THE UNIVERSITY OF MANITOBA

CONTRIBUTIONS TO QUALIMETRY

by

Hansheng Xie

A Thesis

submitted to the Faculty of Graduate Studies

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Statistics

Winnipeg, Manitoba

0-612-45014-7

THE UNIVERSITY OF MANITOBA

FACULTY OF GRADUATE STUDIES
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ABSTRACT

Under the normality assumption. four univariate exponentially moving average single control charts are proposed and they are designed to monitor simultaneously both the process mean and the process variability. The performances of these four charts are evaluated by comparing their average run lengths among themselves as well as to two other competing combination charts. Based on the comparison of the six univariate charts. a multivariate exponentially moving average single control chart is developed as an extension of one of the best univariate charts. This chart performs better than the combination of the two widely used multivariate charts when small changes are of interest.

In dealing with positively-skewed distributed data. the direct logarithmic transformation may result in a control chart with inappropriate control parameters in the application of quality control. When a specific interval for the lognormal mean is given. a new method is introduced to set up two control charts and these two charts can monitor a process for which the underlying distribution of the quality characteristic is lognormal.

ACKNOWLEDGMENTS

My advisors. Dr. S. W. Cheng and Dr. B. D. Macpherson. deserve my sincere appreciation for their caring guidance. continuing encouragement and genuine support in the course of my research. They have provided numerous comments and made many valuable contributions to my research. I would like to thank my committee members. Dr. J. C. Fu and Dr. J. Pai for their valuable suggestions and helpful attitude.

I wish to extend my thanks to two special people. Dr. G. Chen was very helpful with parts of Chapter 3 . Chapter 5 and Chapter 7. His excellent statistical computing ability has always been appreciated. Professor Y. Zhang was my advisor of Master program in China. He first introduced me to the field of statistics. fostered my interest and encouraged me to further pursue in this area. I thank him for having always been a source of inspiration.

The financial support from the Department of Statistics in the form of Teaching & Research Assistantship. from the Faculty of Graduate Studies in the form of UM Graduate Fellowship. from Natural Sciences and Engineering Research Council of Canada through Dr. S. W. Cheng and from Statistical Quality Control Research and Applications Group of the Department of Statistics through Dr. B. D. Macpherson is greatly acknowledged.

Finally. I would like to thank my son. Ming. and my friend. G. G.. for their genuine love. patience. support and encouragement throughout the years.

With my greatest gratitude. I dedicate this thesis to my dearly loving parents without whom I would not be what I am and where I am today.

CHAPTER 1

Introduction

1.1 Overall View

Efforts to improve quality of industrial products brought on the technique of statistical quality control. A major objective of statistical quality control is to monitor an ongoing process and to detect quickly the occurrence of process shifts so that corrective action may be taken. As a result, the process can be kept in a state of control for long periods of time. For this purpose the control chart is one of the most useful tools and has been widely used in quality control since the early 1920s.

If the measurement of the quality characteristic has a continuous scale, the quality characteristic is usually called a variable and the control charts dealing with continuous data are collectively called variables control charts. Using this general type of control chart, the variables mostly controlled are the mean value and the dispersion of the quality characteristic. To determine a state of statistical control, two separate charts are used traditionally: one is an $\bar{X}$ chart and the other is either an R chart or S chart. The $\bar{X}$ chart is used for controlling the central tendency, while the R chart or S chart is used for monitoring the variability of the process.

Since Shewhart [51] introduced control charts, practitioners in quality control have commonly used these well-known Shewhart control charts. Shewhart control charts are simple to construct and easy to understand. If the distribution of the characteristic is approximately normal and the process changes are moderately large, these charts are very effective in detecting mean shifts and variability changes of the process.

However, Shewhart control charts have some disadvantages. Firstly, they only make use of the information about the process contained in the last plotted statistic and they are ineffective in detecting relatively small changes in the process. Secondly, since changes may exist in both process average and process dispersion, it is inconvenient to use two control charts in monitoring the process center and the process spread separately. Thirdly, in some situations the underlying distribution of the quality characteristic may be very different from normal and therefore Shewhart control charts may not be appropriate. Finally, sometimes a process is influnced by simultaneous effects of several quality characteristics and the univariate Shewhart control charts may not be able to control these quality characteristics effectively.

1.2 Outline of the Thesis

As Parr [45] pointed out, as the level of quality maturity of a company increases there will be a corresponding increase in the use of variables control charts. This dissertation is mainly concerned with variables control charts. The overall review of literature in Chapter 2 shows that, to develop effective alternatives to the Shewhart control chart, the latest trends in control charting methodology have focused on two research fields. One is the highly sensitive control chart, which is sensitive to small changes within a process, and the other is the single chart, which employs a single plot to monitor both the mean and the variability of a process.

Several new sensitive single control charts are developed in this dissertation. In Chapters 3, 4, 5 and 6, four univariate Exponentially Weighted Moving Average

(EWMA) single control charts are proposed and their performances are studied. In Chapter 7. the performances of these four charts together with that of two existing combined univariate control charts are compared. Based upon ARL comparisons. diagnostic ability studies for three of the preferred control charts are discussed. In Chapter 8. two existing control charts based on the lognormal distribution are critically examined and a new method is proposed to set up control charts for variables having this lognormal distribution. Based on the comparison of the six univariate charts. a multivariate exponentially moving average single control chart is developed as an extension of one of the best univariate charts in Chapter 9. This chart performs better than the combination of the two widely used multivariate charts when small changes are of interest. Finally. conclusions are drawn and recommendations are given in Chapter 10.

Several computer programs written in FORTRAN 77 code are included in Appendices A and B. These programs are designed to obtain Average Run Lengths (ARL's) for all the new control charts and the existing control charts considered. and to obtain simulations of diagnostic ability studies for the three preferred control charts.

1.3 Notation

The notation below is used throughout the thesis.

SPC	Statistical process control
EWMA	Exponentially weighted moving average
CUSUM	Cumulative Sum
UCL	Upper control limit
CL	Center line of a control chart

LCL	Lower control limit
ARL	Average run length
CDF	Cumulative distribution function
pdf	probability density function
k	Dimensionality of a vector
n_i	Size of the i^{th} sample, $i = 1, 2, \ldots$
n	Equal size for all samples
m	Total number of samples taken from an in-control process
μ	Process mean
$\boldsymbol{\mu}$	Process mean vector
σ^2	Process variance
σ	Process standard deviation
$\boldsymbol{\Sigma}$	A kxk process covariance matrix
$\lvert\boldsymbol{\Sigma}\rvert$	Determinant of the process covariance matrix
ρ	Correlation coefficient of two quality characteristics
$E(\cdot)$	Mean function of a distribution
$Var(\cdot)$	Variance function of a univariate distribution
$Cov(\cdot)$	Covariance function of a multivariate distribution
ARL_0	In-control ARL of a control chart
$\Phi(\cdot)$	Standard normal distribution function
$\phi(\cdot)$	Standard normal density function
$\Phi^{-1}(\cdot)$	Inverse of standard normal distribution function
$H_\nu(\cdot)$	Chi-square distribution function with ν degrees of freedom
$H_{\nu,\delta^2}(\cdot)$	Noncentral chi-square distribution function with ν degrees of freedom and noncentrality parameter δ^2

$h_\nu(\cdot)$	Chi-squared density function with ν degrees of freedom
$h_{\nu,\delta^2}(\cdot)$	Noncentral chi-square density function with ν degrees of freedom and noncentrality parameter δ^2
$F_{\nu_1,\nu_2}(\cdot)$	F distribution function with (ν_1, ν_2) degrees of freedom
$X \sim N(\mu, \sigma^2)$	A random variable X follows the normal distribution with mean μ and variance σ^2
$X \sim LN(\mu, \sigma^2)$	A random variable X follows the lognormal distribution with parameter μ and σ^2
$X \sim \chi^2_\nu$	A random variable X follows the chi-square distribution with ν degrees of freedom
$X \sim \chi^2_{\nu,\delta^2}$	A random variable X follows the non-central chi-square distribution with ν degrees of freedom and noncentrality parameter δ^2
$X \sim F_{\nu_1,\nu_2}$	A random variable X follows the F distribution function with (ν_1, ν_2) degrees of freedom
$\mathbf{X} \sim N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$	A kx1 random vector $\mathbf{X}$ follows multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$
z_α	Percentage point of the standard normal distribution such that $1 - \Phi(z_\alpha) = \alpha$
$\chi^2_{\alpha,\nu}$	Percentage point of the chi-square distribution with ν degrees of freedom such that $1 - H_\nu(\chi^2_{\alpha,\nu}) = \alpha$
f_{α,ν_1,ν_2}	Percentage point of the F distribution function with (ν_1, ν_2) degrees of freedom such that 1 - $F_{\nu_1,\nu_2}(f_{\alpha,\nu_1,\nu_2}) = \alpha$
X_{ij}	Measurement of a quality characteristic on the j^{th} observation in the i^{th} sample, $i = 1, 2, \ldots$ and $j = 1, 2, \ldots, n_i$

$\mathbf{X}_{ij}$	Measurement of a kx1 vector of quality characteristics
$\bar{X}_i$	i^{th} sample mean of a quality characteristic. $=\frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij}$
$\bar{\mathbf{X}}_i$	i^{th} sample mean of a kx1 vector of quality characteristics. $=\frac{1}{n}\sum_{j=1}^{n} \mathbf{X}_{ij}$
R_i	i^{th} sample range of a quality characteristic. $= max\{X_{i1}, X_{i2}, \ldots, X_{in_i}\} - min\{X_{i1}, X_{i2}, \ldots, X_{in_i}\}$
S_i^2	i^{th} sample variance. $= \frac{1}{n_i-1}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^2$
S_i	i^{th} sample standard derivation
S_{12}	Sample covariance between two quality characteristics. $= \frac{1}{n-1}\sum_{i=1}^{n}(X_{i1} - \bar{X}_1)(X_{i2} - \bar{X}_2)$
$\mathbf{S}_i$	i^{th} kxk sample covariance matrix whose elements are sample variaces and covariances
$\lvert\mathbf{S}\rvert$	Sample generalized variance which is the determinant of $\mathbf{S}_i$
$\bar{\bar{X}}$	Grand average of sample means. $= \frac{1}{m}\sum_{i=1}^{m} \bar{X}_i$
$\bar{R}$	Grand average of sample ranges. $= \frac{\sum_{i=1}^{m} n_i R_i}{\sum_{i=1}^{m} n_i}$
$\bar{S^2}$	Grand average of sample variances. $= \frac{\sum_{i=1}^{m}(n_i-1)S_i^2}{\sum_{i=1}^{m}(n_i-1)}$
$\bar{S}$	Grand average of sample standard deviations. $= \sqrt{\bar{S^2}}$
$\bar{\bar{\mathbf{X}}}$	Grand average of sample mean vectors. $= \frac{1}{m}\sum_{i=1}^{m} \bar{\mathbf{X}}_i$
$\bar{\mathbf{S}}_i$	Grand average of kxk covariance matrices. $= \frac{1}{m}\sum_{i=1}^{m} \mathbf{S}_i$
λ	A smoothing constant for EWMA scheme. $0 < \lambda \leq 1$
L	A multiplier of the standard deviation for an EWMA statistic
a	A multiplier of a step-shift in the process mean. i.e., $\mu = \mu_0 + a\sigma_0$
b	A multiplier of a step-change in the process standard deviation . i.e., $\sigma = b\sigma_0$

$[y]$	The largest integer that is smaller than or equal to y
$\bar{n}$	Average of sample sizes. $= \left[\frac{1}{m}\sum_{i=1}^{m} n_i\right]$
$d_2(\bar{n})$	A control chart constant associated with $\bar{R}$
$c_4(\bar{n})$	A control chart constant associated with $\bar{S}$
$+o$	Number of times that only an increase in the mean is detected
$-o$	Number of times that only a decrease in the mean is detected
$o+$	Number of times that only an increase in the variability is detected
$o-$	Number of times that only a decrease in the variability is detected
$++$	Number of times that increases in both the mean and the variability are simultaneously detected
$+-$	Number of times that an increase in the mean and a decrease in the variability are simultaneously detected
$-+$	Number of times that a decrease in the mean and an increase in the variability are simultaneously detected
$--$	Number of times that decreases in both the mean and the variability are simultaneously detected
m+	A plotting character of an out-of-control signal when only the process mean is up
m-	A plotting character of an out-of-control signal when only the process mean is down
v+	A plotting character of an out-of-control signal when only the process variability is up
v-	A plotting character of an out-of-control signal when only the process variability is down
m+v+	A plotting character of an out-of-control signal when both

	the process mean and the process variability are up
m+v-	A plotting character of an out-of-control signal when process mean is up and the process variability is down
m-v+	A plotting character of an out-of-control signal when process mean is down and the process variability is up
m-v-	A plotting character of an out-of-control signal when both the process mean and the process variability are down

CHAPTER 2

Review of the Literature

2.1 History and Evolution of Quality Control

In the earliest of time, ancient people were concerned about quality of products and it is known that rudimentary techniques for quality control must have existed (Wierda [58]). Before industrialization, individual workers inspected the quality of their own work and were responsible for providing the quality the market demanded. Gradually, as the capacity to produce products grew and work became more specialized, inspectors had the responsibility for quality and their job was to inspect the work of others.

The industrial revolution increased productivity as well as the need for standardization of products. In large factories the burden on the inspectors was too heavy and independent inspection departments were created. The method of controlling quality was mass inspection of items produced and the technology of quality control was developed to assist the inspection work. However, at that stage, quality control was merely quality inspection without concern for quality improvement. Quality was kept up only by removing unacceptable items produced by the process, but assignable causes of the defects were neither identified nor eliminated.

In the 1920's, it was realized that, to maintain the quality of products, studying the underlying production process was more effective than inspecting all the finished products. In addition to this important idea, another element for quality control was developed at the same time. Sampling inspection began to be considered in quality control as an alternative to 100 percent inspection. Much of the earliest recorded work in quality control was done in the Bell Telephone System. At Bell Telephone Laboratories. Walter A. Shewhart. with his colleagues. recognized that variation in a process is a statistical phenomenon and developed statistical methods for quality control.

In 1924. Walter A. Shewhart presented to his chief at Bell Telephone Laboratories his first control chart showing the monthly number of percent defective items in some unspecified piece of apparatus. In the December 1925 issue of the *Journal of the American Statistical Association*. he published a paper entitled "The Application of Statistics as an Aid in Maintaining Quality of a Manufactured Product." In this paper. he introduced the control chart. Later in 1931. he published his famous book. *Economic Control of Quality of a Manufactured Product*. outlining the control chart method. This period is generally considered as the beginning of statistical quality control.

In World War II. an immediate need for large quantities of war material increased the productivity of American manufacturing industries. American industry rapidly expanded during the war. and therefore. those new and expanding factories employed many inexperienced people. Quality then became more important since the quality of the products suffered from the lack of skilled workers. As a result. training programs were established by companies and the use of statistical quality control was taught during the training. The massive and widespread training programs launched extensive applications among American manufacturing industries. This brought about widespread use and acceptance of the concepts of statistical quality control in manufacturing industries. At the same time. British firms and others also witnessed similar development of statistical quality control. After World War II, statistical methods for quality control were widely applied in a variety of industries in America as well as in other countries.

2.2 Theoretical Basis of Control Charts

The construction of a control chart is based on statistical principles. The aim of a control chart is to recognize. from a sample. whether the obtained value of a sample statistic deviates too far from a desired condition. If a sample is drawn from a process whose variability is due only to chance causes. a sample statistic will be distributed in an expected pattern and the process is assumed to be in a state of statistical control. or simply in control. Otherwise. if a sample is drawn from a process whose variability is due to assignable causes. the distribution of the statistic is not desirable and the process is said to be out-of-control.

The control chart is a graphical statistical tool for monitoring the control of a process. A typical univariate control chart displays a quality characteristic. which has been measured from a sequence of samples. on a graph. For example. Shewhart control charts contain a center line that represents the target of the quality characteristic. Two control limits. the upper control limit and the lower control limit. are located at a distance from the center line. These control limits give the range of variability to be expected in the sample statistic when the process is in control. Basically. the process is said to be in control when results behave like sampling from a single population. A commonly used criterion for the in-control state is that some sample points fall within the control limits and behave in a random manner. However. if a single sample point falls outside of the control limits or all the sample points behave in a systematic manner. this is an indication that the process could be out of control. When such an indication occurs. taking action to find and eliminate some assignable causes will keep the process in a state of statistical control.

The statistical theory employed in control charts is the theory of hypothesis testing. Applying a control chart can be considered as doing repeated tests of the statistical hypothesis that the process is in a state of statistical control. When a sample point is plotted on the control chart. the hypothesis of statistical control is to be tested based on the information obtained from the sample. A point falling between the control limits is equivalent to accepting the hypothesis. and a point falling outside the control limits is equivalent to rejecting the hypothesis. As in hypothesis testing. there are also two types of error for a control chart. The probability of type I error represents the probability that the control chart will give an out-of-control signal when. in fact. the process is in control. The probability of type II error represents the probability that the control chart will not detect some assignable causes when the process is actually out of control. An optimal design of a control chart is to achieve the smallest probability of type II error when a desired probability of type I error is given.

There are at least five reasons for the use of control charts (Montgomery [38])

1. Control charts are a proven technique for improving productivity:
2. Control charts are effective in defect prevention:
3. Control charts prevent unnecessary process adjustments:
4. Control charts provide diagnostic information:
5. Control charts provide information about process capability.

2.3 Developments in Control Chart Techniques

2.3.1 Shewhart Control Charts

Since Shewhart originated the concept of statistical control and the control chart technique in 1920's, his $\bar{X}$, R and S charts have become the mostly commonly used control charts in practice for variables data.

Suppose that a quality characteristic X follows a normal distribution with mean μ and standard deviation σ approximately. Random samples of size n are drawn from a population produced by the process. The $\bar{X}$ chart, based on the distribution of the sample mean obtained from the process, shows the variation in sample averages. The R chart and the S chart show the general variability of a process, and they respectively employ sample ranges or sample standard deviations to monitor the process variability. It is customary to set control limits for these charts at some multiple of the standard deviation of the statistic being plotted. The most common multiple is 3 and they are called 3-sigma limits. In practice, μ and σ are usually unknown, and they have to be estimated with preliminary samples drawn from an in-control process.

It is important to maintain control over both the process mean and the process variability, because the output from a process may be attributable to a shift in the mean and/or a change in the dispersion. It is then necessary to monitor both the shifts in the mean and the changes in the variability. Under normality assumption, two control charts are often employed to separately monitor the mean and the variability. Because people without sophisticated statistical knowledge can easily understand the concept of measuring variability with the range, one of the most commonly used pairs of charts is the combination of the $\bar{X}$ chart and the R chart. But the range ignores all information between the two most extreme values, and it becomes less efficient. as a measure of variability. for large sample size. The sample standard deviation makes use of all information available and can provide a better estimate of the process variability than the range. Thus. the combination of the $\bar{X}$ chart and the S chart is also used in quality control.

Shewhart control charts have many advantages. such as. their simplicity and their effectiveness under certain circumstances. However. Shewhart control charts also have some disadvantages. To improve on these. great efforts in research on control charts have been made. and various modifications to Shewhart control charts have been developed. Some modifications of Shewhart control charts are still based on the statistics used by Shewhart. but others make use of new statistics.

2.3.2 High Sensitive Control Charts

To increase the sensitivity to small shifts in process mean. Weindling et al. [56] modified Shewhart control charts and established a pair of warning limits for the Shewhart charts. The warning limits are located inside the conventional control limits. Corrective action will be taken when a run of a specified number of consecutive sample statistics falls between the warning limit and the control limit. The mean action time is a function of shifts in the process mean. and it is used to measure the sensitivity of the modified chart. Small shifts are detected by means of the occurrence of critical run accumulations in the warning regions: large shifts are detected by means of a single sample statistic outside the conventional control limits. Compared with the Shewhart control chart. the modified chart is more sensitive for small and moderate shifts in the process mean. However. for the modified chart. the drawback is that the false alarm rate will increase.

Page [44] proposed the cumulative sum (CUSUM) control chart as an alternative to Shewhart control charts. Plotting the cumulative sums of the deviations from a target value, the CUSUM chart directly incorporates all of the information in the sequence of sample values. There are two forms for the CUSUM charts, the tabular CUSUM, and the V-mask form of the CUSUM. The tabular CUSUM employs two sample statistics: one is one-sided upper CUSUM that accumulates the deviations above the target, and the other is one-sided lower CUSUM that accumulates the deviations below the target. Being similar to Shewhart charts, the tabular CUSUM still has two straight lines as its control limits. Instead of conventional control limits, the V-mask form of the CUSUM requires the use of a mobile V-shaped mask to decide whether a shift occurs. Of the two forms, the tabular CUSUM is preferable due to its esier applicability. It is possible to devise cumulative-sum schemes for a statistic that follows a non-normal distribution to monitor process variability. Because the CUSUM chart combines information from several samples, it is more effective than Shewhart control charts for detecting small process shifts.

Another form of control chart, which has recently received a great deal of attention and has gained extensive applicability, is the exponentially weighted moving average (EWMA) control chart. Robert [47] first developed an EWMA control chart to detect shifts in the process mean. An exponentially weighted moving average gives the greatest weight to the most recent observation and the decreasing weights to all previous observations in geometric progression from the most recent to the first. As interest centers on early detection of smaller and smaller changes, the appropriate smoothing value decreases from unity and becomes smaller and smaller, and more information is gained from the past data.

Robert compared the EWMA control chart with a special kind of modified Shewhart control chart and with the ordinary moving average control chart. The modified Shewhart control chart is based on a Runs Test that prescribes rejection of the null hypothesis if a single sample point falls outside 3.13σ limits or ten consecutive sample points fall on one side of the central line of the control chart. For the ordinary moving average control chart. the statistic is based on the average of a set of sample means. To get the latest average. the oldest sample mean is dropped and the newest one added to the set. Both of these two control charts are more effective in detecting small shifts in the process mean than the Shewhart control chart. Roberts showed that the EWMA control chart compare most favorably with the modified Shewhart control chart and the moving average control chart with regard to charting and statistical properties. Hence. the EWMA control chart has high potential for on-line automatic sensing and control of manufacturing process.

Further research has also provided evidences that the EWMA control chart is a useful process monitoring and control tool. Robinson and Ho [48]. Crowder [13]. and Lucas and Saccucci [36] gave numerical procedures which make the properties of EWMA schemes easy to investigate. Roberts [47] and Chantraine [8] presented graphical methods. which make the EWMA much easier to be applied in industries. Hunter [25] viewed the EWMA as a compromise between the Shewhart and the cumulative sum charting procedures. and promoted the EWMA as a method for establishing real-time dynamic control of industrial processes.

In addition to many of the EWMA control charts constructed for monitoring the mean of a process. Wortham and Ringer [59] suggested the use of the EWMA to construct a control chart to monitor the variance of a process. Sweet [55] modified Wortham and Ringer's model. and proposed two models to construct simultaneous control charts to monitor the mean and the variance of a process. For the same purpose, Ng and Case [42] discussed the methodologies to construct coupled control charts of the EWMA of the sample mean and the sample range.

To detect increases in process variability, Crowder and Hamilton [14] developed an EWMA control chart based on the log transformation of the sample variance. They showed that the EWMA control chart is superior to the Shewhart control charts in term of its ability to quickly detect small increases in the standard deviation of a normal process.

2.3.3 Single Control Charts for both Center and Spread

Using two control charts to separately monitor the process mean and the process variability is usually inconvenient and time-consuming. In dealing with theoretical issues as well as practical concerns, efforts have been made to design a single control chart to achieve the same purpose as Shewhart control charts for variables data.

White and Schroeder [57] introduced a simultaneous control chart. Through the use of resistant measures and a modified box plot, this single chart controls the process level and variability. Iglewicz and Hoaglin [26] extended and refined the techniques discussed by White and Schroeder. It is argued that the simultaneous control chart provides more effective decision making than the Shewhart control charts. But the information contained in a single plot can be confusing due to its complexity, and it may be ineffective for small sample size. Chan, Cheng and Spiring [6] provided an alternative to the box-plot style of simultaneous control charts that had added advantage of performing equally for both large and small sample sizes. Except for the added advantage of appearing on a single chart, the techniques used are similar to that of Shewhart control charts. However, this chart is not simple since it requires plotting two types of quantities separately on a chart.

Domangue and Patch [18] discussed some EWMA statistical process monitoring schemes. The control charts discussed are sensitive to changes in the mean and/or the variability, but cannot indicate whether the change has actually occurred in the process mean or in the process variability. Cheng and Li [10] proposed a single variable T chart, which plots the sum of magnitudes of deviation of the extreme values in the sample from the target value. Nuland [43] described the "circle technique", in which a circle is always involved, as "an effective and simple statistical technique for insuring compliance with ISO 9000". From the idea of the "circle technique", Chao and Cheng [5] developed a semicircle control chart to jointly combine the detection of the location shift and the dispersion deterioration into one chart. This chart is essentially a 2-dimensional chart and much easier to use. Chen and Cheng [9] designed a single control chart, the Max chart, to monitor both the center and the spread for variables data. The Max chart is shown to be just as effective when it is compared with the combination of Shewhart control charts. However, these single control charts, except for Domangue and Patch's charts, are not sensitive to relatively small changes within a process.

2.3.4 Control Charts for Non-normal Data

A fundamental assumption in the use of Shewhart control charts is that the underlying distribution of the quality characteristic is normal. In many situations, this assumption may be violated. Whenever the data indicate the normality assumption is inappropriate, difficulties are probably encountered and satisfactory results may not be obtained using Shewhart control charts. Even if the form of the underlying distribution is known in some cases, it could still be difficult to derive the sampling distribution of some statistics and to obtain exact probability limits for the control charts.

Morrison [41] described difficulties in the application of the traditional statistical control chart technique to some real data in industry. In the radio valve industry it has been found that much of the thermionic valve test data are positively skewed, some to a very marked degree. For fitting these kinds of positively skewed data, the lognormal distribution is widely applied. Morrison discussed the generation of these skew distributions in theoretical and practical terms. He introduced a modified quality control scheme for these cases: the geometric sample mean is used instead of the arithmetic sample mean as a measure of the process center; ratio of maximum to minimum sample values is used instead of sample range as a measure of the process variation; and the logarithmic transformation is used to calculate control limits.

Morrison's paper offers a new method and a knowledgeable discussion for dealing with non-normal data in the field of quality control. However, the control limits are obtained directly from the direct transformation of their normal counterparts and may be inaccurate for the lognormal distribution.

2.3.5 Multivariate Control Charts

When a process is simultaneously characterized by more than one related quality characteristics, a separate Shewhart control chart for each character can give misleading results. Multivariate quality control techniques will take advantage of the multivariate nature generated by the process. Suppose that several related quality characteristics approximately follow a multivariate normal distribution. As an extension of univariate control charts, many of the statistical techniques used in univariate quality control have been modified and extended to multivariate quality control.

Hotelling [24] first proposed a multivariate approach to quality control. and applied his procedure to bombsight production process during World War II. He introduced the χ^2 control chart, an extension of $\bar{X}$ chart, as a technique for monitoring a multivariate process. Hicks [23], Jackson [27] [28], and Montgomery and Wadsworth [40] continued the research on control procedures for several related quality characteristics.

As in the univariate situation, several alternative control charts to the χ^2 control chart have also been developed and some of them are more powerful in detecting small shifts in the mean vector. Healy [22] and Smith [53] used the fact that a cumulative sum control chart can be viewed as a sequential probability ratio test to develop a multivariate cumulative control chart. In the univariate CUSUM scheme, the n^{th} sample statistic is shrunk toward 0 by a constant. Crosier [12] generalized the univariate shrinking method to the multivariate situation by replacing the scalar quantities of the univariate cumulative sum into vectors. Lowry et al. [35] proposed a multivariate EWMA control chart, and showed that the properties of the multivariate EWMA control chart are more similar to and often even better than those of the multivariate cumulative sum control charts.

Because the concept of covariance is complicated and it is difficult to deal with the changes in the covariance, very few papers are published on multivariate control charts for dispersion. One approach is due to Alt [2], Alt and Bedewi [3], and Alt and Smith [4]. They proposed three control charts. The first chart uses a statistic that is the negative of twice the natural logarithm of the likelihood ratio test statistic, which is slightly modified in order to make the test unbiased. The second chart and the third chart employ the same statistic, namely, the generalized sample variance $|\mathbf{S}|$, which is the determinant of the sample covariance matrix of n new observations. But the proposed control limits for these two charts are different.

The control limits of the second chart are derived with the desired probability. however. that of the third chart are determined by the expectation of $|\mathbf{S}|$ plus or minus three times the standard deviation.

Hotelling [24]. Jackson and Hearne [30]. and Jackson [29] proposed another line of approach. They used a generalized measure of the sample dispersion around the sample mean to develop a control chart. This chart is a multivariate analog of the univariate S^2 chart. and it is easier to construct than the charts developed from the first approach because the exact distribution for the statistic is known and the computation is simple. For a multivariate control chart. it is difficult to determine which of monitored variables is responsible for the out-of-control signal. Alt [2]. Doganaksay. Faltin. and Tucker [17] and Fuchs and Benjamini [19] suggested using univariate control chart for variability as a supplement to the multivariate control chart. A disadvantage of this method is that it is unable to detect a change in correlation structure.

Assuming the in-control covariance matrix is unknown. Wierda [58] presented four tests for the covariance matrix and concluded that the control chart based on the modified likelihood ratio test performs very well. He recommended using a hierarchical procedure that divides the modified likelihood ratio test statistic into three components. A control chart is used for each component and the univariate control charts are also consulted. These control charts can indicate what happened with the covariance matrix when a signal occurs. and the univariate control charts are able to detect changes in only the variances. However. this procedure is complicated because a large quantity of computation is required.

Runger. Alt and Montgomery [49] developed a diagnostic. which is analogous to measures of influences in regression modeling, for a χ^2 chart. Utilizing the correlation between the variables, the diagnostic effectively determines the root cause of an out-of-control signal. In addition to the simplicity of computation and interpretation. this diagnostic could be useful for the signal from other control charts. which are based on quadratic forms of the observed vector.

Based on the univariate semi-circle control chart. Cheng and Mao [11] extended the single control chart to multivariate situation and proposed a multivariate semi-circle control chart for variables data. Based on the ARL performance. it is shown that the control chart performs quite favorably relative to the combination of the χ^2 control chart and the $|\mathbf{S}|$ control chart. As an extension of the alternative variables control chart for univariate case. Spiring and Cheng [54] proposed a single chart for the multivariate situation. The multivariate procedure is similar to the traditional Hotelling χ^2 style of charts but results in a control chart that provides information regarding the process proximity to the target value as well as the overall variability. However. similarly to their univariate counterparts. these two single charts are insensitive to relatively small changes within a process.

CHAPTER 3

The Max-EWMA Chart

3.1 Introduction

As illustrated in Chapter 2, there is an abundance of new developments in control chart techniques. Much attention has been focused on developing two kinds of control charts: the high sensitive chart, and the single chart. Among those high sensitive charts, the EWMA-type chart is one of the most effective in detecting small changes in the process mean and variability. However, to monitor both the process mean and variability, two EWMA charts are usually required. For the single charts that have been developed, they are capable of monitoring both the process mean and variability, but are not sensitive in detecting small changes within a process.

In this chapter, a new control chart, the Max-EWMA chart, is proposed. This chart can simultaneously monitor both the process mean and the process variability, and detect the source and the direction of an out-of-control signal. It is also sensitive in detecting small changes within a process, and capable of handling the case of varying sample size. A design strategy using optimal λ and L is introduced, and an example is given to illustrate the implementation of the new chart.

3.2 The New Control Chart

Assume that a series of random samples $X_{ij} \sim N(\mu, \sigma^2), i = 1, 2, \ldots$ and $j = 1, 2, \ldots, n_i$. Let μ_0 be the nominal process mean and σ_0 be a known value of the process standard deviation, and assume that the process parameters μ and σ

can be expressed as $\mu = \mu_0 + a\sigma_0$ and $\sigma = b\sigma_0$, where a and b are constants and $b > 0$. The process is in control when $a = 0$ and $b = 1$; otherwise the process has changed.

It is known that $\bar{X}_i$ and S_i^2 are uniformly minimum variance unbiased estimators, which have many good features, for the process mean and variance respectively. These two statistics are independent, but they follow different distributions. When sample sizes are different, further complications arise since varying sample sizes can cause varying control limits for the same statistics. To deal with the above two situations, two transformed statistics from $\bar{X}_i$ and S_i^2 are defined below:

$$Z_i = \frac{\bar{X}_i - \mu_0}{\sigma_0/\sqrt{n_i}} \tag{3.1}$$

$$W_i = \Phi^{-1}\{H_{n_i-1}\left[\frac{(n_i-1)S_i^2}{\sigma_0^2}\right]\} \tag{3.2}$$

It is apparent that Z_i and W_i are independent. When $a = 0$ and $b = 1$, both Z_i and W_i follow the standard normal distribution and they don't depend on the sample size n_i.

The two EWMA statistics, based on Z_i and W_i, are given by

$$U_i = (1-\lambda)U_{i-1} + \lambda Z_i \qquad i = 1, 2, \ldots \tag{3.3}$$

$$V_i = (1-\lambda)V_{i-1} + \lambda W_i \qquad i = 1, 2, \ldots \tag{3.4}$$

with Z_0 and W_0 as the respective starting values, $0 < \lambda \leq 1$.

Because U_i and V_i follow the same distribution, a new statistic for the single control chart can be defined as

$$M_i = max\{|U_i|, |V_i|\} \tag{3.5}$$

Notice that M_i is the maximum of the absolute values of the two EWMA statistics. It is natural to name this new chart the Max-EWMA chart. Construction of a Max-EWMA chart involves computing the value of $M_i, i = 1, 2, \ldots,$ and plotting these points on a control chart. The statistic M_i will be observed over time and its values can indicate the state of a process. A large value of M_i means that the process mean has drifted away from μ_0 and/or the process variability has changed. On the other hand, a small value of M_i implies the process mean and variability remain close to their nominal values respectively. Specifically, each M_i is only compared against an UCL since M_i is a non-negative statistic. A value of M_i that is greater than UCL would cause an out-of-control signal. Otherwise, it indicates that the process is in-control.

3.3 Derivation of the UCL

When $a = 0$ and $b = 1$, Z_i and W_i independently follow the standard normal distribution $N(0, 1)$. Assuming that $U_0 = V_0 = 0$, U_i and V_i can be rewritten as:

$$U_i = \lambda \sum_{j=1}^{i-1} (1-\lambda)^j Z_{i-j} \tag{3.6}$$

$$V_i = \lambda \sum_{j=1}^{i-1} (1-\lambda)^j W_{i-j} \tag{3.7}$$

Then, it can be seen that

$$U_i \sim N(0, \sigma^2_{U_i}) \tag{3.8}$$

and

$$V_i \sim N(0, \sigma^2_{V_i}) \tag{3.9}$$

where $\sigma^2_{U_i} = \sigma^2_{V_i} = \frac{\lambda[1-(1-\lambda)^{2i}]}{2-\lambda}$

Because U_i and V_i are independent, the in-control cumulative distribution function (CDF) of M_i is found to be

$$\begin{aligned} F(y; \sigma_{U_i}) &= P(M_i \leq y) \\ &= P(|U_i| \leq y, |V_i| \leq y) \\ &= P(|U_i| \leq y)P(|V_i| \leq y) \\ &= \left[2\Phi(\frac{y}{\sigma_{U_i}})\right]^2, \qquad y \geq 0 \end{aligned} \tag{3.10}$$

The corresponding probability density function (pdf) of M_i is the derivative of $F(y; \sigma_{U_i})$ given by

$$f(y; \sigma_{U_i}) = \frac{4}{\sigma^2_{U_i}}\phi(\frac{y}{\sigma_{U_i}})[2\Phi(\frac{y}{\sigma_{U_i}}) - 1] \tag{3.11}$$

Using numerical computation, the mean and variance of M_i are obtained as

$$\begin{aligned} E(M_i) &= \int_0^\infty y f(y; \sigma_{U_i}) dy \\ &= 1.128379\sigma_{U_i} \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} Var(M_i) &= \int_0^\infty y^2 f(y; \sigma_{U_i}) dy \\ &= 0.363381\sigma_{U_i}^2 \end{aligned} \tag{3.13}$$

respectively.

Therefore, the UCL is defined in a traditional way given by

$$\begin{aligned} UCL &= E(M_i) + L\sqrt{Var(M_i)} \\ &= \sigma_{U_i}(1.128379 + L\sqrt{0.363380}\ ) \\ &= \sqrt{\frac{\lambda[1-(1-\lambda)^{2i}]}{2-\lambda}}(1.128379 + 0.602810L) \end{aligned} \tag{3.14}$$

As i gets larger, the UCL will approach the steady-state value given by

$$UCL = \sqrt{\frac{\lambda}{2-\lambda}}(1.128379 + 0.602810L) \tag{3.15}$$

The design parameters of the Max-EWMA chart, L and λ, control the performance of the chart.

3.4 Design of a Max-EWMA chart

On the basis of the theoretical studies by Crowder [13] and Lucas and Saccucci [36], the criterion for designing an optimal Max-EWMA chart is to make the chart have the best ARL performance. For a desired in-control ARL, if one wants to detect a specified pair of changes in the process mean and variability quickly, the combination of (λ, L) for the optimal design provides the desired in-control ARL and minimizes the out-of-control ARL for the specified changes in the mean and variability. For the Max-EWMA chart. there is no direct way to compute the ARL. so each ARL value is obtained using 10.000 simulations.

For a given in-control ARL of 250. and $\lambda = 0.05(0.005)1$. the corresponding L is found such that the $(\lambda. L)$ combination gives the desired in-control ARL. Using 191 such combinations. each out-of-control ARL is calculated with respect to a pair of specified a and b. The optimal $(\lambda. L)$ combination. for a pair of specified a and b. is the one which leads to the smallest value of 191 out-of-control ARL's.

The approximate UCL in Equation (3.15) is the steady-state value that will be approached after the Max-EWMA chart has been running for several time periods. which is approximately taken as 5. It is more likely that the process will stay in the in-control state for some period of time before it drifts to the out-of-control state. Using the approximate UCL. Table 3.1 contains some representative optimal values of $(\lambda. L)$ and the corresponding out-of-control ARL's for n = 5 and for various changes in the process mean and the process variability. with the in-control ARL of 250 and the starting values $Z_0 = W_0 = 0$. For example. if one wants to have an in-control ARL of 250 and to guard against one quarter unit increase in the mean and one quarter unit increase in the variability. i.e.. $a = 0.25$. $b = 1.25$. the optimal parameter values are $\lambda = 0.185$ and L = 4.028.

As illustrated in Table 3.1. smaller values of λ are more effective in detecting small changes in the mean and/or the variability. Although Table 3.1 contains ARL's only for the desired in-control ARL of 250. the performance at other in-control ARL's is nearly the same as when the in-control ARL is 250.

MacGregor and Harris [37] investigated properties of the EWMA chart. and concluded that using the exact variance of the EWMA statistic leads to a natural fast initial response for an EWMA chart. This means that initial out-of-control conditions can be detected more quickly using the exact UCL in Equation (3.14).

Table 3.1: (λ, L) combinations and the corresponding ARL's for optimal Max-EWMA control schemes in a steady state and $n = 5$.

$ARL_0 = 250$									
		a							
b		0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
	λ	0.4900	0.4900	0.4900	0.5150	0.9500	1.0000	1.0000	1.0000
0.25	L	3.22112	3.22112	3.22112	3.22692	3.25645	3.25811	3.25811	3.25811
	ARL	2.27	2.27	2.27	2.16	1.13	1.00	1.00	1.00
	λ	0.2200	0.2200	0.2200	0.4250	0.8750	1.0000	1.0000	1.0000
0.50	L	3.07464	3.07464	3.07464	3.20021	3.2573	3.25811	3.25811	3.25811
	ARL	5.41	5.41	5.22	2.76	1.35	1.00	1.00	1.00
	λ	0.9900	0.0650	0.1600	0.4450	0.7350	0.9050	1.0000	1.0000
1.00	L	3.25728	2.65411	2.98588	3.20021	3.25629	3.25778	3.25811	3.25811
	ARL	249.93	24.57	8.58	2.94	1.56	1.09	1.01	1.00
	λ	0.1500	0.1850	0.2450	0.4700	0.7700	0.9050	0.9800	1.0000
1.25	L	2.96515	3.02835	3.09836	3.21647	3.28794	3.25778	3.25844	2.79345
	ARL	17.79	12.79	7.10	2.86	1.60	1.15	1.02	1.00
	λ	0.3600	0.4100	0.4650	0.6150	0.8050	0.8950	0.9950	1.0000
1.50	L	3.17218	3.19524	3.21481	3.24252	3.25728	3.25828	3.25745	3.2581
	ARL	6.28	5.69	4.51	2.52	1.59	1.19	1.05	1.01
	λ	0.7800	0.8050	0.7400	0.8100	0.8600	0.8750	0.9600	1.0000
2.00	L	3.25811	3.25728	3.25629	3.25794	3.25960	3.25728	3.25745	3.25811
	ARL	2.50	2.45	2.28	1.82	1.44	1.21	1.08	1.03
	λ	0.8550	0.8550	0.8500	0.8550	0.8650	0.8800	0.9500	0.9550
2.50	L	3.25944	3.25944	3.2590	3.25944	3.25861	3.25761	3.25645	3.25612
	ARL	1.84	1.81	1.75	1.56	1.36	1.20	1.11	1.05
	λ	0.8550	0.8550	0.8550	0.8550	0.8650	0.8800	0.8800	0.8800
3.00	L	3.25944	3.25944	3.25944	3.25944	3.25861	3.25761	3.25761	3.25761
	ARL	1.66	1.64	1.60	1.50	1.36	1.24	1.14	1.08

Table 3.2: (λ, L) combinations and the corresponding ARL's for optimal Max-EWMA control schemes in an initial state and $n = 5$.

		$ARL_0 = 250$							
		a							
b		0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
	λ	0.0500	0.0500	0.0500	0.0500	0.0500	1.0000	1.0000	1.0000
0.25	L	2.53632	2.53632	2.53632	2.53632	2.53632	3.25778	3.25778	3.25778
	ARL	1.58	1.58	1.58	1.58	1.00	1.00	1.00	1.00
	λ	0.0500	0.0500	0.0500	0.0500	0.0500	0.0600	1.0000	1.0000
0.50	L	2.53632	2.53632	2.53632	2.53632	2.53632	2.62010	3.25778	3.25778
	ARL	3.68	3.68	3.48	1.88	1.08	1.00	1.00	1.00
	λ	0.4950	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.1450
1.00	L	3.22261	2.53632	2.53632	2.53632	2.53632	2.53632	2.53632	2.95719
	ARL	249.84	18.58	5.97	2.05	1.26	1.04	1.00	1.00
	λ	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0650
1.25	L	2.53632	2.53632	2.53632	2.53632	2.53632	2.53632	2.53632	2.65411
	ARL	12.54	9.02	4.87	2.06	1.32	1.07	1.01	1.00
	λ	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
1.50	L	2.53632	2.53632	2.53632	2.53632	2.53632	2.53632	2.53632	2.53632
	ARL	4.41	4.03	3.20	1.90	1.34	1.10	1.02	1.00
	λ	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
2.00	L	2.53632	2.53632	2.53632	2.53632	2.53632	2.53632	2.53632	2.53632
	ARL	1.94	1.88	1.78	1.48	1.25	1.11	1.04	1.01
	λ	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
2.50	L	2.53632	2.53632	2.53632	2.53632	2.53632	2.53632	2.53632	2.53632
	ARL	1.46	1.44	1.39	1.28	1.17	1.09	1.04	1.02
	λ	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
3.00	L	2.53632	2.53632	2.53632	2.53632	2.53632	2.53632	2.53632	2.53632
	ARL	1.27	1.26	1.24	1.18	1.11	1.07	1.04	1.02

The ARL performance of the Max-EWMA chart agrees with MacGregor and Harris' conclusion. Using the exact UCL, Table 3.2 also contains some representative optimal values of (λ, L) and the corresponding out-of-control ARL's for n = 5 and for various changes in the process mean and the process variability, with the in-control ARL of 250 and the starting values $Z_0 = W_0 = 0$. It shows that all the ARL's in Table 3.2 are smaller than or equal to the corresponding ones in Table 3.1. An interesting phenomenon is that 0.05, the smallest value for λ in the chosen set, is the optimal value for λ even if the changes in the process mean and/or the variability are large. This is because, for a small λ value, the exact UCL is much smaller than the approximate one during the initial stage. In the initial stage, using the exact UCL will improve the performance of the Max-EWMA chart in detecting an initial out-of-control condition.

To detect small to moderate changes in the mean and the variability, the recommended λ values are in the range 0.05 to 0.30 because using smaller λ values can detect smaller changes. When using small λ values, occurrence of an inertia problem is the worst state for the EWMA-type control charts. For example, in the worst-case situation, M_i will be very near the UCL when a large change in the other direction occurs, resulting in a slow reaction to the large change if a small λ value is used. To guard against this problem, one can simultaneously use an additional Max-EWMA chart with $\lambda = 1$. Montgomery [38] and Lowry [34] discussed the inertia problem associated with the EWMA-type control charts.

Notice that when $\lambda = 1$, this particular Max-EWMA chart is equivalent to the Max chart which is a useful alternative to the common practice of using $\bar{X}$ and S (or R) charts.

For given in-control ARL's of 185 and 250, and for some commonly used λ values, the corresponding L is found such that the (λ, L) combination gives the desired in-control ARL's when using the approximate UCL and the starting values $Z_0 = W_0 = 0$. Table 3.3 lists these (λ, L) combinations.

Table 3.3: (λ, L) combinations for Max-EWMA control schemes in a steady state when sample size n = 5.

$ARL_0 = 185$										
λ	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	0.80	1.00
L	2.24	2.60	2.78	2.86	2.93	2.97	3.03	3.06	3.10	3.10
$ARL_0 = 250$										
λ	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	0.80	1.00
L	2.46	2.79	2.96	3.04	3.10	3.14	3.19	3.22	3.26	3.26

3.5 Charting Procedure

The charting procedure of a Max-EWMA chart is similar to that of any other EWMA charts commonly used. The successive $M_i's$ are plotted on a chart versus the sample number or time. However, to identify the source and the direction of an assignable cause, several plotting characters must be used along with sample points.

The procedure can be briefly summarized in the following steps:

1. If μ_0 is unknown, substitute $\bar{\bar{X}}$ for μ_0. If σ_0 is unknown, substitute $\bar{S}/c_4$ (or $\bar{R}/d_2$) for σ_0 and $\bar{S^2}$ for σ_0^2.

2. For each sample, compute Z_i and W_i.

3. To detect specified changes of the process mean and variability in an initial stage, choose the optimal (λ, L) combination from Table 3.2. Calculate U_i and V_i with $Z_0 = W_0 = 0$ as starting values, and construct UCL according to Equation (3.14).

4. To detect specified changes of the process mean and variability in a steady-state. choose the optimal $(\lambda. L)$ combination from Table 3.1: if it is not apparent what changes in the process mean and the process variability should be guarded against. choose the desired $(\lambda. L)$ combination from Table 3.3. Calculate U_i and V_i with $Z_0 = W_0 = 0$ as starting values. and construct UCL according to Equation (3.15).

5. Compute M_i and compare it with the UCL's.

6. Plot a sample point against the sample number i when $M_i \leq UCL$.

7. Plot a plotting character against the sample number i when $M_i > UCL$. For the case of only $|U_i| > UCL$. plot "m+" if $U_i > 0$ and plot "m-" if $U_i < 0$: For the case of only $|V_i| > UCL$. plot "v+" if $V_i > 0$. and plot "v-" if $V_i < 0$: For the case of both $|U_i| > UCL$ and $|V_i| > UCL$. plot " m+v+" if $U_i > 0$ and $V_i > 0$. plot "m+v-" if $U_i > 0$ and $V_i < 0$: plot "m-v+" if $U_i < 0$ and $V_i > 0$: plot "m-v-" if $U_i < 0$ and $V_i < 0$.

8. Investigate the cause(s) associated with each out-of-control signal.

3.6 An Illustrative Example

This example is taken from DeVor. Chang and Sutherland [16]. where data from the first 35 samples of size five were collected every half an hour. The measurements represent the inside diameter of cylinder bores in an engine block and are made to 1/10000 of an inch. such as 3.5205. 3.5202. 3.5204. .... For simplicity. the last three digits in the measurements are given in Table 3.4.

Table 3.4: Cylinder diameter data.

Sample i	X_1	X_2	X_3	X_4	X_5	Sample i	X_1	X_2	X_3	X_4	X_5
1	205	202	204	207	205	19	207	206	194	197	201
2	202	196	201	198	202	20	200	204	198	199	199
3	201	202	199	197	196	21	203	200	204	199	200
4	205	203	196	201	197	22	196	203	197	201	194
5	199	196	201	200	195	23	197	199	203	200	196
6	203	198	192	217	196	24	201	197	196	199	207
7	202	202	198	203	202	25	204	196	201	199	197
8	197	196	196	200	204	26	206	206	199	200	203
9	199	200	204	196	202	27	204	203	199	199	197
10	202	196	204	195	197	28	199	201	201	194	200
11	205	204	202	208	205	29	201	196	197	204	200
12	200	201	199	200	201	30	203	206	201	196	201
13	205	196	201	197	198	31	203	197	199	197	201
14	202	199	200	198	200	32	197	194	199	200	199
15	200	200	201	205	201	33	200	201	200	197	200
16	201	187	209	202	200	34	199	199	201	201	201
17	202	202	204	198	203	35	200	204	197	197	197
18	201	198	204	201	201						

Suppose that based on past experience an operator wanted to guard against the changes a = 1.50 and b = 1.50 with in-control ARL = 250. To use the Max-EWMA chart to monitor the cylinder production process, μ_0 is estimated by $\bar{\bar{X}}$ = 200.24 and σ_0 is estimated by $\bar{S}/c_4$ = 3.30. Using these estimates, the first Max-EWMA chart, consisting of the first five points for the initial stage with λ = 0.05 and L = 2.536 and the other thirty points for the steady-state stage with λ = 0.805 and L = 3.258, is shown in Figure 3.1. As indicated in Figure 3.1, there are three points above the UCL. U_1, V_6 and V_{16} are respectively greater than UCL. Sample 1 is related to an increased shift in the process mean while sample 6 and sample 16 are related to increased changes in the process variability. According to DeVor, Chang and Sutherland [16], sample 1 occurred at 8:00 a.m., corresponding roughly to the startup of the production line in the morning when the machine was cold. An investigation reveals that sample 6 and sample 16 corresponded to the time when the regular operator was absent. and a less-experienced relief operator was in charge of the production line. When these three samples are excluded. new estimates are obtained as $\bar{\bar{X}} = 200.10$ and $\bar{S}/c_4 = 2.96$. To guard against the changes $a = 1.50$ and $b = 1.00$. the optimal values are $\lambda = 0.735$ and $L = 3.256$ for the steady-state stage and the second chart is plotted in Figure 3.2. As seen from the plot. one point (sample 11 in the original data set) is found to be above the UCL. U_{11} is greater than UCL and it is related to an increased shift in the process mean. Sample 11 was produced at 1:00 p.m.. corresponding roughly to the startup of the production line directly after the lunch hour. when the machine was shut down for tool changing. Once the machine warmed up. in about 10 minutes the problem seems to disappear. When this sample is removed. the two estimates are given by $\bar{\bar{X}} = 199.94$ and $\bar{S}/c_4 = 2.98$. To detect small changes with $a = 0.50$ and $b = 1.25$. the optimal value of $(\lambda. L)$ is $(0.245. 3.098)$. The display of the third chart in Figure 3.3 shows there is no out-of-control signal.

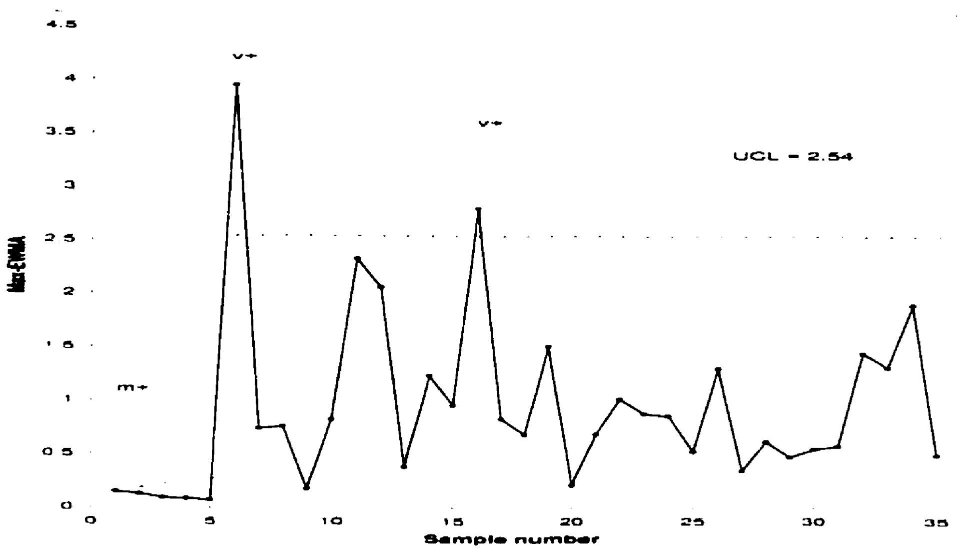

Figure 3.1: The first Max-EWMA chart for the cylinder diameter data

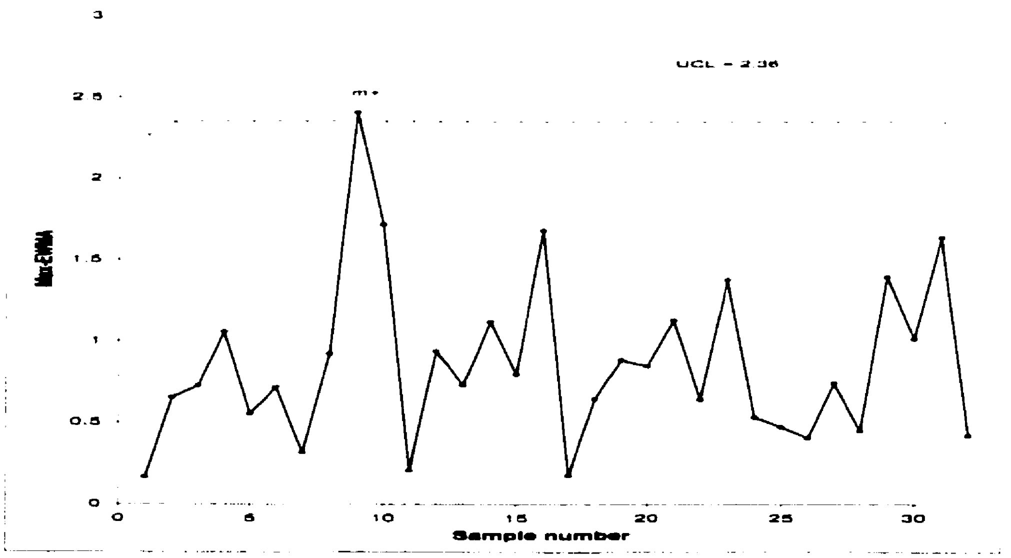

Figure 3.2: The second Max-EWMA chart for the cylinder diameter data

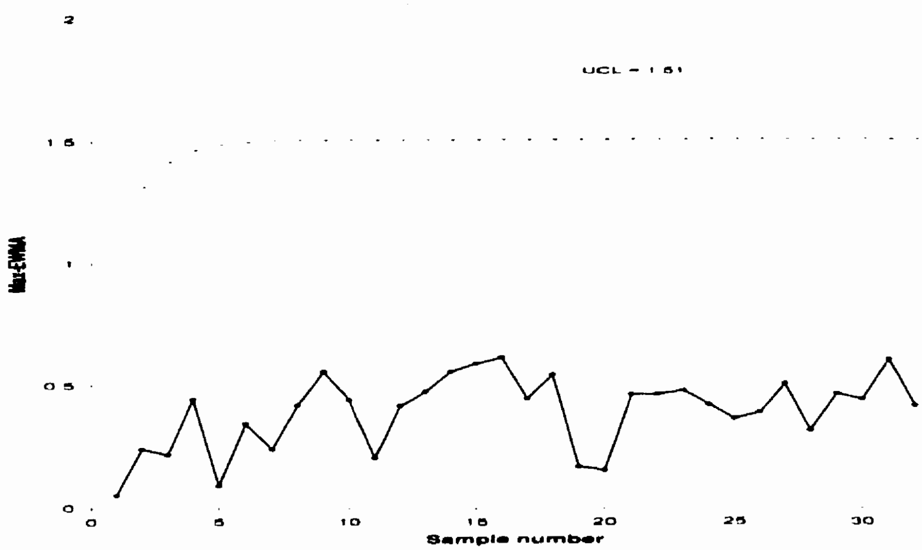

Figure 3.3: The third Max-EWMA chart for the cylinder diameter data

CHAPTER 4

The SS-EWMA Chart

4.1 Introduction

In addition to the efficiency of a control chart. it is important. especially in a job-shop manufacturing environment. to make the charting procedure easy to implement. Chao and Cheng [5] developed a single control chart. the semicircle chart. which is simple to use and easy to understand. One of the most impressive features of the semicircle chart is that it is easy to attribute an out-of-control signal to the cause of the mean shift or/and variability change.

Combining this thought with the EWMA technique. a new single control chart. SS-EWMA Chart. is proposed. The properties of this chart is similar to those of the Max-EWMA chart. but it has an added advantage of charting more easily in practical application. which allows for greater flexibility than the usual approach. As done in Chapter 3. designs are also made using the optimal values of λ and L. and the implementation of the new chart is illustrated through an example.

4.2 The New Control Chart

Under the same assumptions as in Chapter 3, the formulas for Z_i. W_i. U_i and V_i in this chapter are defined in the same way as given in Chapter 3.

Based on (3.3) and (3.4), the new statistic for this single chart is defined as

$$SS_i = U_i^2 + V_i^2 \qquad i = 1, 2, \ldots \tag{4.1}$$

A large value of SS_i results from a shift in the process mean and/or a change in the process variability. otherwise, the value of SS_i will be small and the process is in-control. Because the statistic SS_i for this new chart is the sum of squares of two EWMA statistics, it is natural to name this chart as SS-EWMA chart , with only an UCL needed due to the non-negative nature of SS_i.

Because $\frac{U_i}{\sigma_{U_i}}$ and $\frac{V_i}{\sigma_{U_i}}$ independently follow the identical standard normal distribution, $N(0, 1)$, it is obvious that

$$\frac{SS_i}{\sigma_{U_i}^2} = \frac{U_i^2}{\sigma_{U_i}^2} + \frac{V_i^2}{\sigma_{U_i}^2} \sim \chi_2^2 \tag{4.2}$$

Hence,

$$E(SS_i) = 2\sigma_{U_i}^2 \tag{4.3}$$

$$Var(SS_i) = 4\sigma_{U_i}^4 \tag{4.4}$$

Therefore, the UCL is given by

$$\begin{aligned} UCL &= E(SS_i) + L\sqrt{Var(SS_i)} \\ &= 2\sigma_{U_i}^2(1 + L) \\ &= \frac{2\lambda[1 - (1 - \lambda)^{2i}]}{2 - \lambda}(1 + L) \end{aligned} \tag{4.5}$$

As i gets larger, the UCL approaches

$$UCL = \frac{2\lambda}{2 - \lambda}(1 + L) \tag{4.6}$$

where L and λ, the design parameters of SS-EWMA Chart, can control the performance of the chart.

4.3 Design of a SS-EWMA Chart

For the SS-EWMA Chart. the same design strategy proposed for the Max-EWMA chart can be used to find the optimal $(\lambda. L)$ combination with respect to the desired in-control ARL and specified changes in the process mean and variability. Because there is still no direct way to compute the ARL. each ARL value is obtained using 10.000 simulations.

For a given in control ARL of 250 and $\lambda = 0.05(0.005)1$. there are 191 $(\lambda. L)$ combinations and the optimal one leads to the smllest out-of-control ARL. Using the approximate UCL and the exact UCL. Tables 4.1 and 4.2 respectively contain representative optimal values of $(\lambda. L)$ and the corresponding out-of-control ARL's for n = 5 and for changes in the process mean and the process variability. with the starting values $Z_0 = 0$ and $W_0 = 0$.

In Table 4.1. the results. similar to the results obtained in Table 3.1. suggest that smaller values of λ are more effective in detecting small changes in the mean and/or the variability. Again similar to the results obtained in Table 3.2. the results in Table 4.2 show that using exact UCL. the smallest λ in the chosen set is mostly likely to be the optimal value even if the changes within the process are large. It is interesting to see that all the ARL's in Table 4.2 are smaller or equal to the corresponding ones in Table 4.1.

Table 4.1: (λ, L) combinations and the corresponding ARL's for optimal SS-EWMA control schemes in a steady state and $n = 5$.

		$ARL_0 = 250$							
		a							
b		0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
	λ	0.430	0.460	0.510	0.840	1.000	1.000	1.000	1.000
0.25	L	4.428	4.441	4.465	4.522	4.528	4.528	4.528	4.528
	ARL	2.51	2.42	2.19	1.48	1.00	1.00	1.00	1.00
	λ	0.195	0.205	0.265	0.485	0.905	1.000	1.000	1.000
0.50	L	4.074	4.103	4.240	4.460	4.523	4.528	4.528	4.528
	ARL	5.96	5.37	4.17	2.32	1.26	1.00	1.00	1.00
	λ	0.090	0.060	0.170	0.415	0.760	0.925	0.975	1.000
1.00	L	3.510	3.141	3.991	4.415	4.523	4.524	4.528	4.528
	ARL	249.91	24.40	8.83	3.07	1.62	1.11	1.01	1.00
	λ	0.175	0.190	0.265	0.555	0.815	0.905	1.000	1.000
1.25	L	4.008	4.060	4.240	4.482	4.517	4.523	4.528	4.528
	ARL	17.11	11.67	6.58	2.78	1.59	1.15	1.02	1.00
	λ	0.415	0.415	0.460	0.690	0.795	0.925	0.985	1.000
1.50	L	4.415	4.415	4.441	4.516	4.519	4.524	4.528	4.528
	ARL	5.93	5.26	4.08	2.33	1.52	1.17	1.04	1.01
	λ	0.780	0.770	0.755	0.845	0.895	0.940	0.950	0.960
2.00	L	4.521	4.522	4.522	4.523	4.524	4.527	4.527	4.527
	ARL	2.17	2.12	1.98	1.62	1.34	1.16	1.06	1.02
	λ	0.875	0.900	0.915	0.915	0.900	0.910	0.950	0.960
2.50	L	4.524	4.523	4.522	4.522	4.523	4.524	4.525	4.526
	ARL	1.45	1.43	1.40	1.30	1.20	1.12	1.06	1.02
	λ	0.950	0.930	0.955	0.920	0.960	0.975	0.960	0.995
3.00	L	4.524	4.525	4.526	4.523	4.526	4.528	4.526	4.528
	ARL	1.21	1.20	1.20	1.16	1.12	1.08	1.05	1.03

Table 4.2: (λ, L) combinations and the corresponding ARL's for optimal SS-EWMA control schemes in an initial state and $n = 5$.

$ARL_0 = 250$									
		a							
b		0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
	λ	0.050	0.050	0.050	0.050	0.105	1.000	1.000	1.000
0.25	L	3.105	3.105	3.105	3.105	3.698	4.529	4.529	4.529
	ARL	1.75	1.70	1.54	1.07	1.00	1.00	1.00	1.00
	λ	0.050	0.050	0.050	0.050	0.050	0.065	1.000	1.000
0.50	L	3.105	3.105	3.105	3.105	3.105	3.322	4.529	4.529
	ARL	4.06	3.64	2.80	1.60	1.05	1.00	1.00	1.00
	λ	0.205	0.050	0.050	0.050	0.050	0.050	0.050	0.070
1.00	L	4.126	3.105	3.105	3.105	3.105	3.105	3.105	3.382
	ARL	249.92	18.85	6.11	2.09	1.28	1.04	1.00	1.00
	λ	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.090
1.25	L	3.105	3.105	3.105	3.105	3.105	3.105	3.105	3.580
	ARL	12.11	8.30	4.54	2.00	1.31	1.07	1.01	1.00
	λ	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050
1.50	L	3.105	3.105	3.105	3.105	3.105	3.105	3.105	3.105
	ARL	4.20	3.78	2.93	1.79	1.29	1.09	1.02	1.00
	λ	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050
2.00	L	3.105	3.105	3.105	3.105	3.105	3.105	3.105	3.105
	ARL	1.78	1.74	1.64	1.40	1.21	1.09	1.03	1.01
	λ	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050
2.50	L	3.105	3.105	3.105	3.105	3.105	3.105	3.105	3.105
	ARL	1.30	1.29	1.27	1.20	1.13	1.07	1.04	1.01
	λ	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050
3.00	L	3.105	3.105	3.105	3.105	3.105	3.105	3.105	3.105
	ARL	1.14	1.14	1.13	1.11	1.08	1.05	1.03	1.02

Noting that small λ values are more effective in detecting small changes. small λ values in the range 0.05 to 0.30 can be used to detect small to moderate changes in the mean and/or variability. When using small λ values, one can simultaneously use an additional SS-EWMA chart with $\lambda = 1$ to guard against a possible inertia problem.

Table 4.3 lists some commonly used (λ, L) combinations using the approximate UCL and the starting values $Z_0 = W_0 = 0$.

Table 4.3: (λ, L) combinations for SS-EWMA control schemes in a steady-state when sample size n = 5.

					$ARL_0 = 185$					
λ	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	0.80	1.00
L	2.58	3.23	3.55	3.74	3.87	3.96	4.08	4.14	4.21	4.22
					$ARL_0 = 250$					
λ	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	0.80	1.00
L	2.96	3.60	3.91	4.09	4.22	4.30	4.40	4.47	4.52	4.53

4.4 Charting Procedure

The charting procedure of a SS-EWMA chart differs from most of the EWMA charts commonly used. Instead of a summary statistic SS_i, successive pairs of (U_i, V_i)'s are plotted on a chart. The position of a point (U_i, V_i) on the plane can directly indicate the source and the direction of an assignable cause.

The control region for the SS-EWMA chart is a circle because Equation (4.1) is a circle. The circle is centered at (0, 0) with radius $\sqrt{UCL}$ and each sample point is plotted with coordinates (U, V). The circular control region is useful in indicating the source and direction for a detected change. If a sample point is out of the circle, the process is likely out of control. It will indicate that the change is due to a shift in the process mean when the point deviates sufficiently from the V axis. It will indicate that the change is due to a change in the process variability when the point deviates sufficiently from the U axis. It will indicate that the change is due to a combination effect of both the process mean and the process variability when the point is close to either the line U + V = 0 or the line U - V = 0. For an out-of-control signal, it is also easy to identify the changing directions for the process mean and the process variability from the position of the sample point. For example, a point in the first quadrant indicates that both the process mean and the process variability have increased.

The procedure can be briefly summarized in the following steps:

1. If μ_0 is unknown, substitute $\bar{\bar{X}}$ for μ_0. If σ_0 is unknown, substitute $\bar{S}/c_4$ (or $\bar{R}/d_2$) for σ_0 and $\bar{S}^2$ for σ_0^2.

2. For each sample, compute Z_i and W_i.

3. In an initial stage, if one wants to quickly detect specified changes in the process mean and the process variability for the desired in-control ARL of 250, choose the optimal (λ, L) combination from Table 4.2. Let $U_i' = U_i\sqrt{\frac{2-\lambda}{\lambda[1-(1-\lambda)^{2i}]}}$ and $V_i' = V_i\sqrt{\frac{2-\lambda}{\lambda[1-(1-\lambda)^{2i}]}}$. To avoid drawing several concentric circles, compute U_i' and V_i', i=1,2,3,4,5, where $Z_0 = 0$ and $W_0 = 0$ are starting values. Compute $\sqrt{2(1+L)}$, which is the radius of the circular control region for the first five samples.

4. In a steady-state, if one wants to quickly detect specified changes in the process mean and the process variability for the desired in-control ARL of 250, choose the optimal (λ, L) combination from Table 4.1; if it is not apparent what changes in the process mean and the process variability should guard

against. choose the desired (λ, L) combination from Table 4.3. Compute $\sqrt{UCL}$ according to Equation (4.6).

5. In an initial stage. draw a circle centered at (0. 0) with radius $\sqrt{2(1+L)}$: in a steady-state. draw a circle centered at (0. 0) with radius $\sqrt{UCL}$.

6. In an initial stage. plot sample points (U_i', V_i'). i=1.2.3.4.5 : in a steady-state. plot sample points (U_i, V_i).

7. Check if any point is outside of its corresponding circle. For an out-of-control signal. identity the source and the direction according to the positon of the point on the chart. and indicate the source and the direction using plotting characters.

8. Investigate the cause(s) associated with each out-of-control signal.

4.5 An Example

For the data given in Table 3.4. suppose that based on past experience an operator wanted to guard against the changes $a = 2.00$ and $b = 1.50$. To use the SS-EWMA chart to monitor the cylinder production process. μ_0 is estimated by $\bar{\bar{X}} = 200.24$ and σ_0 is estimated by $\bar{S}/c_4 = 3.30$. Using these estimates for the initial stage. the first SS-EWMA chart with in-control ARL = 250. $\lambda = 0.05$ and L = 3.105 is shown in Figure 4.1. There is one point (sample 1) out of the circle.

As indicated in Figure 4.1. the point deviates far from the V axis and hence the change is related to the process mean. According to DeVor. Chang and Sutherland [16], sample 1 occurred at 8:00 a.m., corresponding roughly to the startup of the production line in the morning when the machine was cold. After sample 1 is removed, estimates are recalculated as $\bar{\bar{X}} = 200.12$ and $\bar{S}/c_4 = 3.35$.

As the process is already in a steady-state. the SS-EWMA chart is applied with in-control ARL = 250. $\lambda = 0.925$ and L = 4.524. The second chart is given in Figure 4.2. This time. two points (sample 6 and sample 16 in the original data set) are outside the circle. Since these two points are located far from the U axis. both of them are related to the process variability. An investigation reveals that sample 6 and sample 16 corresponded to the time when the regular operator was absent. and a relief operator. who was less experienced. was in charge of the production line. When these two samples are excluded. estimates are obtained as $\bar{\bar{X}} = 200.10$ and $\bar{S}/c_4 = 2.97$ and the third chart is plotted in Figure 4.3. As seen from the plot. one point (sample 11 in the original data set) is found to be outside the circle. and it is related to the process mean since it deviates sufficiently from the V axis. Sample 11 was produced at 1:00 p.m.. corresponding roughly to the startup of the production line directly after the lunch hour. when the machine was shut down for tool changing. Once the machine warmed up. in about 10 minutes. the problem seems to disappear. When this sample is further removed. the two estimates are given by $\bar{\bar{X}} = 199.95$ and $\bar{S}/c_4 = 2.99$. and the display of the fourth chart in Figure 5.4 shows there is no point falling outside the circle.

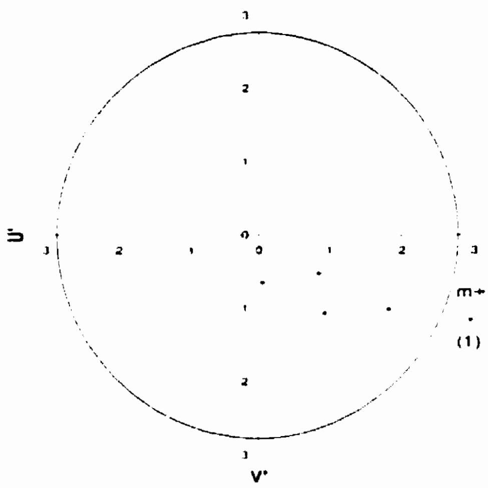

Figure 4.1: The first SS-EWMA chart for the cylinder diameter data

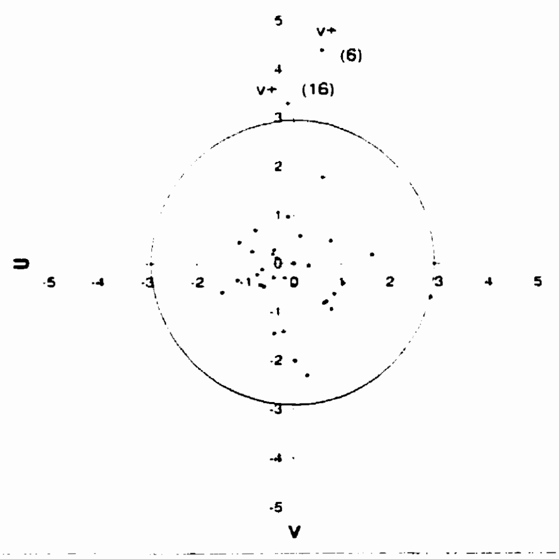

Figure 4.2: The second SS-EWMA chart for the cylinder diameter data

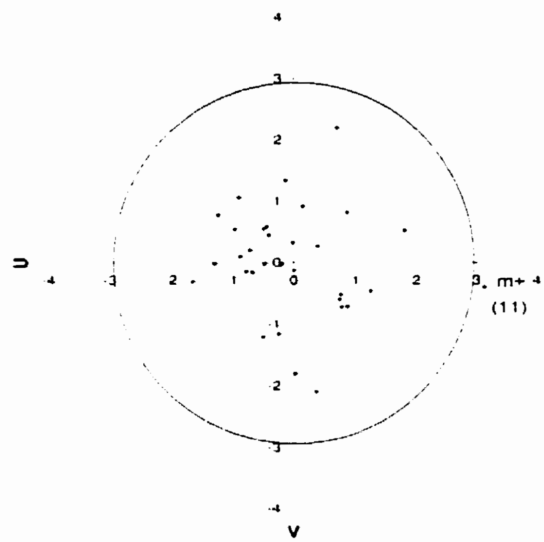

Figure 4.3: The third SS-EWMA chart for the cylinder diameter data

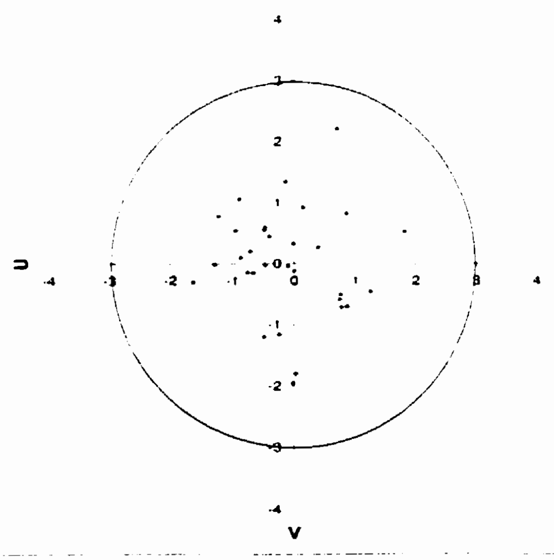

Figure 4.4: The fourth SS-EWMA chart for the cylinder diameter data

CHAPTER 5

The EWMA-Max Chart

5.1 Introduction

The Max Chart proposed by Chen and Cheng [9] is a single control chart which is essentially equivalent to a combination of the $\bar{X}$ chart and the S chart. It has the main advantage of simultaneously monitoring the process mean and the process variability. however. similar to Shewhart charts. this chart is not sensitive to small changes of a process.

To improve the sensitivity of the Max chart. the EWMA techniques are directly applied to the Max statistic. and a new single control chart. the EWMA-Max chart. is proposed. This chart can simultaneously monitor the process mean and the process variability. moreover. it is capable of detecting small changes in the mean and/or the variability. Another advantage is that an integral equation method could be used to compute ARL's of the EWMA-Max chart. The integral equation approach makes the ARL calculation much easier than the simulation approach does. Therefore. in addition to an optimal design of in-control ARL = 250. optimal designs of two other in-control ARL's are also given.

5.2 The New Control Chart

Under the same assumptions of Chapter 3. the formulas Z_i and W_i in this chapter are the same as those used in Chapter 3. The statistic for the Max chart is defined as

$$G_i = max\{|Z_i|, |W_i|\} \tag{5.1}$$

The EWMA statistic Y_i is computed from the sequence of G_i's given by

$$Y_i = (1-\lambda)Y_{i-1} + \lambda G_i \qquad i = 1, 2, \ldots \tag{5.2}$$

with Y_0 as the starting value.

Because the EWMA technique is applied to the Max statistic, this chart is named as EWMA-Max chart, with only a UCL needed due to the non-negative nature of Y_i.

Similar to the derivation of the UCL in Chapter 3, The UCL of this chart is given by

$$\begin{aligned} UCL &= E(Y_i) + L\sqrt{Var(Y_i)} \\ &= E(G_i) + L\sqrt{\frac{\lambda[1-(1-\lambda)^{2i}]}{2-\lambda}}\sqrt{Var(G_i)} \\ &= 1.128379 + 0.602810L\sqrt{\frac{\lambda[1-(1-\lambda)^{2i}]}{2-\lambda}} \end{aligned} \tag{5.3}$$

As i gets larger, the UCL will approach the steady-state value given by

$$UCL = 1.128379 + 0.602810L\sqrt{\frac{\lambda}{2-\lambda}} \tag{5.4}$$

5.3 The Integral Equation Approach for Computing ARL's

Three methods that are often used to compute ARL's of an EWMA chart are Markov chain method, integral equation method and simulation method. As was done in Chapter 3 and Chapter 4. simulation had to be used because it was impossible to use the first two approaches. Of the first two approaches. Champ and Rigdon [7] showed that the integral equation approach is preferable wherever an integral can be found. This is the case for the EWMA-Max chart. and the integral equation approach proposed by Crowder [13] is used to compute the ARL's. To do this. it is necessary to find the PDF of G_i. From the independence between Z_i and W_i. the CDF of G_i is found to be:

$$\begin{aligned} F_G(t; n_i, a, b) &= P(G_i \le t) \\ &= P(|Z_i| \le t, |W_i| \le t) \\ &= P(|Z_i| \le t)P(|W_i| \le t) \\ &= [\Phi(\frac{t}{b} - \frac{a}{b}\sqrt{n_i}) - \Phi(-\frac{t}{b} - \frac{a}{b}\sqrt{n_i})] \\ &\quad \cdot [H_{n_i-1}(\frac{H_{n_i-1}^{-1}(\Phi(t))}{b^2}) - H_{n_i-1}(\frac{H_{n_i-1}^{-1}(\Phi(-t))}{b^2})], t \ge 0 \end{aligned} \tag{5.5}$$

The pdf of G_i is the derivative of the CDF given by

$$\begin{aligned} f_G(t; n_i, a, b) &= \frac{1}{b}[\phi(\frac{t}{b} - \frac{a}{b}\sqrt{n_i}) + \phi(-\frac{t}{b} - \frac{a}{b}\sqrt{n_i})] \\ &\quad \cdot [H_{n_i-1}(\frac{H_{n_i-1}^{-1}(\Phi(t))}{b^2}) - H_{n_i-1}(\frac{H_{n_i-1}^{-1}(\Phi(-t))}{b^2}) \\ &\quad + \frac{1}{b^2}[\Phi(\frac{t}{b} - \frac{a}{b}\sqrt{n_i}) - \Phi(-\frac{t}{b} - \frac{a}{b}\sqrt{n_i})] \\ &\quad \cdot [\frac{h_{n_i-1}(\frac{H_{n_i-1}^{-1}(\Phi(t))}{b^2})\phi(t)}{h_{n_i-1}(H_{n_i-1}^{-1}(\Phi(t))} \\ &\quad + \frac{h_{n_i-1}(\frac{H_{n_i-1}^{-1}(\Phi(-t))}{b^2})\phi(-t)}{h_{n_i-1}(H_{n_i-1}^{-1}(\Phi(-t))}] \end{aligned} \tag{5.6}$$

Let $L(Y_0)$ denote the ARL with the starting point $Y_0 = E(Y_i)$. When samples have the same size, the integral equation for th ARL is a Fredholm integral equation of the second kind. and it is given by

$$L(Y_0) = 1 + \frac{1}{\lambda}\int_0^{UCL} L(t) f_G\left(\frac{t-(1-\lambda)Y_0}{\lambda}\right)dt \tag{5.7}$$

To find accurate of ARL's. 64-point Gaussian quadrature is used to numerically solve this integral equation. Notice that ARL can be denoted as a function of the starting point for the EWMA-Max chart.

5.4 Design of an EWMA-Max Chart

For the EWMA-Max chart. the design strategy is the same as that for the Max-EWMA chart. In a steady state. each ARL value is obtained using Crowder's method to solve Equation (5.7).

Given the respective in-control ARL's of 250. 370. and 500. Tables 5.1. 5.2 and 5.3 representatively contain some optimal values of $(\lambda. L)$ and the corresponding out-of-control ARL's using the approximate UCL for n=5 with the starting value $Y_0 = 1.128379$. Provided that the in-control ARL = 250. Table 5.4 gives the related results obtained from simulations when the exact UCL is used. Based on the results in these two tables. the same conclusions as seen in Chapters 3 and 4 can be drawn.

It is also recommended that λ values in the range 0.05 to 0.30 be used to detect small to moderate changes in the mean and the variability. and one more EWMA-Max chart with $\lambda = 1$ be used simultaneously to guard against the possible inertia problem. Notice that when $\lambda = 1$, the EWMA-Max chart is equivalent to the Max chart.

Table 5.5 lists some commonly used (λ, L) combinations using the approximate UCL and the starting values $Y_0 = 1.128379$.

Table 5.1: (λ. L) combinations and the corresponding ARL's for optimal EWMA-Max control schemes in a steady state when ARL_0 = 250 and n = 5.

		a							
b		0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
	λ	0.3500	0.3500	0.3500	0.3800	0.9150	0.9900	0.3100	0.0700
0.25	L	2.92782	2.92782	2.92782	2.96419	3.24743	3.25379	2.87517	2.12800
	ARL	2.54	2.54	2.54	2.32	1.13	1.00	1.00	1.00
	λ	0.0700	0.0700	0.0700	0.2750	0.7850	0.9900	0.9900	0.9850
0.50	L	2.12800	2.12800	2.12800	2.82531	3.21732	3.25379	3.25379	3.25636
	ARL	12.63	11.63	8.74	3.25	1.35	1.00	1.00	1.00
	λ	0.0550	0.0700	0.0700	0.3100	0.6500	0.9000	0.9800	0.9900
1.00	L	2.67344	2.12800	2.12800	2.87517	3.15977	3.24495	3.25342	3.25379
	ARL	249.96	70.94	15.81	3.46	1.57	1.09	1.01	1.00
	λ	0.0700	0.0700	0.1700	0.4300	0.6500	0.8550	0.9600	0.9900
1.25	L	2.12800	2.12800	2.60789	3.01085	3.15977	3.23623	3.25211	3.25379
	ARL	16.20	13.03	8.14	3.08	1.62	1.15	1.02	1.00
	λ	0.2850	0.2850	0.3100	0.4800	0.6700	0.8450	0.9450	0.9800
1.50	L	2.84036	2.84036	2.87517	3.05198	3.17031	3.23392	3.25091	3.25342
	ARL	5.92	5.45	4.41	2.54	1.59	1.19	1.04	1.01
	λ	0.5700	0.5700	0.5900	0.6500	0.7550	0.8450	0.9250	0.9600
2.00	L	3.11456	3.11456	3.12689	3.15977	3.20714	3.22392	3.24862	3.25221
	ARL	2.25	2.20	2.08	1.73	1.41	1.19	1.08	1.02
	λ	0.7250	0.7250	0.7350	0.7650	0.8200	0.8650	0.9250	0.9550
2.50	L	3.19584	3.19584	3.19988	3.21058	3.22769	3.23839	3.24862	3.25180
	ARL	1.49	1.48	1.45	1.35	1.24	1.14	1.07	1.03
	λ	0.8250	0.8250	0.8450	0.8550	0.8650	0.9000	0.9250	0.9550
3.00	L	3.22909	3.22909	3.23392	3.23623	3.23839	3.24495	3.24862	3.25180
	ARL	1.23	1.23	1.22	1.18	1.14	1.09	1.06	1.03

Table 5.2: (λ, L) combinations and the corresponding ARL's for optimal EWMA-Max control schemes in a steady state when $ARL_0 = 370$ and $n = 5$.

		a							
b		0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
	λ	0.3100	0.3100	0.3100	0.3650	0.8650	0.9950	1.0000	1.0000
0.25	L	3.05993	3.05933	3.05933	3.13304	3.42842	3.44425	3.44431	3.44431
	ARL	2.74	2.74	2.74	2.50	1.26	1.00	1.00	1.00
	λ	0.0700	0.0700	0.0700	0.2750	0.7250	0.9850	0.9950	0.9950
0.50	L	2.31427	2.31427	2.31427	3.01377	3.38468	3.44399	3.44425	3.44425
	ARL	14.22	13.03	9.68	3.55	1.47	1.00	1.00	1.00
	λ	0.0600	0.0700	0.0700	0.3100	0.6500	0.8950	0.9800	0.9950
1.00	L	3.95007	2.31427	2.31427	3.05993	3.34669	3.43441	3.44379	3.44425
	ARL	369.82	91.53	18.06	3.74	1.66	1.09	1.01	1.00
	λ	0.0700	0.0700	0.1700	0.3650	0.6500	0.8450	0.9550	0.9900
1.25	L	2.31427	2.31427	2.79609	3.13304	3.34669	3.42387	3.44220	3.44413
	ARL	18.52	14.73	9.03	3.32	1.70	1.18	1.03	1.00
	λ	0.2300	0.2750	0.3100	0.4800	0.6500	0.8200	0.9250	0.9800
1.50	L	2.93914	3.01377	3.05993	3.23778	3.34669	3.41757	3.43900	3.44379
	ARL	6.49	5.97	4.80	2.72	1.67	1.22	1.05	1.01
	λ	0.5350	0.5350	0.5700	0.6300	0.7250	0.8450	0.9000	0.9600
2.00	L	3.27865	3.27865	3.30054	3.33614	3.38468	3.42387	3.43523	3.44262
	ARL	2.38	2.33	2.20	1.81	1.46	1.22	1.09	1.03
	λ	0.6950	0.6950	0.7250	0.7550	0.8100	0.8650	0.9000	0.9550
2.50	L	3.37106	3.37106	3.38468	3.39632	3.41485	3.42842	3.43523	3.44220
	ARL	1.54	1.53	1.50	1.39	1.27	1.16	1.08	1.04
	λ	0.8200	0.8200	0.8200	0.8450	0.8650	0.9000	0.9250	0.9550
3.00	L	3.41757	3.41757	3.41757	3.42387	3.42842	3.43523	3.43900	3.44220
	ARL	1.26	1.25	1.24	1.20	1.16	1.11	1.06	1.04

Table 5.3: (λ, L) combinations and the corresponding ARL's for optimal EWMA-Max control schemes in a steady state when $ARL_0 = 500$ and $n = 5$.

		a							
b		0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
	λ	0.3100	0.3100	0.3100	0.3500	0.8450	0.9950	1.0000	1.0000
0.25	L	3.19607	3.19607	3.19607	3.25148	3.56561	3.58648	3.58654	3.58654
	ARL	2.89	2.89	2.89	2.64	1.41	1.00	1.00	1.00
	λ	0.0700	0.0700	0.0700	0.2300	0.6950	0.9800	0.9950	0.9950
0.50	L	2.46947	2.46947	2.46947	3.07745	3.51099	3.58605	3.58648	3.58648
	ARL	15.55	14.23	10.50	3.77	1.57	1.01	1.00	1.00
	λ	0.0550	0.0700	0.0700	0.2300	0.6950	0.9800	0.9950	0.9950
1.00	L	3.42779	2.50623	2.46947	3.19607	3.47435	3.57026	3.58605	3.58648
	ARL	499.56	110.28	19.72	3.97	1.73	1.13	1.01	1.00
	λ	0.0700	0.0700	0.1550	0.3100	0.6300	0.8450	0.9800	0.9950
1.25	L	2.46947	2.46947	2.89870	3.25148	3.47435	3.56561	3.58447	3.58637
	ARL	0.1700	0.2300	0.2850	0.4800	0.6500	0.8100	0.9250	0.9800
	λ	0.1700	0.2300	0.2850	0.4800	0.6500	0.8100	0.9250	0.9800
1.50	L	2.93686	3.07745	3.16284	3.37537	3.48484	3.55464	3.58120	3.58605
	ARL	6.94	6.37	5.09	2.85	1.73	1.24	1.06	1.01
	λ	0.5350	0.5350	0.5700	0.6300	0.6950	0.8200	0.8950	0.9550
2.00	L	3.41651	3.41651	3.43836	3.47435	3.51099	3.55926	3.57649	3.58447
	ARL	2.49	2.43	2.29	1.88	1.50	1.24	1.10	1.03
	λ	0.6700	0.6700	0.6950	0.7550	0.7950	0.8550	0.9000	0.9450
2.50	L	3.49711	3.49711	3.51099	3.53735	3.55163	3.56804	3.57733	3.58358
	ARL	1.59	1.57	1.54	1.43	1.29	1.17	1.09	1.04
	λ	0.7950	0.8100	0.8200	0.8450	0.8650	0.8950	0.9250	0.9550
3.00	L	3.55163	3.55646	3.55926	3.56561	3.57026	3.57649	3.58120	3.58447
	ARL	1.28	1.27	1.26	1.22	1.17	1.12	1.07	1.04

Table 5.4: (λ, L) combinations and the corresponding ARL's for optimal EWMA-Max control schemes in an initial state and $n = 5$.

		$ARL_0 = 250$							
		a							
b		0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
	λ	0.0500	0.0500	0.0500	0.0500	0.0500	1.0000	1.0000	1.0000
0.25	L	2.0572	2.0572	2.0572	2.0572	2.0572	3.2576	3.2576	3.2576
	ARL	1.82	1.82	1.82	1.82	1.82	1.00	1.00	1.00
	λ	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	1.0000	1.0000
0.50	L	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572	3.2576	3.2576
	ARL	9.76	8.82	6.33	2.25	1.15	1.00	1.00	1.00
	λ	1.0000	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
1.00	L	3.2576	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572
	ARL	249.93	66.23	12.51	2.53	1.35	1.05	1.00	1.00
	λ	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
1.25	L	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572
	ARL	12.98	10.21	6.02	2.36	1.41	1.10	1.01	1.00
	λ	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
1.50	L	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572
	ARL	4.42	4.10	3.35	2.04	1.40	1.13	1.03	1.00
	λ	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
2.00	L	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572
	ARL	1.94	1.91	1.81	1.55	1.31	1.15	1.06	1.02
	λ	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
2.50	L	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572
	ARL	1.51	1.50	1.47	1.37	1.25	1.15	1.08	1.04
	λ	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
3.00	L	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572	2.0572
	ARL	1.42	1.41	1.40	1.34	1.25	1.17	1.10	1.06

Table 5.5: (λ, L) combinations for EWMA-Max control schemes in a steady state when sample size n = 5.

					$ARL_0 = 250$					
λ	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	0.80	1.00
L	2.43	2.37	2.58	2.70	2.79	2.86	2.99	3.07	3.22	3.25
					$ARL_0 = 370$					
λ	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	0.80	1.00
L	2.64	2.57	2.78	2.90	2.98	3.05	3.18	3.25	3.41	3.44
					$ARL_0 = 500$					
λ	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	0.80	1.00
L	2.75	2.72	2.93	3.05	3.12	3.19	3.32	3.39	3.55	3.59

5.5 Charting Procedure

The charting procedure of an EWMA-Max chart is similar to that of a Max-EWMA chart. It can be summarized as follows:

1. If μ_0 is unknown, substitute $\bar{\bar{X}}$ for μ_0. If σ_0 is unknown, substitute $\bar{S}/c_4$ (or $\bar{R}/d_2$) for σ_0 and $\bar{S^2}$ for σ_0^2.

2. For each sample, compute Z_i and W_i.

3. To detect specified changes of the process mean and variability in an initial stage, choose the optimal (λ, L) combination from Table 5.4. Calculate Y_i with $Y_0 = 1.128379$. Set up UCL according to Equation (5.3) and compare Y_i with the UCL.

4. To detect specified changes of the process mean and variability in a steady-state, choose the optimal (λ, L) combination from one of Table 5.1, 5.2 and 5.3; if it is not apparent what changes in the process mean and the process variability should guard against, choose the desired (λ, L) combination from

Table 5.5. Calculate Y_i with $Y_0 = 1.128379$. Set up UCL according to Equation (5.4) and compare Y_i with the UCL.

5. Plot a sample point against the sample number i when $Y_i \leq UCL$.

6. Calculate $O_i = (1 - \lambda)Y_{i-1} + \lambda|Z_i|$ and $Q_i = (1 - \lambda)Y_{i-1} + \lambda|W_i|$.

7. Plot a plotting character against the sample number i when $Y_i > UCL$. For the case of only $O_i > UCL$. plot "m+" if $Z_i > 0$ and plot "m-" if $Z_i < 0$: For the case of only $Q_i > UCL$. Plot "v+" if $W_i > 0$. and plot "v-" if $W_i < 0$: For the case of both $O_i > UCL$ and $Q_i > UCL$. plot " m+v+" if $Z_i > 0$ and $W_i > 0$. plot "m+v-" if $Z_i > 0$ and $W_i < 0$: Plot "m-v+" if $Z_i < 0$ and $W_i > 0$: plot "m-v-" if $Z_i < 0$ and $W_i < 0$.

8. Investigate the cause(s) associated with each out-of-control signal.

5.6 An Example

For the data given in Table 3.4. suppose that based on the past experience an operator wanted to guard against the changes $a = 1.50$ and $b = 1.50$. To use the EWMA-Max chart to monitor the cylinder production process. μ_0 is estimated by $\bar{\bar{X}} = 200.24$ and σ_0 is estimated by $\bar{S}/c_4 = 3.30$. Using these estimates. the first EWMA-Max chart, consisting of the first five points for the initial stage with $\lambda = 0.05$ and L = 2.057 and the other thirty points for the steady-state stage with $\lambda = 0.67$ and L = 3.170, is shown in Figure 5.1. As indicated in Figure 5.1 , there are three points above the UCL. Since O_1 is greater than UCL and Z_1 is greater than 0, sample 1 is related to an increased shift in the process mean. However, sample 6 and sample 16 are related to increased changes in the process variability because both Q_6 and Q_{16} are greater than UCL and both W_6 and W_{16} are great than 0. When these three samples are excluded, estimates are obtained as $\bar{\bar{X}} = 200.10$ and $\bar{S}/c_4 = 2.96$. To guard against the same changes for the steady-state stage, the second chart is plotted in Figure 5.2. As seen from the plot, sample 11 in the original data set is found to be above the UCL. Since O_{11} is greater than UCL and Z_{11} is greater than 0, it is related to an increased shift in the process mean. When this sample is further removed, the two estimates are given by $\bar{\bar{X}} = 199.94$ and $\bar{S}/c_4 = 2.98$. The third chart in Figure 5.3 indicates that there is no point above the UCL.

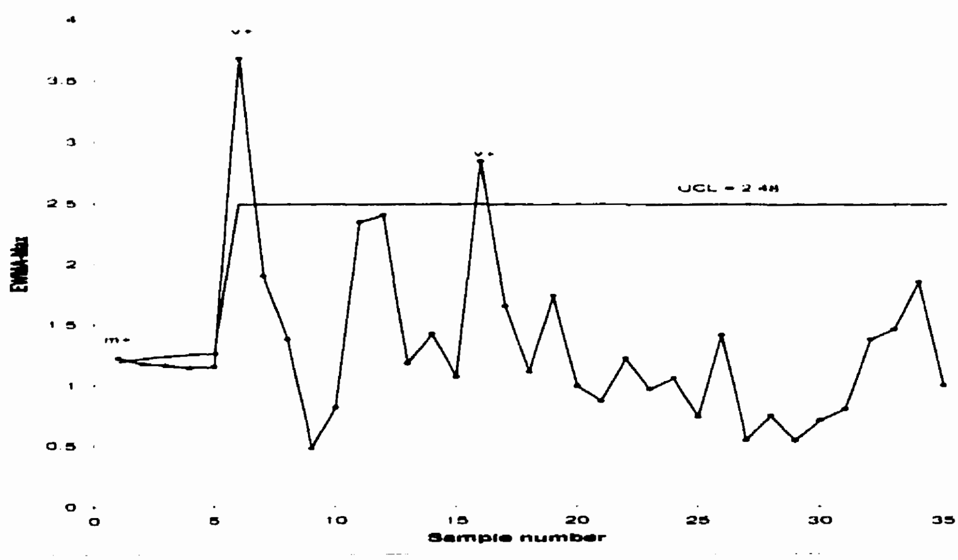

Figure 5.1: The first EWMA-Max chart for the cylinder diameter data

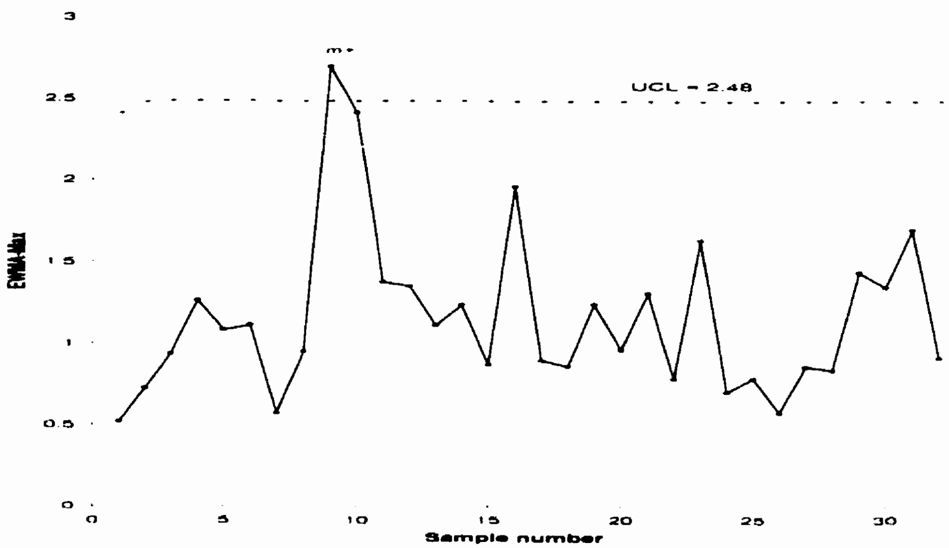

Figure 5.2: The second EWMA-Max chart for the cylinder diameter data

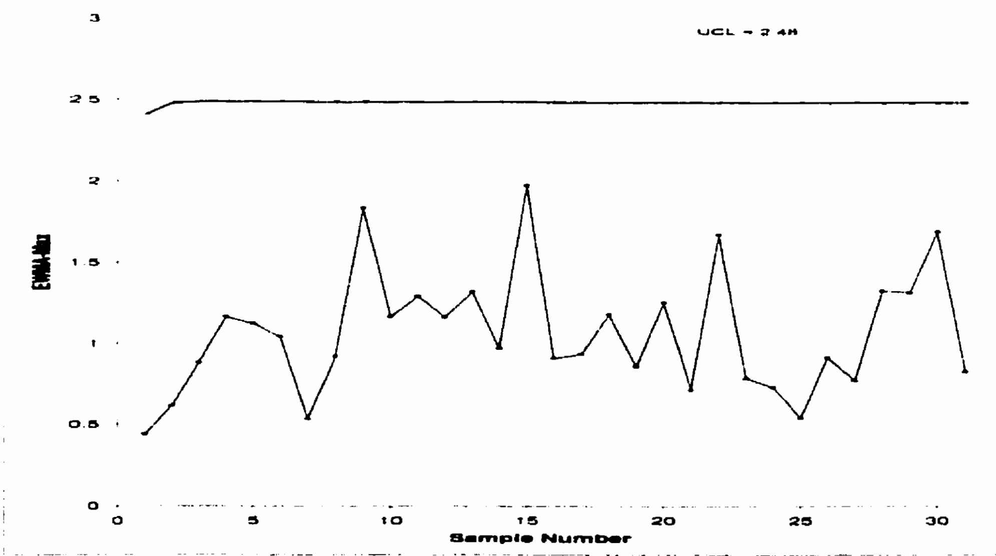

Figure 5.3: The third EWMA-Max chart for the cylinder diameter data

CHAPTER 6

The EWMA-SC Chart

6.1 Introduction

The semicircle (SC) chart proposed by Chao and Cheng [5] is another useful alternative to a combination of the $\bar{X}$ chart and S chart. As described in Chapter 2. this single control chart is essentially a 2-dimensional chart that is very easy to use. However. the SC chart is also insensitive to small changes of a process.

To make the SC chart sensitive to small changes. the EWMA techniques are directly applied to the statistic employed in the SC chart. and a new single control chart. the EWMA-SC chart. is proposed. With high sensitivity to small changes. this chart is capable of simultaneously monitoring the process mean and the increased process variability. Moreover. it preserves the good feature of the SC chart in charting procedure. from which the source of an out-of-control signal can easily be detected. Similar to the ARL calculation for the EWMA-Max Chart. Crowder's integral equation approach is used to provide three optimal designs. with in-control ARL' of 250. 370. and 500 respectively.

6.2 The New Control Chart

Under the same assumptions of Chapter 3. the statistic of the SC chart is defined as

$$T_i = (\bar{X}_i - \mu_0)^2 + \frac{n-1}{n} S_i^2 \qquad i = 1, 2, \dots \tag{6.1}$$

Let $T_i^* = \frac{n}{\sigma_0^2} T_i$. The EWMA statistic Q_i is computed from

$$Q_i = (1-\lambda)Q_{i-1} + \lambda T_i^* \qquad i = 1, 2, \ldots \tag{6.2}$$

with $Q_0 = n$.

Because $T_i^* \sim \chi_n^2$ when $a = 0$ and $b = 1$, it is easy to show that

$$\begin{aligned} E(Q_i) &= E(T_i^*) \\ &= n \end{aligned} \tag{6.3}$$

$$\begin{aligned} Var(Q_i) &= \frac{\lambda[1-(1-\lambda)^{2i}]}{2-\lambda} Var(T_i^*) \\ &= \frac{2n\lambda[1-(1-\lambda)^{2i}]}{2-\lambda} \end{aligned} \tag{6.4}$$

Equation (6.2) can also be rewritten as

$$Q_i = U_i + V_i + n \tag{6.5}$$

where

$$U_i = (1-\lambda)U_{i-1} + \lambda[\frac{n(\bar{X}_i - \mu_0)^2}{\sigma_0^2} - 1]$$

$$V_i = (1-\lambda)V_{i-1} + \lambda[(n-1)(\frac{S_i^2}{\sigma_0^2} - 1)]$$

with $U_0 = V_0 = 0$.

Because this EWMA chart is based on the statistic of the SC chart, this chart is named as EWMA-SC chart. Since Q_i is non-negative. only a UCL is needed. The UCL. corresponding to Equation (6.2), is given by

$$\begin{aligned} UCL_1 &= E(Q_i) + L\sqrt{Var(Q_i)} \\ &= n + L\sqrt{\frac{2n\lambda[1-(1-\lambda)^{2i}]}{2-\lambda}} \end{aligned} \tag{6.6}$$

The UCL. corresponding to Equation (6.5). is given by

$$UCL_2 = L\sqrt{\frac{2n\lambda[1-(1-\lambda)^{2i}]}{2-\lambda}} \tag{6.7}$$

As i gets larger. the UCL will approach the steady-state value. Equation (6.6) will become

$$UCL_1 = n + L\sqrt{\frac{2n\lambda}{2-\lambda}} \tag{6.8}$$

Equation (6.7) will become

$$UCL_2 = L\sqrt{\frac{2n\lambda}{2-\lambda}} \tag{6.9}$$

6.3 ARL Computation

To use the integral equation approach for ARL computation. the PDF of T_i^* has to be found. T_i^* can be decomposed as

$$T_i^* = b^2(T_{i_1}^* + T_{i_2}^*)$$

where

$$T^*_{i_1} = \frac{n}{b^2\sigma_0^2}[(\bar{X}_i - \mu) + a\sigma_0]^2$$
$$T^*_{i_2} = \frac{n-1}{b^2\sigma_0^2}S_i^2 \tag{6.10}$$

Because $T_i^* \sim \chi^2_{1,\delta^2}$, $T_i^* \sim \chi^2_{n-1}$, and they are independent, $T^*_{i_1} + T^*_{i_2} \sim \chi^2_{n,\delta^2}$, where $\delta^2 = n\frac{a^2}{b^2}$.

A Gaussian approximation for CDF of T_i^* (see Jensen and Solomon [31]) is given by

$$H_{n,\delta^2}(y) \approx \Phi\{\frac{1}{d}[(\frac{y}{n+\delta^2})^r - c]\} \tag{6.11}$$

where

$$r = \frac{1}{3}[1 + \frac{2\delta^4}{(n+2\delta^2)^2}]$$
$$c = 1 + \frac{r(r-1)(n+2\delta^2)}{(n+\delta^2)^2}$$

and

$$d = \frac{r\sqrt{2(n+2\delta^2)}}{n+\delta^2}$$

Differentiating (6.11) with respect to y, the Gaussian approximation for $h_{n,\delta^2}(y)$ is found to be

$$h_{n,\delta^2}(y) \approx \frac{r}{d(n+\delta^2)^2}\phi\{\frac{1}{d}[(\frac{y}{n+\delta^2})^r - c]\} \tag{6.12}$$

Hence, the pdf of T_i^* is given by

$$f_{T^*}(y) = \frac{1}{b^2} h_{n,\delta^2}(\frac{y}{b^2}), \qquad y \geq 0 \tag{6.13}$$

Therefore, the integral equation for the ARL of the EWMA-SC chart is found to be

$$L(Q_0) = 1 + \frac{1}{\lambda} \int_0^{UCL} L(y) f_{T^*}(\frac{y - (1-\lambda)Q_0}{\lambda}) dy \tag{6.14}$$

The ARL is solved from this integral equation using the 64 point Gaussian quadrature.

6.4 Design of an EWMA-SC Chart

For the EWMA-SC chart, the design strategy is the same as that for the Max-EWMA chart and the EWMA-Max chart. In a steady state, each ARL value is obtained using Crowder's method to solve Equation (6.14).

Given the respective in-control ARL's of 250, 370 and 500, Tables 6.1, 6.2, and 6.3 contain some representative optimal values of (λ, L) and the corresponding out-of-control ARL's using the approximate UCL for n = 5 with the starting value $Q_0 = 5$. Provided that the in-control ARL = 250, Table 6.4 gives the related results obtained from simulations when the exact UCL is used. Based on the results in these four tables, the same conclusions as seen in Chapters 3, 4 and 5 can be drawn.

Table 6.5 lists some commonly used (λ, L) combinations using the approximate UCL and the starting values $Q_0 = 5$.

Table 6.1: (λ. L) combinations and the corresponding ARL's for optimal EWMA-SC control schemes in a steady state when $ARL_0 = 250$ and$n = 5$.

		a							
b		0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
	λ	0.0500	0.0500	0.0500	0.0500	0.4100	0.7000	0.9200	0.9800
1.00	L	1.87996	1.87996	1.87996	1.87996	3.40589	3.75687	3.87189	3.88108
	ARL	250.00	73.46	22.60	5.16	2.11	1.25	1.03	1.00
	λ	0.0500	0.500	0.500	0.3000	0.4650	0.7000	0.9000	0.9700
1.25	L	1.87996	1.87996	1.87996	3.18646	3.49091	3.75687	3.86667	3.88021
	ARL	9.43	8.50	6.51	3.22	1.79	1.23	1.05	1.00
	λ	0.2250	0.2650	0.2650	0.4100	0.5700	0.7300	0.8850	0.9600
1.50	L	3.07814	3.10506	3.10506	3.40589	3.63012	3.78053	3.86209	3.87907
	ARL	4.17	3.96	3.44	2.30	1.56	1.20	1.06	1.01
	λ	0.4650	0.5000	0.5000	0.5900	0.7000	0.7950	0.8850	0.9500
2.00	L	3.49091	3.54163	3.54163	3.65165	3.75687	3.82226	3.86209	3.87766
	ARL	1.87	1.84	1.76	1.52	1.30	1.14	1.06	1.02
	λ	0.6550	0.6550	0.7000	0.7000	0.7950	0.8600	0.9100	0.9350
2.50	L	3.71688	3.71688	3.75687	3.82226	3.85326	3.85328	3.86948	3.87510
	ARL	1.34	1.33	1.31	1.24	1.16	1.10	1.05	1.02
	λ	0.7950	0.7950	0.7950	0.8250	0.8600	0.8850	0.9350	0.9350
3.00	L	3.82226	3.82226	3.82226	3.83783	385328	3.86209	3.87510	3.87510
	ARL	1.16	1.16	1.15	1.13	1.09	1.06	1.04	1.02

Table 6.2: (λ, L) combinations and the corresponding ARL's for optimal EWMA-SC control schemes in a steady state when $ARL_0 = 370$ and $n = 5$.

		a							
b		0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
	λ	0.0500	0.0500	0.0500	0.0500	0.3650	0.6550	0.9000	0.9800
1.00	L	1.94790	1.94790	1.94790	1.94790	3.57945	3.99578	4.15768	4.14281
	ARL	369.99	78.75	23.66	5.33	2.26	1.30	1.04	1.00
	λ	0.0500	0.0500	0.0500	0.2650	0.4650	0.6750	0.8850	0.9750
1.25	L	1.94790	1.94790	1.94790	3.34585	3.75513	4.01538	4.15283	4.17241
	ARL	9.80	8.80	6.73	3.47	1.89	1.27	1.06	1.01
	λ	0.0500	0.0500	0.2750	0.3650	0.5550	0.7000	0.8700	0.9500
1.50	L	1.94790	1.94790	3.37492	3.57945	3.88463	4.03916	4.14728	4.17076
	ARL	4.48	4.28	3.71	2.44	1.64	1.24	1.07	1.01
	λ	0.4650	0.4650	0.4850	0.5550	0.6550	0.7850	0.8800	0.9500
2.00	L	3.75513	3.75513	3.78798	3.88463	3.99578	4.10416	4.15108	4.16931
	ARL	1.96	1.93	1.84	1.58	1.33	1.16	1.07	1.03
	λ	0.6550	0.6550	0.6550	0.6750	0.7550	0.8450	0.8800	0.9800
2.50	L	3.99578	3.99578	3.99578	4.01538	4.08420	4.13634	4.15108	4.16554
	ARL	1.38	1.35	1.37	1.27	1.18	1.11	1.06	1.02
	λ	0.7750	0.7850	0.7850	0.8150	0.8450	0.8800	0.9300	0.9300
	L	4.09778	4.10416	4.10416	4.12147	4.13634	4.15108	4.16554	4.16554
3.00	ARL	1.18	1.17	1.17	1.14	1.10	1.07	1.04	1.02

Table 6.3: (λ. L) combinations and the corresponding ARL's for optimal EWMA-SC control schemes in a steady state when $ARL_0 = 500$ and $n = 5$.

		a							
b		0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
	λ	0.0750	0.0500	0.0500	0.0500	0.3250	0.6550	0.8950	0.9850
1.00	L	4.81295	1.98485	1.98485	1.98485	3.67776	4.20688	4.37768	4.39549
	ARL	499.99	80.96	24.08	5.42	2.37	1.34	1.05	1.00
	λ	0.0500	0.0500	0.0500	0.0500	0.4650	0.6550	0.8550	0.9750
1.25	L	1.98485	1.98485	1.98485	1.98485	3.95253	4.20688	4.36167	4.39474
	ARL	9.95	8.93	6.82	3.63	1.97	1.30	1.07	1.01
	λ	0.0500	0.0500	0.0500	0.3250	0.5000	0.6750	0.8450	0.9600
1.50	L	1.98485	1.98485	1.98485	3.67776	4.01069	4.22752	4.35674	4.39305
	ARL	4.54	4.34	3.85	2.56	1.69	1.26	1.08	1.02
	λ	0.4650	0.4650	0.4650	0.5700	0.6750	0.7750	0.8450	0.9300
2.00	L	3.95253	3.95253	3.95253	4.10935	4.22752	4.31610	4.35674	4.38762
	ARL	2.04	2.00	1.90	1.63	1.36	1.18	1.07	1.02
	λ	0.6300	0.6300	0.6550	0.6750	0.7500	0.8150	0.9000	0.9300
2.50	L	4.18027	4.18027	4.20688	4.22752	4.29857	4.34115	4.37936	4.38762
	ARL	1.41	1.40	1.38	1.29	1.20	1.12	1.06	1.03
	λ	0.7300	0.7750	0.7850	0.7850	0.8450	0.8450	0.9050	0.9300
3.00	L	4.28130	4.31610	4.32300	4.32300	4.35674	4.35674	4.38117	4.38762
	ARL	1.19	1.19	1.18	1.15	1.11	1.08	1.05	1.03

Table 6.4: (λ. L) combinations and the corresponding ARL's for optimal EWMA-SC control schemes in an initial state and $n = 5$.

		a							
b		0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
	λ	0.40	0.05	0.05	0.05	0.05	0.05	0.05	0.05
1.00	L	3.4054	2.1180	2.1180	2.1180	2.1180	2.1180	2.1180	2.1180
	ARL	249.94	66.30	21.21	3.61	1.61	1.12	1.01	1.00
	λ	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
1.25	L	2.1180	2.1180	2.1180	2.1180	2.1180	2.1180	2.1180	2.1180
	ARL	7.53	6.61	4.77	2.38	1.46	1.13	1.02	1.00
	λ	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
1.50	L	2.1180	2.1180	2.1180	2.1180	2.1180	2.1180	2.1180	2.1180
	ARL	3.07	2.92	2.57	1.82	1.35	1.12	1.03	1.00
	λ	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
2.00	L	2.1180	2.1180	2.1180	2.1180	2.1180	2.1180	2.1180	2.1180
	ARL	1.56	1.54	1.49	1.34	1.20	1.10	1.04	1.01
	λ	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
2.50	L	2.1180	2.1180	2.1180	2.1180	2.1180	2.1180	2.1180	2.1180
	ARL	1.23	1.22	1.21	1.17	1.11	1.06	1.03	1.01
	λ	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
3.00	L	2.1180	2.1180	2.1180	2.1180	2.1180	2.1180	2.1180	2.1180
	ARL	1.11	1.11	1.10	1.09	1.06	1.04	1.02	1.01

Values of λ in the range 0.05 to 0.30 are usually used to detect small to moderate changes in the mean and the variability, and an additional EWMA-SC chart with $\lambda = 1$ is simultaneously used to guard against the possible inertia problem. It is worth noting that when $\lambda = 1$, this particular EWMA-SC chart is equivalent to SC chart which is a useful alternative to the common practice of using the $\bar{X}$ and the S (or R) charts.

Table 6.5: (λ, L) combinations for EWMA-SC control schemes in a steady state when sample size n = 5.

					$ARL_0 = 250$					
λ	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	0.80	1.00
L	1.88	2.45	2.74	2.93	3.08	3.19	3.39	3.54	3.82	3.88
					$ARL_0 = 370$					
λ	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	0.80	1.00
L	1.95	2.65	2.98	3.17	3.32	3.44	3.65	3.81	4.11	4.17
					$ARL_0 = 500$					
λ	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	0.80	1.00
L	1.98	2.78	3.16	3.36	3.51	3.62	3.84	4.01	4.33	4.40

6.5 Charting Procedure

For the EWMA-SC chart there are two ways to plot the statistic: plotting Q_i against i, and plotting (U_i, V_i) on the two dimensional chart. Of the two ways, the latter is preferable because the source of an assignable cause can be identified directly from the location of the plotted sample point on the chart. On a U-V coordinate plane, the control region, $\{(U_i, V_i) : U_i + V_i \leq UCL_2\}$, consists of all the points on or below the line $U_i + V_i = UCL_2$. If a sample point is above the line, it will indicate that the change is due to a shift in the process mean when the point deviates sufficiently from V axis: it will indicate that the change is due to a change in the process variability when the point deviates sufficiently from U axis, and it will indicate that the change is due to a combination effect of both the process mean and variability when the point is close to one of the two lines: $U_i - V_i = 0$ or $U_i + V_i = 0$. For an out-of-control signal, it is also easy to identify the direction of a shift in the process mean from the position of the sample point. A point in the right half plane indicates that the process mean is increased, otherwise, the process mean is decreased.

The procedure can be briefly summarized in the following steps:

1. If μ_0 is unknown. substitute $\bar{\bar{X}}$ for μ_0. If σ_0 is unknown. substitute $\bar{S}/c_4$ (or $\bar{R}/d_2$) for σ_0 and $\bar{S^2}$ for σ_0^2.

2. For each sample. compute U_i and V_i with $U_0 = V_0 = 0$.

3. In an initial stage. if one wants to quickly detect specified changes in the process mean and the process variability ARL of 250. choose the optimal $(\lambda. L)$ combination from Table 6.4. Let $U_i' = U_i\sqrt{\frac{2-\lambda}{\lambda[1-(1-\lambda)^{2i}](2n)}}$ and $V_i' = V_i\sqrt{\frac{2-\lambda}{\lambda[1-(1-\lambda)^{2i}](2n)}}$. To avoid drawing several parallel lines. compute U_i' and V_i'. i=1.2.3.4.5. and plot them on U'-V' coordinate plane. Draw the line $U_i' + V_i' = L$ as the boundary of the control region.

4. To detect specified changes of the process mean and variability in a steady-state. choose the optimal $(\lambda. L)$ combination from one of Table 6.1. 6.2 and 6.3: if it is not apparent what changes in the process mean and the process variability should be guarded against. choose the desired $(\lambda. L)$ combination from Table 6.5. Plot U_i and V_i on U-V coordinate plane with the line $U+V = L\sqrt{\frac{2n\lambda}{2-\lambda}}$ as the boundary of the control region.

5. Check if any point is outside of its control region. For an out-of-control signal. identify the source of the signal the direction of a mean shift according to the location of the point on the chart, and indicate the source and the direction using plotting characters.

6. Investigate the cause(s) associated with each out-of-control signal.

6.6 An Example

For the data given in Table 3.1, suppose that, based on past experience, an operator wanted to guard against the changes $a = 1.50$ and $b = 1.50$. To use the EWMA-SC chart to monitor the cylinder production process, μ_0 is estimated by $\bar{\bar{X}} = 200.24$ and σ_0 is estimated by $\bar{S}/c_4 = 3.30$. Using these estimates, the first EWMA-SC chart, consisting of the first five points for the initial stage with $\lambda = 0.05$ and L = 2.118 and the other thirty points for the steady-state stage with $\lambda = 0.57$ and L = 3.630, is shown in Figure 6.1. As indicated in Figure 6.1, there is no point above the line $U' + V' = 2.12$ although the point of sample 1 is very close to the control bound. This is because the EWMA-SC chart is designed for detecting a mean shift accompanying increased variability, but decreased variability may affect the ability to detect an increased mean shift.

The second chart is given in Figure 6.2. As the process is already in a steady-state, the EWMA-SC chart is applied with in-control ARL = 250 , $\lambda = 0.57$ and L = 3.630. This time, two points are above the line $U + V = 7.25$. Since these two points are located far from the U axis, both of them are related to the process variability. When these two samples are excluded, estimates are obtained as $\bar{\bar{X}} = 200.22$ and $\bar{S}/c_4 = 2.93$. To guard against the same changes for the steady-state stage, the third chart is plotted in Figure 6.3 . As seen from the plot, sample 11 in the original data set is found to be above the line. It is related to the process mean since it deviates sufficiently from the V axis. When this sample is further removed, the two estimates are given by $\bar{\bar{X}} = 200.08$ and $\bar{S}/c_4 = 2.95$. The fourth chart in Figure 6.4 indicates that there is no point above the line.

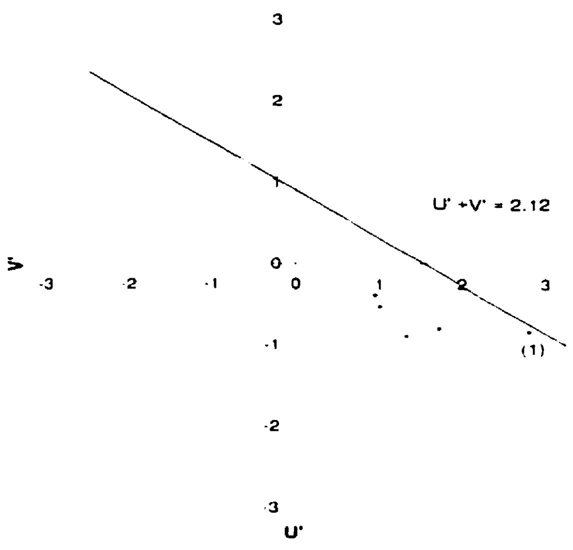

Figure 6.1: The first EWMA-SC chart for the cylinder diameter data

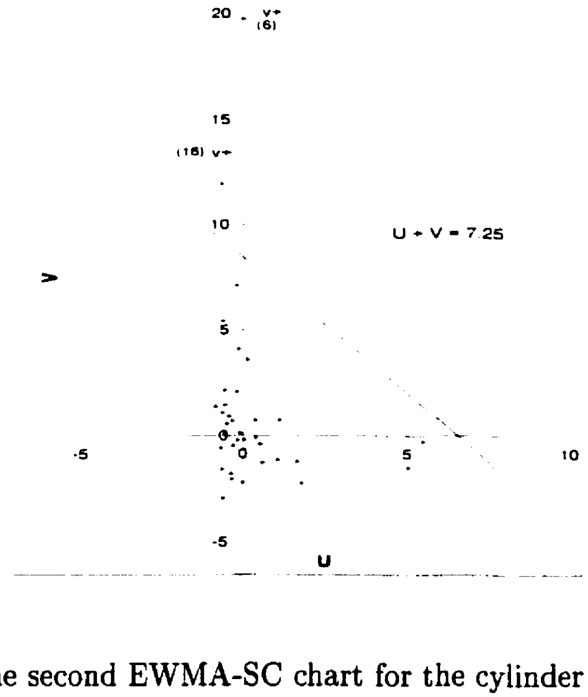

Figure 6.2: The second EWMA-SC chart for the cylinder diameter data

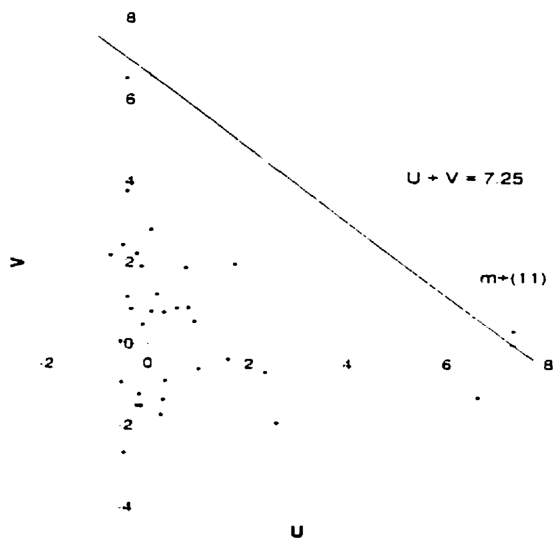

Figure 6.3: The third EWMA-SC chart for the cylinder diameter data

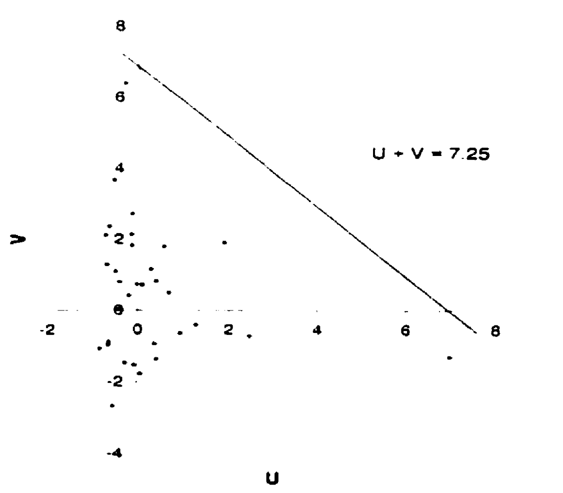

Figure 6.4: The fourth EWMA-SC chart for the cylinder diameter data

CHAPTER 7

Comparisons of Several Control Charts

7.1 Introduction

To assess the performance of a control chart. a common way is to evaluate the ARL characteristics of the control chart. The concept of ARL refers to the number of sample points that are plotted until an out-of-control signal is received. If performances of two control charts with the same in-control ARL are compared. the better chart is the one that has smaller out-of-control ARL.

Assuming that μ_0 and σ_0 are known. and sample sizes n_i are all equal to n. different control charts are compared under the same assumptions of Chapter 3. ARL comparisons are carried out among the four new charts. the combination of the two Shewhart charts and the combination of the two EWMA charts in the steady state. i.e.. all types of the EWMA charts are using the approximate control limits. The two new charts. the Max-EWMA chart and the SS-EWMA chart. demonstrate good overall ARL performances. In addition to the ARL comparisons. diagnostic abilities are further studied on the Max-EWMA chart. the SS-EWMA chart and the combination of the two standard EWMA charts in the steady-state.

7.2 The Two Combination Charts

7.2.1 The Combination of the Two Shewhart Charts

The two Shewhart charts, the $\bar{X}$ chart and the S chart, have been considered as the most important and useful on-line SPC techniques since Shewhart [51] introduced the control chart theory in the 1920's.

To monitor the process mean, a $\bar{X}$ chart has the following control limits:

$$UCL_{\bar{X}} = \mu_0 + \frac{Z_{\alpha/2}\sigma_0}{\sqrt{n}}$$
$$CL_{\bar{X}} = \mu_0$$
$$LCL_{\bar{X}} = \mu_0 - \frac{Z_{\alpha/2}\sigma_0}{\sqrt{n}}$$

Letting $\alpha = 0.002$ and $n = 5$, then $Z_{\alpha/2} = 3.090$ and this specific $\bar{X}$ chart has a Type I error probability 0.002 when the process is in control.

To monitor the process variability, an S chart with probability limits is used. A probability 0.001 is assigned to each tail and the control limits are given by

$$UCL_S = \sigma_0\sqrt{\frac{\chi^2_{n-1,\alpha/2}}{n-1}}$$
$$LCL_S = \sigma_0\sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n-1}}$$

Letting $\alpha = 0.002$ and $n = 5$, then $\chi^2_{n-1,\alpha/2} = 18.466$, $\chi^2_{n-1,1-\alpha/2} = 0.908$ and the type I error probability is also 0.002 when the process is in control.

The combination of the $\bar{X}$ chart and the S chart has a combined Type I error probability $1-(1-0.002)^2 \approx 0.004$, which is equivalent to have an in-control ARL of 250.

Let $p_{\bar{X}}$ be the probability of an out-of-control signal detected by the $\bar{X}$ chart. Let p_S be the probability of an out-of-control signal detected by the S chart. Let p be the probability of an out-of-control signal detected by the combination of the $\bar{X}$ chart and the S chart. For various changes in the process mean and/or the process variability, we have

$$p_{\bar{X}} = 2 - 2\Phi[\frac{1}{b}(3.090 - a\sqrt{5})]$$
$$p_S = 1 - H_4\left(\frac{18.466}{b^2}\right) + H_4\left(\frac{0.091}{b^2}\right)$$
$$p = 1 - (1 - p_{\bar{X}})(1 - p_S)$$

Because $\bar{X}_1, \bar{X}_2, \ldots$ are independent and so are $S_1, S_2, \ldots$, the ARL for the combination of the $\bar{X}$ chart and the S chart is $1/p$ with respect to a and b.

7.2.2 The Combination of the Two Standard EWMA Charts

EWMA charts are known to be effective in detecting small changes in the process mean and/or variability. Of the two EWMA charts employed in the combination, one is the usual EWMA $\bar{X}$ proposed by Robert [47], and the other is a modified EWMA $ln(S^2)$ chart in Crowder and Hamilton [14]. In the following discussion, it is assumed that the sample size n is equal to five.

To monitor the process mean, the EWMA $\bar{X}$ chart has the following control limits:

$$UCL_1 = \mu_1 + L_1\sqrt{\frac{\lambda_1}{2-\lambda_1}}\sigma_1$$
$$CL_1 = \mu_1$$
$$LCL_1 = \mu_1 - L_1\sqrt{\frac{\lambda_1}{2-\lambda_1}}\sigma_1$$

where λ_1 and L_1 are the parameters that control the performance of the EWMA $\bar{X}$ chart, $\mu_1 = \mu_0$ and $\sigma_1 = \frac{\sigma_0}{\sqrt{n}}$.

The plotting statistics are

$$Q_i = (1 - \lambda_1)Q_{i-1} + \lambda_1 \bar{X}_i \qquad i = 1, 2, \ldots$$

where $Q_0 = \mu_1$.

To monitor the process variability, the original EWMA $ln(S^2)$ chart is modified into a two-sided chart since the original one is primarily designed to detect an increase in the variability. For the modified EWMA $ln(S^2)$ chart, the control limits are

$$UCL_2 = \mu_2 + L_2\sqrt{\frac{\lambda_2}{2 - \lambda_2}}\sigma_2$$

$$CL_2 = \mu_2$$

$$LCL_2 = \mu_2 - L_2\sqrt{\frac{\lambda_2}{2 - \lambda_2}}\sigma_2$$

where λ_2 and L_2 are the parameters that control the performance of the EWMA $ln(S^2)$ chart. μ_2 is the approximation mean of $ln(S^2)$ given by

$$\mu_2 = ln(\sigma_0^2) - \frac{1}{n-1} - \frac{1}{3(n-1)^2} + \frac{2}{15(n-1)^5}$$

and σ_2^2 is the approximate variance of $ln(S^2)$ given by

$$\sigma_2^2 = \frac{2}{n-1} + \frac{2}{(n-1)^2} + \frac{4}{3(n-1)^3} - \frac{16}{15(n-1)^5}$$

The plotting statistics are

$$Y_i = (1 - \lambda_2)Y_{i-1} + \lambda_2 ln(S_i^2) \qquad i = 1, 2, \ldots$$

where $Y_0 = \mu_2$.

Based on the numerical evaluation of run-length distributions of EWMA charts. Crowder [13] concluded that. for the in-control large ARL cases. the run-length approximately follows a geometric distribution with parameter as the reciprocal of the in-control ARL. Because Q_i and Y_i are independent. the in-control ARL for the combination of the two EWMA charts can be obtained in the same way as that for the combination of the two Shewhart charts in term of the in-control ARL's of the two EWMA charts.

Given in-control ARL of 500 for both the EWMA $\bar{X}$ chart and the EWMA $ln(S^2)$ chart. Table 7.1 contains the optimal (λ_1, L_1) values for specified shifts in the process mean. the optimal (λ_2, L_2) values for specified changes in the process variability and the corresponding smallest ARL's. which are obtained from solving integral equations in the same way as that in Crowder [13] and Crowder and Hamilton [14].

Table 7.1: The optimal parameter values used for the EWMA $\bar{X}$ chart and the EWMA $ln(S^2)$ chart when $ARL_0 = 500$ and n = 5.

EWMA $\bar{X}$ Chart								
	a							
	0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
λ_1	0.655	0.055	0.160	0.430	0.765	0.940	0.995	1.000
L_1	3.084	2.645	2.920	3.060	3.088	3.090	3.090	3.090
ARL_1	500.0	24.4	9.6	3.0	1.6	1.1	1.0	1.0
EWMA $ln(S^2)$ Chart								
	b							
	0.25	0.50	1.00	1.25	1.50	2.00	2.50	3.00
λ_2	0.445	0.180	0.435	0.050	0.110	0.200	0.270	0.330
L_2	3.575	3.095	3.561	2.633	2.900	3.143	3.292	3.402
ARL_2	2.0	5.0	500.0	24.5	10.3	4.8	3.3	2.6

For the combination of the EWMA $\bar{X}$ chart and the EWMA $ln(S^2)$ chart.

the in-control ARL is given by

$$\begin{aligned} ARL_0 &\approx \frac{ARL_{01} ARL_{02}}{ARL_{01} + ARL_{02} - 1} \\ &= \frac{250000}{500 + 500 - 1} \\ &= 250 \end{aligned}$$

where ARL_0 is the in-control ARL for the combination chart. ARL_{01} is the in-control ARL for the EWMA $\bar{X}$ chart and ARL_{02} is the in-control ARL for the EWMA $ln(S^2)$ chart.

Because there is no direct way to compute the out-of-control ARL. each of the ARL value has to be estimated from 10.000 simulated run lengths.

7.3 ARL Comparisons

To compare the performance of the various control chart schemes on an equal footing. each scheme is calibrated so that the in-control ARL is approximately equal to 250. Since an EWMA-type chart is controlled by the $(\lambda. L)$ combination. there are many possible ARL's for an out-of-control condition and two approaches can be employed to make comparisons among the five EWMA-type charts. The first approach compares the best ARL performance of each EWMA-type chart with that of the other EWMA-type charts. and the combination of the $\bar{X}$ chart and the S chart.

For a pair of specified changes in the process mean and the process variability as given by specified a and b. the smallest out-of-control ARL for each EWMA-type chart is obtained using its optimal parameters, whereas the combination of the $\bar{X}$ chart and the S chart only has one out-of-control ARL. Tables 7.2, 7.3 and 7.4

Table 7.2: Optimal ARL Values of Max-EWMA chart and EWMA-Max chart when n = 5 and ARL_0 = 250.

	Max-EWMA Chart							
	a							
b	0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
0.25	2.27	2.27	2.27	2.16	1.13	1.00	1.00	1.00
0.50	5.41	5.41	5.22	2.76	1.35	1.00	1.00	1.00
1.00	249.93	23.95	8.58	2.94	1.56	1.09	1.01	1.00
1.25	17.79	12.80	7.10	2.86	1.60	1.15	1.02	1.00
1.50	6.28	5.69	4.51	2.52	1.59	1.19	1.05	1.01
2.00	2.50	2.45	2.28	1.82	1.44	1.21	1.08	1.03
2.50	1.84	1.81	1.75	1.59	1.36	1.20	1.11	1.05
3.00	1.66	1.64	1.60	1.50	1.36	1.24	1.14	1.08
	EWMA-Max Chart							
	a							
b	0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
0.25	2.54	2.54	2.54	2.32	1.13	1.00	1.00	1.00
0.50	13.39	12.29	9.04	3.26	1.35	1.00	1.00	1.00
1.00	250.02	79.44	16.76	3.46	1.57	1.09	1.01	1.00
1.25	17.07	13.52	8.14	3.08	1.62	1.15	1.02	1.00
1.50	5.92	5.45	4.41	2.54	1.59	1.19	1.05	1.01
2.00	2.25	2.20	2.08	1.73	1.41	1.20	1.08	1.02
2.50	1.49	1.48	1.45	1.35	1.24	1.14	1.07	1.03
3.00	3.00	1.23	1.22	1.18	1.14	1.09	1.06	1.03

display results of such comparisons for the six charts. The entries for the four new charts are taken from Tables 3.1, 4.1, 5.1 and 6.1. The entries for the combination of the EWMA $\bar{X}$ chart and the EWMA $ln(S^2)$ chart are obtained based on the optimal parameter values for the combinations of the (λ_1, L_1) and (λ_2, L_2), which are given in Table 7.1. The entries for the combination of the $\bar{X}$ chart and the S chart are calculated as $1/p$ with respect to various changes of the process.

The second approach is to make comparisons when the five EWMA-type charts use the same values for the weight, i.e., $\lambda = \lambda_1 = \lambda_2$. Tables 7.5, 7.6, and 7.7 display the respective out-of-control ARL's of the six charts for various changes in

Table 7.3: Optimal ARL values of SS-EWMA chart and Combination of the two EWMA charts when n = 5 and $ARL_0 = 250$.

	SS-EWMA Chart							
	a							
b	0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
0.25	2.51	2.42	2.19	1.48	1.00	1.00	1.00	1.00
0.50	5.96	5.37	4.17	2.32	1.26	1.00	1.00	1.00
1.00	249.91	24.40	8.83	3.07	1.62	1.11	1.01	1.00
1.25	17.11	11.67	6.58	2.78	1.59	1.15	1.02	1.00
1.50	5.93	5.26	4.08	2.33	1.52	1.17	1.04	1.00
2.00	2.17	2.12	1.98	1.62	1.34	1.16	1.06	1.02
2.50	1.44	1.43	1.40	1.30	1.20	1.11	1.06	1.02
3.00	1.21	1.20	1.20	1.16	1.12	1.08	1.05	1.02
	EWMA Combination Chart							
	a							
b	0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
0.25	1.99	1.99	1.99	1.94	1.20	1.00	1.00	1.00
0.50	4.99	4.99	4.84	2.77	1.37	1.00	1.00	1.00
1.00	250.13	23.53	8.47	2.94	1.55	1.09	1.01	1.00
1.25	20.83	16.52	7.99	2.94	1.61	1.15	1.02	1.00
1.50	9.26	8.58	6.40	2.88	1.66	1.21	1.05	1.01
2.00	3.83	4.64	4.00	2.54	1.70	1.31	1.12	1.04
2.50	2.57	3.21	2.88	2.16	1.64	1.36	1.18	1.08
3.00	2.04	2.58	2.34	1.89	1.56	1.37	1.21	1.12

Table 7.4: Optimal ARL values of EWMA-SC chart and Combination of the two Shewhart charts when n = 5 and ARL_0 = 250.

	EWMA-SC Chart							
	a							
b	0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
1.00	250.00	73.46	22.60	5.16	2.11	1.25	1.03	1.00
1.25	9.43	8.50	6.51	3.22	1.79	1.23	1.05	1.00
1.50	4.17	3.96	3.44	2.30	1.56	1.20	1.06	1.01
2.00	1.87	1.84	1.76	1.52	1.30	1.14	1.06	1.02
2.50	1.34	1.33	1.31	1.24	1.16	1.10	1.05	1.02
3.00	1.16	1.16	1.15	1.13	1.09	1.06	1.04	1.02
	Shewhart Combination Chart							
	a							
b	0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
0.25	6.06	6.06	6.06	6.05	1.14	1.00	1.00	1.00
0.50	68.39	68.39	68.20	17.31	1.42	1.00	1.00	1.00
1.00	250.21	128.14	38.08	5.05	1.65	1.09	1.01	1.00
1.25	30.92	23.85	13.20	3.82	1.69	1.15	1.02	1.00
1.50	8.30	7.50	5.78	2.90	1.65	1.20	1.08	1.02
2.00	2.43	2.38	2.22	1.80	1.43	1.20	1.08	1.02
2.50	1.52	1.50	1.47	1.37	1.24	1.14	1.07	1.03
3.00	1.24	1.23	1.22	1.19	1.14	1.09	1.06	1.03

Table 7.5: ARL's of Max-EWMA chart and EWMA-Max chart when n = 5 and ARL_0 = 250.

		Max-EWMA chart					EWMA-Max chart				
		a					a				
	b	0.00	0.25	0.50	1.00	2.00	0.00	0.25	0.50	1.00	2.00
	0.25	3.9	3.9	3.9	3.9	2.1	4.0	4.0	4.0	3.7	2.0
$\lambda = 0.05$	0.50	6.9	6.9	6.8	4.5	2.3	16.7	15.3	11.4	4.8	2.0
$L_{ME} = 2.46$	1.00	250.1	23.9	9.9	4.6	2.4	250.0	96.7	21.0	5.1	2.0
$L_{EM} = 2.43$	1.50	8.7	8.3	7.1	4.6	2.4	8.1	7.6	6.4	4.0	2.0
	2.00	5.1	5.0	4.7	3.9	2.4	3.6	3.5	3.4	2.9	1.9
	0.25	3.2	3.2	3.2	3.2	2.0	3.0	3.0	3.0	2.8	1.4
$\lambda = 0.10$	0.50	5.9	5.9	5.8	3.7	2.0	13.4	12.3	9.0	3.6	1.4
$L_{ME} = 2.79$	1.00	250.0	24.9	8.8	3.9	2.0	250.0	79.4	16.8	3.9	1.5
$L_{EM} = 2.37$	1.50	7.5	7.0	5.9	3.7	2.0	6.3	5.9	4.9	3.1	1.5
	2.00	4.1	4.0	3.8	3.1	2.0	2.8	2.7	2.6	2.2	1.5
	0.25	2.7	2.7	2.7	2.7	2.0	2.7	2.7	2.7	2.5	1.0
$\lambda = 0.20$	0.50	5.4	5.4	5.2	3.1	1.8	15.7	90.4	10.3	3.3	1.1
$L_{ME} = 3.04$	1.00	249.9	31.3	8.6	3.3	1.7	250.1	18.5	3.6	3.0	1.3
$L_{EM} = 2.70$	1.50	6.6	6.1	5.0	3.1	1.7	6.0	5.5	4.6	2.8	1.4
	2.00	3.4	3.3	3.1	2.5	1.7	2.5	2.4	2.3	2.0	1.3
	0.25	2.4	2.4	2.4	2.4	1.2	2.6	2.6	2.6	2.4	1.0
$\lambda = 0.30$	0.50	5.6	5.6	5.4	2.9	1.3	18.9	17.6	12.8	3.3	1.0
$L_{ME} = 3.14$	1.00	249.9	40.1	9.4	3.0	1.4	250.1	96.8	20.3	3.5	1.2
$L_{EM} = 2.86$	1.50	6.3	5.7	4.6	2.8	1.5	5.9	5.5	4.4	2.6	1.3
	2.00	3.0	2.9	2.7	2.2	1.5	2.4	2.3	2.2	1.8	1.3
	0.25	2.3	2.3	2.3	2.2	1.0	2.6	2.6	2.6	2.4	1.0
$\lambda = 0.50$	0.50	8.2	8.2	8.0	2.8	1.0	29.0	27.7	22.1	3.8	1.0
$L_{ME} = 3.22$	1.00	250.0	61.2	12.8	3.0	1.2	250.1	108.2	24.9	3.6	1.2
$L_{EM} = 3.07$	1.50	6.4	5.8	4.5	2.4	1.2	6.3	5.7	4.5	2.5	1.2
	2.00	2.6	2.6	2.4	1.9	1.2	2.3	2.2	2.1	1.8	1.2
	0.25	2.9	2.9	2.9	2.8	1.0	3.7	3.7	3.7	3.5	1.0
$\lambda = 0.80$	0.50	26.7	26.7	2.65	5.1	1.0	51.4	50.4	46.4	7.7	1.0
$L_{ME} = 3.26$	1.00	250.1	102.2	24.6	3.6	1.1	250.1	121.5	32.8	4.2	1.1
$L_{EM} = 3.22$	1.50	7.4	6.6	5.1	2.6	1.2	7.3	6.6	5.1	2.7	1.2
	2.00	2.5	2.5	2.3	1.8	1.2	2.3	2.3	2.1	1.7	1.2
	0.25	6.0	6.0	6.0	6.0	1.0	6.1	6.1	6.1	6.1	1.0
$\lambda = 1.00$	0.50	68.2	68.2	68.1	17.4	1.0	69.0	68.7	68.3	17.3	1.0
$L_{ME} = 3.26$	1.00	250.1	129.3	38.2	5.0	1.1	250.1	128.2	38.1	5.1	1.1
$L_{EM} = 3.25$	1.50	8.4	7.7	5.8	2.9	1.2	8.3	7.5	5.8	2.9	1.2
	2.00	2.6	2.6	2.4	1.9	1.2	2.4	2.4	2.2	1.8	1.2

Table 7.6: ARL's of the SS-EWMA chart and the combination of two EWMA charts when n = 5 and in-control $ARL_0 = 250$.

		SS-EWMA chart					Combination Chart				
		a					a				
	b	0.00	0.25	0.50	1.00	2.00	0.00	0.25	0.50	1.00	2.00
	0.25	4.2	4.1	3.9	3.1	2.0	3.0	3.0	3.1	3.1	2.1
$\lambda = 0.05$	0.50	7.4	6.9	5.8	4.0	2.4	6.1	6.1	60.0	4.4	2.3
$L = 2.96$	1.00	250.0	24.5	10.3	4.9	2.6	249.6	24.2	9.9	4.7	2.4
$L_1 = 2.61$	1.50	8.7	7.9	6.5	4.3	2.5	10.7	10.1	8.0	4.7	2.5
$L_2 = 2.60$	2.00	4.3	4.2	4.0	3.3	2.3	5.9	5.8	5.5	4.3	2.5
	0.25	3.5	3.4	3.2	2.8	2.0	2.5	2.6	2.5	2.5	2.0
$\lambda = 0.10$	0.50	6.4	5.9	4.9	3.3	2.0	5.3	5.3	5.2	3.6	2.0
$L = 3.60$	1.00	250.0	25.4	9.1	4.0	2.1	252.1	25.1	8.8	3.9	2.0
$L_1 = 2.81$	1.50	7.3	6.6	5.4	3.5	2.1	9.7	8.8	6.9	3.9	2.1
$L_2 = 2.86$	2.00	3.5	3.5	3.3	2.7	1.9	5.0	4.8	4.5	3.5	2.1
	0.25	2.9	2.8	2.6	2.0	1.7	2.2	2.2	2.2	2.2	1.9
$\lambda = 0.20$	0.50	6.0	5.4	4.3	2.8	1.9	5.0	5.0	4.8	3.0	1.8
$L = 3.91$	1.00	250.0	31.8	8.9	3.4	1.8	250.9	31.5	8.6	3.3	1.7
$L_1 = 2.96$	1.50	6.4	5.7	4.6	2.9	1.7	10.2	8.7	6.3	3.3	1.7
$L_2 = 3.14$	2.00	2.9	2.9	2.7	2.2	1.5	4.4	4.3	3.9	2.9	1.7
	0.25	2.6	2.6	2.3	2.0	1.0	2.1	2.1	2.1	2.0	1.1
$\lambda = 0.30$	0.50	6.5	5.6	4.2	2.5	1.3	5.5	5.4	5.2	2.8	1.3
$L = 4.30$	1.00	250.0	40.6	9.7	3.2	1.5	249.4	40.2	9.4	3.0	1.4
$L_1 = 3.02$	1.50	6.0	5.4	4.3	2.6	1.4	12.8	10.1	6.5	3.0	1.5
$L_2 = 3.35$	2.00	2.6	2.5	2.4	2.0	1.4	4.3	4.2	3.8	2.7	1.5
	0.25	2.5	2.4	2.2	1.8	1.0	2.0	2.0	2.0	1.9	1.0
$\lambda = 0.50$	0.50	11.0	8.7	5.3	2.3	1.0	8.4	8.2	8.0	2.8	1.0
$L = 4.47$	1.00	250.0	61.3	13.4	3.1	1.2	248.0	61.7	12.8	3.0	1.2
$L_1 = 3.07$	1.50	6.0	5.3	4.1	2.4	1.3	22.9	15.0	7.5	2.9	1.3
$L_2 = 3.64$	2.00	2.3	2.2	2.1	1.7	1.2	5.2	4.9	4.0	2.6	1.4
	0.25	3.9	3.6	2.7	1.5	1.0	2.4	2.5	2.4	2.4	1.0
$\lambda = 0.80$	0.50	45.5	35.0	16.6	3.0	1.0	20.6	20.6	20.8	4.9	1.0
$L = 4.52$	1.00	250.0	99.3	25.0	3.9	1.1	249.4	99.5	24.2	3.7	1.1
$L_1 = 3.09$	1.50	6.5	5.8	4.4	2.4	1.2	25.6	17.4	9.2	3.1	1.2
$L_2 = 3.88$	2.00	2.2	2.1	2.0	1.6	1.2	7.8	6.8	5.2	2.8	1.3
	0.25	11.3	9.8	6.3	1.7	1.0	3.7	3.6	3.7	3.7	1.0
$\lambda = 1.00$	0.50	128.1	103.0	53.5	6.1	1.0	36.1	36.1	35.4	14.2	1.0
$L = 4.53$	1.00	250.0	126.6	38.4	5.4	1.1	251.3	127.5	38.0	5.1	1.1
$L_1 = 3.09$	1.50	7.5	6.7	5.0	2.6	1.2	25.2	18.6	10.3	3.6	1.2
$L_2 = 3.95$	2.00	3.5	3.4	2.9	2.0	1.2	8.1	7.3	5.6	3.0	1.3

Table 7.7: ARL's of the EWMA-SC chart and the combination of two Shewhart charts when n = 5 and in-control ARL = 250.

		EWMA-SC chart					Combination Chart				
		a					a				
	b	0.00	0.25	0.50	1.00	2.00	0.00	0.25	0.50	1.00	2.00
$\lambda = 0.05$	1.00	250.0	73.5	22.6	5.2	1.6	250.2	128.1	38.1	5.1	1.1
$L_{ES} = 1.88$	1.50	4.3	4.2	3.7	2.6	1.4	8.3	7.5	5.8	2.9	1.2
	2.00	2.2	2.1	2.0	1.8	1.3	2.4	2.4	2.2	1.8	1.2
$\lambda = 0.10$	1.00	250.0	109.2	27.5	5.2	1.5	250.2	128.1	38.1	5.1	1.1
$L_{ES} = 2.45$	1.50	4.3	4.1	3.6	2.5	1.4	8.3	7.5	5.8	2.9	1.2
	2.00	2.1	2.1	2.0	1.7	1.2	2.4	2.4	2.2	1.8	1.2
$\lambda = 0.20$	1.00	250.0	128.4	34.2	5.2	1.4	250.2	128.1	38.1	5.1	1.1
$L_{ES} = 2.93$	1.50	4.2	4.0	3.5	2.4	1.3	8.3	7.5	5.8	2.9	1.2
	2.00	2.0	1.9	1.9	1.6	1.2	2.4	2.4	2.2	1.8	1.2
$\lambda = 0.30$	1.00	250.0	134.8	39.3	5.4	1.3	250.2	128.1	38.1	5.1	1.1
$L_{ES} = 3.19$	1.50	4.2	4.0	3.4	2.3	1.2	8.3	7.5	5.8	2.9	1.2
	2.00	1.9	1.9	1.8	1.6	1.2	2.4	2.4	2.2	1.8	1.2
$\lambda = 0.50$	1.00	250.0	146.6	49.5	6.3	1.3	250.2	128.1	38.1	5.1	1.1
$L_{ES} = 3.54$	1.50	4.5	4.2	3.6	2.3	1.2	8.3	7.5	5.8	2.9	1.2
	2.00	1.9	1.8	1.8	1.5	1.2	2.4	2.4	2.2	1.8	1.2
$\lambda = 0.80$	1.00	250.0	157.5	61.5	8.4	1.3	250.2	128.1	38.1	5.1	1.1
$L_{ES} = 3.82$	1.50	5.2	4.9	4.0	2.5	1.2	8.3	7.5	5.8	2.9	1.2
	2.00	1.9	1.9	1.8	1.5	1.1	2.4	2.4	2.2	1.8	1.2
$\lambda = 1.00$	1.00	250.0	161.4	67.2	10.0	1.3	250.2	128.1	38.1	5.1	1.1
$L_{ES} = 3.88$	1.50	5.7	5.4	4.4	2.6	1.2	8.3	7.5	5.8	2.9	1.2
	2.00	2.0	1.9	1.8	1.6	1.2	2.4	2.4	2.2	1.8	1.2

the process mean and the process variability.

Similar results are shown in Tables 7.2 - 7.7. It should be noted that. in term of detecting small changes in the mean and/or the variability. the performance of the combination of the $\bar{X}$ chart and the S chart is very poor in comparison to others. It is also seen that. in term of detecting shifts in the mean alone ($b = 1$). the Max-EWMA chart. the SS-EWMA Chart and the combination of the EWMA $\bar{X}$ chart and the EWMA $ln(S^2)$ chart yield smaller ARL's than the other charts. and they perform almost equally well. One interesting point to be noted is that the Max-EWMA chart and the SS-EWMA chart have similar ARL performances. In term of detecting mean shifts that are accompanied with variability changes when the variability is decreased. the combination of the EWMA $\bar{X}$ chart and the EWMA $ln(S^2)$ chart performs better than the others but is only slightly superior to the Max-EWMA chart and the SS-EWMA chart. However. when the variability is increased. the four new charts perform better than the combination of the EWMA $\bar{X}$ chart and the EWMA $ln(S^2)$. This difference in performance can be explained by the fact that symmetric control limits are used in the EWMA $ln(S^2)$ chart. but the distribution of $ln(S^2)$ is not symmetric.

For the comparisons among the four new charts. the SS-EWMA chart have the smallest ARL's when a mean shift accompanies a decreased variability change. while the EWMA-SC chart yields the smallest ARL's when a mean shift accompanies an increased variability change. Another interesting result is that even for large changes of a process, most optimal ARL values of the four new charts are smaller than the ARL values of the combination of the $\bar{X}$ chart and the S chart.

The overall ARL performance for these charts shows that the Max-EWMA chart and the SS-EWMA chart appear to be better control schemes than others for detecting various shifts in the process mean and/or changes in the process variability because in different situations their ARL values are at least close to the smallest ones. If a mean shift accompanies a non-increased variability change, the combination of the EWMA $\bar{X}$ chart and the EWMA $ln(S^2)$ chart has good ARL performance. When a mean shift accompanies an increased variability change, the EWMA-SC chart performs well. In general, when an EWMA-type chart is used, smaller λ values give better performance for detecting smaller changes in the mean and/or the variability.

It is important to note that, for the comparisons of these control charts, the in-control ARL of 250 is only one of many possible choices and it is a value between 185 and 370, which are two often used values for the in-control ARL in quality control. If any other value is chosen for the in-control ARL, the results would most likely be the same as those when the in-control ARL is 250.

7.4 Diagnostic Ability Studies

From the results of the last section, the Max-EWMA chart and the SS-EWMA chart demonstrate overall good ARL performances. Since our main objective is to provide alternatives to the use of combination of the existing charts, further comparisons of the diagnostic abilities are made among the Max-EWMA chart, the SS-EWMA chart and the combination of the EWMA $\bar{X}$ chart and the EWMA $ln(S^2)$ chart (referred to as the combination chart in the following discussion). To identify the source and the direction of the detected changes, 1,000 out-of-control signals are simulated. Each chart, with in control ARL of 250, is applied to the same set of the 1,000 signals using the approximate UCL. Out-of-control signals are counted according to the charting procedure of each chart.

Table 7.8 contains some comparative results of the Max-EWMA chart and the combination chart, where $(\lambda, L) = (0.10, 2.79)$, $(\lambda_1, L_1) = (0.10, 2.81)$ and $(\lambda_2, L_2) = (0.10, 2.86)$. Table 7.9 contains some comparative results for the SS-EWMA chart and the combination chart, where $(\lambda, L) = (0.10, 3.60)$, $(\lambda_1, L_1) = (0.10, 2.81)$ and $(\lambda_2, L_2) = (0.10, 2.86)$. Notation in Tables 7.8 and 7.9 is defined as follows: $+o$ denotes the number of times that an increase in the mean alone is detected; $-o$ denotes the number of times that a decrease in the mean alone is detected; $o+$ denotes the number of times that an increase in the variability alone is detected; $o-$ denotes the number of times that a decrease in the variability alone is detected; $++$ denotes the number of times that an increase in both the mean and the variability is detected simultaneously; $+-$ denotes the number of times that an increase in the mean and a decrease in the variability are simultaneously detected; $-+$ denotes the number of times that a decrease in the mean and an increase in the variability are simultaneously detected, and $--$ denotes the number of times that a decrease in both the mean and the variability is detected simultaneously.

It can be seen from Tables 7.8 and 7.9 that, in the in-control case ($a = 0, b = 1$), the Max-EWMA chart and the SS-EWMA chart give a balanced performance while the combination chart gives an unbalanced performance. Also, it can be seen that, in the out-of-control cases, the two new charts perform as well or nearly as well as the combination chart does when the variability is decreased, however, both of the two new charts perform better than the combination chart does when the variability is increased. The two new charts, especially the SS-EWMA chart, seem to be more effective than the combination chart for detecting simultaneous changes in both the mean and the variability. For example, when $a = 1.00$ and $b = 0.25$, out of the 1,000 simultaneous change out-of-control signals, the Max-EWMA chart identifies 304 signals, the SS-EWMA chart 651 signals and the combination chart 105 signals. In term of the accuracy of detecting the source and direction of an

Table 7.8: A comparison of the diagnostic abilities between the Max-EWMA chart and the EWMA combination chart.

		Max-EWMA chart					Combination Chart				
		a					a				
b		0.00	0.25	0.50	1.00	2.00	0.00	0.25	0.50	1.00	2.00
	$+o$	0	0	0	43	973	0	0	0	2	543
	$-o$	0	0	0	0	0	0	0	0	0	0
	$o+$	0	0	0	0	0	0	0	0	0	0
0.25	$o-$	1000	1000	1000	653	0	1000	1000	1000	893	8
	$++$	0	0	0	0	0	0	0	0	0	0
	$+-$	0	0	0	304	27	0	0	0	105	449
	$-+$	0	0	0	0	0	0	0	0	0	0
	$--$	0	0	0	0	0	0	0	0	0	0
	$+o$	0	0	73	858	999	0	1	59	705	979
	$-o$	0	0	0	0	0	0	0	0	0	0
	$o+$	0	0	0	0	0	0	0	0	0	0
0.50	$o-$	1000	1000	844	27	0	1000	999	874	108	0
	$++$	0	0	0	0	0	0	0	0	0	0
	$+-$	0	0	83	115	1	0	0	67	187	21
	$-+$	0	0	0	0	0	0	0	0	0	0
	$--$	0	0	0	0	0	0	0	0	0	0
	$+o$	252	985	994	1000	1000	245	961	996	998	999
	$-o$	237	0	0	0	0	236	0	0	0	0
	$o+$	252	17	3	0	0	73	5	0	0	0
1.00	$o-$	258	22	2	0	0	444	30	3	2	0
	$++$	0	0	0	0	0	0	0	0	0	0
	$+-$	0	3	1	0	0	1	4	1	0	1
	$-+$	0	0	0	0	0	0	0	0	0	0
	$--$	1	0	0	0	0	1	0	0	0	0
	$+o$	22	148	420	824	977	54	258	606	944	999
	$-o$	30	4	1	1	0	53	2	0	0	0
	$o+$	935	812	491	93	4	874	696	322	28	0
1.50	$o-$	0	0	0	0	0	0	0	0	0	0
	$++$	5	36	88	83	19	10	44	72	28	1
	$+-$	0	0	0	0	0	0	0	0	0	0
	$-+$	8	0	0	0	0	9	0	0	0	0
	$--$	0	0	0	0	0	0	0	0	0	0
	$+o$	33	95	198	427	812	50	137	278	651	963
	$-o$	32	16	2	1	0	52	17	5	0	0
	$o+$	903	855	712	384	38	859	792	613	208	6
2.00	$o-$	0	0	0	0	0	0	0	0	0	0
	$++$	16	33	88	188	150	22	47	104	141	31
	$+-$	0	0	0	0	0	0	0	0	0	0
	$-+$	16	1	0	0	0	0	17	7	0	0
	$--$	0	0	0	0	0	0	0	0	0	0

Table 7.9: A comparison of the diagnostic abilities between the SS-EWMA chart and the EWMA combination chart.

		SS-EWMA chart					Combination Chart				
				a					a		
b		0.00	0.25	0.50	1.00	2.00	0.00	0.25	0.50	1.00	2.00
	$+o$	0	0	0	3	994	0	0	0	2	543
	$-o$	6	0	0	0	0	0	0	0	0	0
	$o+$	0	0	0	0	0	0	0	0	0	0
0.25	$o-$	994	911	701	346	0	1000	1000	1000	893	8
	$++$	0	0	0	0	0	0	0	0	0	0
	$+-$	0	89	299	651	6	0	0	0	105	449
	$-+$	0	0	0	0	0	0	0	0	0	0
	-	0	0	0	0	0	0	0	0	0	0
	$+o$	0	0	1	294	990	0	1	59	705	979
	$-o$	27	0	0	0	0	0	0	0	0	0
	$o+$	0	0	0	0	0	0	0	0	0	0
0.50	$o-$	973	570	164	5	0	1000	999	874	108	0
	$++$	0	0	0	0	0	0	0	0	0	0
	$+-$	0	430	835	701	10	0	0	67	187	21
	$-+$	0	0	0	0	0	0	0	0	0	0
	$--$	0	0	0	0	0	0	0	0	0	0
	$+o$	254	930	983	997	1000	245	961	996	998	999
	$-o$	247	0	0	0	0	236	0	0	0	0
	$o+$	244	27	7	2	0	73	5	0	0	0
1.00	$o-$	255	43	10	1	0	444	30	3	2	0
	$++$	0	0	0	0	0	0	0	0	0	0
	$+-$	0	0	0	0	0	1	4	1	0	1
	$-+$	0	0	0	0	0	0	0	0	0	0
	$--$	0	0	0	0	0	1	0	0	0	0
	$+o$	41	54	189	541	892	54	258	606	944	999
	$-o$	49	1	0	0	0	53	2	0	0	0
	$o+$	905	483	249	44	1	874	696	322	28	0
1.50	$o-$	5	9	9	5	3	0	0	0	0	0
	$++$	0	417	549	410	100	10	44	72	28	1
	$+-$	0	0	1	0	4	0	0	0	0	0
	$-+$	0	36	3	0	0	9	0	0	0	0
	$--$	0	0	0	0	0	0	0	0	0	0
	$+o$	32	37	93	245	691	50	137	278	651	963
	$-o$	41	4	1	0	0	52	17	5	0	0
	$o+$	862	640	531	246	29	859	792	613	208	6
2.00	$o-$	61	82	87	82	41	0	0	0	0	0
	$++$	3	181	269	425	238	22	47	104	141	31
	$+-$	0	0	0	0	1	0	0	0	0	0
	$-+$	1	56	19	2	0	17	7	0	0	0
	$--$	0	0	0	0	0	0	0	0	0	0

out-of-control signal. the Max-EWMA chart is the best among the three charts. and the SS-EWMA chart produces a few more incorrect signals than the other two in some cases. Overall. the Max-EWMA chart seems to have the highest diagnostic ability of the three charts. These results are consistent with that of Section 7.3.

CHAPTER 8

Discussion on Lognormal Quality Control

8.1 Introduction

The lognormal distribution $LN(\mu, \sigma^2)$, is defined as the distribution of a random variable whose logarithm follows the normal distribution $N(\mu, \sigma^2)$. Since many kinds of data in real life have a positively-skewed distribution, the lognormal distribution has been widely applied in many areas.

Morrison [41] first applied the lognormal distribution to quality control, and proposed a modified quality control scheme that can process skewed data in the original scale of measurement when the assumption of normality can not be made. Ferrell [20] also suggested using this control scheme for computing and plotting control charts when data are from a badly skewed distribution which can be approximated by a lognormal distribution.

Based on the fundamental relationship between normal and lognormal distributions, Morrison derived control limits for the lognormal variable from the corresponding control limits for a normal variable, using the inverse logarithmic transformation. However, because of the complexity of the lognormal distribution, its application to quality control cannot be referred to that of the normal distribution by simply taking the direct transformation, which may result in a control chart with inappropriate control parameters.

For simplicity, in this chapter a lognormal process refers to a process in which its characteristic follows a lognormal distribution and a normal process refers to a process in which its characteristic follows a normal distribution.

To monitor a lognormal process, the corresponding normal process is obtained through the logarithmic transformation and a new quality control scheme is developed. When it is given that the lognormal process mean lies in a specific interval, then two control charts are set up for the lognormal process. The control of a complex lognormal process is simplified to that of a normal process, for which good control schemes are available and it is much easier to implement.

8.2 A Modified Quality Control Scheme

Suppose that $X_{ij}, i = 1, 2, \ldots$ and $j = 1, 2, \ldots, n$ represent the quality characteristic of a process, and they follow lognormal distribution $LN(\mu, \sigma^2)$. Let $Y_{ij} = lnX_{ij}$, then $Y_{ij}, i = 1, 2, \ldots$ and $j = 1, 2, \ldots, n$, follow normal distribution $N(\mu, \sigma^2)$.

In the modified quality control scheme proposed by Morrison [41], the statistics for a lognormal process and the corresponding 3σ control limits can be obtained from the following derivations. Notice that

$$
\begin{aligned}
& P\left(|\frac{\ddot{Y} - E(\ddot{Y})}{\sigma_{\ddot{Y}}}| \leq 3\right) \\
= \quad & P\left(exp(\mu - 3\sigma_{\ddot{Y}}) \leq exp(\ddot{Y}) \leq exp(\mu - 3\sigma_{\ddot{Y}})\right) \\
= \quad & P\left(exp(\mu - 3\sigma_{\ddot{Y}}) \leq \sqrt{X_{i(1)}X_{i(n)}} \leq exp(\mu + 3\sigma_{\ddot{Y}})\right) \qquad (8.1)
\end{aligned}
$$

where $\ddot{Y}$ is the sample midrange for a normal process, and $X_{i(1)}$ and $X_{i(n)}$ are the minimum and maximum of the i^{th} sample, respectively.

Similarly,

$$\begin{aligned}
& P\left(|\frac{R_i - E(R_i)}{\sigma_R}| \leq 3\right) \\
= \; & P\left(exp((d_2 - 3d_3)\sigma) \leq exp(R_i) \leq exp((d_2 + 3d_3)\sigma)\right) \\
= \; & P\left(exp((d_2 - 3d_3)\sigma) \leq \frac{X_{i(n)}}{X_{i(1)}} \leq exp((d_2 + 3d_3)\sigma\right) \qquad (8.2)
\end{aligned}$$

where R_i is the sample range for a normal process, and d_2 and d_3 are control chart constants.

Thus, from (8.1), the geometric sample mean $\sqrt{X_{i(1)}X_{i(n)}}$ is used as a measure of the lognormal process mean and the exact control limits are $exp(\mu \pm 3\sigma_{\ddot{Y}})$. With the available sample results, the control limits are estimated by

$$exp(\ddot{Y})\bar{r}^{\pm A_2} = \sqrt{X_{(1)}X_{(nm)}}\left(\frac{1}{m}\sum_{i=1}^{m}\frac{X_{i(n)}}{X_{i(1)}}\right)^{\pm A_2} \qquad (8.3)$$

where $X_{(1)}$ and $X_{(nm)}$ are the minimum and maximum in nm sample values for a lognormal process, $\bar{r}$ is the average ratio of the maximum to the minimum from m samples of size n for a lognormal process, and A_2 is a control chart constant.

Similarly, from (8.2), the sample ratio $\frac{X_{i(n)}}{X_{i(1)}}$ is used as a measure of the lognormal process variability and the exact control limits are $exp((d_2 + 3d_3)\sigma)$, which are estimated by

$$\bar{r}^{D_3} = \left(\frac{1}{m}\sum_{i=1}^{m}\frac{X_{i(n)}}{X_{i(1)}}\right)^{D_3} \qquad (8.4)$$

and

$$\bar{r}^{D_4} = \left(\frac{1}{m}\sum_{i=1}^{m}\frac{X_{i(n)}}{X_{i(1)}}\right)^{D_4} \tag{8.5}$$

where D_3 and D_4 are control chart constants.

From the derivations. it is seen that the in-control probability of the derived statistics for a lognormal process is the same as that for its normal counterpart. but the control limits for lognormal control charts are inaccurate. Because the control parameters for lognormal and normal processes are different. the direct transformations may not assure that the statistical state of a lognormal process is the same as that of the corresponding normal process. Morrison's chart actually sets the target as the normal process mean. and therefore a normal process is monitored through its lognormal counterpart. Moreover. because the parameter estimators of (8.3). (8.4) and (8.5) resulting from the inverse logarithmic transformation are biased. the control charts neither have proper probability nor proper 3-sigma control limits.

8.3 New Control Charts for Lognormal Processes

It is difficult to directly construct a control chart for a lognormal process since sampling properties associated with the lognormal statistics are not easy to derive. Making use of the relationship between normal and lognormal distributions and having been given a specific interval for the lognormal mean to a lognormal process, a new method is proposed to avoid the complexity of the lognormal distribution. The two control charts for lognormal distribution can be constructed to monitor a lognormal process.

8.3.1 The Logarithmic Transformation

In statistical analysis. a logarithmic transformation is often applied to a set of positively-skewed distributed data before proceeding with the analysis. This approach works well for usual statistical analysis. However. the direct logarithmic transformation may result in a control chart with inappropriate control parameters in the application of quality control. For a lognormal process. it is of interest to control the parameters μ_* and σ_*, while. for a normal process. μ and σ are of interest. It can be shown that. for a specified significance level α. the control limits for individual measurements of a lognormal process is different from that of the corresponding normal process. Without loss of generality. assume that $X \sim LN(0, 1)$. then $Y = ln(X) \sim N(0, 1)$. For $\alpha = 0.0027$. it follows from

$$P(X > x_{0.00135}) = 0.00135$$

that the upper percentile $x_{0.00135}$ can be found as

$$x_{0.00135} = exp(3)$$

Hence. the upper control limit for the X chart is

$$\begin{aligned} UCL_X &= \mu_* + x_{0.00135}\sigma_* \\ &= exp(0.5) + exp(3.5)\sqrt{exp(1) - 1} \\ &= 45.06 \end{aligned}$$

and $ln(UCL_X) = 3.81$.

The upper control limit for the Y chart is

$$\begin{aligned} UCL_Y &= \mu + z_{0.00135}\sigma \\ &= 3.00 \end{aligned}$$

Obviously, $ln(UCL_X)$ is not equal to UCL_Y so that the direct transformation may result in different control state for corresponding normal process. Therefore, some standards have to be given in order to guarantee the control state of the corresponding process is equivalent to that of the original lognormal process.

8.3.2 A Specific Interval for the Lognormal Process Mean

Suppose that m samples are randomly drawn from a lognormal process, and μ_* is known to lie in an interval:

$$(\mu_{*L}, \mu_{*U})$$

and it is given for the process according to technical specifications. It could be either a given margin of error or specification limits for a single measurement. The margin of error is give by

$$-a_1 \le X_{ij} - \mu_* \le a_2 \tag{8.6}$$

$i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$
where a_1 and a_2 are known positive constants.

Equation (8.6) can be written as

$$-a_2 + X_{ij} \le \mu_* \le a_1 + X_{ij}$$

$i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$

which is equivalent to

$$X_{(mn)} - a_2 \leq \mu_* \leq X_{(1)} + a_1$$

$$X_{(mn)} - X_{(1)} \leq a_1 + a_2$$

Hence, an interval for possible values of μ_* is

$$\mu_{*U} = X_{(1)} + a_1 \tag{8.7}$$

$$\mu_{*L} = X_{(mn)} - a_2 \tag{8.8}$$

If specification limits are available, the upper and lower specification limits can be used as μ_{*U} and μ_{*L}, respectively.

8.3.3 Derivation of Intervals for Parameters

The control parameters for a lognormal process are μ_* and σ_*^2. The control parameters for the corresponding normal process are μ and σ. The parameters μ_* and σ_*^2 are functions of μ and σ given by

$$\mu_* = exp(\mu + 0.5\sigma^2) \tag{8.9}$$

$$\sigma_*^2 = exp(2\mu + \sigma^2)[exp(\sigma^2) - 1] \tag{8.10}$$

M preliminary samples collected from the in-control process can be used to estimate σ^2 by $\hat{\sigma}^2 = \bar{S}^2$. From (8.9), an inteval for μ is obtained as below:

$$\mu_U = ln(\mu_{*U}) - 0.5\hat{\sigma}^2 \tag{8.11}$$

$$\mu_L = ln(\mu_{*L}) - 0.5\hat{\sigma}^2 \tag{8.12}$$

From (8.10), an inteval for σ_*^2 is obtained as below:

$$\sigma_{*U}^2 = exp(2\mu_U + \hat{\sigma}^2)[exp(\hat{\sigma}^2) - 1] \tag{8.13}$$

$$\sigma_{*L}^2 = exp(2\mu_L + \hat{\sigma}^2)[exp(\hat{\sigma}^2) - 1] \tag{8.14}$$

Because the normal distibution is symmetric about mean, the target for the corresponding normal process is

$$\begin{aligned} \mu_0 &= 0.5(\mu_U + \mu_L) \\ &= 0.5ln(\mu_{*U}\mu_{*L}) - 0.5\hat{\sigma}^2 \end{aligned} \tag{8.15}$$

which implies that the target for the lognormal process is the geometric mean of μ_{*U} and μ_{*L}, i.e., $\mu_{*0} = \sqrt{\mu_{*U}\mu_{*L}}$.

8.3.4 Constructing Control Charts for Lognormal Processes

When m preliminary samples are taken from a lognormal process, the logarithms of each observation form the m initial samples of the corresponding normal process. To determine whether the process variability is stabilized, an S chart can be set up with control limits:

$$UCL_s = \sqrt{\frac{\chi^2_{\alpha_4}}{n-1}}\frac{\bar{S}}{c_4} \tag{8.16}$$

$$LCL_s = \sqrt{\frac{\chi^2_{\alpha_3}}{n-1}}\frac{\bar{S}}{c_4} \tag{8.17}$$

where α_3 and α_4 are Type I error probabilities for lower and upper tails respectively.

If all the standard deviations of these samples plot inside the control limits. then the process variability appears to be in control. Otherwise. each of the out-of-control points for which assignable causes can be found is discarded and the control limits are recalculated. Then these control limits can be used for controlling current or future production and σ^2 is estimated from the formula $\hat{\sigma}^2 = \bar{S}^2$.

The percentiles for $\bar{Y}$ chart can be obtained by setting

$$\mu_U - \mu_0 = z_{\alpha_2}\sigma \tag{8.18}$$

$$\mu_L - \mu_0 = z_{\alpha_1}\sigma \tag{8.19}$$

Then.

$$z_{\alpha_1} = \frac{\mu_L - \mu_0}{\sigma} \tag{8.20}$$

$$z_{\alpha_2} = \frac{\mu_U - \mu_0}{\sigma} \tag{8.21}$$

where α_1 and α_2 are Type I error probabilities for lower and upper tails respectively.

A $\bar{Y}$ chart can be set up with the following control limits:

$$\begin{aligned} CL_{\bar{Y}} &= \mu_0 & (8.22) \\ UCL_{\bar{Y}} &= \mu_0 + \frac{z_{\alpha_2}\sigma}{\sqrt{n}} \\ &= (1 - \frac{1}{\sqrt{n}})\mu_0 + \frac{\mu_U}{\sqrt{n}} & (8.23) \\ LCL_{\bar{Y}} &= \mu_0 + \frac{z_{\alpha_1}\sigma}{\sqrt{n}} \\ &= (1 - \frac{1}{\sqrt{n}})\mu_0 + \frac{\mu_L}{\sqrt{n}} & (8.24) \end{aligned}$$

8.4 Properties of the New Control Charts

When a specific interval for μ_* is given, the derivations of control limits for the control charts monitoring the two related processes are reversible and hence the statistical control state of the lognormal process can refer to that of the corresponding normal process. As a result, it is necessary to study properties of the two control charts for the normal process and effects of normal parameter changes on the lognormal parameters.

8.4.1 The ARL Calculations

Assume that $Y_{ij} \sim N(\mu_0, \sigma_0^2)$ independently, where $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$. Suppose that the normal process mean changes from μ_0 to $\mu_0 + a\sigma_0$ and the normal process standard deviation changes from σ_0 to $b\sigma_0$.

The probability of type II error for $\bar{Y}$ chart can be computed from

$$\begin{aligned} \beta_{\bar{Y}} &= P(LCL_{\bar{Y}} \leq \bar{Y} \leq UCL_{\bar{Y}} | \mu = \mu_0 + a\sigma_0; \sigma = b\sigma_0) \\ &= \Phi[\frac{1}{2\sigma_0} ln(\frac{\mu_{*U}}{\mu_{*L}}) - a\sqrt{n}] - \Phi[-\frac{1}{2\sigma_0} ln(\frac{\mu_{*U}}{\mu_{*L}}) - a\sqrt{n}] & (8.25) \end{aligned}$$

When $a = 0$. the probability of type I error for $\bar{Y}$ chart is

$$\begin{aligned} \alpha_{\bar{Y}} &= 1 - \beta_{\bar{Y}} \\ &= 2\Phi[-\frac{1}{2\sigma_0} ln(\frac{\mu_{*U}}{\mu_{*L}})] \end{aligned} \tag{8.26}$$

from which it is noted that $\alpha_{\bar{Y}}$ is a function of $\frac{\mu_{*U}}{\mu_{*L}}$ and σ_0. To achieve a small $\alpha_{\bar{Y}}$. $\frac{\mu_{*U}}{\mu_{*L}}$ and σ_0 are usually not larger for high precision products so that the process variability has to be small. while. σ_0 is allowed to be a little bit large for medium or low precision products.

The ARL's for $\bar{Y}$ chart can be easily obtained from

$$ARL_{\bar{Y}} = \frac{1}{1 - \beta_{\bar{Y}}} \tag{8.27}$$

When $\alpha_{\bar{Y}}$ is fixed. ARL will decrease as a and n increase.

The ARL's for S chart can be computed from

$$ARL_S = \frac{1}{1 - \beta_S} \tag{8.28}$$

where $\beta_S = H(\frac{\chi^2_{\alpha_4, n-1}}{b^2}) - H(\frac{\chi^2_{\alpha_3, n-1}}{b^2})$ and $\alpha_S = \alpha_3 + \alpha_4$.

8.4.2 Effects of Changes in Parameters

When there are changes in the normal process mean and process variability. the lognormal parameters will be changed to

$$\mu_{1*} = exp[\mu_0 + a\sigma_0 + 0.5(b\sigma_0)^2]$$

$$\sigma^2_{1*} = exp[2(\mu_0 + a\sigma_0) + (b\sigma_0)^2][exp((b\sigma_0)^2) - 1]$$

where $a \neq 0$ and $b > 0$.

Because the derivatives of μ_{1*} with respect to a and b are

$$\frac{\partial \mu_{1*}}{\partial a} = \sigma_0 \mu_{1*} > 0 \tag{8.29}$$

$$\frac{\partial \mu_{1*}}{\partial b} = b\sigma_0^2 \mu_{1*} > 0 \tag{8.30}$$

Notice that μ_{1*} is a monotone increasing function of a and b. and σ_{1*} can be written as a monotone increasing function of μ_{1*}:

$$\sigma_{1*} = \mu_{1*}\sqrt{exp((b\sigma_0)^2) - 1} \tag{8.31}$$

since $\sqrt{exp((b\sigma_0)^2) - 1} > 0$. Then σ_{1*} is also a monotone increasing function of a and b.

Thus, the direction of an-out-of-control signal from a lognormal process can be identified from the corresponding normal process.

8.5 Charting Procedure and Example

The steps to set up the two charts are summarized below:

1. Determine the values of μ_{*U} and μ_{*L}.

   (a) Use values provided by technical specifications. or if not available.

   (b) use μ_{*U} and μ_{*L} obtained from preliminary data as follows:
   if the overall range of the data is less or equal to $a_1 + a_2$, calculate μ_{*U} and μ_{*L}; however, if the overall range is greater than $a_1 + a_2$, remove the possible outliers $X_{(nm)}, X_{(1)}, \ldots,$ until the overall range is less or equal to $a_1 + a_2$, and then calculate μ_{*U} and μ_{*L}.

2. Transform data using $Y = ln(X)$.

3. Construct an S chart and estimate σ^2 by $\bar{S}^2$ when the process variability is in control.

4. Compute μ_U, μ_L, μ_0, σ_{*U}^2 and σ_{*L}^2.

5. Construct a $\bar{Y}$ chart.

6. For a sample point that plots outside one of the control limits, calculate $\hat{\mu}_{*i}$ and $\hat{\sigma}_{*i}^2$ using $\bar{Y}_i$ as the estimate of μ and S_i^2 as the estimate of σ^2. Plot 'm+' or 'm-' against sample number if only $\hat{\mu}_{*i} > \mu_{*U}$ or $\hat{\mu}_{*i} < \mu_{*L}$; plot 'v+' or 'v-' against sample number if only $\hat{\sigma}_{*i} > \sigma_{*U}$ or $\hat{\sigma}_{*i} < \sigma_{*U}$; plot 'm+v+', 'm+v-','m-v+' or 'm-v-' against sample number according to the sources and the directions of an out-of-control signal.

7. Examine the assignable cause(s).

An example is given to illustrate how to apply the new control scheme to lognormal distributed data. The data, consisting of 34 samples of size 5, are given in Table 8.1. The first 30 samples are taken from Morrison [41], where it was stated that they were collected from a process in the valves industry. The last 4 samples are added to simulate an out-of-control process. For the measurement of individual values, the upper and lower specification limits are 1 and 10.

A probability plot of the real data in Figure 8.1 suggests that the observations do not behave as though arising from a normal distribution. To adjust for non-normality, lognormal transformation is applied to the original data. A probability plot of the transformed data in Figure 8.2 shows that a lognormal distribution curve can be fitted quite well, suggesting that lognormal quality control

Table 8.1: Valve data.

Sample i	X_1	X_2	X_3	X_4	X_5	Sample i	X_1	X_2	X_3	X_4	X_5
1	4.55	4.99	3.62	3.52	3.77	18	2.85	4.16	3.17	2.50	3.91
2	1.93	3.95	4.10	4.16	1.61	19	3.16	3.70	2.61	2.65	3.42
3	2.22	1.73	5.10	4.52	4.06	20	2.54	4.77	1.63	2.64	3.59
4	2.71	2.45	4.6	2.09	1.90	21	3.61	2.13	5.08	2.01	1.92
5	2.91	5.68	4.33	3.51	3.24	22	3.16	4.20	2.32	2.44	1.62
6	2.20	5.66	3.71	3.35	1.61	23	2.96	6.09	3.78	2.29	4.16
7	2.82	5.22	3.75	3.50	3.31	24	2.47	3.49	3.38	4.45	2.61
8	2.76	4.4	3.13	1.55	3.70	25	3.55	3.35	3.18	4.75	8.72
9	4.98	4.05	4.00	7.20	3.18	26	1.35	2.50	2.51	4.20	3.50
10	4.88	2.71	3.51	3.15	4.81	27	2.30	2.26	2.22	1.60	9.70
11	4.50	1.95	3.41	2.87	1.90	28	3.71	3.06	1.53	2.45	6.40
12	3.07	4.02	4.17	4.33	4.06	29	9.48	1.72	4.20	3.37	5.58
13	2.39	2.91	3.09	3.15	2.52	30	1.90	2.56	4.28	3.18	1.94
14	2.92	4.25	3.02	2.26	5.72	31	8.88	9.35	9.75	9.98	14.9
15	2.56	4.38	1.24	2.62	1.92	32	2.55	1.32	9.21	8.22	3.15
16	2.53	4.16	3.78	3.77	1.72	33	1.63	15.90	9.62	8.58	9.96
17	3.41	3.10	6.02	1.09	2.92	34	1.12	9.78	8.50	7.96	9.20

scheme should be employed in this case. Suppose that the first 20 samples in Table 8.1 are used as preliminary samples. After applying logarithmic transformation. an S chart is set up with $\alpha_S = 0.0027$ and it is shown in Figure 8.3. When the 20 sample standard deviations are plotted on this chart. there is no indication of an out-of-control condition. Then σ^2 is estimated by $\bar{S}^2 = 0.1263$.

Since $z_{\alpha_1} = -3.0449$ and $z_{\alpha_2} = 3.0449$. a $\bar{Y}$ chart can be set up with $\alpha_{\bar{Y}} = 0.0023$ and it is shown in Figure 8.4. When the 20 sample means are plotted on this chart, there is also no indication of an out-of-control condition. Since the S and $\bar{Y}$ charts constructed using the first 20 samples indicate that both the process variability and the process mean are in control, the control limits obtained can be used in on-line statistical process control.

Assuming that there is a $2\sigma_0$ shift in the process mean and a 3 times change in the process standard deviation, the probability of detecting the mean shift on the first subsequence sample is 0.9236, and the probability of detecting the variability change on the first subsequence sample is 0.7398. Hence, the expected number of samples taken before the shift is detected is 1.0827, and the expected number of samples taken before the change is detected is 1.3517. When the last 14 sample means are plotted on the $\bar{Y}$ chart shown in Figure 8.5 and the last 14 sample standard deviations are plotted on the S chart shown in Figure 8.6 , it is seen that the last 4 points are abobe at least one of the UCL's. This indicates that the lognormal process is out of control with an increase in μ_* and σ_*^2. To identify the sources of these out-of-control signals, $\hat{\mu}_{*i}$ and $\hat{\sigma}_{*i}^2$ are calculated. It is found that $\hat{\mu}_{*31}$ and $\hat{\mu}_{*33}$ are greater than μ_{*U}, and $\hat{\sigma}_{*32}^2$, $\hat{\sigma}_{*33}^2$ and $\hat{\sigma}_{*34}^2$ exceed $\hat{\sigma}_{*U}^2$, which is equal to 13.4651. This diagnosis is supported by reference back to the individual measurements of the last 4 samples, since there are individuals exceeding μ_{*U} in sample 31 and 33 and greater variability within sample 32, 33, and 34. It should be noted that, for the last sample, only the lognormal process variability is out of control although both of the corresponding normal process mean and variability are out of control.

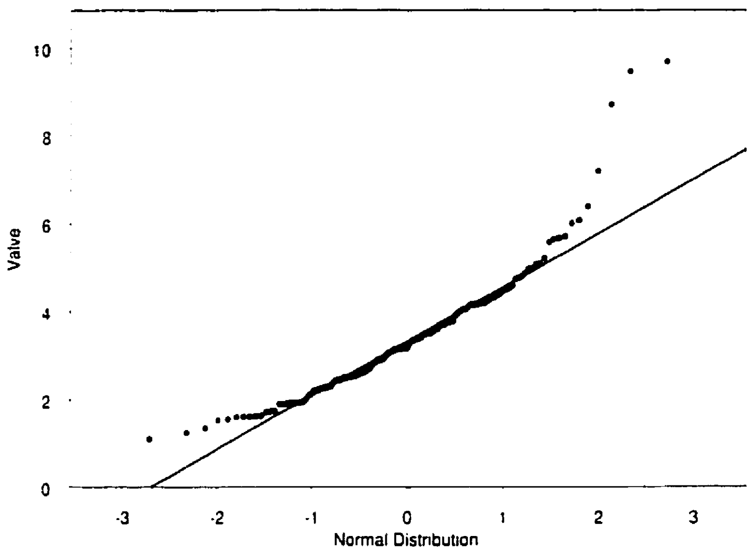

Figure 8.1: The probability plot for the valve data

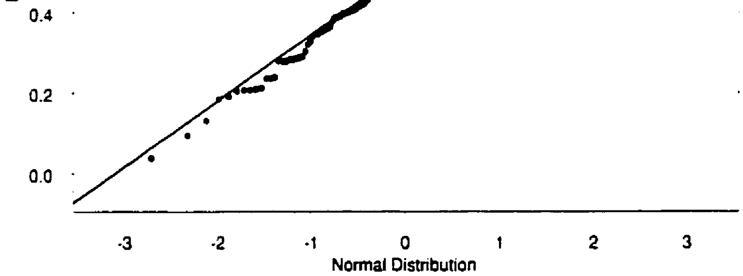

Figure 8.2: The probability plot for the logarithm of valve data

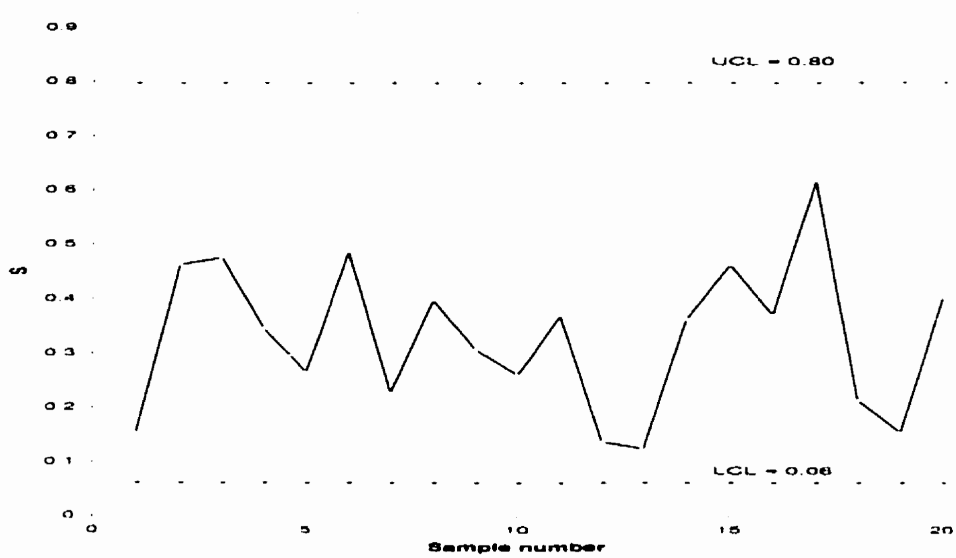

Figure 8.3: The first S chart for the valve data

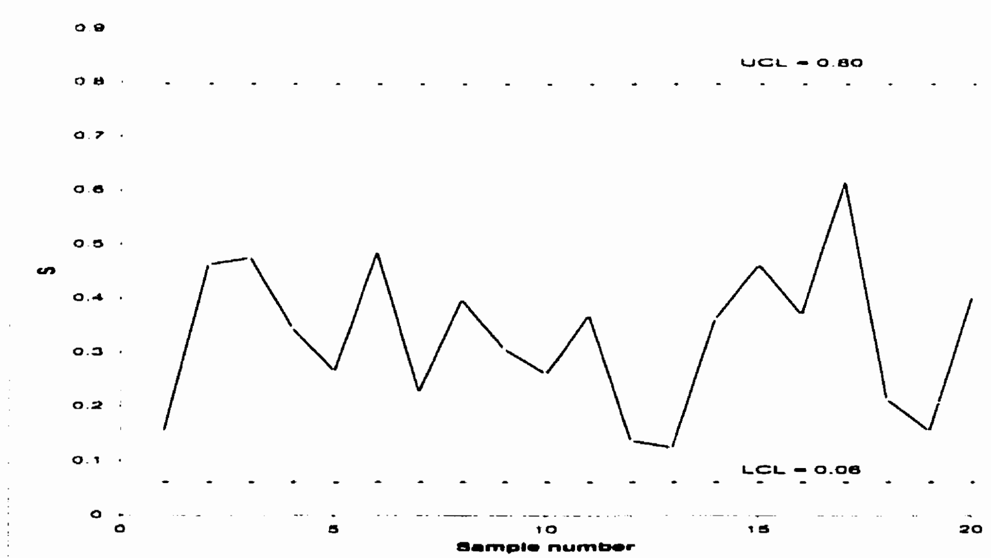

Figure 8.4: The first $\bar{Y}$ chart for the valve data

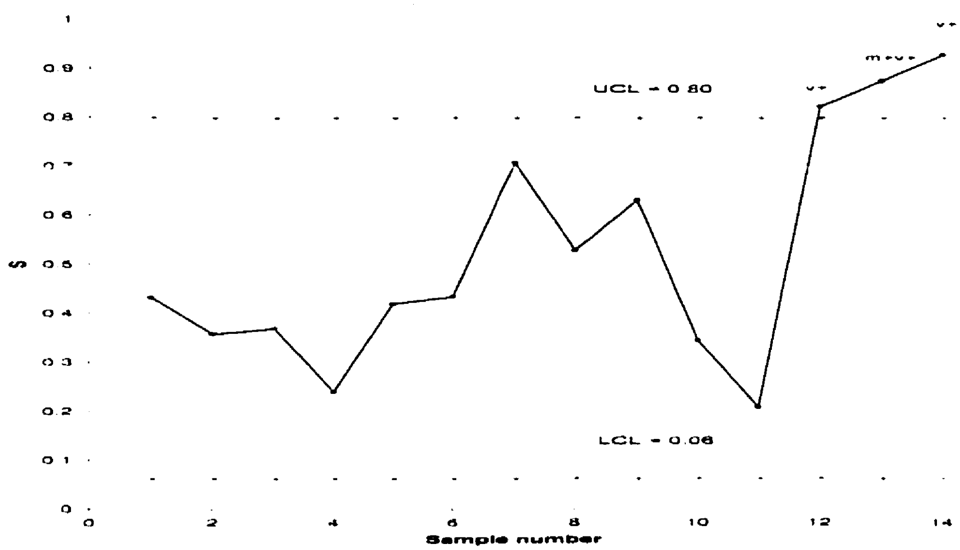

Figure 8.5: The second S chart for the valve data

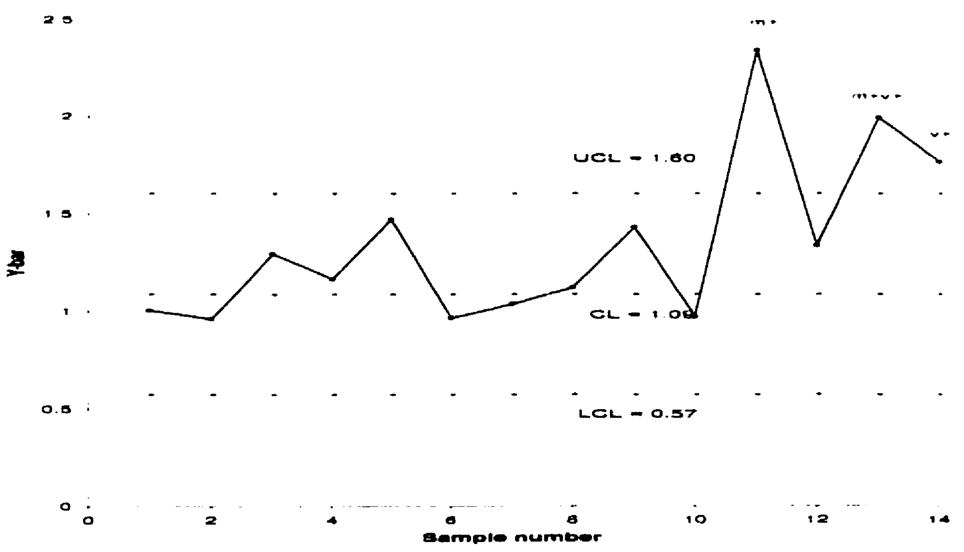

Figure 8.6: The second $\bar{Y}$ chart for the valve data

CHAPTER 9

A Multivariate Max-EWMA Control Chart

9.1 Introduction

There are many situations in which a process is simultaneously characterized by more than one related quality characteristic. Because these quality characteristics are correlated. quality control requires a multivariate approach. that is. it is necessary to simultaneously control these related quality characteristics.

As shown in Chapter 3 and Chapter 7. the univariate Max-EWMA chart has high capability of detecting small changes in the process mean and/or variability as well as identifying the source and the direction of an out-of-control signal. In this chapter. the technique used in the univariate Max-EWMA chart is extended to multivariate quality control and a multivariate Max-EWMA Chart is proposed. This new chart can be used to simultaneously monitor both the process mean vector and process variability in the multivariate case as well as identify the source and the direction of an out-of-control signal. Once an out-of-control signal is detected. a diagnostic developed by Runger. Alt and Montgomery [49] is employed to investigate which quality characteristic is responsible for the out-of-control condition. ARL properties are studied and Monte Carlo simulation is used to evaluate the ARL performance. Compared with the combination of the χ^2 and the $|\mathbf{S}|$ charts. the new chart is more sensitive in detecting small changes of a process. An example is given to illustrate the implementation of the new chart.

9.2 The New Control Chart

Assume that a process consists of k quality characteristics denoted by $\mathbf{X}$, where $\mathbf{X} \sim N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\mathbf{X}_{i_1}, \mathbf{X}_{i_2}, \ldots, \mathbf{X}_{i_n}, i = 1, 2, \ldots$ are the i^{th} sample of size n drawn from the process. Let $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$ be the mean vector and the standard covariance matrix, respectively.

To monitor the process mean vector, a statistic proposed by Rigdon and Champ [46] is given by

$$\mathbf{Z}_i = (1 - \lambda)\mathbf{Z}_{i-1} + \lambda(\bar{\mathbf{X}}_i - \boldsymbol{\mu}_0) \tag{9.1}$$

with $\mathbf{Z}_0$ as the starting point.

Because

$$E(\mathbf{Z}_i) = \boldsymbol{\mu} - \boldsymbol{\mu}_0 \tag{9.2}$$

and

$$Cov(\mathbf{Z}_i) = \frac{\lambda[(1 - \lambda)^{2i}]}{n(2 - \lambda)}\boldsymbol{\Sigma} \tag{9.3}$$

It is found that

$$\mathbf{Z}_i \sim N_k(\boldsymbol{\mu} - \boldsymbol{\mu}_0, Cov(\mathbf{Z}_i)) \tag{9.4}$$

and hence

$$T_i = \frac{n(2 - \lambda)}{\lambda[(1 - \lambda)^{2i}]}\mathbf{Z}_i'\boldsymbol{\Sigma}^{-1}\mathbf{Z}_i \sim \chi^2_{k,\delta^2} \tag{9.5}$$

where $\delta^2 = n(\boldsymbol{\mu} - \boldsymbol{\mu}_0)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu}_0)$. Then a statistic for monitoring the process mean vector is defined as

$$U_i = \Phi^{-1}[H_k\{\frac{n(2-\lambda)}{\lambda[(1-\lambda)^{2i}]}\mathbf{Z}_i'\boldsymbol{\Sigma}_0{}^{-1}\mathbf{Z}_i\}] \tag{9.6}$$

To monitor the process variability, a statistic that is a multivariate analog of the univariate S^2 is given by

$$W_i = \sum_{j=1}^{n}(\mathbf{X}_{ij} - \bar{\mathbf{X}}_i)'\boldsymbol{\Sigma}_0^{-1}(\mathbf{X}_{ij} - \bar{\mathbf{X}}_i) \tag{9.7}$$

Obviously, $W_i \sim \chi^2_{k(n-1)}$ when $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$. An EWMA statistic is defined as

$$Y_i = (1-\lambda)Y_{i-1} + \lambda\Phi^{-1}\{H_{k(n-1)}(W_i)\} \tag{9.8}$$

with Y_0 as the starting point. It is noted that if $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$, then

$$E(Y_i) = 0 \tag{9.9}$$

and

$$Var(Y_i) = \frac{\lambda[1-\lambda)^{2i}]}{2-\lambda} \tag{9.10}$$

A statistic for monitoring the process variability can be formed as

$$V_i = \sqrt{\frac{2-\lambda}{\lambda[(1-\lambda)^{2i}]}}Y_i \tag{9.11}$$

It is apparent that U_i and V_i are independent. When $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$, both U_i and V_i follow the standard normal distribution. Thus, based on U_i and V_i, a new statistic for the multivariate single chart would be defined as below:

$$M_i = max\{|U_i|, |V_i|\} \tag{9.12}$$

Notice that M_i is the maximum of the absolute values of the two multivariate EWMA statistics. It is natural to name the new chart the Max-MEWMA chart.

Similar to the univariate Max-EWMA chart, a large value of M_i, for the Max-MEWMA chart, means that the process mean vector has drifted away from $\boldsymbol{\mu}_0$ and/or the process variability has changed. On the other hand, a small value of M_i implies that the process mean vector and variability have remained close to their nominal values.

9.3 Derivation of the UCL

Because U_i and V_i independently follow the standard normal distributions, and given $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$, the in-control CDF of M_i is found to be

$$\begin{aligned} F(y; \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) &= P(M_i \le y | \boldsymbol{\mu} = \boldsymbol{\mu}_0, \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0) \\ &= P(|U_i| \le y, |V_i| \le y) \\ &= P(|U_i| \le y) P(|V_i| \le y) \\ &= [2\Phi(y) - 1]^2, \qquad y \ge 0 \end{aligned} \tag{9.13}$$

The corresponding pdf of M_i is given by

$$f(y; \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) = 4\phi(y)[2\Phi(y) - 1]^2 \tag{9.14}$$

Table 9.1: (λ, L) combinations for Max-MEWMA control schemes in a steady state when ARL_0=200.

K =2 and n = 2							
λ	0.05	0.10	0.20	0.30	0.60	0.80	1.00
L	2.7710	2.8563	2.9380	2.9730	3.0170	3.0230	3.0245
K = 2 and n = 5							
λ	0.05	0.10	0.20	0.30	0.60	0.80	1.00
L	2.7722	2.8659	2.9928	3.0592	3.1330	3.1397	3.1436
K =3 and n = 2							
λ	0.05	0.10	0.20	0.30	0.60	0.80	1.00
L	2.7573	2.8751	3.0025	3.0634	3.1330	3.1430	3.1453

Then, through numerical computation it is easy to find that $E(M_i) = 1.128379$ and $Var(M_i) = 0.363381$.

Therefore, the UCL is given by

$$\begin{aligned} UCL &= E(M_i) + L\sqrt{Var(M_i)} \\ &= 1.128379 + 0.602811L \qquad (9.15) \end{aligned}$$

where L is a multiplier and controls the performance of the chart with λ for a specified value of in-control ARL.

Table 9.1 lists some commonly used (λ, L) for the starting values $\mathbf{Z}_0 = \mathbf{0}$ and $Y_0 = 0$.

9.4 Properties of the ARL

The ARL study for a multivariate control chart could be very complicated if changes of the process covariance matrix are concerned. Even for the widely used $|\mathbf{S}|$ chart, it seems that no one has evaluated its ARL performance, resulting from the complex structure of the process covariance matrix. To simplify the ARL study of the Max-MEWMA Chart. two properties are given in this section.

Property 1. *Let* $\mathbf{X}_{1_i}, i = 1, 2, \ldots,$ *be independent sample mean vectors from pdf 1 and* $\mathbf{X}_{2_i}, i = 1, 2, \ldots,$ *be independent sample mean vectors from pdf 2. Let pdf 1 be multivariate normal with mean* $\boldsymbol{\mu}_1$ *and covariance matrix* $\boldsymbol{\Sigma}_1/n$ *and pdf 2 be multivariate normal with mean* $\boldsymbol{\mu}_2$ *and covariance matrix* $\boldsymbol{\Sigma}_2/n$. *If* $\boldsymbol{\mu}_1'\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2'\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\mu}_2$, *then* $\forall$ *integer m.* $f_1(U_i, i = 1, 2, \ldots, m) = f_2(U_i, i = 1, 2, \ldots, m)$, *where* $f_1(U_i, i = 1, 2, \ldots, m)$ *is the joint distribution of* $U_i, i = 1, 2, \ldots,$ *given pdf 1 and* $f_2(U_i, i = 1, 2, \ldots)$ *is the joint distribution of* $U_i, i = 1, 2, \ldots,$ *given pdf 2.*

Property 1 is based on Lowry's [34] results where she showed the joint distribution of T_i and thus the ARL depends on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ only through the value of the noncentrality parameter δ^2. Because U_i is defined upon T_i through two one-to-one transformation functions, $H(\cdot)$ and $\Phi^{-1}(\cdot)$, Property 1 can be obtained directly from Theorem 2 in Lowry [34]. Property 1 implies that, for different mean vectors as well as different covariance matrices, the joint distribution of U_i is still the same. Then, to investigate the property of shifts in the process mean vector, one only needs to look at the magnitude of δ^2 and does not have to consider each possible direction of the shift in one mean vector and each possible covariance matrix separately. This fundamental property is very useful in the evaluation of the ARL performance of the Max-MEWMA chart. Without it, too many possible situations would make the evaluation much more difficult.

Property 2. *Suppose that* $\boldsymbol{\Sigma} = b^2\boldsymbol{\Sigma}_0$. *When* $k = 2$ *and for specified values of* b *(or changes for variances), n and* $\boldsymbol{\mu}-\boldsymbol{\mu}_0$, *the value of noncentrality parameter* δ^2 *is only related to the magnitude of* σ_{12}, *the covariance of the two quality characteristics.*

From Property 1, joint distribution of $U_i, i = 1, 2, \ldots,$ does not depend on the direction of $\boldsymbol{\mu} - \boldsymbol{\mu}_0$ and the special from of $\boldsymbol{\Sigma}_0$.

Without loss of generality. let $\boldsymbol{\mu} - \boldsymbol{\mu}_0 = \begin{pmatrix} d_1 \\ 0 \end{pmatrix}$ and $\boldsymbol{\Sigma}_0 = \begin{pmatrix} 1 & \sigma_{12} \\ \sigma_{12} & 1 \end{pmatrix}$ where d_1 is a constant. Then. the noncentrality parameter is

$$\begin{aligned} \delta^2 &= n(\boldsymbol{\mu} - \boldsymbol{\mu}_0)'(b^2\boldsymbol{\Sigma}_0)^{-1}n(\boldsymbol{\mu} - \boldsymbol{\mu}_0) \\ &= \frac{nd_1^2}{b^2(1-\sigma_{12}^2)} \end{aligned} \tag{9.16}$$

Because the values of b and n and d_1 are fixed. δ^2 is only related to the magnitude of σ_{12}.

Property 2 indicates that for $k = 2$. increases and decreases of the same magnitude in σ_{12} result in the same ARL provided that the values of b and n and d_1 are specified. The importance of this property is clear when one realizes that the number of ARL values required for the evaluation can be reduced to half of the original ones.

9.5 The ARL Performance

Even though the burden of the ARL evaluation is partially alleviated by using Property 1 and Property 2. it still requires much work to evaluate the ARL performance in detail because of the complex nature for a multivariate chart. especially for a multivariate EWMA single chart. In this section. detailed discussion of the ARL performance for the Max-MEWMA chart is given with respect to different values of λ. k. n, the shift in the process mean vector and the change in the process covariance matrix. For the Max-MEWMA chart, there is no direct way to compute the ARL, so each ARL value is obtained using 10,000 simulations.

9.5.1 Computation Set-up

According to Property 1. the Max-MEWMA chart is directional invariate in the mean shift. Without loss of generality. assume that for $k = 2$.

$$\boldsymbol{\mu}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{\mu} = \begin{pmatrix} 0 \\ a \end{pmatrix} \text{ and } \boldsymbol{\Sigma}_0 = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \quad -1 < \rho < 1;$$

for $k = 3$.

$$\boldsymbol{\mu}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \text{ and } \boldsymbol{\Sigma}_0 = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}, \quad -1 < \rho < 1;$$

That is. for $k = 2$. $\sigma_{01}^2 = \sigma_{02}^2 = 1$. and $\sigma_{12}^2 = \sigma_{21}^2 = \rho$. the correlation between the two quality characteristics; for $k = 3$. $\sigma_{01}^2 = \sigma_{02}^2 = \sigma_{03}^2 = 1$. $\sigma_{12}^2 = \sigma_{13}^2 = \sigma_{21}^2 = \sigma_{31}^2 = \sigma_{23}^2 = \sigma_{32}^2 = \rho$. the correlation between any two of the three quality characteristics.

For a given in-control ARL of 200. ARL's are simulated. It is seen. from Property 1. that δ^2 only depends the magnitude of a and thus only positive values are needed to be considered for a. Similarly. when $k = 2$. only positive values are needed for ρ based on Property 2. It should be noted that. in order to get a positive definite matrix $\boldsymbol{\Sigma}$. ρ can only take limited vales in (-1. 1).

To calculate ARL's of the Max-MEWMA chart, we consider three forms of the changed covariance matrix:

I. $b^2\boldsymbol{\Sigma}_0$ $(b > 0)$

In this case, the correlation between the two quality characteristics is still equal to ρ after the covariance matrix has changed. The combinations are set as below:

1. $k = 2; n = 2$:

   $\lambda = 0.05, 0.1(0.1)1.0$;

   $a = 0.0(0.5)3.0$;

   $b = 0.0(0.5)3.0$;

   $\rho = 0.0(0.3)0.9$.

2. $k = 2; n = 5$:

   $\lambda = 0.05, 0.1(0.1)1.0$;

   $a = 0.0(0.5)3.0$;

   $b = 0.0(0.5)3.0$;

   $\rho = 0.0(0.3)0.9$.

3. $k = 3; n = 2$:

   $\lambda = 0.05, 0.1, 0.2, 0.3$;

   $a = 0.0(0.5)3.0$;

   $b = 0.0(0.5)3.0$;

   $\rho = -0.3(0.3)0.9$.

$$II.\ \mathbf{\Sigma} = \begin{pmatrix} \rho & \sigma_{12} \\ \sigma_{12} & \rho \end{pmatrix}$$

Let ρ' be the correlation between the two quality characteristics after the covariance has changed, then $\rho' = \frac{\rho}{\sigma_1\sigma_2}$, which results in $\rho \in (-1, 1) \cap (-\sigma_1\sigma_2, \sigma_1\sigma_2)$. The combinations are set as below:

For $k = 2$, $n = 2$, or $n = 5$; $\lambda = 0.05, 0.1(0.1)1.0$, and $a = 0.0(0.5)3.0$.

1. Both σ_1 and σ_2 increase with $\rho' < \rho$.

$\sigma_1 = 1.25$ and $\sigma_2 = 2.0$:

$\rho = 0.0(0.3)0.9$.

2. Only one σ increases with $\rho' < \rho$.

   $\sigma_1 = 1.25$ and $\sigma_2 = 1.0$:

   $\rho = 0.0(0.3)0.9$.

3. Both σ_1 and σ_2 decrease with $\rho' > \rho$.

   $\sigma_1 = \sigma_2 = 0.6$:

   or $\sigma_1 = 0.6$ and $\sigma_2 = 0.8$:

   $\rho = 0.0, 0.3$.

4. Only σ increases with $\rho' > \rho$.

   $\sigma_1 = 1.25$ and $\sigma_2 = 1.0$:

   $\rho = 0.0, 0.3$.

5. One σ increases but the other decreases. $\sigma_1 = 0.5$ and $\sigma_2 = 1.5$ with $\rho' > \rho$.

   $\rho = 0.0, 0.3, 0.6$: $\sigma_1 = 0.5$ and $\sigma_2 = 2.0$ with $\rho' = \rho$.

   or $\sigma_1 = 0.5$ and $\sigma_2 = 2.5$ with $\rho' < \rho$.

   $\rho = 0.0(0.3)0.9$.

$$III.\ \mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}$$

In this case, $\rho' = \rho'_{12} = \rho'_{13} = \rho'_{21} = \rho'_{31} = \rho/\sigma_1$ and $\rho'_{23} = \rho'_{32} = \rho$, where $\rho \in (-0.5, 1) \cap (\frac{\sigma_1^2 - \sqrt{8+\sigma_1^2}\sigma_1}{4}, \frac{\sigma_1^2 + \sqrt{8+\sigma_1^2}\sigma_1}{4})$. For $k = 3, n = 2$, $\lambda = 0.05, 0.1, 0.2, 0.3$, and $a = 0.0(0.5)3.0$.

1. σ_1 decreases with $\rho' > \rho$.

   $\sigma_1 = 0.75$. $\rho = -0.3, 0.0, 0.3$.

2. σ_1 increases with $\rho' < \rho$.

   $\sigma_1 = 1.25$; or $\sigma_1 = 1.50$; or $\sigma_1 = 2.0$;

   $\rho = -0.3(0.3)0.9$.

Because too many tables are required to list all the results. Tables 9.2-9.5 display the ARL performance for the various changes in the process mean vector and variability when $\lambda = 0.2$. which is one of the popular choices in practice for EWMA-type control charts.

9.5.2 Discussion

When $\mathbf{\Sigma} = b^2\mathbf{\Sigma}_0$, the distribution of U_i is given by

$$P(U_i \leq y) = H_{k,\delta^2}\left(\frac{H^{-1}(\Phi(y))}{b^2}\right) \tag{9.17}$$

where $\delta^2 = \frac{n}{b^2}(\boldsymbol{\mu} - \boldsymbol{\mu}_0)'\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu}_0)$.

Then the ARL performance. in term of the capacity for detecting the shift in the mean vector. depends on $\boldsymbol{\mu} - \boldsymbol{\mu}_0$ and $\mathbf{\Sigma}$ only through the value of δ^2.

For $k = 2$, $\delta^2 = \frac{na^2}{b^2(1-\rho^2)}$.

For example. if $\lambda = 0.20$,$n = 2$.and $b = 0.5$, two cases are chosen as

$a_1 = 2.5$, $\rho_1 = 0.0$; $a_2 = 2.0$, $\rho_2 = 0.6$

so that $\delta_1^2 \approx \delta_2^2$. As seen in Table 9.2, $ARL_1 = ARL_2 = 2.3$.

For $k = 3$, $\delta^2 = \frac{na^2(1+\rho)}{b^2(1+\rho-2\rho^2)}$.

For example, if $\lambda = 0.20$,$n = 2$,and $b = 1.5$, two cases are chosen as

$a_1 = 2.5$, $\rho_1 = 0.3$; $a_2 = 2.0$, $\rho_2 = 0.6$

Table 9.2: ARL values of Max-MEWMA chart when $k = 2$, $n = 2$, $L = 2.9380$ and $\lambda = 0.20$ in Case I.

$ARL_0 = 200$								
		a						
	ρ	0.00	0.50	1.00	1.50	2.00	2.50	3.00
$b = 0.50$	0.0	10.5	10.9	7.4	4.2	3.0	2.3	2.0
	0.3	10.4	11.0	6.9	3.9	2.9	2.1	2.0
	0.6	10.5	10.7	5.4	3.2	2.3	2.0	1.9
	0.9	10.5	6.1	2.6	2.0	1.1	1.0	1.0
$b = 1.00$	0.0	202.7	30.6	7.6	4.2	3.0	2.4	2.0
	0.3	202.2	27.6	7.1	4.0	2.8	2.3	2.0
	0.6	202.8	18.8	5.4	3.2	2.4	2.0	1.7
	0.9	202.2	6.2	2.6	1.9	1.3	1.0	1.0
$b = 1.50$	0.0	10.1	8.2	5.5	3.8	2.9	2.4	2.0
	0.3	10.1	8.1	5.3	3.7	2.8	2.3	1.9
	0.6	10.0	7.5	4.5	3.1	2.4	1.9	1.7
	0.9	10.1	4.9	2.6	1.8	1.3	1.1	1.0
$b = 2.00$	0.0	4.3	4.1	3.6	3.1	2.6	2.2	1.9
	0.3	4.3	4.1	3.5	3.0	2.5	2.2	1.9
	0.6	4.3	4.0	3.3	2.7	2.2	1.9	1.6
	0.9	4.3	3.5	2.4	1.7	1.4	1.1	1.0
$b = 2.50$	0.0	2.9	2.9	2.7	2.5	2.2	2.0	1.8
	0.3	2.9	2.9	2.7	2.5	2.2	2.0	1.8
	0.6	2.9	2.8	2.6	2.3	2.0	1.8	1.6
	0.9	2.9	2.6	2.1	1.7	1.4	1.2	1.1
$b = 3.00$	0.0	2.3	2.3	2.2	2.1	2.0	1.9	1.7
	0.3	2.3	2.3	2.2	2.1	2.0	1.8	1.7
	0.6	2.3	2.3	2.2	2.0	1.9	1.7	1.5
	0.9	2.3	2.2	1.9	1.6	1.4	1.2	1.1

Table 9.3: ARL values of Max-MEWMA chart when $k = 3$, $n = 2$, $L = 3.0025$ and $\lambda = 0.20$ in Case I.

		$ARL_0 = 200$						
		a						
	ρ	0.00	0.50	1.00	1.50	2.00	2.50	3.00
$b = 0.50$	-0.3	6.3	6.8	5.9	3.8	2.8	2.1	2.0
	0.0	6.3	6.8	6.5	4.5	3.2	2.5	2.1
	0.3	6.3	6.9	6.2	4.2	3.0	2.3	2.0
	0.6	6.3	6.9	5.1	3.2	2.3	2.0	1.9
	0.9	6.4	5.3	2.4	2.0	1.0	1.0	1.0
$b = 1.00$	-0.3	199.2	25.5	6.6	3.7	2.7	2.2	1.9
	0.0	200.4	36.6	8.4	4.5	3.2	2.4	2.1
	0.3	200.3	30.0	7.4	4.2	2.9	2.3	2.0
	0.6	200.1	18.2	5.3	3.2	2.3	2.0	1.7
	0.9	201.3	5.5	2.4	1.8	1.2	1.0	1.0
$b = 1.50$	-0.3	7.5	6.4	4.6	3.4	2.6	2.1	1.8
	0.0	7.5	6.6	5.1	3.8	3.0	2.3	2.0
	0.3	7.5	6.5	4.9	3.6	2.8	1.9	1.6
	0.6	7.5	6.1	4.1	3.0	2.3	1.7	1.2
	0.9	7.5	4.2	2.3	2.2	1.9	1.0	1.0
$b = 2.00$	-0.3	3.4	3.3	3.0	2.6	2.3	2.0	1.7
	0.0	3.4	3.3	3.1	2.8	2.5	2.1	1.8
	0.3	3.4	3.3	3.0	2.7	2.4	1.9	1.8
	0.6	3.4	3.2	2.9	2.4	2.1	1.8	1.6
	0.9	3.4	2.9	2.1	1.6	1.2	1.1	1.0
$b = 2.50$	-0.3	2.4	2.4	2.3	2.1	2.0	1.8	1.6
	0.0	2.4	2.4	2.3	2.2	2.1	1.9	1.7
	0.3	2.4	2.4	2.3	2.2	2.0	1.7	1.6
	0.6	2.4	2.4	2.2	2.1	1.9	1.7	1.5
	0.9	2.4	2.2	1.9	1.5	1.2	1.0	1.0
$b = 3.00$	-0.3	2.0	2.0	1.9	1.8	1.7	1.6	1.5
	0.0	2.0	2.0	2.0	1.9	1.8	1.7	1.6
	0.3	2.0	2.0	1.9	1.9	1.8	1.5	1.4
	0.6	2.0	2.0	1.9	1.8	1.7	1.1	1.0
	0.9	2.0	1.9	1.7	1.5	1.2	1.0	1.0

Table 9.4: ARL values of Max-MEWMA chart when $k = 2$, $n = 2$, $L = 2.9380$ and $\lambda = 0.20$ in Case II.

$ARL_0 = 200$								
		a						
	ρ	0.00	0.50	1.00	1.50	2.00	2.50	3.00
	0.0	64.7	22.6	7.1	4.1	2.9	2.3	2.0
$\sigma_1 = 1.25$	0.3	57.3	19.9	6.7	3.9	2.8	2.2	2.0
$\sigma_2 = 1.00$	0.6	37.2	13.5	5.1	3.2	2.3	2.0	1.7
	0.9	8.6	4.8	2.6	1.8	1.4	1.1	1.0
	0.0	28.2	15.2	6.8	4.1	3.0	2.4	2.0
$\sigma_1 = 1.25$	0.3	24.8	13.5	6.3	3.9	2.8	2.3	1.9
$\sigma_2 = 1.25$	0.6	15.5	9.5	4.9	3.2	2.4	2.0	1.7
	0.9	4.7	3.7	2.5	1.8	1.4	2.2	1.1
	0.0	7.0	6.3	4.8	3.7	2.9	2.4	2.1
$\sigma_1 = 1.25$	0.3	6.6	5.7	4.6	3.5	2.8	2.3	2.0
$\sigma_2 = 2.00$	0.6	4.8	4.4	3.6	2.9	2.3	2.0	1.7
	0.9	2.4	2.3	2.0	1.8	1.5	1.3	1.2
$\sigma_1 = 0.60$	0.3	18.4	18.7	8.0	4.2	3.0	2.3	2.0
$\sigma_2 = 0.60$	0.6	11.1	10.5	6.9	3.9	2.8	2.1	2.0
$\sigma_1 = 0.60$	0.3	36.5	28.8	8.1	4.3	3.0	2.3	2.0
$\sigma_2 = 0.80$	0.6	24.8	22.0	7.4	4.0	2.8	2.2	2.0

Table 9.5: ARL values of Max-MEWMA chart when $k = 3$, $n = 2$, $L = 3.0025$ and $\lambda = 0.20$ in Case II.

$ARL_0 = 200$								
		a						
	ρ	0.00	0.50	1.00	1.50	2.00	2.50	3.00
$\sigma_1 = 0.75$	-0.3	107.7	28.7	6.8	3.8	2.7	2.2	1.9
	0.0	156.7	38.4	8.7	4.6	3.2	2.5	2.1
	0.3	129.3	33.0	7.6	4.2	3.0	2.3	2.0
$\sigma_1 = 1.25$	-0.3	57.9	18.5	6.3	3.7	2.7	2.2	1.9
	0.0	80.3	26.8	78.0	4.4	3.1	2.5	2.1
	0.3	68.3	22.4	7.0	4.0	2.9	2.3	2.0
	0.6	39.0	13.5	5.0	3.1	2.3	1.9	1.7
	0.9	7.7	4.2	2.3	1.7	1.2	1.0	1.0
$\sigma_1 = 1.50$	-0.3	20.0	12.2	5.7	3.6	2.6	2.1	1.9
	0.0	29.3	17.5	7.2	4.3	3.1	2.4	2.1
	0.3	24.3	14.7	6.4	3.9	2.8	2.3	2.0
	0.6	13.5	8.9	4.6	3.0	2.3	1.9	1.7
	0.9	4.03	3.2	2.1	1.6	1.2	1.0	1.0
$\sigma_1 = 2.00$	-0.3	7.3	6.3	4.5	3.2	2.5	2.1	1.8
	0.0	9.7	8.5	5.6	3.9	2.9	2.4	2.0
	0.3	8.4	7.3	5.0	3.5	2.7	2.2	1.9
	0.6	5.6	5.0	3.7	2.7	2.2	1.8	1.6
	0.9	2.5	2.3	1.9	1.5	1.2	1.0	1.0

so that $\delta_1^2 \approx \delta_2^2$. As seen in Table 9.3. $ARL_1 = ARL_2 = 2.3$.

When $\boldsymbol{\Sigma}$ has the form as defined in II or III. It is impossible to directly get the distribution of U_i. However. since this new chart is mainly designed for detecting small changes of a process. the changes of σ_1 and/or σ_2 are small. The difference between $\delta_0^2 = (\boldsymbol{\mu} - \boldsymbol{\mu}_0)'\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu}_0)$ and $\delta^2 = (\boldsymbol{\mu} - \boldsymbol{\mu}_0)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu}_0)$ is usually small. and δ_0^2 is an approximate noncentrality parameter for δ.

For $k = 2$. $\delta_0^2 = \frac{na^2}{1-\rho^2}$.

For example. if $\lambda = 0.20, n = 2$. $\sigma_1 = 1.25$ and $\sigma_2 = 1.00$. two cases are chosen as

$a_1 = 2.5$. $\rho_1 = 0.0$; $a_2 = 2.0$. $\rho_2 = 0.6$

so that $\delta_1^2 \approx \delta_2^2$. As seen in Table 9.4. $ARL_1 = ARL_2 = 2.3$.

For $k = 3$. $\delta^2 = \frac{na^2(1+\rho)}{1+\rho-2\rho^2}$.

For example. if $\lambda = 0.20, n = 2$.and $\sigma_1 = 1.25$. two cases are chosen as

$a_1 = 2.5$. $\rho_1 = 0.3$; $a_2 = 2.0$. $\rho_2 = 0.6$

so that $\delta_1^2 \approx \delta_2^2$. As seen in Table 9.5. $ARL_1 = ARL_2 = 2.3$.

The results displayed in all the tables show that ARL's become smaller when a increases and/or b (or $\sigma's$) has a big change. It is noted that. when ρ increases. the highly correlated quality characteristics result in shorter ARL's since more information is available in the highly correlated data. One interesting phenomenon is that. as k increases. ARL's increase for a shift in the mean vector alone. but ARL's decrease for a change in the variability alone. This can be explained by the fact that an increasing amount of noise associated with the higher dimensions makes it harder to detect a shift in the mean vector. but it is more sensitive to detect a change in the variability. Another notable fact is that, for a small change within the process, the smallest ARL is obtained corresponding to small values for weakly correlated data, however, the optimal value for λ will become slightly larger when data are highly correlated. For an increasing sample size, it is well known that ARL's become smaller since more information is contained in a bigger sample.

9.6 One Combination Chart

Two control charts. the χ^2 chart and the $|\mathbf{S}|$ chart. have been widely used in multivariate quality control.

To monitor the process mean vector. the test statistic plotted on the χ^2 chart is

$$T^2 = n(\bar{\mathbf{X}}_i - \boldsymbol{\mu}_0)'\boldsymbol{\Sigma}_0^{-1}(\bar{\mathbf{X}}_i - \boldsymbol{\mu}_0) \tag{9.18}$$

with $UCL_1 = \chi^2_{\alpha_1,k}$.

To monitor the process variability. $|\mathbf{S}|$ is the plotted test statistic. When $k = 2$. $\frac{2(n-1)|\mathbf{S}_i|^{\frac{1}{2}}}{|\boldsymbol{\Sigma}_0|^{\frac{1}{2}}} \sim \chi^2_{2n-4}$. Then the control limits for the $|\mathbf{S}_i|$ chart are given by

$$UCL_2 = \frac{|\boldsymbol{\Sigma}_0|(\chi^2_{6,\alpha_2})^2}{4(n-1)^2} \tag{9.19}$$

$$LCL_2 = \frac{|\boldsymbol{\Sigma}_0|(\chi^2_{6,1-\alpha_2})^2}{4(n-1)^2} \tag{9.20}$$

For $k = 2$, $n = 5$. $\alpha_1 = 0.0025$, and $\alpha_2 = 0.00125$. the combination of the χ^2 chart and the $|\mathbf{S}|$ chart has a combined Type I error probability $1-(1-0.0025)^2 \approx 0.005$. which is equivalent to an in-control ARL of 200.

Let $\boldsymbol{\mu}_0$, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}_0$ be the same as those defined in Section 9.4.1. To calculate ARL's of the combination chart, two forms of the changed covariance matrix are considered.

I. $b^2\boldsymbol{\Sigma}_0$ $(b > 0)$

Let p_1 be the probability of an out-of-control signal detected by the χ^2 chart. Let p_2 be the probability of an out-of-control signal detected by the $|\mathbf{S}|$ chart. Let p be the probability of an out-of-control signal detected by the combination chart.

For various changes in the mean vector and/or the variability. We have

$$
\begin{aligned}
p_1 &= 1 - H_{2,\delta^2}\left(\frac{\chi^2_{2,0.0025}}{b^2}\right) \\
p_2 &= 1 - H_6\left(\frac{\chi^2_{2,0.00125}}{b^2}\right) + H_6\left(\frac{\chi^2_{2,0.99875}}{b^2}\right) \\
p &= 1 - (1 - p_1)(1 - p_2)
\end{aligned}
$$

Because $T_1^2, T_2^2, \ldots$ are independent and so are $|\mathbf{S}_1|, |\mathbf{S}_2|, \ldots$, the ARL for the combination chart is $1/p$ with respect to a, b and ρ.

$$
II.\ \mathbf{\Sigma} = \begin{pmatrix} \rho & \sigma_{12} \\ \sigma_{12} & \rho \end{pmatrix}
$$

In this case, because the distributions of T_i and $|\mathbf{S}_i|$ can not be obtained directly, simulations have to be used. The combinations chosen are the same as those in Section 9.4.1.

For various changes in the mean vector alone, in the variability alone, and in both the mean vector and the variability, ARL's for the combination chart are calculated. Some representative results are given in Tables 9.6-9.8 and compared with the ARL's obtained from the Max-MEWMA chart with respect to different λ values. As expected, the Max-MEWMA chart yields smaller ARL's than the combination chart. Thus, it is more sensitive than the combination chart in detecting small to moderate changes in the mean vector and/or the variability. In general, smaller λ values give better ARL performance.

Table 9.6: ARL's of the Max-MEWMA chart and the combination of χ^2 chart and $|\mathbf{S}|$ chart when $k = 2$, $n = 5$, $\lambda = 0.60$ and $L = 2.7722$ in Case I.

		Max-MEWMA chart					Combination Chart				
		a					a				
	ρ	0.00	0.25	0.50	1.00	2.00	0.00	0.25	0.50	1.00	2.00
	0.0	3.2	3.2	2.9	1.1	1.0	19.6	19.6	16.8	1.0	1.0
$b = 0.50$	0.3	3.2	3.2	2.8	1.1	1.0	19.6	19.6	15.1	1.0	1.0
	0.6	3.2	3.2	2.3	1.0	1.0	19.6	19.6	6.6	1.0	1.0
	0.9	3.2	2.6	1.0	1.0	1.0	19.6	10.8	1.0	1.0	1.0
	0.0	201.3	28.4	4.1	1.3	1.0	200.2	48.4	6.6	1.1	1.0
$b = 1.00$	0.3	199.3	24.9	3.7	3.0	1.0	200.2	43.6	5.6	1.1	1.0
	0.6	199.3	14.9	2.6	2.5	1.0	200.2	28.2	3.2	1.0	1.0
	0.9	199.3	3.1	1.1	1.0	1.0	200.2	4.2	1.0	1.0	1.0
	0.0	3.6	3.2	2.3	1.3	1.0	5.1	4.1	2.6	1.2	1.2
$b = 1.50$	0.3	3.6	3.2	2.3	1.2	1.0	5.1	4.0	2.4	1.2	1.2
	0.6	3.6	3.0	2.0	1.1	1.0	5.1	3.7	2.0	1.1	1.1
	0.9	3.6	2.1	1.2	1.0	1.0	5.1	2.2	1.1	1.1	1.0
	0.0	1.5	1.5	1.4	1.2	1.0	1.7	1.6	1.4	1.1	1.0
$b = 2.00$	0.3	1.5	1.5	1.4	1.1	1.0	1.7	1.6	1.4	1.1	1.0
	0.6	1.5	1.5	1.3	1.1	1.0	1.7	1.6	1.3	1.1	1.0
	0.9	1.5	1.3	1.1	1.0	1.0	1.7	1.4	1.1	1.1	1.0
	0.0	1.1	1.1	1.1	1.1	1.0	1.2	1.2	1.1	1.1	1.0
$b = 2.50$	0.3	1.1	1.1	1.1	1.0	1.0	1.2	1.2	1.1	1.1	1.0
	0.6	1.1	1.1	1.1	1.0	1.0	1.2	1.2	1.1	1.0	1.0
	0.9	1.1	1.1	1.0	1.0	1.0	1.2	1.1	1.0	1.0	1.0
	0.0	1.0	1.0	1.0	1.0	1.0	1.1	1.1	1.1	1.0	1.0
$b = 3.00$	0.3	1.0	1.0	1.0	1.0	1.0	1.1	1.1	1.1	1.0	1.0
	0.6	1.0	1.0	1.0	1.0	1.0	1.1	1.1	1.0	1.0	1.0
	0.9	1.0	1.0	1.0	1.0	1.0	1.1	1.0	1.0	1.0	1.0

Table 9.7: ARL's of the Max-MEWMA chart and the combination of χ^2 chart and $|\mathbf{S}|$ chart when $k = 2$, $n = 5$, $\lambda = 0.20$ and $L = 2.7722$ in Case I.

		Max-MEWMA chart					Combination Chart				
				a					a		
	ρ	0.00	0.25	0.50	1.00	2.00	0.00	0.25	0.50	1.00	2.00
	0.0	3.2	3.2	3.1	2.0	1.1	19.6	19.6	16.8	1.0	1.0
$b = 0.50$	0.3	3.2	3.2	3.0	2.0	1.0	19.6	19.6	15.1	1.0	1.0
	0.6	3.2	3.2	2.8	1.8	1.0	19.6	19.6	6.6	1.0	1.0
	0.9	3.2	2.9	2.0	1.0	1.0	19.6	10.8	1.0	1.0	1.0
	0.0	199.4	11.5	3.9	2.0	1.2	200.2	48.4	6.6	1.1	1.0
$b = 1.00$	0.3	199.5	11.5	3.7	1.9	1.1	200.2	43.6	5.7	1.1	1.0
	0.6	199.5	7.7	3.0	1.6	1.0	200.2	28.2	3.2	1.0	1.0
	0.9	199.5	3.3	1.8	1.0	1.0	200.2	4.2	1.0	1.0	1.0
	0.0	4.2	3.8	3.1	1.9	1.3	5.1	4.1	2.6	1.2	1.2
$b = 1.50$	0.3	4.2	3.8	3.0	1.8	1.2	5.1	4.0	2.4	1.2	1.2
	0.6	4.2	3.7	2.7	1.6	1.0	5.1	3.7	2.0	1.1	1.1
	0.9	4.2	2.8	1.7	1.0	1.0	5.1	2.2	1.1	1.0	1.0
	0.0	2.3	2.3	2.2	1.8	1.3	1.7	1.6	1.4	1.1	1.0
$b = 2.00$	0.3	2.3	2.3	2.1	1.7	1.3	1.7	1.6	1.4	1.1	1.0
	0.6	2.3	2.3	2.1	1.5	1.1	1.7	1.6	1.3	1.1	1.0
	0.9	2.3	2.1	1.6	1.0	1.0	1.7	1.4	1.1	1.0	1.0
	0.0	2.0	2.0	1.9	1.7	1.3	1.2	1.2	1.1	1.1	1.0
$b = 2.50$	0.3	2.0	2.0	1.9	1.6	1.3	1.2	1.2	1.1	1.1	1.0
	0.6	2.0	2.0	1.9	1.5	1.1	1.2	1.2	1.1	1.0	1.0
	0.9	2.0	1.9	1.5	1.0	1.0	1.2	1.1	1.0	1.0	1.0
	0.0	1.9	1.9	1.8	1.6	1.3	1.1	1.1	1.1	1.0	1.0
$b = 3.00$	0.3	1.9	1.9	1.8	1.6	1.3	1.1	1.1	1.1	1.0	1.0
	0.6	1.9	1.8	1.7	1.4	1.2	1.1	1.1	1.0	1.0	1.0
	0.9	1.9	1.8	1.5	1.1	1.1	1.1	1.0	1.0	1.0	1.0

Table 9.8: ARL's of the Max-MEWMA chart and the combination of χ^2 chart and $|\mathbf{S}|$ chart when $k = 2$, $n = 5$, $\lambda = 0.20$ and $L = 2.7722$ in Case II.

		Max-MEWMA chart					Combination Chart				
				a					a		
	ρ	0.00	0.25	0.50	1.00	2.00	0.00	0.25	0.50	1.00	2.00
$\sigma_1 = 0.60$	0.0	4.4	4.5	3.6	2.0	1.1	49.0	48.4	21.6	1.0	1.0
$\sigma_2 = 0.60$	0.3	3.5	3.6	3.2	2.0	1.0	13.0	12.8	10.2	1.0	1.0
	0.0	28.2	9.5	3.9	2.0	1.2	53.5	24.7	5.4	1.1	1.0
$\sigma_1 = 0.60$	0.3	24.6	8.8	3.6	1.9	1.3	47.6	21.3	4.7	1.1	1.0
$\sigma_2 = 0.80$	0.6	14.3	6.5	3.0	1.6	1.0	30.0	11.7	2.8	1.0	1.0
	0.9	3.7	2.8	1.7	1.0	1.0	5.6	2.4	1.2	1.0	1.0
	0.0	10.3	7.0	3.7	1.9	1.2	16.7	11.3	4.0	1.2	1.0
$\sigma_1 = 1.25$	0.3	9.0	6.4	3.5	1.9	1.2	17.1	9.7	3.5	1.2	1.0
$\sigma_2 = 1.00$	0.6	5.9	4.6	2.9	1.6	1.0	10.1	5.8	2.3	1.1	1.0
	0.9	2.4	2.2	1.7	1.0	1.0	2.3	1.7	1.2	1.0	1.0
	0.0	3.2	3.0	2.7	1.9	1.3	3.5	3.1	2.3	1.3	1.0
$\sigma_1 = 1.25$	0.3	3.0	2.9	2.5	1.8	1.3	3.2	2.8	2.1	1.2	1.0
$\sigma_2 = 1.25$	0.6	2.5	2.4	2.2	1.6	1.2	2.3	2.1	1.6	1.1	1.0
	0.9	1.8	1.8	1.6	1.2	1.0	1.2	1.2	1.1	1.0	1.0
	0.0	4.9	4.3	3.3	2.0	1.4	11.3	7.1	3.6	1.4	1.1
$\sigma_1 = 0.50$	0.3	4.5	4.0	3.1	1.9	1.3	10.3	6.6	3.3	1.4	1.0
$\sigma_2 = 2.00$	0.6	3.5	3.2	2.5	1.6	1.1	7.2	4.8	2.5	1.2	1.0
	0.9	2.0	1.9	1.6	1.1	1.0	2.9	2.2	1.5	1.0	1.0

However, as in univariate case, occurrence of an inertia problem is the worst-state for a EWMA-type control chart because the EWMA-type control chart may not react to a large change quickly. To prevent possible delays in detecting large changes, one can use an additional Max-MEWMA chart with $\lambda = 1$ simultaneously.

9.7 Charting Procedure and Example

The charting procedure of a Max-MEWMA chart is similar to that of a Max-EWMA chart except that a diagnostic has to be used to identify which variable(s) contributed to an out-of-control signal.

The procedure can be briefly summarized in the following steps:

1. If $\boldsymbol{\mu}_0$ is unknown, substitute $\bar{\bar{\mathbf{X}}}$ for $\boldsymbol{\mu}_0$. If $\boldsymbol{\Sigma}_0$ is unknown, substitute $\bar{\mathbf{S}}$ for $\boldsymbol{\Sigma}_0$.

2. For each sample, compute $\mathbf{Z_i}$ with $\mathbf{Z}_0 = \mathbf{0}$ as starting value; calculate Y_i with $Y_0 = 0$ as starting value.

3. Calculate U_i and V_i and construct UCL according to (9.15).

4. Compute M_i and compare it with the UCL.

5. Plot a sample point against the sample number i when $M_i \leq UCL$.

6. Plot a plotting character against the sample number i when $M_i > UCL$. For the case of only $|U_i| > UCL$, plot "m+" if $U_i > 0$ and plot "m-" if $U_i < 0$; For the case of only $|V_i| > UCL$, Plot "v+" if $V_i > 0$, and plot "v-" if $V_i < 0$; For the case of both $|U_i| > UCL$ and $|V_i| > UCL$, plot " m+v+" if $U_i > 0$ and $V_i > 0$, plot "m+v-" if $U_i > 0$ and $V_i < 0$; Plot "m-v+" if $U_i < 0$ and $V_i > 0$; plot "m-v-" if $U_i < 0$ and $V_i < 0$.

7. When $|U_i| > UCL$. calculate $C_i = \frac{n(2-\lambda)}{\lambda}\mathbf{Z}_i'\mathbf{\Sigma}^{-1}\mathbf{Z}_i$ and then calculate $D_j = C_i - C_{i(j)}$. $j = 1, 2, \ldots, k$. where $C_{i(j)}$ denotes the value of C_i obtained from the $k-1$ variables omitting $\bar{X}_j$. Compare the relative magnitudes of each $D_j's$. A large value could potentially identify an assignable cause for the mean vector.

8. When $|V_i| > UCL$. calculate $E_j = |V_i| - |V_i|_{(j)}$. $j = 1, 2, \ldots, k$. where $|V_i|_{(j)}$ is the value of $|V_i|$ obtained from the $k-1$ variables omitting $(\mathbf{X}_{ij} - \bar{\mathbf{X}}_i)'\mathbf{\Sigma}_0^{-1}(\mathbf{X}_{ij} - \bar{\mathbf{X}}_i)$. Compare the relative magnitudes of each $E_j's$. A large one could potentially identify an assignable cause for the variability.

9. Investigate the cause(s) associated with each out-of-control signal.

An example is given to illustrate how to apply the Max-MEWMA chart to multivariate normally distributed data. The data. consisting of 12 samples of size 5. are given in Table 9.9. The first 10 samples are taken from Cheng and Mao [11]. where it was stated that they were collected every half an hour from a spring process in a spring manufacture company.

According to historical information from the company.

$$\boldsymbol{\mu}_0 = \begin{pmatrix} 28.29 \\ 45.85 \end{pmatrix} \text{ and } \mathbf{\Sigma}_0 = \begin{pmatrix} 0.0035 & -0.0046 \\ -0.0046 & 0.0226 \end{pmatrix}.$$ The last 2 samples are added to simulate an out-of-control process. One quality characteristic is the inner diameter of the spring with specification of 28.30±0.10 and another quality characteristic is the elasticity of the spring with specification of 46.0±0.50. With $ARL_0 = 200$ and $\lambda = 0.2$, a Max-EWMA chart is set up to monitor the spring process and it is shown in Figure 9.1. As seen from the plot, the last two sample points are above the UCL. Becase U_{11}, U_{12} and V_{12} are greater than UCL, sample

Table 9.9: Spring data

X_1					X_2				
28.14	28.31	28.27	28.20	28.26	46.32	45.79	45.88	45.88	45.80
28.50	28.35	28.3	28.32	28.20	45.85	45.91	45.80	45.91	45.93
28.29	28.30	28.29	28.38	28.29	45.83	45.75	45.75	45.52	45.58
28.22	28.26	28.27	28.27	28.28	45.81	45.99	45.78	46.02	45.85
28.30	28.36	28.27	28.32	28.30	45.77	45.94	46.04	45.77	45.67
28.34	28.29	28.32	28.27	28.19	45.77	45.93	45.77	45.92	46.04
28.24	28.32	28.31	28.36	28.41	45.90	45.83	45.69	45.78	45.72
28.23	28.36	28.34	28.31	28.33	45.75	45.89	45.66	45.84	45.74
28.25	28.39	28.31	28.35	28.32	45.59	46.10	45.87	45.57	45.87
28.31	28.28	28.31	28.36	28.32	45.70	45.75	45.78	45.89	45.90
28.37	28.38	28.35	28.45	28.39	45.82	45.35	45.76	45.81	45.88
28.17	28.22	28.28	28.12	28.35	45.30	45.25	45.73	45.81	45.88

11 is related to an increased shift in the mean vector. while sample 12 are related to increased changes in both mean vector and variability matrix. To investigate which quality characteristic is responsible for each out-of-control condition. the relative magnitudes of each $D_j's$ and $E_j's$ are compared. It is found that. for sample 11. the inner diameter of the spring provides an out-of-control signal on the control chart which may indicate an assignable cause in the process resulting in a shift in the mean vecter. For sample 12. the elasticity of the spring shows an out-of-control signal on the control chart which may indicate an assignable cause in the process resulting in changs in both the mean vector and the variability matrix.

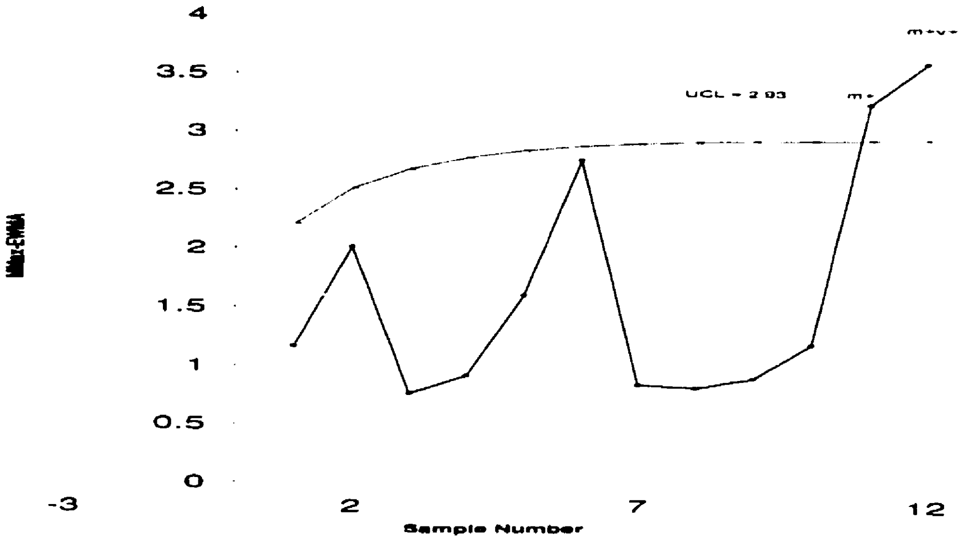

Figure 9.1: The Max-MEWMA chart for the spring data

CHAPTER 10

Conclusions

10.1 Summary

The major goal of this study is to develop EWMA single control charts. which are designed to simultaneously and effectively monitor both the process mean and the process variability when small changes are of interest. Under normality assumption. five new control charts of this type are presented in this thesis. Among them. four are univariate charts and one is a multivariate chart.

There are three main properties of a good control chart: high capability in detecting out-of-control conditions. identifying the source of an out-of-control signal and indicating the direction of an out-of-control signal.

ARL performance is an indication of the ability of a control chart to detect out-of-control conditions for a process. In the univariate case. the ARL comparisons in Chapter 7 show that. if overall performance is considered. the Max-EWMA chart and the SS-EWMA chart perform better than the two other new charts. the combination of the two Shewhart charts and the combination of the two EWMA charts. However. if a mean shift accompanies an increased variability change. the EWMA-SC chart has the best performance of all the charts considered and all the four new charts yield smaller ARL's than the two combination charts. In the multivariate case. the ARL comparison in Chapter 9 indicates that. in term of detecting small changes within a process. the performance of the Max-MEWMA chart is better than that of combination of the χ^2 chart and the $|\mathbf{S}|$ chart.

Diagnosis is an indication of the ability to identify the source and the direction of an out-of-control signal. All the five new charts have this ability except that the EWMA-Max chart may be insensitive in identifying the source of a small change. For the Max-EWMA chart. the EWMA-Max chart and the Max-MEWMA chart. plotting characters are used to indicate the source and direction of a detected change. For the SS-EWMA chart and the EWMA-SC chart. the position of a plotted point can directly tell the source and the direction of an out-of-control signal and a number is required to indicate the sample number.

Another goal of this research is to elaborate upon and propose control chart when the underlying distribution of the quality characteristic is lognormal. Although the lognormal quality control was considered quite some time ago. the literature seems to be in error. Based on the basic relationship of normal and lognormal distributions. the corresponding normal process is obtained through logarithmic transformation after a specific interval for the lognormal mean is given to a lognormal process. Then two control charts are set up for the lognormal process and the complicated lognormal process can be monitored through its simpler normal counterpart.

It should be noted that. in this dissertation. an important assumption is that of independence among the observations. If this assumption is not met. the control charts studied here may signal too many false alarms. For correlated data, Montgomery and Mastrangelo [39] proposed an approximation of the exact time-series model approach based on EWMA technique, which is based on an independently distributed sequence of one-step-ahead prediction errors.

10.2 Areas for Future Research

Some topics below are worth further research:

1. Since all the new EWMA single charts are under the normality assumption,

it is necessary to determine the effect of departures from normality upon these charts.

2. Develop multivariate analogue of the SS-EWMA chart. The performance of the multivariate procedure would most likely be similar to its univariate counterpart.

3. Compute values for optimal Max-EWMA and SS-EWMA charts at other in-control ARL values such as 370 and 500. The ARL properties are expected to be the same as those when the in-control ARL is 250.

4. Compute ARL values of the Max-MEWMA chart at other in-control ARL values such as 370 and 500. The ARL properties would most likely be similar to those when the in-control ARL is 200.

5. Consider applications of the five new charts to correlated data.

6. Compare the new control charts for lognormal population with other charts using simulation studies.

7. Propose new control charts for other skewed distributions.

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APPENDIX A

Computer Programs for New Control charts

A.1 Programs for Max-EWMA Chart

A.1.1 ARL Computation

This program computes ARL's of a Max-EWMA chart for controlling both the mean and the variance of a normal process. For a given in-control ARL of 250. each ARL value is obtained using 10.000 simulations generated with IMSL Fortran Subroutines. A1 and B1 are changes in the process for the mean and the variance respectively and they are expressed as a multiple of the standard deviation of a normal random variable.

Program Listing

```
      REAL LA.ARL.MU.SD.H.A(10000000).XA1.XA(10000000).A1.B1.
     &X.T.Z(0:10000000).U(10000000).Y(10000000).V(10000000).
     &UCL.S1.W(0:10000000)
      INTEGER L.SIZE.COUNT.NOUT.NR.C.ISEED
      EXTERNAL RNNOA.RNNOA.RNSET.UMACH
      MU=0
      SD=1
      WRITE(*.2)
2     FORMAT(5X.'ARL''S FOR AN MAX-EWMA CHART (ARL0=250):')
      LA=0.10D0
      DO 3 N=1.12
      B1=N*.25
      DO 5 M=0.12
      A1=M*.25
      SIZE=5
      ARL=0
      COUNT=0
      ISEED=723459
6     CALL UMACH (2.NOUT)
      NR=10000000
      CALL RNSET (ISEED)
      CALL RNNOA(NR.A)
      C=0
7     T=0
      UCL= 2.85
      Z(0)=MU
      W(0)=0.0
      I=1
10    IF (T .LE. UCL) THEN
```

```
      XA1=0.0
      DO 20 J=C+1,C+5
      A(J)=A1+A(J)*B1
      XA1=XA1+A(J)/5
20    CONTINUE
      XA(I)=XA1
      S=0.0
      DO 30 J=C+1,C+5
      S=S+(A(J)-XA(I))*(A(J)-XA(I))
30    CONTINUE
      S1=S/(SD*SD)
      Z(I)=(1-LA)*Z(I-1)+LA*XA(I)
      U(I)=SQRT(5*(2-LA)/LA)*(Z(I)-MU)/SD
      X=H(S1)
      CALL INVNORM(X,X0)
      Y(I)=X0
      W(I)=(1-LA)*W(I-1)+LA*Y(I)
      V(I)=SQRT((2-LA)/LA)*W(I)
      T=MAX(ABS(U(I)),ABS(V(I)))
      I=I+1
      C=C+5
      GOTO 10
      END IF
      ARL=ARL+(I-1)
      COUNT=COUNT+1
      IF ( C .GE. 9990000) THEN
      ISEED=123459
      GOTO 6
      END IF
      IF (COUNT .LT. 10000) THEN
      GOTO 7
      ELSE
      ARL=ARL/10000
END IF
      WRITE(*,50) SIZE,A1,B1,LA,UCL,ARL
50    FORMAT(5X,'SIZE = ',I2,2X,'A1=',F4.2,2X,'B1=',F4.2,
      & 2X,'LAMBDA = ',F4.2,2X,'UCL=',F6.4,2X,'ARL =',F12.5)
5     CONTINUE
3     CONTINUE
      END
      FUNCTION H(X)
      REAL X
      H=1-EXP(-X/2)*(1+X/2)
      RETURN
      END
```

A.1.2 Simulations of Diagnostic Study

For a given in-control ARL of 250, this program simulates 1000 out-of-control signals with respect to a pair of specified A1 and B1, which are changes in the process for the mean and the variance respectively. To identify the source and the direction of the detected changes, Max-EWMA chart is applied and the out-of-control signals are counted according to the charting procedure of the chart.

Program Listing

```
      REAL LA,MU,SD,H,A(10000000),XA1,XA(10000000),A1,B1,
     &X,T,Z(0:10000000),U(10000000),Y(10000000),V(10000000),
     &UCL,S1,W(0:10000000),U01,V01,U02,V02
      INTEGER COUNT,NOUT,NR,C,ISEED,M1,M2,V1,V2,
     &MV11,MV12,MV21,MV22
      EXTERNAL RNNOA,RNNOA,RNSET,UMACH
      MU=0
      SD=1
      WRITE(*,2)
2     FORMAT(5X,'DIAGNOSTIC ABILITIES FOR THE MAX-EWMA CONTROL CHART
     &(ARL0=250; SIZE=5):')
      UCL=2.81232
      DO 3 N=0,4
      B1=N*.5D0
      DO 5 M=1,1
      A1=M*.25
      COUNT=0
      M1=0
      M2=0
      V1=0
      V2=0
      MV11=0
      MV12=0
      MV21=0
      MV22=0
      ISEED=723459
6     CALL UMACH (2,NOUT)
      NR=10000000
      CALL RNSET (ISEED
      CALL RNNOA(NR,A)
      C=0
7     T=0
      Z(0)=MU
      W(0)=0.0
      I=1
10    IF (T .LE. UCL) THEN
      XA1=0.0
      DO 20 J=C+1,C+5
      A(J)=A1+A(J)*B1
      XA1=XA1+A(J)/5
20    CONTINUE
      XA(I)=XA1
      S=0.0
      DO 30 J=C+1,C+5
```

```
      S=S+(A(J)-XA(I))*(A(J)-XA(I))
30    CONTINUE
      S1=S/(SD*SD)
      Z(I)=(1-LA)*Z(I-1)+LA*XA(I)
      U(I)=SQRT(5*(2-LA)/LA)*(Z(I)-MU)/SD
      X=H(S1)
      CALL INVNORM(X,X0)
      Y(I)=X0
      W(I)=(1-LA)*W(I-1)+LA*Y(I)
      V(I)=SQRT((2-LA)/LA)*W(I)
      T=MAX(ABS(U(I)),ABS(V(I)))
      U01=U(I)
      V01=V(I)
      U02=ABS(U01)
      V02=ABS(V01)
      I=I+1
      C=C+5
      GOTO 10
      END IF
      COUNT=COUNT+1
      IF ((U02 .GT. UCL) .AND. (V02 .GT. UCL)) THEN
      IF (U01 .GT. 0) THEN
      IF (V01 .GT. 0) THEN
      MV11=MV11+1
      ELSE
      MV12=MV12+1
      END IF
      ELSE
      IF (V01 .GT. 0) THEN
      MV21=MV21+1
      ELSE
      MV22=MV22+1
      END IF
      END IF
      ELSE
      IF (U02 .GT. UCL) THEN
      IF (U01 .GT. 0) THEN
      M1=M1+1
      ELSE
      M2=M2+1
      END IF
      ELSE
      IF (V02 .GT. UCL) THEN
      IF (V01 .GT. 0) THEN
      V1=V1+1
      ELSE
      V2=V2+1
      END IF
      END IF
      END IF
      END IF
      IF ( C .GE. 9990000) THEN
```

```
      ISEED=123459
      GOTO 6
      END IF
      IF (COUNT .LT. 1000) THEN
      GOTO 7
      END IF
      WRITE(*,50) LA,A1,B1,M1,M2,V1,V2,MV11,MV12,MV21,MV22
50    FORMAT(2X,'LAMBDA = ',F4.2,2X,'A =',F4.2,2X,'B =',F4.2,
     &2X,'M+ =',I4,2X,'M- =',I4,2X,'V+ =',I4,2X,'V- =',I4,
     &10X,'M+V+ =',I4,2X,'M+V- = ',I4,2X,'M-V+ =',I4,2X,'M-V- =',I4)
5     CONTINUE
3     CONTINUE
      END
      FUNCTION H(X)
      REAL X
      H=1-EXP(-X/2)*(1+X/2)
      RETURN
      END
```

A.2 Programs for SS-EWMA Chart

A.2.1 ARL Computation

This program computes ARL's of a SS-EWMA chart for controlling both the mean and the variance of a normal process. For a given in-control ARL of 250, each ARL value is obtained using 10,000 simulations generated with IMSL Fortran Subroutines. A1 and B1 are changes in the process for the mean and the variance respectively and they are expressed as a multiple of the standard deviation of a normal random variable.

Program Listing

```
      REAL LA,ARL,MU,SD,H,A(10000000),XA1,XA(10000000),A1,B1,
     &X,T,Z(0:1000000),U(10000000),Y(10000000),V(10000000),
     &UCL,S1,W(0:10000000)
      INTEGER I,J,SIZE,COUNT,NR,C,ISEED
      EXTERNAL RNNOA,RNNOA,RNSET,UMACH
      ISEED=723459
      LA=.3
      MU=0.0
      SD=1.0
      SIZE=5
      ARL=0
      COUNT=0
6     CALL UMACH (2,NOUT)
      NR=10000000
      CALL RNSET (ISEED
```

```
      CALL RNNOA(NR,A)
      C=0
      A1=0.0
      B1=1.0
1     T=0
      UCL=9.92435
      Z(0)=MU
      W(0)=0.0
      I=1
10    IF (T .LE. UCL) THEN
      XA1=0.0
      DO 20 J=C+1,C+5
      A(J)=A1+A(J)*B1
      XA1=XA1+A(J)/5
20    CONTINUE
      XA(I)=XA1
      S=0.0
      DO 30 J=C+1,C+5
      S=S+(A(J)-XA(I))*(A(J)-XA(I))
30    CONTINUE
      S1=S/(SD*SD)
      Z(I)=(1-LA)*Z(I-1)+LA*XA(I)
      U(I)=SQRT(5*(2-LA)/LA)*(Z(I)-MU)/SD
      X=H(S1)
      CALL INVNORM(X,X0)
      Y(I)=X0
      W(I)=(1-LA)*W(I-1)+LA*Y(I)
      V(I)=SQRT((2-LA)/LA)*W(I)
      T=U(I)*U(I)+V(I)*V(I)
      I=I+1
      C=C+5
      GOTO 10
      END IF
      ARL=ARL+(I-1)
      COUNT=COUNT+1
      IF ( C .GE. 9990000) THEN
      ISEED=123459
      GOTO 6
      END IF
      IF (COUNT .LT. 10000) THEN
      GOTO 1
      ELSE
      ARL=ARL/10000
      END IF
      WRITE(*,50) SIZE,LA,ARL
50    FORMAT(5X,'SAMPLE SIZE = ',I4,2X,'LAMBDA = ',F5.3,
      &2X,'ARL =',F15.2)
      END
      FUNCTION H(X)
      REAL X
      H=1-EXP(-X/2)*(1+X/2)
      RETURN
```

```
END
SUBROUTINE INVNORM(P,X0)
REAL C(14),P,Q,R,X,X0
DATA C/2.5066282,-18.6150006,41.3911977,-25.4410605,-8.4735109,
& 23.0833674,-21.0622410,3.1308291,-2.7871893,-2.2979648,
& 4.8501401413,2.3212128,3.5438892,1.6370678/
Q=P-.5D0
IF (ABS(Q) .GE. .42D0) THEN
R=P
IF (Q .GE. 0.D0) THEN
R=1-P
R=SQRT(-LOG(R))
X=(((C(12)*R+C(11))*R+C(10))*R+C(9))
X0=X/((C(14)*R+C(13))*R+1)
ELSE
R=SQRT(-LOG(R))
X=(((C(12)*R+C(11))*R+C(10))*R+C(9))
X0=-X/((C(14)*R+C(13))*R+1)
END IF
ELSE
R=Q*Q
X=Q*(((C(4)*R+C(3))*R+C(2))*R+C(1))
X0=X/((((C(8)*R+C(7))*R+C(6))*R+C(5))*R+1)
END IF
RETURN
END
```

A.2.2 Simulations of Diagnostic Study

For a given in-control ARL of 250, this program simulates 1000 out-of-control signals with respect to a pair of specified A1 and B1, which are changes in the process for the mean and the variance respectively. To identify the source and the direction of the detected changes, SS-EWMA chart is applied and the out-of-control signals are counted according to the charting procedure of the chart.

Program Listing

```
      REAL LA,MU,SD,H,A(10000000),XA1,XA(10000000),A1,B1,
     &X,T,Z(0:10000000),U(10000000),Y(10000000),V(10000000),
     &UCL,S1,W(0:10000000),U01,V01,U02,V02
      INTEGER COUNT,NOUT,NR,C,ISEED,M1,M2,V1,V2,
     &MV11,MV12,MV21,MV22
      EXTERNAL RNNOA,RNNOA,RNSET,UMACH
      MU=0
      SD=1
      WRITE(*,2)
2     FORMAT(5X,'DIAGNOSTIC ABILITIES FOR THE SS-EWMA CONTROL CHART
```

```
      &(ARL0=250; SIZE=5):')
      UCL=9.19573
      UCL1=SQRT(UCL)
      DO 3 N=0,4
      B1=N*.5D0
      DO 5 M=1,1
      A1=M*.25
      COUNT=0
      M1=0
      M2=0
      V1=0
      V2=0
      MV11=0
      MV12=0
      MV21=0
      MV22=0
      ISEED=723459
6     CALL UMACH (2,NOUT)
      NR=10000000
      CALL RNSET (ISEED
      CALL RNNOA(NR,A)
      C=0
7     T=0
      Z(0)=MU
      W(0)=0.0
      I=1
10    IF (T .LE. UCL) THEN
      XA1=0.0
      DO 20 J=C+1,C+5
      A(J)=A1+A(J)*B1
      XA1=XA1+A(J)/5
20    CONTINUE
      XA(I)=XA1
      S=0.0
      DO 30 J=C+1,C+5
      S=S+(A(J)-XA(I))*(A(J)-XA(I))
30    CONTINUE
      S1=S/(SD*SD)
      Z(I)=(1-LA)*Z(I-1)+LA*XA(I)
      U(I)=SQRT(5*(2-LA)/LA)*(Z(I)-MU)/SD
      X=H(S1)
      CALL INVNORM(X,X0)
      Y(I)=X0
      W(I)=(1-LA)*W(I-1)+LA*Y(I)
      V(I)=SQRT((2-LA)/LA)*W(I)
      T=U(I)*U(I)+V(I)*V(I)
      U0=U(I)
      V0=V(I)
      I=I+1
      C=C+5
      GOTO 10
      END IF
```

```
COUNT=COUNT+1
IF ( C .GE. 9990000) THEN
ISEED=123459
GOTO 6
END IF
IF (U0 .GE. 0) THEN
IF (U0 .GT. UCL1) THEN
IF (V0 .GT. UCL1) THEN
MV11=MV11+1
ELSE
IF (V0 .LT. -UCL1) THEN
MV12=MV12+1
ELSE
M1=M1+1
END IF
END IF
ELSE
IF (V0 .GT. 0) THEN
IF (V0 .LE. UCL1) THEN
IF ((A1 .EQ. 0) .OR. (B1 .EQ. 1)) THEN
IF (V0 .EQ. U0) THEN
MV11=MV11+1
ELSE
IF (U0 .GT. V0) THEN
M1=M1+1
ELSE
V1=V1+1
END IF
END IF
ELSE
MV11=MV11+1
END IF
ELSE
V1=V1+1
END IF
ELSE
IF (V0 .GE. -UCL1) THEN
IF ((A1 .EQ. 0) .OR. (B1 .EQ. 1)) THEN
IF (U0 .EQ. -V0) THEN
MV12=MV12+1
ELSE
IF (U0 .GT. -V0) THEN
M1=M1+1
ELSE
V2=V2+1
END IF
END IF
ELSE
MV12=MV12+1
END IF
ELSE
IF (V0 .GE. -UCL1) THEN
```

```
MV12=MV12+1
ELSE
V2=V2+1
END IF
END IF
END IF
END IF
ELSE
IF (U0 .LT. -UCL1) THEN
IF (V0 .LT. -UCL1) THEN
MV22=MV22+1
ELSE
IF (V0 .GT. UCL1) THEN
MV21=MV21+1
ELSE
M2=M2+1
END IF
END IF
ELSE
IF (V0 .LT. 0) THEN
IF (V0 .GE. -UCL1) THEN
IF ((A1 .EQ. 0) .OR. (B1 .EQ. 1)) THEN
IF (U0 .EQ. V0) THEN
MV22=MV22+1
ELSE
IF (U0 .GT. V0) THEN
M2=M2+1
ELSE
V2=V2+1
END IF
END IF
ELSE
MV22=MV22+1
END IF
ELSE
V2=V2+1
END IF
ELSE
IF (V0 .LE. UCL1) THEN
IF ((A1 .EQ. 0) .OR. (B1 .EQ. 1)) THEN
IF (U0 .EQ.-V0) THEN
MV21=MV21+1
ELSE
IF (U0 .LT.-V0) THEN
M2=M2+1
ELSE
V1=V1+1
END IF
END IF
ELSE
MV21=MV21+1
END IF
```

```
      ELSE
      V1=V1+1
      END IF
      END IF
      END IF
      END IF
      IF (COUNT .LT. 1000) THEN
      GOTO 7
      END IF
      WRITE(*,50) LA,A1,B1,M1,M2,V1,V2,MV11,MV12,MV21,MV22
50    FORMAT(2X,'LAMBDA = ',F4.2,2X,'A =',F4.2,2X,'B =',F4.2,
     &2X,'M+ =',I4,2X,'M- =',I4,2X,'V+ =',I4,2X,'V- =',I4,
     &10X,'M+V+ =',I4,2X,'M+V- =',I4,2X,'M-V+ =',I4,2X,'M-V- =',I4)
5     CONTINUE
3     CONTINUE
      END
      FUNCTION H(X)
      REAL X
      H=1-EXP(-X/2)*(1+X/2)
      RETURN
      SUBROUTINE INVNORM(P,X0)
      REAL C(14),P,Q,R,X,X0
      DATA C/2.5066282,-18.6150006,41.3911977,-25.4410605,-8.4735109,
     & 23.0833674,-21.0622410,3.1308291,-2.7871893,-2.2979648,
     & 4.8501401413,2.3212128,3.5438892,1.6370678/
      Q=P-.5D0
      IF (ABS(Q) .GE. .42D0) THEN
      R=P
      IF (Q .GE. 0.D0) THEN
      R=1-P
      R=SQRT(-LOG(R))
      X=(((C(12)*R+C(11))*R+C(10))*R+C(9))
      X0=X/((C(14)*R+C(13))*R+1)
      ELSE
      R=SQRT(-LOG(R))
      X=(((C(12)*R+C(11))*R+C(10))*R+C(9))
      X0=-X/((C(14)*R+C(13))*R+1)
      END IF
      ELSE
      R=Q*Q
      X=Q*(((C(4)*R+C(3))*R+C(2))*R+C(1))
      X0=X/((((C(8)*R+C(7))*R+C(6))*R+C(5))*R+1)
      END IF
      RETURN
      END
```

A.3 Program for EWMA-Max Chart

This program computes ARL's of a EWMA-Max chart for controlling both the mean and the variance of a normal process. For a given in-control ARL of 250, a numerical method of using integral equations is used to computer ARL's. A0 and B0 are changes in the process for the mean and the variance respectively and they are expressed as a multiple of the standard deviation of a normal random variable.

Program Listing

```
DOUBLE PRECISION LA,ARL,ARG,K0,A(64,64),B(64),W(64),P(64),
&X(64),H,WK(64),F,T,A0,B0
EXTERNAL F
INTEGER IPIVOT(64),IFLAG
P( 1)= 0.99930504173577D0
P( 2)= 0.99634011677196D0
P( 3)= 0.99101337147674D0
P( 4)= 0.98333625388463D0
P( 5)= 0.97332682778991D0
P( 6)= 0.96100879965205D0
P( 7)= 0.94641137485840D0
P( 8)= 0.92956917213194D0
P( 9)= 0.91052213707850D0
P( 10)= 0.88931544599511D0
P( 11)= 0.86599939815409D0
P( 12)= 0.84062929625258D0
P( 13)= 0.81326531512280D0
P( 14)= 0.78397235894334D0
P( 15)= 0.75281990726053D0
P( 16)= 0.71988185017161D0
P( 17)= 0.68523631305423D0
P( 18)= 0.64896547125466D0
P( 19)= 0.61115535517239D0
P( 20)= 0.57189564620263D0
P( 21)= 0.53127946401989D0
P( 22)= 0.48940314570705D0
P( 23)= 0.44636601725346D0
P( 24)= 0.40227015796399D0
P( 25)= 0.35722015833767D0
P( 26)= 0.31132287199021D0
P( 27)= 0.26468716220877D0
P( 28)= 0.21742364374001D0
P( 29)= 0.16964442042399D0
P( 30)= 0.12146281929612D0
P( 31)= 7.2993121787799D-02
P( 32)= 2.4350292663424D-02
W( 1)= 1.7832807216964D-03
W( 2)= 4.1470332605625D-03
W( 3)= 6.5044579689784D-03
```

```
      W( 4)= 8.8467598263639D-03
      W( 5)= 1.1168139460131D-02
      W( 6)= 1.3463047896719D-02
      W( 7)= 1.5726030476025D-02
      W( 8)= 1.7951715775697D-02
      W( 9)= 2.0134823153530D-02
      W( 10)= 2.2270173808383D-02
      W( 11)= 2.4352702568711D-02
      W( 12)= 2.6377469715055D-02
      W( 13)= 2.8339672614259D-02
      W( 14)= 3.0234657072402D-02
      W( 15)= 3.2057928354852D-02
      W( 16)= 3.3805161837142D-02
      W( 17)= 3.5472213256882D-02
      W( 18)= 3.7055128540240D-02
      W( 19)= 3.8550153178616D-02
      W( 20)= 3.9953741132720D-02
      W( 21)= 4.1262563242624D-02
      W( 22)= 4.2473515123654D-02
      W( 23)= 4.3583724529323D-02
      W( 24)= 4.4590558163757D-02
      W( 25)= 4.5491627927418D-02
      W( 26)= 4.6284796581314D-02
      W( 27)= 4.6968182816210D-02
      W( 28)= 4.7540165714830D-02
      W( 29)= 4.7999388596458D-02
      W( 30)= 4.8344762234803D-02
      W( 31)= 4.8575467441503D-02
      W( 32)= 4.8690957009140D-02
      DO 1 I=1,32
      P(65-I)=-P(I)
      W(65-I)=W(I)
1     CONTINUE
      k0=1.99265D0
      LA=.05D0
      H=1.128379D0+k0*DSQRT(0.3633808D0*LA/(2.0D0-LA))
      DO 2 I=1,64
      W(I)=H*W(I)/2.0D0
      P(I)=H*(P(I)+1.D0)/2.0D0
2     CONTINUE
      DO 12 N=1,7
      B0=N*.25D0
      DO 3 M=0,6
      A0=M*.25D0
      DO 10 I=1,64
      B(I)=-1.0D0
      DO 20 J=1,64
      ARG=(P(J)-(1.0D0-LA)*P(I))/LA
      IF(ARG .LE. .0D0) THEN
      T = 0.0D0
      ELSE
      T = F(A0,B0,ARG)
```

```
      END IF
      IF (I .EQ. J) THEN
      A(I,J)=W(I)*T/LA-1.0D0
      ELSE
      A(I,J)=W(J)*T/LA
      END IF
20    CONTINUE
10    CONTINUE
      CALL FACTOR(A,64,WK,IPIVOT,IFLAG)
      IF (IFLAG .EQ. 0) THEN
      WRITE(6,50)
      END IF
      CALL SUBST(A,IPIVOT,B,64,X)
      ARL=0.0D0
      DO 30 I=1,64
      ARG=(P(I)-(1.0D0-LA)*1.128379D0)/LA
      IF(ARG .LE. 0.0D0) THEN
      T = 0.0D0
      ELSE
      T = F(A0,B0,ARG)
      END IF
      ARL=ARL + W(I)*X(I)*T
30    CONTINUE
      ARL=1.0D0+ARL/LA
      WRITE(6,60) LA,K0,H,A0,B0,ARL
60    FORMAT(5X,'LAMBDA=',F4.2,2X,'k0=',F7.5,2x,'H=',F6.4,2X,
     &'A0=',F4.2,2X,'B0=',F4.2,2X,'ARL=',F15.5)
50    FORMAT(5X,'ZERO DETERMINANT FOR LINEAR SYSTEM')
3     CONTINUE
12    CONTINUE
      END
      DOUBLE PRECISION FUNCTION F(A0,B0,X)
      DOUBLE PRECISION X
      DOUBLE PRECISION CF,CF1,CF2,CF3,Y1,Y2,T,B1,B2,B3,B4,B5,P,Z,Z1,Z2,
     &H0,H1,H2,H3,H4,DH,H01,H02,A0,B0,SR,PPCHI,T1,T2
      EXTERNAL H0,DH
      SR=DSQRT(5.D0)
      B1=0.319381530D0
      B2=-0.356563782D0
      B3=1.781477937D0
      B4=-1.821255978D0
      B5=1.330274429D0
      P=0.2316419D0
      T=1.0D0/(1.0D0+P*DABS(X))
      T1=1.0D0/(1.D0+P*DABS((X-A0*SR)/B0))
      T2=1.0D0/(1.D0+P*DABS((-X-A0*SR)/B0))
      Z=(3.989422804014327D-1)*DEXP(-.5D0*X*X)
      IF (X .LE. 8.D0) THEN
      Z1=(3.989422804014327D-1)*DEXP(-.5D0*((X-A0*SR)/B0)**2)
      ELSE
      Z1=0.D0
      END IF
```

```
IF (X .LE. 4.5D0) THEN
Z2=(3.989422804014327D-1)*DEXP(-.5D0*((-X-A0*SR)/B0)**2)
ELSE
Z2=0.D0
END IF
CF=1.0D0-Z*(B1*T+B2*T**2+B3*T**3+B4*T**4+B5*T**5)
IF ((X-A0*SR)/B0 .GT. 0.D0) THEN
CF1=1.D0-Z1*(B1*T1+B2*T1**2+B3*T1**3+B4*T1**4+B5*T1**5)
ELSE
CF1=Z1*(B1*T1+B2*T1**2+B3*T1**3+B4*T1**4+B5*T1**5)
END IF
IF ((-X-A0*SR)/B0 .LE.-3.D0) THEN
CF2=0.D0
ELSE
CF2=Z2*(B1*T2+B2*T2**2+B3*T2**3+B4*T2**4+B5*T2**5)
END IF
CF3=1.D0-CF
IF ( CF .EQ. 1) THEN
Y1=55
ELSE
CALL INVH(CF,PPCHI)
Y1=PPCHI
END IF
IF (CF3 .EQ. 0) THEN
Y2=.0000001
ELSE
CALL INVH(CF3,PPCHI)
Y2=PPCHI
END IF
H1=DH(Y1/(B0*B0))
H2=DH(Y2/(B0*B0))
H3=DH(Y1)
H4=DH(Y2)
H01=H0(Y1/(B0*B0))
H02=H0(Y2/(B0*B0))
F = (Z1+Z2)*(H01-H02)/B0 + (CF1-CF2)*(H1*Z/H3+H2*Z/H4)/(B0*B0)
RETURN
END
DOUBLE PRECISION FUNCTION H0(X)
DOUBLE PRECISION X
H0=1-DEXP(-X/2)*(1+X/2)
RETURN
END
DOUBLE PRECISION FUNCTION DH(X)
DOUBLE PRECISION X
DH=X*DEXP(-X/2)/4.D0
RETURN
END
SUBROUTINE INVH(P,PPCHI)
DOUBLE PRECISION P,X(100000),EPS,PPCHI
I=1
X(1)=-2*DLOG(1-P)
```

```
      EPS=.5D-6
1     I=I+1
      X(I)=X(1)+2*DLOG(1+X(I-1)/2)
      IF (DABS(X(I)-X(I-1)) .GT. EPS) THEN
      GOTO 1
      END IF
      PPCHI=X(I)
      RETURN
      END
      SUBROUTINE SUBST(W1,IPIVOT,B,N,X2)
      INTEGER IPIVOT(64),I,IP,J
      DOUBLE PRECISION B(64),W1(64,64),X2(64),SUM
      IF (N .LE. 1) THEN
      X2(1)=B(1)/W1(1,1)
      RETURN
      END IF
      IP=IPIVOT(1)
      X2(1)=B(IP)
      DO 15 I=2,N
      SUM=.0D0
      I1=I-1
      DO 14 J=1,I1
      SUM=W1(I,J)*X2(J) + SUM
14    CONTINUE
      IP=IPIVOT(I)
      X2(I)=B(IP) - SUM
15    CONTINUE
      X2(N)=X2(N)/W1(N,N)
      I2=N-1
      DO 20 ISTEP=1,I2
      I=N-ISTEP
      SUM=.0D0
      I3=I+1
      DO 19 J=I3,N
      SUM=W1(I,J)*X2(J) + SUM
19    CONTINUE
      X2(I)=(X2(I)-SUM)/W1(I,I)
20    CONTINUE
      RETURN
      END
      SUBROUTINE FACTOR(W1,N,D1,IPIVOT,IFLAG)
      DOUBLE PRECISION D1(64),W1(64,64),AWIKOD,
      COLMAX,RATIO,ROWMAX,TEMP
      INTEGER IFLAG,IPIVOT(64),I,ISTAR,J,K
      IFLAG=1
      DO 9 I=1,N
      IPIVOT(I)=I
      ROWMAX=.0D0
      DO 5 J=1,N
      ROWMAX=DMAX1(ROWMAX,DABS(W1(I,J)))
5     CONTINUE
      IF (ROWMAX .EQ. .0D0) THEN
```

```
      IFLAG=0
      ROWMAX=1.0D0
      END IF
      D1(I)=ROWMAX
9     CONTINUE
      IF (N .LE. 1) RETURN
      N1=N-1
      DO 20 K=1,N1
      COLMAX=DABS(W1(K,K))/D1(K)
      ISTAR=K
      K1=K+1
      DO 13 I=K1,N
      AWIKOD=DABS(W1(I,K))/D1(K)
      IF (AWIKOD .GT. COLMAX) THEN
      COLMAX=AWIKOD
      ISTAR=I
      END IF
13    CONTINUE
      IF (COLMAX .EQ. .0D0) THEN
      IFLAG=0
      ELSE
      IF (ISTAR .GT. K) THEN
      IFLAG=-IFLAG
      I=IPIVOT(ISTAR)
      IPIVOT(ISTAR)=IPIVOT(K)
      IPIVOT(K)=I
      TEMP=D1(ISTAR)
      D1(ISTAR)=D1(K)
      D1(K)=TEMP
      DO 15 J=1,N
      TEMP=W1(ISTAR,J)
      W1(ISTAR,J)=W1(K,J)
      W1(K,J)=TEMP
15    CONTINUE
      END IF
      K2=K+1
      DO 19 I=K2,N
      W1(I,K)=W1(I,K)/W1(K,K)
      RATIO=W1(I,K)
      K3=K+1
      DO 18 J=K3,N
      W1(I,J)=W1(I,J)-RATIO*W1(K,J)
18    CONTINUE
19    CONTINUE
      END IF
20    CONTINUE
      IF (W1(N,N) .EQ. .0D0) IFLAG=0
      RETURN
      END
```

A.4 Program for EWMA-SC Chart

This program computes ARL's of a EWMA-SC chart for controlling both the mean and the variance of a normal process. For a given in-control ARL of 250, a numerical method of using integral equations is used to computer ARL's. A0 and B0 are changes in the process for the mean and the variance respectively and they are expressed as a multiple of the standard deviation of a normal random variable.

Program Listing

```
DOUBLE PRECISION LA,ARL,ARG,K,A(64,64),B(64),W(64),P(64),
&X(64),F,UCL,WK(64),N,T,A0,B0
EXTERNAL F
INTEGER IPIVOT(64),IFLAG
K=3.37455D0
N=5.0D0
LA=.065D0
A0=0.D0
B0=1.D0
UCL=N+K*DSQRT(2*N*LA/(2.0D0-LA))
P( 1)= 0.99930504173577D0
P( 2)= 0.99634011677196D0
P( 3)= 0.99101337147674D0
P( 4)= 0.98333625388463D0
P( 5)= 0.97332682778991D0
P( 6)= 0.96100879965205D0
P( 7)= 0.94641137485840D0
P( 8)= 0.92956917213194D0
P( 9)= 0.91052213707850D0
P( 10)= 0.88931544599511D0
P( 11)= 0.86599939815409D0
P( 12)= 0.84062929625258D0
P( 13)= 0.81326531512280D0
P( 14)= 0.78397235894334D0
P( 15)= 0.75281990726053D0
P( 16)= 0.71988185017161D0
P( 17)= 0.68523631305423D0
P( 18)= 0.64896547125466D0
P( 19)= 0.61115535517239D0
P( 20)= 0.57189564620263D0
P( 21)= 0.53127946401989D0
P( 22)= 0.48940314570705D0
P( 23)= 0.44636601725346D0
P( 24)= 0.40227015796399D0
P( 25)= 0.35722015833767D0
P( 26)= 0.31132287199021D0
P( 27)= 0.26468716220877D0
P( 28)= 0.21742364374001D0
P( 29)= 0.16964442042399D0
```

```
      P( 30)= 0.12146281929612D0
      P( 31)= 7.2993121787799D-02
      P( 32)= 2.4350292663424D-02
      W( 1)= 1.7832807216964D-03
      W( 2)= 4.1470332605625D-03
      W( 3)= 6.5044579689784D-03
      W( 4)= 8.8467598263639D-03
      W( 5)= 1.1168139460131D-02
      W( 6)= 1.3463047896719D-02
      W( 7)= 1.5726030476025D-02
      W( 8)= 1.7951715775697D-02
      W( 9)= 2.0134823153530D-02
      W( 10)= 2.2270173808383D-02
      W( 11)= 2.4352702568711D-02
      W( 12)= 2.6377469715055D-02
      W( 13)= 2.8339672614259D-02
      W( 14)= 3.0234657072402D-02
      W( 15)= 3.2057928354852D-02
      W( 16)= 3.3805161837142D-02
      W( 17)= 3.5472213256882D-02
      W( 18)= 3.7055128540240D-02
      W( 19)= 3.8550153178616D-02
      W( 20)= 3.9953741132720D-02
      W( 21)= 4.1262563242624D-02
      W( 22)= 4.2473515123654D-02
      W( 23)= 4.3583724529323D-02
      W( 24)= 4.4590558163757D-02
      W( 25)= 4.5491627927418D-02
      W( 26)= 4.6284796581314D-02
      W( 27)= 4.6968182816210D-02
      W( 28)= 4.7540165714830D-02
      W( 29)= 4.7999388596458D-02
      W( 30)= 4.8344762234803D-02
      W( 31)= 4.8575467441503D-02
      W( 32)= 4.8690957009140D-02
      DO 1 I=1,32
      P(65-I)=-P(I)
      W(65-I)=W(I)
1     CONTINUE
      DO 2 I=1,64
      W(I)=UCL*.5D0*W(I)
      P(I)=UCL*.5*(P(I)+1.D0)
2     CONTINUE
      DO 10 I=1,64
      B(I)=-1.0D0
      DO 20 J=1,64
      ARG=(P(J)-(1.0D0-LA)*P(I))/LA
      IF(ARG .LE. 0.0D0) THEN
      T = 0.0D0
      ELSE
      T = F(N,A0,B0,ARG)
      END IF
```

```
      IF (I .EQ. J) THEN
      A(I,J)=W(I)*T/LA-1.0D0
      ELSE
      A(I,J)=W(J)*T/LA
      END IF
20    CONTINUE
10    CONTINUE
      CALL FACTOR(A,64,WK,IPIVOT,IFLAG)
      IF (IFLAG .EQ. 0) THEN
      WRITE(6,50)
      STOP
      END IF
      CALL SUBST(A,IPIVOT,B,64,X)
      DO 9 I=1,64
      IF (X(I) .LE. 0) THEN
      X(I)=0
      END IF
9     CONTINUE
      ARL=.0D0
      DO 30 I=1,64
      ARG=(P(I)-(1.D0-LA)*N)/LA
      IF(ARG .LE. 0.0D0) THEN
      T = 0.0D0
      ELSE
      T = F(N,A0,B0,ARG)
      END IF
      ARL=ARL + W(I)*X(I)*T
30    CONTINUE
      ARL=1.0D0+ARL/LA
      WRITE(6,60) LA,N,K,UCL,ARL
60    FORMAT(5X,'LAMBDA=',F6.4,2X,'N=',F5.2,2X,'K=',F7.5,
     &2X,'UCL=',F10.5,2X,'ARL=',F15.5)
50    FORMAT(5X,'ZERO DETERMINANT FOR LINEAR SYSTEM')
      END
      DOUBLE PRECISION FUNCTION F(N,A0,B0,X)
      DOUBLE PRECISION X,H,DE,A1,B1,A0,B0,N,B
      B=B0**2
      DE=N*A0**2/B0**2
      H=(1.D0/3.D0)*(1.D0+2.D0*(DE**2)/(N+2.D0*DE)**2)
      A1=1.D0+H*(H-1)*(N+2*DE)/(N+DE)**2
      B1=H*DSQRT(2*(N+2*DE))/(N+DE)
      IF (A0 .EQ. 0) THEN
      F=DSQRT(.5D0*X/(3.14156D0*B))*(X/B)*DEXP(-X/(2.D0*B))/3.D0
      ELSE
      F=H*(X**(H-1))*DEXP(-((((X/((B0**2)*(N+DE)))**H-A1)/B1)**2)/2.D0)/
     & (B1*(B0**(2*H))*((N+DE)**H)*DSQRT(2.D0*3.14156D0))
      END IF
      RETURN
      END
      SUBROUTINE SUBST(W1,IPIVOT,B,N,X2)
      INTEGER IPIVOT(64),I,IP,J
      DOUBLE PRECISION B(64),W1(64,64),X2(64),SUM
```

```
      IF (N .LE. 1) THEN
      X2(1)=B(1)/W1(1,1)
      RETURN
      END IF
      IP=IPIVOT(1)
      X2(1)=B(IP)
      DO 15 I=2,N
      SUM=.0D0
      I1=I-1
      DO 14 J=1,I1
      SUM=W1(I,J)*X2(J) + SUM
14    CONTINUE
      IP=IPIVOT(I)
      X2(I)=B(IP) - SUM
15    CONTINUE
      X2(N)=X2(N)/W1(N,N)
      I2=N-1
      DO 20 ISTEP=1,I2
      I=N-ISTEP
      SUM=.0D0
      I3=I+1
      DO 19 J=I3,N
      SUM=W1(I,J)*X2(J) + SUM
19    CONTINUE
      X2(I)=(X2(I)-SUM)/W1(I,I)
20    CONTINUE
      RETURN
      END
      SUBROUTINE FACTOR(W1,N,D1,IPIVOT,IFLAG)
      DOUBLE PRECISION D1(64),W1(64,64),AWIKOD,
      COLMAX,RATIO,ROWMAX,TEMP
      INTEGER IFLAG,IPIVOT(64),I,ISTAR,J,K
      IFLAG=1
      DO 9 I=1,N
      IPIVOT(I)=I
      ROWMAX=.0D0
      DO 5 J=1,N
      ROWMAX=DMAX1(ROWMAX,DABS(W1(I,J)))
5     CONTINUE
      IF (ROWMAX .EQ. .0D0) THEN
      IFLAG=0
      ROWMAX=1.0D0
      END IF
      D1(I)=ROWMAX
9     CONTINUE
      IF (N .LE. 1) RETURN
      N1=N-1
      DO 20 K=1,N1
      COLMAX=DABS(W1(K,K))/D1(K)
      ISTAR=K
      K1=K+1
      DO 13 I=K1,N
```

```
      AWIKOD=DABS(W1(I,K))/D1(K)
      IF (AWIKOD .GT. COLMAX) THEN
      COLMAX=AWIKOD
      ISTAR=I
      END IF
13    CONTINUE
      IF (COLMAX .EQ. .0D0) THEN
      IFLAG=0
      ELSE
      IF (ISTAR .GT. K) THEN
      IFLAG=-IFLAG
      I=IPIVOT(ISTAR)
      IPIVOT(ISTAR)=IPIVOT(K)
      IPIVOT(K)=I
      TEMP=D1(ISTAR)
      D1(ISTAR)=D1(K)
      D1(K)=TEMP
      DO 15 J=1,N
      TEMP=W1(ISTAR,J)
      W1(ISTAR,J)=W1(K,J)
      W1(K,J)=TEMP
15    CONTINUE
      END IF
      K2=K+1
      DO 19 I=K2,N
      W1(I,K)=W1(I,K)/W1(K,K)
      RATIO=W1(I,K)
      K3=K+1
      DO 18 J=K3,N
      W1(I,J)=W1(I,J)-RATIO*W1(K,J)
18    CONTINUE
19    CONTINUE
      END IF
20    CONTINUE
      IF (W1(N,N) .EQ. .0D0) IFLAG=0
      RETURN
      END
```

A.5 Program for Max-MEWMA Chart

This program computes ARL's of a Max-MEWMA chart for controlling both the mean vector and the covariance matrix of a multivariate normal process in Case I. For a given in-control ARL of 200, each ARL value is obtained using 10,000 simulations generated with IMSL Fortran Subroutines.

Program Listing

```
      INTEGER LDZ1,LDB,LDC1,NCZ1,NCB,NCC1,NRZ1,NRB,NRC1,NCC2,NRC2,
```

```
      & LDC2,LDC3,NCC3,NRC3,IRANK,ISEED,LDR,LDRSIG,NOUT,NR,LDCOV,C,
      & L,COUNT,L0,L1,L2,L3
      PARAMETER (LDZ1=1,LDB=2,LDC1=1,NCZ1=2,NCB=2,NCC1=2,NRZ1=1,
      & NRB=2,NRC1=1,LDC2=2,NRC2=2,NCC2=1,LDC3=1,NRC3=1,NCC3=1,
      &LDCOV=2,LDRSIG=2,LDR=10000000,N1=2,LDC4=1,NCC4=2,NRC4=1,LDC5=2,
      &NRC5=2,NCC5=1,LDC6=1,NRC6=1,NCC6=1,LDS=1,NRS=1,NCS=2,LDCOV0=2)
      REAL XA(1,2),B(LDB,NCB),C1(LDC1,NCC1),COV(2,2),R(LDR,2),Z1(1,2),
      & RSIG(2,2),S(1,2),Y,T,X(2),C2(LDC2,NCC2),C3(LDC3,NCC3),COV0(2,2),
      & C4(LDC4,NCC4),C5(LDC5,NCC5),C6(LDC6,NCC6),P1,P2,
      & W(10000000),K1,K2,U(0:10000000),V(0:10000000),ARL,ME,
      & LA(0:20),RO,V1(10000000),UCL,A1,B1,Z(10,0:10000000)
      EXTERNAL MRRRR,TRNRR,CHFAC,RNMVN,RNSET,UMACH,LINRG,WRRRN,
      & CHIDF,ANORIN
      WRITE(*,1)
1     FORMAT(5X,'ARL"S FOR AN MAX-MEWMA CHART (ARL0=200):')
      UCL=2.771
      LA(0)=0.05
      DO 12 L1=0,6
      DO 13 L2=1,2
      DO 14 L3=3,6
      A1=L1*0.5
      B1=L2*0.5
      RO=-0.9+L3*0.3
      CALL UMACH (2,NOUT)
      NR=10000000
      N=2
      K=2
      COUNT=0
      COV0(1,1)=1.0
      COV0(1,2)=RO
      COV0(2,1)=RO
      COV0(2,2)=1.0
      DO 3 I=1,K
      DO 4 J=1,K
      COV(I,J)=B1*B1*COV0(I,J)
4     CONTINUE
3     CONTINUE
      CALL LINRG (N1,COV0,LDCOV0,B,LDB)
      CALL CHFAC (K,COV,LDCOV,0.00001,IRANK,RSIG,LDRSIG)
      ISSD=723459
6     C=0
      CALL RNSET (ISEED)
      CALL RNMVN (NR,K,RSIG,LDRSIG,R,LDR)
7     L=1
      V(0)=0.0
      ME=0.0
      DO 8 M=1,K
      Z(M,0)=0.0
8     CONTINUE
9     IF (ME .LE. UCL(L0)) THEN
      DO 5 I=C+1,C+N
      R(I,K)=R(I,K)+A1
```

```
5     CONTINUE
      DO 10 M=1,K
      X(M)=0.0
      DO 20 I=C+1,C+N
      X(M)=X(M)+R(I,M)/N
20    CONTINUE
      XA(1,M)=X(M)
      Z(M,L)=(1-LA(L0))*Z(M,L-1)+LA(L0)*XA(1,M)
      Z1(1,M)=Z(M,L)
10    CONTINUE
      CALL MRRRR (NRZ1,NCZ1,Z1,LDZ1,NRB,NCB,B,LDB,NRC1,NCC1,C1,LDC1)
      CALL TRNRR (NRC1,NCC1,C1,LDC1,NRC2,NCC2,C2,LDC2)
      CALL MRRRR (NRZ1,NCZ1,Z1,LDZ1,NRC2,NCC2,C2,LDC2,NRC3,
      &NCC3,C3,LDC3)
      Y=N*C3(1,1)*(2.0-LA(L0))/LA(L0)
      T=0.0
      DO 15 I=C+1,C+N
      DO 25 M=1,K
      S(1,M)=R(I,M)-XA(1,M)
25    CONTINUE
      CALL MRRRR (NRS,NCS,S,LDS,NRB,NCB,B,LDB,NRC4,NCC4,C4,LDC4)
      CALL TRNRR (NRC4,NCC4,C4,LDC4,NRC5,NCC5,C5,LDC5)
      CALL MRRRR (NRS,NCS,S,LDS,NRC5,NCC5,C5,LDC5,NRC6,NCC6,C6,LDC6)
      T=T+C6(1,1)
15    CONTINUE
      K1=2.0
      K2=K1*(N-1)
      P1=CHIDF(Y,K1)
      IF (P1 .LE. 0.00001) THEN
      P1=0.00001
      ELSE
      IF (P1 .GE. .99999) THEN
      P1=0.99999
      ELSE
      P1=CHIDF(Y,K1)
      END IF
      END IF
      P2=CHIDF(T,K2)
      IF (P2 .LE. 0.00001) THEN
      P2=0.00001
      ELSE
      IF (P2 .GE. .99999) THEN
      P2=0.99999
      ELSE
      P2=CHIDF(T,K2)
      END IF
      END IF
      U(L)=ANORIN(P1)
      W(L)=ANORIN(P2)
      V(L)=(1.0-LA(L0))*V(L-1)+LA(L0)*W(L)
      V1(L)=SQRT((2.0-LA(L0))/LA(L0))*V(L)
      ME=MAX(ABS(U(L)),ABS(V1(L)))
```

```
      L=L+1
      C=C+N
      GOTO 9
      END IF
      ARL=ARL+(L-1)
      COUNT=COUNT+1
      IF ( C .GE. 9990000) THEN
      ISEED=123459
      GOTO 6
      END IF
      IF (COUNT .LT. 10000) THEN
      GOTO 7
      ELSE
      ARL=ARL/10000
      END IF
      WRITE (*,50) LA(L0),A1,B1,RO,UCL(L0),ARL
50    FORMAT (2X,'LA =',F5.2,2X,'A1 =',F4.2,2X,'B1 =',F4.2,2X,
     & 'ROU =',F5.2,2X,'UCL =',F6.4,2X,'ARL =',F8.2)
14    CONTINUE
13    CONTINUE
12    CONTINUE
      END
```

APPENDIX B

Computer Programs for Combination Control Charts

B.1 Program for Combination of Two Shewhart Charts

This program computes optimal ARL's of the combination of $\bar{x}$ and S charts for controlling both the mean and the variance of a normal process. For a given in-control ARL of 250, each ARL value is obtained from the function of the two PDF's. A0 and B0 are changes in the process for the mean and the variance respectively and they are expressed as a multiple of the standard deviation of a normal random variable.

Program Listing

```
      DOUBLE PRECISION X1,X2,Y1,Y2,A0,B0,F1,F2,P1,P2,
      & ARL,N
      WRITE(6,3)
3     FORMAT(5X,'ARL"S OF COMBINED X BAR & S CHART FOR ARL0=250 & n=5:
      & ')
      N=5.D0
      DO 1 I=1,12
      B0=I*.25D0
      DO 5 J=0,12
      A0=J*.25D0
      X1=-3.0902/B0-A0*DSQRT(N)/B0
      X2=3.0902/B0-A0*DSQRT(N)/B0
      Y1=.0908D0/B0**2
      Y2=18.466D0/B0**2
      P1=1-F1(X2)+F1(X1)
      P2=1-F2(Y2)+F2(Y1)
      ARL=1/(1-(1-P1)*(1-p2))
      WRITE(6,10) A0,B0,ARL
10    FORMAT(5X,'A0=',F4.2,2X,'B0=',F4.2,4X,'ARL=',F10.5)
5     CONTINUE
1     CONTINUE
      END
      DOUBLE PRECISION FUNCTION F1(X)
      DOUBLE PRECISION X
      DOUBLE PRECISION T,B1,B2,B3,B4,B5,P,Z
      B1=0.319381530D0
      B2=-0.356563782D0
      B3=1.781477937D0
      B4=-1.821255978D0
      B5=1.330274429D0
      P=0.2316419D0
      T=1.0D0/(1.0D0+P*DABS(X))
      Z=(3.989422804014327D-1)*DEXP(-.5D0*(X**2))
      IF (X .GT. 0) THEN
      F1=1.0D0-Z*(B1*T+B2*T**2+B3*T**3+B4*T**4+B5*T**5)
      ELSE
```

```
F1=Z*(B1*T+B2*T**2+B3*T**3+B4*T**4+B5*T**5)
END IF
RETURN
END
DOUBLE PRECISION FUNCTION F2(X)
DOUBLE PRECISION X
F2=1-DEXP(-X/2)*(1+X/2)
RETURN
END
```

B.2 Programs for Combination of Two EWMA Charts

B.2.1 ARL Computation

This program computes optimal ARL's of the combination of the two EWMA charts for controlling both the mean and the variance of a normal process. For a given in-control ARL of 250, each ARL value is obtained using 10,000 simulations generated with IMSL Fortran Subroutines. A0 and B0 are changes in the process for the mean and the variance respectively and they are expressed as a multiple of the standard deviation of a normal random variable.

Program Listing

```
DOUBLE PRECISION LA,K(0:20),L(0:20),ARL,A0,B0,
& ARL1,ARL2
L( 0)= 2.70143
L( 1)= 2.80055
L( 2)= 2.85931
L( 3)= 2.89800
L( 4)= 2.92499
L( 5)= 2.94449
L( 6)= 2.95890
L( 7)= 2.96968
L( 8)= 2.97782
L( 9)= 2.98397
L(10)= 2.98862
L(11)= 2.99210
L(12)= 2.99470
L(13)= 2.99660
L(14)= 2.99796
L(15)= 2.99891
L(16)= 2.99952
L(17)= 2.99987
L(18)= 2.99998
K( 0)= 2.73430
K( 1)= 2.88050
```

```
      K( 2)= 2.99657
      K( 3)= 3.09790
      K( 4)= 3.18869
      K( 5)= 3.27030
      K( 6)= 3.34353
      K( 7)= 3.40918
      K( 8)= 3.46797
      K( 9)= 3.52047
      K(10)= 3.56713
      K(11)= 3.60825
      K(12)= 3.64399
      K(13)= 3.67445
      K(14)= 3.69965
      K(15)= 3.71956
      K(16)= 3.73409
      K(17)= 3.74315
      K(18)= 3.74660
      DO 5 I=0,18
      LA=.1D0+I*.05D0
      DO 10 J=1,8
      B0=J*.25D0
      CALL CRL2(LA,B0,K(I),ARL2)
      DO 15 M=0,8
      A0=M*.25D0
      CALL CRL1(LA,A0,L(I),ARL1)
      IF ((A0 .EQ. 0) .AND. (B0 .EQ. 1)) THEN
      ARL=(ARL1*ARL2)/(ARL1+ARL2-1)
      ELSE
      ARL=DMIN1(ARL1,ARL2)
      END IF
      WRITE(6,20) LA,A0,B0,ARL
20    FORMAT(5X,'LAMBDA=',F4.2,2X'A0=',F4.2,2X,'B0=',F4.2,4X,'ARL OF
      &COMBINED EWMA CHARTS = ',F10.5)
15    CONTINUE
10    CONTINUE
5     CONTINUE
      END
      SUBROUTINE CRL1(LA,A0,K,ARL)
      DOUBLE PRECISION LA,ARL,ARG,K,A(24,24),B(24),W(24),P(24),
      &X(24),D,F1,H,WK(24),A0
      INTEGER IPIVOT(24),IFLAG
      H=DSQRT(LA/(2.0D0-LA))*K
      P(1)=.9951872199970213D0
      P(2)=.9747285559713095D0
      P(3)=.9382745520027327D0
      P(4)=.8864155270044010D0
      P(5)=.8200019859739029D0
      P(6)=.7401241915785543D0
      P(7)=.6480936519369755D0
      P(8)=.5454214713888395D0
      P(9)=.4337935076260451D0
      P(10)=.3150426796961634D0
```

```
      P(11)=.1911188674736163D0
      P(12)=.0640568928626056D0
      W(1)=.0123412297999872D0
      W(2)=.0285313886289337D0
      W(3)=.0442774388174198D0
      W(4)=.0592985849154368D0
      W(5)=.0733464814110803D0
      W(6)=.0861901615319533D0
      W(7)=.0976186521041139D0
      W(8)=.1074442701159656D0
      W(9)=.1155056680537256D0
      W(10)=.1216704729278034D0
      W(11)=.1258374563468283D0
      W(12)=.1279381953467521D0
      DO 1 I=1,12
      P(25-I)=-P(I)
      W(25-I)=W(I)
1     CONTINUE
      DO 2 I=1,24
      W(I)=H*W(I)
      P(I)=H*P(I)
2     CONTINUE
      D=A0
      DO 10 I=1,24
      B(I)=-1.0D0
      DO 20 J=1,24
      ARG=(P(J)-(1.0D0-LA)*P(I))/LA
      IF (I .EQ. J) THEN
      A(I,J)=(1.0D0/LA)*W(I)*F1(ARG-D)-1.0D0
      ELSE
      A(I,J)=(1.0D0/LA)*W(J)*F1(ARG-D)
      END IF
20    CONTINUE
10    CONTINUE
      CALL FACTOR(A,24,WK,IPIVOT,IFLAG)
      IF (IFLAG .EQ. 0) THEN
      WRITE(6,50)
      STOP
      END IF
      CALL SUBST(A,IPIVOT,B,24,X)
      ARL=.0D0
      DO 30 I=1,24
      ARG=P(I)/LA
      ARL=ARL + W(I)*X(I)*F1(ARG-D)
30    CONTINUE
      ARL=1.0D0+ARL/LA
50    FORMAT(5X,'ZERO DETERMINANT FOR LINEAR SYSTEM')
      RETURN
      END
      DOUBLE PRECISION FUNCTION F1(X)
      DOUBLE PRECISION X
      F1=(3.989422804014327D-1)*DEXP(-.5D0*X*X)
```

```
      RETURN
      END
      SUBROUTINE CRL2(LA,B0,K,ARL)
      DOUBLE PRECISION LA,ALPHA,BETA,ARL,ARG,A(24,24),B0,
     &B(24),W(24),P(24),X(24),F2,STD,UCL,LCL,V,U,N,K,WK(24),T
      INTEGER IPIVOT(24),IFLAG
      N=5.D0
      U=2/(15*(N-1)**4)-1/(N-1)-1/(3*(N-1)**2)
      V=2/(N-1)+2/(N-1)**2+4/(3*(N-1)**3)-16/(15*(N-1)**5)
      UCL=U+DSQRT((V*LA)/(2.0D0-LA))*K
      LCL=U-DSQRT((V*LA)/(2.0D0-LA))*K
      ALPHA=(N-1)/2.D0
      P(1)=.9951872199970213D0
      P(2)=.9747285559713095D0
      P(3)=.9382745520027327D0
      P(4)=.8864155270044010D0
      P(5)=.8200019859739029D0
      P(6)=.7401241915785543D0
      P(7)=.6480936519369755D0
      P(8)=.5454214713888395D0
      P(9)=.4337935076260451D0
      P(10)=.3150426796961634D0
      P(11)=.1911188674736163D0
      P(12)=.0640568928626056D0
      W(1)=.0123412297999872D0
      W(2)=.0285313886289337D0
      W(3)=.0442774388174198D0
      W(4)=.0592985849154368D0
      W(5)=.0733464814110803D0
      W(6)=.0861901615319533D0
      W(7)=.0976186521041139D0
      W(8)=.1074442701159656D0
      W(9)=.1155056680537256D0
      W(10)=.1216704729278034D0
      W(11)=.1258374563468283D0
      W(12)=.1279381953467521D0
      DO 1 I=1,12
      P(25-I)=-P(I)
      W(25-I)=W(I)
1     CONTINUE
      DO 2 I=1,24
      W(I)=(UCL-LCL)*W(I)/2.D0
      P(I)=LCL+(UCL-LCL)*(P(I)+1)/2.D0
2     CONTINUE
      STD=B0
      BETA=ALPHA/(STD**2)
      DO 10 I=1,24
      B(I)=-1.0D0
      DO 20 J=1,24
      ARG=(P(J)-(1.0D0-LA)*P(I))/LA
      IF (ARG .GT. 2.2D0) THEN
      T=0.0D0
```

```
      ELSE
      T=F2(ARG,ALPHA,BETA)
      END IF
      IF (I .EQ. J) THEN
      A(I,J)=(1.0D0/LA)*W(I)*T-1.0D0
      ELSE
      A(I,J)=(1.0D0/LA)*W(J)*T
      END IF
20    CONTINUE
10    CONTINUE
      CALL FACTOR(A,24,WK,IPIVOT,IFLAG)
      IF (IFLAG .EQ. 0) THEN
      WRITE(6,50)
      STOP
      END IF
      CALL SUBST(A,IPIVOT,B,24,X)
      ARL=.0D0
      DO 30 I=1,24
      ARG=(P(I)-(1.0D0-LA)*U)/LA
      IF (ARG .GT. 2.2D0) THEN
      T=0.0D0
      ELSE
      T=F2(ARG,ALPHA,BETA)
      END IF
      ARL=ARL + W(I)*X(I)*T
30    CONTINUE
      ARL=1.0D0+ARL/LA
50    FORMAT(5X,'ZERO DETERMINANT FOR LINEAR SYSTEM')
      RETURN
      END
      DOUBLE PRECISION FUNCTION F2(X,ALPHA,BETA)
      DOUBLE PRECISION ALPHA, ARG,BETA,GAMMA,F2,X,Y
      Y=X
      ARG=ALPHA*Y-BETA*DEXP(Y)
      F2=BETA**ALPHA/GAMMA(ALPHA)*DEXP(ARG)
      RETURN
      END
      DOUBLE PRECISION FUNCTION LGAMMA(X)
      DOUBLE PRECISION F,LGAMMA,X,Y,Z
      Y=X
      IF(Y.LT.7.D0) THEN
      F=1.D0
      Z=Y-1.D0
1     Z=Z+1.D0
      IF(Z.LT.7.D0) THEN
      Y=Z
      F=F*Z
      GO TO 1
      END IF
      Y=Y+1.D0
      F=-DLOG(F)
      ELSE
```

```
      F=0.D0
      END IF
      Z=1.D0/Y**2
      LGAMMA=F+(Y-.50)*DLOG(Y)-Y+.918938533204673+
     &(((-.000595238095238D0*Z+.000793650793651D0)
     &*Z-.002777777777778D0)*Z+.083333333333333D0)/Y
      RETURN
      END
      DOUBLE PRECISION FUNCTION GAMMA(X)
      DOUBLE PRECISION GAMMA,LGAMMA,X,Y
      Y=LGAMMA(X)
      GAMMA=DEXP(Y)
      RETURN
      END
      SUBROUTINE SUBST(W1,IPIVOT,B,N,X2)
      INTEGER IPIVOT(64),I,IP,J
      DOUBLE PRECISION B(64),W1(64,64),X2(64),SUM
      IF (N .LE. 1) THEN
      X2(1)=B(1)/W1(1,1)
      RETURN
      END IF
      IP=IPIVOT(1)
      X2(1)=B(IP)
      DO 15 I=2,N
      SUM=.0D0
      I1=I-1
      DO 14 J=1,I1
      SUM=W1(I,J)*X2(J) + SUM
14    CONTINUE
      IP=IPIVOT(I)
      X2(I)=B(IP) - SUM
15    CONTINUE
      X2(N)=X2(N)/W1(N,N)
      I2=N-1
      DO 20 ISTEP=1,I2
      I=N-ISTEP
      SUM=.0D0
      I3=I+1
      DO 19 J=I3,N
      SUM=W1(I,J)*X2(J) + SUM
19    CONTINUE
      X2(I)=(X2(I)-SUM)/W1(I,I)
20    CONTINUE
      RETURN
      END
      SUBROUTINE FACTOR(W1,N,D1,IPIVOT,IFLAG)
      DOUBLE PRECISION D1(64),W1(64,64),AWIKOD,
      COLMAX,RATIO,ROWMAX,TEMP
      INTEGER IFLAG,IPIVOT(64),I,ISTAR,J,K
      IFLAG=1
      DO 9 I=1,N
      IPIVOT(I)=I
```

```
      ROWMAX=.0D0
      DO 5 J=1,N
      ROWMAX=DMAX1(ROWMAX,DABS(W1(I,J)))
5     CONTINUE
      IF (ROWMAX .EQ. .0D0) THEN
      IFLAG=0
      ROWMAX=1.0D0
      END IF
      D1(I)=ROWMAX
9     CONTINUE
      IF (N .LE. 1) RETURN
      N1=N-1
      DO 20 K=1,N1
      COLMAX=DABS(W1(K,K))/D1(K)
      ISTAR=K
      K1=K+1
      DO 13 I=K1,N
      AWIKOD=DABS(W1(I,K))/D1(K)
      IF (AWIKOD .GT. COLMAX) THEN
      COLMAX=AWIKOD
      ISTAR=I
      END IF
13    CONTINUE
      IF (COLMAX .EQ. .0D0) THEN
      IFLAG=0
      ELSE
      IF (ISTAR .GT. K) THEN
      IFLAG=-IFLAG
      I=IPIVOT(ISTAR)
      IPIVOT(ISTAR)=IPIVOT(K)
      IPIVOT(K)=I
      TEMP=D1(ISTAR)
      D1(ISTAR)=D1(K)
      D1(K)=TEMP
      DO 15 J=1,N
      TEMP=W1(ISTAR,J)
      W1(ISTAR,J)=W1(K,J)
      W1(K,J)=TEMP
15    CONTINUE
      END IF
      K2=K+1
      DO 19 I=K2,N
      W1(I,K)=W1(I,K)/W1(K,K)
      RATIO=W1(I,K)
      K3=K+1
      DO 18 J=K3,N
      W1(I,J)=W1(I,J)-RATIO*W1(K,J)
18    CONTINUE
19    CONTINUE
      END IF
20    CONTINUE
      IF (W1(N,N) .EQ. .0D0) IFLAG=0
```

```
      RETURN
      END
```

B.2.2 Simulations of Diagnostic Study

For a given in-control ARL of 250. this program simulates 1000 out-of-control signals with respect to a pair of specified A1 and B1. which are changes in the process for the mean and the variance respectively. To identify the source and the direction of the detected changes. the combination of the two EWMA charts is applied and the out-of-control signals are counted according to the charting procedure of the chart.

Program Listing

```
      REAL MU.SD.A(10000000).XA1.XA(1000000).A1.B1.
     &Z(0:10000000).Y(10000000).
     &W(0:10000000).L.K.LA.KA.UCL.
     &LCL.H.U0.V0.N.S.T1.T2
      INTEGER COUNT.NOUT.NR.C.ISEED.B2.M1.M2.V1.V2.
     &MV11.MV12.MV21.MV22
      EXTERNAL RNNOA.RNNOA.RNSET.UMACH
      K=2.86
      LA=0.1
      KA=0.1
      N=5.0
      U0=2/(15*(N-1)**4)-1/(N-1)-1/(3*(N-1)**2)
      V0=2/(N-1)+2/(N-1)**2+4/(3*(N-1)**3)-16/(15*(N-1)**5)
      MU=0
      SD=1
      WRITE(*.1)
1     FORMAT(5X.'DIAGNOSTIC ABILITIES FOR THE COMBINATION OF
     & CROWDER"S CONTROL CHARTS (ARL0=250: SAMPLE SIZE=5):')
      DO 3 B2=1.4
      B1=B2*.5
      UCL=U0+SQRT((V0*KA)/(2.0-KA))*K
      LCL=U0-SQRT((V0*KA)/(2.0-KA))*K
      DO 5 M=1.1
      A1=M*.25D0
      H=DSQRT(LA/(2.0D0-LA)/5.D0)*L
      M1=0
      M2=0
      V1=0
      V2=0
      MV11=0
      MV12=0
MV21=0
      MV22=0
      COUNT=0
```

```
      ISEED=123457
6     CALL UMACH (2,NOUT)
      NR=10000000
      CALL RNSET (ISEED
      CALL RNNOA(NR,A)
      C=0
7     Z(0)=MU
      W(0)=U0
      I=1
      T1=0.0
      T2=U0
10    IF (((T1 .LE. H) .AND. (T1 .GE. -H)) .AND.
      & ((T2 .LE. UCL) .AND. (T2 .GE. LCL))) THEN
      XA1=0.0
      DO 20 J=C+1,C+5
      A(J)=A1+A(J)*B1
      XA1=XA1+A(J)/5
20    CONTINUE
      XA(I)=XA1
      S=0.0
      DO 30 J=C+1,C+5
      S=S+(A(J)-XA(I))*(A(J)-XA(I))
30    CONTINUE
      Z(I)=(1-LA)*Z(I-1)+LA*XA(I)
      T1=Z(I)
      Y(I)=LOG(S/4.0)
      W(I)=(1-KA)*W(I-1)+KA*Y(I)
      T2=W(I)
      I=I+1
      C=C+5
      GOTO 10
      END IF
      COUNT=COUNT+1
      IF (T1 .GT. H) THEN
      IF (T2 .GT. UCL) THEN
      MV11=MV11+1
      ELSE
      IF (T2 .LT. LCL) THEN
      MV12=MV12+1
      ELSE
      M1=M1+1
      END IF
      END IF
      ELSE
      IF (T1 .LT. -H) THEN
      IF (T2 .GT. UCL) THEN
      MV21=MV21+1
      ELSE
      IF (T2 .LT. LCL) THEN
      MV22=MV22+1
      ELSE
      M2=M2+1
```

```
      END IF
      END IF
      ELSE
      IF (T2 .GT. UCL) THEN
      V1=V1+1
      ELSE
      IF (T2 .LT. LCL) THEN
      V2=V2+1
      END IF
      END IF
      END IF
      END IF
      IF (C .GE. 9990000) THEN
      ISEED=123459
      GOTO 6
      END IF
      IF (COUNT .LT. 1000) THEN
      GOTO 7
      END IF
      WRITE(*,50) LA,A1,B1,M1,M2,V1,V2,MV11,MV12,MV21,MV22
50    FORMAT(2X,'LAMBDA = ',F4.2,2X,'A =',F4.2,2X,'B =',F4.2,
     &2X,'M+ =',I4,2X,'M- =',I4,2X,'V+ =',I4,2X,'V- =',I4,
     &10X,'M+V+ =',I4,2X,'M+V- =',I4,2X,'M-V+ =',I4,2X,'M-V- =',I4)
5     CONTINUE
3     CONTINUE
      END
```

B.3 Program for Combination of Two Multivariate Charts

This program computes ARL's of the combination of χ^2 and $|\mathbf{S}|$ charts for controlling both the mean vector and the covariance matrix of a multivariate normal process in Case I. For a given in-control ARL of 200, each ARL value is obtained using 10,000 simulations generated with IMSL Fortran Subroutines.

Program Listing

```
      INTEGER LDXA,LDS1,LDS2,LDS3,NCXA,NRXA,LDCOV0,
     & IRANK,ISEED,LDR,LDRSIG,NOUT,NR,LDCOV,C,LDFAC,
     & L,COUNT,L1,I1,I2,IPVT(2),LDT,L0,L2,LDC1,NCC1,NRC1,
     & LDC2,NCC2,NRC2,LDC3,NCC3,NRC3,LDB,NRB,NCB
      PARAMETER (LDXA=1,NCXA=2,NRXA=1,LDT=2,LDFAC=2,LDB=2,NRB=2,
     &LDCOV=2,LDRSIG=2,LDR=10000000,LDS1=1,NCS1=2,NRS1=1,LDS2=2,
     &NRS2=2,NCS2=1,LDS3=2,NRS3=2,NCS3=2,LDCOV0=2,LDC1=1,NCC1=2,
     &NRC1=1,LDC2=2,NCC2=1,NRC2=2,LDC3=1,NCC3=1,NRC3=1,NCB=2,N1=2)
      REAL XA(LDXA,NCXA),COV(2,2),R(LDR,2),COV0(2,2),FAC(2,2),
     & RSIG(2,2),X(2),RO,LCL2,UCL2,ARL,DS,B1,A1,
```

```
     & S1(LDS1,NCS1),S2(LDS2,NCS2),S3(LDS3,NCS3),UCL1,
     & T(2,2),CH1,CH2,DET1,DET2,CH3,Y,C1(LDC1,NCC1),
     & C2(LDC2,NCC2),C3(LDC3,NCC3)
     EXTERNAL MRRRR,TRNRR,CHFAC,RNMVN,RNSET,UMACH,WRRRN,LFTRG,
     & LFDRG,CHIIN,LINRG
     WRITE(*,1)
1    FORMAT(5X,'ARLs OF THE COMBINED CHI-SQUARE
     & and |S| CHART (N=5,K=2):'
     CALL UMACH (2,NOUT)
     NR=10000000
     N=5
     K=2
     CH1=CHIIN(0.9975,2.0)
     CH3=CHIIN(0.00125,6.0)
     CH2=CHIIN(0.99875,6.0)
     DO 1 L0=1,3
     A1=L0*0.5
     DO 2 L2=1,1
     B1=L2*.5
     DO 3 L1=3,6
     RO=-0.9+L1*0.3
     COUNT=0
     ARL=0.0
     COV0(1,1)=1.0
     COV0(1,2)=RO
     COV0(2,1)=RO
     COV0(2,2)=1.0
     DO 4 I1=1,K
     DO 5 I2=1,K
     COV(I1,I2)=COV0(I1,I2)*B1**2
5    CONTINUE
4    CONTINUE
     CALL LFTRG (K,COV0,LDCOV0,FAC,LDFAC,IPVT)
     CALL LFDRG (K,FAC,LDFAC,IPVT,DET1,DET2)
     UCL1=CH1
     LCL2=(DET1*(10**DET2))*CH3**2/64.0
     UCL2=(DET1*(10**DET2))*CH2**2/64.0
     CALL LINRG (N1,COV0,LDCOV0,B,LDB)
     CALL CHFAC (K,COV,LDCOV,0.00001,IRANK,RSIG,LDRSIG)
     ISSD=723459
6    C=0
     CALL RNSET (ISEED)
     CALL RNMVN (NR,K,RSIG,LDRSIG,R,LDR)
7    L=1
     Y=UCL1/2.0
     DS=(UCL2+LCL2)/2.0
9    IF ((Y .LE. UCL1) .AND.
     &((DS .LE. UCL2) .AND. (DS .GT. LCL2))) THEN
     DO 11 I=C+1,C+N
     R(I,K)=R(I,K)+A1
11   CONTINUE
     DO 10 M=1,K
```

```
      X(M)=0.0
      DO 20 I=C+1,C+N
      X(M)=X(M)+R(I,M)/N
20    CONTINUE
      XA(1,M)=X(M)
10    CONTINUE
      CALL MRRRR (NRXA,NCXA,XA,LDXA,NRB,NCB,B,LDB,NRC1,
     &NCC1,C1,LDC1)
      CALL TRNRR (NRC1,NCC1,C1,LDC1,NRC2,NCC2,C2,LDC2)
      CALL MRRRR (NRXA,NCXA,XA,LDXA,NRC2,NCC2,C2,LDC2,
     &NRC3,NCC3,C3,LDC3)
      Y=N*C3(1,1)
      DO 12 I1=1,K
      DO 14 I2=1,K
      T(I1,I2)=0.0
14    CONTINUE
12    CONTINUE
      DO 15 I=C+1,C+N
      DO 25 M=1,K
      S1(1,M)=R(I,M)-XA(1,M)
25    CONTINUE
      CALL TRNRR (NRS1,NCS1,S1,LDS1,NRS2,NCS2,S2,LDS2)
      CALL MRRRR (NRS2,NCS2,S2,LDS2,NRS1,NCS1,S1,LDS1,
     &NRS3,NCS3,S3,LDS3)
      DO 16 I1=1,K
      DO 18 I2=1,K
      T(I1,I2)=T(I1,I2)+S3(I1,I2)
18    CONTINUE
16    CONTINUE
15    CONTINUE
      DS=ABS((T(1,1)*T(2,2)-T(1,2)*T(2,1))/(N-1)**2)
      L=L+1
      C=C+N
      GOTO 9
      END IF
      ARL=ARL+(L-1)
      COUNT=COUNT+1
      IF ( C .GE. 9990000) THEN
      ISEED=123459
      GOTO 6
      END IF
      IF (COUNT .LT. 10000) THEN
      GOTO 7
      ELSE
      ARL=ARL/10000.0
      END IF
      WRITE (*,30) A1,B1,RO,ARL
30    FORMAT(2X,'A1 =',F5.2,2X,'B1 =',F5.2,2X,'RO= ',F5.2,3X,
     &'ARL =',F8.2)
3     CONTINUE
2     CONTINUE
1     CONTINUE
```

END