ON THE NUMBER OF ESSENTIALLY n-ARY POLYNOMIALS OF IDEMPOTENT ALGEBRAS

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INTRODUCTION

A <u>universal algebra</u>, or briefly, <u>algebra</u> $\mathfrak{A}$ is an ordered pair $\langle A;F \rangle$ where A is a non-empty set and F is a family of finitary operations on A. For each natural number n, we can consider the set $P^{(n)}(\mathfrak{A})$ of n-ary polynomials of $\mathfrak{A}$ which are certain functions from A^n to A built up from the variables x_i, $i = 1,2,\ldots,n$ by substituting them in the operations f, $f \in F$, successively in a finite number of steps.

An n-ary polynomial p over $\mathfrak{A}$ is said to <u>depend on</u> x_i if there exist $a_1,\ldots,a_i,a_i',\ldots,a_n$ in A such that

$$p(a_1,\ldots,a_i,\ldots,a_n) \neq p(a_1,\ldots,a_i',\ldots,a_n).$$

By an <u>essentially n-ary polynomial</u> over $\mathfrak{A}$ is meant a n-ary polynomial over $\mathfrak{A}$ which depends on each variable x_i, $i = 1,\ldots,n$. For $n > 1$, let $p_n(\mathfrak{A})$ designate the number of essentially n-ary polynomials over $\mathfrak{A}$. We denote by $p_1(\mathfrak{A})$ and $p_o(\mathfrak{A})$ the number of non-constant unary polynomials excluding x_1 and the number of constant unary polynomials respectively. Thus, with any algebra $\mathfrak{A}$, there is associated an ω-sequence of cardinals $\langle p_o(\mathfrak{A}), p_1(\mathfrak{A}),\ldots,p_n(\mathfrak{A}),\ldots \rangle$.

Let $\underset{\sim}{C}$ be a class of algebra. A sequence $\langle p_o,p_1,\ldots,p_n,\ldots\rangle$ of cardinals is said to <u>representable in</u> $\underset{\sim}{C}$ if there exists an algebra $\mathfrak{A}$ in $\underset{\sim}{C}$ with $p_n = p_n(\mathfrak{A})$ for each $n \geq 0$. If $\underset{\sim}{C}$ is the class of all algebras, then we say that the sequence $\langle p_o,p_1,\ldots,p_n,\ldots\rangle$ is <u>representable</u>. An algebra $\mathfrak{A} = \langle A;F \rangle$

is said to be <u>idempotent</u> if $f(x,\ldots,x) = x$, for any f in F. Thus, it is easy to see that an algebra $\mathcal{O}\!\mathcal{L}$ is idempotent if and only if $p_0(\mathcal{O}\!\mathcal{L}) = p_1(\mathcal{O}\!\mathcal{L}) = 0$. We shall say that an algebra $\mathcal{O}\!\mathcal{L} = \langle A;F \rangle$ can be <u>represented</u> as an algebra $\mathcal{O}\!\mathcal{L}^* = \langle A; f_1, f_2, \ldots \rangle$ of type τ if it is possible to choose a sequence $(f_1, f_2, \ldots)$ of polynomials from F in such a way that the sequence of the arities of the f_i equals τ. Note that the set of polynomials $\{f_i\}$ can be taken as a set of operations in $\mathcal{O}\!\mathcal{L}$.

Our basic problem is to study and characterize representable sequences. An easy combinatorial argument shows that this problem is equivalent to Problem 42 in [6] which can be stated as follows : Let $\underset{\sim}{K}$ be an equational class, and let F_n denote the cardinal of the free algebra over $\underset{\sim}{K}$ on n generators. Characterize the sequence $\langle F_n \rangle$.

The development of the study of $\langle p_n \rangle$ sequence may briefly be divided into three stages.

The period that started in 1910 may be considered as the initial stage. In this period, even though there were no significant contributions to the theory, the idea was foreshadowed by the work of S. Sierpinski. He published a series of articles between 1918 and 1945 for the purpose of investigating the composition of functions. One of his typical results (see [41]) says that given any set A and any function $f : A^n \longrightarrow A$, f can be obtained by an appropriate composition of binary functions. Recently, R.W. Quackenbush studied the corresponding problem for idempotent functions. He proved [37] that every idempotent function on a given set A can be obtained by composition of binary idempotent functions provided $|A| > 2$. For $|A| = 2$, the role of binary functions is replaced by ternary functions.

The explicit formulation of the basic problem, given by E. Marczewski in 1963 - 1964, may be considered as the beginning of the second stage. Since it was considered too difficult to deal with explicitly; E. Marczewski and his colleagues in Wroclaw studied only problems associated with it. In particular, he himself defined for each $\mathfrak{A}$, the <u>zero set</u>

$$Z(\mathfrak{A}) = \{ n \,/\, p_n(\mathfrak{A}) = 0 \}$$

and showed, for instance, in [22] that for algebras without constants and with one essentially n-ary symmetry (or even quasi-symmetric) polynomial the complement of the zero set $Z(\mathfrak{A})$ contains the arithmetical progression $n + k(n - 1)$, $(k = 0,1,\ldots)$. This generalizes a result of J. Płonka [28] for $n = 2$. One of the deepest results was obtained by K. Urbanik [42] who gave a complete description of all possible sets $Z(\mathfrak{A})$.

It was in 1968 that the present stage began with a systematic and intensive investigation of the basic problem in G. Grätzer's seminar at the University of Manitoba. Influenced by the first paper due to G. Grätzer, J. Płonka and A. Sekanina

[11], a steady flow of contributions to it, by the members of the seminar, has appeared in 1968 - 1969. G. Grätzer, J. Płonka and R. Padmanabhan have especially enriched and clarified the subject. The results were summarized by G. Grätzer [8] who gave a survey lecture at the Conference on Universal Algebras held at Queen's University in October, 1969.

In this period, the investigation of the basic problem was naturally split into two categories : (1). Study the basic problem for non-idempotent algebras; (2). Study the same for idempotent algebras. The first case was attacked by G. Grätzer, J. Płonka, A. Sekanina in [11], [12] and [33]. Some of their results were of the type that sequences satisfying some mild condition (e.g. $p_0 > 0$) are all representable, and so the p_i are independent. However, the situation completely changes when we deal with the idempotent case. As a matter of fact, the cardinals $p_n(\mathfrak{A})$, for idempotent algebra $\mathfrak{A}$, turn out to be quite strongly interrelated (see, for instance, [13] and [14]). Because of this extremely interesting fact, recently, most papers were devoted to the study of idempotent algebra ; this study can be separated roughly into two parts : (A). Investigate the behaviour and the maximum asymptotic rate of growth of the general sequence $\langle p_n \rangle$; (B). Description of all algebras representing a given sequence with application to the Minimal Extension Property (for definitin, see Chapter two of Part IV).

The purpose of this thesis is to provide some results for the second part of category 2 in a systematic way with emphasis on applications to the Minimal Extension Property. The equivalent problem of $\langle p_n \rangle$ sequences in Group Theory, the so called growth function of free groups, has been extensively studied for equational classes of groups by British mathematicians ; for instance, G. Higman (see [16] and [23]), P. Neumann and his students. The application of $\langle p_n \rangle$ sequences to semilattices was considered in G. Grätzer and J. Płonka [15] while the case of idempotent semigroups was settled by J.A. Gerhard [5] . In this thesis, we make a first attack in applying the $\langle p_n \rangle$ sequences to Lattice Theory.

This thesis falls into four parts with nine chapters altogether. Since a short description of the content is given at the beginning of each chapter, we shall include here only a brief outline. Part I, which consists of three chapters, is devoted to study idempotent algebras with one essentially binary polynomial. The sequence $\langle 0,0,1,2 \rangle$ has, in particular, very interesting properties. Thus, we restrict our attention to this sequence in the first two chapters. Some results of Part I are generalized to Part II in which we consider idempotent algebras with one essentially m-ary polynomial for $m \geqslant 2$. All of these are applied to derive the function $F(n,k)$ with the property that $F(n,k)$ is the least value such that the sequence $\langle 0,0,\overbrace{1,\ldots,1}^{k},n,F(n,k) \rangle$ is representable by symmetric algebra. Part III consists of two chapters. Płonka's basic lemmas are used in Chapter 1 to derive some results about the sequences ⟨ 0,0,2,m ⟩ . The considerations of Chapter 2 center around the algebras representing ⟨ 0,0,3,m ⟩ . Part IV consists of two chapters. Previous results are applied to Lattice Theory in Chapter 1 and the Minimal Extension Property in Chapter 2.

Cross references are given in the form (III,1,2) where III stands for Part III, 1 for Chapter 1 and 2 for section 2. The part and chapter numerals will be omitted in case the reference is made in the same chapter.

For those basic concepts and notations, we refer to G. Grätzer's books [6] and [9] .

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IDEMPOTENT ALGEBRAS WITH ONE ESSENTIALLY BINARY POLYNOMIAL

CHAPTER 1

ALGEBRAS REPRESENTING $\langle 0,\underline{0},\underline{1},\underline{2} \rangle$

The sequence $\langle 0,0,1,1 \rangle$ is, evidently, representable. This can be seen simply by taking a non-trivial semilattice. To go one step further, we are interested in the case where $p_3 = 2$. Thus, the following questions naturally arises :

(1). Is the sequence $\langle 0,0,1,2 \rangle$ representable ?

(2). If the answer to (1) is in the affirmative, what can we say about those algebras representing $\langle 0,0,1,2 \rangle$?

It is the main object of this chapter to provide solutions to the above questions. We shall see that the sequence $\langle 0,0,1,2 \rangle$ is indeed representable. As a matter of fact, it is shown that there exist exactly two equational classes of algebras $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$ such that an algebra $\mathfrak{A}$ represents $\langle 0,0,1,2 \rangle$ if and only if $\mathfrak{A}$ can be represented as an algebra belonging to either $\underset{\sim}{K}_1$ or $\underset{\sim}{K}_2$.

1. Basic Lemmas.

Let $\mathfrak{A}$ be an algebra representing $\langle 0,0,1 \rangle$. Then $\mathfrak{A}$ has one and only one essentially binary polynomial which is commutative and idempotent. There are two possible cases, namely, the binary polynomial is either associative or non-associative. Lemma 2.2 gives a sufficient condition for the former case to be happen. We need the following:

Lemma 1.1 (J. Płonka [28]).

let $\mathfrak{A}$ be an algebra without constants. If $p(x_0, x_1)$ is an essentially commutative binary polynomial over $\mathfrak{A}$, then $p(x_0, p(x_1, (\ldots, p(x_{n-2}, x_{n-1}) \ldots)))$ is essential n-ary, for each $n=2,3, \ldots$.

Lemma 1.2 .

Let $\mathfrak{A}$ be an algebra representing $\langle 0,0,1 \rangle$. If there exist $n \in \{3,4,5,\ldots\}$ such that $p_n(\mathfrak{A}) < \frac{1}{3}(2^n - (-1)^n)$, then $\mathfrak{A}$ has a semilattice operation.

Proof: Suppose that the binary Operation " $\circ$ " is non-associative. We claim that $p_n(\mathfrak{A}) \geqslant \frac{1}{3}(2^n - (-1)^n)$ for each $n=3,4, \ldots$.

First of all, consider the following ternary polynomials: $(xy)z$, $(yz)x$, $(zx)y$.

It follows from Lemma 1.1 that they are all essential. By the commutativity of "$\circ$", it is easy to see that the equality of any two would imply the associativity of "$\circ$", which contradicts our assumption. Thus, we have $p_3(\mathfrak{A}) \geqslant 3 = \frac{1}{3}(2^3 - (-1)^3)$.

Since $p_3(\mathfrak{A}) \geqslant 3 > 2$, we can apply a result (Theorem 4 of [10]) and obtain $p_n(\mathfrak{A}) \geqslant \frac{1}{3}(2^n - (-1)^n)$, for $n \geqslant 4$, as required.

Hence, "$\circ$" must be associative and therefore $\mathfrak{A}$ has a semilattice operation.

Suppose that $\mathfrak{A}$ is an algebra representing $\langle 0,0,1,2\rangle$. By Lemma 1.2, $\mathfrak{A}$ has a semilattice operation "$\circ$". By Lemma 1.1, we have already one essentially ternary polynomial $x\circ y\circ z$ over $\mathfrak{A}$. Thus, if $p_3(\mathfrak{A})=2$, there must exist one and only one essentially ternary polynomial $f(x,y,z)$ which is distinct from xyz. Our aim, here, is to investigate the general properties of the polynomial $f(x,y,z)$.

Clearly, we have

(1) $f(x,y,z)$ is idempotent.

Observe that if $p= p(x_0, \ldots, x_{n-1})$ is an essentially n-ary polynomial over $\mathfrak{A}$, then so is $p = p(x_{0\alpha}, \cdots, x_{(n-1)\alpha})$ for each $\alpha \in S(n)$, where $S(n)$ is the symmetric group on n symbols. Thus, $f(y,x,z)$ is essentially ternary. If $f(y,x,z) = xyz$, then $f(x,y,z) = yxz = xyz$, a contradiction. Hence, it follows that

(2) $f(x,y,z)$ is symmetric.

By identifying any two variables in $f(x,y,z)$, the resulting polynomial is binary. The following crucial result shows that it is essentially binary.

(3) $f(x,y,y) = xy$

<u>Proof</u>: As $\mathfrak{A}$ represents $\langle 0,0,1\rangle$, we have only the following three cases:

$$f(x,y,y) = \begin{cases} x \\ y \\ xy \end{cases}$$

<u>Case 1</u>. $f(x,y,y) = x$

First of all, we claim that the following polynomials

(*) $f(x,y,z)x$, $f(x,y,z)y$, $f(x,y,z)z$

are pairwise distinct.

Assume that $f(x,y,z)x = f(x,y,z)y$. Setting $x=z$, we get $yx=y$, a contradiction. By symmetricity of $f(x,y,z)$ it follows that polynomials in (*) are pairwise distinct.

Next, we assert that each polynomial in (*) is essentially ternary. By symmetry, we need only check for $f(x,y,z)x$. Clearly, $f(x,y,z)x$ depends on x. Moreover, it depends on y if, and only if it depends on z. Thus, if $f(x,y,z)x$ is not essentially ternary, we then get

$$f(x,y,z)x = x .$$

Setting $x=y$, it follows from (2) that $zx=x$, which is impossible. Hence $f(x,y,z)x$ is essentially ternary, as was to be shown.

Accordingly, if $f(x,y,y) = x$ holds, we would have $p_3(\mathfrak{A}) \geq 3$, a contradiction.

<u>Case 2</u>. $f(x,y,y) = y$

In analogy to case 1, we claim that the polynomials in (*) are pairwise distinct.

For this purpose, assume that $f(x,y,z)x = f(x,y,z)y$. Setting $x=y$, we obtain $x=xy$, a contradiction. Thus, they are pairwise distinct.

If one of them is essentially ternary, then so are the other two and hence $p_3(\mathfrak{A}) \geqslant 3$, a contradiction. Since, for example, $f(x,y,z)x$ is not essentially ternary we have

$$f(x,y,z)x = \begin{cases} x \\ y \\ z \\ xy \\ yz \\ zx \end{cases}$$

From the fact that $f(x,y,z)x$ depends on x and is symmetric with respect to y,z, it follows that

$$f(x,y,z)x = x \; .$$

Set y=z. Then we obrain $yx = x$, a contradiction.

Thus, we conclude that the case $f(x,y,y) = y$ is impossible.

Therefore, it is necessary that $f(x,y,y) = xy$, proving (3).

(4) $f(x,y,z)$ is not diagonal.

Proof : If $f(x,y,z)$ were diagonal, we would have

$$\begin{aligned} f(x,y,z) &= f(f(x,y,z),\ f(x,y,z),\ f(x,y,z)) && (1) \\ &= f(f(x,y,z),\ f(y,x,z),\ f(y,z,x)) && (2) \\ &= f(x,x,x) && \text{(diagonality)} \\ &= x\ , && (1) \end{aligned}$$

which is a contradiction.

The following result will be of great use in deriving other identities.

(5) $f(xy,x,y) = xy$.

Proof : As $p_0(\mathfrak{A})=0$, $f(xy,x,y)$ is not a constant. However, by symmetry, $f(xy,x,y)$ depends on x if, and only if it depends on y. Hence $f(xy,x,y)$ is essentially binary and thus $f(xy,x,y) = xy$, since $p_2(\mathfrak{A}) = 1$.

From (5), we obtain

(6) $\quad f(xyz,xy,z) = xyz$.

(7) $\quad f(xyz,xy,xz) = xyz$.

Consider the following ternary polynomial:

$$f(xy,y,z) \text{ .}$$

It is easy to check that $f(xy,y,z)$ is essentially ternary. Thus we have the two possible cases:

$$f(xy,y,z) = \begin{cases} f(x,y,z) \\ xyz \end{cases}$$

Suppose $f(xy,y,z) = f(x,y,z,)$ ——————— (A)

We observe that

$$\begin{aligned} xyz &= f(xyz,xy,xz) && (7) \\ &= f(z\cdot xy,xy,xz) && \\ &= f(z,xy, xz) && (A) \\ &= f(xz,z,xy) && (2) \\ &= f(x,z,xy) && (A) \\ &= f(yx,x,z) && (2) \\ &= f(x,y,z) && (A), (2) \end{aligned}$$

which is impossible. Thus, we have

(8) $\quad f(xy,y,z) = xyz$.

The following are immediate consequences of the above identities:

(9) $f(xy,xz,z) = xyz$.

(10) $f(xy,xz,x) = xyz$.

(11) $f(xyz,y,z) = xyz$.

(12) $f(xyz,xy,y) = xyz$.

Though most of the identities of $f(x,y,z)$ are easy consequences of the previous ones, the following seems to be an exception.

(13) $f(xy,yz,zx) = xyz$.

Proof : Consider $f(xy,yz,zx)$. Set $x=y$. Then we obtain $f(x,xz,xz) = xz$ by (3). Thus, $f(xy,yz,zx)$ depends on z. By symmetry, it also depends on x and y . Hence $f(xy,yz,zx)$ is essentially ternary.

If $f(xy,yz,zx) = f(x,y,z)$ ——————— (B)

then $xyz = f(xyz,xyz,xyz)$ (1)

$= f(xy \cdot yz,\ yz \cdot zx,\ zx \cdot xy)$

$= f(xy,yz,zx)$ (B)

$= f(x,y,z)$ (B)

which is a contradiction. Therefore, (13) follows.

A ternary operation f is associative if the following property holds:

$$f(f(x,y,z),u,v) = f(x,f(y,z,u),v) = f(x,y,f(z,u,v)).$$

Clearly, we have

(14) $f(x,y,z)$ is non-associative.

2. Two Types of Algebras.

In this section, we continue our study of ternary polynomials built up from "$\circ$" and "f". As a result, we obtain two types of algebras which are both compatible with our hypothesis.

To begin with, let us consider the polynomial $f(x,y,z)\circ x$. It turns out that $f(x,y,z)x$ is essentially ternary. Thus, we have

$$f(x,y,z)x = \begin{cases} f(x,y,z) & \text{———— I} \\ xyz & \text{———— II} \end{cases}$$

From now on, we shall naturally split our investegation into two parts, each of which deals with each of the two possibilities in detail. We shall call those algebras satisfying the identity I, <u>Type I algebras</u> and those satisfying II, <u>Type II algebras.</u>

<u>TYPE I</u>. $\quad f(x,y,z)x = f(x,y,z)$ ———————— I

In this case, by the symmetry of $f(x,y,z)$, we get

(15) $\quad f(x,y,z)x = f(x,y,z)xy = f(x,y,z)xyz = f(x,y,z)$.

Consider the polynomial $f(f(x,y,z),y,z)$. We have

$$\begin{aligned} f(f(x,y,z),y,z) &= f(f(x,y,z)y,y,z) && (15) \\ &= f(x,y,z)yz && (8) \\ &= f(x,y,z) && (15) \end{aligned}$$

Thus, it follows that

(16) $\quad f(f(x,y,z),y,z) = f(x,y,z)$.

By applying the same argument, using (8) and (15) the following identities can be derived immediately.

(17) $f(f(x,y,z),xy,z) = f(x,y,z)$.

(18) $f(f(x,y,z),xyz,z) = f(x,y,z)$.

(19) $f(f(x,y,z),xy,x) = f(x,y,z)$.

(20) $f(f(x,y,z),xy,xz) = f(x,y,z)$.

(21) $f(f(x,y,z),xyz,xy) = f(x,y,z)$.

<u>TYPE II.</u> $f(x,y,z)x = xyz$ ——————————— II

In this case, as $f(x,y,z)$ is symmetric, we have

(22) $f(x,y,z)x = f(x,y,z)y = f(x,y,z)z = xyz$.

Now, consider the polynomial $f(f(x,y,z),y,z)$. It can be easily checked that it is essentially ternary.

If $f(f(x,y,z),y,z) = f(x,y,z)$ ————— (C)

then
$$\begin{aligned} f(x,y,z) &= f(x,y,z)\cdot f(x,y,z) \\ &= f(f(x,y,z),y,z)\, f(x,y,z) && (C) \\ &= f(x,y,z)yz && (22) \\ &= xyz && (22) \end{aligned}$$

which is a contradiction. Thus it follows that

(23) $f(f(x,y,z),y,z) = xyz$.

Similarly, we get

(24) $f(f(x,y,z),xy,z) = xyz$.

(25) $f(f(x,y,z),xy,xz) = xyz$.

Observe that

$$\begin{aligned} f(f(x,y,z),xyz,z) &= f(f(x,y,z),xy\cdot z,z) \\ &= xyz\, f(x,y,z) && (8) \\ &= xyz && (22) \end{aligned}$$

Thus, we have

(26) $f(f(x,y,z),xyz,z) = xyz$.

Similar arguments can be applied to yield the following:

(27) $f(f(x,y,z),xy,x) = xyz$.

(28) $f(f(x,y,z),xyz,xy) = xyz$.

Let $\mathfrak{A}$ be an algebra representing $\langle 0,0,1,2 \rangle$. Let $p(x,y,z)$ be an arbitrary ternary polynomial over $\mathfrak{A}$. Then $p(x,y,z)$ is built up from the set of symbols $\{x,y,z\}$ by substituting them in two operation symbols "∘" and "f" . If $\mathfrak{A}$ is a Type I algebra, then by making use of those identities hold in $\mathfrak{A}$, $p(x,y,z)$ can be reduced to one of the ternary polynomials $\{xyz, f(x,y,z)\}$. If $\mathfrak{A}$ is a Type II algebra, the same situation holds. For clarity, we now give the following list:

		TYPE I	TYPE II
A	f(xy,y,z) f(xy,xz,x) f(xy,yz,z) f(xy,yz,zx) f(xyz,y,z) f(xyz,xy,z) f(xyz,xy,y) f(xyz,xy,yz)	= xyz	= xyz
B	f(x,y,z)x f(x,y,z)xy f(x,y,z)xyz	= f(xy,z)	= xyz
C	f(f(x,y,z),y,z) f(f(x,y,z),xy,z) f(f(x,y,z),xy,x) f(f(x,y,z), xy,xz) f(f(x,y,z),xyz,z) f(f(x,y,z),xyz,xy)	= f(x,y,z)	= xyz

3. Characterization Theorem and Applications.

We are now in a position to establish some of the main results of this chapter. Summarizing all the results in the previous sections, we arrive at the following

<u>Theorem 3.1</u>

Let $\mathfrak{A}$ be an algebra representing $\langle 0,0,1,2\rangle$. Then $\mathfrak{A}$ can be represented as an algebra $\langle A; \circ, f\rangle$ of type $\langle 2,3\rangle$ where "$\circ$" is the semilattice operation belonging to one of the equational classes $\underset{\sim}{K}_1$, $\underset{\sim}{K}_2$ of algebras where

$$\mathrm{Id}(\underset{\sim}{K}_1) = \begin{cases} (1) & f(x,y,z) = f(y,x,z) = f(y,z,x) \\ (2) & f(xy,y,z) = xyz \\ (3) & f(xy,yz,zx) = xyz \\ (4) & f(x,y,z)x = f(x,y,z) \end{cases}$$

$$\mathrm{Id}(\underset{\sim}{K}_2) = \begin{cases} (1) & f(x,y,z) = f(y,x,z) = f(y,z,x) \\ (2) & f(xy,y,z) = xyz \\ (3) & f(xy,yz,zx) = xyz \\ (4) & f(x,y,z)x = xyz \\ (5) & f(f(x,y,z),y,z) = xyz \\ (6) & f(f(x,y,z),xy,z) = xyz \\ (7) & f(f(x,y,z),xy,xz) = xyz \end{cases}$$

Moreover, if $\mathfrak{A} \in \underset{\sim}{K}_1$ and $p(x,y,z)$ is an essentially ternary polynomial over $\mathfrak{A}$ then

$$p(x,y,z) = \begin{cases} f(x,y,z) & \text{if the whole factor } f(x,y,z) \text{ appears} \\ & \text{in } p(x,y,z) \\ xyz & \text{otherwise} \end{cases}$$

If $\mathfrak{A} \in \underset{\sim}{K}_2$ and $p(x,y,z)$ is an essentially ternary polynomial over $\mathfrak{A}$ then

$$p(x,y,z) = \begin{cases} f(x,y,z) & \text{if } p(x,y,z) \text{ is of the form } f(x,y,z) \\ xyz & \text{otherwise} \end{cases}$$

We will now prove ~~the~~ a converse of Theorem 3.1 . The two types of algebras will be considered separately.

<u>Theorem 3.2 (Type I).</u>

Let $\mathfrak{A} = \langle A; \cdot, f \rangle$ be an algebra of type $\langle 2,3 \rangle$ where "$\cdot$" is the semilattice operation and $f(x,y,z)$ is the ternary operation satisfying $Id(\underset{\sim}{K}_1)$ of Theorem 3.1 . Then $\mathfrak{A}$ represents $\langle 0,0,1,2 \rangle$.

<u>Proof</u> : Since "$\cdot$" is idempotent and $f(x,x,x) = x$ by (2) of $Id(K_1)$, it follows that $\mathfrak{A}$ is idempotent. This is equivalent to saying that $p_0(\mathfrak{A}) = p_1(\mathfrak{A}) = 0$.

(2) of $Id(\underset{\sim}{K}_1)$ implies $f(x,y,y) = xy$ and $f(xy,x,y) = xy$. Combine these with (1) Of $Id(\underset{\sim}{K}_1)$. Then it follows that 'xy' is the only essentially binary polynomial over $\mathfrak{A}$. Thus, $p_2(\mathfrak{A}) = 1$.

Finally, we have to prove that $p_3(\mathfrak{A}) = 2$. Since $f(x,y,z) \neq xyz$, $p_3(\mathfrak{A}) \geqslant 2$. On the other hand, according to the results in sections 1 and 2, we see that (1),(2) amd (3) of $Id(\underset{\sim}{K}_1)$ imply that all the forms of ternary polynomials in category A (see section 2) are the same and equal to xyz.

Moreover, from (2) and (4) of Id($\underset{\sim}{K}_1$), it follows that all the forms of the ternary polynomials in categories B and C are the same and equal to f(x,y,z). Hence, $p_3(\mathfrak{A}) = 2$, proving our theorem.

Theorem 3.3 (Type II).

Let $\mathfrak{A} = \langle A; \cdot, f \rangle$ be an algebra of type $\langle 2,3 \rangle$ where "." is the semilattice operation and f(x,y,z) is the ternary operation satisfying Id($\underset{\sim}{K}_2$) of Theorem 3.1. Then $\mathfrak{A}$ represents $\langle 0,0,1,2 \rangle$.

Proof : In analogy to the proof of Theorem 3.2, we see that (1) and (2) of Id($\underset{\sim}{K}_2$) imply that $\mathfrak{A}$ represents $\langle 0,0,1 \rangle$. To prove that $p_3(\mathfrak{A}) = 2$, observe that (1),(2) and (3) guarantee that all the forms of the ternary polynomials in category A are all the same and equal to xyz. Furthermore, (4) of Id($\underset{\sim}{K}_2$) implies that all the forms of the ternary polynomials in category B are all the same and equal to xyz. Finally, (2),(4), (5),(6) and (7) imply that all the forms of the ternary polynomials in category C are all the same and again equal to xyz. Hence, $p_3(\mathfrak{A}) = 2$, as was to be shown.

Combining the above three results, we have the following characterization theorem.

Theorem 3.4

There exist two equational classes of algebras $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$

such that an algebra $\mathfrak{A}$ represents the sequence $\langle 0,0,1,2 \rangle$ if, and only if $\mathfrak{A}$ can be represented as an algebra $\langle A; \cdot, f \rangle$ of type $\langle 2,3 \rangle$ where "$\cdot$" is the semilattice operation belonging to either $\underset{\sim}{K}_1$ or $\underset{\sim}{K}_2$.

Applying our main Theorem, some simple results which show the behavior of the sequence $\langle 0,0,1,2, p_4, \dots, p_n, \dots \rangle$ can easily be derived.

In [13], G. Grätzer and J. Płonka proved the following result: Let $\mathfrak{A}$ be an idempotent algebra having a commutative and associative binary polynomial. If $p_n(\mathfrak{A}) \neq 1$ $(n \geqslant 2)$ then $p_{n+1}(\mathfrak{A}) \geqslant p_n(\mathfrak{A}) + 1 + \max.\{p_n(\mathfrak{A}), n+1\}$

From this, we have

<u>Corollary 3.5</u>

Let $\mathfrak{A}$ be an algebra representing $\langle 0,0,1,2 \rangle$. Then $p_n(\mathfrak{A}) \geqslant 2^{n-1} - 1$ for all $n \geqslant 4$.

<u>Proof:</u> We prove the corollary by induction on n.

If $n=4$, then $p_4(\mathfrak{A}) \geqslant p_3(\mathfrak{A}) + 1 + \max.\{p_3(\mathfrak{A}), 4\}$ $= 7 = 2^{4-1} - 1$.

Assume the statment is true for n=k, that is $p_k(\mathfrak{A}) \geqslant 2^{k-1} - 1$. Consider the case when n=k+1. We have

$$\begin{aligned} p_{k+1}(\mathfrak{A}) &\geq p_k(\mathfrak{A}) + 1 + \max.\{p_k(\mathfrak{A}), k+1\} \\ &= 2p_k(\mathfrak{A}) + 1 \\ &\geqslant 2(2^{k+1} - 1) + 1 \\ &= 2^k - 1. \end{aligned}$$

Hence the Corollary follows.

Corollary 3.6 .

Let $\mathfrak{A}$ be an algebra representing $\langle 0,0,1,2\rangle$. Then for each integer k, $p_n(\mathfrak{A}) > k$ for all but finitely many n.

Corollary 3.7 .

The sequence $\langle 0,0,1,2,p_4(\mathfrak{A}),p_5(\mathfrak{A}), \ldots,p_n(\mathfrak{A}),\ldots\rangle$ is unbounded.

Following [13], we say that a sequence $\langle p_i\rangle$ is conditionally strictly increasing (C.S.I) if $1 < p_i < \aleph_0$ implies $p_i < p_{i+1}$. Thus, we have

Corollary 3.8 .

Let $\mathfrak{A}$ be an algebra representing $\langle 0,0,1,2\rangle$. Then $\langle p_n(\mathfrak{A}),p_{n+1}(\mathfrak{A}), \ldots\ldots\rangle$ is C.S.I. for each $n \geqslant 1$.

CHAPTER 2.

ALGEBRAS REPRESENTING $\langle 0,0,1,2\rangle$ (CONTINUED)

The study of the two equational classes of algebras representing $\langle 0,0,1,2\rangle$ is continued in this chapter. By making use of previous results, the structures of those algebras representing $\langle 0,0,1,2\rangle$ will be considered here.

This chapter falls into five sections. Though Theorem 3.4 (I,1) characterizes those algebras representing $\langle 0,0,1,2\rangle$, whether these exist algebras in $\underset{\sim}{K}_1$ or $\underset{\sim}{K}_2$ has not so far been discussed. In section 1, we establish two Existence Theorems, one for each equational class. Several algebras representing $\langle 0,0,1,2\rangle$ will be furnished and some relations between them will be indicated in section 2. The major result of this chapter states that if $\mathfrak{A}$ is an algebra representing $\langle 0,0,1,2\rangle$ then $\mathfrak{A}$ contains one of the eight algebras as a subalgebra. This is shown in section 3. Applying this main result, we are able to provide in section 4 a lower bound for $\langle p_n(\mathfrak{A})\rangle$ which is much stronger than that in Corollary 3.5 (I,1). Finally, finite subdirectly irreducible algebras in $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$ will be studied in section 5.

1. Existence Theorems.

Let $K(\tau)$ be the class of all algebras of type τ. The n-ary polynomial algebra,

$$\mathcal{B}^{(n)}(\tau) = \langle P^{(n)}(\tau);\ F \rangle$$

where the underlying set $P^{(n)}(\tau)$ is the set of all n-ary polynomial symbols and the operations on $P^{(n)}(\tau)$ are defined in a natural way (see [6]), is known to be an element of $K(\tau)$.

Let us now confine ourselves to a special case where $\tau = \langle 2,3 \rangle$ and n=3. In this situation, we have an algebra in $K(\langle 2,3\rangle)$, namely,

$$\mathcal{B}^{(3)}(\langle 2,3 \rangle)$$

Denote by "$\circ$" and "f" the binary and ternary operations of $\mathcal{B}^{(3)}(\langle 2,3 \rangle)$ respectively. Consider the following set Σ_1 of identities:

$$\Sigma_1 : \begin{cases} (1) & x \circ x = x \\ (2) & (x \circ y)z = (yz)x \\ (3) & f(x,y,z) = f(y,x,z) = f(y,z,x) \\ (4) & f(xy,y,z) = xyz \\ (5) & f(xy,yz,zx) = xyz \\ (6) & f(x,y,z)x = f(x,y,z) \end{cases}$$

Remark : It is proved in [24] that the two identities (1) and (2) of Σ_1 characterize the semilattice operation.

We shall now define a binary relation Θ on $\mathcal{B}^{(3)}(\langle 2,3 \rangle)$ as follows : For any two elements p, q in $P^{(3)}(\langle 2,3\rangle)$, we put $p \equiv q(\Theta)$ if, and only if the identity p=q is provable from the set Σ_1.

It turns out that Θ is an equivalence relation on $P^{(3)}(\langle 2,3 \rangle)$.

Furthermore, one can check that Ⓗ has the Substitution Property. Thus, Ⓗ is a congruence relation on the algebra $\mathcal{B}^{(3)}(\langle 2,3\rangle)$. From this, we get a quotient algebra, namely,

$$\mathcal{B}^{(3)}(\langle 2,3\rangle)/Ⓗ$$

Now, by making use of the results in Chapter 1 (I) and the identities of Σ_1 we can describe the set of all ternary polynomials over the algebra $\mathcal{B}^{(3)}(\langle 2,3\rangle)/Ⓗ$ which turns out to be the following:

$$\begin{cases} x \quad y \quad z \\ xy \quad yz \quad zx \\ xyz \quad f(x,y,z) \end{cases}$$

Evidently, xyz and f(x,y,z) are the only essentially ternary polynomials. Thus $p_3(\mathcal{B}^{(3)}(\langle 2,3\rangle)/Ⓗ) = 2$. The only essentially binary polynomial is xy and there are no constants and no unary polynomials which are distinct from the projections. Consequently, we have the following :

<u>Theorem 1.1 (Existence Theorem of Type 1 Algebras)</u>

The algebra $\mathcal{B}^{(3)}(\langle 2,3\rangle)/Ⓗ$ represents the sequence $\langle 0,0,1,2\rangle$. Moreover, $\mathcal{B}^{(3)}(\langle 2,3\rangle)/Ⓗ \in \underset{\sim}{K}_1$.

On the other hand, instead of the set Σ_1, let us take the following:

$$\Sigma_2 : \begin{cases} (1) & x \circ x = x \\ (2) & (x \circ y)z = (yz)x \\ (3) & f(x,y,z) = f(y,x,z) = f(y,z,x) \\ (4) & f(xy,y,z) = xyz \\ (5) & f(xy,yz,zx) = xyz \\ (6) & f(x,y,z)x = xyz \\ (7) & f(f(x,y,z),y,z) = xyz \\ (8) & f(f(x,y,z),xy,z) = xyz \\ (9) & f(f(x,y,z),xy,xz) = xyz. \end{cases}$$

Moreover, instead of Ⓗ, we define a binary relation Φ on $\mathcal{B}^{(3)}(\langle 2,3\rangle)$ as follows: For any elements p , q in $P^{(3)}(\langle 2,3\rangle)$ we put $p \equiv q$ (Φ) if, and only if the identity p = q is provable from the set Σ_2.

In analogy to the first case, it follows that Φ is a congruence relation on $\mathcal{B}^{(3)}(\langle 2,3\rangle)$ and the algebra $\mathcal{B}^{(3)}(\langle 2,3\rangle)/\Phi$ has the following properties.

<u>Theorem 1.2. (Existence Thorem of Type 2 Algebras).</u>

The algebra $\mathcal{B}^{(3)}(\langle 2,3\rangle)/\Phi$ represents the sequence $\langle 0,0,1,2\rangle$. Moreover $\mathcal{B}^{(3)}(\langle 2,3\rangle)/\Phi \in \underset{\sim}{K}_2$.

2. Other Examples.

In this section, we shall construct eight algebras I(j), II(j), j = 1,2,3,4, where $5 \leq |I(j)|, |II(j)| \leq 8$, four for each equational class $\underset{\sim}{K}_i$, i = 1,2 and each of which represents the sequence $\langle 0,0,1,2 \rangle$.

(A) Examples in $\underset{\sim}{K}_1$.

1) Algebra I(1) where $|I(1)| = 5$

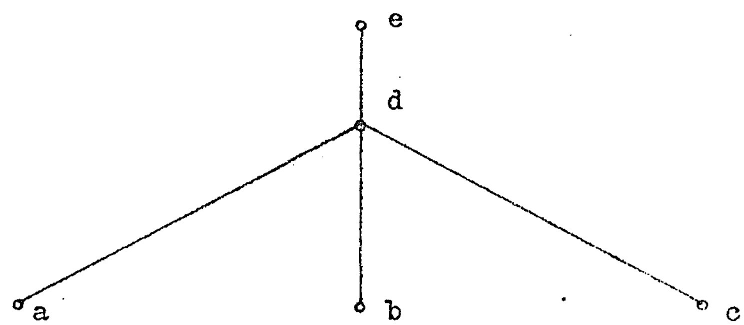


Fig. 1

2) Algebra I(2) where $|I(2)| = 6$

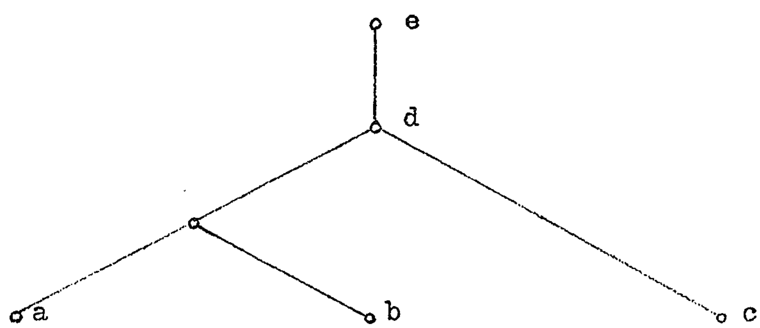


Fig. 2

3) Algebra I(3) where $|I(3)| = 7$

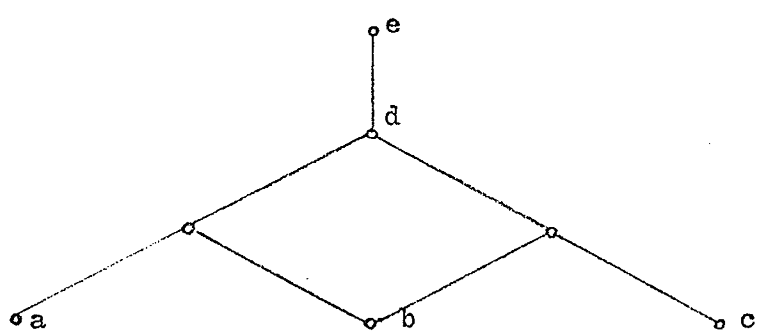


Fig. 3

4) Algebra $I(4)$ where $|I(4)| = 8$.

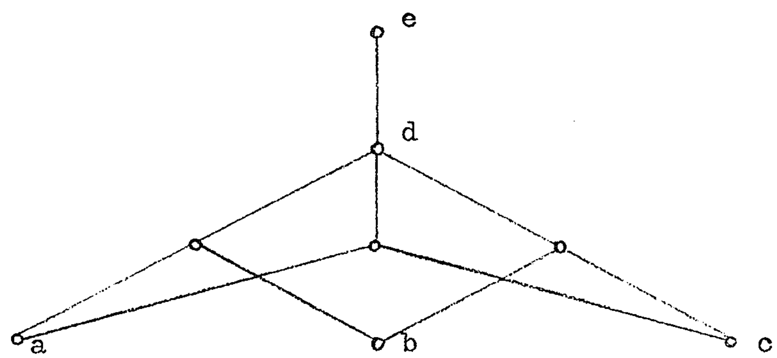


Fig. 4

For each $j = 1,2,3,4$, $I(j)$ is an algebra of type $\langle 2,3 \rangle$ where the base set is shown in Fig.j . The binary operation "." in $I(j)$ regarded as a join semilattice operation while the ternary operation "f" is defined as follows:

$$f(x,y,z) = \begin{cases} e & \text{if } \{x,y,z\} = \{a,b,c\} \\ xyz & \text{otherwise} \end{cases}.$$

It follows immediately from the above definition of f that $f(x,y,z)$ is an essentially ternary polynomial over $I(j)$ and $f(x,y,z) \neq xyz$. To show that each algebra $I(j)$, $j = 1,2,3,4$ is an element in $\underset{\sim}{K}_1$, we have to show by Theorem 3.2 (I,1) that the ternary operation "f" defined above satisfies the set $Id(\underset{\sim}{K}_1)$ of Thorem 3.1 (I,1). We shall now give a proof for the **algebra** $I(4)$. The other three can be proved in a similar way.

Clearly, $f(x,y,z)$ is symmetric. Thus (1) of $Id(\underset{\sim}{K}_1)$ holds. To see that $f(xy,y,z) = xyz$, we note that $\{xy,y,z\}(S) \neq \{a,b,c\}$ for any substitution S. For if $\{xy,y,z\}(S) = \{a,b,c\}$ then $(xy)(S) = a$ say, and we would get $y(S) = a$, a contradiction.

Thus by definition, $f(xy,y,z) = xy \circ y \circ z = xyz$, which was to be shown. Similarly, we have $f(xy,yz,zx) = xyz$. Finally, we claim that $f(x,y,z)x = f(x,y,z)$. To this end, observe that if $\{x,y,z\}(S) = \{a,b,c\}$, then $f(a,b,c)a = ea = e = f(a,b,c)$. If $\{x,y,z\}(S) \neq \{a,b,c\}$, then $(f(x,y,z)x)(S) = [(xyz)x](S) = (xyz)(S) = f(x,y,z)(S)$. Thus the identity $f(x,y,z)x = f(x,y,z)$ follows.

Hence, we have

Theorem 2.1.

For each $j = 1,2,3,4$, $I(j) \in \underset{\sim}{K}_1$.

(B) Example in $\underset{\sim}{K}_2$.

1) Algebra II(1) where $|II(1)| = 5$

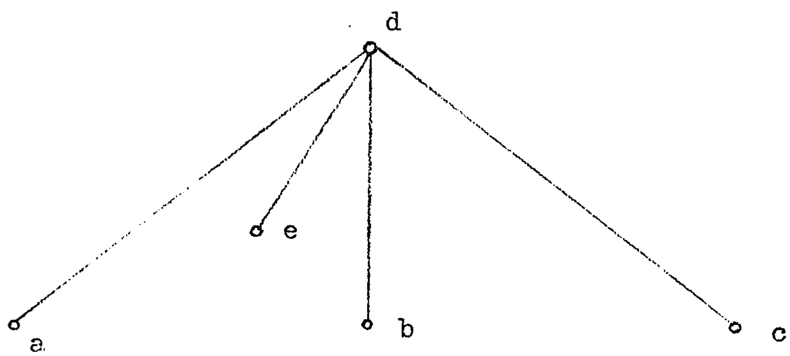


Fig. 5.

2) Algebra II(2) where $|II(2)| = 6$

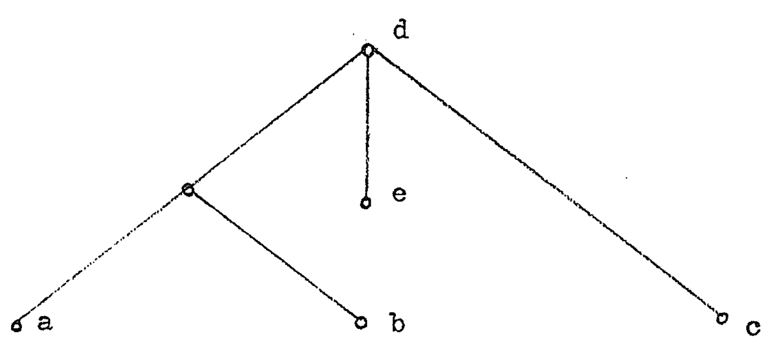


Fig. 6.

3) Algebra II(3) where $|II(3)| = 7$

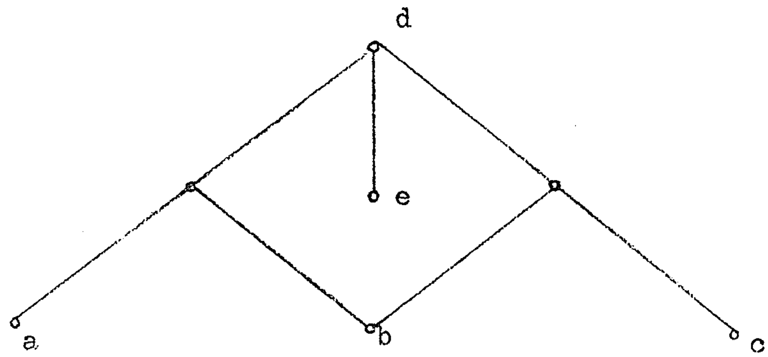


Fig. 7.

4) Algebra II(4) where $|II(4)| = 8$

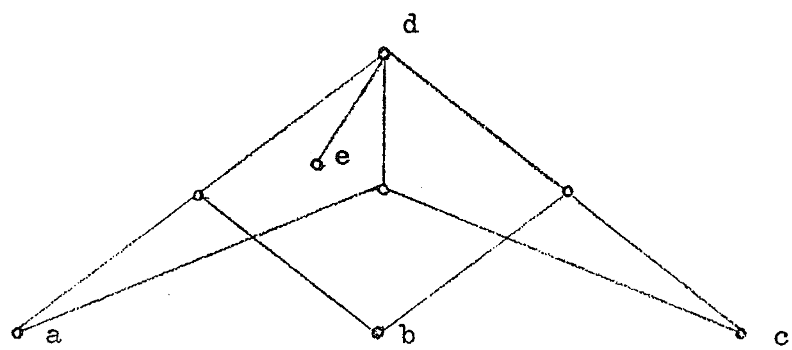


Fig. 8.

For each j = 1,2,3,4, II(j) is an algebra of type $\langle 2,3\rangle$ where the base set is shown in Fig(4+j). The binary operation "." in II(j) is regarded as a join semilattice operation while the ternary operation is defined as follows:

$$f(x,y,z) = \begin{cases} e & \text{if } \{x,y,z\} = \{a,b,c\} \\ xyz & \text{otherwise} \end{cases} .$$

Clearly, f(x,y,z) is an essentially ternary polynomial over II(j) and $f(x,y,z) \neq xyz$. By using the similar argument as before, it can be shown that the set $Id(\underset{\sim}{K}_2)$ of Theorem 3.1 (I ,1) is satisfied by "f" and thus we get

Theorem 2.2.

$$\text{For each } j = 1,2,3,4,\ II(j) \in \underset{\sim}{K}_2.$$

It is quite interesting to note that the structures of the four algebras, for each equational class $\underset{\sim}{K}_i$, $i = 1,2$ are closely related. Thus, it is perhaps worthwhile to point out some relationships between them.

Remark. 1

Observe that the cardinality of the algebra $\mathcal{B}^{(3)}(\langle 2,3\rangle)/\Theta$ shown in section 1 is eight. In fact, this algebra is isomorphic to the algebra I(4) and it turns out that both of them are the free algebra over $\underset{\sim}{K}_1$ with the free generating set which consists three unordered elements. In notation,

$$\mathcal{B}^{(3)}(\langle 2,3\rangle)/\Theta \cong F_{\underset{\sim}{K}_1}(3) \cong I(4) .$$

Similarly, we have

$$\mathcal{B}^{(3)}(\langle 2,3\rangle)/\Phi \cong F_{\underset{\sim}{K}_2}(3) \cong II(4)$$

Remark 2.

It is easily seen that for each $j = 1,2,3,4$, $I(j) - \{e\}$, considered as a semilattice, is a homomorphic image of the free semilattice $I(4) - \{e\}$. Moreover, they are the only homomorphic images of $I(4) - \{e\}$.

The same relation holds for $II(j) - \{e\}$ and $II(4) - \{e\}$.

Remark 3.

If we consider only the semilattice structure, then it is clear that I(1) is isomorphic to a sub-semilattice of I(4).

Indeed, I(1) can be embedded in I(4) and is isomorphic to the five-element sub-semilattic $\{ab,bc,ca,d,e\}$ of I(4).
However I(1) is no longer a subalgebra of I(4) if we consider both of them as algebras of type $\langle 2,3\rangle$, for $f(ab,bc,ca) = abc = d$ in I(4) while in I(1), we have $f(a,b,c) = e$.

The same remark is true for II(1) and II(4).

3. A Theorem on Subalgebras.

The eight algebras I(j), II(j), j = 1,2,3,4, play central roles in the study of the equational classes $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$. In Lattice Theory, it is well-known that a lattice is non-distributive if and only if it contains M_5 or N_5 as sublattices. In our case, we have a similar result for the "only if" part. This can be seen from the following

Theorem 3.1.

Let $\mathfrak{A}$ be an algebra of type $\langle 2,3\rangle$ which represents $\langle 0,0,1,2\rangle$. If $\mathfrak{A} \in \underset{\sim}{K}_1$, then $\mathfrak{A}$ contains one of the I(j), j = 1,2,3,4 as subalgebra. If $\mathfrak{A} \in \underset{\sim}{K}_2$, then $\mathfrak{A}$ contains one of the II(j), j = 1,2,3,4 as subalgebra.

Proof: Let f be the ternary operation in $\mathfrak{A}$. Since $f(x,y,z) \neq xyz$, there exist $a,b,c \in A$ such that

$$f(a,b,c) \neq abc \quad \text{in } A.$$

Claim 1. a,b,c are pairwise distinct.

If this is not the case, say a = b, then we would have

$$f(a,b,c) = f(a,a,c)$$
$$= ac$$
$$= abc$$

which is a contradiction. Hence a,b and c are pairwise distinct.

A partial ordering can be defined in A in a natural way, namely, $p \leqslant q$ if and only if $pq = q$.

<u>Claim 2</u>. The set $\{a,b,c\}$ is unordered.

Otherwise, say $b \leqslant a$, i.e., $ab = a$, then it follows that

$$f(a,b,c) = f(ab,b,c) = abc,$$

a contradiction. The other possible cases can be proved similarly. Thus, $\{a,b,c\}$ is unordered.

Now, if we set $d = abc$, $e = f(a,b,c)$, we assert that d and e are comparable.

Indeed, if $\mathfrak{A} \in \underset{\sim}{K}_1$, then $f(x,y,z)xyz = f(x,y,z)$ holds in $\mathfrak{A}$. From this, it follows that $e.d = e$ and so $e > d$. If $\mathfrak{A} \in \underset{\sim}{K}_2$, then $f(x,y,z)xyz = xyz$ holds in $\mathfrak{A}$. Thus we get $ed = d$, i.e., $e < d$, as required.

Now, $\mathfrak{A}$, being an algebra containing a, b and c, must contain the subalgebra generated by $\{a,b,c\}$.

If $\mathfrak{A} \in \underset{\sim}{K}_1$, then $e > d$ and the subalgebra generated by $\{a,b,c\}$ must be one of the $I(j)$, $j = 1,2,3,4$.

If $\mathfrak{A} \in \underset{\sim}{K}_2$, then $e < d$. Observe that e and each of the elements in $\{a,b,c\}$ are incomparable. For :

$$f(a,b,c)a = abc = d$$

i.e., $$ea = d$$

If e and a are comparable then either $e = d$ or $e = a$ which are impossible. Thus, knowing this fact, it is easy to see that the subalgebra generated by $\{a,b,c\}$ is one of the II(j), $j = 1,2,3,4$. This completes the proof of the theorem.

The converse of Theorem 3.1 is tempting, however false, in general. In what follows, we shall construct two counter examples.

<u>Example 1.</u>

Consider the following algebra $\mathcal{A} = \langle A; \cdot, f \rangle$ (see Fig. 9) where "$\cdot$" is the semilattice operation and f is the ternary operation defined by

$$f(x,y,z) = \begin{cases} e & \text{if } \{x,y,z\} = \{a,b,c\} \\ e' & \text{if } \{x,y,z\} = \{a',b,c\} \\ xyz & \text{otherwise} \end{cases}$$

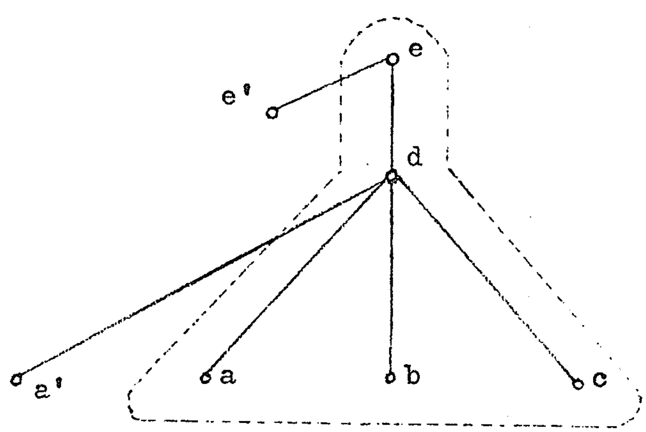


Fig. 9.

It is easy to check that $\mathcal{A}$ contains I(1) as subalgebra. But $\mathcal{A} \notin K_1$ since

$$f(a',b,c)a' = e'a' = e \neq e' = f(a',b,c),$$

i.e., $f(x,y,z)x = f(x,y,z)$ does not hold in $\mathfrak{A}$.

Example 2.

Consider the following algebra $\mathfrak{A} = \langle A;\cdot,f\rangle$ (see Fig. 10) where "$\cdot$" is the semilattice operation and the ternary operation f is defined as follows:

$$f(x,y,z) = \begin{cases} e & \text{if } \{x,y,z\} = \{a,b,c\} \\ e' & \text{if } \{x,y,z\} = \{a',b,c\} \\ xyz & \text{otherwise.} \end{cases}$$

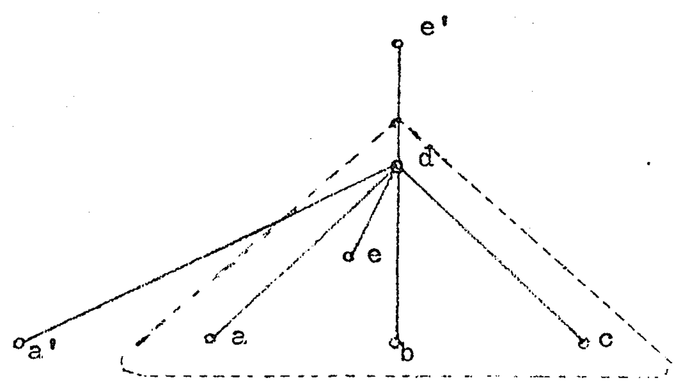


Fig. 10.

Clearly, II(1) is a subalgebra of $\mathfrak{A}$. However, $\mathfrak{A} \notin \underset{\sim}{K}_2$

since $$f(a',b,c)a' = e'a' = e' \neq d = a'bc,$$

i.e., $f(x,y,z)x = xyz$ does not hold in $\mathfrak{A}$.

In view of Theorem 3.1, in order to get some information about the sequence $\langle p_n(\mathfrak{A})\rangle$ where $\mathfrak{A}$ is an algebra representing $\langle 0,0,1,2\rangle$, it is necessary to study the relationship between the sequences $\{p_n(I(j)), p_n(II(j))\}$, $j = 1,2,3,4$.

There are redundant essential polynomials over I(j) and II(j).

Indeed, if for $n \geqslant 2$, we let $p(x_1,\ldots,x_n)$ be a non-trivial n-ary polynomial (i.e., with exactly n variables). Then we have

Lemma 3.2.

For each $j = 1,2,3,4$, $p(x_1,\ldots,x_n)$ is an essentially n-ary polynomial over I(j) and II(j).

Proof : It suffices to prove that $p(x_1,\ldots,x_n)$ depends on x_1. Observe that if we put $x_1 = \ldots = x_n = a$ (see Fig. 1 - 8), then by the idempotence of "." and "f", it follows that $p(a,\ldots,a) = a$. On the other hand, if we set $x_1 = d$, $x_2 = \ldots = x_n = a$, then by using an inductive argument, we will get $p(d,a,\ldots,a) = d$ in I(j) and II(j). Thus, as $p(a,a,\ldots,a) \neq p(d,a,\ldots,a)$, $p(x_1,\ldots,x_n)$ depends on x_1, as required.

Corollary 3.3.

Let $\mathfrak{A}$ be an algebra representing $\langle 0,0,1,2\rangle$. Then $p(x_1,\ldots,x_n)$ is essential over $\mathfrak{A}$.

Proof : Let us note that if $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then if p is an essentially n-ary polynomial over $\mathfrak{B}$, so is p over $\mathfrak{A}$. Combining this fact with Theorem 3.1 and Lemma 3.2, the Corollary follows.

Let p be a polynomial over $\mathfrak{A}$. For simplicity, we denote by $p_{\mathfrak{A}}(S)$, an element in A which is obtained from p under the substitution S.

Lemma 3.4.

Let $\mathcal{B}$ be a homomorphic image of $\mathcal{A}$. Then $p_n(\mathcal{B}) \leqslant p_n(\mathcal{A})$, for each $n = 0,1,2,\ldots$.

Proof : Let $\varphi: \mathcal{A} \longrightarrow \mathcal{B}$ be a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$. Let $p(x_1,\ldots,x_n)$ be an essentially n-ary polynomial over $\mathcal{B}$. In what follows, we shall prove that $p(x_1,\ldots,x_n)$ is an essentially n-ary polynomial over $\mathcal{A}$. It suffices to prove that p depends on x_1. By assumption, there exist $b_1, b_1', b_2,\ldots,b_n \in B$ such that $p_{\mathcal{B}}(b_1,b_2,\ldots,b_n) \neq p_{\mathcal{B}}(b_1',b_2,\ldots,b_n)$. As φ is onto, there exist $a_1,a_1',a_2,\ldots,a_n \in A$ with $a_i\varphi = b_i$ $i = 1,\ldots,n$ and $a_1'\varphi = b_1'$. Thus $p_{\mathcal{A}}(a_1,\ldots,a_n)\varphi = p_{\mathcal{B}}(a_1\varphi,\ldots,a_n\varphi)$ $= p_{\mathcal{B}}(b_1,b_2,\ldots,b_n) \neq p_{\mathcal{B}}(b_1',b_2,\ldots,b_n) = p_{\mathcal{B}}(a_1'\varphi,a_2\varphi,\ldots,a_n\varphi)$ $= p_{\mathcal{A}}(a_1',a_2,\ldots,a_n)\varphi$. Hence $p_{\mathcal{A}}(a_1,a_2,\ldots,a_n) \neq p_{\mathcal{A}}(a_1',a_2,\ldots,a_n)$, as was to be shown.

Now, let p , q be two distinct essentially n-ary polynomials over $\mathcal{B}$. Then there is a substitution S such that $p_{\mathcal{B}}(S) \neq q_{\mathcal{B}}(S)$. Let $T \subseteq A$ with $T\varphi = S$. Then $p_{\mathcal{A}}(T)\varphi = p_{\mathcal{B}}(T\varphi) = p_{\mathcal{B}}(S) \neq q_{\mathcal{B}}(S)$ $= q_{\mathcal{B}}(T\varphi) = q_{\mathcal{A}}(T)\varphi$. Hence $p_{\mathcal{A}}(T) \neq q_{\mathcal{A}}(T)$ and so $p \neq q$ over $\mathcal{A}$.

From these, it follows that $p_n(\mathcal{B}) \leqslant p_n(\mathcal{A})$ for each n, proving Lemma 3.4.

Theorem 3.5.

(1) $p_n(I(1)) = p_n(I(2)) = p_n(I(3)) = p_n(I(4))$;

(2) $p_n(II(1)) = p_n(II(2)) = p_n(II(3)) = p_n(II(4))$,

for each $n = 0,1,2,\ldots$.

Proof : We will prove only (2). The proof of (1) is similar. It is easy to check that for $i \leqslant j$, $i,j = 1,2,3,4$, $II(i)$ is a homomorphic image of $II(j)$. Thus, invoking Lemma 3.4, we have $p_n(II(1)) \leqslant p_n(II(2)) \leqslant p_n(II(3)) \leqslant p_n(II(4))$, for each n. Hence, to get (2), it suffices to prove that

$$p_n(II(4)) \leqslant p_n(II(1)), \text{ for each } n.$$

To this end, let p , q be any two essentially n-ary polynomials with $p \neq q$ over II(4). By Corollary 3.3, p and q are essentially n-ary polynomials over I(1). Thus, what we have to prove is that $p \neq q$ over I(1).

As $p \neq q$ over II(4), there exist a non-trivial substitution S such that $p_{II(4)}(S) \neq q_{II(4)}(S)$ (see Fig. 8). Observe that $p_{II(4)}(S), q_{II(4)}(S) \notin \{a,b,c\}$ since S is non-trivial. Hence, by symmetry, we have only four possible cases:

$$\begin{cases} p_{II(4)}(S) \\ q_{II(4)}(S) \end{cases} = \begin{cases} (1) & \begin{cases} d \\ ab \end{cases} \\ (2) & \begin{cases} e \\ ab \end{cases} \\ (3) & \begin{cases} ab \\ ac \end{cases} \\ (4) & \begin{cases} d \\ e \end{cases} \end{cases}$$

Suppose (1) holds, i.e.,

$$p_{II(4)}(S) = d$$
$$q_{II(4)}(S) = ab.$$

Then, since $q_{II(4)}(S) = ab$, we have $S \subseteq \{a,b,ab\}$. However, in this case, $p_{II(4)}(S) < d$, which is a contradiction. Thus, (1) is impossible.

Suppose (2) holds, i.e.,

$$p_{II(4)}(S) = e$$
$$q_{II(4)}(S) = ab.$$

Since $p_{II(4)}(S) = e$ and S is non-trivial; it follows that

$$\{a,b,c\} \subseteq S.$$

On the other hand, $q_{II(4)}(S) = ab$ implies

$$S \subseteq \{a,b,ab\}.$$

Combining the two inclusions, we have

$$\{a,b,c\} \subseteq \{a,b,ab\},$$

which is a contradiction. Thus, (2) is impossible.

Suppose (3) holds, i.e.,

$$p_{II(4)}(S) = ab$$
$$q_{II(4)}(S) = ac.$$

Clearly, $p_{II(4)}(S) = ab$ implies $S \subseteq \{a,b,ab\}$ and

$q_{II(4)}(S) = ac$ implies $S \subseteq \{a,c,ac\}$

Hence, $S \subseteq \{a,b,ab\} \cap \{a,c,ac\} = \{a\}$ and so $S = \{a\}$, which contradicts the fact that S is non-trivial. Thus, (3) is impossible.

In conclusion, we must have (4), i.e.,

$$p_{II(4)}(S) = d$$
$$q_{II(4)}(S) = e.$$

Since $q_{II(4)}(S) = e$ and S is non-trivial, it follows that

$$S \subseteq \{a,b,c,e\} .$$

In this situation, we can use the same substitution S for II(1) as $\{a,b,c,e\} \subseteq II(1)$. Clearly, we have

$$p_{II(1)}(S) = d$$

and $$q_{II(1)}(S) = e .$$

Hence $p \neq q$ over II(1), which was to be shown.

Theorem 3.6.

Let $p(x_1,\ldots,x_n)$, $q(x_1,\ldots,x_n)$ be two n-ary polynomials. Then $p \neq q$ over $\mathfrak{A}$, for each algebra $\mathfrak{A}$ representing $\langle 0,0,1,2 \rangle$ if and only if $p \neq q$ over I(1) and II(1).

Proof : One implication is trivial. Thus, assume that $p \neq q$ over I(1) and II(1). Let $\mathfrak{A}$ be an arbitrary algebra representing $\langle 0,0,1,2 \rangle$, we shall prove that $p \neq q$ over $\mathfrak{A}$.

By Theorem 3.1, $\mathfrak{A}$ contains one of the algebras I(j),II(j) j = 1,2,3,4 as subalgebra. Thus, if we know that

$$p \neq q \text{ over } I(j), II(j), j = 1,2,3,4,$$

it is clear that $p \neq q$ over $\mathfrak{A}$.

Observe that I(1),II(1) are homomorphic images of I(j), II(j), j = 1,2,3,4 respectively. Hence, if $p \neq q$ over I(1) and II(1), then $p \neq q$ over I(j) and II(j) for each j = 1,2,3,4, by Lemma 3.4. Therefore the proof of Theorem 3.6 is complete.

Theorem 3.7.

Let $\mathcal{O}\!\mathcal{C}$ be an algebra representing $\langle 0,0,1,2\rangle$. Then $p_n(\mathcal{O}\!\mathcal{C}) \geqslant \min. \{ p_n(I(1)), p_n(II(1)) \}$ for each $n = 0,1,2, \ldots$.

Proof : If $\mathcal{O}\!\mathcal{C} \in \underset{\sim}{K}_1$ then $\mathcal{O}\!\mathcal{C}$ contains $I(j)$ as a subalgebra, for some $j = 1,2,3,4$. Thus

$$p_n(\mathcal{O}\!\mathcal{C}) \geqslant p_n(I(j)) = p_n(I(1)) \geqslant \min \{ p_n(I(1)); p_n(II(1)) \}$$

If $\mathcal{O}\!\mathcal{C} \in \underset{\sim}{K}_2$, then there exists a $j = 1,2,3,4$ such that

$$p_n(\mathcal{O}\!\mathcal{C}) \geqslant p_n(II(j)) = p_n(II(1)) \geqslant \min. \{ p_n(I(1), p_n(II(1)) \}.$$

Hence the theorem follows.

4. A Lower Bound for $p_n(\mathcal{O}\!\mathcal{C})$.

Though Theorem 3.7 gives us a lower bound for $\langle p_n(\mathcal{O}\!\mathcal{C})\rangle$ in terms of $\langle p_n(I(1))\rangle$ and $\langle p_n(II(1))\rangle$, we still do not know the exact rate of increase of the sequence. In this section, we shall fill this gap by providing a lower bound for $\langle p_n(\mathcal{O}\!\mathcal{C})\rangle$. It turns out that this lower bound is much stronger than that in Corollary 3.5(II,1).

Our main result is the following:

Theorem 4.1.

Let $\mathcal{O}\!\mathcal{C}$ be an algebra representing the sequence $\langle 0,0,1,2\rangle$. Then $p_n(\mathcal{O}\!\mathcal{C}) \geqslant 11 \cdot \frac{n!}{4!}$ for each $n \geqslant 4$.

The proof of Theorem 4.1 is based on the following construction of polynomials and Lemmas 4.2, 4.3.

Construction of Polynomials.

It is a simple matter to check that the following eleven polynomials are essential and distinct over I(1) and II(1). Thus, by Theorem 3.6, it follows that for each algebra representing $\langle 0,0,1,2 \rangle$, $\mathfrak{A}$ has at least eleven essentially 4 - ary polynomials.

$$x_1\, x_2\, x_3\, x_4,$$

$$f(x_1,x_2,x_3)x_4,\ f(x_2,x_3,x_4)x_1,\ f(x_3,x_4,x_1)x_2,\ f(x_4,x_1,x_2)x_3$$

$$f(x_1x_2,x_3,x_4),\ f(x_1x_3,x_2,x_4),\ f(x_1x_4,x_2,x_3)$$

$$f(x_2x_3,x_1,x_4),\ f(x_2x_4,x_1,x_3),\ f(x_3x_4,x_1,x_2)$$

Now, for each polynomial listed above, we claim that we can construct at least five 5-ary polynomials. For instance,

1) From $x_1x_2x_3x_4$, we construct

$$\left[\begin{array}{ll} x_1x_2x_3x_4\underline{x_5} & \\ f(x_1,x_2x_3x_4,\underline{x_5}) & f(x_1x_2,x_3x_4,\underline{x_5}) \\ f(x_2,x_1x_3x_4,\underline{x_5}) & f(x_1x_3,x_2x_4,\underline{x_5}) \\ f(x_3,x_1x_2x_4,\underline{x_5}) & f(x_1x_4,x_2x_3,\underline{x_5}) \\ f(x_4,x_1x_2x_3,\underline{x_5}) & \end{array}\right.$$

2) From $f(x_1,x_2,x_3)x_4$, we construct

$$\left[\begin{array}{l} f(x_1, x_2, x_3)x_4\underline{x_5} \\ f(x_1\underline{x_5}, x_2, x_3)x_4 \\ f(x_1, x_2\underline{x_5}, x_3)x_4 \\ f(x_1, x_2, x_3\underline{x_5})x_4 \\ f(f(x_1, x_2, x_3), x_4, \underline{x_5}) \end{array}\right.$$

3) From $f(x_1x_2, x_3, x_4)$, we construct

$$\left[\begin{array}{l} f(x_1 \cdot x_2, x_3, x_4)\underline{x_5} \\ f(x_1x_2\underline{x_5}, x_3, x_4) \\ f(x_1x_2, x_3\underline{x_5}, x_4) \\ f(x_1x_2, x_3, x_4\underline{x_5}) \\ f(f(x_1, x_2, \underline{x_5}), x_3, x_4) \end{array}\right.$$

Similar constructions can be given for the other polynomials.

Suppose that we are given a 5-ary polynomial $p(x_1, \ldots, x_5)$ which is obtained from one of the eleven 4-ary polynomials by using the above Construction. Applying the same argument, it is not difficult to construct six 6-ary polynomials out of p. For example,

1) If $p = x_1x_2x_3x_4x_5$, we construct

$$\left[\begin{array}{ll} \prod_{i=1}^{5} x_i \cdot \underline{x_6} & \\ f(x_1, \prod_{i=2}^{5} x_i, \underline{x_6}), & f(x_1x_2, x_3x_4x_5, \underline{x_6}) \\ f(x_2, x_1x_3x_4x_5, x_6), & f(x_1x_3, x_2x_4x_5, \underline{x_6}) \\ \vdots & \vdots \end{array}\right.$$

2) If $p = f(x_1,x_2,x_3)x_4x_5$, we construct

$$\left[\begin{array}{l} f(x_1,x_2,x_3)x_4x_5\underline{x_6} \\ f(x_1\underline{x_6},x_2,x_3)x_4x_5 \\ f(x_1,x_2\underline{x_6},x_3)x_4x_5 \\ f(x_1,x_2,x_3\underline{x_6})x_4x_5 \\ f(f(x_1,x_2,x_3),x_4x_5,\underline{x_6}) \\ f(x_4,f(x_1x_2x_3)x_5,x_6) \\ f(x_5,f(x_1,x_2,x_3)x_4,\underline{x_6}) \end{array}\right.$$

We are now in a position to construct, by using an inductive argument, at least n + 1 (n+1)-ary polynomials out of a given n-ary polynomial.

Let $p(x_1,\ldots,x_n)$ be such a n-ary polynomials. Consider the following three types of constructions :

(A) $p(x_1,\ldots,x_n)\cdot x_{n+1}$;

(B) If there is a factor $f(A,B,C)$ in p, we set

1) $f(A\cdot\underline{x}_{n+1},B,C)$,

2) $f(A,B\cdot\underline{x}_{n+1},C)$,

3) $f(A,B,C\cdot\underline{x}_{n+1})$ in $p(x_1,\ldots,x_n)$.

(C) If there is a product $\prod_{\alpha\in\Lambda} A_\alpha$ in $p(x_1,\ldots,x_n)$ where $|\Lambda| > 1$ and Λ is maximal (in the sense that if there is a product $\prod_{\beta\in\Delta} A_\beta$ in $p(x_1,\ldots,x_n)$ with $\Lambda \subseteq \Delta$ then $\Lambda = \Delta$), we set

$$f\left(\prod_{\alpha\in I} A_\alpha, \prod_{\alpha\in J} A_\alpha, \underline{x_{n+1}}\right) \text{ in } p(x_1,\ldots,x_n)$$

where $\{I,J\}$ is a partition of the index set Λ .

Let us note that under these constructions, those (n+1)-ary polynomials have the following properties:

1) Let $p(x_1,\ldots,x_{n+1})$ be such a (n+1)-ary polynomial, the number of occurrences of each variable x_i in p is exactly one.

2) By Corollary 3.3, all such (n+1)-ary polynomial are essential.

3) All such (n+1)-ary polynomials are of different forms.

Lemma 4.2.

Let $p(x_1,\ldots,x_n)$ be an n-ary polynomial. Then the number of (n+1)-ary polynomials obtained by using the constructions (A), (B) and (C) is greater than or equal to n+1.

Proof : Let r be the number of occurrences of the symbol "f" in $p(x_1,\ldots,x_n)$. Then, by applying the constructions (A) and (B) we have at least 1 + 3r (n+1)-ary polynomials.

If $r \geqslant \frac{n}{3}$, then $1+3r \geqslant 1+n$, and the proof is complete.

Thus, we may assume $r < \frac{n}{3}$, in other words, $3r < n$.

In order to have n+1 (n+1)-ary polynomials, we need $(n+1) - (3r+1) = n-3r$ more (n+1)-ary polynomials. Observe that we have at most 3r positions in all the forms f(, ,), therefore at least n-3r variables appear in the product form $\prod A_i$ in p. Hence, by construction (C) we get at least n-3r (n+1)-ary polynomials. This completes the proof of Lemma 4.2.

For a given n-ary polynomial which is constructed by induction, we obtain, by Lemma 4.2, at least n+1 (n+1)-ary polynomials different in forms. The question arises: are

these(n+1)-ary polynomials distinct over algebra representing $\langle 0,0,1,2 \rangle$? The answer to this question is in the affirmative. Indeed, this can be seen from the following somewhat stronger version:

Lemma 4.3.

Let $\mathfrak{A}$ be an algebra representing $\langle 0,0,1,2 \rangle$. Let p , q be two essentially n-ary polynomials over $\mathfrak{A}$ such that

1) The number of occurrences of each variable x_i $(i=1,\ldots,n)$ in both p and q is exactly one;

2) p and q are in different forms.

Then $p \neq q$ over $\mathfrak{A}$.

Proof: In view of Theorem 3.6, it suffices to show that $p \neq q$ over I(1) and II(1).

If $p = \prod_{i=1}^{n} x_i$, then by condition 2), there is a factor f(A,B,C) in q . Let us construct a substitution S for the variables x_i's with resprect to the algebra II(1) in such a way that

$$f(A,B,C)(S) = f(a,b,c)$$

and $\quad x_i(S) = e$ for every x_i not in $f(A,B,C)$.

Observe that according to the condition 1) such a substitution exists.

In this situation, we get

$$p_{II(1)}(S) = \begin{Bmatrix} abc \\ \text{or} \\ abce \end{Bmatrix} = d;$$

while $q_{II(1)}(S) = e$.

Thus, $p \neq q$ over II(1).

To show that $p \neq q$ over I(1), we construct a substitution T for the variables x_i's with respect to the algebra I(1) in such a way that

$$f(A,B,C)(T) = f(a,b,c)$$

and $\quad x_i(T) = d$ for every x_i not in $f(A,B,C)$.

Again, such a T exists by 1). In this case, we get

$$p_{I(1)}(T) = \begin{Bmatrix} abc \\ \text{or} \\ abcd \end{Bmatrix} = d ;$$

while $\quad q_{I(1)}(T) = e \circ d = e$.

Hence, $p \neq q$ over I(1), which was to be shown.

Now, we may assume that both p and q are different from $\prod_{i=1}^{n} x_i$. In other words, both p and q include f(, ,) as a factor. By assumption 2) there is a factor $f_*(A_1,A_2,A_3)$ in p, say, such that for each factor $f(B_1, B_2, B_3)$ in q , $A_j \neq B_i$ for some $i, j = 1, 2, 3$. Let us choose such a $f_*(A_1, A_2, A_3)$ in which the number of occurrences of variables is minimum. If we construct a substitution S for the variables x_i's w.r.t II(1) in such a way that

$$f_*(A_1, A_2, A_3)(S) = f(a, b, c)$$

and $\quad x_i(S) = e$ if x_i is not in $f_*(A_1, A_2, A_3)$,

then we get $\quad p_{II(1)}(S) = e$

while

$$q_{II(1)}(S) = \left\{ \begin{array}{c} abc \\ \text{or} \\ abce \end{array} \right\} = d \ .$$

(Note that there is no factor $f(a,b,c)$in $q_{II(1)}(S)$; for otherwise, if $f_\Delta(B_1, B_2, B_3)$ exists in q such that $f_\Delta(B_1, B_2, B_3)(S) = f(a, b, c)$, we would change the roles of f_Δ and f_*).

Hence, $p \neq q$ over II(1).

To show that $p \neq q$ over I(1), we use the same argument as about but instead of the substitution S, we construct T in such a way that

$$f_*(A_1, A_2, A_3)(T) = f(a, b, c)$$

and $\quad x_i(T) = d$ if x_i is not in $f_*(A_1, A_2, A_3)$.

It then follows that

$$p_{I(1)}(T) = e \circ d = e$$

while $\quad q_{I(1)}(T) = d$.

Thus, $p \neq q$ over I(1).

The proof of Lemma 4.2 is therefore complete.

Proof of Theorem 4.1 : For $n \geqslant 4$, let $\overline{p_n(\mathfrak{A})}$ be the set of all essentially n-ary polynomials over $\mathfrak{A}$. If N is the set of essentially n-ary polynomials constructed by induction, let R(N) denote the set of all essentially (n+1)-ary polynomials induced by some element in N by using the constructions (A),(B) or (C). Then by Lemmas 4.2 and 4.3, we have

$$|R(N)| \geqslant (n+1)\,|N|.$$

Evidently,

$$\overline{p_{n+1}(\mathfrak{A})} \supseteq \underbrace{R(R(\ldots(R}_{n-3}(E))\ldots)),$$

where E is the set of those eleven 4-ary polynomials.

From this, it follows that

$$\begin{aligned} p_{n+1}(\mathfrak{A}) &= |\overline{p_{n+1}(\mathfrak{A})}| \\ &\geqslant (n+1)n(n-1)\ldots 5|E| \\ &= 11.\frac{(n+1)!}{4!} \qquad \text{for each } n \geqslant 3. \end{aligned}$$

This completes the proof of our main Theorem.

5. Finite Subdirectly Irreducible Algebras.

An algebra $\mathfrak{A}$ is called <u>subdirectly irreducible</u> if the relation $\wedge(\Theta_i \mid i \in I) = \omega$ implies that $\Theta_i = \omega$ for some $i \in I$, where for each $i \in I$, $\Theta_i \in C(\mathfrak{A})$. Equivalently, $\mathfrak{A}$ is <u>subdirectly irreducible</u> if there exist $u, v \in A$ such that $u \neq v$ and $u \equiv v(\Theta)$ for each $\Theta > \omega$.

From Birkhoff's classical result (see [1]) on subdirect decompositions which asserts that every algebra having more than one element is a subdirect product of subdirectly irreducible algebras, it thus follows that subdirectly irreducible algebras play an important role in the study of structure of algebras.

The purpose of this section is to give a characterization theorem for finite subdirectly irreducible algebras in the equational class $\underset{\sim}{K}_1$. To this end, we shall first define some notation and basic concepts.

Let $\mathfrak{A} \in \underset{\sim}{K}_1$ such that $|A|$ is finite. Consider the following set of triples:

$$T = \{(a_i,b_i,c_i)/a_i,b_i,c_i \in A,\ f(a_i,b_i,c_i) \neq a_ib_ic_i\} .$$

Let us make the following elementary observations:

1) Since $f(x,y,z) \neq xyz$ in $\mathfrak{A}$, thus, $T \neq \emptyset$ and $|T|$ is finite.

2) As "f" and "." are symmetric, hence if $(a,b,c) \in T$, so does $(a\alpha,b\alpha,c\alpha)$, where α is any permutation on the set $\{a,b,c\}$.

3) If $(a,b,c) \in T$, then it follows from the proof of Theorem 3.1 that $\{a,b,c\}$ are pairwise distinct and incomparable. Thus $abc > a,b,c$.

For each i and for each $(a_i,b_i,c_i) \in T$, denote

$$e_i = f(a_i,b_i,c_i) \quad ; \quad d_i = a_ib_ic_i.$$

Since $f(x,y,z)xyz = f(x,y,z)$ holds in $\underset{\sim}{K}_1$, we have $e_i > d_i$, for each i . Let

$$E = \{ e_i / (a_i, b_i, c_i) \in T \}.$$

Then E is a finite subposet of A. Thus, the set of maximal elements in E is not empty. By the Axiom of Chóice, let e* be a maximal element of E.

Let $d_j^* = a_j b_j c_j$ where $\{a_j, b_j, c_j\}$ is a triple of elements of A such that $f(a_j, b_j, c_j) = e^*$. Let d* be one of the maximal elements in $\{d_j^*\}$. We have $e^* > d^*$.

Let $a \in A$. Then a is said to be a component of an element of T if there exist $b, c \in A$ such that $(a,b,c) \in T$. Thus, it follows that if a is not a component of any element of T, then

$$f(a,u,v) = auv \text{ for all } u,v \in A.$$

We are, now, ready to establish the following

Theorem 5.1.

Let $\mathfrak{A}$ be a finite algebra in $\underset{\sim}{K}_1$. Then $\mathfrak{A}$ is subdirectly irreducible if and only if $\mathfrak{A}$ satisfies all the following conditions:

1) $e^* \succ d^*$

2) Let $u,v \in A$ such that $u \prec v$ and $\{u,v\} \neq \{d^*, e^*\}$. If both u and v are not components of elements of T, then there exists $p \in A$ such that $u < p$ but $v \not< p$.

3) For each c, c' in A such that i) $c \succ c'$ and ii) $(a,b,c) \in T$ if, and only if $(a,b,c') \in T$, if $f(a,b,c) = f(a,b,c')$ for $(a,b,c) \in T$, then there

exists $p \in A$ such that $c' \prec p$ but $c \not\preccurlyeq p$.

<u>Proof</u> : Let $\mathfrak{A}$ be a finite subdirectly irreducible algebra in $\underset{\sim}{K}_1$. We shall show that conditions 1), 2) and 3) hold in $\mathfrak{A}$. Firstly, we have <u>Claim</u>: e^* is the maximum element of $\mathfrak{A}$.

Suppose that this is not the case, let m be the largest element of $\mathfrak{A}$. Then $m > e^*$. If

(1) There exist $m_1, m_2 \in A$, $m_1 \neq m_2$ such that $m_1 \prec m$, $m_2 \prec m$ (see Fig. 11), then consider the following two equivalence relations

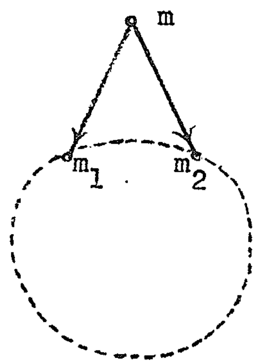


Fig. 11

$$\Phi_1 = \{(x,x) / x \in A\} \cup \{(m_1,m),(m,m_1)\} ,$$

$$\Phi_2 = \{(x,x) / x \in A\} \cup \{(m_2,m),(m,m_2)\} .$$

It is a simple matter to check that both m_1 and m_2 cannot be components of any elements of T. Thus, from the previous observation 3), it follows that Φ_1 and Φ_2 have the Substitution Property with respect to "." and "f" . Hence, Φ_1 and Φ_2 are congruence relations of $\mathfrak{A}$. Clearly, $\Phi_1, \Phi_2 > \omega$, but $\Phi_1 \wedge \Phi_2 = \omega$. This contradicts the fact that $\mathfrak{A}$ is subdirectly irreducible. Therefore (1) is not the case, but then we have

(2) there exist $m_1, m_2 \in A$ such that $m_2 \prec m_1 \prec m$ (see Fig. 12)

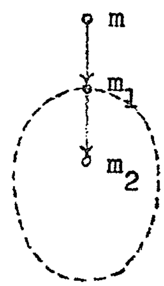


Fig. 12

Consider the following two equivalence relations :

$$\Phi_1 = \{ (x,x)/x \in A \} \cup \{(m,m_1),(m_1,m)\} ,$$
$$\Phi_2 = \{ (x,x)/x \in A \} \cup \{(m_1,m_2),(m_2,m_1)\} .$$

Clearly, m_1 cannot be a component of elements of T. The same is true of m_2. For, if it were, then there exist b, c $\in$ A such that $(m_2,b,c) \in T$. So, $e_2 = f(m_2,b,c) > m_2bc > m_2$. Since $m > e^*$, thus $m > e_2$ and so $m_1 \geq e_2$ (note that we assume (1) is not the case !). We have

$$m > m_1 \geq e_2 > m_2bc > m_2 ,$$

which contradicts the fact that $m_1 \succ m_2$.

As m_1 and m_2 are not components of elements of T, it follows that Φ_1 and Φ_2 have the Substitution Property with respect to "f" and ".". Thus, $\Phi_1, \Phi_2 \in C(\mathcal{A})$. However, the fact that $\Phi_1, \Phi_2 > \omega$ and $\Phi_1 \wedge \Phi_2 = \omega$ contradicts our assumption that $\mathcal{A}$ is subdirectly irreducible.

Hence, we conclude that e* is the largest element in $\mathcal{A}$, as was to be shown.

By applying the same argument as in the first part, it follows that if $e^* \succ p$ for some p, then $e^* > x$ implies $p \geqslant x$. In other words, the algebra $\mathfrak{A}$ is illustrated in Fig. 13.

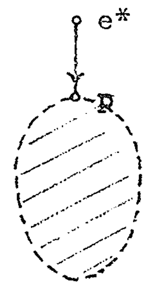


Fig. 13

We are now in a position to prove the statements 1), 2) and 3). Assume that 1) is not the case, i.e., $e^* \not\succ d^*$, then there exist $p_1, p_2 \in [d^*, e^*)$ such that $e^* \succ p_1 \succ p_2$. Clearly, p_1 is not a component of elements of T. The same is true of p_2. For, if it were, then there exist $b, c \in A$ such that $(p_2, b, c) \in T$. Thus $e_2 = f(p_2, b, c) > p_2bc > p_2$. If $e_2 = e^*$, then $p_2bc = d_2^* > p_2 \geqslant d^*$, contradicting the fact that d^* is maximal in $\{d_j^*\}$. If $e_2 < e^*$, then by the above observation, $e_2 \leqslant p_1$ and so $e^* \succ p_1 \geqslant e_2 > p_2bc > p_2$, which contradicts $p_1 \succ p_2$. Consequently, p_2 is not a component of elements of T. From this, it follows that equivalence relations

$$\Phi_1 = \{(x,x) / x \in A\} \cup \{(e^*, p_1), (p_1, e^*)\} ,$$
$$\Phi_2 = \{(x,x) / x \in A\} \cup \{(p_1, p_2), (p_2, p_1)\}$$

have the Substitution Property with respect to "f" and ".". Thus $\Phi_1, \Phi_2 \in C(\mathfrak{A})$. As $\Phi_1 \wedge \Phi_2 = \omega$, while $\Phi_1, \Phi_2 > \omega$, we get a contradiction. Hence, $e^* \succ d^*$, proving 1).

To prove 2), suppose to the contrary that there exist u, v $\in$ A such that $u \prec v$, $\{u,v\} \neq \{d^*, e^*\}$, both u and v are not components of elements of T and that $u < x$ implies $v \leq x$ for each $x \in A$.

Let us consider the following equivalence relation

$$\Phi = \{(x,x) / x \in A\} \cup \{(u,v),(v,u)\} .$$

Claim : $\Phi \in C(\mathfrak{A})$.

We first prove that Φ has the Substitution Property with respect to ".". It suffices to show that $u \equiv v$ (Φ) and $x \equiv x$ (Φ) imply $ux \equiv vx$ (Φ). (1).If $x \leq u$, then $x < v$. Thus $ux = u \equiv v = vx$ (Φ). (2).If $x \parallel u$, then $ux > u$. By assumption, $ux \geq v$. Thus, $ux \geq vx$. On the other hand, $u < v$ implies $ux \leq vx$. Hence $ux = vx$ and so $(ux,vx) \in \Phi$. (3).If $x > u$, then by assumption $x \geq v$. Thus, $ux = x = vx$ and so $(ux,vx) \in \Phi$.

It remains to prove that Φ has the Substitution Property with respect to "f". Observe that $u \equiv v$ (Φ), $v \equiv u$ (Φ) and $x \equiv x$ (Φ) imply that $f(u,v,x) \equiv f(v,u,x)$ (Φ), since $f(u,v,x) = f(v,u,x)$. Thus, it suffices to show that $u \equiv v$ (Φ), $x \equiv x$ (Φ) and $y \equiv y$ (Φ) imply $f(u,x,y) \equiv f(v,x,y)$ (Φ). To see this, we note that as u and v are not components of elements of T, by a previous observation, we obtain

$$f(u,x,y) = uxy \text{ and } f(v,x,y) = vxy.$$

Hence, $f(u,x,y) = uxy \equiv vxy = f(v,x,y)$ (Φ), as required. Therefore, $\Phi \in C(\mathfrak{A})$.

Consider the following congruence relation of $\mathfrak{A}$,

$$Ⓗ(e^*,d^*) = \{(x,x)/x \in A\} \cup \{(e^*,d^*),(d^*,e^*)\}.$$

Clearly, $Ⓗ(e^*,d^*)$, $\Phi > \omega$, while $Ⓗ(e^*,d^*) \wedge \Phi = \omega$, since $\{u,v\} \neq \{d^*,e^*\}$ and $u \prec v$, $d^* \prec e^*$. Thus, $\mathfrak{A}$ is subdirectly reducible, a contradiction. Hence, 2) follows.

Again, assume that 3) is not the case. Then there exist c, c' in A such that $c \succ c'$, $(a,b,c) \in T$ iff $(a,b,c') \in T$ and $f(a,b,c) = f(a,b,c')$ for $(a,b,c) \in T$, but $c' < x$ implies $c \leq x$ for each $x \in A$.

In what follows, we shall prove that the following equivalence relation $\Phi = \{(x,x) / x \in A\} \cup \{(c,c'),(c',c)\}$ is in $C(\mathfrak{A})$. The method used above can be applied again to show that Φ has Substitution Property with respect to "$\circ$". To show the same for "f", we need only check that

$$\left.\begin{array}{l} c \equiv c' \ (\Phi) \\ x \equiv x \ (\Phi) \\ y \equiv y \ (\Phi) \end{array}\right\} \quad \text{imply } f(x,y,c) \equiv f(x,y,c') \ (\Phi).$$

This is, indeed, the case for (1) If $(x,y,c) \in T$, then $(x,y,c') \in T$ and $f(x,y,c) = f(x,y,c')$ by assumption. Thus, $f(x,y,c) \equiv f(x,y,c') \ (\Phi)$. (2) If $(x,y,c) \notin T$, then $(x,y,c') \notin T$ by assumption. Hence, $f(x,y,c) = xyc$ and $f(x,y,c') = xyc'$. From the fact that $xyc \equiv xyc' \ (\Phi)$, it follows that $f(x,y,c) \equiv f(x,y,c') \ (\Phi)$. Hence $\Phi \in C(\mathfrak{A})$.

Since e^* and d^* cannot be components of elements of T and $\{e^*,d^*\} \neq \{c,c'\}$; thus we have

$$\Theta(e^*,d^*) > \omega, \ \Phi > \omega \text{ but } \Theta(e^*,d^*) \wedge \Phi = \omega,$$

which is a contradiction. Therefore 3) follows.

Conversely, let $\mathfrak{A}$ be a finite algebra in $\underset{\sim}{K}_1$ satisfying the conditions 1), 2), and 3). What we are going to prove is that is subdirectly irreducible. To achieve this, we prove, for each Ⓗ $\in C(\mathfrak{A})$, Ⓗ $> \omega$, that $d^* \equiv e^*$ (Ⓗ) always holds.

For preparation, we make the following observation:

<u>Claim 1.</u> e^* is the greatest element of $\mathfrak{A}$.

Suppose that this is false. Ley m be the greatest. Then $m > e^*$. Let $s \in A$ be such that $m \succ s \geqslant e^*$. Evidently, m, s cannot be components of elements of T and $\{m,s\} \neq \{e^*,d^*\}$. Thus, by virtue of 2), there exists p in A such that $s < p$ but $m \nleq p$. The fact that m is the greatest element implies $p < m$. Thus, we have $s < p < m$, contradicting the fact that $s \prec m$. Hence, e^* is greatest.

<u>Claim 2.</u> If $x < e^*$, then $x \leqslant d^*$ for each x in A.

Suppose to the contrary that there is a y in A such that $y < e^*$ but $y \nleq d^*$. Let q be in A such that $y \leqslant q \prec e^*$. Clearly, $q \nleq d^*$. Furthermore, $\{q,e^*\} \neq \{d^*,e^*\}$ and both q and e^* are not components of elements of T. Thus, according to the condition 2), there is a p in A with $q < p$ but $e^* \nleq p$. Since e^* is the greatest element of A, it follows that $e^* > p > q$. This, however, contradicts the fact that $e^* \succ q$. Thus, Claim 2 follows.

From the above two observations, it is now clear that $\mathfrak{A}$ is illustrated in Fig. 13.

Let Ⓗ $\in C(\mathfrak{A})$ such that Ⓗ $> \omega$. Then there exist s, t in A such that $s \neq t$ and $s \equiv t$ (Ⓗ). Since, (1) $s \equiv t$ (Ⓗ) implies $st \equiv t$ (Ⓗ) and (2) $s \equiv t$ (Ⓗ) implies $x \equiv y$ (Ⓗ) for each x, y in $[s,t]$ if $s < t$; hence, without loss of generality, we may assume $s \prec t$.

Suppose that $e^* \not\equiv d^*$ (Ⓗ). Then $\{s,t\} \not\equiv \{d^*,e^*\}$. In fact, $s \prec t \leq d^*$. Let us choose a pair $\{s,t\}$ such that s is maximal. (Note that this is possible as $\mathcal{O}\!\mathcal{L}$ is finite.)

We have the following four cases :

<u>Case I.</u> Both s and t are not components of elements of T.

<u>Case II.</u> s is a component of an element of T but t not.

<u>Case III.</u> t is a component of an element of T but s not.

<u>Case IV.</u> Both s and t are components of elements of T.

<u>Case I.</u> In this case, by 2), there is a p in A such that $s < p$ but $t \not< p$, (see Fig. 14). Now, $s \equiv t$ (Ⓗ) implies

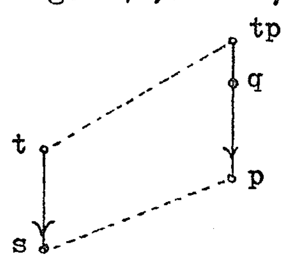


Fig. 14

that $sp \equiv tp$ (Ⓗ) and so $p \equiv tp$ (Ⓗ). Clearly, $tp > p > s$. Let $q \in (p,tp]$ such that $q \succ p$. As $p \equiv q$ (Ⓗ), it follows that $\{p,q\} \not\equiv \{d^*,e^*\}$ where $p > s$.

<u>Case II.</u> Since s is a component of elements of T; thus, there exist b, c in A such that $(s,b,c) \in T$. We have

$$e_s = f(s,b,c) > sbc.$$

As t is not a component of any element of T; thus, $(t,b,c) \notin T$ and so $f(t,b,c) = tbc$.

Observe that $s \equiv t\ (\textcircled{H})$ implies $f(s,b,c) \equiv f(t,b,c)\ (\textcircled{H})$, i.e., $e_s \equiv tbc\ (\textcircled{H})$. If (1) $e_s \neq tbc$, then as $e_s > s$ and $tbc \geqslant t > s$, we can choose p, q in A such that $p \prec q$, $p \equiv q$ where $p, q \in [\mathrm{minimal}(e_s, tbc), e_s \cdot (tbc)]$. Obviously, $p > s$ and $\{p,q\} \neq \{d^*, e^*\}$. If (2) $e_s = tbc$, then as $tbc = e_s >$ sbc and $t \equiv s\ (\textcircled{H})$ implies $tbc \equiv sbc\ (\textcircled{H})$, there exist p, q in $[sbc, tbc]$ such that $p \prec q$ and $p \equiv q\ (\textcircled{H})$. Observe that $(s,b,c) \in T$ implies $sbc > s$ and thus $p > s$.

Case III. Since t is a component of element of T; there exist b,c in A such that $(t,b,c) \in T$. Thus, $e_t = f(t,b,c) >$ tbc. Because s is not a component of any element of T, we obtain $f(s,b,c) = sbc$. As $s \equiv t\ (\textcircled{H})$ implies $f(s,b,c) \equiv f(t,b,c)\ (\textcircled{H})$, we get $sbc \equiv e_t\ (\textcircled{H})$. Observe that $sbc > s$. For, if $sbc = s$ then $bc \leqslant s$ and so $tbc \leqslant ts = t$. On the other hand, as $(t,b,c) \in T$, it follows that $tbc > t$, a contradiction. Hence $sbc > s$, as required. From this, we have (see Fig. 15)

$$e_t > tbc \geqslant sbc > s\ .$$

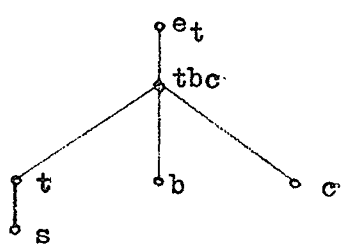


Fig. 15

Let $p, q \in [sbc, e_t]$ such that $p \prec q$. Clearly, $p \equiv q\ (\textcircled{H})$ and $\{p,q\} \neq \{d^*, e^*\}$ where $p > s$.

Case IV. In this case, we assume that both s and t are components of elements of T.

If (1) there exist a, b in A such that $(a,b,s) \in T$ but $(a,b,t) \notin T$, then the same argument as in Case II can be applied. If (2) there exist a, b in A such that $(a,b,t) \in T$ but $(a,b,s) \notin T$, then the situation is same as Case III. If (1) and (2) are not the case, we have (3), $(a,b,s) \in T$ iff $(a,b,t) \in T$ for $a, b \in A$.

If $f(a,b,s) \neq f(a,b,t)$ for some $a,b \in A$, then $s \equiv t$ (Ⓗ) implies $f(a,b,s) \equiv f(a,b,t)$ (Ⓗ) where

$$f(a,b,s) > abs > s$$

$$\text{and} \quad f(a,b,t) > abt > t > s.$$

Let p, q be in [minimal $\{f(a,b,s),f(a,b,t)\}$, $f(a,b,s)f(a,b,t)$] such that $p \prec q$. Clearly, $p \equiv q$ (Ⓗ), $\{p,q\} \neq \{d^*,e^*\}$ and $p > s$.

Now, if $f(a,b,s) = f(a,b,t)$ for all such a, b in A, then by 3) there is a p in A such that $s < p$ but $t \not\leq p$ (see Fig 16).

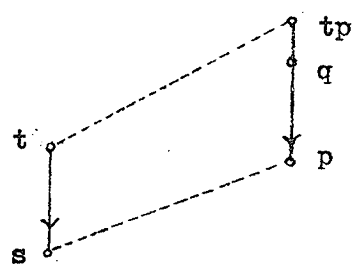


Fig. 16

Clearly, $s \equiv t$ (Ⓗ) implies $sp \equiv tp$ (Ⓗ), i.e., $p \equiv tp$ (Ⓗ).
As $tp > p$, let $q \in (p,tp]$ such that $p \prec q$. We have $p \equiv q$ (Ⓗ), $\{p,q\} \neq \{d^*,e^*\}$ and $p > s$.

Hence, in any case, we obtain a pair $\{p,q\}$ with $p \prec q$, $\{p,q\} \neq \{d^*,e^*\}$, $p \equiv q$ (Ⓗ) and $p > s$. This, however, contradicts the maximality of s. Therefore the assumption that

$e^* \not\equiv d^*$ (Ⓗ) is false. Consequently, for each Ⓗ $\in C(\mathfrak{A})$, Ⓗ $> \omega$ we have $e^* \equiv d^*$ (Ⓗ). Hence, $\mathfrak{A}$ is subdirectly irreducible, which was to be shown.

Thus, the proof of Theorem 5.1 is complete.

Following from the proof of Theorem 5.1, we have

Corollary 5.2.

Let $\mathfrak{A}$ be a finite algebra in $\underset{\sim}{K}_1$. If $\mathfrak{A}$ is subdirectly irreducible, then e^* is the greatest element of $\mathfrak{A}$. Moreover, $\mathfrak{A}$ has one and only one dual atom d^* which contains every element other than e^*.

Let $\mathfrak{A} \in \underset{\sim}{K}_1$. By Theorem 3.1, $\mathfrak{A}$ contains one of the four algebras $I(j)$, $j = 1,2,3,4$ as subalgebra. In particular, however, if $\mathfrak{A}$ is assumed to be a finite subdirectly irreducible algebra, then by virtue of the Characterization Theorem 5.1, we are able to prove that $\mathfrak{A}$ must contain $I(1)$ as subalgebra. Indeed, we have the following :

Corollary 5.3.

The algebra $I(1)$ is subdirectly irreducible. The algebra $I(j)$ is subdirectly reducible, for each $j = 2,3,4$.

Corollary 5.4.

Let $\mathfrak{A}$ be a finite algebra in $\underset{\sim}{K}_1$. If $\mathfrak{A}$ is subdirectly irreducible, then $\mathfrak{A}$ contains $I(1)$ as subalgebra.

Proof : Let $\mathfrak{A}$ be a finite subdirectly irreducible algebra in $\underset{\sim}{K}_1$.

In view of Corollary 5.2, $\mathfrak{A}$ can be represented by Fig. 17.

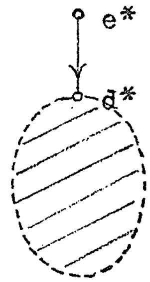


Fig. 17

Let us choose a triple (a^*,b^*,c^*) in T such that

$$f(a^*,b^*,c^*) = e^* \text{ and } a^*b^*c^* = d^*.$$

If $a^*b^* = a^*c^* = b^*c^* = d^*$, then, clearly, $\mathfrak{A}$ contains the algebra of Fig. 18 as subalgebra. In this case, the proof is complete.

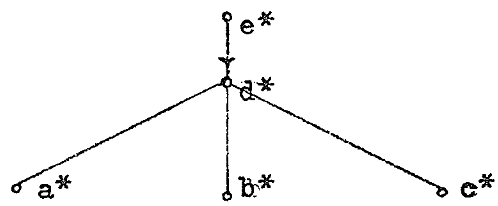


Fig. 18

Thus, we may assume, say, $a^*b^* < d^*$. Let s be in $[a^*b^*,d^*)$ such that $s \prec d^*$ (see Fig. 19).

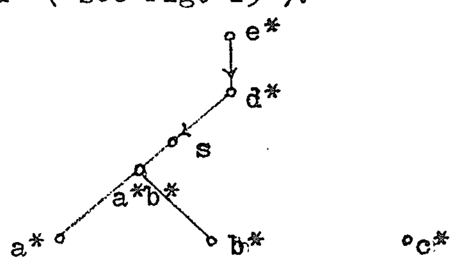


Fig. 19

If s is not a component of any element of T, then by condition 2) of Theorem 5.1, there exists p in A such that $s < p$ but $d^* \not< p$. This implies that $p \neq e^*$;i.e., $p < e^*$. As $p \neq d^*$, by Corollary 5.2, $p < d^*$. Thus, it follows that $s < p < d^*$, which contradicts the fact that $s \prec d^*$.

Therefore, s must be a component of some element of T, and so there exist b_s, c_s in A such that $(s, b_s, c_s) \in T$ (see Fig. 20).

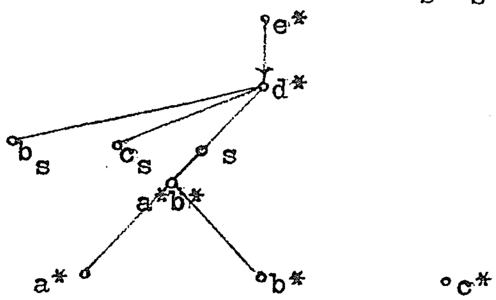


Fig. 20

Evidently, $f(s,b_s,c_s) > sb_sc_s > s$. Moreover, as b_s, c_s $\neq e^*$, we have b_s, $c_s \leqslant d^*$. Thus, b_s, $c_s < d^*$ since d^* cannot be a component of any element of T. Clearly, $sb_sc_s \leqslant d^*$. If $sb_sc_s < d^*$, we would obtain $s < sb_sc_s < d^*$, contradicting the fact that $s \prec d^*$. Hence, it follows that $sb_sc_s = d^*$. But then $f(s,b_s,c_s) > sb_sc_s = d^*$ and therefore $f(s,b_s,c_s) = e^*$. Now, if the elements s,b_s,c_s are such that $sb_s = sc_s = b_sc_s = d^*$, then contains the algebra of Fig. 21 as subalgebra. In this case, the proof is complete.

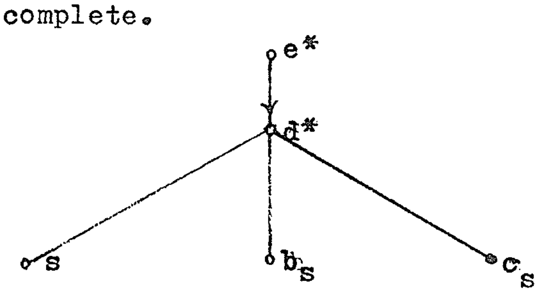


Fig. 21

Thus, we may assume $b_sc_s < d^*$ (note that as $s \prec d^*$, $sb_s = sc_s = d^*$!). Following the same argument, we will abtain elements t_1, t_2, $t_3 \in$ A such that $(t_1,t_2,t_3) \in T$, $f(t_1,t_2,t_3) = e^* > t_1t_2t_3 = d^* > t_i$, $i = 1,2,3$ and $d^* \succ t_1 \geqslant b_sc_s$. If

$t_1 t_2 = t_2 t_3 = t_3 t_1 = d^*$, the proof is complete. Otherwise, we would continue this process. Since $\mathfrak{A}$ is finite, the process will stop after finitely many steps. Hence, the Corollary follows.

The following examples 1 and 2 are subdirectly irreducible algebras in $\underset{\sim}{K}_1$ while example 3 is an algebra in $\underset{\sim}{K}_1$ which is subdirectly reducible.

Example 1.

The algebra is the semilattice of Fig. 22 with the ternary operation "f" defined as follows :

$$f(x,y,z) = \begin{cases} e_i & \text{if } \{x,y,z\} = \{a_i, b_i, c_i\} \\ xyz & \text{otherwise} \end{cases}$$

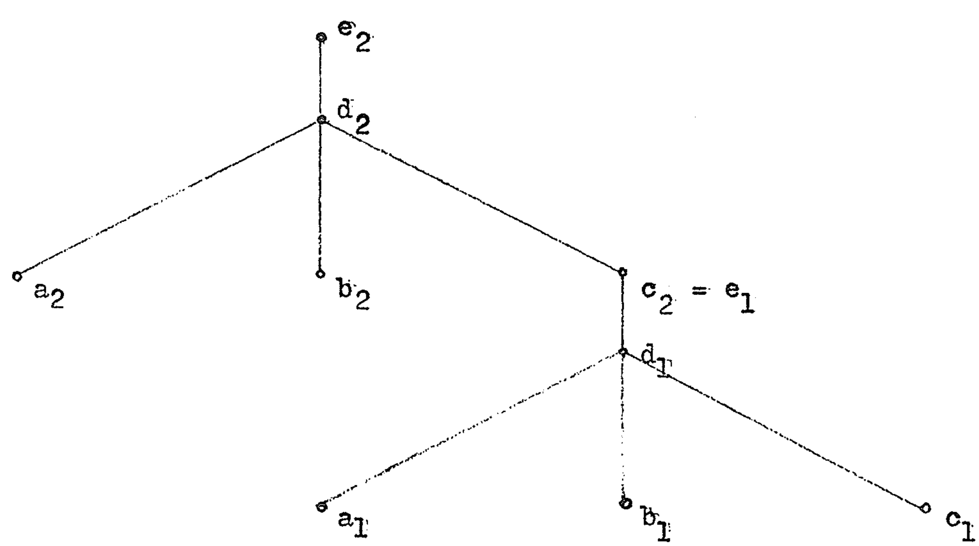


Fig. 22

Example 2.

The algebra is the semilattice of Fig. 23 with the ternary operation "f" defined as above.

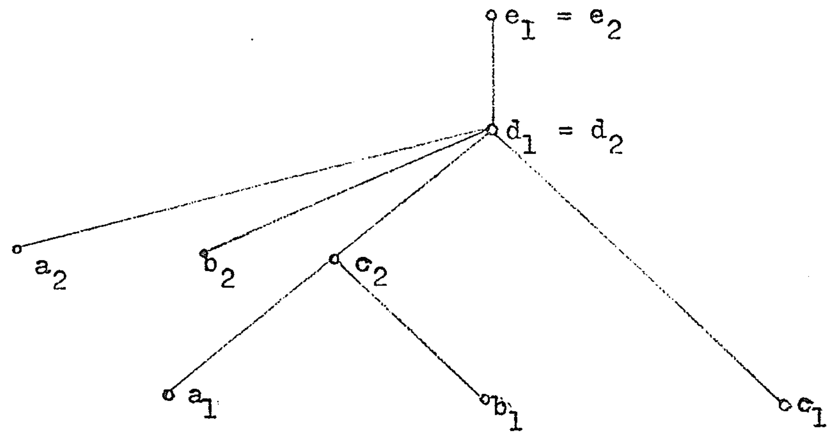


Fig. 23

Example 3.

The algebra is the semilattice of Fig. 24 with the ternary operation "f" defined as above.

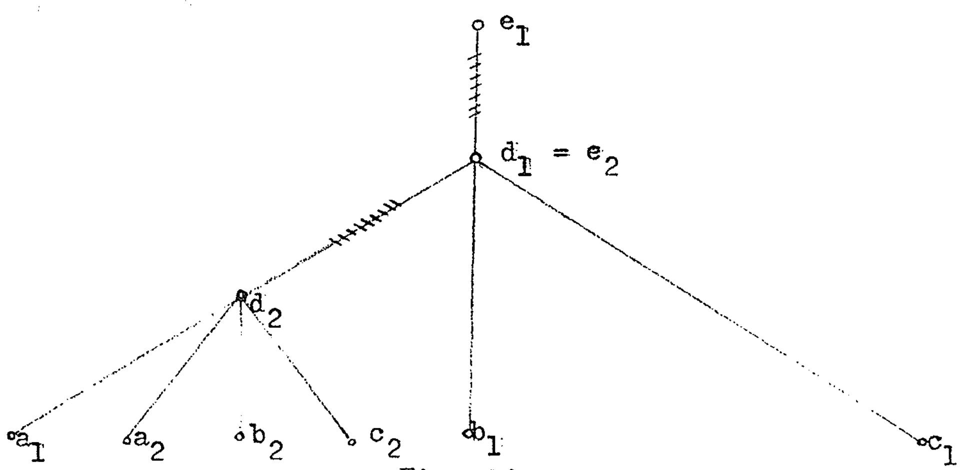


Fig. 24

Remark.

It is conjectured that the corresponding results on finite subdirectly irreducible algebras can similarly be obtained in the equational class $\underset{\sim}{K}_2$. In this case, of course, the role that I(1) plays in $\underset{\sim}{K}_1$ will be replaced by II(1) in $\underset{\sim}{K}_2$.

CHAPTER 3

ALGEBRAS REPRESENTING $\langle 0,0,1,m\rangle$

In this chapter, we shall deal with the representability of the sequence $\langle 0,0,1,m\rangle$, for an arbitrary natural number m. The case m = 1 and m = 2 have been considered before. By a result of G. Grätzer and R. Padmanabhan (see [10]), it is known that if $\mathfrak{A}$ is the idempotent reduct of an abelian group of exponent 3, then $p(\mathfrak{A}) = \langle 0,0,p_2,p_3, \ldots\ldots, p_n, \ldots\ldots\rangle$ where $p_n = \frac{1}{3}(2^n - (-1)^n)$. Thus, in particular, the sequence $\langle 0,0,1,3\rangle$ is representable. It is, therefore, natural to ask : given an arbitrary natural m, is the sequence $\langle 0,0,1,m\rangle$ always representable ?

Let $\mathfrak{A}$ be an algebra representing $\langle 0,0,1\rangle$. Then $\mathfrak{A}$ has one and only one idempotent, commutative essentially binary polynomial. If $p_3(\mathfrak{A}) \geqslant 3$, then the binary polynomial is not necessary associative. However, if we assume that it is associative, then for each natural number m, it is possible to find such algebra $\mathfrak{A}$ such that $p_3(\mathfrak{A}) = m$. More precisely, we are able to construct a semilattice with certain ternary operations defined on it so that the resulting algebra has exactly m essentially ternary polynomials. Thus, the sequence $\langle 0,0,1,m\rangle$ is, a fortiori, representable, for each natural number m, which solves the above problem.

1. Construction of Algebras.

In this section, we start to construct algebras which will be shown to meet the requirement in next section.

For each natural number m, let us consider the four types of semilattices $\langle A; \circ \rangle$ described by Fig. 25——28

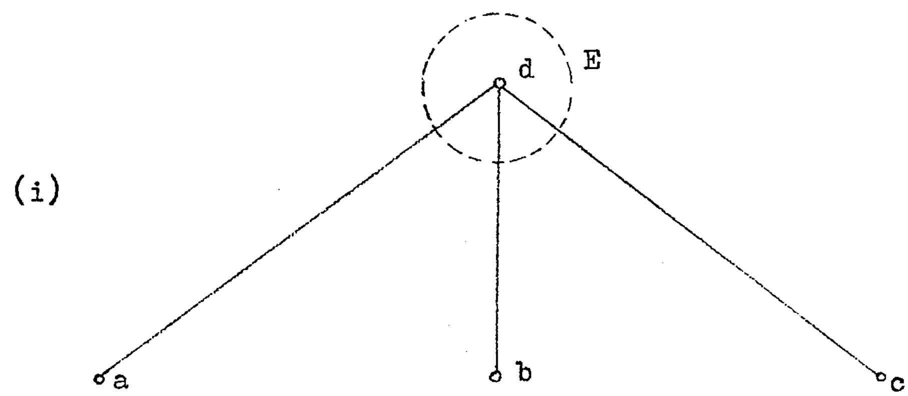


Fig. 25

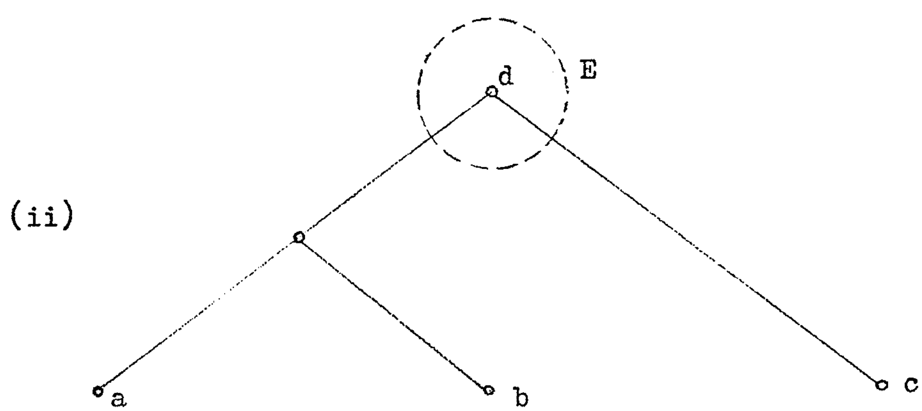


Fig. 26

(iii)

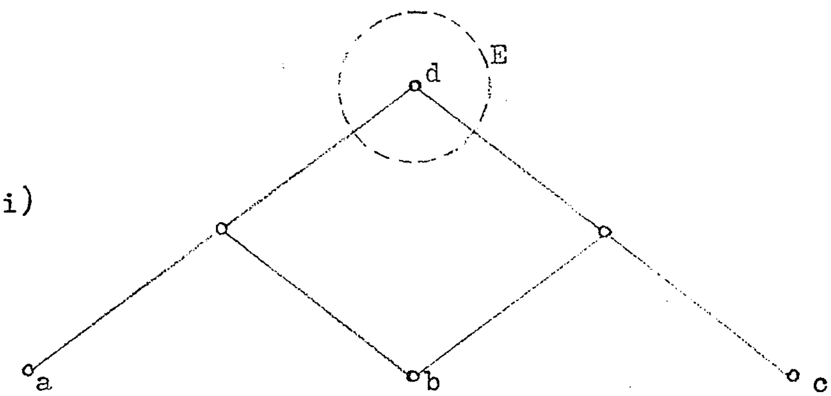


Fig. 27

(iv)

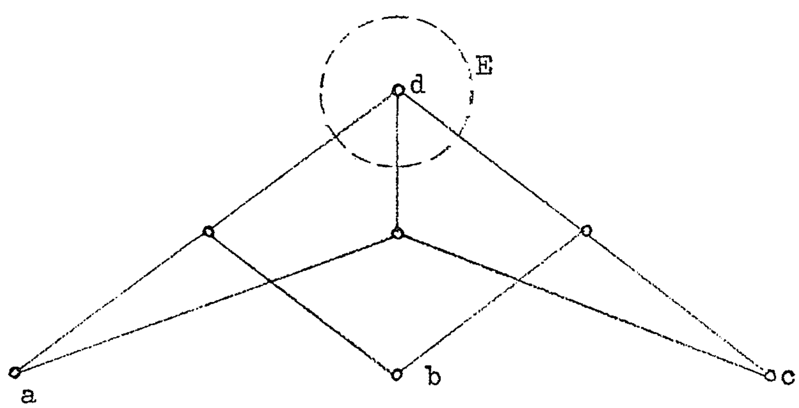


Fig. 28

For each type of semilattice, the subset

$$E = \{ d, e_1, e_2, \ldots, e_{m-1} \}$$

is a m-element subsemilattice such that for each $i = 1, 2, \ldots, m-1$,

either (a) $e_i > d$;

or (b) $e_i < d$ but e_i is incomparable with every element of $A - E$.

Or (c) e_i is incomparable with every element of $(A - E) \cup \{d\}$.

Given any of the semilattices $\langle A; \circ \rangle$, we shall define, for each $i = 1,2,\ldots,m-1$, a ternary operation f_i on A as follows:

$$f_i(x,y,z) = \begin{cases} e_i & \text{if } \{x,y,z\} = \{a,b,c\} \\ xyz & \text{otherwise} \end{cases}$$

Evidently, we have the following elementary observations:

(1) The polynomial $f_i(x,y,z)$ is essentially ternary, for each $i = 1,2,\ldots,m-1$.

(2) $f_i(x,x,x) = x$ for each $i = 1,2,\ldots,m-1$.

(3) $f_i(x,y,z)$ is symmetry, for each $i = 1,2,\ldots,m-1$

(4) If $i \neq j$ then $f_i(x,y,z) \neq f_j(x,y,z)$ over A.

Now, to any of the semilattices $\langle A; \cdot \rangle$, we can associate an idempotent algebra

$$\mathfrak{A} = \langle A; F \rangle$$

where $F = \{\cdot, f_1, f_2, \ldots, f_{m-1}\}$ consists of one semilattice operation and m-1 ternary operations which are defined as above.

Remark.

If $m = 2$, there are exactly eight different algebras (up to isomorphism) obtained from the construction. They are isomorphic to I(j), II(j) j = 1,2,3,4 respectively.

If $m = 3$, we obtain exactly twenty different algebras from the construction. For instance, the following are constructed from the first type of semilattice,

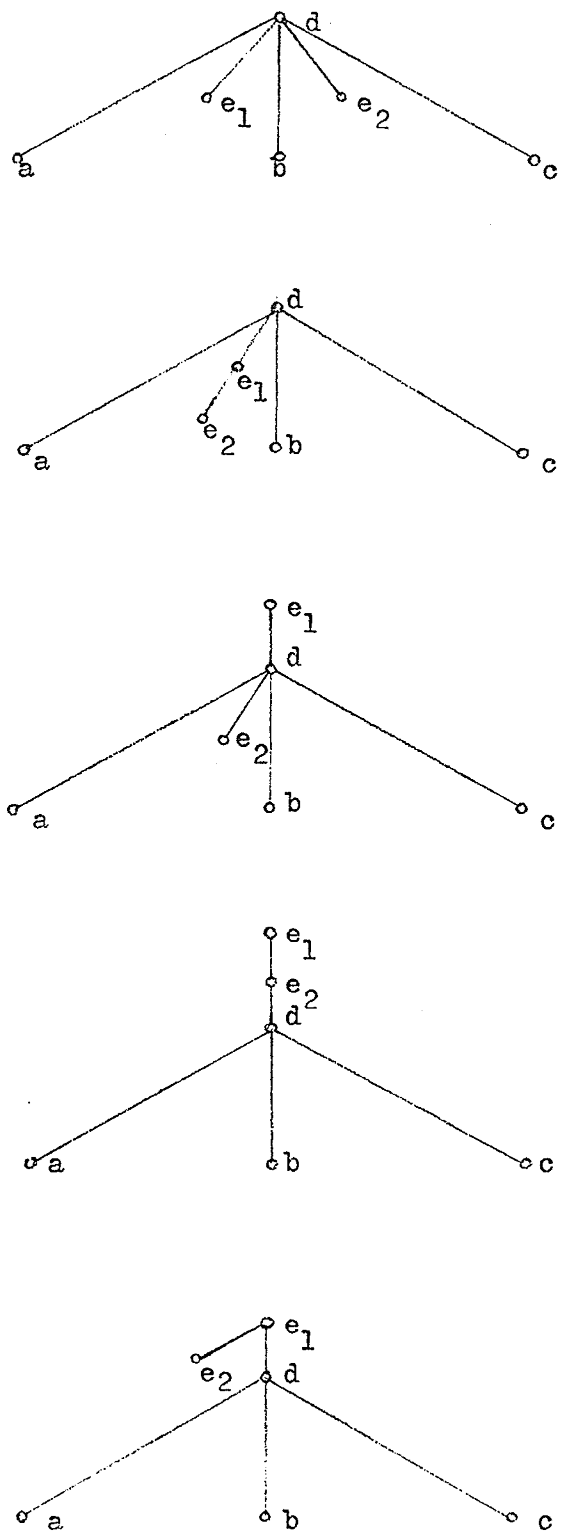


Fig. 29

Where for each $i = 1,2$, f_i is defined as above.

2. Main Theorem.

We are now in a position to prove the following

Theorem 2.1.

For each natural number m, any one of the algebras where $|E| = m$ as constructed in section 1 has exactly m essentially ternary polynomials.

Corollary 2.2.

Let $\underset{\sim}{K}$ be the class of all idempotent algebras with a semilattice operation. Then, for each natural number m, the sequence $\langle 0,0,1,m \rangle$ is representable in $\underset{\sim}{K}$.

Proof of Theorem 2.1 : Let $\mathfrak{A}$ be such a given algebra with $|E| = m$. Then $\mathfrak{A}$ has at least the following m essentially ternary polynomials:

$$xyz,\ f_1(x,y,z),\ \ldots\ldots\ldots,\ f_{m-1}(x,y,z).$$

Thus, to show that $p_3(\mathfrak{A}) = m$, we have to show that the collection of all these m essentially ternary polynomials is closed under substitution. In other words, let $p(x,y,z)$ be an arbitrary essentially ternary polynomial over $\mathfrak{A}$. Our purpose is to show that $p(x,y,z)$ is the same as one of the above m essentially ternary polynomials.

To this end, let $p(x,y,z)$ be given. First of all, we claim that: $p(a,b,c) \in E$.

Suppose that this is not the case. Then $p(a,b,c) \in A - E \subseteq \{a,b,c,ab,ac,bc\}$ (note that the later set depends on $\mathfrak{A}$). However, by definition of $f_i(x,y,z)$ and the fact that $abc = d$, the above inclusion is impossible. Hence, $p(a,b,c) \in E$, as required.

Similarly, $p(a\alpha,b\alpha,c\alpha)$ is an element of E for any permutation α on the set $\{a,b,c\}$.

Next, we prove that $p(x,y,z)$ is symmetric over $\mathfrak{A}$. It suffices to show that $p(x,y,z) = p(y,x,z)$. The other cases can be shown similarly.

Let S be a substitution for $\{x,y,z\}$.

If $S \neq \{a,b,c\}$, then evidently by definition of f_i, we have

$$p(x,y,z)(S) = (xyz)(S) = (yxz)(S) = p(y,x,z)(S).$$

If $S = \{a,b,c\}$, say $x(S) = a$, $y(S) = b$, $z(S) = c$, we have to prove $p(a,b,c) = p(b,a,c)$.

For simplicity, let us make the following conventions:

(1) If there is a factor $f_i(a,b,c)$ in $p(a,b,c)$, we denote $f_i(a,b,c)$ by e_i;

(2) If there is a factor $f_i(u,v,w)$ in $p(a,b,c)$ where $\{u,v,w\} \neq \{a,b,c\}$, we denote $f_i(u,v,w)$ by uvw.

(3) If there is a factor abc in $p(a,b,c)$, we denote abc by d.

Thus, from these and the fact that "." is a semilattice operation, it follows immediately that

$$p(a,b,c) = \begin{cases} \prod_{i\in I} e_i \\ \text{or} \quad (\prod_{i\in I} e_i)a \; [\text{or } (\prod_{i\in I} e_i)b \text{ or } (\prod_{i\in I} e_i)c] \\ \text{or} \quad (\prod_{i\in I} e_i)ab \; [\text{or } (\prod_{i\in I} e_i)bc \text{ or } (\prod_{i\in I} e_i)ac] \\ \text{or} \quad (\prod_{i\in I} e_i)d \end{cases}$$

where I is a finite subset of $1,2,\ldots,m-1$, possibly empty.

If $p(a,b,c) = \prod_{i\in I} e_i$ or $(\prod_{i\in I} e_i)d$, then from the fact that $f_i(x,y,z)$ and xyz are symmetric, the result follows.

If $p(a,b,c) = (\prod_{i\in I} e_i)a$, then $p(b,a,c) = (\prod_{i\in I} e_i)b$. We shall prove that $(\prod_{i\in I} e_i)a = (\prod_{i\in I} e_i)b$.

For simplicity, write $e = \prod_{i\in I} e_i$. Since E is a subsemilattice of A. Thus, $e_i \in E$ imply $e \in E$

Case 1. $e \prec d$.

In this case, as $e \in E$ and $e \prec d$ we have

$$ea = d = eb.$$

Case 2. $e \not\prec d$

In this case, clearly, we have

$$ea = ed = eb.$$

Thus, $p(a,b,c) = p(b,a,c)$, as was to be shown.

For the other possible values of $p(a,b,c)$, the proof is similar . Hence, we conclude that $p(x,y,z)$ is, indeed, symmetric.

Since $p(a,b,c) \in E$, it follows that $p(a,b,c) = d$ or $p(a,b,c) = e_i$, for some $i = 1,\ldots,m-1$.

(1) If $p(a,b,c) = d$, we claim that $p(x,y,z) = xyz$.

Let S be a substitution for the variables x,y,z. If $S \neq \{a,b,c\}$ then clearly

$$p(x,y,z)(S) = (xyz)(S).$$

If $S = \{a,b,c\}$, then by symmetry of p(x,y,z), we have

$$p(x,y,z)(S) = p(a,b,c) = d = abc = (xyz)(S).$$

Hence $p(x,y,z) = xyz$, as required.

(2) If $p(a,b,c) = e_i$, for some $i = 1,\ldots,m-1$. We claim that $p(x,y,z) = f_i(x,y,z)$.

Let S be a substitution for the variables x,y,z. If $S \neq \{a,b,c\}$, then clearly,

$$p(x,y,z)(S) = (xyz)(S) = (f_i(x,y,z))(S).$$

If $S = \{a,b,c\}$, then by symmetry of p(x,y,z), we have

$$\begin{aligned} p(x,y,z)(S) &= p(a,b,c) \\ &= e_i \\ &= f_i(a,b,c) \\ &= f_i(x,y,z)(S). \end{aligned}$$

Hence $p(x,y,z) = f_i(x,y,z)$.

Therefore, we conclude that

$$p(x,y,z) = \begin{cases} xyz & \text{if } p(a,b,c) = d \\ f_i(x,y,z) & \text{if } p(a,b,c) = e_i \end{cases}$$

which completes the proof of Theorem 2.1.

PART II

IDEMPOTENT ALGEBRAS WITH ONE ESSENTIALLY m-ARY POLYNOMIAL, $m \geq 2$

CHAPTER 1

ALGEBRAS REPRESENTING $\langle 0,0,a_1 \ldots, a_k, m \rangle$ WITH $a_1 = \ldots = a_k = 1$

In this chapter, we shall, naturally, deal with the same problems as in Part I for the more general seqence $\langle 0,0,a_1 \ldots, a_k, m \rangle$ with $a_1 = \ldots = a_k = 1$ where k,m are arbitrary positive integers. It turns out that almost all the results in Part I can be extended to this general case.

The case that m = 2 will be examined in section 1. We obtain a generalization of Theorem 3.4(I,1). Moreover, a lower bound for sequence $\langle p_n \rangle$ is provided in this case. In section 2, the representability Theorem for the sequence $\langle 0,0,a_1,\ldots,a_k,m \rangle$ with $a_1 = \ldots = a_k = 1$, k,m are arbitrary possitive integers, is established. It is applied, in section 3, to prove a characterization Theorem about the sequence $\langle 0,0,1,..1,m,n \rangle$

1. The Case m = 2.

Let $\mathfrak{A}$ be an algebra representing the sequence $\langle 0,0,\overbrace{1,\ldots,1}^{k},2 \rangle$ where k is a positive integer (we may assume $k > 1$). By Lemma 1.2(I,1), $\mathfrak{A}$ has a unique semilattice operation. Since $p_{k+2}(\mathfrak{A}) = 2$, it follows that there is one and only one essentially (k+2)-ary polynomial, denoted by $f(x_1,\ldots,x_{k+2})$, which is distinct from $\prod_{i=1}^{k+2} x_i$ over $\mathfrak{A}$. For the sake of simplicity, we write $n = k+2$. Clearly, f is idempotent and symmetric. If we identify $x_1 = x_2$ in f, we obtain the polynomial $f(x_2,x_2,x_3,\ldots,x_n)$.

As $\mathcal{O}\!\mathcal{L}$ represents $\langle 0,0,1,\ldots,1\rangle$ and f is symmetric, we have

$$f(x_2,x_2,x_3,\ldots,x_n) = \begin{cases} x_2 \\ \prod_{i=3}^{n} x_i \\ \prod_{i=2}^{n} x_i \end{cases}$$

If $f(x_2,x_2,x_3,\ldots,x_n) = x_2$, setting $x_3 = x_4$, we obtain $x_3 = x_2$, which is impossible. If $f(x_2,x_2,x_3,\ldots,x_n) = \prod_{i=3}^{n} x_i$, setting $x_3 = x_4$, it follows that $x_2 \cdot \prod_{i=5}^{n} x_i = \prod_{i=4}^{n} x_i$. Set $x_4 = \ldots = x_n$. We have $x_2 \cdot x_n = x_n$, a contradiction. Thus it is necessary that

(1) $f(x_2,x_2,x_3\ldots,x_n) = \prod_{i=2}^{n} x_i$.

From this, it follows immediately that

(2) $f(x_1x_2,x_1,x_2,x_4,\ldots,x_n) = x_1x_2\prod_{i=4}^{n} x_i$.

Let us now consider the polynomial $p = f(x_1x_2,x_2,x_3,\ldots,x_n)$. Setting $x_1 = x_2$, we obtain by (1) that $p = \prod_{i=2}^{n} x_i$. Thus, p depends on x_i, for each $i = 3,\ldots,n$. Setting $x_3 = x_4$, we have $p = x_1x_2\prod_{i=4}^{n} x_i$. Thus p depends on x_1 and x_2. Hence p is essentially n-ary.

If $f(x_1x_2,x_2,\ldots,x_n) = f(x_1,\ldots,x_n)$ ——— (A),

then $\prod_{i=1}^{n} x_i = f(\prod_{i=1}^{n} x_i, \prod_{i=2}^{n} x_i, x_1\prod_{i=3}^{n} x_i, x_1x_2\prod_{i=4}^{n} x_i, \ldots, \prod_{i=1}^{n-2} x_i \cdot x_n)$ (2)

$= f(x_1, \prod_{i=2}^{n} x_i, x_1\prod_{i=3}^{n} x_i, x_1x_2\prod_{i=4}^{n} x_i, \ldots, \prod_{i=1}^{n-2} x_i \cdot x_n)$ (A)

$$= f(x_1, x_2, \ldots, x_i x_j, \ldots, x_n) \qquad (A)$$

$$= f(x_1, x_2, \ldots, x_j, \ldots, x_n), \qquad (A)$$

which is a contradiction. Thus, we obtain

(3) $f(x_1 x_2, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i$.

Let us write $\prod_j x_i = x_1 x_2 \cdots x_{j-1} x_{j+1} \cdots x_n$ for each $j = 1, \ldots, n$. The polynomial $f(\prod_1 x_i, \prod_2 x_i, \ldots, \prod_n x_i)$ is clearly essentially n-ary.

If $f(\prod_1 x_i, \prod_2 x_i, \ldots, \prod_n x_i) = f(x_1, x_2, \ldots, x_n) \qquad (B)$

then $\prod_{i=1}^{n} x_i = f(\prod_{i=1}^{n} x_i, \ldots, \prod_{i=1}^{n} x_i)$

$$= f(\prod_2 x_i \cdot \prod_3 x_i \cdot \cdots \prod_n x_i, \ \ldots\ldots, \prod_1 x_i \cdot \prod_2 x_i \cdot \cdots \prod_{n-1} x_i)$$

$$= f(\prod_1 x_i, \prod_2 x_i, \ldots, \prod_n x_i)$$

$$= f(x_1, x_2, \ldots, x_n), \qquad (B)$$

a contradiction. Thus, it follows that

(4) $f(\prod_1 x_i, \prod_2 x_i, \ldots, \prod_n x_i) = \prod_{i=1}^{n} x_i$.

Because of what we have now proved, it is evident that we can apply arguments similar to those of Chapter 1 of Part I to derive several identities that hold in $\mathfrak{A}$. In this way it can be shown that there are two types of algebras satisfying the identities

$f(x_1, \ldots, x_n) x_1 = f(x_1, \ldots, x_n)$ ———————— I

$f(x_1, \ldots, x_n) x_1 = \prod_{i=1}^{n} x_i$ ———————— II

respectively.

In conclusion, we arrive at the following result.

Theorem 1.1.

For each positive integer $k = 1,2,\ldots,$ there exist two equational classes of algebra $\underset{\sim}{K}_{1k}$ and $\underset{\sim}{K}_{2k}$ such that an algebra $\mathfrak{A}$ represents the sequence $\langle 0,0,\overbrace{1,\ldots,1}^{k},2\rangle$ if and only if $\mathfrak{A}$ can be represented as an algebra of type $\langle 2, k+2\rangle$ belonging to either $\underset{\sim}{K}_{1k}$ or $\underset{\sim}{K}_{2k}$.

If $k = 2$, the following examples can easily be checked to represent the sequence $\langle 0,0,1,1,2\rangle$. Let us note that the free semilattice on four generators consists of 15 elements, and that therefore , there are eleven non-isomorphic semilattices generated by four elements, the homomorphic images of the free one. Hence, we have the following twenty two algebras, (see Fig. 30)

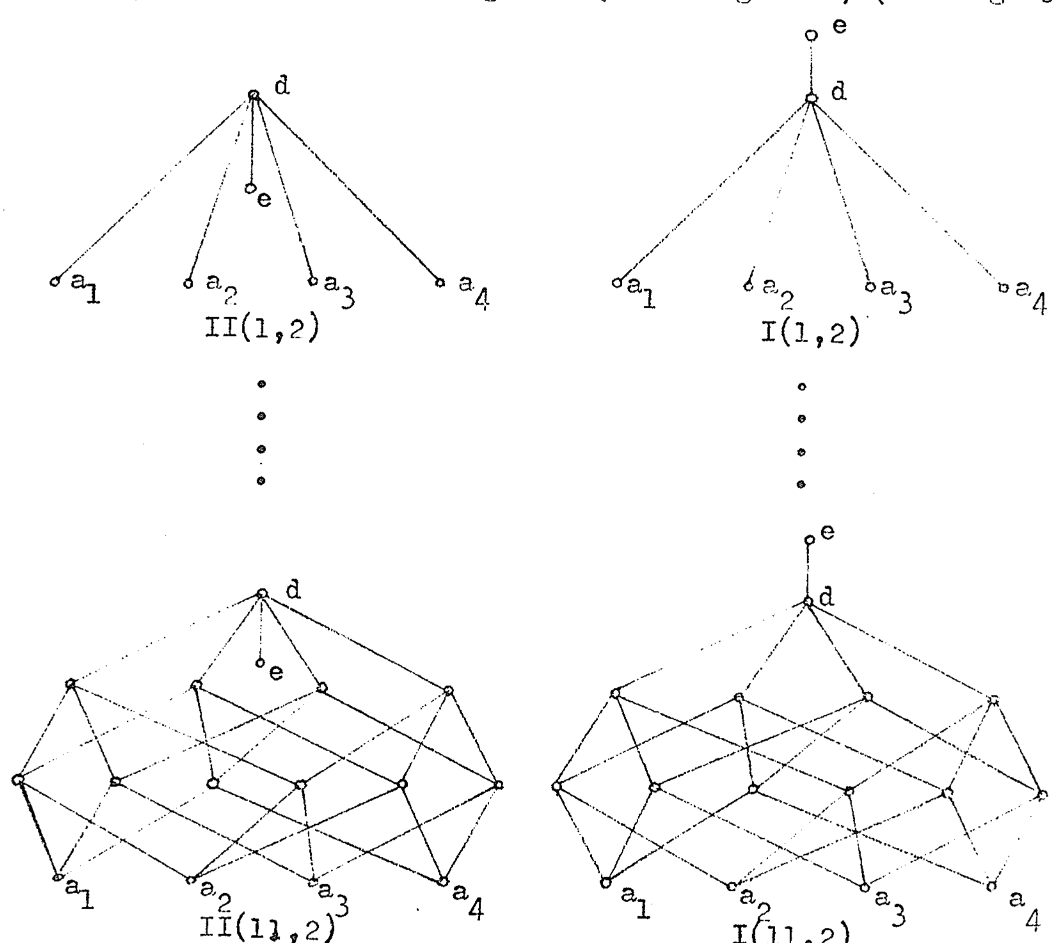


Fig. 30

where each algebra I(j,2), II(j,2) is a semilattice shown as above and the 4-ary operation f is defined by

$$f(x_1,x_2,x_3,x_4) = \begin{cases} e & \text{if } \{x_1,x_2,x_3,x_4\} = \{a_1,a_2,a_3,a_4\}, \\ \prod_{i=1}^{4} x_i & \text{otherwise}. \end{cases}$$

In general, for each positive integer k, the free semilattice on k+2 generators consists of $2^{k+2} - 1$ elements. Hence, there are $2^{k+2} - (k+3)$ non-isomorphic semilattice generated by k+2 elements which are the homomorphic images of the free one. Therefore, by using the same idea, we can contruct $2^{k+3} - 2(k+3)$ algebras i.e., I(j,k), II(j,k), $j = 1,2,\ldots, 2^{k+2} - (k+3)$ each of which represents the sequence $\langle 0,0,\overbrace{1,\ldots,1}^{k},2\rangle$.

The following result can similarly be shown.

<u>Theorem 1.2.</u>

Let $\mathfrak{A}$ be an algebra representing the sequence $\langle 0,0,\overbrace{1,\ldots,1}^{k},2\rangle$ where k is an arbitrary positive integer. Then $\mathfrak{A}$ can be represented as an algebra of type $\langle 2,k+2\rangle$ which contains one of the algebras I(j,k), II(j,k), $j = 1,2,\ldots,2^{k+2} - (k+3)$ as a subalgebra.

Let $\mathfrak{A}$ be an algebra in $\underset{\sim}{K}_{1k}$ or $\underset{\sim}{K}_{2k}$. Let p, q be two essentially n-ary polynomials over $\mathfrak{A}$. By Theorem 1.2, it follows that if $p \neq q$ over I(1,k) and II(1,k), then $p \neq q$ over $\mathfrak{A}$. In view of this fact, we can now provide a lower bound for the sequence $\langle p_n(\mathfrak{A})\rangle$.

Theorem 1.3.

Let $\mathcal{O}\!\mathcal{L}$ be an algebra representing the sequence $\langle 0,0,\overbrace{1,\ldots,1}^{k},2\rangle$ for $k \geqslant 1$. Then $p_n(\mathcal{O}\!\mathcal{L}) \geqslant \left[1 + \frac{1}{2}(k+3)(k+4)\right](k+3)^{n-(k+3)}$, for each $n \geqslant k+3$.

Proof : It is not difficult to check that the following $1 + \frac{1}{2}(k+3)(k+4)$ polynomials are distinct and essential over the algebras $I(1,k)$ and $II(1,k)$. Thus, by the above observation, $\mathcal{O}\!\mathcal{L}$ has at least $1 + \frac{1}{2}(k+3)(k+4)$ essentially $(k+3)$-ary polynomials, namely :

$$\prod_{i=1}^{k+3} x_i,$$

$$f(x_1,\ldots,x_{k+2})x_{k+3},\quad f(x_2,\ldots,x_{k+3})x_1,\quad \ldots,\quad f(x_{k+3},x_1,\ldots,x_{k+1})x_{k+2},$$

$$f(x_1,\ldots,x_{k+1},x_{k+2},x_{k+3}),\ldots\ldots,\ f(x_2,\ldots,x_{k+2},x_1x_{k+3})$$

We shall now construct new polynomials inductively from the above polynomials in the following manner :

Let $p(x_1,\ldots,x_n)$ be an n-ary polynomial obtained by induction. With respect to p, we produce the following (n+1)-ary polynomials by using three types of constructions :

(A) $p(x_1,\ldots,x_n)\cdot\underline{x}_{n+1}$;

(B) If there is a factor $f(A_1,\ldots,A_{k+2})$ in p, we then set

1) $f(A_1\underline{x}_{n+1},A_2,\ldots,A_{k+2})$,

2) $f(A_1,A_2\underline{x}_{n+1},\ldots,A_{k+2})$,

⋮

k+2) $f(A_1,A_2,\ldots,A_{k+2}\underline{x}_{n+1})$;

(C) If there is a product $\prod_{j \in J} A_j$ in p where J is maximal (in the sense of Theorem 4.1(I,2)), we set

$$f(\prod_{j \in J(1)} A_j, \ldots, \prod_{j \in J(k+1)} A_j, \underline{x}_{n+1})$$ in p (if this is possible) where $\{ J(1), \ldots, J(k+1) \}$ is a partition of the index set J.

Now, observe that if there is an occurrence of "f" in p, then by constructions (A) and (B), we have at least k + 3 (n+1)-ary polynomials. If there is no occurrence of "f" in $p(x_1, \ldots, x_n)$, then since $n \geqslant k + 3$, we obtain at least k + 3 (n+1)-ary polynomials by using the constructions (A) and (C). Applying the same argument as in Theorem 4.1(I,2), it follows that all such (n+1)-ary polynomials are essential and distinct over $\mathfrak{A}$. Therefore, the theorem is proved.

For each positive interger k, let $\underset{\sim}{K}_{1k}$, $\underset{\sim}{K}_{2k}$ be the two equational classes of algebras as shown in Theorem 1.1. We have the following

Theorem 1.4.

There is a one-to-one but not onto mapping from $\underset{\sim}{K}_{is}$ to $\underset{\sim}{K}_{it}$ for each i = 1,2, if s < t.

Proof : We consider the case i = 1. The other case is similar.

For simplicity, we replace s, t by s - 2 and t - 2 respectively. Let $\mathfrak{A} \in \underset{\sim}{K}_{1(s-2)}$. Set

$$D = \left\{ (a_1, \ldots, a_s) \,/\, a_i \in A,\ f(a_1, \ldots, a_s) \neq \prod_{i=1}^{s} a_i \right\}.$$

Since $f(x_1x_2, x_2, \ldots, x_s) = \prod_{i=1}^{s} x_i$ and

$f(x_1,x_1,x_3,\ldots,x_s) = x_1 \prod_{i=3}^{s} x_i$ hold in $\mathfrak{A}$, it follows that for each $(a_1,\ldots,a_s) \in D$, the elements a_i's are pairwise distinct and incomparable.

To each $(a_1,\ldots,a_s) \in D$, let us adjoin a set of new elements $\{a_{s+1},\ldots,a_t\}$ in such a way that

1) The set $\{a_1,\ldots,a_s,a_{s+1},\ldots,a_t\}$ is pairwise distinct and incomparable;

2) $a_j \prec \prod_{i=1}^{s} a_i$, for each $j = s+1,\ldots,t$;

3) $a_j < c$ iff $\prod_{i=1}^{s} a_i \leqslant c$ for $c \in A$, where $j = s+1,\ldots,t$;

4) $a_j \,||\, b$, for each $b \not\geq \prod_{i=1}^{s} a_i$ in A, $j = s+1,\ldots,t$;

5) If $(a_1,\ldots,a_s)$, $(b_1,\ldots,b_s) \in D$ and $\{a_1,\ldots,a_s\} \neq \{b_1,\ldots,b_s\}$, then $a_j \,||\, b_r$ for $j, r = s+1,\ldots,t$.

Let $A^* = A \cup \bigcup \left\{ \{a_{s+1},\ldots,a_s\} \,/\, (a_1,\ldots,a_s) \in D \right\}$. Then A^* is a semilattice. Define a t-ary operation g in A^* as follows :

$$g(x_1,x_2,\ldots,x_t) = \begin{cases} f(a_1,\ldots,a_s) & \text{if } \{x_1,\ldots,x_t\} = \{a_1,\ldots,a_s,a_{s+1},\ldots,a_t\} \\ & \text{and } (a_1,\ldots,a_s) \in D \,; \\ \prod_{i=1}^{t} x_i & \text{otherwise .} \end{cases}$$

From the fact that $\mathfrak{A} \in \underset{\sim}{K}_{1(s-2)}$, it is easy to show that the algebra $\mathfrak{A}^* = \langle A^*;\cdot,g\rangle$ belongs to $\underset{\sim}{K}_{1(t-2)}$. Moreover, it follows from our construction that if $\mathfrak{A} \not\cong \mathfrak{B}$, then $\mathfrak{A}^* \not\cong \mathfrak{B}^*$. Thus, the mapping $\varphi : \underset{\sim}{K}_{1(s-2)} \longrightarrow \underset{\sim}{K}_{1(t-2)}$ defined by $\varphi(\mathfrak{A}) = \mathfrak{A}^*$ for $\mathfrak{A}$ in $\underset{\sim}{K}_{1(s-2)}$ is one-to-one. Clearly, φ is not onto since there does not exist an algebra $\mathfrak{A}$ in $\underset{\sim}{K}_{1(s-2)}$ such that $\mathfrak{A}^* = I(2^t-(t+1),t-2)$.

Corollary 1.5.

For each i = 1, 2 and $k \geq 1$, we have $|\underset{\sim}{K}_{ik}| \leqslant |\underset{\sim}{K}_{i(k+1)}|$.

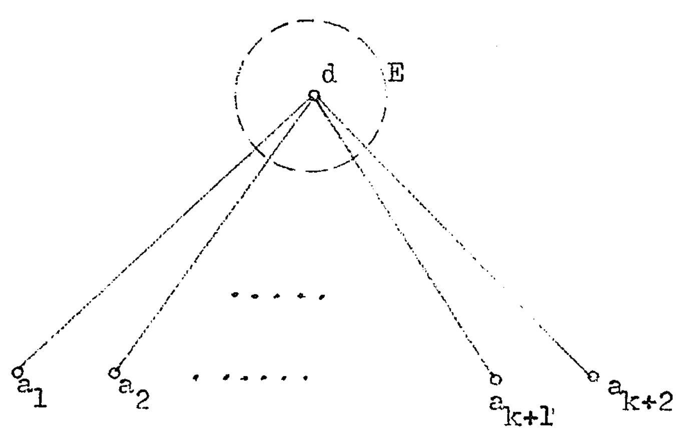
2. The Representability Theorem.

In this section, our purpose is to extend the results of Chapter 3 of Part I to the much more general situation. We will observe that by expanding those algebras representing $\langle 0,0,1,m \rangle$ in a suitable way, we can construct algebras representing the sequence $\langle 0,0,\overbrace{1,\ldots,1}^{k},m \rangle$ for a given pair of positive integers k, m.

One remark should be mentioned is that for $k > 1$, if $\mathfrak{A}$ is an algebra representing $\langle 0,0,\overbrace{1,1,\ldots,1}^{k},m \rangle$, then $\mathfrak{A}$ has a unique semilattice operation. For k = 1, this is, however, not necessary in general.

Construction.

For each pair of positive integers $\langle m,k \rangle$, let us consider the following types of semilattice $\langle A;\cdot \rangle$ (see Fig. 31)



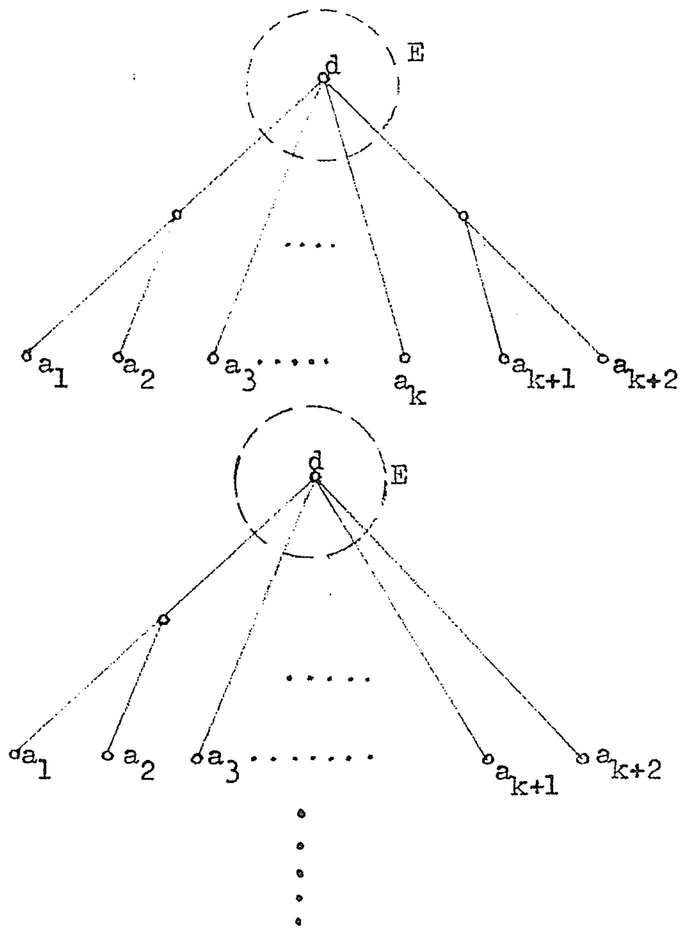


Fig. 31

where $E = \{d, e_1, e_2, \ldots, e_{m-1}\}$ is a m-element sub-semilattice of $\langle A; \circ \rangle$ and for each type, $\langle (A-E) \cup \{d\} ; \circ \rangle$, considered as a semilattice, is a homomorphic image of the free semilattice generated by $\{a_1, a_2, \ldots, a_{k+2}\}$. The elements of E satisfy the following conditions : for each $i = 1, 2, \ldots, m-1$,

either (a) $e_i > d$;

or (b) $e_i < d$ and e_i is incomparable with every element of $A - E$;

or (c) e_i is incomparable with every element of $(A - E) \cup \{d\}$.

Given any one of the semilattices $\langle A; \circ \rangle$, we shall define, for each $i = 1,2,\ldots,m-1$, a $(k+2)$-ary operation f_i on A as follows:

$$f_i(x_1,x_2,\ldots,x_{k+2}) = \begin{cases} e_i & \text{if } \{x_1,\ldots,x_{k+2}\} = \{a_1,\ldots,a_{k+2}\} \\ \prod_{j=1}^{k+2} x_j & \text{otherwise} \end{cases}.$$

Note that for each $i = 1,2,\ldots,m-1$

(1) $f_i(x_1,\ldots,x_{k+2})$ is essentially $(k+2)$-ary;

(2) $f_i(x_1,\ldots,x_{k+2}) \neq \prod_{j=1}^{k+2} x_j$;

(3) $f_i(x_1,\ldots,x_{k+2})$ is idempotent and symmetric;

(4) $f_i(x_1,\ldots,x_{k+2}) \neq f_j(x_1,\ldots,x_{k+2})$ if $i \neq j$.

Thus, each of the semilattices $\langle A; \circ \rangle$, is associated with an idempotent algebra $\mathfrak{A} = \langle A; F \rangle$ where $F = \{\circ, f_1,\ldots,f_{m-1}\}$ consists of a semilattice operation and $m-1$ $(k+2)$-ary operations defined as above.

Theorem 2.1.

For each pair of positive integers $\langle m,k \rangle$, any one of the algebras with $|E| = m$ constructed above has exactly m essentially $(k+2)$-ary polynomials.

Corollary 2.2.

For each pair of positive integers $\langle m,k \rangle$, the sequence $\langle 0,0,\overbrace{1,\ldots,1}^{k},m \rangle$ is representable in $\underset{\sim}{K}$ where $\underset{\sim}{K}$ is the class of all idempotent algebras with a semilattice operation.

The proof of Theorem 2.1 can be carried out by modifying that of Theorem 2.1(I,3).

3. The Characterization Theorem.

Consider the following sequence

$$(*) \quad \langle 0,0,\overbrace{1,\ldots,1}^{k},m,n \rangle$$

where k is a natural number, m and n are positive integrs. Theorem 2.1 says that if m=1 then the sequence (*) is representable for each n = 1,2,.... In this section, we are going to prove the converse of this result; namely , we show that if the sequence (*) is representable for each n = 1,2,..., then it is necessary that m = 1.

First of all , we shall state the following Lemma which is a slight generalization of the Lemma 3 in J. Płonka [34]. Its proof is in fact identical with the proof of that Lemma.

Lemma 3.1.

Let $\mathfrak{A}$ be an algebra such that $p_0(\mathfrak{A}) = p_1(\mathfrak{A}) = 0$, $p_2(\mathfrak{A}) > 1$. If $p_3(\mathfrak{A}) > 0$ then $p_3(\mathfrak{A}) \geqslant 3$.

Remark.

If, in addition, we assume $p_2(\mathfrak{A}) = 2$, then this is Płonka's Lemma.

Lemma 3.2.

Suppose that the sequence

$$(*) : \quad \langle 0,0,\overbrace{1,\ldots,1}^{k},m,n \rangle$$

is representable. Then any one of the following conditions implies that m = 1.

(1) $k > 1$ and $n \leqslant m + \max.\{m, k+3\}$;

(2) $k = 1$ and $n \leqslant 4$;

(3) $k = 0$, $m > 0$ and $0 < n \leqslant 2$.

<u>Proof</u> : Assume (1) holds. Since $k > 1$; $p_2 = p_3 = 1$ and hence it follows that the only binary polynomial is a semilattice polynomial. If $m \neq 1$, then we can apply a result of G. Grätzer and J. Plonka to yield the following:

$$\begin{aligned} n = p_{k+3} &\geqslant p_k + 1 + \max.\{p_{k+2}, k+3\} \\ &= m + 1 + \max.\{m, k+3\} \\ &> m + \max.\{m, k+3\}, \end{aligned}$$

which contradicts our assumption. Thus, $m = 1$, as required.

Assume (2) holds. As $k = 1$, the sequence becomes

$$(*): \quad \langle 0,0,1,m,n \rangle .$$

If $p_4 = n \leqslant 4 < \frac{1}{3}(2^4 - (-1)^4)$, then by Lemma 1.2(I,1), it follows that there is a semilattice polynomial over any algebra $\mathfrak{A}$ representing the sequence (*). Thus, if $m > 1$, we obtain

$$\begin{aligned} n = p_4(\mathfrak{A}) &\geqslant p_3(\mathfrak{A}) + 1 + \max.\{p_3(\mathfrak{A}), 4\} \\ &= m + 1 + \max.\{m, 4\} \\ &> 1 + 4 = 5, \end{aligned}$$

which is a contradiction. Hence $m = 1$.

Finally, suppose that the condition (3) holds. In this case the sequence becomes (*) : $\langle 0,0,m,n \rangle$.

If $m \neq 1$ then by (3), $p_2 = m > 1$. Since $p_3 = n > 0$, by invoking Lemma 3.1, it follows that $p_3 = n \geqslant 3$. But this contradicts our assumption that $n \leqslant 2$. Hence we must have $m = 1$, completing the proof of Lemma 3.2.

In view of this Lemma and Corollary 2.2, we arrive at the following result.

<u>Theorem 3.3.</u>

Let k be a non-negative integer and m, a positive integer. The sequence $\langle 0,0,\overbrace{1,\ldots,1}^{k},m,n \rangle$ is representable for every positive integer n if and only if $m = 1$.

CHAPTER 2

THE FUNCTION $F(n,k)$

In this chapter, we are devoted to the following problem : Given a positive integer n, what is the minimum value of m so that the sequence $\langle 0,0,1,n,m \rangle$ is representable ? In other words, our object is to search for the number m^* such that

(1) the sequence $\langle 0,0,1,n,m^* \rangle$ is representable;

(2) if the sequence $\langle 0,0,1,n,m \rangle$ is representable, then $m \geqslant m^*$.

The result we obtain is the following : Let $\underset{\sim}{K}(1)$ be the class of all idempotent algebras with a semilattice operation such that all the essentially ternary polynomials are symmetric. For each n, let $F(n)$ be the smallest integer such that the sequence $\langle 0,0,1,n,F(n) \rangle$ is representable by algebras in $\underset{\sim}{K}(1)$. Then $F(n) = 10n - 9$.

Furthermore, by applying similar techniques, we are able to extend the above result to a more general situation and obtain the following result : For each pair of positive integers $\langle n,k \rangle$, $k > 1$, let $F(n,k)$ be the smallest value such that the sequence $\langle 0,0,\overbrace{1,\ldots,1}^{k},n,F(n,k) \rangle$ is representable in $\underset{\sim}{K}(k)$ where $\underset{\sim}{K}(k)$ is the class of all idempotent algebras such that all the essentially $(k+2)$-ary polynomials are symmetric. Then

$$F(n,k) = 1 + \tfrac{1}{2}(n-1)(k+3)(k+4).$$

1. Some Preliminary Results.

Throughout this chapter, we shall adopt the following notation. Let $\mathcal{A}(1) = \langle A;\circ\rangle$ be the four-element semilattice (see Fig. 32)

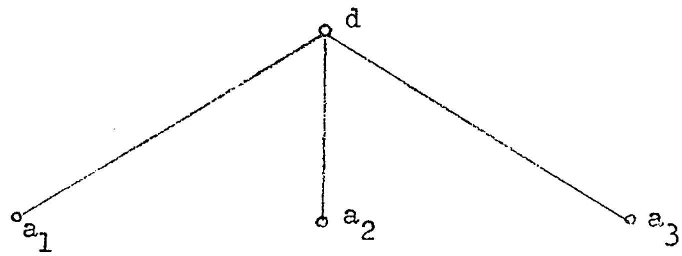


Fig. 32

Let $\mathcal{A}(2) = \langle A_2;\circ,f_1\rangle$ be the algebra II(1) (see Fig. 5). Inductively, for each positive integer n, let $\mathcal{A}(n) = \langle A_n;\circ,f_1,\ldots,f_{n-1}\rangle$ be the algebra (see Fig. 33)

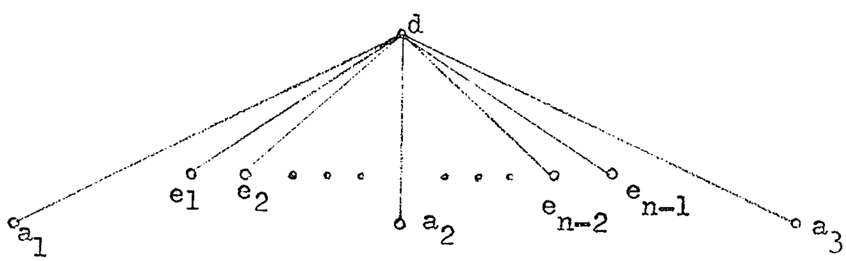


Fig. 33

where $|A_n| = n + 3$; "$\circ$" is the join semilattice operation and for each $i = 1,2,\ldots,n-1$, f_i is a ternary operation defined as follows :

$$f_i(x,y,z) = \begin{cases} e_i & \text{if } \{x,y,z\} = \{a_1,a_2,a_3\} \\ xyz & \text{otherwise.} \end{cases}$$

According to the results of Chapter 1, it is known that for each positive integer n, the algebra $\mathcal{A}(n)$ represents the sequence $\langle 0,0,1,n\rangle$. We shall show that $p_4(\mathcal{A}(n)) = 10n - 9$. To this end, we prove the following lemmas. For simplicity, if p = q is an identity, we write L for p and R for q (L, R stand for left and right respectively).

Lemma 1.1.

In the algebra $\mathcal{O}\!\mathcal{L}(n)$, $f_i(x_1x_2,x_3,x_4) = f_i(x_1,x_3,x_4)f_i(x_2,x_3,x_4)$, for each $i = 1,2,\dots,n-1$.

Proof : It suffices to consider the case $i = 1$.

Let S be a substitution such that $L(S) = e_1$, i.e.,

$$f_1(x_1x_2,x_3,x_4)(S) = e_1.$$

This can happen only if either (1) $(x_1x_2)(S) = x_3(S) = x_4(S) = e_1$ or (2) $(x_1x_2)(S) = a_1$, $x_3(S) = a_2$, $x_4(S) = a_3$ (or symmetrically).

Assume (1) holds. We obtain $x_i(S) = e_1$, for each $i = 1,2,3,4$. Thus, $R(S) = e_1$.

If (2) is the case, then $x_1(S) = x_2(S) = a_1$, $x_3(S) = a_2$, $x_4(S) = a_3$. Thus, $R(S) = f_1(a_1,a_2,a_3)f_1(a_1,a_2,a_3) = e_1$.

Hence, $L(S) = e_1$ implies that $R(S) = e_1$.

Let S be a substitution such that $L(S) = e_k$, $k = 2,\dots,n-1$, i.e., $f_1(x_1x_2,x_3,x_4)(S) = e_k$. Since $k \neq 1$, we have $(x_1x_2)(S) = x_3(S) = x_4(S) = e_k$, by definition of f_1. Thus, $x_i(S) = e_k$, for each $i = 1,2,3,4$ and hence $R(S) = e_k$. Thus, $L(S) = e_k$ implies $R(S) = e_k$.

Conversely, let S be a substitution such that $R(S) = e_1$. Then $f_1(x_1,x_3,x_4)(S) = f_1(x_2,x_3,x_4)(S) = e_1$. We have either (1) $x_i(S) = e_1$, for each $i = 1,2,3,4$ or (2) $x_1(S) = x_2(S) = a_1$, $x_3(S) = a_2$, $x_4(S) = a_3$ (or symmetrically). Clearly, in each case, $L(S) = e_1$.

Let S be a substitution such that $R(S) = e_k$, $k = 2,\dots,n-1$, i.e., $f_1(x_1,x_3,x_4)(S) = f_1(x_2,x_3,x_4)(S) = e_k$. Since $k \neq 1$, it follows by definition that $x_i(S) = e_k$, for each $i = 1,2,3,4$. Thus, $L(S) = e_k$. Hence, we prove that $R(S) = e_i$ implies $L(S) = e_i$, for each $i = 1,2,\ldots,n-1$.

Therefore, we conclude that $L = R$, as required.

The following Lemma can be proved easily.

<u>Lemma 1.2.</u>

In the algebra $\mathfrak{A}(n)$, $f_i(x_1,x_2,x_3)x_1 = x_1x_2x_3$, for each $i = 1,2,\ldots,n-1$.

<u>Lemma 1.3.</u>

In $\mathfrak{A}(n)$, $f_i(f_i(x_1,x_2,x_3),x_4,x_1) = \prod_{j=1}^{4} x_j$, for each $i = 1,\ldots,n-1$.

<u>Proof</u> : We need only prove the lemma for $i = 1$.

Let S be a substitution such that $L(S) = e_1$. Then either (1) $f_1(x_1,x_2,x_3)(S) = x_4(S)=x_1(S)=e_1$ or (2) $f_1(x_1,x_2,x_3)(S) = a_1$, $x_4(S) = a_2$, $x_1(S) = a_3$ (or symmetrically). Evidently, (2) is impossible as $f_1(a_3,x_2,x_3)(S) \neq a_1$. Thus, we have (1). But then it follows that $x_j(S) = e_1$, for each $j = 1,2,3,4$. Therefore, $R(S) = (\prod_{j=1}^{4} x_j)(S) = \prod e_1 = e_1$.

Let S be a substitution such that $L(S) = e_k$, $k = 2,\ldots,n-1$. Then $f_1(f_1(x_1,x_2,x_3),x_4,x_1)(S) = e_k$. As $k \neq 1$, we obtain

$$f_1(x_1,x_2,x_3)(S) = x_4(S) = x_1(S) = e_k.$$

Thus, $x_j(S) = e_k$ for each $j = 1,2,3,4$ and so $R(S) = e_k$. Hence, $L(S) = e_j$ implies $R(S) = e_j$, for each $j = 1,2,\ldots,n-1$.

Conversely, it is clear that if S is a substitution and $R(S) = e_j$, then $L(S) = e_j$, for each $j = 1,\ldots,n-1$.

From these, we thus conclude that $L = R$ over $\mathfrak{C}(n)$.

Lemma 1.4.

In $\mathfrak{C}(n)$, $f_i(f_j(x_1,x_2,x_3),x_4,x_1) = \prod_{t=1}^{4} x_t$, where $i,j = 1,2,\ldots,n-1$, and $i \neq j$.

Proof : Without loss of generality, we may assume $i = 1$, $j = 2$.

Let S be a substitution such that $L(S) = e_1$, i.e.,

$$f_1(f_2(x_1,x_2,x_3),x_4,x_1)(S) = e_1$$

Then either (1) $f_2(x_1,x_2,x_3)(S) = x_4(S) = x_1(S) = e_1$ or (2) $f_2(x_1,x_2,x_3)(S) = a_1$, $x_4(S) = a_2$, $x_1(S) = a_3$ (or symmetrically)

Observe that (2) is impossible as $f_2(a_3,x_2,x_3)(S) \neq a_1$. Hence, we have (1). But this implies that $x_t(S) = e_1$, for each $t = 1,2,3,4$. Thus, $R(S) = e_1$.

Let S be such that $L(S) = e_2$. Then we get

$$f_2(x_1,x_2,x_3)(S) = x_4(S) = x_1(S) = e_2.$$

Since $f_2(e_2,x_2,x_3)(S) = e_2$ implies $x_t(S) = e_2$, $t = 1,2,3,4$, it follows that $R(S) = e_2$.

Let S be such that $L(S) = e_k$, $k = 3,\ldots,n-1$. Then we get $x_t(S) = e_k$, $t = 1,2,3,4$. Thus, $R(S) = e_k$.

Conversely, if S is a substitution such that $R(S) = e_k$, $k = 1,2,\ldots,n-1$, then it follows immediately that $L(S) = e_k$.

Hence, we have $L(S) = e_k$ if and only if $R(S) = e_k$, $k = 1, 2,\ldots,n-1$, in $\mathfrak{C}(n)$. Therefore, $L = R$, which completes the proof of Lemma 4.

Lemma 1.5.

Let $f_i(p,q,r)$ be an essentially 4-ary polynomial over $\mathfrak{A}(n)$ where p, q and r are pairwise distinct polynomials over $\mathfrak{A}(n)$ which contain no sub-polynomials of the product form $x_s x_t$. Then

$$f_i(p,q,r) = \prod_{j=1}^{4} x_j \quad \text{in} \quad \mathfrak{A}(n).$$

Proof : Let i = 1, and let S be a substitution such that L(S) $= e_1$. Then we have either (1) $p(S) = a_1$, $q(S) = a_2$, $r(S) = a_3$ (or symmetrically) or (2) $p(S) = q(S) = r(S) = e_1$.

Claim : (1) is impossible.

Observe that since $f_1(p,q,r)$ is essentially 4-ary, it consists of four distinct variables. As $f_j(x_1,x_2,x_2) = x_1x_2$ holds in $\mathfrak{A}(n)$ and there is no product of the form $x_s x_t$ occuring in p, q and r, it follows that at least one of the $\{p,q,r\}$ must contain a factor $f_j(\ ,\ ,\ )$ consisting of at least three distinct variables. Therefore, at least two of $\{p,q,r\}$ have a variable in common. Suppose that (1) is the case. Then all variables in p, q and r must be substituted by a_1, a_2 and a_3 respectively (or symmetrically). This can be happen only if any two of $\{p,q,r\}$ have no variables in common, a contradiction. Thus, (1) is impossible, as required.

Hence, we have (2). It then follows immediately from the assumption that $x_i(S) = e_1$, for each i = 1,2,3,4. Thus, R(S) $= \prod e_1 = e_1$.

Let S be such that $f_1(p,q,r)(S) = e_k$, k = 2,...,n-1. Then, $p(S) = q(S) = r(S) = e_k$ by definition. From this, it follows that $x_i(S) = e_k$, for each $i = 1,2,3,4$. Thus, $R(S) = \prod e_k = e_k$.

Conversely, it is clear that $R(S) = e_j$ implies $L(S) = e_j$, for each $j = 1,2,\ldots,n-1$. Hence we conclude that $L = R$, as was to be shown.

Remark.

Let us note that Lemmas 1.3 and 1.4 are indeed special cases of Lemma 1.5.

Lemma 1.6.

Let $\prod_{\alpha\in\Lambda} f_i(x_{\alpha(1)},x_{\alpha(2)},x_{\alpha(3)})$ be an essentially 4-ary polynomial over $\mathcal{O}(n)$, where $i = 1,2,\ldots,n-1$ is a fixed index.

Then $\prod_{\alpha\in\Lambda} f_i(x_{\alpha(1)},x_{\alpha(2)},x_{\alpha(3)}) = \prod_{j=1}^{4} x_j$ in $\mathcal{O}(n)$ if $|\Lambda| \geq 3$.

Proof : Let $i = 1$. We may assume that for each $\alpha \in \Lambda$, the set $\{x_{\alpha(1)},x_{\alpha(2)},x_{\alpha(3)}\}$ of variables is pairwise distinct. For otherwise, using the identities

$$f_1(x_1,x_2,x_2) = x_1x_2$$

and

$$f_1(x_1,x_2,x_3,)x_1 = x_1x_2x_3$$

that hold in $\mathcal{O}(n)$, it is easy to check that $\prod_{\alpha\in\Lambda} f_1(x_{\alpha(1)},x_{\alpha(2)},x_{\alpha(3)}) = \prod_{j=1}^{4} x_j$.

If $|\Lambda| = 3$, then $\prod_{\alpha\in\Lambda} f_1(x_{\alpha(1)},x_{\alpha(2)},x_{\alpha(3)})$
$= f_1(x_{\alpha(1)},x_{\alpha(2)},x_{\alpha(3)})f_1(x_{\beta(1)},x_{\beta(2)},x_{\beta(3)})f_1(x_{\gamma(1)},x_{\gamma(2)},x_{\gamma(3)})$, say.

In order that the variables in each factor $f_1(\ , \ , \ )$ are pairwise distinct, the number of occurrences of each of the four variables x_1,x_2,x_3,x_4 is at least two and at most three in the polynomial. Thus the partition of nine positions should be

$$2 : 2 : 2 : 3$$

Without loss of generality, we may assume the following distribution (See Fig. 34)

x_1	x_2	x_3	x_4
3	2	2	2

Fig. 34

Thus, we have $\prod_{\alpha\in\Lambda} f_1(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)})$

$$= f_1(x_1,x_2,x_3)f_1(x_1,x_2,x_4)f_1(x_1,x_3,x_4).$$

To prove that $f_1(x_1,x_2,x_3)f_1(x_1,x_2,x_4)f_1(x_1,x_3,x_4) = \prod_{j=1}^{4} x_j$ in $\mathcal{O}\mathcal{C}(n)$, let S be a substitution such that $L(S) = e_1$. This implies

$$f_1(x_1,x_2,x_3)(S) = f_1(x_1,x_2,x_4)(S) = f_1(x_1,x_3,x_4)(S) = e_1 .$$

From this, it follows that $x_i(S) = e_1$, for each $i = 1,2,3,4$. Thus $R(S) = e_1$. If $L(S) = e_k$, $k = 2,\dots,n-1$ then $x_i(S) = e_k$, for each $i = 1,2,3,4$. Hence, $R(S) = e_k$.

Conversely, it is clear that $R(S) = e_j$ implies $L(S) = e_j$, for each $j = 1,\dots,n-1$. Hence we conclude that for $|\Lambda| = 3$

$$\prod_{\alpha\in\Lambda} f_1(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}) = \prod_{j=1}^{4} x_j .$$

If $|\Lambda| > 3$, then

$$\prod_{\alpha\in\Lambda} f_1(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}) = f_1(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}) f_1(x_{\beta(1)}, x_{\beta(2)}, x_{\beta(3)})$$

$$f_1(x_{\gamma(1)}, x_{\gamma(2)}, x_{\gamma(3)}) \prod_{\delta} f_1(x_{\delta(1)}, x_{\delta(2)}, x_{\delta(3)})$$

$$= x_1x_2x_3x_4 \prod_{\delta} f_1(x_{\delta(1)}, x_{\delta(2)}, x_{\delta(3)})$$

$$= \prod_{j=1}^{4} x_j .$$

Hence, the proof of Lemma1.6 is complete.

Lemma 1.7.

In $\mathcal{O}\mathcal{L}(n)$, $f_i(x_1,x_2,x_3)f_j(x_2,x_3,x_4) = \prod_{t=1}^{4} x_t$ where $i,j = 1,2,\ldots,n-1$ and $i \neq j$.

Proof : We may assume $i = 1$, $j = 2$.

Let S be a substitution such that $L(S) = e_1$. Then we get $f_1(x_1,x_2,x_3)(S) = f_2(x_2,x_3,x_4)(S) = e_1$. Hence, $x_i(S) = e_1$, for each $i = 1,2,3,4$. This implies $R(S) = e_1$. Symmetrically, $L(S) = e_2$ implies $R(S) = e_2$.

Let S be such that $L(S) = e_k$, $k = 3,4,\ldots,n-1$. Clearly, we obtain $x_i(S) = e_k$. Thus, $R(S) = e_k$.

The converse is trivial. Hence Lemma 7 follows.

We are now in a position to establish the following

Proposition 1.8.

There are exactly 10n-9 distinct essentially 4-ary polynomials over $\mathcal{O}\mathcal{L}(n)$. They are :

$$(*)\quad\left\{\begin{array}{l} x_1x_2x_3x_4\ ; \\ f_i(x_1,x_2,x_3)x_4,\ f_i(x_2,x_3,x_4)x_1,\ f_i(x_3,x_4,x_1)x_2,\ f_i(x_4,x_1,x_2)x_3; \\ f_i(x_1,x_2,x_3)f_i(x_1,x_2,x_4),\quad f_i(x_1,x_3,x_2)f_i(x_1,x_3,x_4), \\ f_i(x_1,x_4,x_2)f_i(x_1,x_4,x_3),\quad f_i(x_2,x_3,x_1)f_i(x_2,x_3,x_4), \\ f_i(x_2,x_4,x_1)f_i(x_2,x_4,x_3),\quad f_i(x_3,x_4,x_1)f_i(x_3,x_4,x_2), \end{array}\right.$$

For each $i = 1,2,\ldots,n-1$.

Proof : First of all, it is routine to check that the 4-ary polynomials in (*) are distinct and essential over $\mathfrak{A}(n)$.

Now, let $p(x_1,x_2,x_3,x_4)$ be an essentially 4-ary polynomial over $\mathfrak{A}(n)$. By Lemma 1.1, p can be written as

$$p(x_1,x_2,x_3,x_4) = \prod f_1(A_{\alpha 1},B_{\alpha 1},C_{\alpha 1}) \prod f_2(A_{\alpha 2},B_{\alpha 2},C_{\alpha 2}) \cdots\cdots$$
$$\cdots\cdots \prod f_{n-1}(A_{\alpha(n-1)},B_{\alpha(n-1)},C_{\alpha(n-1)}) \prod x_k,$$

where, for each α and i, $A_{\alpha i}$, $B_{\alpha i}$, $C_{\alpha i}$ are polynomials which consist of no sub-polynomials of the product of the form $x_i x_j$.

We may assume that p cannot be reduced to a simpler form. By Lemmas 1.3, 1.4, 1.5, we have

$$p(x_1,x_2,x_3,x_4) = \prod f_1(x_{\alpha 1},x_{\beta 1},x_{\gamma 1}) \prod f_2(x_{\alpha 2},x_{\beta 2},x_{\gamma 2}) \cdots\cdots$$
$$\cdots\cdots \prod f_{n-1}(x_{\alpha(n-1)},x_{\beta(n-1)},x_{\gamma(n-1)}) \prod x_k.$$

If p has no "f_i" factors, for each $i = 1,\ldots,n-1$, then

$$p = \prod_{j=1}^{4} x_j.$$

If p has a factor "f_i" for some i, then by Lemma1.7, it follows that

$$p = \prod_{\Lambda} f_i(x_{\alpha i},x_{\beta i},x_{\gamma i}) \prod x_k.$$

<u>Case 1</u>. $|\Lambda| = 1$.

In this case, $p = f_i(x_\alpha,x_\beta,x_\gamma)x_\delta$, where $\{\alpha,\beta,\gamma,\delta\} = \{1,2,3,4\}$ by Lemma 1.2.

<u>Case 2</u>. $|\Lambda| = 2$.

In this case, $p = f_i(x_\alpha,x_\beta,x_\gamma)f_i(x_\alpha,x_\beta,x_\delta) \prod x_k$

$$= f_i(x_\alpha,x_\beta,x_\gamma)f_i(x_\alpha,x_\beta,x_\delta),$$

where $\{\alpha,\beta,\gamma,\delta\} = \{1,2,3,4\}$, by Lemma 1.2.

<u>Case 3</u>. $|\Lambda| \geqslant 3$.

If this is the case, then by Lemma 1.6, $p = \prod_{j=1}^{4} x_j$.

Hence, any essentially 4-ary polynomial over $\mathfrak{A}(n)$ must be equal to one of the polynomials in (*). The number of the polynomials in (*) is $1 + (n-1)\binom{4}{1} + (n-1)\binom{4}{1} = 10n - 9$. This completes the proof of Proposition 1.8.

2. The Value of F(n,1).

Let $\underset{\sim}{K}(1)$ be the class of all idempotent algebras with a semilattice operation such that all the essentially ternary polynomials are symmetric. Thus, for instance, $\mathfrak{A}(n)$ is an element in $\underset{\sim}{K}(1)$, for $n = 1,2,\ldots\ldots$. For each positive integer n, let $\mathfrak{A}$ be an algebra in $\underset{\sim}{K}(1)$ representing $\langle 0,0,1,n\rangle$. Since $\mathfrak{A}$ has one semilattice operation "$\circ$"; we have one essentially ternary polynomial $x_1x_2x_3$. As $p_3(\mathfrak{A}) = n$, let

$$\begin{cases} g_1(x_1,x_2,x_3) \\ g_2(x_1,x_2,x_3) \\ \vdots \\ g_{n-1}(x_1,x_2,x_3) \end{cases}$$

denote the other n-1 essentially ternary polynomials over $\mathfrak{A}$. Note that as $\mathfrak{A} \in \underset{\sim}{K}(1)$, $g_i(x_1,x_2,x_3)$ is symmetric, for each $i = 1,2,\ldots,n-1$.

In this section, our purpose is to prove that, corresponding to the $10n - 9$ distinct essentially 4-ary polynomials of $\mathfrak{A}(n)$ which were described in Proposition 1.8, the following $10n - 9$ 4-ary polynomials are distinct and essential over $\mathfrak{A}$:

$$(\Delta)\quad\left\{\begin{array}{l} x_1x_2x_3x_4, \\ g_i(x_1,x_2,x_3)x_4,\ g_i(x_2,x_3,x_4)x_1,\ g_i(x_3,x_4,x_1)x_2,\ g_i(x_4,x_1,x_2)x_3, \\ g_i(x_1,x_2,x_3)g_i(x_1,x_2,x_4),\ g_i(x_1,x_3,x_2)g_i(x_1,x_3,x_4), \\ g_i(x_1,x_4,x_2)g_i(x_1,x_4,x_3),\ g_i(x_2,x_3,x_4)g_i(x_2,x_3,x_1), \\ g_i(x_2,x_4,x_1)g_i(x_2,x_4,x_3),\ g_i(x_3,x_4,x_1)g_i(x_3,x_4,x_2), \end{array}\right.$$

for each $i = 1,2,\ldots,n-1$.

If $n = 1$, we have nothing to prove.

If $n = 2$, then $\mathfrak{A}$, being an algebra representing $\langle 0,0,1,2\rangle$, must belong to one of the equational classes $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$. If $\mathfrak{A} \in \underset{\sim}{K}_1$, then $p_4(\mathfrak{A}) \geqslant p_4(I(1))$. If $\mathfrak{A} \in \underset{\sim}{K}_2$, then $p_4(\mathfrak{A}) \geqslant p_4(II(1)) = p_4(\mathfrak{A}(2))$. Thus, it suffices to show that the above eleven polynomials are distinct and essential over $I(1)$. However, it is clear that this is, indeed, the case. Hence, from now on, we may assume that $n \geqslant 3$.

We need the following Lemmas :

<u>Lemma 2.1.</u>

In $\mathfrak{A}$, $g_i(x_1,x_2,x_2) = x_1x_2$, for each $i = 1,2,\ldots,n-1$.

<u>Proof</u> : We need only prove the lemma for $i = 1$. Since $\mathfrak{A}$ represents $\langle 0,0,1\rangle$; we have only three possible cases :

$$g_1(x_1,x_2,x_2) = \left\{\begin{array}{l} x_1 \\ x_2 \\ x_1x_2 \end{array}\right.$$

<u>Case 1.</u> $g_1(x_1,x_2,x_2) = x_1$.

We first claim that the following polynomials are pairwise distinct and essentially ternary.

$$\begin{cases} g_1(x_1,x_2,x_3)x_1,\ g_1(x_1,x_2,x_3)x_2,\ g_1(x_1,x_2,x_3)x_3 \\ g_2(x_1,x_2,x_3) \\ \vdots \\ g_{n-1}(x_1,x_2,x_3) \end{cases}$$

Observe that if $g_1(x_1,x_2,x_3)x_1 = g_1(x_1,x_2,x_3)x_2$, then setting $x_2 = x_3$, we obtain $x_1 = x_1x_2$, a contradiction. Hence by symmetry, the polynomials $g_1(x_1,x_2,x_3)x_1$, $g_1(x_1,x_2,x_3)x_2$, $g_1(x_1,x_2,x_3)x_3$ are pairwise distinct.

If $g_1(x_1,x_2,x_3)x_1 = g_i(x_1,x_2,x_3)$ for some $i = 2,\ldots,n-1$, then $g_i(x_2,x_3,x_1) = g_1(x_2,x_3,x_1)x_2$. From this, it follows that

$$\begin{aligned} g_1(x_1,x_2,x_3)x_1 &= g_i(x_2,x_3,x_1) \\ &= g_1(x_2,x_3,x_1)x_2 \\ &= g_1(x_1,x_2,x_3)x_2, \end{aligned}$$

which is a contradiction. Hence we conclude that the above n+1 ternary polynomials are pairwise distinct.

Consider $p = g_1(x_1,x_2,x_3)x_1$. Set $x_2 = x_3$. We have

$$p = g_1(x_1,x_2,x_2)x_1 = x_1.$$

Hence, p depends on x_1. Setting $x_1 = x_3$, it follows that $p = x_2x_1$. Thus, p depends on x_2. As p is symmetric with repect to x_2 and x_3, p also depends on x_3. Thus p is essentially ternary. Similarly, $g_1(x_1,x_2,x_3)x_2$ and $g_1(x_1,x_2,x_3)x_3$ are essential.

Hence, if we assume $g_1(x_1,x_2,x_2) = x_1$, we have $p_3(\mathfrak{A}) \geq n+1$, which is a contradiction. Thus case 1 is impossible.

Case 2. $g_1(x_1,x_2,x_2) = x_2$.

Again, we claim that the polynomials as shown in case 1 are distinct and essential.

Note that if $g_1(x_1,x_2,x_3)x_1 = g_1(x_1,x_2,x_3)x_2$, then setting $x_1 = x_3$, it follows that $x_1 = x_1x_2$, a contradiction. If $g_1(x_1,x_2,x_3)x_1 = g_i(x_1,x_2,x_3)$ for some $i = 2,3,\ldots,n-1$, then, as in case 1, we would have

$$g_1(x_1,x_2,x_3)x_1 = g_1(x_1,x_2,x_3)x_2,$$

which is impossible. Hence we have n + 1 distinct polynomials.

Now, observe that if one of the polynomials $g_1(x_1,x_2,x_3)x_i$, $i = 1,2,3$ is essentially ternary, then, by the symmetry of $g_1(x_1,x_2,x_3)$, so are the other two. In this situation, we would have $p_3(\mathfrak{A}) \geqslant n + 1$, which is a contradiction. Hence, $g_1(x_1,x_2,x_3)x_1$ cannot be essentially ternary. As $\mathfrak{A}$ represents $\langle 0,0,1 \rangle$; we have the following possibilities :

$$g_1(x_1,x_2,x_3)x_1 = \begin{cases} x_1 \\ x_2 \\ x_3 \\ x_1x_2 \\ x_1x_3 \\ x_2x_3 \end{cases}$$

Since $g_1(x_1,x_2,x_3)x_1$ is symmetric with respect to x_2 and x_3; it follows that

$$g_1(x_1,x_2,x_3)x_1 = \begin{cases} x_1 \\ x_2x_3 \end{cases}$$

If $g_1(x_1,x_2,x_3)x_1 = x_1$, then setting $x_2 = x_3$, we obtain $x_2x_1 = x_1$, a contradiction. If $g_1(x_1,x_2,x_3)x_1 = x_2x_3$, setting $x_2 = x_3$, it follows that $x_2x_1 = x_2$, which is a contradiction.

These arguments show that the assumption $g_1(x_1,x_2,x_2) = x_2$ is impossible. Therefore, we must have

$$g_1(x_1,x_2,x_2) = x_1x_2,$$

as was to be shown.

Lemma 2.2.

In $\mathfrak{A}$, $g_i(x_1x_2,x_1,x_2) = x_1x_2$, for each $i = 1,2,\ldots,n-1$.

Proof : Since $p_0(\mathfrak{A}) = 0$, $g_i(x_1x_2,x_1,x_2)$ is not a constant. However, if $g_i(x_1x_2,x_1,x_2)$ depends on x_1, it depends on x_2 by symmetry. Thus, $g_i(x_1x_2,x_1,x_2)$ is essentially binary and hence

$$g_i(x_1x_2,x_1,x_2) = x_1x_2,$$

as $p_2(\mathfrak{A}) = 1$.

Lemma 2.3.

In $\mathfrak{A}$, $g_i(x_1x_2,x_2,x_3) = x_1x_2x_3$, for each $i = 1,2,\ldots,n-1$.

Proof : Assume $i = 1$. Observe that the polynomial $p = g_1(x_1x_2,x_2,x_3)$ is essentially ternary. For, if we set $x_1 = x_2$, we obtain $p = x_1x_3$, by Lemma 2.1. Thus, p depends on x_3. Setting $x_1 = x_3$, we have $p = x_1x_2$, by Lemma 2.2. Thus, p depends on x_2. Setting $x_2 = x_3$, it follows that $p = x_1x_2$, by Lemma 2.1. Thus, p depends on x_1. Hence, p is an essentially ternary polynomial.

Since $p_3(\mathfrak{A}) = n$, it follows that

$$g_1(x_1x_2,x_2,x_3) = \begin{cases} g_1(x_1,x_2,x_3) \\ g_k(x_1,x_2,x_3), \quad k \neq 1 \\ x_1x_2x_3 \end{cases}$$

If $g_1(x_1x_2,x_2,x_3) = g_1(x_1,x_2,x_3)$ ——————— (1)

then observe that

$$\begin{aligned} x_1x_2x_3 &= g_1(x_1x_2x_3,\ x_1x_2,\ x_1x_3) && \text{(Lemma 2.2)} \\ &= g_1(x_3(x_1x_2),\ x_1x_2,\ x_1x_3) \\ &= g_1(x_3,\ x_1x_2,\ x_1x_3) && \text{(by (1))} \\ &= g_1(x_1,x_3,x_1x_2) && \text{(by (1))} \\ &= g_1(x_1,x_2,x_3) && \text{(by (1))} \end{aligned}$$

which is a contradiction. Hence (1) is impossible.

If $g_1(x_1x_2,x_2,x_3) = g_k(x_1,x_2,x_3)$, for some $k = 2,3,\ldots,n-1$ —— (2)

then note that $g_k(x_1x_2,x_2,x_3) = g_1((x_1x_2)x_2,x_2,x_3)$ (by (2))

$$= g_1(x_1x_2,x_2,x_3)$$

$$= g_k(x_1,x_2,x_3) \quad \text{(by (2))}$$

Thus, we have $g_k(x_1x_2,x_2,x_3) = g_k(x_1,x_2,x_3)$ ——————— (3)

Now, observe that

$$\begin{aligned} x_1x_2x_3 &= g_1(x_1x_2x_3,x_1x_2,x_1x_3) && \text{(Lemma 2.2)} \\ &= g_k(x_3,x_1x_2,x_1x_3) && \text{(by (2))} \\ &= g_k(x_1,x_3,x_1x_2) && \text{(by (3))} \\ &= g_k(x_2,x_1,x_3) && \text{(by (3)} \\ &= g_k(x_1,x_2,x_3), \end{aligned}$$

a contradiction. Hence, (2) is impossible. It therefore follows that $g_1(x_1x_2,x_2,x_3) = x_1x_2x_3$,

proving Lemma 2.3.

With the aid of these Lemmas, we are now in a position to prove the following result.

Proposition 2.4.

Let $\mathfrak{A}$ be an algebra in $\underset{\sim}{K}(1)$ representing $\langle 0,0,1,n \rangle$. Then $p_4(\mathfrak{A}) \geqslant 10n-9$.

Proof : It suffices to consider $n \geqslant 3$.

By Lemma 2.1, it is easy to prove that the polynomials in (Δ) are essentially 4-ary Over $\mathfrak{A}$. For instance, take

$$p = g_i(x_1,x_2,x_3)x_4.$$

Clearly, p depends on x_4. Setting $x_1 = x_2$, we have $p = x_1x_3x_4$. Thus, p depends on x_3. By symmetry, p depends on every variable. Hence, p is essentially 4-ary. Next, consider

$$p = g_i(x_1,x_2,x_3)g_i(x_1,x_2,x_4).$$

Set $x_3 = x_4$. We obtain $p = g_i(x_1,x_2,x_3)$ which is essentially ternary. Thus, p depends on x_1 and x_2. Set $x_1 = x_2$. It follows that $p = g_i(x_1,x_1,x_3)g_i(x_1,x_1,x_4) = x_1x_3x_4$, which depends on x_3 and x_4. Hence p is an essentially 4-ary polynomial.

It remains to prove that all the polynomials in (Δ) are distinct. By symmetry, we have only to prove that the twenty-one essential polynomials with i = 1,2 are distinct.

In preparation, we make the following observations.

Since $x_1x_2x_3 \neq g_1(x_1,x_2,x_3)$; there exist $c_1,c_2,c_3 \in A$ such that $c_1c_2c_3 \neq g_1(c_1,c_2,c_3)$. Let

$$c^* = g_1(c_1,c_2,c_3)$$

and $$\bar{c} = c_1c_2c_3.$$

Observe that if $c_2 = c_3$ (or symmetrically), then by Lemma 2.1, $g_1(c_1,c_2,c_3) = g_1(c_1,c_2,c_2) = c_1c_2 = c_1c_2c_3$, a contradiction. If $c_2 \geq c_3$ then $c_2c_3 = c_2$. It follows by Lemma 2.3 that $c_1c_2c_3 = g_1(c_1,c_3,c_2c_3) = g_1(c_1,c_3,c_2)$, which is impossible. Hence we conclude that (1) c_1,c_2,c_3 are pairwise distinct;

(2) c_1,c_2,c_3 are pairwise incomparable (Thus,

$\bar{c} > c_i$, $i = 1,2,3$).

As $g_1(x_1,x_2,x_3) \neq g_2(x_1,x_2,x_3)$; there exist $b_1,b_2,b_3 \in A$ such that $g_1(b_1,b_2,b_3) \neq g_2(b_1,b_2,b_3)$. Let

$$b^* = g_1(b_1,b_2,b_3)$$

and $$b' = g_2(b_1,b_2,b_3).$$

Note that if $b_1 \geq b_2$ then $b_1b_2 = b_1$. Thus, by Lemma 2.3, we have $g_1(b_1,b_2,b_3) = g_1(b_1b_2,b_2,b_3) = b_1b_2b_3 = g_2(b_1b_2,b_2,b_3) = g_2(b_1,b_2,b_3)$, a contradiction. Hence we conclude that the elements b_1, b_2 and b_3 are pairwise distinct and incomparable.

We are now ready to prove that the polynomials are distinct.

(1) $x_1x_2x_3x_4 \neq g_1(x_1,x_2,x_3)x_4$.

There are several cases to consider :

(a) $\bar{c} > c^*$.

Let S be a substitution such that $x_i(S) = c_i$, $i = 1,2,3$, $x_4(S) = c^*$. Then $L(S) = \bar{c}c^* = \bar{c}$ while $R(S) = c^*c^* = c^*$. Thus, $L(S) \neq R(S)$.

(b) $\bar{c} < c^*$.

Let S be such that $x_i(S) = c_i$, $i = 1,2,3$, $x_4(S) = c_1$. Then $R(S) = c^*c_1 = (c^*\bar{c})c_1 = c^*(\bar{c}c_1) = c^*\bar{c} = c^*$ while $L(S) = c_1c_2c_3c_1$

$= \bar{c}$. Thus, $L(S) \neq R(S)$.

(c) $\bar{c} \parallel c^*$.

Let S be the same as that in (a). We have $R(S) = c^*$ while $L(S) = \bar{c}c^* > c^*$. Thus, $L(S) \neq R(S)$.

$$(2) \quad x_1x_2x_3x_4 \neq g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4).$$

If (2) is not the case, then setting $x_3 = x_4$, we obtain $x_1x_2x_3 = g_1(x_1,x_2,x_3)$, a contradiction. Thus, (2) follows.

Before carrying on, let us prove the following assertion.

(W) In $\mathfrak{R}$, if $c^* < \bar{c}$, then $c_i \not\leqslant c^*$ for each $i = 1,2,3$.

To this end, we need only prove that $c_1 \not\leqslant c^*$.

Assume that $c^* < \bar{c}$. Consider the polynomial $g_1(x_1,x_2,x_3)x_1$. Evidently, it is essential. Thus we have

$$g_1(x_1,x_2,x_3)x_1 = \begin{cases} g_1(x_1,x_2,x_3) \\ x_1x_2x_3 \\ g_k(x_1,x_2,x_3), \ k \neq 1. \end{cases}$$

The case $g_1(x_1,x_2,x_3)x_1 = g_1(x_1,x_2,x_3)$ is impossible. For equality implies, by symmetry, that $g_1(x_1,x_2,x_3)x_1x_2x_3 = g_1(x_1,x_2,x_3)$. Thus, we obtain $c^*\bar{c} = c^*$, i.e., $c^* \geqslant \bar{c}$, a contradiction.

If $g_1(x_1,x_2,x_3)x_1 = x_1x_2x_3$, we have $c^*c_1 = \bar{c} > c^*$. Thus, if $c_1 \leqslant c^*$, it follows that $c^* = c^*c_1 > c^*$, a contradiction. Hence $c^* \not\geqslant c_1$, as required.

If $g_1(x_1,x_2,x_3)x_1 = g_k(x_1,x_2,x_3)$, $k \neq 1$, then by symmetry, $g_1(x_1,x_2,x_3)x_1x_2x_3 = g_k(x_1,x_2,x_3)$. Thus, $g_k(c_1,c_2,c_3) = c^*\bar{c}$

$= \bar{c} > c^*$. If $c_1 \leq c^*$, then $c^* = c^*c_1 = g_1(c_1,c_2,c_3)c_1 = g_k(c_1,c_2,c_3) > c^*$, a contradiction. Hence, $c_1 \not\leq c^*$, as required. This proves (W).

$$(3) \quad g_1(x_1,x_2,x_3)x_4 \neq g_1(x_2,x_3,x_4)x_1.$$

There are three cases to consider :

(a) $c^* < \bar{c}$.

Let S be such that $x_i(S) = c_i$, $i = 1,2,3$, $x_4(S) = c^*$. Then $L(S) = c^*c^* = c^*$, $R(S) = g_1(c_2,c_3,c^*)c_1$. Note that $R(S) \neq c^*$. For, if it were, then $g_1(c_2,c_3,c^*)c_1 = c^*$ implies $c^* \geq c_1$. This, however, contradicts the assertion (W).

(b) $c^* > \bar{c}$.

Let S be such that $x_i(S) = c_i$, $i = 1,2,3$, $x_4(S) = c_3$. Then $L(S) = c^*c_3 = c^*$ while $R(S) = \bar{c}$ by Lemma 2.1.

(c) $c^* \,||\, \bar{c}$.

Let S be that of (b). Then $L(S) \geq c^*$, $R(S) = \bar{c}$.

$$(4) \quad g_1(x_1,x_2,x_3)x_4 \neq g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4).$$

(a) $c^* < \bar{c}$.

Let S be such that $x_i(S) = c_i$, $i = 1,2,3$, $x_4(S) = c_3$. Then $R(S) = c^*$, $L(S) = c^*c_3$. If $R(S) = L(S)$, we would have $c^* = c^*c_3$ which implies that $c^* \geq c_3$. This, however, contradicts the assertion (W). Thus, $L(S) \neq R(S)$.

(b) $c^* > \bar{c}$.

Let S be such that $x_i(S) = c_i$, $i = 1,2$, $x_3(S) = c_1$, $x_4(S) = c_3$. Then by Lemma 2.1, $R(S) = c_1c_2c^* = c^*$, $L(S) = \bar{c}$.

(c) $c^* \,||\, \bar{c}$.

Apply the same argument as in (b).

$$(5) \quad g_1(x_1,x_2,x_3)x_4 \neq g_1(x_1,x_4,x_2)g_1(x_1,x_4,x_3).$$

If (5) is not the case, then setting $x_2 = x_3$, we have by Lemma 2.1, $x_1x_2x_4 = g_1(x_1,x_4,x_2)$, a contradiction.

$$(6) \quad g_1(x_1,x_2,x_3)x_4 \neq g_2(x_1,x_2,x_3)x_4.$$

If (6) is not the case, setting $x_4 = g_1(x_1,x_2,x_3)$, we obtain $g_1(x_1,x_2,x_3) = g_2(x_1,x_2,x_3)g_1(x_1,x_2,x_3)$. Thus, by symmetry, it follows that $g_1(x_1,x_2,x_3) = g_2(x_1,x_2,x_3)$, a contradiction.

$$(7) \quad g_1(x_1,x_2,x_3)x_4 \neq g_2(x_2,x_3,x_4)x_1.$$

(a) $c^* < \bar{c}$.

Let S be such that $x_i(S) = c_i$, $i = 1,2,3$, $x_4(S) = c^*$. Then $L(S) = c^*$, $R(S) = g_2(c_2,c_3,c^*)c_1$. If $L(S) = R(S)$, then $c^* \geqslant c_1$, contradicting the assertion (W).

(b) $c^* \not< \bar{c}$.

Let S be such that $x_i(S) = c_i$, $i = 1,2,3$, $x_4(S) = c_2$. Then $L(S) = c^*c_2 \geqslant c^*$, $R(S) = \bar{c}$ by Lemma 2.1. If $L(S) = R(S)$, we would have $\bar{c} > c^*$, a contradiction.

$$(8) \quad g_1(x_1,x_2,x_3)x_4 \neq g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4).$$

(a) $b^* < b'$.

Let S be such that $x_i(S) = b_i$, $i = 1,2,3$, $x_4(S) = b^*$. Then $L(S) = b^*$ while $R(S) = b'g_2(b_1,b_2, b^*) \geqslant b'$. Thus, $R(S) \geqslant b' > b^* = L(S)$.

(b) $b^* \not< b'$

Let S be such that $x_i(S) = b_i$, $i = 1,2,3$, $x_4(S) = b_3$. Then

$R(S) = b'$, $L(S) = b^*b_3 \geqslant b^*$. Thus, if $L(S) = R(S)$, we would have $b' > b^*$, a contradiction.

(9) $g_1(x_1,x_2,x_3)x_4 \neq g_2(x_3,x_4,x_1)g_2(x_3,x_4,x_2)$.

If $g_1(x_1,x_2,x_3)x_4 = g_2(x_3,x_4,x_1)g_2(x_3,x_4,x_2)$, setting $x_1 = x_2$, we would have $x_1x_3x_4 = g_2(x_3,x_4,x_1)$, a contradiction.

(10) $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) \neq g_1(x_2,x_4,x_1)g_1(x_2,x_4,x_3)$.

(a) $c^* < \bar{c}$.

Let S be such that $x_i(S) = c_i$, $i = 1,2$, $x_i(S) = c_3$, $i = 3,4$. Then $L(S) = c^*$ and $R(S) = c^*c_2c_3$. If $L(S) = R(S)$, we would have $c^* \geqslant c_2c_3 > c_2$, which contradicts the assertion (W).

(b) $c^* \nless \bar{c}$.

Let S be such that $x_i(S) = c_i$, $i = 1,2,3$, $x_4(S) = c_2$. Then $L(S) = c^*c_1c_2 \geqslant c^*$ and $R(S) = \bar{c}$ by Lemma 2.1. If $L(S) = R(S)$, it follows that $\bar{c} > c^*$, a contradiction.

(11) $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) \neq g_1(x_3,x_4,x_1)g_1(x_3,x_4,x_2)$.

If (11) is not the case, setting $x_3 = x_4$, we obtain $g_1(x_1,x_2,x_3) = x_1x_2x_3$, a contradiction.

(12) $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) \neq g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4)$.

If (12) is not the case, setting $x_3 = x_4$, we have $g_1(x_1,x_2,x_3) = g_2(x_1,x_2,x_3)$, a contradiction.

(13) $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) \neq g_2(x_1,x_3,x_2)g_2(x_1,x_3,x_4)$.

(a) $b^* > b'$.

Let S be such that $x_1(S) = b_1$, $x_2(S) = x_4(S) = b_3$, $x_3(S) = b_2$. Then $R(S) = b'$ while $L(S) = b^*b_1b_3$. Thus, $L(S) \geqslant b^* > b' = R(S)$.

(b) $b^* \not> b'$.

Let S be such that $x_i(S) = b_i$, $i = 1,2,3$, $x_4(S) = b_3$. Then $L(S) = b^*$ and $R(S) = b'b_1b_3 \geqslant b'$. If $L(S) = R(S)$, it follows that $b^* > b'$, a contradiction.

(14) $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) \neq g_2(x_3,x_4,x_1)g_2(x_3,x_4,x_2)$.

If (14) is not the case, setting $x_3 = x_4$, we obtain $x_1x_2x_3 = g_1(x_1,x_2,x_3)$ by Lemma 2.1, which is impossible.

Now, it is a simple matter to check that all the other possible cases are just permutations of the above fourteen cases. Thus, a similar argument can be applied for them.

It therefore follows that the polynomials in (Δ) are essential and distinct over $\mathfrak{A}$. Hence, we have $p_4(\mathfrak{A}) \geqslant 10n-9$. This completes the proof of Proposition 2.4.

We shall now establish the following main result.

Theorem 2.5.

For each positive integer n, let F(n,1) be the smallest value such that the sequence $\langle 0,0,1,n,F(n,1) \rangle$ is representable in $\underset{\sim}{K}(1)$. Then $F(n,1) = 10n-9$.

Proof : By Proposition 1.8 and the above result.

3. F(n,k) Described.

We shall extend the results in the previous sections to a more general case in this section. Instead of the sequence $\langle 0,0,1,n,F(n,1) \rangle$, we shall consider the sequence

$\langle 0,0,\overbrace{1,\ldots,1}^{k},n,F(n,k)\rangle$ where $k > 1$.

Throughout this section, the following notations will be adopted.

For each positive integer $k > 1$, let $\mathcal{A}(1,k) = \langle A(1,k);\cdot\rangle$ be the $(k+3)$-element semilattice (see Fig. 35).

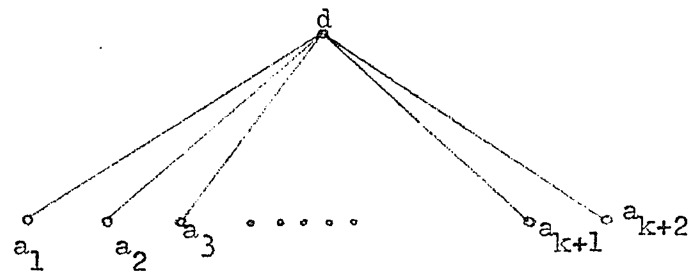


Fig. 35

For each pair of positive integers n,k , let (n,k) $=$ $A(n,k);F$ be the algebra (see Fig. 36)

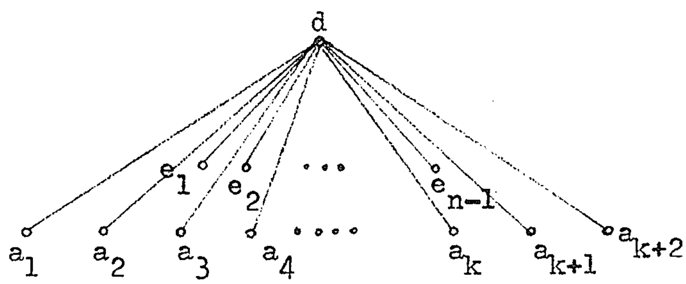


Fig. 36

where $|A(n,k)| = n+k+2$ and the set of the operations $F = \{\cdot,f_1,\ldots,f_{n-1}\}$ consists of one join semilattice operation and $n-1$ $(k+2)$-ary operation f_i such that for each $i = 1,\ldots,n-1$, f_i is defined by the following rule :

$$f_i(x_1,\ldots,x_{k+2}) = \begin{cases} e_i & \text{if } \{x_1,\ldots,x_{k+2}\} = \{a_1,\ldots,a_{k+2}\} \\ \prod_{j=1}^{k+2} x_j & \text{otherwise .} \end{cases}$$

According to a previous result, it is known that the algebra $\mathfrak{A}(n,k)$ represents the sequence $\langle 0,0,\overbrace{1,\ldots,1}^{k},n\rangle$. Furthermore, applying an argument similar to one used in the previous sections, we can prove that the following identities hold in $\mathfrak{A}(n,k)$.

(1) $f_i(x_1,\ldots,x_{k+1},x_{k+2}x_{k+3})$
$= f_i(x_1,\ldots,x_{k+1},x_{k+2})f_i(x_1,\ldots,x_{k+1},x_{k+3})$;

(2) $f_i(x_1,p_1,\ldots,p_{k+2})x_1 = x_1 \prod_{j=1}^{k+2} p_j$ where p_j is a polynomial over $\mathfrak{A}(n,k)$;

(3) $f_i(f_j(x_1,\ldots,x_{k+1},x_{k+2}),x_{k+3},x_1,\ldots,x_k) = \prod_{j=1}^{k+3} x_j$, $i,j = 1,2,\ldots,n-1$;

(4) $f_i(p_1,\ldots,p_{k+2}) = \prod_{j=1}^{k+3} x_j$, where $f_i(p_1,\ldots,p_{k+2})$ is an essentially (k+3)-ary polynomial; the p_j's are pairwise distinct polynomials over $\mathfrak{A}(n,k)$ containing no subpolynomials of the product of the form $x_t x_s$.

(5) $f_i(x_1,\ldots,x_{k+2})f_j(x_2,\ldots,x_{k+2},x_{k+3}) = \prod_{t=1}^{k+3} x_t$, where $i \neq j$;

(6) $\prod_{\Lambda} f_i(x_{\alpha(1)},\ldots,x_{\alpha(k+2)}) = \prod_{j=1}^{k+3} x_j$ if $|\Lambda| \geqslant 3$.

With the aid of the identities (1)---(6), we are able to arrive at the following

<u>Proposition 3.1.</u>

In the algebra $\mathfrak{A}(n,k)$, only the following (k+3)-ary polynomials are distinct and essential :

$$
(\#)\quad \left\{\begin{array}{l}
\prod_{j=1}^{k+3} x_j \\
f_i(x_1,\dots,x_{k+2})x_{k+3},\ f_i(x_2,\dots,x_{k+3})x_1,\dots, \\
\dots,\dots,\ f_i(x_{k+3},x_1,\dots,x_{k+1})x_{k+2}, \\
f_i(x_1,\dots,x_{k+1},x_{k+2})f_i(x_1,\dots,x_{k+1},x_{k+3}), \\
\vdots \\
f_i(x_2,\dots,x_{k+2},x_1)f_i(x_2,\dots,x_{k+2},x_{k+3}),
\end{array}\right.
$$

for $i = 1,2,\dots,n-1$.

Thus, $p_{k+3}(\mathfrak{A}(n,k)) = 1 + \frac{1}{2}(n-1)(k+3)(k+4)$.

<u>Proof</u> : It is trivial that all the (k+3)-ary polynomials in (#) are distinct and essential over $\mathfrak{A}(n,k)$.

On the other hand, let p be an essentially (k+3)-ary polynomial over $\mathfrak{A}(n,k)$. By (1), p can be expressed as

$$p = \prod f_1(A_1,\dots,A_{k+2}) \prod f_2(B_1,\dots,B_{k+2})\cdots \prod f_{n-1}(M_1,\dots,M_{k+2}) \prod x_k$$

where $A_i,\dots,M_i$ are polynomials which contain no sub-polynomials of the product of the form $x_i x_j$ and the A_i's are pairwise distinct over $\mathfrak{A}(n,k)$. The same is true for B_i's,..., and M_i's.

By (3) and (4), we have

$$p = \prod f_1(x_{\alpha(1)},\dots,x_{\alpha(k+2)})\cdots\cdots \prod f_{n-1}(x_{\beta(1)},\dots,x_{\beta(k+2)}) \prod x_j.$$

If there is no occurrence of f_i in p, then

$$p = \prod_{j=1}^{k+3} x_j.$$

If p has a factor "f_i", for some $i = 1,\dots,n-1$, then by (5),

$$p = \prod_{\gamma\in\Lambda} f_i(x_{\gamma(1)},\dots,x_{\gamma(k+2)}) \prod x_j .$$

By (6), the number of occurrences of the symbol "f_i" is either one or two. If the number is one, then by (2),

$$p = f_i(x_1,\ldots,x_{k+2})x_{k+3}, \text{ or symmetrically.}$$

If the number is two, then, again by (2), we have

$$p = f_i(x_1,\ldots,x_{k+1},x_{k+2})f_i(x_1,\ldots,x_{k+1},x_{k+3}), \text{ and so on.}$$

Thus, the polynomials in (#) are the only distinct and essentially (k+3)-ary polynomials over $\mathfrak{A}(n,k)$. From this, it follows that

$$p_{k+3}(\mathfrak{A}(n,k)) = 1 + (n-1)\binom{k+3}{1} + (n-1)\binom{k+3}{2}$$
$$= 1 + \tfrac{1}{2}(n-1)(k+3)(k+4),$$

which was to be shown.

Now, for each $k > 1$, let us denote by $\underset{\sim}{K}(k)$ the class of all idempotent algebras such that all the essentially (k+2)-ary polynomials are symmetric. Evidently, $\mathfrak{A}(n,k) \in \underset{\sim}{K}(k)$ for each positive integer n. Let $\mathfrak{A}$ be an algebra in $\underset{\sim}{K}(k)$ representing the sequence $\langle 0,0,\overbrace{1,\ldots,1}^{k},n \rangle$. Since $k > 1$; it is easily seen that the only essentially binary polynomial over $\mathfrak{A}$ must be a semilattice polynomial. Thus, we have already one essentially (k+2)-ary polynomial, namely, $\prod_{j=1}^{k+2} x_j$. As $p_{k+2}(\mathfrak{A}) = n$, let g_i, $i = 1,\ldots,n-1$ denote the remaining n-1 essentially (k+2)-ary polynomials. Since $\mathfrak{A} \in \underset{\sim}{K}(k)$, g_i is symmetric for each $i = 1,2,\ldots,n-1$.

In what follows, we are going to prove that corresponding to the $1+\frac{1}{2}(n-1)(k+3)(k+4)$ distinct essentially (k+3)-ary polynomials of $\mathfrak{A}(n,k)$ which were described in Proposition 3.1, the following (k+3)-ary polynomials are distinct and essential over $\mathfrak{A}$:

$$(\theta)\quad \begin{cases} \prod_{j=1}^{k+3} x_j ; \\ g_i(x_1,\dots,x_{k+2})x_{k+3}, g_i(x_2,\dots,x_{k+3})x_1, \dots\dots \\ \dots\dots, g_i(x_{k+3},x_1,\dots,x_{k+1})x_{k+2} ; \\ g_i(x_1,\dots,x_{k+1},x_{k+2})g_i(x_1,\dots,x_{k+1},x_{k+3}) \\ \vdots \\ g_i(x_2,\dots,x_{k+2},x_1)g_i(x_2,\dots,x_{k+2},x_{k+3}) \end{cases}$$

where $i = 1,2,\dots,n-1$

By generalizing the technique which was applied in proving Lemmas 2.1, 2.2 and 2.3 in a suitable way, we have the following basic Lemmas:

<u>Lemma 3.2.</u>

In $\mathfrak{A}$, $g_i(x_1,\dots,x_{k+1},x_{k+1}) = \prod_{j=1}^{k+1} x_j$, for each $i = 1,2,\dots,n-1$.

<u>Lemma 3.3.</u>

In $\mathfrak{A}$, $g_i(x_1,x_2,\dots,x_{k-1},x_k,x_{k+1},x_kx_{k+1}) = \prod_{j=1}^{k+1} x_j$, for each $i = 1,2,\dots,n-1$.

<u>Lemma 3.4.</u>

In $\mathfrak{A}$, $g_i(x_1,x_2,\dots,x_k,x_{k+1},x_{k+1}x_{k+2}) = \prod_{j=1}^{k+2} x_j$, for each $i = 1,2,\dots,n-1$.

With the help of these Lemmas, we have

<u>Proposition 3.5.</u>

Let $\mathfrak{A}$ be angebra in K(k) representing $\langle 0,0,\overbrace{1,\dots,1}^{k},n\rangle$.

Then $\quad p_{k+3}(\mathfrak{A}) \geqslant 1 + \frac{1}{2}(n-1)(k+3)(k+4)$.

Proof : Lemma 3.2 implies that the polynomials in (θ) are essentially (k+3)-ary over $\mathfrak{A}$. Our proof will be complete if we can show that the polynomials in (θ) are distinct. By making use of Lemmas 3.2, 3.3 and 3.4, and following the same arguments as in the proof of Proposition 2.4, it can be shown that this is, indeed, the case. For instance, to check that

$$\prod_{j=1}^{k+3} x_j \neq g_1(x_1,\ldots,x_{k+2})x_{k+3},$$

choose a subset $\{c_1,\ldots,c_{k+2}\}$ of A such that

$$\prod_{j=1}^{k+2} c_j = \bar{c} \neq c^* = g_1(c_1,\ldots,c_{k+2}).$$

By Lemmas 3.2 and 3.4, it follows that $c_1,\ldots,c_{k+2}$ are pairwise distinct and incomparable.

Case 1. $\bar{c} < c^*$

Let S be a substitution such that $x_i(S) = c_i$, $i = 1,2,\ldots,k+2$, and $x_{k+3}(S) = c_1$. Then we have $L(S) = \prod_{j=1}^{k+2} c_j = \bar{c}$ while $R(S)$ $= g_1(c_1,\ldots,c_{k+2})c_1 = c^*c_1 = (c^*\bar{c})c_1 = c^*(\bar{c}c_1) = c^*\bar{c} = c^*$. Thus $R(S) \neq L(S)$.

Case 2. $\bar{c} \not< c^*$

Let S be such that $x_i(S) = c_i$, $i = 1,2,\ldots,k+2$ and $x_{k+3}(S) = c^*$. Then $L(S) = \bar{c}c^* \geq \bar{c}$, $R(S) = c^*c^* = c^*$. If $L(S) = R(S)$, it follows that $c^* > \bar{c}$, which is impossible. Thus, $L(S) \neq R(S)$.

Hence, it follows that $\prod_{j=1}^{k+3} x_j \neq g_1(x_1,\ldots,x_{k+2})x_{k+3}$, as required.

By combining this with Proposition 3.1, we obtain the following main result.

Theorem 3.6.

For each integer $k > 1$, let $\underset{\sim}{K}(k)$ be the class of all idempotent algebras such that all the essentially $(k+2)$-ary polynomials are symmetric. For each positive integer n, let $F(n,k)$ be the smallest value such that the sequence $\langle 0,0,\overbrace{1,\ldots,1}^{k},n,F(n,k)\rangle$ is representable in $\underset{\sim}{K}(k)$. Then

$$F(n,k) = 1 + \tfrac{1}{2}(n-1)(k+3)(k+4).$$

PART III

IDEMPOTENT ALGEBRAS WITH TWO OR THREE ESSENTIALLY BINARY POLYNOMIALS

CHAPTER I

THE SEQUENCE $\langle 0,0,2,m \rangle$

Idempotent algebras with exactly one essentially binary polynomial have been discussed in Part I and Part II. In this chapter, we start to consider idempotent algebras with two essentially binary polynomials, i.e., algebras representing $\langle 0,0,2 \rangle$.

It is known that the sequence $\langle 0,0,2,0 \rangle$ is representable. For instance, if $\mathfrak{A}$ is a diagonal semigroup, then $p_2(\mathfrak{A}) = 2$ and $p_n(\mathfrak{A}) = 0$, for each $n \neq 2$. J. Płonka proved in [34] that if the sequence $\langle 0,0,2,k \rangle$ is representable and $k > 0$, then $k \geqslant 3$. The case $k = 3$ is possible. It turns out that algebras representing $\langle 0,0,2,3 \rangle$ can be classified into four equational classes of algebras which are described in [35]. It has been pointed out by J. Gerhard in [5] that the sequence $\langle 0,0,2,6 \rangle$ is representable. In fact, if $\mathfrak{A}$ is an idempotent semigroup satisfying $aba = ab$, then $p_n(\mathfrak{A}) = n!$ for each $n \geqslant 2$.

It is, perhaps, important to note that there is a common feature in all the algebras shown above. That is, each of them has one and only one essentially binary polynomial which is non-commutative. Thus, in order to continue their investigations, it is natural to study the opposite case; namely, algebras with two distinct commutative essentially binary polynomials. The main object of this chapter is to deal with this case.

In section 1, the binary polynomials are studied in detail. The results are summarized in Proposition 1.4 which will be of great use in developing the Characterization Theorems in Chapter 1 of Part IV. Some results of Płonka are mentioned in section 2 which are applied to prove our main results in sections 3 and 4.

1. Binary Polynomials.

Throughout the remaining chapters, let $\underset{\sim}{K}$ be the equational classes of algebras defined by the following two identities:

$$\begin{cases} x + y = y + x \\ xy = yx \end{cases}$$

where x + y, xy are two distinct essentially binary polynomials over algebras in $\underset{\sim}{K}$. ~~Thus, if $\mathfrak{A}$ is an algebra of type $\langle 2,2 \rangle$ and $p_3(\mathfrak{A}) = 2$, then $\mathfrak{A} \in \underset{\sim}{K}$.~~

Let $\mathfrak{A} \in \underset{\sim}{K}$ represent the sequence $\langle 0,0,2 \rangle$. Then clearly, x + y and xy are the only two idempotent, commutative essentially binary polynomials over $\mathfrak{A}$.

Consider the binary polynomials x + xy, x(x + y). As $p_3(\mathfrak{A}) = 2$, we have the following possibilities :

$$x + xy = \begin{cases} x \\ y \\ xy \\ x + y \end{cases} \qquad x(x + y) = \begin{cases} x \\ y \\ xy \\ x + y \end{cases}$$

Thus, there are sixteen cases for $x + xy$ and $x(x + y)$ in general. However, we shall show in Proposition 1.4 that there are, in fact, only four. To this end, we first establish the following Lemmas.

Lemma 1.1.

Let $\mathfrak{A} \in \underset{\sim}{K}$ represent the sequence $\langle 0,0,2 \rangle$. Then

$$x + xy \neq y \quad \text{and} \quad x(x + y) \neq y .$$

Proof : Assume that $x(x + y) = y$ ———————— (a)

We have the following four possible cases.

Case 1. $x + xy = x$ ———————————————— (b)

Observe that $x + y = x + x(x + y)$ (by (a))

$= x$ (by (b))

which is a contradiction.

Case 2. $x + xy = y$ ———————————————— (c)

We have $x + y = y(y + (x + y))$ (by (a))

$= y((x + y) + y)$

$= y((x + y) + x(x + y))$ (by (a))

$= y((x + y) + (x + y)x)$

$= yx$ (by (c))

which is a contradiction.

Case 3. $x + xy = xy$ ———————————————— (d)

Observe that $x = (xy)((xy) + x)$ (by (a))

$= (xy)(xy)$ (by (d))

$= xy,$

which is a contradiction.

Case 4. $\qquad x + xy = x + y$ ———————— (e)

Note that $xy = x(x + xy)$ $\qquad$ (by (a))

$\qquad = x(x + y)$ $\qquad$ (by (e))

$\qquad = y$ $\qquad$ (by (a)),

which is a contradiction.

Hence, if (a) holds in $\mathfrak{A}$, we have no choice for $x + xy$. Therefore, $x(x + y) \neq y$, as required. Similarly, $x + xy \neq y$.

Lemma 1.2.

Let $\mathfrak{A} \in \underset{\sim}{K}$ represent the sequence $\langle 0,0,2 \rangle$. Then $x + xy = x$ if and only if $x(x + y) = x$.

Proof : By symmetry, it suffices to prove that $x + xy = x$ implies $x(x + y) = x$. Thus, assume

$x + xy = x$ ———————— (a)

holds in $\mathfrak{A}$. By Lemma 1.1, we have three possible cases:

$$x(x + y) = \begin{cases} x \\ xy \\ x + y \end{cases}$$

Case 1. $\qquad x(x + y) = xy$ ———————— (b)

Observe that $x + y = (x + y) + (x + y)y$ $\qquad$ (by (a))

$\qquad = (x + y) + xy$ $\qquad$ (by (b))

Thus, we have $x + y = (x + y) + xy$ ———————— (c)

Moreover, $x + y = (x + y)(x + y)$

$\qquad = ((x + y) + xy)(x + y)$ $\qquad$ (by (c))

$\qquad = (x + y)(xy)$ $\qquad$ (by (b))

i.e., $\quad x + y = (x + y)(xy)$ ———————— (d).

From these, we obtain

$$x + y = (x + y) + xy \qquad (\text{ by (c) })$$

$$= (xy) + (xy)(x + y) \qquad (\text{ by (d) })$$

$$= xy \qquad (\text{ by (a) })$$

a contradiction.

Case 2. $x(x + y) = x + y.$ ——————— (e)

If this is the case, we would have

$$x = x + xy \qquad (\text{ by (a) })$$

$$= (xy)(xy + x) \qquad (\text{ by (e) })$$

$$= (xy)x \qquad (\text{ by (a) })$$

i.e., $x = (xy)x$ ——————— (f)

Now, it follows that $x + y = x(x + y)$ (by (e))

$$= (x(x + y))x \qquad (\text{ by (e) })$$

$$= x \qquad (\text{ by (f) })$$

which is impossible.

Thus, we must have $x(x + y) = x$, completing the proof of Lemma 1.2.

Lemma 1.3.

Let $\mathfrak{A} \in K$ represent the sequence $\langle 0,0,2 \rangle$. Then

1) $x + xy = x + y$ implies $x(x + y) = x + y$;

2) $x(x + y) = xy$ implies $x + xy = xy$.

Proof: By symmetry, it suffices to prove 1).

Thus, assume that $x + xy = x + y$ ——————— (a)

By Lemmas 1.1 and 1.2, we have

$$x(x + y) = \begin{cases} xy \\ x + y. \end{cases}$$

If $x(x + y) = xy$, ——————————— (b)

we would have $x(xy) = x(x + xy)$ (by (b))

$= x(x + y)$ (by (a))

$= xy$, (by (b))

i.e., $x(xy) = xy$ ——————————— (c).

Similarly, $x + (x + y) = x + x(x + y)$ (by (a))

$= x + xy$ (by (b))

$= x + y$ (by (a)),

i.e., $x + (x + y) = x + y$ ——————————— (d)

Furthermore, $(xy)(x + y) = (xy)(x + xy)$ (by (a))

$= (xy)x$ (by (b))

$= xy$ (by (c))

i.e., $(xy)(x + y) = xy$ ——————————— (e)

Similarly, by (a), (b), (d), we have

$(x + y) + xy = x + y$ ——————————— (f)

It thus follows that $x + y = xy + (x + y)$ (by (f))

$= xy + (xy)(x + y)$ (by (a))

$= xy + xy$ (by (e))

$= xy$,

which is a contradiction.

which is (1). Moreover, it follows that

$$(x + y) + x = x(x + y) + x \qquad (\text{ by } (a) \text{ })$$
$$= x(x + y) \qquad (\text{ by } (b) \text{ })$$
$$= x + y \qquad (\text{ by } (a) \text{ })$$

which is (2).

To prove (3), consider $(x + y) + xy$ and $(x + y)(xy)$. Clearly, both of them are symmetric with respect to x and y. Thus, if they depend on x, they must depend on y simultaneously and vice versa. As $p_0(\mathcal{O}\!\mathcal{L}) = p_1(\mathcal{O}\!\mathcal{L}) = 0$, we have

$$(x + y) + xy = \begin{cases} x + y \\ xy \end{cases}, \qquad (x + y)(xy) = \begin{cases} x + y \\ xy \end{cases}$$

If $(x + y) + xy = x + y$ ———————————— (c)

then $(x + y)(xy) = ((x + y) + xy)(xy)$ (by (c))

$= xy + (x + y)$ (by (a))

$= x + y$ (by (c)).

Thus, $(x + y) + xy = x + y$ implies that $(x + y)(xy) = x + y$.

On the other hand, if $(x + y) + xy = xy$ ———————— (d)

then $(x + y)(xy) = (x + y)((x + y) + xy)$ (by (d))

$= (x + y) + xy$ (by (a))

$= xy$ (by (d)).

Thus, $(x + y) + xy = xy$ implies $(x + y)(xy) = xy$.

From these, (3) follows.

2. Płonka's Basic Lemmas.

Let $\mathcal{O}\!\mathcal{L}$ be an algebra having two distinct binary polynomials denoted by $x + y$ and xy. We shall introduce the following notation :

$$\begin{cases} f_{11} = (x + y) + z & f_{21} = (xy)z \\ f_{12} = (y + z) + x & f_{22} = (yz)x \\ f_{13} = (z + x) + y & f_{23} = (zx)y \\ & \\ f_{31} = (x + y)z & f_{41} = xy + z \\ f_{32} = (y + z)x & f_{42} = yz + x \\ f_{33} = (z + x)y & f_{43} = zx + y \end{cases}$$

The following Lemmas are stated without proofs. In fact, they are included in J. Płonka [29].

<u>Lemma 2.1.</u>

If $x + y$ and xy are both commutative and idempotent, then f_{ik} is an essentially ternary polynomials for each $i,k = 1,2,3,4$.

<u>Lemma 2.2.</u>

If $x + y$ and xy are both idempotent and commutative, then f_{ik} and f_{jt} are distinct for $i \neq j$.

The next Lemma says that if we have two idempotent and commutative binary polynomials in an algebra we then have at least eight distinct essentially ternary polynomials.

<u>Lemma 2.3.</u>

Let $x + y$ and xy be both idempotent and commutative.

(1) If $x + y$ and xy are not associative, then f_{1k}, f_{2k}, f_{31},

f_{41}, k = 1,2,3 are distinct;

(2) If x + y is associative but xy not, then f_{11}, f_{2k}, f_{3k}, f_{41}, k = 1,2,3 are distinct;

(3) If xy is associative but x + y not, then f_{1k}, f_{21}, f_{31}, f_{4k}, k = 1,2,3 are distinct.

Let $\mathfrak{A} = \langle A;+,\circ\rangle$ be an algebra where "+" and "$\circ$" satisfy the idempotent, commutative and associative laws. It is known that $\mathfrak{A}$ is not a lattice in general since the absorption laws are independent of those mentioned above. However, if we assume that $|A| \geqslant 2$ and $p_2(\mathfrak{A}) = 2$, then we can prove that $\mathfrak{A}$ is, indeed, a lattice. This can be seen from the following.

Lemma 2.4.

Let $\mathfrak{A} = \langle A;+,\circ\rangle$ be an algebra such that (1) $|A| \geqslant 2$; (2) x + y and xy are the two distinct idempotent, commutative, associative essentially binary polynomials in $\mathfrak{A}$; (3) no other essentially j-ary polynomials for j = 1,2 except x, y, x + y, xy. Then the absorption laws $x(x + y) = x + xy = x$ hold in $\mathfrak{A}$.

Lemma 2.5.

If x + y and xy are distinct idempotent, cammutative essentially binary polynomials in $\mathfrak{A}$ with $|A| \geqslant 2$ and the polynomial $f_{31} = (x + y)z$ is symmetric, then there exists in $\mathfrak{A}$ an essentially binary polynomial which is different from x + y and xy. The same statement is true if we replace the polynomial f_{31} by $f_{41} = xy + z$.

3. The Smallest Value of m.

In this section, to begin with, we bring forward the following problem : Find the greatest integer m^* such that if a sequence $\langle 0,0,2,m \rangle$ is representable in $\underset{\sim}{K}$, then $m \geqslant m^*$. Our first result reveals that the required integer m^* is "9". Thus, to proceed, it is natural to investigate whether there is any algebra representing $\langle 0,0,2,9 \rangle$. Our next result provides a positive answer to this.

Theorem 3.1.

If the sequence $\langle 0,0,2,m \rangle$ is representable in $\underset{\sim}{K}$, then $m \geqslant 9$.

Proof : Let $\mathfrak{A}$ be an algebra in $\underset{\sim}{K}$ representing the sequence $\langle 0,0,2,m \rangle$. Let $x + y$ and xy be the only two distinct idempotent, commutative essentially binary polynomials. Since $x + y \neq xy$, $|A| \geqslant 2$. The following are the only four possible cases :

(1) $x + y$ and xy are non-associative ;

(2) $x + y$ is associative but xy not ;

(3) xy is associative but $x + y$ not ;

(4) $x + y$ and xy are associative.

If (1) is the case, then according to Lemmas 2.1 and 2.3(1), the following f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{31}, f_{41}, f_{23} are distinct essentially ternary polynomials over $\mathfrak{A}$.

Consider the polynomial $f_{31} = (x + y)z$. If it is symmetric, then by Lemma 2.5, we obtain $p_2(\mathfrak{A}) > 2$, a contradiction.

The case for $f_{41} = xy + z$ is similar. Thus, f_{31} and f_{41} are not symmetric. Hence the following essentially ternary polynomials

$$\begin{matrix} f_{31} & f_{32} & f_{33} \\ f_{41} & f_{42} & f_{43} \end{matrix}$$

are pairwise distinct. For example, if $f_{31} = f_{32}$, then $f_{33} = (z + x)y = (x + z)y = (z + y)x = (y + z)x = f_{32}$; i.e., $f_{31} = f_{32} = f_{33}$, a contradiction.

By Lemmas 2.1 and 2.2, it follows that $m = p_3(\mathfrak{A}) \geqslant 12 > 9$.

If (2) is the case, then according to Lemmas 2.1 and 2.3(2), we have the following distinct essentially ternary polynomials :

$$f_{11},\ f_{21},\ f_{22},\ f_{23},\ f_{31},\ f_{32},\ f_{33},\ f_{41}\ .$$

If f_{41} is symmetric, then by Lemma 2.5, we would have $p_2(\mathfrak{A}) > 2$, a contradiction. So f_{41}, f_{42}, f_{43} are pairwise distinct. By Lemmas 2.1 and 2.2, we have

$$m = p_3(\mathfrak{A}) \geqslant 10 > 9.$$

The case (3) is symmetric to (2). The proof is similar.

Thus, it remains to consider case (4). In this situation, observe that as $\mathfrak{A}$ represents $\langle 0,0,2 \rangle$, all the three conditions of Lemma 2.4 are fulfilled. Accordingly, the two absorption laws hold in $\mathfrak{A}$. In other words, $\mathfrak{A}$ is a lattice. Now, let us look at the following polynomials :

$$\begin{array}{ll} xyz & x + y + z \\ xy + z & (x + y)z \\ yz + x & (y + z)x \\ zx + y & (z + x)y \end{array}$$

$$xy + yz + zx$$

It is a simple matter to check that they are distinct and essentially ternary over $\mathcal{O}\!\mathcal{C}$. For instance, if $xy + z = xy + yz + zx$, setting $x = y$, we have $x + z = x + xz = x$, a contradiction. Therefore, we obtain $m = p_3(\mathcal{O}\!\mathcal{C}) \geqslant 9$. The proof of Theorem 3.1 is thus complete.

The following result shows that the case m = 9 is possible.

Theorem 3.2.

Every distributive lattice with more that one element represents the sequence $\langle 0,0,2,9 \rangle$.

Proof : Let $\mathcal{O}\!\mathcal{C}$ be a given distributive lattice with more than one element. Let us consider $P^{(n)}(\mathcal{O}\!\mathcal{C})$, the set of n-ary polynomials over $\mathcal{O}\!\mathcal{C}$. Being a lattice, it is well-known that $P^{(n)}(\mathcal{O}\!\mathcal{C})$ is a free distributive lattice on n generators. Clearly, $P^{(0)}(\mathcal{O}\!\mathcal{C}) = \emptyset$; $P^{(1)}(\mathcal{O}\!\mathcal{C})$ is one element lattice ; $P^{(2)}(\mathcal{O}\!\mathcal{C})$ is the four-element lattice C_2^2. Moreover, the lattice $P^{(3)}(\mathcal{O}\!\mathcal{C})$ consists of eighteen elements, and its diagram is given below (see Fig. 37).

Let $\underset{\sim}{C}$ be an equational class of algebras, and let F_n denote the cardinality of the free algebra over C on n generators. Let $\mathcal{B}$ be the free algebra over $\underset{\sim}{C}$ on ω generators, $p_k = p_k(\mathcal{B})$.

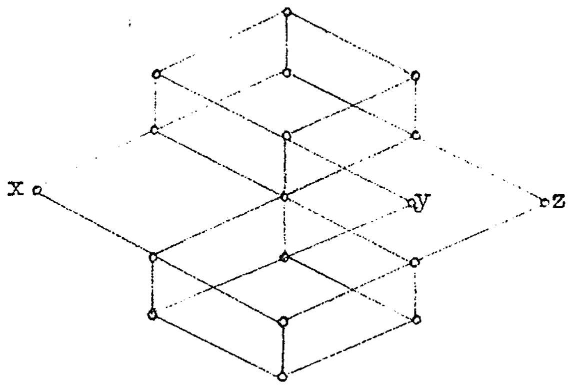


Fig. 37

Then we have the following nice formula (see [21])

$$F_n = n + \sum_{k=0}^{n} \binom{n}{k} p_k .$$

Invoking this, a direct computation shows that $\mathfrak{A}$ represents the sequence $\langle 0,0,2,9 \rangle$.

Summarizing all the results about the sequence $\langle 0,0,2,m \rangle$, we have

Corollary 3.3.

Let $\mathfrak{A}$ be an algebra representing the sequence $\langle 0,0,2,m \rangle$.
The following are the only two possible cases :

(1) If $\mathfrak{A}$ has a non-commutative essentially binary polynomial, then $m = 0$ or $m \geq 3$.

(2) If $\mathfrak{A}$ has two distinct commutative essentially binary polynomials, then $m \geq 9$.

4. Algebras Representing $\langle 0,0,2,10 \rangle$.

It was shown in section 3 that the sequence $\langle 0,0,2,9 \rangle$ is representable. In this section, we study the sequence

$\langle 0,0,2,10 \rangle$.

Let us consider the following algebra $\mathcal{L}(4) = \langle L(4);+,\cdot \rangle$ where $\langle L(4);\cdot \rangle$ is the following four-element join semilattice (see Fig. 38)

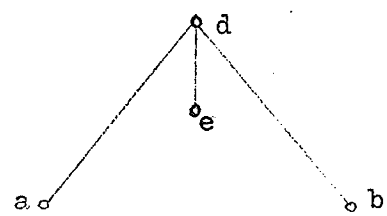


Fig. 38

and the binary operation "+" is defined as follows :

$$x + y = \begin{cases} e & \text{if } \{x,y\} = \{a,b\} \\ xy & \text{otherwise} . \end{cases}$$

For the sake of convenience, let us denote $x + y$ by $f(x,y)$. It is a simple matter to prove that the following identities hold in $\mathcal{L}(4)$.

(1) $f(x,y)x = xy$

(2) $f(xy,z) = f(x,z)f(y,z)$

(3) $f(f(x,y),f(x,z)) = f(x,y)f(x,z)$

(4) $f(f(f(x,y),z),x) = xyz$

(5) $f(f(f(x,y),z),z) = f(x,y)z$

(6) $f(f(f(x,y),z),f(x,y)) = f(x,y)z$

(7) $f(f(f(x,y),z),f(x,z)) = f(x,z)f(y,z)$

(8) $f(f(x,y),z)x = xyz$

(9) $f(f(x,y),z)f(x,z) = f(x,z)f(y,z)$

(10) $f(f(x,y),z)f(f(x,z),y) = xyz$

(11) $f(x,y)f(y,z)f(z,x) = xyz$

It follows from (1) and (2) that the polynomials $f(x,y)$, xy are the only two essentially binary polynomials over $\mathcal{L}(4)$.

Let p be an arbitrary essentially ternary polynomial over $\mathcal{L}(4)$. By (2), p can be written as

$$p = \prod_{i\in I} f(q_i, r_i) \prod x_j$$

where q_i, r_i are polynomials containing no sub-polynomials of the form A.B and $x_j \in \{x,y,z\}$.

Assume that p cannot be reduced to any simpler form. Then according to the above identities, the case $|I| \geqslant 3$ is impossible. If $|I| = 2$, we have $f(x,y)f(x,z)$ and the polynomials formed from it by symmetry. If $|I| = 1$, we have $f(x,y)z$, $f(f(x,y),z)$ and the polynomials formed from them by symmetry. If $|I| = 0$, we obtain xyz.

Thus, the following are the ten and only ten distinct essentially ternary polynomials over $\mathcal{L}(4)$:

$$\begin{cases} xyz \\ (x + y) + z, \quad (y + z) + x, \quad (z + x) + y, \\ (z + x)(z + y) \quad (= z + xy), \\ (x + y)(x + z), \\ (y + x)(y + z), \\ (x + y)z, \quad (y + z)x, \quad (z + x)y. \end{cases}$$

Consequently, we have the following

Theorem 4.1.

The algebra $\mathcal{L}(4)$ represents the sequence $\langle 0,0,2,10 \rangle$.

As a matter of fact, the algebra $\mathcal{L}(4)$ plays a central role in the class of algebras representing $\langle 0,0,2,10 \rangle$. This is a consequence of the following Theorem.

Theorem 4.2

Let $\mathcal{O}\!\mathcal{C}$ be an algebra of $\underset{\sim}{K}$ representing $\langle 0,0,2,10 \rangle$. Then $\mathcal{O}\!\mathcal{C}$ can be represented as an algebra which contains $\mathcal{L}(4)$ as a subalgebra.

Proof : Let $x + y$, xy be the two commutative essentially binary polynomials over $\mathcal{O}\!\mathcal{C}$.

Case 1. $x + y$ and xy are non-associative.

In this case, as in the proof of Theorem 3.1, we have $p_3(\mathcal{O}\!\mathcal{C}) \geqslant 12$, a contradiction.

Case 2. $x + y$ and xy are associative.

In this case, it follows that $\langle A;+,\cdot \rangle$ is a lattice. Thus, by Theorem 3.3 (IV,1), either $p_3(\mathcal{O}\!\mathcal{C}) = 9$ or $p_3(\mathcal{O}\!\mathcal{C}) \geqslant 19$, which is impossible.

Accordingly, one of them, $x + y$ say is associative and the other is not. In this situation, as in the proof of Theorem 3.1, we have at least the following ten distinct essentially ternary polynomials :

$$(*) \quad \begin{cases} xyz \\ (x + y) + z, \quad (y + z) + x, \quad (z + x) + y, \\ x + yz, \quad y + zx, \quad z + xy, \\ (x + y)z, \quad (y + z)x, \quad (z + x)y \end{cases}$$

In virtue of Proposition 1.4, we have the following four possible cases :

$$\begin{cases} (x + y)x \\ x + xy \end{cases} = \begin{cases} \begin{cases} x + y \\ x + y \end{cases} & \text{--------- (1)} \\ \begin{cases} x + y \\ xy \end{cases} & \text{--------- (2)} \\ \begin{cases} x \\ x \end{cases} & \text{--------- (3)} \\ \begin{cases} xy \\ xy \end{cases} & \text{--------- (4)} \end{cases}$$

The case (1) is impossible, for, if it were the case, we would have $xy = xy + xy = xy + (xy)y = xy + y = x + y$, a contradiction.

Assume (2) holds, i.e., $(x + y)x = x + y$ and $x + xy = xy$. We claim that the polynomial $p = (x + y)(y + z)(z + x)$ is essentially ternary and is distinct from the polynomials in (*).

It is trivial that p is essential. If $p = xyz$, setting $y = z$, we have $(x + y)y = xy$, i.e., $x + y = xy$, a contradiction. Thus $p \neq xyz$. If $p = (x + y) + z$, then as p is symmetric, it follows that $(x + y) + z = (y + z) + x = (z + x) + y$, which is impossible. Thus $p \neq (x + y) + z$. The proofs of the other cases are similar. Accordingly, we have $p_3(\mathcal{A}) \geqslant 11$, which contradicts our assumption.

Assume (3) holds, i.e., $(x + y)x = x = x + xy$. In analogy to the previous case, we claim that $p = (x + y)(y + z)(z + x)$ is essential and distinct from the polynomials in (*).

Clearly, p is essentially. If $p = xyz$, putting $z = x$, we have $(x + y)x = xy$, i.e., $x = xy$, a contradiction. Thus

$p \neq xyz$. The proofs of other cases are similar to the above. Hence, we have $p_3(\mathfrak{A}) \geqslant 11$, which is impossible.

Therefore, if $\mathfrak{A} \in \underset{\sim}{K}$ represents $\langle 0,0,2,10 \rangle$, it is necessary that

$$(x + y)x = xy = x + xy \qquad \text{------------} \qquad (\#)$$

Consider the algebra $\mathfrak{A}^* = \langle A;+,\circ \rangle$. As $x + y \neq xy$, there exist a, $b \in A$ such that $a + b \neq ab$. In view of (#), we have $a \| b$. Set $e = a + b$, $d = ab$. From the fact that $(x + y)xy = xy$, it follows that $e < d$. Moreover, by (#), $e \| a$, $e \| b$. Hence $\mathfrak{A}^*$ contains $\mathcal{L}(4)$ as a subalgebra, as was to be shown.

Some further results along this line will be shown in (IV,1).

CHAPTER 2.

THE SEQUENCE $\langle 0,0,3,m\rangle$.

Suppose that we are given an arbitrary sequence $\langle 0,0,n,m\rangle$. Let us consider the following general problem: For each n, what is the smallest value of m such that the sequence $\langle 0,0,n,m\rangle$ is representable. If n = 1, the corresponding value of m is, of course, equal to one. The case that n = 2 has just been dealt with in Corollary 3.3(III,1). In this chapter, we go one step further by considering the sequence $\langle 0,0,3,m\rangle$.

This chapter falls into three sections. In the first section, we shall develop a series of fundamental results concerning ternary polynomials on which our main results are based. The answer to the above problem (when n = 3) will be given in section 2. In section 3, we study the sequence $\langle 0,0,3,m^*\rangle$ where m* is the minimum value in the above sense.

1. Fundamental Results.

Let $\mathfrak{A}$ be an algebra representing the sequence $\langle 0,0,3\rangle$. Then there are exactly three distinct idempotent essentially binary polynomials over $\mathfrak{A}$. The following are the only two possible cases :

Case 1. Each of the three essentially binary polynomials is commutative.

Case 2. There is one commutative essentially binary polynomial while the other one is non-commutative.

Observe that if Case 1 holds in $\mathfrak{A}$, then in view of Lemma 2.3 (III,1), it follows that $p_3(\mathfrak{A}) \geqslant 8$.

From now on, in order to obtain some information about $p_3(\mathfrak{A})$, we shall be interested in Case 2. So, let $x + y$ be the commutative idempotent essentially binary polynomial and xy be the non-commutative one. Thus $x + y$, xy and yx are exactly the three essentially binary polynomials.

We shall use the following notation :

$$g_{11} = (x + y) + z \qquad g_{12} = (y + z) + x \qquad g_{13} = (z + x) + y$$
$$g_{21} = (x + y)z \qquad g_{22} = (y + z)x \qquad g_{23} = (z + x)y$$
$$g_{31} = z(x + y) \qquad g_{32} = x(y + z) \qquad g_{33} = y(z + x)$$
$$g_{41} = xy + z \qquad g_{42} = yz + x \qquad g_{43} = zx + y .$$

Lemma 1.1.

In $\mathfrak{A}$, g_{ij} is essentially ternary for each $i = 1,2,3,4$, $j = 1,2,3$.

Proof : g_{1j} is clearly essentially ternary by Lemma 2.1(III,1).

To prove that g_{21} is essential, note that by setting $x = y$, we have $g_{21} = x \circ z$. Thus g_{21} depends on z and one of x and y. However as x and y are symmetric, it follows that g_{21} is essentially ternary. Similarly, g_{2j}, g_{3j} are essential for $j = 1,2,3$.

It remains to consider the polynomial $g_{41} = xy + z$. Setting

$x = y$, we obtain $g_{41} = x + z$. Thus g_{41} depends on z and either x or y. If g_{41} depends on x but not y, then we would have

$$g_{41} = xy + z = x + z.$$

Setting $z = xy$, we obtain $xy = x + xy$. Setting $z = x$, we have $xy + x = x$. Thus, it follows that $xy = x + xy = x$, a contradiction. If g_{41} depends on y but not x, then we would have $\quad g_{41} = xy + z = y + z.$

Setting $z = xy$, we have $xy = y + xy$. Setting $z = y$, we obtain $xy + y = y$. Thus, it follows that $xy = y$, which is a contradiction. Hence g_{41} is essentially ternary. The proof is similar for g_{4j}.

<u>Lemma 1.2.</u>

In $\mathfrak{A}$, $g_{1j} \neq g_{i1}$, $j = 1,2,3$, $i = 2,3,4$.

<u>Proof</u>: If $g_{11} = (x + y) + z = (x + y)z = g_{21}$, setting $x = y$, we have $x + z = xz$, a contradiction. Thus $g_{11} \neq g_{21}$.

If $g_{12} = (y + z) + x = (x + y)z = g_{21}$, setting $x = y$, we have

$$(x + z) + x = xz \quad \text{———————} \quad (1)$$

Again, setting $y = z$ and $z = x$ respectively, we obtain

$$y + x = (x + y)y \quad \text{———————} \quad (2)$$

$$(y + x) + x = (x + y)x \quad \text{———————} \quad (3)$$

It thus follows that
$$\begin{aligned} xy &= (x + y) + x && (\text{ by } (1)\ ) \\ &= (x + y)x && (\text{ by } (3)\ ) \\ &= x + y && (\text{ by } (2)\ ) \end{aligned}$$

which is impossible. Thus $g_{12} \neq g_{21}$.

If $g_{13} = (z + x) + y = (x + y)z = g_{21}$, we have
$g_{21} = (x + y)z = (y + x)z = (z + y) + x = (y + z) + x = g_{12}$,
a contradiction. Thus, $g_{13} \neq g_{21}$.

If $g_{11} = (x + y) + z = z(x + y) = g_{31}$, setting $x = y$, we have $x + z = zx$, a contradiction. Thus, $g_{11} \neq g_{31}$.

If $g_{12} = (y + z) + x = z(x + y) = g_{31}$, then setting $x = y$, $y = z$ and $z = x$, we have, respectively,

$$(x + z) + x = zx \quad \text{———} \quad (1)$$

$$y + x = y(x + y) \quad \text{———} \quad (2)$$

$$(y + x) + x = x(x + y) \quad \text{———} \quad (3)$$

From these, it follows that

$$\begin{aligned} yx &= (x + y) + x && (\text{ by } (1)\ ) \\ &= x(x + y) && (\text{ by } (3)\ ) \\ &= x + y && (\text{ by } (2)\ ) \end{aligned}$$

which is a contradiction. Thus, $g_{12} \neq g_{31}$.

Similarly, $g_{13} \neq g_{31}$.

If $g_{11} = (x + y) + z = xy + z = g_{41}$, setting $z = xy$, we obtain $(x + y) + xy = xy$. Hence if we set $z = x + y$, we have $x + y = xy + (x + y) = xy$, a contradiction. Thus, $g_{11} \neq g_{41}$.

If $g_{12} = (y + z) + x = xy + z = g_{41}$, then setting $y = z$, $x = y$ and $z = xy$, we have, respectively,

$$y + x = xy + y \quad \text{———} \quad (1)$$

$$(x + z) + x = x + z \quad \text{———} \quad (2)$$

$$(y + xy) + x = xy \quad \text{———} \quad (3)$$

Hence, $xy = (y + xy) + x = (y + x) + x = x + y$, by (1),

(2) and (3), which is impossible. Thus, $g_{12} \neq g_{41}$.

Similarly, we have $g_{13} \neq g_{41}$.

Lemma 1.3.

In $\mathfrak{A}$, the following polynomials g_{11}, g_{21}, g_{31}, g_{41} are pairwise distinct.

Proof : The proof is trivial. For instance, if $g_{21} = g_{31}$, setting $x = y$, we obtain $xz = zx$, a contradiction. If $g_{31} = g_{41}$, setting $x = y$, we have $zx = x + z$, which is again a contradiction.

Lemma 1.4.

In $\mathfrak{A}$, $g_{2i} \neq g_{3j}$, for each i, j = 1,2,3.

Proof : By Lemma 1.3 and the commutativity of $x + y$, it suffices to prove that $g_{22} \neq g_{31}$. If $g_{22} = (y + z)x = z(x + y) = g_{31}$, then setting $y = z$, $x = y$, $x = z$, we have, respectively,

$$yx = y(x + y),$$

$$(x + z)x = zx,$$

$$(y + x)x = x(x + y).$$

Thus, $xy = (y + x)y = y(x + y) = yx$, a contradiction. Hence, $g_{22} \neq g_{31}$, as required.

Lemma 1.5.

In $\mathfrak{A}$, $g_{22} \neq g_{41}$.

Proof : If $g_{22} = (y + z)x = xy + z = g_{41}$, setting $x = y$, we have

$$(x + z)x = x + z \qquad \text{———} \qquad (1)$$

Again, setting $x = y + z$, we obtain by (1) that

$$y + z = (y + z) + z \qquad \text{——— (2)}$$

Set $z = yz$. It follows that

$$(y + yx)x = xy + yx \qquad \text{——— (3)}$$

Moreover, putting $y = z$, we obtain

$$yx = xy + y \qquad \text{——— (4)}$$

As $g_{22} = g_{41}$; $xy + z = (y + z)x = (z + y)x = xz + y$. Thus, if we set $x = z$, we have

$$xy + x = x + y \qquad \text{——— (5)}$$

Now, observe that

$$\begin{aligned} x + y &= (x + y)x && (\text{ by (1) }) \\ &= (y + x)x \\ &= (yx + y)x && (\text{ by (5) }) \\ &= xy + yx && (\text{ by (3) }) \\ &= xy + (xy + y) && (\text{ by (4) }) \\ &= xy + y && (\text{ by (2) }) \\ &= yx && (\text{ by (4) }). \end{aligned}$$

This is impossible. Thus, $g_{22} \neq g_{41}$, as was to be shown.

2. The Smallest value of m.

In this section, we are going to search for the smallest value of m such that the sequence $\langle 0,0,3,m \rangle$ is representable. For preparation, we shall first establish the following lemmas where all algebras are assumed to have one commutative and one non-commutative essentially binary polynomials denoted by $x + y$ and xy respectively.

Lemma 2.1.

Let $\mathfrak{A}$ be an algebra representing $\langle 0,0,3 \rangle$. If $x + y$ is associative, then $p_3(\mathfrak{A}) \geqslant 7$.

Proof : We claim that the following polynomials

$$g_{11},\ g_{21},\ g_{22},\ g_{23},\ g_{31},\ g_{32},\ g_{33}$$

are pairwise distinct over $\mathfrak{A}$.

By Lemmas 1.2 and 1.4, it suffices to show that $g_{2i} \neq g_{2j}$, $g_{3i} \neq g_{3j}$ for $i \neq j$. By the commutativity of $x + y$, it remains to prove that $g_{21} \neq g_{22}$.

Assume that $g_{21} = g_{22}$. Then $(x + y)z = (y + z)x$. If we set $z = x + y$, we have $x + y = (y + (x + y))x = (x + y)x$ by the associativity of $x + y$. Again, setting $z = y$, we obtain $yx = (x + y)y$. Thus, it follows that

$$x + y = (x + y)x = (y + x)x = xy,$$

which is impossible. Hence $g_{21} \neq g_{22}$, as required. Similarly, $g_{31} \neq g_{32}$. Thus, Lemma 2.1 follows.

In view of Lemma 2.1, we shall now deal with the case that $x + y$ is non-associative in the following Lemmas 2.2---2.6.

Since it is assumed that $\mathfrak{A}$ represents the sequence $\langle 0,0,3 \rangle$; we have the following five possible cases :

$$(x + y)x = \begin{cases} x \\ y \\ x + y \\ xy \\ yx \end{cases}$$

Lemma 2.2.

If $(x + y)x = x$ holds in $\mathfrak{A}$, then the following polynomials g_{1i}, g_{2i}, g_{31}, g_{41}, for each $i = 1,2,3$, are distinct over $\mathfrak{A}$.

Proof : By assumption, we have $(x + y)x = x$ ——— (A).

By virtue of Lemmas 1.1---1.5, our proof will be complete if we can show that $g_{21} \neq g_{22}$ and $g_{23} \neq g_{41}$.

However, this is indeed the case. For, if

$$g_{21} = (x + y)z = (y + z)x = g_{22},$$

then setting $x = y$ gives $xz = x$ by (A). This is impossible.

Again, if $g_{23} = (z + x)y = xy + z = g_{41}$, setting $x = y$, we have $x = x + z$ by (A). Thus, $g_{23} \neq g_{41}$, as required.

Lemma 2.3.

If $(x + y)x = y$ holds in $\mathfrak{A}$, then the following polynomials g_{1i}, g_{2i}, g_{31}, g_{41}, for each $i = 1,2,3$, are distinct over $\mathfrak{A}$.

Proof : By assumption, we have $(x + y)x = y$ ——— (B).

In analogy to Lemma 2.2, it suffices to prove that

$$g_{21} \neq g_{22} \text{ and } g_{23} \neq g_{41}.$$

However, if $g_{21} = g_{22}$, setting $x = y$, we obtain by (B) that $xz = z$. If $g_{23} = g_{41}$, setting $x = y$ again, we have $z = x + z$. Thus, the Lemma follows.

Lemma 2.4.

Suppose that $(x + y)x = x + y$ holds in $\mathfrak{A}$. Then either (1) $g_{23} = g_{41}$ and the polynomials g_{1i}, g_{21}, g_{3i}, g_{41} are distinct for each $i = 1,2,3$;

or (2) $g_{23} \neq g_{41}$ and the polynomials g_{1i}, g_{2i}, g_{31}, g_{41} are distinct for each $i = 1,2,3$.

<u>Proof</u> : By hypothesis, we have $(x + y)x = x + y$ ——— (C)

Assume first that $g_{23} = (z + x)y = xy + z = g_{41}$. We shall show that g_{1i}, g_{21}, g_{3i}, g_{41}, $i = 1,2,3$ are distinct. According to Lemmas 1.2---1.4, we need only prove that

$$g_{31} \neq g_{32}, \quad g_{32} \neq g_{41}, \quad g_{33} \neq g_{41}.$$

Since $g_{23} = g_{41}$, setting $z = x$, $y = x + z$ and $y = z$, we have, respectively,

$$xy = xy + x \qquad \text{——— (1)}$$

$$x + z = x(x + z) + z \qquad \text{——— (2)}$$

$$(y + x)y = xy + y \qquad \text{——— (3)}$$

Now, if $g_{31} = g_{32}$, then $z(x + y) = x(y + z)$. Setting $x = y$, we have

$$zx = x(x + z) \qquad \text{——— (4).}$$

Thus, it follows that

$$\begin{aligned} x + z &= x(x + z) + z && \text{(by (2))} \\ &= zx + z && \text{(by (4))} \\ &= zx && \text{(by (1)),} \end{aligned}$$

which is a contradiction.

If $g_{32} = g_{41}$, then $x(y + z) = xy + z$. Setting $y = z$, we obtain

$$xy = xy + y \qquad \text{——— (5).}$$

Thus,

$$\begin{aligned} x + y &= (y + x)y && \text{(by (C))} \\ &= xy + y && \text{(by (3))} \\ &= xy && \text{(by (5))} \end{aligned}$$

which is impossible.

If $g_{33} = g_{41}$, then $y(z + x) = xy + z$. Setting $z = x$, we have $yx = xy + x = xy$ by (1), a contradiction.

Hence, $g_{31} \neq g_{32}$, $g_{32} \neq g_{41}$ and $g_{33} \neq g_{41}$, as required.

Thus, we may assume now that $g_{23} \neq g_{41}$. In this situation, we claim that the following g_{1i}, g_{2i}, g_{31}, g_{41}, $i = 1,2,3$ are distinct. Observe that by Lemmas 1.2---1.5 and the assumption $g_{23} \neq g_{41}$, we need only prove that $g_{21} \neq g_{22}$. However, this is trivial. For, if $g_{21} = (x + y)z = (y + z)x = g_{22}$, then setting $y = z$, we obtain, by (C), that $x + y = (x + y)y = yx$, a contradiction.

The proof of Lemma 2.4 is thus complete.

Lemma 2.5.

Suppose that $(x + y)x = xy$ holds in $\mathcal{R}$. Then either (1) $g_{31} = g_{32}$ and the polynomials g_{1i}, g_{21}, g_{31}, g_{4i} are distinct for each $i = 1,2,3$;

Or (2) $g_{31} \neq g_{32}$ and the polynomials g_{1i}, g_{21}, g_{3i}, g_{41} are distinct for each $i = 1,2,3$.

Proof : By assumption, we have $(x + y)x = xy$ ———— (D)

Assume that $g_{31} = g_{32}$ (and hence $g_{31} = g_{32} = g_{33}$). We shall prove that g_{1i}, g_{21}, g_{31}, g_{4i}, $i = 1,2,3$ are distinct. By Lemmas 1.2---1.4, it suffices to prove that

$g_{21} \neq g_{42}$, $g_{21} \neq g_{43}$, $g_{31} \neq g_{42}$, $g_{31} \neq g_{43}$ and

$g_{4i} \neq g_{4j}$ for $i \neq j$.

If $g_{21} = g_{42}$, then $(x + y)z = yz + x$. Setting $y = z$, we obtain, by (D), that $yx = (x + y)y = y + x$, a contradiction. Thus $g_{21} \neq g_{42}$. Similarly, $g_{21} \neq g_{43}$.

If $g_{31} = g_{42}$, then $z(x + y) = yz + x$. Setting $x = y$, $y = z$ and $z = x$, we have, respectively,

$$zx = xz + x \quad \text{———} \quad (1)$$

$$y(x + y) = y + x \quad \text{———} \quad (2)$$

and by (2),

$$x + y = yx + x \quad \text{———} \quad (3).$$

Again setting $z = x + y$ and using (2), we have

$$x + y = (x + y) + x \quad \text{———} \quad (4)$$

Now, observe that

$$\begin{aligned} x + y &= yx + x && (\text{ by } (3)\ ) \\ &= (xy + x) + x && (\text{ by } (1)\ ) \\ &= xy + x && (\text{ by } (4)\ ) \\ &= yx && (\text{ by } (1)\ ). \end{aligned}$$

This is impossible. Thus, $g_{31} \neq g_{42}$ (Note that this is true in general).

If $g_{31} = g_{43}$, then $z(x + y) = zx + y$. Setting $x = y$, we have

$$zx = zx + x \quad \text{———} \quad (5).$$

Setting $z = x + y$, it follows that

$$\begin{aligned} x + y &= (x + y)x + y \\ &= xy + y && (\text{ by (D) }) \\ &= xy && (\text{ by } (5)\ ). \end{aligned}$$

Thus, $g_{31} \neq g_{43}$, as required.

It remains to prove that $g_{4i} \neq g_{4j}$, $i \neq j$. By assumption, we have $g_{31} = z(x + y) = x(y + z) = g_{32}$. If we set $y = z$ and

$y = xz$, we obtain, respectively,

$$y(x + y) = xy \quad \text{———} \quad (6)$$

$$z(x + xz) = x(xz + z) \quad \text{———} \quad (7).$$

If $g_{41} = g_{42}$, then $xy + z = yz + x$. Setting $x = y$ and $x = z$, we have, respectively,

$$x + z = xz + x \quad \text{———} \quad (8)$$

and by (8), $$x + y = yx + x \quad \text{———} \quad (9).$$

It thus follows that

$$\begin{aligned} xz &= z(x + z) && (\text{ by } (6)\ ) \\ &= z(x + xz) && (\text{ by } (8)\ ) \\ &= x(xz + z) && (\text{ by } (7)\ ) \\ &= x(x + z) && (\text{ by } (9)\ ) \\ &= zx && (\text{ by } (6)\ ), \end{aligned}$$

which is impossible. Hence, $g_{41} \neq g_{42}$.

If $g_{41} = g_{43}$, then $xy + z = zx + y$. Setting $x = y$ and $z = x$, we have, respectively,

$$x + z = zx + x \quad \text{———} \quad (10)$$

$$xy + x = x + y \quad \text{———} \quad (11).$$

Accordingly,

$$\begin{aligned} xy &= y(x + y) && (\text{ by } (6)\ ) \\ &= y(xy + x) && (\text{ by } (11)\ ) \\ &= y(x + xy) && \\ &= x(xy + y) && (\text{ by } (7)\ ) \\ &= x(x + y) && (\text{ by } (10)\ ) \\ &= yx && (\text{ by } (6)\ ). \end{aligned}$$

This is impossible. Thus $g_{41} \neq g_{43}$, and similarly, $g_{42} \neq g_{43}$. This proves the first part.

Hence, we may assume that $g_{31} \neq g_{32}$ (and so $g_{3i} \neq g_{3j}$, $i \neq j$). In this case, we claim that g_{1i}, g_{21}, g_{3i}, g_{41} are distinct for $i = 1,2,3$. We need only prove that $g_{32} \neq g_{41}$ and $g_{33} \neq g_{41}$.

If $g_{32} = g_{41}$, then $x(y + z) = xy + z$. Setting $y = z$, we have

$$xy = xy + y \qquad (1).$$

Setting $x = y$, we have $x(x + z) = x + z$ ——— (2).

Setting $x = z$, we have by (2), $x + y = xy + x$ ——— (3).

Again, if we put $z = xy$, we obtain, by (1), that

$$x(xy) = xy \qquad (4).$$

Moreover,

$$\begin{aligned} y(xy) &= y(xy + y) && \text{(by (1))} \\ &= y + xy && \text{(by (2))} \\ &= xy && \text{(by (1)).} \end{aligned}$$

Thus, $y(xy) = xy = x(xy)$ ——— (5).

Observe that

$$\begin{aligned} xy &= x(xy) && \text{(by (4))} \\ &= x((x + y)x) && \text{(by (D))} \\ &= (x + y)((x + y)x) && \text{(by (5))} \\ &= (x + y)(xy) && \text{(by (D))} \\ &= (xy + x)(xy) && \text{(by (3))} \\ &= (xy)x && \text{(by (D))} \end{aligned}$$

Thus, $xy = (xy)x$ ——— (6).

From these, we obtain

$$\begin{aligned} xy &= (x + y)x && \text{(by (D))} \\ &= ((x + y)x)(x + y) && \text{(by (6))} \\ &= (xy)(x + y) && \text{(by (D))} \\ &= (xy)(xy + x) && \text{(by (3))} \end{aligned}$$

$$= xy + x \qquad (\text{ by } (2))$$

$$= x + y \qquad (\text{ by } (3)),$$

which is a contradiction. Hence, $g_{32} \neq g_{41}$.

If $g_{33} = g_{41}$, then $y(z + x) = xy + z$. Setting $z = x$ and $x = y$, we have, respectively,

$$yx = xy + x \text{ ———— } (1)$$

$$x(z + x) = x + z \text{ ———— } (2).$$

Setting $y = z$ and invoking (2), it follows that

$$x + y = xy + y \text{ ———— } (3).$$

By (D) and (3), we obtain $yx = (y + x)y = (x + y)y$ $= (xy + y)y = (y + xy)y = y(xy)$, i.e.,

$$y(xy) = yx \text{ ———— } (4).$$

From (1) and (4) we have $(yx)x = x(yx) + x = xy + x = yx$, i.e.,

$$(yx)x = yx \text{ ———— } (5)$$

Again, by (D) and (5), we have $(xy)x = ((x + y)x)x$ $= (x + y)x = xy$, i.e.,

$$(xy)x = xy \text{ ———— } (6)$$

Hence, it follows that

$$\begin{aligned} xz &= (x + z)x & (\text{ by } (D)) \\ &= (x(z + x))x & (\text{ by } (2)) \\ &= x(z + x) & (\text{ by } (6)) \\ &= x + z & (\text{ by } (2)). \end{aligned}$$

This is impossible. Thus, $g_{33} \neq g_{41}$.

The proof of Lemma 2.5 is therefore complete.

<u>Lemma 2.6.</u>

If $(x + y)x = yx$ holds in $\mathfrak{A}$, then the following polynomials $g_{1i}, g_{2i}, g_{31}, g_{41}$ for $i = 1,2,3$ are distinct.

<u>Proof</u> : By hypothesis, we have $(x + y)x = yx$ ——— (E).

According to the Lemmas in section 1, it suffices to prove that $g_{2i} \neq g_{2j}$, $i \neq j$ and $g_{23} \neq g_{41}$.

If $g_{21} = g_{22}$, then $(x + y)z = (y + z)x$. Setting $y = z$, we have $yx = (x + y)y = (y + x)y = xy$ by (E), a contradiction.

If $g_{23} = g_{41}$, then $(z + x)y = xy + z$. Setting $x = y$, it follows that $x + z = (z + x)x = (x + z)x = zx$ by (E), a contradiction. Hence, Lemma 2.6 follows.

Summarizing the results of Lemmas 2.2---2.6, we arrive at the following

<u>Proposition 2.7</u>.

Let $\mathfrak{A}$ be an algebra representing the sequence $\langle 0,0,3 \rangle$. If there is a commutative essentially binary polynomial which is non-associative, then $p_3(\mathfrak{A}) \geqslant 8$.

<u>Theorem 2.8</u>.

If the sequence $\langle 0,0,3,m \rangle$ is representable, then $m \geqslant 7$.

<u>Proof</u> : Let $\mathfrak{A}$ be an arbitrary algebra representing the sequence $\langle 0,0,3,m \rangle$. If each of the three essentially binary polynomials is commutative, then by Lemma 2.3(III,1), $m \geqslant 8$. Otherwise, there is one commutative and one non-commutative essentially binary polynomials. If the commutative one is associative, according to Lemma 2.1, we have $m \geqslant 7$. Otherwise, by Proposition 2.7, it follows that $m \geqslant 8$.

3. The Best Lower Bound for $\langle p_n \rangle$.

In this section, we shall first furnish an example to show that the sequence $\langle 0,0,3,7 \rangle$ is representable. After that, we will be interested in algebras representing $\langle 0,0,3,7 \rangle$. Finally, the best lower bound for the sequence

$$\langle 0,0,3,7,p_4,\ldots\ldots,p_n,\ldots\ldots \rangle$$

will be derived.

In [13], G. Grätzer and J. Płonka consider the algebra $\mathfrak{A} = \langle A;+,\cdot \rangle$ of type $\langle 2,2 \rangle$ where "+" is a semilattice operation and "." is a partition function (see [32]) in the sense that

1) $xx = x, \quad x(yz) = (xy)z, \quad x(yz) = x(zy)$;

2) $(x + y)z = xz + yz, \quad x(y + z) = xy + xz$;

3) $(x + y)x = x + y$.

They showed that for $n \geqslant 2$, $p_n(\mathfrak{A}) = 2^n - 1$, and $p_0(\mathfrak{A}) = p_1(\mathfrak{A}) = 0$. In fact, it can be shown that for each $p(x_1,\ldots,x_n) \neq x_1 + \ldots\ldots + x_n$, $p(x_1,\ldots,x_n)$ is an essentially n-ary polynomial over $\mathfrak{A}$ if and only if p has a unique representation in the form $(x_{i(1)} + \ldots + x_{i(k)})(x_{i(k+1)} + \ldots + x_{i(n)})$ where $(\{x_{i(1)},\ldots,x_{i(k)}\} , \{x_{i(k+1)},\ldots,x_{i(n)}\})$ is a partition of the set $\{x_1,\ldots,x_n\}$. Thus, in particular, $\mathfrak{A}$ represents $\langle 0,0,3,7 \rangle$.

On the other hand, suppose that $\mathfrak{A}$ is an algebra representing $\langle 0,0,3,7 \rangle$. It is not known that whether $\mathfrak{A}$ can be represented as an algebra of type $\langle 2,2 \rangle$ satisfying the above conditions.

However, following the proof of Theorem 2.8, we do have the following interesting result.

Theorem 3.1.

Let $\mathfrak{A}$ be an algebra representing the sequence $\langle 0,0,3,7 \rangle$. Then $\mathfrak{A}$ has a unique semilattice polynomial.

As a consequence, we have

Corollary 3.2.

Let $\mathfrak{A}$ be an algebra representing the sequence $\langle 0,0,3,7 \rangle$. Then $p_n(\mathfrak{A}) \geqslant 2^n - 1$ for each $n \geqslant 2$.

Proof : Clearly, the above algebra represents the sequence $\langle 0,0,3,7,\ldots,2^n - 1,\ldots \rangle$. On the other hand, if $\mathfrak{A}$ represents $\langle 0,0,3,7 \rangle$; invoking Theorem 3.1, it follows that there is a semilattice polynomial over $\mathfrak{A}$. Hence, if $p_n(\mathfrak{A}) \neq 1$ ($n \geqslant 2$), we have

$$p_{n+1}(\mathfrak{A}) \geqslant p_n(\mathfrak{A}) + 1 + \max \{ p_n(\mathfrak{A}), n + 1 \}$$
$$\geqslant 2\, p_n(\mathfrak{A}) + 1.$$

We shall prove that $p_n(\mathfrak{A}) \geqslant 2^n - 1$ for each $n \geqslant 2$ by induction. The inequality is true for $n = 2,3$. Moreover, $p_4(\mathfrak{A}) \geqslant 2\, p_3(\mathfrak{A}) + 1 = 2.7 + 1 = 15 = 2^4 - 1$; it thus true for $n = 4$. Assume that $p_k(\mathfrak{A}) \geqslant 2^k - 1$. Then for $n = k + 1$, we have $p_{k+1}(\mathfrak{A}) \geqslant 2\, p_k(\mathfrak{A}) + 1 \geqslant 2.(2^k - 1) + 1 = 2^{k+1} - 1$. From this, the Theorem follows.

PART IV

APPLICATIONS.

CHAPTER 1

APPLICATIONS OF $\langle p_n \rangle$ SEQUENCES TO LATTICE THEORY.

~~Let $\underset{\sim}{K}$ be the class of algebras of type $\langle 2,2 \rangle$~~. Let $\underset{\sim}{L}$, $\underset{\sim}{D}$, $\underset{\sim}{M}$ be the equational classes of lattices, distributive lattices and modular lattices respectively. Then $\underset{\sim}{D} \subset \underset{\sim}{M} \subset \underset{\sim}{L} \subset \underset{\sim}{K}$.

Let $\mathfrak{A} = \langle A;+,\cdot \rangle$ be a lattice. As the two operations are idempotent, it follows that $p_0(\mathfrak{A}) = p_1(\mathfrak{A}) = 0$. Clearly, $p^{(2)}(\mathfrak{A})$ consists of four elements namely x_1, x_2, $x_1 + x_2$, x_1x_2. However, only $x_1 + x_2$ and x_1x_2 are essential. Thus, $p_2(\mathfrak{A}) = 2$. From now on, $p_n(\mathfrak{A})$ depends, of course, on the specific lattice in question.

Suppose, on the other hand, that we are given an algebra $\mathfrak{A}$ of $\underset{\sim}{K}$ which represents $\langle 0,0,2 \rangle$. Then $\mathfrak{A}$ is evidently not necessary a lattice. In fact, the class of all T-lattices (see [4]) furnish other examples of algebras which represent $\langle 0,0,2 \rangle$. Thus, it is interesting to enquire as to what conditions should be imposed on $\mathfrak{A}$ so that if $\mathfrak{A}$ represents $\langle 0,0,2 \rangle$, then $\mathfrak{A} \in \underset{\sim}{L}$, $\underset{\sim}{D}$ or $\underset{\sim}{M}$.

In this chapter, we shall attack this problem by using two different methods. The first one which will be studied in section one is by imposing identities while the second one which will be dealt with in section two and three is by considering the cardinal p_3.

1. Characterizations by Identities without Absorption Laws.

In general, the difficult part of proving an algebra $\mathfrak{A}$ of type $\langle 2,2 \rangle$ to be a lattice is to prove the absorption laws. Thus, at least one of them is always assumed to hold in $\mathfrak{A}$. (See, for instance, [25], [26], [40].) However, as we shall see, this crucial assumption is not necessary in our treatment.

Theorem 1.1.

Let $\mathfrak{A} = \langle A;+,\cdot \rangle$ be an algebra in $\underset{\sim}{K}$. Then $\mathfrak{A} \in \underset{\sim}{L}$ if and only if 1) $(x + y) + z = x + (y + z)$

2) $(xy)z = x(yz)$

3) $\mathfrak{A}$ represents $\langle 0,0,2 \rangle$.

Proof : The necessity is obvious. To prove the sufficiency, observe that as $\mathfrak{A}$ satisfies 1), 2) and 3), it follows that (a) $|A| \geqslant 2$; (b) $x + y$ and xy are the two distinct idempotent, commutative, associative essentially binary polynomials over $\mathfrak{A}$:(c) there are no essentially j-ary polynomials for $j = 1,2$ except x, y, $x + y$, xy. Thus, by Lemma 2.4 (III,1), we have $x(x + y) = x = x + xy$. Hence $\mathfrak{A}$ is a lattice.

Recently, R. Padmanabhan [27] proved that the following three identities :

(1) $(x + y)z = zx + (z(y + x))$

(2) $xy + z = (z + x)(xy + z)$

(3) $xy + y = y$

characterize lattices.

Keeping (1) and (2), replacing (3) by requiring the algebra representing $\langle 0,0,2 \rangle$, we are able to prove the same.

Theorem 1.2.

Let $\mathfrak{A} = \langle A;+,\cdot \rangle$ be an algebra in $\underset{\sim}{K}$. Then $\mathfrak{A}$ is a lattice if and only if

1) $(x + y)z = zx + z(y + x)$

2) $xy + z = (z + x)(xy + z)$

3) $\mathfrak{A}$ represents $\langle 0,0,2 \rangle$.

Proof : It is enough to prove the sufficiency. To this end, we need only prove that $xy + y = y$. Since $\mathfrak{A}$ represents $\langle 0,0,2 \rangle$, $x + y$ and xy are idempotent and symmetric. Thus, according to Proposition 1.4(III,1), we have

$$\begin{cases} x(x+y) \\ x + xy \end{cases} = \begin{cases} \begin{cases} x + y \\ x + y \end{cases} \\ \begin{cases} xy \\ xy \end{cases} \\ \begin{cases} x + y \\ xy \end{cases} \\ \begin{cases} x \\ x \end{cases} \end{cases}$$

If $x(x + y) = x + y$ ——————————— (a)

$x + xy = x + y$ ——————————— (b)

then setting $z = xy$ in 2), we have $xy = xy + xy = (xy + x)(xy) = (x + y)(xy)$ by (b), i.e.,

$$(x + y)(xy) = xy \quad \text{———————} \quad (1).$$

By (1) and (b), it follows that $xy = xy + xy = (x + y)(xy) + xy$ $= (x + y) + xy$, i.e.,

$$(x + y) + xy = xy \quad \text{———} \quad (2).$$

However, if we set $z = y$ in 1), we have, by (a) and (2),

$$x + y = (x + y)y = yx + y(y + x) = xy + (x + y) = xy,$$

a contradiction.

If $x(x + y) = xy$ ——— (c)

$x + xy = xy$ ——— (d)

then setting $z = x + y$ in 1), we have $x + y = xy + (x + y)$.
Setting $z = y$ in 2), it follows that $xy + y = (y + x)(xy + y)$.
Thus, $x + y = (x + y)(x + y) = (x + y)(xy + (x + y))$ $= (xy)(x + y) = (y + x)(xy + y) = xy + y = xy$, a contradiction.

If $x(x + y) = x + y$ ——— (e)

$x + xy = xy$ ——— (f)

then setting $y = z$ in 1), we obtain $x + y = (x + y)y$ $= yx + y(y + x) = yx + (x + y)$. Thus, invoking Lemma 1.5(3) (III,1), it follows that

$$(yx)(x + y) = x + y \quad \text{———} \quad (1)$$

However, if we set $y = z$ in 2) and apply (f) and (1), we obtain $xy = xy + y = (y + x)(xy + y) = (x + y)(xy) = x + y$, a contradiction.

Hence, it is necessary that $x(x + y) = x + xy = x$, proving Theorem 1.2.

<u>Theorem 1.3.</u>

Let $\mathcal{A} = \langle A;+,\circ\rangle$ be an algebra in $\underset{\sim}{K}$. Then $\mathcal{A}$ is a lattice in which the identity $f(y_1,y_2,\dots,y_n) = g(y_1,y_2,\dots,y_n)$ holds if and only if

1) $\mathfrak{A}$ represents the sequence $\langle 0,0,2 \rangle$

2) $((xf)z + u) + v = ((gz)x + v) + ((t + u)u)$ holds in $\mathfrak{A}$.

Proof : According to a result of R. Padmanabhan [26], it suffices to show that $x + xy = x$.

Firstly, as $x + y$ and xy are idempotent, if we put $y = y_1 = \dots = y_n$, 2) becomes

3) $((xy)z + u) + v = ((yz)x + v) + ((t + u)u)$.

In view of Proposition 1.4(III,1), we have four possible cases.

If $\begin{cases} x(x + y) = x + y & \text{——— (a)} \\ x + xy = x + y & \text{——— (b)} \end{cases}$

3) becomes $((xy)z + u) + v = ((yz)x + v) + (t + u)$ —— (4)

In preparation, we claim that $(xy)y = x + y = (x + y) + y$ holds in this case.

Clearly, $(xy)y \in \{x,y,xy,x + y\}$. If $(xy)y = x$, then $x = (x(x + y))(x + y) = x + y$, a contradiction. If $(xy)y = y$, then $y = ((x + y)y)y = x + y$, a contradiction. If $(xy)y = xy$, then $xy = xy + xy = (xy)y + xy = xy + y = x + y$, again a contradiction. Thus, we have

$$(xy)y = x + y \qquad \text{——— (5)}$$

Next, observe that $(x + y) + y = (x + y) + (x + y)y$ $= (x + y) + (x + y) = x + y$ by (a) and (b). Hence, we have

$$(x + y) + y = x + y \qquad \text{——— (6)}$$

Now, if we put $z = u = v = xy$ and $t = x + y$ in (4), then by using (a), (b), (2) and (6), we have $xy = ((xy)(xy) + xy) + xy$ $= ((y(xy))x + xy) + ((x + y) + xy) = ((x + y)x + xy) + ((x + y) + xy)$ $= ((x + y) + xy) + ((x + y) + xy) = (x + y) + xy = (x + xy) + xy$ $= x + xy = x + y$, which is impossible.

If
$$\begin{cases} x(x + y) = xy & \text{——— (c)} \\ x + xy = xy & \text{——— (d)} \end{cases}$$
then 3) becomes

$$((xy)z + u) + v = ((yz)x + v) + tu \quad \text{——— (7).}$$

Observe that $x + (x + y) \in \{x, y, x + y, xy\}$. In this case, it is clear that $x + (x + y) \notin \{x, y\}$. If $x + (x + y)$ $= x + y$, then we have $x + y = (x + y)(x + y) = (x + (x + y))(x + y)$ $= x(x + y) = xy$, a contradiction. Thus,

$$x + (x + y) = xy \quad \text{——— (8).}$$

Setting $x = y = z$ in (7), we have

$$(x + u) + v = (x + v) + tu.$$

Putting $v = x + u$, $t = u$, it follows from (d) and (8) that

$$x + u = (x + (x + u)) + u = xu + u = xu,$$

a contradiction.

If
$$\begin{cases} x(x + y) = x + y & \text{——— (e)} \\ x + xy = xy & \text{——— (f)} \end{cases}$$
then 3) becomes

$$((xy)z + u) + v = ((yz)x + v) + (t + u) \quad \text{—— (9).}$$

Now, if we put $z = v = u = xy$, $t = x + y$ in (9), then by using Lemma 1.5(1),(2) (III,1), we have

$$xy = ((xy)(xy) + xy) + xy = ((y(xy))x + xy) + ((x + y) + xy)$$
$$= (xy + xy) + ((x + y) + xy) = xy + ((x + y) + xy)$$
$$= (x + y) + xy,$$

i.e., $$xy = (x + y) + xy \qquad (10).$$

On the other hand, putting $x = y = z$, $v = x + u$ and $t = xu$ in (9), we get $x + u = (x + u) + (x + u) = (x + (x + u)) + (xu + u) = (x + u) + xu$, by Lemma 1.5(2) (III,1) and (f). From this and (10), it follows that $x + y = (x + y) + xy = xy$, a contradiction.

Hence, we must have $x(x + y) = x = x + xy$, proving Theorem 1.3.

<u>Corollary 1.4</u>.

Let $\mathfrak{A} = \langle A;+,\cdot\rangle$ be an algebra in $\underset{\sim}{K}$. Then $\mathfrak{A} \in \underset{\sim}{L}$ if and only if 1) $\mathfrak{A}$ represents $\langle 0,0,2 \rangle$

2) $((xy)z + u) + v = ((yz)x + v) + ((t + u)u)$ holds in $\mathfrak{A}$.

Consider the following equational classes of algebras :

$\underset{\sim}{K}(D) = \{ \mathfrak{A} \in \underset{\sim}{K} / x(y + z) = xy + xz,\ x + yz = (x + y)(x + z)$ hold in $\mathfrak{A} \}$,

$\underset{\sim}{K}(\cdot) = \{ \mathfrak{A} \in \underset{\sim}{K} / x(y + z) = xy + xz,\ x(xy) = xy$ hold in $\mathfrak{A} \}$.

In what follows, we shall characterize distributive lattices in terms of $\underset{\sim}{K}(D)$ and $\underset{\sim}{K}(\cdot)$. We need the following result :

<u>Sholander's Theorem</u> ([40])

Let $\mathfrak{A} = \langle A;+,\cdot\rangle$ ~~$\in \underset{\sim}{K}$~~ be an algebra of type <2,2>. Then $\mathfrak{A} \in \underset{\sim}{D}$ if and only if

1) $x = x(x + y)$

2) $x(y + z) = zx + yx$ hold in $\mathfrak{A}$.

Theorem 1.5.

Let $\mathfrak{A} = \langle A;+,\cdot\rangle \in \underset{\sim}{K}$. Then $\mathfrak{A} \in \underset{\sim}{D}$ if and only if 1) $\mathfrak{A}$ represents $\langle 0,0,2\rangle$; 2) $\mathfrak{A} \in \underset{\sim}{K}(D)$.

Proof : Let $\mathfrak{A} \in \underset{\sim}{K}$ be an algebra satisfying 1) and 2). Thus, $x + y$, xy are idempotent, commutative essentially binary polynomials over $\mathfrak{A}$. By Proposition 1.4 (III,1), we have four possible cases.

If
$$\begin{cases} x(x + y) = xy & \text{——— (a)} \\ x + xy + xy & \text{——— (b)} \end{cases}$$

then, since $(x + y)z = xz + yz$ holds in $\mathfrak{A}$, setting $z = x + y$, we have $x + y = (x + y)(x + y) = x(x + y) + y(x + y) = xy$, a contradiction.

If
$$\begin{cases} x(x + y) = x + y & \text{——— (c)} \\ x + xy = x + y & \text{——— (d)} \end{cases}$$

then, since $z + xy = (z + x)(z + y)$ holds in $\mathfrak{A}$, setting $z = xy$, we obtain $xy = x + y$, a contradiction.

If
$$\begin{cases} x(x + y) = x + y & \text{——— (e)} \\ x + xy = xy & \text{——— (f)} \end{cases}$$

then setting $z = xy$ in $(x + y)z = xz + yz$ and invoking Lemma 1.5(1) (III,1), we obtain

$$(x + y)(xy) = x(xy) + y(xy) = xy + xy = xy,$$

i.e., $$(x + y)(xy) = xy \quad \text{——— (1)}$$

Setting $z = x + y$ in $z + xy = (z + x)(z + y)$ and making use of Lemma 1.5(2) (III,1), it follows that

$$(x + y) + xy = ((x + y) + x)((x + y) + y) = x + y.$$

However, this, together with (1) contradict Lemma 1.5(3) (III,1).

Hence, we have $x(x + y) = x = x + xy$. By Sholander's Theorem, $\mathfrak{A} \in \underset{\sim}{D}$. The necessity is obvious. This completes the proof of Theorem 1.5.

By a <u>distributive quasi-lattice</u> (see [36]) is meant an algebra $\mathfrak{A} = \langle A;+,\cdot \rangle$ of $\underset{\sim}{K}$ whose operations "+" and "." are idempotent, commutative, associative and satisfying two identities $x(y + z) = xy + xz$, $x + yz = (x + y)(x + z)$. As a consequence of Theorem 1.5, we have

<u>Corollary 1.6.</u>

Let $\mathfrak{A}$ be a distributive quasi-lattice. Then $\mathfrak{A} \in \underset{\sim}{D}$ if and only if $p_2(\mathfrak{A}) = 2$.

<u>Theorem 1.7.</u>

Let $\mathfrak{A} \in \underset{\sim}{K}$. Then $\mathfrak{A} \in \underset{\sim}{D}$ if and only if 1) $\mathfrak{A}$ represents $\langle 0,0,2 \rangle$ and 2) $\mathfrak{A} \in \underset{\sim}{K}(\cdot)$.

<u>Proof</u> : Let $\mathfrak{A}$ be an algebra of $\underset{\sim}{K}$ with properties 1) and 2). Then the following identities hold in $\mathfrak{A}$

$$x(y + z) = xy + xz \quad \text{------------} \quad (1)$$

$$x(xy) = xy \quad \text{------------------} \quad (2)$$

If $x(x + y) = xy$ and $x + xy = xy$, then setting $x = y + z$ in (1), we have $y + z = yz$, a contradiction.

If $x(x + y) = x + y$ and $x + xy = x + y$, observe that, as $x + y$ is commutative, we have $x(x + y) = y + xy$. Thus,

$$x(x + xy) = xy + x(xy) \qquad (3)$$

Accordingly, we obtain $x + y = x + xy = x(x + xy) = xy + x(xy) = xy + xy = xy$, a contradiction.

If $\begin{cases} x(x + y) = x + y \\ x + xy = xy, \end{cases}$

then setting $z = x + y$ in (1), we obtain, by Lemma 1.5(2) (III,1) that $x + y = x(x + y) = x(y + (x + y)) = xy + x(x + y) = xy + (x + y)$, i.e., $xy + (x + y) = x + y$.

On the other hand, if we set $x = yz$ in (1), we have by Lemma 1.5(1) (III,1) that $(yz)(y + z) = (yz)y + (yz)z = yz$, i.e., $(xy)(x + y) = xy$. Thus, we have

$$\begin{cases} xy + (x + y) = x + y \\ (x + y)(xy) = xy, \end{cases}$$

which contradicts Lemma 1.5(3) (III,1). Hence, by Proposition 1.4 (III,1), the two absorption laws $x(x + y) = x = x + xy$ hold in $\mathfrak{A}$. By Sholander's Theorem, $\mathfrak{A} \in \underset{\sim}{D}$. The necessity is trivial. The proof is thus complete.

Let us consider the following three identities :

$$xy + xz = x(y + xz) \qquad (1)$$

$$x(y + z) = x(y(x + z) + z) \qquad (2)$$

$$x + yz = x + ((y + xz)z) \qquad (3).$$

Let $\underset{\sim}{K}(M) = \{ \mathfrak{A} \in \underset{\sim}{K} \,/\, (1), (2) \text{ and } (3) \text{ hold in } \mathfrak{A} \}$. We shall now characterize $\underset{\sim}{M}$ in terms of $\underset{\sim}{K}(M)$.

Theorem 1.8.

Let $\mathfrak{A} \in \underline{K}$. Then $\mathfrak{A} \in \underset{\sim}{M}$ if and only if 1) $\mathfrak{A}$ represents $\langle 0,0,2 \rangle$ and 2) $\mathfrak{A} \in \underset{\sim}{K}(M)$.

Proof : Since every modular lattice satisfies (1), (2) and (3) ; the necessity is trivial. Thus, assume that $\mathfrak{A}$ is an algebra of $\underline{K}$ with properties 1) and 2).

If $x(x + y) = xy$ and $x + xy = xy$, then putting $x = z$ in (2), we have $x(y + x) = x(yx + x)$. Thus,

$$xy = x(xy) \qquad \text{(4)}.$$

If we set $y = z$ in (3), we have, by using (4), that

$$x + y = x + ((y + xy)y) = x + (xy)y = x + xy = xy,$$

a contradiction.

If $x(x + y) = x + y = x + xy$, then setting $x = z$ in (3), we obtain

$$x + y = x + yx = x + ((y + x)x) = x + (x + y),$$

i.e.,

$$x + (x + y) = x + y \qquad \text{(5)}.$$

Putting $y = z$ in (2) and making use of (5), we have

$$xy = x(y + y) = x(y(x + y) + y) = x((x + y) + y)$$
$$= x(x + y) = x + y,$$

a contradiction.

If $x(x + y) = x + y$ and $x + xy = xy$, then setting $y = z$ in (3), we have

$$x + y = x + ((y + xy)y) = x + (y + xy) = x + xy = xy,$$

a contradiction.

Thus, by Proposition 1.4 (III,1), it follows that

$$x(x + y) = x = x + xy$$

holds in $\mathcal{O}\!\mathcal{L}$. Our proof that $\mathcal{O}\!\mathcal{L} \in \underset{\sim}{M}$ will be complete if we can show that x + y and xy are associative. To this end, let

$$p = (x + y) + z, \quad q = x + (y + z).$$

Firstly, we claim that $y + (y + z) = y + z$. Indeed, by the absorption laws, we have

$$y + (y + z) = y(y + z) + (y + z) = y + z \quad \text{——} \ (6).$$

Next, observe that

$$\begin{aligned} y(x + (y + z)) &= y(x(y + (y + z)) + (y + z)) \quad (\text{ by } (2)\ ) \\ &= y(x(y + z) + (y + z)) \quad (\text{ by } (6)\ ) \\ &= y(y + z) \\ &= y. \end{aligned}$$

Thus, $$y(x + (y + z)) = y \quad \text{————} \quad (7).$$

Similarly, $$z(x + (y + z)) = z \quad \text{————} \quad (8).$$

From this, we have

$$\begin{aligned} p &= (x + y) + z \\ &= \left[x(x + (y + z)) + y(x + (y + z)) \right] + z(x + (y + z)) \\ &\qquad\qquad (\text{ by } (7),(8)\ ) \\ &= (x + (y + z))(x + y(x + (y + z))) + z(x + (y + z)) \\ &\qquad\qquad (\text{ by } (1)\ ) \\ &= (x + (y + z))(\ x + y) + z(x + (y + z)) \quad (\text{ by } (7)\ ) \\ &= (x + (y + z))((x + y) + z(x + (y + z))) \quad (\text{ by } (1)\ ) \\ &= (x + (y + z))((x + y) + z) \qquad (\text{ by } (8)\ ) \\ &= q \cdot p. \end{aligned}$$

Interchanging the roles of x and z, we get $p \circ q = q$.
Thus, by the symmetry of xy, it follows that p = q, i.e.,

$$(x + y) + z = x + (y + z).$$

J. Riečan (see [38]) proved the following result :

Let $\mathfrak{A} = \langle A;+,\circ\rangle$ be an algebra in $\underset{\sim}{K}$. Then $\mathfrak{A} \in \underset{\sim}{M}$ if and only if (a) $xy + xz = (zx + y)x$,

(b) $(x + (y + z))z = z$ hold in $\mathfrak{A}$.

Evidently, by the commutativity of x + y and xy, (a) follows from (1). To prove (b), observe that, by the associativity of x + y and the absorption law, we have

$$(x + (y + z))z = ((x + y) + z)z = z.$$

Hence, we conclude that $\mathfrak{A} \in \underset{\sim}{M}$.

2. Distributive Lattices and $\langle p_n \rangle$ Sequences.

In Theorem 3.2 (III,1), we proved that every distributive lattice represents the sequence $\langle 0,0,2,9 \rangle$. In what follows, we shall show that this sequence indeed characterizes distributive lattices.

<u>Theorem 2.1.</u>

Let $\mathfrak{A}$ be an algebra in $\underset{\sim}{K}$. Then $\mathfrak{A} \in \underset{\sim}{D}$ if and only if $\mathfrak{A}$ represents the sequence $\langle 0,0,2,9 \rangle$.

<u>Proof</u> : It suffices to prove the sufficiency.

Let $\mathfrak{A} = \langle A;+,\circ\rangle$ be an algebra of $\underset{\sim}{K}$ representing $\langle 0,0,2,9\rangle$. Then x + y and xy are the only two distinct idempotent, commutative essentially binary polynomials over $\mathfrak{A}$. As $x + y \neq xy$,

$|A| \geqslant 2$.

In analogy to Theorem 3.1 (III,1), there are four and only four cases : If $x + y$, xy are non-associative, we would have $p_3(\mathcal{O\!\!\!\!\!\mathcal{C}}) \geqslant 12$, a contradiction. If $x + y$ is associative but xy not, we obtain $p_3(\mathcal{O\!\!\!\!\!\mathcal{C}}) \geqslant 10$, which is impossible. The case that xy is associative and $x + y$ not is similar to the previous one. Thus, it is necessary that $x + y$ and xy are associative. According to Theorem 1.1, it follows immediately that $\mathcal{O\!\!\!\!\!\mathcal{C}} \in \underset{\sim}{L}$.

Suppose now to the contrary that $\mathcal{O\!\!\!\!\!\mathcal{C}}$ is non-distributive. We claim that $p_3(\mathcal{O\!\!\!\!\!\mathcal{C}}) > 9$. Indeed, let us consider the following ternary polynomial $p = (x + y)(y + z)(z + x)$.

As $p_0(\mathcal{O\!\!\!\!\!\mathcal{C}}) = 0$, p is not constant. Evidently, p is symmetric. Thus, if p depends on x, it depends on y and z simultaneously. Hence, p is essentially ternary. Direct verifications show that p is distinct from the first eight polynomials listed in the proof of Theorem 3.1 (III,1). Moreover, if

$$p = xy + yz + zx,$$

then $\mathcal{O\!\!\!\!\!\mathcal{C}}$ satisfies the so-called " Self-Dual Median Law " and hence is distributive, a contradiction. Thus, $p_3(\mathcal{O\!\!\!\!\!\mathcal{C}}) > 9$, as required. The proof is therefore complete.

Let us denote by $F_{\underset{\sim}{D}}(n)$ the free distributive lattice on n generators. It is an out-standing problem in Lattice Theory to determine the cardinal of $F_{\underset{\sim}{D}}(n)$, for each n. Though the problem has been considered for a long time, it is far from complete yet. In this section, we shall point out a remark to it in Proposition 2.2. Instead of "+" and ".", the notations "$\vee$" and "$\wedge$" for join and meet will be adopted.

Let $p(x_1,\ldots,x_n)$ be a distributive lattice polynomial. Then p can be written as

$$p = (\bigwedge_{i\in I(1)} x_i) \vee (\bigwedge_{i\in I(2)} x_i) \vee \cdots \vee (\bigwedge_{i\in I(m)} x_i) ,$$

where $I(j) \subseteq \{1, 2,\ldots, n\}$ for $j = 1,2,\ldots,m$

If $I(j) \subseteq I(k)$ for some $j, k = 1,\ldots,m$, then

$$\bigwedge_{i\in I(j)} x_i \geqslant \bigwedge_{i\in I(k)} x_i$$

and so the factor $(\bigwedge_{i\in I(k)} x_i)$ is redundant in $p(x_1,\ldots,x_n)$. Thus, without loss of generality, we may assume

$$(*) \quad I(j) \not\subseteq I(k) \quad \text{for all } j,k \in \{1, 2,\ldots,m\}.$$

<u>Proposition 2.2.</u>

Let $\mathcal{C} \in \underset{\sim}{D}$. Then $p(x_1,\ldots,x_n)$ is essentially n-ary polynomial over $\mathcal{C}$ if and only if $\bigcup_{j=1}^{m} I(j) = \{1, 2,\ldots,n\}$.

<u>Proof</u> : The necessity is trivial.

To prove the sufficiency, we have to prove that p depends on x_i, for each $i = 1, 2,\ldots, n$. However, it suffices to prove for $i = 1$.

Let $M = \{1, 2,\ldots, m\}$ and $J = \{k\in M / 1 \in I(k)\}$. Since $\bigcup_{j=1}^{m} I(j) = \{1, 2,\ldots, n\}$, $1 \in I(j)$ for some j. Thus $J \neq \emptyset$.

<u>Case 1.</u> $J = M$.

We have $p = (x_1 \wedge \bigwedge_{i\in I(1)-\{1\}} x_i) \vee \cdots\cdots \vee (x_1 \wedge \bigwedge_{i\in I(m)-\{1\}} x_i)$

$$= x_1 \wedge ((\bigwedge_{i\in I(1)-\{1\}} x_i) \vee \cdots \vee (\bigwedge_{i\in I(m)-\{1\}} x_i)) .$$

Thus, p depends on x_1.

<u>Case 2</u>. $J \subset M$.

Asseme $J = \{1, 2, \ldots, j\}$ for some $j < m$. Then

$$p = \left[x_1 \wedge ((\bigwedge_{i\in I(1)-\{1\}} x_i) \vee \cdots \vee (\bigwedge_{i\in I(j)-\{1\}} x_i))\right]$$
$$\vee \left[(\bigwedge_{i\in I(j+1)} x_i) \vee \cdots \vee (\bigwedge_{i\in I(m)} x_i)\right]$$
$$= (x_1 \wedge y) \vee z,$$

where $y = (\bigwedge_{i\in I(1)-\{1\}} x_i) \vee \cdots \vee (\bigwedge_{i\in I(j)-\{1\}} x_i)$

$z = (\bigwedge_{i\in I(j+1)} x_i) \vee \cdots \vee (\bigwedge_{i\in I(m)} x_i)$.

Evidently, $x_1 \neq y$, $x_1 \neq z$.

<u>Claim</u> : $y \neq z$.

For simplicity, set $I'(k) = I(k) - \{1\}$, for each $k = 1, 2, \ldots, j$. If $y = z$, then we have

$$\bigwedge_{i\in I'(1)} x_i \leq (\bigwedge_{i\in I(j+1)} x_i) \vee \cdots \vee (\bigwedge_{i\in I(m)} x_i).$$

Since $\bigwedge_{i\in I'(1)} x_i$ is $\vee$-irreducible in $P_{\underset{\sim}{D}}(n)$, thus by a property of distributive lattice, we get

$$\bigwedge_{i\in I'(1)} x_i \leq \bigwedge_{i\in I(t)} x_i \text{ , for some } t = j+1, \ldots, m.$$

Hence, $I'(1) \supseteq I(t)$ and so

$$I(1) = I'(1) \cup \{1\} \supset I(t),$$

contradicting the assumption (*). Thus, $y \neq z$, as was to be shown.

Now, observe that as x_1, y and z are distinct, the ternary polynomial $(x_1 \wedge y) \vee z$ is essential over $\mathcal{R}$. Thus, p depends on x_1, as required.

Consequently, $p(x_1,\ldots,x_n)$ is essentially n-ary over $\mathcal{O}\!\mathcal{L}$.

In view of this Proposition, it seems likely that an upper or a lower bound for the cardinal p_n of essentially n-ary polynomials over L, the free distributive lattice on ω generators can be provided. In fact, this computation would be combinatorial in natural. If this is the case, then invoking the formula

$$F_n = n + \sum_{k=0}^{n} \binom{n}{k} p_k ,$$

the cardinal of $F_{\underset{\sim}{D}}(n)$ can be determined approximately.

3. Modular Lattices and $\langle p_n \rangle$ Sequences.

The relation between distributive lattices and $\langle p_n \rangle$ sequence has been studied in section two. It is our intention in this section to study the same for modular lattices.

Consider the following equational classes :

$\underset{\sim}{K}(1) = \{ \mathcal{O}\!\mathcal{L} \in \underset{\sim}{K} \,/\, (x + y) + z = x + (y + z),\ (xy)z = x(yz)$ hold in $\mathcal{O}\!\mathcal{L} \}$.

$\underset{\sim}{K}(2) = \{ \mathcal{O}\!\mathcal{L} \in \underset{\sim}{K} \,/\, (x + y)z = zx + z(y + x),\ xy + z = (z + x)(xy + z)$ hold in $\mathcal{O}\!\mathcal{L} \}$.

$\underset{\sim}{K}(3) = \{ \mathcal{O}\!\mathcal{L} \in \underset{\sim}{K} \,/\, ((xy)z + u) + v = ((yz)x + v) + ((t + u)u)$ holds in $\mathcal{O}\!\mathcal{L} \}$.

Let $\underset{\sim}{K}^* = \underset{\sim}{K}(1) \cup \underset{\sim}{K}(2) \cup \underset{\sim}{K}(3)$. Then we have $\underset{\sim}{L} \subset \underset{\sim}{K}^* \subset \underset{\sim}{K}$. Instead of working within the class $\underset{\sim}{K}$, as we did for distributive lattices, we shall, restrict ourselves to the subclass $\underset{\sim}{K}^*$ of $\underset{\sim}{K}$, for modular lattices.

Let $\mathfrak{A} = \langle A;+,\cdot\rangle$ be a modular lattice which is non-distributive. Thus, by Theorem 3.1 (III,1) and Theorem 2.1, it follows that $p_3(\mathfrak{A}) > 9$. Our object now is to compute the exact value of $p_3(\mathfrak{A})$.

It is well-known (see [9]) that the free modular lattice on three generators consists of 28 elements. Thus, by making use of the formula for F_n, we obtain $p_3(\mathfrak{A}) = 19$. In fact, one can check that the 19 distinct essentially ternary polynomials over $\mathfrak{A}$ are exactly the following :

$$(*)\quad\left\{\begin{array}{ccc} & xyz & x + y + z \\ xy + z & yz + x & zx + y \\ (x + y)z & (y + z)x & (z + x)y \\ (x + z)(y + z) & (y + x)(z + x) & (z + y)(x + y) \\ xz + yz & yx + zx & zy + xy \\ & (x + y)(y + z)(z + x) & xy + yz + zx \end{array}\right.$$

$$(\#)\quad\left\{\begin{array}{l} (x + y)(z + xy) \\ (z + x)(y + zx) \\ (y + z)(x + yz). \end{array}\right.$$

Therefore, we arrive at the conclusion that if $\mathfrak{A} \in \underset{\sim}{M} - \underset{\sim}{D}$, then $\mathfrak{A}$ represents the sequence $\langle 0,0,2,19\rangle$.

Suppose, now, that an algebra $\mathfrak{A}$ of $\underset{\sim}{K}^*$ is given which represents $\langle 0,0,2,19\rangle$. It is natural to ask whether it is true that $\mathfrak{A}$ must be a modular lattice. We are now going to prove that this is exactly the case and hence, from this, it follows that the class $\underset{\sim}{M} - \underset{\sim}{D}$ can be characterized from $\underset{\sim}{K}^*$ by the sequence $\langle 0,0,2,19 \rangle$.

To this end, we first establish the following :

Lemma 3.1.

If $\mathfrak{A}$ is a non-distributive lattice, then $p_3(\mathfrak{A}) \geqslant 19$.

Proof : We shall prove that the polynomials in (*) and (#) are essential and distinct over $\mathfrak{A}$.

For instance, take $p = (x + y)(z + xy)$. Setting $x = y$, $x = z$ and $y = z$, we have $p = x$, $p = x$ and $p = y$ respectively. Thus, if p is not an essentially ternary polynomial, then as $\mathfrak{A}$ represents $\langle 0,0,2 \rangle$, we have

$$p = \begin{cases} x \\ y \\ z \\ xy \\ xz \\ yz \\ x + y \\ y + z \\ z + x \end{cases}.$$

Since p is symmetric with respect to x and y; it is thus reduced to the following

$$p = \begin{cases} z \\ xy \\ x + y \end{cases}.$$

However, if $p = z$, setting $x = y$, we obtain $x = z$, a contradiction. If $p = xy$, setting $y = z$, we have $y = xy$, a contradiction. If $p = x + y$, setting $y = z$, it follows that $y = x + y$, which is impossible. Thus, it is necessary that p is an essentially ternary polynomial. The essentiality of the other polynomials can similarly be proved.

It is routine to check that the polynomials in (*) are distinct over $\mathcal{O}\!\ell$.

Now, consider the polynomial $p = (x + y)(z + xy)$. We shall show that it is distinct from those listed in (*). For, if say, $p = yx + zx$, setting $y = z$, we have $y = yx$, which is impossible. Thus, $p \neq yx + zx$.

To prove that $p \neq (x + y)(y + z)(z + x)$, we first note that as $\mathcal{O}\!\ell$ is non-distributive, it contains M_5 or N_5 as sublattice (see Fig. 39)

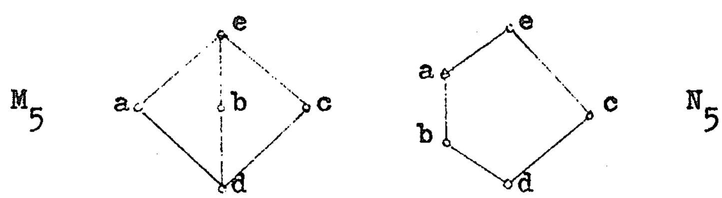


Fig. 39

If $\mathcal{O}\!\ell$ contains M_5 as sublattice, putting $x = a$, $y = b$, $z = c$, we have $(x + y)(z + xy) = ec = c$, $(x + y)(y + z)(z + x) = eee = e$. If $\mathcal{O}\!\ell$ contains N_5 as sublattice, setting $x = a$, $y = c$ and $z = b$, we obtain $(x + y)(z + xy) = eb = b$,

$$(x + y)(y + z)(z + x) = eea = a.$$

Thus, it follows that $p \neq (x + y)(y + z)(z + x)$, which was to be shown.

Furthermore, we claim that three polynomials in (#) are distinct. To this end, by symmetry, it suffices to prove that

$$p = (x + y)(z + xy) \neq (z + x)(y + zx) = q.$$

Observe that if $\mathcal{A}$ contains N_5 as a sublattice, setting $x = a$, $y = b$ and $z = c$, we have $p = a(c + b) = a$ and $q = e(b + d) = b$. If $\mathcal{A}$ contains M_5 as a sublattice, setting $x = a$, $y = b$ and $z = c$, we obtain $p = e(c + d) = ec = c$ while $q = e(b + d) = eb = b$. Thus, $p \neq q$, as required.

It should be noted that any polynomial in (#) is distinct from any one of (*) over $\mathcal{A}$. Hence, we obtain $p_3(\mathcal{A}) \geqslant 19$.

<u>Lemma 3.2.</u>

If $\mathcal{A}$ is a non-modular lattice, then $p_3(\mathcal{A}) > 19$.

<u>Proof</u> : As $\mathcal{A}$ is non-distributive, by Lemma 3.1, $p_3(\mathcal{A}) \geqslant 19$.

Now, consider the ternary polynomial $r = x(y + xz)$. Putting $y = z$, we obtain $r = xy$, thus r depends on x. Moreover, if we set $x = y$, $x = z$, we have $r = x$.

Since $\mathcal{A}$ represents $\langle 0,0,2 \rangle$ and r depends on x, if r is not essential, we then have

$$r = \begin{cases} x \\ x + y \\ x + z \\ xy \\ xz \end{cases}$$

If $r = x$, setting $y = z$, we have $xy = x$, a contradiction. If $r = x + y$, setting $y = z$, we obtain $xy = x + y$, which is impossible. Again, if $r = x + z$, putting $y = z$, it follows that $xy = x + y$, a contradiction. If $r = xy$, setting $x = z$, we have $x = xy$, a contradiction. Finally, if $r = xz$, putting $x = y$, we obtain $x = xz$, which is impossible. Hence r must be essentially ternary.

Our proof will be complete if we can prove that r is distinct from the polynomials listed in (*) and (#). However, this is indeed the case. For instance, if

$$r = xy + yz + zx,$$

then setting $y = z$, we have $xy = y$, a contradiction. If

$$r = (x + y)(y + z)(z + x),$$

setting $y = z$, we obtain $xy = y$, which is impossible. If

$$r = (x + y)z,$$

setting $x = y$, we have $x = xy$, a contradiction. If

$$r = xz + yz,$$

setting $y = z$, we have $xy = y$, a contradiction. If

$$r = (y + x)(z + x),$$

setting $y = z$, we obtain $xy = y$, a contradiction. If

$$r = (x + y)(z + xy),$$

setting $y = z$, we have again $xy = y$, a contradiction. If

$$r = yx + zx,$$

then it follows that $\mathfrak{A}$ is a modular lattice, a contradiction.

The other possible cases can be checked similarly. Thus, r is distinct from the polynomials listed in (*) and (#), as was to be shown. Hence, it follows that $p_3(\mathfrak{A}) \geqslant 20 > 19$, proving Lemma 3.2.

We are now in a position to establish the following :

Theorem 3.3.

If the sequence $\langle 0,0,2,m \rangle$ is representable in $\underset{\sim}{K}^*$, then either 1) $m = 9$ or 2) $m \geqslant 19$.

Proof : Let $\mathcal{O}\!\mathcal{C}$ be an arbitrary algebra of K^* representing $\langle 0,0,2,m \rangle$. If $\mathcal{O}\!\mathcal{C} \in \underset{\sim}{D} \subset \underset{\sim}{K}^*$, then by Theorem 2.1, we have $m = 9$. Thus, we may assume that $\mathcal{O}\!\mathcal{C} \in \underset{\sim}{K}^* - \underset{\sim}{D}$. By Theorems 1.1, 1.2 and Corollary 1.4, it follows that $\mathcal{O}\!\mathcal{C}$ is a lattice. If $\mathcal{O}\!\mathcal{C}$ is modular, then $m = p_3(\mathcal{O}\!\mathcal{C}) = 19$. If $\mathcal{O}\!\mathcal{C}$ is non-modular, then by Lemma 3.2, we have $m > 19$. Hence Theorem 3.3 follows.

Corollary 3.4.

If the sequence $\langle 0,0,2,m \rangle$ is representable in $\underset{\sim}{K}^* - \underset{\sim}{D}$, then $m \geqslant 19$.

Theorem 3.5.

Let $\mathcal{O}\!\mathcal{C}$ be an algebra in $\underset{\sim}{K}^*$. Then $\mathcal{O}\!\mathcal{C}$ represents the sequence $\langle 0,0,2,19 \rangle$ if and only if $\mathcal{O}\!\mathcal{C} \in \underset{\sim}{M} - \underset{\sim}{D}$.

Remarks.

(1) It is well-known that the free modular lattice with four generators is infinite (see [1]). Thus, it follows that the cardinal p_4 in the representable sequence $\langle 0,0,2,19,p_4 \rangle$ with respect to the class $\underset{\sim}{K}^*$ cannot be finite.

(2) It was pointed out by A. Waterman ([1] , p.150) that the free lattice on three generators over the equational class $\underset{\sim}{N_5}$ generated by N_5 consists of 99 elements. Thus,

it follows immediately that $p_3(\mathfrak{A}) = 90$, for each $\mathfrak{A}$ of $\underset{\sim}{N}_5$. Hence, following the same reasoning as in distributive and modular lattices, we have the following conjecture :

Let $\mathfrak{A} \in \underset{\sim}{K}^*$. Then $\mathfrak{A} \in \underset{\sim}{N}_5$ if and only if $\mathfrak{A}$ represents the sequence $\langle 0,0,2,90 \rangle$.

CHAPTER 2

THE MINIMAL EXTENSION PROPERTY

In 1969, G. Grätzer (see [7]) introduced the following concept.

Definition. A finite sequence $\langle p_0, p_1, \ldots, p_n \rangle$ of cardinals is said to have the Minimal Extension Property (M.E.P.) with respect to the class $\underset{\sim}{C}$ if the following conditions hold :

1) there exists an algebra $\mathfrak{A}^*$ in $\underset{\sim}{C}$ such that $p_k(\mathfrak{A}^*) = p_k$ for $0 \leq k \leq n$,

2) if $\mathfrak{A}$ is an algebra in $\underset{\sim}{C}$ satisfying $p_k(\mathfrak{A}) = p_k$, for $0 \leq k \leq n$, then $p_k(\mathfrak{A}^*) \leq p_k(\mathfrak{A})$ for each $k = 0, 1, 2, \ldots, \ldots$

If $\underset{\sim}{C}$ is the class of all algebras, then we say that the sequence $\langle p_0, p_1, \ldots, p_n \rangle$ has the Minimal Extension Property.

Thus, if a finite sequence $\langle p_0, p_1, \ldots, p_n \rangle$ has the M.E.P. with the minimal extension sequence $\langle p_j(\mathfrak{A}) \rangle$, it is clear that any sequence $\langle p_j \rangle$ with $p_t < p_t(\mathfrak{A})$ for some $t > n$ cannot be representable. Therefore, to get some significant result, we need only to investigate the sequences $\langle p_j \rangle$ with $p_k \geq p_k(\mathfrak{A})$, for each $k > n$.

In this chapter, we shall show by applying previous results that some finite sequences do possess this property.

1. Examples.

In order to clarify this concept, we shall now give some examples.

Example 1. The sequence $\langle 0,0, \rangle$ has the M.E.P. with the minimal extension $\langle 0,0,\ldots,0,\ldots \rangle$ which is represented by any trivial algebra (i.e., a non-empty set with no operations).

Example 2. The sequence $\langle 0,0,1 \rangle$ has the M.E.P. with the minimal extension $\langle 0,0,1,1,\ldots,1,1,\ldots \rangle$ which is represented by any non-trivial semilattice.

Indeed, if $\mathfrak{A}$ is an arbitrary algebra representing $\langle 0,0,1 \rangle$, then $\mathfrak{A}$ has one commutative essentially binary polynomial. As $\mathfrak{A}$ has no conatants, it follows from Lemma 1.1 (I,1) that $p_n(\mathfrak{A}) \geqslant 1$.

Example 3. The sequence $\langle 0,0,2 \rangle$ has the M.E.P. with the minimal extension $\langle 0,0,2,0,\ldots,0,\ldots \rangle$ which is represented by any non-trivial diagonal semigroup.

Example 4. (G. Grätzer and R. Padmanabhan [10]) The sequence $\langle 0,0,1,3 \rangle$ has the M.E.P. with the minimal extension $\langle 0,0,1,3,5,11,\ldots,\frac{1}{3}(2^n - (-1)^n),\ldots \rangle$ which is represented by an idempotent reduct $\langle G;\circ \rangle$ of abelian group $\langle G;+ \rangle$ of exponent three (i.e., $3x = 0$).

Example 5. (J. Płonka [35]) The sequence $\langle 0,0,2,3 \rangle$ has the M.E.P. with the minimal extension $\langle 0,0,2,3,4,5,\ldots,n,\ldots \rangle$ which is represented by any algebra belonging to one of the four equational classes described in [35] .

Example 6. The sequence $\langle 0,0,0,1 \rangle$ has the M.E.P. with the minimal extension $\langle 0,0,0,1,0,1,0,1,0,\ldots,\ldots \rangle$ which is represented by an idempotent reduct $\langle B;g \rangle$ of a Boolean group $\langle B;+ \rangle$ (a group in which all elements different from the zero element are of order two) where $g(x,y,z) = x + y + z$.

In fact, it has been shown by J. Płonka [31] and K. Urbanik [42] that $\langle B;g \rangle$ represents the sequence $\langle 0,0,0,1,0,1,0,\ldots \rangle$. On the other hand, let $\mathfrak{A}$ be an arbitrary algebra representing $\langle 0,0,0,1 \rangle$. Then $\mathfrak{A}$ has a symmetry essentially ternary polynomial. Thus, by a result of E. Marczewski [22], we have $p_{3+2k}(\mathfrak{A}) \geqslant 1$, for each $k = 0,1,2,\ldots$. Therefore, $p_n(\mathfrak{A}) \geqslant p_n(\mathfrak{B})$, for each $n = 0,1,2,\ldots$.

2. Main Results.

In this section, some special finite sequences described in the previous chapters will be shown to have the M.E.P. .

First of all, as a consequence of the results in Chapters 1 and 2 (I), we have the following :

Theorem 2.1.

The sequence $\langle 0,0,1,2,11 \rangle$ has the M.E.P. with the minimal extension $\langle 0,0,1,2,11,136,\ldots,p_n(II(1)),\ldots \rangle$.

Proof : According to Proposition 1.8 (II,2), we have $p_3(II(1)) = 11$. Thus, the algebra II(1) represents the sequence

$\langle 0,0,1,2,11 \rangle$.

Let $\mathcal{A}$ be an algebra representing $\langle 0,0,1,2,11 \rangle$. As $\mathcal{A}$ represents $\langle 0,0,1,2 \rangle$, it follows by Theorem 3.1 (I,1) that $\mathcal{A}$ can be represented as an algebra belonging to either $\underset{\sim}{K}_1$ or $\underset{\sim}{K}_2$.

If $\mathcal{A} \in \underset{\sim}{K}_1$, by Theorem 3.1 (II,1), $\mathcal{A}$ contains I(j) as a subalgebra for j = 1,2,3,4. It is a simple matter to check the following 4-ary polynomials are essential and distinct over I(1) :

$$\left[\begin{array}{l} x_1x_2x_3x_4 , \\ f(x_1,x_2,x_3)x_4, \ f(x_2,x_3,x_4)x_1, \ f(x_3,x_4,x_1)x_2, \\ f(x_4,x_1,x_2)x_3, \\ f(x_1x_2,x_3,x_4), \ f(x_1x_3,x_2,x_4), \ f(x_1x_4,x_2,x_3), \\ f(x_2x_3,x_4,x_1), \ f(x_2x_4,x_1,x_3), \ f(x_3x_4,x_1,x_2), \\ f(x_1,x_3,x_4)f(x_2,x_3,x_4) . \end{array}\right.$$

Thus, $p_4(\mathcal{A}) \geqslant p_4(I(j)) = p_4(I(1)) > 11$, by Theorem 3.5 (I,2), which is a contradiction.

Hence, it is necessary that $\mathcal{A} \in \underset{\sim}{K}_2$. However, by Theorem 3.1 (II,1) again, $\mathcal{A}$ contains II(j) as subalgebra for some j = 1,2,3,4. Thus, $p_n(\mathcal{A}) \geqslant p_n(II(j)) = p_n(II(1))$ by Theorem 3.5(I,2). This completes the proof of our Theorem.

<u>Remark.</u>

The fact that $p_5(II(1)) = 136$ will be shown in Appendix.

By using a similar argument, it is not difficult to extend Theorem 2.1 to the following result.

Theorem 2.2.

For each positive integer k, the sequence $\langle 0,0,\overbrace{1,\ldots,1}^{k},2,1+\frac{1}{2}(k+3)(k+4)\rangle$ has the M.E.P. with the minimal extension $\langle 0,0,\overbrace{1,\ldots,1}^{k},2,1+\frac{1}{2}(k+3)(k+4),\ldots,p_n(\mathfrak{A}(k+2)),\ldots\rangle$.

Theorem 2.3.

The sequence $\langle 0,0,2,9\rangle$ has the M.E.P. with respect to the class $\underset{\sim}{K}$.

Proof : Consider the two-element lattice C_2. Since C_2 is distributive ; by Theorem 3.2 (III,1), C_2 represents the sequence $\langle 0,0,2,9\rangle$. If $\mathfrak{A}$ is an algebra of $\underset{\sim}{K}$ representing $\langle 0,0,2,9\rangle$, then by Theorem 2.1 (IV,1), $\mathfrak{A}$ is a distributive lattice. Since $p_2(\mathfrak{A}) = 2$, $|A| \geqslant 2$. Hence, $\mathfrak{A}$ contains C_2 as a sublattice. Thus, $p_n(\mathfrak{A}) \geqslant p_n(C_2)$, for each $n = 0,1,2,\ldots,\ldots$. By definition, $\langle 0,0,2,9\rangle$ has the M.E.P. with respect to $\underset{\sim}{K}$.

Theorem 2.4.

The sequence $\langle 0,0,2,10\rangle$ has the M.E.P. with respect to the class $\underset{\sim}{K}$.

Proof : By Theorem 4.1 (III,1), the algebra $\mathcal{L}(4)$ represents the sequence $\langle 0,0,2,10\rangle$.

Let $\mathfrak{A}$ be an algebra of $\underset{\sim}{K}$ which represents $\langle 0,0,2,10\rangle$. Then, by Theorem 4.2 (III,1), $\mathfrak{A}$ contains $\mathcal{L}(4)$ as a subalgebra. Thus, $p_n(\mathfrak{A}) \geqslant p_n(\mathcal{L}(4))$, for each n. Hence, Theorem 2.4 follows.

Theorem 2.5.

The sequence $\langle 0,0,2,19 \rangle$ has the M.E.P. with respect to the class $\underset{\sim}{K}^*$.

Proof : Consider the lattice M_5. It is non-distributive but modular. Thus, invoking Theorem 3.5 (IV,1), M_5 represents $\langle 0,0,2,19 \rangle$.

Let $\mathcal{O}\!\mathcal{L}$ be an algebra of $\underset{\sim}{K}^*$ representing $\langle 0,0,2,19 \rangle$. Then, again, by Theorem 3.5 (IV,1), $\mathcal{O}\!\mathcal{L}$ is a non-distributive but modular lattice. Since $\mathcal{O}\!\mathcal{L}$ is non-distributive ; it contains M_5 or N_5 as sublattice. Since $\mathcal{O}\!\mathcal{L}$ is modular ; N_5 cannot be a sublattice of $\mathcal{O}\!\mathcal{L}$. Therefore, $\mathcal{O}\!\mathcal{L}$ must contain M_5 as sublattice. Thus, we have $p_n(\mathcal{O}\!\mathcal{L}) \geqslant p_n(M_5)$, for each $n = 0,1,2,\ldots$, proving that $\langle 0,0,2,19 \rangle$ has the M.E.P. with respect to $\underset{\sim}{K}^*$.

It is known by Example 3 in section 1 that the sequence $\langle 0,0,2 \rangle$ has the M.E.P. . By restricting ourselves to some special class of $\underline{K}$, it can be shown that $\langle 0,0,2 \rangle$ still has the M.E.P. with respect to such class of algebras. Indeed, we have

Theorem 2.6.

1) The sequence $\langle 0,0,2 \rangle$ has the M.E.P. with respect to the class $\underset{\sim}{K}^*$. The algebra representing this sequence with minimal number of essential polynomials is the two-element chain.

2) The sequence $\langle 0,0,2 \rangle$ has the M.E.P. with respect to

the class $\underset{\sim}{M} - \underset{\sim}{D}$. The algebra in $\underset{\sim}{M} - \underset{\sim}{D}$ representing this sequence with minimal number of essential polynomials is the M_5 lattice.

Finally, we have the following :

Theorem 2.7.

The sequence $\langle 0,0,3,7 \rangle$ has the M.E.P. with the minimal extension $\langle 0,0,3,7,15,\ldots,2^n - 1,\ldots,\ldots \rangle$.

Proof : Let $\mathcal{O}\!\mathcal{l}^* = \langle A;+,\circ \rangle$ be an algebra where "+" is a semilattice operation and "∘" is a partition function (see (III,2,3)). Thus, it is known that $p_o(\mathcal{O}\!\mathcal{l}^*) = p_1(\mathcal{O}\!\mathcal{l}^*) = 0$ and $p_n(\mathcal{O}\!\mathcal{l}^*) = 2^n - 1$, for $n \geqslant 2$.

Let $\mathcal{O}\!\mathcal{l}$ be an algebra representing $\langle 0,0,3,7 \rangle$. Then, by Corollary 3.2 (III,2), we have

$$p_n(\mathcal{O}\!\mathcal{l}) \geqslant 2^n - 1 = p_n(\mathcal{O}\!\mathcal{l}^*), \text{ for } n \geqslant 2.$$

Hence, Theorem 2.7 follows.

As a conclusion, we give the following table which shows all the results on this topic.

	The sequence	With respect to the class	The minimal extension sequence
1	$\langle 0,0 \rangle$		$\langle 0,0,0,\ldots,0,\ldots \rangle$
2	$\langle 0,0,0,1 \rangle$		$\langle 0,0,0,1,0,1,0,1,0,\ldots \rangle$
3	$\langle 0,0,1 \rangle$		$\langle 0,0,1,1,1,\ldots,1,\ldots \rangle$
4	$\langle 0,0,1,2,11 \rangle$		$\langle 0,0,1,2,11,136,\ldots \rangle$
5	$\langle 0,0,\overbrace{1,\ldots,1}^{k},2,1+\frac{1}{2}(k+3)(k+4) \rangle$		
6	$\langle 0,0,1,3 \rangle$		$\langle 0,0,1,3,\ldots,\frac{1}{3}(2^n-(-1)^n),\ldots \rangle$
7	$\langle 0,0,2 \rangle$		$\langle 0,0,2,0,\ldots,0,\ldots \rangle$
8	$\langle 0,0,2 \rangle$	$\underset{\sim}{K}^*$	$\langle 0,0,2,9,\ldots,\ldots \rangle$
9	$\langle 0,0,2 \rangle$	$\underset{\sim}{M} - \underset{\sim}{D}$	$\langle 0,0,2,19,\ldots,\ldots \rangle$
10	$\langle 0,0,2,3 \rangle$		$\langle 0,0,2,3,4,\ldots,n,\ldots \rangle$
11	$\langle 0,0,2,9 \rangle$	$\underset{\sim}{K}$	$\langle 0,0,2,9,\ldots,\ldots \rangle$
12	$\langle 0,0,2,10 \rangle$	$\underset{\sim}{K}$	$\langle 0,0,2,10,\ldots,\ldots \rangle$
13	$\langle 0,0,2,19 \rangle$	$\underset{\sim}{K}^*$	$\langle 0,0,2,19,\ldots,\ldots \rangle$
14	$\langle 0,0,3,7 \rangle$		$\langle 0,0,3,7,\ldots,2^n-1,\ldots \rangle$

Algebra representing the minimal extension sequence	Reference	
Trivial algebra		1
Idempotent reduct of Boolean group		2
Semilattice		3
II(1)	see Theorem 2.1(IV,2)	4
$\mathcal{A}(k,2)$	see Theorem 2.2(IV,2)	5
Idempotent reduct of abelian group with 3x = 0	see G.Grätzer and R. Padmanabhan [10]	6
Diagonal semigroup		7
Two-element chain	see Theorem 2.6(IV,2)	8
M_5	see Theorem 2.6(IV,2)	9
Algebra in one of the four equational classes	see J.Płonka [35]	10
Two-element chain	see Theorem 2.3	11
$\mathcal{L}(4)$	see Theorem 2.4	12
M_5	see Theorem 2.5	13
Semilattice with a partition function	see Theorem 2.7	14

CONCLUSION

Since the appearance of G. Grätzer, J. Płonka and A. Sekanina's paper [11] , a number of important developments have taken place in the study of $\langle p_n \rangle$ sequences (see the Bibliography). As a result of all this pioneer work, the topic not only provides a new line of research but also has become recognized as a substantial branch of Universal Algebra.

In the investigation of this topic, one will owe much to one's close acquaintance with the properties of operations and polynomials. In view of this fact, the intensive and systematic study of operations and polynomials becomes very necessary and important. As a result of the consideration of the connections between these basic concepts, it is naturally expected that several new algebras will be discovered and described, which, undoubtly, will enlarge the contents of Universal Algebra.

There are a good few disections that research can take in order to enrich the theory. The results established in this thesis present some of them. As a conclusion, we would like to mention some related problems on which further research is called for.

Problem 1. Prove that the sequence $\langle 0,0,1,2 \rangle$ has the M.E.P. .

It is surprising that even though we considered algebras representing $\langle 0,0,1,2 \rangle$ in detail, we could only prove the weaker result, that the sequence $\langle 0,0,1,2,11 \rangle$ has the M.E.P. . According to some results in (I,2) and the fact that $p_4(II(1)) < p_4(I(1))$, it is easy to see that the sequence $\langle 0,0,1,2 \rangle$ has the M.E.P. if and only if $p_n(II(1)) \leqslant p_n(I(1))$, for all n. Let p and q be two distinct essentially n-ary polynomials over II(1). It can be shown that if $n < 7$, then p and q are again distinct essentially n-ary polynomials over I(1). However, if $n \geqslant 7$, this situation may fail to be occur. For instance, take

$$p = f(f(x_1,x_2,x_3),f(x_4,x_5,x_6),x_7)f(x_1,x_4,x_7)$$

and

$$q = f(x_1,x_2,x_3)f(x_4,x_5,x_6)f(x_1,x_4,x_7) .$$

Therefore, the natural method cannot be applied to prove that $p_n(II(1)) \leqslant p_n(I(1))$. In spite of this, we strongly suspect that $p_n(II(1)) \leqslant p_n(I(1))$. In this situation, of course, some new techniques are needed to attack this somewhat difficult problem.

Problem 2. Classify the algebras representing $\langle 0,0,2,m \rangle$ for $0 \leqslant m \leqslant 20$.

For $0 \leqslant m \leqslant 20$, the sequences $\langle 0,0,2,m \rangle$ which are known to be representable so far are only the following : (see (III,1) and (IV,1)) $\langle 0,0,2,0 \rangle$, $\langle 0,0,2,3 \rangle$, $\langle 0,0,2,6 \rangle$, $\langle 0,0,2,9 \rangle$, $\langle 0,0,2,10 \rangle$ and $\langle 0,0,2,19 \rangle$. Algebras representing the first three sequences have one non-commutative essentially binary polynomial while those representing the last three sequences have two commutative essentially binary ones. Are there any relations between these two categories ? Do there exist other new sequences $\langle 0,0,2,m \rangle$, $0 \leqslant m \leqslant 20$, which are representable ?

Problem 3. Let Σ be a set of identities including one of the absorption laws such that $\Sigma^* = \underset{\sim}{L}$. Let $\underset{\sim}{C}$ be the class of algebras satisfying $\Sigma - \{\text{absorption laws}\}$ and representing $\langle 0,0,2 \rangle$. Is it true that $\underset{\sim}{C} = \underset{\sim}{L}$?

In (IV,I,1), we proved the above conjecture for some special sets of identities. Observe that one inclusion $\underset{\sim}{C} \supseteq \underset{\sim}{L}$ is obvious.

Problem 4. Let $\mathfrak{A} \in \underline{K}$. Is it true that $\mathfrak{A} \in \underset{\sim}{M}$ if and only if $\mathfrak{A}$ represents $\langle 0,0,2,19 \rangle$?

We have proved in (IV,1,3) the above result for $\mathfrak{A} \in \underset{\sim}{K}^*$. It is hopefully the case that the class $\underset{\sim}{K}^*$ can be extended to $\underline{K}$.

Problem 5. Prove or disprove that the sequence $\langle 0,0,3 \rangle$ has the M.E.P. .

It was shown in (IV,2) that the sequence $\langle 0,0,3,7 \rangle$ has the M.E.P. with minimal extension $\langle 0,0,3,7,15,\ldots,2^n - 1,\ldots \rangle$. It is still not known whether the sequence $\langle 0,0,3,7 \rangle$ can be reduced to $\langle 0,0,3 \rangle$.

Problem 6. Study the sequence $\langle 0,0,4,m \rangle$.

The sequences $\langle 0,0,1,m \rangle$, $\langle 0,0,2,m \rangle$ and $\langle 0,0,3,m \rangle$ have been dealt with in this thesis. It is natural to go one step further by considering $\langle 0,0,4,m \rangle$. Of course, the larger the cardinal p_2, the more complicated situation we have. Observe that if $\mathcal{A}$ represents $\langle 0,0,4 \rangle$, then $\mathcal{A}$ has either (1) two non-commutative essentially binary polynomials ; or (2) two commutative essentially binary polynomials and one non-commutative essentially binary polynomial ; or (3) four commutative essentially binary polynomials. We have no informations about any algebras representing $\langle 0,0,4,m \rangle$ except the following result due to J. Gerhard (see [5]) : Let $\mathcal{A}$ be an idempotent semigroup satisfying $abcd = acbd$. Then $p_n(\mathcal{A}) = n^2$, for $n \geqslant 2$.

Problem 7. Let $\langle p_n \rangle$ be a sequence of cardinals which is represented by an idempotent, equationally complete algebra. Is it true that there exists a "k" such that the sequence $\langle 0,0,p_1,p_2,\ldots,p_k \rangle$ has the M.E.P. with the minimal extension $\langle p_n \rangle$?

The above conjecture is false for general algebras. In fact, E. Fried has recently found an algebra $\mathcal{A}$ of finite type which is the three-element tournament lattice such that for each k, the sequence $\langle 0,0,p_1(\mathcal{A}),\ldots,p_k(\mathcal{A}) \rangle$ has no M.E.P. with the minimal extension $\langle p_n(\mathcal{A}) \rangle$. Another example of such a sequence is $\langle p_n(\infty) \rangle$ of J.A. Gerhard [5] . However, it is still not known for algebra which is equationally complete.

Problem 8. Does there exist a finite sequence $\langle 0,0,p_1,\ldots\ldots,p_n \rangle$ which is representable but has no M.E.P. ?

Problem 9. Characterizes the projective and injective algebras in the equational classes $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$.

APPENDIX

With the help of the results in (II,2,1), it can be shown that there are exactly 136 distinct essentially 5-ary polynomials over the algebra II(1). In fact, if, for simplicity, we write i for x_i, ij for $x_i x_j$ and (A,B,C) for f(A,B,C), then these 136 polynomials are precisely the following :

12345

(1,2,3)45 (1,2,4)35 (1,2,5)34 (1,3,4)25 (1,3,5)24
(1,4,5)23 (2,3,4)15 (2,3,5)14 (2,4,5)13 (3,4,5)12

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