operation with a non-zero control is minimized in region II(a) by using the control $u = -1$, as was shown in Chapter IV. Furthermore, switching the control to $u = 0$ at the earliest possible time will minimize the time of operation with a non-zero control even more as well as minimizing the difference $|z_1(o)-z_2(\tau)|$, where τ is the switch time. The same strategy will minimize the fuel consumed from a point inside region II(b). The only real difference between regions II(a) and II(b) is the fuel-optimal control from the time-restricted boundaries, which comes about because the time-optimal control is singular in region II(b). The fuel-optimal control from the time-restricted boundary of region II(a) is, therefore, $u^* = -1$. From the time-restricted boundary of region II(b) it is $u^* = [-1, 0]$ where the switch to zero occurs when the $u = -1$ trajectory reaches the $u = 0$ trajectory to $z_1 = 0$, $z_2 = -1$.

By similar reasoning, the fuel-optimal control sequence from points in and on the time-restricted boundary of region III is $u^* = [0, +1]$, as for region III of example 1.

Region IV of this example corresponds to region V of example 1, where the fuel-optimal control was found to be $u^* = +1$. Since $z_1(o)$ and $z_1(\tau) = 0$ are fixed, the fuel-optimal solution in and on the boundaries of region IV, except $z_2(t) = 0$, is the time-optimal solution, $u^* = +1$.

Solutions will not exist outside the regions just covered, except of course for their mirror images, where the fuel-optimal controls are identical, but with reversed polarities.

One final point to be remembered is that $u^*(T) = - z_2^*(T)$ must hold for controls from all regions in order for $\underline{y}(T) = \underline{0}$ to hold as required.

Regions II and III and the $z_2 = 0$ axis are the singular regions of this problem. The $z_2 = 0$ axis is singular because any control of the proper polarity will drive the system to the origin with the same fuel. The time limitation will, of course, restrict the range of controls, but infinitely many are still possible. Regions II and III are also singular, but because the fuel-optimal controls violate the non-switching criterion of equation 33. In accordance with the concept of a singular fuel-optimal control, the polarities of fuel-optimal controls in regions II and III do not change. It should also be noted that all singular fuel-optimal controls from these regions which reach the interior of the target set, take exactly T_f seconds, as required by the singular condition derived near the beginning of this example.

CHAPTER VI

CONCLUSIONS

It was seen, during the solution of examples 1 and 2 of the fuel-optimal problem, that a change in a few system parameters necessitated the adoption of a completely different approach in order to obtain the problem solutions. This is in direct contrast to the normal problem, where the approach to a given optimization problem is fairly straightforward.

The presence of numerator dynamics complicates the optimal solution, over and above the introduction of the possibility of a singular solution. This complication, the occurrence of derivatives of the control function in the state equations, is eliminated by the state equation transformation given in Chapter II. Expansion of the target set, from a point to a line in this case, is the additional complexity which arises upon use of this transformation.

The main conclusion reached is that singularity and uncontrollability are very closely tied, in general, and inseparable for the single-input system. These ties were illustrated in section I of Chapter III and made even more evident during the solution of both the time-optimal and fuel-optimal problems. The singular regions for the time-optimal problem were due entirely to the uncontrollability of z_2. It was also shown that some other cost function could be minimized in the singular regions. A comparison of the fuel-optimal examples also clearly illustrates the restrictions uncontrollability places on the allowable initial states. It can also be seen that system parameter drift could affect the problem enough so that a previously allowable initial state becomes one for which no solution is possible.

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