Risk Management with Reinsurance Policies

by

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Abstract

Businesses face various risks that may negatively influence their operations, therefore implementing strategies to deal with risks is important. In recent years, risk management has become an active area of research in finance and insurance. The primary goals of risk management include identifying, assessing and controlling risks to minimize their potential impact. For insurance companies, reinsurance is an effective risk management tool to control risks. As a natural measure of risk, we consider the ruin probability of an insurance business. Our ultimate objective is to evaluate the impact of reinsurance in risk management, particularly in minimizing the ruin probability, and to find the corresponding optimal reinsurance policies.

We first study the problem of minimizing the ruin probability in a discrete-time risk model with unknown parameters. A proportional reinsurance is purchased to control the ruin probability. We formulate the problem as a Markov decision process and solve this problem by means of discrete-time dynamic programming. The Bayesian approach is applied to address the issue of parameter uncertainty. We obtain the explicit expressions of minimum ruin probabilities and the corresponding optimal reinsurance strategies. Some structural properties of ruin probabilities are investigated under certain conditions.

We also consider an optimization problem by joint decisions of excess-of-loss reinsurance and investment in a continuous-time financial market. The reserve may be invested in a financial market consisting of a risk-free asset

and a risky asset with the price process follows geometric Brownian motion. Borrowing is allowed, however, the interest rate of borrowing is higher than the return rate of risk-free. Meanwhile, an excess-of-loss reinsurance is purchased. We apply stochastic control theory and Hamilton-Jacobi-Bellman equation to find the optimal strategy of joint reinsurance and investment decisions, and derive the closed form expression of the minimum ruin probability function.

Our results are illustrated numerically. Both theoretical and numerical results show that reinsurance has a significant effect in alleviating the risk of ruin.

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This thesis is dedicated to my parents
for their love, support
and encouragement.

Contents

Chapter 1

Introduction

Risk theory is traditionally considered as a central topic of actuarial mathematics. Risk management with the objective of modelling, measuring, managing financial risks, and achieving marketing goals has become a challenging field of research. One ultimate goal of risk management is to control risks and prevent ruin of insurance companies. As an important risk measure, the ruin probability is the main quantity of interest in an insurance risk model. In recent years research in optimal control strategies, particularly under a risk model with a reinsurance control, that minimize the ruin probability has attracted a lot of attention. In this chapter, we introduce the backgrounds of risk management, risk theory and reinsurances policies.

1.1 Risk management

All businesses face numerous risks. Risks can be generally divided into following categories: market risk, credit risk, liquidity risk, operational risk, legal and regulatory risk, business risk, strategic risk, and reputation risk. Risk arises from the randomness of an entity's future return. There is a natural trade-off between risk and reward. Generally speaking, the greater the risk taken, the greater the potential reward.

In the financial industry, risk is defined as uncertainty that adversely affects income or wealth, or is related only to negative outcomes. In simplified terms, risk is seen as the potential of loss due to uncertainty. Even though forecasting the systematic events with major financial impact may not be realistic, financial risks caused by uncertainty can still be managed.

The first step of managing uncertainty is to identify risks. Then risks need to be quantified and estimated before suitable methods are found to alleviate risks. On the other hand, overall impact of risk exposures need to be determined or a cost-benefit analysis of risk transfer methods need to be performed. At the same time, developing risk alleviation strategies (i.e., avoid, transfer, mitigate or take risks), evaluating performance and modifying risk alleviation strategies may be needed. The process of this sequence of activities is called risk management.

Generally speaking, the occurrence of a large loss is not necessarily caused by the failure of risk management. It could be due to bad luck or immeasurable risks. In order to distinguish possible causes of losses, we need to classify losses.

1. The first category of loss is called the expected loss. It is the amount that an entity expects to lose in the normal business and can usually be calculated relatively easily in advance.

2. The second category of loss is called the unexpected loss. It could be an extreme event of the expected loss or a loss caused by a risk that already exists but has not been properly measured.

3. The last category of loss is caused by the Knightian uncertainty proposed by economist Frank Knight in 1921. This type of loss is due to immeasurable risks. Knight refers to variability that cannot be quantified as uncertainty and variability that can be quantified according to statistical science as "true" risk.

The main task of risk managers is to properly measure unexpected losses, and evaluate the corresponding returns. The expected loss can be regarded as variable costs or predictable expenses, rather than risks or uncertainty. But distinguishing between unexpected and expected losses is not an easy task. This is also the key to understand modern risk management. Risk managers should be responsible for various risks, not just measurable risks. Although the loss from Knightian uncertainty cannot be quantified, its existence and impact cannot be ignored. Therefore, risk managers could manage Knightian uncertainty through avoidance or other forms of risk management.

Although risk management is a practical activity, it should be based on theoretical research on risk. The development of theoretical research on risk in the past 70 years can be summarized with the following four risk models:

1. The foundations of modern risk analysis called Modern Portfolio Theory (MPT) was proposed by Harry Markowitz in 1952. MPT provides investors with a portfolio framework that maximizes expected return for a given risk where the risk is defined as the standard deviation of the rate of return. Markowitz believes that investors should not put all their assets in one investment, but diversify them into multiple investments. Therefore, he considers the expected return and risk of a portfolio instead of individual investment. Based on the correlation coefficient between each investment, he obtained the expected returns and risks under different weight combinations. Investors determine the weight of each investment that best suits their needs. For this research, Markowitz received the Nobel Prize in Economics in 1990.

2. Based on Harry Markowitz's early work on MPT, the Capital Asset Pricing Model (CAPM) was introduced by William Sharpe, John Lintner and Jan Mossin independently in the mid-1960s. They introduced the expected return $\mathbb{E}(R_i)$ of an asset R_i, defined as

$$\mathbb{E}(R_i) = R_f + \beta_i\big[\mathbb{E}(R_m) - R_f\big],$$

where R_f is the risk-free rate, $\mathbb{E}(R_m)$ denotes the expected return of the market and $\beta_i = \frac{\mathrm{Cov}(R_i, R_m)}{\mathrm{Var}(R_m)}$ is the sensitivity of the expected excess asset returns to the expected excess market returns. The CAPM describes the relationship between expected returns and risk for a given asset. It provides a useful measure to help investors determine how much return they deserve from their investments in exchange for putting their money at risk. Like other theoretical models, CAPM is inseparable from a large number of assumptions. Some of the assumptions behind the CAPM have proven to be unrealistic. However, there is still some value in using CAPM as a tool to assess the reasonableness or comparison of future expectations. Therefore, the CAPM is still widely used in the finance industry. In 1990, Sharpe was awarded the Nobel Prize for this research.

3. In 1973, Fisher Black, Myron Scholes and Robert Merton developed a classic model, called Black-Scholes or Black-Scholes-Merton model, for pricing stock options. This is a mathematical model of a financial derivative market. It gives a theoretical estimate for pricing the European option and shows that the price is unique. Briefly, they believed that the price of the stock can be described by a Geometric Brownian motion, so at a fixed time point in the future, the price of the stock follows the log-normal distribution. According to the no-arbitrage criterion, the investor's expected return minus the cost is the price of the option. The emergence of Black-Scholes-Merton model has led to a significant increase in options trading. Now, the Black-Scholes-Merton formula has been widely adopted by traders and investors. In the past 40 years, a large number of extensions and evolutions of the Black-Scholes-Merton model have been developed. This also presents challenges and opportunities for risk managers. In 1997, Merton and Scholes were awarded the Nobel Prize for their research.

4. In 1958, Franco Modigliani and Merton Miller developed the Modigliani-

Miller Theorem or M&M Theorem. This theorem is not relatated to financial markets but to the capital structure. Modigliani and Miller point out that in the perfect capital market (no taxes, no bankruptcy costs, no agency costs, no information asymmetry), the company's capital structure has no effect on the company's value. According to M&M theory, risk management does not increase firm's value. When a firm's financial risk is increasing, equity holders will claim compensation for this risk and expect a higher rate of return. This gives investors an attitude to risk in the normal capital market: investors do not expect higher returns, but expect higher risk-adjusted returns. In 1985, Modigliani and Miller were awarded the Nobel Prize in Economics.

Theoretical research on risk is the core of risk management. In the 1980s, financial derivatives were starting to become important risk management tools. By the 1990s, with the frequent and large fluctuations in interest rates, exchange rates, stock prices and commodity prices, the financial derivatives market had been developed rapidly. At the same time, risks brought by the market had also attracted great attention from companies. Some of these companies were very successful in their areas of expertise, but have a significant reduction in profits due to a lack of understanding of market risks. For example, a well-run shipping company lost money because of a sudden increase in the cost of crude oil. Other companies that had great competitive advantages in risk management had made risk management their main business. Because these companies had a competitive advantage in risk management, they even increased their risk exposure to profit. In this sense, risk management is managing risk and creating economic value, not reducing risk. Risk management and risk taking are not opposites, but two sides of the same coin. The two together become the core of the management process of all successful companies.

Risk management can improve the company's tax structure, reduce the probability of bankruptcy, increase the ability to liabilities and reduce the

cost of liabilities. These benefits can keep the company's cash flow and income stable.

As a formal discipline, risk management has been developed rapidly in the past 30 years. A large number of institutions with risk management as their main business are established and they have achieved some extraordinary successes in risk management. With the development of credit derivatives, risk management is also recognized as the most creative force in the financial market. At the same time, there have been some failure cases in risk management, such as the bankruptcy of Long-Term Capital Management (LTCM) in 1998 and a series of financial scandals related to the stock market. These failures have prompted researchers to look more closely at the liquidity and complexity of the risk itself, and to determine whether any changes in the company's risk profile are in the interests of stakeholders.

1.2 Risk theory

As a part of actuarial mathematics, risk theory refers to mathematical techniques developed to model, measure and manage risks for a business. One primary goal of risk theory is to evaluate the risk of a portfolio associated with insurance contracts.

We start with formulating a usual risk model. Consider an insurance contract with initial reserve x. Let N_t denote the number of claims during the time interval (0, t]. Specifically, let T_i denote the elapsed time between the $(i-1)^{th}$ and the i^{th} claim and Y_i denote the i^{th} claim amount with a distribution function F_i. The waiting time until the n^{th} claim is defined as $W_n = \sum_{i=1}^{n} T_i$. Suppose that the insurance company collects premiums continuously with a constant rate c, then the reserve of the insurance company at time t is

$$\hat{R}_t^A = x + ct - \sum_{i=1}^{N_t} Y_i, \text{ for all } t \geqslant 0. \tag{1.2.1}$$

This is a standard risk model for the evolution of the reserve over periods of time. One important problem considered in risk theory is to study the ruin probability, that is, the probability that the reserve becomes negative.

1.2.1 Cramér-Lundberg risk model

One of the most important risk model in the risk theory is developed by Lundberg (1903) and Cramér (1930), and is called the Cramér-Lundberg risk model or classical risk model. The Cramér-Lundberg risk model $\{\hat{R}_t\}_{t\geqslant 0}$ can be generalized by imposing the following assumptions on model (1.2.1):

(i) $Y_1, Y_2, \cdots$ are independent and identically distributed random variables having a common distribution F with mean $\mu < \infty$;

(ii) $\{N_t\}_{t\geqslant 0}$ is a Poisson process with intensity $\lambda > 0$;

(iii) N_t and $Y_1, Y_2, \cdots$ are independent.

As a natural measure of risk, we consider the ruin probability. Let $\tau = \inf\{t > 0 : \hat{R}_t < 0\}$ denote the time of ruin. Suppose that the initial reserve is $x \geqslant 0$. The ultimate ruin probability $\psi(x)$, which is also referred to as the ruin probability with infinite horizon, is the probability that the reserve $\hat{R}_t$ ever drops below zero. That is,

$$\psi(x) = \mathbb{P}\big(\inf_{t>0} \hat{R}_t < 0 | \hat{R}_0 = x\big) = \mathbb{P}\big(\tau < \infty | \hat{R}_0 = x\big).$$

One of the main concerns is the premium calculation. Determining the premium is not a trivial task. Intuitively, the premium can not be too cheap since this will cause bankruptcy. On the other hand, the premium can not be too expensive due to the consideration of market competition.

Suppose that risk process $\{\hat{R}_t\}_{t\geqslant 0}$ is given by the Cramér-Lundberg model, then it can be showed that (see Ibe (2013))

$$\frac{\hat{R}_t}{t} \to c - \mu\lambda \quad \text{almost surely (a.s.)}, \quad \text{as} \quad t \to \infty. \tag{1.2.2}$$

By result 1.2.2, we have the following results:

(i) If $c > \lambda\mu$, then $\lim\limits_{t\to\infty} \hat{R}_t = \infty$ a.s. and $\psi(x) = \mathbb{P}\big(\inf\limits_{t>0} \hat{R}_t < 0 | \hat{R}_0 = x\big) < 1$;

(ii) If $c < \lambda\mu$, then $\lim\limits_{t\to\infty} \hat{R}_t = -\infty$ a.s. and $\psi(x) = \mathbb{P}\big(\inf\limits_{t>0} \hat{R}_t < 0 | \hat{R}_0 = x\big) = 1$;

(iii) If $c = \lambda\mu$, then $\limsup_t \hat{R}_t = \infty$ and $\liminf_t \hat{R}_t = -\infty$. Thus, $\psi(x) = 1$.

Consider the non-ruin probability $\bar{\psi}(x) = 1 - \psi(x)$. It is obvious that we should take the premium which leads to a positive non-ruin probability. Thus, we need the condition $c > \lambda\mu$, which is called the net profit condition (NPC). In practice, we use the term safety loading (or security loading) to describe the relative amount of premium rate c which exceeds the average cost of claims $\lambda\mu$. Let η denote the safety loading, then we have

$$\eta = \frac{c - \lambda\mu}{\lambda\mu}.$$

From the mathematical point of view, $\eta > 0$ is required. Moreover, insurance companies need a positive safety loading to gain extra income for running business.

To investigate the ruin probability, it is usually more convenient to work with the non-ruin probability $\bar{\psi}(x)$. Under the Cramér-Lundberg risk model, $\bar{\psi}(x)$ can be expressed as the following integral equation

$$\bar{\psi}(x) = \bar{\psi}(0) + \frac{\lambda}{c}\int_0^x \bar{F}_Y(y)\bar{\psi}(x-y)dy, \tag{1.2.3}$$

where $\bar{F}_Y(y) = 1 - F_Y(y)$.

In general, it is difficult to solve the equation (1.2.3) in a closed form under the Cramér-Lundberg risk model. However, there are some classical results that are summarized below. Assuming that there exists a constant $R > 0$ such that

$$\frac{\lambda}{c}\int_0^\infty e^{Ry}(1 - F_Y(y))dy = 1, \tag{1.2.4}$$

then we have the following results:

1. The ruin probability with zero initial reserve is

$$\psi(0) = \frac{1}{1+\eta}.$$

   By the result of 1.2.2, $\inf_{t>0} \hat{R}_t$ is finite almost surely with $\hat{R}_0 = x$. Thus, we have $\psi(\infty) = 0$. Taking the limit of $x \to \infty$ on both sides of equation (1.2.3) and apply the Monotone Convergence Theorem, we have

$$\begin{aligned}\bar{\psi}(\infty) =& \bar{\psi}(0) + \frac{\lambda}{c}\int_0^\infty \bar{F}_{Y_1}(y)\bar{\psi}(\infty)dy \\ =& \bar{\psi}(0) + \frac{\lambda\mu}{c}\bar{\psi}(\infty),\end{aligned}$$

   where $c = (1+\eta)\lambda\mu$. Therefore, we obtain $\psi(0) = \frac{1}{1+\eta}$. As we see, $\psi(0)$ depends on the mean μ of distribution F only. Thus, we say $\psi(0)$ is insensitive with respect to the shape of F.

2. If the claim sizes follow an exponential distribution with mean μ, then we have an explicit form of the ruin probability

$$\psi(x) = \frac{1}{1+\eta}e^{-\frac{\eta x}{(1+\eta)\mu}}.$$

3. The Cramér-Lundberg approximation

$$\lim_{x\to\infty} e^{Rx}\psi(x) = \frac{\eta\mu}{h'(R) - p/\lambda},$$

where $h(R) = \int_0^\infty e^{Ry} dF_Y(y) - 1$ and R, which is the positive solution of the equation $h(R) = \frac{c}{\lambda}R$, is called the Lundberg exponent. Since $\lim_{x\to\infty} \psi(x) = 0$, a natural question is how fast it converges to 0. To answer this question, we study the asymptotic behaviour of $\psi(x)$ as $x \to \infty$. To obtain this result, we consider the equation (1.2.3) as a renewal type equation and then we apply the Key Renewal Theorem. The result shows that $\psi(x) \to 0$ as $x \to \infty$ at exponential rate.

4. The Lundberg inequality is given by

$$\psi(x) \leqslant e^{-Rx}.$$

To obtain the result, we consider an event A={ruin occurs with initial reserve x} and another event A_n={ ruin occurs before or at the n^{th} claim occurs with initial reserve x} for $n = 1, 2, \cdots$. Obviously, $A_n \subseteq A_{n+1}$ and $\lim_{n\to\infty} A_n = A$. By the continuity of probability, $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A) = \psi(x)$. Thus, it suffices to prove that

$$\mathbb{P}(A_n) \leqslant e^{-Rx}$$

for each $n \geqslant 1$. By the method of mathematical induction, we obtain the result. Moreover, if claim sizes are bounded from above by a constant K, then we have more accurate upper bond such that $\psi(x) \leqslant (1+\eta)^{-x/K}$.

The ruin probability $\psi(x)$ can also be represented as a compound geometric probability which is called Pollaczek-Khinchin Formula, given as

$$\psi(x) = (1 - \frac{1}{1+\eta}) \sum_{n=1}^{\infty} (\frac{1}{1+\eta})^n \bar{F}_I^{*n}(x), \tag{1.2.5}$$

where F_I is the equilibrium distribution function (e.d.f.) of F and is defined as

$$F_I(x) = \frac{1}{\mu}\int_0^x \bar{F}(z)dz, \quad x \geqslant 0,$$

F_I^{*n} is the n^{th} convolution of the distribution F_I.

The infinite series representation for $\psi(x)$ is particularly useful for theoretical considerations. However, it can also be used for numerical approximations of the ruin probability since the Pollaczek-Khinchin formula shows that $\psi(x)$ is the distribution function of a compound geometric.

1.2.2 Ruin theory for heavy-tailed distributions

The asymptotic behaviours of the ruin function $\psi(x)$ depend on the existence of the Lundberg exponent R. From equation (1.2.4), this assumption is equivalent to that the tail of the claim size distribution function decreases at least exponentially fast. However, such an assumption is unrealistic in many cases. For instance, data from catastrophe insurance (earthquakes, flooding etc.) or fire insurance clearly show heavy tail behaviours. Particularly, lognormal, loggamma and Pareto distributions are popular in actuarial mathematics.

Heavy-tailed distributions are probability distributions whose tails are heavier than the exponential distribution and the two most important classes in risk theory are subexponential class $\mathcal{S}$ and its subclass, regularly varying class $\mathcal{R}$. We introduce definitions of these classes of heavy-tailed distributions as follows.

Definition 1.2.1. ***(Subexponential distributions)***
Let F be a distribution function on $(0, \infty)$. We say that F is the distribution function of a subexponential distribution, written as $F \in \mathcal{S}$, if

$$\lim_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n,$$

for all $n \geqslant 2$, where F^{*n} is the n^{th} convolution of F and $\overline{F}(x) = 1 - F(x)$.

Definition 1.2.2. ***(Regularly varying)***
A measurable function $f : (0, \infty) \to (0, \infty)$ is called regularly varying with index α ($\alpha \in \mathbb{R}$) if

$$\lim_{x \to \infty} \frac{f(tx)}{f(x)} = t^{\alpha} \quad \textit{for all} \quad t > 0.$$

We write $f \in \mathcal{R}_{\alpha}$. If $\alpha = 0$, f is said to be slowly varying.

Definition 1.2.3. ***(Regularly varying distributions)***
Let F be a distribution function on $(0, \infty)$. We say F is a regularly varying distribution function with index α if $\overline{F}$ is regularly varying with index $-\alpha$.

Embrechts and Veraverbeke (1982) obtain the asymptotic ruin probability $\psi(x)$ under the condition that the equilibrium distribution function F_I of claim sizes belongs to the subexponential class $\mathcal{S}$, that is

$$\psi(x) \sim \frac{1}{\eta} \overline{F}_I(x), \quad \text{as} \quad x \to \infty,$$

where $f(x) \sim g(x)$ if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$. It can be showed that $\overline{F} \in \mathcal{R}_{-\alpha}$ if and only if $\overline{F}_I \in \mathcal{R}_{-\alpha+1}$. Therefore, if $\overline{F} \in \mathcal{R}_{-\alpha}$ with $\alpha > 1$, then we have

$$\psi(x) \sim \frac{1}{\mu\eta} \int_x^{\infty} \bar{F}(z) dz, \quad x \to \infty.$$

In practice, risk problems in insurance are often treated within an economic environment involving interest, discounting or indexing. We consider the Cramér-Lundberg risk model $\hat{R}_t = x + ct - \sum_{i=1}^{N_t} Y_i$. Assuming that all

income earn the interest with a constant interest force $r > 0$. Let the aggregate claims process be $S_t = \sum_{i=1}^{N_t} Y_i$, then the total reserve up to time t, represented by $\hat{R}_t^r$, satisfies the equation

$$\hat{R}_t^r = xe^{rt} + c\int_0^t e^{zt}dz - \int_0^t e^{r(t-z)}dS_t, \quad t \geqslant 0.$$

Let $\psi_r(x)$ denote the ultimate ruin probability for this risk process. Klüppelberg and Stadtmüller (1998) establish an asymptotic formula for the ruin probability $\psi_r(u)$ with $r > 0$, given by

$$\psi_r(u) \sim \frac{\lambda}{r}\int_u^\infty \bar{F}(z)\frac{dz}{z}, \quad u \to \infty,$$

with the condition that $\overline{F} \in \mathcal{R}_{-\alpha}$ with $\alpha > 1$. Asmussen et al. (2002) obtain the same asymptotic formula for the ruin probability $\psi_r(u)$ with the condition that $\overline{F}_I \in \mathcal{A}$ where $\mathcal{A}$ is a subclass of subexponetial class which is wider than the regularly varying class. Although $\mathcal{A}$ is a subclass of $\mathcal{S}$, it covers almost all the well-known subexponential distributions such as Pareto, log-normal, Weibull, Burr, Benktander I and II distributions.

1.3 Reinsurance

Reinsurance is an essential activity of almost all insurance companies. A reinsurance operation is a contractual arrangement in which the reinsurer agrees to indemnify the insurer (or the cedent) for a specified part of its underwritten insurance risk. In turn, the insurer pays a reinsurance premium to the reinsurer for this service. That is, through reinsurance the insurance company seeks the possibility of paying a fixed premium to replace parts of its future losses. Therefore, reinsurance is referred to as the "insurance for the insurer".

While reinsurance naturally shares various common features with insurance, it is also quite different from primary insurance in many ways, including types of risks, types of data for risk analysis and regulatory guidelines.

1.3.1 The importance of reinsurance

Similar to other financial institutions, insurance companies leverage the capital provided by shareholders to increase the liability in business operations. The financial liability would typically have pre-determined date and value, however, this is not the case for insurance companies. The primary business of an insurance company is selling insurance policies to insureds, which makes the liability very risky since claims may occur suddenly with unexpected sizes. The risk may generate profit, but it may also cause considerable financial losses and even bankruptcy. Therefore, the demand for risk management has greatly increased. Reinsurance is an available risk management tool primarily used to disperse risk. Risk transfer through reinsurance can improve the insurance company's overall risk profile.

The main objective of reinsurance is to reduce the probability of suffering financial losses. Other than this motivation, some of the main reasons for the insurer to buy reinsurance are outlined below.

1. Stabilizing business volatility.

   The volatility is an important risk measure. Through reinsurance, random losses are replaced by a deterministic premium payment, and thus the volatility of an insurance company's financial result can be reduced to a desired level. Therefore, for an insurance portfolio, reinsurance can be a tool of hedge to steer the volatility.

2. Reducing required capital.

   Entering a reinsurance contract reduces the aggregate risk and then reduces the required capital to bear risks. Concretely, if the reduction of

capital is greater than the reinsurance premium together with the corresponding administration costs, the reinsurance is desirable. Due to the risk-based regulation, the capital management has become a central issue for insurance companies. In fact, reinsurance should be understood as a substitute for capital.

3. Increasing underwriting capacity.

   Under a reinsurance contract, the insurer assumes only a part of the risk and hence the insurance company can underwrite more and larger policies. This gives the insurer an opportunity to test and enter new markets, to gain experience, and to improve liquidity.

For these reasons, reinsurance has greatly improved the stability and liquidity of insurance companies. Moreover, the efficiency of the market has been increased by reinsurance on the society level.

1.3.2 Types of reinsurance

Let X denote the risk, which can be the individual claim size or the aggregate claim size. For the event $\{X = x\}$, the reinsurance policy is a function $R : \mathbb{R}_+ \to \mathbb{R}_+$ such that $0 \leqslant R(x) \leqslant x$. The reinsurer assumes $R(x)$ and the rest $x - R(x)$ is paid by the insurer. There are various types of reinsurance, which are suitable for different categories of insurance risk. We now discuss some common types of reinsurance policies.

1. The simplest and most popular type of reinsurance is the quota-share (QS) reinsurance or proportional reinsurance. Let x denote the individual claim size, then the risk assumed by the reinsurer for this individual claim is

$$R(x) = (1 - b)x,$$

where $0 \leqslant b \leqslant 1$ is a proportionality factor denoting the retention level. Due to its simple form and administrative simplicity, the QS reinsurance is widely used in almost all insurance companies. One of the main advantages of QS reinsurance is that it improves the premium-to-surplus ratio which can be used to measure the capacity of an insurance company for underwriting new policies. According to accounting principles, all relevant expenses connected to a policy should be immediately included in the balance sheet when the policy is issued. However, the respective premium can only be added to the balance sheet gradually. The QS reinsurance will improve the premium-to-surplus ratio since it reduces the premium and expenses simultaneously. One of the main disadvantages of QS reinsurance is that all claims will be involved in the reinsurance policy. This is not an ideal situation, as small claims could be easily borne by the insurer alone. Another main disadvantage is that it exposes all information of the insurance policy.

2. A proportional reinsurance form with the proportionality factor depending on the sum insured is called the surplus reinsurance. Under this type of policy, the insurer will determine the maximum loss of claim it can retain, which is a fixed retention line M. Let L_i denote the sum insured of the individual claim X_i, then the risk assumed by the reinsurer for this claim is

$$R(x_i) = \Big(1 - \frac{M}{L_i}\Big) x_i \mathbb{1}_{(M,\infty)}(L_i).$$

Obviously, this is also a proportion type of reinsurance. However, the proportion is not fixed. Particularly, in the case of an insured sum L_i below M, the claim X_i is fully borne by the insurer. It is clear that the surplus reinsurance only takes proportional reinsurance for larger claims. Consequently, it keeps main advantages of the QS reinsurance and improves disadvantages of QS reinsurance.

3. Now we move on to non-proportional types of reinsurance. The simplest case is the excess-of-loss (XL) reinsurance. Suppose the duration of policy is a time interval $[0, t]$. Let N_t denote the number of claims during the policy period and x_i denote the individual claim size. The pre-defined maximum retained claim size M is referred to as the retention level. Under this type of policy, the reinurer pays for the excess over M. The reinsurance coverage up to time t is then defined by

$$R(t) = \sum_{i=1}^{N_t} \big(x_i - M\big)_+,$$

where $(x_i - M)_+ = \max\{x_i - M, 0\}$. Excess-of-loss reinsurance is widely used in casualty and fire insurance. It is effective in reducing the exposure of the ceding company, but the calculation of premiums is more complicated than proportional reinsurance. From a ceding company's perspective, this type of reinsurance is very suitable for the case of small number of claims but the distribution of the individual claim size is heavy-tailed.

4. A stop-loss (SL) reinsurance is also a form of non-proportional coverage. Instead of considering the individual loss in XL reinsurance, the ceding company will consider the aggregate loss over the policy period. Suppose the maximum retained size of the aggregate claim over the time interval is C and the reinsurer pays for the excess, then the risk assumed by the reinsurer up to time t is

$$R(t) = \Big(\sum_{i=1}^{N_t} x_i - C\Big)_+.$$

Although SL reinsurance is an alternative to XL reinsurance which completely eliminates tail risks, it amplifies some problems of XL reinsurance such as moral hazard issues.

5. An intuitive treaty to deal with the risk of large claims for the ceding company is large claim reinsurance. Suppose N_t is the number of claims during the time interval $[0, t]$ and claims are ordered as $x_{1:N_t} \leqslant x_{2:N_t} \leqslant \cdots \leqslant x_{N_t:N_t}$, where $x_{i:N_t}, 1 \leqslant i \leqslant N_t$ denotes the i^{th} smallest claim. Under a large claim reinsurance contract, the reinsurer will cover the r largest claims, that is up to time t,

$$R(t) = \sum_{i=N_t-r+1}^{N_t} x_{i:N_t},$$

where $r \geqslant 1$ is fixed. This type of reinsurance is not very applicable due to complex mathematical details and considerable model risks. There is another form of large claim reinsurance called ECOMOR (Excédent du Coût Moyen Relatif). In this form of reinsurance contract, the risk covered by the reinsurer up to time t is

$$R(t) = \sum_{i=1}^{N_t} (x_i - x_{N_t-r:N_t})_+.$$

Clearly, this is an XL-type treaty. The retention level is the $(r+1)^{th}$ largest claim and the reinsurer pays for the excess over this retention. So far, due to its mathematical challenges, the ECOMOR treaty is popular in academics, but this is also the reason that ECOMOR reinsurance is currently not used in practice. However, there are some reinsurance treaties that mimic ECOMOR's features to some extent. For example, the SL reinsurance covers all claims greater than a certain threshold.

In practice, in addition to these common types of reinsurance, some non-standard reinsurance forms such as the combination of proportional and non-proportional reinsurance are also very common.

1.4 Structure of the thesis

This chapter provides a review of the background for our research. We have introduced the development of risk management, the ruin problem in risk theory, as well as different types of reinsurance policies. The primary objective of our research is to assess the impact of reinsurance in risk management, particularly in minimizing the ruin probability of an insurance company, and to determine the optimal reinsurance strategies.

The thesis is structured as follows. In Chapter 2, we introduce the mathematical preliminaries. We briefly review the relevant issues in stochastic control theory and main methods of solving stochastic optimal control problems.

In Chapter 3, we consider the problem of minimizing ruin probability by reinsurance decisions for a discrete-time risk process with an unknown parameter. We formulate a Markov decision process and apply the method of dynamic programming to solve this optimal control problem.

Chapter 4 is devoted to study a continuous-time risk model which can be controlled by both reinsurance and investment. We use the Hamilton-Jacobi-Bellman (HJB) approach to investigate optimal reinsurance and investment strategies to minimize the ruin probability.

Finally, in Chapter 5, we conclude our research findings and discuss some future research problems.

Chapter 2

Mathematical preliminaries

One important objective in risk management is to implement strategies to control risks. Stochastic control is one of the most powerful techniques to deal with optimal risk control problems in the insurance and finance world. In the literature, stochastic control problems and methods are well established, see for example, Fleming and Rishel (1975), Fleming and Soner (1993), and Schmihli (2008).

Main problems considered in stochastic control include investigating the optimal value of the objective function, which is also called the value function, and finding an optimal control which attains the value function. A classical way to solve this kind of problems is to apply the dynamic programming principle (DPP). In continuous time stochastic control, the dynamic programming method leads to an associated differential equation for the value function, the so called Hamilton-Jacobi-Bellman (HJB) equation. To solve the stochastic control problem, we have to prove that a possible solution of this HJB equation is the value function. We also need to show the existence of optimal control strategies.

In this chapter, we review some important results and general approaches of stochastic control. These form the foundation of our research. Most of the results can also be found in Schmihli (2008). For stochastic control problems

in continuous time, we also show that the dynamic programming equation converges to a HJB equation. Although a general introduction can be found in the literature, we present a more detailed proof for the derivation of the HJB equation. This equation is the key in a latter chapter to solve stochastic control problems.

2.1 Definitions of terms

In order to introduce standard issues and results, we include some general definitions in this section.

Definition 2.1.1. ***(Sample space)***
The sample space Ω is a set of all possible outcomes $\omega \in \Omega$ of some random experiment or phenomenon.

Definition 2.1.2. ***(σ-algebra of sets and measurable space)***
Let Ω be a set and $\mathcal{F}$ a collection of its subsets. We say that $\mathcal{F}$ is a σ-algebra (or σ-field) if

(i) $\mathcal{F}$ contains the sample space: $\Omega \in \mathcal{F}$,

(ii) $\mathcal{F}$ is closed under complements: $A \in \mathcal{F} \implies A^c \in \mathcal{F}$,

(iii) $\mathcal{F}$ is closed under countable unions: $A_1, A_2, ... \in \mathcal{F} \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$,

where A^c is the complement of A in Ω and $\bigcup_{n=1}^{\infty} A_n$ is the countable union. In the case when $\mathcal{F}$ is a σ-field the couple $(\Omega, \mathcal{F})$ is called a measurable space, and elements of $\mathcal{F}$ are called events.

Definition 2.1.3. ***(Probability measure and probability space)***
A probability measure defined on $(\Omega, \mathcal{F})$ is a real valued set function $\mathbb{P} : \mathcal{F} \to [0, 1]$ satisfying

(i) $0 \leqslant \mathbb{P}(A) \leqslant 1$, *for all* $A \in \mathcal{F}$,

(ii) $\mathbb{P}(\Omega) = 1$,

(iii) for every sequence of pairwise disjoint $A_1, A_2, ..., \in \mathcal{F}$, *we have*

$$\mathbb{P}\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ *is called a probability space.*

Definition 2.1.4. ***(Filtration, right continuous filtration and filtered probability space)***

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. A filtration on* $(\Omega, \mathcal{F}, \mathbb{P})$ *is an increasing family* $\{\mathcal{F}_t\}_{t\geqslant 0}$ *of sub-*σ*-algebras of* $\mathcal{F}$. *In other words, for each* t, $\mathcal{F}_t$ *is a sub-*σ*-algebra of* $\mathcal{F}$ *and* $\mathcal{F}_s \subseteq \mathcal{F}_t$ *if* $s \leqslant t$. *The filtration* $\{\mathcal{F}_t\}_{t\geqslant 0}$ *is called right continuous filtration if* $\mathcal{F}_t = \mathcal{F}_{t^+}$, *where* $\mathcal{F}_{t^+} = \bigcap_{s>t} \mathcal{F}_s$. *A probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *endowed with a filtration* $\{\mathcal{F}_t\}_{t\geqslant 0}$ *is called a filtered probability space, denoted as* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$. *In discrete time, it is denoted as* $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}^+}, \mathbb{P})$.

Definition 2.1.5. ***(Random variable)***

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. A mapping* $X : \Omega \to \mathbb{R}$ *is called a (real-valued) random variable if for each* $B \in \mathcal{B}(\mathbb{R})$, *we have* $X^{-1}(B) \in \mathcal{F}$, *where* $\mathbb{R}$ *is the set of all real numbers and* $\mathcal{B}(\mathbb{R})$ *is the Borel* σ*-algebra on* $\mathbb{R}$.

Definition 2.1.6. ***(Stochastic process)***

A stochastic process is a parametrized collection of random variables X_t $(t \in \mathcal{T})$ *defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *assuming values in* $\mathbb{R}$. *In other words, it is a family of random variables* X_t *parameterized by time* $t \in \mathcal{T}$, *where* $\mathcal{T}$ *is the parameter set of the stochastic process.*

Definition 2.1.7. ***(Progressively measurable process)***
Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space. A stochastic process $\{X_t\}_{t\geqslant 0}$ is called progressively measurable if for every $t \geqslant 0$, the function $X(s,\omega) : [0,t]\times\Omega \to \mathbb{R}$ is measurable with respect to the σ-algebra $\mathcal{B}([0,t])\times\mathcal{F}_t$, where $\mathcal{B}([0,t])$ is the Borel σ-algebra on the closed interval $[0,t]$.

Definition 2.1.8. ***(Adapted process)***
A stochastic process $\{X_t\}_{t\geqslant 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ is adapted to the filtration $\mathcal{F}_t$ if, for each $t \geqslant 0$, X_t is $\mathcal{F}_t$-measurable.

Definition 2.1.9. ***(Discrete time predictable process)***
Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}^+}, \mathbb{P})$ be a filtered probability space, where $\mathbb{N}^+ = \{1, 2, \cdots\}$. A stochastic process $\{X_n\}_{n\in\mathbb{N}^+}$ is called predictable if X_n is measurable with respect to $\mathcal{F}_{n-1}$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Definition 2.1.10. ***(Continuous time predictable process)***
Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space. A stochastic process $\{X_t\}_{t\geqslant 0}$ is called predictable if X_t is measurable with respect to the predictable σ-algebra $\mathcal{P}$, where $\mathcal{P}$ is the smallest σ-algebra on $\mathbb{R}^+\times\Omega$ making all left-continuous and adapted processes measurable.

Definition 2.1.11. ***(Markov process)***
A stochastic process $\{X_t\}_{t\geqslant 0}$ with a state space $\mathbb{S}$ and a corresponding Borel σ-field $\mathcal{B}$ of subsets of $\mathbb{S}$ is a Markov process with respect to a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ if it is adapted to $\{\mathcal{F}_t\}_{t\geqslant 0}$ and for any $t, s \geqslant 0$ and $A \in \mathcal{B}$, we have

$$\mathbb{P}(X_{t+s} \in A|\mathcal{F}_t) = \mathbb{P}(X_{t+s} \in A|X_t).$$

Definition 2.1.12. ***(Stopping time)***
Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space. A random variable $\tau : \Omega \to [0,\infty]$ is a stopping time if $\{\tau \leqslant t\} \in \mathcal{F}_t$ for all t.

Definition 2.1.13. ***(Infinitesimal generator)***

The infinitesimal generator $\mathfrak{A}$ *of a Markov process* $\{X_t\}_{t\geqslant 0}$ *with* $X_0 = x$ *is an operator defined on the space of continuously differentiable functions is given by*

$$\mathfrak{A}f(x) = \lim_{t\to 0} \frac{\mathbb{E}\left[f(X_t)\right] - f(x)}{t}.$$

Definition 2.1.14. ***(Submartingale, Supermartingale and Martingale)***

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ *be a filtered probability space. A stochastic process* $\{X_t\}_{t\geqslant 0}$ *of random variables is said to be a submartingale with respect to a filtration* $\{\mathcal{F}_t\}_{t\geqslant 0}$ *if*

(i) $X_t \in \mathcal{F}_t$ *for every* $t \geqslant 0$,

(ii) $\mathbb{E}|X_t| < \infty$ *for every* $t \geqslant 0$,

(iii) $\mathbb{E}(X_t|\mathcal{F}_s) \geqslant X_s$ *for every* $0 \leqslant s < t$.

The process is called supermartingale if the inequality (iii) *is reserved. The process is called martingale if the inequality* (iii) *becomes equality.*

Definition 2.1.15. ***(Brownian motion process)***

A Brownian motion process is a stochastic process $\{B_t\}_{t\geqslant 0}$, *which satisfies*

(i) $B_0 = 0$,

(ii) The increments $B_t - B_s$ *are normally distributed with mean zero and variance* $|t - s|$.

(iii) For every pair of disjoint time intervals $[t_1, t_2]$ *and* $[t_3, t_4]$, *with* $0 \leqslant t_1 < t_2 \leqslant t_3 < t_4$, *the increments* $B_{t_4} - B_{t_3}$ *and* $B_{t_2} - B_{t_1}$ *are independent random variables.*

Definition 2.1.16. ***(Class C^k)***

A function is of class C^k if it is differentiable k times and the k^{th} derivative is continuous.

Definition 2.1.17. ***(Variation and quadratic variation)***

Let $f : [a, b] \to \mathbb{R}$ be a real-valued function defined on interval $[a, b]$ and $\Pi_n = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$. Define the mesh of the partition of Π_n by

$$\Pi_n^* = \max_{1 \leqslant i \leqslant n} (t_i - t_{i-1}).$$

The variation of the function f on interval $[a, b]$ is defined as

$$V_f([a, b]) = \lim_{\Pi_n^* \to 0} \sum_{i=1}^{n} \big| f(t_i) - f(t_{i-1}) \big|.$$

The quadratic variation of the function f on interval $[a, b]$ is defined as

$$Q_f([a, b]) = \lim_{\Pi_n^* \to 0} \sum_{i=1}^{n} \big[f(t_i) - f(t_{i-1}) \big]^2.$$

Definition 2.1.18. ***(Topology and topological space)***

Let Ω be a be a non-empty set and 2^Ω is the set of all possible subsets of Ω. A family of subsets $\mathcal{T} \subseteq 2^\Omega$ is a topology on Ω if the following properties hold:

(i) $\emptyset, \Omega \in \mathcal{T}$.

(ii) $\mathcal{T}$ *is closed under finite intersections:* $A \in \mathcal{T}, B \in \mathcal{T} \implies A \cap B \in \mathcal{T}$.

(iii) $\mathcal{T}$ *is closed under arbitrary unions:* $\mathcal{F} \subseteq \mathcal{T} \implies \bigcup_{A \in \mathcal{F}} A \in \mathcal{T}$.

The space $(\Omega, \mathcal{T})$ is called a topological space.

Definition 2.1.19. ***(Polish space)***

A Polish space is a separable completely metrizable topological space. That is, a space homeomorphic to a complete metric space that has a countable dense subset. One example of the Polish space is the real line.

2.2 Itô integral

Let $\mathbf{L}^2$ denote the space of stochastic processes $\{g_t\}_{t\geqslant 0}$ such that

$$\mathbb{E}\Big(\int_0^T g_t^2 dt\Big) < \infty, \quad \text{for every} \quad T > 0. \tag{2.2.1}$$

The Itô integral is denoted as

$$I_T(g) = \int_0^T g_t dB_t, \tag{2.2.2}$$

where $\{B_t\}_{t\geqslant 0}$ is a Brownian motion process and the stochastic process $\{g_t\}_{t\geqslant 0} \in \mathbf{L}^2$ is adapted to the same filtration as that for $\{B_t\}_{t\geqslant 0}$.

Since $I_T(g)$ is obtained by integrating with respect to a Brownian motion, $I_T(g)$ is assumed to hold the properties of Brownian motion including the martingale property and the Markov property. On the other hand, the paths of a Brownian motion are continuous but no-where differentiable. Moreover, the paths of Brownian motion do not have bounded variation which is the sufficient condition for Riemann–Stieltjes integral. Therefore, we have to construct $I_T(g)$ in another way.

We start with $\{g_t\}_{0\leqslant t\leqslant T} \in \mathbf{L}^2$ which is a simple adapted process. Then there exist times $0 = t_0 < t_1 < ... < t_n = T$ and random variables $X_0, X_1, ..., X_{n-1}$, where X_i is $\mathcal{F}_{t_i}$ measurable and $\mathbb{E}(X_i) < \infty$, such that

$$g_t = \sum_{i=0}^{n-2} X_i \mathbb{1}_{[t_i, t_{i+1})}(t) + X_{n-1}\mathbb{1}_{[t_{n-1}, t_n]}(t), \quad \text{for} \quad i = 0, 1, ..., n-1.$$

The Itô integral of the simple adapted process $\{g_t\}_{0\leqslant t\leqslant T}$ is then defined as

$$I_T(g) = \int_0^T g_t dB_t = \sum_{i=0}^{n-1} X_i\big(B_{t_{i+1}} - B_{t_i}\big).$$

Next, we extend the Itô integral of a simple adapted process to the Itô integral of progressively measurable processes. Suppose $\{g_t\}_{0\leqslant t\leqslant T} \in \mathbf{L}^2$ is adapted to the same filtration as that for $\{B_t\}_{t\geqslant 0}$. It can be shown that there exists a sequence of simple adapted processes $\{\{g_t^n\}_{t\geqslant 0}\}_{n\in\mathbb{N}^+}$ for every $t \in [0, T]$, such that

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_0^T \big(g_t^n - g_t\big)^2 dt\Big] = 0.$$

Thus, the Itô integral (2.2.2) is defined as

$$I_T(g) = \int_0^T g_t dB_t = \lim_{n\to\infty} \int_0^T g_t^n dB_t.$$

Theorem 2.2.1. ***(Properties of the Itô Integral)***

(i) ***Linearity:*** *Suppose $\{f_t\}_{0\leqslant t\leqslant T}$ and $\{g_t\}_{0\leqslant t\leqslant T}$ are progressively measurable processes which satisfying equation (2.2.1) and α and β are some constants, then*

$$\int_0^T \big(\alpha f_t + \beta g_t\big) dB_t = \alpha \int_0^T f_t dB_t + \beta \int_0^T g_t dB_t.$$

(ii) ***Martingale property:*** *$\{I_t(g)\}_{t\geqslant 0}$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, i.e.,*

$$\mathbb{E}\big(I_t(g)|\mathcal{F}_s\big) = I_s(g), \quad \textit{for all} \quad 0 \leqslant s \leqslant t.$$

(iii) ***Continuity:*** *With probability one, the trajectory $T \to \int_0^T g_t dB_t$ is a continuous function of T.*

(iv) ***Itô isometry:*** $\mathbb{E}\big(\int_0^T g_t dB_t\big)^2 = \int_0^T \mathbb{E}\big(g_t^2\big) dt$.

2.3 Stochastic differential equation (SDE)

A differential equation is called a stochastic differential equation (SDE) if at least one of the terms is a stochastic process. Let $\{X_t\}_{t\geqslant 0}$ be a stochastic process. The general form of the stochastic differential equation is

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \tag{2.3.1}$$

with $X_0 = x$, where $x \in \mathbb{R}$, $b : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0, \infty) \times \mathbb{R} \to \mathbb{R}$. Both functions b and σ are deterministic functions and B_t is a standard Brownian motion. We say that the stochastic process $\{X_t\}_{t\geqslant 0}$ is the solution of above equation if it satisfies

$$X_t - X_0 = \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s. \tag{2.3.2}$$

In general, the solution of the stochastic differential equation (2.3.1) may not be unique or may not even exist. To obtain a unique solution, we need Lipschtiz and linear conditions for the functions b and σ.

Theorem 2.3.1. *For all $t \in [0, \infty)$ and $x, y \in \mathbb{R}$, if there exist constants C and D such that*

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leqslant C|x - y|$$

and

$$|b(t, x)| + |\sigma(t, x)| \leqslant D(1 + |x|),$$

then the stochastic differential equation (2.3.1) has a unique solution $\{X_t\}_{t\geqslant 0}$, which is a stochastic process. Furthermore, $\{X_t\}_{t\geqslant 0}$ is called the Itô process.

Suppose $\{X_t\}_{t\geqslant 0}$ is the unique solution of a stochastic differential equation (2.3.1). One of the most important properties of $\{X_t\}_{t\geqslant 0}$ is the Markov property. That is, given the past and present states, the future state of the process depends on the present state only.

If the functions b and σ in the stochastic differential equation (2.3.1) are independent of t and satisfy Lipschtiz and linear conditions, then X_t is called the Itô diffusion. It can be shown that the Itô diffusion is a homogeneous Markov process.

Recall that the stochastic process involves stochastic term B_t which is continuous everywhere but nowhere differentiable. To derive rules of differentiation and integration of a stochastic process, the fundamental theorem of calculus does not work, so we use the fundamental theorem of stochastic integration, which is called the Itô Formula.

Lemma 2.3.2. ***(Itô Formula)*** *Consider the stochastic process* $\{X_t\}_{t\geqslant 0}$ *with the stochastic differential equation*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t.$$

Suppose that $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ *is continuous, and* $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ *and* $\frac{\partial^2 f}{\partial x^2}$ *exist and are continuous. Then*

$$df(t, X_t) = \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2 + \frac{\partial f}{\partial t}(t, X_t)dt,$$

where $(dX_t)^2 = (dX_t)(dX_t)$ *is obtained by the rules* $(dB_t)^2 = dt$, $(dt)^2 = 0$ *and* $(dB_t)(dt) = (dt)(dB_t) = 0$.

The Itô formula is considered as the stochastic version of the chain rule from deterministic calculus. It is a fundamental tool to solve stochastic differential and integration problems. Given a stochastic process which solves a certain stochastic differential equation and objective function, the Itô formula gives us the differentiation of the objective function on that process.

2.4 Stochastic control

In mathematics, control theory describes how to deal with the behaviour of dynamical systems in some optimal manner. One subfield of control theory

is called stochastic control or stochastic optimal control. It aims to design the time path of the controlled variables in order to maximize some expected reward (or minimize cost) over some future time period. The term of "stochastic" is used to describe the stochastic uncertainty of the states of the dynamical system. Suppose the uncertainties can be described by probabilistic terms and the objective functions expressed in terms of expected values are well defined, then the related problems are called stochastic control problems.

Stochastic control in discrete time and stochastic control in continuous time are two categories of stochastic control, which are distinguished by the expression of time.

2.4.1 Stochastic control in discrete time

We start with considering stochastic control in discrete time $t = 0, 1, 2, ..., T$. Let E denote the state space with a σ-algebra $\mathcal{E}$ and elements (or states) are denoted by $x \in E$. We make a decision at each time point $n \in \mathbb{N}$. The decision is modelled as a control variable U_n from a space $\mathcal{U}$. The control variable U is called an admissible control if $U(X) \in \mathcal{U}(E)$, for all $X \in E$. An admissible strategy $\pi = \{U_n : n \in \mathbb{N}\}$ is a sequence of admissible controls made at each time point. Let Π denote the set of all admissible strategies. Suppose $Y_1, Y_2, \cdots$ are independent and identically distributed random variables on the Polish space $(E_Y, \mathcal{E}_\mathcal{Y})$. Let $X_0 = x$ be the initial state of the process. The process at time $n + 1$ is

$$X_{n+1} = f(X_n, U_n, Y_{n+1}),$$

where $f : E \times \mathcal{U} \times E_Y \to E$ is a measurable function, which is called state transition equation.

At time point n, there is an instantaneous cost (or reward) $c(X_n, U_n)$ if the current state is X_n and the decision U_n is taken. Given a particular admissible

strategy π, the expected value of the total discounted cost is

$$J_T(x,\pi) = \mathbb{E}\Big[\sum_{n=0}^{T} c(X_n^{\pi}, U_n)e^{-\delta n}\Big],$$

where $\delta \geqslant 0$ is a discounting parameter. An admissible strategy π is composed of admissible controls U_n, $n = 0, 1, ..., T$. In general, we only consider controls which satisfy certain conditions such as requirements of integrability or smoothness, constraints on the state of control and so on. Different problems have different definitions of admissibility.

Our objective is to find the value function $V_T(x) = \inf_{\pi\in\Pi} J_T(x,\pi)$. For $T = \infty$, we write $V(x)$ and $J(x,\pi)$ instead. We assume that $\delta > 0$ so that $V(x,\pi)$ is finite for all $\pi \in \Pi$.

It is not feasible to calculate $J_T(x,\pi)$ for every strategy π, particularly when E and T are infinite. To find $V_T(x)$ or $V(x)$, we use the dynamic programming principle to derive the so called Bellman's equation.

Lemma 2.4.1. *Suppose that $V_t(x)$ is finite, then it satisfies the dynamic programming principle*

$$V_t(x) = \inf_{u\in\mathcal{U}} \big\{c(x,u) + e^{-\delta}\mathbb{E}\big[V_{t-1}\big(f(x,u,Y)\big)\big]\big\}, \quad \textit{for all} \quad t = 0, 1, \cdots, T,$$

where $V_{-1}(x) = 0$ and Y is a generic random variable with the same distribution as Y_n. If $T = \infty$, the dynamic programming principle becomes

$$V(x) = \inf_{u\in\mathcal{U}} \big\{c(x,u) + e^{-\delta}\mathbb{E}\big[V(f(x,u,Y))\big]\big\}.$$

According to the Bellman's equation, we have to minimize the current cost plus the discounted, expected future cost in order to find the optimal value function $V_T(x)$ or $V(x)$. We make optimal decisions based on the Bellman's equation at each time point, then we can achieve the optimal value overall. In practice, we obtain the optimal strategy π by backward induction.

2.4.2 Stochastic control in continuous time

We define all processes on a filtered probability space$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$. Suppose $\{X_t\}_{t\geqslant 0}$ is a homogeneous Markov process which solves the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

with $X_0 = x$ and the state $X \in \mathbb{R}$. Let U_t from the space $\mathcal{U} = \mathbb{R}$ be the control at time $t \in [0, \infty)$. The control can influence the next state X by modifying the functions b and σ such that

$$dX_t = b(X_t, U_t)dt + \sigma(X_t, U_t)dB_t \tag{2.4.1}$$

with $X_0 = x$. Because the control process U is non-anticipative with respect to $\{B_t\}_{t\geqslant 0}$, we restrict U to be $\mathcal{F}_t$-adapted. We also restrict U to be a stationary Markov control which means $U_t = u(X_t)$ for all $t \geqslant 0$, with a measurable function $u : \mathbb{R} \to \mathbb{R}$. That is, U_t depends on the current state only. Due to the control variable U_t, we need to modify Lipschtiz condition and linear condition to make sure that a unique solution of the stochastic differential equation (2.4.1) exits.

Theorem 2.4.2. *For all $\alpha \in \mathcal{U}$ and $x, y \in \mathbb{R}$, if there exist constants C and D such that*

$$|b(x, \alpha) - b(y, \alpha| + |\sigma(x, \alpha) - \sigma(y, \alpha)| \leqslant C|x - y|$$

and

$$|b(x, \alpha)| + |\sigma(x, \alpha)| \leqslant D(1 + |x|),$$

then the stochastic differential equation (2.4.1) has a unique solution $\{X_t\}_{t\geqslant 0}$, which is a stochastic process. Furthermore, the process $\{X_t\}_{t\geqslant 0}$ is still a homogeneous Markov process.

At time t, the instantaneous cost under control u with state X is denoted as $c\big(X_t, u(X_t)\big)$. We may be interested in minimizing the total cost

$$\int_0^\infty e^{-\delta s} c\big(X_s, u(X_s)\big) ds,$$

where $\delta > 0$ is a discounting parameter. However, X_t is a random variable for every fixed time t. Instead, our realistic objective is to minimize the expected total cost

$$J(x, \pi) = \mathbb{E}\Big[\int_0^\infty e^{-\delta s} c\big(X_s^\pi, u(X_s)\big) ds\Big].$$

The above quantity represents the expected total cost with initial state $X_0 = x$ under an admissible strategy $\pi = \{u(X_t)\}_{t \geqslant 0}$, where $\pi = \{u(X_t)\}_{t \geqslant 0}$ is composed of admissible controls $u(X_t)$. Let Π denote the set of all admissible strategies. Our objective is to find the value function

$$V(x) = \inf_{\pi \in \Pi} J(x, \pi). \tag{2.4.2}$$

Lemma 2.4.3. *The value function $V(x)$ satisfies the dynamic programming principle*

$$V(x) = \inf_{\pi \in \Pi} \mathbb{E}\Big[\int_0^t e^{-\delta s} c(X_s^\pi, U_s) ds + e^{-\delta t} V(X_t^\pi)\Big], \quad \textit{for all} \quad t \geqslant 0. \tag{2.4.3}$$

The intuition of the dynamic programming principle is as follows. Instead of considering the time interval $[0, \infty)$ as a whole, we consider the time interval $[0, t)$ and $[t, \infty)$ separately. Suppose we implement an arbitrary admissible strategy π in the time interval $[0, t)$, then we obtain the expected cost value $\mathbb{E}\big[\int_0^t e^{-\delta s} c(X_s^\pi, U_s^\pi) ds\big]$ over $[0, t)$ and the state becomes X_t^π at time t. Then we treat X_t^π as our initial state over $[t, \infty)$ to obtain the value function $V(X_t^\pi)$. Since the initial state $V(X_t^\pi)$ is a random variable, we need to take

its expected value. Finally, we search for a strategy π such that the total of $\mathbb{E}\big[\int_0^t e^{-\delta s}c(X_s^\pi, U_s^\pi)ds\big]$ and $\mathbb{E}\big[V(X_t^\pi)\big]$ is minimized.

The dynamic programming principle provides a method to obtain the optimal strategy. However, it is not realistic to implement. Suppose $\pi = \{u(X_s)\}_{0\leqslant s<t}$ is an arbitrary admissible strategy in the time interval $[0,t)$. Since the time is continuous and $u(X_s)$ are admissible controls for $0 \leqslant s < t$, it can be shown that the admissible strategy $\pi = \{u(X_s)\}_{0\leqslant s<t}$ is continuous in the time interval $[0,t)$. Therefore, it is not feasible to obtain $u(X_s)$ for all $0 \leqslant s < t$ pointwisely.

In order to obtain the optimal strategy, we shall study the most important equation in continuous time stochastic control which is called Hamilton-Jacobi-Bellman (HJB) equation. The name Bellman comes from the equation (2.4.3) and Hamilton-Jacobi comes from physics. The idea behind the HJB approach is letting $t \to 0$, the computation of the optimal control then becomes a pointwise minimization. Therefore, the HJB equation is the infinitesimal version of the dynamic programming principle, which is also called dynamic programming equation.

Lemma 2.4.4. *Suppose the stochastic process $\{X_t\}_{t\geqslant 0}$ is the unique continuous solution of*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

with $X_0 = x$ and the value function $V(x) \in C^2(\mathbb{R})$, where $V(x)$ is defined as equation (2.4.2). Then the value function $V(x)$ fulfils the Hamilton-Jacobi-Bellman equation

$$\inf_{u\in\mathcal{U}}\big\{\mathfrak{A}_u V(x) - \delta V(x) + c\big(x, u(x)\big)\big\} = 0,$$

where $\mathfrak{A}_u$ denotes the infinitesimal generator of process $\{X_t\}_{t\geqslant 0}$ controlled by decision u.

Proof. Suppose $dX_t = b\big(X_t, u(X_t)\big)dt + \sigma\big(X_t, u(X_t)\big)dB_t$, where u is an arbitrary Markov control. Let $f(t,x) = e^{-\delta t}V(x)$. Since $V(x) \in C^2(\mathbb{R})$, we apply the Itô formula and obtain

$$\begin{aligned}
&d\big(e^{-\delta t}V(X_t)\big)\\
=&e^{-\delta t}V^{'}(X_t)dX_t + \frac{1}{2}e^{-\delta t}V^{''}(X_t)(dX_t)^2 - \delta e^{-\delta t}V(X_t)dt\\
=&e^{-\delta t}\big[V^{'}(X_t)b\big(X_t, u(X_t)\big) + \frac{1}{2}V^{''}(X_t)\sigma^2\big(X_t, u(X_t)\big) - \delta V(X_t)\big]dt\\
&+ e^{-\delta t}V^{'}(X_t)\sigma\big(X_t, u(X_t)\big)dB_t,
\end{aligned}$$

where $(dB_t)^2 = dt$, $(dt)^2 = 0$ and $(dB_t)(dt) = (dt)(dB_t) = 0$. It follows that

$$\begin{aligned}
&e^{-\delta t}V(X_t) - V(x)\\
=&\int_0^t e^{-\delta s}\Big(V^{'}(X_s)b\big(X_s, u(X_s)\big) + \frac{1}{2}V^{''}(X_s)\sigma^2\big(X_s, u(X_s)\big) - \delta V(X_s)\Big)ds\\
+&\int_0^t e^{-\delta s}V^{'}(X_s)\sigma\big(X_s, u(X_s)\big)dB_s. \qquad (2.4.4)
\end{aligned}$$

We add $\int_0^t e^{-\delta s}c(X_s, u(X_s))ds$ on both sides of equation (2.4.4) and then take expectation. We find that the last term vanishes because it is an Itô integral which is a martingale. It follows that

$$\begin{aligned}
0 \leqslant \mathbb{E}\Big[\int_0^t e^{-\delta s}\Big(&V^{'}(X_s)b\big(X_s, u(X_s)\big) + \frac{1}{2}V^{''}(X_s)\sigma^2\big(X_s, u(X_s)\big)\\
&-\delta V(X_s) + c(X_s, u(X_s))\Big)ds\Big].
\end{aligned}$$

The inequality is true for any Markov control u and the equality holds if u is

the optimal control. Divided by t on both sides and let $t \to 0$, we have

$$\begin{aligned} 0 = \inf_{u\in\mathcal{U}} \lim_{t\to 0} \mathbb{E}\Big[\frac{1}{t}\int_0^t e^{-\delta s}\Big(V^{'}(X_s)b\big(X_s,u(X_s)\big) + \frac{1}{2}V^{''}(X_s)\sigma^2\big(X_s,u(X_s)\big) \\ - \delta V(X_s) + c(X_s,u(X_s))\Big)ds\Big] \\ = \inf_{u\in\mathcal{U}} \lim_{t\to 0} \mathbb{E}\Big(\frac{1}{t}\int_0^t W_s ds\Big), \end{aligned}$$

where

$$\begin{aligned} W_s =& e^{-\delta s}\big[V^{'}(X_s)b\big(X_s,u(X_s)\big) + \frac{1}{2}V^{''}(X_s)\sigma^2\big(X_s,u(X_s)\big) \\ & - \delta V(X_s) + c(X_s,u(X_s))\big]. \end{aligned}$$

Obviously, the random variables $\frac{1}{t}\int_0^t W_s ds, t > 0$, are uniformly bounded. Therefore, we can interchange the limit and the expectation operator according to the Dominated Convergence Theorem. Then we have

$$\begin{aligned} 0 = \inf_{u\in\mathcal{U}} \mathbb{E}\Big[\lim_{t\to 0}\frac{1}{t}\int_0^t e^{-\delta s}\Big(V^{'}(X_s)b\big(X_s,u(X_s)\big) + \frac{1}{2}V^{''}(X_s)\sigma^2\big(X_s,u(X_s)\big) \\ - \delta V(X_s) + c(X_s,u(X_s))\Big)ds\Big] \\ = \inf_{u\in\mathcal{U}} \big[V^{'}(x)b\big(x,u(x)\big) + \frac{1}{2}V^{''}(x)\sigma^2\big(x,u(x)\big) - \delta V(x) + c\big(x,u(x)\big)\big]. \end{aligned}$$

On the other hand, we apply the Itô's Formula to the function $V(x)$ and take expectation. This implies that

$$\mathbb{E}\big(V(X_t)\big) - V(x) = \mathbb{E}\int_0^t \big[V^{'}(X_s)b\big(X_s,u(X_s)\big) + \frac{1}{2}V^{''}(X_s)\sigma^2\big(X_s,u(X_s)\big)\big]ds.$$

Divided by t on both sides and let $t \to 0$, we have

$$\begin{aligned} \mathfrak{A}_u V(x) &= \lim_{t\to 0}\frac{\mathbb{E}\big(V(X_t)\big) - V(x)}{t} \\ &= V^{'}(x)b\big(x,u(x)\big) + \frac{1}{2}V^{''}(x)\sigma^2\big(x,u(x)\big). \end{aligned}$$

Thus, we obtain the desired equation

$$\inf_{u\in\mathcal{U}}\left\{\mathfrak{A}_u V(x) - \delta V(x) + c\big(x, u(x)\big)\right\} = 0.$$

□

For continuous time stochastic control problem, the HJB approach provides a classical way to investigate the value function and the associated optimal control. However, the solution to the HJB equation may not be the desired value function. Thus, we have to show that the solution found from the corresponding HJB equation is the desired value function, which can be dealt with by proving a so-called verification theorem.

We will discuss the HJB approach and the verification arguments in details for our research problem introduced in Chapter 4. The presented methods are developed for diffusion processes. We note that for problems containing a diffusion part, the value function must be continuous and twice differentiable so that the Itô formula can be applied.

In many cases, we consider the diffusion approximations for classical risk models due to mathematical convenience. We now provide the general approach of the approximations for future introduction.

2.5 Diffusion approximations

Consider the classical Cramér-Lundberg risk model in which the insurer's reserve $\hat{R}_t$ at time t is given by

$$\hat{R}_t = x + pt - \sum_{i=1}^{N_t} Y_i, \quad t \geqslant 0, \tag{2.5.1}$$

where x is an initial reserve, $\{N_t\}_{t\geqslant 0}$ is a Poisson process with an intensity rate $\lambda > 0$, claim sizes $Y_1, Y_2, \cdots$, are independent and identically distributed,

positive random variables with finite first moment μ and finite variance σ^2, $c = (1+\eta)\mathbb{E}(Y)\lambda$ is the constant gross premium rate calculated based on the expected value principle, and $\eta > 0$ is the relative safety loading of the insurer.

In the classical risk model, it is often difficult to calculate the exactly quantity of interest, such as the ruin probability. One therefore looks for approximations. A simple idea, quite successful in queueing theory, is diffusion approximation. The idea is to consider a sequence of classical risk models $\{\hat{R}_t^{(n)}\}_{n\in\mathbb{N}^+}$ with the same first and second moments as $\hat{R}_t$ defined in (2.5.1). Furthermore, $\{\hat{R}_t^{(n)}\}$ converges weakly to a diffusion process as $n \to \infty$.

Let $\eta^{(n)} = \frac{\eta}{\sqrt{n}}$, $Y^{(n)} = \frac{Y}{\sqrt{n}}$ and $\{N_t^{(n)}\}_{t\geqslant 0}$ be a Poisson process with intensity $n\lambda > 0$. The sequence of rescaled reserve process:

$$\begin{aligned}\hat{R}_t^{(n)} &= x + (1+\eta^{(n)})E(Y^{(n)})n\lambda t - \sum_{i=1}^{N_t^{(n)}} Y_i^{(n)} \\ &= x + \eta\lambda\mu t + \frac{n\lambda\mu t - \sum_{i=1}^{N_t^{(n)}} Y_i}{\sqrt{n}}, \quad t \geqslant 0.\end{aligned}$$

It can be show that

$$\left\{\frac{n\lambda\mu t - \sum_{i=1}^{N_t^{(n)}} Y_i}{\sqrt{n}}\right\}_{t\geqslant 0} \xrightarrow{D} \left\{\sqrt{\lambda(\mu^2+\sigma^2)}B_t\right\}_{t\geqslant 0}, \quad \text{as} \quad n \to \infty,$$

where $\{B_t\}_{t\geqslant 0}$ is a standard Brownian motions process (see Iglehart (1969)). Therefore,

$$\left\{\hat{R}_t^{(n)}\right\}_{t\geqslant 0} \xrightarrow{D} \left\{x + \eta\lambda\mu t + \sqrt{\lambda(\mu^2+\sigma^2)}B_t\right\}_{t\geqslant 0}, \quad \text{as} \quad n \to \infty.$$

Let $\{R_t^{(DP)}\}_{t\geqslant 0}$ denote the limit process of the above approximation, then we have

$$\{R_t^{(DP)}\}_{t\geqslant 0} = \{x + \eta\lambda\mu t + \sqrt{\lambda(\mu^2+\sigma^2)}B_t\}_{t\geqslant 0}.$$

The ruin time of $R_t^{(DP)}$ is its first passage time of reaching zero.

On the other hand,

$$\sqrt{n}\hat{R}_t^{(n)} = \sqrt{n}x + (1+\eta^{(n)})\mathbb{E}(Y)n\lambda t - \sum_{i=1}^{N_t^{(n)}} Y_i.$$

It is easy to see

$$\mathbb{P}\Big(\inf_{t>0} \hat{R}_t^{(n)} < 0 | \hat{R}_0^{(n)} = x\Big) = \mathbb{P}\Big(\inf_{t>0} \hat{R}_t' < 0 | \hat{R}_0 = \sqrt{n}x\Big),$$

where

$$\hat{R}_t' = \sqrt{n}x + (1+\frac{\eta}{\sqrt{n}})\mu n\lambda t - \sum_{i=1}^{N_t^{(n)}} Y_i, \quad t \geqslant 0.$$

Therefore, diffusion approximations work well when the safety loading is small, initial reserve is large and the intensity rate is high.

Chapter 3

Control of ruin probabilities by reinsurance in a risk model with unknown parameters

3.1 Introduction

Reinsurance is an important technique for insurance companies to control risk by limiting its consequences. With reinsurance for insurance companies, the insurer (the cedent) transfers a part of the risk to a reinsurer and pays premium to the reinsurer for this coverage. Reinsurance can reduce the probability and severity of suffering significant losses, diminish the impact of risk and hence is referred as "insurance for insurers". Proportional reinsurance is one of the most important reinsurance arrangements, with which the insurer covers a certain proportion when the claim occurs and the remaining proportion is covered by the reinsurer. To find the optimal proportional reinsurance strategy, the proportion is chosen based on a certain optimization criterion such as minimizing the ruin probability of the insurance company.

Researches in determining the optimal reinsurance strategy for an insurance company have been done by many authors. Schmidli (2001) considers dynamic optimal proportional reinsurance policies for a diffusion risk model

and a compound Poisson risk model. Taksar and Markussen (2003) determine the optimal reinsurance policy which minimizes the ruin probability for an insurance company whose reserve is relatively large compared to the size of individual claim. Works combining proportional and other types of reinsurance strategy for the diffusion model are presented in Zhang, Zhou and Guo (2007). Bi and Zhang (2015) study the optimization problem with both investment and proportional reinsurance control under the assumption that the reserve process is represented by a pure diffusion process. Li, Zhou and Yin (2015) investigate a dynamic optimal reinsurance problem with both proportional and fixed transaction costs for an insurance company whose reserve process is modelled by a Brownian motion with positive drift. In addition, Schäl (2003, 2004, 2005) applies results from discrete-time dynamic programming for minimizing costs by formulating the ruin probability as some total cost. Under this approach, the problem of minimizing ruin probability is solved by means of Markov decision processes.

Although the problem of finding an optimal reinsurance strategy to minimize the ruin probability has been discussed a lot in the literature, optimal control problems based on the model with unknown parameters are usually not considered. In practice, the distribution and parameters of claim sizes are not always known, thus the analysis of ruin problems under uncertainty is of great importance.

In this chapter, we investigate the ruin probability of a discrete-time risk model with unknown parameters. To reduce the risk, the insurance company may purchase a proportional reinsurance. As in the classical risk model, the claim process is described by a compound process, however, the distribution of claim sizes involves unknown parameters. Our ultimate objective is to find the proportional reinsurance strategy which minimizes the ruin probability. We apply the Bayesian approach to address the issue of the parameter uncertainty. The problem of controlling ruin probabilities is formulated as a Markov

decision process. By means of dynamic programming, explicit expressions of minimum ruin probabilities are obtained for the first two time periods. The corresponding optimal reinsurance strategies are also provided. Within one time period, we investigate structural properties of the minimum ruin probability under certain conditions. Our results are illustrated numerically.

3.2 Model formulation

Suppose that all random processes in this chapter are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{N}^+}, \mathbb{P})$.

Consider the following discrete-time risk process

$$U_t = U_{t-1} + [P_{t-1} - (1-b_t)R_{t-1}] - b_t S_t, \qquad t = 1, 2, \cdots,$$
$$U_0 = x,$$

where U_t stands for the insurer's reserve at the end of period t, P_{t-1} is the premium for the insurer and R_{t-1} is the premium for the reinsurer in time $(t-1, t]$. We assume that $R_{t-1} > P_{t-1}$. Otherwise, the insurer could reinsure the whole portfolio and make a risk-free profit. The claim process is described by a compound process $S_t = \sum_{i=1}^{\hat{N}_t} X_i$ where $\hat{N}_t$ is the number of claims occurring during the time interval $(t-1, t]$, and X_i represents the amount of the i^{th} claim, independent of $\hat{N}_t$. It is assumed that $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables from a certain distribution with an unknown parameter. However, we follow the Bayesian approach and assume a prior distribution of the unknown parameter, which provides our initial information about the parameter.

We assume that the insurer uses the expected value principle for the premium calculation, then we have

$$P_{t-1} = (1+\eta)\mathbb{E}(S_t),$$

where $\eta > 0$ is the relative safety loading of the insurer. Furthermore, we assume that the reinsurer also uses the expected value principle with a positive constant safety loading θ. Then we have

$$R_{t-1} = (1+\theta)\mathbb{E}(S_t).$$

To be realistic, we assume that $0 < \eta < \theta < 1$. The net premium for the insurer in time $(t-1, t]$ with reinsurance retention level b_t becomes

$$\begin{aligned} & P_{t-1} - (1-b_t)R_{t-1} \\ =&(1+\eta)\mathbb{E}(S_t) - (1-b_t)(1+\theta)\mathbb{E}(S_t) \\ =&\big[b_t(1+\theta) - (\theta-\eta)\big]\mathbb{E}(S_t). \end{aligned}$$

The reserve process becomes

$$\begin{aligned} & U_t = U_{t-1} + [b_t(1+\theta) - (\theta-\eta)]\mathbb{E}(S_t) - b_t S_t, \qquad t = 1, 2, \cdots, \\ & U_0 = x. \end{aligned}$$

In order to fulfill the net profit condition, we need to assume that

$$[b_t(1+\theta) - (\theta-\eta)]\mathbb{E}(S_t) \geqslant b_t\mathbb{E}(S_t).$$

It follows that

$$b_t \geqslant 1 - \frac{\eta}{\theta}, \qquad t = 1, 2, \cdots.$$

Ruin occurs when the insurance company's reserve becomes negative for the first time. Once in state of ruin, the system moves to the absorbing state, denoted as $-\infty$. Suppose that $U_0 = x \geqslant 0$, we have the following law of motion for the reserve:

$$\begin{aligned} & U_t = U_{t-1} + [P_{t-1} - (1-b_t)R_{t-1}] - b_t S_t, \text{ for } U_{t-1} \geqslant 0, \\ & U_t = -\infty, \text{ for } U_{t-1} < 0. \end{aligned}$$

Let b_t be the reinsurance decision at time t, where b_t is the retention level in the proportional reinsurance. A strategy π is a sequence of reinsurance decisions $\{b_t\}_{t\in\mathbb{N}^+}$. The strategy π is called admissible if it is predictable with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{N}^+}$ and satisfies $1-\frac{\eta}{\theta}\leqslant b_t\leqslant 1$ for all $t\in\mathbb{N}^+$. The strategy is stationary if $b_t=b$ for all $t\in\mathbb{N}^+$. Let Π denote the set of all admissible strategies.

Suppose that the insurance company has a nonnegative initial reserve x and a strategy π, then the risk process can be written as $\{U_t^{x,\pi}, t=1,2,\cdots\}$. Now we define the ruin probability within n periods as

$$\Psi_n^{\pi}(x)=\mathbb{P}(U_t^{x,\pi}<0 \text{ for some } t\leqslant n),$$

and the ultimate ruin probability as

$$\Psi^{\pi}(x)=\mathbb{P}(U_t^{x,\pi}<0 \text{ for some } t\geqslant 1).$$

A strategy π^* is called optimal if

$$\Psi_n^{\pi^*}(x)=\inf_{\pi\in\Pi}\Psi_n^{\pi}(x),$$

or

$$\Psi^{\pi^*}(x)=\inf_{\pi\in\Pi}\Psi^{\pi}(x).$$

Furthermore, we define the optimal equation as

$$\Psi_n(x)=\inf_{\pi\in\Pi}\Psi_n^{\pi}(x).$$

Suppose that the optimal strategy π^* exists, we have

$$\Psi_n(x)=\Psi^{\pi^*}(x)=\inf_{\pi\in\Pi}\Psi_n^{\pi}(x).$$

We assume that one unit of cost has to be paid when entering the ruin state, and define the one-period cost function $g(u)=I_{(-\infty,0)}(u)$. Since a ruin

state can be visited at most once, the cost of one unit is paid at most once. Then the ruin probability within n periods can be written as

$$\Psi_n^{\pi}(x) = \mathbb{E}\Big[\sum_{m=1}^{n} g(U_m)\Big], \qquad n \leqslant \infty. \tag{3.2.1}$$

Define $\mathcal{B} = [1 - \frac{\eta}{\theta}, 1]$, by the equation (3.2.1) and dynamic programming principle lemma 2.4.1,

$$\begin{aligned}\Psi_n(x) &= \inf_{\pi\in\Pi} \mathbb{E}\Big[\sum_{m=1}^{n} g(U_m)\Big] \\ &= \inf_{b_1\in\mathcal{B}} \mathbb{E}\Big[g(U_1) + \Psi_{n-1}(U_1)\Big],\end{aligned}$$

where $U_1 = x + [P_0 - (1 - b_1)R_0] - b_1 S_1$.

3.3 Exponential claims with unknown parameters

To investigate the ruin probability, we first consider the distribution of the total amount of claims. Suppose that for the aggregate claim $S_1 = \sum_{i=1}^{\hat{N}_1} X_i$, $\{\hat{N}_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables from the first success distribution $Fs(\delta)$ with the probability mass function

$$\mathbb{P}(\hat{N}_i = k) = \delta(1 - \delta)^{k-1}, \quad \text{for} \quad k = 1, 2, \cdots.$$

Suppose that $X_1, X_2, \cdots$ are independent and identically distributed with the exponential distribution $Exp(\lambda)$, with $\mathbb{E}(X_i) = \frac{1}{\lambda}$. The parameter λ is unknown but follows a Gamma distribution $Gamma(\alpha, \beta)$ with the probability density function

$$f_\lambda(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda\beta}, \quad \text{for} \quad \lambda > 0.$$

Then we have

$$\begin{aligned} M_{S_1|\lambda}(t) =& \mathbb{E}(e^{tS_1}|\lambda) \\ =& \mathbb{E}\Big(\mathbb{E}(e^{tX_1}|\lambda)\Big)^{\hat{N}_1} \\ =& \mathbb{E}\Big[\Big(\frac{\lambda}{\lambda - t}\Big)^{\hat{N}_1}\Big] \\ =& \frac{\delta\frac{\lambda}{\lambda-t}}{1-(1-\delta)\frac{\lambda}{\lambda-t}} \\ =& \frac{\delta\lambda}{\delta\lambda - t}. \end{aligned}$$

Therefore, $S_1|\lambda \sim Exp(\delta\lambda)$. So we have

$$\mathbb{P}(S_1 < z|\lambda) = 1 - e^{-(\delta\lambda)z},$$

and

$$\begin{aligned} \mathbb{E}(S_1) =& \mathbb{E}[\mathbb{E}(S_1|\lambda)] \\ =& \mathbb{E}\big(\frac{1}{\delta\lambda}\big) \\ =& \int_0^\infty \frac{1}{\delta\lambda} f_\lambda(\lambda) d\lambda \\ =& \frac{\beta}{\delta(\alpha-1)}. \end{aligned}$$

To be realistic, we should have $\alpha > 1$ so that $\mathbb{E}(S_1) > 0$.

In the Bayesian probability theory, the Gamma distribution as a prior distribution describes initial knowledge about the unknown parameter λ. We now determine the posterior distribution of λ. The joint likelihood function of $X_1, X_2, \cdots, X_n$, given λ, is

$$f_{X_1,X_2,...,X_n|\lambda}(x_1, x_2, ..., x_n) = \lambda^n e^{-\lambda\sum_{i=1}^n x_i}, \quad \text{for} \quad x_1, x_2, ..., x_n > 0.$$

From the Bayes' theorem, the posterior distribution of λ, given observations $X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n$, is

$$\begin{aligned} f_{\lambda|x_1,x_2,\cdots,x_n}(\lambda) =& \frac{f_{X_1,X_2,\cdots,X_n|\lambda}(x_1,x_2,\cdots,x_n)f_\lambda(\lambda)}{\int_0^\infty f_{X_1,X_2,\cdots,X_n|\lambda}(x_1,x_2,\cdots,x_n)f_\lambda(\lambda)d\lambda} \\ \propto& f_{X_1,X_2,\cdots,X_n|\lambda}(x_1,x_2,\cdots,x_n)f_\lambda(\lambda) \\ \propto& \lambda^{\alpha+n-1}e^{-\lambda(\beta+\sum_{i=1}^n x_i)}, \quad \text{for} \quad x_1,x_2,\cdots,x_n > 0. \end{aligned}$$

Therefore, the posterior distribution of λ is again a Gamma distribution $Gamma(\alpha+n, \beta+\sum_{i=1}^n x_i)$. It is clear that with the choice of a gamma prior distribution, a conjugate family forms, giving an algebraic convenience.

Now we consider the reserve process

$$\begin{aligned} &U_n = U_{n-1} + [P_{n-1} - (1-b_n)R_{n-1}] - b_n S_n, \quad \text{for} \quad n = 1,2,\cdots, \\ &U_0 = x. \end{aligned}$$

Let $\Psi_n(x,\alpha,\beta)$ denote the minimum ruin probability within n periods with an initial reserve x and a conjugate prior $Gamma(\alpha,\beta)$ for unknown parameter λ in the claim size distribution.

Theorem 3.3.1. *Under the above setting, we have*

$$\Psi_1(x,\alpha,\beta) = \left(\frac{\beta}{\beta + \frac{\delta x + \delta[P_0 - (1-b_1^*)R_0]}{b_1^*}}\right)^\alpha,$$

where b_1^* is the optimal reinsurance strategy. Moreover, $b_1^* = 1$ if $R_0 - P_0 > x$; $b_1^* = 1 - \frac{\eta}{\theta}$ if $R_0 - P_0 < x$; Any admissible control is the optimal control if $R_0 - P_0 = x$.

Proof.

$$\begin{aligned}
\Psi_1(x,\alpha,\beta) &= \min_{b_1\in\mathcal{B}} \mathbb{E}\Big(g(U_1)\Big) \\
&= \min_{b_1\in\mathcal{B}} \mathbb{P}(U_1<0) \\
&= \min_{b_1\in\mathcal{B}} \int_0^\infty \mathbb{P}(S_1 > \frac{x+[P_0-(1-b_1)R_0]}{b_1}|\lambda) f(\lambda) d\lambda \\
&= \min_{b_1\in\mathcal{B}} \int_0^\infty e^{-(\delta\lambda)\frac{x+[P_0-(1-b_1)R_0]}{b_1}} f(\lambda) d\lambda \\
&= \min_{b_1\in\mathcal{B}} \left(\frac{\beta}{\beta+\frac{\delta x+\delta[P_0-(1-b_1)R_0]}{b_1}}\right)^\alpha \\
&= \min_{b_1\in\mathcal{B}} R(b_1,\alpha,\beta),
\end{aligned}$$

where $R(b_1,\alpha,\beta) = \left(\frac{\beta}{\beta+\frac{\delta x+\delta[P_0-(1-b_1)R_0]}{b_1}}\right)^\alpha$.

Let $k(b_1) = \frac{\delta x+\delta[P_0-(1-b_1)R_0]}{b_1}$. Taking the first derivative of $k(b_1)$ with respect to b_1, we have

$$\frac{\partial k}{\partial b_1} = \frac{1}{b_1^2}[-\delta x + \delta(R_0-P_0)].$$

We consider the following cases.

(i) If we are given $R_0 - P_0 > x$, then we have $\frac{\partial k}{\partial b_1} > 0$ and $R(b_1,\alpha,\beta)$ decreases in b_1. To minimize $\Psi_1(x,\alpha,\beta)$, we take $b_1^* = 1$ which means no reinsurance is purchased.

(ii) If we are given $R_0 - P_0 < x$, then we have $\frac{\partial k}{\partial b_1} < 0$ and $R(b_1,\alpha,\beta)$ increases in b_1. In this case, the minimum is achieved at $b_1^* = 1 - \frac{\eta}{\theta}$, which is the minimum value of b_1 we can take.

(iii) If we are given $R_0 - P_0 = x$, the function $R(b_1, \alpha, \beta)$ is independent of b_1. Therefore, any admissible control is the optimal control.

□

Furthermore, we are concerned with structural properties of $\Psi_1(x, \alpha, \beta)$ with respect to α and β. Before we introduce the results, we need the following lemmas.

Lemma 3.3.2. *Suppose that* $\alpha > 1$. *Let* $u(\alpha) = t(\alpha) - \ln\big(\frac{1}{1+h(\alpha)}\big)$, *where* $h(\alpha) = \frac{\delta x}{b\beta} + \frac{b(1+\theta)-(\theta-\eta)}{b(\alpha-1)}$, $t(\alpha) = \frac{\alpha}{1+h(\alpha)}\frac{\partial}{\partial\alpha}h(\alpha)$, $b \geqslant 1 - \frac{\eta}{\theta}$ *and* $0 < \eta < \theta < 1$. *For any constant* $c > 1$,

(i) if $u(c) > 0$, *then we have* $u(\alpha) > 0$, *for all* $\alpha \geqslant c$.

(ii) if $u(c) < 0$, *then there exists a unique* α^* *such that* $\alpha^* > c$ *and* $u(\alpha^*) = 0$.

(iii) if α^* *exists, then we have* $\big(\frac{1}{1+h(\alpha^*)}\big)^{\alpha^*} \geqslant \big(\frac{1}{1+h(\alpha)}\big)^{\alpha}$, *for all* $\alpha > 1$.

Proof. (i) Since $u(\alpha)$ is a continuous function in α, taking derivative of $u(\alpha)$ with respect to α, we have

$$\frac{\partial}{\partial\alpha}u(\alpha) = \frac{\partial}{\partial\alpha}t(\alpha) - \frac{\partial}{\partial\alpha}\ln\big(\frac{1}{1+h(\alpha)}\big).$$

Taking derivative of $\ln\big(\frac{1}{1+h(\alpha)}\big)$ with respect to α, we have

$$\frac{\partial}{\partial\alpha}\ln\big(\frac{1}{1+h(\alpha)}\big) = \frac{b(1+\theta)-(\theta-\eta)}{(\alpha-1)^2 b + \frac{(\alpha-1)^2\delta x}{\beta} + (\alpha-1)[b(1+\theta)-(\theta-\eta)]}.$$

Since $b \geqslant 1 - \frac{\eta}{\theta}$, we have $\frac{\partial}{\partial \alpha} \ln\left(\frac{1}{1+h(\alpha)}\right) > 0$ for $\alpha > 1$. Also, taking derivative of $t(\alpha)$ with respect to α, we have

$$\begin{aligned}
&\frac{\partial}{\partial \alpha} t(\alpha) \\
=&\frac{\left[b(1+\theta) - (\theta - \eta)\right]\left[(\alpha^2-1)b + (\alpha^2-1)\frac{\delta x}{\beta} + b(1+\theta) - (\theta-\eta)\right]}{\left((\alpha-1)^2 b + \frac{(\alpha-1)^2 \delta x}{\beta} + (\alpha-1)[b(1+\theta) - (\theta-\eta)]\right)^2} \\
=&\left(\frac{\partial}{\partial \alpha} \ln\left(\frac{1}{1+h(\alpha)}\right)\right) \frac{(\alpha^2-1)b + (\alpha^2-1)\frac{\delta x}{\beta} + b(1+\theta) - (\theta-\eta)}{(\alpha-1)^2 b + \frac{(\alpha-1)^2 \delta x}{\beta} + (\alpha-1)[b(1+\theta) - (\theta-\eta)]}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\frac{\partial}{\partial \alpha} u(\alpha) \\
=&\left(\frac{\partial}{\partial \alpha} \ln\left(\frac{1}{1+h(\alpha)}\right)\right) \left[\frac{(\alpha^2-1)b + (\alpha^2-1)\frac{\delta x}{\beta} + [b(1+\theta) - (\theta-\eta)]}{(\alpha-1)^2 b + \frac{(\alpha-1)^2 \delta x}{\beta} + (\alpha-1)[b(1+\theta) - (\theta-\eta)]} - 1\right].
\end{aligned}$$

Now,

$$\begin{aligned}
&(\alpha^2-1)b + (\alpha^2-1)\frac{\delta x}{\beta} + [b(1+\theta) - (\theta-\eta)] \\
=&[(\alpha-1)^2 + 2(\alpha-1)]b + (\alpha^2-1)\frac{\delta x}{\beta} + [b(1+\theta) - (\theta-\eta)] \\
=&(\alpha-1)^2 b + (\alpha^2-1)\frac{\delta x}{\beta} + (\alpha-1)2b + [b(1+\theta) - (\theta-\eta)].
\end{aligned}$$

It is easy to see that $\alpha^2 - 1 > (\alpha-1)^2$ for $\alpha > 1$. Since $0 < \eta < \theta < 1$, we have $2b > b(1+\theta) - (\theta - \eta)$. It follows that

$$\frac{(\alpha^2-1)b + (\alpha^2-1)\frac{\delta x}{\beta} + [b(1+\theta) - (\theta-\eta)]}{(\alpha-1)^2 b + \frac{(\alpha-1)^2 \delta x}{\beta} + (\alpha-1)[b(1+\theta) - (\theta-\eta)]} - 1 > 0.$$

Therefore, we have

$$\frac{\partial}{\partial \alpha} u(\alpha) > 0, \text{ for } \alpha > 1.$$

Suppose that there exists a constant c such that $c > 1$ and $u(c) > 0$, then we have

$$u(\alpha) > u(c) > 0, \text{ for all } \alpha \geqslant c.$$

(ii) First of all, we have

$$\begin{aligned}
&\lim_{\alpha\to\infty} t(\alpha) \\
=&\lim_{\alpha\to\infty} \frac{\alpha}{1+h(\alpha)}\frac{\partial}{\partial\alpha}h(\alpha) \\
=&\lim_{\alpha\to\infty} \frac{-\alpha[b(1+\theta)-(\theta-\eta)]}{(1+\frac{\delta x}{b\beta})b(\alpha-1)^2+[b(1+\theta)-(\theta-\eta)](\alpha-1)} \\
=&0
\end{aligned}$$

and

$$\lim_{\alpha\to\infty} \ln\big(\frac{1}{1+h(\alpha)}\big) = \ln\Big(\frac{1}{1+\frac{\delta x}{b\beta}}\Big) < 0.$$

It follows that

$$\lim_{\alpha\to\infty} u(\alpha) = \lim_{\alpha\to\infty} t(\alpha) - \lim_{\alpha\to\infty} \ln\big(\frac{1}{1+h(\alpha)}\big) > 0.$$

Suppose that there exists a constant c such that $c > 1$ and $u(c) < 0$. Since $u(\alpha)$ is a continuously increasing function in α for $\alpha \geqslant c$ and $\lim_{\alpha\to\infty} u(\alpha) > 0$, there exists a unique α^* such that $\alpha^* > c$ and $u(\alpha^*) = 0$.

(iii)

$$\frac{\partial}{\partial\alpha}\big(\frac{1}{1+h(\alpha)}\big)^\alpha = \big(\frac{1}{1+h(\alpha)}\big)^\alpha\big[-u(\alpha)\big].$$

First of all, we have $\big(\frac{1}{1+h(\alpha)}\big)^\alpha > 0$, for $\alpha > 1$. Since α^* exists, we have $u(\alpha^*) = 0$. Therefore,

$$\frac{\partial}{\partial\alpha}\big(\frac{1}{1+h(\alpha)}\big)^\alpha\Big|_{\alpha=\alpha^*} = 0.$$

To verify that α^* maximize $\big(\frac{1}{1+h(\alpha)}\big)^\alpha$, we apply the second derivative test

$$\begin{aligned}&\frac{\partial^2}{\partial\alpha^2}\big(\frac{1}{1+h(\alpha)}\big)^\alpha\\ =&\big(\frac{1}{1+h(\alpha)}\big)^\alpha\Big\{\big[-u(\alpha)\big]^2-2\frac{t(\alpha)}{\alpha}-\alpha\frac{\partial}{\partial\alpha}\big(\frac{t(\alpha)}{\alpha}\big)\Big\}.\end{aligned}$$

Define

$$\begin{aligned}\mathcal{A}_\alpha=&\Big\{\alpha:-2\frac{t(\alpha)}{\alpha}-\alpha\frac{\partial}{\partial\alpha}\big(\frac{t(\alpha)}{\alpha}\big)\leqslant 0\Big\}\\ =&\Big\{\alpha:\frac{-2[\frac{\partial}{\partial\alpha}h(\alpha)]\big(1+h(\alpha)\big)-\alpha[\frac{\partial^2}{\partial\alpha^2}h(\alpha)]\big(1+h(\alpha)\big)+\alpha[\frac{\partial}{\partial\alpha}h(\alpha)]^2}{\big(1+h(\alpha)\big)^2}\leqslant 0\Big\}\\ =&\Big\{\alpha:-2\frac{\partial h(\alpha)}{\partial\alpha}\big(1+h(\alpha)\big)-\alpha\frac{\partial^2h(\alpha)}{\partial\alpha^2}\big(1+h(\alpha)\big)+\alpha\Big(\frac{\partial h(\alpha)}{\partial\alpha}\Big)^2\leqslant 0\Big\}\\ =&\Big\{\alpha:\alpha\geqslant\frac{b+\frac{\delta x}{\beta}-[b(1+\theta)-(\theta-\eta)]}{b+\frac{\delta x}{\beta}-\frac{1}{2}[b(1+\theta)-(\theta-\eta)]}\Big\}.\end{aligned}$$

It is easy to see

$$\frac{b+\frac{\delta x}{\beta}-[b(1+\theta)-(\theta-\eta)]}{b+\frac{\delta x}{\beta}-\frac{1}{2}[b(1+\theta)-(\theta-\eta)]}<1.$$

Therefore, we have $\alpha^*\in\mathcal{A}_\alpha$ and

$$\frac{\partial^2}{\partial\alpha^2}\big(\frac{1}{1+h(\alpha)}\big)^\alpha\Big|_{\alpha=\alpha^*}<0.$$

Thus, α^* maximizes $\big(\frac{1}{1+h(\alpha)}\big)^\alpha$.

□

Theorem 3.3.3. *Let* $u(\alpha)=t(\alpha)-\ln\big(\frac{1}{1+h(\alpha)}\big)$, *where* $h(\alpha)=\frac{\delta x}{b^*\beta}+\frac{b^*(1+\theta)-(\theta-\eta)}{b^*(\alpha-1)}$, $t(\alpha)=\frac{\alpha}{1+h(\alpha)}\frac{\partial}{\partial\alpha}h(\alpha)$. *For any constant* $c>1$, *if*

$$u(c)>0,$$

the function $\Psi_1(x,\alpha,\beta)$ *is decreasing in* α *for* $\alpha \geqslant c$ *and increasing in* β *for any given* $\alpha > 1$.

Proof. First of all, we have

$$\Psi_1(x,\alpha,\beta) = \min_{b_1\in\mathcal{B}} \left(\frac{\beta}{\beta + \frac{\delta x + \delta[P_0 - (1-b_1)R_0]}{b_1}} \right)^{\alpha}$$

Since $[P_0 - (1-b_1)R_0] = [b_1(1+\theta) - (\theta-\eta)]\mathbb{E}(S_1) = \frac{[b_1(1+\theta)-(\theta-\eta)]\beta}{\delta(\alpha-1)}$, we have

$$\begin{aligned}\Psi_1(x,\alpha,\beta) &= \min_{b_1\in\mathcal{B}} \left(\frac{\beta}{\beta + \frac{\delta x}{b_1} + \frac{[b_1(1+\theta)-(\theta-\eta)]\beta}{b_1(\alpha-1)}} \right)^{\alpha} \\ &= \left(\frac{1}{1 + \frac{\delta x}{\beta b_1^*} + \frac{[b_1^*(1+\theta)-(\theta-\eta)]}{b_1^*(\alpha-1)}} \right)^{\alpha}.\end{aligned}$$

By Theorem 3.3.1, we have $b_1^* = 1$ if $\alpha < \frac{(\theta-\eta)\beta}{\delta x}$, $b_1^* = 1 - \frac{\eta}{\theta}$ if $\alpha > \frac{(\theta-\eta)\beta}{\delta x}$ and any admissible control is the optimal control if $\alpha = \frac{(\theta-\eta)\beta}{\delta x}$. By the safety loading assumption, we have $0 < \eta < \theta < 1$.

Suppose that $c \leqslant \alpha \leqslant \frac{(\theta-\eta)\beta}{\delta x}$, we have $b_1^* = 1$. Taking derivative of $\Psi_1(x,\alpha,\beta)$ with respect to α, we have

$$\frac{\partial}{\partial\alpha}\Psi_1(x,\alpha,\beta) = -u(\alpha)\Psi_1(x,\alpha,\beta).$$

Since $u(c) > 0$, by Lemma 3.3.2(i) , we have $u(\alpha) > 0$ for all $\alpha \geqslant c$. Therefore, we have

$$\frac{\partial}{\partial\alpha}\Psi_1(x,\alpha,\beta) = -u(\alpha)\Psi_1(x,\alpha,\beta) < 0,$$

for all $c \leqslant \alpha \leqslant \frac{(\theta-\eta)\beta}{\delta x}$.

Suppose that $c \leqslant \frac{(\theta-\eta)\beta}{\delta x} \leqslant \alpha$ or $\frac{(\theta-\eta)\beta}{\delta x} \leqslant c \leqslant \alpha$, then $b_1^* = 1 - \frac{\eta}{\theta}$. Using ideas similar to above, we have the same result for $c \leqslant \frac{(\theta-\eta)\beta}{\delta x} \leqslant \alpha$.

Suppose that $\alpha = \frac{(\theta-\eta)\beta}{\delta x}$, any admissible control is the optimal control. Therefore, $\Psi_1(x,\alpha,\beta)$ is decreasing in α for $\alpha \geqslant c$.

Suppose that $\beta \leqslant \frac{\alpha\delta x}{\theta-\eta}$, then we have $b_1^* = 1 - \frac{\eta}{\theta}$. Taking derivative of $\Psi_1(x,\alpha,\beta)$ with respect to β, we have

$$\frac{\partial}{\partial\beta}\Psi_1(x,\alpha,\beta) = \frac{\alpha\delta x}{\beta^2 b_1^*}\left(\frac{1}{1+\frac{\delta x}{\beta b_1^*}+\frac{[b_1^*(1+\theta)-(\theta-\eta)]}{b_1^*(\alpha-1)}}\right)^{\alpha+1} > 0.$$

Suppose that $\beta \geqslant \frac{\alpha\delta x}{\theta-\eta}$, then we have $b_1^* = 1$. Using ideas similar to above, we have the same result. When $\beta = \frac{\alpha\delta x}{\theta-\eta}$, any admissible control is the optimal control. Therefore, $\Psi_1(x,\alpha,\beta)$ is increasing in β for any given $\alpha > 1$. □

Corollary 3.3.4. *Let* $u(\alpha) = t(\alpha) - \ln\left(\frac{1}{1+h(\alpha)}\right)$, *where* $h(\alpha) = \frac{\delta x}{b^*\beta} + \frac{b^*(1+\theta)-(\theta-\eta)}{b^*(\alpha-1)}$, $t(\alpha) = \frac{\alpha}{1+h(\alpha)}\frac{\partial}{\partial\alpha}h(\alpha)$. *For any constant* $c > 1$, *if*

$$u(c) < 0,$$

then there exists a unique $\alpha^* > c$ *such that the function* $\Psi_1(x,\alpha,\beta)$ *is increasing in* α *for* $1 < \alpha \leqslant \alpha^*$ *and decreasing in* α *for* $\alpha \geqslant \alpha^*$, *and increasing in* β *for any* $\alpha > 1$.

Proof. By using ideas similar to those in Theorem 3.3.3, the result is derived from Lemma 3.3.2(*ii*) and Lemma 3.3.2(*iii*). □

By means of dynamic programming, we also derive $\Psi_2(x,\alpha,\beta)$.

Theorem 3.3.5. *Assume that random variables* λ *and* $\hat{N}_1$ *are independent.*

Then we have

$$\Psi_2(x,\alpha,\beta)$$

$$=\min_{b_{(1,2)}\in\mathcal{B}}\Bigg\{R(b_{(1,2)},\alpha,\beta)+\mathbb{E}_{\hat{N}_1}\Big[\int_{\frac{\beta}{x_1+\beta}}^{1}\Psi_1\Big(b_{(1,2)}(x_1-\beta(\frac{1-t}{t})),\alpha+\hat{N}_1,\frac{\beta}{t}\Big)$$

$$f_{T|\hat{N}_1}(t)dt\Big]\Bigg\}$$

$$=R(b^*_{(1,2)},\alpha,\beta)+\mathbb{E}_{\hat{N}_1}\Big[\int_{\frac{\beta}{x_1^*+\beta}}^{1}\Psi_1\Big(b^*_{(1,2)}(x_1^*-\beta(\frac{1-t}{t})),\alpha+\hat{N}_1,\frac{\beta}{t}\Big)f_{T|\hat{N}_1}(t)dt\Big],$$

where $R(b_{(1,2)},\alpha,\beta)=\left(\frac{\beta}{\beta+\frac{\delta x+\delta[P_0-(1-b_{(1,2)})R_0]}{b_{(1,2)}}}\right)^{\alpha}$, $x_1^*=\frac{x+[P_0-(1-b^*_{(1,2)})R_0]}{b^*_{(1,2)}}$, $T|\hat{N}_1=n_1\sim Beta(\alpha,n_1)$, $\hat{N}_1\sim Fs(\delta)$ *and* $b^*_{(1,2)}$ *is the optimal reinsurance decision for the first time period.*

Proof. By the dynamic programming principle, we have

$$\begin{aligned}\Psi_2(x,\alpha,\beta)&=\min_{b_{(1,2)}\in\mathcal{B}}\big\{\mathbb{E}[g(U_1)]+\mathbb{E}[\Psi_1(U_1,\alpha+\hat{N}_1,\beta+S_1)]\big\}\\&=\min_{b_{(1,2)}\in\mathcal{B}}\big\{\mathbb{P}(S_1>x_1)+\mathbb{E}[\Psi_1(U_1,\alpha+\hat{N}_1,\beta+S_1)]\big\}\\&=\min_{b_{(1,2)}\in\mathcal{B}}\big\{R(b_1,\alpha,\beta)+\mathbb{E}[\Psi_1(U_1,\alpha+\hat{N}_1,\beta+S_1)]\big\},\end{aligned}$$

where $U_1=b_{(1,2)}(x_1-S_1)$ and $x_1=\frac{x+[P_0-(1-b_{(1,2)})R_0]}{b_{(1,2)}}$.

Furthermore,

$$
\begin{aligned}
&\mathbb{E}\big[\Psi_1(U_1, \alpha + \hat{N}_1, \beta + S_1)\big] \\
=&\mathbb{E}\Big[\mathbb{E}\big[\Psi_1(U_1, \alpha + \hat{N}_1, \beta + S_1)\big|\hat{N}_1\big]\Big] \\
=&\sum_{n_1=1}^{\infty} \mathbb{E}\big[\Psi_1(U_1, \alpha + \hat{N}_1, \beta + S_1)\big|\hat{N}_1 = n_1\big]\delta(1-\delta)^{n_1-1} \\
=&\sum_{n_1=1}^{\infty} \mathbb{E}\big[\Psi_1(b_{(1,2)}(x_1 - \sum_{i=1}^{n_1} X_i), \alpha + n_1, \beta + \sum_{i=1}^{n_1} X_i)\big]\delta(1-\delta)^{n_1-1} \\
=&\sum_{n_1=1}^{\infty} \mathbb{E}\Big[\mathbb{E}\big[\Psi_1(b_{(1,2)}(x_1 - \sum_{i=1}^{n_1} X_i), \alpha + n_1, \beta + \sum_{i=1}^{n_1} X_i)\big|\sum_{i=1}^{n_1} X_i\big]\Big]\delta(1-\delta)^{n_1-1}.
\end{aligned}
$$

Since $X_1, X_2, \cdots$ are independent and identically distributed with the exponential distribution $Exp(\lambda)$ with $\mathbb{E}(X_i) = \frac{1}{\lambda}$, and λ follows a gamma distribution $Gamma(\alpha, \beta)$ with $\mathbb{E}(\lambda) = \frac{\alpha}{\beta}$, we have $S_1\big|(\hat{N}_1 = n_1, \lambda) = \sum_{i=1}^{n_1} X_i|\lambda$ follows a gamma distribution $Gamma(n_1, \lambda)$ with $\mathbb{E}(\sum_{i=1}^{n_1} X_i|\lambda) = \frac{n_1}{\lambda}$. Since

λ and $\hat{N}_1$ are independent, then we have

$$\sum_{n_1=1}^{\infty} \mathbb{E}\Big[\mathbb{E}\big[\Psi_1(b_{(1,2)}(x_1 - \sum_{i=1}^{n_1} X_i), \alpha + n_1, \beta + \sum_{i=1}^{n_1} X_i)\big| \sum_{i=1}^{n_1} X_i\big]\Big]\delta(1-\delta)^{n_1-1}$$

$$= \sum_{n_1=1}^{\infty} \int_0^{\infty} \mathbb{E}\big[\Psi_1(b_{(1,2)}(x_1 - s_1), \alpha + n_1, \beta + s_1)\big] f_{S_1|\hat{N}_1=n_1}(s_1) ds_1 \delta(1-\delta)^{n_1-1}$$

$$= \sum_{n_1=1}^{\infty} \int_0^{\infty} \Psi_1(b_{(1,2)}(x_1 - s_1), \alpha + n_1, \beta + s_1)$$

$$\int_0^{\infty} f_{S_1|\hat{N}_1=n_1,\lambda}(s_1) f_{\lambda}(\lambda) d\lambda ds_1 \delta(1-\delta)^{n_1-1}$$

$$= \sum_{n_1=1}^{\infty} \int_0^{x_1} \Psi_1(b_{(1,2)}(x_1 - s_1), \alpha + n_1, \beta + s_1)$$

$$\int_0^{\infty} \frac{\lambda^{n_1}}{\Gamma(n_1)} s_1^{n_1-1} e^{-\lambda s_1} \frac{\beta^{\alpha}}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} d\lambda ds_1 \delta(1-\delta)^{n_1-1}$$

$$= \sum_{n_1=1}^{\infty} \int_0^{x_1} \Psi_1(b_{(1,2)}(x_1 - s_1), \alpha + n_1, \beta + s_1) \frac{s_1^{n_1-1}\beta^{\alpha}}{\Gamma(n_1)\Gamma(\alpha)}$$

$$\int_0^{\infty} \lambda^{\alpha+n_1-1} e^{-\lambda(s_1+\beta)} d\lambda ds_1 \delta(1-\delta)^{n_1-1}$$

$$= \sum_{n_1=1}^{\infty} \int_0^{x_1} \Psi_1(b_{(1,2)}(x_1 - s_1), \alpha + n_1, \beta + s_1) \frac{s_1^{n_1-1}\beta^{\alpha}\Gamma(\alpha+n_1)}{\Gamma(n_1)\Gamma(\alpha)(s_1+\beta)^{\alpha+n_1}} ds_1$$

$$\delta(1-\delta)^{n_1-1}$$

$$= \sum_{n_1=1}^{\infty} \int_0^{x_1} \Psi_1(b_{(1,2)}(x_1 - s_1), \alpha + n_1, \beta + s_1)$$

$$\frac{\Gamma(\alpha+n_1)}{\Gamma(\alpha)\Gamma(n_1)} (\frac{s_1}{s_1+\beta})^{n_1-1} (\frac{\beta}{s_1+\beta})^{\alpha-1} \frac{\beta}{(s_1+\beta)^2} ds_1 \delta(1-\delta)^{n_1-1}.$$

Let $t = \frac{\beta}{s_1+\beta}$, then we have

$$\begin{aligned}
&\mathbb{E}\big[\Psi_1(U_1, \alpha + \hat{N}_1, \beta + S_1)\big] \\
=&\sum_{n_1=1}^{\infty} \int_{\frac{\beta}{x_1+\beta}}^{1} \Psi_1(b_{(1,2)}(x_1 - \beta(\frac{1-t}{t})), \alpha + n_1, \frac{\beta}{t}) \\
&\qquad \frac{\Gamma(\alpha+n_1)}{\Gamma(\alpha)\Gamma(n_1)} t^{\alpha-1}(1-t)^{n_1-1} dt \delta(1-\delta)^{n_1-1} \\
=&\mathbb{E}_{\hat{N}_1}\Big[\int_{\frac{\beta}{x_1+\beta}}^{1} \Psi_1\Big(b_{(1,2)}(x_1 - \beta(\frac{1-t}{t})), \alpha + \hat{N}_1, \frac{\beta}{t}\Big) f_{T|\hat{N}_1}(t)dt\Big],
\end{aligned}$$

where $T|\hat{N}_1 = n_1 \sim Beta(\alpha, n_1)$ and $\hat{N}_1 \sim Fs(\delta)$. It follows that

$$\begin{aligned}
&\Psi_2(x, \alpha, \beta) \\
=&\min_{b_{(1,2)} \in \mathcal{B}} \Bigg\{ R(b_{(1,2)}, \alpha, \beta) + \mathbb{E}_{\hat{N}_1}\Big[\int_{\frac{\beta}{x_1+\beta}}^{1} \Psi_1\Big(b_{(1,2)}(x_1 - \beta(\frac{1-t}{t})), \alpha + \hat{N}_1, \frac{\beta}{t}\Big) \\
&\qquad f_{T|\hat{N}_1}(t)dt\Big]\Bigg\} \\
=&R(b^*_{(1,2)}, \alpha, \beta) + \mathbb{E}_{\hat{N}_1}\Big[\int_{\frac{\beta}{x_1^*+\beta}}^{1} \Psi_1\Big(b^*_{(1,2)}(x_1 - \beta(\frac{1-t}{t})), \alpha + \hat{N}_1, \frac{\beta}{t}\Big) f_{T|\hat{N}_1}(t)dt\Big].
\end{aligned}$$

□

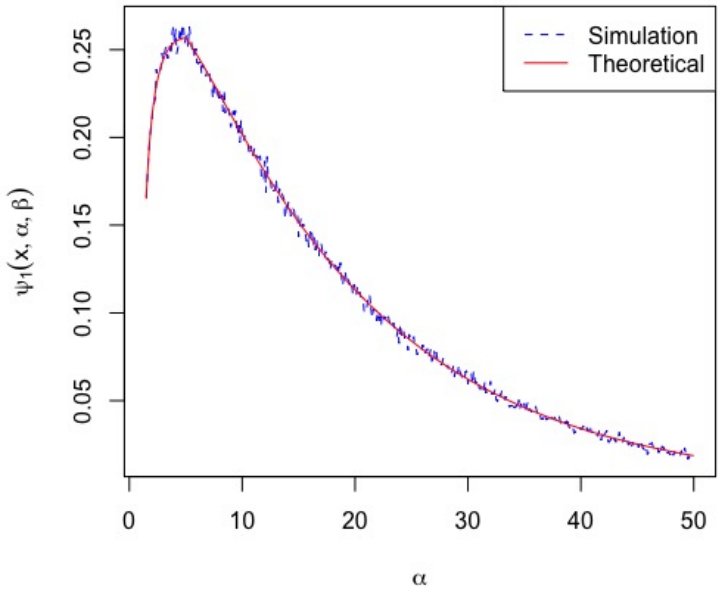
So far, explicit expressions of the ruin probability for the first two time periods have been derived. Due to the complexity of computation, we do not work on expressions of the ruin probability for longer time periods. However, the minimum ruin probability for any time period may be obtained numerically.

3.4 Numerical results

Some numerical calculations are derived in this section to illustrate the model. We simulate and plot minimum ruin probabilities within one period of time

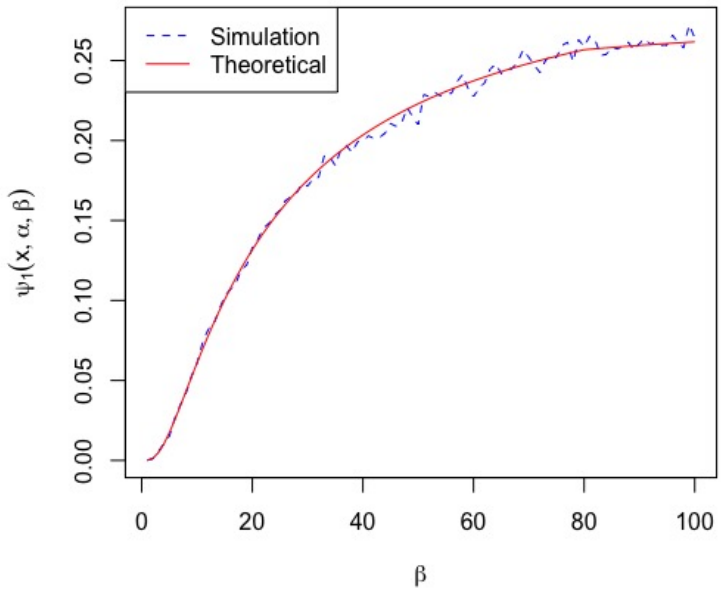
when the distribution of claim sizes is exponential with a random parameter following a Gamma distribution (α, β). We observe values of $\Psi_1(x, \alpha, \beta)$ on changing values of α and β, when $\eta = 0.15, \theta = 0.25, \delta = 0.4$, and $x = 5$. We also compute and plot minimum ruin probabilities with same values of parameters based on our results in Section 3.3.

Figure 3.1: Minimum ruin probabilities, $(\eta, \theta, \delta, \beta, x) = (0.15, 0.25, 0.4, 80, 5)$



In Figure 3.1, we set $(\eta, \theta, \delta, \beta, x) = (0.15, 0.25, 0.4, 80, 5)$ and change values of α from 1.5 to 50. For the given parameters, since $u(1.5) < 0$, there exists a unique $\alpha^* > 1.5$ such that the minimum ruin probability function $\Psi_1(x, \alpha, \beta)$ is increasing in α for $\alpha \leqslant \alpha^*$ and decreasing in α for $\alpha \geqslant \alpha^*$ based on results from Section 3.3. In Figure 3.1, we have observed this property of $\Psi_1(x, \alpha, \beta)$ on changing values of α.

Figure 3.2: Minimum ruin probabilities, $(\eta, \theta, \delta, \alpha, x) = (0.15, 0.25, 0.4, 5, 5)$



In Figure 3.2, we set $(\eta, \theta, \delta, \alpha, x) = (0.15, 0.25, 0.4, 5, 5)$ and change values of β from 1 to 100. It can be seen from Figure 3.2 that the minimum ruin probability function $\Psi_1(x, \alpha, \beta)$ overall is increasing in β, which coincides with our theoretical results in Section 3.3.

Next, we simulate and plot the minimum ruin probability $\Psi_2(x, \alpha, \beta)$ as well as the corresponding optimal reinsurance strategy b_1^* on changing values of α, when $\eta = 0.15, \theta = 0.25, \delta = 0.4, \beta = 80$ and $x = 5$. We also use Monte Carlo method to estimate $\Psi_2(x, \alpha, \beta)$ and b_1^* as our calculation results. We have observed that $\Psi_2(x, \alpha, \beta)$ has the same property as $\Psi_1(x, \alpha, \beta)$ on changing values of α.

Figure 3.3: Minimum ruin probabilities, $(\eta, \theta, \delta, \beta, x) = (0.15, 0.25, 0.4, 80, 5)$

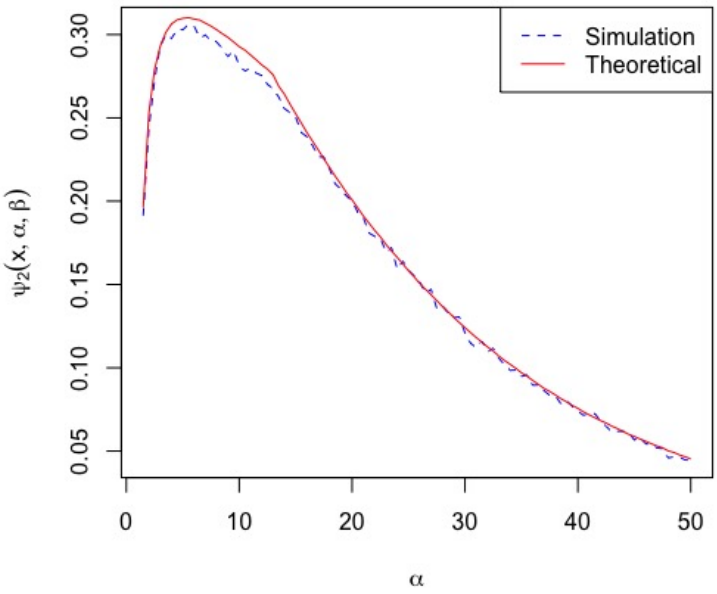
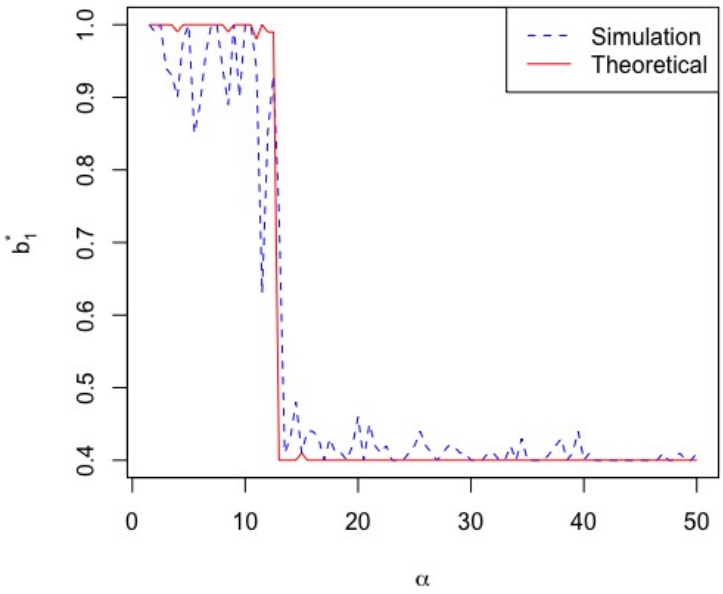


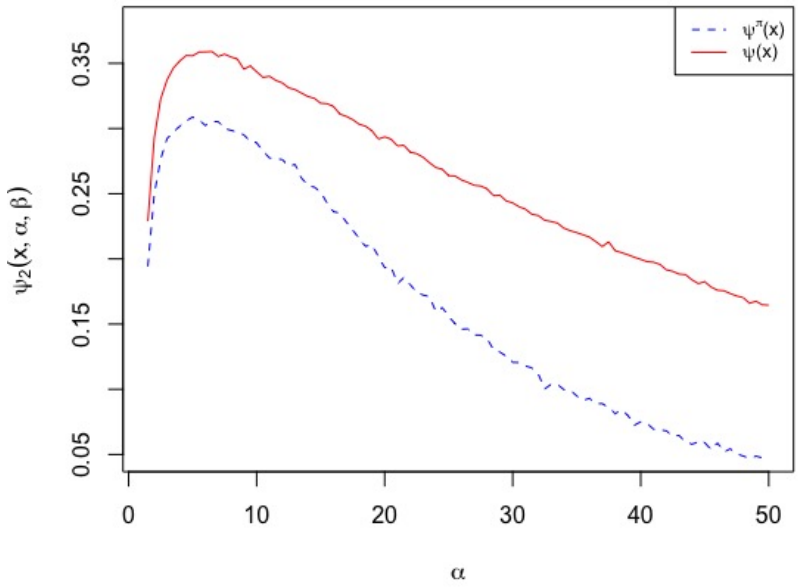
Figure 3.4: Optimal reinsurance strategies, $(\eta, \theta, \delta, \beta, x) = (0.15, 0.25, 0.4, 80, 5)$



Overall, all theoretical and simulation results coincide in Figures 3.1, 3.2, 3.3 and 3.4.

Finally, we simulate the ruin probability with our derived optimal reinsurance strategies and compare them with the ruin probability without taking reinsurance, when $\eta = 0.15, \theta = 0.25, \delta = 0.4, \beta = 80, x = 5$ and α ranges from 1.5 to 50. Figure 3.5 shows a lower ruin probability when optimal reinsurance strategy is applied.

Figure 3.5: Comparison of ruin probability with or without reinsurance



3.5 Conclusion

We have considered the problem of minimizing the ruin probability by proportional reinsurance in a discrete-time risk model. Although the literature on the optimal reinsurance strategy is increasing rapidly, but none of the studies considers this optimization problem under the assumption that the distribution of claim sizes with unknown parameters. In this case, we solve the problem by the Bayesian approach. Moreover, we formulate the problem as a Markov decision process and use the method of discrete-time dynamic programming to solve the model. The optimal proportional reinsurance strategy is characterized and structural properties of the minimum ruin probability are discussed. In addition, explicit expressions of the minimum ruin probability is obtained for the first two time periods. Our theoretical results are illustrated with

numerical computations and simulations.

Chapter 4

Minimizing ruin probability with joint strategies of investment and reinsurance

4.1 Introduction

There has been a considerable development on stochastic control in insurance since the first few papers (e.g. Brockett and Xia 1995; Browne 1995) appeared. The main problems considered in this field include minimizing the ruin probability and maximizing the expected discounted dividend payments, where the risk may be controlled in several ways. One possible way is to dynamically invest part of the reserve in a financial market, and another possibility is to purchase a reinsurance, by which the insurer transfers a part of its risk to a reinsurance company.

As an essential objective of any insurance company, we concentrate on minimizing ruin probability. We apply stochastic control theory to deal with the problem of optimal control by investment and reinsurance. In the literature, the problem of minimizing ruin probability with joint or sole investment and reinsurance decisions has been studied by many authors. Hipp and Plum (2000) find the optimal investment strategy to minimize ruin probability when

the reserve is partially invested in a risky asset. Schmidli (2001) studies the strategy of a proportional reinsurance and determines the optimal reinsurance strategy when the reserve is modeled by a geometric Brownian motion with drift. Promislow and Young (2005) find joint investment and proportional reinsurance decisions to minimize ruin probability in a diffusion approximation model. Luo (2008) considers the same optimal control problem of proportional reinsurance and investment subject to borrowing constraints. Liang and Young (2018) investigate excess-of-loss reinsurance and derive jointly optimal investment and reinsurance strategies. They demonstrate the optimality of excess-of-loss reinsurance under certain model assumptions.

In this chapter, we characterize and derive jointly optimal investment and reinsurance decisions to minimize ruin probability. We allow excess-of-loss reinsurance and extend Liang and Young's work by assuming an interest rate of borrowing that is higher than the interest rate of the risk-free asset. The reserve process is modeled by a diffusion process. The reserve can be invested in a financial market consisting of a risky asset with the price process follows geometric Brownian motion and a risk-free asset. We formulate the problem and characterize the HJB equation, then we derive main results under a constraint on the interest rate for borrowing. The closed form expression of the minimum ruin probability function is obtained, and results are illustrated numerically.

4.2 Formulation of the problem

Suppose that all random processes in this chapter are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ and all filtrations are right continuous filtrations.

Consider the classical Cramér-Lundberg risk model in which the insurer's

reserve $\hat{R}_t$ at time t is given by

$$\hat{R}_t = x + ct - \sum_{i=1}^{N_t} Y_i, \quad t \geqslant 0,$$

where x is an initial reserve, $\{N_t\}_{t\geqslant 0}$ is a Poisson process with an intensity rate $\lambda > 0$, claim sizes $Y_1, Y_2, \cdots$, are independent and identically distributed, positive random variables with finite first and second moments, $c = (1 + \eta)\mathbb{E}(Y)\lambda$ is the constant gross premium rate calculated based on the expected value principle, and $\eta > 0$ is the relative safety loading of the insurer.

Suppose that the insurer may purchase an excess-of-loss reinsurance with a fixed reinsurance retention level d. Let $Y_i \wedge d$ be the portion of the i^{th} claim retained by the insurer, then $Y_i - Y_i \wedge d$ is covered by reinsurance. The reinsurance premium rate is also calculated by the expected value principle and given by $c^R = (1+\theta)\mathbb{E}(Y - Y \wedge d)\lambda$, where θ is the relative safety loading for reinsurance. To be realistic, we assume that $\theta > \eta$.

Let $\hat{R}_t^R$ denote the reserve with a fixed reinsurance retention level d, the insurer's reserve process becomes

$$\hat{R}_t^R = x + c^N t - \sum_{i=1}^{N_t} (Y_i \wedge d), \quad t \geqslant 0,$$

where c^N is called the net premium which can be calculated as

$$\begin{aligned} c^N =& c - c^R \\ =& (1+\eta)\mathbb{E}(Y)\lambda - (1+\theta)\mathbb{E}(Y - Y \wedge d)\lambda \\ =& (1+\theta)\mathbb{E}(Y \wedge d)\lambda - (\theta - \eta)\mathbb{E}(Y)\lambda. \end{aligned}$$

To fulfill the net profit condition, we assume that

$$\mathbb{E}\big[c^N t - \sum_{i=1}^{N_t} (Y_i \wedge d)\big] \geqslant 0.$$

It follows that

$$\big[\theta\mathbb{E}(Y\wedge d)-(\theta-\eta)\mathbb{E}(Y)\big]\geqslant 0.$$

The expectation $\mathbb{E}(Y\wedge d)=\int_0^d S(y)dy$ is increasing and continuous in d, where $S(y)$ is the survival function of Y. Moreover, we have $\big[\theta\mathbb{E}(Y\wedge 0)-(\theta-\eta)\mathbb{E}(Y)\big]<0$ and $\big[\theta\mathbb{E}(Y\wedge\infty)-(\theta-\eta)\mathbb{E}(Y)\big]>0$. Therefore, there exists a unique $\hat{d}$ such that $\big[\theta\mathbb{E}(Y\wedge\hat{d})-(\theta-\eta)\mathbb{E}(Y)\big]=0$. Hence, we assume that $d\geqslant\hat{d}$.

Following the results from Chapter 2, the reserve process $\{\hat{R}_t^R\}_{t\geqslant 0}$ can be approximated by the diffusion process $\{R_t^R\}_{t\geqslant 0}$ which satisfies the stochastic differential

$$\begin{aligned} dR_t^R =&\big[\theta\mathbb{E}(Y\wedge d)\lambda-(\theta-\eta)\mathbb{E}(Y)\lambda\big]dt+\sqrt{\mathbb{E}(Y\wedge d)^2\lambda}dB_t^R, \qquad (4.2.1)\\ R_0^R =&x, \end{aligned}$$

where $\{B_t^R\}_{t\geqslant 0}$ is a standard Brownian motion process.

Meanwhile, the insurer invests the whole reserve in a financial market consisting of a risk-free asset whose price process $\{S_t^0\}_{t\geqslant 0}$ is given by

$$dS_t^0=\alpha S_t^0 dt$$

and a risky asset whose price process $\{S_t\}_{t\geqslant 0}$ follows geometric Brownian motion which is given by

$$dS_t=\mu_1 S_t dt+\sigma_1 S_t dB_t^I,$$

where μ_1 is the mean of the return rate which is greater then α, σ_1 is the volatility of return rate and $\{B_t^I\}_{t\geqslant 0}$ is a standard Brownian motion. We assume that the individual investment does not affect the financial market. Thus, $\{B_t^I\}_{t\geqslant 0}$ and $\{B_t^R\}_{t\geqslant 0}$ are independent.

Suppose that the borrowing is allowed with the rate β. To be realistic, we assume that the borrowing rate β is higher than the risk-free rate α. Also, we assume that the short-selling is not allowed. Let b be the fraction of the reserve invested in the risky asset, then we have $b \in (0, \infty)$ and $(1-b)_+ = \max\{1-b, 0\}$ is the fraction of the reserve invested in the risk-free asset.

Let r be a constant which takes value α when $0 < b \leqslant 1$, or value β when $b > 1$. Let R_t denote the reserve with reinsurance retention level d and b denotes the fraction of reserve invested in the risky asset, the insurer's reserve process follows the stochastic differential equation

$$\begin{aligned} dR_t =& \big[\theta\lambda\mathbb{E}(Y \wedge d) - (\theta - \eta)\lambda\mathbb{E}(Y) + r(1-b)R_t + b\mu_1 R_t\big]dt \\ &+ \sqrt{\mathbb{E}(Y \wedge d)^2\lambda}dB_t^R + \sigma_1 b R_t dB_t^I, \\ R_0 =& x. \end{aligned}$$

It is easy to see

$$\begin{cases} dB_t^R = (B_{t+dt}^R - B_t^R) = \sqrt{dt}Z_1 \\ dB_t^I = (B_{t+dt}^I - B_t^I) = \sqrt{dt}Z_2, \end{cases}$$

where Z_1 and Z_2 are standard normal random variables. Since $\{B_t^R\}_{t\geqslant 0}$ and $\{B_t^I\}_{t\geqslant 0}$ are independent standard Brownian motions processes, we have

$$\begin{aligned} \sqrt{\mathbb{E}(Y \wedge d)^2\lambda}dB_t^R + \sigma_1 b R_t dB_t^I =& \Big(\sqrt{\mathbb{E}(Y \wedge d)^2\lambda + (\sigma_1 b R_t)^2}\Big)\sqrt{dt}Z_3 \\ =& \sqrt{\mathbb{E}(Y \wedge d)^2\lambda + (\sigma_1 b R_t)^2}dB_t, \end{aligned}$$

where Z_3 is a standard normal random variable and $\{B_t\}_{t\geqslant 0}$ is a standard Brownian motion process, and the equality means equal in distribution. It follows that

$$\begin{aligned} dR_t =& \big[\theta\lambda\mathbb{E}(Y \wedge d) - (\theta - \eta)\lambda\mathbb{E}(Y) + r(1-b)R_t + b\mu_1 R_t\big]dt \\ &+ \sqrt{\mathbb{E}(Y \wedge d)^2\lambda + (\sigma_1 b R_t)^2}dB_t, \\ R_0 =& x. \end{aligned}$$

Note, as the reserve becomes large enough, we do not require that $d \geqslant \hat{d}$ because the investment makes the drift to be positive.

Let $\pi_t = (b_t, d_t)$ be the joint investment and reinsurance decisions at time t, where b_t is the fraction of reserve invested in the risky asset at time t and d_t is the retention level of excess-of-loss reinsurance at time t. A strategy $\pi = \{\pi_t\}_{t\geqslant 0}$ is a sequence of joint decisions of investment and reinsurance. Moreover, a strategy π is called admissible if it is predictable with respect to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ and satisfies $0 \leqslant d_t < \infty$ and $\int_0^t b_s^2 ds < \infty$ almost surely for all $t \geqslant 0$. We restrict the control d_t from the space $\mathcal{D} = \mathbb{R}^+$ to be a stationary Markov control which means $d_t = d(R_t)$ for all $t \geqslant 0$, with a function $d : \mathbb{R}^+ \to \mathbb{R}^+$. We also restrict b_t to be a stationary Markov control which means $b_t = b(R_t)$ for all $t \geqslant 0$, with a function $b : \mathbb{R}^+ \to \mathbb{R}^+$.

Let Π denotes the set of all admissible strategies. Under a particular strategy $\pi \in \Pi$, the insurer's reserve process R_t^π fulfils the stochastic differential equation

$$\begin{aligned} dR_t^\pi =& \big[\theta\lambda\mathbb{E}(Y \wedge d_t) - (\theta - \eta)\lambda\mathbb{E}(Y) + r(1 - b_t)R_t^\pi + b_t\mu_1 R_t^\pi\big]dt \\ &+ \sqrt{\mathbb{E}(Y \wedge d_t)^2\lambda + (\sigma_1 b_t R_t^\pi)^2}dB_t, \qquad (4.2.2) \\ R_0^\pi =& x. \end{aligned}$$

A natural question is whether the stochastic differential equation (4.2.2) has a unique solution $\{R_t^\pi\}_{t\geqslant 0}$. To answer this question, we have the following two lemmas.

Lemma 4.2.1. *Suppose $a > b > 0$. For any $x \geqslant 0$,*

$$l(x) = \sqrt{x + a} - \sqrt{x + b}$$

is decreasing in x.

Proof. This is evident from

$$\frac{d}{dx}l(x) = \frac{1}{2}\Big(\frac{1}{\sqrt{x + a}} - \frac{1}{\sqrt{x + b}}\Big) < 0.$$

□

Lemma 4.2.2. *R_t^π is the unique solution of stochastic differential equation (4.2.2).*

Proof. Let α denote the joint control (b, d). By Theorem 2.4.2, we only need to check Lipschtiz and linear conditions. Now,

$$\begin{aligned}
&|b(x,\alpha) - b(y,\alpha)| + |\sigma(x,\alpha) - \sigma(y,\alpha)| \\
=&|r(1-b) + b\mu_1||x-y| + \left|\sqrt{\mathbb{E}(Y\wedge d)^2\lambda + \sigma_1^2 b^2 x^2} - \sqrt{\mathbb{E}(Y\wedge d)^2\lambda + \sigma_1^2 b^2 y^2}\right|.
\end{aligned}$$

By Lemma 4.2.1,

$$\left|\sqrt{\mathbb{E}(Y\wedge d)^2\lambda + \sigma_1^2 b^2 x^2} - \sqrt{\mathbb{E}(Y\wedge d)^2\lambda + \sigma_1^2 b^2 y^2}\right| \leqslant \left|\sqrt{\sigma_1^2 b^2 x^2} - \sqrt{\sigma_1^2 b^2 y^2}\right|.$$

It follows that

$$\begin{aligned}
&|b(x,\alpha) - b(y,\alpha)| + |\sigma(x,\alpha) - \sigma(y,\alpha)| \\
\leqslant&|r(1-b) + b\mu_1||x-y| + \left|\sqrt{\sigma_1^2 b^2 x^2} - \sqrt{\sigma_1^2 b^2 y^2}\right| \\
=&|r(1-b) + b\mu_1||x-y| + |\sigma_1 b|\big||x| - |y|\big| \\
\leqslant&\Big(|r(1-b) + b\mu_1| + |\sigma_1 b|\Big)|x-y|.
\end{aligned}$$

Since r, b, μ_1 and σ_1 are all finite constants, there must exist a constant C such that

$$|b(x,\alpha) - b(y,\alpha)| + |\sigma(x,\alpha) - \sigma(y,\alpha)| \leqslant C|x-y|.$$

On the other hand,

$$\begin{aligned}
&|b(x,\alpha)| + |\sigma(x,\alpha)| \\
=&\left|\theta\lambda\mathbb{E}(Y\wedge d) - (\theta-\eta)\lambda\mathbb{E}(Y) + r(1-b)x + b\mu_1 x\right| + \left|\sqrt{\mathbb{E}(Y\wedge d)^2\lambda + \sigma_1^2 b^2 x^2}\right| \\
\leqslant&\left|\theta\lambda\mathbb{E}(Y\wedge d) - (\theta-\eta)\lambda\mathbb{E}(Y)\right| + |r(1-b) + b\mu_1||x| + \sqrt{\mathbb{E}(Y\wedge d)^2\lambda} + |\sigma_1 b||x| \\
=&\left|\theta\lambda\mathbb{E}(Y\wedge d) - (\theta-\eta)\lambda\mathbb{E}(Y)\right| + \sqrt{\mathbb{E}(Y\wedge d)^2\lambda} + \Big(|r(1-b) + b\mu_1| + |\sigma_1 b|\Big)|x|
\end{aligned}$$

Since all parameters are constants and the expectations exist, there must exist a constant D such that

$$|b(x,\alpha)| + |\sigma(x,\alpha)| \leqslant D(1+|x|).$$

□

Given any admissible strategy π, the time of ruin is defined by

$$\tau^\pi = \inf\{t > 0 : R_t^\pi \leqslant 0\}.$$

Lemma 4.2.3. *τ^π is a stopping time.*

Proof. We say that $\tau^\pi < t$ if for some n, $\tau^\pi \leqslant t - \frac{1}{n}$.
Thus,

$$\begin{aligned}\{\tau^\pi < t\} &= \bigcup_{n=1}^{\infty} \{\tau^\pi \leqslant t - \frac{1}{n}\} \\ &= \bigcup_{n=1}^{\infty} \{R^\pi_{t-\frac{1}{n}} \leqslant 0\}.\end{aligned}$$

It is known that $\{R^\pi_{t-\frac{1}{n}}\} \in \mathcal{F}_{t-\frac{1}{n}} \subset \mathcal{F}_t$, therefore, $\{\tau^\pi < t\} \in \mathcal{F}_t$. Now,

$$\{\tau^\pi \leqslant t\} = \bigcap_{n=1}^{\infty} \{\tau^\pi < t + \frac{1}{n}\}.$$

Since $\{\tau^\pi < t + \frac{1}{n}\} \in \mathcal{F}_{t+\frac{1}{n}}$, we have $\{\tau^\pi \leqslant t\} \in \mathcal{F}_{t+\frac{1}{n}}$ as well. By the right-continuity of $\{\mathcal{F}_t\}_{t\geqslant 0}$, we obtain $\{\tau^\pi \leqslant t\} \in \mathcal{F}_t$ □

We also define that $R_t^\pi = 0$ for $t > \tau^\pi$. Thus under strategy π, the ruin probability with an initial reserve x is

$$\psi^\pi(x) = P(\tau^\pi < \infty | R_0^\pi = x).$$

Our objective is to find the minimum ruin probability

$$\psi(x) = \inf_{\pi \in \Pi} \psi^{\pi}(x). \tag{4.2.3}$$

Now, we have the following properties of $\psi(x)$.

Lemma 4.2.4. *Under any admissible strategy π, the ruin probability $\psi^{\pi}(0) = 1$.*

Proof. Let $d_0 = \lim_{t \to 0} d_t$. If $d_0 = 0$, we have $\tau^{\pi} = 0$ since $\theta > \eta$. If $d_0 > 0$, we also have $\tau^{\pi} = 0$ because the Brownian motion crosses the t-axis infinitely often in the small interval. □

Lemma 4.2.5. *We define $x^* = \frac{(\theta-\eta)\lambda E(Y)}{\alpha}$. When $x \geqslant x^*$, we have $\pi_t^* = (0, 0)$ and $\psi^{\pi^*}(x) = 0$.*

Proof. If $x \geqslant x^*$, then we have

$$x\alpha \geqslant (\theta - \eta)\lambda \mathbb{E}(Y).$$

Thus, all risk can be reinsured and the ruin probability becomes zero. □

As a result, we need only to consider the case of $x \in (0, x^*)$.

Lemma 4.2.6. *When $x \in (0, x^*)$, $\psi(x)$ is a decreasing function in x.*

Proof. Suppose $0 < x_1 < x_2 < x^*$, the ruin probability under an arbitrary strategy π_1 with initial reserve x_1 is $\psi^{\pi_1}(x_1)$ and the ruin probability under strategy π_2 with initial reserve x_2 is $\psi^{\pi_2}(x_2)$. We consider π_2 in the following way,

(i) the reserve x_2 can be divided into x_1 and $x_2 - x_1$,

(ii) we implement the same strategy π_1 for x_1 and implement another arbitrary strategy $\pi_1^{'}$ for reserve $x_2 - x_1$.

Therefore, the ruin probability under strategy π_2 with initial reserve x_2 is given by

$$\psi^{\pi_2}(x_2) = \psi^{\pi_1}(x_1)\psi^{\pi_1^{'}}(x_2 - x_1).$$

Since $0 < \psi^{\pi_1^{'}}(x_2 - x_1) < 1$ and π_1 is an arbitrary strategy, we have $\psi(x_2) < \psi(x_1)$. □

4.3 HJB equation and verification theorem

The process of obtaining the minimum ruin probability and the corresponding optimal strategy is as follows:

1. Establish the dynamic programming principle regarding $\psi(x)$.
2. Derive the HJB equation for $\psi(x)$.
3. Show that the solution of this HJB equation is the desired value function by verification arguments.

Theorem 4.3.1. *The minimum ruin probability $\psi(x)$ defined by (4.2.3) is the desired value function and satisfies the dynamic programming principle.*

Proof. Consider the stochastic differential equation defined in (4.2.2). The instantaneous cost function under the joint investment and reinsurance decisions π_t at time t is defined as

$$c(R_t^{\pi}, \pi_t) = \begin{cases} 0 & \text{for} \quad R_t^{\pi} > 0, \\ 1 & \text{for} \quad R_t^{\pi} \leqslant 0. \end{cases}$$

Our objective is to minimize the expected total cost

$$J(x,\pi)=\mathbb{E}\Big[\int_0^{\tau^\pi} c(R_s^\pi,\pi_s)ds+c(R_{\tau^\pi})\Big],$$

where $c(R_{\tau^\pi})$ is the final cost depending on the final state only. It is easy to see

$$\int_0^{\tau^\pi} c(R_s^\pi,\pi_s)ds+c(R_{\tau^\pi})=\begin{cases}0 & \text{if} \quad \tau^\pi=\infty,\\ 1 & \text{if} \quad \tau^\pi<\infty.\end{cases}$$

Therefore, the expected total cost

$$\begin{aligned}J(x,\pi)=&\mathbb{E}\Big[\int_0^{\tau^\pi} c(R_s^\pi,\pi_s)ds+c(R_{\tau^\pi})\Big]\\ =&\psi^\pi(x).\end{aligned}$$

Thus, our objective is to find the value function

$$\inf_{\pi\in\Pi} J(x,\pi)=\inf_{\pi\in\Pi}\psi^\pi(x)=\psi(x).$$

By Lemma 2.4.3, we have

$$\psi(x)=\inf_{\pi\in\Pi}\mathbb{E}\left[\int_0^{t\wedge\tau^\pi} c(R_s^\pi,\pi_s)ds+\psi(R_{t\wedge\tau^\pi}^\pi)\right],\quad \text{for all}\quad t\geqslant 0. \tag{4.3.1}$$

□

Before we derive the HJB equation for the value function $\psi(x)$, we need the following lemma.

Lemma 4.3.2. *Assume that $\psi(x)$ defined by (4.2.3) is a twice continuously differentiable function on $(0,x^*)$. The stochastic integral*

$$\int_0^{t\wedge\tau^\pi}\Big[\sqrt{\mathbb{E}(Y\wedge d_s)^2\lambda+\sigma_1^2 b_s^2(R_s^\pi)^2}\Big]\psi'(R_s^\pi)dB_s$$

is a martingale, where τ^π is the ruin time under strategy π.

Proof. Clearly we have

$$\int_0^{t\wedge\tau^\pi} \left[\sqrt{\mathbb{E}(Y\wedge d_s)^2\lambda+\sigma_1^2 b_s^2(R_s^\pi)^2}\right]\psi^{'}(R_s^\pi)dB_s$$

$$=\int_0^{t} \left[\sqrt{\mathbb{E}(Y\wedge d_s)^2\lambda+\sigma_1^2 b_s^2(R_s^\pi)^2}\right]\psi^{'}(R_s^\pi)\mathbb{1}_{(0,\tau^\pi)}(s)dB_s.$$

From Lemma 4.2.3, $\{\tau^\pi \leqslant s\} \in \mathcal{F}_s$, then we have $\{\tau^\pi \leqslant s\}^c = \{s < \tau^\pi\} \in \mathcal{F}_s$. Thus, $\mathbb{1}_{(0,\tau^\pi)}(s)$ is adapted to the filtration $\mathcal{F}_s$. Also, the function $\psi^{'}$ is continuous since the second derivative $\psi^{''}$ exists and R_s^π is a homogeneous Markov process. Therefore,

$$\left\{\sqrt{\mathbb{E}(Y\wedge d_s)^2\lambda+\sigma_1^2 b_s^2(R_s^\pi)^2}\psi^{'}(R_s^\pi)\mathbb{1}_{(0,\tau^\pi)}(s)\right\}_{0\leqslant s\leqslant t}$$

is a continuous adapted process which is progressively measurable.

Meanwhile, we have

$$\int_0^t \left\{\sqrt{\mathbb{E}(Y\wedge d_s)^2\lambda+\sigma_1^2 b_s^2(R_s^\pi)^2}\psi^{'}(R_s^\pi)\mathbb{1}_{(0,\tau^\pi)}(s)\right\}^2 d_s < \infty.$$

Therefore, by Theorem 2.2.1, we obtain the result. □

Theorem 4.3.3. *Assume that $\psi(x)$ defined by (4.2.3) is a twice continuously differentiable decreasing convex function on $(0, x^*)$, then $\psi(x)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation*

$$0 = \inf_{0<b<\infty, 0\leqslant d<\infty}\{[\theta\mathbb{E}(Y\wedge d)\lambda-(\theta-\eta)\mathbb{E}(Y)\lambda+r(1-b)x+b\mu_1 x]\psi^{'}(x)$$

$$+\frac{1}{2}[\mathbb{E}(Y\wedge d)^2\lambda+b^2\sigma_1^2x^2]\psi^{''}(x)\}, \tag{4.3.2}$$

with boundary conditions $\psi(0)=1$ and $\psi(x^*)=0$.

Proof. Suppose $\pi^{'}$ is an arbitrary strategy. Since $R_0^{\pi^{'}} = x > 0$ and $\tau^{\pi^{'}} = \inf\{t > 0 : R_t^{\pi^{'}} \leqslant 0\}$, we have $\tau^{\pi^{'}} > 0$. Therefore, we can partition the time

interval $(0, \tau^{\pi'}]$ into $(0, h]$ and $(h, \tau^{\pi'}]$. Let $\pi = \{(b_t, d_t)\}_{0<t\leqslant\tau^{\pi}}$ be a strategy that implements an arbitrary strategy for $t \in (0, h]$ and then follows the optimal strategy for $t \in (h, \tau^{\pi}]$. Since $\psi(x)$ is a twice continuously differentiable function, by Itô Formula, we have

$$\begin{aligned}
&\psi(R_h^\pi) - \psi(x) \\
=&\int_0^h [\theta\lambda\mathbb{E}(Y \wedge d_s) - (\theta-\eta)\lambda\mathbb{E}(Y) + r(1-b_s)R_s^\pi + b_s\mu_1 R_s^\pi]\psi^{'}(R_s^\pi\ )ds \\
&+ \int_0^h \frac{1}{2}\big[\mathbb{E}(Y \wedge d_s)^2\lambda + (\sigma_1 b_s R_s^\pi)^2\big]\psi^{''}(R_s^\pi)ds \\
&+ \int_0^h \Big[\sqrt{\mathbb{E}(Y \wedge d_s)^2\lambda + (\sigma_1 b_s R_s^\pi)^2}\Big]\psi^{'}(R_s^\pi)dB_s.
\end{aligned}$$

Taking expectations on both sides, we have

$$\begin{aligned}
&\mathbb{E}\Big(\psi(R_h^\pi)\Big) - \psi(x) \\
=&\mathbb{E}\int_0^h \Big[\Big(\theta\lambda\mathbb{E}(Y \wedge d_s) - (\theta-\eta)\lambda\mathbb{E}(Y) + r(1-b_s)R_s^\pi + b_s\mu_1 R_s^\pi\Big)\psi^{'}(R_s^\pi) \\
&+ \frac{1}{2}\Big(E(Y \wedge d_s)^2\lambda + (\sigma_1 b_s R_s^\pi)^2\Big)\psi^{''}(R_s^\pi)\Big]ds \\
&+ \mathbb{E}\int_0^h \Big[\sqrt{\mathbb{E}(Y \wedge d_s)^2\lambda + (\sigma_1 b_s R_s^\pi)^2}\Big]\psi^{'}(R_s^\pi)dB_s.
\end{aligned}$$

By Lemma 4.3.2, we have

$$\mathbb{E}\int_0^h \Big[\sqrt{\mathbb{E}(Y \wedge d_s)^2\lambda + (\sigma_1 b_s R_s^\pi)^2}\Big]\psi^{'}(R_s^\pi)dB_s = 0.$$

Adding $\int_0^h c(R_s^\pi, \pi_s)ds$ on both sides, we have

$$\psi^\pi(x) - \psi(x) = \mathbb{E}\Big[\int_0^h \Big(W_s^\pi + c(R_s^\pi, \pi_s)\Big)ds\Big],$$

where

$$W_s^\pi = \Big[\Big(\theta\lambda\mathbb{E}(Y\wedge d_s) - (\theta-\eta)\lambda\mathbb{E}(Y) + r(1-b_s)R_s^\pi + b_s\mu_1 R_s^\pi\Big)\psi^{'}(R_s^\pi) + \frac{1}{2}\Big(\mathbb{E}(Y\wedge d_s)^2\lambda + (\sigma_1 b_s R_s^\pi)^2\Big)\psi^{''}(R_s^\pi)\Big].$$

Since an arbitrary strategy for $t \in (0,h]$ is taken under π, then we have

$$0 \leqslant \mathbb{E}\Big[\int_0^h \Big(W_s^\pi + c(R_s^\pi, \pi_s)\Big)ds\Big].$$

The inequality is true for any Markov control $\{\pi_s\}_{0<s\leqslant h}$ and the equality holds if $\{\pi_s\}_{0<s\leqslant h}$ is the optimal control. Divided by h on both sides and letting $h \to 0$, we have

$$0 \leqslant \lim_{h\to 0}\mathbb{E}\Big[\frac{1}{h}\int_0^h \Big(W_s^\pi + c(R_s^\pi, \pi_s)\Big)ds\Big].$$

Since the random variable $\frac{1}{h}\int_0^h \Big(W_s^\pi + c(R_s^\pi, \pi_s)\Big)ds$ is uniformly bounded, we can interchange the limit and the expectation operator according to the Dominated Convergence Theorem. Thus, we have

$$0 \leqslant \big[\theta\lambda\mathbb{E}(Y\wedge d) - (\theta-\eta)\lambda\mathbb{E}(Y) + r(1-b)x + b\mu_1 x\big]\psi^{'}(x) + \frac{1}{2}\big[\mathbb{E}(Y\wedge d)^2\lambda + \sigma_1^2 b^2 x^2\big]\psi^{''}(x).$$

The joint decisions (b,d) can be thought as the decision at time 0^+ with state $R_{0^+}^\pi = x$. This inequality holds for all joint decisions (b,d) at time 0^+ with state x and the equality holds if the joint decisions (b,d) is the optimal decision at time 0^+ with state x. Therefore, we obtain our desired HJB equation

$$0 = \inf_{0<b<\infty, 0\leqslant d<\infty}\Big\{\big[\theta\lambda\mathbb{E}(Y\wedge d) - (\theta-\eta)\lambda\mathbb{E}(Y) + r(1-b)x + b\mu_1 x\big]\psi^{'}(x) + \frac{1}{2}\big[\mathbb{E}(Y\wedge d)^2\lambda + \sigma_1^2 b^2 x^2\big]\psi^{''}(x)\Big\}.$$

□

Theorem 4.3.3 provides a way to find the value function and the corresponding optimal strategy. However, in order to derive the HJB equation, we made several assumptions which need to be verified. Also, it may occur that a solution to the HJB equation is not really the desired value function. Thus, we have to show that the solution found from the corresponding HJB equation is the desired value function, which can be dealt with by a technique known as verification.

Generally speaking, the verification technique works backwards. Suppose we have found a solution according to the HJB equation, then we show that this solution coincides with the desired value function. At the same time, a guess for the optimal control is provided. We need to verify that this control is indeed optimal.

Theorem 4.3.4. ***(Verification Theorem)*** *Assume that $V(x)$ is a twice continuously differentiable decreasing convex function on $(0, x^*)$ and satisfies the following Hamilton-Jacobi-Bellman (HJB) equation*

$$0 = \inf_{0<b<\infty, 0\leqslant d<\infty} \{[\theta\mathbb{E}(Y \wedge d)\lambda - (\theta-\eta)\mathbb{E}(Y)\lambda + r(1-b)x + b\mu_1 x]V^{'}(x) + \frac{1}{2}[\mathbb{E}(Y \wedge d)^2\lambda + b^2\sigma_1^2 x^2]V^{''}(x)\} \tag{4.3.3}$$

with boundary conditions $V(0) = 1$ and $V(x^*) = 0$, then the value function $\psi(x)$ defined by (4.2.3) coincides with $V(x)$. Furthermore, the strategy $\pi^* = (b^*, d^*)$ such that

$$\begin{aligned} 0 =& \{[\theta\mathbb{E}(Y \wedge d^*)\lambda - (\theta-\eta)\mathbb{E}(Y)\lambda + r(1-b^*)x + b^*\mu_1 x]V^{'}(x) \\ &+ \frac{1}{2}[\mathbb{E}(Y \wedge d^*)^2\lambda + (b^*)^2\sigma_1^2 x^2]V^{''}(x)\}, \end{aligned} \tag{4.3.4}$$

for every fixed $x \in (0, x^*)$ is the corresponding optimal strategy.

Proof. Suppose $R_0 = x$ and π is an arbitrary strategy. By Itô Formula,

$$\begin{aligned}
&V(R^{\pi}_{t\wedge\tau^{\pi}}) - V(x) \\
&= \int_0^{t\wedge\tau^{\pi}} \Big[\theta\lambda\mathbb{E}(Y\wedge d_s) - (\theta-\eta)\lambda\mathbb{E}(Y) + \big(r(1-b_s) + b_s\mu_1\big)R^{\pi}_s\Big]V^{'}(R^{\pi}_s)ds \\
&\quad + \int_0^{t\wedge\tau^{\pi}} \frac{1}{2}\big[\mathbb{E}(Y\wedge d_s)^2\lambda + \sigma_1^2 b_s^2 (R^{\pi}_s)^2\big]V^{''}(R^{\pi}_s)ds \\
&\quad + \int_0^{t\wedge\tau^{\pi}} \Big[\sqrt{\mathbb{E}(Y\wedge d_s)^2\lambda + \sigma_1^2 b_s^2 (R^{\pi}_s)^2}\Big]V^{'}(R^{\pi}_s)dB_s \\
&\geqslant \int_0^{t\wedge\tau^{\pi}} \Big[\sqrt{\mathbb{E}(Y\wedge d_s)^2\lambda + \sigma_1^2 b_s^2 (R^{\pi}_s)^2}\Big]V^{'}(R^{\pi}_s)dB_s.
\end{aligned}$$

Note that the inequality follows from equation (4.3.3). Taking expected values on both sides, we obtain

$$\mathbb{E}\Big(V(R^{\pi}_{t\wedge\tau^{\pi}})\Big) - V(x) \geqslant \mathbb{E}\int_0^{t\wedge\tau^{\pi}} \Big[\sqrt{E(Y\wedge d_s)^2\lambda + \sigma_1^2 b_s^2 (R^{\pi}_s)^2}\Big]V^{'}(R^{\pi}_s)dB_s.$$

By Lemma 4.3.2, we have

$$\mathbb{E}\Big(\int_0^{t\wedge\tau^{\pi}} \Big[\sqrt{E(Y\wedge d_s)^2\lambda + \sigma_1^2 b_s^2 (R^{\pi}_s)^2}\Big]V^{'}(R^{\pi}_s)dB_s\Big) = 0.$$

It follows that

$$\mathbb{E}\Big(V(R^{\pi}_{t\wedge\tau^{\pi}})\Big) \geqslant V(x).$$

The above inequality shows that $\{V(R^{\pi}_{t\wedge\tau^{\pi}})\}_{t\geqslant 0}$ is a submartingale. Letting $t\to\infty$, we have $\mathbb{E}\Big(V(R^{\pi}_{\tau^{\pi}})\Big) \geqslant V(x)$. Since $R^{\pi}_{\tau^{\pi}} = 0$ and $V(0) = 1$, then we have

$$\mathbb{E}\Big(V(R^{\pi}_{\tau^{\pi}})\Big) = \mathbb{E}\big(\mathbb{1}_{(0,\infty)}(\tau^{\pi})\big) = \psi^{\pi}(x) \geqslant V(x).$$

Since π is an arbitrary strategy, we have

$$\inf_{\pi\in\Pi}\psi^{\pi}(x) \geqslant V(x). \tag{4.3.5}$$

Suppose we implement the optimal strategy π^* with $R_0 = x$.
By Itô Formula,

$$
\begin{aligned}
&V(R^{\pi^*}_{t\wedge\tau^{\pi^*}}) - V(x) \\
&= \int_0^{t\wedge\tau^{\pi^*}} [\theta\lambda\mathbb{E}(Y\wedge d^*) - (\theta-\eta)\lambda\mathbb{E}(Y) + r(1-b^*)R^{\pi^*}_s + b^*\mu_1 R^{\pi^*}_s]V^{'}(R^{\pi^*}_s)ds \\
&\quad + \int_0^{t\wedge\tau^{\pi^*}} \frac{1}{2}\big[\mathbb{E}(Y\wedge d^*)^2\lambda + \sigma_1^2(b^*_s R^{\pi^*}_s)^2\big]V^{''}(R^{\pi^*}_s)ds \\
&\quad + \int_0^{t\wedge\tau^{\pi^*}} \Big[\sqrt{\mathbb{E}(Y\wedge d^*_s)^2\lambda + \sigma_1^2(b^*_s R^{\pi^*}_s)^2}\Big]V^{'}(R^{\pi^*}_s)dB_s.
\end{aligned}
$$

According to equation (4.3.4), it follows that

$$
V(R^{\pi^*}_{t\wedge\tau^{\pi^*}}) - V(x) = \int_0^{t\wedge\tau^{\pi^*}} \Big[\sqrt{\mathbb{E}(Y\wedge d^*_s)^2\lambda + \sigma_1^2(b^*_s R^{\pi^*}_s)^2}\Big]V^{'}(R^{\pi^*}_s)dB_s.
$$

Taking expected values on both sides, we have

$$
\mathbb{E}\Big(V(R^{\pi}_{t\wedge\tau^{\pi}})\Big) = V(x).
$$

This shows that $\{V(R^{\pi^*}_{t\wedge\tau^{\pi^*}})\}_{t\geqslant 0}$ is a martingale.
Letting $t\to\infty$, we have

$$
\psi^{\pi^*}(x) = V(x).
$$

Now we have demonstrated that the value function $\psi(x)$ defined by equation (4.2.3) coincides with $V(x)$ and π^* is the optimal strategy. □

4.4 Joint investment and reinsurance decisions

In the HJB equation (4.3.2), since r takes value α if $0 < b \leqslant 1$, or value β if $b > 1$, our objective is to obtain the minimum ruin probability and the corresponding optimal strategies for the two cases.

In this section, we derive the HJB equations for $0 < b < 1$, $b > 1$ and $b = 1$. Then we find the solution of each HJB equation. Finally, we use verification theorem to obtain the minimum ruin probability and determine its associated optimal joint investment and reinsurance decisions (b^*, d^*) for any given reserve $x \in (0, x^*)$.

4.4.1 Jointly optimal decisions when $0 < b < 1$

To determine jointly optimal decisions of investment and reinsurance when $0 < b < 1$, we take $r = \alpha$ and the HJB equation (4.3.2) becomes

$$0 = \inf_{0<b<\infty, 0\leqslant d<\infty} \Big\{ \big[\theta\mathbb{E}(Y \wedge d)\lambda - (\theta - \eta)\mathbb{E}(Y)\lambda + \alpha((1-b)x) + b\mu_1 x\big]\psi^{'}(x) + \frac{1}{2}\big[\mathbb{E}(Y \wedge d)^2\lambda + b^2\sigma_1^2 x^2\big]\psi^{''}(x)\Big\}. \tag{4.4.1}$$

Lemma 4.4.1. *Suppose $\psi(x)$ is a twice continuously differentiable decreasing convex function on $(0, x^*)$, then $b^*_{\alpha,x} = \frac{\mu_1-\alpha}{\sigma_1^2\theta x} d^*_{\alpha,x}$ and $d^*_{\alpha,x} = -\theta\frac{\psi^{'}(x)}{\psi^{''}(x)}$ jointly minimize the function*

$$\begin{aligned} h(b,d) =& [\theta\mathbb{E}(Y \wedge d)\lambda - (\theta - \eta)\mathbb{E}(Y)\lambda + \alpha(1-b)x + b\mu_1 x]\psi^{'}(x) \\ &+ \frac{1}{2}[\mathbb{E}(Y \wedge d)^2\lambda + b^2\sigma_1^2 x^2]\psi^{''}(x). \end{aligned}$$

Moreover, $d^*_{\alpha,x}$ is the solution of

$$\theta\lambda\mathbb{E}(Y \wedge d^*_{\alpha,x}) - \frac{1}{2d^*_{\alpha,x}}\theta\lambda\mathbb{E}(Y \wedge d^*_{\alpha,x})^2 + \frac{(\mu_1-\alpha)^2}{\sigma_1^2}\frac{d^*_{\alpha,x}}{2\theta} = (\theta - \eta)\mathbb{E}(Y)\lambda - \alpha x. \tag{4.4.2}$$

Proof. Recall that $\mathbb{E}(Y \wedge d) = \int_0^d S(y)dy$ and $\mathbb{E}(Y \wedge d)^2 = \int_0^d 2yS(y)dy$, where $S(y)$ is the survival function of Y. Suppose $d^*_{\alpha,x}$ and $b^*_{\alpha,x}$ jointly minimize the

function

$$h(b,d) = \big[\theta\mathbb{E}(Y\wedge d)\lambda - (\theta-\eta)\mathbb{E}(Y)\lambda + \alpha(1-b)x + b\mu_1 x\big]\psi^{'}(x) + \frac{1}{2}\big[\mathbb{E}(Y\wedge d)^2\lambda + b^2\sigma_1^2x^2\big]\psi^{''}(x).$$

Setting partial derivatives of h with respect to b and d to 0, we have

$$\begin{cases} \frac{\partial}{\partial b}h(b,d) = (\mu_1-\alpha)x\psi^{'}(x) + b\sigma_1^2x^2\psi^{''}(x) = 0, \\ \frac{\partial}{\partial d}h(b,d) = \theta\lambda\psi^{'}(x)S(d) + \lambda\psi^{''}(x)dS(d) = 0. \end{cases}$$

This gives us $b^*_{\alpha,x} = \frac{\mu_1-\alpha}{\sigma_1^2\theta x}d^*_{\alpha,x}$ and $d^*_{\alpha,x} = -\theta\frac{\psi^{'}(x)}{\psi^{''}(x)}$. To verify that $b^*_{\alpha,x}$ and $d^*_{\alpha,x}$ jointly minimize $h(b,d)$, we apply the second derivative test

$$\begin{cases} \frac{\partial^2}{\partial b^2}h(b,d) = \sigma_1^2x^2\psi^{''}(x), \\ \frac{\partial^2}{\partial d^2}h(b,d) = -\theta\lambda\psi^{'}(x)f(d) + \lambda\psi^{''}(x)[S(d) - df(d)], \\ \frac{\partial^2}{\partial b\partial d}h(b,d) = 0. \end{cases}$$

When $b = b^*_{\alpha,x}$ and $d = d^*_{\alpha,x}$, we have $\frac{\partial^2}{\partial b^2}h(b,d)\frac{\partial^2}{\partial d^2}h(b,d) - (\frac{\partial^2}{\partial b\partial d}h(b,d))^2 > 0$. Therefore, $b^*_{\alpha,x} = \frac{\mu_1-\alpha}{\sigma_1^2\theta x}d^*_{\alpha,x}$ and $d^*_{\alpha,x} = -\theta\frac{\psi^{'}(x)}{\psi^{''}(x)}$ minimize $h(b,d)$, globally.

Since $\psi(x)$ is a twice continuously differentiable decreasing convex function on $(0,x^*)$, by Theorem 4.3.3, we have

$$0 = \big[\theta\mathbb{E}(Y\wedge d^*_{\alpha,x})\lambda - (\theta-\eta)\mathbb{E}(Y)\lambda + \frac{(\mu_1-\alpha)^2}{\sigma_1^2}\frac{d^*_{\alpha,x}}{\theta} + \alpha x\big]\psi^{'}(x) + \frac{1}{2}\big[\mathbb{E}(Y\wedge d^*_{\alpha,x})^2\lambda + \frac{(\mu_1-\alpha)^2}{\sigma_1^2}\frac{d^{*2}_{\alpha,x}}{\theta^2}\big]\psi^{''}(x).$$

Dividing by $\psi^{'}(x)$ on both sides, we have

$$\theta\lambda\mathbb{E}(Y\wedge d^*_{\alpha,x}) - \frac{1}{2d^*_{\alpha,x}}\theta\lambda\mathbb{E}(Y\wedge d^*_{\alpha,x})^2 + \frac{(\mu_1-\alpha)^2}{\sigma_1^2}\frac{d^*_{\alpha,x}}{2\theta} = (\theta-\eta)\mathbb{E}(Y)\lambda - \alpha x.$$

□

Lemma 4.4.2. *The function* $g_\alpha(d) = \theta\lambda\mathbb{E}(Y \wedge d) - \frac{1}{2d}\theta\lambda\mathbb{E}(Y \wedge d)^2 + \frac{(\mu_1-\alpha)^2}{\sigma_1^2}\frac{d}{2\theta}$ *increases as* d *increases.*

Proof. This is evident from $\frac{\partial g_\alpha(d)}{\partial d} = \frac{\theta\lambda}{2d^2}\mathbb{E}(Y \wedge d)^2 + \frac{(\mu_1-\alpha)^2}{\sigma_1^2}\frac{1}{2\theta} > 0.$ □

Lemma 4.4.3. *For any given reserve* $x \in (0, x^*)$*, the optimal decision* $d^*_{\alpha,x}$ *exists and is unique.*

Proof. When $x \in (0, x^*)$, we have $(\theta - \eta)\mathbb{E}(Y)\lambda - \alpha x > 0$. Now, $g_\alpha(d)$ is increasing and continuous in d. Moreover, we have $g_\alpha(0) = 0 < (\theta-\eta)\mathbb{E}(Y)\lambda - \alpha x$ and $g_\alpha(\infty) = \infty > (\theta - \eta)\mathbb{E}(Y)\lambda - \alpha x$, for any $x \in (0, x^*)$. Hence $d^*_{\alpha,x} > 0$ exists and is unique. □

Equation (4.4.2) says that $k = g_\alpha(d^*_{\alpha,x}) = (\theta - \eta)\mathbb{E}(Y)\lambda - \alpha x$ is decreasing in x. But $d^*_{\alpha,x} = g_\alpha^{-1}(k)$ is increasing in k, so $d^*_{\alpha,x}$ is decreasing in $x \in (0, x^*)$. At x^*, we have $k = 0$. Moreover, $g_\alpha(0) = 0$. Given that $g_\alpha(d)$ is increasing and continuous in d, there is a solution x^*_α such that $\frac{\mu_1-\alpha}{\sigma_1^2\theta}d^*_{\alpha,x^*_\alpha} = x^*_\alpha$. Furthermore, $0 < x^*_\alpha < x^*$.

Lemma 4.4.4. *When* $x \in (x^*_\alpha, x^*)$*, we have* $0 < b < 1$*. Therefore, we may invest in the savings account as well as in the risky asset.*

Proof. If $x^*_\alpha < x < x^*$, then $\frac{\mu_1-\alpha}{\sigma_1^2\theta}d^*_{\alpha,x} < \frac{\mu_1-\alpha}{\sigma_1^2\theta}d^*_{\alpha,x^*_\alpha} = x^*_\alpha < x$ and so $b^*_{\alpha,x} < 1$. □

Lemma 4.4.5. *Suppose* $\psi(x)$ *is a twice continuously differentiable decreasing convex function on* $(0, x^*)$*. If* $\psi(x)$ *solves*

$$\psi'(x) + \frac{d^*_{\alpha,x}}{\theta}\psi''(x) = 0,$$

then $\psi(x)$ *solves the HJB equation (4.3.2) for* $x \in (x^*_\alpha, x^*)$*.*

Proof. When $b^*_{\alpha,x} = \frac{\mu_1-\alpha}{\sigma_1^2\theta x} d^*_{\alpha,x}$ and $d^*_{\alpha,x} = -\theta\frac{\psi^{'}(x)}{\psi^{''}(x)}$, we simplify the equation (4.3.2) and obtain the result. □

4.4.2 Jointly optimal decisions when $b > 1$

When borrowing is allowed, we take $r = \beta$ and the HJB equation (4.3.2) becomes

$$0 = \inf_{0<b<\infty, 0\leqslant d<\infty} \Big\{ \big[\theta\mathbb{E}(Y\wedge d)\lambda - (\theta-\eta)\mathbb{E}(Y)\lambda + \beta((1-b)x) + b\mu_1 x\big]\psi^{'}(x) + \frac{1}{2}\big[\mathbb{E}(Y\wedge d)^2\lambda + b^2\sigma_1^2x^2\big]\psi^{''}(x)\Big\}.$$

Let $d^*_{\beta,x}$ and $b^*_{\beta,x}$ be values of d and b that jointly minimize the function

$$k(b,d) = \big[\theta\mathbb{E}(Y\wedge d)\lambda - (\theta-\eta)\mathbb{E}(Y)\lambda + \beta(1-b)x + b\mu_1 x\big]\psi^{'}(x) + \frac{1}{2}\big[\mathbb{E}(Y\wedge d)^2\lambda + b^2\sigma_1^2x^2\big]\psi^{''}(x).$$

Using ideas similar to that in Section 4.4.1, the solutions $b^*_{\beta,x} = \frac{\mu_1-\beta}{\sigma_1^2\theta x} d^*_{\beta,x}$ and $d^*_{\beta,x} = -\theta\frac{\psi^{'}(x)}{\psi^{''}(x)}$ exist and are unique. Furthermore, $d^*_{\beta,x}$ decreases as x increases. Suppose $\psi(x)$ is a twice continuously differentiable decreasing convex function on $(0, x^*)$, then $d^*_{\beta,x}$ is the solution of

$$\theta\lambda\mathbb{E}(Y\wedge d^*_{\beta,x}) - \frac{1}{2d^*_{\beta,x}}\theta\lambda\mathbb{E}(Y\wedge d^*_{\beta,x})^2 + \frac{(\mu_1-\beta)^2}{\sigma_1^2}\frac{d^*_{\beta,x}}{2\theta} = (\theta-\eta)\mathbb{E}(Y)\lambda - \beta x. \tag{4.4.3}$$

Define $g_\beta(d) = \theta\lambda\mathbb{E}(Y\wedge d) - \frac{1}{2d}\theta\lambda\mathbb{E}(Y\wedge d)^2 + \frac{(\mu_1-\beta)^2}{\sigma_1^2}\frac{d}{2\theta}$. Then it increases as d increases. Let $g_\beta(d) = l$. The $d = g_\beta^{-1}(l)$ exists and is increasing in l. Since $l = (\theta-\eta)\mathbb{E}(Y)\lambda - \beta x$ is decreasing in x when $d = d^*_{\beta,x}$, the function $d^*_{\beta,x} = g_\beta^{-1}(l)$ is decreasing in x.

Define $x^{**} = \frac{(\theta-\eta)E(Y)\lambda}{\beta}$. Then $x^{**} < x^*$ and at $x = x^{**}$, we have $l = 0$ and hence $d^*_{\beta,x^{**}} = 0$. When $x = 0$, we have $d^*_{\beta,0} > 0$. Therefore, there is an unique solution x^*_β such that $\frac{\mu_1-\beta}{\sigma_1^2\theta} d^*_{\beta,x^*_\beta} = x^*_\beta$. Furthermore, $0 < x^*_\beta < x^*$.

Lemma 4.4.6. *When $x \in (0, x^*_\beta)$, we have $b > 1$. Therefore, we may borrow and invest all the reserve in the risky asset.*

Proof. If $0 < x < x^*_\beta$, then $\frac{\mu_1-\beta}{\sigma^2\theta} d^*_{\beta,x} > \frac{\mu_1-\beta}{\sigma^2\theta} d^*_{\beta,x^*_\beta} = x^*_\beta > x$. So $b^*_{\alpha,x} > 1$. □

Lemma 4.4.7. *Suppose $\psi(x)$ is a twice continuously differentiable decreasing convex function on $(0, x^*)$. If $\psi(x)$ solves*

$$\psi^{'}(x) + \frac{d^*_{\beta,x}}{\theta}\psi^{''}(x) = 0,$$

then $\psi(x)$ solves the HJB equation (4.3.2) for $x \in (0, x^*_\beta)$.

Proof. When $b^*_{\beta,x} = \frac{\mu_1-\beta}{\sigma_1^2\theta x} d^*_{\beta,x}$ and $d^*_{\beta,x} = -\theta\frac{\psi^{'}(x)}{\psi^{''}(x)}$, we simplify the equation (4.3.2) and obtain the result. □

4.4.3 Jointly optimal decisions when $b = 1$

When $b = 1$, we take $r = \alpha$ and the HJB equation (4.3.2) becomes

$$0 = \inf_{0<b<\infty}\Big\{\big[\theta\mathbb{E}(Y\wedge d)\lambda - (\theta-\eta)\mathbb{E}(Y)\lambda + \mu_1 x\big]\psi^{'}(x) + \frac{1}{2}\big[\mathbb{E}(Y\wedge d)^2\lambda + \sigma_1^2x^2\big]\psi^{''}(x)\Big\}.$$

We can easily check that $d^*_{\alpha,\beta,x} = -\theta\frac{\psi^{'}(x)}{\psi^{''}(x)}$ minimizes the function

$$\big[\theta\mathbb{E}(Y\wedge d)\lambda - (\theta-\eta)\mathbb{E}(Y)\lambda + \mu_1 x\big]\psi^{'}(x) + \frac{1}{2}\big[\mathbb{E}(Y\wedge d)^2\lambda + \sigma_1^2x^2\big]\psi^{''}(x).$$

Suppose $\psi(x)$ is a twice continuously differentiable decreasing convex function on $(0, x^*)$, then $d^*_{\alpha,\beta,x}$ is the solution of

$$\theta\lambda\mathbb{E}(Y \wedge d^*_{\alpha,\beta,x}) - \frac{\theta}{2d^*_{\alpha,\beta,x}}\lambda\mathbb{E}(Y \wedge d^*_{\alpha,\beta,x})^2 - \frac{\theta}{2d^*_{\alpha,\beta,x}}\sigma_1^2 x^2 = (\theta - \eta)\mathbb{E}(Y)\lambda - \mu_1 x. \tag{4.4.4}$$

Let $g_{\alpha,\beta}(d) = \theta\lambda\mathbb{E}(Y \wedge d) - \frac{\theta}{2d}\lambda\mathbb{E}(Y \wedge d)^2 - \frac{\theta}{2d}\sigma_1^2 x^2$. Then it is increasing in d. Moreover, $g_{\alpha,\beta}(0) = -\infty < (\theta - \eta)\mathbb{E}(Y)\lambda - \mu_1 x$ and $g_{\alpha,\beta}(\infty) = \theta\lambda\mathbb{E}(Y) > (\theta - \eta)\mathbb{E}(Y)\lambda - \mu_1 x$ for all $x \in (0, x^*)$. Therefore, $d^*_{\alpha,\beta,x}$ exists and is unique for any $x \in (0, x^*)$.

For any given $x \in (0, x^*)$, we have shown the existence of two critical values x^*_α and x^*_β for jointly optimal decisions. From Lemma 4.4.4 and Lemma 4.4.6, we have $b \geqslant 1$ for $x \in (0, x^*_\alpha]$ and $b \leqslant 1$ for $x \in [x^*_\beta, x^*)$ respectively. Therefore, we have $b = 1$ for $x \in \big((0, x^*_\alpha] \cap [x^*_\beta, x^*)\big)$. Now,

Lemma 4.4.8. *For any given $x \in (0, x^*)$, we have $x^*_\alpha > x^*_\beta$.*

Proof. Suppose $x^*_\alpha \leqslant x^*_\beta$. Then $\frac{\mu_1-\alpha}{\sigma_1^2\theta} d^*_{\alpha,x^*_\alpha} \leqslant \frac{\mu_1-\beta}{\sigma_1^2\theta} d^*_{\beta,x^*_\beta}$. Since $\alpha < \beta$, we have $d^*_{\beta,x^*_\beta} > d^*_{\alpha,x^*_\alpha}$. But $d^*_{\alpha,x}$ is decreasing in x and $x^*_\alpha \leqslant x^*_\beta$, so $d^*_{\alpha,x^*_\alpha} \geqslant d^*_{\alpha,x^*_\beta}$. Putting together, we have $d^*_{\beta,x^*_\beta} > d^*_{\alpha,x^*_\beta}$.

However, from Equation (4.4.3) we see that $d^*_{\beta,x}$ is decreasing in β for any given x, so $\alpha < \beta$ implies that $x^*_{\beta,x^*_\beta} < x^*_{\alpha,x^*_\beta}$. This creates a contradiction. □

Thus, we have $b = 1$ for $x \in [x^*_\beta, x^*_\alpha]$.

Lemma 4.4.9. *Suppose $\psi(x)$ is a twice continuously differentiable decreasing convex function on $(0, x^*)$. If $\psi(x)$ solves*

$$\psi'(x) + \frac{d^*_{\alpha,\beta,x}}{\theta}\psi''(x) = 0,$$

then $\psi(x)$ solves the HJB equation (4.3.2) for $x \in [x^*_\beta, x^*_\alpha]$.

Proof. When $d^*_{\alpha,\beta,x} = -\theta\frac{\psi^{'}(x)}{\psi^{''}(x)}$, we simplify the equation (4.3.2) and obtain the result. □

4.4.4 Summary of results

Define the sets

$$\begin{aligned}\mathcal{A}_x =&\{x : x^*_\alpha < x < x^*\},\\ \mathcal{B}_x =&\{x : x^*_\beta \leqslant x \leqslant x^*_\alpha\},\\ \mathcal{C}_x =&\{x : 0 < x < x^*_\beta\}.\end{aligned}$$

It is easy to see that $\{\mathcal{A}_x, \mathcal{B}_x, \mathcal{C}_x\}$ is partition of the interval $(0, x^*)$.

Theorem 4.4.10. *Assume that $V(x)$ is a twice continuously differentiable decreasing convex function on $(0, x^*)$ and satisfies*

$$V^{'}(x) + \frac{d^*_{\beta,x}}{\theta}V^{''}(x) = 0 \quad for \quad x \in (0, x^*_\beta), \tag{4.4.5}$$

$$V^{'}(x) + \frac{d^*_{\alpha,\beta,x}}{\theta}V^{''}(x) = 0 \quad for \quad x \in [x^*_\beta, x^*_\alpha], \tag{4.4.6}$$

$$V^{'}(x) + \frac{d^*_{\alpha,x}}{\theta}V^{''}(x) = 0 \quad for \quad x \in (x^*_\alpha, x^*), \tag{4.4.7}$$

with boundary conditions $V(0) = 1$ and $V(x^*) = 0$, then the value function $\psi(x)$ defined by (4.2.3) coincides with $V(x)$. Furthermore, the strategy (b^*_x, d^*_x) such that

$$(b^*_x, d^*_x) = \begin{cases}(b^*_{\beta,x}, d^*_{\beta,x}) & for \quad x \in (0, x^*_\beta),\\ (1, d^*_{\alpha,\beta,x}) & for \quad x \in [x^*_\beta, x^*_\alpha],\\ (b^*_{\alpha,x}, d^*_{\alpha,x}) & for \quad x \in (x^*_\alpha, x^*),\end{cases}$$

for every fixed x is the corresponding optimal strategy.

Proof. This theorem immediately follows the Lemma 4.4.5, Lemma 4.4.7, Lemma 4.4.9 and Theorem 4.3.4. □

Theorem 4.4.11. *Given any reserve $x \in (0, x^*)$, the value function $\psi(x)$ defined by (4.2.3) is*

$$\psi(x) = \begin{cases} 1 - \frac{H_\beta(x)}{H_\beta(x^*_\beta)+h_\beta(x^*_\beta)H_{\alpha,\beta}(x^*_\alpha)+h_\beta(x^*_\beta)h_{\alpha,\beta}(x^*_\alpha)H_\alpha(x^*)} & for \quad x \in (0, x^*_\beta), \\ 1 - \frac{H_\beta(x^*_\beta)+h_\beta(x^*_\beta)H_{\alpha,\beta}(x)}{H_\beta(x^*_\beta)+h_\beta(x^*_\beta)H_{\alpha,\beta}(x^*_\alpha)+h_\beta(x^*_\beta)h_{\alpha,\beta}(x^*_\alpha)H_\alpha(x^*)} & for \quad x \in [x^*_\beta, x^*_\alpha], \\ 1 - \frac{H_\beta(x^*_\beta)+h_\beta(x^*_\beta)H_{\alpha,\beta}(x^*_\alpha)+h_\beta(x^*_\beta)h_{\alpha,\beta}(x^*_\alpha)H_\alpha(x)}{H_\beta(x^*_\beta)+h_\beta(x^*_\beta)H_{\alpha,\beta}(x^*_\alpha)+h_\beta(x^*_\beta)h_{\alpha,\beta}(x^*_\alpha)H_\alpha(x^*)} & for \quad x \in (x^*_\alpha, x^*), \end{cases}$$

where

$$\begin{cases} H_\beta(x) = \int_0^x exp\{-\int_0^u \frac{\theta}{d^*_{\beta,v}}dv\}du, \\ H_{\alpha,\beta}(x) = \int_{x^*_\beta}^x exp\{-\int_{x^*_\beta}^u \frac{\theta}{d^*_{\alpha,\beta,v}}dv\}du, \\ H_\alpha(x) = \int_{x^*_\alpha}^x exp\{-\int_{x^*_\alpha}^u \frac{\theta}{d^*_{\alpha,v}}dv\}du, \\ h_\beta(x) = exp\{-\int_0^x \frac{\theta}{d^*_{\beta,v}}dv\}, \\ h_{\alpha,\beta}(x) = exp\{-\int_{x^*_\beta}^x \frac{\theta}{d^*_{\alpha,\beta,v}}dv\}, \\ h_\alpha(x) = exp\{-\int_{x^*_\alpha}^x \frac{\theta}{d^*_{\alpha,v}}dv\}. \end{cases}$$

The corresponding optimal investment and reinsurance decisions (b^*_x, d^*_x) are given by

$$(d^*_x, b^*_x) = \begin{cases} (d^*_{\beta,x}, b^*_{\beta,x}) & for \quad x \in (0, x^*_\beta), \\ (d^*_{\alpha,\beta,x}, 1) & for \quad x \in [x^*_\beta, x^*_\alpha], \\ (d^*_{\alpha,x}, b^*_{\alpha,x}) & for \quad x \in (x^*_\alpha, x^*). \end{cases}$$

Proof. By Theorem 4.4.10, our objective is to find the function $V(x)$ which satisfies the assumption of Theorem 4.4.10. From equations (4.4.5), (4.4.6) and (4.4.7), we can easily see that the function $V(x)$ can be written as the

following form

$$V(x)=\begin{cases}-K_1\int_0^x\exp\{-\int_0^u\frac{\theta}{d^*_{\beta,v}}dv\}du+K_2 & \text{for} \quad x\in(0,x^*_\beta),\\ -K_3\int_{x^*_\beta}^x\exp\{-\int_{x^*_\beta}^u\frac{\theta}{d^*_{\alpha,\beta,v}}dv\}du+K_4 & \text{for} \quad x\in[x^*_\beta,x^*_\alpha],\\ -K_5\int_{x^*_\alpha}^x\exp\{-\int_{x^*_\alpha}^u\frac{\theta}{d^*_{\alpha,v}}dv\}du+K_6 & \text{for} \quad x\in(x^*_\alpha,x^*),\end{cases}$$

where K_i, $i=1,2,\cdots,6$ are constants to be determined. Then

$$V^{'}(x)=\begin{cases}-K_1\exp\{-\int_0^x\frac{\theta}{d^*_{\beta,v}}dv\} & \text{for} \quad x\in(0,x^*_\beta),\\ -K_3\exp\{-\int_{x^*_\beta}^x\frac{\theta}{d^*_{\alpha,\beta,v}}dv\} & \text{for} \quad x\in[x^*_\beta,x^*_\alpha],\\ -K_5\exp\{-\int_{x^*_\alpha}^x\frac{\theta}{d^*_{\alpha,v}}dv\} & \text{for} \quad x\in(x^*_\alpha,x^*).\end{cases}$$

Since $V(x)$ is continuous at $x=x^*_\beta$ and $x=x^*_\alpha$ with the boundary assumptions $V(0)=1$ and $V(x^*)=1$, we have

$$\begin{cases}K_2=1,\\ K_4=1-K_1\int_0^{x^*_\beta}\exp\{-\int_0^u\frac{\theta}{d^*_{\beta,v}}dv\}du,\\ K_6=1-K_1\int_0^{x^*_\beta}\exp\{-\int_0^u\frac{\theta}{d^*_{\beta,v}}dv\}du-K_3\int_{x^*_\beta}^{x^*_\alpha}\exp\{-\int_{x^*_\beta}^u\frac{\theta}{d^*_{\alpha,\beta,v}}dv\}du,\\ 1=K_1\int_0^{x^*_\beta}\exp\{-\int_0^u\frac{\theta}{d^*_{\beta,v}}dv\}du+K_3\int_{x^*_\beta}^{x^*_\alpha}\exp\{-\int_{x^*_\beta}^u\frac{\theta}{d^*_{\alpha,\beta,v}}dv\}du\\ \quad +K_5\int_{x^*_\alpha}^{x^*}\exp\{-\int_{x^*_\alpha}^u\frac{\theta}{d^*_{\alpha,v}}dv\}du.\end{cases}$$

Furthermore, $V^{'}(x)$ is continuous at $x=x^*_\beta$ and $x=x^*_\alpha$ as well. Then we have

$$\begin{cases}K_1\exp\{-\int_0^{x^*_\beta}\frac{\theta}{d^*_{\beta,v}}dv\}=K_3,\\ K_3\exp\{-\int_{x^*_\beta}^{x^*_\alpha}\frac{\theta}{d^*_{\alpha,\beta,v}}dv\}=K_5.\end{cases}$$

Therefore, we have

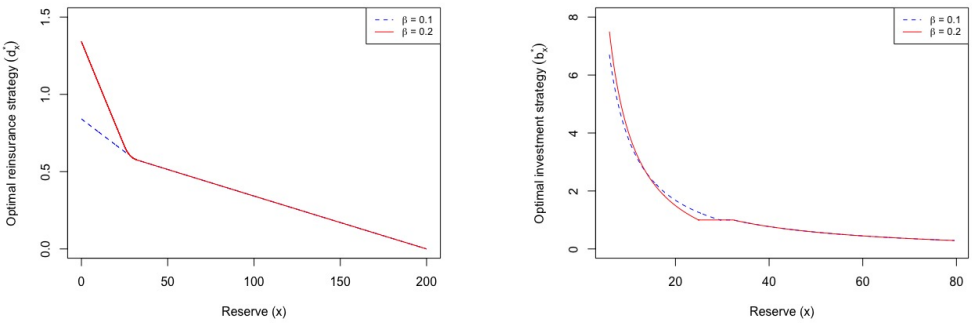
$$\begin{cases} K_1 = \frac{1}{H_\beta(x_\beta^*)+h_\beta(x_\beta^*)H_{\alpha,\beta}(x_\alpha^*)+h_\beta(x_\beta^*)h_{\alpha,\beta}(x_\alpha^*)H_\alpha(x^*)}, \\ K_2 = 1, \\ K_3 = \frac{h_\beta(x_\beta^*)}{H_\beta(x_\beta^*)+h_\beta(x_\beta^*)H_{\alpha,\beta}(x_\alpha^*)+h_\beta(x_\beta^*)h_{\alpha,\beta}(x_\alpha^*)H_\alpha(x^*)}, \\ K_4 = \frac{h_\beta(x_\beta^*)H_{\alpha,\beta}(x_\alpha^*)+h_\beta(x_\beta^*)h_{\alpha,\beta}(x_\alpha^*)H_\alpha(x^*)}{H_\beta(x_\beta^*)+h_\beta(x_\beta^*)H_{\alpha,\beta}(x_\alpha^*)+h_\beta(x_\beta^*)h_{\alpha,\beta}(x_\alpha^*)H_\alpha(x^*)}, \\ K_5 = \frac{h_\beta(x_\beta^*)h_{\alpha,\beta}(x_\alpha^*)}{H_\beta(x_\beta^*)+h_\beta(x_\beta^*)H_{\alpha,\beta}(x_\alpha^*)+h_\beta(x_\beta^*)h_{\alpha,\beta}(x_\alpha^*)H_\alpha(x^*)}, \\ K_6 = \frac{h_\beta(x_\beta^*)h_{\alpha,\beta}(x_\alpha^*)H_\alpha(x^*)}{H_\beta(x_\beta^*)+h_\beta(x_\beta^*)H_{\alpha,\beta}(x_\alpha^*)+h_\beta(x_\beta^*)h_{\alpha,\beta}(x_\alpha^*)H_\alpha(x^*)}. \end{cases}$$

Thus, we obtain the value function $\psi(x)$ defined by (4.2.3) and the corresponding optimal investment and reinsurance decisions (b_x^*, d_x^*). □

4.5 Numerical results

In this Section, we illustrate the jointly optimal investment and reinsurance decisions with numerical results. Our goals are twofold. On one hand, we compute and plot jointly optimal decisions of investment and reinsurance and compare their structural properties with changing values of the parameters. On the other hand, we simulate the ruin probability of our derived jointly optimal decisions and compare with the ruin probability when no control is applied. Random claim sizes are assumed to be independent and identically distributed and follow the exponential distribution with a mean of 5.

Figure 4.1: Jointly optimal decisions, $(\theta, \eta, \mu_1, \alpha, \lambda, \sigma_1) = (0.2, 0.1, 0.5, 0.05, 20, 0.2)$



(a) (b)

In Figure 4.2(a), we set $(\theta, \eta, \mu_1, \alpha, \lambda, \sigma_1) = (0.2, 0.1, 0.5, 0.05, 20, 0.2)$ and compare, side-by-side, the values of the optimal reinsurance decisions $d^*_{\beta=0.1,x}$ and $d^*_{\beta=0.2,x}$. For these given parameters, we have found that $x^* = 200$, $x^*_\alpha = 32.35$, $x^*_{\beta=0.1} = 29.52$ and $x^*_{\beta=0.2} = 24.93$. Figure 4.2(a) shows that the calculated values of $d^*_{\beta=0.1,x}$ and $d^*_{\beta=0.2,x}$ coincide when $29.52 \leqslant x < 200$ but $d^*_{\beta=0.2,x} > d^*_{\beta=0.1,x}$ when $0 < x < 29.52$.

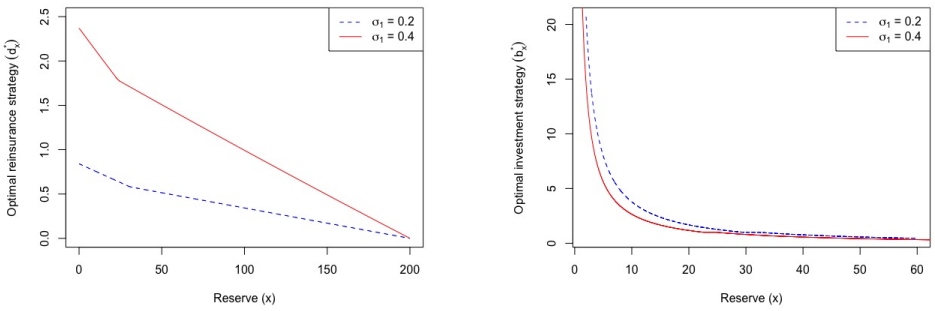
In Figure 4.2(b), we compare values of optimal investment decisions $b^*_{\beta=0.1,x}$ and $b^*_{\beta=0.2,x}$, side-by-side. Based on our numerical calculation, the investment strategy changes between $\beta = 0.1$ and $\beta = 0.2$ when $x = 13.79$. Figure 4.2(b) shows that the calculated values of $b^*_{\beta_1,x}$ and $b^*_{\beta_2,x}$ coincide when $29.52 \leqslant x < 200$ but $b^*_{\beta=0.2,x} < b^*_{\beta=0.1,x}$ when $13.79 \leqslant x < 29.52$ and $b^*_{\beta=0.2,x} > b^*_{\beta=0.1,x}$ when $0 < x \leqslant 13.79$. When the reserve is more than 32.35, it is optimal to partially invest in the savings account, and the higher the reserve, the more the investment in the savings account.

Generally speaking, the strategy is radical which means increasing the risk exposure to improve profit when the reserve is small. However, as the

reserve increases, the strategy becomes more conservative which means the risk exposure is reduced. It can be noticed from 4.2(b) that the investment strategy changes between $\beta = 0.1$ and $\beta = 0.2$ when $x = 13.79$. A plausible explanation is that we are seeking joint decisions to minimize the ruin probability. When $0 < x \leqslant 13.79$, with a higher borrowing rate, the insurance company should borrow more amount and invest it into risky asset to cover the extra cost caused by the higher borrowing rate. When $13.79 \leqslant x$, the strategy becomes mild which means the insurance company still considers increasing the risk exposure to improve profit. However, the insurance company does not want to borrow more amount due to the expensive borrowing rate.

Another reason is that the risky investment volatility $\sigma_1 = 0.2$ which is not large. If the risky investment volatility σ_1 becomes larger, say $\sigma_1 = 1$, $b^*_{\beta=0.1,x}$ may always greater than $b^*_{\beta=0.2,x}$ even the reserve is small. That because the risk come from risky investment already large enough, the insurance company does not need to assume the extra risk from expansive borrowing rate.

Figure 4.2: Jointly optimal decisions, $(\theta, \eta, \mu_1, \alpha, \beta, \lambda) = (0.2, 0.1, 0.5, 0.05, 0.1, 20)$

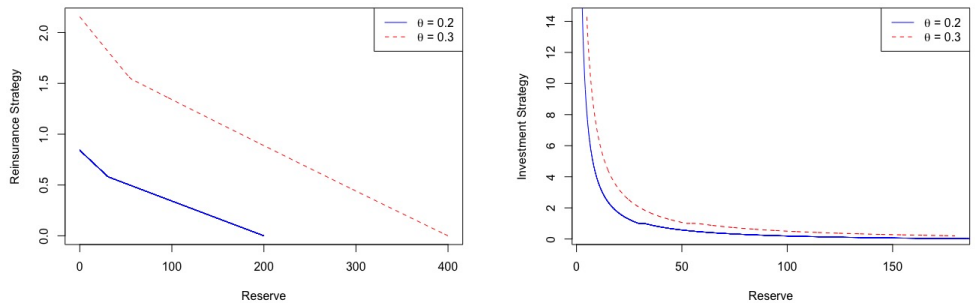


(a) (b)

We set $(\theta, \eta, \mu_1, \alpha, \beta, \lambda) = (0.2, 0.1, 0.5, 0.05, 0.1, 20)$ in Figure 4.2 and observe the impact of the market volatility σ_1 from the risky asset on the jointly

optimal decisions of investment and reinsurance. For the given parameters, we calculate that $x^* = 200$. When the reserve is less than 200, the higher the volatility, the higher the reinsurance retention level, and the lesser investment in the risky asset. This is intuitive because a high volatility means the price of the risky asset fluctuates a lot, and hence implies a high risk. To control the risk, we have to invest less reserve in the risky asset and buy less reinsurance.

Figure 4.3: Jointly optimal decisions, $(\eta, \mu_1, \alpha, \beta, \lambda, \sigma_1) = (0.1, 0.5, 0.05, 0.1, 20, 0.2)$



(a) (b)

Figure 4.3 shows similar structural results with respect to the relative safety loading θ for reinsurance. Based on our calculations, $x^* = 200$ when $\theta = 0.2$ and $x^* = 400$ when $\theta = 0.3$. Moreover, the higher the reinsurance safety loading, the higher the retention level of reinsurance, and the larger the proportion of investment in the risky asset.

The numerical results show that the insurance company should borrow money and invest them in the risky asset when the reserve is small. However, there is a minimum cash reserve policy mandated by the government. Also, there are some regulatory rules which may prevent investment in assets with high risks.

Finally, we simulate the performance of our jointly optimal decisions with

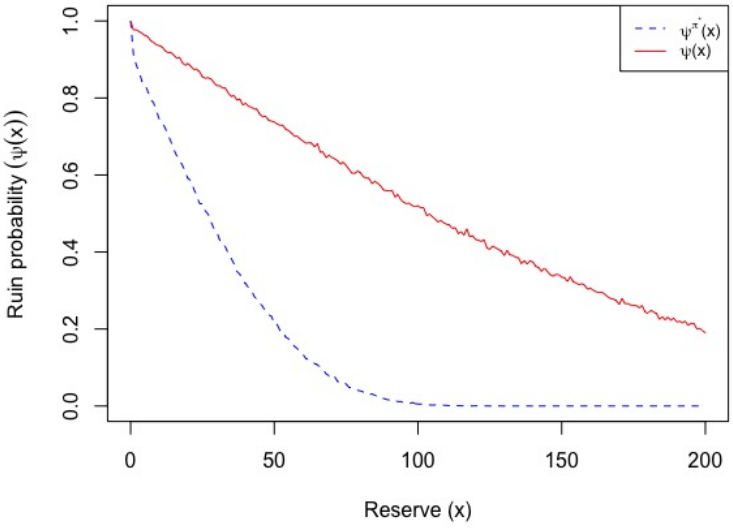
random claims and random investment returns. Each scenario of a given initial reserve is simulated 10000 runs. Within each run, we simulate 1000 joint decisions of investment and reinsurance within a period of 25 years. Parameters for the simulation are θ=0.2, η=0.1, μ_1=0.5, α=0.05, β=0.1, $\lambda = 20$ and σ_1=0.2.

We simulate and compare the ruin probability with the jointly optimal control decisions of investment and reinsurance against the ruin probability when no control whatsoever is imposed. Table 4.1 shows a lower ruin probability when jointly optimal decisions of investment and reinsurance are applied, and the difference becomes larger when the reserve is higher. This is further evident from Figure 4.4 in which the dashed curve corresponds to the ruin probability of the optimal strategy π^* of joint investment and reinsurance decisions.

Table 4.1: Ruin probability with or without controls

Initial reserve x	Ruin probability	
	Without controls	With jointly optimal controls
$x = 10$	0.9358	0.7443
$x = 50$	0.7374	0.2132
$x = 100$	0.5197	0.0051
$x = 150$	0.3349	0
$x = 180$	0.2405	0

Figure 4.4: Comparison of ruin probability with or without controls



4.6 Conclusion

We have considered an optimization problem of risk control for an insurance company whose reserve process is modeled by a diffusion process. To minimize its ruin probability, the insurance company can purchase excess-of-loss reinsurance while investing its reserve in a financial market consisting of a risky asset and a savings account. Specifically, we study the model under a constraint that the interest rate of borrowing is higher than the interest rate of savings. We develop the corresponding HJB equation and drive explicit expressions for the minimum ruin probability function and its corresponding jointly optimal investment and reinsurance decisions. Theoretical properties are illustrated with numerical results. To the best of our knowledge, this borrowing constraint is considered for the first time in the literature, when joint decisions of investment and excess-of-loss reinsurance are allowed. Our results may be potentially useful for decision making in real practice.

Chapter 5

Conclusion

5.1 Summary of achievements

This thesis is focused on risk management with reinsurance polices. Specifically, we study the problem of implementing reinsurance strategies to minimize the ruin probability. To deal with this optimal control problem, the major part of our research is dedicated to stochastic control method in risk management and insurance applications.

We first consider a stochastic control problem in discrete time. For the purpose of minimizing ruin probability, a proportional reinsurance is purchased. In particular, the case of unknown parameters in the claim size distribution is discussed. A classical way to solve optimal control problems is to apply the dynamic programming principle. We formulate the problem as a Markov decision process and calculate the objective function as a minimization criterion linked to cost. The optimal proportional reinsurance strategy is characterized and structural properties of the minimum ruin probability are also provided. It seems that this is the first time stochastic control problems with unknown parameters are discussed and solved.

For a continuous time stochastic control problem, we consider the case of minimizing ruin probability with joint investment and reinsurance decisions.

Particularly, we study the model under a constraint that the interest rate of borrowing is higher than the interest rate of savings. This is a realistic constraint, however, it is not considered in the current literature when joint decisions of investment and excess-of-loss reinsurance are allowed. By solving the HJB equation, we drive explicit expressions for the minimum ruin probability function and its corresponding jointly optimal investment and reinsurance decisions. A detailed discussion of the basic approach to find a HJB equation and the verification arguments is also provided.

In this thesis, we apply stochastic control techniques to solve different practical problems in insurance. We show that reinsurance can diminish the risk and we also provide methods to investigate the optimal reinsurance policies. The methods and results discussed can be potentially useful for decision making in risk management.

5.2 Future research

In risk management, initially we are concerned with the problem of minimizing the ruin probability. However, for a more realistic consideration the management of the business should also look for maximizing the profit. Therefore, our research problem can be extended in the following directions. Subject to investment and reinsurance, we consider the optimization problem with the objective of maximizing utility for insurance companies. In the setting of the diffusion approximation, we want to determine the optimal investment and excess-of-loss reinsurance strategy by the criterion of utility maximization. We will consider the case that trading the asset is subject to proportional translation costs. By applying stochastic control theory, we expect to derive the optimal strategy.

In addition, we will investigate the relation between these two optimization criteria: maximizing the expected utility and minimizing the probability of

ruin. We are searching for an approach to solve the optimization problems by a trade-off of these two criteria and we also want to establish a criterion combining these two objectives in one objective function.

Furthermore, since the most significant loss for an insurance company is caused by unexpected extreme claims, we will also pay attention to the case of heavy-tailed claims. In the long run, we expect to extend our research to investigate the impact of reinsurance, especially the large claims reinsurance treaties such as ECOMOR in risk management. It is known that reinsurance is an important technique to diminish the risk of large claims. However, the study of implementing reinsurance strategies to control risks in the heavy-tailed case has been neglected in the literature.

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