COMPOSITE ESTIMATORS OF SMALL-AREA MEANS AND THE INTRACLASS CORRELATION COEFFICIENT

By

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A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University of Manitoba in partial fulfillment of the requirement of the degree

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To my God in Heaven, Mary and Ashley

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ABSTRACT

Parameter estimation is one of the major tasks for a statistician. An efficient estimator is always desired. For the same parameter, there are usually several possible estimators. Some estimators may be more efficient than other estimators in one situation, but less efficient in another. For example, in the context of the random intercept statistical model, or the nested-error regression model, one estimator may be more efficient if the intraclass correlation is small and less efficient if the intraclass correlation is large. In the ideal case that the intraclass correlation is known, the choice is obvious. However, life is full of uncertainties and in most cases, the true values of the variance components are not known in advance. This thesis intends to provide a cutting-edge technique for estimation. The proposed approach makes use of a composite estimator that is a linear combination of two estimators, one that is believed to be more efficient for a small intraclass correlation coefficient and the other more efficient for a large intraclass correlation. In the case of a small intraclass correlation, the estimator that is more efficient in that situation would be dominant and when the intraclass correlation is large, the other estimator becomes dominant. This approach

is applied in two important fields of applied statistics: small-area estimation in survey sampling and intraclass correlation estimation in statistical genetics. In both fields, the proposed estimator is compared to the existing estimators including the maximum likelihood estimators under the normality assumption. The comparisons are based on four very common distributions of the random effects: the double exponential distribution, the exponential distribution and the uniform distribution in addition to the normal distribution. Through Monte Carlo simulation studies for the full range of values of the intraclass correlation, the proposed estimators are more efficient than their competitors. This is the case especially when the intraclass correlation is small. For large intraclass correlation, the estimators have similar performances. Although in the case of the normal distribution, the restricted maximum likelihood estimator and the ordinary maximum likelihood estimator may be slightly more efficient, the advantage of this is offset by the computer intensive nature of the calculations. Therefore, the proposed estimators are prefered.

Chapter 1

INTRODUCTION

1.1 Introduction

Life is full of uncertainties. We are living in a world full of random events or risks. It happens everyday to everyone and there are absolutely no exceptions. Some people treat them positively as exciting elements of their lives. Some strive to avoid them, however, without success. No matter what, it cannot be denied that these random sparkles are important elements in our lives. Human beings learned this a long long time ago. Prediction has a long history; it has been ongoing since men existed. Of course, prediction in ancient times would not have been scientific in terms of present day standards. Loosely speaking, fortune telling was a kind of prediction. Ancient people still believed in their predictions or at least they tried to do so although they did not have accurate predictions. Moreover, there may not have been tools to verify the validity of their methods of predictions. Some of them were also impossible to verify. Perhaps nobody wanted any solid proofs at that time. They may not have

been interested in the accuracy of the predictions because,in their point of view, a prediction, like fortune telling, was more a kind of consolation more than a scientific study of the association of events. Nowadays, statisticians in some ways predict the future events as did people in ancient times. However, the methods of predictions or estimations are so much more advanced and have scientific support as a result of many years of research.

Statistics is about working with data, whether in the data collection stage or data analysis stage. A sound design in survey sampling or an appropriate design of an experiment increases the precision and accuracy of the information derived from the data. Statistics is also concerned about techniques to extract information from data scientifically and systematically so that it provides a solid supporting argument to interpret the data. It could be as simple as describing the data set by a single value such as a sample average or a sample median. However, when more technical mathematical and statistical theories are applied to study the validity of a hypothesis about a parameter, the statistical methodology becomes more complicated. Estimation plays a very important role in statistics. When interested in a particular characteristic of a population, a parameter used to describe that characteristic would be estimated by a statistic based on a random sample drawn from the same population. For example, if the average income of the Canadian population is of interest, the sample mean

of a random sample drawn from the population of Canada is a natural estimate of the average income of the Canadian population. In many situations, variance components and intraclass correlation coefficients are among the important parameters of interest. In others, these are merely nuisance statistics that complicate inference. Statistics can also be used to do prediction based on existing information. One of the most common examples is weather forecasting. It uses current weather conditions and other auxiliary weather related information together with dynamical and statistical models to predict the weather. When responses are observed, together with measurements of some related factors, statistical prediction of the responses could be made for different levels of the related factors. Suppose the yield of a certain crop is related to some factors including the moisture level in the soil, the amount of fertilizer used and the average daily duration of sunlight. It is possible to predict the yield of the crop with new data on the levels of these factors.

One of the commonly used techniques to study the relationship of a response with its auxiliary variables is multiple linear regression, which assumes additive effects of the factors. By linear model, this means that it is linear in the variable coefficients that are considered to be parameters in the model. The factors themselves can be nonlinear. Linear models are widely used in many research areas and practical studies because of their capability to describe the relationship between the response and the

effects. The interpretation is not always so simple. When all the effects are assumed to be constants, the model is called a fixed effects model. On the other hand, if all the effects are considered to be random, the model is a random effects model. The linear mixed model is a general model in which the parameters are a mixture of fixed and random.

1.2 Fixed Effects Models

There is a wide variety of applications for linear models. Linear models are used in situations where a variable is thought to be related to other variables in a study or an experiment using a linear function. One example would be to use the information on age, weight and skin-fold thickness to estimate a person's free fatty acid level. Another example would be the prediction of the class average score $\overline{y}$ based on the abilities of a teacher to present course material x_1 and to stimulate students' interest x_2 in that course. For the ith student, the relationship between the dependent variable which is his score in the class and the independent variable which is the abilities of the teacher can be expressed as

$$y_i = \alpha + \beta_1 x_{1i} + \beta_1 x_{2i} + \epsilon_i \tag{1.2.1}$$

where α and β_i are unknown but assumed to be fixed constants. One objective would be to estimate these constants such that the model best describes the relationship of the variables based on the information extracted from the data. On average, the

class average is $\overline{y}_i$ for a teacher who has abilities x_{1i} and x_{2i} to present the course material and to stimulate the ith students respectively. Therefore, the expectation of equation (1.2.1) is given by

$$\mathrm{E}[y_i] = \alpha + \beta_1 x_{1i} + \beta_1 x_{2i} \tag{1.2.2}$$

provided $\mathrm{E}[\epsilon_i] = 0$. The deviation ϵ_i which is a random error defined in (1.2.1) represents how much an observed score y_i differs from its expectation $\mathrm{E}[y_i]$. Usually the random error ϵ_i is assumed to follow a normal distribution with expected value zero and variance σ_ϵ^2 and the covariance between ϵ_i and ϵ_j is zero for $i \neq j$. Suppose there are N observations with k covariates, a general matrix representation of the model can be expressed as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \tag{1.2.3}$$

where $\mathbf{y} = (y_1, \ldots, y_N)'$, $\boldsymbol{\beta} = (\alpha, \beta_1, \ldots, \beta_k)'$ is a $k+1$ vector, $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_N)'$ and $\mathbf{X}$ is a $N \times (k+1)$ matrix where the first column has elements all equal to one and the jth column is $(x_{1j}, \ldots, x_{Nj})'$ for $j = 2, \ldots, k+1$. Suppose the ϵ_i's have common variance σ_ϵ^2 and are independent, the least squares estimator of $\boldsymbol{\beta}$ is

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \tag{1.2.4}$$

assuming the inverse of $\mathbf{X}'\mathbf{X}$ exists. This result does not require the normality assumption of ϵ_i. Using the generalized least squares regression approach, an estimator

of $\boldsymbol{\beta}$ is given by

$$\hat{\boldsymbol{\beta}}_{GLS} = \left(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y} \tag{1.2.5}$$

where $\mathbf{V}$ is the variance of $\mathbf{y}$ (or $\boldsymbol{\epsilon}$ since all regression coefficients are constants). If $\mathbf{V}$ is known, $\hat{\boldsymbol{\beta}}_{GLS}$ is also the best linear unbiased estimator (BLUE) of $\boldsymbol{\beta}$. Again if the ϵ_i's have common variance σ_ϵ^2 and are independent, the general least squares estimator of $\boldsymbol{\beta}$ in equation (1.2.5) reduces to equation (1.2.4). If the random term is also assumed to follow normal distribution with mean zero and variance σ_ϵ^2, it is well known that the maximum likelihood estimator of $\boldsymbol{\beta}$ is also given by equation (1.2.4) when the variance of $\boldsymbol{\epsilon}$ is $\sigma_\epsilon^2\mathbf{I}$. The predicted response $\hat{\mathbf{y}}$ is simply given by

$$\hat{\mathbf{y}} = \mathbf{X}\hat{\boldsymbol{\beta}}_{GLS}\ , \tag{1.2.6}$$

in general, which is the best linear unbiased predictor (BLUP). The model in equation (1.2.3) is called a fixed effect model where all the regression coefficients are assumed to be fixed constants with only one random term ϵ_i.

1.3 Random Effects Models

Fixed effects models have been widely used in many disciplines of research. In elementary statistics courses, statistical linear models are mainly fixed effects models. The simple linear regression model where there is an intercept and a slope (corresponding to one independent variable x) is one of the simplest cases of the fixed models.

However, the assumption that the regression coefficients are all fixed constants may not be justified in some situations. The fixed effects models cannot be applied in certain situations where the effects may not be fixed at all. Consider the following example that studies the maternal ability of mice using weights of ten-day-old pups as a measure of maternal ability, given on page 377 of Searle [53]. The study includes six pups from each of four female mice. One appropriate model is given by

$$y_{ij} = \mu + \nu_i + \epsilon_{ij} \tag{1.3.1}$$

where y_{ij} is the weight of the jth pup from the ith sow and ν_i is the effect due to the ith sow, which has mean zero. The four sows represent a random sample of four from a population of female mice. It is unlikely that these four mice were chosen specifically for this experiment. The differences among these mice are also not of main importance to the experiment. However the maternal ability, an inherited trait, throughout the whole population of female mice is of interest. Therefore, the terms ν_i's are considered to be a random sample from the population of ν. They should be considered to be random and are called random effects. Hence, the model in equation (1.3.1) is called a random effects model. In equation (1.3.1), it is not hard to realize that the total expectation of an observation is μ and the weights of the six pups from the same mother are not independent. Suppose the variance of ν_i is σ_ν^2, so that the total variance of y_{ij} is $\sigma_y^2 = \sigma_\nu^2 + \sigma_\epsilon^2$. Usually, ν_i and ϵ_{ij} are assumed to be independent of each other and observations from different mothers should also

be independent. However, in the random effects model here, observations within the same group are correlated and the covariance between y_{ij} and $y_{ij'}$ where $j \neq j'$ is σ_ν^2. This is related to an important concept useful in genetics called heritability. As defined in Scheffé [50], the measure of heritability is the intraclass correlation ρ which is the ordinary correlation coefficient between any two observations in the same group:

$$\rho = \frac{\sigma_\nu^2}{\sigma_\nu^2 + \sigma_\epsilon^2} \ . \tag{1.3.2}$$

Note that the overall expectation of the weights is

$$\mathrm{E}[y_{ij}] = \mu \ .$$

However, the expected weight within group i (given ν_i) is

$$\mathrm{E}[y_{ij} \mid \nu_i] = \mu + \nu_i$$

where the breeding value ν_i of the ith mother is unobservable. When the sample sizes of the groups are large, it is possible to obtain a reliable estimate of the population group mean. However, if the group sample sizes are small, a special procedure to adjust the estimate is needed which is conceptually related to using small-area estimation techniques in survey sampling applications.

1.4 Linear Mixed Models

The effects in each of the models in sections 1.2 and 1.3 are all either fixed or random. There are some circumstances where some of the effects are fixed and some are random. Such models are called mixed models. In general, fixed and random effects models can be regarded as special cases of linear mixed models. The term linear mixed model usually refers to models with the presence of both fixed and random effects.

Consider the same maternal ability example from Searle [53] where three specially prepared diets were used for comparison. The model in (1.3.1) can be re-written as

$$y_{ijk} = \mu + \nu_i + \phi_j + \gamma_{ij} + \epsilon_{ijk} \tag{1.4.1}$$

where y_{ijk} represents the weight of the kth pup from the ith mother receiving diet j. The effects ϕ_j for those diets are assumed fixed. The term γ_{ij} is a gene-environment interaction term coming from the interactions between the mothers and the diets. It is reasonable to consider the interaction effects to be random too. The random effects ν_j and γ_{ij} are assumed to have means zero and variances σ^2_ν and σ^2_γ respectively. It is reasonable to assume that all the random effects including the ϵ_{ijk} are mutually independent.

Suppose the random effects can be represented by a t-vector $\boldsymbol{\nu}$ where t is the number of random effects and the fixed effects by $\boldsymbol{\beta}$ as defined following equation (1.2.3), a general expression of the linear mixed model can be written as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\nu} + \boldsymbol{\epsilon} \tag{1.4.2}$$

where $\mathbf{Z}$ for $\boldsymbol{\nu}$ is similiarly defined as $\mathbf{X}$ for $\boldsymbol{\beta}$ and $\mathbf{y}$ and $\boldsymbol{\epsilon}$ have the usually definitions. Suppose the variance matrix of $\boldsymbol{\nu}$ is given by $\mathbf{G}$ and that of the $\boldsymbol{\epsilon}$ is $\mathbf{R}$, the total variance of the random observations $\mathbf{y}$ is given by

$$\mathbf{V} = \mathrm{Var}(\mathbf{y}) = \mathrm{Var}(\mathbf{Z}\boldsymbol{\nu}) + \mathbf{R} = \mathbf{Z}\mathbf{G}\mathbf{Z}' + \mathbf{R}\ . \tag{1.4.3}$$

Given that the variance of $\mathbf{y}$ is known and of full rank, the best linear unbiased estimator (BLUE) of $\boldsymbol{\beta}$ and the best linear unbiased predictor (BLUP) of $\boldsymbol{\nu}$ are given by

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y} \tag{1.4.4}$$

and

$$\hat{\boldsymbol{\nu}} = \mathbf{G}\mathbf{Z}'\mathbf{V}^{-1}\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}\right)\ , \tag{1.4.5}$$

respectively. If $\mathbf{V}$ is not of full rank, $\mathbf{V}^{-1}$ can be replaced by a generalized inverse usually denoted by $\mathbf{V}^{-}$.

The diets in the maternal ability example are clearly fixed effects and the female

mice are randomly chosen. In survey sampling, the effect from stratification in a stratified sample is considered to be fixed if the observations are classified into stata based on information prior to the collection of data. The result as to whether an observation belongs to a statum is already known (that is, fixed). However, if the observations are grouped together into different strata based on information collected from the survey itself, it is reasonable to believe the effect from the stratification is random. One would not know whether an observation belongs to a certain group or strata unless it is actually chosen. This kind of stratification is called post-stratification, or subsetting, because the observations are classified into strata after they are drawn to form a sample.

There are many situations in which the effects are not so easily classified. An interesting question then arises when considering which effects are to be fixed and which are to be considered random. Suppose there are three laboratory technicians caring for the mice. The effects of their interactions with the mice could be considered to be random when they are from a random sample of a group of technicians. But this may not always be true. What happens if they are the only available technicians of the laboratory and the differences between them were of interest. In such a case, the technician effects would be considered to be fixed effects according to Searle [53]. When the effects are hard to classify as being fixed or random, whether an effect

should be considered fixed or random, Searle in [53] on page 383 suggested that one should be concerned about how the data were gathered and about the source environment:

> "... when inferences are going to be confined to the effects in the model the effects are considered fixed; and when inferences will be made about a population of effects from which those in the data are considered to be a random sample then the effects are considered as random."

1.5 Fixed and Random Effects Estimation with Unknown Variances

In estimation using equations (1.4.4) and (1.4.5), an important assumption is that the variance components in the model are known. The estimator $\hat{\boldsymbol{\beta}}$ and the predictor $\hat{\boldsymbol{\nu}}$ are best linear unbiased provided this assumption is valid as seen in the equations. However, in many situations, the variance components are simply not known. One simple and straightforward technique is to replace these unknown values by their estimates. Instead of estimating $\boldsymbol{\beta}$ and $\boldsymbol{\nu}$ by equations (1.4.4) and (1.4.5), the following estimator and predictor are used

$$\hat{\boldsymbol{\beta}}_E = \left(\mathbf{X}'\hat{\mathbf{V}}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\hat{\mathbf{V}}^{-1}\mathbf{y} \tag{1.5.1}$$

and

$$\hat{\nu}_E = \hat{\mathbf{G}}\mathbf{Z}'\hat{\mathbf{V}}^{-1}(\mathbf{y} - \mathbf{X}\hat{\beta}_E) \tag{1.5.2}$$

where $\hat{\mathbf{G}}$ and $\hat{\mathbf{V}}$ are matrices with estimated components.

There are many different variance estimation methods available. Some of the commonly used ones are the maximum likelihood method, the restricted maximum likelihood method and Henderson's method. In this thesis, a new approach for estimating the variance components is proposed. This is based on the framework of the weighted least-squares regression. Since the goals of this thesis are to predict the small-area mean and to estimate the intraclass correlation efficiently, an approach to shrink the mean square errors of the estimators will be used with two weighted variance estimators. From Monte Carlo studies it will be shown that the proposed estimators are easily calculated and are more efficient than existing estimators. Detailed discussions of the approach will be given in the following chapters. In the following sections, a brief introduction to small-area estimation and the estimation of the intraclass correlation coefficient will be given.

1.6 Small-Area Estimation

Small-area estimation has gained more attention in the last decade as more efficient estimators have been developed. A small area is not necessarily an area. It could refer

to a group of units in the population. However, there is no universal agreement on the size of a small area. One small area may contain 20 units whereas another small area might contain a hundred. The size of an area or a group is considered to be small relative to the size of the population to which it belongs. Because the sizes of the small areas can vary, direct estimators are not efficient and there is a need to improve the efficiency of estimation within the small areas. One popular approach is to borrow information from similar areas to increase efficiency. It is reasonable to assume similar areas would have similar attributes and therefore could provide certain information about the primary area of interest. Under the linear mixed model context, small-area means can be estimated more efficiently using BLUP. Henderson [23] has shown that for known variances, the best linear unbiased estimator of $\mu = \mathbf{l}'\boldsymbol{\beta} + \mathbf{m}'\boldsymbol{\nu}$ is given by

$$\begin{aligned} \dot{\mu}_E &= \mathbf{l}'\hat{\boldsymbol{\beta}} + \mathbf{m}'\hat{\boldsymbol{\nu}} \\ &= \mathbf{l}'\hat{\boldsymbol{\beta}} + \mathbf{m}'\mathbf{G}\mathbf{Z}'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}) \end{aligned} \tag{1.6.1}$$

based on the model in (1.4.2) with $\hat{\boldsymbol{\beta}}$ from equation (1.4.4).

As mentioned previously, the variance components are usually unknown and need to be estimated. With the estimated variance components replacing the variance components in equation (1.6.1), an empirical BLUP, or EBLUP, is given by

$$\ddot{\mu}_E = \mathbf{l}'\hat{\boldsymbol{\beta}}_E + \mathbf{m}'\hat{\boldsymbol{\nu}}_E \ . \tag{1.6.2}$$

This estimated predictor $\ddot{\mu}_E$ is derived from a purely practical point of view. It is unbiased for the small-area mean μ as $\tilde{\mu}$ in equation (1.6.1) provided the variance estimators are unbiased, translation invariant and the variance matrix of $\mathbf{y}$ is symmetric. However, with the estimated parameters, no statistical rationale such as minimum variance has been established for the small-area mean estimators (McCulloch & Searle [35]). Moreover, the unbiasedness of EBLUP does not guarantee that it should be a best linear unbiased predictor. It is possible that there exists an unbiased predictor of the small-area mean μ more efficient than $\ddot{\mu}_E$ with variance estimators using conventional methods such as Henderson's method of fitting constants. This thesis suggests that the small-area mean can be estimated more efficiently in two steps. The proposed estimator is a shrinkage estimator combining two ELBUPs with the same form in equation (1.6.2). The variance components in each of the two EBLUPs are estimated by weighted least squares. Detailed discussion on the estimation method will be given in chapter 2.

1.7 Intraclass Correlation Coefficient

The intraclass correlation coefficient defined in equation (1.3.2) is an ordinary correlation coefficient between any two observations in the same group. It is commonly used in genetics and epidemiology to measure the degree of resemblance within the same family with respect to some biological or physiological attribute. It is also used

in agriculture in the analysis of experiments and in survey sampling to assess homogeneity of clusters. The intraclass correlation is often estimated directly by replacing the variance parameters with their estimates. Usually, the estimation of the variance components assumes a one-way random effects model similar to equation (1.3.1) where covariates do not exist. In such a case, the estimation of the variances is purely based on the observed values of the responses. Many forms of estimators of intraclass correlation have been suggested. Pearson [39] proposed the pairwise intraclass correlation. Fieller & Smith [11] generalized the ANOVA estimator of Fisher [12] to the unbalanced situation. Karlin et al. [27] suggested a weighted product-moment estimator. Srivastava & Ng [65] suggested a weighted sum of two estimators in an intercept-free regression model. More recently, Keen [28] introduced a method of combining pairs of weighted estimators of intraclass correlation that is fully efficient over nearly the complete range of values of the intraclass parameter. However, none of these estimators, except the weighted product-moment estimator suggested by Srivastava & Ng [65], adjust for the covariates although they may have a certain impact on the response.

This thesis proposes a cutting-edge approach to estimate the intraclass correlation in the presence of the covariates. Similar to the procedures used in small-area estimation, the intraclass correlation coefficient estimation uses the same variance estimators developed based on the weighted least squares regression method and the

optimal estimator is constructed by a convex combination of the two estimators of two separate weighting schemes. Detailed discussion of the procedures are given in Chapters 2 and 4.

Chapter 2

VARIOUS ESTIMATORS IN SMALL AREA AND INTRACLASS CORRELATION PROBLEMS

2.1 Introduction

Linear mixed models have numerous applications in many research areas including genetics, genetic epidemiology, biological sciences and survey sampling. Unknown intercept and regression coefficients in the models are always of interest. Given the estimates of these unknown parameters, the response could then be predicted for different levels of the covariates. These parameters can be estimated by different methods. Ordinary least squares regression and generalized least squares regression are two popular ones. However, when the variance structure of the responses is not of an identity matrix form, that is when it is not $\sigma^2\mathbf{I}$, the best estimators of the regression coefficients in the models are weighted functions of the variance components, which

are usually unknown. For example, the generalized least squares estimator of the coefficient vector $\boldsymbol{\beta}$ in model (1.4.2) is given by

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y} \tag{2.1.1}$$

where the variance $\mathbf{V}$ of the response vector $\mathbf{y}$ usually involves unknown components that need to be estimated. In linear mixed models with the presence of the random group effects, the variance $\mathbf{V}$ can also be expressed in terms of the intraclass correlation.

This thesis focuses on a two-stage estimation procedure. The first stage is the estimation of the variances components in the linear model. The second stage is the estimation of linear functions of the regression parameters of interest in which the estimators of the regression parameters are functions of the variance estimators. The estimator of the regression coefficient is a weighted estimator based on the ordinary linear regression model. The proposed estimator is a shrinkage estimator that combines two estimators of the same form. One of these two estimators makes use of one weighting scheme for the variance estimator that is known to be efficient for small intraclass correlations and the other one uses another weighting scheme known to be efficient for large intraclass correlations.

2.2 The Linear Mixed Model

Consider the linear mixed model as in equation (1.4.2)

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\nu} + \boldsymbol{\epsilon} \tag{2.2.1}$$

where $\mathbf{y}$ is an n-vector of observations, $\mathbf{X}$ is an $n \times (k+1)$ matrix of known covariates. Usually the elements in the first column of $\mathbf{X}$ are equal to 1 and there are k concomittant variables, or covariates. The unknown but fixed parameter vector $\boldsymbol{\beta}$ is a $(k+1)$-vector represents the fixed effects. The first element of $\boldsymbol{\beta}$ is the intercept in the model and the other elements are the regression coefficients. In the second term of (2.2.1), $\mathbf{Z}$ is an $n \times t$ known matrix and the random vector $\boldsymbol{\nu}$ is a t-vector of random effects. The error term $\boldsymbol{\epsilon}$ is an n-dimensional random vector as defined in the ordinary regression models. The random effects $\boldsymbol{\nu}$ and the error term $\boldsymbol{\epsilon}$ are assumed to be mutually independent $N(\mathbf{0}, \mathbf{G})$ and $N(\mathbf{0}, \mathbf{R})$, respectively. Therefore the variance of $\mathbf{y}$ is simply

$$\mathbf{V} = \mathbf{Z}\mathbf{G}\mathbf{Z}' + \mathbf{R}\,. \tag{2.2.2}$$

In the following sections, two popular methods for the estimation of the variance components, Henderson's method 3 and the maximum likelihood method, will be studied. A proposed method for estimating the variance components based on ordinary least squares regression is also presented.

2.3 Henderson's Method

Henderson's Method 3, also called the "method of fitting constants," is useful in finding the estimators of the variance components namely, σ_ϵ^2 and σ_ν^2. It makes use of the difference in the reduction in sum of squares of full models and reduced models to estimate the variance components. Consider the same linear mixed model in (2.2.1) with variance-covariance matrix in equation (2.2.2) assuming $\mathbf{G} = \sigma_\nu^2\mathbf{I}$ and $\mathbf{R} = \sigma_\epsilon^2\mathbf{I}$, the variance of $\mathbf{y}$ can be expressed as

$$\mathbf{V} = \sigma_\nu^2\mathbf{Z}\mathbf{Z}' + \sigma_\epsilon^2\mathbf{I} \; . \tag{2.3.1}$$

For simplicity, let $\mathbf{M} = (\mathbf{X}\ \mathbf{Z})$ and $\mathbf{B}' = (\boldsymbol{\beta}', \boldsymbol{\nu}')$, the mixed model in equation (2.2.1) can now be expressed as

$$\mathbf{y} = \mathbf{M}\mathbf{B} + \boldsymbol{\epsilon} \; . \tag{2.3.2}$$

Let $\mathbf{R}(\mathbf{B})$ be the reduction in sum of squares based on treating the random effects as if they were fixed effects in the full model; it can be represented by

$$\mathbf{R}(\mathbf{B}) = \mathbf{y}' \left[\mathbf{M}\left(\mathbf{M}'\mathbf{M}\right)^{-}\mathbf{M}'\right]\mathbf{y} \; . \tag{2.3.3}$$

Hence, the sum of squares of error is

$$\begin{aligned} \mathbf{y}'\mathbf{y} - \mathbf{R}\left(\mathbf{B}\right) &= \mathbf{y}'\mathbf{y} - \mathbf{y}'\left[\mathbf{M}\left(\mathbf{M}'\mathbf{M}\right)^{-}\mathbf{M}'\right]\mathbf{y} \\ &= \mathbf{y}'\left[\mathbf{I} - \mathbf{M}(\mathbf{M}'\mathbf{M})^{-}\mathbf{M}'\right]\mathbf{y} \\ &= \sum_{i=1}^{t}\sum_{j=1}^{n_i}\hat{\epsilon}_{ij}^2 \end{aligned} \tag{2.3.4}$$

where $\hat{\epsilon}_{ij} = y_{ij} - \hat{y}_{ij}$, with $\hat{y}_{ij} = \mathbf{M}\hat{\mathbf{B}}$ where $\hat{\mathbf{B}}$ is the ordinary least regression estimator of $\mathbf{B}$ based on the full model in (2.3.2).

The expectation of the sum of squares of error is

$$\begin{aligned} \mathrm{E}\left[\sum_{i=1}^{t}\sum_{j=1}^{n_i}\hat{\epsilon}_{ij}^2\right] &= \sigma_\epsilon^2 \,\mathrm{tr}\left[\mathbf{I} - \mathbf{M}\left(\mathbf{M}'\mathbf{M}\right)^{-}\mathbf{M}'\right] \\ &= \sigma_\epsilon^2\left[n - \mathrm{r}\left(\mathbf{M}\right)\right] \end{aligned} \tag{2.3.5}$$

where $\mathrm{r}(\mathbf{M})$ is the rank of $\mathbf{M}$ and n is the total number of observations in all i areas or subgroups. Equate (2.3.4) and (2.3.5) after substituting $\tilde{\sigma}_\epsilon^2$ for σ_ϵ^2, and we obtain an estimator of σ_ϵ^2:

$$\tilde{\sigma}_\epsilon^2 = \frac{\mathbf{y}'\left[\mathbf{I} - \mathbf{M}(\mathbf{M}'\mathbf{M})^{-}\mathbf{M}'\right]\mathbf{y}}{n - \mathrm{r}(\mathbf{M})} \,. \tag{2.3.6}$$

Now, consider the expectation in the reduction in sum of squares of the full model;

$$\begin{aligned} \mathrm{E}\left[\mathbf{R}\left(\mathbf{B}\right)\right] &= \mathrm{tr}\left[\left(\mathbf{M}'\mathbf{M}\right)\mathrm{E}\left(\mathbf{B}\mathbf{B}'\right)\right] + \sigma_\epsilon^2 \mathrm{r}\left(\mathbf{M}\right) \\ &= \mathrm{tr}\left[\begin{pmatrix}\mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{Z} \\ \mathbf{Z}'\mathbf{X} & \mathbf{Z}'\mathbf{Z}\end{pmatrix}\mathrm{E}\left(\mathbf{B}\mathbf{B}'\right)\right] + \sigma_\epsilon^2 \mathrm{r}\left(\mathbf{M}\right) \,. \end{aligned} \tag{2.3.7}$$

The reduction in sum of squares of the reduced model with $\boldsymbol{\beta}$ only and ignoring the presence of the random effect $\boldsymbol{\nu}$ is given by

$$\mathbf{R}\left(\boldsymbol{\beta}\right) = \mathbf{y}'\left[\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-}\mathbf{X}'\right]\mathbf{y} \tag{2.3.8}$$

with expectation

$$\begin{aligned} \mathrm{E}\left[\mathbf{R}\left(\boldsymbol{\beta}\right)\right] &= \mathrm{tr}\left[\mathbf{M}'\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-}\mathbf{X}'\mathbf{M}\,\mathrm{E}\left(\mathbf{B}\mathbf{B}'\right)\right] + \sigma_\epsilon^2 \mathrm{r}\left(\mathbf{X}\right) \\ &= \mathrm{tr}\left[\begin{pmatrix}\mathbf{X}'\mathbf{X} \\ \mathbf{Z}'\mathbf{X}\end{pmatrix}\left(\mathbf{X}'\mathbf{X}\right)^{-}\left(\mathbf{X}'\mathbf{X}\;\mathbf{X}'\mathbf{Z}\right)\,\mathrm{E}\left(\mathbf{B}\mathbf{B}'\right)\right] + \sigma_\epsilon^2 \mathrm{r}\left(\mathbf{X}\right) \,. \end{aligned} \tag{2.3.9}$$

where $\mathrm{r}(\mathbf{X})$ is the rank of $\mathbf{X}$. Since $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{X} = \mathbf{X}$, this expectation becomes

$$\mathrm{E}\left[\mathbf{R}\left(\boldsymbol{\beta}\right)\right] = \mathrm{tr}\left[\begin{pmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{Z} \\ \mathbf{Z}'\mathbf{X} & \mathbf{Z}'\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-}\mathbf{X}'\mathbf{Z} \end{pmatrix} \mathrm{E}\left(\mathbf{B}\mathbf{B}'\right)\right] + \sigma_{\epsilon}^{2}\mathrm{r}\left(\mathbf{X}\right) \ . \tag{2.3.10}$$

The reduction in sum of squares of the conditional model based on $\boldsymbol{\nu}$ given $\boldsymbol{\beta}$ already in the model is

$$\begin{aligned} \mathbf{R}\left(\boldsymbol{\nu}|\boldsymbol{\beta}\right) &= \mathbf{R}\left(\mathbf{B}\right) - \mathbf{R}\left(\boldsymbol{\beta}\right) \\ &= \mathbf{y}'\mathbf{M}\left(\mathbf{M}'\mathbf{M}\right)^{-}\mathbf{M}'\mathbf{y} - \mathbf{y}'\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-}\mathbf{X}'\mathbf{y} \\ &= \mathbf{y}'\left[\mathbf{I} - \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-}\mathbf{X}'\right]\mathbf{y} - \mathbf{y}'\left[\mathbf{I} - \mathbf{M}\left(\mathbf{M}'\mathbf{M}\right)^{-}\mathbf{M}'\right]\mathbf{y} \\ &= \sum_{i=1}^{t}\sum_{j=1}^{n_i}\hat{u}_{ij}^{2} - \sum_{i=1}^{t}\sum_{j=1}^{n_i}\hat{\epsilon}_{ij}^{2} \end{aligned} \tag{2.3.11}$$

where $\sum_{i=1}^{t}\sum_{j=1}^{n_i}\hat{u}_{ij}^{2} = \mathbf{y}'\left[\mathbf{I} - \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-}\mathbf{X}'\right]\mathbf{y}$ which is the sum of squares of the residuals of an ordinary least squares regression of $\mathbf{y}$ on $\mathbf{X}$ based on a reduced model without $\mathbf{Z}$. The expectation of the reduction in sum of squares is

$$\begin{aligned} \mathrm{E}\left[\mathbf{R}\left(\boldsymbol{\nu}|\boldsymbol{\beta}\right)\right] &= \mathrm{E}\left[\mathbf{R}\left(\mathbf{B}\right) - \mathbf{R}\left(\boldsymbol{\beta}\right)\right] \\ &= \mathrm{tr}\left[\left(\mathbf{Z}'\mathbf{Z} - \mathbf{Z}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Z}\right)\ \mathrm{E}\left(\boldsymbol{\nu}\boldsymbol{\nu}'\right)\right] + \sigma_{\epsilon}^{2}\left[\mathrm{r}(\mathbf{M}) - \mathrm{r}(\mathbf{X})\right] \\ &= \mathrm{tr}\left[\left(\mathbf{Z}'\mathbf{Z} - \mathbf{Z}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Z}\right)\right]\sigma_{\nu}^{2} + \sigma_{\epsilon}^{2}\left[\mathrm{r}(\mathbf{M}) - \mathrm{r}(\mathbf{X})\right] \ . \end{aligned} \tag{2.3.12}$$

Equate the reduction in sum of squares of the conditional model with its expectation after substituting $\tilde{\sigma}_{\epsilon}^{2}$ for σ_{ϵ}^{2}, $\tilde{\sigma}_{\nu}^{2}$ for σ_{ν}^{2} , and $(n - \mathrm{r}(\mathbf{M}))\tilde{\sigma}_{\epsilon}^{2}$ for $\sum_{i=1}^{t}\sum_{j=1}^{n_i}\hat{\epsilon}_{ij}^{2}$, and rearranging the terms, we obtained an estimator of σ_{ν}^{2} :

$$\tilde{\sigma}_{\nu}^{2} = \frac{\sum_{i=1}^{t}\sum_{j=1}^{n_i}\hat{u}_{ij}^{2} - (n - \mathrm{r}(\mathbf{X}))\tilde{\sigma}_{\epsilon}^{2}}{\mathrm{tr}\left[(\mathbf{Z}'\mathbf{Z} - \mathbf{Z}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Z})\right]} \ . \tag{2.3.13}$$

Note that for the nested-error regression model where $\mathbf{Z} = \text{diag}(\mathbf{1}_{n_1}, \ldots \mathbf{1}_{n_t})$, the estimator of σ_ν^2 can be reduced to

$$\tilde{\sigma}_\nu^2 = \frac{\sum_{i=1}^{t}\sum_{j=1}^{n_i}\hat{u}_{ij}^2 - (n - \text{r}(\mathbf{X}))\tilde{\sigma}_\epsilon^2}{n - \text{tr}\left[(\mathbf{X}'\mathbf{X})^{-}\sum_{i=1}^{t} n_i^2 \overline{\mathbf{x}}_i \overline{\mathbf{x}}_i'\right]} . \tag{2.3.14}$$

2.4 Maximum Likelihood Approach

The maximum likelihood approach is another popular way to estimate the variance components. Suppose the variance of the random vector $\mathbf{y}$ in model (2.2.1) is given by (2.3.1). It is possible to write the variance of the response as a function of two parameters $\sigma_\tau^2 = \sigma_\epsilon^2 + \sigma_\nu^2$ and $\rho = \sigma_\nu^2/\sigma_\tau^2$. With this parametrization, the likelihood function given the observations $\mathbf{y}_i$ and the covariates $\mathbf{X}_i$ in the ith area is given by

$$L_i\left(\sigma_\tau^2, \boldsymbol{\beta}, \rho\right) = \frac{\exp\left\{-\left(\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta}\right)'\mathbf{V}_i^{-1}\left(\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta}\right)/2\right\}}{(2\pi)^{n_i/2}\left|\mathbf{V}_i\right|^{1/2}} . \tag{2.4.1}$$

The total likelihood is then

$$\begin{aligned} L &= \prod_{i=1}^{t} L_i\left(\sigma_\tau^2, \boldsymbol{\beta}, \rho\right) \\ &= \frac{\exp\left\{-\sum_{i=1}^{t}\left(\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta}\right)'\mathbf{V}_i^{-1}\left(\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta}\right)/2\right\}}{(2\pi)^{N/2}\prod_{i=1}^{t}\left|\mathbf{V}_i\right|^{1/2}} . \end{aligned} \tag{2.4.2}$$

In Appendix A.16, it is shown that the derivatives of $-2\log L$ with respect to σ_τ^2 and to ρ are, respectively, given by

$$\frac{\partial(-2\log L)}{\partial\sigma_\tau^2} = \frac{N}{\sigma_\tau^2} - \frac{\sum_{i=1}^{t} SSI_i}{(1-\rho)\,\sigma_T^4} + \frac{\rho}{(1-\rho)\,\sigma_T^4}\sum_{i=1}^{t}\frac{SSJ_i}{1+(n_i-1)\,\rho} \tag{2.4.3}$$

and

$$\frac{\partial\left(-2\log L\right)}{\partial\rho} = -\frac{N-t}{1-\rho}+\sum_{i=1}^{t}\frac{n_i-1}{1+(n_i-1)\rho}+\frac{\sum_{i=1}^{t}SSI_i}{(1-\rho)^2\sigma_\tau^2} -\frac{1}{(1-\rho)^2\sigma_\tau^2}\sum_{i=1}^{t}\frac{SSJ_i\left[1+(n_i-1)\rho^2\right]}{\left[1+(n_i-1)\,\rho\right]^2} \tag{2.4.4}$$

where $SSI_i=\left(\mathbf{y}_i-\mathbf{X}_i\boldsymbol{\beta}\right)'\mathbf{I}_{n_i}\left(\mathbf{y}_i-\mathbf{X}_i\boldsymbol{\beta}\right)$, $SSJ_i=\left(\mathbf{y}_i-\mathbf{X}_i\boldsymbol{\beta}\right)'\mathbf{J}_{n_i}\left(\mathbf{y}_i-\mathbf{X}_i\boldsymbol{\beta}\right)$ with $\mathbf{I}_{n_i}$ an $n_i\times n_i$ identity matrix and $\mathbf{J}_{n_i}$ an $n_i\times n_i$ matrix of 1's. Setting equation (2.4.3) to zero, an expression of the maximum likelihood estimator of σ_τ^2 is given by

$$\hat{\sigma}_\tau^2=\frac{1}{N\left(1-\rho\right)}\sum_{i=1}^{t}\left(SSI_i-\frac{\rho\,SSJ_i}{1+\left(n_i-1\right)\rho}\right)\ . \tag{2.4.5}$$

Setting equation (2.4.4) to zero, and replacing σ_τ^2 by its maximum likelihood estimator $\hat{\sigma}_\tau^2$ in (2.4.5), we obtain an expression for the estimating equation for ρ as in equation (A.16.10) in the Appendix:

$$N\left[\sum_{i=1}^{t}\left(SSI_i-\frac{\rho\,SSJ_i}{1+\left(n_i-1\right)\rho}\right)\right]^{-1} = \sum_{i=1}^{t}\left(SSI_i-\frac{SSJ_i\left[1+(n_i-1)\rho^2\right]}{\left[1+\left(n_i-1\right)\rho\right]^2}\right) - \sum_{i=1}^{t}\frac{n_i\left(n_i-1\right)\rho}{1+(n_i-1)\rho}=0\ . \tag{2.4.6}$$

From this estimating equation, the maximum likelihood estimator of the intraclass correlation ρ does not have a closed form. To obtain an estimate of ρ, we can solve the above estimating equation numerically. One of the commonly used method is Newton-Raphson's method, which is introduced in the Appendix A.17.

2.5 Restricted Maximum Likelihood (REML)

One major criticism of the maximum likelihood estimator is that it is not unbiased. When estimating the variance components, it does not take into account the loss of degrees of freedom for the estimation of fixed effects in the model (McCulloch & Searle [35], Smyth & Verbyla [58]). Therefore, there is a preference for the modified maximum likelihood procedure introduced by Patterson & Thompson [37] which attempts to account for the loss of degrees of freedom by maximizing the likelihood of linear independent combinations of elements of the responses $\mathbf{y}$. This modified maximum likelihood is also called residual maximum likelihood because it can be presented as the marginal likelihood of the residual contrasts or error contrasts. A contrast is defined to be a linear function of the parameters with the sum of the coefficients equal to zero [31]. Suppose the contrasts among responses can be divided into the contrasts between the treatments totals and the contrasts with zero mean which are the called error contrasts; the contrasts between the treatments totals can then be excluded from the likelihood because they provide no information on the errors as long as the treatments effects are considered to be fixed [37].

The residual maximum likelihood is also known as the restricted maximum likelihood (REML). According to Searle et al. [55], the term residual maximum likelihood is more popular in Europe. For estimating the variance components, the restricted

maximum likelihood procedure (REML) maximizes the likelihood of linear combinations $\mathbf{k}'\mathbf{y}$ of the elements of the response vector $\mathbf{y}$ such that the expectation of $\mathbf{k}'\mathbf{y} = 0$. This is the case when $\mathbf{k}'\mathbf{X} = \mathbf{0}$ which means $\mathbf{k}'\mathbf{y}$ does not contain any of the fixed effects in $\boldsymbol{\beta}$. The maximum number of linearly independent error contrasts $\mathbf{k}'\mathbf{y}$ is $N - \mathrm{r}(\mathbf{X})$ where N is the total number of observations and $\mathrm{r}(\mathbf{X})$ is the rank of the matrix $\mathbf{X}$. A particular set of linearly independent error contrasts is $\mathbf{K}'\mathbf{X}$ where $\mathbf{K} = (\mathbf{k}_1, \ldots, \mathbf{k}_{N-\mathrm{r}(\mathbf{X})})$. The one used in Patterson & Thompson [37] is a special case in which $\mathbf{K} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Given that the random effects are normally distributed, that is $\mathbf{y} \sim N(\mathbf{X}\boldsymbol{\beta}, \mathbf{V})$, with $\mathbf{K}'\mathbf{X} = \mathbf{0}$, consider the REML function used by McCulloch & Searle [35]:

$$l_S = c_1 - \frac{1}{2}\log|\mathbf{K}'\mathbf{V}\mathbf{K}| - \frac{1}{2}\mathbf{y}'\mathbf{P}\mathbf{y} \tag{2.5.1}$$

where $\mathbf{P} = \mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}\left(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{V}^{-1} = \mathbf{K}\left(\mathbf{K}'\mathbf{V}\mathbf{K}\right)^{-1}\mathbf{K}'$, and $\mathbf{V} = \sum_{i=1}^{r}\sigma_i^2\mathbf{Z}_i\mathbf{Z}_i'$. Note that

$$\frac{\partial \mathbf{V}}{\partial \sigma_i^2} = \mathbf{Z}_i\mathbf{Z}_i'$$

$$\frac{\partial \mathbf{V}^{-1}}{\partial \sigma_i^2} = -\mathbf{V}^{-1}\mathbf{Z}_i\mathbf{Z}_i'\mathbf{V}^{-1}$$

$$\frac{\partial \log|\mathbf{V}|}{\partial \sigma_i^2} = \mathrm{tr}\left(\mathbf{V}^{-1}\mathbf{Z}_i\mathbf{Z}_i'\right)$$

$$\frac{\partial \mathbf{P}}{\partial \sigma_i^2} = -\mathbf{P}\mathbf{Z}_i\mathbf{Z}_i'\mathbf{P}$$

and

$$\frac{\partial \mathbf{K}'\mathbf{V}\mathbf{K}}{\partial \sigma_i^2} = \mathbf{K}'\mathbf{Z}_i\mathbf{Z}_i'\mathbf{K} \ .$$

Now differentiating the likelihood in equation (2.5.1) yields:

$$\begin{aligned} \frac{\partial l_S}{\partial \sigma_i^2} &= -\frac{1}{2}\mathrm{tr}\left[\left(\mathbf{K}'\mathbf{V}\mathbf{K}\right)^{-1}\mathbf{K}'\mathbf{Z}_i\mathbf{Z}_i'\mathbf{K}\right] + \frac{1}{2}\mathbf{y}'\mathbf{P}\mathbf{Z}_i\mathbf{Z}_i'\mathbf{P}\mathbf{y} \\ &= -\frac{1}{2}\mathrm{tr}\left(\mathbf{P}\mathbf{Z}_i\mathbf{Z}_i'\right) + \frac{1}{2}\mathbf{y}'\mathbf{P}\mathbf{Z}_i\mathbf{Z}_i'\mathbf{P}\mathbf{y} \ . \end{aligned} \tag{2.5.2}$$

Harville [18] gave another expression of the REML function:

$$l_H = c_2 - \frac{1}{2}\log|\mathbf{V}| - \frac{1}{2}\log|\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}| - \frac{1}{2}\left(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}\right)'\mathbf{V}^{-1}\left(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}\right) \ . \tag{2.5.3}$$

Note that

$$\left(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}\right)'\mathbf{V}^{-1}\left(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}\right) = \left(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}\right)'\mathbf{V}^{-1}\mathbf{V}\mathbf{V}^{-1}\left(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}\right) = \mathbf{y}'\mathbf{P}\mathbf{V}\mathbf{P}\mathbf{y} = \mathbf{y}'\mathbf{P}\mathbf{y} \ . \tag{2.5.4}$$

The derivative of the likelihood by Harville in equation (2.5.3) is given by

$$\begin{aligned} \frac{\partial l_H}{\partial \sigma_i^2} &= -\frac{1}{2}\mathrm{tr}\left(\mathbf{V}^{-1}\mathbf{Z}_i\mathbf{Z}_i'\right) + \frac{1}{2}\mathrm{tr}\left[\left(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{Z}_i\mathbf{Z}_i'\mathbf{V}^{-1}\mathbf{X}'\right] \\ &\quad + \frac{1}{2}\mathbf{y}'\mathbf{P}\mathbf{Z}_i\mathbf{Z}_i'\mathbf{P}\mathbf{y} \\ &= -\frac{1}{2}\mathrm{tr}\left[\left(\mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}\left(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{V}^{-1}\right)\mathbf{Z}_i\mathbf{Z}_i'\right] \\ &\quad + \frac{1}{2}\mathbf{y}'\mathbf{P}\mathbf{Z}_i\mathbf{Z}_i'\mathbf{P}\mathbf{y} \ . \end{aligned} \tag{2.5.5}$$

The first derivative of the likelihood functions (2.5.1) and that of (2.5.3) with respect to σ_i^2 are identical and therefore either likelihood function gives the same set of solutions for the variance estimates.

2.6 Variance Estimation: An OLS Regression Approach

One focus of this thesis is to design a shrinkage estimator that can efficiently estimate the small-area means and another focus is to design a shrinkage estimation of the intraclass correlation coefficient in terms of their mean square errors. In both small-area estimation and intraclass correlation estimation, shrinkage estimators are weighted linear functions of two estimators involving the variance estimators. Therefore, the variance estimation, or rather the ratio of variance components, has a direct impact on the second stage estimation of the parameters. Consider the random intercept model with random intercept, α_i, for the ith small area, or small group:

$$y_{ij} = \alpha_i + x_{ij1}\beta_1 + \ldots + x_{ijk}\beta_k + \epsilon_{ij}, \quad i = 1, \ldots, t, \quad j = 1, \ldots, n_i, \quad (2.6.1)$$

where y_{ij} is the observed response for the jth sampled unit in the ith group, x_{ijl} is the lth covariate and β_l is the lth unknown regression coefficient for $l = 1, \ldots, k$. There are t groups, or small areas, and for the ith group, the sample size is n_i. The total number of observations in the sample is $N = \sum_{i=1}^{t} n_i$. The random effects $\{\alpha_i\}$ are identically distributed with mean α and variance σ_ν^2, the random errors $\{\epsilon_{ij}\}$ are

identically distributed with mean 0 and variance σ_ϵ^2 with $\{\alpha_i\}$ and $\{\epsilon_{ij}\}$ mutually independent. Let $\boldsymbol{\theta} = (\beta_1, \ldots, \beta_k, \alpha_1, \ldots, \alpha_t)'$, $\boldsymbol{\epsilon} = (\epsilon_{11}, \ldots, \epsilon_{tn_t})'$, and

$$\mathbf{X}_\alpha = \begin{pmatrix} x_{111} & \cdots & x_{11k} & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{1n_11} & \cdots & x_{1n_1k} & 1 & 0 & \cdots & 0 \\ x_{211} & \cdots & x_{21k} & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{2n_21} & \cdots & x_{2n_2k} & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{t11} & \cdots & x_{t1k} & 0 & 0 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{tn_t1} & \cdots & x_{tn_tk} & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{2.6.2}$$

the above model can be expressed in matrix form:

$$\mathbf{y} = \mathbf{X}_\alpha \boldsymbol{\theta} + \boldsymbol{\epsilon}\,. \tag{2.6.3}$$

Let $\mathbf{D}_\alpha = (\mathbf{0}_{t\times k} \mid \mathbf{I}_t)$ with $\mathbf{0}_{t\times k}$ a matrix of zeros and $\mathbf{I}_t$ a $(t \times t)$ identity matrix. For the model in (2.6.3), the expectation and variance of the response vector $\mathbf{y}$ are, respectively,

$$\mathrm{E}(\mathbf{y}) = \mathbf{X}_\alpha \boldsymbol{\theta}_o$$

and

$$\mathrm{Var}(\mathbf{y}) = \sigma_\epsilon^2 \mathbf{I} + \sigma_\nu^2 \mathbf{X}_\alpha \mathbf{D}_\alpha' \mathbf{D}_\alpha \mathbf{X}_\alpha' \tag{2.6.4}$$

where $\boldsymbol{\theta}_o = (\beta_1, \ldots, \beta_k, \alpha\mathbf{1}')'$. It can be shown that this variance is the same as the variance in equation (2.3.1).

The parameter σ_ν^2 can be estimated by an unbiased weighted estimator, using ordinary least squares, based on the random intercept model. Let $\hat{\alpha}_i$ be the ordinary least squares estimator of α_i for the ith area and $\hat{\alpha} = N_\omega^{-1} \sum_{i=1}^{t} \omega_i \hat{\alpha}_i$ where ω_i is the weight associated for the ith group and $N_\omega = \sum_{i=1}^{t} \omega_i$. Taking the expectation of this weighted sum of squares with some algebraic manipulation, an unbiased weighted estimator of σ_ν^2 can be expressed as

$$s_\nu^2 = \frac{\sum_{i=1}^{t} \omega_i(\hat{\alpha}_i - \hat{\alpha})^2 - s_\epsilon^2 \operatorname{tr}\left((\boldsymbol{\Omega} - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\right)}{(t-1)\left(\overline{\omega} - \frac{s_\omega^2}{t\,\overline{\omega}}\right)} \tag{2.6.5}$$

with

$$s_\epsilon^2 = \frac{\mathbf{y}'\left[\mathbf{I} - \mathbf{X}_\alpha\left(\mathbf{X}_\alpha'\mathbf{X}_\alpha\right)^{-1}\mathbf{X}_\alpha'\right]\mathbf{y}}{N - \mathrm{r}(\mathbf{X}_\alpha)} \tag{2.6.6}$$

where $\boldsymbol{\omega}' = (\omega_1, \omega_2, \ldots \omega_t)$, $\boldsymbol{\Omega} = \operatorname{diag}(\omega_1, \omega_2, \ldots \omega_t)$, $\overline{\omega} = t^{-1}\sum_{i=1}^{t}\omega_i$, $s_\omega^2 = (t-1)^{-1}\sum_{i=1}^{t}(\omega_i - \overline{\omega})^2$, and $\mathrm{r}(\mathbf{X}_\alpha)$ is the rank of $\mathbf{X}_\alpha$. In Appendix A.1 and A.2, both

estimators s_ϵ^2 and s_ν^2 are shown to be unbiased and their variances are given by

$$\begin{aligned}
\mathrm{Var}(s_\nu^2) \;=\; & \left[(t-1)\left(\overline{\omega} - \frac{s_\omega^2}{t\,\overline{\omega}}\right)\right]^{-2} \\
& \times\Bigg\{ 2\sigma_\epsilon^4 \mathrm{tr}\Big((\boldsymbol{\Omega} - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\Big)^2 \\
& +4\sigma_\epsilon^2\sigma_\nu^2 \mathrm{tr}\Big((\boldsymbol{\Omega} - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')^2\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\Big) \\
& +2\sigma_\nu^4 \mathrm{tr}\Big(\boldsymbol{\Omega} - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}'\Big)^2 \\
& +\mathrm{Var}(s_\epsilon^2)\Big[\mathrm{tr}\Big((\boldsymbol{\Omega} - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\Big)\Big]^2\Bigg\}
\end{aligned} \tag{2.6.7}$$

and

$$\mathrm{Var}(s_\epsilon^2) = \frac{2\sigma_\epsilon^4}{N - \mathrm{r}(\mathbf{X}_\alpha)}\,, \tag{2.6.8}$$

respectively. The value of s_ν^2 can be adjusted through the weights $\boldsymbol{\omega}' = (\omega_1, \omega_2, \ldots\, \omega_t)$. The estimation of s_ν^2 can be more efficient with weights thought to be best for a certain value of the intraclass correlation coefficient. The uniform weighting scheme where $\{\omega_i = 1\}$ is thought to be best for a large intraclass correlation ρ and is shown to be such in Keen [28] in a similar situation but without covariates. The weights $\{\omega_i\}$ equal to $n_i(n_i - 1)$ in a pairwise weighting scheme that is thought to be best for a small intraclass correlation ρ, based on the simulations in Keen [28] without covariates.

2.7 Small-Area Mean Estimation

There are three major groups of small-area estimators, namely, direct estimators, indirect estimators and model-based estimators. An example of the direct estimator

of the small-area mean μ_i of the ith area is its sample counterpart $\tilde{\mu}_i^{(sm)} = \overline{y}_i = n_i^{-1} \sum_{j=1}^{n_i} y_{ij}$, which is conditionally unbiased for a non-empty area with fixed size n_i. Indirect estimators include synthetic estimators that borrow strength from related small areas using supplementary variables with known mean totals. For example, a ratio synthetic estimator makes use of the assumption that the population ratios across different groups are constant and can be estimated by $\hat{R} = \overline{y}/\overline{x}$ so that the estimator of the ith small-area mean is given by $\tilde{\mu}_i^{(syn)} = \hat{R}X_i/n_i$. However, the estimator can be heavily biased when the population ratios are different, which is nearly always the case. Another example of an indirect estimator is a composite estimator which is a weighted average of two estimates given by, for example, $\tilde{\mu}_i^{(comp)} = a_i\tilde{\mu}_i^{(sm)} + (1-a_i)\tilde{\mu}_i^{(syn)}$. Model based estimators such as empirical best linear unbiased predictor (EBLUP) assume the existence of a statistical model that includes random area effects. Rao & Choudhry [46] compared the average relative efficiencies of the direct estimators and the indirect estimators, such as one suggested by Prasad & Rao [41] and showed that the EBLUP is significantly better than them in terms of the mean square errors. This thesis focuses on the development of an efficient estimator that is indeed a combined indirect model-based estimator and has a better performance profile than the existing EBLUP of the small-area mean. Regarding the model-based estimator, it means that the estimation is done entirely based on a statistical model without considering the survey design. Therefore, design weights

are not used in any stage of the model-based estimation.

As mentioned before, the best linear unbiased predictor (BLUP) of $\mu = \mathbf{l}'\beta + \mathbf{m}'\nu$ is given by

$$\dot{\mu} = \mathbf{l}'\tilde{\beta} + \mathbf{m}'\mathbf{G}\mathbf{Z}'\mathbf{V}^{-1}\left(\mathbf{y} - \mathbf{X}\tilde{\beta}\right) \tag{2.7.1}$$

Note that the random intercept model in equation (2.6.1) can be written as

$$y_{ij} = \mathbf{x}_{ij}'\beta + \nu_i + \epsilon_{ij}, \qquad i = 1, \ldots, t, \quad j = 1, \ldots, n_i, \tag{2.7.2}$$

where $\mathbf{x}_{ij} = (1, x_{ij1}, \cdots, x_{ijk})'$ is a $(k+1)$-vector of corresponding covariates and $\beta = (\alpha, \beta_1, \cdots, \beta_k)'$ represents a vector of unknown but fixed regression coefficients and ν the random elements from the intercept. For the nested error regression model in equation (2.7.2), the small-area mean for the ith area is

$$\mu_i = \overline{\mathbf{X}}_i'\beta + \nu_i \tag{2.7.3}$$

where $\overline{\mathbf{X}}_i'$ is the vector of known means of $\mathbf{x}_{ij}$ for the ith group and the sample mean vector $\overline{\mathbf{x}}_i = n_i^{-1}\sum_{j=1}^{n_i}\mathbf{x}_{ij}$.

Unfortunately, for the small-area mean estimator vector in equation (2.7.1), the variance parameters are nearly always unknown. Usually, the EBLUP would be used, which is given by an equation similar to (2.7.1), but with the variances replaced by

their estimators. That is, the variances σ_ν^2 and σ_ϵ^2 in $\mathbf{V}$ and in $\tilde{\boldsymbol{\beta}}$ in (2.7.1) would be estimated. Because the variances are unknown and estimated, the optimality property of the EBLUP that it is the most efficient estimator with the smallest mean square error among all linear estimators is no longer true. Moreover, the efficiency of the EBLUP depends on the choice of estimators for the variance components. To assure efficient estimation of the small-area mean, a shrinkage estimator is proposed that is a convex combination of two empirical predictors (EBLUPs) of the same expression in (2.7.1) with different estimators for σ_ν^2 given by $s_{\nu 1}^2$ and $s_{\nu 2}^2$ each with its own weighting scheme as outlined in section 2.6. Therefore, the small-area mean estimator for the ith small-area mean can be expressed as

$$\tilde{\mu}_i^c \;=\; (1-a_i)\,\tilde{\mu}_i^{(1)} + a_i\,\tilde{\mu}_i^{(2)} \tag{2.7.4}$$

where, $\tilde{\mu}_i^{(1)}$ is an EBLUP for the ith small-area mean similar to the ith small-area mean estimator in equation (2.7.1), with $s_{\nu 2}^2$ and $\tilde{\mu}_i^{(2)}$ similarly defined. The two weighting schemes for small and large intraclass correlations used for estimating σ_ν^2 are thought to be useful for estimating the small-area mean with $\tilde{\mu}_i^c$ in (2.7.4) because with the proper choice of a_i, the estimator $\tilde{\mu}_i^c$ would be nearly uniformly efficient for small or large intraclass correlations. The choice of a_i is one such that the mean square error of $\tilde{\mu}_i^c$ would be minimal. It is shown in Appendix A.3 that the value of

a_i for the ith small-area mean estimator is given by

$$a_i = \frac{\mathrm{MSE}(\tilde{\mu}_i^{(1)}) - \mathrm{E}\Big((\tilde{\mu}_i^{(1)} - \mu_i)(\tilde{\mu}_i^{(2)} - \mu_i)\Big)}{\mathrm{MSE}(\tilde{\mu}_i^{(1)}) + \mathrm{MSE}(\tilde{\mu}_i^{(2)}) - 2\mathrm{E}\Big((\tilde{\mu}_i^{(1)} - \mu_i)(\tilde{\mu}_i^{(2)} - \mu_i)\Big)} . \qquad (2.7.5)$$

where $\mathrm{MSE}(\tilde{\mu}_i^{(1)})$ and $\mathrm{MSE}(\tilde{\mu}_i^{(2)})$ are the mean square errors of $\tilde{\mu}_i^{(1)}$ and $\tilde{\mu}_i^{(2)}$ respectively and $\mathrm{E}\Big((\tilde{\mu}_i^{(1)} - \mu_i)(\tilde{\mu}_i^{(2)} - \mu_i)\Big)$ is the covariance of $\tilde{\mu}_i^{(1)}$ and $\tilde{\mu}_i^{(2)}$ around μ_i. Hence, the mean square error of $\tilde{\mu}_i^c$ is given by

$$\mathrm{MSE}(\tilde{\mu}_i^c) = \frac{\mathrm{MSE}(\tilde{\mu}_i^{(1)})\mathrm{MSE}(\tilde{\mu}_i^{(2)}) - \Big[\mathrm{E}\Big((\tilde{\mu}_i^{(1)} - \mu_i)(\tilde{\mu}_i^{(2)} - \mu_i)\Big)\Big]^2}{\mathrm{MSE}(\tilde{\mu}_i^{(1)}) + \mathrm{MSE}(\tilde{\mu}_i^{(2)}) - 2\mathrm{E}\Big((\tilde{\mu}_i^{(1)} - \mu_i)(\tilde{\mu}_i^{(2)} - \mu_i)\Big)} . \qquad (2.7.6)$$

Since the value of μ_i is not known, the value of a_i would be estimated by the approximated values of $\mathrm{MSE}(\tilde{\mu}_i^{(1)})$, $\mathrm{MSE}(\tilde{\mu}_i^{(2)})$ and $\mathrm{E}\Big((\tilde{\mu}_i^{(1)} - \mu_i)(\tilde{\mu}_i^{(2)} - \mu_i)\Big)$ given in Appendices A.6 and A.7. The value of a_i is bounded between 0 and 1 inclusive so that the estimates of μ_i by $\tilde{\mu}_i^c$ is bounded to fall between $\tilde{\mu}_i^{(1)}$ and $\tilde{\mu}_i^{(2)}$. When $a_i = 0$, $\tilde{\mu}_i^{(1)}$ gets all of the weight and $\tilde{\mu}_i^c$ is equal to $\tilde{\mu}_i^{(1)}$. On the other hand, when $a_i = 1$, $\tilde{\mu}_i^{(2)}$ gets all of the weight and $\tilde{\mu}_i^c$ is then the same as $\tilde{\mu}_i^{(2)}$.

The estimator $\tilde{\mu}_i^c$ is also an indirect estimator. When the sample size of a group is large, usually the parameter can be efficiently estimated based on the responses of the group alone. However, the problem arises when little is known about the group because of the small sample size. Direct estimation using the responses of the small sample always produces unsatisfactory results with large standard errors. In such circumstances, the efficiency can be improved by estimating the parameters

indirectly with additional information from other groups with similar characteristics. The estimator $\tilde{\mu}_i^c$ is one such estimator which uses information from other groups to estimate the parameters for more efficient estimation performance.

2.8 Intraclass Correlation Estimation

Our focus is on the random intercept model in (2.6.1) , and its special case: the nested-error regression model in (2.7.2) – which follows directly from the more general case of the random intercept model. For either model with variance $\mathbf{V}$ in (2.6.4), the intraclass correlation ρ is equal to $\sigma_\nu^2/(\sigma_\nu^2 + \sigma_\epsilon^2)$. A technique similar to small-area estimation is used for the estimation of the intraclass correlation in animal breeding experiments, for example. The variance components σ_ν^2 and σ_ϵ^2 are estimated using equation (2.6.5) and (2.6.6). Suppose r_1 denotes an intraclass correlation estimator by direct substitution with s_ν^2 and s_ϵ^2 using the uniform weighting scheme and r_2 is similarly defined using the other weighting scheme where $\{\omega_i\} = n_i(n_i - 1)$, that is

$$r_1 = \frac{s_{\nu 1}^2}{s_{\nu 1}^2 + s_\epsilon^2} \tag{2.8.1}$$

and

$$r_2 = \frac{s_{\nu 2}^2}{s_{\nu 2}^2 + s_\epsilon^2} \,. \tag{2.8.2}$$

As in the small-area estimation, a shrinkage estimator is proposed to estimate the intraclass correlation efficiently and it is given by

$$r_c = (1-a)r_1 + a\, r_2\ . \tag{2.8.3}$$

As expected, the choice of a would be important for the efficiency of r_c. It is chosen so that the mean square error of r_c is minimal and it is a function of the mean square errors of r_1 and r_2 given by

$$a \;=\; \frac{\mathrm{MSE}[r_1] - \mathrm{E}\Big[(r_1-\rho)(r_2-\rho)\Big]}{\mathrm{MSE}[r_1] + \mathrm{MSE}[r_2] - 2\mathrm{E}\Big[(r_1-\rho)(r_2-\rho)\Big]} \tag{2.8.4}$$

such that the mean square error of r_c can be written as

$$\mathrm{MSE}[r_c] = \frac{\mathrm{MSE}[r_1]\mathrm{MSE}[r_2] - \Big(\mathrm{E}\Big[(r_1-\rho)(r_2-\rho)\Big]\Big)^2}{\mathrm{MSE}[r_1] + \mathrm{MSE}[r_2] - 2\mathrm{E}\Big[(r_1-\rho)(r_2-\rho)\Big]}\ . \tag{2.8.5}$$

Because the true value of ρ is not known, $\mathrm{MSE}[r_1]$, $\mathrm{MSE}[r_2]$ and $\mathrm{E}\Big[(r_1-\rho)(r_2-\rho)\Big]$ will be replaced by their approximations given in Appendix A.10.

Chapter 3

APPLICATION IN SMALL AREA ESTIMATION

3.1 Introduction

Small areas have existed for a long time and so have small-area data. The earliest small-area data can be traced back to 1666 in Canada [40]. The interest in small-area estimation has been growing for the last decade. As economic and social developments have changed drastically, there is an increasing governmental concern with issues of distribution, equity and disparity, especially in Canada [40]. Some regions or subgroups of the population do not keep up with the average and there is a need to identify these areas. Moreover, at the provincial and local levels, governments may want to obtain data of their own areas for the purpose of planning and administration within their jurisdictions. Politicians find statistical information about their regions useful for assessing the impact of proposed policies or legislation. Therefore, the demand for small-area data and analysis is growing.

Attempts at small-area estimation using direct survey estimators are not successful because small sample sizes yield considerably large standard errors. One way to approach this problem is to attempt to increase precision by borrowing information across similar small areas through indirect estimation. However, some of these estimators can be heavily biased in the design-based framework for some of the areas. So there is a demand for unbiased estimators with better efficiency.

All small-area models can be considered as special cases of a general mixed linear model with both fixed and random effects. Small-area means are then expressed as functions of these effects. These small-area means can be efficiently estimated by best linear unbiased predictors if variance-component parameters are known. Moreover, these estimators have the smallest mean square error among all linear unbiased estimators. These variance components can be estimated by a number of methods including the method of moments, the maximum likelihood method, the restricted maximum likelihood method and the method of fitting constants. Prasad & Rao [41] studied estimators in the context of the nested-error regression model, which estimates variance components by the method of fitting constants—also known as Henderson's method 3 [22]. The small-area means are then estimated by empirical best linear unbiased predictors [23] with the variance-component estimates replacing the parameters.

In section 3.3, a weighted unbiased estimator of small-area means is developed. To improve the efficiency of estimating the small-area means, a two-stage small-area estimator that is a linear combination of two small-area estimators with different weights for the estimator of the group effect is presented. It is also an indirect estimator by a combination of two EBLUPs. A formula for its mean square error is also given. Monte Carlo simulations for a variety of distributions typically encountered in survey sampling are presented in section 3.4 for the situation of known covariate means. The results of a Monte Carlo study on their relative efficiencies with respect to the maximum likelihood estimator, in terms of mean square errors, are reported in section 3.4 for the normal distribution. Novel comparisons are presented in section 3.5 when the population means of the covariates are not known. Comparisons of different small-area mean estimators to the maximum likelihood estimator with Monte Carlo simulations have never been studied before. Novel comparisons with the corresponding maximum likelihood estimators, assuming normality, are given in section 3.6. Conclusions and possible future directions are considered in section 3.7.

3.2 The Models

Consider the random intercept model with random intercept, α_i, for the ith small area, or small group as defined in equation (2.6.1):

$$y_{ij} = \alpha_i + x_{ij1}\beta_1 + \ldots + x_{ijk}\beta_k + \epsilon_{ij}, \qquad i = 1, \ldots, t, \quad j = 1, \ldots, n_i \quad (3.2.1)$$

Note that the random intercept, α_i, can be considered as a sum of a fixed effect, α, and a random group effect, ν_i, with zero mean, that is $\alpha_i = \alpha + \nu_i$. The random intercept model in (3.2.1), can be written as the nested-error regression model with a single random effect, ν_i, for the ith small area, or small group, as studied by Battese et al. [4] and by Prasad & Rao [41]:

$$y_{ij} = \mathbf{x}_{ij}'\boldsymbol{\beta} + \nu_i + \epsilon_{ij}, \qquad i = 1, \ldots, t, \quad j = 1, \ldots, n_i, \tag{3.2.2}$$

where $\mathbf{x}_{ij} = (1, x_{ij1}, \cdots, x_{ijk})'$ is a $(k+1)$-vector of corresponding covariates and $\boldsymbol{\beta} = (\alpha, \beta_1, \cdots, \beta_k)'$ is a $(k+1)$-vector of unknown regression coefficients. The group effects $\{\nu_i\}$ are distributed with mean $\mathbf{0}$ and variance-covariance matrix $\mathbf{G} = \sigma_\nu^2\mathbf{I}$. Note that $\{\nu_i\}$ and $\{\epsilon_{ij}\}$ are mutually independent. For convenience, define matrices $\mathbf{X}' = (\mathbf{x}_{11}, \ldots, \mathbf{x}_{tn_t})$ and $\mathbf{Z} = \text{diag}(\mathbf{1}_{n_1}, \ldots, \mathbf{1}_{n_t})$ where $\mathbf{1}_{n_i}$ is a n_i-vector with a value of 1 for each element. Let $\overline{\mathbf{X}}_i'$ be the vector of known means of $\mathbf{x}_{ij}$ for the ith group and let the sample mean vector $\overline{\mathbf{x}}_i = n_i^{-1}\sum_{j=1}^{n_i}\mathbf{x}_{ij}$. This assumption of a known $\overline{\mathbf{X}}_i'$ matrix is reasonable because small-area estimation is mainly used in survey sampling and, most of the time, the population information is available through census data. Define $\boldsymbol{\mu}$ to be the vector of small-area means in the population with component

$$\mu_i = \overline{\mathbf{X}}_i'\boldsymbol{\beta} + \nu_i \tag{3.2.3}$$

for the ith area. Henderson [23] suggested an unbiased estimator of μ_i given by

$$\overline{\mathbf{X}}_i'\tilde{\boldsymbol{\beta}} + \mathbf{m}_i'\mathbf{G}Z'\mathbf{V}^{-1}\left(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}}\right) \tag{3.2.4}$$

where $\mathbf{m}_i$ is a t-vector of zero with the ith element equal to 1. Note that $\tilde{\boldsymbol{\beta}}$ is always a function of the unknown parameters σ_ν^2 and σ_ϵ^2. It is natural to replace them with corresponding unbiased estimators. Prasad & Rao [41] suggested

$$\tilde{\mu}_i = \overline{\mathbf{X}}_i{}'\tilde{\boldsymbol{\beta}} + \mathbf{m}_i'\tilde{\mathbf{G}}\mathbf{Z}'\tilde{\mathbf{V}}^{-1}\left(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}}\right) \tag{3.2.5}$$

where the unknown variance components in $\tilde{\mathbf{G}}$, $\tilde{\mathbf{V}}$ and $\tilde{\boldsymbol{\beta}}$ are estimated with Henderson's Method 3 as outlined in Chapter 2. Equations (3.2.4) and (3.2.5) are basically trivial applications of (E)BLUP theory and Henderson's method 3.

3.3 Estimation Of The Small-Area Means

In most cases, the variance parameters in a regression model are unknown and need to be estimated. An unbiased weighted estimator of σ_ν^2 is developed, using ordinary least squares, based on the random intercept model. Let $\hat{\alpha}_i$ be the ordinary least squares estimator of α_i for the ith area and $\hat{\alpha} = N_\omega^{-1}\sum_{i=1}^t \omega_i\hat{\alpha}_i$ where ω_i is the weight associated for the ith group and $N_\omega = \sum_{i=1}^t \omega_i$. The weighted sum of squares of the α_i's and $\hat{\alpha}$ should be chosen to capture the variability of ν_i to a certain extent. Taking the expectation of this weighted sum of squares with some algebraic manipulation, an unbiased weighted estimator of σ_ν^2 can be expressed as

$$s_\nu^2 = \frac{\sum_{i=1}^t \omega_i(\hat{\alpha}_i - \hat{\alpha})^2 - s_\epsilon^2\,\mathrm{tr}\left((\boldsymbol{\Omega} - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\right)}{(t-1)\left(\overline{\omega} - \frac{s_\omega^2}{t\,\overline{\omega}}\right)} \tag{3.3.1}$$

with

$$s_\epsilon^2 = \frac{\mathbf{y}'\left[\mathbf{I} - \mathbf{X}_\alpha\left(\mathbf{X}_\alpha'\mathbf{X}_\alpha\right)^{-1}\mathbf{X}_\alpha'\right]\mathbf{y}}{N - \mathrm{r}(\mathbf{X}_\alpha)} \tag{3.3.2}$$

where $\boldsymbol{\omega}' = (\omega_1, \omega_2, \ldots \omega_t)$, $\boldsymbol{\Omega} = \mathrm{diag}(\omega_1, \omega_2, \ldots \omega_t)$, $\overline{\omega} = t^{-1}\sum_{i=1}^{t}\omega_i$, $s_\omega^2 = (t-1)^{-1}\sum_{i=1}^{t}(\omega_i - \overline{\omega})^2$, and $\mathrm{r}(\mathbf{X}_\alpha)$ is the rank of $\mathbf{X}_\alpha$. In Appendix (A.2) and (A.1), it is shown that both s_ν^2 and s_ϵ^2 are unbiased estimators of σ_ν^2 and σ_ϵ^2, respectively.

Regarding the choice of weights, for example, with an appropriate adjustment of the weights $\{\omega_i\}$, areas with a single observation may be excluded so that their effects can be limited.

Under the random intercept model of equation (3.2.1), the estimator s_ν^2 in equation (3.3.1) has an advantage in that the weights $\{\omega_i\}$ can compensate for differences among the small areas or equivalently the effects of the random intercept estimators $\{\hat{\alpha}_i\}$. The efficiency of small-area mean estimation can be further improved with a convex combination of two different estimators of σ_ν^2, namely $s_{\nu 1}^2$ and $s_{\nu 2}^2$ with different weights, $\{\omega_i^{(1)}\}$ and $\{\omega_i^{(2)}\}$, respectively, as in equation (3.3.1). Note that the true value of σ_ν^2 and σ_ϵ^2 are not known and neither is the intraclass correlation coefficient. The composite estimator has an advantage in that when the intraclass correlation is large, it puts more weight on the small-area mean estimator using

variance estimator s_ν^2 with $\{\omega_i\}$ appropriate for large intraclass correlation coefficient. If the intraclass correlation is small, it then puts more weight on the s_ν^2, which uses the set of $\{\omega_i\}$ appropriate for small intraclass correlation coefficient. The usual small-area mean estimator has only one σ_ν^2 estimator and makes no adjustment for different intraclass correlation coefficients. An estimator of the mean of the ith small area by this approach is

$$\tilde{\mu}_i^c = (1-a_i)\,\tilde{\mu}_i^{(1)} + a_i\,\tilde{\mu}_i^{(2)} \tag{3.3.3}$$

where $\tilde{\mu}_i^{(1)}$ and $\tilde{\mu}_i^{(2)}$ are estimators of μ_i using $s_{\nu 1}^2$ and $s_{\nu 2}^2$ respectively. Obviously the choice of a_i affects the value of the mean square error of $\tilde{\mu}_i^c$. An efficient estimator of μ_i is the one with the smallest mean square error. The minimum mean square error of $\tilde{\mu}_i^c$

$$\mathrm{MSE}(\tilde{\mu}_i^c) = \frac{\mathrm{MSE}(\tilde{\mu}_i^{(1)})\mathrm{MSE}(\tilde{\mu}_i^{(2)}) - \Big[\mathrm{E}\Big((\tilde{\mu}_i^{(1)}-\mu_i)(\tilde{\mu}_i^{(2)}-\mu_i)\Big)\Big]^2}{\mathrm{MSE}(\tilde{\mu}_i^{(1)}) + \mathrm{MSE}(\tilde{\mu}_i^{(2)}) - 2\mathrm{E}\Big((\tilde{\mu}_i^{(1)}-\mu_i)(\tilde{\mu}_i^{(2)}-\mu_i)\Big)} \tag{3.3.4}$$

is attained when

$$a_i = \frac{\mathrm{MSE}(\tilde{\mu}_i^{(1)}) - \mathrm{E}\Big((\tilde{\mu}_i^{(1)}-\mu_i)(\tilde{\mu}_i^{(2)}-\mu_i)\Big)}{\mathrm{MSE}(\tilde{\mu}_i^{(1)}) + \mathrm{MSE}(\tilde{\mu}_i^{(2)}) - 2\mathrm{E}\Big((\tilde{\mu}_i^{(1)}-\mu_i)(\tilde{\mu}_i^{(2)}-\mu_i)\Big)}\,. \tag{3.3.5}$$

The expressions for $\mathrm{MSE}(\tilde{\mu}_i^{(1)})$, $\mathrm{MSE}(\tilde{\mu}_i^{(2)})$ and $\mathrm{E}\Big((\tilde{\mu}_i^{(1)}-\mu_i)(\tilde{\mu}_i^{(2)}-\mu_i)\Big)$ are given in Appendices A.6 and A.7.

Among many variance estimation methods, Henderson's method 3 was used by

Prasad & Rao [41] for the estimation of σ_ϵ^2 and σ_ν^2 in the nested-error regression model. The estimators by this method are given by

$$\tilde{\sigma}_\epsilon^2 = \frac{\mathbf{y}'\left[\mathbf{I} - \mathbf{M}(\mathbf{M}'\mathbf{M})^{-}\mathbf{M}'\right]\mathbf{y}}{N - \mathrm{r}(\mathbf{M})}, \tag{3.3.6}$$

and

$$\tilde{\sigma}_\nu^2 = \frac{\sum_{i=1}^{t}\sum_{j=1}^{n_i}\hat{u}_{ij}^2 - (n-k)\tilde{\sigma}_\epsilon^2}{n - \mathrm{tr}\left((\mathbf{X}'\mathbf{X})^{-1}\sum_{i=1}^{t} n_i^2 \overline{\mathbf{x}}_i\,\overline{\mathbf{x}}_i'\right)} \tag{3.3.7}$$

where $\mathbf{M} = (\mathbf{X} \mid \mathbf{Z})$, $\mathrm{r}(\mathbf{M})$ is the rank of $\mathbf{M}$ and $\sum_{i=1}^{t}\sum_{j=1}^{n_i}\hat{u}_{ij}^2 = \mathbf{y}'\left[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\right]\mathbf{y}$.

It is possible for $\tilde{\sigma}_\nu^2$ and s_ν^2 to be negative; therefore, it is common practice to take instead $\max(0, \tilde{\sigma}_\nu^2)$ and $\max(0, s_\nu^2)$ as estimators of σ_ν^2. Nevertheless, as sample sizes increase, both $\tilde{\sigma}_\nu^2$ and s_ν^2 tend to be non-negative almost surely.

3.4 Monte Carlo Study With Population $\overline{\mathbf{X}}_i$

A Monte Carlo simulation that mimics the data set of Battese et al. [4] under the nested-error regression model with a single covariate, $y_{ij} = \alpha + \beta x_{ij} + \nu_i + \epsilon_{ij}$, was conducted to study the relative efficiency of the two-stage estimator combining $s_{\nu 1}^2$ and $s_{\nu 2}^2$ with respect to the one with $\tilde{\sigma}_\nu^2$. Let y_{ij} equal acreage of corn cultivation and x_{ij} equal the number of pixels of corn cultivation for the jth unit of county i from satellite imagery. The three counties with only 1 unit were pooled together as done in Prasad & Rao [41]. The number of small areas t was increased to 20 from 10 by duplication

as also done by Prasad & Rao [41]. The value of the intercept α and the slope β were set to the estimates obtained by Battese et al. [4], that is, 5.5 and 0.388, respectively. Ten thousand independent sets of ν_i and ϵ_{ij} with zero means and different intraclass correlation $\rho = \sigma_\nu^2/(\sigma_\nu^2 + \sigma_\epsilon^2)$ from 0.00 to 0.95, at increments of 0.05, were generated to obtain 10,000 sets of y_{ij} using the covariate values of Battese et al. [4] for each of 4 distributional assumptions: normal, double-exponential, exponential and uniform.

In Prasad & Rao [41], the simulation was done based on 10,000 independent sets of variates ϵ_{ij} (with $\sigma_\epsilon^2 = 292$) and ν_i (with $\sigma_\nu^2 = 64$) with the intercept and the slope equal to 5.5 and 0.388 respectively for each of the four distributions above. In other words, the Monte Carlo simulations in this thesis have a more complete coverage for different combinations of σ_ϵ^2 and σ_ν^2 and for the domain of the intraclass correlation coefficient ρ. With $\sigma_\epsilon^2 = 292$ and $\sigma_\nu^2 = 64$, the intraclass correlation coefficient is equal to $\rho = \sigma_\nu^2/(\sigma_\nu^2 + \sigma_\epsilon^2) = 0.18$. Theoretically speaking, in the context of cluster sampling, this intraclass correlation coefficient is desired to be zero which means that the subjects within each domain in the cluster survey design of the population are uncorrelated.

The normal distribution has a wide range of applications. It is very common in health-related research, for instance, as in health related variables such as blood

pressure (systolic or diastolic), weight, and height, etc. The exponential distribution is another important distribution. It is commonly used in the field of life-testing to represent the life-time or life characteristic [25]. In survey sampling, the distribution is also used to model the income data of a population. The double exponential distribution will also be used in the Monte Carlo simulations for the study of the small-area estimators. The double exponential distribution is also known as the Laplace distribution. Its tails are thicker than the tails of the normal distribution and it has sharper peak in the center than the normal distribution. One application of the double exponential distribution is for the strength of flaws in materials [2]. Another application could be the modelling of the difference of two independent exponential variables, for example, two independent income variables following the exponential distribution [2]. The uniform distribution is also known as the rectangular distribution. It can be used to form a model of the distribution of traffic along a straight road. Suppose the speed of a vehicle is constant within the road interval, its position is uniformly distributed over the road interval [26].

The weights $\{\omega_i^{(1)}\}$ for the estimator $s_{\nu 1}^2$ in $\tilde{\mu}_i^{(1)}$ are equal to 1. This is the uniform weighting scheme that is thought to be best for a large intraclass correlation ρ and shown to be such in Keen [28] in a similar situation but without covariates. The weights $\{\omega_i^{(2)}\}$ for the estimator $s_{\nu 2}^2$ in $\tilde{\mu}_i^{(2)}$ are equal to $n_i(n_i - 1)$ in a pairwise

weighting scheme that is thought to be best for a small intraclass correlation ρ based on the simulations in Keen [28] without covariates. This latter scheme implicitly deletes single-unit areas when estimating σ_ν^2. Athough Srivastava & Ng [65] also studied a class of composite estimators, they only used it for the estimation of the intraclass correlation coefficient ρ which will be discussed in chapter 4.

The estimated values of the a_i's in equation (3.3.5) presented in Table E.1 in Appendix E based on the simulation agree with this weighting scheme. When the intraclass correlation is large, the value of a is small so more weight is assigned to $\tilde{\mu}_i^{(1)}$ which uses $\{\omega_i^{(1)}\}$ for the estimator $s_{\nu 1}^2$. When the intraclass correlation is small, the value of a is large so more weight is assigned to $\tilde{\mu}_i^{(2)}$ which uses $\{\omega_i^2\}$ for the estimator $s_{\nu 2}^2$. This is true whether the population means of the covariates are known or not for all the small areas.

Figure 3.1 summarizes the comparison of the composite estimator $\tilde{\boldsymbol{\mu}}^c$ to the estimator of Prasad & Rao [41] $\tilde{\boldsymbol{\mu}}$ under different distributions of the random effects ν_i and ϵ_{ij} using the known means of the covariates, $\{\overline{X}_i\}$, as would be available from a census, in different small areas. For the normally distribution random effects, $\tilde{\boldsymbol{\mu}}^c$ is relatively more efficient than $\tilde{\boldsymbol{\mu}}$ for estimating the small-area means when the intraclass correlation coefficient is small. For moderate to large intraclass correlation

coefficient, both estimators have similar efficiencies as shown in Figure 3.1 (a). This is also the case when the random effects have uniform distribution as in Figure 3.1 (d). When the random effects have heavy tail distributions such as the exponential distribution or the double exponential distribution, the estimator $\tilde{\mu}^c$ is also relatively more efficient than $\tilde{\mu}$ for a greater range of small intraclass correlation coefficients. It is also slightly more efficient than $\tilde{\mu}$ for moderate to large intraclass correlation coefficients as presented in Figure 3.1 (b) and (c).

To summarize, under all the normal and three non-normal distributions, the composite estimator $\tilde{\mu}^c$ is observed to be more efficient than $\tilde{\mu}$ especially when the intraclass correlation is small. The relative efficiency for an individual small area can be as high as 142% for the heavily skewed exponential distribution, 129% for the heavy tailed double-exponential distribution and 115% for the normal distribution and 110% for the small-tailed (no-tailed) uniform distribution.

It is reasonable to inquire as to whether the composite estimator offers an improvement or whether s_ν^2 alone in place of σ_ν^2 is sufficient for improvement. Figure 3.2 (a) and (b) show that the relative efficiency of $\tilde{\mu}^c$ with respect to $\tilde{\mu}^{(1)}$ and $\tilde{\mu}^{(2)}$ respectively. These are the extreme cases when the value of a_i is equal to 1 and

0, respectively. For small to moderate intraclass correlation coefficient, the composite estimator $\tilde{\mu}^c$ is always more efficient than the small-area estimator $\tilde{\mu}^{(1)}$ and the relative efficiency can reach as high as 128% for random effects with exponential distribution and 123% for random effects with double exponential distribution. As the intraclass correlation increases, the relative efficiency of the composite estimator with respect to $\tilde{\mu}^{(1)}$ approaches 1. Nevertheless, for large intraclass correlation coefficients, the composite estimator is as efficient as $\tilde{\mu}^{(1)}$. Although the relative efficiency of the composite estimator with respect to $\tilde{\mu}^{(2)}$ is not as dramatic as to $\tilde{\mu}^{(1)}$, it is always slightly more efficient than $\tilde{\mu}^{(2)}$ for random effects with any of the four distributions. This is the case especially when the intraclass correlation coefficient is moderate. The proposed shrinkage estimator is more efficient than either $\tilde{\mu}^{(1)}$ and $\tilde{\mu}^{(2)}$ as expected.

3.5 Monte Carlo Study With Sample $\overline{\mathbf{x}}_i$

When the population means of the covariates are neither known nor available from a nearly time-coincident census, the sample means of covariates $\{\overline{x}_i\}$ are used in place of $\{\overline{X}_i\}$ in equation (3.2.4) at the second stage of estimation. The relative efficiencies based on different distributions are depicted in Figure 3.3. As in the case with $\{\overline{X}_i\}$, the composite estimator $\tilde{\mu}^c$ is more efficient than the estimator $\tilde{\mu}$ of Prasad & Rao [41] for small intraclass correlation for random effects with normal distribution

and uniform distribution. Although the relative efficiency of $\tilde{\mu}^c$ decreases as the intraclass correlation coefficient increases, for moderate to large intraclass correlation coefficients, the composite estimator $\tilde{\mu}^c$ is always as efficient as the estimator $\tilde{\mu}$ of Prasad & Rao [41]. These are shown in Figure 3.3 (a) and (d) The composite estimator $\tilde{\mu}^c$ is more efficient than $\tilde{\mu}$ for a greater range of small intraclass correlation coefficients when the random effects have heavy tail distributions such as the exponential distribution and double exponential distribution based on the simulation studies. In Figure 3.3 (b) and (c), for the exponential and double exponential distributions, the composite estimator $\tilde{\mu}^c$ is always slightly more efficient than the estimator $\tilde{\mu}$ of Prasad & Rao [41] for moderate to large intraclass correlation coefficients.

The relative efficiency of the composite estimator $\tilde{\mu}^c$ can be as high as 119% in some small areas random effects under the exponential distribution, 115% under the double-exponential distribution, 110% under the normal distribution and 107% under the uniform distribution. Although the relative efficiencies of the composite estimator $\tilde{\mu}^c$ with respect to the estimator $\tilde{\mu}$ used in Prasad & Rao [41] decrease as the intraclass correlation coefficient increases for all four distributions, it is at least as efficient as $\tilde{\mu}$. These results are similar to those in the study with population $\overline{\mathbf{X}}_i$ in section 3.4.

Note that $\tilde{\mu}^c$ is more efficient than $\tilde{\mu}$ for small intraclass correlation coefficients. This is an important result. In survey sampling, stratification is set up such that the observations within a stratum are as homogenous as possible. In this case, the intraclass correlation is high and in a perfect situation, it is equal to 1. However, when we use a cluster design, we deliberately set up the observations such that within each cluster, the observations should be as heterogenous as possible. Ideally, the intraclass correlation in this case should be equal to zero. Therefore, in the case of cluster sampling, the intraclass correlation is not a nuisance parameter but it needs to be estimated for checking the validity of the design for the data. If for a cluster sample, a large intraclass correlation estimate is obtained, there might be a need to investigate the design procedure or a redesign of the sampling scheme may be needed.

Regarding the issue of whether the composite estimator offers improvement or whether s_ν^2 alone in place of σ_ν^2 is sufficient for improvement, in the case of unknown population means of covariates, Figure 3.4 shows the advantage of $\tilde{\mu}^c$ compared to either $\tilde{\mu}^{(1)}$ and $\tilde{\mu}^{(2)}$. Note the similarity of Figure 3.4 to Figure 3.2 in the case of known population means for the covariates. The composite estimator $\tilde{\mu}^c$ is more efficient than $\tilde{\mu}^{(1)}$, which is designed for large intraclass correlation, when the intraclass correlation is small in the cases of $\{\overline{X}_i\}$ and $\{\overline{x}_i\}$. The degree of efficiency decreases as the intraclass class correlation increases. Compared to $\tilde{\mu}^{(2)}$, the composite estimator

$\tilde{\mu}^c$ is uniformly more efficient regardless of the value of the intraclass correlation coefficient.

3.6 Monte Carlo Study of Maximum Likelihood Estimators

It is worthwhile to consider the relative efficiencies of the proposed small-area estimator with respect to the corresponding gold-standard, in the asymptotic sense for normal data: the maximum likelihood estimator. It can be shown that for the ith small area, the maximum likelihood estimator $\hat{\mu}_i$ of the small-area mean μ_i is simply the sample mean $\overline{y}_i$ of the response variable, which does not take any covariate in the model into account when estimating the small-area means. If the population small-area means are of interest and the population means of the covariates in the model area are known, which is often the case in survey sampling, then $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \ldots, \hat{\mu}_t)'$ may not be a good estimator because the population means of the covariates are not used for adjusting the estimation of the small-area means. This could be a serious problem when the population means of the covariates are significantly different from those in the sample, and highlights the limitation of straightforward maximum likelihood estimation in this instance. A modification of the maximum likelihood approach that seems reasonable in this setting is to let $\hat{\boldsymbol{\mu}}^{Vm}$ be the BLUP in which the unknown variance components are replaced by their maximum likelihood estimators (MLE's).

Figure 3.5 compares the relative efficiency of the composite estimator $\tilde{\mu}^c$ and $\tilde{\mu}$ of Prasad & Rao [41] with $\hat{\mu}^{Vm}$ in the case of normality assuming known population means for the covariates.

Note that $\tilde{\mu}^c$ is more efficient than $\hat{\mu}^{Vm}$ for almost all values of the intraclass correlation and never less efficient whereas $\tilde{\mu}$ is only more efficient than the MLE when $\rho \geq 0.20$. It is the small values of ρ, as typically considered in cluster sampling designs, for which the proper selection of an estimator is most crucial.

The unknown variance components in the BLUP can also be estimated by the residual likelihood estimator or the restricted likelihood estimator. The empirical best linear unbiased estimator (EBLUP) with the *residual* maximum likelihood estimator denoted by $\hat{\mu}^{(S)}$ is obtained following the approach given by Patterson & Thompson [37] with a special case of the error contrasts constructed by $\mathbf{K} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. The EBLUP with the *restricted* likeliood estimator denoted by $\hat{\mu}^{(T)}$ is obtained by the approach suggested in McCulloch & Searle [35] in which the matrix $\mathbf{K}$ for the construction of the error contrasts does not have to be specified.

Figure 3.6 shows that $\tilde{\mu}^c$ is always more efficient than the EBLUP with the *residual* maximum likelihood estimator $\hat{\mu}^{(S)}$ especially at extreme values of the intraclass

correlation. However, $\tilde{\mu}^c$ is more efficient than the EBLUP with the *restricted* maximum likelihood estimator when the intraclass correlation is less than 0.10 with average relative efficiency as high as 112.5%. For the intraclass correlation between 0.10 and 0.50, $\tilde{\mu}^c$ is less efficient than $\hat{\mu}^{(T)}$ and for the intraclass correlation greater than 0.50, both estimators have similar efficiency. Figure 3.7 shows the relative efficiency of $\tilde{\mu}^c$ with respect to $\hat{\mu}^{(S)}$ and $\hat{\mu}^{(T)}$ when the population mean of the covariates is estimated by its sample counterpart $\overline{\mathbf{x}}_i$. The composite estimator $\tilde{\mu}^c$ is still more efficient than either $\hat{\mu}^{(S)}$ or $\hat{\mu}^{(T)}$ when the intraclass correlation is small.

In Figure 3.8, in the case where sample values $\{\overline{x}_i\}$ are used rather than population values, the naive maximum likelihood estimator $\hat{\mu}$ is inefficient compared to the composite estimator $\tilde{\mu}^c$ and the EBLUP $\tilde{\mu}$ in the estimation of small-area means. Again $\tilde{\mu}^c$ is superior compared to either $\hat{\mu}$ or $\tilde{\mu}$ for small values of the intraclass correlation coefficient.

You & Rao [67] developed a pseudo-EBLUP $\tilde{\mu}^{ps}$ for estimating the small-area means. The pseudo-EBLUP involves the sampling weights and uses the variance estimators by Henderson's method 3 as in Prasad & Rao (1990). It is a linear function of the direct design-based estimator of the ith small-area mean $\overline{y}_{i\omega} = \sum_{j=1}^{n_i} \omega_{ij} y_{ij}$, where ω_{ij} is the sampling weight. The estimator $\overline{y}_{i\omega}$ is a direct estimator for the ith

area because it does not involve observations from other areas. It is also a design-based estimator because it involves the sampling weights. It converges in probability to the population mean of the ith area. Although $\tilde{\mu}^{ps}$ is a design-based estimator, Figure 3.9 and Figure 3.10 show that the composite estimator $\tilde{\mu}^{c}$ is more efficient than the pseudo-EBLUP whether with population or sample mean of covariates based on the simulation studies. In fact, it is obvious that the pseudo-EBLUP is also less efficient that the estimator $\tilde{\mu}$ suggested by Prasad & Rao [41]. Therefore, it is not as competitive as other small-area estimators even when the sampling design is taken into account in the pseudo-EBLUP estimator.

3.7 Conclusions and Future Directions

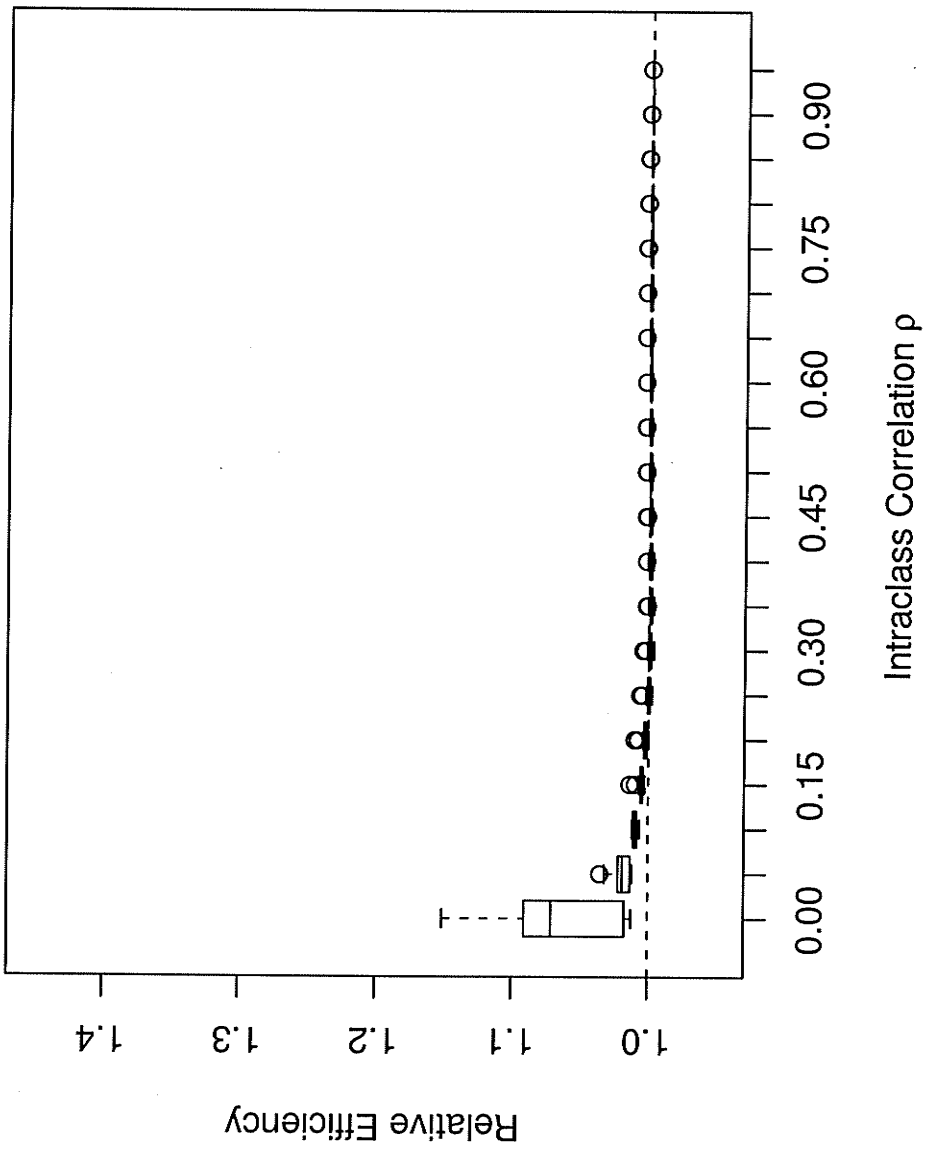
The composite estimator $\tilde{\mu}^{c}$ of the small-area mean, which is implicitly a shrinkage approach, has more flexibility in adjusting for the impact of random effects on the small areas by adaptively balancing two different weighting schemes. It is also a combination of two EBLUPs and an indirect estimator which improves the efficiency of the estimation by integrating the information from other similar areas. Results from the Monte Carlo simulations show that the proposed estimator is more efficient in small samples than the maximum likelihood estimator assuming a normal distribution. On the other hand, the small-area mean estimator, $\tilde{\mu}$, of Prasad & Rao [41] and the Pseudo-EBLUP of You & Rao [67] are relatively inefficient with respect to the

maximum likelihood estimator when the intraclass correlation is small, as is typical in survey sampling situations, as shown in Figures 3.5 and 3.8. More importantly, Figures 3.1 and 3.3 indicate that no matter whether the population means of the covariates are known or not, the proposed estimator $\tilde{\mu}^c$ is more efficient than $\tilde{\mu}$ of Prasad & Rao [41] when the intraclass correlation ρ is less than 0.20 for all four distributions studied in terms of their mean square errors and always better than the Pseudo-EBLUP of You & Rao [67]. Despite whether variation $\{\nu_i\}$ or $\{\epsilon_{ij}\}$ has more impact on the response, $\tilde{\mu}^c$ should be the choice among the estimators studied for small-area estimation.

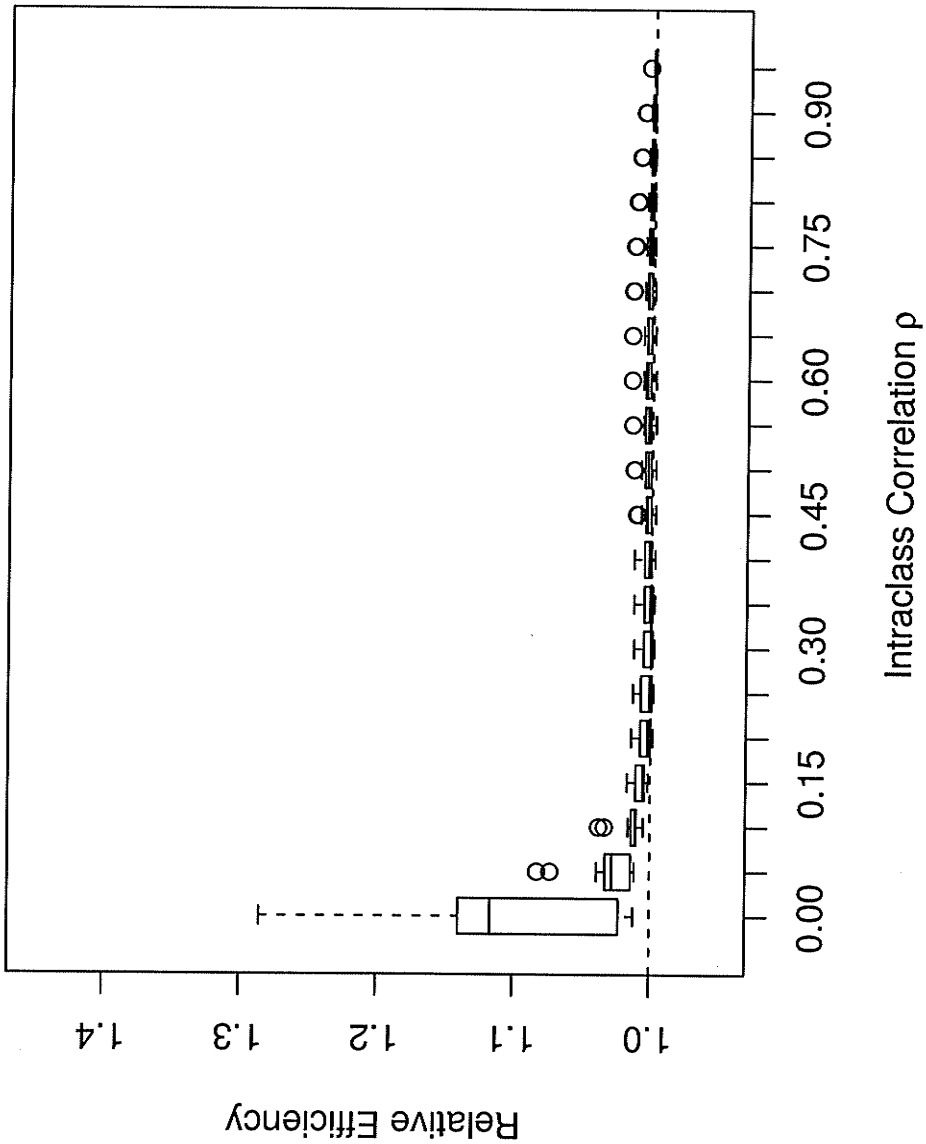
We further propose to extend the application of this new technique of estimating the small-area means by convex combination of two unbiased estimators to other mixed models, including nonlinear mixed models.

Figure 3.1: Boxplots of relative efficiencies of $\tilde{\mu}^c$ with respect to $\tilde{\mu}$ over 20 areas for intraclass correlation coefficient from 0.00 to 0.95 with 4 distributions of random effects assuming the population mean of the covariates in each area is known.

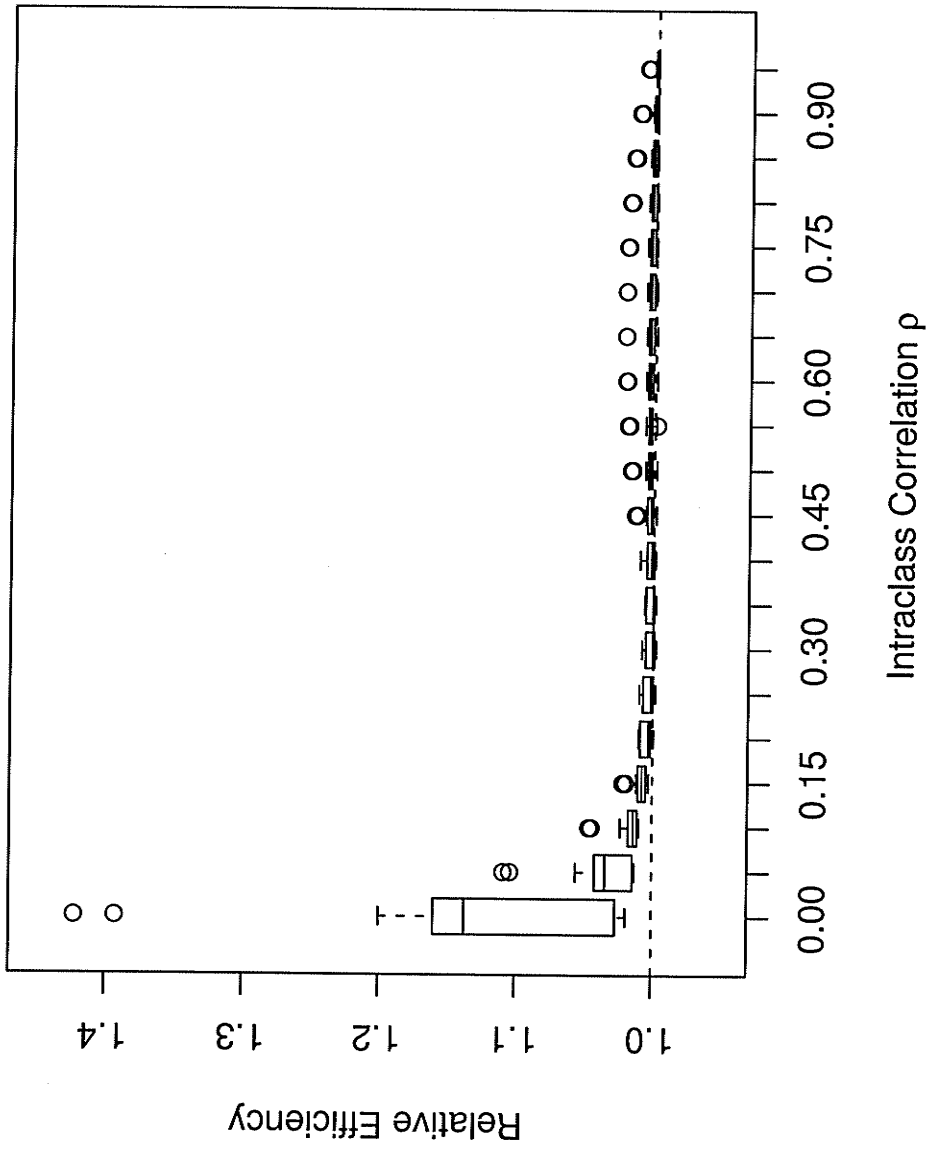
(a) Normal Distribution



(b)Double Exponential distribution



(c) Exponential Distribution



(d) Uniform Distribution

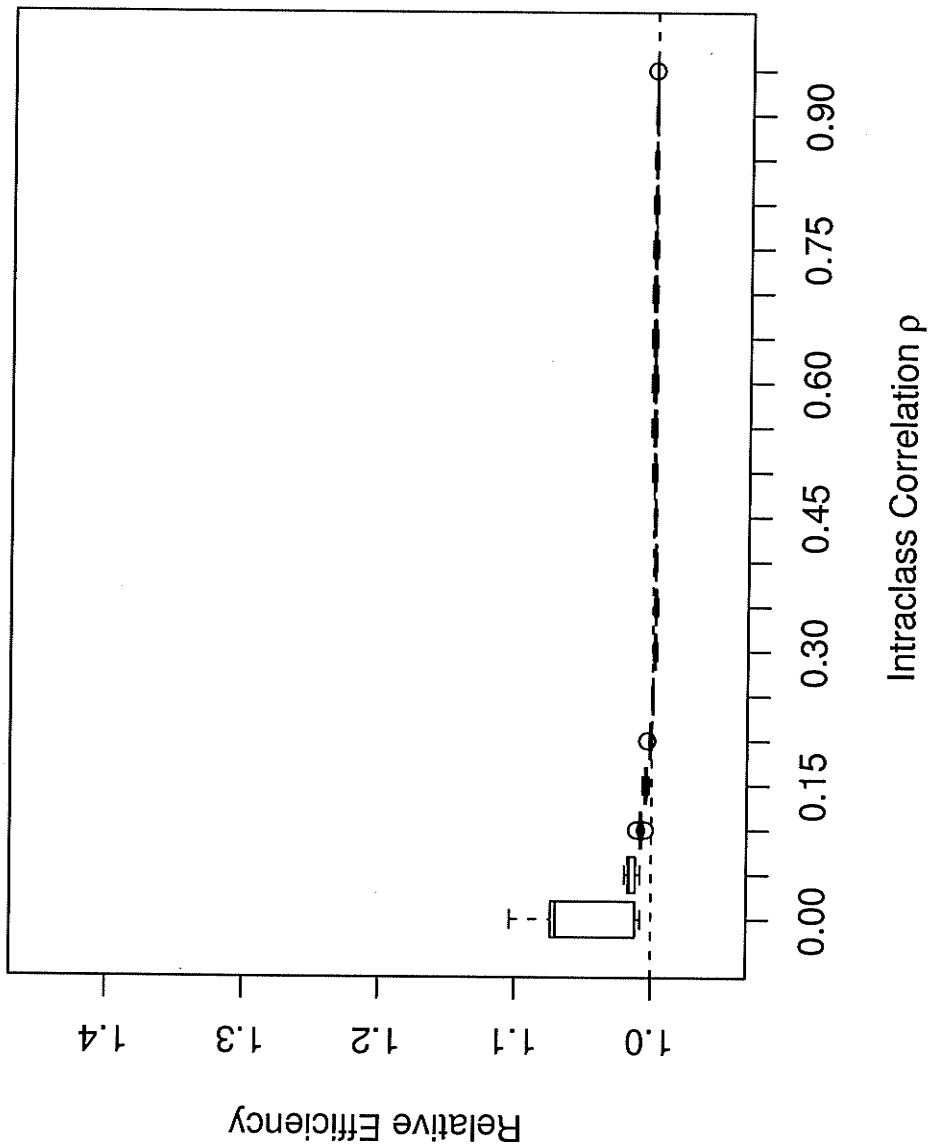
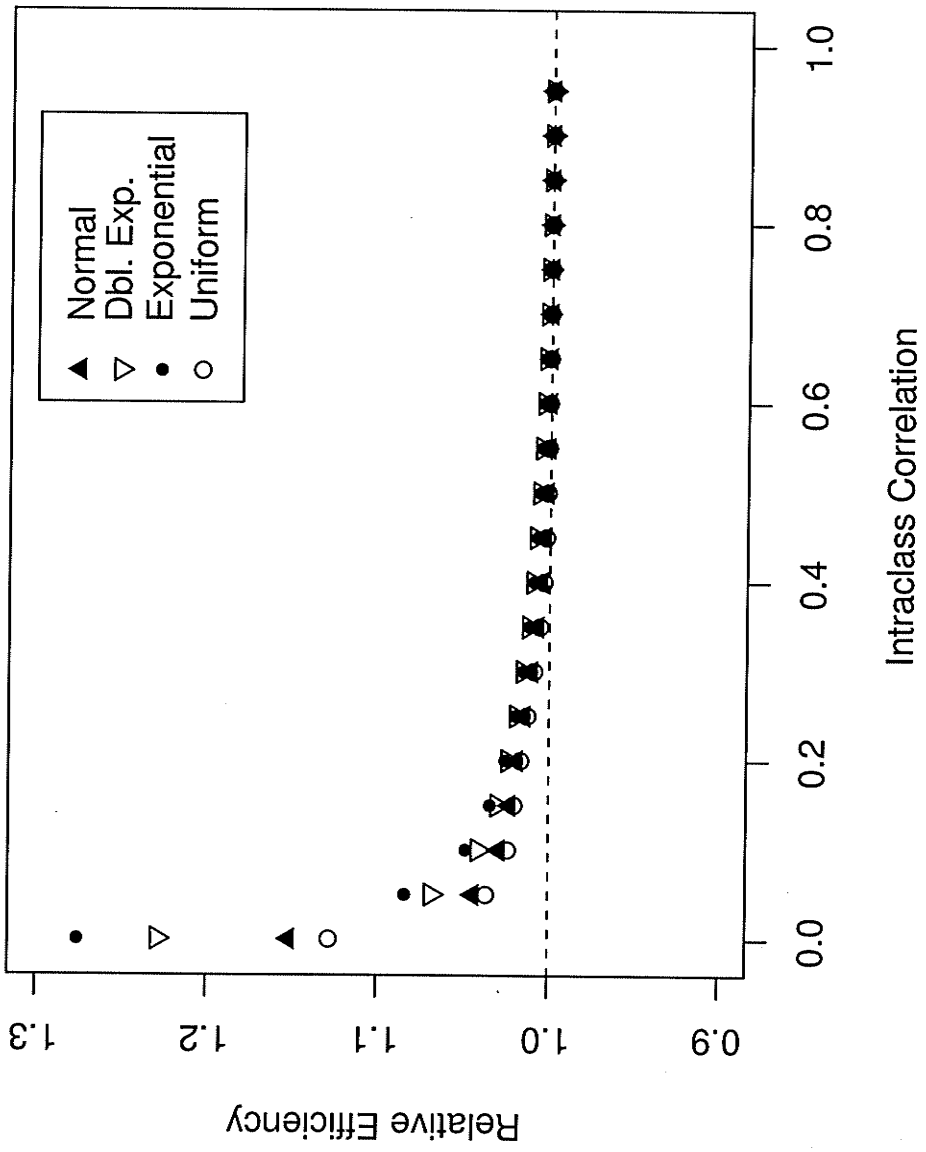


Figure 3.2: Average relative efficiencies of $\tilde{\mu}^c$ with respect to $\tilde{\mu}^{(1)}$ which is when $a_i = 0$ and to $\tilde{\mu}^{(2)}$ which is when $a_i = 1$ for different distributions.

(a) $\tilde{\mu}^{(1)}(a_i = 0)$



(b) $\tilde{\mu}^{(2)}(a_i = 1)$

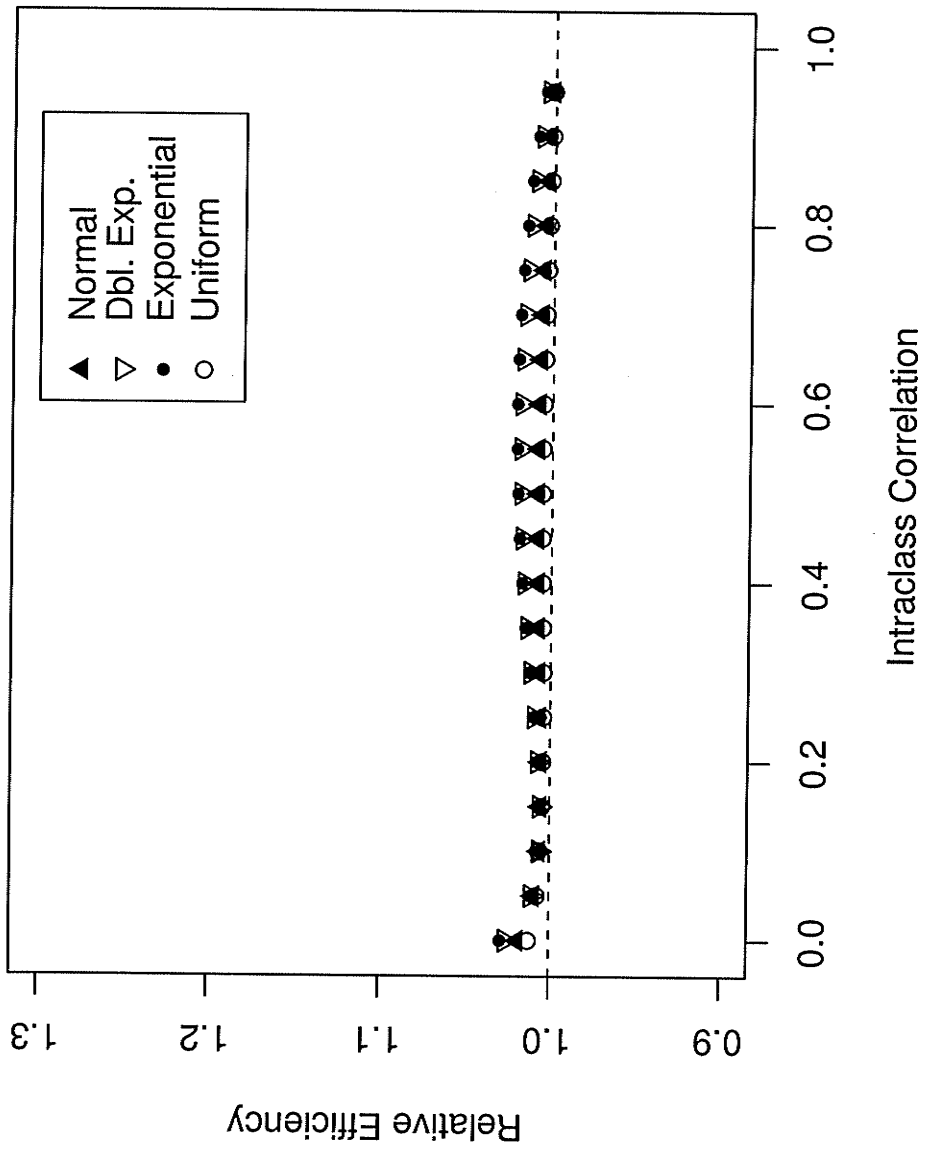
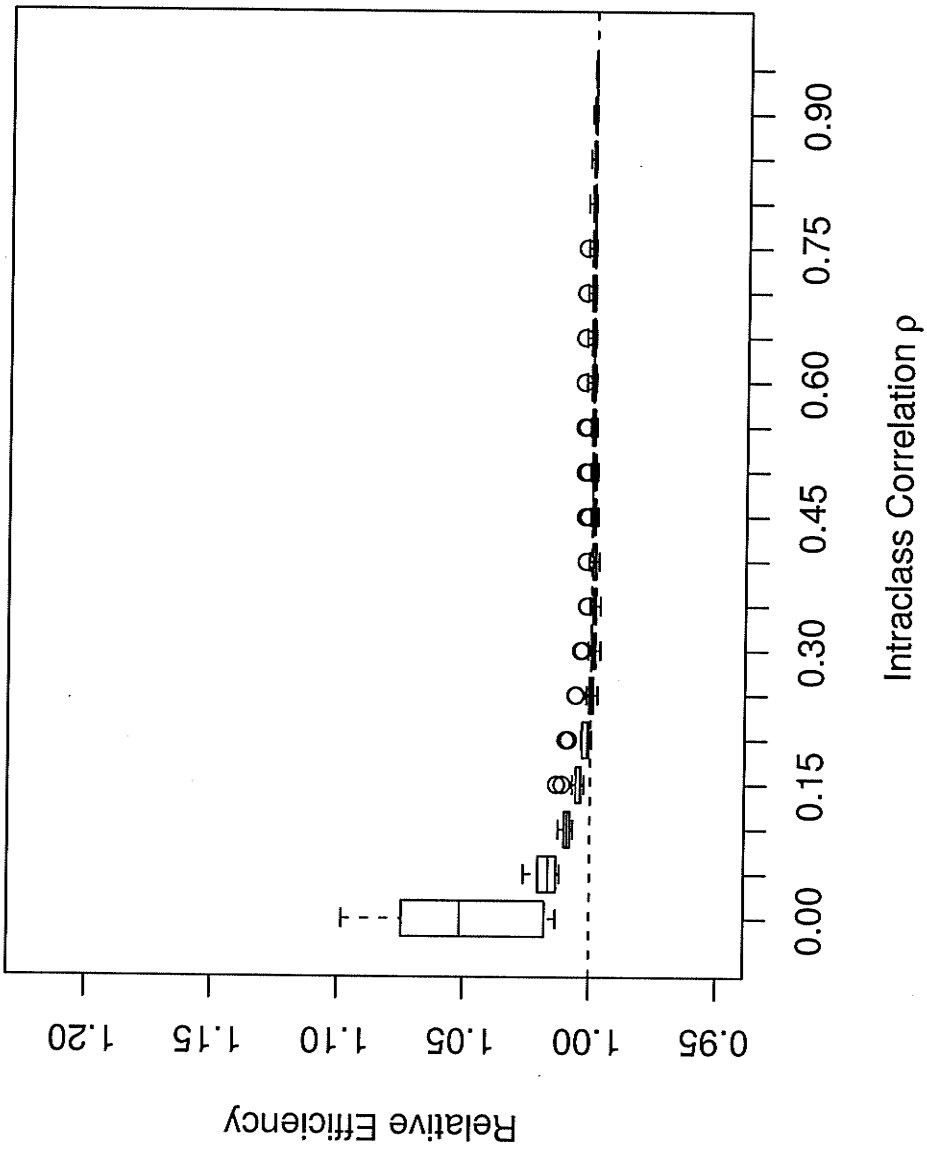
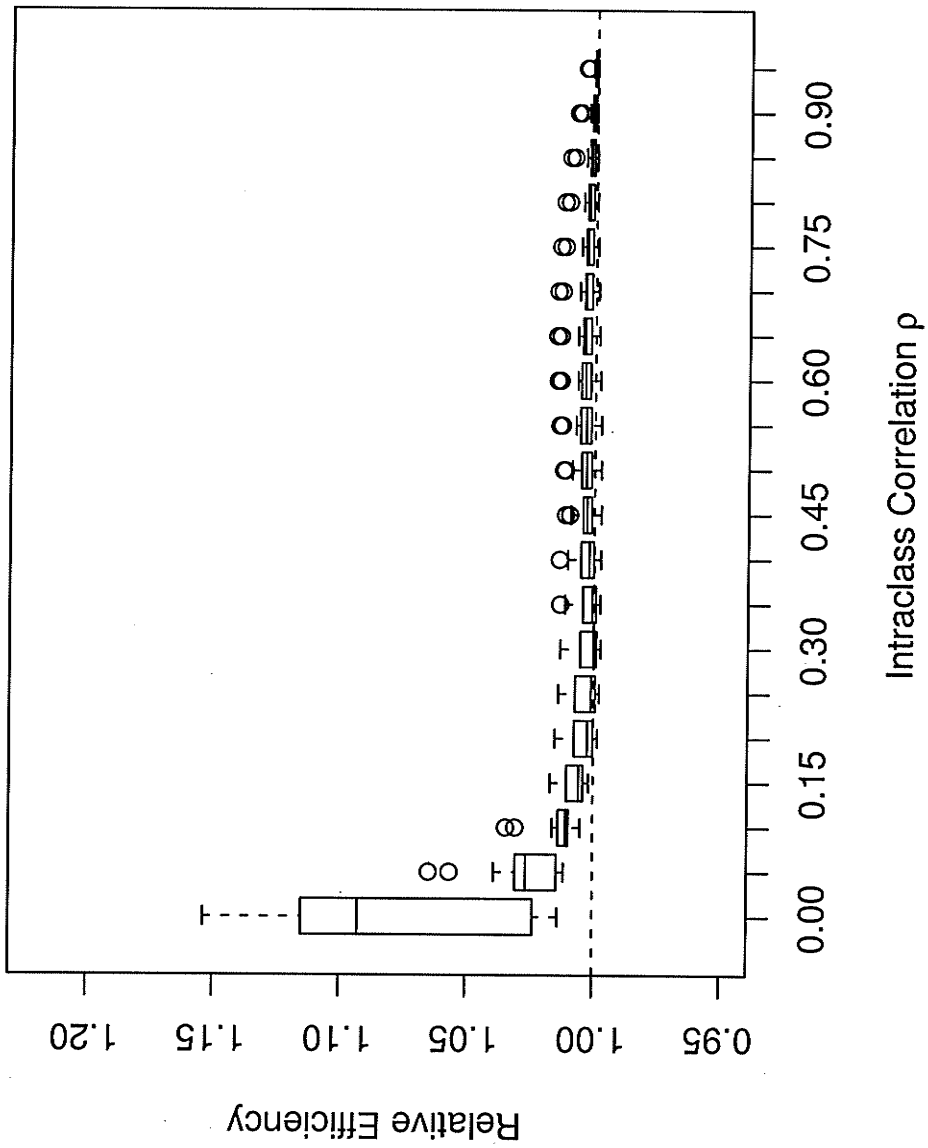


Figure 3.3: Boxplots of relative efficiencies of $\tilde{\mu}^c$ with respect to $\tilde{\mu}$ over 20 areas for intraclass correlation coefficient from 0.00 to 0.95 with 4 distributions of random effects. The population mean of the covariates in each area is not known and is estimated by the sample mean.

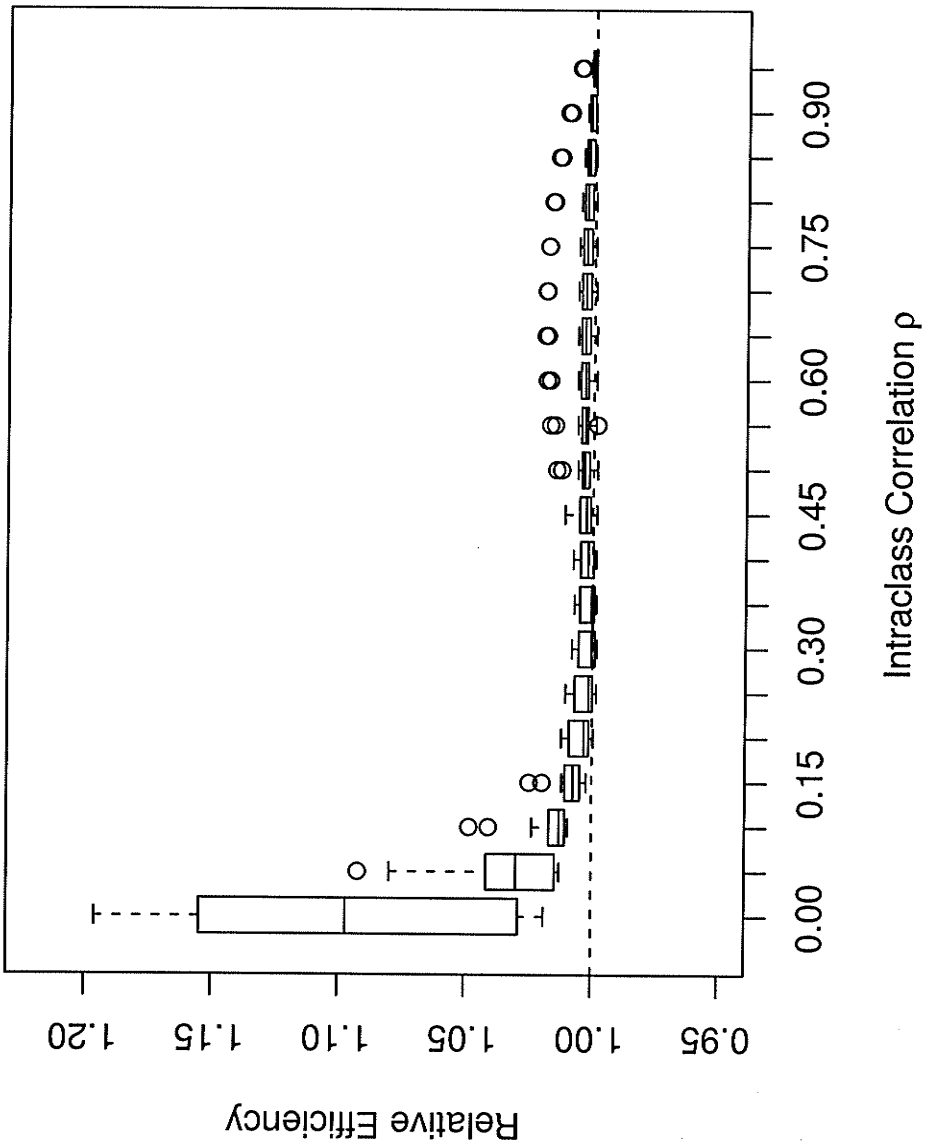
(a) Normal Distribution



(b)Double Exponential distribution



(c) Exponential Distribution



(d) Uniform Distribution

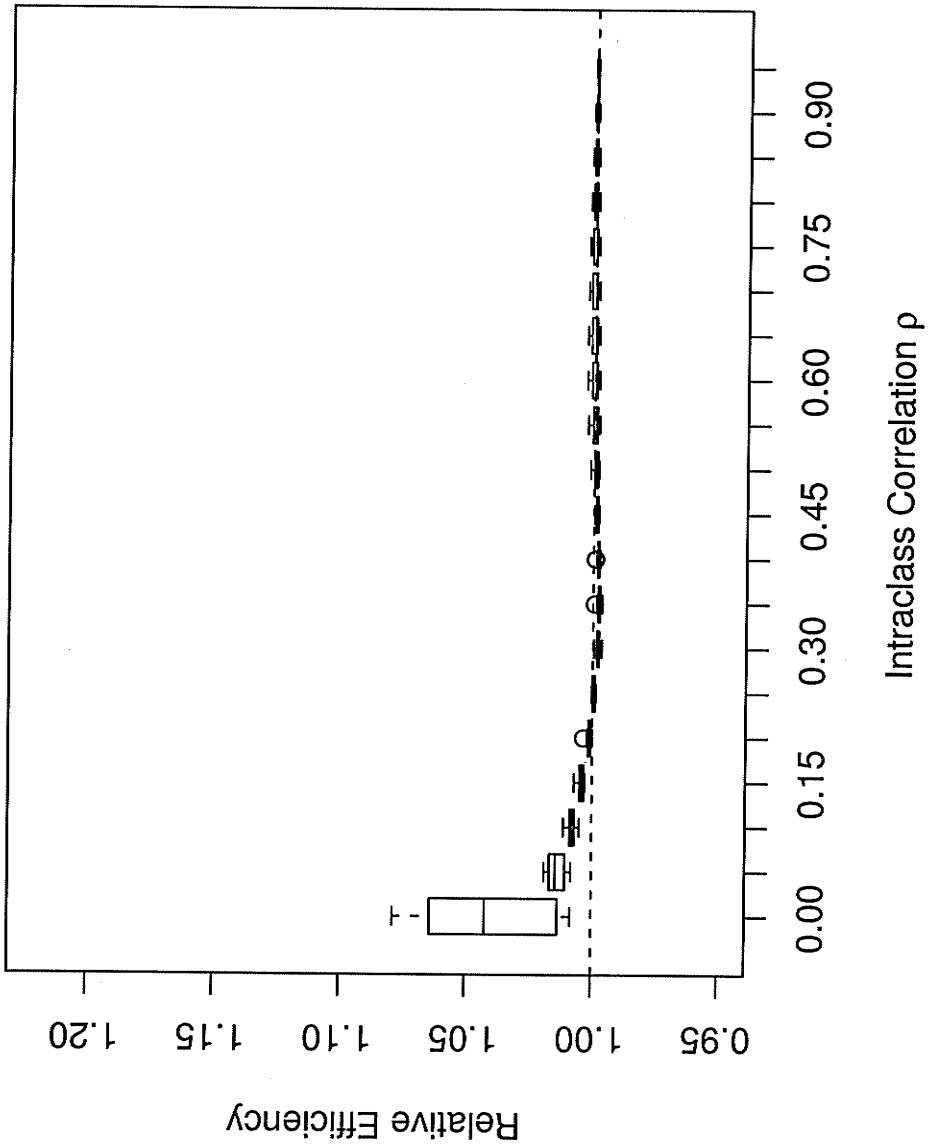
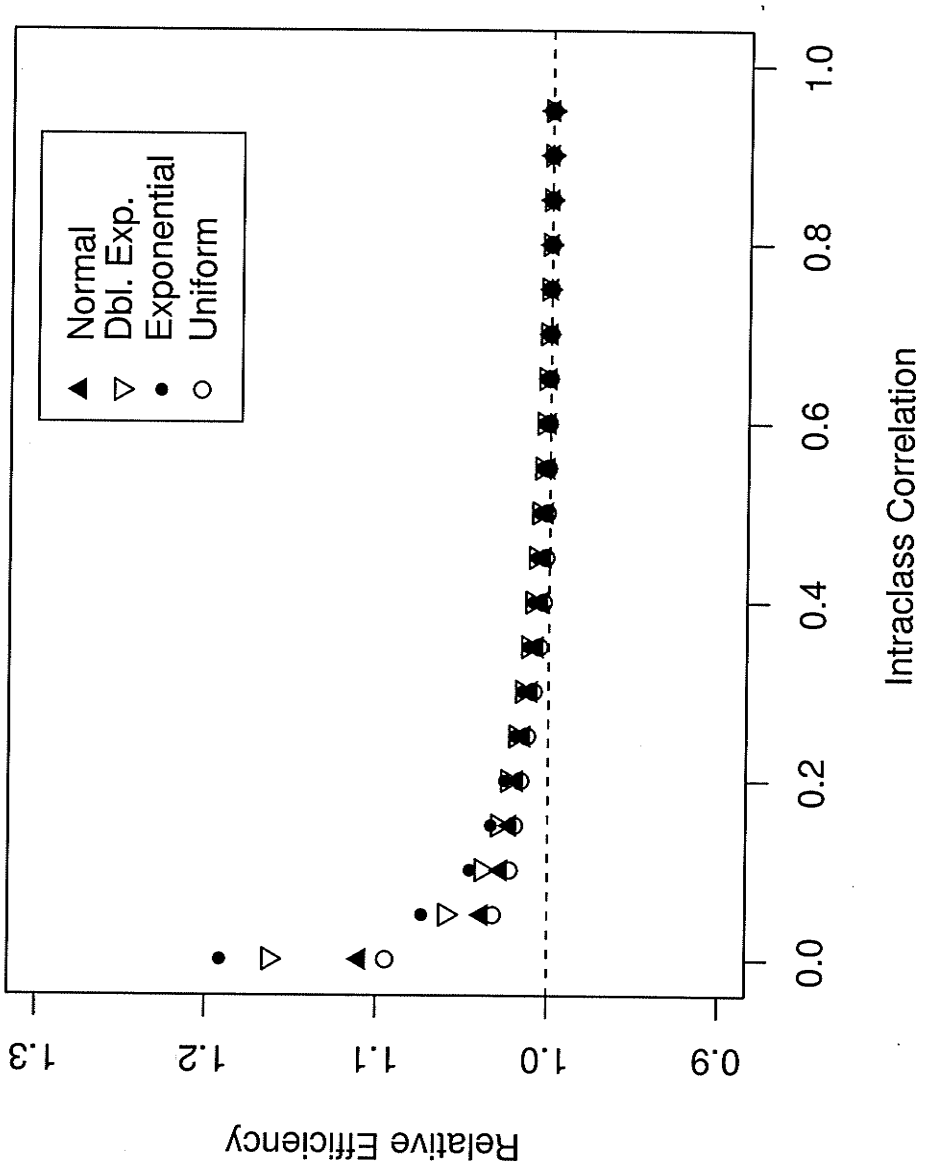


Figure 3.4: Average relative efficiency of $\tilde{\mu}^c$ with respect to $\tilde{\mu}^{(1)}$ which is when $a_i = 0$ and to $\tilde{\mu}^{(2)}$ which is when $a_i = 1$ for different distributions with sample mean of covariates.

(a) $\tilde{\mu}^{(1)}(a_i = 0)$



(b) $\tilde{\mu}^{(2)}(a_i = 1)$

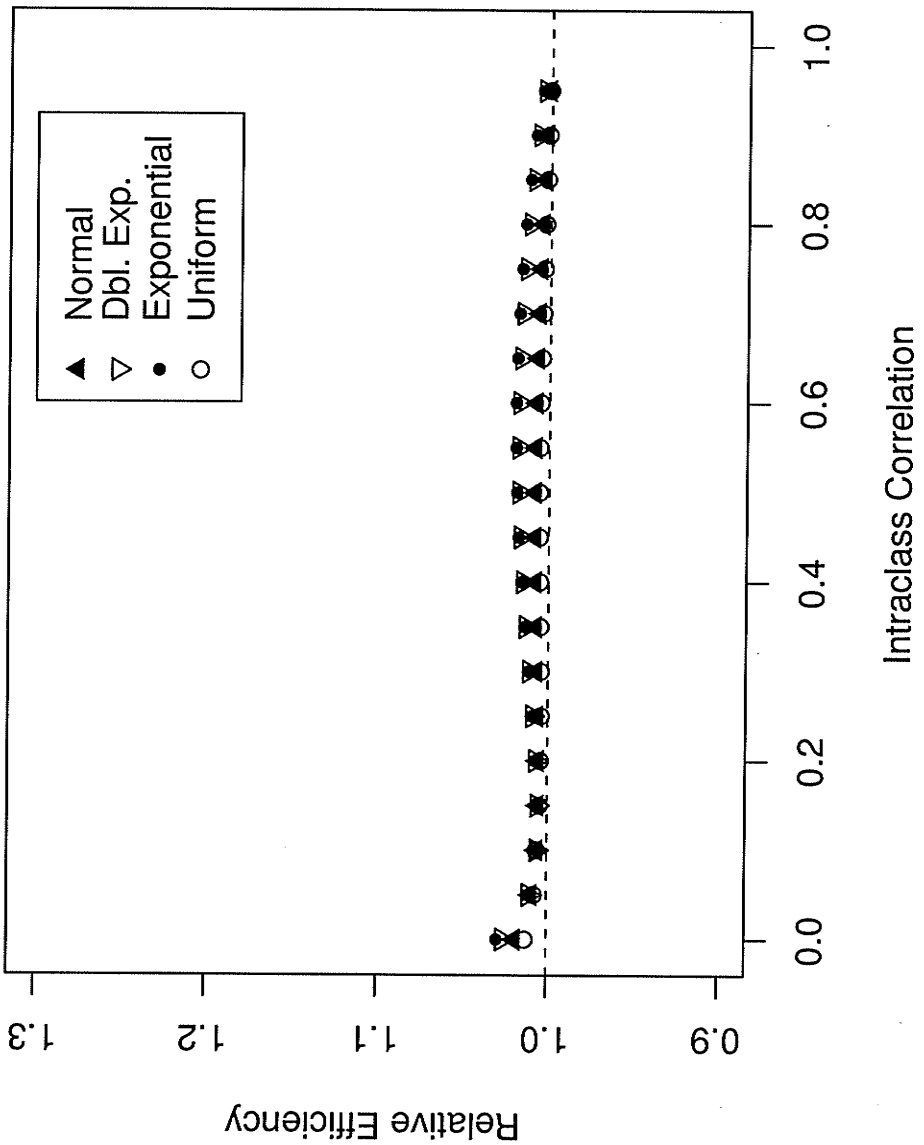
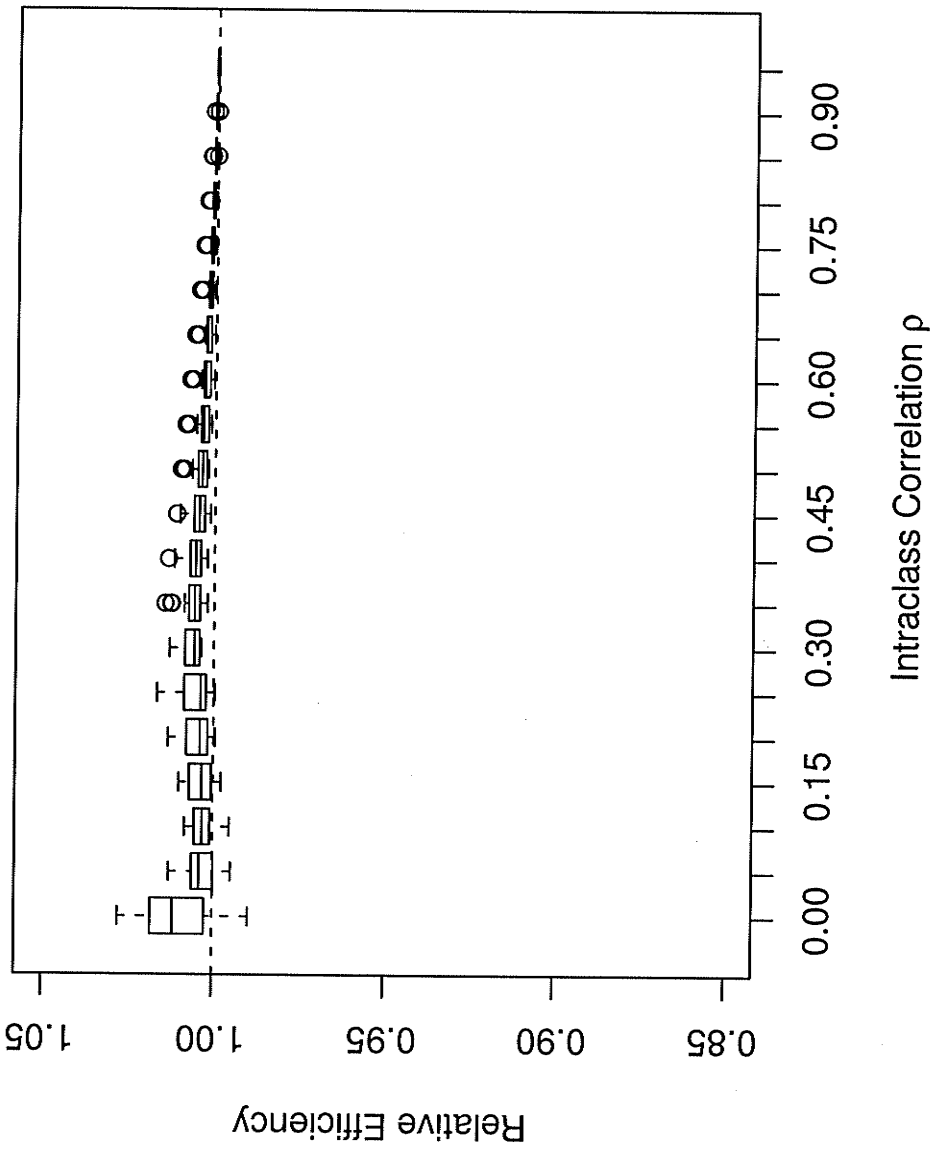


Figure 3.5: Boxplots of relative efficiencies of $\tilde{\mu}^c$ and $\tilde{\mu}$ with respect to $\hat{\mu}^{Vm}$ over 20 areas for intraclass correlation coefficient from 0.00 to 0.95 with Normal distribution of random effects assuming the population mean of the covariates in each area is known.

(a) $\tilde{\mu}^c$ with respect to $\hat{\mu}^{Vm}$



(b) $\tilde{\mu}$ with respect to $\hat{\mu}^{Vm}$

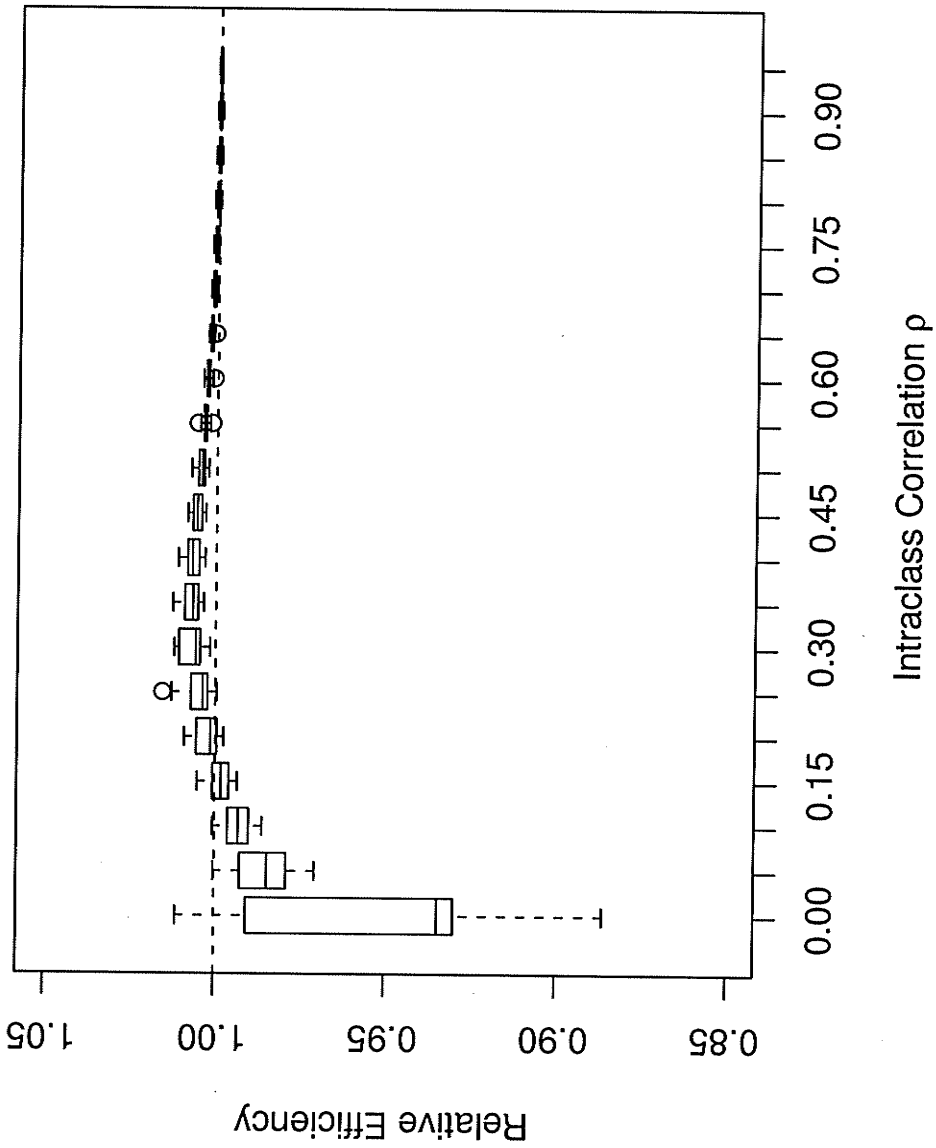
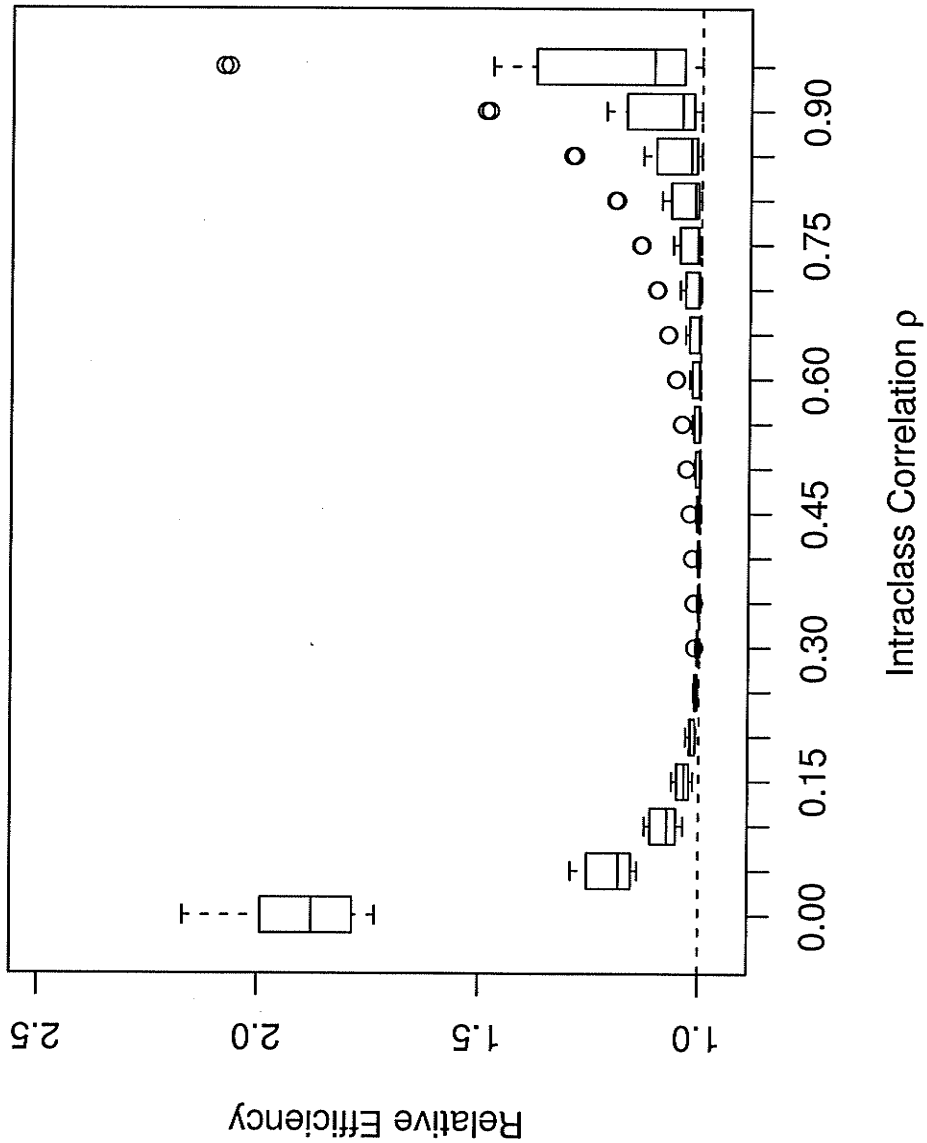


Figure 3.6: Boxplots of relative efficiencies of $\tilde{\mu}^c$ with respect to $\hat{\mu}^{(S)}$ and $\hat{\mu}^{(T)}$ over 20 areas for intraclass correlation coefficient from 0.00 to 0.95 with Normal distribution of random effects. The population mean of the covariates is assumed to be known.

(a) $\tilde{\mu}^c$ with respect to $\hat{\mu}^{(S)}$



(b) $\tilde{\mu}^c$ with respect to $\hat{\mu}^{(T)}$

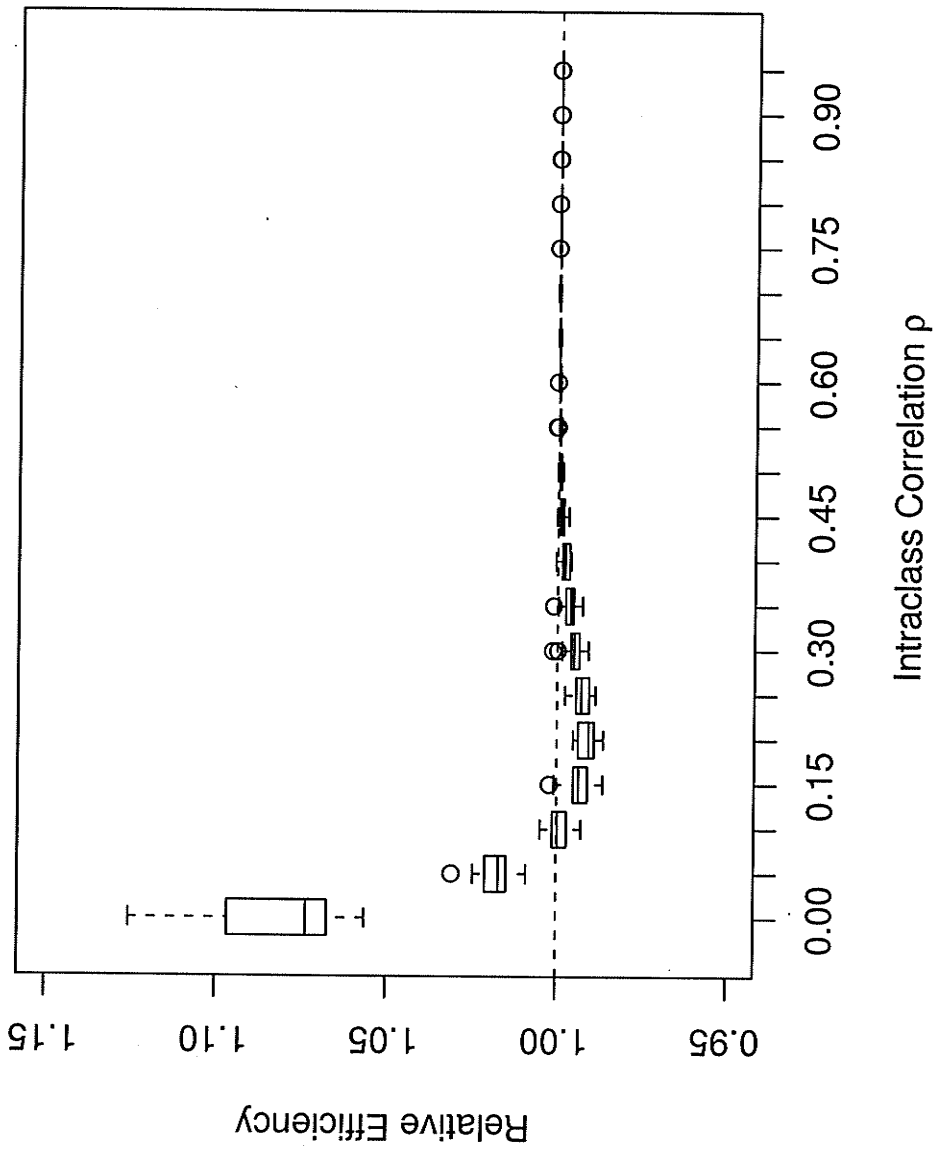
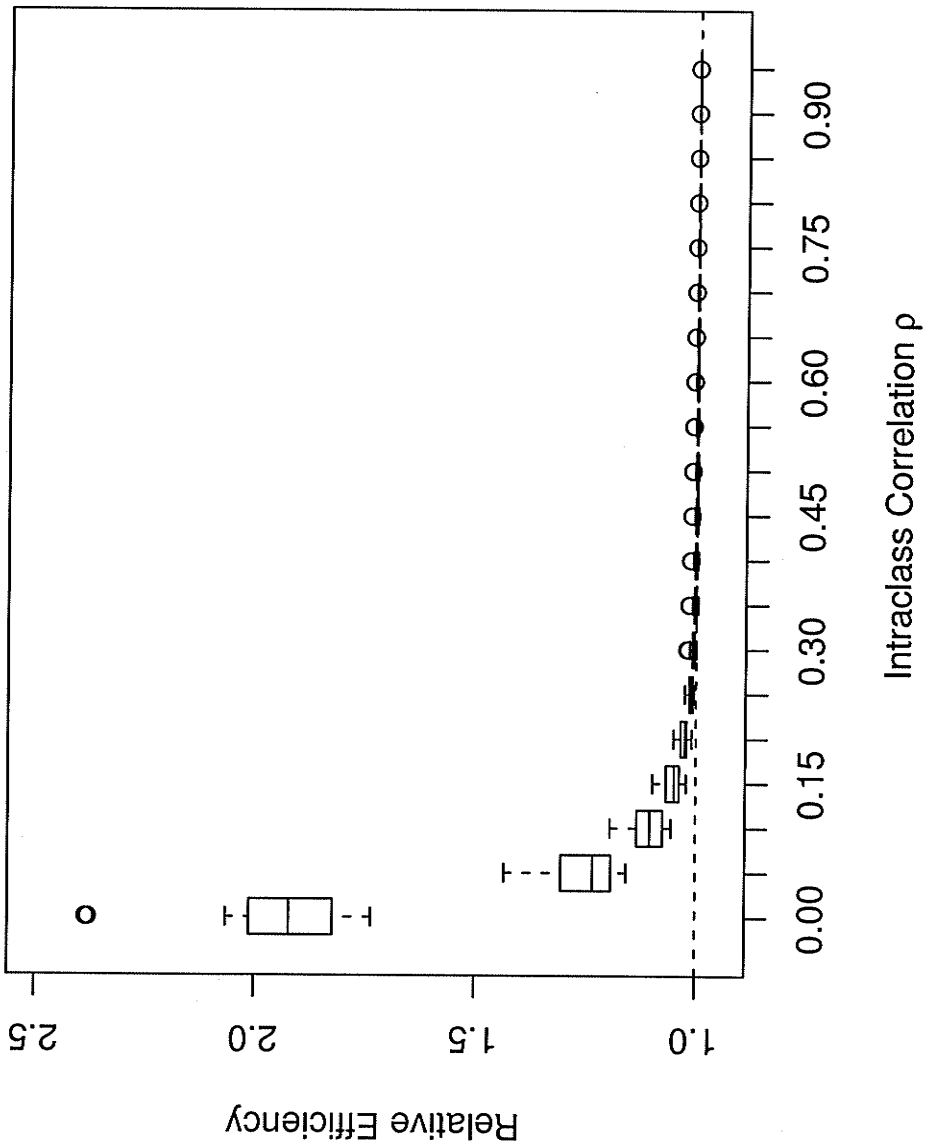


Figure 3.7: Boxplots of relative efficiencies of $\tilde{\mu}^c$ with respect to $\hat{\mu}^{(S)}$ and $\hat{\mu}^{(T)}$ over 20 areas for intraclass correlation coefficient from 0.00 to 0.95 with Normal distribution of random effects. The population mean of the covariates in each area is not known and is estimated by the sample mean.

(a) $\tilde{\mu}^c$ with respect to $\hat{\mu}^{(S)}$



(b) $\tilde{\mu}^c$ with respect to $\hat{\mu}^{(T)}$

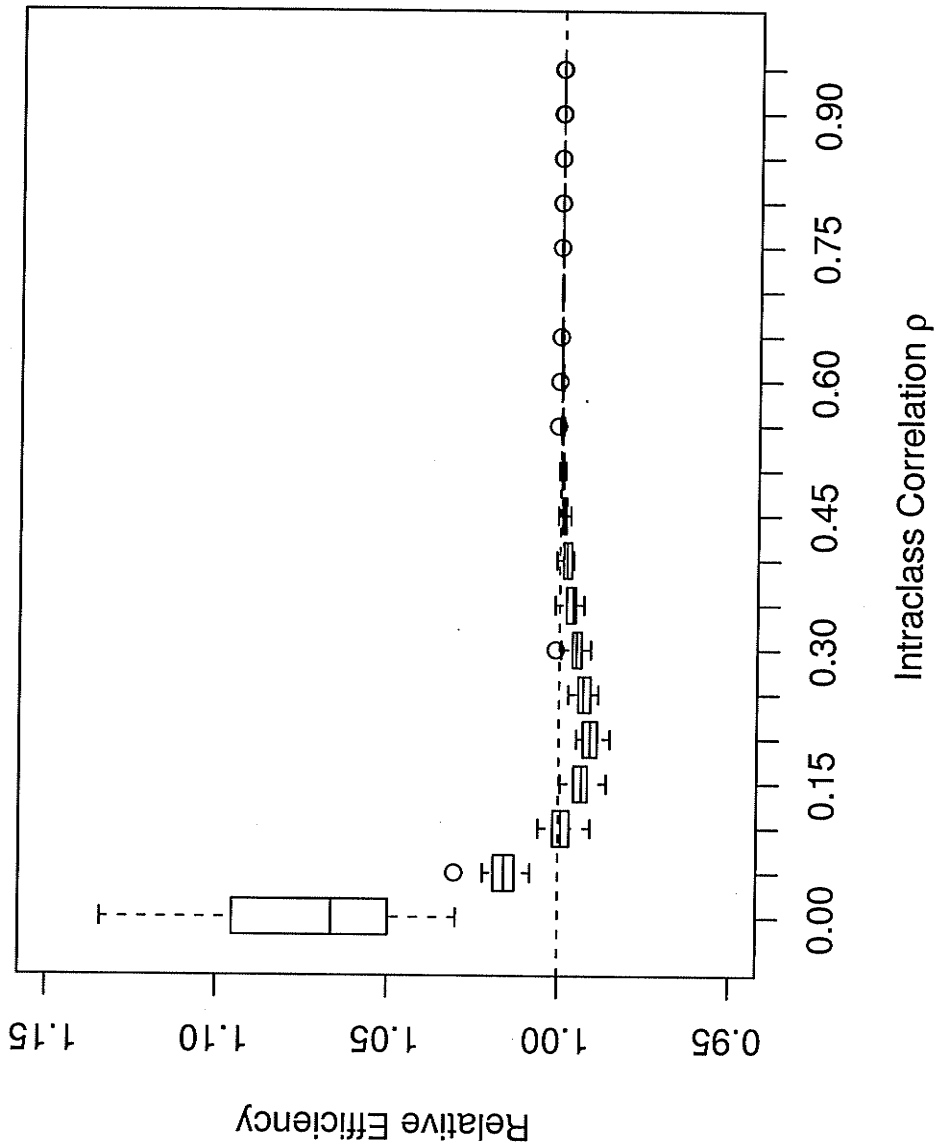
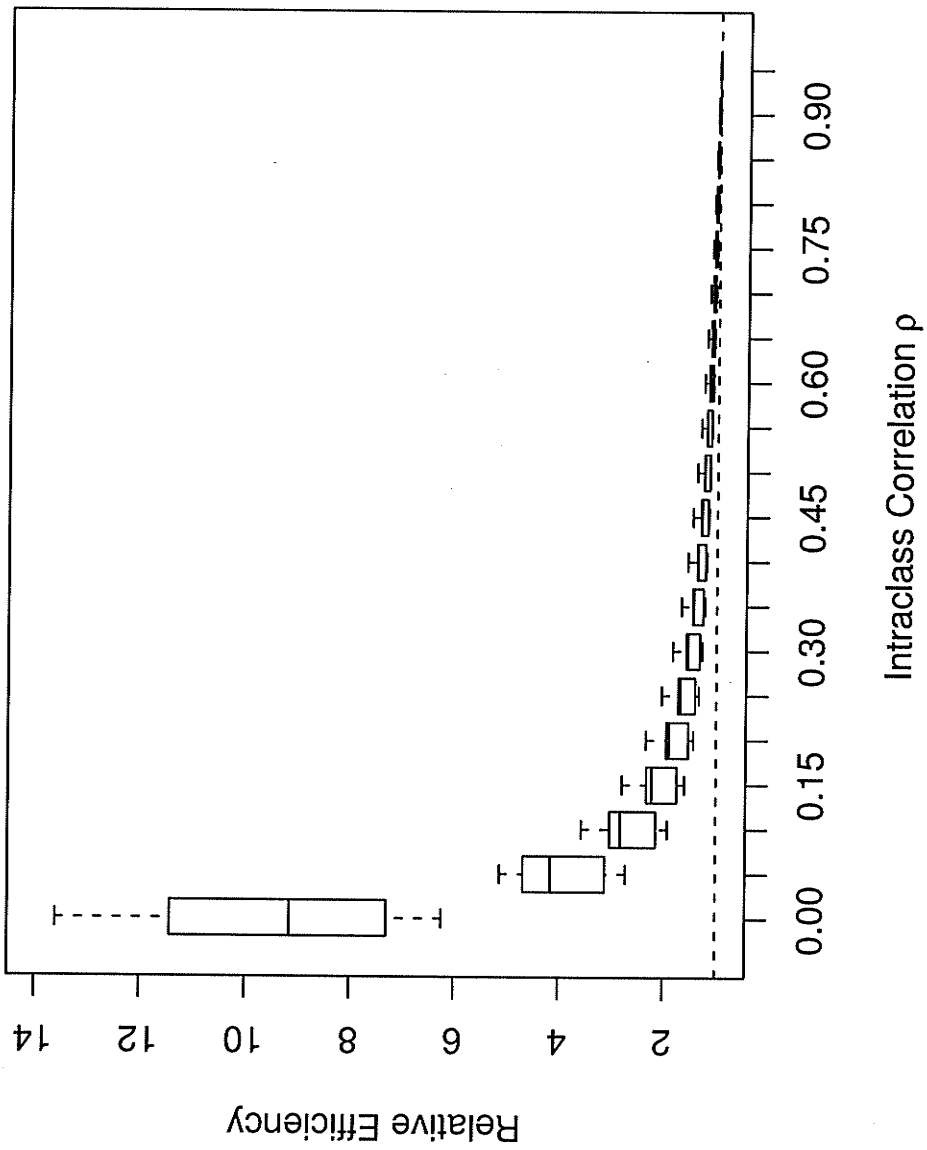


Figure 3.8: Boxplots of relative efficiencies of $\tilde{\mu}^c$ and $\tilde{\mu}$ with respect to $\hat{\mu}$ over 20 areas for intraclass correlation coefficient from 0.00 to 0.95 with Normal distribution of random effects. The population mean of the covariates in each area is not known and is estimated by the sample mean.

(a) $\tilde{\mu}^c$ with respect to $\hat{\mu}$



(b) $\tilde{\mu}$ with respect to $\hat{\mu}$

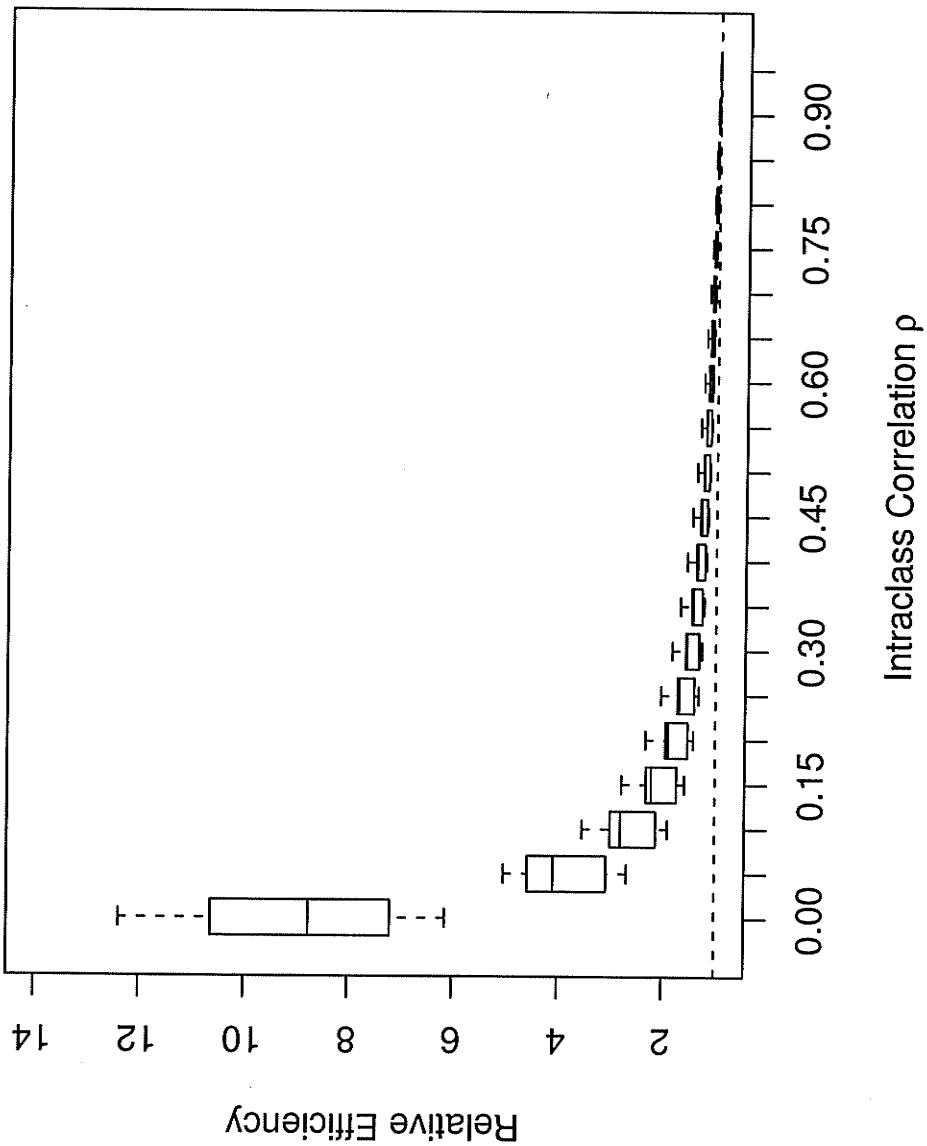
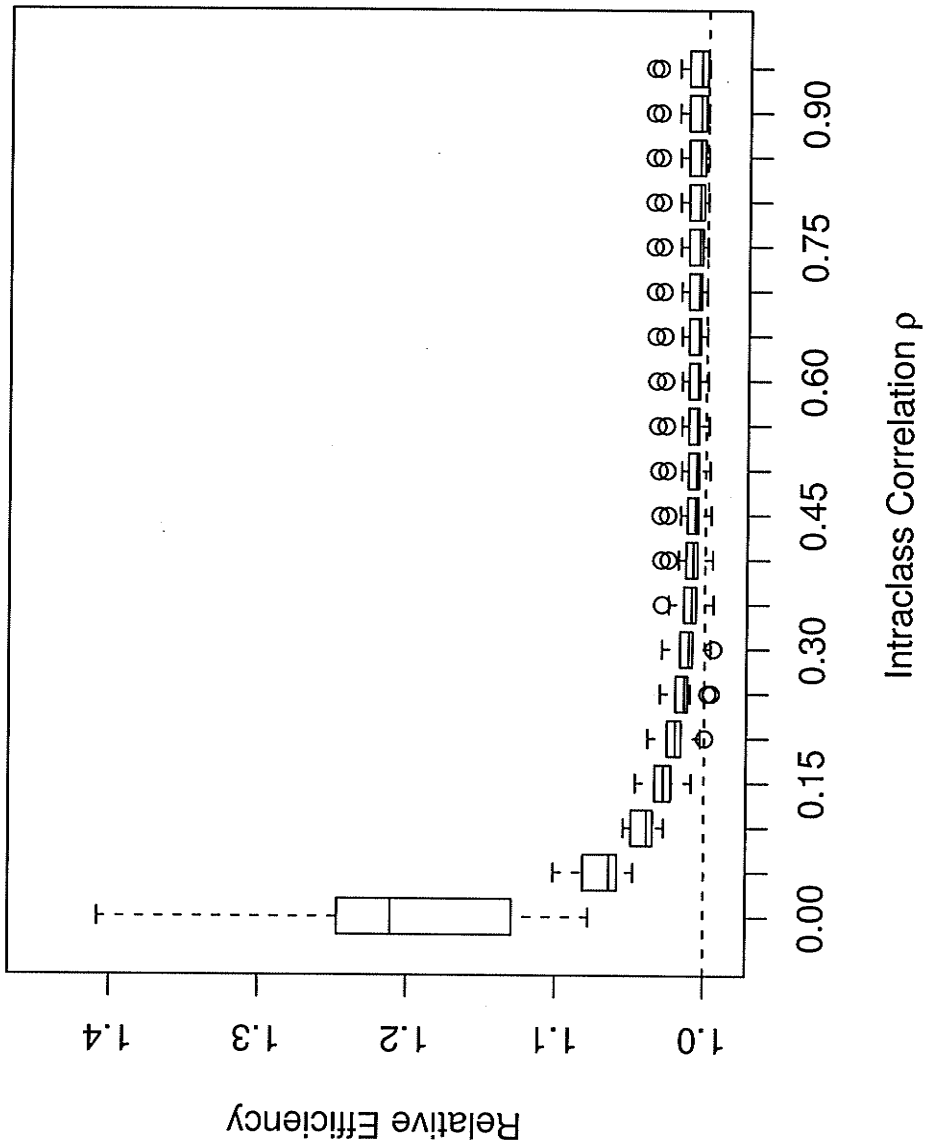
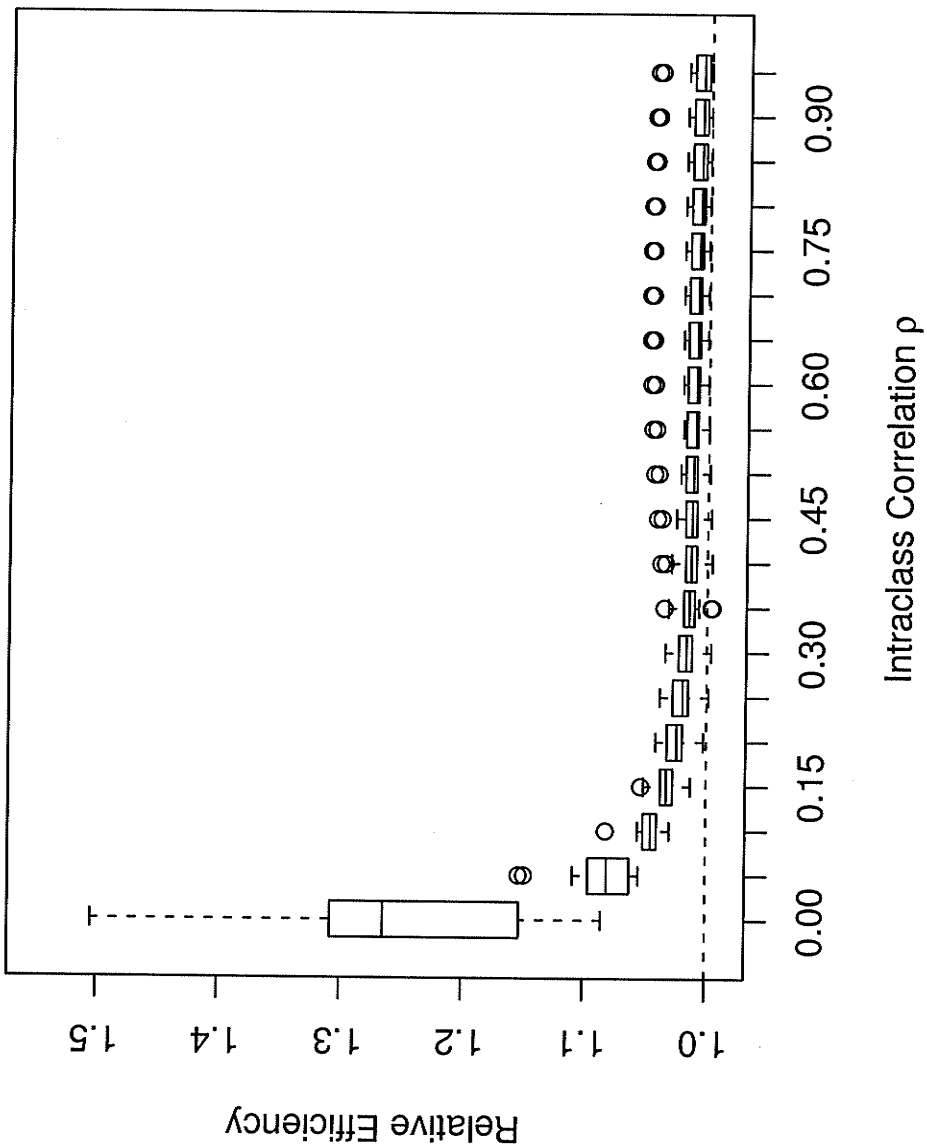


Figure 3.9: Boxplots of relative efficiencies of $\tilde{\mu}^c$ with respect to $\tilde{\mu}^{ps}$ over 20 areas for intraclass correlation coefficient from 0.00 to 0.95 with 4 distributions of random effects assuming the population mean of the covariates in each area is known.

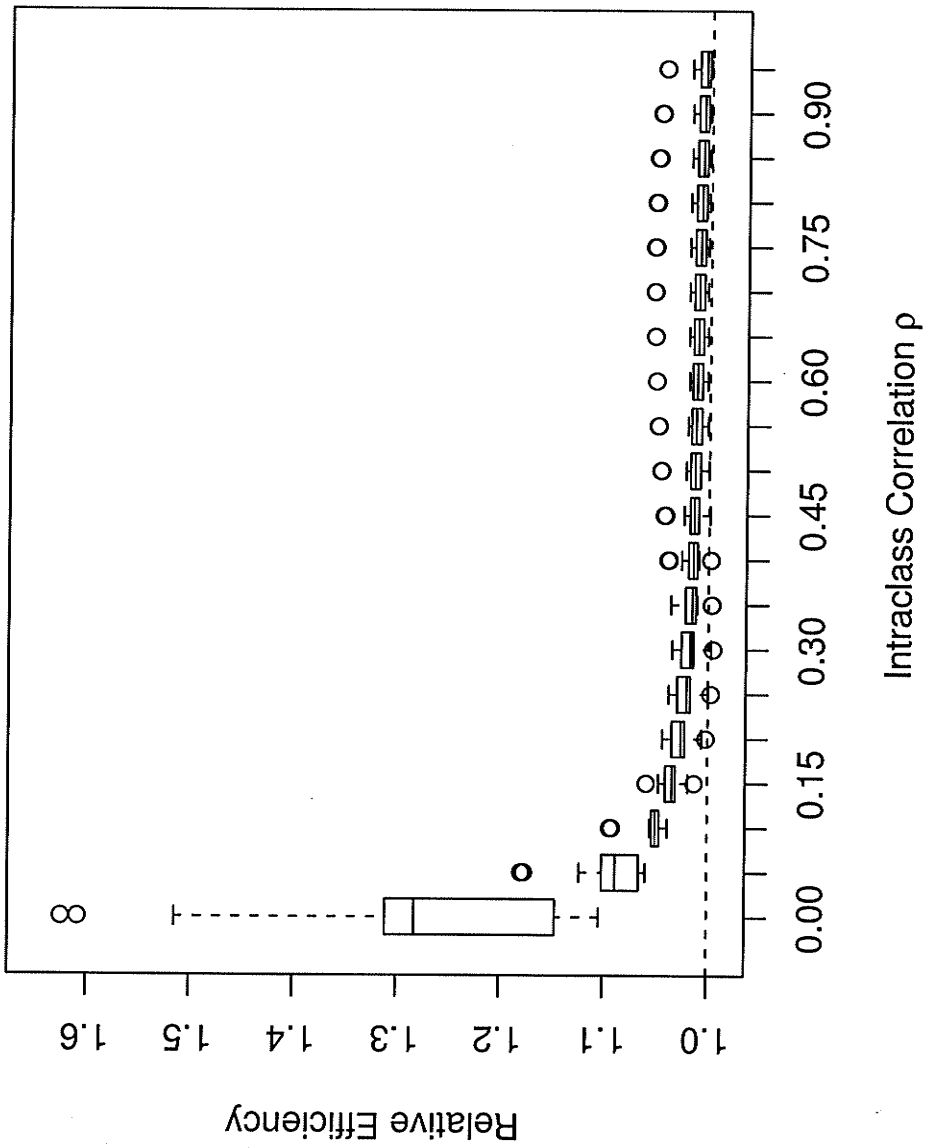
(a) Normal Distribution



(b)Double Exponential distribution



(c) Exponential Distribution



(d) Uniform Distribution

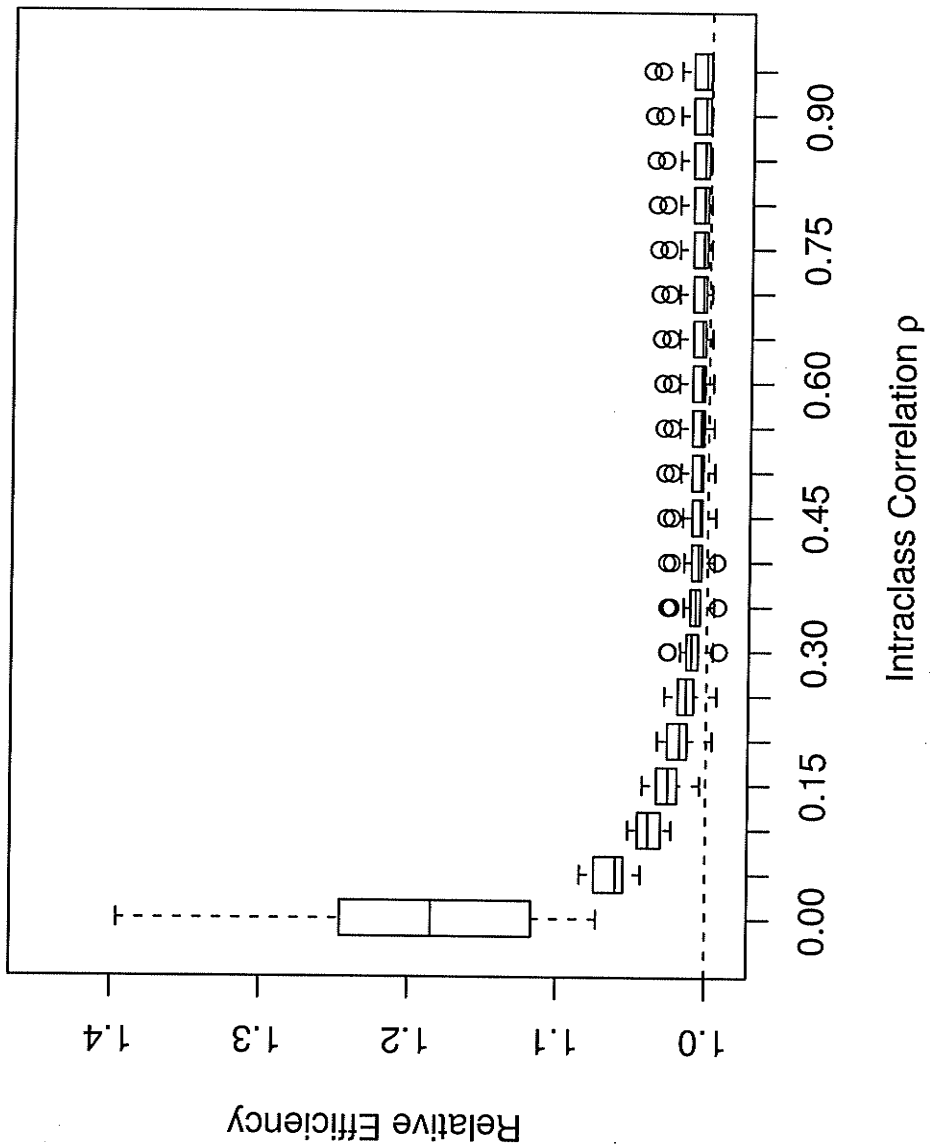
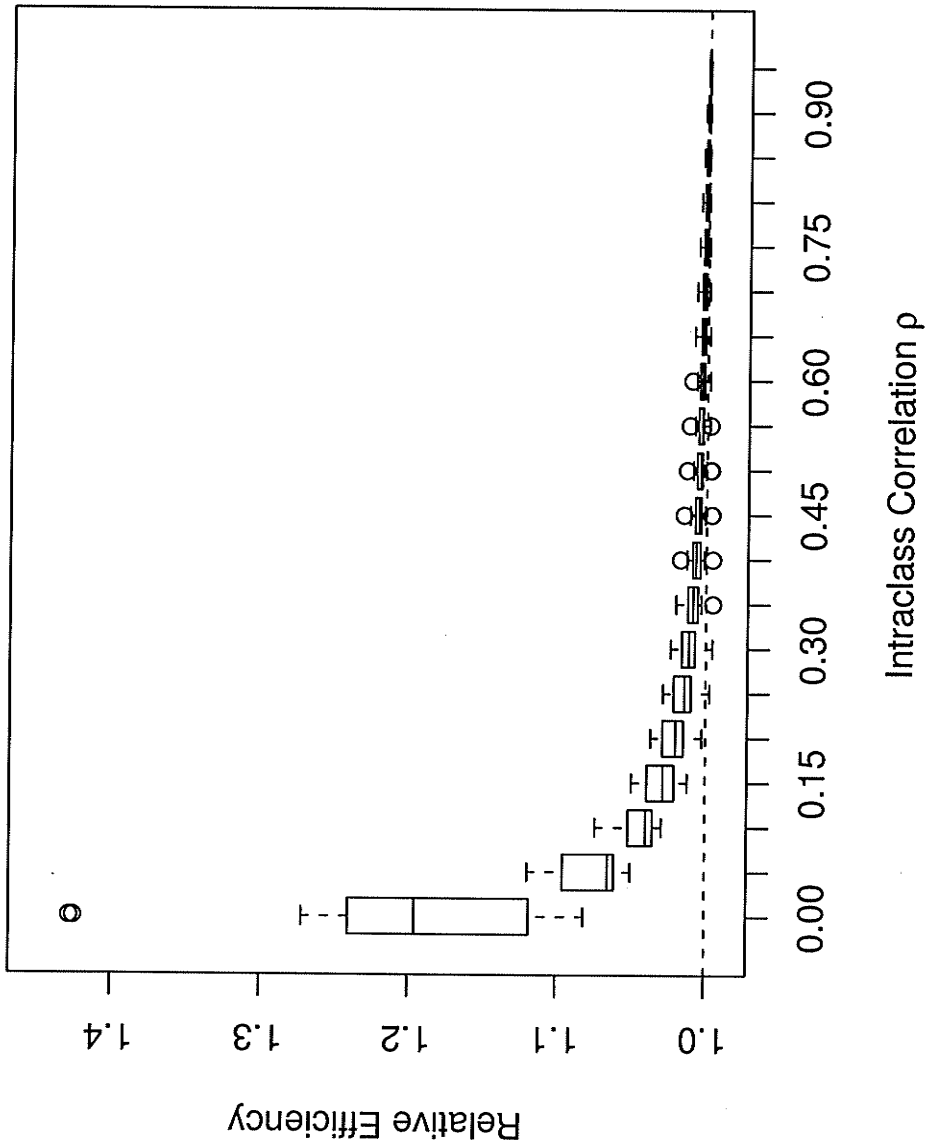
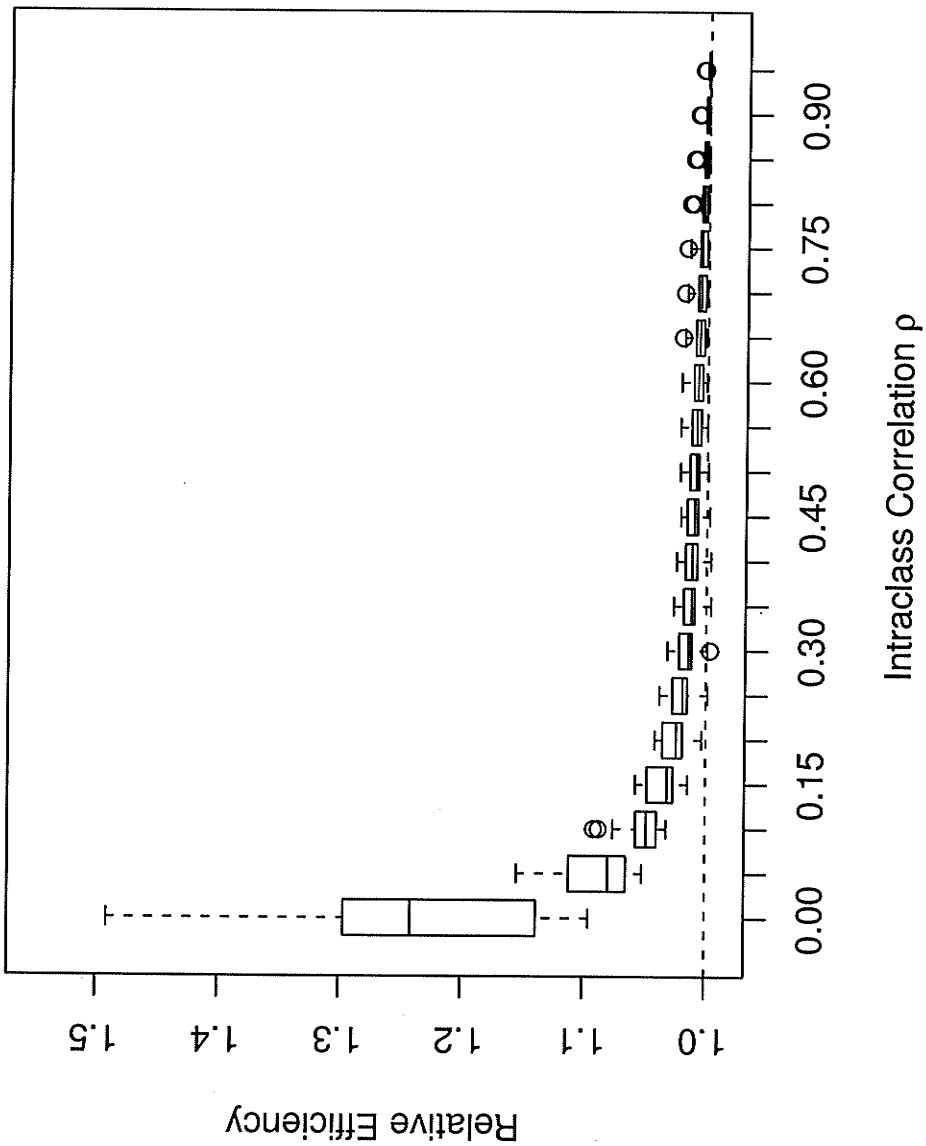


Figure 3.10: Boxplots of relative efficiencies of $\tilde{\mu}^c$ with respect to $\tilde{\mu}^{ps}$ over 20 areas for intraclass correlation coefficient from 0.00 to 0.95 with 4 distributions of random effects using sample mean of the covariates.

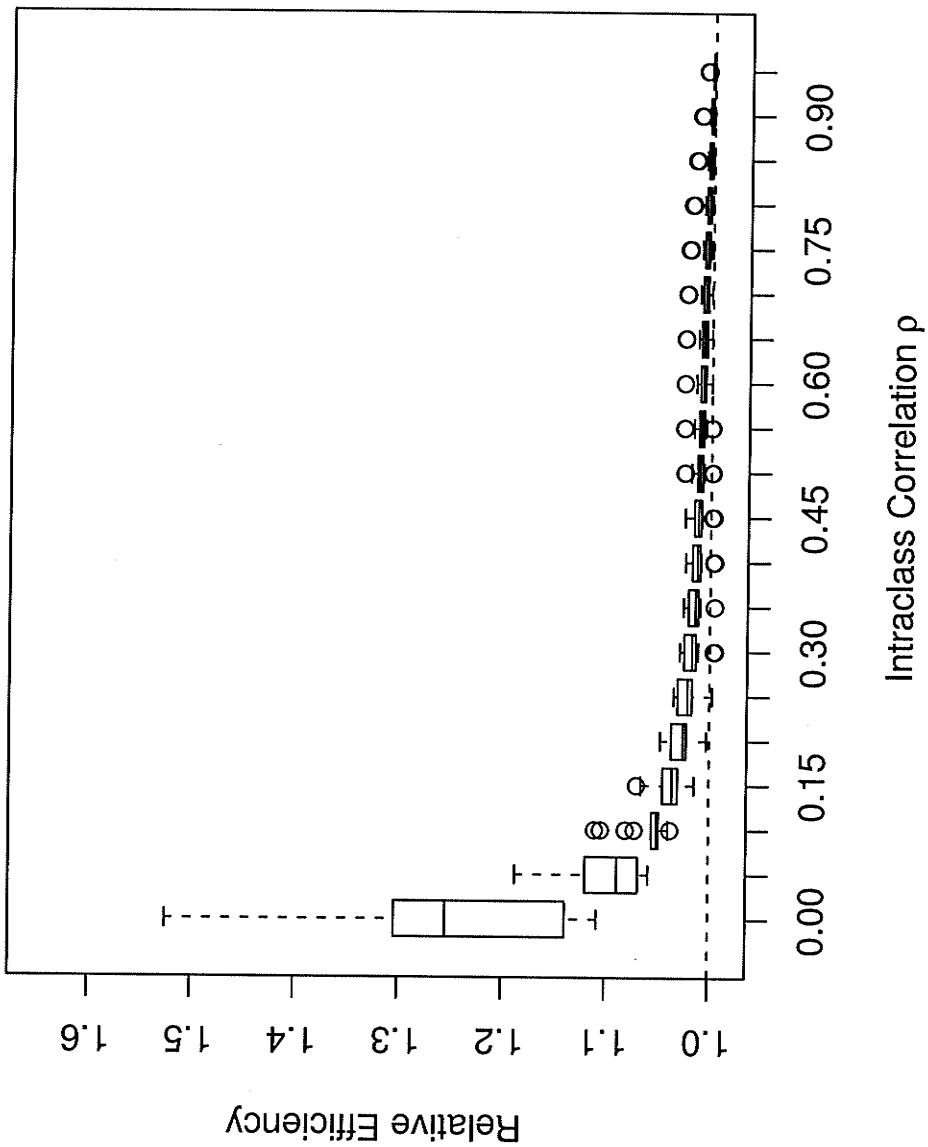
(a) Normal Distribution



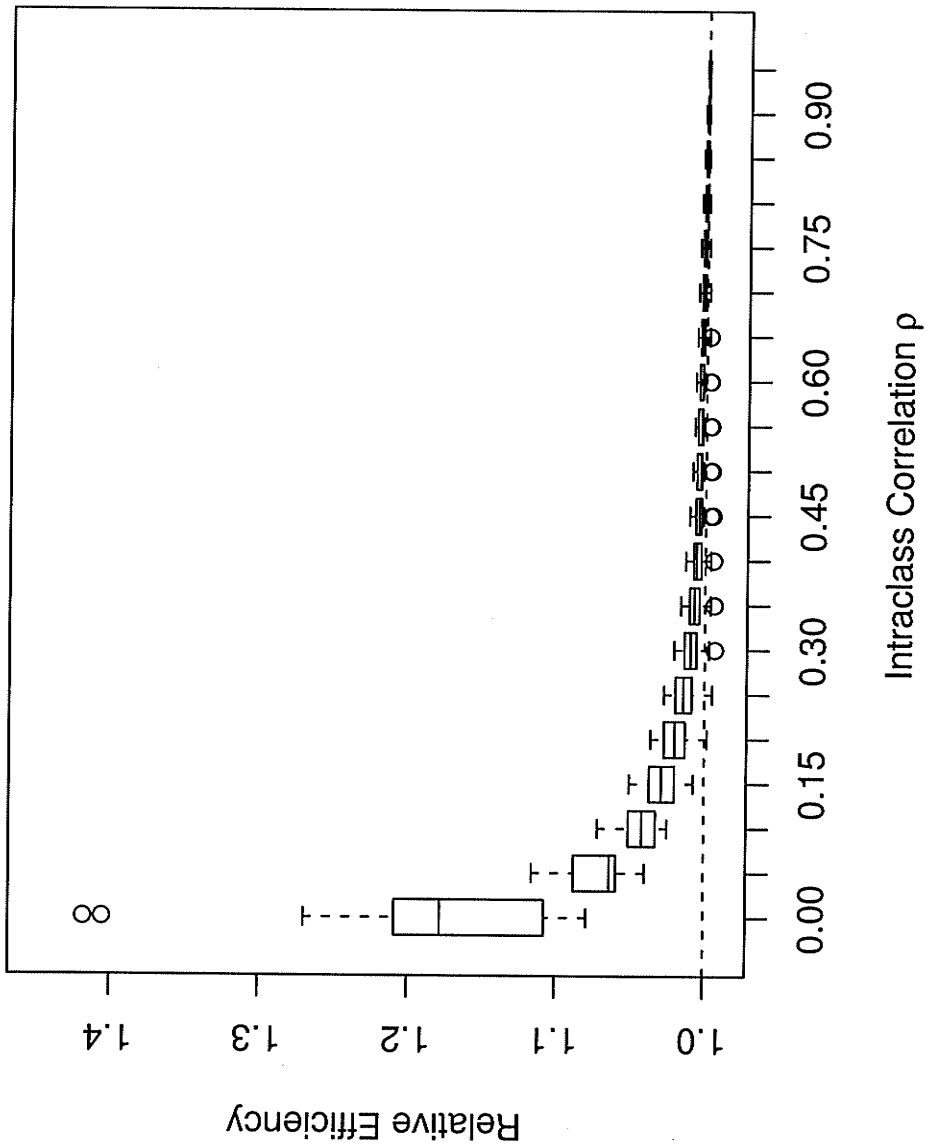
(b)Double Exponential distribution



(c) Exponential Distribution



(d) Uniform Distribution



Chapter 4

INTRACLASS CORRELATION ESTIMATION

4.1 Introduction

Estimation of the intraclass correlation coefficient ρ has long been done in many different fields. It is commonly used in genetics and epidemiology to measure the degree of resemblance within the same family with respect to some biological or physiological attribute. It is used in agriculture in the analysis of variance in experiments and in survey sampling to assess homogeneity of clusters. Over the years, different forms of estimators of intraclass correlation in the one-way random effect model have been suggested. Pearson [39] proposed the pairwise intraclass correlation. Fieller & Smith [11] generalized the ANOVA estimator of Fisher [12] to the unbalanced situation. Karlin et al. [27] suggested a weighted product-moment estimator. More recently, Keen [28]

introduced a method of combining pairs of weighted estimators of intraclass correlation that is fully efficient over nearly the complete range of values of the intraclass parameter. However, none of these estimators adjust for concommittant variables, or so-called covariates. Srivastava & Ng [65] included the covariates in their estimation procedure with the composite estimation approach. Their estimator is not as efficient as the maximum likelihood estimator as noted by the authors [65]. In this chapter, a new estimator of intraclass correlation under the nested-error regression model is proposed. An approximation of the mean square error of this estimator is also derived. Monte Carlo simulations reveal that this estimator has a better efficiency profile than those due to either the maximum likelihood method or the method of fitting constants (Henderson's method 3) based on comparison of their mean square errors for normal, uniform, exponential and double exponential distributions. An example is given in which a new estimator is applied in the study of blood pressure from a survey of cardiovascular disease in 4 municipalities within the area of a study in Finnmark, northern Norway, in 1997 as reported by Brenn [5]. In this example, the intraclass correlation is employed as an estimator of the heritability of systolic blood pressure after taking its linear relationship with diastolic blood pressure into account.

4.2 The Models

Consider the same random intercept model with random intercept in section 3.2:

$$y_{ij} = \alpha_i + x_{ij1}\beta_1 + \ldots + x_{ijk}\beta_k + \epsilon_{ij}, \quad i = 1, \ldots, t, \quad j = 1, \ldots, n_i \ . \quad (4.2.1)$$

The model can be written as a nested-error regression model with a single random effect, ν_i, for the ith group

$$y_{ij} = \mathbf{x}'_{ij}\boldsymbol{\beta} + \nu_i + \epsilon_{ij}, \qquad i = 1, \ldots, t, \quad j = 1, \ldots, n_i \ . \quad (4.2.2)$$

As in section 3.2 the random effects effects $\{\nu_i\}$ are assumed to be identically distributed with mean 0 and variance σ^2_ν. The random effect $\{\nu_i\}$ and the error term $\{\epsilon_{ij}\}$ are mutually independent. Their variances σ^2_ν and σ^2_ϵ can be estimated with the same estimators in chapter 3 by

$$s^2_\nu = \frac{\sum_{i=1}^{t} \omega_i(\hat{\alpha}_i - \hat{\alpha})^2 - s^2_\epsilon \mathrm{tr}\left\{ \left[\Omega - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}'\right] \mathbf{D}_\alpha (\mathbf{X}'_\alpha \mathbf{X}_\alpha)^{-1} \mathbf{D}'_\alpha \right\}}{(t-1)\left[\overline{\omega} - \frac{s^2_\omega}{t\,\overline{\omega}}\right]} \quad (4.2.3)$$

and

$$s^2_\epsilon = \frac{\mathbf{y}'\left[\mathbf{I} - \mathbf{X}_\alpha \left(\mathbf{X}'_\alpha \mathbf{X}_\alpha\right)^{-1} \mathbf{X}'_\alpha\right] \mathbf{y}}{N - \mathrm{r}(\mathbf{X}_\alpha)} \quad (4.2.4)$$

respectively. All the notations and symbols are defined exactly as in chapter 3.

4.3 Estimation of Intraclass Correlation

With the random intercept model (4.2.1), or equivalently the nested-error regression model (4.2.2), the intraclass correlation coefficient ρ is given by

$$\rho = \frac{\sigma_\nu^2}{\sigma_\nu^2 + \sigma_\epsilon^2} \,. \tag{4.3.1}$$

It is natural to express an estimator of the intraclass correlation by substituting σ_ϵ^2 and σ_ν^2 with their estimators.

Let the estimators, $s_{\nu 1}^2$ and $s_{\nu 2}^2$, be of the same class as s_ν^2 in equation (4.2.3) with different weighting schemes, $\boldsymbol{\omega}'_{(1)} = \left(\omega_1^{(1)}, \omega_2^{(1)}, \ldots \omega_t^{(1)}\right)$ best for large ρ and $\boldsymbol{\omega}'_{(2)} = \left(\omega_1^{(2)}, \omega_2^{(2)}, \ldots \omega_t^{(2)}\right)$ best for small ρ respectively. These weights are already defined in section 3.4. Define $r_1 = s_{\nu 1}^2/(s_{\nu 1}^2 + s_\epsilon^2)$ and $r_2 = s_{\nu 2}^2/(s_{\nu 2}^2 + s_\epsilon^2)$. A proposed shrinkage estimator of ρ by an algebraic combination of r_1 and r_2 is given by

$$r_c \;=\; (1-a)r_1 + a\, r_2 \tag{4.3.2}$$

where a is the weight for the combination. The choice of a affects the value of the mean square error of r_c and so a reasonable criterion would be choosing a so that the estimator of r_c has the smallest mean square error. In Appendix A.9, it is shown that an approximation to the mean square error of r_h, $\mathrm{MSE}\left[r_h\right]$, for $h = 1, 2$ is given by

$$\frac{\sigma_\epsilon^4 \mathrm{Var}\left[s_{\nu h}^2\right] - 2\,\sigma_\nu^2 \sigma_\epsilon^2 \mathrm{Cov}\left[s_{\nu h}^2, s_\epsilon^2\right] + \sigma_\nu^4 \mathrm{Var}[s_\epsilon^2]}{(\sigma_\nu^2 + \sigma_\epsilon^2)^{-4}} \,, \tag{4.3.3}$$

and from Appendix A.10, an approximation to $\mathrm{E}\Big[(r_1-\rho)(r_2-\rho)\Big]$ is given by

$$\frac{\sigma_\epsilon^4\mathrm{Cov}\big[s_{\nu 1}^2,s_{\nu 2}^2\big]+\sigma_\nu^4\mathrm{Var}\big[s_\epsilon^2\big]-\sigma_\nu^2\sigma_\epsilon^2\Big(\mathrm{Cov}\big[s_{\nu 1}^2,s_\epsilon^2\big]+\mathrm{Cov}\big[s_{\nu 2}^2,s_\epsilon^2\big]\Big)}{\big(\sigma_\nu^2+\sigma_\epsilon^2\big)^{-4}} \tag{4.3.4}$$

where $\mathrm{Var}\big[s_\epsilon^2\big]=2\sigma_\epsilon^4\Big(N-\mathrm{r}(\mathbf{X}_\alpha)\Big)^{-1}$,

$$\begin{aligned}
\mathrm{Var}(s_\nu^2) \;=\; & \left[(t-1)\left(\overline{\omega}_{(h)}-\frac{s_{\omega(h)}^2}{t\,\overline{\omega}_{(h)}}\right)\right]^{-2} \\
& \times\left\{2\sigma_\epsilon^4\mathrm{tr}\left[\left[\left(\boldsymbol{\Omega}_{(h)}-N_{\omega(h)}^{-1}\boldsymbol{\omega}_{(h)}\boldsymbol{\omega}_{(h)}'\right)\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\right]^2\right]\right. \\
& +4\sigma_\epsilon^2\sigma_\nu^2\mathrm{tr}\left[\left(\boldsymbol{\Omega}_{(h)}-N_{\omega(h)}^{-1}\boldsymbol{\omega}_{(h)}\boldsymbol{\omega}_{(h)}'\right)^2\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\right] \\
& +2\sigma_\nu^4\mathrm{tr}\left[\left(\boldsymbol{\Omega}_{(h)}-N_{\omega(h)}^{-1}\boldsymbol{\omega}_{(h)}\boldsymbol{\omega}_{(h)}'\right)^2\right] \\
& \left.+\mathrm{Var}\big[s_\epsilon^2\big]\left(\mathrm{tr}\left[\left(\boldsymbol{\Omega}_{(h)}-N_{\omega(h)}^{-1}\boldsymbol{\omega}_{(h)}\boldsymbol{\omega}_{(h)}'\right)\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\right]\right)^2\right\},
\end{aligned} \tag{4.3.5}$$

$$\mathrm{Cov}\big[s_\epsilon^2,s_{\nu h}^2\big] \;=\; \frac{-2\sigma_\epsilon^4\mathrm{tr}\left[\left(\boldsymbol{\Omega}_{(h)}-N_{\omega(h)}^{-1}\boldsymbol{\omega}_{(h)}\boldsymbol{\omega}_{(h)}'\right)\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\right]}{(t-1)\Big(N-\mathrm{r}(\mathbf{X}_\alpha)\Big)\left(\overline{\omega}_{(h)}-\frac{s_{\omega(h)}^2}{t\,\overline{\omega}_{(h)}}\right)}, \tag{4.3.6}$$

and

$$\mathrm{Cov}\big[s_{\nu 1}^2,s_{\nu 2}^2\big] \;=\; \frac{2\,\mathrm{tr}\left[\mathbf{C}_1\mathbf{V}\mathbf{C}_2\mathbf{V}\right]+T_1T_2\mathrm{Var}\big[s_\epsilon^2\big]}{(t-1)^2\left(\overline{\omega}_{(1)}-\frac{s_{\omega(1)}^2}{t\,\overline{\omega}_{(1)}}\right)\left(\overline{\omega}_{(2)}-\frac{s_{\omega(2)}^2}{t\,\overline{\omega}_{(2)}}\right)} \tag{4.3.7}$$

with

$$\mathbf{C}_h \;=\; \mathbf{X}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\Big(\boldsymbol{\Omega}_{(h)}-N_{\omega(h)}^{-1}\boldsymbol{\omega}_{(h)}\boldsymbol{\omega}_{(h)}'\Big)\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{X}_\alpha' \tag{4.3.8}$$

and

$$T_h = \mathrm{tr}\left[\left(\boldsymbol{\Omega}_{(h)} - N_{\omega(h)}^{-1}\boldsymbol{\omega}_{(h)}\boldsymbol{\omega}_{(h)}{}'\right)\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\right] \tag{4.3.9}$$

for $h = 1, 2$. The minimum mean square error of r_c

$$\mathrm{MSE}[r_c] = \frac{\mathrm{MSE}[r_1]\ \mathrm{MSE}[r_2] - \Big(\mathrm{E}\Big[(r_1-\rho)(r_2-\rho)\Big]\Big)^2}{\mathrm{MSE}[r_1] + \mathrm{MSE}[r_2] - 2\mathrm{E}\Big[(r_1-\rho)(r_2-\rho)\Big]} \tag{4.3.10}$$

is attained when

$$a = \frac{\mathrm{MSE}[r_1] - \mathrm{E}\Big[(r_1-\rho)(r_2-\rho)\Big]}{\mathrm{MSE}[r_1] + \mathrm{MSE}[r_2] - 2\mathrm{E}\Big[(r_1-\rho)(r_2-\rho)\Big]}\ . \tag{4.3.11}$$

The composite estimator will be compared with the estimator $\tilde{\rho}$ given by

$$\tilde{\rho} \ = \ \frac{\tilde{\sigma}_\nu^2}{\tilde{\sigma}_\nu^2 + \tilde{\sigma}_e^2} \tag{4.3.12}$$

where $\tilde{\sigma}_\nu^2$ and $\tilde{\sigma}_e^2$ are the variance component estimators of Prasad & Rao [41] defined in equation (3.3.6) and (3.3.7) respectively.

For normal distributed $\{\nu_i\}$ and $\{\epsilon_{ij}\}$, from the likelihood equation, an estimating equation for ρ is given by

$$\begin{aligned} & N\left[\sum_{i=1}^{t}\left(SSI_i - \frac{\rho\, SSJ_i}{1+(n_i-1)\,\rho}\right)\right]^{-1} \\ & \times \ \sum_{i=1}^{t}\left(SSI_i - \frac{SSJ_i\,[1+(n_i-1)\rho^2]}{[1+(n_i-1)\,\rho]^2}\right) - \sum_{i=1}^{t}\frac{n_i\,(n_i-1)\rho}{1+(n_i-1)\rho} = 0 \end{aligned} \tag{4.3.13}$$

where $SSI_i = (\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta})' \mathbf{I}_i (\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta})$, $SSJ_i = (\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta})' \mathbf{J}_i (\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta})$ with $\mathbf{I}_i$ an $(i \times i)$ identity matrix and $\mathbf{J}_i$ an $(i \times i)$ square matrix with a value of 1 for each element. The maximum likelihood estimator of ρ has no closed form but the root of the estimating equation can be solved iteratively by a numerical method such as the Newton-Raphson method.

4.4 Results of Monte Carlo Simulations

Monte Carlo simulations that mimic the data set of Battese et al. [4] under the nested-error regression model with a single covariate, $y_{ij} = \alpha + \beta x_{ij} + \nu_i + \epsilon_{ij}$, were conducted to study the relative efficiency of the composite estimator r_c with respect to other intraclass correlation coefficient estimators. The simulations were done in a similar fashion as for the small-area estimation. The details of these simulations have been described in section 3.4. Ten thousand independent sets of ν_i and ϵ_{ij} with zero means and different intraclass correlation ρ from 0.00 to 0.95 were generated at increments of 0.05 to obtain 10,000 sets of y_{ij} using the covariate values of Battese et al. [4] for each of 4 distribution assumptions: normal, double-exponential, exponential and uniform.

The weights $\{\omega_i^{(1)}\}$ for the estimator $s_{\nu 1}^2$ are also equal to 1, which is thought to be best for a large intraclass correlation coefficient and the weights $\{\omega_i^{(2)}\}$ for the

estimator $s^2_{\nu 2}$ are equal to $n_i(n_i-1)$, which is thought to be best for a small intraclass correlation coefficient. This latter scheme implicitly deletes single-member families.

In the case of normally distributed random effects, let $\hat{\rho}_m$ be the maximum likelihood estimator based on the estimating equation with r_c as its initial value. A one-step maximum likelihood estimator $\hat{\rho}_{m1}$ is also studied under normal random effects. This is a version of the maximum likelihood estimator with only one iterative step when solving the estimating equation (4.3.13). We examine this estimator as well because, on the assumption of regularity conditions, it is asymptotically fully efficient.

The estimator $\hat{\rho}_{(S)}$ is constructed with the variance components estimated by the *residual* maximum likelihood estimators following the same approach as in Patterson & Thompson [37]. As described in chapter 3, the matrix $\mathbf{K}$ for the error contrasts is equal to $\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. The estimator $\hat{\rho}_{(T)}$ is constructed similarly with the *restricted* maximum likelihood estimators for the variance parameters following the approach in McCulloch & Searle [35] in which the matrix $\mathbf{K}$ for the construction of the error contrasts do not have to be specified. A good initial value for the maximum likelihood estimators is important. If the initial value is not at the vicinity of the parameter, it may require more iterative steps for the maximum likelihood estimators

to converge. In fact, there is no guarantee that the maximum likelihood estimators will converge. In our study both $\hat{\rho}_{(S)}$ and $\hat{\rho}_{(T)}$ use r_c as their initial value. The mean square errors of the intraclass correlation estimators were estimated by adding the squares of the differences of the estimates from ρ in all samples and then dividing the sum by the total number of samples in the Monte Carlo simulation. That is, for an intraclass correlation coefficient estimator ρ^*, the mean square error is calculated by

$$\text{MSE}[\rho^*] = \frac{\sum_{i=1}^{N_s}(\rho_i^* - \rho)^2}{N_s} \tag{4.4.1}$$

where N_s is the number of simulated samples, which is equal to 10,000 for our simulation studies.

Figure 4.1 shows the bias of each of the intraclass correlation estimators in the example of a small sample for different distribution of the random effects. In Figure 4.1 (a) and (b), with normally distributed random effects, both maximum likelihood estimators $\hat{\rho}_m$ and $\hat{\rho}_{m1}$ are more biased than the other estimators which have similar degrees of biasedness. For the double exponential distribution and the exponential distribution which have long tails, as shown in Figure 4.1 (c) and (d) respectively, the bias of the estimators attains extrema greater than those of the uniform distributions in Figure 4.1 (e). Nevertheless under these four distributions, $\tilde{\rho}$ and r_c have similar small degrees of bias. However, r_c is slightly better than $\tilde{\rho}$ when the true intraclass correlation coefficient is large.

The maximum likelihood estimators $\hat{\rho}_m$ and $\hat{\rho}_{m1}$ and the intraclass correlation estimators $\hat{\rho}_{(S)}$ and $\hat{\rho}_{(T)}$ with residual and restricted maximum likelihood method respectively were only calculated for the normal distribution. This is because the likelihood function was based on the normal distribution therefore these estimators are not valid for the cases with random effects having distributions other than normal.

Figure 4.2 shows the relative efficiencies of the intraclass correlation estimators with respect to $\hat{\rho}_m$ for different values of the parameter. For $\rho > 0.15$, the estimator $\hat{\rho}_{(T)}$ with the restricted maximum likelihood estimation method of McCulloch & Searle [35] is more efficient than $\hat{\rho}_m$. For $\rho > 0.25$, $\tilde{\rho}$, r_c and the estimator $\hat{\rho}_{(S)}$ with the residual maximum likelihood estimation method of Patterson & Thompson [37] are more efficient than $\hat{\rho}_m$. Although the efficiency of $\tilde{\rho}$ and $\hat{\rho}_{(S)}$ start to drop after they attain their highest level around $\rho = 0.55$, the efficiency of r_c continues to increase as ρ increases. The proposed estimator r_c has similar efficiency as $\hat{\rho}_{(T)}$ after $\rho > 0.50$. However, r_c is more efficient than $\hat{\rho}_{(T)}$ and $\hat{\rho}_{(S)}$ when the intraclass correlation is less than 0.10. In fact, over the whole simulation, r_c is more efficient than the residual maximum likelihood estimator $\hat{\rho}_{(S)}$. Compared to $\tilde{\rho}$ and $\hat{\rho}_{(S)}$, r_c has a better efficiency profile. Figure 4.3 shows the relative efficiency of r_c with respect to $\tilde{\rho}$ for the normal and three non-normal distributions we considered. The closer the value of ρ to 0 or 1 is, the higher the relative efficiency of r_c. When the random

effects have double exponential distribution or exponential distribution, r_c is always more efficient than $\tilde{\rho}$. The relative efficiency was observed to be as high as 116% and 124% at $\rho = 0.00$ under the double exponential distribution and the exponential distribution, respectively in Figure 4.3.

Srivastava & Ng [65] also studied the composite estimator that was reported to have relatively low asymptotic variances for both small and large intraclass correlation compared to the intraclass estimators developed by Fuller & Battese [13] and by Srivastava [60]. However, the model they studied is a no-intercept model with normally distributed random effects only. Whereas in this thesis, the model being studied is more general with an intercept and with the random effects not restricted to a normal distribution. The estimator $\tilde{\rho}^{SN}$ of Srivastava & Ng [65] is a combination of two intraclass correlation coefficient estimators t_0 and t_1: they are supposed to have low asymptotic variance when the intraclass correlation coefficients are $\rho = 0$ and $\rho = 1$ respectively. The estimator $\tilde{\rho}^{SN}$ is an self-adjusting estimator because $\tilde{\rho}^{SN} = (1 - \tilde{\rho}^{SN})\, t_0 + \tilde{\rho}^{SN}\, t_1$. In the perfect cases, $\tilde{\rho}^{SN}$ reduces to its special case t_0 or t_1 depending on the true value of ρ. The composite estimator r_c proposed here introduces a cutting edge approach by making use of the intraclass correlation coefficient estimators r_1 and r_2 which are best for $\rho = 1$ and $\rho = 0$ respectively and adjusting the combination weight a of r_1 and r_2 to attain the minimum mean square error possible.

The composite estimator $\tilde{\rho}^{SN}$ of Srivastava & Ng [65] is not as efficient as the maximum likelihood estimator for all values of ρ as noted by Srivastava & Ng [65]. This is not the case with the composite estimator r_c in this thesis because, in the Monte Carlo simulation study, r_c is more efficient than the maximum likelihood estimator when the intraclass correlation is larger than 0.30. (see. Figure 4.2.)

4.5 Example: The cardiovascular disease study in Finnmark, Norway

Brenn [5] reports the study of cardiovascular disease in 4 municipalities within the area of Finnmark, northern Norway, in 1977. The survey included a self-administered questionnaire and measurements of weight, height, blood pressure, non-fasting blood glucose and serum lipids.

In this study, the general population between 23 and 52 years of age as well as a 10% random sample of those aged 20–22 were invited to participate. Of the 6087 men and women invited, 4883 actually participated. From this sample, Brenn [5] considered only selected individuals who were part of nuclear families in which the spouse pairs, if present, aged between 42 and 52 and siblings aged 20–34 years. These restrictions yielded 575 families with 1377 members. For our analysis, we eliminated the parents from consideration, which leaves 348 sibships comprising 683 siblings. We

then eliminated any siblings without measurements for both systolic and diastolic pressure, with a resulting sample of 682 siblings in 348 sibships, of which 120 sibships consist of a single offspring.

For purposes of confidentiality, the data we received were in the form of residuals after adjusting for the following covariates: gender, age, age^2, age^3, sex $\times$ age, sex $\times$ age^2, sex $\times$ age^3, time since last meal (graded 1–5 with increasing time), ethnic group origin (four dichotomous variables generated on the basis of grandparents' ancestry), seasonality (five dichotomous variables denoting the month during which the survey took place), current pregnancy (no, yes), and menopause started (no, uncertain, yes).

In this example, the combined estimator $\tilde{\rho}_c$ intraclass correlation is employed as an estimator of the heritability of systolic blood pressure after taking its linear relationship with diastolic blood pressure into account. The intercept and its standard error are estimated to be -0.003 and 0.039, respectively. The slope and its standard error are estimated to be 0.421 and 0.035, respectively. The combined estimate $\tilde{\rho}_c$ of the intraclass correlation is found to be 0.213 for systolic blood pressure with a root mean square error estimated to be 0.052.

On the other hand, after accounting for the linear relationship with systolic pressure the combined estimate $\tilde{\rho}_c$ of the intraclass correlation coefficient for diastolic blood pressure is found to be 0.194 with a root mean square error estimated to be 0.052. The intercept and its standard error is estimated to be -0.001 and 0.038, respectively. The slope and its standard error is estimated to be 0.421 and 0.035, respectively. Note that the correct estimate of heritability is very complicated. However, it can be approximated by 2 times the intraclass correlation coefficient estimate. With the composite estimator r_c, the approximated heritability estimates of the systolic and diastolic blood pressures are 0.426 and 0.388, respectively. As a comparison, the estimates reported in Brenn [5] are 0.48 and 0.35, respectively.

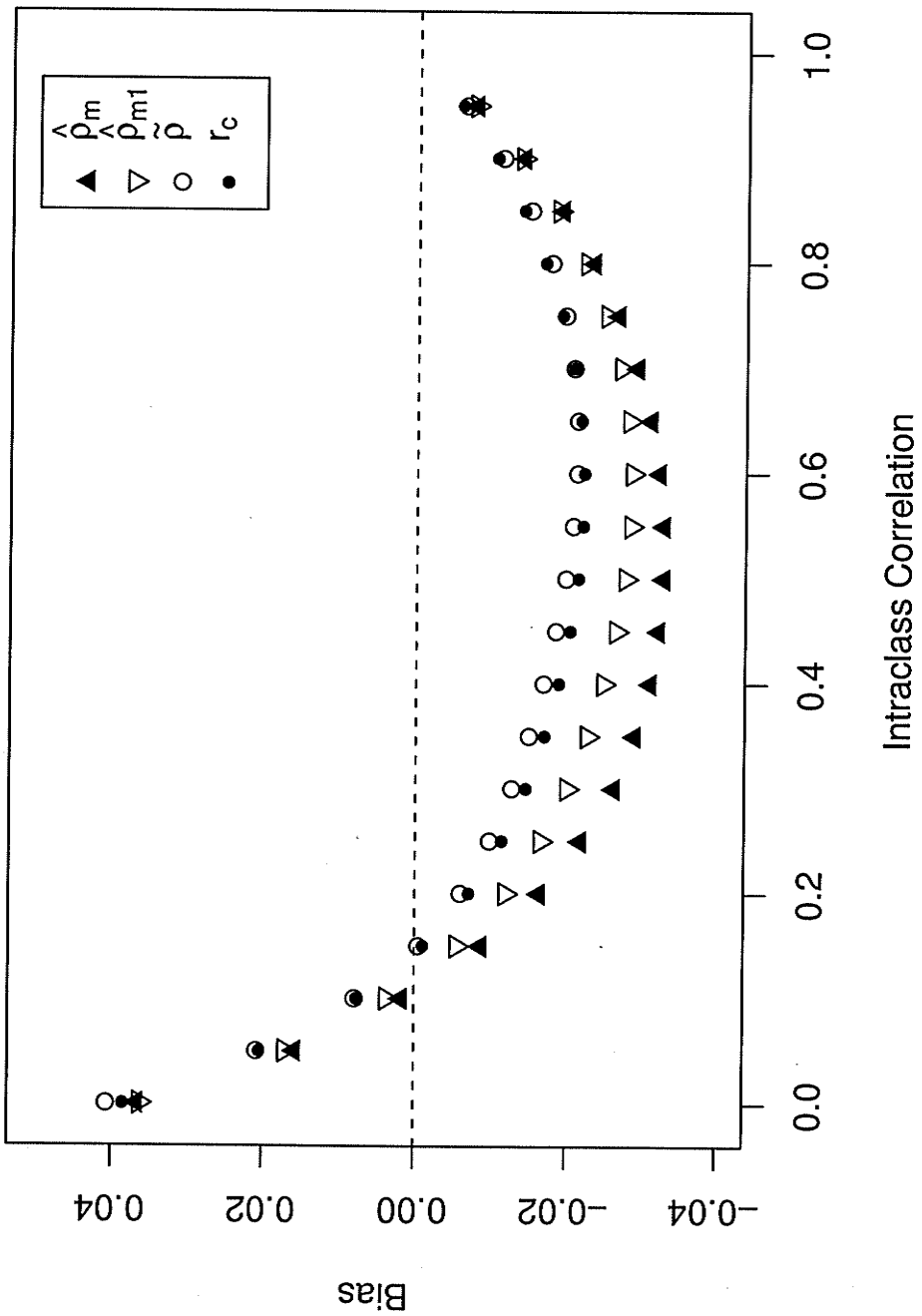
4.6 Conclusion

As an unbiased estimator of the intraclass correlation does not exist in a closed form, all estimators of ρ studied here are biased. Moreover, a biased maximum likelihood estimator does not exist in closed form. The best estimator is one with smallest mean square error. Under the normal distribution with small value of ρ, the maximum likelihood estimator $\hat{\rho}_m$ seems to be the most efficient one among all estimators studied here in small sample Monte Carlo simulations. Nevertheless, all maximum likelihood estimators assume r_c as an initial value. Between $\tilde{\rho}$ and r_c, the estimator r_c has the preferred mean square error profile. When $\tilde{\rho}$ has the smaller mean square error in the

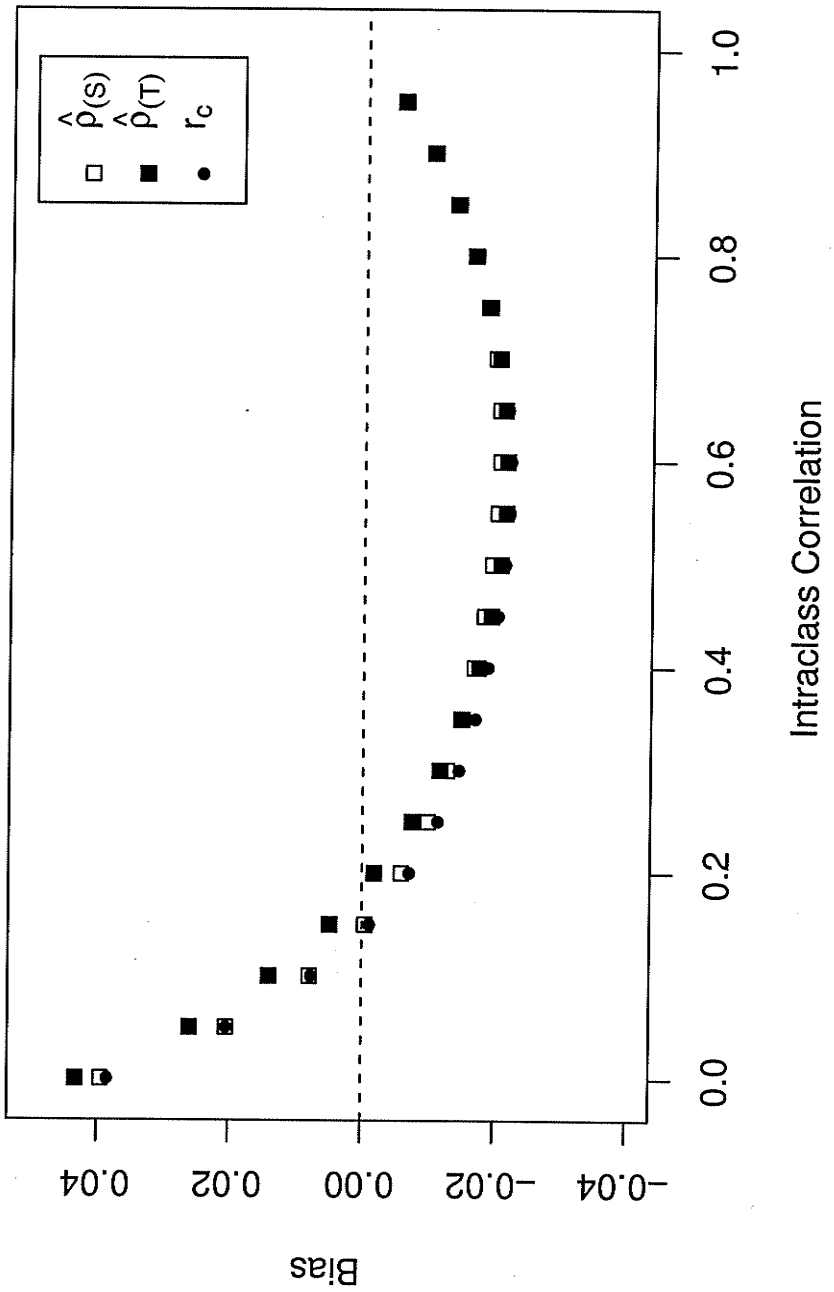
simulations, as in the case for middle range intraclass correlation only for the normal and uniform distributions, the advantage appears to be negligible based on the Monte Carlo simulations. For small intraclass correlations with normally distributed data, as sometimes encountered in survey sampling or more often: biology as in the study of complex diseases in humans, the maximum likelihood estimator with initial values from r_c is recommended. Overall, r_c is a better choice over its competitors with acceptable performance in the case of non-normality.

Figure 4.1: Bias of intraclass correlation estimators based on random effects with four different distributions. (a) - (e)

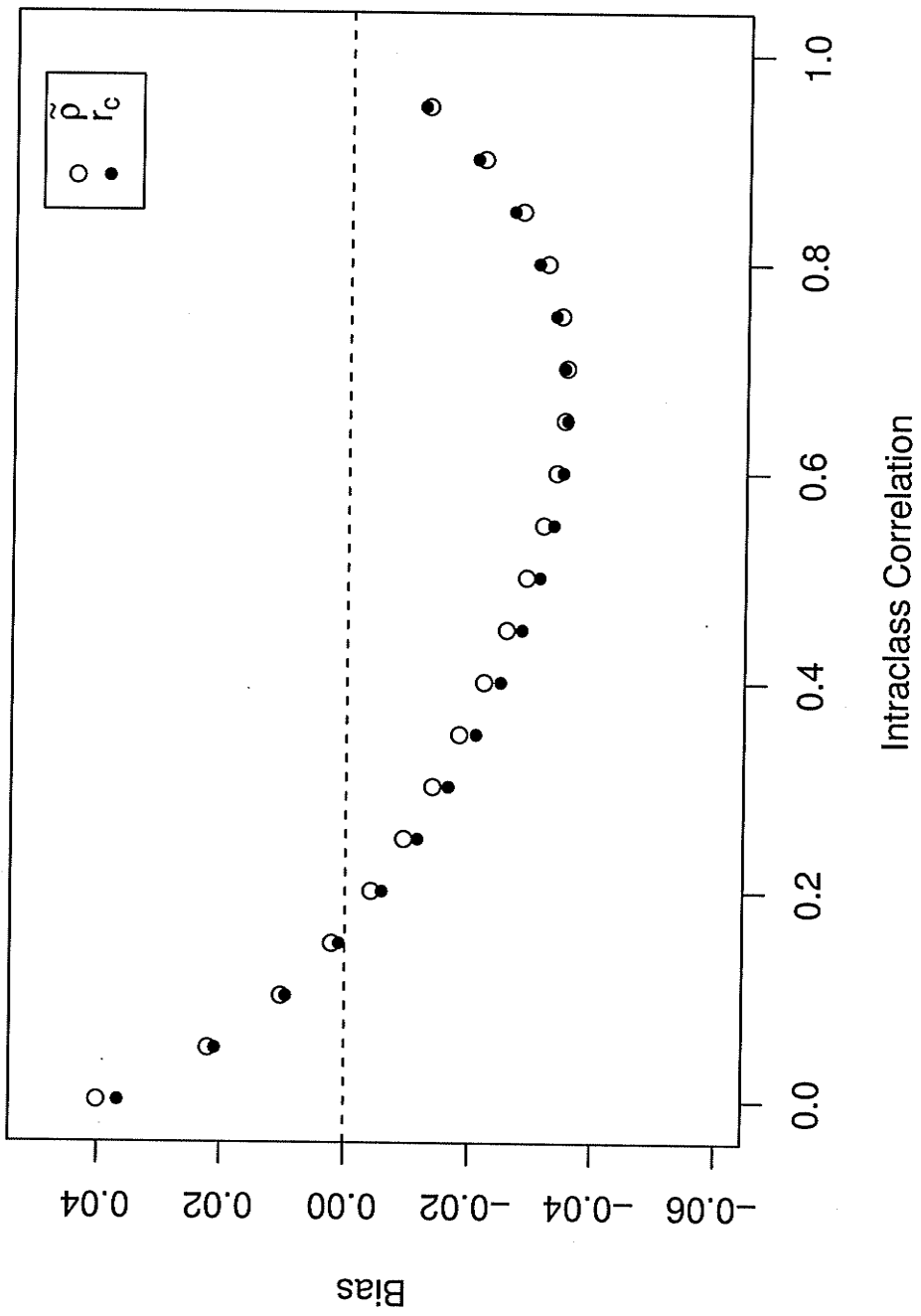
(a) Normal Distribution with $\hat{\rho}_m$, $\hat{\rho}_{m1}$, $\tilde{\rho}$ and r_c



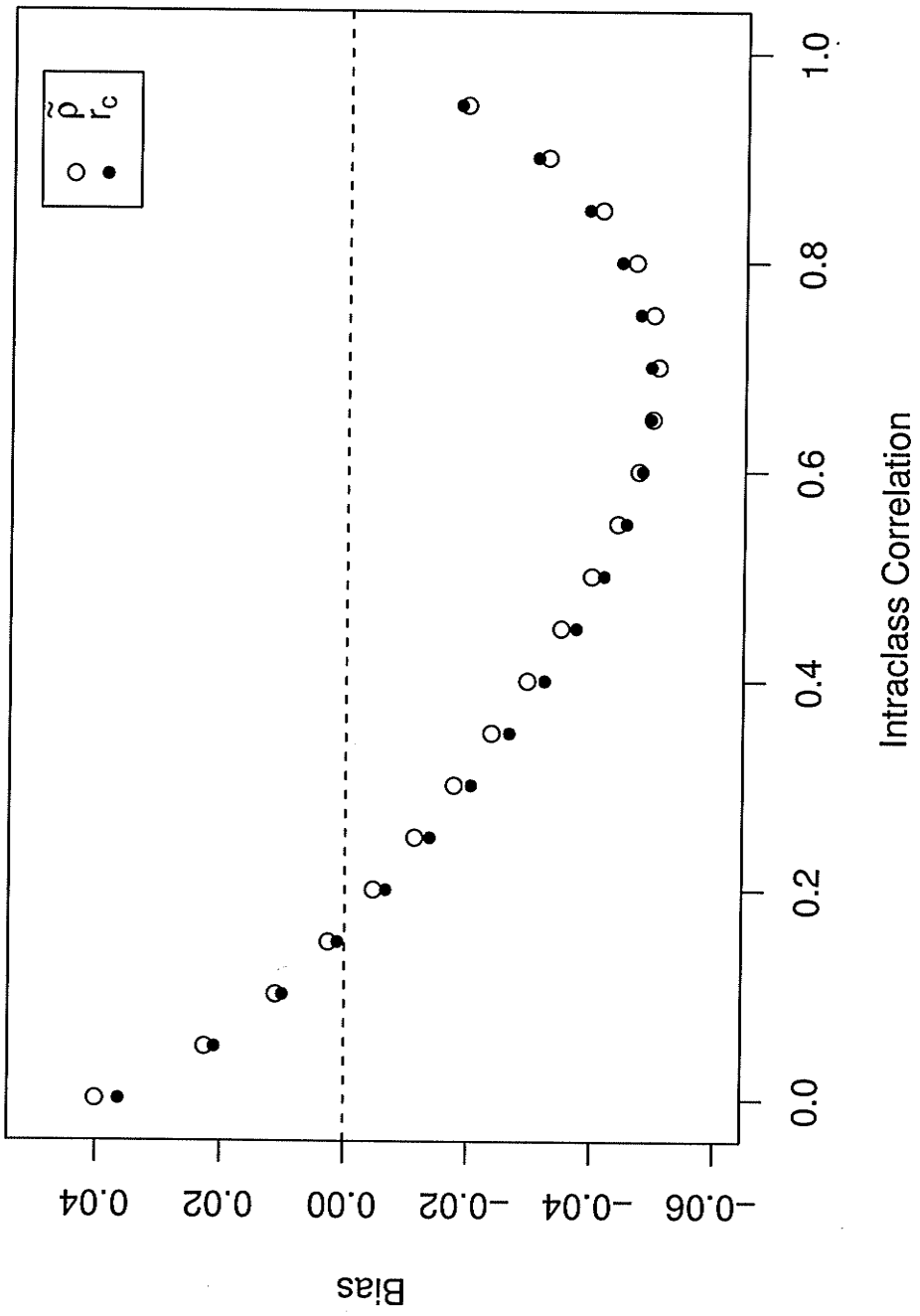
(b) Normal distribution with $\hat{\rho}_{(S)}$, $\hat{\rho}_{(T)}$ and r_c



(c) Double Exponential distribution



(d) Exponential Distribution



(e) Uniform Distribution

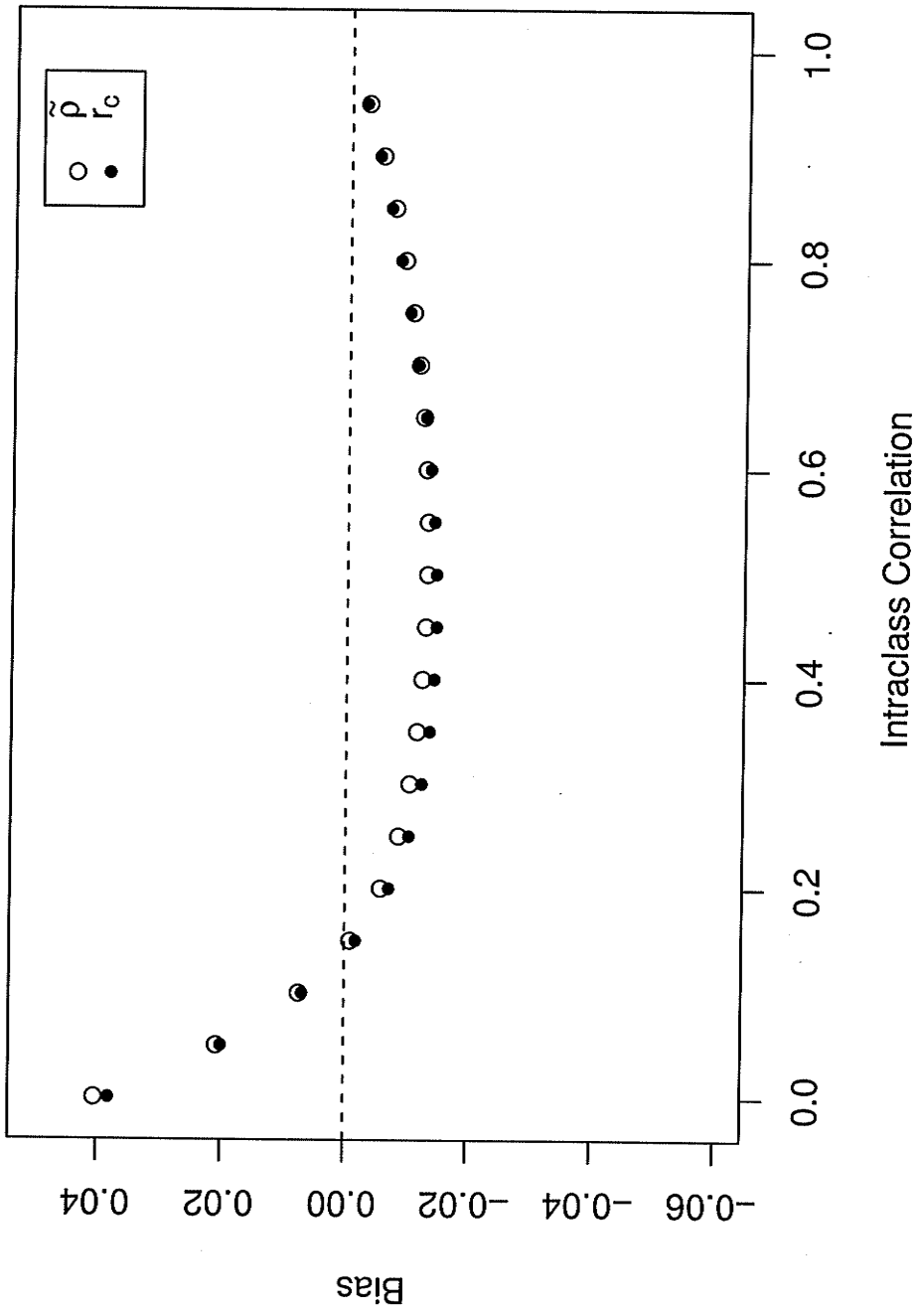
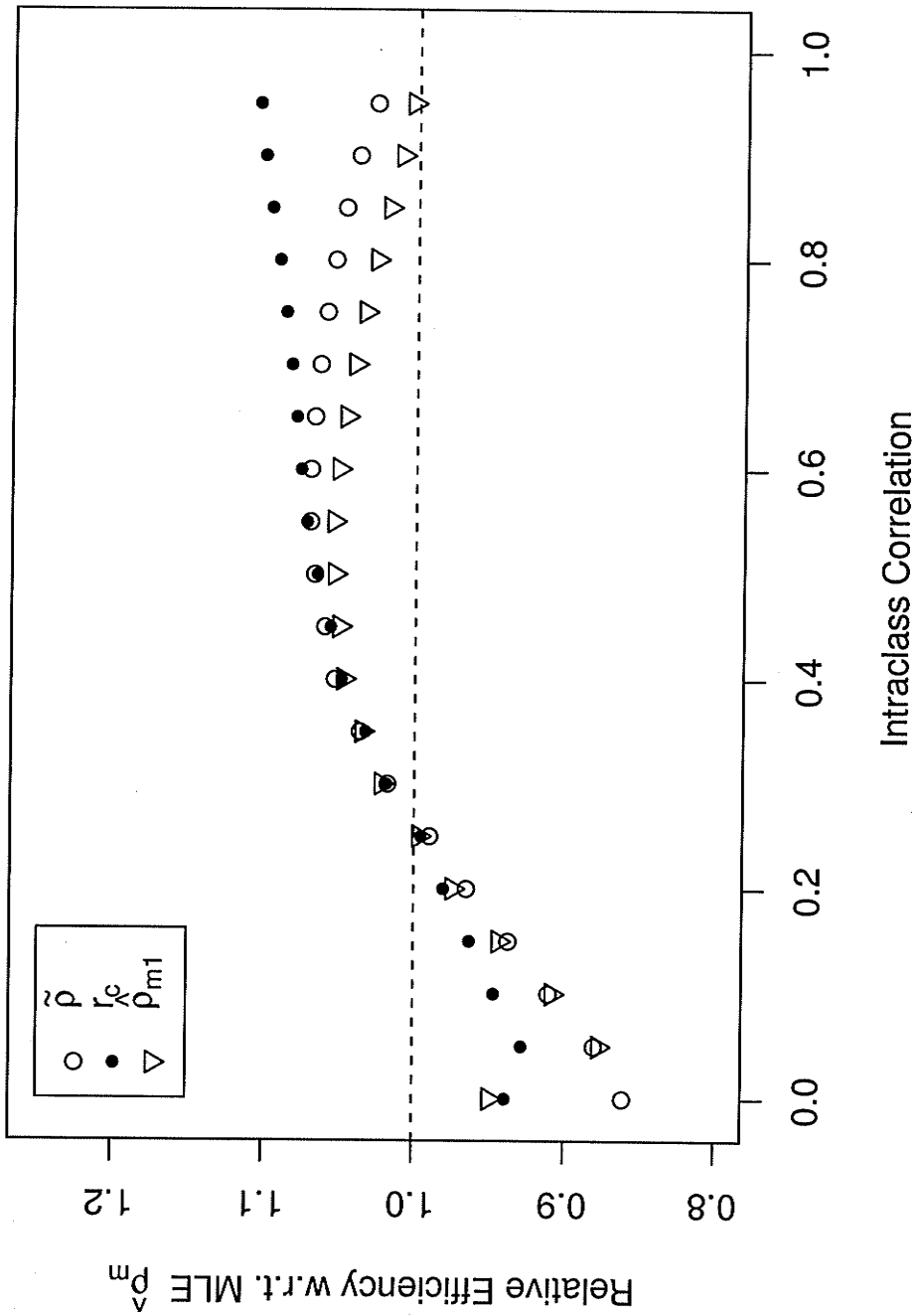


Figure 4.2: Relative efficiencies of $\tilde{\rho}$, r_c, $\hat{\rho}_{m1}$, $\hat{\rho}_{(S)}$ and $\hat{\rho}_{(T)}$ with respect to $\hat{\rho}_m$ with normally distributed $\{\nu_i\}$ and $\{\epsilon_{ij}\}$ (a) - (b)

(a) Relative efficiencies of $\tilde{\rho}$, r_c and $\hat{\rho}_{m1}$



(b) Relative efficiencies of r_c, $\hat{\rho}_{(S)}$ and $\hat{\rho}_{(T)}$

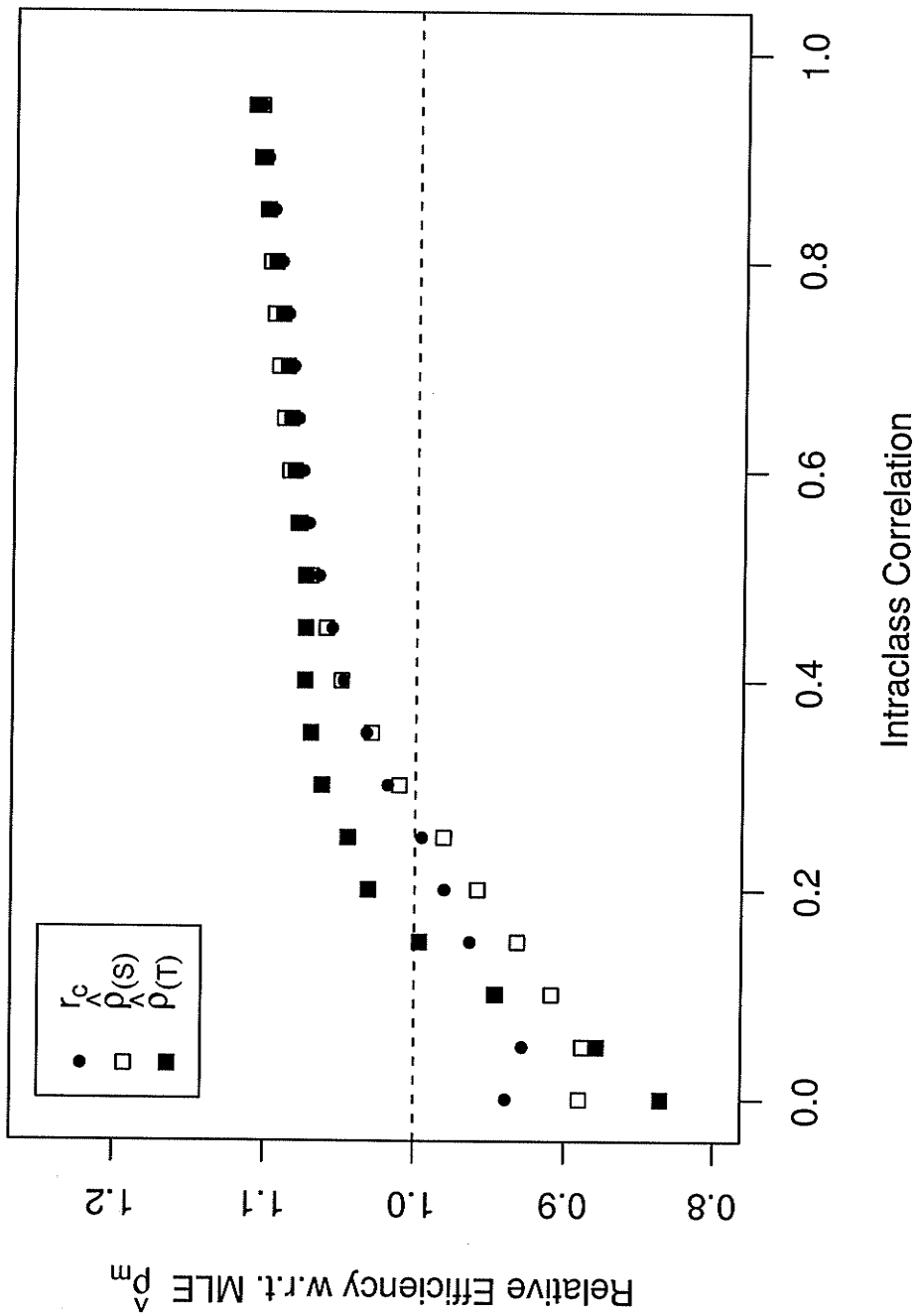
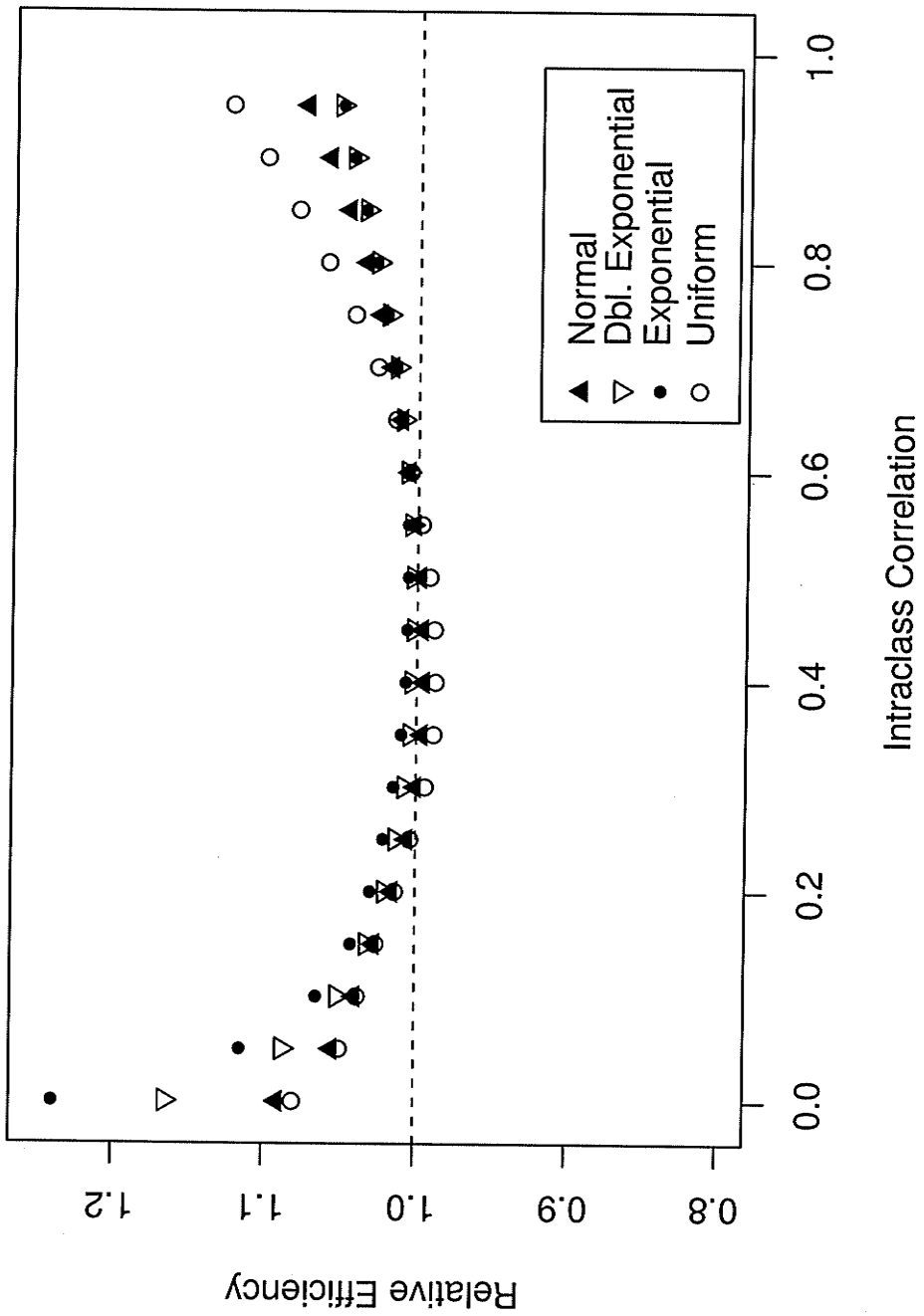


Figure 4.3: Relative efficiencies of r_c with respect to $\tilde{\rho}$ with different distributions of the random effects $\{\nu_i\}$ and $\{\epsilon_{ij}\}$



Chapter 5

CONCLUSION AND FUTURE RESEARCH DIRECTIONS

5.1 Conclusion

Statisticians are trained to deal with uncertainties. It does not necessarily mean that our responsibility is to eliminate the uncertainties. At least with our knowledge, we are working to quantify the uncertainties. One major task of statisticians is to formulate a statistical model for a situation or a phenomenon. The most popular model which many people know about is the linear regression model, which to many is almost like a magic that can be applied to a wide variety of occasions. Statistical models can be used for predicting or estimating the components of the situation. In this thesis, we showed the nested-error regression model is a special case of the mixed linear model used to describe a situation in which two, or more, random components are involved in a situation. In the situation considered, one of the random components

comes from the global variation of the data that affects each of the individuals in the same way and the other comes locally which affects the individuals within the same region, areas, group or family.

Usually, a single estimator is used to estimate the parameter of interest, in contrast to a composite estimator, which could be a multi-stage estimator involving more than one stage of estimation or a (linear) combination of estimators. For example, the EBLUP small-area estimator is a linear combination of a global estimator for the overall mean and an estimator for the local random effect. The latter is an adjustment to the former for a more accurate estimation of the local mean. The adjustment is only a prediction of the random effect within a local area.

The method applied in this thesis for estimation of small-area means and intraclass correlation coefficients is a composite estimator that is a linear combination of two estimators each of which can be used to estimate the parameter of interest. In the small-area estimation, it also produces an indirect estimator through a combination of two EBLUPs for estimating small areas. However, if this is the case, why do we need two estimators which serve the same purpose? This approach follows a successful approach by Keen [28], Srivastava [61], and Srivastava & Ng [65] which used a composite estimator to efficiently estimate the intraclass correlation. The

estimator is a linear combination of two intraclass correlation estimators; one is best for large intraclass correlation and the other is best for small intraclass correlation. In reality, it is unknown whether the intraclass correlation is large or small. It is likely that an estimator efficient for small intraclass correlation is not efficient for large intraclass correlation, and vice versa. One advantage of the method presented here is that it yields an optimal estimate of the parameter by self-adjusting the weight of the combination. The composite estimator will shift the weight to the estimator most appropriate in the extreme case. This has been shown in the Monte Carlo simulation studies of the small-area estimation and intraclass correlation estimation.

Among all the estimators, the composite estimator yields a better efficiency profile in terms of the mean square error than its competitors in small-area estimation for all four distributions of the random effects. The composite estimator $\tilde{\mu}^c$ is relatively more efficient than the estimator $\tilde{\mu}$ of Prasad & Rao [41] and the estimator $\tilde{\mu}^{ps}$ of You & Rao [67] especially when the intraclass correlation is almost zero for the normal and three non-normal distributions. In survey sampling, this is important for the cluster design in which the intraclass correlation coefficient is designed to be zero. An efficient estimator of the intraclass correlation coefficient provides a reliable check for the validity of the cluster design of the sample. When the random effects are normally distributed, the composite estimator $\tilde{\mu}^c$ is slightly more efficient than

the estimator $\hat{\mu}^{Vm}$ (with variance components estimated by the standard maximum likelihood estimator, which actually uses $\tilde{\mu}^c$ as its initial value). This might be due to the fact that the maximum likelihood estimators of the variance components are more biased. For small intraclass correlation coefficients, $\tilde{\mu}^c$ has smaller mean square error than the estimators with variance components estimated by the residual maximum likelihood method, $\tilde{\mu}^{(S)}$, and the restricted likelihood method, $\tilde{\mu}^{(T)}$. For moderate intraclass correlation coefficients, the residual or restricted maximum likelihood method estimators are slightly more efficient than the composite estimator $\tilde{\mu}^c$. However, $\tilde{\mu}^c$, $\tilde{\mu}^{(S)}$ and $\tilde{\mu}^{(T)}$ have similar efficiencies for large intraclass correlation coefficient. It is important to note that $\hat{\mu}^{Vm}$, $\tilde{\mu}^{(S)}$ and $\tilde{\mu}^{(T)}$ are more computer intensive than the composite estimator $\tilde{\mu}^c$ and they need an accurate estimate as their initial value which, in this study, has been the estimate by $\tilde{\mu}^c$.

For intraclass correlation estimation under the normal, double exponential, exponential and uniform distributions of the random effects, the composite estimator r_c is more efficient than $\tilde{\rho}$, which uses Prasad and Rao's estimated variance components. When the random effects are normally distributed, it has been shown that r_c is also competitive to $\hat{\rho}_{(T)}$ and $\hat{\rho}_{(S)}$. For the other three distributions, we only compared r_c with $\tilde{\rho}$ because the maximum likelihood estimators, including the residual and restricted MLEs, are not appropriate due to their assumption of the normal

distribution.

This thesis provides a cohesive approach to the estimation of small-area means and the intraclass correlation coefficient. Prasad & Rao [41] focused on the estimation of small-area means by borrowing information from other small areas while Srivastava & Ng [65] makes use of a composite estimator for the estimation of the intraclass correlation in an intercept-free regression model. This thesis combines the techniques of Keen [28] and Srivastava & Ng [65] for a composite estimator suitable for the estimation of small-area mean and intraclass correlation coefficient. The results obtained by the proposed estimation method yields a relatively more efficient estimator than that of either Prasad & Rao [41] for small-area estimation or Srivastava & Ng [65] for the intraclass correlation coefficient estimation. My work here bridges the survey sampling literature and the biometric literature.

5.2 Future Directions

The composite estimation approach can be used in both small-area estimation and intraclass correlation estimation to yield satisfactory results. This conclusion is based on small sample Monte Carlo simulations. Usually, we are not only interested in the biases of the estimators but also the mean square errors of these estimators. Knowledge of the mean square error of the estimator permits confidence interval

estimation and hypothesis testing and further allows the assessment of the reliability of the estimator itself. A large mean square error implies that the estimator has large variability about the true value of the parameter. Typically, the true value of the parameter is unknown, otherwise estimation would not be needed. In this thesis, using the delta method, we have provided the mathematical formula for the calculation of the estimated mean square errors of the composite estimators for the small-area estimation and for the intraclass correlation estimation. These formulae provide a starting point for higher order approximation of the mean square errors. These will be studied in the future research. More advanced statistical details may be needed for these estimators to give more accurate estimates for the mean square errors in finite populations. This, for example, may be achieved with second-order terms in the Taylor series expansion of the estimators.

In this thesis, the composite estimator, which is a convex combination of two estimators, was applied to the nested-error regression model. The idea of the combination is to make use of two estimators to achieve a more stable and more efficient estimation of the parameter. Each of the two estimators is known to be efficient for extreme situations. In our case, one is good for large intraclass correlation and the other for small intraclass correlation. The composite estimation method is also a novel technique in the context of indirect estimation for the small-area mean in which

the information from other areas are pooled together for more efficient estimation. This is particularly important when the group sizes are small. In the future research, the same technique will be applied to more complicated statistical models such as with non-linear models.

Most studies of the estimators assume infinite population size. We, however, have expended considerable effort in studying the small sample properties of our estimators and the estimators proposed by others. In survey sampling, the population size is finite and we need further to consider the impact of this fact. You & Rao [67] has introduced their estimator to achieve this objective. However, their estimator is not as efficient compared to other small-area mean estimators especially the composite estimator proposed here. I propose to develop a design-based estimator using the composite estimator for more efficient and design consistent estimation. This is not restricted to small-area estimation, but can also be applied to the intraclass correlation in the presence of genetic heterogeneity due to finite sub-population domains.

Last but not least, a future research project will include the study of the accuracy of estimation of the mean square error itself. This is important because it will allow us to evaluate the reliability of a mean square error estimator. With high mean square error reliability, we will be able to perform reliable hypothesis testing on the parameters.

Appendix A

TECHNICAL DETAILS

A.1 Proof of the Unbiasedness of s_ϵ^2

Note that the estimator of σ_ϵ^2 in equation (3.3.2) can be written as

$$s_\epsilon^2 = \mathbf{y}'\mathbf{A}\mathbf{y}$$

where

$$\mathbf{A} = \frac{\mathbf{I} - \mathbf{X}_\alpha (\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1} \mathbf{X}_\alpha'}{N - \mathrm{r}(\mathbf{X}_\alpha)} . \tag{A.1.1}$$

The expectation of s_ϵ^2 is then given by

$$\mathrm{E}(s_\epsilon^2) = \mathrm{tr}[\mathbf{A}\mathbf{V}] + \boldsymbol{\theta}_o'\mathbf{X}_\alpha'\mathbf{A}\mathbf{X}_\alpha\boldsymbol{\theta}_o \tag{A.1.2}$$

where $\mathbf{V} = \mathrm{Var}(\mathbf{y}) = \sigma_\epsilon^2\mathbf{I} + \sigma_\nu^2\mathbf{X}_\alpha\mathbf{D}_\alpha'\mathbf{D}_\alpha\mathbf{X}_\alpha'$ and $\boldsymbol{\theta}_o = (\beta_1, \ldots, \beta_k, \alpha\mathbf{1}')'$. Note that $\boldsymbol{\theta}_o'\mathbf{X}_\alpha'\mathbf{A}\mathbf{X}_\alpha\boldsymbol{\theta}_o = 0$ and

$$\left[\mathbf{I} - \mathbf{X}_\alpha (\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1} \mathbf{X}_\alpha'\right] \mathbf{X}_\alpha\mathbf{D}_\alpha'\mathbf{D}_\alpha\mathbf{X}_\alpha' = \mathbf{X}_\alpha\mathbf{D}_\alpha'\mathbf{D}_\alpha\mathbf{X}_\alpha' - \mathbf{X}_\alpha\mathbf{D}_\alpha'\mathbf{D}_\alpha\mathbf{X}_\alpha' = \mathbf{0} .$$

Hence, s_ϵ^2 is unbiased because

$$\mathrm{tr}(\mathbf{AV}) = \frac{\sigma_\epsilon^2 \,\mathrm{tr}\left(\mathbf{I} - \mathbf{X}_\alpha \left(\mathbf{X}_\alpha'\mathbf{X}_\alpha\right)^{-1}\mathbf{X}_\alpha'\right)}{N - \mathrm{r}(\mathbf{X}_\alpha)} = \sigma_\epsilon^2 \,. \tag{A.1.3}$$

A.2 Proof of the Unbiasedness of s_ν^2

With the expression of s_ν^2 in equation (2.6.5), the expectation of s_ν^2 is given by

$$\begin{aligned}\mathrm{E}\left(s_\nu^2\right) = & \left[(t-1)\left(\overline{\omega} - \frac{s_w^2}{t\,\overline{\omega}}\right)\right]^{-1} \left\{\mathrm{E}\left(\sum_{i=1}^{t}\omega_i(\hat{\alpha}_i - \hat{\alpha})^2\right)\right. \\ & \left. - \sigma_\epsilon^2 \,\mathrm{tr}\left((\boldsymbol{\Omega} - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\right)\right\}\end{aligned} \tag{A.2.1}$$

because s_ϵ^2 is an unbiased estimator of σ_ϵ^2. Consider the sum

$$\sum_{i=1}^{t}\omega_i(\hat{\alpha}_i - \hat{\alpha})^2 = \mathbf{y}'\mathbf{C}\mathbf{y}$$

where

$$\mathbf{C} = \mathbf{X}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\left(\boldsymbol{\Omega} - N_\omega{}^{-1}\boldsymbol{\omega}\boldsymbol{\omega}'\right)\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{X}_\alpha' \,. \tag{A.2.2}$$

The expectation of $\mathbf{y}'\mathbf{C}\mathbf{y}$ is given by

$$\begin{aligned}\mathrm{E}\left(\sum_{i=1}^{t}\omega_i(\hat{\alpha}_i - \hat{\alpha})^2\right) &= \boldsymbol{\theta}_o{}'\mathbf{X}_\alpha'\mathbf{C}\mathbf{X}_\alpha\boldsymbol{\theta}_o + \mathrm{tr}\left(\mathbf{CV}\right) \\ &= \boldsymbol{\theta}_o{}'\mathbf{D}_\alpha'\left(\boldsymbol{\Omega} - N_\omega{}^{-1}\boldsymbol{\omega}\boldsymbol{\omega}'\right)\mathbf{D}_\alpha\boldsymbol{\theta}_o \\ &\quad + \sigma_\epsilon^2 \,\mathrm{tr}\left((\boldsymbol{\Omega} - N_\omega{}^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\right) \\ &\quad + \sigma_\nu^2 \,\mathrm{tr}\left(\mathbf{C}\mathbf{X}_\alpha\mathbf{D}_\alpha'\mathbf{D}_\alpha\mathbf{X}_\alpha'\right) \,.\end{aligned} \tag{A.2.3}$$

Note that

$$\boldsymbol{\theta}_o'\mathbf{D}_\alpha'\left(\boldsymbol{\Omega}-N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}'\right)\mathbf{D}_\alpha\boldsymbol{\theta}_o \;=\; \alpha^2\mathbf{1}'\left(\boldsymbol{\Omega}-N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}'\right)\mathbf{1}=0$$

and $\sigma_\nu^2\mathrm{tr}\left(\mathbf{C}\mathbf{X}_\alpha\mathbf{D}_\alpha'\mathbf{D}_\alpha\mathbf{X}_\alpha'\right)$ can be written as $\sigma_\nu^2\mathrm{tr}\left(\boldsymbol{\Omega}-N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}'\right)$, which is equal to $\sigma_\nu^2\sum_{i=1}^{t}\left(\omega_i-\omega_i^2N_\omega^{-1}\right)$ or $\sigma_\nu^2(t-1)\left[\overline{\omega}-\left(s_\omega^2/t\,\overline{\omega}\right)\right]$. Therefore, substituting these results into equation (A.4) yields $\mathrm{E}\left(s_\nu^2\right)=\sigma_\nu^2$.

A.3 Optimal Choice of a_i for Small-Area Mean Estimator $\tilde{\mu}_i^c$

Let $\tilde{\mu}_i^c=(1-a_i)\,\tilde{\mu}_i^{(1)}+a_i\,\tilde{\mu}_i^{(2)}$ and the mean squared error can be written as

$$\begin{aligned}\mathrm{MSE}\left(\tilde{\mu}_i^c\right) &= \mathrm{E}\left(\left[(1-a_i)\,\tilde{\mu}_i^{(1)}+a_i\,\tilde{\mu}_i^{(2)}-\mu_i\right]^2\right)\\ &= (1-a_i)^2\mathrm{MSE}\left(\tilde{\mu}_i^{(1)}\right)+a_i^2\mathrm{MSE}\left(\tilde{\mu}_i^{(2)}\right)\\ &\quad +2a_i(1-a_i)\mathrm{E}\left((\tilde{\mu}_i^{(1)}-\mu_i)(\tilde{\mu}_i^{(2)}-\mu_i)\right). \qquad \text{(A.3.1)}\end{aligned}$$

Consider the derivative of $\mathrm{MSE}(\tilde{\mu}_i^c)$ with respect to a_i,

$$\begin{aligned}\frac{d}{da_i}\mathrm{MSE}\left(\tilde{\mu}_i^c\right) &= -2(1-a_i)\mathrm{E}\Big((\tilde{\mu}_i^{(1)}-\mu_i)^2\Big)+2a_i\mathrm{E}\Big((\tilde{\mu}_i^{(2)}-\mu_i)^2\Big)\\ &\quad +2(1-2a_i)\mathrm{E}\Big((\tilde{\mu}_i^{(1)}-\mu_i)(\tilde{\mu}_i^{(2)}-\mu_i)\Big).\end{aligned}$$

Setting the above derivative to zero, an expression of a_i such that the MSE$[\tilde{\mu}_i^c]$ is at its minimum is given by

$$a_i = \frac{\mathrm{MSE}(\tilde{\mu}_i^{(1)}) - \mathrm{E}\Big((\tilde{\mu}_i^{(1)} - \mu_i)(\tilde{\mu}_i^{(2)} - \mu_i)\Big)}{\mathrm{MSE}(\tilde{\mu}_i^{(1)}) + \mathrm{MSE}(\tilde{\mu}_i^{(2)}) - 2\mathrm{E}\Big((\tilde{\mu}_i^{(1)} - \mu_i)(\tilde{\mu}_i^{(2)} - \mu_i)\Big)} . \qquad \text{(A.3.2)}$$

A.4 Optimal Choice of a for Intraclass Correlation Estimator r_c

The shrinkage estimator r_c defined in equation (4.3.2) is $(1-a)\,r_1 + a\,r_2$ with mean squared error is given by

$$\begin{aligned}
\mathrm{MSE}[r_c] &= \mathrm{E}\left[\Big((1-a)\,r_1 + a\,r_2 - \rho\Big)^2\right] \\
&= \mathrm{E}\Big[(1-a)^2(r_1-\rho)^2 + a^2(r_2-\rho)^2 + 2a(1-a)(r_1-\rho)(r_2-\rho)\Big] \\
&= (1-a)^2\mathrm{E}\Big[(r_1-\rho)^2\Big] + a^2\mathrm{E}\Big[(r_2-\rho)^2\Big] \qquad \text{(A.4.1)} \\
&\quad +2a(1-a)\mathrm{E}\Big[(r_1-\rho)(r_2-\rho)\Big] \\
&= (1-a)^2\mathrm{MSE}[r_1] + a^2\mathrm{MSE}[r_2] + 2a(1-a)\mathrm{E}\Big[(r_1-\rho)(r_2-\rho)\Big] .
\end{aligned}$$

Consider the derivative of MSE$[r_c]$ with respect to a,

$$\begin{aligned}
\frac{d}{da}\mathrm{MSE}[r_c] &= -2(1-a)\mathrm{E}\Big[(r_1-\rho)^2\Big] + 2a\mathrm{E}\Big[(r_2-\rho)^2\Big] \qquad \text{(A.4.2)} \\
&\quad +2(1-2a)\mathrm{E}\Big[(r_1-\rho)(r_2-\rho)\Big] .
\end{aligned}$$

Setting the above derivative to zero, an expression of a such that the MSE$[r_c]$ is at its minimum is given by

$$a = \frac{\text{MSE}[r_1] - \text{E}\Big[(r_1-\rho)(r_2-\rho)\Big]}{\text{MSE}[r_1] + \text{MSE}[r_2] - 2\text{E}\Big[(r_1-\rho)(r_2-\rho)\Big]} . \qquad \text{(A.4.3)}$$

A.5 Mean Squared Error of $\tilde{\mu}_i^c$

From equation (A.3.1), the mean squared error of $\tilde{\mu}_i^c$ can be written as

$$\begin{aligned}\text{MSE}\,(\tilde{\mu}_i^c) &= (1-a_i)^2\text{MSE}(\tilde{\mu}_i^{(1)}) + a_i^2\text{MSE}(\tilde{\mu}_i^{(2)}) \\ &\quad +2a_i(1-a_i)\text{E}\Big((\tilde{\mu}_i^{(1)}-\mu_i)(\tilde{\mu}_i^{(2)}-\mu_i)\Big) . \qquad \text{(A.5.1)}\end{aligned}$$

Substituting the expression of a_i from equation (A.3.2) into equation (A.3.1), with some algebraic manipulation, the mean squared error of $\tilde{\mu}_i^c$ can be expressed as

$$\text{MSE}(\tilde{\mu}_i^c) = \frac{\text{MSE}(\tilde{\mu}_i^{(1)})\text{MSE}(\tilde{\mu}_i^{(2)}) - \Big[\text{E}\Big((\tilde{\mu}_i^{(1)}-\mu_i)(\tilde{\mu}_i^{(2)}-\mu_i)\Big)\Big]^2}{\text{MSE}(\tilde{\mu}_i^{(1)}) + \text{MSE}(\tilde{\mu}_i^{(2)}) - 2\text{E}\Big((\tilde{\mu}_i^{(1)}-\mu_i)(\tilde{\mu}_i^{(2)}-\mu_i)\Big)} . \qquad \text{(A.5.2)}$$

A.6 Approximation of the Mean Squared Error of $\tilde{\mu}_i^{(p)}$ for $p = 1, 2$

For the nested-error regression model in (3.2.2), the best linear unbiased estimator of the small-area mean for the ith area

$$\mu_i = \overline{\mathbf{X}}_i'\boldsymbol{\beta} + \nu_i \qquad \text{(A.6.1)}$$

is given by

$$\tilde{\mu}_i = \overline{\mathbf{X}}_i'\tilde{\boldsymbol{\beta}} + \mathbf{m}_i'\mathbf{G}Z'\mathbf{V}^{-1}\left(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}}\right) \tag{A.6.2}$$

which can be reduced to equation (3.3) of Prasad & Rao [41]:

$$\tilde{\mu}_i = \overline{\mathbf{X}}_i\tilde{\boldsymbol{\beta}} + \gamma_i\left(\overline{y}_i - \overline{\mathbf{x}}_i\tilde{\boldsymbol{\beta}}\right) \tag{A.6.3}$$

where $\gamma_i = \sigma_\nu^2(\sigma_\nu^2 + \sigma_\epsilon^2 n_i^{-1})^{-1}$ and $\tilde{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}$. Let $s_{\nu p}^2$ be similarly defined as in equation (3.3.1) for weights $\{\omega_i^{(p)}\}$. Thus, the two-stage estimator $\tilde{\mu}_i^{(p)}$ of μ_i for the ith area is given by

$$\tilde{\mu}_i^{(p)} = \overline{\mathbf{X}}_i'\tilde{\boldsymbol{\beta}}_p + \tilde{\gamma}_{ip}\left(\overline{y}_i - \overline{\mathbf{x}}_i\tilde{\boldsymbol{\beta}}_p\right) \tag{A.6.4}$$

where $\tilde{\gamma}_{ip} = s_{\nu p}^2(s_{\nu p}^2 + s_\epsilon^2 n_i^{-1})^{-1}$ and $\tilde{\boldsymbol{\beta}}_p = \left(\mathbf{X}'\tilde{\mathbf{V}}_p^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\tilde{\mathbf{V}}_p^{-1}\mathbf{y}$ with $\tilde{\mathbf{V}}_p^{-1}$ an estimator of $\mathbf{V}^{-1}$ obtained by replacing σ_ν^2 and σ_ϵ^2 by $s_{\nu p}^2$ and s_ϵ^2 respectively. Note that $\mathbf{X}$ is an $N \times (k+1)$ design matrix and $\tilde{\mathbf{V}}_p^{-1} = \mathrm{diag}(\tilde{\mathbf{V}}_{1p}^{-1}, \ldots, \tilde{\mathbf{V}}_{tp}^{-1})$ where $\tilde{\mathbf{V}}_{ip}^{-1} = s_\epsilon^{-2}\mathbf{I}_{n_i} - (n_i s_\epsilon^2)^{-1}\tilde{\gamma}_{ip}\mathbf{1}_{n_i}\mathbf{1}_{n_i}'$. For convenience, define $\tilde{\mathbf{W}}_{ip}^{-1} = \mathbf{I}_{n_i} - n_i^{-1}\tilde{\gamma}_{ip}\mathbf{1}_{n_i}\mathbf{1}_{n_i}'$ and $\tilde{\mathbf{W}}_p^{-1} = \mathrm{diag}(\tilde{\mathbf{W}}_{1p}^{-1}, \ldots, \tilde{\mathbf{W}}_{tp}^{-1})$. Hence, the regression coefficients estimator $\tilde{\boldsymbol{\beta}}_p$ can be written as

$$\tilde{\boldsymbol{\beta}}_p = \left(\mathbf{X}'\tilde{\mathbf{W}}_p^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\tilde{\mathbf{W}}_p^{-1}\mathbf{y}\,. \tag{A.6.5}$$

It can be shown that

$$\frac{\partial\tilde{\gamma}_{ip}}{\partial s_\epsilon^2} = -\frac{s_{\nu p}^2/n_i}{(s_{\nu p}^2 + s_\epsilon^2/n_i)^2} = \frac{-\tilde{\gamma}_{ip}^2}{n_i s_{\nu p}^2}$$

and

$$\frac{\partial \tilde{\gamma}_{ip}}{\partial s^2_{\nu p}} = \frac{s^2_\epsilon / n_i}{(s^2_{\nu p} + s^2_\epsilon / n_i)^2} = \frac{s^2_\epsilon \tilde{\gamma}^2_{ip}}{n_i s^4_{\nu p}} .$$

Also note that

$$\begin{aligned}\frac{\partial \tilde{\boldsymbol{\beta}}_p}{\partial s^2_\epsilon} = & -\left(\mathbf{X}'\tilde{\mathbf{W}}_p^{-1}\mathbf{X}\right)^{-1}\left(\mathbf{X}'\frac{\partial \tilde{\mathbf{W}}_p^{-1}}{\partial s^2_\epsilon}\mathbf{X}\right)\left(\mathbf{X}'\tilde{\mathbf{W}}_p^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\tilde{\mathbf{W}}_p^{-1}\mathbf{y} \\ & + \left(\mathbf{X}'\tilde{\mathbf{W}}_p^{-1}\mathbf{X}\right)^{-1}\left(\mathbf{X}'\frac{\partial \tilde{\mathbf{W}}_p^{-1}}{\partial s^2_\epsilon}\mathbf{y}\right) \qquad \text{(A.6.6)}\end{aligned}$$

and similarly,

$$\begin{aligned}\frac{\partial \tilde{\boldsymbol{\beta}}_p}{\partial s^2_{\nu p}} = & -\left(\mathbf{X}'\tilde{\mathbf{W}}_p^{-1}\mathbf{X}\right)^{-1}\left(\mathbf{X}'\frac{\partial \tilde{\mathbf{W}}_p^{-1}}{\partial s^2_{\nu p}}\mathbf{X}\right)\left(\mathbf{X}'\tilde{\mathbf{W}}_p^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\tilde{\mathbf{W}}_p^{-1}\mathbf{y} \\ & + \left(\mathbf{X}'\tilde{\mathbf{W}}_p^{-1}\mathbf{X}\right)^{-1}\left(\mathbf{X}'\frac{\partial \tilde{\mathbf{W}}_p^{-1}}{\partial s^2_{\nu p}}\mathbf{y}\right) \qquad \text{(A.6.7)}\end{aligned}$$

where

$$\begin{aligned}\frac{\partial \tilde{\mathbf{W}}_p^{-1}}{\partial s^2_\epsilon} &= \operatorname{diag}\left(\frac{\partial \tilde{\mathbf{W}}_{1p}^{-1}}{\partial s^2_\epsilon}, \ldots, \frac{\partial \tilde{\mathbf{W}}_{tp}^{-1}}{\partial s^2_\epsilon}\right) \\ &= \frac{1}{s^2_{\nu p}}\operatorname{diag}\left(\frac{\tilde{\gamma}^2_{1p}}{n_1^2}\mathbf{1}_{n_1}\mathbf{1}'_{n_1}, \ldots, \frac{\tilde{\gamma}^2_{tp}}{n_t^2}\mathbf{1}_{n_t}\mathbf{1}'_{n_t}\right) \qquad \text{(A.6.8)}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \tilde{\mathbf{W}}_p^{-1}}{\partial s^2_{\nu p}} &= \operatorname{diag}\left(\frac{\partial \tilde{\mathbf{W}}_{1p}^{-1}}{\partial s^2_{\nu p}}, \ldots, \frac{\partial \tilde{\mathbf{W}}_{tp}^{-1}}{\partial s^2_{\nu p}}\right) \\ &= \frac{-s^2_\epsilon}{s^4_{\nu p}}\operatorname{diag}\left(\frac{\tilde{\gamma}^2_{1p}}{n_1^2}\mathbf{1}_{n_1}\mathbf{1}'_{n_1}, \ldots, \frac{\tilde{\gamma}^2_{tp}}{n_t^2}\mathbf{1}_{n_t}\mathbf{1}'_{n_t}\right) . \qquad \text{(A.6.9)}\end{aligned}$$

The total differential of $\hat{\mu}_i^{(p)}$ is given by

$$d\tilde{\mu}_i^{(p)} = \frac{\partial \tilde{\mu}_i^{(p)}}{\partial s^2_\epsilon} ds^2_\epsilon + \frac{\partial \tilde{\mu}_i^{(p)}}{\partial s^2_{\nu p}} ds^2_{\nu p} \qquad \text{(A.6.10)}$$

where

$$\frac{\partial \tilde{\mu}_i^{(p)}}{\partial s_\epsilon^2} = \left(\overline{\mathbf{X}}_i - \tilde{\gamma}_{ip}\overline{\mathbf{x}}_i\right)' \frac{\partial \tilde{\boldsymbol{\beta}}_p}{\partial s_\epsilon^2} + \left(\overline{y}_i - \overline{\mathbf{x}}_i'\tilde{\boldsymbol{\beta}}_p\right) \frac{\partial \tilde{\gamma}_{ip}}{\partial s_\epsilon^2} \qquad \text{(A.6.11)}$$

and

$$\frac{\partial \tilde{\mu}_i^{(p)}}{\partial s_{\nu p}^2} = \left(\overline{\mathbf{X}}_i - \tilde{\gamma}_{ip}\overline{\mathbf{x}}_i\right)' \frac{\partial \tilde{\boldsymbol{\beta}}_p}{\partial s_{\nu p}^2} + \left(\overline{y}_i - \overline{\mathbf{x}}_i'\tilde{\boldsymbol{\beta}}_p\right) \frac{\partial \tilde{\gamma}_{ip}}{\partial s_{\nu p}^2} \,. \qquad \text{(A.6.12)}$$

Replacing d by δ and treating the differential as a finite difference and evaluating the difference about the point $(\sigma_\nu^2, \sigma_\epsilon^2)$ yield

$$\delta\tilde{\mu}_i^{(p)} = \left.\frac{\partial \tilde{\mu}_i^{(p)}}{\partial s_\epsilon^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)} \delta s_\epsilon^2 + \left.\frac{\partial \tilde{\mu}_i^{(p)}}{\partial s_{\nu p}^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)} \delta s_{\nu p}^2 \,. \qquad \text{(A.6.13)}$$

Note that

$$\begin{aligned}\left(\delta\tilde{\mu}_i^{(p)}\right)^2 = &\left(\left.\frac{\partial \tilde{\mu}_i^{(p)}}{\partial s_\epsilon^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)}\right)^2 \left(\delta s_\epsilon^2\right)^2 + \left(\left.\frac{\partial \tilde{\mu}_i^{(p)}}{\partial s_{\nu p}^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)}\right)^2 \left(\delta s_{\nu p}^2\right)^2 \\ &+2\left(\left.\frac{\partial \tilde{\mu}_i^{(p)}}{\partial s_\epsilon^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)}\right)\left(\left.\frac{\partial \tilde{\mu}_i^{(p)}}{\partial s_{\nu p}^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)}\right) \delta s_\epsilon^2 \, \delta s_{\nu p}^2 \,. \qquad \text{(A.6.14)}\end{aligned}$$

Substituting the finite differences $\delta\tilde{\mu}_i^{(p)} = \tilde{\mu}_i^{(p)} - \mu_i$, $\delta s_\epsilon^2 = s_\epsilon^2 - \sigma_\epsilon^2$ and $\delta s_{\nu p}^2 = s_{\nu p}^2 - \sigma_\nu^2$ into the above expression and taking an expectation, leads to the following expression for the approximation of the mean squared error of $\tilde{\mu}_i^{(p)}$ for $p = 1, 2$:

$$\begin{aligned}\text{MSE}\left(\tilde{\mu}_i^{(p)}\right) &= \text{E}\left((\tilde{\mu}_i^{(p)} - \mu_i)^2\right) \\ &\approx \left(\left.\frac{\partial \tilde{\mu}_i^{(p)}}{\partial s_\epsilon^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)}\right)^2 \text{Var}(s_\epsilon^2) + \left(\left.\frac{\partial \tilde{\mu}_i^{(p)}}{\partial s_{\nu p}^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)}\right)^2 \text{Var}(s_{\nu p}^2) \\ &\quad +2\left(\left.\frac{\partial \tilde{\mu}_i^{(p)}}{\partial s_\epsilon^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)}\right)\left(\left.\frac{\partial \tilde{\mu}_i^{(p)}}{\partial s_{\nu p}^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)}\right) \text{Cov}(s_\epsilon^2, s_{\nu p}^2) \,. \qquad \text{(A.6.15)}\end{aligned}$$

To simplify, let $\boldsymbol{\beta}_{d\epsilon,p} = \partial\tilde{\boldsymbol{\beta}}_p/\partial s^2_\epsilon$ and $\boldsymbol{\beta}_{d\nu,p} = \partial\tilde{\boldsymbol{\beta}}_p/\partial s^2_{\nu p}$. The approximation of the mean squared error of $\tilde{\mu}_i^{(p)}$ for $p = 1, 2$ can be expressed as

$$\begin{aligned}
\text{MSE}\left(\tilde{\mu}_i^{(p)}\right) \approx\ & \text{Var}(s^2_\epsilon)\left[\left(\overline{\mathbf{X}}_i - \tilde{\gamma}_{ip}\overline{\mathbf{x}}_i\right)'\boldsymbol{\beta}_{d\epsilon,p} - \left(\overline{y}_i - \overline{\mathbf{x}}_i'\tilde{\boldsymbol{\beta}}_p\right)\frac{\tilde{\gamma}^2_{ip}}{n_i s^2_{\nu p}}\right]^2 \\
& + \text{Var}(s^2_{\nu p})\left[\left(\overline{\mathbf{X}}_i - \tilde{\gamma}_{ip}\overline{\mathbf{x}}_i\right)'\boldsymbol{\beta}_{d\nu,p} + \left(\overline{y}_i - \overline{\mathbf{x}}_i'\tilde{\boldsymbol{\beta}}_p\right)\frac{s^2_\epsilon\tilde{\gamma}^2_{ip}}{n_i s^4_{\nu p}}\right]^2 \\
& + 2\ \text{Cov}(s^2_\epsilon, s^2_{\nu p})\left\{\left[\left(\overline{\mathbf{X}}_i - \tilde{\gamma}_{ip}\overline{\mathbf{x}}_i\right)'\boldsymbol{\beta}_{d\epsilon,p} - \left(\overline{y}_i - \overline{\mathbf{x}}_i'\tilde{\boldsymbol{\beta}}_p\right)\frac{\tilde{\gamma}^2_{ip}}{n_i s^2_{\nu p}}\right]\right. \\
& \left.\times\left[\left(\overline{\mathbf{X}}_i - \tilde{\gamma}_{ip}\overline{\mathbf{x}}_i\right)'\boldsymbol{\beta}_{d\nu,p} + \left(\overline{y}_i - \overline{\mathbf{x}}_i'\tilde{\boldsymbol{\beta}}_p\right)\frac{s^2_\epsilon\tilde{\gamma}^2_{ip}}{n_i s^4_{\nu p}}\right]\right\}. \qquad \text{(A.6.16)}
\end{aligned}$$

A.7 Approximation of $\text{E}\left((\tilde{\mu}_i^{(1)} - \mu_i)(\tilde{\mu}_i^{(2)} - \mu_i)\right)$

Following the similar approach in the previous section, we obtain

$$\delta\tilde{\mu}_i^{(1)} = \left.\frac{\partial\tilde{\mu}_i^{(1)}}{\partial s^2_\epsilon}\right|_{(\sigma^2_\nu,\sigma^2_\epsilon)}\delta s^2_\epsilon + \left.\frac{\partial\tilde{\mu}_i^{(1)}}{\partial s^2_{\nu 1}}\right|_{(\sigma^2_\nu,\sigma^2_\epsilon)}\delta s^2_{\nu 1} \qquad \text{(A.7.1)}$$

and

$$\delta\tilde{\mu}_i^{(2)} = \left.\frac{\partial\tilde{\mu}_i^{(2)}}{\partial s^2_\epsilon}\right|_{(\sigma^2_\nu,\sigma^2_\epsilon)}\delta s^2_\epsilon + \left.\frac{\partial\tilde{\mu}_i^{(2)}}{\partial s^2_{\nu 2}}\right|_{(\sigma^2_\nu,\sigma^2_\epsilon)}\delta s^2_{\nu 2}\,. \qquad \text{(A.7.2)}$$

Substituting the finite differences $\delta\tilde{\mu}_i^{(1)} = \tilde{\mu}_i^{(1)} - \mu_i$, $\delta\tilde{\mu}_i^{(2)} = \tilde{\mu}_i^{(2)} - \mu_i$, $\delta s^2_\epsilon = s^2_\epsilon - \sigma^2_\epsilon$, $\delta s^2_{\nu 1} = s^2_{\nu 1} - \sigma^2_\nu$ and $\delta s^2_{\nu 2} = s^2_{\nu 2} - \sigma^2_\nu$ into the above expression and taking an expectation

leads to the following approximation:

$$\begin{aligned}
\mathrm{E}\left((\tilde{\mu}_i^{(1)}-\mu_i)(\tilde{\mu}_i^{(2)}-\mu_i)\right) \approx\ & \left.\frac{\partial\tilde{\mu}_i^{(1)}}{\partial s_\epsilon^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)} \left.\frac{\partial\tilde{\mu}_i^{(2)}}{\partial s_\epsilon^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)} \mathrm{Var}(s_\epsilon^2) \\
& + \left.\frac{\partial\tilde{\mu}_i^{(1)}}{\partial s_{\nu1}^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)} \left.\frac{\partial\tilde{\mu}_i^{(2)}}{\partial s_{\nu2}^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)} \mathrm{Cov}(s_{\nu1}^2, s_{\nu2}^2) \\
& + \left.\frac{\partial\tilde{\mu}_i^{(2)}}{\partial s_\epsilon^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)} \left.\frac{\partial\tilde{\mu}_i^{(1)}}{\partial s_{\nu1}^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)} \mathrm{Cov}(s_\epsilon^2, s_{\nu1}^2) \\
& + \left.\frac{\partial\tilde{\mu}_i^{(1)}}{\partial s_\epsilon^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)} \left.\frac{\partial\tilde{\mu}_i^{(2)}}{\partial s_{\nu2}^2}\right|_{(\sigma_\nu^2,\sigma_\epsilon^2)} \mathrm{Cov}(s_\epsilon^2, s_{\nu2}^2)\ ,
\end{aligned} \tag{A.7.3}$$

which can be expressed as

$$\begin{aligned}
&\mathrm{E}\left((\tilde{\mu}_i^{(1)}-\mu_i)(\tilde{\mu}_i^{(2)}-\mu_i)\right) \\
&\approx \mathrm{Var}(s_\epsilon^2)\left[\left(\overline{\mathbf{X}}_i-\tilde{\gamma}_{i1}\overline{\mathbf{x}}_i\right)'\boldsymbol{\beta}_{d\epsilon,1}-\left(\overline{y}_i-\overline{\mathbf{x}}_i'\tilde{\boldsymbol{\beta}}_1\right)\frac{\tilde{\gamma}_{i1}^2}{n_i s_{\nu1}^2}\right] \\
&\quad\times\left[\left(\overline{\mathbf{X}}_i-\tilde{\gamma}_{i2}\overline{\mathbf{x}}_i\right)'\boldsymbol{\beta}_{d\epsilon,2}-\left(\overline{y}_i-\overline{\mathbf{x}}_i'\tilde{\boldsymbol{\beta}}_2\right)\frac{\tilde{\gamma}_{i2}^2}{n_i s_{\nu2}^2}\right] \\
&\quad+\mathrm{Cov}(s_{\nu1}^2, s_{\nu2}^2)\left[\left(\overline{\mathbf{X}}_i-\tilde{\gamma}_{i1}\overline{\mathbf{x}}_i\right)'\boldsymbol{\beta}_{d\nu,1}+\left(\overline{y}_i-\overline{\mathbf{x}}_i'\tilde{\boldsymbol{\beta}}_1\right)\frac{s_\epsilon^2\tilde{\gamma}_{i1}^2}{n_i s_{\nu1}^4}\right] \\
&\quad\times\left[\left(\overline{\mathbf{X}}_i-\tilde{\gamma}_{i2}\overline{\mathbf{x}}_i\right)'\boldsymbol{\beta}_{d\nu,2}+\left(\overline{y}_i-\overline{\mathbf{x}}_i'\tilde{\boldsymbol{\beta}}_2\right)\frac{s_\epsilon^2\tilde{\gamma}_{i2}^2}{n_i s_{\nu2}^4}\right] \\
&\quad+\mathrm{Cov}(s_{\nu1}^2, s_\epsilon^2)\left[\left(\overline{\mathbf{X}}_i-\tilde{\gamma}_{i2}\overline{\mathbf{x}}_i\right)'\boldsymbol{\beta}_{d\epsilon,2}-\left(\overline{y}_i-\overline{\mathbf{x}}_i'\tilde{\boldsymbol{\beta}}_2\right)\frac{\tilde{\gamma}_{i2}^2}{n_i s_{\nu2}^2}\right] \\
&\quad\times\left[\left(\overline{\mathbf{X}}_i-\tilde{\gamma}_{i1}\overline{\mathbf{x}}_i\right)'\boldsymbol{\beta}_{d\epsilon,1}-\left(\overline{y}_i-\overline{\mathbf{x}}_i'\tilde{\boldsymbol{\beta}}_1\right)\frac{s_\epsilon^2\tilde{\gamma}_{i1}^2}{n_i s_{\nu1}^4}\right] \\
&\quad+\mathrm{Cov}(s_{\nu2}^2, s_\epsilon^2)\left[\left(\overline{\mathbf{X}}_i-\tilde{\gamma}_{i1}\overline{\mathbf{x}}_i\right)'\boldsymbol{\beta}_{d\epsilon,1}-\left(\overline{y}_i-\overline{\mathbf{x}}_i'\tilde{\boldsymbol{\beta}}_1\right)\frac{\tilde{\gamma}_{i1}^2}{n_i s_{\nu1}^2}\right] \\
&\quad\times\left[\left(\overline{\mathbf{X}}_i-\tilde{\gamma}_{i2}\overline{\mathbf{x}}_i\right)'\boldsymbol{\beta}_{d\epsilon,2}-\left(\overline{y}_i-\overline{\mathbf{x}}_i'\tilde{\boldsymbol{\beta}}_2\right)\frac{s_\epsilon^2\tilde{\gamma}_{i2}^2}{n_i s_{\nu2}^4}\right]\ .
\end{aligned} \tag{A.7.4}$$

A.8 Mean Squared Error of r_c

From equation (A.4.1), the mean squared error of the shrinkage estimator r_c is given by

$$(1-a)^2\text{MSE}[r_1] + a^2\text{MSE}[r_2] + 2a(1-a)\text{E}\Big[(r_1-\rho)(r_2-\rho)\Big]$$

Substituting the expression of a from equation (A.4.3) into equation (A.4.1), with some algebraic manipulation, the mean squared error of r_c can be expressed as

$$\text{MSE}[r_c] = \frac{\text{MSE}[r_1]\text{MSE}[r_2] - \Big(\text{E}\Big[(r_1-\rho)(r_2-\rho)\Big]\Big)^2}{\text{MSE}[r_1] + \text{MSE}[r_2] - 2\text{E}\Big[(r_1-\rho)(r_2-\rho)\Big]} \,. \tag{A.8.1}$$

A.9 Approximation of the Mean Squared Error of r_h for $h = 1, 2$

Suppose $r = f(s_\nu^2, s_\epsilon^2) = (1 + s_\epsilon^2/s_\nu^2)^{-1}$ is an estimator of the intraclass correlation ρ where s_ν^2 and s_ϵ^2 are unbiased estimators of σ_ν^2 and σ_ϵ^2. The total differentiation of the function $r = f(s_\nu^2, s_\epsilon^2)$ is given by

$$\begin{aligned} dr &= \frac{\partial f}{\partial s_\nu^2}\, d\, s_\nu^2 + \frac{\partial f}{\partial s_\epsilon^2}\, d\, s_\epsilon^2 \\ &= -\left(1+\frac{s_\epsilon^2}{s_\nu^2}\right)^{-2} \frac{1}{s_\nu^2}\, d\, s_\epsilon^2 + \left(1+\frac{s_\epsilon^2}{s_\nu^2}\right)^{-2} \frac{s_\epsilon^2}{s_\nu^4}\, d\, s_\nu^2 \\ &= r^2\left(\frac{-1}{s_\nu^2}\, d\, s_\epsilon^2 + \frac{s_\epsilon^2}{s_\nu^4}\, d\, s_\nu^2\right) \,. \end{aligned}$$

Replacing d with δ and treat this differential as a finite difference, we have

$$\delta r = r^2\left(\frac{-1}{s_\nu^2}\, \delta s_\epsilon^2 + \frac{s_\epsilon^2}{s_\nu^4}\, \delta s_\nu^2\right) \,.$$

Evaluating this differential about the point $(\sigma_\epsilon^2, \sigma_\nu^2)$ yields

$$\begin{aligned}\delta r &= \rho^2 \left(\frac{-1}{\sigma_\nu^2} \, \delta s_\epsilon^2 + \frac{\sigma_\epsilon^2}{\sigma_\nu^4} \, \delta s_\nu^2 \right) \\ &= \frac{\sigma_\epsilon^2}{\sigma_\nu^2} \rho^2 \left(\frac{\delta s_\nu^2}{\sigma_\nu^2} - \frac{\delta s_\epsilon^2}{\sigma_\epsilon^2} \right) .\end{aligned}$$

Note that

$$(\delta r)^2 = \frac{\sigma_\epsilon^4}{\sigma_\nu^4} \rho^4 \left[\left(\frac{\delta s_\nu^2}{\sigma_\nu^2} \right)^2 - 2 \left(\frac{\delta s_\nu^2}{\sigma_\nu^2} \right) \left(\frac{\delta s_\epsilon^2}{\sigma_\epsilon^2} \right) + \left(\frac{\delta s_\epsilon^2}{\sigma_\epsilon^2} \right)^2 \right] .$$

Substituting the finite differences $\delta r = r - \rho$, $\delta s_\epsilon^2 = s_\epsilon^2 - \sigma_\epsilon^2$ and $\delta s_\nu^2 = s_\nu^2 - \sigma_\nu^2$ into the above expression and taking an expectation leads to the following expression for the approximation of the mean squared error of r_h for $h = 1, 2$ in equation (4.3.10):

$$\begin{aligned}\mathrm{MSE}[r_h] &= \mathrm{E}\left[(r_h - \rho)^2 \right] \\ &\approx \frac{\sigma_\epsilon^4}{\sigma_\nu^4} \rho^4 \left(\frac{\mathrm{Var}\left[s_{\nu h}^2 \right]}{\sigma_\nu^4} - \frac{2\mathrm{Cov}\left[s_{\nu h}^2, s_\epsilon^2 \right]}{\sigma_\nu^2 \sigma_\epsilon^2} + \frac{\mathrm{Var}\left[s_\epsilon^2 \right]}{\sigma_\epsilon^4} \right) \\ &\approx \left(\sigma_\nu^2 + \sigma_\epsilon^2 \right)^{-4} \\ &\quad \times \left(\sigma_\epsilon^4 \mathrm{Var}\left[s_{\nu h}^2 \right] - 2 \, \sigma_\nu^2 \sigma_\epsilon^2 \mathrm{Cov}\left[s_{\nu h}^2, s_\epsilon^2 \right] + \sigma_\nu^4 \mathrm{Var}[s_\epsilon^2] \right) . \qquad \text{(A.9.1)}\end{aligned}$$

A.10 Approximation of $\mathrm{E}\Big[(r_1 - \rho)(r_2 - \rho) \Big]$

Following the similar approach in the previous section, we have

$$\delta r_1 = \frac{\sigma_\epsilon^2}{\sigma_\nu^2} \rho^2 \left(\frac{\delta s_{\nu 1}^2}{\sigma_\nu^2} - \frac{\delta s_\epsilon^2}{\sigma_\epsilon^2} \right)$$

and

$$\delta r_2 = \frac{\sigma_\epsilon^2}{\sigma_\nu^2}\rho^2\left(\frac{\delta s_{\nu 2}^2}{\sigma_\nu^2} - \frac{\delta s_\epsilon^2}{\sigma_\epsilon^2}\right)$$

and hence

$$\begin{aligned}\delta r_1 \delta r_2 \;=\; & \frac{\sigma_\epsilon^4}{\sigma_\nu^4}\rho^4\left[\left(\frac{\delta s_{\nu 1}^2}{\sigma_\nu^2}\right)\left(\frac{\delta s_{\nu 2}^2}{\sigma_\nu^2}\right) - \left(\frac{\delta s_{\nu 1}^2}{\sigma_\nu^2}\right)\left(\frac{\delta s_\epsilon^2}{\sigma_\epsilon^2}\right)\right. \\ & \left. - \left(\frac{\delta s_{\nu 2}^2}{\sigma_\nu^2}\right)\left(\frac{\delta s_\epsilon^2}{\sigma_\epsilon^2}\right) + \left(\frac{\delta s_\epsilon^2}{\sigma_\epsilon^2}\right)^2\right].\end{aligned}$$

Substituting the finite differences $\delta r_1 = r_1 - \rho$, $\delta r_2 = r_2 - \rho$, $\delta s_\epsilon^2 = s_\epsilon^2 - \sigma_\epsilon^2$, $\delta s_{\nu 1}^2 = s_{\nu 1}^2 - \sigma_\nu^2$ and $\delta s_{\nu 2}^2 = s_{\nu 2}^2 - \sigma_\nu^2$ into the above expression and taking an expectation leads to the following expression for the approximation of $\mathrm{E}\Big[(r_1-\rho)(r_2-\rho)\Big]$:

$$\begin{aligned}\mathrm{E}\Big[(r_1-\rho)(r_2-\rho)\Big] \;\approx\; & \left(\sigma_\nu^2+\sigma_\epsilon^2\right)^{-4}\left\{\sigma_\epsilon^4\mathrm{Cov}\big[s_{\nu 1}^2, s_{\nu 2}^2\big] + \sigma_\nu^4\mathrm{Var}\big[s_\epsilon^2\big]\right. \\ & \left. - \sigma_\nu^2\sigma_\epsilon^2\Big(\mathrm{Cov}\big[s_{\nu 1}^2, s_\epsilon^2\big] + \mathrm{Cov}\big[s_{\nu 2}^2, s_\epsilon^2\big]\Big)\right\}. \quad \text{(A.10.1)}\end{aligned}$$

A.11 Proof that s_ϵ^2 and $\sum_{i=1}^t \omega_i(\hat{\alpha}_i - \hat{\alpha})^2$ are uncorrelated

The estimator of σ_ϵ^2 in equation (3.3.2) can be written as

$$s_\epsilon^2 \;=\; \mathbf{y}'\mathbf{A}\mathbf{y}$$

where $\mathbf{A}$ is defined in equation (A.1.1). The sum $\sum_{i=1}^t \omega_i(\hat{\alpha}_i - \hat{\alpha})^2$ can be written as $\mathbf{y}'\mathbf{C}\mathbf{y}$ where $\mathbf{C}$ is defined in equation (A.2.2). Note that $\mathbf{A}$ and $\mathbf{C}$ are symmetric matrices and $\mathbf{AC} = \mathbf{0}$ so that s_ϵ^2 and $\sum_{i=1}^t \omega_i(\hat{\alpha}_i - \hat{\alpha})^2$ are uncorrelated.

A.12 Variance of s_ϵ^2

Defining $\mathbf{A}_\alpha = [\mathbf{I} - \mathbf{X}_\alpha(\mathbf{X}'_\alpha\mathbf{X}_\alpha)^{-1}\mathbf{X}'_\alpha]$, the variance of s_ϵ^2 can be written as

$$\begin{aligned}\mathrm{Var}(s_\epsilon^2) &= \frac{\mathrm{Var}(\mathbf{y}'\mathbf{A}_\alpha\mathbf{y})}{\Big[N - \mathrm{r}(\mathbf{X}_\alpha)\Big]^2} \\ &= \frac{2\left[\mathrm{tr}\left(\mathbf{A}_\alpha\mathbf{V}\mathbf{A}_\alpha\mathbf{V}\right)^2 + 4\,\boldsymbol{\theta}'_o\mathbf{X}'_\alpha\mathbf{A}_\alpha\mathbf{V}\mathbf{A}'_\alpha\mathbf{X}_\alpha\boldsymbol{\theta}_o\right]}{\Big[N - \mathrm{r}(\mathbf{X}_\alpha)\Big]^2}\,. \end{aligned} \tag{A.12.1}$$

Note that $\mathbf{A}_\alpha\mathbf{V} = \sigma_\epsilon^2\mathbf{A}_\alpha$ and because $\mathbf{A}_\alpha$ is an idempotent matrix, $\mathbf{A}_\alpha\mathbf{V}\mathbf{A}_\alpha\mathbf{V} = \sigma_\epsilon^4\mathbf{A}_\alpha$ and $\mathbf{A}_\alpha\mathbf{V}\mathbf{A}'_\alpha = \sigma_\epsilon^2\mathbf{A}_\alpha$. Also note that the last term in the above expression is $\mathbf{0}$, and hence the variance of s_ϵ^2 is

$$\begin{aligned}\mathrm{Var}(s_\epsilon^2) &= \frac{2\sigma_\epsilon^4}{\Big[N - \mathrm{r}(\mathbf{X}_\alpha)\Big]^2}\,\mathrm{tr}(\mathbf{A}_\alpha) \\ &= \frac{2\sigma_\epsilon^4}{N - \mathrm{r}(\mathbf{X}_\alpha)} \end{aligned} \tag{A.12.2}$$

because the trace of the matrix $\mathbf{I} - \mathbf{X}_\alpha(X'_\alpha X_\alpha)^{-1}X'_\alpha$ is equal to $N - \mathrm{r}(\mathbf{X}_\alpha)$.

A.13 Variance of s_ν^2

The variance of s_ν^2 can be expressed as

$$\begin{aligned}\mathrm{Var}(s_\nu^2) &= \left[(t-1)\left(\overline{\omega} - \frac{s_\omega^2}{t\,\overline{\omega}}\right)\right]^{-2}\left\{\mathrm{Var}\left(\sum_{i=1}^{t}\omega_i(\hat{\alpha}_i - \hat{\alpha})^2\right)\right. \\ &\quad \left. +\mathrm{Var}(s_\epsilon^2)\left[\mathrm{tr}\left((\boldsymbol{\Omega} - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{D}_\alpha(\mathbf{X}'_\alpha\mathbf{X}_\alpha)^{-1}\mathbf{D}'_\alpha\right)\right]^2\right\} \end{aligned} \tag{A.13.1}$$

because $\sum_{i=1}^{t}\omega_i(\hat{\alpha}_i-\hat{\alpha})^2$ and s_ϵ^2 are uncorrelated. The variance of the sum in the numerator is

$$\begin{aligned}\mathrm{Var}\Big(\sum_{i=1}^{t}\omega_i(\hat{\alpha}_i-\hat{\alpha})^2\Big) &= \mathrm{Var}\big(\mathbf{y}'\mathbf{C}\mathbf{y}\big)\\ &= 2\,\mathrm{tr}\,(\mathbf{CVCV})^2+4\,\boldsymbol{\theta}_o'\mathbf{X}_\alpha'\mathbf{CVC}'\mathbf{X}_\alpha\boldsymbol{\theta}_o \quad \text{(A.13.2)}\end{aligned}$$

where $\mathbf{C}$ is defined in equation (A.2.2). If we expand $\mathbf{CV}$, we have

$$\begin{aligned}\mathbf{CV} &= \sigma_\epsilon^2\mathbf{X}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'(\boldsymbol{\Omega}-N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{X}_\alpha'\\ &\quad+\sigma_\nu^2\mathbf{X}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'(\boldsymbol{\Omega}-N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{D}_\alpha\mathbf{X}_\alpha' \quad \text{(A.13.3)}\end{aligned}$$

and hence, we obtain $\mathbf{CVCV}$ as

$$\begin{aligned}\mathbf{CVCV} &= \sigma_\epsilon^4\mathbf{X}_\alpha\Big[(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\left(\boldsymbol{\Omega}-N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}'\right)\mathbf{D}_\alpha\Big]^2(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{X}_\alpha'\\ &\quad+\sigma_\epsilon^2\sigma_\nu^2\mathbf{X}_\alpha\Big[(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\left(\boldsymbol{\Omega}-N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}'\right)\mathbf{D}_\alpha\Big]^2\mathbf{X}_\alpha'\\ &\quad+\sigma_\epsilon^2\sigma_\nu^2\mathbf{X}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\left(\boldsymbol{\Omega}-N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}'\right)^2\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{X}_\alpha'\\ &\quad+\sigma_\nu^4\mathbf{X}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\left(\boldsymbol{\Omega}-N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}'\right)^2\mathbf{D}_\alpha\mathbf{X}_\alpha'\,. \quad \text{(A.13.4)}\end{aligned}$$

After some matrix manipulation, the trace of $\mathbf{CVCV}$ can be shown as

$$\begin{aligned}\mathrm{tr}(\mathbf{CVCV})^2 &= \sigma_\epsilon^4\mathrm{tr}\left((\boldsymbol{\Omega}-N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\right)^2\\ &\quad+2\sigma_\epsilon^2\sigma_\nu^2\mathrm{tr}\left((\boldsymbol{\Omega}-N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')^2\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\right) \quad \text{(A.13.5)}\\ &\quad+\sigma_\nu^4\mathrm{tr}\left(\boldsymbol{\Omega}-N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}'\right)^2\,.\end{aligned}$$

Note that $\mathbf{CVC}'$ can be written as

$$\begin{aligned}\mathbf{CVC}' &= \sigma_\epsilon^2 \mathbf{X}_\alpha \Big[(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'(\boldsymbol{\Omega} - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{D}_\alpha\Big]^2 (\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{X}_\alpha' \\ &\quad + \sigma_\nu^2 \mathbf{X}_\alpha (\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'(\boldsymbol{\Omega} - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')^2\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{X}_\alpha' \quad \text{(A.13.6)}\end{aligned}$$

and

$$\begin{aligned}\mathbf{X}'\mathbf{CVC}'\mathbf{X} &= \sigma_\epsilon^2 \mathbf{D}_\alpha'(\boldsymbol{\Omega} - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'(\boldsymbol{\Omega} - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{D}_\alpha \\ &\quad + \sigma_\nu^2 \mathbf{D}_\alpha'(\boldsymbol{\Omega} - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')^2\mathbf{D}_\alpha \quad \text{(A.13.7)}\end{aligned}$$

also

$$4\,\boldsymbol{\theta}_o'\mathbf{X}'\mathbf{CVC}'\mathbf{X}\boldsymbol{\theta}_o = 0\,.$$

Combining these results, we obtain an expression of the variance of s_ν^2 as

$$\begin{aligned}\mathrm{Var}(s_\nu^2) &= \left[(t-1)\left(\overline{\omega} - \frac{s_\omega^2}{t\,\overline{\omega}}\right)\right]^{-2} \\ &\quad \times \Bigg\{ 2\sigma_\epsilon^4 \mathrm{tr}\Big((\boldsymbol{\Omega} - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\Big)^2 \\ &\quad + 4\sigma_\epsilon^2\sigma_\nu^2 \mathrm{tr}\Big((\boldsymbol{\Omega} - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')^2\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\Big) \quad \text{(A.13.8)} \\ &\quad + 2\sigma_\nu^4 \mathrm{tr}\Big(\boldsymbol{\Omega} - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}'\Big)^2 \\ &\quad + \mathrm{Var}(s_\epsilon^2)\Big[\mathrm{tr}\Big((\boldsymbol{\Omega} - N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\Big)\Big]^2 \Bigg\}\,.\end{aligned}$$

A.14 Covariance of s_ϵ^2 and s_ν^2

It has been shown that $\sum_{i=1}^{t}\omega_i(\hat{\alpha}_i-\hat{\alpha})^2$ and s_ϵ^2 are uncorrelated, therefore, the covariance of s_ϵ^2 and s_ν^2 is

$$\begin{aligned}\mathrm{Cov}(s_\epsilon^2,s_\nu^2) &= \frac{-\mathrm{tr}\Big((\Omega-N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\Big)}{(t-1)(\overline{\omega}-\frac{s_\omega^2}{t\,\overline{\omega}})}\,\mathrm{Var}(s_\epsilon^2)\\ &= \frac{-2\sigma_\epsilon^4\mathrm{tr}\Big((\Omega-N_\omega^{-1}\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\Big)}{\Big[N-\mathrm{r}(\mathbf{X}_\alpha)\Big]\,(t-1)\left(\overline{\omega}-\frac{s_\omega^2}{t\,\overline{\omega}}\right)}. \qquad \text{(A.14.1)}\end{aligned}$$

A.15 Covariance of $s_{\nu1}^2$ and $s_{\nu2}^2$

Both $s_{\nu1}^2$ and $s_{\nu2}^2$ are unbiased estimators of σ_ν^2 with weights $\boldsymbol{\omega}_{(1)}{}' = \left(\omega_1^{(1)},\omega_2^{(1)},\ldots\,\omega_t^{(1)}\right)$ and $\boldsymbol{\omega}_{(2)}{}' = \left(\omega_1^{(2)},\omega_2^{(2)},\ldots\,\omega_t^{(2)}\right)$ respectively. Define $\hat{\alpha}_{(p)} = N_{\omega(p)}^{-1}\sum_{i=1}^{t}\omega_i^{(p)}\hat{\alpha}_i$ for $p=1,2$. The sums $\sum_{i=1}^{t}\omega_i^{(1)}\left(\hat{\alpha}_i-\hat{\alpha}_{(1)}\right)^2$ and $\sum_{i=1}^{t}\omega_i^{(2)}\left(\hat{\alpha}_i-\hat{\alpha}_{(2)}\right)^2$ are independent of s_ϵ^2 and hence the covariance of $s_{\nu1}^2$ and $s_{\nu2}^2$ can be written as

$$\begin{aligned}\mathrm{Cov}(s_{\nu1}^2,s_{\nu2}^2) &= \left[(t-1)^2\left(\overline{\omega}_{(1)}-\frac{s_{\omega(1)}^2}{t\,\overline{\omega}_{(1)}}\right)\left(\overline{\omega}_{(2)}-\frac{s_{\omega(2)}^2}{t\,\overline{\omega}_{(2)}}\right)\right]^{-1}\\ &\quad\times\left[\mathrm{Cov}\left(\sum_{i=1}^{t}\omega_i^{(1)}\left(\hat{\alpha}_i-\hat{\alpha}_{(1)}\right)^2,\sum_{j=1}^{t}\omega_j^{(2)}\left(\hat{\alpha}_j-\hat{\alpha}_{(2)}\right)^2\right)\right.\\ &\quad\left.+T_1T_2\,\mathrm{Var}(s_\epsilon^2)\right] \qquad \text{(A.15.1)}\end{aligned}$$

where $T_p=\mathrm{tr}\Big((\Omega_{(p)}-N_{\omega(p)}^{-1}\boldsymbol{\omega}_{(p)}\boldsymbol{\omega}_{(p)}{}')\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{D}_\alpha'\Big)$ for $p=1,2$. The covariance

between the two sums can be expressed as

$$\begin{aligned}\mathrm{Cov}\Big(\sum_{i=1}^{t}\omega_i^{(1)}\left(\hat{\alpha}_i-\hat{\alpha}_{(1)}\right)^2,\sum_{j=1}^{t}\omega_j^{(2)}\left(\hat{\alpha}_j-\hat{\alpha}_{(2)}\right)^2\Big)\\ =2\mathrm{tr}\left(\mathbf{C}_1\mathbf{V}\mathbf{C}_2\mathbf{V}\right)+4\,\boldsymbol{\theta}_o'\mathbf{X}_\alpha'\mathbf{C}_1\mathbf{V}\mathbf{C}_2\mathbf{X}_\alpha\boldsymbol{\theta}_o \qquad \text{(A.15.2)}\end{aligned}$$

where $\mathbf{C}_1$ and $\mathbf{C}_2$ are similarly defined in equation (A.2.2). The second term in equation (A.15.2) is equal to zero because, for p =1, 2,

$$\boldsymbol{\theta}_o'\mathbf{X}_\alpha'\mathbf{C}_p \;=\; \boldsymbol{\theta}_o'\mathbf{D}_\alpha'(\boldsymbol{\Omega}_{(p)}-N_{\omega(p)}^{-1}\boldsymbol{\omega}_{(p)}\boldsymbol{\omega}_{(p)}')\mathbf{D}_\alpha(\mathbf{X}_\alpha'\mathbf{X}_\alpha)^{-1}\mathbf{X}_\alpha'=\mathbf{0}'$$

and the covariance of $s_{\nu 1}^2$ and $s_{\nu 2}^2$ is then given by

$$\mathrm{Cov}(s_{\nu 1}^2,s_{\nu 2}^2) \;=\; \frac{2\ \mathrm{tr}\left(\mathbf{C}_1\mathbf{V}\mathbf{C}_2\mathbf{V}\right)+T_1T_2\mathrm{Var}(s_\epsilon^2)}{(t-1)^2\left(\overline{\omega}_{(1)}-\frac{s_{\omega(1)}^2}{t\ \overline{\omega}_{(1)}}\right)\left(\overline{\omega}_{(2)}-\frac{s_{\omega(2)}^2}{t\ \overline{\omega}_{(2)}}\right)}\,. \qquad \text{(A.15.3)}$$

A.16 Maximum Likelihood Estimator of Intraclass Correlation ρ

The intraclass correlation coefficient is usually estimated by its maximum likelihood estimator. In the context of nested-error regression model, let $\sigma_\tau^2=\sigma_\epsilon^2+\sigma_\nu^2$, the intraclass correlation can be expressed as $\rho=\sigma_\nu^2/\sigma_\tau^2$. The likelihood function given the observations $\mathbf{y}_i$ and the covariates $\mathbf{X}_i$ in the ith area is

$$L_i\Big(\sigma_\tau^2,\boldsymbol{\beta},\rho\Big)=\frac{\exp\left\{-\Big(\mathbf{y}_i-\mathbf{X}_i\boldsymbol{\beta}\Big)'\mathbf{V}_i^{-1}\Big(\mathbf{y}_i-\mathbf{X}_i\boldsymbol{\beta}\Big)/2\right\}}{(2\pi)^{n_i/2}\Big|\mathbf{V}_i\Big|^{1/2}} \qquad \text{(A.16.1)}$$

where $\boldsymbol{\beta} = (\alpha, \beta_1, \ldots, \beta_k)'$ and $\mathbf{V}$ is a variance matrix of $\mathbf{y}_i$ given by

$$\mathbf{V}_i = \sigma_\epsilon^2 \mathbf{I}_{n_i} + \sigma_\nu^2 \mathbf{J}_{n_i} = \sigma_\tau^2 \Big[(1-\rho)\mathbf{I}_{n_i} + \rho \mathbf{J}_{n_i} \Big] . \tag{A.16.2}$$

with inverse

$$\mathbf{V}_i^{-1} = \frac{1}{(1-\rho)\,\sigma_\tau^2} \left[\mathbf{I}_{n_i} - \frac{\rho}{1+(n_i-1)\rho} \mathbf{J}_{n_i} \right] \tag{A.16.3}$$

and determinant

$$\Big| \mathbf{V}_i \Big| = \Big(\sigma_\tau^2 \Big)^{n_i} (1-\rho)^{n_i - 1} \Big[1 + (n_i - 1)\rho \Big] . \tag{A.16.4}$$

The total likelihood is

$$\begin{aligned} L &= \prod_{i=1}^{t} L_i \Big(\sigma_\tau^2, \boldsymbol{\beta}, \rho \Big) \\ &= \frac{\exp \Big\{ - \sum_{i=1}^{t} \Big(\mathbf{y}_i - \mathbf{X}_i \boldsymbol{\beta} \Big)' \mathbf{V}_i^{-1} \Big(\mathbf{y}_i - \mathbf{X}_i \boldsymbol{\beta} \Big) / 2 \Big\}}{(2\pi)^{N/2} \prod_{i=1}^{t} \Big| \mathbf{V}_i \Big|^{1/2}} . \end{aligned} \tag{A.16.5}$$

Consider the expression of $-2 \log L$:

$$\begin{aligned} -2 \log L &= N \log 2\pi + \sum_{i=1}^{t} n_i \log \sigma_\tau^2 + \sum_{i=1}^{t} (n_i - 1) \log(1-\rho) + \sum_{i=1}^{t} \log \Big[1 + (n_i - 1)\rho \Big] \\ &\quad + \sum_{i=1}^{t} \frac{\Big(\mathbf{y}_i - \mathbf{X}_i \boldsymbol{\beta} \Big)' \mathbf{I}_{n_i} \Big(\mathbf{y}_i - \mathbf{X}_i \boldsymbol{\beta} \Big)}{(1-\rho)\,\sigma_\tau^2} - \sum_{i=1}^{t} \rho \frac{\Big(\mathbf{y}_i - \mathbf{X}_i \boldsymbol{\beta} \Big)' \mathbf{J}_{n_i} \Big(\mathbf{y}_i - \mathbf{X}_i \boldsymbol{\beta} \Big)}{(1-\rho) \Big[1 + (n_i - 1)\rho \Big] \sigma_\tau^2} \\ &= N \log \sigma_\tau^2 + (N-t) \log(1-\rho) + \sum_{i=1}^{t} \log \Big[1 + (n_i - 1)\rho \Big] \\ &\quad + \frac{1}{(1-\rho)\,\sigma_T^2} \sum_{i=1}^{t} SSI_i - \frac{\rho}{(1-\rho)\,\sigma_T^2} \sum_{i=1}^{t} \frac{SSJ_i}{1 + (n_i - 1)\,\rho} + c \end{aligned}$$

where N is the total number of observations in all areas and is equal to $\sum_{i=1}^{t} n_i$, $SSI_i = \left(\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta}\right)' \mathbf{I}_{n_i} \left(\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta}\right)$, $SSJ_i = \left(\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta}\right)' \mathbf{J}_{n_i} \left(\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta}\right)$ with $\mathbf{I}_{n_i}$ an $n_i \times n_i$ identity matrix and $\mathbf{J}_{n_i}$ an $n_i \times n_i$ matrix of 1's and $c = N \log 2\pi$ which is a constant. It is well known that the maximum likelihood estimator of $\boldsymbol{\beta}$ is

$$\hat{\boldsymbol{\beta}} = \left(\sum_{i=1}^{t} \mathbf{X}_i' \mathbf{V}_i^{-1} \mathbf{X}_i\right)^{-1} \sum_{i=1}^{t} \mathbf{X}_i' \mathbf{V}_i^{-1} \mathbf{y}_i \,. \tag{A.16.6}$$

For the maximum likelihood estimator of σ_τ^2, consider the derivative of $-2 \log L$ with respect to σ_τ^2,

$$\frac{\partial - 2 \log L}{\partial \sigma_\tau^2} = \frac{N}{\sigma_\tau^2} - \frac{\sum_{i=1}^{t} SSI_i}{(1-\rho)\,\sigma_T^4} + \frac{\rho}{(1-\rho)\,\sigma_T^4} \sum_{i=1}^{t} \frac{SSJ_i}{1 + (n_i - 1)\,\rho} \,. \tag{A.16.7}$$

Setting this partial derivative to zero and solving for σ_τ^2, we obtain an expression of the maximum likelihood estimator of σ_τ^2,

$$\hat{\sigma}_\tau^2 = \frac{1}{N\,(1-\rho)} \sum_{i=1}^{t} \left(SSI_i - \frac{\rho\, SSJ_i}{1 + (n_i - 1)\,\rho}\right) \,. \tag{A.16.8}$$

For the maximum likelihood estimator of ρ, consider the derivative of $-2 \log L$ with respect to ρ,

$$\begin{aligned} \frac{\partial - 2 \log L}{\partial \rho} = & -\frac{N - t}{1 - \rho} + \sum_{i=1}^{t} \frac{n_i - 1}{1 + (n_i - 1)\rho} + \frac{\sum_{i=1}^{t} SSI_i}{(1-\rho)^2 \sigma_\tau^2} \\ & - \frac{1}{(1-\rho)^2 \sigma_\tau^2} \sum_{i=1}^{t} \frac{SSJ_i \left[1 + (n_i - 1)\rho^2\right]}{\left[1 + (n_i - 1)\,\rho\right]^2} \,. \end{aligned} \tag{A.16.9}$$

Setting this partial derivative equal to zero and replacing σ_τ^2 by its maximum likelihood estimator, the estimating equation for ρ is

$$N\left[\sum_{i=1}^{t}\left(SSI_i - \frac{\rho\, SSJ_i}{1+(n_i-1)\,\rho}\right)\right]^{-1} \qquad \sum_{i=1}^{t}\left(SSI_i - \frac{SSJ_i\Big[1+(n_i-1)\rho^2\Big]}{\Big[1+(n_i-1)\,\rho\Big]^2}\right)$$
$$- \sum_{i=1}^{t}\frac{n_i\,(n_i-1)\rho}{1+(n_i-1)\rho} = 0\ . \qquad \text{(A.16.10)}$$

Thus the maximum likelihood estimate of ρ can be obtained by solving this numerically, for example by Newton-Raphson method.

A.17 Newton-Raphson Method for the MLE of ρ

In appendix A.16, the maximum likelihood estimate of ρ is the root of equation (A.16.10), which can be solve numerically, for example by Newton-Raphson's iterative method. This method involves the first derivatives of the estimating function with respect to the variables in the function. Let $f(\rho)$ be the expression on the left hand side of the estimating function in equation A.16.10. Now let

$$\phi_1(\rho) \;=\; \sum_{i=1}^{t}\left(SSI_i - \frac{SSJ_i\Big[1+(n_i-1)\rho^2\Big]}{\Big[1+(n_i-1)\,\rho\Big]^2}\right) \qquad \text{(A.17.1)}$$

and

$$\phi_2(\rho) \;=\; \sum_{i=1}^{t}\left(SSI_i - \frac{\rho\, SSJ_i}{1+(n_i-1)\,\rho}\right) \qquad \text{(A.17.2)}$$

so that

$$f(\rho) = N\frac{\phi_1(\rho)}{\phi_2(\rho)} - \sum_{i=1}^{t}\frac{n_i\,(n_i-1)\rho}{1+(n_i-1)\rho}\,. \tag{A.17.3}$$

Differentiating $f(\rho)$ with respective to ρ, we have,

$$\begin{aligned} f'(\rho) &= N\left(\frac{1}{\phi_2(\rho)}\frac{d\,\phi_1(\rho)}{d\,\rho} - \frac{\phi_1(\rho)}{\phi_2^2(\rho)}\frac{d\,\phi_2(\rho)}{d\,\rho}\right) \\ &\quad -\sum_{i=1}^{t}\frac{n_i(n_i-1)}{\Big[1+(n_i-1)\rho\Big]^2} \end{aligned} \tag{A.17.4}$$

where

$$\begin{aligned} \frac{d\,\phi_1(\rho)}{d\,\rho} &= \frac{d}{d\,\rho}\sum_{i=1}^{t}\left(SSI_i - \frac{SSJ_i\Big[1+(n_i-1)\rho^2\Big]}{\Big[1+(n_i-1)\,\rho\Big]^2}\right) \\ &= \sum_{i=1}^{t}\left(\frac{d\,SSI_i}{d\,\rho} - \frac{d}{d\,\rho}\frac{SSJ_i\Big[1+(n_i-1)\rho^2\Big]}{\Big[1+(n_i-1)\,\rho\Big]^2}\right) \\ &= \sum_{i=1}^{t}\frac{d\,SSI_i}{d\,\rho} + \sum_{i=1}^{t}\frac{2\,SSJ_i\Big[1+(n_i-1)\rho^2\Big](n_i-1)}{\Big[1+(n_i-1)\,\rho\Big]^3} \\ &\quad -\sum_{i=1}^{t}\frac{\frac{d}{d\rho}SSJ_i\Big[1+(n_i-1)\rho^2\Big]}{\Big[1+(n_i-1)\,\rho\Big]^2} \\ &= \sum_{i=1}^{t}\frac{d\,SSI_i}{d\,\rho} + \sum_{i=1}^{t}\frac{2\,SSJ_i\Big[1+(n_i-1)\rho^2\Big](n_i-1)}{\Big[1+(n_i-1)\,\rho\Big]^3} \\ &\quad -\sum_{i=1}^{t}\Big[1+(n_i-1)\,\rho\Big]^{-2}\left[\frac{d\,SSJ_i}{d\,\rho}\Big[1+(n_i-1)\rho^2\Big] + 2\,(n_i-1)\rho\,SSJ_i\right] \\ &= \sum_{i=1}^{t}\frac{d\,SSI_i}{d\,\rho} - \sum_{i=1}^{t}\left[\frac{1+(n_i-1)\rho^2}{\Big[1+(n_i-1)\,\rho\Big]^2}\frac{d\,SSJ_i}{d\,\rho} - \frac{2\,(n_i-1)(1-\rho)\,SSJ_i}{\Big[1+(n_i-1)\,\rho\Big]^3}\right], \end{aligned}$$

and

$$\begin{aligned}
\frac{d\,\phi_2(\rho)}{d\,\rho} &= \frac{d}{d\,\rho}\sum_{i=1}^{t}\left(SSI_i - \frac{\rho\,SSJ_i}{1+(n_i-1)\,\rho}\right) \\
&= \sum_{i=1}^{t}\frac{d\,SSI_i}{d\,\rho} - \sum_{i=1}^{t}\frac{d}{d\,\rho}\frac{\rho\,SSJ_i}{1+(n_i-1)\,\rho} \\
&= \sum_{i=1}^{t}\frac{d\,SSI_i}{d\,\rho} - \sum_{i=1}^{t}\left(\frac{\rho\frac{d\,SSJ_i}{d\,\rho}+SSJ_i}{1+(n_i-1)\,\rho} - \frac{(n_i-1)\rho\,SSJ_i}{\left[1+(n_i-1)\,\rho\right]^2}\right) \\
&= \sum_{i=1}^{t}\frac{d\,SSI_i}{d\,\rho} - \sum_{i=1}^{t}\left(\frac{\rho\frac{d\,SSJ_i}{d\,\rho}}{1+(n_i-1)\,\rho} + \frac{\left[1+(n_i-1)\,\rho\right]SSJ_i-(n_i-1)\rho\,SSJ_i}{\left[1+(n_i-1)\,\rho\right]^2}\right) \\
&= \sum_{i=1}^{t}\frac{d\,SSI_i}{d\,\rho} - \sum_{i=1}^{t}\left(\frac{\rho\frac{d\,SSJ_i}{d\,\rho}}{1+(n_i-1)\,\rho} + \frac{SSJ_i}{\left[1+(n_i-1)\,\rho\right]^2}\right) .
\end{aligned}$$

The derivative of SSI_i, SSJ_i and $\boldsymbol{\beta}$ can be found following the procedures below. Define a variable $\mathbf{W} = \mathrm{diag}(\mathbf{W}_1, \ldots, \mathbf{W}_t)$ such that $\mathbf{V}_i = \sigma_\tau^2\mathbf{W}_i$ and

$$\mathbf{W}_i^{-1} = \sigma_\tau^2\mathbf{V}_i^{-1} = \frac{1}{(1-\rho)}\left[\mathbf{I}_{n_i} - \frac{\rho}{1+(n_i-1)\rho}\mathbf{J}_{n_i}\right] . \qquad \text{(A.17.5)}$$

Note that the maximum likelihood estimator of $\boldsymbol{\beta}$ is

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y} = \left(\mathbf{X}'\mathbf{W}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{W}^{-1}\mathbf{y} . \qquad \text{(A.17.6)}$$

Replacing $\boldsymbol{\beta}$ by $\hat{\boldsymbol{\beta}}$ in SSI_i, we have

$$\begin{aligned}
\frac{d\,SSI_i}{d\,\rho} &= \frac{d}{d\,\rho}\left(\mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}\right)'\left(\mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}\right) \\
&= \left[\frac{d}{d\,\rho}\left(\mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}\right)'\right]\left(\mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}\right) + \left(\mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}\right)'\left[\frac{d}{d\,\rho}\left(\mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}\right)\right] \\
&= -\left(\mathbf{X}_i\frac{d\,\hat{\boldsymbol{\beta}}}{d\,\rho}\right)'\left(\mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}\right) - \left(\mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}\right)'\left(\mathbf{X}_i\frac{d\,\hat{\boldsymbol{\beta}}}{d\,\rho}\right) .
\end{aligned}$$

Similarly, the derivative of SSJ_i is

$$\begin{aligned}
\frac{d\,SSJ_i}{d\,\rho} &= \frac{d}{d\,\rho}\left(\mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}\right)'\mathbf{J}_{n_i}\left(\mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}\right) \\
&= \left[\frac{d}{d\,\rho}\left(\mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}\right)'\right]\mathbf{J}_{n_i}\left(\mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}\right) + \left(\mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}\right)'\mathbf{J}_{n_i}\left[\frac{d}{d\,\rho}\left(\mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}\right)\right] \\
&= -\left(\mathbf{X}_i\frac{d\,\hat{\boldsymbol{\beta}}}{d\,\rho}\right)'\mathbf{J}_{n_i}\left(\mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}\right) - \left(\mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}\right)'\mathbf{J}_{n_i}\left(\mathbf{X}_i\frac{d\,\hat{\boldsymbol{\beta}}}{d\,\rho}\right) .
\end{aligned}$$

Both derivatives of SSI_i and SSJ_i are functions of the derivative of $\hat{\boldsymbol{\beta}}$ with respect to ρ which is

$$\begin{aligned}
\frac{d\,\hat{\boldsymbol{\beta}}}{d\,\rho} &= \left[\frac{d}{d\,\rho}\left(\mathbf{X}'\mathbf{W}^{-1}\mathbf{X}\right)^{-1}\right]\mathbf{X}'\mathbf{W}^{-1}\mathbf{y} + \left(\mathbf{X}'\mathbf{W}^{-1}\mathbf{X}\right)^{-1}\left[\frac{d}{d\,\rho}\mathbf{X}'\mathbf{W}^{-1}\mathbf{y}\right] \\
&= -\left(\mathbf{X}'\mathbf{W}^{-1}\mathbf{X}\right)^{-1}\left[\frac{d}{d\,\rho}\left(\mathbf{X}'\mathbf{W}^{-1}\mathbf{X}\right)\right]\left(\mathbf{X}'\mathbf{W}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{W}^{-1}\mathbf{y} \\
&\quad + \left(\mathbf{X}'\mathbf{W}^{-1}\mathbf{X}\right)^{-1}\left[\frac{d}{d\,\rho}\mathbf{X}'\mathbf{W}^{-1}\mathbf{y}\right].
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{d}{d\,\rho}\mathbf{W}_i^{-1} &= \frac{d}{d\,\rho}\frac{1}{(1-\rho)}\left(\mathbf{I}_{n_i} - \frac{\rho}{1+(n_i-1)\rho}\mathbf{J}_{n_i}\right) \\
&= \frac{1}{(1-\rho)^2}\left(\mathbf{I}_{n_i} - \frac{\left[1+(n_i-1)\rho^2\right]}{\left[1+(n_i-1)\,\rho\right]^2}\mathbf{J}_{n_i}\right)
\end{aligned}$$

Differentiating $\mathbf{X}'\mathbf{W}^{-1}\mathbf{X}$ with respect to ρ, we obtain

$$\frac{d}{d\rho}\mathbf{X}'\mathbf{W}^{-1}\mathbf{X} = \mathbf{X}'\mathbf{W}_d^{-1}\mathbf{X}$$

where $\mathbf{W}_d^{-1} = \text{diag}\left(\frac{d}{d\rho}\mathbf{W}_1^{-1}, \ldots, \frac{d}{d\rho}\mathbf{W}_t^{-1}\right)$. Similarly, the derivative of $\mathbf{X}'\mathbf{W}^{-1}\mathbf{y}$ is given by

$$\frac{d}{d\rho}\mathbf{X}'\mathbf{W}^{-1}\mathbf{y} = \mathbf{X}'\mathbf{W}_d^{-1}\mathbf{y}$$

Appendix B

TABLES: BIASES OF SMALL AREA MEAN ESTIMATES

(1) with population covariate means (from p. 143 to 152)

(2) with sample covariate means (from p. 153 to 162)

a_i: weight defined in equation (2.7.4)

$\tilde{\mu}^{(1)}$: EBLUP defined in equation (2.7.4)

$\tilde{\mu}^{(2)}$: EBLUP defined in equation (2.7.4)

$\tilde{\mu}^{c}$: composite estimator defined in equation (2.7.4)

$\tilde{\mu}$: estimator defined in equation (3.2.5)

$\tilde{\mu}^{Vm}$: EBLUP with M.L.E. variance estimators

$\tilde{\mu}^{(S)}$: EBLUP with Residual M.L.E. variance estimators

$\tilde{\mu}^{(T)}$: EBLUP with Restricted M.L.E. variance estimators

$\tilde{\mu}^{ps}$: PSEUDO-EBLUP by You & Rao

Table B.1: Bias of small-area mean estimators with known population covariates under normal random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	−0.0027	−0.0031	−0.0032	−0.0042	−0.0055	−0.0066	−0.0076	−0.0085	−0.0106	−0.0129
	Q1	−0.0023	−0.0015	−0.0017	−0.0027	−0.0028	−0.0023	−0.0019	−0.0022	−0.0026	−0.0030
	median	−0.0013	−0.0006	−0.0001	−0.0001	0.0001	0.0005	0.0010	0.0012	0.0012	0.0014
	Q3	−0.0008	0.0001	0.0008	0.0018	0.0027	0.0034	0.0043	0.0051	0.0059	0.0068
	max	0.0015	0.0022	0.0028	0.0038	0.0048	0.0059	0.0073	0.0088	0.0105	0.0123
$\tilde{\mu}^{(2)}$	min	−0.0028	−0.0029	−0.0030	−0.0038	−0.0046	−0.0055	−0.0066	−0.0084	−0.0105	−0.0129
	Q1	−0.0021	−0.0015	−0.0017	−0.0023	−0.0021	−0.0018	−0.0021	−0.0022	−0.0024	−0.0028
	median	−0.0017	−0.0008	−0.0004	0.0000	0.0002	0.0006	0.0012	0.0011	0.0013	0.0018
	Q3	−0.0010	0.0000	0.0009	0.0020	0.0026	0.0035	0.0045	0.0055	0.0062	0.0068
	max	0.0020	0.0022	0.0025	0.0035	0.0047	0.0059	0.0074	0.0091	0.0108	0.0126
$\tilde{\mu}^{c}$	min	−0.0028	−0.0030	−0.0029	−0.0037	−0.0049	−0.0061	−0.0072	−0.0084	−0.0105	−0.0129
	Q1	−0.0021	−0.0016	−0.0017	−0.0024	−0.0023	−0.0018	−0.0017	−0.0021	−0.0025	−0.0029
	median	−0.0017	−0.0008	−0.0003	0.0000	0.0002	0.0006	0.0011	0.0011	0.0012	0.0015
	Q3	−0.0010	0.0000	0.0008	0.0020	0.0025	0.0035	0.0045	0.0054	0.0061	0.0067
	max	0.0016	0.0022	0.0025	0.0036	0.0048	0.0059	0.0073	0.0089	0.0106	0.0124
$\tilde{\mu}$	min	−0.0028	−0.0032	−0.0031	−0.0038	−0.0049	−0.0061	−0.0071	−0.0085	−0.0106	−0.0130
	Q1	−0.0021	−0.0015	−0.0017	−0.0027	−0.0024	−0.0019	−0.0018	−0.0021	−0.0025	−0.0029
	median	−0.0016	−0.0009	−0.0002	−0.0002	0.0001	0.0006	0.0011	0.0011	0.0011	0.0016
	Q3	−0.0008	0.0001	0.0008	0.0021	0.0025	0.0035	0.0045	0.0053	0.0061	0.0067
	max	0.0017	0.0022	0.0027	0.0038	0.0048	0.0060	0.0074	0.0090	0.0107	0.0125

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{Vm}$	min	−0.0029	−0.0021	−0.0025	−0.0035	−0.0041	−0.0053	−0.0065	−0.0083	−0.0103	−0.0130
	Q1	−0.0020	−0.0007	−0.0013	−0.0024	−0.0018	−0.0015	−0.0016	−0.0019	−0.0023	−0.0030
	median	−0.0014	−0.0001	−0.0001	−0.0002	0.0003	0.0005	0.0010	0.0010	0.0012	0.0015
	Q3	−0.0007	0.0010	0.0014	0.0020	0.0030	0.0038	0.0046	0.0055	0.0062	0.0065
	max	0.0026	0.0027	0.0037	0.0042	0.0056	0.0066	0.0077	0.0093	0.0109	0.0125
$\tilde{\mu}^{(S)}$	min	−0.0029	−0.0025	−0.0024	−0.0034	−0.0044	−0.0060	−0.0073	−0.0088	−0.0107	−0.0131
	Q1	−0.0016	−0.0009	−0.0011	−0.0020	−0.0017	−0.0019	−0.0022	−0.0024	−0.0027	−0.0032
	median	−0.0013	−0.0003	0.0003	0.0001	0.0003	0.0003	0.0005	0.0005	0.0009	0.0014
	Q3	−0.0003	0.0007	0.0015	0.0022	0.0031	0.0033	0.0041	0.0051	0.0059	0.0065
	max	0.0032	0.0027	0.0036	0.0044	0.0055	0.0062	0.0074	0.0090	0.0108	0.0124
$\tilde{\mu}^{(T)}$	min	−0.0020	−0.0030	−0.0028	−0.0037	−0.0052	−0.0059	−0.0071	−0.0086	−0.0106	−0.0130
	Q1	−0.0014	−0.0016	−0.0014	−0.0022	−0.0022	−0.0017	−0.0016	−0.0023	−0.0026	−0.0030
	median	−0.0011	−0.0007	0.0000	0.0001	−0.0003	0.0006	0.0011	0.0009	0.0012	0.0015
	Q3	−0.0001	0.0001	0.0012	0.0020	0.0026	0.0036	0.0047	0.0053	0.0060	0.0066
	max	0.0028	0.0024	0.0030	0.0041	0.0048	0.0062	0.0076	0.0090	0.0107	0.0124
$\tilde{\mu}^{ps}$	min	−0.0028	−0.0033	−0.0033	−0.0036	−0.0048	−0.0060	−0.0070	−0.0085	−0.0107	−0.0130
	Q1	−0.0019	−0.0013	−0.0013	−0.0025	−0.0025	−0.0019	−0.0018	−0.0021	−0.0025	−0.0029
	median	−0.0013	−0.0008	−0.0008	0.0000	0.0001	0.0005	0.0010	0.0013	0.0013	0.0016
	Q3	−0.0006	0.0000	0.0000	0.0020	0.0026	0.0033	0.0043	0.0053	0.0062	0.0071
	max	0.0024	0.0026	0.0026	0.0038	0.0046	0.0057	0.0070	0.0085	0.0103	0.0125

Table B.2: Bias of small-area mean estimators with known population covariates under normal random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	−0.0155	−0.0185	−0.0218	−0.0257	−0.0304	−0.0361	−0.0435	−0.0540	−0.0708	−0.1074
	Q1	−0.0034	−0.0039	−0.0044	−0.0051	−0.0058	−0.0066	−0.0076	−0.0090	−0.0112	−0.0135
	median	0.0020	0.0028	0.0041	0.0057	0.0075	0.0091	0.0107	0.0122	0.0147	0.0226
	Q3	0.0074	0.0089	0.0104	0.0121	0.0140	0.0159	0.0183	0.0212	0.0267	0.0392
	max	0.0145	0.0172	0.0202	0.0238	0.0279	0.0331	0.0397	0.0491	0.0642	0.0967
$\tilde{\mu}^{(2)}$	min	−0.0155	−0.0185	−0.0218	−0.0257	−0.0304	−0.0361	−0.0435	−0.0540	−0.0708	−0.1074
	Q1	−0.0033	−0.0038	−0.0043	−0.0049	−0.0057	−0.0065	−0.0075	−0.0090	−0.0113	−0.0136
	median	0.0026	0.0036	0.0049	0.0064	0.0080	0.0094	0.0109	0.0123	0.0148	0.0227
	Q3	0.0076	0.0087	0.0103	0.0120	0.0138	0.0157	0.0181	0.0208	0.0269	0.0389
	max	0.0147	0.0173	0.0202	0.0237	0.0278	0.0329	0.0395	0.0489	0.0639	0.0965
$\tilde{\mu}^{c}$	min	−0.0155	−0.0185	−0.0218	−0.0257	−0.0304	−0.0361	−0.0435	−0.0540	−0.0708	−0.1074
	Q1	−0.0034	−0.0039	−0.0044	−0.0051	−0.0058	−0.0066	−0.0076	−0.0090	−0.0112	−0.0136
	median	0.0021	0.0029	0.0042	0.0057	0.0075	0.0091	0.0107	0.0122	0.0147	0.0227
	Q3	0.0075	0.0089	0.0104	0.0121	0.0140	0.0159	0.0183	0.0212	0.0267	0.0389
	max	0.0146	0.0172	0.0202	0.0238	0.0279	0.0331	0.0397	0.0491	0.0642	0.0967
$\tilde{\mu}$	min	−0.0156	−0.0185	−0.0219	−0.0258	−0.0304	−0.0361	−0.0436	−0.0540	−0.0708	−0.1074
	Q1	−0.0034	−0.0038	−0.0044	−0.0050	−0.0058	−0.0066	−0.0076	−0.0090	−0.0112	−0.0136
	median	0.0022	0.0033	0.0046	0.0061	0.0077	0.0093	0.0107	0.0122	0.0147	0.0226
	Q3	0.0075	0.0089	0.0104	0.0121	0.0139	0.0158	0.0183	0.0211	0.0268	0.0391
	max	0.0148	0.0174	0.0204	0.0239	0.0280	0.0331	0.0398	0.0491	0.0641	0.0967

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{Vm}$	min	−0.0154	−0.0183	−0.0217	−0.0256	−0.0302	−0.0360	−0.0434	−0.0539	−0.0708	−0.1073
	Q1	−0.0034	−0.0039	−0.0044	−0.0050	−0.0057	−0.0066	−0.0076	−0.0090	−0.0112	−0.0135
	median	0.0021	0.0031	0.0044	0.0058	0.0076	0.0091	0.0107	0.0122	0.0147	0.0226
	Q3	0.0074	0.0088	0.0104	0.0120	0.0139	0.0158	0.0182	0.0212	0.0266	0.0392
	max	0.0144	0.0171	0.0201	0.0236	0.0278	0.0330	0.0396	0.0490	0.0641	0.0967
$\tilde{\mu}^{(S)}$	min	−0.0157	−0.0187	−0.0221	−0.0260	−0.0307	−0.0364	−0.0438	−0.0543	−0.0712	−0.1079
	Q1	−0.0036	−0.0041	−0.0047	−0.0053	−0.0061	−0.0069	−0.0079	−0.0093	−0.0117	−0.0142
	median	0.0020	0.0028	0.0041	0.0056	0.0074	0.0091	0.0109	0.0126	0.0152	0.0226
	Q3	0.0073	0.0087	0.0103	0.0120	0.0139	0.0160	0.0185	0.0213	0.0256	0.0371
	max	0.0144	0.0171	0.0200	0.0235	0.0277	0.0328	0.0395	0.0489	0.0640	0.0966
$\tilde{\mu}^{(T)}$	min	−0.0155	−0.0185	−0.0219	−0.0258	−0.0304	−0.0361	−0.0435	−0.0540	−0.0708	−0.1074
	Q1	−0.0034	−0.0039	−0.0044	−0.0051	−0.0058	−0.0066	−0.0076	−0.0090	−0.0112	−0.0135
	median	0.0021	0.0030	0.0043	0.0058	0.0076	0.0091	0.0107	0.0122	0.0147	0.0226
	Q3	0.0074	0.0089	0.0104	0.0121	0.0139	0.0159	0.0183	0.0212	0.0267	0.0392
	max	0.0146	0.0172	0.0203	0.0238	0.0279	0.0331	0.0397	0.0491	0.0642	0.0967
$\tilde{\mu}^{ps}$	min	−0.0157	−0.0186	−0.0220	−0.0259	−0.0306	−0.0363	−0.0437	−0.0542	−0.0710	−0.1076
	Q1	−0.0034	−0.0039	−0.0044	−0.0051	−0.0058	−0.0067	−0.0077	−0.0091	−0.0114	−0.0138
	median	0.0022	0.0032	0.0044	0.0060	0.0078	0.0095	0.0111	0.0126	0.0150	0.0226
	Q3	0.0078	0.0087	0.0104	0.0122	0.0141	0.0162	0.0186	0.0213	0.0264	0.0382
	max	0.0148	0.0174	0.0204	0.0239	0.0280	0.0331	0.0397	0.0491	0.0641	0.0967

Table B.3: Bias of small-area mean estimators with known population covariates under double exponential random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	−0.0040	−0.0047	−0.0057	−0.0068	−0.0078	−0.0087	−0.0095	−0.0102	−0.0108	−0.0113
	Q1	−0.0020	−0.0020	−0.0027	−0.0027	−0.0033	−0.0042	−0.0047	−0.0050	−0.0052	−0.0054
	median	−0.0016	−0.0018	−0.0013	−0.0012	−0.0016	−0.0013	−0.0007	0.0000	0.0008	0.0011
	Q3	−0.0004	−0.0002	−0.0001	0.0009	0.0015	0.0021	0.0031	0.0038	0.0040	0.0042
	max	0.0003	0.0026	0.0049	0.0075	0.0101	0.0130	0.0161	0.0192	0.0223	0.0258
$\tilde{\mu}^{(2)}$	min	−0.0040	−0.0043	−0.0052	−0.0063	−0.0071	−0.0078	−0.0085	−0.0090	−0.0096	−0.0107
	Q1	−0.0019	−0.0019	−0.0023	−0.0023	−0.0026	−0.0033	−0.0039	−0.0043	−0.0047	−0.0051
	median	−0.0014	−0.0013	−0.0012	−0.0017	−0.0013	−0.0010	−0.0005	0.0000	0.0006	0.0011
	Q3	−0.0006	−0.0002	0.0000	0.0010	0.0015	0.0024	0.0031	0.0036	0.0040	0.0043
	max	0.0006	0.0027	0.0046	0.0073	0.0099	0.0126	0.0154	0.0183	0.0215	0.0248
$\tilde{\mu}^{c}$	min	−0.0040	−0.0044	−0.0054	−0.0066	−0.0075	−0.0084	−0.0093	−0.0100	−0.0107	−0.0113
	Q1	−0.0019	−0.0019	−0.0023	−0.0024	−0.0028	−0.0034	−0.0041	−0.0046	−0.0050	−0.0053
	median	−0.0015	−0.0014	−0.0014	−0.0016	−0.0012	−0.0008	−0.0003	0.0003	0.0010	0.0012
	Q3	−0.0005	−0.0002	0.0000	0.0010	0.0015	0.0023	0.0032	0.0038	0.0041	0.0043
	max	0.0002	0.0024	0.0047	0.0073	0.0100	0.0128	0.0159	0.0190	0.0223	0.0258
$\tilde{\mu}$	min	−0.0043	−0.0047	−0.0058	−0.0069	−0.0078	−0.0086	−0.0093	−0.0099	−0.0104	−0.0109
	Q1	−0.0020	−0.0022	−0.0023	−0.0028	−0.0031	−0.0040	−0.0043	−0.0046	−0.0049	−0.0052
	median	−0.0014	−0.0015	−0.0014	−0.0014	−0.0014	−0.0010	−0.0004	0.0001	0.0007	0.0010
	Q3	−0.0005	0.0000	0.0001	0.0010	0.0017	0.0024	0.0033	0.0037	0.0040	0.0043
	max	0.0003	0.0028	0.0051	0.0077	0.0103	0.0130	0.0160	0.0190	0.0222	0.0256
$\tilde{\mu}^{ps}$	min	−0.0039	−0.0040	−0.0040	−0.0061	−0.0070	−0.0078	−0.0085	−0.0092	−0.0098	−0.0102
	Q1	−0.0019	−0.0017	−0.0017	−0.0020	−0.0024	−0.0031	−0.0035	−0.0038	−0.0042	−0.0045
	median	−0.0013	−0.0009	−0.0009	−0.0007	−0.0007	−0.0002	0.0003	0.0007	0.0013	0.0016
	Q3	−0.0002	0.0008	0.0008	0.0015	0.0022	0.0029	0.0038	0.0043	0.0046	0.0048
	max	0.0005	0.0031	0.0031	0.0081	0.0109	0.0137	0.0166	0.0196	0.0228	0.0262

Table B.4: Bias of small-area mean estimators with known population covariates under double exponential random effects for ρ from 0.50 to 0.95

ρ		0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	−0.0123	−0.0144	−0.0169	−0.0198	−0.0233	−0.0276	−0.0332	−0.0410	−0.0537	−0.0809
	Q1	−0.0056	−0.0062	−0.0069	−0.0078	−0.0088	−0.0098	−0.0109	−0.0125	−0.0150	−0.0202
	median	0.0015	0.0023	0.0031	0.0033	0.0035	0.0038	0.0041	0.0050	0.0078	0.0116
	Q3	0.0044	0.0047	0.0054	0.0066	0.0084	0.0106	0.0133	0.0169	0.0230	0.0353
	max	0.0295	0.0334	0.0377	0.0425	0.0482	0.0552	0.0642	0.0767	0.0967	0.1400
$\tilde{\mu}^{(2)}$	min	−0.0124	−0.0144	−0.0168	−0.0196	−0.0230	−0.0272	−0.0328	−0.0406	−0.0533	−0.0806
	Q1	−0.0055	−0.0061	−0.0069	−0.0078	−0.0085	−0.0094	−0.0106	−0.0122	−0.0147	−0.0200
	median	0.0018	0.0025	0.0034	0.0037	0.0039	0.0042	0.0045	0.0048	0.0076	0.0119
	Q3	0.0046	0.0048	0.0054	0.0062	0.0079	0.0101	0.0129	0.0170	0.0227	0.0353
	max	0.0285	0.0324	0.0366	0.0415	0.0473	0.0543	0.0633	0.0759	0.0961	0.1395
$\tilde{\mu}^{c}$	min	−0.0123	−0.0144	−0.0169	−0.0198	−0.0233	−0.0276	−0.0332	−0.0410	−0.0537	−0.0809
	Q1	−0.0055	−0.0061	−0.0069	−0.0078	−0.0088	−0.0098	−0.0109	−0.0125	−0.0150	−0.0200
	median	0.0019	0.0027	0.0035	0.0037	0.0039	0.0041	0.0044	0.0052	0.0079	0.0119
	Q3	0.0045	0.0047	0.0054	0.0066	0.0084	0.0106	0.0133	0.0169	0.0230	0.0353
	max	0.0295	0.0334	0.0377	0.0425	0.0482	0.0552	0.0642	0.0767	0.0967	0.1400
$\tilde{\mu}$	min	−0.0124	−0.0145	−0.0170	−0.0198	−0.0232	−0.0275	−0.0331	−0.0409	−0.0536	−0.0808
	Q1	−0.0056	−0.0063	−0.0070	−0.0079	−0.0088	−0.0097	−0.0109	−0.0124	−0.0149	−0.0202
	median	0.0017	0.0025	0.0032	0.0035	0.0037	0.0040	0.0043	0.0049	0.0077	0.0117
	Q3	0.0046	0.0047	0.0053	0.0064	0.0082	0.0104	0.0132	0.0170	0.0229	0.0353
	max	0.0293	0.0331	0.0374	0.0423	0.0480	0.0550	0.0640	0.0765	0.0966	0.1399
$\tilde{\mu}^{ps}$	min	−0.0119	−0.0141	−0.0165	−0.0194	−0.0229	−0.0272	−0.0328	−0.0407	−0.0533	−0.0807
	Q1	−0.0052	−0.0059	−0.0067	−0.0075	−0.0086	−0.0095	−0.0107	−0.0123	−0.0148	−0.0201
	median	0.0023	0.0030	0.0036	0.0039	0.0041	0.0043	0.0046	0.0053	0.0081	0.0119
	Q3	0.0049	0.0053	0.0057	0.0068	0.0085	0.0106	0.0132	0.0169	0.0227	0.0348
	max	0.0298	0.0337	0.0380	0.0429	0.0486	0.0556	0.0645	0.0771	0.0971	0.1404

Table B.5: Bias of small-area mean estimators with known population covariates under exponential random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	−0.0092	−0.0100	−0.0097	−0.0102	−0.0112	−0.0122	−0.0134	−0.0156	−0.0180	−0.0205
	Q1	−0.0050	−0.0061	−0.0048	−0.0042	−0.0026	−0.0036	−0.0037	−0.0050	−0.0051	−0.0060
	median	0.0031	0.0029	0.0025	0.0013	0.0012	0.0000	0.0006	0.0007	0.0011	0.0017
	Q3	0.0068	0.0071	0.0062	0.0053	0.0051	0.0049	0.0052	0.0048	0.0047	0.0043
	max	0.0166	0.0172	0.0173	0.0171	0.0169	0.0166	0.0163	0.0163	0.0164	0.0169
$\tilde{\mu}^{(2)}$	min	−0.0016	−0.0019	−0.0044	−0.0068	−0.0092	−0.0114	−0.0134	−0.0152	−0.0168	−0.0191
	Q1	−0.0005	−0.0012	−0.0025	−0.0036	−0.0053	−0.0066	−0.0076	−0.0085	−0.0092	−0.0102
	median	0.0002	−0.0005	−0.0011	−0.0027	−0.0035	−0.0041	−0.0044	−0.0046	−0.0049	−0.0050
	Q3	0.0008	−0.0002	0.0002	0.0006	0.0001	0.0007	0.0015	0.0024	0.0034	0.0045
	max	0.0031	0.0034	0.0079	0.0118	0.0153	0.0184	0.0209	0.0229	0.0247	0.0263
$\tilde{\mu}^{c}$	min	−0.0078	−0.0061	−0.0048	−0.0067	−0.0088	−0.0108	−0.0128	−0.0149	−0.0171	−0.0196
	Q1	−0.0026	−0.0028	−0.0025	−0.0028	−0.0026	−0.0032	−0.0035	−0.0038	−0.0041	−0.0044
	median	−0.0003	−0.0014	−0.0012	−0.0006	0.0004	0.0010	0.0015	0.0019	0.0024	0.0029
	Q3	0.0006	−0.0003	0.0001	0.0008	0.0018	0.0029	0.0038	0.0049	0.0056	0.0061
	max	0.0031	0.0031	0.0037	0.0076	0.0118	0.0153	0.0182	0.0205	0.0223	0.0240
$\tilde{\mu}$	min	−0.0066	−0.0063	−0.0062	−0.0071	−0.0090	−0.0111	−0.0131	−0.0152	−0.0175	−0.0199
	Q1	−0.0025	−0.0033	−0.0025	−0.0021	−0.0027	−0.0039	−0.0052	−0.0056	−0.0061	−0.0073
	median	0.0019	0.0016	0.0005	0.0004	−0.0004	−0.0007	−0.0004	0.0000	−0.0004	−0.0005
	Q3	0.0049	0.0047	0.0036	0.0031	0.0020	0.0024	0.0018	0.0020	0.0030	0.0039
	max	0.0081	0.0083	0.0079	0.0073	0.0075	0.0105	0.0131	0.0153	0.0172	0.0191
$\tilde{\mu}^{ps}$	min	−0.0089	−0.0085	−0.0085	−0.0085	−0.0101	−0.0119	−0.0136	−0.0155	−0.0176	−0.0199
	Q1	−0.0048	−0.0054	−0.0054	−0.0035	−0.0038	−0.0048	−0.0057	−0.0058	−0.0062	−0.0071
	median	−0.0001	−0.0005	−0.0005	−0.0011	−0.0015	−0.0015	−0.0009	−0.0004	−0.0005	−0.0005
	Q3	0.0030	0.0027	0.0027	0.0016	0.0009	0.0016	0.0013	0.0017	0.0028	0.0039
	max	0.0063	0.0063	0.0063	0.0057	0.0065	0.0097	0.0125	0.0149	0.0170	0.0189

Table B.6: Bias of small-area mean estimators with known population covariates under exponential random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	−0.0232	−0.0262	−0.0295	−0.0334	−0.0379	−0.0435	−0.0508	−0.0611	−0.0779	−0.1144
	Q1	−0.0066	−0.0072	−0.0079	−0.0086	−0.0094	−0.0104	−0.0116	−0.0133	−0.0159	−0.0216
	median	0.0021	0.0018	0.0011	0.0005	−0.0001	−0.0006	−0.0012	−0.0017	−0.0023	−0.0033
	Q3	0.0047	0.0051	0.0055	0.0060	0.0067	0.0076	0.0088	0.0112	0.0156	0.0256
	max	0.0177	0.0188	0.0203	0.0224	0.0251	0.0288	0.0339	0.0415	0.0542	0.0822
$\tilde{\mu}^{(2)}$	min	−0.0216	−0.0244	−0.0276	−0.0313	−0.0358	−0.0413	−0.0487	−0.0591	−0.0761	−0.1130
	Q1	−0.0119	−0.0131	−0.0142	−0.0155	−0.0166	−0.0171	−0.0176	−0.0180	−0.0186	−0.0232
	median	−0.0047	−0.0042	−0.0037	−0.0041	−0.0043	−0.0039	−0.0034	−0.0028	−0.0023	−0.0040
	Q3	0.0054	0.0061	0.0065	0.0066	0.0063	0.0071	0.0103	0.0130	0.0156	0.0239
	max	0.0278	0.0293	0.0309	0.0325	0.0344	0.0366	0.0394	0.0433	0.0498	0.0688
$\tilde{\mu}^{c}$	min	−0.0222	−0.0251	−0.0284	−0.0321	−0.0367	−0.0423	−0.0498	−0.0603	−0.0774	−0.1143
	Q1	−0.0047	−0.0051	−0.0057	−0.0063	−0.0071	−0.0079	−0.0097	−0.0122	−0.0150	−0.0232
	median	0.0027	0.0023	0.0020	0.0018	0.0016	0.0015	0.0013	0.0006	−0.0009	−0.0040
	Q3	0.0065	0.0067	0.0072	0.0078	0.0085	0.0093	0.0103	0.0123	0.0164	0.0239
	max	0.0255	0.0269	0.0281	0.0294	0.0309	0.0328	0.0353	0.0415	0.0542	0.0822
$\tilde{\mu}$	min	−0.0225	−0.0255	−0.0288	−0.0326	−0.0371	−0.0426	−0.0500	−0.0604	−0.0772	−0.1139
	Q1	−0.0084	−0.0092	−0.0099	−0.0106	−0.0113	−0.0118	−0.0126	−0.0144	−0.0169	−0.0223
	median	−0.0006	−0.0007	−0.0008	−0.0008	−0.0007	−0.0007	−0.0011	−0.0016	−0.0025	−0.0045
	Q3	0.0049	0.0056	0.0063	0.0070	0.0078	0.0087	0.0099	0.0116	0.0146	0.0248
	max	0.0209	0.0226	0.0244	0.0264	0.0286	0.0312	0.0345	0.0389	0.0463	0.0763
$\tilde{\mu}^{ps}$	min	−0.0224	−0.0253	−0.0285	−0.0322	−0.0367	−0.0423	−0.0496	−0.0600	−0.0769	−0.1136
	Q1	−0.0080	−0.0087	−0.0093	−0.0099	−0.0104	−0.0107	−0.0121	−0.0137	−0.0163	−0.0217
	median	−0.0005	−0.0006	−0.0006	−0.0005	−0.0004	−0.0006	−0.0010	−0.0015	−0.0025	−0.0046
	Q3	0.0050	0.0057	0.0065	0.0072	0.0081	0.0090	0.0102	0.0118	0.0152	0.0254
	max	0.0208	0.0226	0.0244	0.0264	0.0286	0.0312	0.0344	0.0389	0.0477	0.0776

Table B.7: Bias of small-area mean estimators with known population covariates under uniform random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	−0.0014	−0.0032	−0.0046	−0.0056	−0.0067	−0.0084	−0.0104	−0.0125	−0.0148	−0.0171
	Q1	−0.0002	−0.0016	−0.0026	−0.0031	−0.0035	−0.0045	−0.0055	−0.0066	−0.0077	−0.0084
	median	0.0006	−0.0005	−0.0009	−0.0013	−0.0018	−0.0023	−0.0025	−0.0025	−0.0025	−0.0025
	Q3	0.0018	0.0008	0.0014	0.0022	0.0029	0.0032	0.0028	0.0024	0.0020	0.0025
	max	0.0038	0.0053	0.0061	0.0070	0.0081	0.0091	0.0102	0.0114	0.0128	0.0142
$\tilde{\mu}^{(2)}$	min	−0.0014	−0.0039	−0.0050	−0.0062	−0.0075	−0.0087	−0.0106	−0.0125	−0.0146	−0.0167
	Q1	−0.0001	−0.0013	−0.0022	−0.0027	−0.0036	−0.0046	−0.0054	−0.0061	−0.0069	−0.0079
	median	0.0008	−0.0001	−0.0009	−0.0012	−0.0016	−0.0019	−0.0020	−0.0021	−0.0022	−0.0023
	Q3	0.0020	0.0012	0.0018	0.0023	0.0029	0.0032	0.0030	0.0026	0.0022	0.0025
	max	0.0034	0.0043	0.0052	0.0065	0.0077	0.0088	0.0099	0.0111	0.0124	0.0139
$\tilde{\mu}^{c}$	min	−0.0014	−0.0035	−0.0048	−0.0060	−0.0073	−0.0085	−0.0103	−0.0122	−0.0143	−0.0167
	Q1	−0.0002	−0.0013	−0.0023	−0.0028	−0.0036	−0.0045	−0.0054	−0.0062	−0.0073	−0.0082
	median	0.0008	−0.0002	−0.0008	−0.0015	−0.0019	−0.0021	−0.0022	−0.0023	−0.0023	−0.0024
	Q3	0.0020	0.0008	0.0016	0.0023	0.0029	0.0032	0.0029	0.0025	0.0021	0.0025
	max	0.0034	0.0044	0.0055	0.0067	0.0078	0.0089	0.0100	0.0113	0.0126	0.0141
$\tilde{\mu}$	min	−0.0016	−0.0034	−0.0048	−0.0058	−0.0070	−0.0084	−0.0103	−0.0123	−0.0145	−0.0168
	Q1	−0.0002	−0.0014	−0.0024	−0.0029	−0.0036	−0.0045	−0.0054	−0.0063	−0.0074	−0.0082
	median	0.0006	−0.0001	−0.0008	−0.0014	−0.0018	−0.0021	−0.0022	−0.0022	−0.0023	−0.0023
	Q3	0.0019	0.0010	0.0015	0.0021	0.0028	0.0031	0.0029	0.0026	0.0022	0.0025
	max	0.0036	0.0046	0.0057	0.0068	0.0079	0.0090	0.0101	0.0113	0.0126	0.0141
$\tilde{\mu}^{ps}$	min	−0.0019	−0.0034	−0.0034	−0.0058	−0.0068	−0.0085	−0.0104	−0.0124	−0.0146	−0.0168
	Q1	−0.0006	−0.0018	−0.0018	−0.0031	−0.0037	−0.0049	−0.0057	−0.0067	−0.0078	−0.0084
	median	0.0002	−0.0005	−0.0005	−0.0018	−0.0022	−0.0021	−0.0022	−0.0021	−0.0021	−0.0022
	Q3	0.0014	0.0007	0.0007	0.0019	0.0028	0.0030	0.0028	0.0024	0.0021	0.0025
	max	0.0026	0.0039	0.0039	0.0063	0.0075	0.0086	0.0099	0.0112	0.0126	0.0141

Table B.8: Bias of small-area mean estimators with known population covariates under uniform random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	−0.0195	−0.0221	−0.0250	−0.0283	−0.0321	−0.0367	−0.0426	−0.0507	−0.0636	−0.0912
	Q1	−0.0091	−0.0103	−0.0115	−0.0130	−0.0147	−0.0169	−0.0192	−0.0222	−0.0271	−0.0376
	median	−0.0025	−0.0029	−0.0036	−0.0045	−0.0057	−0.0071	−0.0088	−0.0111	−0.0148	−0.0202
	Q3	0.0034	0.0043	0.0054	0.0064	0.0074	0.0086	0.0101	0.0123	0.0156	0.0217
	max	0.0158	0.0175	0.0194	0.0216	0.0242	0.0273	0.0314	0.0370	0.0459	0.0650
$\tilde{\mu}^{(2)}$	min	−0.0189	−0.0214	−0.0241	−0.0274	−0.0313	−0.0360	−0.0419	−0.0501	−0.0631	−0.0909
	Q1	−0.0087	−0.0100	−0.0115	−0.0129	−0.0146	−0.0168	−0.0193	−0.0223	−0.0271	−0.0377
	median	−0.0024	−0.0029	−0.0036	−0.0045	−0.0056	−0.0071	−0.0089	−0.0112	−0.0149	−0.0203
	Q3	0.0033	0.0042	0.0053	0.0065	0.0074	0.0086	0.0102	0.0123	0.0156	0.0217
	max	0.0154	0.0172	0.0191	0.0213	0.0239	0.0271	0.0311	0.0368	0.0457	0.0649
$\tilde{\mu}^{c}$	min	−0.0192	−0.0218	−0.0248	−0.0281	−0.0319	−0.0365	−0.0424	−0.0506	−0.0635	−0.0912
	Q1	−0.0089	−0.0102	−0.0114	−0.0129	−0.0146	−0.0168	−0.0191	−0.0222	−0.0271	−0.0377
	median	−0.0025	−0.0029	−0.0036	−0.0045	−0.0057	−0.0071	−0.0088	−0.0111	−0.0148	−0.0203
	Q3	0.0033	0.0043	0.0053	0.0064	0.0074	0.0086	0.0101	0.0123	0.0156	0.0217
	max	0.0156	0.0174	0.0193	0.0215	0.0241	0.0273	0.0313	0.0369	0.0459	0.0649
$\tilde{\mu}$	min	−0.0192	−0.0219	−0.0248	−0.0280	−0.0319	−0.0365	−0.0424	−0.0505	−0.0634	−0.0911
	Q1	−0.0090	−0.0102	−0.0115	−0.0129	−0.0146	−0.0168	−0.0192	−0.0222	−0.0271	−0.0377
	median	−0.0024	−0.0029	−0.0036	−0.0045	−0.0057	−0.0071	−0.0088	−0.0111	−0.0148	−0.0202
	Q3	0.0033	0.0043	0.0054	0.0065	0.0074	0.0086	0.0102	0.0123	0.0156	0.0217
	max	0.0156	0.0174	0.0193	0.0215	0.0241	0.0273	0.0313	0.0369	0.0459	0.0650
$\tilde{\mu}^{ps}$	min	−0.0192	−0.0218	−0.0246	−0.0279	−0.0316	−0.0362	−0.0421	−0.0502	−0.0631	−0.0908
	Q1	−0.0094	−0.0106	−0.0118	−0.0133	−0.0151	−0.0169	−0.0191	−0.0221	−0.0270	−0.0376
	median	−0.0023	−0.0028	−0.0035	−0.0044	−0.0055	−0.0069	−0.0087	−0.0110	−0.0147	−0.0202
	Q3	0.0034	0.0044	0.0054	0.0067	0.0077	0.0089	0.0105	0.0126	0.0159	0.0221
	max	0.0157	0.0175	0.0194	0.0217	0.0243	0.0275	0.0316	0.0372	0.0462	0.0653

Table B.9: Bias of small-area mean estimators with sample covariates under normal random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	−0.0027	−0.0032	−0.0033	−0.0043	−0.0056	−0.0068	−0.0077	−0.0086	−0.0105	−0.0128
	Q1	−0.0023	−0.0015	−0.0016	−0.0026	−0.0026	−0.0022	−0.0021	−0.0022	−0.0025	−0.0030
	median	−0.0013	−0.0006	0.0000	−0.0002	0.0003	0.0007	0.0012	0.0011	0.0012	0.0015
	Q3	−0.0008	0.0001	0.0008	0.0018	0.0026	0.0034	0.0042	0.0050	0.0058	0.0066
	max	0.0015	0.0021	0.0032	0.0044	0.0056	0.0067	0.0081	0.0096	0.0112	0.0130
$\tilde{\mu}^{(2)}$	min	−0.0029	−0.0031	−0.0030	−0.0037	−0.0045	−0.0056	−0.0067	−0.0082	−0.0104	−0.0128
	Q1	−0.0021	−0.0015	−0.0017	−0.0021	−0.0020	−0.0016	−0.0020	−0.0022	−0.0024	−0.0028
	median	−0.0017	−0.0009	−0.0001	0.0001	0.0005	0.0009	0.0011	0.0011	0.0014	0.0019
	Q3	−0.0008	0.0000	0.0008	0.0018	0.0025	0.0036	0.0046	0.0054	0.0061	0.0066
	max	0.0019	0.0020	0.0031	0.0043	0.0056	0.0068	0.0083	0.0099	0.0116	0.0134
$\tilde{\mu}^{c}$	min	−0.0029	−0.0031	−0.0031	−0.0038	−0.0050	−0.0062	−0.0074	−0.0083	−0.0104	−0.0128
	Q1	−0.0021	−0.0015	−0.0016	−0.0023	−0.0022	−0.0017	−0.0017	−0.0021	−0.0025	−0.0029
	median	−0.0017	−0.0008	−0.0001	−0.0001	0.0004	0.0008	0.0011	0.0011	0.0012	0.0015
	Q3	−0.0009	0.0001	0.0008	0.0018	0.0027	0.0036	0.0045	0.0053	0.0060	0.0066
	max	0.0016	0.0021	0.0031	0.0043	0.0056	0.0068	0.0082	0.0097	0.0113	0.0131
$\tilde{\mu}$	min	−0.0028	−0.0033	−0.0033	−0.0039	−0.0051	−0.0062	−0.0073	−0.0084	−0.0105	−0.0129
	Q1	−0.0021	−0.0014	−0.0016	−0.0025	−0.0022	−0.0019	−0.0018	−0.0021	−0.0025	−0.0029
	median	−0.0016	−0.0008	−0.0002	−0.0002	0.0004	0.0008	0.0010	0.0010	0.0012	0.0017
	Q3	−0.0008	0.0001	0.0008	0.0019	0.0026	0.0036	0.0045	0.0053	0.0060	0.0066
	max	0.0017	0.0020	0.0033	0.0045	0.0056	0.0068	0.0082	0.0098	0.0114	0.0132

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{Vm}$	min	−0.0112	−0.0114	−0.0116	−0.0121	−0.0125	−0.0130	−0.0134	−0.0138	−0.0151	−0.0128
	Q1	−0.0057	−0.0040	−0.0035	−0.0036	−0.0042	−0.0044	−0.0045	−0.0047	−0.0048	−0.0048
	median	−0.0014	0.0000	−0.0016	−0.0016	−0.0013	−0.0009	−0.0001	0.0007	0.0016	0.0025
	Q3	0.0012	0.0030	0.0039	0.0050	0.0058	0.0064	0.0069	0.0076	0.0084	0.0093
	max	0.0127	0.0141	0.0147	0.0153	0.0157	0.0162	0.0167	0.0172	0.0177	0.0137
$\tilde{\mu}^{(S)}$	min	−0.0037	−0.0028	−0.0026	−0.0034	−0.0044	−0.0062	−0.0075	−0.0087	−0.0106	−0.0130
	Q1	−0.0019	−0.0009	−0.0011	−0.0020	−0.0017	−0.0021	−0.0022	−0.0024	−0.0027	−0.0031
	median	−0.0007	−0.0003	0.0004	0.0001	0.0003	0.0006	0.0005	0.0005	0.0010	0.0015
	Q3	0.0003	0.0007	0.0015	0.0022	0.0031	0.0035	0.0042	0.0051	0.0059	0.0064
	max	0.0023	0.0031	0.0041	0.0045	0.0055	0.0071	0.0085	0.0096	0.0114	0.0132
$\tilde{\mu}^{(T)}$	min	−0.0020	−0.0031	−0.0029	−0.0037	−0.0052	−0.0060	−0.0072	−0.0085	−0.0105	−0.0129
	Q1	−0.0014	−0.0015	−0.0015	−0.0022	−0.0022	−0.0016	−0.0016	−0.0023	−0.0026	−0.0030
	median	−0.0011	−0.0007	0.0002	0.0001	−0.0002	0.0007	0.0011	0.0009	0.0011	0.0015
	Q3	−0.0002	0.0001	0.0012	0.0020	0.0026	0.0038	0.0047	0.0053	0.0060	0.0065
	max	0.0029	0.0022	0.0037	0.0042	0.0050	0.0067	0.0082	0.0095	0.0113	0.0131
$\tilde{\mu}^{ps}$	min	−0.0030	−0.0037	−0.0037	−0.0039	−0.0051	−0.0063	−0.0073	−0.0083	−0.0104	−0.0128
	Q1	−0.0021	−0.0013	−0.0013	−0.0021	−0.0020	−0.0018	−0.0018	−0.0020	−0.0024	−0.0029
	median	−0.0013	−0.0006	−0.0006	0.0000	0.0008	0.0012	0.0012	0.0012	0.0013	0.0018
	Q3	−0.0003	0.0005	0.0005	0.0021	0.0027	0.0036	0.0045	0.0053	0.0060	0.0066
	max	0.0020	0.0028	0.0028	0.0053	0.0064	0.0075	0.0089	0.0104	0.0120	0.0137

Table B.10: Bias of small-area mean estimators with sample covariates under normal random effects for ρ from 0.50 to 0.95

ρ		0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	−0.0154	−0.0184	−0.0218	−0.0257	−0.0303	−0.0361	−0.0435	−0.0540	−0.0708	−0.1074
	Q1	−0.0034	−0.0039	−0.0044	−0.0051	−0.0058	−0.0066	−0.0076	−0.0090	−0.0112	−0.0136
	median	0.0021	0.0028	0.0042	0.0057	0.0074	0.0090	0.0106	0.0122	0.0147	0.0226
	Q3	0.0075	0.0090	0.0104	0.0120	0.0139	0.0158	0.0182	0.0212	0.0267	0.0391
	max	0.0149	0.0172	0.0202	0.0238	0.0279	0.0331	0.0397	0.0491	0.0642	0.0967
$\tilde{\mu}^{(2)}$	min	−0.0154	−0.0184	−0.0218	−0.0257	−0.0303	−0.0361	−0.0435	−0.0540	−0.0708	−0.1074
	Q1	−0.0032	−0.0037	−0.0043	−0.0049	−0.0057	−0.0065	−0.0075	−0.0090	−0.0112	−0.0136
	median	0.0026	0.0037	0.0049	0.0064	0.0078	0.0093	0.0108	0.0123	0.0148	0.0227
	Q3	0.0075	0.0088	0.0103	0.0119	0.0137	0.0156	0.0180	0.0208	0.0269	0.0389
	max	0.0153	0.0174	0.0202	0.0237	0.0278	0.0329	0.0395	0.0488	0.0639	0.0965
$\tilde{\mu}^{c}$	min	−0.0154	−0.0184	−0.0218	−0.0257	−0.0303	−0.0361	−0.0435	−0.0540	−0.0708	−0.1074
	Q1	−0.0033	−0.0039	−0.0044	−0.0050	−0.0058	−0.0065	−0.0076	−0.0090	−0.0112	−0.0136
	median	0.0021	0.0029	0.0042	0.0057	0.0074	0.0090	0.0107	0.0122	0.0147	0.0227
	Q3	0.0075	0.0089	0.0104	0.0120	0.0139	0.0158	0.0182	0.0212	0.0267	0.0389
	max	0.0149	0.0172	0.0202	0.0238	0.0279	0.0331	0.0397	0.0491	0.0642	0.0967
$\tilde{\mu}$	min	−0.0155	−0.0185	−0.0218	−0.0257	−0.0304	−0.0361	−0.0435	−0.0540	−0.0709	−0.1074
	Q1	−0.0033	−0.0038	−0.0044	−0.0050	−0.0057	−0.0065	−0.0076	−0.0090	−0.0112	−0.0136
	median	0.0023	0.0033	0.0046	0.0061	0.0076	0.0092	0.0107	0.0122	0.0147	0.0226
	Q3	0.0076	0.0089	0.0104	0.0120	0.0138	0.0158	0.0182	0.0211	0.0268	0.0390
	max	0.0151	0.0174	0.0204	0.0239	0.0280	0.0331	0.0398	0.0491	0.0641	0.0967

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{Vm}$	min	−0.0199	−0.0227	−0.0259	−0.0295	−0.0339	−0.0393	−0.0464	−0.0565	−0.0729	−0.1088
	Q1	−0.0046	−0.0044	−0.0048	−0.0053	−0.0059	−0.0067	−0.0077	−0.0092	−0.0115	−0.0141
	median	0.0036	0.0047	0.0059	0.0069	0.0082	0.0097	0.0108	0.0123	0.0150	0.0228
	Q3	0.0103	0.0114	0.0126	0.0140	0.0156	0.0172	0.0194	0.0223	0.0277	0.0400
	max	0.0195	0.0220	0.0248	0.0281	0.0319	0.0367	0.0430	0.0519	0.0665	0.0984
$\tilde{\mu}^{(S)}$	min	−0.0156	−0.0186	−0.0219	−0.0258	−0.0305	−0.0362	−0.0436	−0.0541	−0.0709	−0.1075
	Q1	−0.0036	−0.0041	−0.0047	−0.0053	−0.0059	−0.0067	−0.0077	−0.0092	−0.0113	−0.0137
	median	0.0020	0.0029	0.0042	0.0056	0.0071	0.0087	0.0104	0.0119	0.0145	0.0225
	Q3	0.0074	0.0088	0.0102	0.0119	0.0137	0.0156	0.0181	0.0210	0.0266	0.0390
	max	0.0151	0.0171	0.0200	0.0235	0.0277	0.0328	0.0395	0.0489	0.0640	0.0966
$\tilde{\mu}^{(T)}$	min	−0.0155	−0.0184	−0.0218	−0.0257	−0.0303	−0.0361	−0.0435	−0.0540	−0.0708	−0.1074
	Q1	−0.0034	−0.0039	−0.0044	−0.0050	−0.0058	−0.0065	−0.0076	−0.0090	−0.0112	−0.0136
	median	0.0022	0.0031	0.0043	0.0058	0.0075	0.0091	0.0106	0.0122	0.0147	0.0226
	Q3	0.0074	0.0089	0.0104	0.0120	0.0139	0.0158	0.0182	0.0212	0.0267	0.0391
	max	0.0150	0.0172	0.0203	0.0238	0.0279	0.0331	0.0397	0.0491	0.0642	0.0967
$\tilde{\mu}^{ps}$	min	−0.0154	−0.0184	−0.0218	−0.0257	−0.0303	−0.0361	−0.0435	−0.0540	−0.0708	−0.1074
	Q1	−0.0033	−0.0038	−0.0044	−0.0050	−0.0057	−0.0065	−0.0075	−0.0090	−0.0112	−0.0136
	median	0.0024	0.0033	0.0046	0.0061	0.0075	0.0091	0.0106	0.0121	0.0147	0.0226
	Q3	0.0077	0.0090	0.0104	0.0121	0.0138	0.0158	0.0182	0.0210	0.0268	0.0390
	max	0.0155	0.0175	0.0204	0.0239	0.0280	0.0331	0.0398	0.0491	0.0641	0.0967

Table B.11: Bias of small-area mean estimators with sample covariates under double exponential random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	−0.0049	−0.0054	−0.0065	−0.0075	−0.0085	−0.0094	−0.0101	−0.0109	−0.0115	−0.0119
	Q1	−0.0025	−0.0022	−0.0023	−0.0029	−0.0031	−0.0036	−0.0040	−0.0043	−0.0045	−0.0048
	median	−0.0015	−0.0016	−0.0017	−0.0018	−0.0019	−0.0018	−0.0016	−0.0010	−0.0002	0.0006
	Q3	−0.0006	−0.0003	0.0003	0.0005	0.0013	0.0023	0.0029	0.0032	0.0034	0.0036
	max	0.0010	0.0037	0.0061	0.0086	0.0113	0.0141	0.0172	0.0202	0.0234	0.0268
$\tilde{\mu}^{(2)}$	min	−0.0046	−0.0050	−0.0059	−0.0070	−0.0078	−0.0085	−0.0091	−0.0097	−0.0101	−0.0106
	Q1	−0.0026	−0.0028	−0.0024	−0.0027	−0.0030	−0.0032	−0.0029	−0.0036	−0.0040	−0.0044
	median	−0.0018	−0.0015	−0.0018	−0.0017	−0.0020	−0.0019	−0.0020	−0.0016	−0.0009	−0.0001
	Q3	−0.0001	−0.0004	0.0002	0.0008	0.0009	0.0015	0.0022	0.0028	0.0033	0.0037
	max	0.0009	0.0034	0.0057	0.0084	0.0111	0.0137	0.0165	0.0194	0.0225	0.0259
$\tilde{\mu}^{c}$	min	−0.0046	−0.0050	−0.0061	−0.0073	−0.0083	−0.0091	−0.0099	−0.0107	−0.0114	−0.0119
	Q1	−0.0026	−0.0024	−0.0024	−0.0028	−0.0031	−0.0032	−0.0030	−0.0037	−0.0043	−0.0046
	median	−0.0016	−0.0014	−0.0016	−0.0017	−0.0019	−0.0018	−0.0017	−0.0011	−0.0003	0.0006
	Q3	−0.0003	−0.0005	0.0003	0.0006	0.0008	0.0017	0.0024	0.0030	0.0034	0.0037
	max	0.0009	0.0034	0.0058	0.0085	0.0112	0.0139	0.0170	0.0201	0.0234	0.0268
$\tilde{\mu}$	min	−0.0048	−0.0054	−0.0065	−0.0076	−0.0085	−0.0092	−0.0099	−0.0105	−0.0110	−0.0114
	Q1	−0.0027	−0.0026	−0.0024	−0.0028	−0.0031	−0.0032	−0.0035	−0.0039	−0.0042	−0.0047
	median	−0.0016	−0.0015	−0.0016	−0.0018	−0.0020	−0.0018	−0.0017	−0.0012	−0.0005	0.0003
	Q3	−0.0001	−0.0001	0.0004	0.0004	0.0010	0.0019	0.0027	0.0030	0.0034	0.0037
	max	0.0011	0.0039	0.0063	0.0088	0.0115	0.0142	0.0171	0.0201	0.0232	0.0267
$\tilde{\mu}^{ps}$	min	−0.0043	−0.0045	−0.0045	−0.0066	−0.0075	−0.0083	−0.0090	−0.0097	−0.0102	−0.0107
	Q1	−0.0023	−0.0017	−0.0017	−0.0020	−0.0025	−0.0023	−0.0029	−0.0033	−0.0037	−0.0043
	median	−0.0014	−0.0010	−0.0010	−0.0009	−0.0008	−0.0007	−0.0006	−0.0004	0.0003	0.0011
	Q3	0.0002	0.0005	0.0005	0.0009	0.0016	0.0026	0.0033	0.0038	0.0041	0.0044
	max	0.0008	0.0041	0.0041	0.0091	0.0119	0.0146	0.0175	0.0205	0.0236	0.0270

Table B.12: Bias of small-area mean estimators with sample covariates under double exponential random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	−0.0124	−0.0144	−0.0169	−0.0198	−0.0233	−0.0276	−0.0332	−0.0410	−0.0536	−0.0809
	Q1	−0.0053	−0.0059	−0.0067	−0.0076	−0.0086	−0.0098	−0.0111	−0.0127	−0.0151	−0.0204
	median	0.0015	0.0021	0.0029	0.0035	0.0037	0.0040	0.0043	0.0054	0.0081	0.0117
	Q3	0.0039	0.0045	0.0059	0.0070	0.0081	0.0096	0.0122	0.0159	0.0215	0.0338
	max	0.0305	0.0344	0.0387	0.0435	0.0492	0.0561	0.0651	0.0776	0.0976	0.1408
$\tilde{\mu}^{(2)}$	min	−0.0124	−0.0144	−0.0168	−0.0195	−0.0229	−0.0272	−0.0327	−0.0406	−0.0533	−0.0806
	Q1	−0.0051	−0.0058	−0.0066	−0.0075	−0.0086	−0.0096	−0.0108	−0.0124	−0.0149	−0.0202
	median	0.0009	0.0019	0.0031	0.0039	0.0042	0.0044	0.0047	0.0052	0.0080	0.0120
	Q3	0.0040	0.0045	0.0056	0.0070	0.0082	0.0095	0.0118	0.0158	0.0213	0.0337
	max	0.0295	0.0334	0.0376	0.0425	0.0482	0.0552	0.0643	0.0768	0.0970	0.1403
$\tilde{\mu}^{c}$	min	−0.0124	−0.0144	−0.0169	−0.0198	−0.0233	−0.0276	−0.0332	−0.0410	−0.0536	−0.0808
	Q1	−0.0051	−0.0058	−0.0066	−0.0075	−0.0086	−0.0098	−0.0111	−0.0127	−0.0151	−0.0202
	median	0.0015	0.0025	0.0032	0.0039	0.0041	0.0044	0.0046	0.0056	0.0083	0.0120
	Q3	0.0039	0.0045	0.0059	0.0071	0.0082	0.0096	0.0122	0.0159	0.0215	0.0337
	max	0.0305	0.0344	0.0387	0.0435	0.0492	0.0561	0.0651	0.0776	0.0976	0.1408
$\tilde{\mu}$	min	−0.0124	−0.0145	−0.0170	−0.0198	−0.0232	−0.0275	−0.0331	−0.0409	−0.0536	−0.0808
	Q1	−0.0053	−0.0060	−0.0068	−0.0076	−0.0087	−0.0099	−0.0110	−0.0126	−0.0151	−0.0203
	median	0.0013	0.0022	0.0030	0.0037	0.0039	0.0042	0.0045	0.0053	0.0081	0.0118
	Q3	0.0040	0.0045	0.0058	0.0070	0.0081	0.0095	0.0121	0.0159	0.0214	0.0337
	max	0.0303	0.0342	0.0384	0.0433	0.0489	0.0559	0.0649	0.0774	0.0974	0.1407
$\tilde{\mu}^{ps}$	min	−0.0119	−0.0140	−0.0165	−0.0194	−0.0228	−0.0271	−0.0327	−0.0406	−0.0533	−0.0806
	Q1	−0.0050	−0.0057	−0.0065	−0.0073	−0.0084	−0.0096	−0.0107	−0.0123	−0.0149	−0.0202
	median	0.0020	0.0027	0.0034	0.0040	0.0042	0.0044	0.0047	0.0056	0.0083	0.0120
	Q3	0.0047	0.0052	0.0062	0.0073	0.0085	0.0100	0.0126	0.0163	0.0218	0.0340
	max	0.0306	0.0345	0.0387	0.0436	0.0493	0.0562	0.0652	0.0777	0.0977	0.1409

Table B.13: Bias of small-area mean estimators with sample covariates under exponential random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	−0.0084	−0.0093	−0.0093	−0.0104	−0.0113	−0.0123	−0.0134	−0.0157	−0.0180	−0.0205
	Q1	−0.0053	−0.0055	−0.0045	−0.0038	−0.0023	−0.0034	−0.0036	−0.0049	−0.0051	−0.0060
	median	0.0021	0.0023	0.0022	0.0013	0.0011	0.0000	0.0006	0.0006	0.0011	0.0017
	Q3	0.0070	0.0074	0.0063	0.0054	0.0051	0.0049	0.0052	0.0048	0.0047	0.0043
	max	0.0143	0.0152	0.0156	0.0156	0.0157	0.0158	0.0158	0.0160	0.0164	0.0170
$\tilde{\mu}^{(2)}$	min	−0.0015	−0.0019	−0.0044	−0.0066	−0.0087	−0.0109	−0.0130	−0.0148	−0.0167	−0.0190
	Q1	−0.0007	−0.0012	−0.0025	−0.0034	−0.0049	−0.0061	−0.0070	−0.0078	−0.0084	−0.0094
	median	0.0002	−0.0005	−0.0009	−0.0020	−0.0027	−0.0033	−0.0042	−0.0044	−0.0045	−0.0046
	Q3	0.0008	−0.0003	0.0001	0.0003	0.0000	0.0003	0.0010	0.0017	0.0027	0.0037
	max	0.0033	0.0033	0.0077	0.0114	0.0148	0.0177	0.0200	0.0219	0.0235	0.0250
$\tilde{\mu}^{c}$	min	−0.0070	−0.0057	−0.0049	−0.0067	−0.0088	−0.0108	−0.0128	−0.0149	−0.0171	−0.0195
	Q1	−0.0029	−0.0024	−0.0024	−0.0027	−0.0027	−0.0030	−0.0033	−0.0036	−0.0040	−0.0042
	median	−0.0005	−0.0013	−0.0011	−0.0007	0.0005	0.0011	0.0016	0.0020	0.0024	0.0028
	Q3	0.0007	−0.0003	0.0001	0.0005	0.0017	0.0024	0.0031	0.0041	0.0047	0.0051
	max	0.0033	0.0031	0.0038	0.0075	0.0115	0.0149	0.0176	0.0198	0.0216	0.0233
$\tilde{\mu}$	min	−0.0061	−0.0059	−0.0063	−0.0071	−0.0090	−0.0111	−0.0131	−0.0152	−0.0174	−0.0199
	Q1	−0.0027	−0.0029	−0.0023	−0.0021	−0.0027	−0.0038	−0.0048	−0.0053	−0.0056	−0.0060
	median	0.0014	0.0013	0.0007	0.0004	−0.0004	−0.0005	−0.0003	0.0000	−0.0003	−0.0007
	Q3	0.0052	0.0049	0.0037	0.0028	0.0020	0.0024	0.0017	0.0018	0.0027	0.0036
	max	0.0074	0.0076	0.0072	0.0070	0.0075	0.0103	0.0127	0.0148	0.0167	0.0185
$\tilde{\mu}^{ps}$	min	−0.0084	−0.0082	−0.0082	−0.0086	−0.0100	−0.0119	−0.0136	−0.0154	−0.0175	−0.0198
	Q1	−0.0050	−0.0052	−0.0052	−0.0035	−0.0038	−0.0047	−0.0055	−0.0057	−0.0058	−0.0061
	median	−0.0005	−0.0008	−0.0008	−0.0011	−0.0015	−0.0015	−0.0009	−0.0003	−0.0004	−0.0006
	Q3	0.0033	0.0028	0.0028	0.0013	0.0009	0.0016	0.0012	0.0016	0.0026	0.0036
	max	0.0055	0.0057	0.0057	0.0054	0.0064	0.0096	0.0123	0.0146	0.0166	0.0185

Table B.14: Bias of small-area mean estimators with sample covariates under exponential random effects for ρ from 0.50 to 0.95

ρ		0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
		Rho50	Rho55	Rho60	Rho65	Rho70	Rho75	Rho80	Rho85	Rho90	Rho95
$\tilde{\mu}^{(1)}$	min	−0.0232	−0.0262	−0.0295	−0.0333	−0.0379	−0.0435	−0.0508	−0.0611	−0.0779	−0.1144
	Q1	−0.0064	−0.0070	−0.0076	−0.0082	−0.0089	−0.0099	−0.0110	−0.0127	−0.0153	−0.0209
	median	0.0021	0.0020	0.0013	0.0008	0.0003	−0.0002	−0.0007	−0.0012	−0.0017	−0.0027
	Q3	0.0047	0.0051	0.0055	0.0060	0.0066	0.0075	0.0090	0.0116	0.0161	0.0261
	max	0.0180	0.0194	0.0211	0.0234	0.0263	0.0301	0.0354	0.0431	0.0559	0.0840
$\tilde{\mu}^{(2)}$	min	−0.0216	−0.0244	−0.0276	−0.0313	−0.0357	−0.0413	−0.0486	−0.0591	−0.0760	−0.1129
	Q1	−0.0107	−0.0116	−0.0126	−0.0134	−0.0138	−0.0143	−0.0147	−0.0152	−0.0164	−0.0217
	median	−0.0045	−0.0042	−0.0038	−0.0041	−0.0045	−0.0050	−0.0045	−0.0039	−0.0033	−0.0054
	Q3	0.0049	0.0057	0.0056	0.0054	0.0073	0.0093	0.0111	0.0127	0.0153	0.0250
	max	0.0264	0.0278	0.0293	0.0309	0.0326	0.0348	0.0376	0.0415	0.0481	0.0728
$\tilde{\mu}^{c}$	min	−0.0221	−0.0250	−0.0283	−0.0321	−0.0366	−0.0423	−0.0497	−0.0603	−0.0773	−0.1143
	Q1	−0.0045	−0.0049	−0.0055	−0.0061	−0.0068	−0.0076	−0.0091	−0.0114	−0.0143	−0.0217
	median	0.0027	0.0022	0.0018	0.0015	0.0013	0.0012	0.0008	−0.0002	−0.0008	−0.0054
	Q3	0.0057	0.0063	0.0070	0.0076	0.0083	0.0091	0.0101	0.0127	0.0169	0.0250
	max	0.0248	0.0262	0.0276	0.0290	0.0306	0.0327	0.0354	0.0431	0.0559	0.0840
$\tilde{\mu}$	min	−0.0225	−0.0255	−0.0288	−0.0325	−0.0370	−0.0426	−0.0499	−0.0603	−0.0772	−0.1138
	Q1	−0.0067	−0.0074	−0.0080	−0.0085	−0.0092	−0.0104	−0.0117	−0.0133	−0.0158	−0.0212
	median	−0.0010	−0.0012	−0.0013	−0.0014	−0.0015	−0.0017	−0.0021	−0.0026	−0.0035	−0.0043
	Q3	0.0045	0.0054	0.0061	0.0068	0.0076	0.0085	0.0097	0.0114	0.0155	0.0256
	max	0.0201	0.0218	0.0235	0.0254	0.0276	0.0301	0.0333	0.0378	0.0493	0.0790
$\tilde{\mu}^{ps}$	min	−0.0224	−0.0252	−0.0284	−0.0322	−0.0366	−0.0422	−0.0495	−0.0600	−0.0768	−0.1136
	Q1	−0.0068	−0.0074	−0.0079	−0.0083	−0.0088	−0.0100	−0.0113	−0.0129	−0.0154	−0.0210
	median	−0.0008	−0.0009	−0.0010	−0.0010	−0.0010	−0.0013	−0.0017	−0.0022	−0.0032	−0.0040
	Q3	0.0047	0.0057	0.0064	0.0071	0.0080	0.0089	0.0101	0.0117	0.0158	0.0259
	max	0.0202	0.0220	0.0238	0.0257	0.0279	0.0304	0.0336	0.0381	0.0498	0.0795

Table B.15: Bias of small-area mean estimators with sample covariates under uniform random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	−0.0016	−0.0029	−0.0044	−0.0054	−0.0066	−0.0084	−0.0104	−0.0122	−0.0143	−0.0164
	Q1	−0.0002	−0.0014	−0.0025	−0.0037	−0.0048	−0.0059	−0.0069	−0.0077	−0.0083	−0.0096
	median	0.0003	−0.0006	−0.0009	−0.0013	−0.0016	−0.0021	−0.0021	−0.0021	−0.0021	−0.0022
	Q3	0.0020	0.0007	0.0011	0.0018	0.0027	0.0033	0.0029	0.0025	0.0022	0.0028
	max	0.0041	0.0058	0.0066	0.0075	0.0086	0.0096	0.0108	0.0120	0.0134	0.0149
$\tilde{\mu}^{(2)}$	min	−0.0012	−0.0034	−0.0045	−0.0057	−0.0072	−0.0090	−0.0109	−0.0129	−0.0149	−0.0171
	Q1	−0.0003	−0.0011	−0.0018	−0.0028	−0.0039	−0.0050	−0.0061	−0.0073	−0.0083	−0.0094
	median	0.0006	−0.0001	−0.0010	−0.0015	−0.0016	−0.0016	−0.0016	−0.0017	−0.0017	−0.0018
	Q3	0.0020	0.0008	0.0013	0.0017	0.0025	0.0033	0.0031	0.0028	0.0024	0.0027
	max	0.0034	0.0047	0.0057	0.0070	0.0082	0.0093	0.0105	0.0117	0.0131	0.0145
$\tilde{\mu}^{c}$	min	−0.0012	−0.0030	−0.0045	−0.0055	−0.0070	−0.0088	−0.0107	−0.0126	−0.0146	−0.0167
	Q1	−0.0003	−0.0012	−0.0020	−0.0032	−0.0043	−0.0054	−0.0065	−0.0076	−0.0083	−0.0095
	median	0.0005	−0.0002	−0.0009	−0.0014	−0.0017	−0.0017	−0.0018	−0.0019	−0.0019	−0.0021
	Q3	0.0020	0.0009	0.0012	0.0017	0.0026	0.0033	0.0031	0.0027	0.0023	0.0027
	max	0.0034	0.0049	0.0059	0.0072	0.0083	0.0094	0.0106	0.0119	0.0133	0.0147
$\tilde{\mu}$	min	−0.0015	−0.0029	−0.0044	−0.0054	−0.0069	−0.0087	−0.0106	−0.0125	−0.0146	−0.0167
	Q1	−0.0002	−0.0013	−0.0022	−0.0032	−0.0044	−0.0055	−0.0066	−0.0077	−0.0084	−0.0095
	median	0.0002	−0.0005	−0.0010	−0.0014	−0.0018	−0.0018	−0.0018	−0.0018	−0.0019	−0.0020
	Q3	0.0021	0.0007	0.0011	0.0017	0.0026	0.0034	0.0031	0.0027	0.0023	0.0027
	max	0.0036	0.0051	0.0062	0.0073	0.0084	0.0095	0.0107	0.0119	0.0133	0.0147
$\tilde{\mu}^{ps}$	min	−0.0017	−0.0033	−0.0033	−0.0056	−0.0068	−0.0086	−0.0105	−0.0124	−0.0144	−0.0165
	Q1	−0.0007	−0.0017	−0.0017	−0.0031	−0.0039	−0.0050	−0.0062	−0.0073	−0.0082	−0.0092
	median	0.0000	−0.0008	−0.0008	−0.0018	−0.0021	−0.0020	−0.0020	−0.0020	−0.0019	−0.0020
	Q3	0.0014	0.0005	0.0005	0.0018	0.0027	0.0031	0.0029	0.0026	0.0022	0.0027
	max	0.0027	0.0041	0.0041	0.0065	0.0078	0.0090	0.0102	0.0116	0.0130	0.0145

Table B.16: Bias of small-area mean estimators with sample covariates under uniform random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	−0.0188	−0.0214	−0.0243	−0.0276	−0.0313	−0.0359	−0.0418	−0.0499	−0.0628	−0.0905
	Q1	−0.0106	−0.0116	−0.0127	−0.0139	−0.0152	−0.0169	−0.0191	−0.0221	−0.0270	−0.0376
	median	−0.0023	−0.0028	−0.0034	−0.0042	−0.0054	−0.0068	−0.0086	−0.0109	−0.0146	−0.0203
	Q3	0.0036	0.0046	0.0056	0.0069	0.0081	0.0093	0.0108	0.0130	0.0162	0.0224
	max	0.0164	0.0182	0.0201	0.0223	0.0249	0.0281	0.0321	0.0378	0.0467	0.0657
$\tilde{\mu}^{(2)}$	min	−0.0193	−0.0218	−0.0245	−0.0275	−0.0311	−0.0355	−0.0411	−0.0493	−0.0623	−0.0901
	Q1	−0.0106	−0.0117	−0.0130	−0.0141	−0.0155	−0.0171	−0.0192	−0.0222	−0.0271	−0.0376
	median	−0.0022	−0.0028	−0.0034	−0.0042	−0.0053	−0.0068	−0.0087	−0.0110	−0.0147	−0.0203
	Q3	0.0035	0.0044	0.0055	0.0067	0.0081	0.0093	0.0108	0.0130	0.0163	0.0224
	max	0.0161	0.0179	0.0198	0.0220	0.0246	0.0278	0.0319	0.0375	0.0465	0.0656
$\tilde{\mu}^{c}$	min	−0.0190	−0.0214	−0.0241	−0.0273	−0.0311	−0.0358	−0.0417	−0.0498	−0.0627	−0.0904
	Q1	−0.0105	−0.0116	−0.0126	−0.0138	−0.0152	−0.0169	−0.0191	−0.0221	−0.0270	−0.0376
	median	−0.0023	−0.0028	−0.0034	−0.0042	−0.0054	−0.0068	−0.0087	−0.0109	−0.0147	−0.0203
	Q3	0.0035	0.0045	0.0056	0.0068	0.0080	0.0092	0.0108	0.0129	0.0162	0.0224
	max	0.0163	0.0181	0.0200	0.0223	0.0249	0.0280	0.0321	0.0377	0.0466	0.0657
$\tilde{\mu}$	min	−0.0190	−0.0214	−0.0241	−0.0273	−0.0311	−0.0357	−0.0416	−0.0497	−0.0627	−0.0904
	Q1	−0.0106	−0.0117	−0.0128	−0.0140	−0.0153	−0.0170	−0.0191	−0.0222	−0.0270	−0.0376
	median	−0.0022	−0.0028	−0.0034	−0.0042	−0.0053	−0.0068	−0.0087	−0.0109	−0.0147	−0.0203
	Q3	0.0036	0.0045	0.0056	0.0068	0.0081	0.0093	0.0108	0.0130	0.0163	0.0224
	max	0.0163	0.0181	0.0200	0.0222	0.0248	0.0280	0.0321	0.0377	0.0467	0.0657
$\tilde{\mu}^{ps}$	min	−0.0188	−0.0213	−0.0241	−0.0274	−0.0311	−0.0357	−0.0416	−0.0497	−0.0626	−0.0903
	Q1	−0.0103	−0.0114	−0.0127	−0.0139	−0.0152	−0.0169	−0.0191	−0.0221	−0.0270	−0.0375
	median	−0.0022	−0.0028	−0.0034	−0.0042	−0.0053	−0.0068	−0.0086	−0.0109	−0.0147	−0.0202
	Q3	0.0035	0.0045	0.0056	0.0068	0.0081	0.0093	0.0109	0.0130	0.0163	0.0224
	max	0.0162	0.0179	0.0199	0.0222	0.0248	0.0280	0.0321	0.0378	0.0467	0.0658

Appendix C

TABLES: MEAN SQUARE ERRORS OF SMALL AREA MEAN ESTIMATES

(1) with population covariate means (from p. 164 to 173)

(2) with sample covariate means (from p. 174 to 183)

a_i: weight defined in equation (2.7.4)

$\tilde{\mu}^{(1)}$: EBLUP defined in equation (2.7.4)

$\tilde{\mu}^{(2)}$: EBLUP defined in equation (2.7.4)

$\tilde{\mu}^{c}$: composite estimator defined in equation (2.7.4)

$\tilde{\mu}$: estimator defined in equation (3.2.5)

$\tilde{\mu}^{Vm}$: EBLUP with M.L.E. variance estimators

$\tilde{\mu}^{(S)}$: EBLUP with Residual M.L.E. variance estimators

$\tilde{\mu}^{(T)}$: EBLUP with Restricted M.L.E. variance estimators

$\tilde{\mu}^{ps}$: PSEUDO-EBLUP by You & Rao

Table C.1: MSE of small-area mean estimators with known population covariates under normal random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	0.0270	0.0648	0.0930	0.1122	0.1251	0.1338	0.1396	0.1441	0.1477	0.1508
	Q1	0.0291	0.0672	0.0986	0.1219	0.1390	0.1513	0.1600	0.1668	0.1722	0.1768
	median	0.0296	0.0729	0.1114	0.1442	0.1710	0.1931	0.2116	0.2275	0.2413	0.2534
	Q3	0.0311	0.0748	0.1134	0.1453	0.1728	0.1954	0.2140	0.2304	0.2446	0.2568
	max	0.0337	0.0840	0.1314	0.1745	0.2137	0.2491	0.2815	0.3112	0.3386	0.3640
$\tilde{\mu}^{(2)}$	min	0.0217	0.0648	0.0907	0.1079	0.1201	0.1290	0.1358	0.1412	0.1457	0.1496
	Q1	0.0245	0.0681	0.0979	0.1200	0.1364	0.1485	0.1577	0.1650	0.1711	0.1762
	median	0.0268	0.0699	0.1092	0.1428	0.1700	0.1929	0.2122	0.2288	0.2432	0.2557
	Q3	0.0292	0.0709	0.1105	0.1438	0.1720	0.1953	0.2152	0.2323	0.2472	0.2598
	max	0.0315	0.0758	0.1254	0.1709	0.2122	0.2497	0.2838	0.3149	0.3435	0.3700
$\tilde{\mu}^{c}$	min	0.0217	0.0635	0.0898	0.1075	0.1199	0.1289	0.1357	0.1411	0.1456	0.1494
	Q1	0.0245	0.0662	0.0967	0.1192	0.1359	0.1482	0.1574	0.1647	0.1707	0.1758
	median	0.0263	0.0695	0.1086	0.1419	0.1689	0.1915	0.2104	0.2267	0.2408	0.2531
	Q3	0.0285	0.0709	0.1102	0.1428	0.1707	0.1937	0.2130	0.2298	0.2442	0.2566
	max	0.0298	0.0758	0.1252	0.1700	0.2104	0.2469	0.2799	0.3101	0.3379	0.3636
$\tilde{\mu}$	min	0.0248	0.0643	0.0908	0.1090	0.1211	0.1297	0.1363	0.1414	0.1457	0.1494
	Q1	0.0268	0.0671	0.0974	0.1198	0.1364	0.1483	0.1574	0.1645	0.1705	0.1755
	median	0.0278	0.0708	0.1095	0.1424	0.1691	0.1913	0.2102	0.2264	0.2405	0.2528
	Q3	0.0295	0.0724	0.1113	0.1434	0.1708	0.1932	0.2126	0.2294	0.2440	0.2565
	max	0.0308	0.0785	0.1266	0.1706	0.2106	0.2468	0.2800	0.3103	0.3383	0.3643

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{Vm}$	min	0.0221	0.0637	0.0899	0.1083	0.1210	0.1302	0.1365	0.1419	0.1462	0.1515
	Q1	0.0249	0.0670	0.0972	0.1199	0.1365	0.1486	0.1581	0.1654	0.1712	0.1763
	median	0.0265	0.0696	0.1085	0.1418	0.1690	0.1920	0.2113	0.2278	0.2422	0.2544
	Q3	0.0284	0.0709	0.1102	0.1436	0.1713	0.1941	0.2144	0.2315	0.2460	0.2582
	max	0.0299	0.0762	0.1257	0.1715	0.2125	0.2500	0.2834	0.3140	0.3418	0.3748
$\tilde{\mu}^{(S)}$	min	0.0439	0.0771	0.1001	0.1139	0.1232	0.1303	0.1364	0.1418	0.1465	0.1508
	Q1	0.0462	0.0801	0.1055	0.1241	0.1384	0.1489	0.1575	0.1645	0.1706	0.1758
	median	0.0475	0.0812	0.1148	0.1446	0.1713	0.1933	0.2112	0.2271	0.2410	0.2532
	Q3	0.0508	0.0839	0.1167	0.1478	0.1731	0.1947	0.2136	0.2303	0.2448	0.2574
	max	0.0621	0.0920	0.1299	0.1727	0.2119	0.2482	0.2822	0.3139	0.3436	0.3721
$\tilde{\mu}^{(T)}$	min	0.0232	0.0648	0.0892	0.1068	0.1193	0.1281	0.1351	0.1407	0.1454	0.1493
	Q1	0.0264	0.0681	0.0968	0.1185	0.1343	0.1471	0.1564	0.1639	0.1701	0.1753
	median	0.0285	0.0704	0.1086	0.1404	0.1670	0.1896	0.2091	0.2257	0.2401	0.2525
	Q3	0.0306	0.0719	0.1100	0.1422	0.1689	0.1922	0.2119	0.2289	0.2436	0.2564
	max	0.0327	0.0777	0.1258	0.1704	0.2092	0.2463	0.2799	0.3093	0.3375	0.3636
$\tilde{\mu}^{ps}$	min	0.0291	0.0672	0.0943	0.1125	0.1245	0.1328	0.1390	0.1438	0.1479	0.1515
	Q1	0.0302	0.0697	0.0998	0.1220	0.1382	0.1499	0.1587	0.1657	0.1715	0.1764
	median	0.0312	0.0747	0.1126	0.1440	0.1704	0.1927	0.2114	0.2274	0.2414	0.2539
	Q3	0.0341	0.0761	0.1146	0.1466	0.1739	0.1960	0.2151	0.2316	0.2458	0.2583
	max	0.0361	0.0835	0.1320	0.1766	0.2171	0.2541	0.2880	0.3191	0.3480	0.3748

Table C.2: MSE of small-area mean estimators with known population covariates under normal random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	0.1537	0.1563	0.1588	0.1612	0.1633	0.1654	0.1673	0.1691	0.1709	0.1724
	Q1	0.1808	0.1844	0.1876	0.1905	0.1932	0.1956	0.1978	0.1998	0.2016	0.2032
	median	0.2641	0.2739	0.2832	0.2916	0.2994	0.3062	0.3126	0.3184	0.3238	0.3284
	Q3	0.2678	0.2778	0.2868	0.2949	0.3023	0.3091	0.3154	0.3215	0.3273	0.3326
	max	0.3878	0.4099	0.4307	0.4501	0.4683	0.4853	0.5012	0.5162	0.5301	0.5431
$\tilde{\mu}^{(2)}$	min	0.1530	0.1560	0.1588	0.1612	0.1635	0.1656	0.1675	0.1693	0.1711	0.1726
	Q1	0.1806	0.1844	0.1877	0.1907	0.1934	0.1958	0.1980	0.1999	0.2017	0.2032
	median	0.2667	0.2764	0.2856	0.2939	0.3015	0.3084	0.3144	0.3198	0.3247	0.3290
	Q3	0.2709	0.2808	0.2898	0.2978	0.3050	0.3116	0.3175	0.3232	0.3285	0.3333
	max	0.3945	0.4171	0.4381	0.4576	0.4755	0.4920	0.5072	0.5210	0.5336	0.5449
$\tilde{\mu}^{c}$	min	0.1528	0.1558	0.1585	0.1609	0.1632	0.1653	0.1673	0.1691	0.1709	0.1724
	Q1	0.1801	0.1839	0.1873	0.1904	0.1931	0.1955	0.1978	0.1998	0.2016	0.2032
	median	0.2639	0.2738	0.2831	0.2916	0.2994	0.3062	0.3126	0.3184	0.3238	0.3290
	Q3	0.2677	0.2778	0.2868	0.2949	0.3023	0.3091	0.3154	0.3215	0.3273	0.3333
	max	0.3875	0.4098	0.4306	0.4501	0.4683	0.4853	0.5012	0.5162	0.5301	0.5431
$\tilde{\mu}$	min	0.1527	0.1557	0.1584	0.1609	0.1632	0.1653	0.1673	0.1692	0.1709	0.1725
	Q1	0.1799	0.1837	0.1872	0.1902	0.1930	0.1955	0.1977	0.1997	0.2016	0.2032
	median	0.2637	0.2738	0.2832	0.2917	0.2996	0.3065	0.3128	0.3186	0.3239	0.3285
	Q3	0.2676	0.2778	0.2869	0.2952	0.3026	0.3094	0.3157	0.3218	0.3275	0.3327
	max	0.3885	0.4110	0.4320	0.4516	0.4699	0.4869	0.5027	0.5174	0.5311	0.5436

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{Vm}$	min	0.1531	0.1560	0.1587	0.1611	0.1633	0.1654	0.1673	0.1692	0.1709	0.1724
	Q1	0.1806	0.1843	0.1877	0.1906	0.1933	0.1957	0.1979	0.1999	0.2017	0.2032
	median	0.2650	0.2749	0.2840	0.2923	0.2999	0.3067	0.3129	0.3186	0.3240	0.3285
	Q3	0.2693	0.2788	0.2877	0.2957	0.3030	0.3096	0.3157	0.3218	0.3274	0.3327
	max	0.4061	0.4265	0.4456	0.4634	0.4802	0.4959	0.5108	0.5248	0.5381	0.5434
$\tilde{\mu}^{(S)}$	min	0.1548	0.1587	0.1625	0.1664	0.1706	0.1755	0.1816	0.1906	0.2009	0.2086
	Q1	0.1804	0.1845	0.1883	0.1919	0.1957	0.1995	0.2042	0.2105	0.2208	0.2486
	median	0.2643	0.2743	0.2838	0.2927	0.3007	0.3082	0.3147	0.3208	0.3271	0.3355
	Q3	0.2683	0.2783	0.2874	0.2957	0.3040	0.3127	0.3219	0.3317	0.3469	0.3837
	max	0.4196	0.4470	0.4757	0.5071	0.5437	0.5902	0.6575	0.7781	1.1144	1.1317
$\tilde{\mu}^{(T)}$	min	0.1528	0.1558	0.1585	0.1610	0.1633	0.1654	0.1673	0.1691	0.1709	0.1724
	Q1	0.1799	0.1839	0.1873	0.1903	0.1931	0.1956	0.1978	0.1998	0.2016	0.2032
	median	0.2637	0.2737	0.2830	0.2915	0.2993	0.3062	0.3125	0.3184	0.3238	0.3284
	Q3	0.2676	0.2777	0.2867	0.2949	0.3023	0.3091	0.3154	0.3215	0.3272	0.3326
	max	0.4032	0.4240	0.4434	0.4616	0.4787	0.4949	0.5101	0.5244	0.5379	0.5431
$\tilde{\mu}^{ps}$	min	0.1548	0.1577	0.1604	0.1630	0.1653	0.1675	0.1695	0.1715	0.1733	0.1751
	Q1	0.1806	0.1844	0.1878	0.1908	0.1935	0.1959	0.1982	0.2002	0.2020	0.2037
	median	0.2648	0.2747	0.2840	0.2924	0.3001	0.3069	0.3131	0.3189	0.3242	0.3288
	Q3	0.2694	0.2794	0.2883	0.2964	0.3038	0.3109	0.3175	0.3237	0.3295	0.3349
	max	0.3999	0.4233	0.4452	0.4656	0.4847	0.5026	0.5193	0.5349	0.5494	0.5629

Table C.3: MSE of small-area mean estimators with known population covariates under double exponential random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	0.0249	0.0619	0.0877	0.1061	0.1191	0.1284	0.1354	0.1409	0.1455	0.1493
	Q1	0.0278	0.0650	0.0939	0.1157	0.1320	0.1444	0.1534	0.1603	0.1661	0.1712
	median	0.0305	0.0733	0.1096	0.1393	0.1642	0.1849	0.2025	0.2184	0.2323	0.2446
	Q3	0.0315	0.0753	0.1129	0.1445	0.1708	0.1929	0.2120	0.2282	0.2420	0.2543
	max	0.0403	0.0897	0.1354	0.1768	0.2143	0.2481	0.2791	0.3078	0.3345	0.3596
$\tilde{\mu}^{(2)}$	min	0.0211	0.0623	0.0852	0.1010	0.1129	0.1219	0.1291	0.1350	0.1399	0.1443
	Q1	0.0240	0.0657	0.0928	0.1126	0.1271	0.1386	0.1478	0.1555	0.1619	0.1675
	median	0.0252	0.0686	0.1065	0.1384	0.1649	0.1873	0.2065	0.2235	0.2380	0.2506
	Q3	0.0287	0.0703	0.1094	0.1424	0.1703	0.1939	0.2142	0.2319	0.2473	0.2607
	max	0.0307	0.0736	0.1216	0.1672	0.2091	0.2478	0.2836	0.3168	0.3474	0.3758
$\tilde{\mu}^{c}$	min	0.0211	0.0606	0.0842	0.1006	0.1128	0.1219	0.1291	0.1350	0.1399	0.1443
	Q1	0.0240	0.0637	0.0915	0.1119	0.1269	0.1385	0.1478	0.1555	0.1619	0.1675
	median	0.0245	0.0685	0.1061	0.1371	0.1627	0.1841	0.2021	0.2183	0.2322	0.2446
	Q3	0.0270	0.0702	0.1091	0.1416	0.1687	0.1915	0.2111	0.2279	0.2419	0.2543
	max	0.0289	0.0736	0.1216	0.1669	0.2077	0.2442	0.2771	0.3070	0.3344	0.3596
$\tilde{\mu}$	min	0.0246	0.0614	0.0852	0.1020	0.1144	0.1235	0.1307	0.1367	0.1417	0.1458
	Q1	0.0267	0.0646	0.0924	0.1131	0.1283	0.1399	0.1490	0.1564	0.1628	0.1682
	median	0.0274	0.0707	0.1072	0.1374	0.1626	0.1838	0.2021	0.2184	0.2327	0.2449
	Q3	0.0283	0.0724	0.1104	0.1424	0.1690	0.1916	0.2109	0.2278	0.2426	0.2554
	max	0.0321	0.0788	0.1262	0.1697	0.2091	0.2451	0.2782	0.3086	0.3370	0.3634
$\tilde{\mu}^{ps}$	min	0.0278	0.0641	0.0883	0.1052	0.1175	0.1263	0.1333	0.1391	0.1440	0.1480
	Q1	0.0296	0.0674	0.0953	0.1160	0.1310	0.1424	0.1512	0.1584	0.1646	0.1698
	median	0.0307	0.0746	0.1103	0.1402	0.1652	0.1862	0.2038	0.2195	0.2336	0.2456
	Q3	0.0343	0.0764	0.1142	0.1461	0.1727	0.1951	0.2144	0.2308	0.2450	0.2576
	max	0.0376	0.0840	0.1316	0.1754	0.2153	0.2519	0.2856	0.3168	0.3459	0.3731

Table C.4: MSE of small-area mean estimators with known population covariates under double exponential random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	0.1523	0.1550	0.1575	0.1599	0.1620	0.1640	0.1659	0.1677	0.1695	0.1711
	Q1	0.1755	0.1792	0.1827	0.1860	0.1890	0.1918	0.1944	0.1969	0.1994	0.2017
	median	0.2553	0.2649	0.2737	0.2819	0.2894	0.2965	0.3031	0.3093	0.3152	0.3209
	Q3	0.2654	0.2756	0.2849	0.2934	0.3013	0.3086	0.3154	0.3220	0.3283	0.3341
	max	0.3829	0.4050	0.4259	0.4457	0.4646	0.4825	0.4996	0.5160	0.5317	0.5467
$\tilde{\mu}^{(2)}$	min	0.1481	0.1516	0.1548	0.1576	0.1602	0.1626	0.1649	0.1670	0.1690	0.1709
	Q1	0.1724	0.1768	0.1809	0.1845	0.1878	0.1909	0.1938	0.1965	0.1991	0.2016
	median	0.2617	0.2715	0.2804	0.2883	0.2954	0.3019	0.3078	0.3131	0.3179	0.3223
	Q3	0.2726	0.2831	0.2923	0.3005	0.3078	0.3144	0.3205	0.3262	0.3312	0.3357
	max	0.4021	0.4266	0.4491	0.4698	0.4888	0.5059	0.5212	0.5345	0.5457	0.5545
$\tilde{\mu}^{c}$	min	0.1481	0.1516	0.1548	0.1576	0.1602	0.1626	0.1649	0.1670	0.1690	0.1709
	Q1	0.1724	0.1768	0.1808	0.1845	0.1878	0.1909	0.1938	0.1965	0.1991	0.2016
	median	0.2553	0.2649	0.2737	0.2819	0.2894	0.2965	0.3031	0.3093	0.3152	0.3223
	Q3	0.2654	0.2756	0.2849	0.2934	0.3013	0.3086	0.3154	0.3220	0.3283	0.3357
	max	0.3829	0.4050	0.4259	0.4457	0.4646	0.4825	0.4996	0.5160	0.5317	0.5467
$\tilde{\mu}$	min	0.1493	0.1526	0.1556	0.1583	0.1608	0.1631	0.1652	0.1673	0.1692	0.1710
	Q1	0.1729	0.1772	0.1811	0.1847	0.1880	0.1910	0.1939	0.1966	0.1992	0.2017
	median	0.2558	0.2656	0.2746	0.2828	0.2904	0.2974	0.3040	0.3101	0.3158	0.3212
	Q3	0.2669	0.2772	0.2865	0.2950	0.3028	0.3099	0.3165	0.3230	0.3290	0.3345
	max	0.3880	0.4110	0.4325	0.4527	0.4716	0.4894	0.5059	0.5214	0.5357	0.5489
$\tilde{\mu}^{ps}$	min	0.1514	0.1547	0.1577	0.1604	0.1628	0.1652	0.1674	0.1695	0.1714	0.1734
	Q1	0.1745	0.1786	0.1825	0.1860	0.1893	0.1925	0.1954	0.1981	0.2008	0.2033
	median	0.2564	0.2662	0.2752	0.2834	0.2910	0.2980	0.3045	0.3107	0.3164	0.3219
	Q3	0.2690	0.2793	0.2885	0.2971	0.3053	0.3128	0.3199	0.3265	0.3326	0.3383
	max	0.3985	0.4224	0.4449	0.4661	0.4860	0.5048	0.5224	0.5390	0.5544	0.5687

Table C.5: MSE of small-area mean estimators with known population covariates under exponential random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	0.0250	0.0586	0.0842	0.1012	0.1132	0.1222	0.1293	0.1351	0.1398	0.1439
	Q1	0.0268	0.0629	0.0907	0.1112	0.1264	0.1383	0.1479	0.1556	0.1620	0.1675
	median	0.0319	0.0744	0.1096	0.1385	0.1627	0.1835	0.2017	0.2173	0.2311	0.2431
	Q3	0.0345	0.0768	0.1125	0.1418	0.1662	0.1870	0.2048	0.2204	0.2346	0.2475
	max	0.0462	0.0958	0.1402	0.1796	0.2149	0.2468	0.2761	0.3031	0.3281	0.3514
$\tilde{\mu}^{(2)}$	min	0.0220	0.0586	0.0827	0.0972	0.1083	0.1173	0.1248	0.1309	0.1363	0.1409
	Q1	0.0244	0.0631	0.0882	0.1068	0.1213	0.1329	0.1426	0.1506	0.1575	0.1635
	median	0.0259	0.0685	0.1052	0.1358	0.1616	0.1838	0.2033	0.2204	0.2356	0.2491
	Q3	0.0279	0.0705	0.1088	0.1404	0.1669	0.1896	0.2092	0.2263	0.2412	0.2545
	max	0.0306	0.0747	0.1235	0.1678	0.2083	0.2456	0.2801	0.3123	0.3423	0.3704
$\tilde{\mu}^{c}$	min	0.0220	0.0571	0.0812	0.0965	0.1079	0.1170	0.1245	0.1308	0.1362	0.1408
	Q1	0.0242	0.0613	0.0873	0.1064	0.1211	0.1328	0.1426	0.1506	0.1575	0.1635
	median	0.0254	0.0684	0.1050	0.1351	0.1602	0.1817	0.2004	0.2166	0.2309	0.2431
	Q3	0.0266	0.0705	0.1086	0.1393	0.1647	0.1861	0.2044	0.2202	0.2344	0.2474
	max	0.0299	0.0747	0.1235	0.1672	0.2063	0.2414	0.2730	0.3017	0.3277	0.3514
$\tilde{\mu}$	min	0.0248	0.0580	0.0821	0.0975	0.1090	0.1180	0.1253	0.1315	0.1367	0.1414
	Q1	0.0273	0.0621	0.0884	0.1076	0.1223	0.1340	0.1437	0.1516	0.1584	0.1645
	median	0.0284	0.0710	0.1065	0.1357	0.1603	0.1816	0.2002	0.2164	0.2308	0.2437
	Q3	0.0303	0.0732	0.1099	0.1398	0.1649	0.1862	0.2046	0.2208	0.2350	0.2479
	max	0.0337	0.0829	0.1292	0.1709	0.2085	0.2426	0.2741	0.3033	0.3305	0.3557
$\tilde{\mu}^{ps}$	min	0.0278	0.0606	0.0854	0.1008	0.1121	0.1209	0.1281	0.1341	0.1391	0.1437
	Q1	0.0295	0.0651	0.0917	0.1109	0.1255	0.1369	0.1463	0.1540	0.1606	0.1665
	median	0.0320	0.0745	0.1099	0.1391	0.1637	0.1843	0.2020	0.2182	0.2325	0.2454
	Q3	0.0356	0.0779	0.1137	0.1428	0.1670	0.1878	0.2062	0.2224	0.2369	0.2502
	max	0.0390	0.0879	0.1349	0.1771	0.2153	0.2501	0.2823	0.3122	0.3401	0.3661

Table C.6: MSE of small-area mean estimators with known population covariates under exponential random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	0.1476	0.1509	0.1540	0.1568	0.1594	0.1617	0.1639	0.1659	0.1677	0.1693
	Q1	0.1725	0.1770	0.1810	0.1846	0.1880	0.1910	0.1939	0.1966	0.1991	0.2014
	median	0.2544	0.2648	0.2744	0.2835	0.2920	0.3000	0.3076	0.3147	0.3215	0.3279
	Q3	0.2592	0.2700	0.2800	0.2894	0.2982	0.3065	0.3144	0.3218	0.3288	0.3354
	max	0.3733	0.3941	0.4138	0.4326	0.4506	0.4680	0.4849	0.5015	0.5179	0.5348
$\tilde{\mu}^{(2)}$	min	0.1451	0.1489	0.1523	0.1554	0.1582	0.1608	0.1632	0.1654	0.1675	0.1693
	Q1	0.1689	0.1738	0.1783	0.1824	0.1861	0.1896	0.1928	0.1958	0.1986	0.2012
	median	0.2612	0.2718	0.2814	0.2897	0.2976	0.3050	0.3118	0.3181	0.3238	0.3290
	Q3	0.2664	0.2771	0.2868	0.2959	0.3043	0.3120	0.3191	0.3258	0.3319	0.3373
	max	0.3965	0.4207	0.4434	0.4643	0.4835	0.5008	0.5163	0.5296	0.5405	0.5485
$\tilde{\mu}^{c}$	min	0.1451	0.1488	0.1523	0.1554	0.1582	0.1608	0.1632	0.1654	0.1674	0.1692
	Q1	0.1689	0.1738	0.1783	0.1824	0.1861	0.1896	0.1928	0.1958	0.1986	0.2012
	median	0.2543	0.2647	0.2744	0.2835	0.2920	0.3000	0.3076	0.3147	0.3215	0.3290
	Q3	0.2592	0.2700	0.2800	0.2894	0.2982	0.3065	0.3144	0.3218	0.3288	0.3373
	max	0.3733	0.3941	0.4138	0.4326	0.4506	0.4680	0.4849	0.5015	0.5179	0.5348
$\tilde{\mu}$	min	0.1455	0.1492	0.1525	0.1556	0.1584	0.1609	0.1633	0.1655	0.1674	0.1692
	Q1	0.1698	0.1746	0.1790	0.1830	0.1866	0.1900	0.1931	0.1960	0.1987	0.2012
	median	0.2550	0.2652	0.2750	0.2842	0.2927	0.3006	0.3081	0.3152	0.3218	0.3280
	Q3	0.2598	0.2708	0.2810	0.2905	0.2993	0.3076	0.3154	0.3228	0.3296	0.3358
	max	0.3793	0.4015	0.4224	0.4420	0.4605	0.4779	0.4943	0.5098	0.5245	0.5386
$\tilde{\mu}^{ps}$	min	0.1478	0.1514	0.1547	0.1577	0.1604	0.1630	0.1654	0.1676	0.1696	0.1714
	Q1	0.1717	0.1764	0.1807	0.1846	0.1881	0.1914	0.1945	0.1973	0.2000	0.2025
	median	0.2571	0.2675	0.2772	0.2861	0.2944	0.3022	0.3095	0.3163	0.3227	0.3288
	Q3	0.2622	0.2732	0.2834	0.2928	0.3016	0.3099	0.3176	0.3249	0.3318	0.3383
	max	0.3905	0.4134	0.4352	0.4557	0.4750	0.4933	0.5107	0.5272	0.5429	0.5582

Table C.7: MSE of small-area mean estimators with known population covariates under uniform random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	0.0277	0.0659	0.0953	0.1159	0.1293	0.1380	0.1437	0.1474	0.1507	0.1537
	Q1	0.0288	0.0674	0.0989	0.1228	0.1398	0.1515	0.1598	0.1660	0.1713	0.1757
	median	0.0290	0.0730	0.1122	0.1460	0.1736	0.1963	0.2149	0.2307	0.2445	0.2562
	Q3	0.0300	0.0752	0.1146	0.1489	0.1765	0.1994	0.2189	0.2348	0.2485	0.2606
	max	0.0327	0.0773	0.1232	0.1665	0.2065	0.2431	0.2769	0.3081	0.3369	0.3636
$\tilde{\mu}^{(2)}$	min	0.0219	0.0660	0.0944	0.1134	0.1266	0.1358	0.1424	0.1473	0.1514	0.1549
	Q1	0.0247	0.0677	0.0977	0.1207	0.1374	0.1496	0.1588	0.1661	0.1721	0.1770
	median	0.0268	0.0703	0.1099	0.1440	0.1722	0.1953	0.2144	0.2304	0.2442	0.2563
	Q3	0.0290	0.0721	0.1122	0.1466	0.1745	0.1983	0.2181	0.2347	0.2489	0.2613
	max	0.0318	0.0730	0.1202	0.1647	0.2054	0.2424	0.2763	0.3075	0.3362	0.3628
$\tilde{\mu}^{c}$	min	0.0219	0.0648	0.0932	0.1127	0.1261	0.1354	0.1419	0.1466	0.1504	0.1536
	Q1	0.0247	0.0663	0.0968	0.1201	0.1370	0.1491	0.1583	0.1653	0.1710	0.1757
	median	0.0264	0.0701	0.1095	0.1434	0.1714	0.1944	0.2136	0.2294	0.2432	0.2552
	Q3	0.0282	0.0720	0.1119	0.1461	0.1739	0.1974	0.2172	0.2336	0.2477	0.2600
	max	0.0298	0.0729	0.1196	0.1635	0.2038	0.2407	0.2746	0.3058	0.3347	0.3615
$\tilde{\mu}$	min	0.0242	0.0655	0.0940	0.1134	0.1265	0.1353	0.1414	0.1461	0.1500	0.1534
	Q1	0.0264	0.0670	0.0975	0.1207	0.1373	0.1491	0.1578	0.1648	0.1707	0.1756
	median	0.0275	0.0711	0.1104	0.1439	0.1715	0.1943	0.2131	0.2290	0.2428	0.2549
	Q3	0.0298	0.0732	0.1127	0.1467	0.1741	0.1973	0.2168	0.2330	0.2472	0.2596
	max	0.0307	0.0742	0.1203	0.1638	0.2040	0.2407	0.2744	0.3056	0.3344	0.3612
$\tilde{\mu}^{ps}$	min	0.0285	0.0677	0.0978	0.1169	0.1299	0.1385	0.1444	0.1488	0.1525	0.1558
	Q1	0.0300	0.0699	0.1010	0.1242	0.1407	0.1521	0.1605	0.1672	0.1728	0.1775
	median	0.0306	0.0748	0.1128	0.1451	0.1721	0.1945	0.2134	0.2295	0.2432	0.2551
	Q3	0.0336	0.0761	0.1161	0.1496	0.1768	0.1999	0.2191	0.2352	0.2493	0.2616
	max	0.0350	0.0786	0.1256	0.1691	0.2099	0.2473	0.2819	0.3140	0.3439	0.3717

Table C.8: MSE of small-area mean estimators with known population covariates under uniform random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	0.1564	0.1589	0.1613	0.1634	0.1654	0.1672	0.1688	0.1702	0.1715	0.1726
	Q1	0.1797	0.1834	0.1867	0.1897	0.1925	0.1950	0.1973	0.1994	0.2013	0.2031
	median	0.2666	0.2760	0.2845	0.2922	0.2992	0.3056	0.3115	0.3169	0.3219	0.3265
	Q3	0.2713	0.2811	0.2900	0.2980	0.3054	0.3120	0.3182	0.3238	0.3289	0.3336
	max	0.3884	0.4114	0.4327	0.4525	0.4709	0.4878	0.5035	0.5180	0.5313	0.5434
$\tilde{\mu}^{(2)}$	min	0.1578	0.1604	0.1627	0.1647	0.1665	0.1681	0.1695	0.1708	0.1720	0.1728
	Q1	0.1813	0.1851	0.1884	0.1913	0.1939	0.1962	0.1983	0.2002	0.2019	0.2035
	median	0.2670	0.2765	0.2850	0.2927	0.2996	0.3059	0.3117	0.3170	0.3219	0.3265
	Q3	0.2722	0.2820	0.2909	0.2990	0.3062	0.3128	0.3189	0.3243	0.3294	0.3340
	max	0.3876	0.4105	0.4319	0.4517	0.4701	0.4873	0.5031	0.5178	0.5313	0.5435
$\tilde{\mu}^{c}$	min	0.1564	0.1589	0.1613	0.1634	0.1654	0.1672	0.1688	0.1702	0.1715	0.1726
	Q1	0.1797	0.1834	0.1867	0.1897	0.1925	0.1950	0.1973	0.1994	0.2013	0.2035
	median	0.2659	0.2755	0.2841	0.2919	0.2990	0.3054	0.3113	0.3168	0.3218	0.3265
	Q3	0.2709	0.2808	0.2898	0.2979	0.3053	0.3120	0.3181	0.3238	0.3289	0.3340
	max	0.3864	0.4095	0.4310	0.4509	0.4695	0.4867	0.5026	0.5174	0.5309	0.5433
$\tilde{\mu}$	min	0.1564	0.1591	0.1615	0.1637	0.1656	0.1674	0.1690	0.1704	0.1717	0.1727
	Q1	0.1799	0.1837	0.1871	0.1902	0.1929	0.1954	0.1977	0.1997	0.2016	0.2033
	median	0.2657	0.2753	0.2840	0.2918	0.2989	0.3054	0.3113	0.3168	0.3218	0.3264
	Q3	0.2706	0.2805	0.2895	0.2977	0.3051	0.3119	0.3181	0.3238	0.3290	0.3336
	max	0.3860	0.4091	0.4306	0.4505	0.4691	0.4863	0.5023	0.5171	0.5307	0.5431
$\tilde{\mu}^{ps}$	min	0.1587	0.1613	0.1637	0.1659	0.1679	0.1696	0.1713	0.1728	0.1742	0.1754
	Q1	0.1817	0.1854	0.1887	0.1917	0.1944	0.1969	0.1991	0.2011	0.2030	0.2047
	median	0.2657	0.2752	0.2837	0.2914	0.2985	0.3051	0.3111	0.3167	0.3218	0.3266
	Q3	0.2722	0.2820	0.2909	0.2990	0.3063	0.3130	0.3192	0.3248	0.3298	0.3342
	max	0.3976	0.4218	0.4445	0.4656	0.4853	0.5037	0.5209	0.5368	0.5516	0.5652

Table C.9: MSE of small-area mean estimators with sample covariates under normal random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	0.0269	0.0624	0.0896	0.1084	0.1209	0.1292	0.1347	0.1389	0.1424	0.1453
	Q1	0.0286	0.0671	0.0984	0.1215	0.1384	0.1506	0.1593	0.1659	0.1713	0.1758
	median	0.0305	0.0739	0.1125	0.1448	0.1719	0.1942	0.2128	0.2284	0.2418	0.2536
	Q3	0.0367	0.0786	0.1167	0.1488	0.1754	0.1971	0.2146	0.2301	0.2438	0.2557
	max	0.0618	0.1065	0.1482	0.1857	0.2194	0.2496	0.2770	0.3019	0.3249	0.3461
$\tilde{\mu}^{(2)}$	min	0.0244	0.0631	0.0875	0.1045	0.1165	0.1251	0.1315	0.1366	0.1407	0.1443
	Q1	0.0286	0.0679	0.0977	0.1196	0.1359	0.1479	0.1570	0.1642	0.1702	0.1752
	median	0.0293	0.0704	0.1101	0.1436	0.1716	0.1946	0.2135	0.2297	0.2437	0.2560
	Q3	0.0323	0.0754	0.1140	0.1474	0.1749	0.1970	0.2156	0.2319	0.2461	0.2583
	max	0.0528	0.0992	0.1429	0.1828	0.2186	0.2506	0.2795	0.3057	0.3297	0.3517
$\tilde{\mu}^{c}$	min	0.0244	0.0613	0.0866	0.1041	0.1163	0.1249	0.1314	0.1364	0.1405	0.1441
	Q1	0.0269	0.0663	0.0965	0.1188	0.1354	0.1475	0.1567	0.1639	0.1699	0.1748
	median	0.0285	0.0703	0.1096	0.1425	0.1701	0.1929	0.2118	0.2276	0.2413	0.2533
	Q3	0.0323	0.0753	0.1137	0.1465	0.1737	0.1956	0.2136	0.2295	0.2435	0.2556
	max	0.0528	0.0992	0.1428	0.1819	0.2168	0.2478	0.2758	0.3011	0.3244	0.3458
$\tilde{\mu}$	min	0.0267	0.0621	0.0877	0.1053	0.1173	0.1257	0.1318	0.1367	0.1407	0.1441
	Q1	0.0276	0.0671	0.0972	0.1195	0.1358	0.1477	0.1567	0.1637	0.1696	0.1745
	median	0.0291	0.0718	0.1106	0.1430	0.1703	0.1927	0.2112	0.2270	0.2408	0.2529
	Q3	0.0341	0.0764	0.1147	0.1470	0.1737	0.1953	0.2133	0.2291	0.2432	0.2553
	max	0.0561	0.1018	0.1442	0.1825	0.2169	0.2478	0.2758	0.3013	0.3247	0.3464

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{Vm}$	min	0.1668	0.1668	0.1668	0.1668	0.1668	0.1668	0.1668	0.1668	0.1668	0.1449
	Q1	0.2033	0.2033	0.2033	0.2033	0.2033	0.2033	0.2033	0.2033	0.2033	0.2033
	median	0.3280	0.3280	0.3280	0.3280	0.3280	0.3280	0.3280	0.3280	0.3280	0.3280
	Q3	0.3345	0.3345	0.3345	0.3345	0.3345	0.3345	0.3345	0.3345	0.3345	0.3345
	max	0.5046	0.5046	0.5046	0.5046	0.5046	0.5046	0.5046	0.5046	0.5046	0.3494
$\tilde{\mu}^{(S)}$	min	0.0470	0.0777	0.0985	0.1116	0.1204	0.1270	0.1322	0.1365	0.1405	0.1440
	Q1	0.0511	0.0811	0.1062	0.1243	0.1383	0.1485	0.1568	0.1636	0.1693	0.1744
	median	0.0522	0.0848	0.1169	0.1471	0.1721	0.1941	0.2127	0.2282	0.2418	0.2537
	Q3	0.0600	0.0917	0.1231	0.1523	0.1780	0.1983	0.2158	0.2304	0.2438	0.2561
	max	0.1045	0.1314	0.1609	0.1932	0.2241	0.2533	0.2807	0.3058	0.3286	0.3499
$\tilde{\mu}^{(T)}$	min	0.0262	0.0623	0.0860	0.1030	0.1155	0.1242	0.1308	0.1361	0.1404	0.1440
	Q1	0.0300	0.0680	0.0966	0.1181	0.1338	0.1464	0.1557	0.1631	0.1692	0.1744
	median	0.0307	0.0713	0.1099	0.1416	0.1685	0.1915	0.2106	0.2269	0.2406	0.2528
	Q3	0.0340	0.0761	0.1132	0.1455	0.1722	0.1937	0.2123	0.2285	0.2428	0.2553
	max	0.0546	0.1008	0.1432	0.1818	0.2153	0.2472	0.2757	0.3006	0.3242	0.3459
$\tilde{\mu}^{ps}$	min	0.0289	0.0650	0.0909	0.1083	0.1199	0.1278	0.1335	0.1379	0.1416	0.1449
	Q1	0.0303	0.0697	0.0995	0.1215	0.1375	0.1491	0.1578	0.1647	0.1704	0.1752
	median	0.0351	0.0757	0.1135	0.1448	0.1709	0.1930	0.2118	0.2279	0.2419	0.2542
	Q3	0.0385	0.0803	0.1179	0.1500	0.1763	0.1980	0.2161	0.2314	0.2450	0.2566
	max	0.0656	0.1101	0.1515	0.1890	0.2226	0.2529	0.2803	0.3052	0.3282	0.3494

Table C.10: MSE of small-area mean estimators with sample covariates under normal random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	0.1481	0.1505	0.1529	0.1550	0.1569	0.1588	0.1605	0.1622	0.1637	0.1652
	Q1	0.1798	0.1833	0.1865	0.1894	0.1920	0.1944	0.1966	0.1986	0.2004	0.2021
	median	0.2641	0.2734	0.2821	0.2901	0.2972	0.3036	0.3096	0.3151	0.3200	0.3244
	Q3	0.2663	0.2756	0.2842	0.2920	0.2993	0.3060	0.3123	0.3181	0.3239	0.3296
	max	0.3659	0.3843	0.4014	0.4175	0.4325	0.4465	0.4596	0.4719	0.4834	0.4942
$\tilde{\mu}^{(2)}$	min	0.1475	0.1503	0.1529	0.1551	0.1572	0.1591	0.1609	0.1625	0.1640	0.1654
	Q1	0.1795	0.1833	0.1866	0.1896	0.1922	0.1946	0.1968	0.1987	0.2005	0.2021
	median	0.2666	0.2760	0.2845	0.2923	0.2996	0.3058	0.3114	0.3165	0.3212	0.3250
	Q3	0.2690	0.2785	0.2872	0.2948	0.3019	0.3083	0.3142	0.3196	0.3249	0.3301
	max	0.3719	0.3906	0.4079	0.4238	0.4384	0.4519	0.4643	0.4756	0.4859	0.4953
$\tilde{\mu}^{c}$	min	0.1473	0.1500	0.1526	0.1548	0.1569	0.1588	0.1605	0.1622	0.1637	0.1652
	Q1	0.1791	0.1829	0.1862	0.1892	0.1919	0.1943	0.1965	0.1985	0.2004	0.2021
	median	0.2639	0.2733	0.2820	0.2901	0.2971	0.3036	0.3096	0.3151	0.3200	0.3250
	Q3	0.2662	0.2756	0.2842	0.2920	0.2993	0.3060	0.3123	0.3181	0.3239	0.3301
	max	0.3657	0.3842	0.4014	0.4175	0.4325	0.4465	0.4596	0.4719	0.4834	0.4942
$\tilde{\mu}$	min	0.1472	0.1500	0.1525	0.1548	0.1569	0.1588	0.1606	0.1622	0.1638	0.1652
	Q1	0.1789	0.1827	0.1860	0.1891	0.1918	0.1942	0.1965	0.1985	0.2004	0.2021
	median	0.2637	0.2733	0.2821	0.2902	0.2975	0.3039	0.3099	0.3153	0.3202	0.3245
	Q3	0.2660	0.2756	0.2844	0.2923	0.2996	0.3063	0.3126	0.3183	0.3241	0.3297
	max	0.3665	0.3851	0.4024	0.4186	0.4336	0.4476	0.4606	0.4727	0.4840	0.4945

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{Vm}$	min	0.1668	0.1668	0.1668	0.1668	0.1668	0.1668	0.1668	0.1668	0.1668	0.1668
	Q1	0.2033	0.2033	0.2033	0.2033	0.2033	0.2033	0.2033	0.2033	0.2033	0.2033
	median	0.3280	0.3280	0.3280	0.3280	0.3280	0.3280	0.3280	0.3280	0.3280	0.3280
	Q3	0.3345	0.3345	0.3345	0.3345	0.3345	0.3345	0.3345	0.3345	0.3345	0.3345
	max	0.5046	0.5046	0.5046	0.5046	0.5046	0.5046	0.5046	0.5046	0.5046	0.5046
$\tilde{\mu}^{(S)}$	min	0.1470	0.1499	0.1524	0.1547	0.1568	0.1588	0.1605	0.1622	0.1637	0.1652
	Q1	0.1787	0.1825	0.1860	0.1890	0.1918	0.1942	0.1965	0.1985	0.2004	0.2021
	median	0.2644	0.2739	0.2824	0.2904	0.2976	0.3040	0.3099	0.3153	0.3202	0.3245
	Q3	0.2667	0.2762	0.2850	0.2931	0.3003	0.3069	0.3131	0.3187	0.3241	0.3296
	max	0.3809	0.3975	0.4129	0.4273	0.4408	0.4534	0.4652	0.4763	0.4866	0.4947
$\tilde{\mu}^{(T)}$	min	0.1472	0.1500	0.1525	0.1548	0.1569	0.1588	0.1605	0.1622	0.1637	0.1652
	Q1	0.1789	0.1828	0.1862	0.1892	0.1919	0.1943	0.1965	0.1986	0.2004	0.2021
	median	0.2637	0.2732	0.2819	0.2900	0.2971	0.3036	0.3096	0.3151	0.3200	0.3244
	Q3	0.2660	0.2755	0.2841	0.2920	0.2993	0.3060	0.3123	0.3181	0.3239	0.3296
	max	0.3780	0.3949	0.4106	0.4253	0.4391	0.4520	0.4642	0.4755	0.4862	0.4942
$\tilde{\mu}^{ps}$	min	0.1478	0.1504	0.1528	0.1550	0.1570	0.1589	0.1607	0.1623	0.1638	0.1652
	Q1	0.1794	0.1831	0.1864	0.1893	0.1920	0.1944	0.1965	0.1986	0.2004	0.2020
	median	0.2649	0.2746	0.2831	0.2908	0.2980	0.3045	0.3102	0.3154	0.3202	0.3245
	Q3	0.2669	0.2761	0.2848	0.2928	0.3001	0.3068	0.3129	0.3186	0.3241	0.3296
	max	0.3690	0.3873	0.4043	0.4201	0.4348	0.4485	0.4612	0.4731	0.4841	0.4945

Table C.11: MSE of small-area mean estimators with sample covariates under double exponential random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	0.0249	0.0594	0.0847	0.1027	0.1154	0.1244	0.1312	0.1365	0.1409	0.1445
	Q1	0.0270	0.0644	0.0931	0.1146	0.1307	0.1428	0.1516	0.1584	0.1641	0.1691
	median	0.0320	0.0740	0.1104	0.1404	0.1651	0.1857	0.2033	0.2190	0.2327	0.2447
	Q3	0.0373	0.0799	0.1161	0.1462	0.1711	0.1929	0.2111	0.2270	0.2412	0.2535
	max	0.0671	0.1114	0.1520	0.1883	0.2209	0.2499	0.2763	0.3005	0.3229	0.3437
$\tilde{\mu}^{(2)}$	min	0.0235	0.0602	0.0821	0.0975	0.1092	0.1179	0.1249	0.1306	0.1353	0.1394
	Q1	0.0271	0.0652	0.0920	0.1115	0.1259	0.1371	0.1462	0.1537	0.1599	0.1654
	median	0.0284	0.0699	0.1069	0.1387	0.1653	0.1876	0.2070	0.2238	0.2381	0.2505
	Q3	0.0307	0.0752	0.1129	0.1445	0.1710	0.1941	0.2139	0.2311	0.2462	0.2594
	max	0.0498	0.0952	0.1386	0.1787	0.2155	0.2490	0.2798	0.3083	0.3342	0.3581
$\tilde{\mu}^{c}$	min	0.0235	0.0580	0.0812	0.0972	0.1091	0.1179	0.1249	0.1306	0.1353	0.1394
	Q1	0.0256	0.0632	0.0907	0.1109	0.1257	0.1371	0.1462	0.1537	0.1599	0.1654
	median	.0.0271	0.0698	0.1065	0.1377	0.1635	0.1849	0.2029	0.2189	0.2326	0.2447
	Q3	0.0307	0.0751	0.1125	0.1436	0.1693	0.1915	0.2105	0.2267	0.2411	0.2535
	max	0.0498	0.0952	0.1386	0.1786	0.2145	0.2462	0.2744	0.2998	0.3227	0.3437
$\tilde{\mu}$	min	0.0251	0.0588	0.0822	0.0987	0.1107	0.1195	0.1266	0.1323	0.1371	0.1410
	Q1	0.0269	0.0641	0.0916	0.1120	0.1270	0.1384	0.1473	0.1546	0.1608	0.1661
	median	0.0290	0.0716	0.1078	0.1382	0.1634	0.1845	0.2030	0.2191	0.2329	0.2449
	Q3	0.0339	0.0773	0.1140	0.1443	0.1694	0.1916	0.2103	0.2265	0.2411	0.2539
	max	0.0562	0.1013	0.1434	0.1816	0.2160	0.2470	0.2753	0.3011	0.3250	0.3470
$\tilde{\mu}^{ps}$	min	0.0275	0.0618	0.0855	0.1019	0.1136	0.1220	0.1286	0.1341	0.1386	0.1422
	Q1	0.0297	0.0672	0.0948	0.1151	0.1298	0.1408	0.1493	0.1563	0.1622	0.1672
	median	0.0343	0.0760	0.1114	0.1411	0.1657	0.1865	0.2040	0.2198	0.2337	0.2455
	Q3	0.0381	0.0812	0.1177	0.1478	0.1730	0.1951	0.2134	0.2292	0.2431	0.2553
	max	0.0650	0.1099	0.1513	0.1889	0.2226	0.2531	0.2807	0.3060	0.3294	0.3510

Table C.12: MSE of small-area mean estimators with sample covariates under double exponential random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	0.1473	0.1498	0.1521	0.1543	0.1562	0.1580	0.1598	0.1614	0.1629	0.1644
	Q1	0.1732	0.1768	0.1802	0.1834	0.1863	0.1890	0.1915	0.1940	0.1963	0.1985
	median	0.2552	0.2646	0.2733	0.2813	0.2884	0.2951	0.3013	0.3070	0.3124	0.3176
	Q3	0.2644	0.2743	0.2834	0.2918	0.2991	0.3059	0.3121	0.3180	0.3234	0.3287
	max	0.3629	0.3811	0.3982	0.4143	0.4295	0.4440	0.4577	0.4708	0.4833	0.4952
$\tilde{\mu}^{(2)}$	min	0.1431	0.1463	0.1493	0.1519	0.1543	0.1566	0.1586	0.1606	0.1624	0.1641
	Q1	0.1702	0.1745	0.1784	0.1819	0.1851	0.1881	0.1909	0.1936	0.1960	0.1984
	median	0.2615	0.2712	0.2799	0.2877	0.2946	0.3006	0.3060	0.3109	0.3153	0.3191
	Q3	0.2711	0.2813	0.2904	0.2983	0.3052	0.3114	0.3168	0.3217	0.3259	0.3301
	max	0.3800	0.4003	0.4188	0.4357	0.4511	0.4648	0.4769	0.4873	0.4958	0.5023
$\tilde{\mu}^{e}$	min	0.1431	0.1463	0.1493	0.1519	0.1543	0.1566	0.1586	0.1606	0.1624	0.1641
	Q1	0.1702	0.1745	0.1783	0.1819	0.1851	0.1881	0.1909	0.1936	0.1960	0.1984
	median	0.2551	0.2646	0.2733	0.2813	0.2884	0.2951	0.3013	0.3070	0.3124	0.3191
	Q3	0.2644	0.2743	0.2834	0.2918	0.2991	0.3059	0.3121	0.3180	0.3234	0.3301
	max	0.3629	0.3811	0.3982	0.4143	0.4295	0.4440	0.4577	0.4708	0.4833	0.4952
$\tilde{\mu}$	min	0.1443	0.1474	0.1502	0.1527	0.1550	0.1571	0.1591	0.1609	0.1626	0.1642
	Q1	0.1707	0.1748	0.1786	0.1821	0.1853	0.1883	0.1910	0.1936	0.1961	0.1984
	median	0.2556	0.2653	0.2742	0.2823	0.2896	0.2962	0.3022	0.3078	0.3131	0.3179
	Q3	0.2653	0.2754	0.2846	0.2931	0.3004	0.3071	0.3132	0.3188	0.3240	0.3290
	max	0.3674	0.3864	0.4040	0.4205	0.4358	0.4501	0.4634	0.4756	0.4869	0.4972
$\tilde{\mu}^{ps}$	min	0.1453	0.1482	0.1508	0.1532	0.1554	0.1574	0.1593	0.1610	0.1627	0.1643
	Q1	0.1715	0.1755	0.1791	0.1825	0.1855	0.1884	0.1911	0.1937	0.1961	0.1984
	median	0.2561	0.2657	0.2745	0.2825	0.2899	0.2968	0.3027	0.3083	0.3134	0.3181
	Q3	0.2659	0.2754	0.2845	0.2929	0.3007	0.3078	0.3140	0.3195	0.3244	0.3293
	max	0.3709	0.3894	0.4067	0.4228	0.4378	0.4517	0.4646	0.4766	0.4876	0.4976

Table C.13: MSE of small-area mean estimators with sample covariates under exponential random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	0.0246	0.0577	0.0810	0.0977	0.1094	0.1182	0.1251	0.1305	0.1351	0.1389
	Q1	0.0265	0.0625	0.0897	0.1099	0.1250	0.1369	0.1464	0.1539	0.1602	0.1656
	median	0.0330	0.0755	0.1112	0.1402	0.1641	0.1841	0.2017	0.2172	0.2308	0.2428
	Q3	0.0392	0.0807	0.1155	0.1445	0.1685	0.1890	0.2064	0.2216	0.2350	0.2474
	max	0.0763	0.1199	0.1590	0.1934	0.2241	0.2516	0.2767	0.2996	0.3208	0.3405
$\tilde{\mu}^{(2)}$	min	0.0239	0.0589	0.0796	0.0939	0.1047	0.1135	0.1206	0.1265	0.1317	0.1361
	Q1	0.0267	0.0625	0.0874	0.1058	0.1200	0.1316	0.1411	0.1490	0.1557	0.1616
	median	0.0283	0.0697	0.1070	0.1379	0.1633	0.1848	0.2036	0.2201	0.2347	0.2481
	Q3	0.0321	0.0752	0.1109	0.1415	0.1678	0.1904	0.2100	0.2270	0.2415	0.2543
	max	0.0521	0.0982	0.1413	0.1803	0.2157	0.2482	0.2782	0.3059	0.3316	0.3555
$\tilde{\mu}^{c}$	min	0.0239	0.0565	0.0781	0.0931	0.1043	0.1132	0.1204	0.1264	0.1316	0.1360
	Q1	0.0250	0.0608	0.0863	0.1052	0.1198	0.1315	0.1410	0.1489	0.1557	0.1616
	median	0.0270	0.0697	0.1067	0.1369	0.1618	0.1826	0.2006	0.2164	0.2306	0.2427
	Q3	0.0321	0.0752	0.1105	0.1408	0.1662	0.1877	0.2059	0.2214	0.2350	0.2474
	max	0.0521	0.0982	0.1413	0.1801	0.2147	0.2455	0.2731	0.2978	0.3201	0.3403
$\tilde{\mu}$	min	0.0251	0.0573	0.0790	0.0941	0.1053	0.1141	0.1212	0.1271	0.1321	0.1365
	Q1	0.0264	0.0616	0.0874	0.1063	0.1209	0.1326	0.1421	0.1499	0.1566	0.1625
	median	0.0295	0.0726	0.1085	0.1378	0.1619	0.1824	0.2004	0.2161	0.2304	0.2431
	Q3	0.0355	0.0774	0.1124	0.1420	0.1666	0.1877	0.2059	0.2216	0.2355	0.2480
	max	0.0623	0.1072	0.1482	0.1846	0.2173	0.2469	0.2740	0.2990	0.3221	0.3435
$\tilde{\mu}^{ps}$	min	0.0271	0.0604	0.0824	0.0974	0.1083	0.1167	0.1236	0.1292	0.1339	0.1381
	Q1	0.0294	0.0647	0.0908	0.1097	0.1240	0.1353	0.1446	0.1520	0.1584	0.1641
	median	0.0351	0.0764	0.1121	0.1410	0.1648	0.1849	0.2025	0.2185	0.2325	0.2450
	Q3	0.0406	0.0816	0.1157	0.1441	0.1684	0.1893	0.2073	0.2232	0.2374	0.2501
	max	0.0722	0.1166	0.1569	0.1928	0.2249	0.2539	0.2805	0.3049	0.3275	0.3484

Table C.14: MSE of small-area mean estimators with sample covariates under exponential random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	0.1425	0.1456	0.1485	0.1512	0.1535	0.1557	0.1577	0.1595	0.1612	0.1627
	Q1	0.1704	0.1748	0.1787	0.1822	0.1855	0.1884	0.1913	0.1939	0.1964	0.1986
	median	0.2540	0.2643	0.2738	0.2827	0.2910	0.2988	0.3062	0.3132	0.3196	0.3256
	Q3	0.2584	0.2684	0.2776	0.2866	0.2954	0.3037	0.3115	0.3189	0.3259	0.3323
	max	0.3588	0.3761	0.3925	0.4081	0.4231	0.4374	0.4514	0.4651	0.4788	0.4929
$\tilde{\mu}^{(2)}$	min	0.1401	0.1436	0.1468	0.1497	0.1523	0.1547	0.1570	0.1590	0.1609	0.1626
	Q1	0.1669	0.1716	0.1760	0.1800	0.1836	0.1870	0.1901	0.1931	0.1958	0.1984
	median	0.2602	0.2707	0.2799	0.2884	0.2963	0.3037	0.3104	0.3165	0.3220	0.3270
	Q3	0.2656	0.2757	0.2850	0.2934	0.3011	0.3088	0.3162	0.3229	0.3290	0.3344
	max	0.3775	0.3979	0.4168	0.4342	0.4501	0.4644	0.4770	0.4879	0.4968	0.5035
$\tilde{\mu}^{c}$	min	0.1400	0.1435	0.1468	0.1497	0.1523	0.1547	0.1570	0.1590	0.1609	0.1626
	Q1	0.1669	0.1716	0.1760	0.1800	0.1836	0.1870	0.1901	0.1931	0.1958	0.1984
	median	0.2540	0.2643	0.2738	0.2827	0.2910	0.2988	0.3062	0.3132	0.3196	0.3270
	Q3	0.2583	0.2684	0.2776	0.2866	0.2954	0.3037	0.3115	0.3189	0.3259	0.3344
	max	0.3588	0.3761	0.3925	0.4081	0.4231	0.4374	0.4514	0.4651	0.4788	0.4929
$\tilde{\mu}$	min	0.1405	0.1439	0.1471	0.1499	0.1525	0.1549	0.1571	0.1591	0.1610	0.1626
	Q1	0.1677	0.1724	0.1767	0.1806	0.1841	0.1874	0.1905	0.1933	0.1960	0.1985
	median	0.2543	0.2648	0.2745	0.2835	0.2919	0.2997	0.3070	0.3138	0.3201	0.3260
	Q3	0.2596	0.2698	0.2791	0.2877	0.2965	0.3048	0.3126	0.3199	0.3267	0.3329
	max	0.3633	0.3819	0.3993	0.4156	0.4309	0.4453	0.4589	0.4717	0.4839	0.4958
$\tilde{\mu}^{ps}$	min	0.1418	0.1451	0.1481	0.1508	0.1532	0.1555	0.1576	0.1595	0.1612	0.1627
	Q1	0.1691	0.1736	0.1777	0.1814	0.1848	0.1879	0.1909	0.1937	0.1962	0.1986
	median	0.2563	0.2667	0.2762	0.2850	0.2932	0.3008	0.3079	0.3145	0.3206	0.3263
	Q3	0.2612	0.2714	0.2809	0.2893	0.2974	0.3055	0.3131	0.3202	0.3269	0.3329
	max	0.3678	0.3859	0.4029	0.4188	0.4337	0.4476	0.4608	0.4731	0.4849	0.4964

Table C.15: MSE of small-area mean estimators with sample covariates under uniform random effects for ρ from 0.00 to 0.45

ρ		0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	0.0272	0.0632	0.0913	0.1113	0.1242	0.1326	0.1380	0.1417	0.1446	0.1473
	Q1	0.0288	0.0667	0.0979	0.1216	0.1383	0.1499	0.1581	0.1641	0.1691	0.1734
	median	0.0304	0.0736	0.1131	0.1467	0.1743	0.1967	0.2154	0.2310	0.2444	0.2561
	Q3	0.0369	0.0792	0.1180	0.1516	0.1797	0.2015	0.2195	0.2347	0.2477	0.2592
	max	0.0563	0.0991	0.1400	0.1776	0.2116	0.2432	0.2721	0.2986	0.3229	0.3454
$\tilde{\mu}^{(2)}$	min	0.0248	0.0641	0.0902	0.1088	0.1216	0.1304	0.1367	0.1415	0.1453	0.1486
	Q1	0.0285	0.0669	0.0969	0.1195	0.1359	0.1479	0.1570	0.1641	0.1698	0.1746
	median	0.0292	0.0709	0.1111	0.1448	0.1727	0.1954	0.2143	0.2302	0.2439	0.2559
	Q3	0.0326	0.0758	0.1156	0.1494	0.1782	0.2005	0.2190	0.2346	0.2480	0.2598
	max	0.0513	0.0960	0.1382	0.1768	0.2111	0.2424	0.2713	0.2976	0.3219	0.3442
$\tilde{\mu}^{c}$	min	0.0248	0.0624	0.0891	0.1081	0.1211	0.1300	0.1362	0.1408	0.1443	0.1473
	Q1	0.0274	0.0657	0.0959	0.1189	0.1355	0.1475	0.1564	0.1632	0.1688	0.1733
	median	0.0286	0.0707	0.1106	0.1442	0.1719	0.1946	0.2134	0.2294	0.2431	0.2551
	Q3	0.0326	0.0757	0.1153	0.1489	0.1775	0.1997	0.2180	0.2335	0.2468	0.2586
	max	0.0513	0.0958	0.1374	0.1754	0.2095	0.2409	0.2699	0.2964	0.3208	0.3433
$\tilde{\mu}$	min	0.0265	0.0632	0.0901	0.1089	0.1215	0.1300	0.1359	0.1404	0.1440	0.1472
	Q1	0.0277	0.0662	0.0966	0.1195	0.1359	0.1475	0.1561	0.1628	0.1684	0.1732
	median	0.0292	0.0719	0.1115	0.1447	0.1720	0.1944	0.2131	0.2290	0.2427	0.2547
	Q3	0.0345	0.0771	0.1162	0.1495	0.1777	0.1995	0.2175	0.2329	0.2463	0.2582
	max	0.0530	0.0967	0.1381	0.1759	0.2098	0.2410	0.2698	0.2962	0.3205	0.3430
$\tilde{\mu}^{ps}$	min	0.0292	0.0661	0.0933	0.1118	0.1240	0.1319	0.1373	0.1414	0.1448	0.1477
	Q1	0.0303	0.0694	0.1002	0.1230	0.1390	0.1500	0.1581	0.1646	0.1699	0.1744
	median	0.0344	0.0766	0.1140	0.1463	0.1731	0.1954	0.2139	0.2296	0.2432	0.2551
	Q3	0.0385	0.0808	0.1190	0.1524	0.1805	0.2032	0.2218	0.2365	0.2492	0.2603
	max	0.0617	0.1052	0.1460	0.1832	0.2165	0.2462	0.2735	0.2994	0.3232	0.3453

Table C.16: MSE of small-area mean estimators with sample covariates under uniform random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	0.1498	0.1521	0.1542	0.1561	0.1578	0.1594	0.1609	0.1622	0.1633	0.1644
	Q1	0.1772	0.1806	0.1837	0.1866	0.1892	0.1916	0.1938	0.1958	0.1976	0.1993
	median	0.2664	0.2757	0.2840	0.2915	0.2983	0.3045	0.3103	0.3157	0.3206	0.3252
	Q3	0.2696	0.2790	0.2874	0.2954	0.3027	0.3094	0.3155	0.3211	0.3264	0.3312
	max	0.3662	0.3853	0.4031	0.4195	0.4347	0.4487	0.4617	0.4736	0.4847	0.4948
$\tilde{\mu}^{(2)}$	min	0.1513	0.1537	0.1558	0.1575	0.1591	0.1605	0.1617	0.1628	0.1638	0.1646
	Q1	0.1787	0.1823	0.1854	0.1881	0.1906	0.1928	0.1948	0.1965	0.1982	0.1996
	median	0.2665	0.2759	0.2843	0.2918	0.2987	0.3049	0.3106	0.3159	0.3207	0.3252
	Q3	0.2704	0.2800	0.2887	0.2966	0.3037	0.3102	0.3161	0.3216	0.3266	0.3313
	max	0.3649	0.3840	0.4018	0.4183	0.4336	0.4478	0.4610	0.4732	0.4845	0.4948
$\tilde{\mu}^{c}$	min	0.1498	0.1521	0.1542	0.1561	0.1578	0.1594	0.1609	0.1622	0.1633	0.1644
	Q1	0.1772	0.1806	0.1837	0.1866	0.1892	0.1916	0.1938	0.1958	0.1976	0.1996
	median	0.2657	0.2751	0.2835	0.2911	0.2981	0.3044	0.3102	0.3156	0.3205	0.3252
	Q3	0.2692	0.2787	0.2873	0.2953	0.3026	0.3093	0.3155	0.3211	0.3263	0.3313
	max	0.3641	0.3834	0.4013	0.4178	0.4332	0.4475	0.4607	0.4730	0.4843	0.4946
$\tilde{\mu}$	min	0.1500	0.1525	0.1546	0.1566	0.1583	0.1598	0.1612	0.1624	0.1635	0.1645
	Q1	0.1773	0.1809	0.1842	0.1870	0.1896	0.1920	0.1941	0.1960	0.1978	0.1994
	median	0.2654	0.2748	0.2833	0.2910	0.2980	0.3043	0.3102	0.3156	0.3205	0.3251
	Q3	0.2689	0.2784	0.2873	0.2954	0.3027	0.3094	0.3155	0.3212	0.3264	0.3312
	max	0.3638	0.3831	0.4009	0.4175	0.4329	0.4472	0.4604	0.4727	0.4841	0.4945
$\tilde{\mu}^{ps}$	min	0.1503	0.1527	0.1547	0.1566	0.1583	0.1598	0.1611	0.1624	0.1634	0.1644
	Q1	0.1783	0.1817	0.1848	0.1876	0.1901	0.1923	0.1944	0.1962	0.1979	0.1995
	median	0.2656	0.2750	0.2834	0.2910	0.2979	0.3042	0.3100	0.3153	0.3204	0.3250
	Q3	0.2709	0.2804	0.2887	0.2962	0.3032	0.3098	0.3158	0.3213	0.3265	0.3312
	max	0.3657	0.3846	0.4022	0.4185	0.4336	0.4477	0.4608	0.4729	0.4841	0.4945

Appendix D

RELATIVE EFFICIENCIES OF $\tilde{\mu}^c$

(1) with population covariate means (from p. 185 to 194)

(2) with sample covariate means (from p. 195 to 204)

a_i: weight defined in equation (2.7.4)

$\tilde{\mu}^{(1)}$: EBLUP defined in equation (2.7.4)

$\tilde{\mu}^{(2)}$: EBLUP defined in equation (2.7.4)

$\tilde{\mu}^c$: composite estimator defined in equation (2.7.4)

$\tilde{\mu}$: estimator defined in equation (3.2.5)

$\tilde{\mu}^{Vm}$: EBLUP with M.L.E. variance estimators

$\tilde{\mu}^{(S)}$: EBLUP with Residual M.L.E. variance estimators

$\tilde{\mu}^{(T)}$: EBLUP with Restricted M.L.E. variance estimators

$\tilde{\mu}^{ps}$: PSEUDO-EBLUP by You & Rao

Table D.1: Relative efficiency of $\tilde{\mu}^c$ with respect to other estimators with known population covariates under normal random effects for ρ from 0.00 to 0.45

ρ		0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	1.0059	1.0106	1.0181	1.0141	1.0086	1.0049	1.0025	1.0013	1.0007	1.0003
	Q1	1.0225	1.0197	1.0245	1.0174	1.0120	1.0081	1.0051	1.0030	1.0018	1.0009
	median	1.1760	1.0467	1.0262	1.0216	1.0147	1.0097	1.0063	1.0039	1.0023	1.0013
	Q3	1.2000	1.0531	1.0315	1.0253	1.0225	1.0199	1.0153	1.0115	1.0076	1.0046
	max	1.4450	1.1087	1.0495	1.0442	1.0452	1.0390	1.0298	1.0211	1.0143	1.0091
$\tilde{\mu}^{(2)}$	min	1.0000	1.0000	1.0011	1.0028	1.0014	1.0008	1.0007	1.0008	1.0010	1.0014
	Q1	1.0000	1.0009	1.0030	1.0045	1.0028	1.0024	1.0023	1.0023	1.0026	1.0030
	median	1.0000	1.0021	1.0050	1.0050	1.0060	1.0068	1.0079	1.0089	1.0096	1.0101
	Q3	1.0440	1.0208	1.0095	1.0069	1.0083	1.0095	1.0103	1.0106	1.0110	1.0112
	max	1.0833	1.0293	1.0136	1.0088	1.0107	1.0137	1.0160	1.0176	1.0184	1.0187
$\tilde{\mu}$	min	1.0123	1.0114	1.0063	1.0026	0.9998	0.9975	0.9965	0.9966	0.9971	0.9978
	Q1	1.0176	1.0130	1.0082	1.0040	1.0008	0.9991	0.9984	0.9982	0.9984	0.9987
	median	1.0708	1.0189	1.0094	1.0044	1.0015	0.9999	0.9993	0.9988	0.9986	0.9989
	Q3	1.0904	1.0216	1.0108	1.0056	1.0033	1.0007	0.9999	0.9992	0.9994	0.9997
	max	1.1509	1.0351	1.0118	1.0135	1.0101	1.0064	1.0043	1.0020	1.0022	1.0026
$\tilde{\mu}^{Vm}$	min	0.9896	0.9947	0.9952	0.9977	0.9995	0.9997	1.0037	1.0019	1.0020	0.9962
	Q1	1.0031	0.9999	1.0010	1.0007	1.0018	1.0024	1.0043	1.0042	1.0040	1.0030
	median	1.0116	1.0040	1.0031	1.0033	1.0039	1.0038	1.0057	1.0057	1.0054	1.0045
	Q3	1.0179	1.0060	1.0049	1.0070	1.0073	1.0083	1.0081	1.0071	1.0069	1.0059
	max	1.0277	1.0128	1.0083	1.0100	1.0132	1.0165	1.0129	1.0143	1.0134	1.0308

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(S)}$	min	1.7337	1.1395	1.0352	1.0133	1.0071	1.0035	0.9988	0.9969	0.9977	0.9975
	Q1	1.7871	1.1542	1.0519	1.0235	1.0095	1.0050	1.0023	1.0007	1.0008	1.0003
	median	1.8767	1.1819	1.0715	1.0327	1.0181	1.0068	1.0031	1.0022	1.0019	1.0018
	Q3	1.9772	1.2522	1.1075	1.0481	1.0217	1.0106	1.0055	1.0033	1.0039	1.0067
	max	2.1695	1.2910	1.1227	1.0609	1.0302	1.0133	1.0108	1.0132	1.0172	1.0233
$\tilde{\mu}^{(T)}$	min	1.0563	1.0089	0.9927	0.9864	0.9863	0.9886	0.9908	0.9926	0.9960	0.9968
	Q1	1.0677	1.0152	0.9971	0.9912	0.9892	0.9907	0.9936	0.9953	0.9966	0.9983
	median	1.0734	1.0170	0.9997	0.9936	0.9905	0.9928	0.9950	0.9961	0.9978	0.9987
	Q3	1.0956	1.0208	1.0008	0.9949	0.9936	0.9943	0.9958	0.9975	0.9986	0.9994
	max	1.1255	1.0308	1.0048	1.0022	0.9952	0.9976	1.0013	1.0012	1.0006	1.0002
$\tilde{\mu}^{ps}$	min	1.0778	1.0477	1.0272	1.0087	0.9998	0.9956	0.9941	0.9942	0.9949	0.9962
	Q1	1.1320	1.0591	1.0347	1.0227	1.0158	1.0119	1.0088	1.0062	1.0054	1.0050
	median	1.2107	1.0639	1.0387	1.0276	1.0196	1.0137	1.0107	1.0090	1.0080	1.0070
	Q3	1.2446	1.0808	1.0491	1.0325	1.0246	1.0193	1.0160	1.0129	1.0116	1.0112
	max	1.4083	1.1015	1.0544	1.0466	1.0382	1.0301	1.0289	1.0291	1.0298	1.0308

Table D.2: Relative efficiency of $\tilde{\mu}^c$ with respect to other estimators with known population covariates under normal random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	1.0002	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q1	1.0004	1.0002	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	median	1.0007	1.0004	1.0002	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q3	1.0031	1.0021	1.0014	1.0008	1.0004	1.0002	1.0001	1.0000	1.0000	1.0000
	max	1.0057	1.0036	1.0023	1.0015	1.0010	1.0007	1.0005	1.0004	1.0004	1.0003
$\tilde{\mu}^{(2)}$	min	1.0017	1.0017	1.0015	1.0014	1.0012	1.0010	1.0008	1.0005	1.0003	1.0000
	Q1	1.0031	1.0030	1.0027	1.0025	1.0024	1.0022	1.0019	1.0015	1.0011	1.0007
	median	1.0101	1.0100	1.0094	1.0087	1.0077	1.0066	1.0054	1.0040	1.0026	1.0012
	Q3	1.0110	1.0110	1.0106	1.0099	1.0089	1.0078	1.0066	1.0051	1.0035	1.0019
	max	1.0187	1.0183	1.0176	1.0166	1.0154	1.0139	1.0118	1.0094	1.0065	1.0032
$\tilde{\mu}$	min	0.9981	0.9985	0.9989	0.9993	0.9995	0.9996	0.9997	0.9998	0.9998	0.9999
	Q1	0.9988	0.9993	0.9994	0.9996	0.9998	1.0000	1.0001	1.0001	1.0001	1.0001
	median	0.9995	0.9998	1.0002	1.0005	1.0007	1.0007	1.0007	1.0005	1.0004	1.0002
	Q3	1.0000	1.0005	1.0008	1.0010	1.0011	1.0011	1.0010	1.0008	1.0006	1.0003
	max	1.0029	1.0032	1.0034	1.0035	1.0035	1.0033	1.0030	1.0025	1.0018	1.0009
$\tilde{\mu}^{Vm}$	min	1.0019	1.0011	1.0005	1.0004	1.0003	1.0003	1.0002	1.0001	1.0000	1.0000
	Q1	1.0025	1.0021	1.0017	1.0015	1.0012	1.0010	1.0008	1.0006	1.0004	1.0002
	median	1.0037	1.0036	1.0029	1.0024	1.0020	1.0015	1.0011	1.0007	1.0005	1.0003
	Q3	1.0050	1.0043	1.0035	1.0029	1.0023	1.0017	1.0013	1.0009	1.0006	1.0003
	max	1.0099	1.0087	1.0074	1.0061	1.0049	1.0037	1.0027	1.0018	1.0011	1.0005

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(S)}$	min	0.9975	0.9983	0.9995	0.9998	0.9999	1.0001	1.0001	1.0002	1.0002	1.0002
	Q1	1.0005	1.0009	1.0015	1.0021	1.0039	1.0055	1.0076	1.0113	1.0189	1.0417
	median	1.0022	1.0024	1.0031	1.0041	1.0058	1.0087	1.0144	1.0243	1.0450	1.1094
	Q3	1.0097	1.0138	1.0191	1.0261	1.0354	1.0486	1.0686	1.1025	1.1709	1.3777
	max	1.0412	1.0547	1.0732	1.0988	1.1358	1.1927	1.2890	1.4837	2.0718	2.0836
$\tilde{\mu}^{(T)}$	min	0.9986	0.9991	0.9994	0.9997	0.9998	0.9999	1.0000	1.0000	1.0000	1.0000
	Q1	0.9991	0.9995	0.9997	0.9998	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000
	median	0.9994	0.9997	0.9998	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q3	0.9996	0.9998	0.9999	1.0000	1.0001	1.0001	1.0000	1.0000	1.0000	1.0000
	max	1.0007	1.0005	1.0003	1.0003	1.0004	1.0004	1.0004	1.0004	1.0003	1.0003
$\tilde{\mu}^{ps}$	min	0.9972	0.9981	0.9989	0.9995	1.0000	0.9998	0.9996	0.9995	0.9995	0.9996
	Q1	1.0048	1.0047	1.0045	1.0041	1.0036	1.0030	1.0024	1.0019	1.0013	1.0008
	median	1.0062	1.0058	1.0057	1.0055	1.0053	1.0052	1.0051	1.0049	1.0048	1.0047
	Q3	1.0112	1.0114	1.0116	1.0118	1.0120	1.0122	1.0122	1.0123	1.0123	1.0122
	max	1.0318	1.0328	1.0337	1.0345	1.0352	1.0358	1.0361	1.0363	1.0364	1.0364

Table D.3: Relative efficiency of $\tilde{\mu}^c$ with respect to other estimators with known population covariates under double exponential random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	1.0064	1.0171	1.0228	1.0131	1.0071	1.0034	1.0016	1.0005	1.0000	1.0000
	Q1	1.0300	1.0241	1.0306	1.0202	1.0117	1.0064	1.0031	1.0012	1.0003	1.0000
	median	1.2528	1.0676	1.0344	1.0229	1.0173	1.0120	1.0058	1.0025	1.0010	1.0003
	Q3	1.2904	1.0752	1.0394	1.0369	1.0382	1.0355	1.0316	1.0254	1.0200	1.0159
	max	1.8123	1.2312	1.1132	1.0594	1.0577	1.0538	1.0484	1.0439	1.0402	1.0351
$\tilde{\mu}^{(2)}$	min	1.0000	1.0000	1.0000	1.0018	1.0008	1.0002	1.0000	1.0000	1.0000	1.0000
	Q1	1.0000	1.0001	1.0025	1.0043	1.0030	1.0013	1.0005	1.0005	1.0006	1.0005
	median	1.0000	1.0005	1.0044	1.0054	1.0079	1.0124	1.0152	1.0176	1.0189	1.0202
	Q3	1.0595	1.0244	1.0102	1.0067	1.0101	1.0145	1.0186	1.0222	1.0256	1.0273
	max	1.1088	1.0377	1.0154	1.0141	1.0177	1.0213	1.0233	1.0319	1.0391	1.0452
$\tilde{\mu}$	min	1.0121	1.0114	1.0049	1.0018	0.9985	0.9978	0.9974	0.9976	0.9975	0.9973
	Q1	1.0231	1.0141	1.0101	1.0044	1.0003	0.9992	0.9989	0.9991	1.0000	1.0010
	median	1.1163	1.0278	1.0111	1.0057	1.0024	1.0010	1.0002	1.0009	1.0021	1.0032
	Q3	1.1397	1.0325	1.0134	1.0106	1.0074	1.0071	1.0052	1.0052	1.0049	1.0044
	max	1.2856	1.0824	1.0376	1.0169	1.0142	1.0132	1.0126	1.0126	1.0128	1.0114
$\tilde{\mu}^{ps}$	min	1.0854	1.0549	1.0299	1.0128	1.0024	0.9984	0.9965	0.9959	0.9961	0.9972
	Q1	1.1548	1.0623	1.0404	1.0276	1.0201	1.0152	1.0124	1.0105	1.0096	1.0096
	median	1.2646	1.0809	1.0450	1.0326	1.0241	1.0195	1.0167	1.0146	1.0133	1.0127
	Q3	1.3063	1.0961	1.0513	1.0363	1.0322	1.0275	1.0228	1.0191	1.0173	1.0174
	max	1.5046	1.1534	1.0822	1.0535	1.0413	1.0382	1.0340	1.0350	1.0380	1.0413

Table D.4: Relative efficiency of $\tilde{\mu}^c$ with respect to other estimators with known population covariates under double exponential random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	median	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q3	1.0124	1.0097	1.0078	1.0065	1.0054	1.0043	1.0031	1.0021	1.0012	1.0005
	max	1.0284	1.0228	1.0180	1.0144	1.0113	1.0086	1.0064	1.0045	1.0029	1.0015
$\tilde{\mu}^{(2)}$	min	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q1	1.0005	1.0005	1.0004	1.0002	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001
	median	1.0210	1.0209	1.0203	1.0190	1.0172	1.0150	1.0126	1.0099	1.0068	1.0034
	Q3	1.0274	1.0268	1.0256	1.0239	1.0215	1.0189	1.0158	1.0124	1.0088	1.0048
	max	1.0501	1.0532	1.0544	1.0541	1.0521	1.0485	1.0431	1.0358	1.0264	1.0144
$\tilde{\mu}$	min	0.9975	0.9976	0.9980	0.9985	0.9989	0.9992	0.9995	0.9998	0.9999	1.0000
	Q1	1.0010	1.0016	1.0019	1.0018	1.0016	1.0014	1.0012	1.0010	1.0007	1.0003
	median	1.0032	1.0034	1.0039	1.0041	1.0036	1.0029	1.0024	1.0019	1.0013	1.0007
	Q3	1.0050	1.0055	1.0052	1.0048	1.0045	1.0040	1.0034	1.0027	1.0020	1.0011
	max	1.0137	1.0150	1.0155	1.0157	1.0153	1.0142	1.0126	1.0104	1.0075	1.0040
$\tilde{\mu}^{ps}$	min	0.9985	0.9997	1.0007	1.0013	1.0010	1.0008	1.0006	1.0004	1.0002	1.0000
	Q1	1.0098	1.0090	1.0084	1.0077	1.0071	1.0063	1.0054	1.0044	1.0034	1.0024
	median	1.0120	1.0105	1.0101	1.0097	1.0092	1.0085	1.0074	1.0070	1.0066	1.0062
	Q3	1.0177	1.0177	1.0175	1.0172	1.0166	1.0159	1.0153	1.0149	1.0143	1.0135
	max	1.0440	1.0460	1.0473	1.0480	1.0483	1.0481	1.0474	1.0464	1.0450	1.0432

Table D.5: Relative efficiency of $\tilde{\mu}^c$ with respect to other estimators with known population covariates under exponential random effects for ρ from 0.00 to 0.45

ρ		0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	1.0076	1.0203	1.0239	1.0168	1.0081	1.0038	1.0015	1.0003	1.0000	1.0000
	Q1	1.0312	1.0291	1.0349	1.0219	1.0133	1.0077	1.0039	1.0018	1.0005	1.0000
	median	1.2871	1.0822	1.0398	1.0274	1.0204	1.0153	1.0096	1.0044	1.0017	1.0007
	Q3	1.3202	1.0907	1.0440	1.0396	1.0391	1.0343	1.0297	1.0252	1.0209	1.0169
	max	2.0747	1.2823	1.1350	1.0741	1.0490	1.0441	1.0387	1.0333	1.0288	1.0244
$\tilde{\mu}^{(2)}$	min	1.0000	1.0000	1.0000	1.0027	1.0013	1.0005	1.0002	1.0001	1.0000	1.0000
	Q1	1.0000	1.0000	1.0019	1.0040	1.0038	1.0029	1.0020	1.0014	1.0012	1.0010
	median	1.0000	1.0000	1.0036	1.0057	1.0083	1.0113	1.0140	1.0162	1.0181	1.0191
	Q3	1.0620	1.0250	1.0095	1.0077	1.0104	1.0170	1.0211	1.0241	1.0263	1.0282
	max	1.1482	1.0453	1.0185	1.0103	1.0163	1.0209	1.0297	1.0404	1.0506	1.0595
$\tilde{\mu}$	min	1.0193	1.0127	1.0095	1.0027	0.9995	0.9983	0.9980	0.9982	0.9988	0.9986
	Q1	1.0269	1.0144	1.0111	1.0050	1.0016	1.0000	0.9993	0.9995	0.9997	1.0010
	median	1.1372	1.0344	1.0137	1.0076	1.0032	1.0014	1.0002	1.0003	1.0017	1.0026
	Q3	1.1584	1.0420	1.0172	1.0106	1.0091	1.0071	1.0053	1.0056	1.0050	1.0048
	max	1.4223	1.1087	1.0463	1.0218	1.0103	1.0100	1.0083	1.0067	1.0100	1.0140
$\tilde{\mu}^{ps}$	min	1.1038	1.0594	1.0382	1.0127	1.0019	0.9972	0.9953	0.9964	0.9976	0.9991
	Q1	1.1480	1.0663	1.0467	1.0310	1.0226	1.0173	1.0145	1.0124	1.0110	1.0102
	median	1.2824	1.0882	1.0502	1.0341	1.0258	1.0204	1.0171	1.0154	1.0145	1.0141
	Q3	1.3079	1.0991	1.0532	1.0392	1.0332	1.0284	1.0250	1.0215	1.0188	1.0178
	max	1.6248	1.1790	1.0938	1.0591	1.0436	1.0375	1.0345	1.0360	1.0392	1.0431

Table D.6: Relative efficiency of $\tilde{\mu}^c$ with respect to other estimators with known population covariates under exponential random effects for ρ from 0.50 to 0.95

ρ		0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	median	1.0002	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q3	1.0138	1.0113	1.0091	1.0068	1.0051	1.0034	1.0020	1.0011	1.0004	1.0001
	max	1.0212	1.0185	1.0153	1.0125	1.0101	1.0081	1.0062	1.0045	1.0028	1.0012
$\tilde{\mu}^{(2)}$	min	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q1	1.0009	1.0009	1.0009	1.0010	1.0011	1.0011	1.0012	1.0011	1.0010	1.0008
	median	1.0197	1.0203	1.0204	1.0198	1.0185	1.0167	1.0145	1.0119	1.0086	1.0045
	Q3	1.0291	1.0294	1.0286	1.0277	1.0257	1.0227	1.0193	1.0154	1.0109	1.0061
	max	1.0669	1.0721	1.0751	1.0759	1.0743	1.0704	1.0647	1.0561	1.0435	1.0256
$\tilde{\mu}$	min	0.9984	0.9985	0.9989	0.9990	0.9993	0.9995	0.9996	0.9997	0.9999	0.9999
	Q1	1.0018	1.0025	1.0018	1.0013	1.0012	1.0009	1.0006	1.0003	1.0001	1.0001
	median	1.0034	1.0031	1.0036	1.0035	1.0035	1.0033	1.0029	1.0023	1.0017	1.0009
	Q3	1.0048	1.0047	1.0051	1.0052	1.0049	1.0046	1.0041	1.0034	1.0025	1.0015
	max	1.0175	1.0201	1.0215	1.0219	1.0219	1.0211	1.0194	1.0167	1.0127	1.0072
$\tilde{\mu}^{ps}$	min	1.0006	1.0017	1.0024	1.0027	1.0029	1.0025	1.0020	1.0016	1.0013	1.0009
	Q1	1.0086	1.0081	1.0080	1.0076	1.0069	1.0062	1.0055	1.0047	1.0036	1.0028
	median	1.0136	1.0131	1.0123	1.0115	1.0107	1.0098	1.0088	1.0078	1.0068	1.0057
	Q3	1.0178	1.0172	1.0162	1.0154	1.0149	1.0144	1.0138	1.0133	1.0125	1.0114
	max	1.0468	1.0498	1.0518	1.0533	1.0541	1.0541	1.0532	1.0513	1.0482	1.0438

Table D.7: Relative efficiency of $\tilde{\mu}^c$ with respect to other estimators with known population covariates under uniform random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	1.0047	1.0137	1.0179	1.0154	1.0108	1.0071	1.0047	1.0029	1.0012	1.0001
	Q1	1.0280	1.0181	1.0213	1.0181	1.0131	1.0099	1.0070	1.0045	1.0018	1.0007
	median	1.1634	1.0427	1.0241	1.0193	1.0145	1.0111	1.0087	1.0054	1.0033	1.0020
	Q3	1.1701	1.0457	1.0256	1.0223	1.0205	1.0159	1.0104	1.0067	1.0040	1.0028
	max	1.3188	1.0664	1.0297	1.0283	1.0254	1.0202	1.0129	1.0087	1.0067	1.0060
$\tilde{\mu}^{(2)}$	min	1.0000	1.0005	1.0027	1.0021	1.0014	1.0012	1.0014	1.0016	1.0020	1.0022
	Q1	1.0000	1.0011	1.0035	1.0037	1.0033	1.0030	1.0034	1.0042	1.0044	1.0041
	median	1.0000	1.0020	1.0048	1.0044	1.0040	1.0037	1.0041	1.0050	1.0052	1.0050
	Q3	1.0269	1.0162	1.0088	1.0058	1.0047	1.0048	1.0053	1.0056	1.0062	1.0070
	max	1.0691	1.0262	1.0132	1.0073	1.0077	1.0072	1.0074	1.0078	1.0076	1.0087
$\tilde{\mu}$	min	1.0080	1.0081	1.0046	1.0020	1.0000	0.9989	0.9964	0.9963	0.9972	0.9980
	Q1	1.0122	1.0120	1.0073	1.0034	1.0007	0.9991	0.9977	0.9974	0.9979	0.9984
	median	1.0702	1.0158	1.0080	1.0040	1.0010	0.9994	0.9980	0.9977	0.9982	0.9987
	Q3	1.0735	1.0172	1.0086	1.0050	1.0017	0.9998	0.9986	0.9981	0.9984	0.9990
	max	1.1036	1.0197	1.0110	1.0068	1.0035	1.0004	0.9995	0.9991	0.9991	0.9998
$\tilde{\mu}^{ps}$	min	1.0733	1.0436	1.0231	1.0043	0.9961	0.9933	0.9921	0.9928	0.9937	0.9947
	Q1	1.1170	1.0554	1.0304	1.0198	1.0135	1.0095	1.0067	1.0054	1.0042	1.0040
	median	1.1844	1.0604	1.0387	1.0256	1.0178	1.0136	1.0106	1.0079	1.0062	1.0053
	Q3	1.2446	1.0748	1.0454	1.0326	1.0251	1.0187	1.0134	1.0113	1.0107	1.0107
	max	1.3959	1.0845	1.0523	1.0428	1.0329	1.0283	1.0267	1.0268	1.0273	1.0281

Table D.8: Relative efficiency of $\tilde{\mu}^c$ with respect to other estimators with known population covariates under uniform random effects for ρ from 0.50 to 0.95

ρ		0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q1	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	median	1.0013	1.0008	1.0005	1.0003	1.0002	1.0001	1.0000	1.0000	1.0000	1.0000
	Q3	1.0018	1.0012	1.0008	1.0006	1.0004	1.0003	1.0003	1.0002	1.0002	1.0001
	max	1.0053	1.0048	1.0043	1.0038	1.0034	1.0029	1.0024	1.0020	1.0015	1.0010
$\tilde{\mu}^{(2)}$	min	1.0025	1.0025	1.0021	1.0017	1.0014	1.0011	1.0008	1.0005	1.0002	1.0000
	Q1	1.0039	1.0037	1.0033	1.0028	1.0023	1.0019	1.0014	1.0010	1.0006	1.0002
	median	1.0047	1.0048	1.0046	1.0040	1.0034	1.0029	1.0024	1.0019	1.0012	1.0006
	Q3	1.0071	1.0075	1.0073	1.0067	1.0060	1.0050	1.0040	1.0030	1.0020	1.0011
	max	1.0101	1.0104	1.0099	1.0090	1.0079	1.0068	1.0056	1.0043	1.0031	1.0017
$\tilde{\mu}$	min	0.9983	0.9984	0.9984	0.9985	0.9986	0.9987	0.9989	0.9992	0.9994	0.9998
	Q1	0.9987	0.9990	0.9992	0.9995	0.9997	0.9998	0.9999	0.9999	1.0000	1.0000
	median	0.9992	0.9995	0.9997	0.9999	1.0000	1.0000	1.0001	1.0001	1.0001	1.0001
	Q3	0.9999	1.0007	1.0011	1.0013	1.0013	1.0011	1.0009	1.0007	1.0005	1.0002
	max	1.0014	1.0023	1.0027	1.0028	1.0026	1.0024	1.0021	1.0017	1.0012	1.0007
$\tilde{\mu}^{ps}$	min	0.9956	0.9965	0.9972	0.9978	0.9984	0.9989	0.9993	0.9997	0.9999	0.9998
	Q1	1.0034	1.0031	1.0030	1.0028	1.0026	1.0023	1.0021	1.0017	1.0012	1.0007
	median	1.0050	1.0050	1.0048	1.0046	1.0045	1.0043	1.0041	1.0039	1.0037	1.0035
	Q3	1.0109	1.0110	1.0110	1.0108	1.0110	1.0113	1.0115	1.0116	1.0118	1.0119
	max	1.0291	1.0302	1.0313	1.0325	1.0338	1.0350	1.0363	1.0376	1.0389	1.0403

Table D.9: Relative efficiency of $\tilde{\mu}^c$ with respect to other estimators with sample covariates under normal random effects for ρ from 0.00 to 0.45

ρ		0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	1.0065	1.0104	1.0180	1.0140	1.0085	1.0048	1.0025	1.0013	1.0007	1.0003
	Q1	1.0225	1.0201	1.0237	1.0161	1.0110	1.0073	1.0047	1.0029	1.0017	1.0009
	median	1.1244	1.0412	1.0254	1.0201	1.0130	1.0092	1.0060	1.0037	1.0022	1.0012
	Q3	1.1678	1.0503	1.0309	1.0235	1.0221	1.0194	1.0148	1.0111	1.0073	1.0044
	max	1.2127	1.0734	1.0376	1.0449	1.0455	1.0393	1.0302	1.0217	1.0148	1.0096
$\tilde{\mu}^{(2)}$	min	1.0000	1.0000	1.0010	1.0028	1.0015	1.0009	1.0007	1.0008	1.0009	1.0012
	Q1	1.0000	1.0008	1.0030	1.0043	1.0028	1.0024	1.0023	1.0025	1.0028	1.0032
	median	1.0000	1.0021	1.0048	1.0051	1.0059	1.0066	1.0078	1.0088	1.0096	1.0098
	Q3	1.0425	1.0206	1.0097	1.0067	1.0076	1.0088	1.0097	1.0105	1.0110	1.0111
	max	1.0883	1.0304	1.0135	1.0088	1.0107	1.0138	1.0163	1.0180	1.0188	1.0192
$\tilde{\mu}$	min	1.0133	1.0119	1.0067	1.0025	0.9996	0.9975	0.9965	0.9966	0.9971	0.9978
	Q1	1.0185	1.0132	1.0078	1.0037	1.0008	0.9991	0.9984	0.9982	0.9983	0.9987
	median	1.0513	1.0163	1.0090	1.0042	1.0013	1.0000	0.9991	0.9987	0.9986	0.9989
	Q3	1.0744	1.0202	1.0100	1.0056	1.0032	1.0006	0.9999	0.9992	0.9994	0.9997
	max	1.0982	1.0262	1.0124	1.0133	1.0098	1.0061	1.0041	1.0019	1.0022	1.0027
$\tilde{\mu}^{Vm}$	min	6.2378	2.7227	1.9266	1.6026	1.4350	1.3352	1.2697	1.2230	1.1870	0.9964
	Q1	7.3088	3.1185	2.1522	1.7495	1.5366	1.4078	1.3227	1.2622	1.2164	1.1798
	median	9.1529	4.1587	2.8274	2.2297	1.9013	1.6905	1.5422	1.4339	1.3527	1.2892
	Q3	11.3629	4.6721	3.0204	2.3199	1.9507	1.7225	1.5700	1.4581	1.3742	1.3080
	max	13.6021	5.1301	3.5702	2.7957	2.3392	2.0423	1.8335	1.6778	1.5567	1.0152

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(S)}$	min	1.7358	1.1556	1.0545	1.0218	1.0089	1.0038	0.9991	0.9956	0.9957	0.9963
	Q1	1.8265	1.1929	1.0742	1.0375	1.0205	1.0068	1.0031	1.0006	0.9995	0.9988
	median	1.9214	1.2316	1.1033	1.0484	1.0247	1.0113	1.0052	1.0022	1.0017	1.0015
	Q3	2.0096	1.2982	1.1295	1.0643	1.0337	1.0148	1.0080	1.0047	1.0034	1.0028
	max	2.3849	1.4336	1.1929	1.0973	1.0499	1.0249	1.0204	1.0167	1.0137	1.0118
$\tilde{\mu}^{(T)}$	min	1.0299	1.0081	0.9906	0.9861	0.9851	0.9885	0.9907	0.9928	0.9959	0.9970
	Q1	1.0502	1.0128	0.9968	0.9915	0.9888	0.9910	0.9935	0.9953	0.9967	0.9983
	median	1.0664	1.0158	0.9993	0.9932	0.9908	0.9929	0.9950	0.9960	0.9979	0.9988
	Q3	1.0945	1.0183	1.0015	0.9956	0.9928	0.9942	0.9959	0.9980	0.9988	0.9994
	max	1.1339	1.0304	1.0059	0.9996	0.9948	0.9973	1.0011	1.0013	1.0009	1.0006
$\tilde{\mu}^{ps}$	min	1.0816	1.0501	1.0294	1.0122	1.0023	0.9975	0.9956	0.9954	0.9957	0.9964
	Q1	1.1200	1.0620	1.0358	1.0211	1.0153	1.0103	1.0075	1.0057	1.0044	1.0038
	median	1.1956	1.0654	1.0399	1.0285	1.0202	1.0142	1.0113	1.0087	1.0070	1.0053
	Q3	1.2401	1.0939	1.0512	1.0391	1.0279	1.0209	1.0161	1.0117	1.0090	1.0075
	max	1.4276	1.1197	1.0740	1.0497	1.0371	1.0287	1.0236	1.0201	1.0174	1.0152

Table D.10: Relative efficiency of $\tilde{\mu}^c$ with respect to other estimators with sample covariates under normal random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q1	1.0004	1.0002	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	median	1.0006	1.0003	1.0002	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q3	1.0030	1.0020	1.0014	1.0010	1.0005	1.0002	1.0001	1.0001	1.0000	1.0000
	max	1.0061	1.0040	1.0026	1.0016	1.0010	1.0006	1.0005	1.0005	1.0004	1.0004
$\tilde{\mu}^{(2)}$	min	1.0014	1.0015	1.0015	1.0014	1.0013	1.0011	1.0008	1.0004	1.0002	1.0000
	Q1	1.0031	1.0030	1.0027	1.0024	1.0022	1.0021	1.0017	1.0012	1.0008	1.0005
	median	1.0098	1.0095	1.0089	1.0084	1.0075	1.0064	1.0052	1.0039	1.0025	1.0012
	Q3	1.0111	1.0110	1.0105	1.0098	1.0088	1.0076	1.0063	1.0049	1.0034	1.0018
	max	1.0192	1.0188	1.0180	1.0168	1.0153	1.0135	1.0113	1.0089	1.0061	1.0031
$\tilde{\mu}$	min	0.9981	0.9985	0.9989	0.9992	0.9995	0.9996	0.9997	0.9998	0.9999	0.9999
	Q1	0.9988	0.9992	0.9994	0.9996	0.9998	0.9999	1.0000	1.0001	1.0000	1.0000
	median	0.9996	0.9998	1.0001	1.0005	1.0006	1.0006	1.0006	1.0005	1.0004	1.0002
	Q3	1.0000	1.0004	1.0008	1.0009	1.0010	1.0011	1.0010	1.0008	1.0006	1.0003
	max	1.0031	1.0034	1.0035	1.0035	1.0033	1.0030	1.0026	1.0020	1.0014	1.0007
$\tilde{\mu}^{Vm}$	min	1.1323	1.1094	1.0894	1.0721	1.0571	1.0439	1.0322	1.0219	1.0125	1.0050
	Q1	1.1498	1.1252	1.1041	1.0857	1.0694	1.0553	1.0426	1.0312	1.0202	1.0100
	median	1.2380	1.1958	1.1602	1.1298	1.1034	1.0804	1.0602	1.0419	1.0258	1.0118
	Q3	1.2544	1.2096	1.1720	1.1405	1.1130	1.0887	1.0672	1.0479	1.0305	1.0147
	max	1.3797	1.3134	1.2570	1.2087	1.1667	1.1301	1.0978	1.0692	1.0437	1.0209

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(S)}$	min	0.9969	0.9977	0.9983	0.9987	0.9991	0.9993	0.9995	0.9997	0.9998	0.9999
	Q1	0.9983	0.9987	0.9990	0.9993	0.9996	0.9998	0.9999	1.0000	1.0000	1.0000
	median	1.0012	1.0010	1.0007	1.0005	1.0004	1.0003	1.0003	1.0002	1.0002	1.0002
	Q3	1.0023	1.0022	1.0020	1.0018	1.0017	1.0015	1.0013	1.0010	1.0007	1.0004
	max	1.0110	1.0096	1.0083	1.0070	1.0058	1.0046	1.0036	1.0026	1.0020	1.0013
$\tilde{\mu}^{(T)}$	min	0.9986	0.9992	0.9994	0.9997	0.9998	0.9999	1.0000	1.0000	1.0000	1.0000
	Q1	0.9992	0.9995	0.9997	0.9998	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000
	median	0.9994	0.9997	0.9998	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q3	0.9997	0.9998	1.0000	1.0001	1.0001	1.0001	1.0000	1.0000	1.0000	1.0000
	max	1.0009	1.0007	1.0005	1.0003	1.0003	1.0004	1.0004	1.0004	1.0004	1.0004
$\tilde{\mu}^{ps}$	min	0.9970	0.9978	0.9983	0.9988	0.9991	0.9993	0.9995	0.9995	0.9995	0.9996
	Q1	1.0032	1.0026	1.0021	1.0016	1.0012	1.0009	1.0007	1.0004	1.0002	1.0001
	median	1.0044	1.0036	1.0032	1.0029	1.0026	1.0023	1.0019	1.0014	1.0009	1.0004
	Q3	1.0063	1.0056	1.0050	1.0044	1.0039	1.0032	1.0026	1.0020	1.0013	1.0007
	max	1.0133	1.0117	1.0102	1.0088	1.0075	1.0061	1.0049	1.0036	1.0024	1.0012

Table D.11: Relative efficiency of $\tilde{\mu}^c$ with respect to other estimators with sample covariates under double exponential random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	1.0068	1.0168	1.0228	1.0126	1.0068	1.0032	1.0015	1.0005	1.0000	1.0000
	Q1	1.0294	1.0247	1.0296	1.0188	1.0107	1.0055	1.0027	1.0012	1.0003	1.0000
	median	1.1958	1.0634	1.0319	1.0229	1.0173	1.0117	1.0056	1.0025	1.0009	1.0003
	Q3	1.2493	1.0677	1.0391	1.0366	1.0380	1.0351	1.0316	1.0252	1.0198	1.0158
	max	1.3477	1.1704	1.0962	1.0560	1.0598	1.0555	1.0499	1.0454	1.0416	1.0365
$\tilde{\mu}^{(2)}$	min	1.0000	1.0000	1.0000	1.0007	1.0007	1.0001	1.0000	1.0000	1.0000	1.0000
	Q1	1.0000	1.0001	1.0025	1.0040	1.0030	1.0013	1.0005	1.0005	1.0006	1.0006
	median	1.0000	1.0005	1.0043	1.0052	1.0061	1.0112	1.0148	1.0172	1.0193	1.0206
	Q3	1.0572	1.0238	1.0100	1.0070	1.0103	1.0134	1.0177	1.0210	1.0243	1.0265
	max	1.1143	1.0378	1.0157	1.0138	1.0175	1.0210	1.0230	1.0282	1.0356	1.0419
$\tilde{\mu}$	min	1.0137	1.0114	1.0049	1.0018	0.9985	0.9979	0.9974	0.9976	0.9975	0.9973
	Q1	1.0236	1.0142	1.0097	1.0042	1.0003	0.9994	0.9990	0.9993	1.0004	1.0012
	median	1.0926	1.0263	1.0107	1.0056	1.0023	1.0010	1.0001	1.0006	1.0019	1.0030
	Q3	1.1145	1.0300	1.0135	1.0104	1.0073	1.0072	1.0051	1.0044	1.0050	1.0044
	max	1.1538	1.0645	1.0343	1.0168	1.0151	1.0138	1.0131	1.0133	1.0136	1.0113
$\tilde{\mu}^{ps}$	min	1.0956	1.0515	1.0320	1.0145	1.0031	0.9984	0.9966	0.9960	0.9964	0.9976
	Q1	1.1400	1.0652	1.0401	1.0266	1.0192	1.0154	1.0120	1.0099	1.0081	1.0076
	median	1.2415	1.0794	1.0482	1.0314	1.0240	1.0189	1.0145	1.0121	1.0117	1.0101
	Q3	1.2922	1.1078	1.0551	1.0477	1.0343	1.0273	1.0219	1.0182	1.0169	1.0160
	max	1.4911	1.1544	1.0914	1.0574	1.0415	1.0379	1.0317	1.0268	1.0245	1.0213

Table D.12: Relative efficiency of $\tilde{\mu}^c$ with respect to other estimators with sample covariates under double exponential random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	median	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q3	1.0122	1.0096	1.0078	1.0065	1.0054	1.0043	1.0031	1.0021	1.0012	1.0005
	max	1.0297	1.0240	1.0191	1.0153	1.0121	1.0093	1.0069	1.0049	1.0032	1.0016
$\tilde{\mu}^{(2)}$	min	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q1	1.0005	1.0005	1.0004	1.0002	1.0001	1.0001	1.0001	1.0001	1.0001	1.0000
	median	1.0214	1.0212	1.0205	1.0192	1.0173	1.0151	1.0126	1.0098	1.0066	1.0034
	Q3	1.0272	1.0267	1.0255	1.0237	1.0215	1.0189	1.0158	1.0124	1.0087	1.0045
	max	1.0471	1.0504	1.0519	1.0518	1.0502	1.0469	1.0419	1.0349	1.0258	1.0143
$\tilde{\mu}$	min	0.9975	0.9976	0.9980	0.9985	0.9989	0.9992	0.9995	0.9998	0.9999	1.0000
	Q1	1.0015	1.0016	1.0019	1.0020	1.0017	1.0015	1.0013	1.0010	1.0007	1.0004
	median	1.0033	1.0034	1.0039	1.0040	1.0038	1.0032	1.0025	1.0020	1.0013	1.0007
	Q3	1.0049	1.0056	1.0054	1.0048	1.0043	1.0039	1.0034	1.0027	1.0019	1.0011
	max	1.0123	1.0139	1.0148	1.0150	1.0147	1.0138	1.0123	1.0102	1.0075	1.0041
$\tilde{\mu}^{ps}$	min	0.9990	1.0001	1.0009	1.0011	1.0009	1.0007	1.0005	1.0003	1.0000	0.9998
	Q1	1.0069	1.0055	1.0044	1.0033	1.0023	1.0020	1.0018	1.0015	1.0008	1.0003
	median	1.0091	1.0086	1.0078	1.0067	1.0061	1.0051	1.0041	1.0030	1.0019	1.0009
	Q3	1.0137	1.0128	1.0108	1.0092	1.0081	1.0070	1.0058	1.0045	1.0030	1.0015
	max	1.0219	1.0220	1.0215	1.0206	1.0192	1.0174	1.0150	1.0122	1.0088	1.0048

Table D.13: Relative efficiency of $\tilde{\mu}^c$ with respect to other estimators with sample covariates under exponential random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	1.0083	1.0196	1.0240	1.0155	1.0076	1.0037	1.0015	1.0003	1.0000	1.0000
	Q1	1.0296	1.0293	1.0331	1.0213	1.0127	1.0073	1.0038	1.0017	1.0005	1.0000
	median	1.2058	1.0701	1.0374	1.0273	1.0204	1.0153	1.0100	1.0048	1.0020	1.0007
	Q3	1.3014	1.0843	1.0433	1.0395	1.0385	1.0338	1.0293	1.0248	1.0206	1.0167
	max	1.4645	1.2212	1.1251	1.0739	1.0492	1.0440	1.0403	1.0348	1.0293	1.0248
$\tilde{\mu}^{(2)}$	min	1.0000	1.0000	1.0000	1.0008	1.0014	1.0005	1.0002	1.0000	1.0000	1.0000
	Q1	1.0000	1.0000	1.0019	1.0040	1.0039	1.0029	1.0019	1.0012	1.0011	1.0009
	median	1.0000	1.0001	1.0034	1.0057	1.0061	1.0110	1.0137	1.0162	1.0180	1.0190
	Q3	1.0625	1.0254	1.0095	1.0078	1.0101	1.0136	1.0191	1.0242	1.0264	1.0283
	max	1.1551	1.0470	1.0191	1.0096	1.0151	1.0195	1.0234	1.0333	1.0427	1.0511
$\tilde{\mu}$	min	1.0188	1.0127	1.0094	1.0023	0.9996	0.9985	0.9983	0.9986	0.9988	0.9986
	Q1	1.0289	1.0147	1.0107	1.0049	1.0014	1.0000	0.9992	0.9994	0.9998	1.0011
	median	1.0970	1.0296	1.0127	1.0075	1.0032	1.0014	1.0003	1.0005	1.0019	1.0027
	Q3	1.1537	1.0413	1.0167	1.0104	1.0090	1.0067	1.0051	1.0049	1.0046	1.0054
	max	1.1960	1.0922	1.0483	1.0246	1.0121	1.0105	1.0081	1.0072	1.0077	1.0112
$\tilde{\mu}^{ps}$	min	1.1081	1.0583	1.0375	1.0147	1.0028	0.9975	0.9951	0.9956	0.9959	0.9971
	Q1	1.1406	1.0684	1.0486	1.0307	1.0224	1.0173	1.0141	1.0120	1.0102	1.0093
	median	1.2541	1.0886	1.0507	1.0358	1.0256	1.0210	1.0171	1.0144	1.0129	1.0123
	Q3	1.2983	1.1187	1.0550	1.0446	1.0365	1.0307	1.0244	1.0204	1.0172	1.0157
	max	1.5249	1.1875	1.1104	1.0702	1.0476	1.0345	1.0288	1.0256	1.0240	1.0248

Table D.14: Relative efficiency of $\tilde{\mu}^c$ with respect to other estimators with sample covariates under exponential random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	median	1.0002	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q3	1.0135	1.0110	1.0089	1.0069	1.0051	1.0037	1.0025	1.0015	1.0007	1.0001
	max	1.0216	1.0189	1.0157	1.0129	1.0104	1.0081	1.0062	1.0044	1.0029	1.0015
$\tilde{\mu}^{(2)}$	min	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q1	1.0008	1.0007	1.0006	1.0006	1.0006	1.0006	1.0006	1.0006	1.0006	1.0006
	median	1.0196	1.0201	1.0203	1.0197	1.0185	1.0167	1.0145	1.0119	1.0085	1.0045
	Q3	1.0293	1.0293	1.0285	1.0267	1.0246	1.0217	1.0183	1.0144	1.0102	1.0057
	max	1.0582	1.0635	1.0666	1.0676	1.0664	1.0630	1.0570	1.0489	1.0375	1.0216
$\tilde{\mu}$	min	0.9984	0.9985	0.9989	0.9990	0.9993	0.9995	0.9996	0.9997	0.9999	0.9999
	Q1	1.0018	1.0025	1.0024	1.0018	1.0014	1.0013	1.0009	1.0006	1.0003	1.0001
	median	1.0036	1.0034	1.0036	1.0035	1.0035	1.0033	1.0029	1.0023	1.0017	1.0009
	Q3	1.0046	1.0048	1.0051	1.0051	1.0051	1.0048	1.0042	1.0034	1.0025	1.0014
	max	1.0145	1.0170	1.0186	1.0191	1.0189	1.0181	1.0166	1.0142	1.0107	1.0059
$\tilde{\mu}^{ps}$	min	0.9984	0.9994	1.0002	1.0007	1.0008	1.0010	1.0010	1.0010	1.0005	1.0002
	Q1	1.0084	1.0079	1.0065	1.0052	1.0042	1.0034	1.0028	1.0021	1.0013	1.0008
	median	1.0108	1.0097	1.0085	1.0077	1.0066	1.0054	1.0042	1.0032	1.0022	1.0011
	Q3	1.0133	1.0121	1.0111	1.0102	1.0091	1.0079	1.0066	1.0051	1.0037	1.0019
	max	1.0258	1.0265	1.0265	1.0261	1.0251	1.0233	1.0207	1.0172	1.0127	1.0071

Table D.15: Relative efficiency of $\tilde{\mu}^c$ with respect to other estimators with sample covariates under uniform random effects for ρ from 0.00 to 0.45

	ρ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\tilde{\mu}^{(1)}$	min	1.0051	1.0141	1.0179	1.0123	1.0098	1.0070	1.0046	1.0028	1.0012	1.0002
	Q1	1.0277	1.0177	1.0206	1.0164	1.0124	1.0092	1.0069	1.0044	1.0018	1.0007
	median	1.1064	1.0369	1.0221	1.0186	1.0141	1.0110	1.0080	1.0053	1.0032	1.0019
	Q3	1.1444	1.0410	1.0243	1.0228	1.0208	1.0163	1.0105	1.0064	1.0040	1.0028
	max	1.1737	1.0472	1.0262	1.0292	1.0259	1.0203	1.0132	1.0087	1.0068	1.0062
$\tilde{\mu}^{(2)}$	min	1.0000	1.0004	1.0024	1.0021	1.0014	1.0012	1.0014	1.0016	1.0020	1.0022
	Q1	1.0000	1.0013	1.0035	1.0033	1.0034	1.0030	1.0035	1.0041	1.0040	1.0040
	median	1.0000	1.0019	1.0049	1.0044	1.0040	1.0037	1.0041	1.0049	1.0050	1.0051
	Q3	1.0256	1.0164	1.0087	1.0058	1.0046	1.0043	1.0047	1.0051	1.0062	1.0070
	max	1.0744	1.0275	1.0143	1.0078	1.0077	1.0070	1.0073	1.0078	1.0076	1.0090
$\tilde{\mu}$	min	1.0083	1.0081	1.0050	1.0028	1.0001	0.9987	0.9964	0.9964	0.9973	0.9979
	Q1	1.0140	1.0106	1.0069	1.0032	1.0008	0.9991	0.9976	0.9974	0.9979	0.9983
	median	1.0422	1.0143	1.0080	1.0040	1.0011	0.9994	0.9980	0.9977	0.9981	0.9986
	Q3	1.0633	1.0166	1.0086	1.0048	1.0015	1.0001	0.9986	0.9981	0.9983	0.9991
	max	1.0788	1.0187	1.0112	1.0070	1.0036	1.0004	0.9997	0.9994	0.9993	0.9998
$\tilde{\mu}^{ps}$	min	1.0788	1.0398	1.0247	1.0073	0.9982	0.9948	0.9932	0.9937	0.9944	0.9952
	Q1	1.1078	1.0592	1.0331	1.0201	1.0132	1.0087	1.0056	1.0041	1.0029	1.0029
	median	1.1781	1.0633	1.0417	1.0287	1.0197	1.0140	1.0095	1.0072	1.0056	1.0044
	Q3	1.2064	1.0856	1.0488	1.0367	1.0264	1.0190	1.0135	1.0102	1.0077	1.0065
	max	1.4179	1.1160	1.0715	1.0503	1.0361	1.0271	1.0204	1.0162	1.0130	1.0106

Table D.16: Relative efficiency of $\tilde{\mu}^c$ with respect to other estimators with sample covariates under uniform random effects for ρ from 0.50 to 0.95

	ρ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\tilde{\mu}^{(1)}$	min	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Q1	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	median	1.0012	1.0008	1.0005	1.0003	1.0002	1.0001	1.0001	1.0000	1.0000	1.0000
	Q3	1.0018	1.0012	1.0008	1.0006	1.0004	1.0003	1.0002	1.0002	1.0001	1.0001
	max	1.0057	1.0051	1.0046	1.0040	1.0034	1.0027	1.0023	1.0019	1.0015	1.0010
$\tilde{\mu}^{(2)}$	min	1.0022	1.0017	1.0014	1.0011	1.0009	1.0007	1.0006	1.0003	1.0001	1.0000
	Q1	1.0038	1.0037	1.0034	1.0029	1.0024	1.0019	1.0014	1.0010	1.0006	1.0003
	median	1.0048	1.0048	1.0044	1.0039	1.0034	1.0029	1.0023	1.0017	1.0011	1.0006
	Q3	1.0069	1.0074	1.0072	1.0067	1.0060	1.0051	1.0041	1.0031	1.0020	1.0012
	max	1.0104	1.0107	1.0102	1.0093	1.0081	1.0068	1.0055	1.0041	1.0028	1.0016
$\tilde{\mu}$	min	0.9981	0.9981	0.9982	0.9983	0.9985	0.9987	0.9989	0.9992	0.9995	0.9998
	Q1	0.9987	0.9990	0.9992	0.9994	0.9996	0.9997	0.9998	0.9999	1.0000	1.0000
	median	0.9991	0.9993	0.9996	0.9998	0.9999	1.0000	1.0000	1.0001	1.0001	1.0001
	Q3	0.9999	1.0006	1.0011	1.0013	1.0013	1.0012	1.0010	1.0008	1.0005	1.0003
	max	1.0015	1.0025	1.0029	1.0029	1.0027	1.0024	1.0020	1.0016	1.0011	1.0006
$\tilde{\mu}^{ps}$	min	0.9959	0.9966	0.9972	0.9977	0.9981	0.9985	0.9989	0.9992	0.9995	0.9995
	Q1	1.0028	1.0025	1.0022	1.0015	1.0009	1.0006	1.0004	1.0001	0.9999	0.9999
	median	1.0038	1.0031	1.0024	1.0022	1.0017	1.0013	1.0009	1.0007	1.0004	1.0002
	Q3	1.0060	1.0054	1.0042	1.0036	1.0031	1.0025	1.0020	1.0015	1.0010	1.0005
	max	1.0087	1.0076	1.0070	1.0062	1.0054	1.0045	1.0036	1.0028	1.0019	1.0011

Appendix E

VALUE OF a IN SMALL AREA COMPOSITE ESTIMATOR $\tilde{\mu}_i^c$

Table E.1: The average value of a_i's in the small-area estimator $\tilde{\mu}_i^e$ over 20 small-areas based on the simulation results

	with $\overline{X}_i$				with $\overline{x}_i$			
ρ	Normal	Dbl.Exp.	Exp.	Uniform	Normal	Dbl.Exp.	Exp.	Uniform
0.00	0.7765	0.7693	0.7733	0.8060	0.7759	0.7693	0.7724	0.8051
0.05	0.7293	0.7733	0.8018	0.7269	0.7260	0.7712	0.7986	0.7189
0.10	0.6967	0.7333	0.7570	0.6744	0.6925	0.7319	0.7553	0.6700
0.15	0.6673	0.6846	0.6984	0.6756	0.6631	0.6882	0.7015	0.6733
0.20	0.6400	0.6397	0.6383	0.6619	0.6353	0.6429	0.6411	0.6613
0.25	0.6022	0.5868	0.5824	0.6373	0.5970	0.5892	0.5850	0.6381
0.30	0.5492	0.5327	0.5274	0.5928	0.5439	0.5343	0.5296	0.5947
0.35	0.4925	0.4727	0.4699	0.5190	0.4875	0.4729	0.4717	0.5220
0.40	0.4354	0.4142	0.4132	0.4408	0.4310	0.4128	0.4148	0.4444
0.45	0.3734	0.3689	0.3696	0.3566	0.3695	0.3673	0.3712	0.3601
0.50	0.3165	0.3457	0.3464	0.2859	0.3132	0.3449	0.3467	0.2892
0.55	0.2631	0.3292	0.3327	0.2491	0.2610	0.3301	0.3354	0.2518
0.60	0.2168	0.3195	0.3227	0.2257	0.2163	0.3211	0.3265	0.2286
0.65	0.1714	0.3149	0.3151	0.2099	0.1752	0.3165	0.3204	0.2129
0.70	0.1416	0.3106	0.3069	0.1961	0.1481	0.3122	0.3147	0.1992
0.75	0.1209	0.3052	0.2956	0.1844	0.1291	0.3073	0.3062	0.1876
0.80	0.1056	0.2996	0.2821	0.1757	0.1154	0.3023	0.2958	0.1796
0.85	0.0978	0.2950	0.2630	0.1722	0.1086	0.2982	0.2821	0.1765
0.90	0.1009	0.2882	0.2359	0.1793	0.1170	0.2914	0.2612	0.1837
0.95	0.1305	0.2815	0.1903	0.2049	0.1503	0.2877	0.2160	0.2089

Appendix F

INTRACLASS CORRELATION ESTIMATION RESULTS

(1) with normal distribution (from p. 208 to 209)
(2) with double exponential distribution (p. 210)
(3) with exponential distribution (p. 211)
(4) with uniform distribution (p. 212)

r_c: defined in equation (4.3.2)

$\tilde{\rho}$: defined in equation (4.3.12)

$\hat{\rho}_m$: M.L.E. estimator of ρ

$\hat{\rho}_{REMLs}$: Residual M.L.E. of ρ

$\hat{\rho}_{REMLt}$: Restricted M.L.E. of ρ

Table F.1: Bias of intraclass correlation estimators with normal random effects

ρ	r_c	$\tilde{\rho}$	$\hat{\rho}_m$	$\hat{\rho}_{REMLs}$	$\hat{\rho}_{REMLt}$
0.00	0.0385	0.0406	0.0366	0.0394	0.0433
0.05	0.0204	0.0208	0.0157	0.0204	0.0260
0.10	0.0077	0.0079	0.0018	0.0079	0.0140
0.15	−0.0011	−0.0005	−0.0087	−0.0004	0.0049
0.20	−0.0071	−0.0060	−0.0164	−0.0059	−0.0017
0.25	−0.0113	−0.0098	−0.0218	−0.0099	−0.0075
0.30	−0.0145	−0.0127	−0.0262	−0.0127	−0.0116
0.35	−0.0169	−0.0149	−0.0290	−0.0150	−0.0148
0.40	−0.0189	−0.0168	−0.0310	−0.0168	−0.0174
0.45	−0.0203	−0.0184	−0.0321	−0.0182	−0.0193
0.50	−0.0214	−0.0197	−0.0327	−0.0194	−0.0207
0.55	−0.0220	−0.0206	−0.0327	−0.0202	−0.0214
0.60	−0.0221	−0.0212	−0.0321	−0.0205	−0.0216
0.65	−0.0217	−0.0212	−0.0309	−0.0204	−0.0212
0.70	−0.0207	−0.0207	−0.0291	−0.0198	−0.0203
0.75	−0.0191	−0.0195	−0.0265	−0.0185	−0.0187
0.80	−0.0167	−0.0176	−0.0232	−0.0166	−0.0165
0.85	−0.0139	−0.0148	−0.0190	−0.0139	−0.0137
0.90	−0.0103	−0.0110	−0.0138	−0.0103	−0.0101
0.95	−0.0057	−0.0062	−0.0075	−0.0057	−0.0056

Table F.2: Mean Square Errors of intraclass correlation estimators with normal random effects and relative efficiency of r_c w.r.t. other estimators

	MSE					Rel.Eff.*			
ρ	r_c	$\tilde{\rho}$	$\hat{\rho}_m$	$\hat{\rho}_{REMLs}$	$\hat{\rho}_{REMLt}$	$\tilde{\rho}$	$\hat{\rho}_m$	$\hat{\rho}_{REMLs}$	$\hat{\rho}_{REMLt}$
0.00	0.0052	0.0057	0.0049	0.0055	0.0059	1.0906	0.9390	1.0550	1.1241
0.05	0.0069	0.0073	0.0064	0.0072	0.0073	1.0547	0.9283	1.0450	1.0564
0.10	0.0095	0.0098	0.0090	0.0098	0.0095	1.0399	0.9468	1.0415	1.0007
0.15	0.0120	0.0123	0.0116	0.0124	0.0116	1.0275	0.9631	1.0336	0.9665
0.20	0.0141	0.0143	0.0138	0.0145	0.0134	1.0157	0.9803	1.0230	0.9510
0.25	0.0157	0.0158	0.0156	0.0159	0.0150	1.0060	0.9955	1.0148	0.9530
0.30	0.0167	0.0167	0.0170	0.0168	0.0160	1.0006	1.0183	1.0076	0.9587
0.35	0.0171	0.0171	0.0177	0.0172	0.0165	0.9966	1.0327	1.0032	0.9653
0.40	0.0171	0.0170	0.0179	0.0171	0.0167	0.9956	1.0486	0.9989	0.9762
0.45	0.0167	0.0166	0.0176	0.0166	0.0164	0.9966	1.0562	0.9963	0.9835
0.50	0.0159	0.0159	0.0170	0.0158	0.0158	0.9984	1.0652	0.9948	0.9915
0.55	0.0148	0.0148	0.0159	0.0147	0.0147	1.0021	1.0723	0.9930	0.9943
0.60	0.0134	0.0135	0.0144	0.0133	0.0133	1.0061	1.0764	0.9915	0.9946
0.65	0.0117	0.0119	0.0127	0.0116	0.0117	1.0114	1.0797	0.9909	0.9954
0.70	0.0098	0.0100	0.0106	0.0097	0.0098	1.0179	1.0832	0.9909	0.9961
0.75	0.0078	0.0080	0.0085	0.0077	0.0077	1.0257	1.0871	0.9915	0.9964
0.80	0.0057	0.0059	0.0062	0.0056	0.0056	1.0353	1.0917	0.9931	0.9963
0.85	0.0036	0.0038	0.0040	0.0036	0.0036	1.0470	1.0970	0.9956	0.9961
0.90	0.0018	0.0020	0.0020	0.0018	0.0018	1.0602	1.1017	0.9978	0.9960
0.95	0.0005	0.0006	0.0006	0.0005	0.0005	1.0756	1.1053	0.9994	0.9958

*Relative Efficiencies of r_c with respect to other intraclass correlation estimators

Table F.3: Bias, MSE and relative efficiencies of the intraclass correlation estimators with double exponential random effects

	Bias			MSE			Rel.Eff.*	
ρ	r_c	$\tilde{\rho}$	$\hat{\rho}_m$	r_c	$\tilde{\rho}$	$\hat{\rho}_m$	$\tilde{\rho}$	$\hat{\rho}_m$
0.00	0.0367	0.0401	0.0359	0.0048	0.0056	0.0047	1.1647	0.9789
0.05	0.0209	0.0220	0.0174	0.0070	0.0076	0.0067	1.0874	0.9551
0.10	0.0095	0.0103	0.0046	0.0103	0.0108	0.0098	1.0514	0.9581
0.15	0.0010	0.0022	−0.0051	0.0138	0.0142	0.0133	1.0322	0.9659
0.20	−0.0058	−0.0041	−0.0125	0.0171	0.0174	0.0166	1.0206	0.9733
0.25	−0.0115	−0.0093	−0.0188	0.0199	0.0202	0.0196	1.0133	0.9829
0.30	−0.0164	−0.0139	−0.0248	0.0223	0.0224	0.0222	1.0078	0.9985
0.35	−0.0208	−0.0181	−0.0299	0.0240	0.0241	0.0243	1.0042	1.0126
0.40	−0.0247	−0.0221	−0.0341	0.0251	0.0252	0.0257	1.0032	1.0254
0.45	−0.0281	−0.0256	−0.0378	0.0256	0.0256	0.0266	1.0021	1.0386
0.50	−0.0309	−0.0288	−0.0406	0.0254	0.0255	0.0267	1.0025	1.0498
0.55	−0.0331	−0.0315	−0.0429	0.0246	0.0247	0.0262	1.0041	1.0631
0.60	−0.0346	−0.0335	−0.0440	0.0232	0.0234	0.0248	1.0068	1.0679
0.65	−0.0352	−0.0348	−0.0441	0.0212	0.0214	0.0228	1.0103	1.0727
0.70	−0.0348	−0.0351	−0.0432	0.0186	0.0189	0.0200	1.0149	1.0760
0.75	−0.0332	−0.0342	−0.0409	0.0155	0.0158	0.0167	1.0206	1.0780
0.80	−0.0304	−0.0318	−0.0372	0.0120	0.0123	0.0129	1.0279	1.0811
0.85	−0.0263	−0.0277	−0.0317	0.0082	0.0085	0.0089	1.0359	1.0842
0.90	−0.0202	−0.0214	−0.0240	0.0045	0.0047	0.0049	1.0443	1.0870
0.95	−0.0117	−0.0124	−0.0137	0.0014	0.0015	0.0015	1.0530	1.0898

*Relative Efficiencies of r_c with respect to other intraclass correlation estimators

Table F.4: Bias, MSE and relative efficiencies of the intraclass correlation estimators with exponential random effects

	Bias			MSE			Rel.Eff.*	
ρ	r_c	$\tilde{\rho}$	$\hat{\rho}_m$	r_c	$\tilde{\rho}$	$\hat{\rho}_m$	$\tilde{\rho}$	$\hat{\rho}_m$
0.00	0.0364	0.0401	0.0358	0.0048	0.0060	0.0051	1.2399	1.0545
0.05	0.0210	0.0226	0.0183	0.0074	0.0082	0.0074	1.1153	1.0014
0.10	0.0101	0.0111	0.0060	0.0113	0.0120	0.0111	1.0652	0.9845
0.15	0.0011	0.0026	−0.0037	0.0155	0.0162	0.0153	1.0423	0.9841
0.20	−0.0066	−0.0046	−0.0124	0.0196	0.0202	0.0194	1.0298	0.9903
0.25	−0.0137	−0.0113	−0.0205	0.0233	0.0238	0.0233	1.0214	0.9998
0.30	−0.0204	−0.0176	−0.0276	0.0265	0.0269	0.0267	1.0148	1.0075
0.35	−0.0265	−0.0236	−0.0343	0.0291	0.0294	0.0296	1.0100	1.0176
0.40	−0.0322	−0.0294	−0.0403	0.0311	0.0313	0.0319	1.0072	1.0273
0.45	−0.0373	−0.0348	−0.0456	0.0323	0.0325	0.0334	1.0065	1.0352
0.50	−0.0417	−0.0397	−0.0503	0.0328	0.0330	0.0343	1.0058	1.0450
0.55	−0.0452	−0.0439	−0.0542	0.0325	0.0327	0.0343	1.0062	1.0554
0.60	−0.0478	−0.0472	−0.0568	0.0314	0.0316	0.0333	1.0082	1.0625
0.65	−0.0491	−0.0494	−0.0580	0.0294	0.0297	0.0313	1.0110	1.0668
0.70	−0.0491	−0.0503	−0.0579	0.0265	0.0269	0.0285	1.0154	1.0748
0.75	−0.0473	−0.0494	−0.0558	0.0227	0.0232	0.0245	1.0215	1.0776
0.80	−0.0442	−0.0465	−0.0517	0.0181	0.0186	0.0197	1.0288	1.0867
0.85	−0.0389	−0.0410	−0.0448	0.0129	0.0134	0.0141	1.0364	1.0877
0.90	−0.0305	−0.0322	−0.0346	0.0075	0.0078	0.0081	1.0443	1.0875
0.95	−0.0180	−0.0190	−0.0202	0.0025	0.0026	0.0027	1.0518	1.0865

*Relative Efficiencies of r_c with respect to other intraclass correlation estimators

Table F.5: Bias, MSE and relative efficiencies of the intraclass correlation estimators with uniform random effects

ρ	Bias			MSE			Rel.Eff.*	
	r_c	$\tilde{\rho}$	$\hat{\rho}_m$	r_c	$\tilde{\rho}$	$\hat{\rho}_m$	$\tilde{\rho}$	$\hat{\rho}_m$
0.00	0.0382	0.0405	0.0362	0.0051	0.0056	0.0047	1.0801	0.9163
0.05	0.0200	0.0208	0.0154	0.0068	0.0072	0.0063	1.0495	0.9192
0.10	0.0069	0.0074	0.0004	0.0092	0.0095	0.0086	1.0381	0.9400
0.15	−0.0018	−0.0009	−0.0098	0.0114	0.0116	0.0109	1.0261	0.9575
0.20	−0.0071	−0.0058	−0.0168	0.0129	0.0131	0.0127	1.0136	0.9794
0.25	−0.0104	−0.0087	−0.0215	0.0138	0.0138	0.0138	1.0038	1.0018
0.30	−0.0124	−0.0104	−0.0242	0.0140	0.0140	0.0143	0.9939	1.0180
0.35	−0.0136	−0.0116	−0.0257	0.0138	0.0136	0.0143	0.9885	1.0349
0.40	−0.0143	−0.0124	−0.0266	0.0131	0.0130	0.0138	0.9875	1.0531
0.45	−0.0146	−0.0128	−0.0266	0.0122	0.0121	0.0130	0.9884	1.0633
0.50	−0.0145	−0.0131	−0.0261	0.0111	0.0110	0.0119	0.9917	1.0716
0.55	−0.0142	−0.0131	−0.0251	0.0098	0.0098	0.0106	0.9973	1.0774
0.60	−0.0135	−0.0128	−0.0238	0.0084	0.0084	0.0091	1.0050	1.0818
0.65	−0.0125	−0.0123	−0.0220	0.0070	0.0071	0.0076	1.0149	1.0868
0.70	−0.0113	−0.0115	−0.0199	0.0055	0.0057	0.0060	1.0274	1.0923
0.75	−0.0099	−0.0104	−0.0175	0.0041	0.0043	0.0045	1.0426	1.0985
0.80	−0.0083	−0.0090	−0.0147	0.0028	0.0030	0.0031	1.0607	1.1053
0.85	−0.0066	−0.0073	−0.0115	0.0017	0.0018	0.0019	1.0804	1.1110
0.90	−0.0046	−0.0052	−0.0080	0.0008	0.0009	0.0009	1.1015	1.1152
0.95	−0.0025	−0.0028	−0.0042	0.0002	0.0002	0.0002	1.1245	1.1177

*Relative Efficiencies of r_c with respect to other intraclass correlation estimators

Appendix G

VALUE OF a IN INTRACLASS CORRELATION COMPOSITE ESTIMATOR r_c

Table G.1: The value of a in the intraclass correlation estimator r_c based on the simulation results

ρ	Normal	Dbl.Exp.	Exp.	Uniform
0.00	0.9631	1.0000	1.0000	0.9898
0.05	0.7510	0.7994	0.8284	0.7624
0.10	0.6600	0.6841	0.7021	0.6735
0.15	0.6088	0.6105	0.6284	0.6381
0.20	0.5701	0.5571	0.5764	0.6064
0.25	0.5291	0.5112	0.5335	0.5711
0.30	0.4836	0.4695	0.4935	0.5282
0.35	0.4371	0.4272	0.4511	0.4752
0.40	0.3898	0.3844	0.4054	0.4200
0.45	0.3418	0.3398	0.3583	0.3633
0.50	0.2958	0.2921	0.3099	0.3063
0.55	0.2502	0.2442	0.2582	0.2496
0.60	0.2057	0.1953	0.2013	0.1937
0.65	0.1604	0.1463	0.1435	0.1385
0.70	0.1146	0.0962	0.0824	0.0843
0.75	0.0680	0.0440	0.0171	0.0310
0.80	0.0199	0.0000	0.0000	0.0000
0.85	0.0000	0.0000	0.0000	0.0000
0.90	0.0000	0.0000	0.0000	0.0000
0.95	0.0000	0.0000	0.0000	0.0000

Bibliography

[1] ARORA, V. & LAHIRI, P. (1995). On the superiority of the bayesian method over the BLUP in small-area estimation problems. In *Proceedings of the Survey Methods Section.* Statistical Society of Canada.

[2] BAIN, L. J. & ENGELHARDT, M. (1973). Interval estimation for the two-parameter double exponential distribution. *Technometrics* **15**, 875–887.

[3] BAIN, L. J. & ENGELHARDT, M. (1997). *Introduction to Probability and Mathematical Statistics.* Duxbury Press, 2nd ed.

[4] BATTESE, G. E., HARTER, R. M. & FULLER, W. A. (1988). An error-components model for prediction of county crop areas using survey and satellite data. *Journal of the American Statistical Association* **83**, 28–36.

[5] BRENN, T. (1994). Genetic and environmental effects on coronary heart disease risk factors in northern norway. the cardiovascular diease study in finnmark. *Annals of Human Genetics* **58**, 369–379.

[6] CRAIG, A. T. (1943). Note on the independence of certain quadratic forms. *The Annals of Mathematical Satistics* **14**, 195–197.

[7] CUNNINGHAM, E. P. & HENDERSON, C. R. (1968). An iterative procedure for estimating fixed effects and variance components in mixed model situations. *Biometrics* **24**, 13–25.

[8] DEMPSTER, A. P., SELWYN, M. R., PATEL, C. M. & ROTH, A. J. (1984). Statistical and computational aspects of mixed model analysis. *Applied Statistics* **33**, 203–14.

[9] DONNER, A. & KOVAL, J. J. (1980). The estimation of intraclass correlation in the analysis of family data. *Biometrics* **36**, 19–25.

[10] FAY III, R. E. & HERRIOT, R. A. (1979). Estimates of income for small places: An application of james-stein procedures to census data. *Journal of the American Statistical Association* **74**, 269–277.

[11] FIELLER, E. C. & SMITH, C. A. B. (1951). Note on the analysis of variance and intraclass correlation. *Annals of Eugenics* **16**, 97–105.

[12] FISHER, R. A. (1958). *Statistical Methods for Research Workers.* New York: Hafner Publishing Company Inc., thirteenth ed. ed.

[13] FULLER, W. A. & BATTESE, G. E. (1973). Transformations for estimation of linear models with nested error structure. *Journal of the American Statistical Association* **68**, 626–632.

[14] GHOSH, M., NATARAJAN, K., STROUD, T. W. F. & CARLIN, B. P. (1998). Generalized linear models for small-area estimation. *Journal of the American Statistical Association* **93**, 273–282.

[15] GHOSH, M. & RAO, J. N. K. (1994). Small-area estimation: An appraisal. *Statistical Science* **9**, 55–93.

[16] GRAYBILL, F. A. (1969). *Introduction to Matrices With Applications in Statistics.* Belmont, CA: Wadsworth.

[17] HARVILLE, D. A. (1969). Quadratic unbiased estimaton of variance components for the one-way classification. *Biometrika* **56**, 313–326.

[18] HARVILLE, D. A. (1974). Bayesian inference for variance components using only error contrasts. *Biometrika* **61**, 383–385.

[19] HARVILLE, D. A. (1977). Maximum likelihood approaches to variance component estimation and to related problems. *Journal of the American Statistical Association* **72**, 320–340.

[20] HARVILLE, D. A. (1985). Decomposition of predictor error. *Journal of the American Statistical Association* **80**, 132–138.

[21] HARVILLE, D. A. & JESKE, D. R. (1992). Mean squared error of estimation or prediction under a general linear model. *Journal of the American Statistical Association* **87**, 724–731.

[22] HENDERSON, C. R. (1953). Estimation of variance and covariance components. *Biometrics* **9**, 226–252.

[23] HENDERSON, C. R. (1975). Best linear unbiased estimation and prediction under a selection model. *Biometrics* **31**, 423–447.

[24] HULTING, F. L. & HARVILLE, D. A. (1991). Some bayesian and non-bayesian procedures for the analysis of comparative experiments and for small-area estimation: Computational aspects, frequentist properties, and relationships. *Journal of the American Statistical Association* **86**, 557–568.

[25] JOHNSON, N. L. & KOTZ, S. (1970). *Continuous Univariate Distributions - 1.* Boston: Houghton Mifflin Company.

[26] JOHNSON, N. L. & KOTZ, S. (1970). *Continuous Univariate Distributions - 2.* Boston: Houghton Mifflin Company.

[27] KARLIN, S., CAMERON, E. C. & WILLIAM, P. T. (1981). Sibling and parent-offspring correlation estimation with variable family size. In *Proceedings of the National Academy of Sciences of the United States of America*, vol. 78.

[28] KEEN, K. J. (1996). Limiting the effects of single-member families in the estimation of the intraclass correlation. *Biometrics* **52**, 823–832.

[29] KEMPTHORNE, O. (1957). *An Introduction to Genetic Statistics.* New York: John Wiley & Sons, Inc.

[30] KOTT, P. S. (1989). Robust small domain estimation using random effects modeling. *Survey Methodology* **15**, 3–12.

[31] KOTZ, S. & JOHNSON, N. L., eds. (1982). *Encyclopedia of Statistical Sciences*, vol. 2. John Wiley & Sons.

[32] LITTELL, R. C., MILLIKEN, G. A., STROUP, W. W. & WOLFINGER, R. D. (1996). *SAS System for Mixed Models.* Cary, NC. U.S.A.: SAS Institute Inc.

[33] LONGFORD, N. T. (1995). *Random Coefficient Models.* Oxford University Press Inc., New York.

[34] LONGFORD, N. T. (1999). Multivariate shrinkage estimation of small-area means and proportions. *Journal of Royal Statistical Society A* **162**, 227–245.

[35] MCCULLOCH, C. E. & SEARLE, S. R. (2001). *Generalized, Linear and Mixed Models.* John Wiley & Sons, Inc.

[36] MOURA, F. A. S. & HOLT, D. (1999). Small-area estimation using multilevel models. *Survey Methodology* **25**, 73–80.

[37] PATTERSON, H. D. & THOMPSON, R. (1971). Recovery of inter-block information when block sizes are unequal. *Biometrika* **58**, 545–554.

[38] PAUL, S. R. (1990). Maximum likelihood estimation of intraclass correlation in the analysis of familial data: Estimating equation approach. *Biometrika* **77**, 549–555.

[39] PEARSON, K. (1896). Mathematical contributions to the theory of evolution—III. regression, heredity, and panmixia. *Philosophical Transactions of the Royal Society, Series A* **187**, 253–318.

[40] PLATEK, R., RAO, J. N. K., SÄRNDAL, C. E. & SINGH, M. P. (1987). *Small-Area Statistics: An International Symposium.* John Wiley & Sons, Inc.

[41] PRASAD, N. G. N. & RAO, J. N. K. (1990). The estimation of the mean-squared error of small-area estimators. *Journal of the American Statistical Association* **85**, 163–171.

[42] PRASAD, N. G. N. & RAO, J. N. K. (1999). On robust small-area estimation using a simple random effects model. *Survey Methodology* **25**, 67–72.

[43] RAGHUNATHAN, T. E. (1993). A quasi-empirical bayes method for small-area estimation. *Journal of the American Statistical Association* **88**, 1444–1448.

[44] RAO, C. R. (1972). Estimation of variance and covariance components in linear models. *Journal of the American Statistical Association* **67**, 112–115.

[45] RAO, J. N. K. (1999). Some recent advances in model-based small-area estimation. *Survey Methodology* **25**, 175–186.

[46] RAO, J. N. K. & CHOUDHRY, G. H. (1995). *Small-Area Estimation: Overview and Empirical Study*, chap. 27. Business Survey Methods. John Wiley & Sons, Inc., pp. 527–542.

[47] RICHARDSON, A. M. & WELSH, A. H. (1995). Robust restricted maximum likelihood in mixed linear models. *Biometrics* **51**, 1429–1439.

[48] RIVEST, L.-P. & BELMONTE, E. (2000). A conditional mean squared error of small-area estimators. *Survey Methodology* **26**, 67–78.

[49] ROBINSON, G. K. (1991). That BLUP is a good thing: The estimation of random effects. *Statistical Science* **6**, 15–51.

[50] SCHEFFÉ, H. (1959). *The Analysis of Variance.* John Wiley & Sons, Inc.

[51] SEARLE, S. R. (1958). Sampling variances of estimates of components of variance. *Annals of Mathematical Statistics* **29**, 167–178.

[52] SEARLE, S. R. (1968). Another look at henderson's methods of estimating variance components. *Biometrics* **24**, 749–778.

[53] SEARLE, S. R. (1971). *Linear Models.* John Wiley & Sons, Inc.

[54] SEARLE, S. R. (1971). Topics in variance component estimation. *Biometrics* **27**, 1–76.

[55] SEARLE, S. R., CASELLA, G. & MCCULLOCH, C. E. (1992). *Variance Components.* John Wiley & Sons, Inc.

[56] SINGH, A. C. & STUKEL, D. M. (1998). Bayesian versus frequentist measures of error in small-area estimation. *Journal of Royal Statistical Society B* **60**, 377–396.

[57] SITTER, R. R. (1997). Variance estimation for the regression estimator in two-phase sampling. *Journal of the American Statistical Association* **92**, 780–787.

[58] SMYTH, G. K. & VERBYLA, A. P. (1996). A conditional likelihood approach to residual maximum likelihood estimation in generalized linear models. *Journal of Royal Statistical Society B* **58**, 565–572.

[59] SRIVASTAVA, M. S. (1984). Estimation of interclass correlations in familial data. *Biometrika* **71**, 177–185.

[60] SRIVASTAVA, M. S. (1987). Testing for block effects and analysis of regression models based on survey data. Report 2, University of Toronto.

[61] SRIVASTAVA, M. S. (1993). Estimation of the interclass correlation coefficient. *Annals of Human Genetics* **57**, 159–165.

[62] SRIVASTAVA, M. S. & KATAPA, R. S. (1986). Comparison of estimators of interclass and intraclass correlations from familial data. *The Canadian Journal of Statistics* **14**, 29–42.

[63] SRIVASTAVA, M. S. & KEEN, K. J. (1988). Estimation of the interclass correlation coefficient. *Biometrika* **75**, 731–739.

[64] SRIVASTAVA, M. S., KEEN, K. J. & KATAPA, R. S. (1988). Estimation of interclass and intraclass correlations in multivariate familial data. *Biometrics* **44**, 141–150.

[65] SRIVASTAVA, M. S. & NG, F. K. L. (1990). Estimation of intraclass correlation in regressin models. *Gujarat Statistical Review* **Professor Khatri Memorial Volume**.

[66] WACLAWIW, M. A. & LIANG, K. Y. (1993). Prediction of random effects in the generalized linear model. *Journal of the American Statistical Association* **88**, 171–178.

[67] YOU, Y. & RAO, J. N. K. (2002). A pseudo-empirical best linear unbiased prediction approach to small-area estimation using survey data. *The Canadian Journal of Statistics* **30**, 431–439.

[68] YOU, Y., RAO, J. N. K. & DICK, P. (2002). Benchmarking hierarchical bayes small-area estimators with application in census undercoverage estimation. In *Proceedings of the Survey Methods Section*. Statistical Society of Canada.