

On Elasto-Acoustics of Buried and Submerged Shell
Structures

by

Ako Bahari

A Thesis submitted to the Faculty of Graduate Studies of
The University of Manitoba
in partial fulfilment of the requirements of the degree of

Doctor of Philosophy

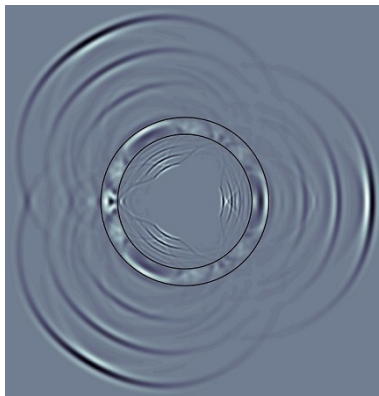
Department of Mechanical Engineering
University of Manitoba
Winnipeg

Copyright © 2019 by Ako Bahari

UNIVERSITY OF MANITOBA
FACULTY OF ENGINEERING
DEPARTMENT OF MECHANICAL ENGINEERING

On Elasto-Acoustics of Buried and Submerged Shell Structures

By
AKO BAHARI





University of Manitoba



University of Manitoba
Faculty of Engineering
Department of Mechanical Engineering
Sound & Vibration Laboratory

On Elasto-Acoustics of Buried and Submerged Shell Structures

PHD THESIS

AKO BAHARI

Advisor: Professor Neil Popplewell

Approved by

Professor Francois Guibault (The external examiner, Polytechnique Montréal)

Professor Neil Popplewell (The supervisor, Mechanical Engineering)

Professor Douglas Ruth (The internal examiner, Mechanical Engineering)

Professor Ahmed Shalaby (The internal examiner, Civil Engineering)

Professor Joe LoVetri (The chair, Electrical Engineering)

Winnipeg, January 15, 2019

Acknowledgements

I would like to express my utmost gratefulness and sincere gratitude to my academic advisor, Professor Neil Popplewell, for all his passion in coaching me all the way through with his incredible patience and dedication. He has been much more than just an academic advisor but also a life mentor towards making me a better person. I just can say, simply, his tireless dedication is one of a kind. He has been always a great support and he never let me down. Professor Popplewell, thank you for everything.

My special thanks goes to Professor Ferdinando Auricchio from the Computational Mechanics Group for which he showed me another level of research and true scientific contribution. He has been and will remain always the true source of motivation in conducting honest research with real impact. I am very grateful for kindly hosting me at University of Pavia. My sincere thanks also goes to Dr. Francesco Calabró for mentoring me on our research in numerical integration.

My special thanks also to Professor Alessandro Reali who have been helping me with his thoughtful comments and supporting me at the time of difficulties while staying at the Computational Mechanics Group, Pavia.

My special thanks also goes to Dr.s Arnaud Mesgouez and Gaëlle Lefeuvre-Mesgouez from LMA for kindly hosting me in Avignon where our research on underground shell structures was conducted. They have been very supportive and overly kind toward me.

My sincere thanks goes to Professors Marchello Imbriani and Livia Visiai at University of Pavia for warmly hosting me in Pavia during my research visit to Salvatore Maugeri Foundation on the diagnostic medical ultrasound.

My sincere thanks also go to my committee members, Professor Douglas Ruth and Professor Ahmed Shalaby, for their review and thoughtful comments throughout. I would like to deeply thank Professor Francois Guibault, my external examiner, for his very constructive comments. Very special thanks also goes to Professor Patrice Masson at *Gaus* group from Sherbrooke University for he kindly allocating significant time to review the manuscript and providing very valuable comments.

Very special thanks to Dr. Pooneh Maghoul for her kind support and warm encouragement, as a true friend, especially toward final stages of my PhD. Her true passion for academic excellence truly has been a source of motivation for me to go further.

Last but not least, I would like to thank my family, they are all what I truly have. My loving parents; my loving fiancé Hana; and my caring brothers Aram and Aras for their truly unconditional love and support. Hana should be acknowledged for giving me hand in editing stage. Their support has made this journey possible.

Winnipeg, January 15, 2019

Ako Bahari

Abstract

Comprehensive elasto-acoustics investigation of submerged and buried shell structures is the main objective. Analyses are carried out in frequency and time domains under a generalized loading condition and various material configurations for different constitutive rules. Major part of this study is performed analytically within the linear domain. The last chapter on cubature formulae, however, is a prelude to numerical analysis when nonlinearities of various types are introduced into the problem.

Primarily, an exhaustive comparative study is carried out on the performance of the most popular shell theories in modelling the elasto-dynamics of a fluid-filled, submerged shell structure subjected to a transient excitation. These models approximate the exact local theory of linear elasticity on the *modal* components. Consequently, a discontinuation in their application, under a general nonstationary excitation, is strongly recommended, henceforth. Subsequently the *total* responses are considered when a centrally symmetric excitation is applied. The corresponding numerical experiments convey practically significant implications.

A more general situation for a buried shell structure, rather than a fluid submerged case, is considered next. The exact local theory of linear elasticity is applied to the both of elastic media whilst the linear acoustical theory for compressible fluids is utilized. Both of generalized traction forces and waves being incident upon a shell's either surfaces are considered as the excitation's sources. Incident waves of shear and compressional nature are included. Origins of these sources are located within the encompassing elastic medium, shell's material or the contained fluid. Extensive sets of numerical examples are performed by using special visualization techniques after which different limiting situations are studied.

The Hook's law for the elastic, isotropic material is considered as constitutive rule in full-space configurations in two previous parts. A generalized Hook's law along with generalized linear piezoelectricity for anisotropic materials, on the other hand, is considered subsequently for the shells submerged in an acoustical wedge medium rather than a full-space configuration.

Analytical solutions are developed for linear domain in previous parts. The last chapter, however, is a prelude of numerical analysis dealing with nonlinearities associated with the kinematics of finite deformation or constitutive rule such as elasto-plasticity, super-elasticity or discontinuities due to a strong shock. Quadrature formulae are presented and implemented numerically for circular cross-sections as it appears, for instance, in spherical and cylindrical shells or inclusions. These integration schemes can be used within a mixed finite element framework or a meshless approach.

Keywords

transient diffraction of elastic and acoustical waves, shell theories, focusing mechanism, cavitation, method of images, addition theorems, acoustical wedge, composites, anisotropy, layered structures, triclinic, spherical incident waves, piezoelectricity, generalized Hooke's law, quadrature integration, cubature formulae, degree of exactness, computational complexity, meshless method

*To my family
for all their love and dedication*

*To the memory of my uncle Faroq
my first teacher
whom will be in my heart forever*

Nomenclature

ϕ	nondimensionalized longitudinal displacement potential
ψ, χ	nondimensionalized secondary (transverse) displacement potentials
c_p	$= \sqrt{(\lambda + 2\mu)/\rho_s}$, primary (longitudinal) wave speed
c_s	$= \sqrt{\mu/\rho_s}$, secondary (shear) wave speed
u^i	nondimensionalized displacement elements where $i = r, \theta$ and φ
v_i	nondimensionalized velocity elements where $i = r, \theta$ and φ
t_0^*	time reference
r	nondimensionalized radial distance from the center of the shell
r^*	radial coordinate
$\delta(t)$	Dirac delta function
$\mathbb{GL}_\mu$	μ -Gauss-Lobatto rule $\mathbb{GL}_\mu$
$\mathbb{G}_\mu$	μ -Gaussian quadrature rule $\mathbb{G}_\mu$
$\mathcal{H}(t)$	Heaviside function
ω_i	quadrature weights
$\text{erfc}(x)$	complementary error function
ζ_i	quadrature nodes
$i_n(z)$	modified spherical Bessel function of the first kind of n^{th} order
$k_n(z)$	modified spherical Bessel function of the second kind of n^{th} order
$N_{(k)}^{OPT}$	minimum number of nodes for exactness degree of k
$N_{\partial\Omega}$	number of points on the boundary
$P_n^m(z)$	associated Legendre function of n^{th} order
α_i	c_i^*/c^* where $i = ex, in, s, p$
σ_0	$\rho_{ex}^* c_{ex}^{*2}$ as a reference stress
σ_{ij}	nondimensionalized stress elements where $i, j = r, \theta$ and φ
ε_{ij}	Green-Lagrange strain tensor

c^i	speeds of sound where $i = ex, in, s, p$
p	nondimensionalized pressure
ex	total wave field within the external fluid field
inc	incident wave
in	internal fluid
sc	scattered or radiated wave
p	compressional wave
sh	elastic shell
s	shear wave

Contents

Acknowledgements	i
Abstract	iii
Preface	xxv
1 Exact Theory of Linear Elasticity versus Shell Theories	1
1.1 Introduction	1
1.2 Mathematical Formulations	5
1.2.1 Problem's Definition	5
1.2.2 Laplace Transform	6
1.2.3 Excitation Sources	7
1.2.4 Fluid Media	8
1.2.5 Elastic Shell	10
1.3 Validations	15
1.4 Numerical Experiments	17
1.4.1 Configurations of The Case Studies	18
1.4.2 The Mode Number Effect	20
1.4.3 The Sources of Excitation	22
1.4.4 The Breathing Mode, $n = 0$	23
1.4.5 The Rigid Body Motion, $n = 1$	24
1.4.6 The Labour Modes, $n > 1$	25
1.4.7 Narrowband versus Boardband Excitations, $\bar{f}(t)$	25
1.4.8 Shell's Material	26
1.5 Conclusions	26
2 Centrally Symmetric Impacted Shell	27
2.1 Introduction	27
2.2 Mathematical Modelling	29
2.2.1 Preliminaries	29
2.2.2 Excitation Source	31
2.2.3 Fluid Media	32
2.2.4 Elastic Medium	33
2.2.5 Fluid-Solid Interaction	34
2.2.6 Construction of Total Response	34
2.3 Results and Discussions	35
2.3.1 Validations	36
2.3.2 Case Studies	38

2.4	Conclusions	46
3	Embedded Sphere: Formulations and Validations	49
3.1	Introduction	49
3.2	Mathematical Formulations	54
3.2.1	Laplace Transform	54
3.2.2	Definition of The Problem	55
3.2.3	Excitation Sources	57
3.2.4	Elastic Media	62
3.2.5	Fluid Field	65
3.2.6	Boundary Conditions	66
3.2.7	Construction of Total Responses	68
3.3	Numerical Procedure and Validations	68
3.3.1	Descriptions of The Limiting Situations	69
3.3.2	Numerical Comparisons	71
3.4	Conclusions	77
4	Embedded Sphere: Numerical Results	79
4.1	Preliminaries	79
4.1.1	Material and Geometrical Configurations	80
4.1.2	Excitation Sources	80
4.2	Numerical Experimentation	83
4.2.1	General Comments	84
4.2.2	Effect of a Shell's Relative Thickness	84
4.2.3	Effect of Relative Material Properties	110
4.2.4	Effect of different excitations	119
4.3	Conclusions	123
5	Anisotropic Cylinder in an Acoustical Wedge: Formulations	133
5.1	Introduction	133
5.2	Mathematical Developments	136
5.2.1	Problem Description	136
5.2.2	Structural Modelling: Elasticity and Piezoelasticity	137
5.2.3	Solution Assembly in Solid Media	139
5.2.4	Assembly of The Ultimate Transfer Matrix	141
5.2.5	Excitation Sources	142
5.2.6	Fluid Media	143
5.2.7	Wedge Domain: Method of Images	145
5.2.8	Shell's Traction Boundary Conditions	148
5.3	Conclusions	149
6	Anisotropic Cylinder in an Acoustical Wedge: Numerical Results	151
6.1	Validations	152
6.1.1	Scattering of Incident Waves	153

6.1.2	Traction Forces	154
6.2	Material Arrangements	157
6.3	Case Studies	159
6.3.1	Incident Wave	159
6.3.2	Surface Traction Force	161
6.4	Conclusions	168
7	Quadrature Formulae For a Circular Domain	171
7.1	Motivations	171
7.2	Literature review and preliminaries	174
7.3	Quadrature methods	175
7.3.1	Product rules	176
7.3.2	Line rules	180
7.4	Numerical evaluations	183
7.4.1	Continuous functions	184
7.4.2	Continuous integrand functions	184
7.4.3	Discontinuous functions	185
	Conclusions	189
	Bibliography	191
	Appendices	207
A	Acoustical Phenomena	209
B	Elastic Waves	211
C	Shell Theories' Operators	215
D	Green's Function	217
E	Spherical Addition Theorems	219
F	$\{^j_k \vartheta_{\theta, \varphi}^i\}$ and $\{^j_k \varsigma_{\theta, \varphi}^i\}$	221
G	Axially Polarized Orthotropic Constitutive Model	225
H	Kinematic Relations	227
I	Conservation Laws	229
J	Modal Coefficient Matrices	231

List of Figures

1.1	Showing the problem's geometry and different excitation sources including traction force(s) and plane incident waves. Azimuthal angle φ is perpendicular to the illustrated plane.	5
1.2	Time histories of the (a) shell's modal displacements, $\bar{u}_r^n, 0 \leq n \leq 7$ at $r = 0.99$; (b) shell's total displacement at $\theta = \pi/2$ as well as the average value for the apex and tail in comparison to the breathing mode displacement (the late time static asymptotic deflection). The water submerged, empty steel spherical shell has a nondimensionalized thickness of $h = 0.02$. It is subjected to a plane incident pressure wave with Heaviside time dependency. (c) inner and outer surface pressure time histories at a spherical shell's apex ($\theta = 0$) and (d) its center point ($r = 0$)(c) Center point, $r = 0$, pressure history when in the later configuration, the shell is filled with water and the thickness reduces to $h = 0.01$	16
1.3	Time histories of the modal pressure at different nondimensionalized distances from the sphere's center ($r = 1.0, 1.5$, and 2) for (a) breathing mode $n = 0$ and (b) second vibratory mode $n = 2$ where an Dirac impulse $\bar{g}(t) = \delta(t)$ applied at the shell's outer surface. Time histories of the modal pressure at different nondimensionalized distances from the sphere's center ($r = 1.0, 1.5$, and 3) for (a) breathing mode $n = 0$ and (b) second vibratory mode $n = 2$ where an Heaviside pulse $\bar{g}(t) = \mathcal{H}(t)$ applied at the shell's inner surface.	17
1.4	Shows the normalized time histories and corresponding frequency spectra of (a) and (b) the Narrowband (NB) as well as (c) and (d) the broadband (BB) excitations, respectively.	20
1.5	Modal ($n = 0, 1, 2, 6, 8, 10$) time histories of the external pressure field at the outer surface of a water filled, steel shell submerged in water when it is subjected to a traction force at its outer surface. The shell wall thickness and the excitation's time-dependency are invariably $h = 0.1$ and the NB Ricker pulse, respectively.	21
1.6	Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at the outer surface of the shell. Three different excitations and four relative wall thicknesses are considered for $n = 10$	2
1.7	Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at the outer surface of the shell. Three different excitations and four relative wall thicknesses are considered for $n = 20$	3
1.8	Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to an incident plane wave.	4
1.9	Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at its outer surface.	5
1.10	Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at its outer surface.	6
1.11	Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at its outer surface.	7

1.12	Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at its outer surface.	8
1.13	Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at its outer surface.	9
1.14	Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at the outer surface of the shell. Three different excitations and four relative wall thicknesses are considered for $n = 20$	10
1.15	Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at the outer surface of the shell. Three different excitations and four relative wall thicknesses are considered for $n = 20$	11
1.16	Showing the breathing mode's time histories in response of a traction force applied at (first row) outer and (second row) inner surfaces of a steel shell where $h = 0.1$ and the excitation is the broadband modulated Gaussian pulse.	12
1.17	Modal time histories of the breathing mode's radial displacement at the outer surface of an air filled steel shell submerged in water when subjected to a surface traction force at the shell's external surface.	13
1.18	Modal time histories of the radial displacement at the outer surface of a water filled steel shell submerged in water when subjected to an plane incident wave.	14
1.19	Modal time histories of the radial displacement at the outer surface of a water filled steel shell submerged in water when subjected to a traction surface load at the shell's outer surface. . . .	15
1.20	Modal time histories of the radial displacement at the outer surface of a water filled steel shell submerged in water when subjected to a traction surface load at the shell's outer surface. . . .	16
1.21	Modal time histories of the radial displacement at the outer surface of an air filled steel shell submerged in water when subjected to an plane incident wave.	17
1.22	Showing modal time histories of the radial displacement at the outer surface of an air filled steel shell submerged in water when subjected to a traction surface load at the shell's outer surface. . .	18
1.23	Showing modal time histories of the radial displacement at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at the shell's outer surface.	19
1.24	Showing modal time histories of the radial displacement at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at the shell's outer surface.	20
1.25	Showing modal time histories of the radial displacement at the outer surface of a water filled steel shell submerged in water when subjected to an plane incident wave.	21
1.26	Showing modal time histories of the surface pressures at the outer surface of a water filled steel shell submerged in water when subjected to a traction surface load for $n = 0$ where the excitation's time-dependencies are in the narrowband range.	22
1.27	Showing modal time histories of the surface pressures at the outer surface of a water filled steel shell submerged in water when subjected to a traction surface load for $n = 0$ where the excitation's time-dependencies are in the broadband range.	23
1.28	Showing modal time histories of the surface pressures at the outer surface of a water filled steel shell submerged in water when subjected to a traction surface load for $n = 0$ where the excitation's time-dependencies are in the UT-broadband range.	24

1.29	Showing modal time histories of the internal (first row) and external (second row) radial displacements of a water filled steel shell submerged in water when subjected to a traction force at its outer surface for $n = 10$. The shell wall-thickness is $h = 0.1$ and the excitation's time-dependency is a broadband Ricker pulse at UT testing range with central frequency of (first column) $\omega_r = 2000$ and (second column) $\omega_r = 6000$	25
1.30	Showing the modal pressure histories associated with the breathing mode at outer (left column) and inner (right column) surfaces for a steel (first row), aluminum (second row) and pyroceram glass (third row) when filled with and submerged in water subjected to a traction force at its outer surface. The shell wall thickness and the excitation's time-dependency are invariably $h = 0.1$ and the NB Ricker pulse, respectively.	26
2.1	depicting the problem's geometry and the double impact excitation. Azimuthal angle φ is perpendicular to the illustrated plane.	30
2.2	External acoustical pressure histories for an empty (air-filled), steel spherical shell with $h = 0.01$ at its apex when subjected to (first row) a Dirac delta pulse, $\delta(t)$ (see Figures 10 through 12 in [Chen and Stepanishen, 1994] and Figure 3b in [Bahari and Popplewell, 2015]) and (second) Heaviside pulse, $\mathcal{H}(t)$ (see Figures 7 and 8 in [Zakout, 2001] as well as Figures 1c and 3b). The first three columns from left gives modal components, whereas the last column illustrates the total responses when the spatial distribution is described by $\theta_c = \pi/3$	36
2.3	Acoustical pressure history at the inner surface of water-filled, steel spherical shell with $h = 0.10$ and 0.20 when subjected to a broadband modulated Gaussian pulse with $\theta_c = \pi/3$	36
2.4a	A series of internal pressure's successive snapshots ($0.1 \leq t \leq 1.1$) in response to an exponentially decaying excitation with $\theta_c = \pi/3$ where (first row) $h = 0.01$ and (second row) $h = 0.03$ in which the Hertzian distribution profile is assumed.	37
2.4b	A series of internal pressure's successive snapshots ($1.5 \leq t \leq 3.3$) in response to an exponentially decaying excitation with $\theta_c = \pi/3$ where (first row) $h = 0.01$ and (second row) $h = 0.03$ in which the Hertzian distribution profile is assumed.	37
2.4c	A series of internal pressure's successive snapshots ($4.1 \leq t \leq 7.1$) in response to an exponentially decaying excitation with $\theta_c = \pi/3$ where (first row) $h = 0.01$ and (second row) $h = 0.03$ in which the Hertzian distribution profile is assumed.	37
2.5	A series of successive snapshots of internal pressure which are evenly spaced in time, $\Delta t = 0.2$, from $t = 7.4$ to $t = 9.6$ in response to an exponentially decaying excitation with $\theta_c = \pi/3$ when $h = 0.01$ in which the Hertzian distribution profile is assumed.	38
2.6	Contour plots of total field response to an explosive pulse (Table 1.2) applied over the region specified by $\theta_0 = \pi/6$ for two shell's wall thicknesses of $h = 0.01$ and $h = 0.06$. (1^{st} row) acoustical energy's density; (2^{nd} row) radial displacement and (3^{rd} row) acoustical energy's density at the shell's entire outer, mid- and inner surfaces, respectively.	39
2.7	Contour plots of density of acoustical energy in response to a <i>broadband modulated Gaussian</i> pulse ($t_0 = 1/200, t_G = 1/2000$ and $\omega_G = 600$) applied over the region specified by $\theta_0 = \pi/6$ for two shell's wall thicknesses of $h = 0.1$ and $h = 0.2$. Load is applied on the shell's external surface in the (1^{st} row) and (2^{nd} row) whereas the shell's inner surface is excited in (3^{rd} row).	40

2.8	Successive snapshots of total field response from left to right in response to a broadband <i>modulated Gaussian</i> pulse with $t_0 = 1/200$ and $t_\omega = 1/2000$ when $\theta_c = \pi/3$ and $h = 0.1$. Each snapshot contains internal and external fluid fields' acoustical pressures as well as the shell's radial displacement u_r	41
2.9	Contour plots of total field response to a Heaviside excitation, $\bar{f}_t(t) = \mathcal{H}(t)$, applied over the region specified by $\theta_0 = \pi/6$ for a shell filled with (a) water (b) air where $h = 0.06$	41
2.10	Pressure histories at the shell's center when $\theta_c = \pi/3$. The excitation are the narrowband Ricker pulse and Heaviside for the external surface pressure and the pressure gradient across the shell's wall, respectively.	42
2.11	Internal pressure histories at the shell's center when $\theta_c = \pi/2$ for four excitations and four shell wall thicknesses.	43
2.12	Successive snapshots of total field response to a concentrated ($f_\theta = \delta(\theta)$) narrow-band <i>Ricker</i> pulse with $t_0 = 1/5$ and $t_\omega = 1/20$. Each snapshot contains internal and external fluid field's acoustical pressures. The radial stress, σ_{rr} is selected as the representation of the elastic shell's response.	43
3.1	Depicting the problem's geometry and different excitation sources including incident waves and traction forces. Azimuthal angle φ is perpendicular to the illustrated plane.	56
3.2	Illustrating r'' for various spherical wavefronts' possible configurations (in red color) when they meet, first, either surfaces of the shell. Two feasible configurations depicted for a source located within the shell. An arbitrary point A on a spherical wavefront originating within the elastic formation ∞ at an arbitrary moment after the initial interaction.	58
3.3	Illustrating an arbitrary point A on a spherical wavefront originating from a source's location, O_i in which O denotes the center of the spherical shell. O_i may denote an excitation originating within the elastic formation ∞ , the shell's material sh or encapsulated fluid as shown in Figure 3.1. The r_i , r and r' are relative distances of O , O_i and A as depicted. If the distance between shell's center and wave's origin becomes very large, $r_i \rightarrow \infty$, a spherical incident becomes a plane wavefront.	58
3.4	(a) illustrates the radial displacement of a buried, empty steel spherical shell's mid-surface due to a Heaviside incident compressional wave where the shell's thickness is $h = 0.19$ ([Reddy and Kierans, 1976]) (b) rigid body radial velocity of a spherical rigid inclusion subjected to an explosive compressional incident wave for different decaying constant where the density of inclusion and elastic formation is the same ([Mow, 1965]).	71
3.5	A spherical cavity subjected to an asymmetric Heaviside torsional loading is considered as the one of [Zakout et al., 2000]. Modal circumferential stress components of $\bar{\sigma}_{r\varphi}^n$ for (a) $n = 1$ (b) $n = 3$ (c) $n = 1 - 14$ and $r = 1.1$ (d) modal circumferential displacement components for $n = 1 - 14$ and $r = 1.1$. Hence, the stress at $r = 1$ is the applied actual stress.	72
3.6	A spherical cavity's response histories to a symmetric pressure which is known a priori as a traction force, at different distances from cavity's surface, r , (a) radial stresses σ_{rr} (b) radial displacements u_r and (c) radial particle velocities v_r time histories are due to a Heaviside $\mathcal{H}(t)$ excitation while (d) depicts particle velocities due to an explosive excitation $e^{-\alpha_d t}$ with a decaying constant of $\alpha_d = 2 \times 10^{-6} s^{-1}$. See [Goldsmith and Allen, 1955].	73

3.7	(a) inner and outer surface pressures at a spherical shell's (b) apex ($\theta = 0$), (c) tail ($\theta = \pi$) and (c) its center-point ($r = 0$). This spherical steel shell is filled and submerged in water which is subjected to an incident plane pressure wave. (d) shell's displacement at $\theta = \pi/2$ as well as the average value between apex and tail in comparison with breathing mode displacement as the late time static asymptotic deflection where $h = 0.01$ [Huang, 1969].	74
3.8	Time histories of the total pressure at the shell's apex ($\theta = 0$) and different nondimensionalized distances from the sphere's center ($r = 1.0, 1.5$, and 3) for (a) the loading case I and Heaviside excitation $\bar{g}(t) = \mathcal{H}(t)$ (b) the loading case II and Heaviside excitation $\bar{g}(t) = \mathcal{H}(t)$ (c) the loading case I and Impulse excitation $\bar{g}(t) = \delta(t)$ and (d) the loading case II and Impulse excitation $\bar{g}(t) = \delta(t)$. The loading case I allows the rigid body motion while the loading case II does not. These results correspond to Figures 2(b,c) and Figures 3(b,c) of [Bahari and Popplewell, 2015] for loading cases I and II, respectively.	75
4.1a	A series of total field successive snapshots of a water filled steel shell embedded in granite in response to a plane compressional incident wave. Relative shell wall thicknesses from top to bottom, respectively, are $h = 0.02, 0.04, 0.20$ and 0.40 . The time dependency of the excitation, $\bar{f}(t)$, is the Ricker pulse. The virtual camera zooms out as time passes. Initial specular reflections and head waves are depicted.	88
4.1b	A series of total field successive snapshots of a water filled steel shell embedded in granite in response to a plane compressional incident wave. Relative shell wall thicknesses from top to bottom, respectively, are $h = 0.02, 0.04, 0.20$ and 0.40 . The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse. The virtual camera zooms out as time passes. The formation of creeping waves in shadow region is illustrated. (continued)	89
4.1c	A series of total field successive snapshots of a water filled steel shell embedded in granite in response to a plane compressional incident wave. Relative shell wall thicknesses from top to bottom, respectively, are $h = 0.02, 0.04, 0.20$ and 0.40 . The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse. The virtual camera zooms out as time passes. Creeping waves' superposition and their movement towards the head region (continued).	90
4.2a	A series of total field successive snapshots of a water filled steel shell embedded in granite in response to a plane compressional incident wave. Relative shell wall thicknesses from top to bottom, respectively, are $h = 0.02, 0.04, 0.20$ and 0.40 . The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse. Strong focusing within the internal pressure filed.	93
4.2b	A series of total field successive snapshots of a water filled steel shell embedded in granite in response to a plane compressional incident wave. Relative shell wall thicknesses from top to bottom, respectively, are $h = 0.02, 0.04, 0.20$ and 0.40 . The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse. Strong focusing within the internal pressure filed. (continue)	94
4.3	Pressure contour plots of a water filled steel shell embedded in granite in response to a plane compressional incident wave for different thicknesses: (1^{st} row) gauge pressure at the shell's entire inner surface (normalized); (2^{nd} row) maximum of positive gauge pressure of all radii at each angle; (3^{rd} row) minimum of negative gauge pressure of all radii at each angle. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.	97

4.4	Maximum absolute values of radial stress within shell material in response to a plane compressional incident wave impinging on a water filled steel shell embedded in granite: maximum (1^{st} row) tensile stress; (2^{nd} row) compressional stress. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.	98
4.5	Radial stress contour plots of a water filled steel shell embedded in granite in response to a plane compressional incident wave for different thicknesses: (1^{st} row) stress at the shell's entire inner surface (normalized); maximum of absolute (2^{nd} row) tensile and (3^{rd} row) compressing stresses of all radii at each angle within the infinite granite formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.	99
4.6	Absolute values of the positive and negative pressures and stresses occurrence through the entire spatial domain of a water filled steel shell embedded in granite. The responses are created by a plane compressional incident wave for different shell wall thicknesses. The time dependency of the excitation, $\bar{f}(t)$, is the Ricker pulse. (1^{st} row) internal pressure; radial stresses within (2^{nd} row) steel shell and (3^{rd} row) infinite granite formation. Columns correspond to $h = 0.02, 0.1, 0.2$ and 0.4 from right to left.	102
4.7	Polar plots of the absolute positive and negative gauge pressures throughout the entire spatial domain of a water filled steel shell embedded in granite. Responses are created by a plane compressional incident wave for different thicknesses. The time dependency of the excitation, $\bar{f}(t)$, is the Ricker pulse.	103
4.8a	A series of total field successive snapshots of a water filled, hollow steel sphere (the top row for $h = 0.80$) and a steel inclusion (bottom row) both embedded in granite in response to a plane compressional incident wave. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse. . . .	106
4.8b	A series of total field successive snapshots of a water filled, hollow steel sphere (the top row for $h = 0.80$) and a steel inclusion (bottom row) both embedded in granite in response to a plane compressional incident wave. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse. (continued)	107
4.8c	A series of total field successive snapshots of a water filled, hollow steel sphere (the top row for $h = 0.80$) and a steel inclusion (bottom row) both embedded in granite in response to a plane compressional incident wave. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse. (continued)	108
4.8d	A series of total field successive snapshots of a water filled, hollow steel sphere (the top row for $h = 0.80$) and a steel inclusion (bottom row) both embedded in granite in response to a plane compressional incident wave. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse. (continued)	109
4.9	Radial stress contours of a water filled steel, hollow sphere ($h = 0.80$) and a steel inclusion both embedded in granite in response to a plane compressional incident wave: (1^{st} row) radial stress at the shell's entire inner surface (normalized); maximum of (2^{nd} row) tensile and (3^{rd} row) compressing stresses of all radii at each angle within the infinite granite formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.	110
4.10	Polar plots of the absolute maximum values of tensile and compressional and negative gauge pressures through the entire spatial domain within (first row) steel and (second row) granite formation in response of a compressional plane, incident wave having the Ricker pulse as its time dependency.	111

4.11a	A series of total field successive snapshots of a water filled shell with shell wall thickness of $h = 0.25$ in response to a plane compressional incident wave. (1^{st} row) steel shell in an pyroceram formation; (2^{nd} row) aluminum shell in an pyroceram formation and (3^{rd} row) steel shell embedded in hard clay. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.	114
4.11b	A series of total field successive snapshots of a water filled shell with shell wall thickness of $h = 0.25$ in response to a plane compressional incident wave. (1^{st} row) steel shell is embedded in an pyroceram formation; (2^{nd} row) aluminum shell is embedded in an pyroceram formation and (3^{rd} row) steel shell embedded in hard clay. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.	115
4.12a	A series of total field successive snapshots of a water filled shell with shell wall thickness of $h = 0.25$ in response to a plane compressional incident wave. (1^{st} row) steel shell is embedded in an pyroceram formation and (2^{nd} row) aluminum shell is embedded in an pyroceram formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.	116
4.12b	A series of total field successive snapshots of a water filled shell with shell wall thickness of $h = 0.25$ in response to a plane compressional incident wave. (1^{st} row) steel shell is embedded in an pyroceram formation and (2^{nd} row) aluminum shell is embedded in an pyroceram formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.	117
4.13	Contour plots of the surface pressures (first column) and radial stresses (second column) of a water filled shell with the wall thickness of $h = 0.24$ due to a plane compressional incident wave: (1^{st} row) a steel shell in an pyroceram formation; (2^{nd} row) an aluminum shell is embedded in an pyroceram formation and (3^{rd} row) steel shell embedded in hard clay formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.	118
4.14	Contour plots of the maximum absolute value of negative gauge pressure of a water filled shell with the wall thickness of $h = 0.24$ due to a plane compressional incident wave: (1^{st} row) a steel shell in an pyroceram formation; (2^{nd} row) an aluminum shell is embedded in an pyroceram formation and (3^{rd} row) steel shell embedded in hard clay formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.	120
4.15	Contour plots of the maximum stresses within the shell material of a water filled shell with the wall thickness of $h = 0.24$ due to a plane compressional incident wave: (1^{st} row) a steel shell in an pyroceram formation; (2^{nd} row) an aluminum shell is embedded in an pyroceram formation and (3^{rd} row) steel shell embedded in hard clay formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.	121
4.16	Absolute values of positive and negative gauge pressures through the entire spatial domain for a water filled shell ($h = 0.24$) in response to a plane compressional incident wave: (1^{st} row) a steel shell embedded in an pyroceram formation; (2^{nd} row) an aluminum shell is embedded in an pyroceram formation and (3^{rd} row) steel shell embedded in hard clay formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.	122
4.17	Absolute values of positive and negative gauge pressures (first column) and radial stresses (second column) within the shell material through the entire spatial domain for a water filled shell ($h = 0.24$) in response to a plane compressional incident wave: (1^{st} row) a steel shell in an epoxy formation; (2^{nd} row) an aluminum shell is embedded in an epoxy formation and (3^{rd} row) steel shell embedded in hard clay formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.	126

4.18	Absolute values of positive and negative gauge pressures through the entire spatial domain for a water filled steel shell ($h = 0.2$) embedded in granite in response to a plane compressional incident wave where the time dependency of the excitation, $\bar{f}(t)$, are (1^{st} column) a Ricker pulse and (2^{nd} column) an explosive load.	127
4.19	Absolute values of compressional and tensile stresses through the entire spatial domain for a water filled steel shell ($h = 0.2$) embedded in granite in response to a plane compressional incident wave where the time dependency of the excitation, $\bar{f}(t)$, are (1^{st} column) a Ricker pulse and (2^{nd} column) an explosive load.	128
4.20	Absolute values of positive and negative gauge pressures through the entire spatial domain for a water filled steel shell ($h = 0.2$) embedded in granite in response to a (1^{st} column) plane compressional and (2^{nd} column) spherical compressional incident wave with $r_{\infty=1.5}$ where the time dependency of the excitation, $\bar{f}(t)$, is a Ricker pulse.	129
4.21	Absolute values of positive and negative gauge pressures through the entire spatial domain for a water filled steel shell ($h = 0.2$) embedded in granite in response to plane (1^{st} column) compressional and (2^{nd} column) shear incident waves where the time dependency of the excitation, $\bar{f}(t)$, is a Ricker pulse.	130
4.22	Contour plots of the surface pressures (first column) and radial stresses (second column) of a water filled shell with the wall thickness of $h = 0.2$ due to a (1^{st} row) plane compressional incident wave and (2^{nd} row) eccentric pressure pulse in granite formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.	131
5.1	Problem Configuration.	137
5.2	Shell position, loading configuration, and measurement points.	145
5.3	Illustrating description of an arbitrary point A with respect to a coordinate origin at O_j and O_λ where $d_{\lambda j}$ is the distance between to centers. The z axis coordinate distinguished by j is translated to the one of λ by using the vector $\mathbf{r}_{\lambda j} \equiv O_j O_\lambda (\mathbf{r}_{\lambda j} = d_{\lambda j})$, while all the axes remain parallel. z axes are perpendicular and outward with respect to the current plane.	145
6.1	Shell position in the external fluid and measurement points A through D and W for the three dimensional acoustical scattering angles are given in rads.	153
6.2	Showing the acoustically transmitted pressure (pa) at point, C , the radiated pressure on the wall at point, W , and the total on-surface acoustical pressure at the apex of the cylinder. The cylinder is the acoustic half space illustrated in Figure 6.1 and it is subjected to an external plane wave.	154
6.3	Frequency response of shell's radial displacements (μm) at the inner surfaces of the cylinder, points B and D when the shell positioned in an acoustical half space when the shell is acted upon by an incident plane wave.	155
6.4	Shell position, loading configuration, and measurement points. (a) Half space (180-degree acoustical wedge) (b) quarter space (90-degree acoustical wedge) (c) 45-degree acoustical wedge.	156
6.5	Pressure's frequency responses at the shell's center-point O_1 , W' and farther from cylinder at F' as well as the radial displacement at D' when the shell is positioned in an acoustical half space and excited by a point load applied radially at the shell's outer surface.	157

6.6	Pressure's frequency responses at the shell's center-point O_1 , W' and farther from cylinder at F' as well as the radial displacement at D' when the shell is positioned in an acoustical quarter space and excited by a point load applied radially at the shell's outer surface.	158
6.7	Pressure's frequency responses at the shell's center-point O_1 , W' and farther from cylinder at F' as well as the radial displacement at D' when the shell is positioned in a 45-degree wedge-shaped acoustical medium and excited by a point load applied radially at the shell's outer surface.	159
6.8	Spatial distributions of Total acoustical pressure, $ p(r_1, \theta_1, z_1, t) $, at selected plane wave frequencies for a half space ($\gamma = \pi$) acoustical medium when the shell is impinged by a plane acoustical incident wave.	162
6.9	Maximum acoustical pressure within the submerging fluid, $ p_{\text{ex}}(r_1, \theta_1, z_1, t) $, when the shell is located in an acoustical half space ($\alpha = \pi$) and it is subjected to a plane acoustical incident wave with the selected frequencies.	163
6.10	Maximum hoop stress, $\sigma_{\theta\theta}$, when the shell is positioned in an acoustical half space, a quarter space and 45-degree acoustical wedge. Three surface loading configurations are considered for three graded profile of the shell's material.	165
6.11	Maximum radial stress, σ_{rr} , when the shell is positioned in an acoustical half space, a quarter space and 45-degree acoustical wedge. Three surface loading configurations are considered for three graded profile of the shell's material.	166
6.12	Center point pressure in frequency range of $1 - 1000\text{Hz}$ for the half space, quarter space, and 45 degree acoustical wedge configurations with three loading situations and three graded material profile.	167
7.1	Stroud's quadrature rule: a product rule based on weighted Gaussian quadratures, both along diameters and angular distances. The nodes are at the vertices of regular polygons.	177
7.2	Peirce's quadrature rule which is a product rule based on Gauss nodes positioned along the diameters and equispaced angles.	179
7.3	Vianello's quadrature rule which is a product rule based on Gauss-Lobatto radial abscissae and equispaced angles.	180
7.4	Bojanov's quadrature rule: a line quadrature where the lines are chosen on the zeros of Chebychev polynomials and on the lines a Gauss-Lobatto rule is chosen.	181
7.5	Hakopian's quadrature rule: a line quadrature based on interpolation in star-like points.	182
7.6	Convergence histories when applying the quadrature formulae to the polynomial test function Equation 7.17 in semi-log scale. It is reminded that S, P, V, B and H denote quadratures due to Stroud, Peirce, Vianello, Bojanov and Hakopian, respectively.	184
7.7	Convergence histories when the quadrature formulae are applied to the regular test function Equation 7.18 in semi-log scale. It is reminded that S, P, V, B and H denote quadratures due to Stroud, Peirce, Vianello, Bojanov and Hakopian, respectively.	185
7.8	Convergence histories when applying the quadrature formulae to the continuous, non-differentiable test function Equation 7.19 in semi-log scale. It is reminded that S, P, V, B and H denote quadratures due to Stroud, Peirce, Vianello, Bojanov and Hakopian, respectively.	185

7.9	Two discontinuous domains are chosen for convergence history tests having a very sharp <i>step-wise</i> jumps.	186
7.10	Convergence histories when applying the quadrature formulae to discontinuous function. In both cases we plot relative error against number of nodes in semi-log scale.	186

List of Tables

1.1	Material properties and shell's thicknesses.	18
1.2	Mathematical Expressions of excitation in time $\bar{g}(t)$, frequency $\check{g}(\omega)$ and Laplace $g(s)$ domains where $\text{erfc}(x)$ denotes the <i>complementary error function</i>	19
2.1	Material properties and geometrical specifications.	35
4.1	Material properties and nondimensionalized shell's relative thicknesses, h	80
4.2	Mathematical expressions of the Ricker pulse and explosion in time, $\bar{f}(t)$, frequency, $\check{f}(\omega)$, and Laplace, $f(s)$, domains where $\text{erfc}(x)$ denotes the <i>complementary error function</i>	81
6.1	Acoustical properties of the constituent materials.	159
6.2	Mechanical and electrical properties of the constituent materials in a non-symmetric (3D) loading situation when the shell is subjected to an oblique incident wave. . .	160
6.3	Mechanical properties of the cylinder's constituent materials for the plane strain case.	160
6.4	Mechanical and electrical properties of the constituent materials when the loading conditions leads to a plane strain situation.	161

Preface

*The mathematics of the full treatment
may be altogether beyond human power in
a reasonable time; nevertheless . . .*

Oliver Heaviside^a

^a [Heaviside, 1893, p. 324]

An overall overview of the current work along with the adapted thesis structure are given briefly here. Comprehensive elasto-acoustics investigations of submerged and buried shell structures is the main theme. The particular situations considered here have been subjected to a growing interest since the mid-twentieth century because of their extensive applications in diverse fields such as seismology, structural and nuclear engineering, petroleum industry as well as biomedical engineering in both diagnosis and treatment stages. Analytical analyses are carried out in frequency and time domains under the most generalized loading conditions and various material configurations. Major part of this study is performed within the linear domain where closed form solutions are developed. The last chapter, however, is focused on the quadrature integration over circular domains as a precursor to the nonlinear numerical analysis. Besides, these quadrature formulae are applied in the second chapter where a Hertzian load distribution over an impacted spherical shell is treated.

More than 20,000 lines of efficiently parallelized codes are developed in MATHEMATICA[®] and MATLAB environments for both stages of the main computations and post processing of the raw results, respectively. Significant amount of time is dedicated to produce high definition, *vector* quality figures and illustrations. The presented mathematical developments and associated numerical results are less than a quarter of the actual achievements during this PhD program. The remaining research works will be the subject of the future communications. An extensive list of references is included here. Book chapters, reports and articles are cited, only, if they have been actually read thoroughly and the corresponding mathematical formulations are re-derived meticulously to ensure their credibility and accuracy. Furthermore, computer codes are written to reproduce the existing pertinent numerical results based on their original solution methods in addition to current proposed approaches. The specific adapted presentation structure is explained next.

Major developments and findings are accumulated in seven chapters corresponding to four main steps. Each chapter is in the style of a stand alone journal paper which includes an abstract, a literature review, an explanation of the problem's state of art, mathematical developments, numerical results and finally the concluding remarks. Part of the mathematical developments which are common among some chapters, are given, more conveniently in the corresponding appendices. They are presented in a way to be helpful for the better understanding the solution procedures. Key findings of the presented works, brief descriptions of the works in progress as

well as the suggestions for the future researches are given at the end. A brief outline of those seven chapters is given next.

- **Chapter 1** *On transient elasto-acoustics of a spherical shell: shell theories versus exact theory of linear elasticity*

An exhaustive, comparative study is carried out in **chapter 1** on the performance of the most popular shell theories which approximate the exact local theory of linear elasticity when the shell is excited by a transient source. Effects of shell's relative wall-thickness, frequency content and the excitation's source and the mode number are studied thoroughly on the accuracy of these approximate theories. It is well demonstrated that these approximate models under a general nonstationary excitation may underestimate or overestimate the exact theory. Consequently the exact local theory of linear elasticity is recommended strongly to be used, henceforth. Besides, the main reason of using these approximations is not justifiable easily anymore, especially when current advances in the computing technology is considered.

- **Chapter 2** *Transient acoustic radiation from a spherical shell filled with and submerged in compressible fluids subjected to a centrally symmetric impact*

The comparative study in **chapter 1** is performed on individual *modal* components. Subsequently, *total* responses based on the exact linear theory are considered in when a centrally symmetric excitation is applied. A *Hertzian* distribution of contact forces rather than commonly used assumption of a uniform profile is considered in **chapter 2**. Such a situation can be a reasonably idealized model for number of significant practical configurations such as head injury and high intensity ultrasound surgery. This loading configuration leads to omission of asymmetric modes leading to a very strong focusing of acoustical energy at the center of the shell. It is demonstrated both quantitatively and qualitatively that under special arrangements, the focal pressure can be magnified, for example, by more than ten folds. Such findings convey practically significant implications in various disciplines.

- **Chapter 3 & 4** *A comprehensive, time-domain elasto-acoustics study of a fluid-filled spherical shell embedded in an elastic medium I & II: Analytical formulations and validations & Numerical results*

Subsequently, **chapter 3** and **chapter 4** are dedicated to a comprehensive investigation on the transient waves propagation through a coupled elasto-acoustic system of a fluid-filled, thick-walled spherical shell embedded in an elastic formation. The exact local theory of linear elasticity is applied to the elastic formation and shell whilst the linear theory of an acoustical phenomenon within a compressible fluid is applied. Both of generalized traction forces and waves being incident upon the shell's either surfaces are considered as the sources of excitation. Extensive sets of numerical examples are carried out by using special visualization techniques after which different limiting situations are studied.

- **Chapter 5 & 6** *A Comprehensive study on elasto-acoustic of fluid-filled anisotropic FGM cylindrical shell in an acoustical wedge with anisotropic piezo-FGM coatings I & II: Formulations & Numerical results*

The Hooke's law for an isotropic material is considered as the constitutive rule in three previous chapters. The generalized Hooke's law, on the other hand, is applied to anisotropic elastic and piezoelectric materials in [chapter 5](#) and [chapter 6](#). Analytical solutions are given a fully coupled an electro-elasto-acoustic system of a multi-layered, piezo-composite cylindrical shell filled with compressible fluid and freely submerged in an acoustical wedge medium rather than a full space fluid medium. Plane incident acoustical waves as well as traction forces on either surfaces of the shell are considered as the sources of excitation. The proposed mathematical developments are capable of treating the most general form of material anisotropy, various wedge angle configurations and different piezoelectric polarization orientations.

- **Chapter 7** *Quadrature formulae for a circular domain*

Analytical analyses for developing the closed form solutions within the linear domain are investigated in previous chapters regardless. The last chapter ([chapter 7](#)), however, is a prelude to nonlinear analysis which is motivated by finite kinematic assumptions as well as the inherent nonlinearity with constitutive laws such as elasto-plasticity, super-elasticity or discontinuities due to a strong shock. Quadrature formulae are presented and implemented for circular domains. These integration formulae are used to integrate the *weak forms* throughout circular cross-sections of cylindrical and spherical shells. These integration schemes can be used, for instance, in a mixed finite element framework or a meshless technique. The particular emphases are on the construction of quadrature rules with a prescribed exactness, having a quasi-uniform distribution of the points within the integration domain which can provide a given number of nodes on the boundary. Several schemes are investigated and numerical experiments are conducted on continuous and discontinuous functions to assess their performance.

The analytical developments given in this work can be used as a benchmark for similar situations when different numerical and experimental methods are employed. In other words, these solutions can be used to calibrate, for example, a FEM model or an experimental set-up. The applications are quite broad: ranging from bio-medicine to NDT (Non-destructive Testing) and seismology. An appropriate output format of these formulae can be used depending on the measured variable. Few common practical examples are as the following. The displacement response is needed to be calculated if geo-phones or laser measurement devices are used. Furthermore, the surface strain or stress responses can be utilized if a conventional ultrasonic probe is used to pick up the system's response. Moreover, the pressure response must be considered if the pulses are recorded by using hydrophones.

The current extensive numerical experimentation indicates significant practical implications in various situations. For instance, special geometrical and material configurations have demonstrated a significantly high level of focusing capability of the acoustical energy of an incident wave which can be used in the design of the acoustical and ultrasonic focal lenses. On the other hand, different kind of arrangement have shown to be highly dispersive which can be used in the design of shock resistant structures.

The outcomes of this research have been disseminated as follows.

- Bahari A., Popplewell N. (2015). Comment on "Transient response of an acoustic medium

by an excited submerged spherical shell". The Journal of the Acoustical Society of America, 137(5), 2966-2969.

- Bahari A., Lefeuvre-Mesgouez G., Mesgouez A., and Popplewell N. (2016). Transient Response of a Fluid-Filled, Thick-Walled Spherical Shell Embedded in an Elastic Medium. E3S (Vol. 12, p. 06004). EDP Sciences.
- Bahari A., Popplewell N., Karimi N., On transient elasto-acoustics of a spherical shell: shell theories versus exact theory of linear elasticity, Journal of Sound and Vibration, Under review.
- Bahari A., Lefeuvre-Mesgouez G., Mesgouez A., and Popplewell N., A comprehensive, time-domain elasto-acoustics study of a fluid-filled spherical shell embedded in an elastic medium I: Analytical formulations and validations, Soil Dynamics and Earthquake Engineering, The revised version is submitted.
- Bahari A., Lefeuvre-Mesgouez G., Mesgouez A., and Popplewell N., A comprehensive, time-domain elasto-acoustics study of a fluid filled spherical shell embedded in an elastic medium II: Numerical results, Soil Dynamics and Earthquake Engineering, The revised version is submitted.
- Bahari A., Rabbani V. and Popplewell N., A Comprehensive Study on Elasto-Acoustic of Fluid-Filled Anisotropic FGM Cylindrical Shell in an Acoustical Wedge with Anisotropic Piezo-FGM Coatings I: Formulations, Submitted to Journal of Sound and Vibration, Under Review.
- Bahari A., Rabbani V. and Popplewell N., Elasto-Acoustics of a Fluid-Filled, Anisotropic FGM Cylindrical Shell in an Acoustical Wedge having Anisotropic Piezo-FGM Coatings II: Numerical Results, Submitted to JSV, Under Review.
- Bahari A., Calabrò F., and Auricchio F., Quadrature Formulae For a Circular Domain, Journal of Applied Mathematical Modelling, Under Review.
- Bahari A., Karimi N., and Popplewell N., Transient response a spherical shell filled with and submerged in compressible fluids subjected to a centrally symmetric excitation, Submitted to ASME Journal of Vibration and Acoustics.

Hence that the volume of the presented numerical results and the level of the details in theoretical developments are more extensive in the thesis compared to the corresponding submitted papers.

Chapter 1

Transient Elasto-Acoustics of A Spherical Shell, Exact Theory of Linear Elasticity versus Shell Theories

Transient radiation and scattering of acoustic waves are considered from a spherical shell subjected to an incident acoustical perturbation or traction force(s) on either of its surfaces. The shell is submerged fully and filled with compressible fluids which are both assumed to be inviscid. An exact linear acoustic wave equation is employed to describe isentropic, isothermal perturbations of an acoustic phenomenon. The elastic shell, on the other hand, is modelled by taking two different categories of kinematic assumptions whereas the generalized Hooke's law for an isotropic elastic medium is applied regardless. A linearized Green-Lagrange (GL) strain tensor is employed first as the exact kinematic description of the local theory of linear elasticity. This exact solution is taken as the reference. Then the two most commonly used shell models (known as *four-* and *five-* mode shell theories) are considered. Their kinematic assumptions are based on midsurface displacements where the radial strain and stresses are neglected. The Green-Lagrange kinematic assumption leads to the *Navier* equation where a *Helmholtz decomposition* is utilized to describe the displacement field from a set of decoupled compressional and shear wave equations. On the other hand, shell theories are applied directly through the decoupled equation of motion which depends only on the radial element of shell's midsurface displacement. Modal expansions of all the field variables are undertaken in terms of *Fourier-Legendre* polynomials. The corresponding unknown modal coefficients are found by satisfying continuity boundary conditions in the exact linear theory within the Laplace space. However the implementation of continuities of the fluid fields and elastic shell is undertaken differently. A modification is proposed here to standard practice by considering the pressure difference between the shell's inner and outer surfaces rather than at the shell's midsurface. Then transient, modal components of the fields' responses are retrieved by applying an appropriate Laplace inversion procedure. The validity and robustness of the current solutions and their associated computer codes are checked against available data from the literature. Numerical cases are selected carefully to examine thoroughly the performance of the shell theory approximations against the exact linear theory. The impact of different parameters are investigated within the internal and external fluids and the shell's material for different excitations and their sources, shell thicknesses and mode numbers. In general, shell theories may overestimate or underestimate unpredictably.

1.1 Introduction

Transient interactions of an acoustical medium coupled with an elastic structure exposed to a mechanical or acoustical excitation, have attracted great attention from the middle of the twentieth century. See, for instance, [Perlman, 1941, DuMond et al., 1946, Primakoff and Keller,

1947, Arons and Yennie, 1948, Anderson, 1955, Payton, 1960, Huang, 1969, Geers, 1972, Gaunaud and Wertman, 1991, Huang and Mair, 1996, Zakout, 2001, Leblond and Sigrist, 2010, Hasheminejad et al., 2011]. This extensive attention has been created by the following problems' widespread practical applications.

- *Diagnostic ultrasound* uses such a solution as an accurate forward kernel in a wave field inversion scheme for imaging cysts and other anomalies ([Grandjean and Leparoux, 2004, Wirgin, 2016]).
- *Nondestructive testing* in ultrasonic immersion testing of submerged structures such as offshore oil and gas containers ([Drinkwater and Wilcox, 2006, Krautkrämer and Krautkrämer, 2013]).
- *Underwater acoustics* and target identifications used in active and passive sonars ([Weston, 1959, Gaunaud and Wertman, 1991, Gaunaud et al., 1995, Bucaro et al., 2008, Fischell and Schmidt, 2015]).
- Interactions of a *shock wave* and shell structures are needed to be considered for safe designs ([Geers, 1972, Sprague and Geers, 1999, Leblond and Sigrist, 2010]).
- *Head injuries* are studied by approximating them as a doubly impacted, fluid filled shell ([Engin, 1969b, Firoozbakhsh and DeSilva, 1975, Bandak, 1997, Zou and Schmiedeler, 2008]).

A numerically efficient approach in modelling such interactions, however, remains problematic. See, for example, [Leblond and Sigrist, 2010, Hasheminejad et al., 2011, Bahari and Popplewell, 2015]. Theoretical investigations of the transient elasto-acoustic behaviour of a spherical shell can be categorized broadly as numerical and analytical studies. Specific numerical methods have been developed for this type of interactions in preference to conventional FEM and BEM approaches. For example, the DAA (Doubly Asymptotic Approximation) was proposed and refined by [Geers, 1972, Geers, 1978, Geers and Zhang, 1994, Sprague and Geers, 1999]. Alternatively [Leblond and Sigrist, 2010] applied a boundary integral approach in conjunction with a shell's *in-vacuo* eigenvectors. This review, however, is restricted to analytical or semi-analytical solutions which have been proposed mostly with the use of Laplace or Fourier transforms. Analytical inversions of the transformed solutions are usually challenging and they have been limited to relatively simple, approximate shell theories. Three such models have been used to describe these coupled elasto-acoustics interactions. They are known as “three-”, “four-” and “five-” mode theories which are distinguished here by 3M, 4M and 5M. See, for instance, [Leissa, 1973, Qatu, 2004]. These particular names relate to the number of independent kinematic variables required to describe the strain fields in a nonaxisymmetric situation. They should not be mistaken as conventional vibration modes. [Lamb, 1882] initially proposed the 3M shell theory for a closed spherical shell in which solely a membrane effect was incorporated. Half a century later, the 4M model was developed by [Federhofer, 1937] based on the general theory of [Love, 1920] by also including bending moments in addition to membrane forces. Subsequently, [Prasad, 1964] proposed a 5M theory by accommodating an additional shear effect in the form of an extra kinematic, rotatory variable. The corresponding general theory was given earlier in [Naghdi, 1962]. Shortly afterwards, [Wilkinson and Kalnins, 1965] improved this 5M model by including rotary inertial coupling terms. This

approach is similar that of [Naghdi and Cooper, 1956] for cylindrical shells. A brief review of the transient elasto-acoustic analyses of a spherical shell is given next in which all these approximate shell theories are outlined.

[Milenkovic and Raynor, 1966] investigated the scattering of a transient plane wave incident on a spherical shell by using the 3M theory. This paper is one of the few studies in which the 3M approach was used to investigate the nonstationary, elasto-acoustics of a spherical shell. [Diercks and Hickling, 1967] is among the earliest authors who utilized the 4M theory. He modelled an incident pressure pulse acting upon a submerged, spherical shell filled with fluid. The temporal response was estimated by integrating the system's harmonic response over a range of frequencies. Huang and colleagues ([Huang, 1969, Hsu and Dranoff, 1987, Huang and Mair, 1996]) also investigated the transient interaction of incident pressure waves with a submerged spherical shell by employing the 4M model. A Laplace transform was used to find the exact modal displacements created by an incident step pulse as well as an exponentially decaying wave. However a Volterra integral was needed to numerically compute the modal pressure histories. This work was extended to include a spherical rather than the former planar front of an incident pressure wave [Huang et al., 1971]. The scattering of a transient, incident pressure pulse from a system of two concentric, spherical shells was considered subsequently in [Huang, 1979]. [Tang and Yen, 1970] considered a similar configuration as [Huang, 1969] but used the 5M theory of [Prasad, 1964]. An early time, asymptotic solution was developed for a plane acoustical step pulse by using the Watson transform. [Engin, 1969a] investigated the dynamic response of a fluid filled, spherical shell subjected to a local radial impulse to represent an impacted human head. A 4M model was utilized to determine the axisymmetric stress and strain fields by using a Laplace transform approach. [Lou and Klosner, 1973] also employed the 4M theory to study the radiated pressure from an empty spherical shell excited by a Heaviside or Delta Dirac impulse. They compared their results with both plane and spherical wave approximations. Later, [Menton and Magrab, 1974] applied a Fourier transform to find the transient, far field pressure radiation from a submerged, fluid filled spherical shell by using the 5M model. Variable frequency increments were used to numerically invert the Fourier transform. [Zhang and Geers, 1993] considered a fluid filled, spherical shell subjected to an incident, step plane wave. The 4M theory was applied with a residual potential approach to retrieve the temporal response. [Chen and Stepanishen, 1994] proposed a generalized approach for thin shells of revolution by using in-vacuo, modal eigenvectors having time dependent coefficients. They analyzed the transient radiation from a fluid loaded, empty spherical shell subjected to an axisymmetric impulse by using the 4M model. Later, [Zakout, 2001] gave an interesting closed form reformulation of the work of [Lou and Klosner, 1973] for a distributed Heaviside excitation. several discrepancies in the numerical results given in [Zakout, 2001] were rectified by [Bahari and Popplewell, 2015]. They also computed the impulse response in the time domain of the radiated field in terms of a Green function.

[Hasheminejad et al., 2011] considered the nonstationary scattering of an acoustical wave by a submerged spherical shell filled with fluid. They applied the exact theory of linear elasticity in conjunction with a Helmholtz decomposition. [Hasheminejad and Mojahed, 2016] analyzed, more recently, the transient acoustical radiation from a hollow sphere located in an acoustical wedge. However, an approximate *transfer matrix* rather than an exact approach was taken. Except for two more recent publications of [Hasheminejad et al., 2011] as well as [Hasheminejad and Mojahed,

2016], a spherical shell's transient response has been investigated by using approximate shell theories rather than the exact kinematic assumptions of the local theory of linear elasticity. Although these approximations are more computationally efficient, they are most likely to be less accurate. However a comparative study has to be performed still because of the continuous, extensive use of the shell theory approximations. Therefore the main objective here is to reassess comprehensively the accuracy ranges of shell theories given today's significantly improved computer technology. Specific highlights and novelties of the current study are listed next.

- Truly comprehensive, comparative evaluations are performed on the performance of approximate shell theories for transient elasto-acoustic interactions.
- The transient acoustical radiation from a spherical shell caused by traction forces is described by employing exact elastic wave equations and solved by using the Helmholtz decomposition.
- The scattering of nonstationary acoustical waves arising from a spherical shell filled with a compressible fluid are evaluated by employing the refined, 5M theory of [Wilkinson and Kalnins, 1965].
- The kinematic continuity of the radial displacements between the interacting elastic shell and both fluid fields is enforced at both surfaces of the shell rather than artificially at its midsurface.
- Universal solutions are given for both the 4M and 5M models and, separately, for the GL approach when mechanical or acoustical excitations are applied.
- The singular point which occurs at the sphere's center and within the encapsulated fluid is accommodated.

The organization of this paper is outlined next. The following section, which gives the mathematical developments, starts with the problem's description followed by the rationale for selecting the particular Laplace inversion procedure which was adopted. Two types of excitation sources are described mathematically subsequently. They include mechanical traction forces as well as incident acoustical perturbations. Acoustical phenomena generated within the internal and external fluid fields are discussed afterwards. The elastic shell domain and its interaction with the fluid media is described first by using the exact local theory of linear elasticity. The shell's dynamics which includes its interactions with fluid fields are described later by using the shell theories. Exact and approximate approaches are compared and fundamental differences are noted. The validity of the all the solutions found currently and the associated computer codes is checked rigorously with published data. Then a truly exhaustive set of carefully selected numerical cases are evaluated. The results presented here are a tiny fraction of those produced and not included here due to space limitations. On the other hand, numerical results are given for four shell wall thicknesses, three shell materials, two types of enclosed fluids, six excitation functions having different frequency spectra and three different sources. The latter category includes two mechanical traction forces, one at each of the two shell's surfaces, and an incident plane wave. Time histories, which include modal pressures and particle displacements, are calculated for the first forty modes which is more extensive than previously.

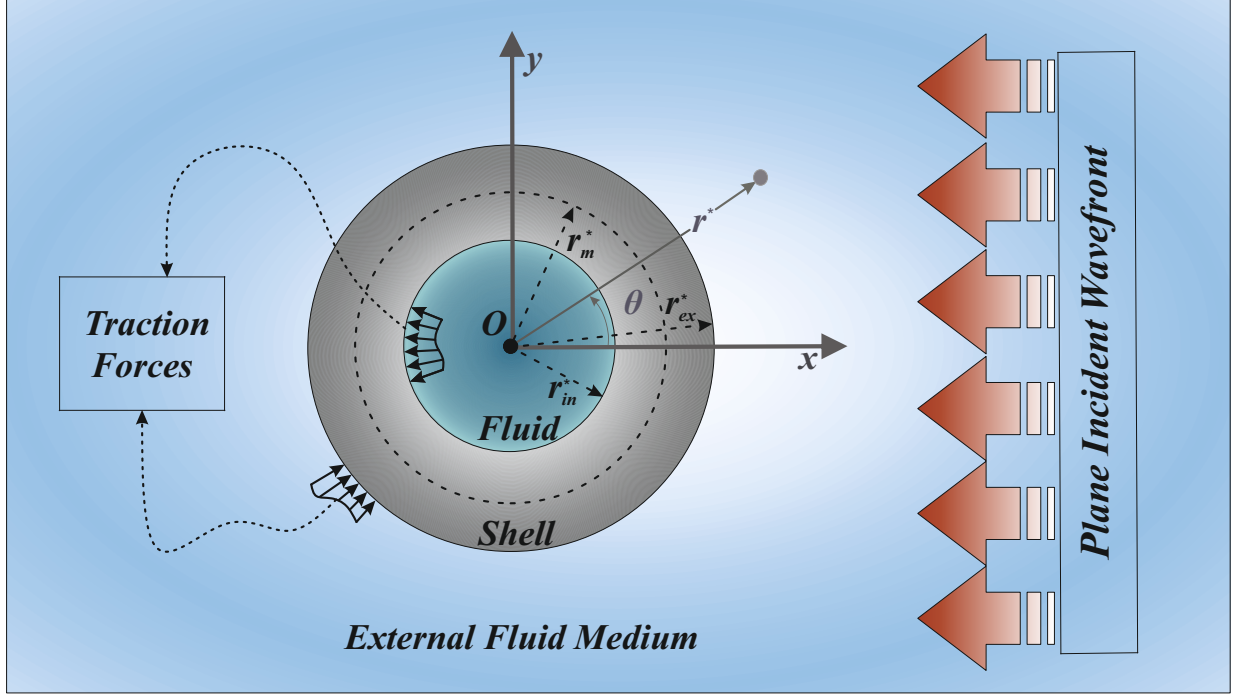


Figure 1.1: Showing the problem's geometry and different excitation sources including traction force(s) and plane incident waves. Azimuthal angle φ is perpendicular to the illustrated plane.

1.2 Mathematical Formulations

The choice of an efficient Laplace inversion procedure is described in this section after which the adopted conventions are given. Then a linear acoustical phenomenon is considered that can occur within each of the fluid media. Subsequently, excitations are introduced in the forms of surface traction forces and acoustical incident waves. Finally, a shell's dynamic responses to each of these excitations are described by using two different approaches, the exact theory of linear elasticity and the class of approximate shell theories.

1.2.1 Problem's Definition

Assume that a spherical shell (denoted by subscript s) is filled internally (in) with a fluid and it is submerged externally (ex) in the same or a different fluid. The fluids are considered to be inviscid but isentropically compressible. The shell itself is subjected to a surface traction force or an incident plane wave both of which are transient. These loading configurations are illustrated in [Figure 1.1](#) along with the shell's dimensions. Excitation amplitudes are always sufficiently weak that the assumption of linearity is reasonable. Consequently wave phenomena in either fluid medium is characterized by an unperturbed mass density and sound speed which are denoted, respectively, by ρ_f^* and c_f^* where subscript f is in or ex . The elastic shell has dimensional inner (outer) radius of r_{in}^* (r_{ex}^*) and it is assumed to be homogeneous as well as isotropic. Its mass density and Lamé constants are ρ_s^* and μ^*, λ^* , respectively. These and all the other parameters are nondimensionalized appropriately. Lengths, which include the displacements and distances from the sphere's center, are taken with respect to r_{ex}^* . Furthermore, stresses and pressures are given relative to $\{\sigma_0^* = \rho_{ex}^* c_{ex}^{*2}\}$ whereas time t^* is nondimensionalized with respect to $\{t_0^* = r_{ex}^* / c_{ex}^*\}$.

The following dimensionless variables (which have no star superscripts) are used throughout

$$t = \frac{t^*}{t_0^*}, r = \frac{r^*}{r_{ex}^*}, \alpha = \frac{c_{ex}^*}{c^*}, \rho = \frac{\rho_{ex}^*}{\rho^*}, p = \frac{p^*}{\sigma_0^*}, \sigma_{ij} = \frac{\sigma_{ij}^*}{\sigma_0^*}, u_i = \frac{u_i^*}{r_{ex}^*}, v_i = \frac{v_i^* t_0^*}{r_{ex}^*} \quad (1.1)$$

where $i, j = r, \theta$ and φ . The t^*, r^*, v_i^* and p^* denote the real time, radial coordinate, fluid's particle velocity and pressure whilst u_i^* and σ_{ij}^* refer to displacement and stress components, respectively. Details of the algorithm used to numerically invert the Laplace transform are presented next.

1.2.2 Laplace Transform

The pertinent literature indicated that the Laplace transform is a versatile tool in analytically treating transient wave interactions with a structure. For example, [Huang, 1969], [Zhang and Geers, 1993], [Huang and Mair, 1996] as well as [Bahari and Popplewell, 2015] employed this approach to solve the transient acoustical scattering or radiation from a spherical shell. Consequently the one-sided Laplace transform is applied here to all temporal quantities. A Laplace transform for an arbitrary function, $\tilde{\Lambda}(t)$, exists if $|\tilde{\Lambda}(t)| < +\infty$ as $t \rightarrow +\infty$. This transform and its complex inversion are defined, respectively, as

$$\Lambda(s) = \int_0^\infty \tilde{\Lambda}(t) e^{-s t} dt \quad \text{and} \quad \tilde{\Lambda}(t) = \frac{1}{2\pi i} \lim_{\eta \rightarrow \infty} \int_{\gamma - i\eta}^{\gamma + i\eta} \Lambda(s) e^{s t} ds. \quad (1.2)$$

where η is an arbitrary real number which is greater than the real parts of all the singularities of $\Lambda(s)$. The analytical inversion (in the context of the current problem) has been limited previously to simplified shell models as in [Huang, 1969], [Tang and Yen, 1970] and [Bahari and Popplewell, 2015]. However [Hasheminejad et al., 2011] and [Hasheminejad and Mojahed, 2016] concluded that it is essentially impossible for more realistic models to perform an analytical inversion so that a numerical inversion algorithm is required. Of the many numerical methods suggested, the one proposed by [Durbin, 1974] has been selected here because it was shown by [Naeni et al., 2015] to be stable and reasonably accurate. This algorithm uses a Fourier series based approach in conjunction with a trapezoidal integration rule. Then the exact inversion integration in Equation 1.2 becomes

$$\tilde{\Lambda}(t) = \lim_{N_d \rightarrow \infty} \Lambda_{N_d}(t, a, N_d, T_0) + \sum_{j=1}^{\infty} e^{-2j(a T_0)} \tilde{\Lambda}(2jT_0 + t), \quad t \in [0, 2T_0] \quad (1.3)$$

in which the second term on the right is the discretization error. The N_d , a , and T_0 are the truncation number, a free parameter (to be assigned) and an arbitrary time instant, respectively. The main deficiency of all Fourier series based approaches, however, is their slow convergence and the presence of nonphysical oscillations known as the Gibb's phenomenon. However a filter $\varrho(j)$ can be applied to mitigate these difficulties. Subsequently the truncated estimate, $\bar{\Lambda}_{N_d}(t)$, of the inversion $\tilde{\Lambda}(t)$ becomes

$$\bar{\Lambda}_{N_d} = \frac{e^{at}}{T_0} \left(\frac{\Lambda(a)}{2} + \sum_{j=1}^{N_d} \varrho(j) \left[\Re\{\tilde{\Lambda}_j(t)\} \cos b_j(t) + \Im\{\tilde{\Lambda}_j(t)\} \sin b_j(t) \right] \right) \quad (1.4)$$

where

$$\bar{\vartheta}_j(t) = \Lambda [a + i b_j(t)], \quad b_j(t) = \frac{j\pi}{T_0}t, \quad \varrho(j) = \sin\left(\frac{j\pi}{N_d}\right) / \left(\frac{j\pi}{N_d}\right) \quad \text{and} \quad t \in [0, 2T_0].$$

The $\varrho(j)$ is the filter proposed by *Lanzcos*. As discussed by [Honig and Hirdes, 1984] the optimal choice of the free parameter $\{a T_0\}$ occurs when the absolute values of the discretization and truncation errors are equal. [Durbin, 1974, Honig and Hirdes, 1984, Valkó and Abate, 2004] suggested $5 \leq a T_0 \leq 10$. The choice of $5 \leq er \leq 6$ meets this criterion and provided reasonable results in our preliminary tests. The different excitation sources considered are formulated next.

1.2.3 Excitation Sources

Two kinds of excitation sources are considered. The first kind is a traction force which is normal to either the shell's inner or outer surface; the other is a travelling plane wave which impinges on the shell's outer surface. Regardless, the temporal origin is adjusted so that it always coincides with the initial interaction between an excitation and the shell. The general vector form of a traction force is

$$\tilde{\mathbf{f}}^i \equiv \tilde{\mathbf{f}}^i(\theta, \varphi, t) = \tilde{f}_r^i(\theta, \varphi, t)\mathbf{e}_r + \tilde{f}_\theta^i(\theta, \varphi, t)\mathbf{e}_\theta + \tilde{f}_\varphi^i(\theta, \varphi, t)\mathbf{e}_\varphi \quad (1.5)$$

However, if the traction force is axisymmetric as well as normal to a shell's surface, Equation 1.5 simplifies to

$$\tilde{\mathbf{f}}^i \equiv \tilde{\mathbf{f}}^i(\theta, t) = \tilde{f}_r^i(\theta, t)\mathbf{e}_r \quad (1.6)$$

which is independent of the azimuth coordinate (φ). As the traction force is presumed to be time and spatially separable, the radial traction force can be expressed as

$$\tilde{f}_r^i(\theta, t) = \eta^i(\theta)\tilde{g}(t), \quad i = in, ex. \quad (1.7)$$

The $\eta^i(\theta)$ in this equation can be expanded in terms of the orthogonal Legendre polynomials $P_n(\cos \theta)$ as

$$\eta^i(\theta) = \sum_{n=0}^{\infty} \eta_n^i P_n(\cos \theta) \quad (1.8)$$

where η_n^i is the n^{th} modal component of $\eta^i(\theta)$ that is given by

$$\eta_n^i = \frac{2n+1}{2} \int_0^\pi \eta^i(\theta) P_n(\cos \theta) \sin \theta d\theta. \quad (1.9)$$

By combining Equation 1.7 through Equation 1.9, the Laplace transform of $\tilde{f}_r^i(\theta, t)$ becomes

$$f_r^i(\theta, s) = \sum_{n=0}^{\infty} \xi_n^i P_n(\cos \theta), \quad \text{where} \quad \xi_n^i = g(s)\eta_n^i. \quad (1.10)$$

The ξ_n^i is the n^{th} modal component of $\tilde{f}_r^i(\theta, t)$. The other excitation source, an incident plane wave, is considered next. It takes the mathematical form

$$\tilde{P}_r^{inc}(r, \theta, t) = \sigma_0 \tilde{g}(r \cos \theta - 1 + t) \mathcal{H}(r \cos \theta - 1 + t) \quad (1.11)$$

where $\mathcal{H}(t)$ is the Heaviside unit step function. The $\tilde{P}_r^{inc}(r, \theta, t)$ can be expressed in terms of spherical harmonics by using the translational addition theorem detailed in [Hasheminejad et al., 2011]. Then

$$p^{inc}(r, \theta, s) = \sum_{n=0}^{\infty} p_n^{inc}(r, s) P_n(\cos \theta) \quad (1.12)$$

where

$$p_n^{inc}(r, s) = (2n + 1)g(s)e^s i_n(rs) \quad (1.13)$$

and $i_n(rs)$ is the modified spherical Bessel function of the first kind. Of course, the superposition principle can be applied straightforwardly if multiple excitations happen simultaneously. The total pressure at any point in the fluid medium surrounding the sphere is the sum of the incident component, given by Equation 1.12, and a scattered component. The scattered component as well as the pressure radiated from the shell into the internal fluid are described next.

1.2.4 Fluid Media

The acoustical pressure perturbation, $\tilde{p}(r, \theta, t)$, generated within an internal or external fluid is governed by Equation A.3 which is given more conveniently in Appendix A. The Fourier-Legendre expansion of this acoustical pressure in the spherical coordinate system illustrated in Figure 1.1 becomes

$$\tilde{p}(r, \theta, t) = \sum_{n=0}^{\infty} \tilde{p}_n(r, t) P_n(\cos \theta) \quad (1.14)$$

Inserting Equation 1.14 into Equation A.3 leads to

$$\frac{d^2 p_n}{dr^2} + \frac{2}{r} \frac{dp_n}{dr} - \frac{n(n+1)}{r^2} p_n = \alpha_i^2 s^2 p_n, \quad i = in, inc, sct, ex \quad (1.15)$$

in the Laplace domain where $p_n \equiv p_n(r, s)$ is the n^{th} modal component of the acoustical pressure. Solutions of Equation 1.15 must be bounded as (i) $r \rightarrow \infty$ for the scattered or radiated pressure field in the fluid surrounding the shell (p^{sct}) and (ii) as $r \rightarrow 0$ for the radiated or refracted pressure into the contained fluid (p^{in}). See, for example, [Junger and Feit, 1986] as well as [Bahari and Popplewell, 2015]. Consequently

$$p_n^{in}(r, s) = A_n(s) i_n(\alpha_{in} rs) \quad \text{and} \quad p_n^{sct}(r, s) = B_n(s) k_n(rs) \quad (1.16)$$

in which the $i_n(x)$ and $k_n(x)$ are the modified spherical Bessel functions of the first and second kinds, respectively. The $A_n(s)$ and $B_n(s)$ are the unknown n^{th} modal coefficients which are determined when the required boundary conditions are specified and satisfied. The total acoustical

pressure in the fluid surrounding the shell, p^{ex} , which arises from an incident wave, is the sum of the pressures generated by the incident and scattered waves. They are given by Equation 1.13 and Equation 1.16, respectively, so that

$$p_n^{ex}(r, s) = B_n(s) k_n(\alpha_{in}rs) + (2n + 1)g(s) e^s i_n(rs). \quad (1.17)$$

The corresponding fluid's particle velocities, which are represented by vector $\mathbf{v}$, are determined in the Laplace domain by substituting the acoustical pressure found from Equation 1.16 and Equation 1.17 into Equation A.1b. Then the nondimensional radial component of $\mathbf{v}$ within the internal and external fluids, respectively, become

$$v_{r,n}^{in} = -A_n(s) \frac{\rho_{in} \alpha_{in}}{s} \left[i_{n+1}(\alpha_{in}rs) + \frac{n}{\alpha_{in}rs} i_n(\alpha_{in}rs) \right] \quad (1.18)$$

and

$$v_{r,n}^{ex} = B_n(s) \frac{1}{s} \left[-k_{n+1}(rs) + \frac{n}{rs} k_n(rs) \right] + v_{r,n}^{inc} \quad (1.19a)$$

where

$$v_{r,n}^{inc} = -(2n + 1)g(s) \frac{e^s}{s} [i_{n+1}(rs) + \frac{n}{rs} i_n(rs)]. \quad (1.19b)$$

The modal pressure and velocity of the hypothetical internal fluid particle located at the center of the shell can be shown by using Equation 1.16 to become indeterminate. This singularity is treated separately for $n = 0$ and $n > 0$. The well known *l'Hôpital's* rule is applied once for $n = 0$ as $r \rightarrow 0$ which leads to

$$p_0^{in}(r, s)|_{r=0} = \lim_{r \rightarrow 0} A_n(s) \frac{\sinh(\alpha_{in}rs)}{\alpha_{in}rs} \equiv A_n(s) \quad (1.20a)$$

and

$$\begin{aligned} v_0^{in}(r, s)|_{r=0} &= \lim_{r \rightarrow 0} A_n(s) \rho_{in} \frac{-rs\alpha_{in} \cosh(\alpha_{in}rs) + \sinh(\alpha_{in}rs)}{\alpha_{in}(rs)^2} \\ &\equiv -\frac{A_n(s)}{2} \rho_{in} \lim_{r \rightarrow 0} \sinh(\alpha_{in}rs) = 0. \end{aligned} \quad (1.20b)$$

By using the limit form of $i_n(z)$ given in §10.52.1 of [Olver, 2010] when $n > 0$, the pressure and particle velocity the sphere's center respectively become

$$p_n^{in}(r, s)|_{r=0}^{n>1} = \lim_{r \rightarrow 0} A_n(s) i_n(\alpha_{in}rs) \equiv A_n(s) \lim_{r \rightarrow 0} \frac{(\alpha_{in}rs)^n}{(2n + 1)!!} = 0 \quad (1.21a)$$

and

$$\begin{aligned}
 v_n^{in}(r, s)|_{r=0}^{n>1} &\equiv -\lim_{r \rightarrow 0} A_n(s) \rho_{in} \frac{n i_n(\alpha_{in} r s) + \alpha_{in} r s i_{n+1}(\alpha_{in} r s)}{r s} \\
 &\equiv -A_n(s) \rho_{in} \alpha_{in} \lim_{r \rightarrow 0} (r s \alpha_{in})^{n-1} \left(\frac{n}{(2n+1)!!} + \frac{(r s \alpha_{in})^2}{(2n+3)!!} \right) \\
 &= 0.
 \end{aligned} \tag{1.21b}$$

An immediate physical interpretation of Equation 1.20a through Equation 1.21a is that the breathing mode ($n = 0$) is the only nonzero modal contribution at the center of the shell. This observation is expected due the complete symmetry about this center. Couplings of the fluid media to the elastic shell are addressed next.

1.2.5 Elastic Shell

The dynamic response of a shell can be described by the exact theory of linear elasticity or by approximate shell theories. Each of these approaches involves different kinematic assumptions. Furthermore they each require an individual treatment to define the fluid-shell interactions.

1.2.5.1 Kinematic Assumptions: Green-Lagrange Strain Tensor

The *equation of motion* ($\mathcal{EOM}$) of an elastic body which is based on the local theory of linear elasticity is described by using the exact Green-Lagrange (GL) kinematic assumptions. It was formulated first by Navier. However the derivation is outlined for completeness in Equation B.1b through Equation B.2. For zero initial conditions and no body force, the Navier equation is

$$\mathbf{L} \mathbf{u} = \mathbf{0} \quad \text{where} \quad \mathbf{L} = -\mu \nabla \times \nabla \times + (\lambda + 2\mu) \nabla \nabla \cdot - \rho_s s^2 \tag{1.22}$$

in the Laplace domain. In spherical coordinates

$$\mathbf{u} \equiv \mathbf{u}(r, \theta, \varphi, s) \quad \text{and} \quad \nabla \equiv \frac{\partial}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \mathbf{e}_\varphi$$

are the displacement vector and gradient operator, respectively. The $\mathbf{e}_r, \mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ are the unit vectors in the radial, angular and meridional directions, respectively. The solution of Equation 1.22 is reduced to finding the three dimensionless displacement potentials, $\phi(r, \theta, \varphi, s)$, $\psi(r, \theta, \varphi, s)$ and $\chi(r, \theta, \varphi, s)$ by using the Helmholtz decomposition given explicitly in Equation B.3. These potentials are governed by

$$\nabla^2 \phi = s^s \alpha_p^2 \phi, \quad \nabla^2 \psi = s^s \alpha_s^2 \psi \quad \text{and} \quad \nabla^2 \chi = s^s \alpha_s^2 \chi \tag{1.23}$$

where $\alpha_p = c_{ex}/c_p$ and $\alpha_s = c_{ex}/c_s$ whereas $c_p = \sqrt{(\lambda + 2\mu)/\rho_s}$ and $c_s = \sqrt{\mu/\rho_s}$ are the compressional and shear wave velocities, respectively. For a configuration having axisymmetric, non-torsional traction boundary conditions, these displacement potentials are independent of the azimuth coordinate, φ , so that $\chi = 0$. The two remaining wave potentials are expressed in the

following Fourier-Legendre series

$$\phi(r, \theta, s) = \sum_{n=0}^{\infty} \phi_n(r, s) P_n(\cos \theta) \quad \text{and} \quad \psi(r, \theta, s) = \sum_{n=0}^{\infty} \psi_n(r, s) P_n(\cos \theta). \quad (1.24)$$

The substitution of the ϕ and ψ from Equation 1.23 into Equation 2.22 leads to

$$\frac{d^2 \varpi_n}{dr^2} + \frac{2}{r} \frac{d\varpi_n}{dr} - \frac{n(n+1)}{r^2} \varpi_n = s^2 \alpha^2 \varpi_n \quad (1.25)$$

where ϖ_n and α are respectively the abbreviated versions, for convenience, of ϕ_n and α_l as well as ψ_n and α_s . Then the succinctly written solution of Equation 1.25 is

$$\begin{Bmatrix} \phi_n(r, s) \\ \psi_n(r, s) \end{Bmatrix} = \begin{Bmatrix} C_n(s) i_n(\alpha_l r s) + D_n(s) k_n(\alpha_l r s) \\ E_n(s) i_n(\alpha_s r s) + F_n(s) k_n(\alpha_s r s) \end{Bmatrix}. \quad (1.26)$$

Six independent boundary conditions (BCs) are required to find the six unknown coefficients $A_n(s)$ through $F_n(s)$ used in Equation 1.16 and Equation 1.26. The BCs are formulated most readily by employing the continuity of two sets of vectors which need to be evaluated at both surfaces of the shell. One vector relates to the *traction stress*, $\mathbf{t}$, and the other vector to the velocity $\mathbf{v} \equiv \mathbf{s}\mathbf{u}$ in the Laplace domain. Vector $\mathbf{t}$ is determined by projecting the Cauchy-Green stress *tensor*, $\boldsymbol{\sigma}$, in the direction of $\mathbf{e}_n$ - the unit vector normal to a given traction surface. Consequently

$$\mathbf{t}^{(i)} \equiv \boldsymbol{\sigma} \cdot \mathbf{e}_n, \quad i = in, ex. \quad (1.27)$$

As the centres of the spherical coordinate frame and shell coincide, $\mathbf{e}_n \equiv \mathbf{e}_r$ so that

$$\mathbf{t}^{(i)} = \sigma_{rr} \mathbf{e}_r + \sigma_{r\theta} \mathbf{e}_\theta + \sigma_{r\phi} \mathbf{e}_\phi. \quad (1.28)$$

In an axisymmetric configuration, however, the normal component of $\mathbf{t}^{(i)}$ (i.e. σ_{rr}) and only one shear component, $\sigma_{r\theta}$ or $\sigma_{r\phi}$, are needed, in addition to the radial component of the surface velocity, v_r , to satisfy the BCs. By substituting Equation 1.26 into Equation B.8 it follows that

$$\begin{Bmatrix} v_r(r, \theta, s) \\ \sigma_{rr}(r, \theta, s) \\ \sigma_{r\theta}(r, \theta, s) \end{Bmatrix} = \sum_{n=0}^{\infty} \begin{Bmatrix} v_n^r(r, s) \\ \sigma_n^{rr}(r, s) \\ \sigma_n^{r\theta}(r, s) \frac{\partial}{\partial \theta} \end{Bmatrix} P_n(\cos \theta). \quad (1.29)$$

Corresponding expressions for the modal components $v_n^r \equiv s u_n^r(r, s)$, $\sigma_n^{rr} \equiv \sigma_n^{rr}(r, s)$ and $\sigma_n^{r\theta} \equiv \sigma_n^{r\theta}(r, s)$ are detailed in Equation B.9 of Appendix B. As the fluid is assumed to be inviscid, both the shear stresses, $\sigma_{r\theta}$ and $\sigma_{r\phi}$, vanish at the traction surfaces. Consequently three independent

scalar equations of continuity are produced at each traction surface so that

$$\begin{pmatrix} \sigma_{rr}|_{r=1} + p^{sct}|_{r=1} \\ \sigma_{rr}|_{r=r_{in}} + p^{in}|_{r=r_{in}} \\ \sigma_{r\theta}|_{r=1} \\ \sigma_{r\theta}|_{r=r_{in}} \\ v_r^{sct}|_{r=1} \\ v_r^{in}|_{r=r_{in}} \end{pmatrix} = \begin{pmatrix} -p^{inc}|_{r=1} + f_r^{ex} \\ -f_r^{in} \\ 0 \\ 0 \\ -v_r^{inc}|_{r=1} + s u|_{r=1} \\ \rho_{in} s u|_{r=r_{in}} \end{pmatrix} \quad (1.30)$$

A fluid's acoustical pressure p^i and the corresponding radial component of the particle velocity, v_r^i where $i = \{inc, sct, in\}$, are found by using Equation 1.16 through Equation 1.19a. The resulting set of linear equations can be written more succinctly as

$$\mathbf{\Gamma}_n \boldsymbol{\omega}_n = \boldsymbol{\xi}_n \quad (1.31)$$

for each mode n . The $\mathbf{\Gamma}_n$ is a 6×6 matrix whose six unknown modal coefficients $\boldsymbol{\omega}_n^T$ and the excitation vector $\boldsymbol{\xi}_n$ are $\boldsymbol{\omega}_n^T = \{A_n(s), \dots, F_n(s)\}$ and $\boldsymbol{\xi}_n^T = [\zeta_n^{ex}(s) \ \xi_n^{in}(s) \ 0 \ 0 \ 0 \ 0]$ where $\zeta_n^{ex} = \xi_n^{ex}(s) + (2n+1)g(s)e^s i_n(rs)$. The ξ_n^{ex} and ξ_n^{in} are given in Equation 1.10.

The following subsection outlines two of the most commonly used shell theories which approximate the previously described GL approach. These approximations, however, require the interactions between a shell and fluid to be treated differently.

1.2.5.2 Approximate Shell Theories

A major difference between the shell theories and GL approach arises from their kinematic assumptions. A midsurface based displacement assumption and conjectured zero radial strains and stresses are employed invariably in all the shell theories. Consequently the first branch of the dispersion's frequency spectrum found from the GL approach is highly unlikely to be captured. See, for example, [Prasad, 1964] and [Qatu, 2004] § 2.34. Therefore the $\mathcal{EOM}$ in a shell theory is independent of the radial coordinate, r unlike the GL based Equation 1.22. As a result, an $\mathcal{EOM}$ for all shell theories can be written generally as

$$\mathbf{L}\bar{\mathbf{u}} = \mathbf{f} \quad \text{where} \quad \bar{\mathbf{u}} \equiv \mathbf{u}|_{r=r_{mid}} \quad \text{and} \quad r_{mid} = \frac{1+r_{in}}{2} \quad (1.32)$$

in the Laplace domain when initial conditions are zero. The linear operator $\mathbf{L}$ is *independent* of the radial coordinate. The detailed expressions of $\mathbf{L}$ are given in Appendix C. Moreover the loading vector, $\mathbf{f}$, used in Equation 1.32 represents all the traction forces which include the mechanical surface forces and those fluid pressures which act on one or both of the shell's surfaces. Therefore

$$\mathbf{f}|_{r=r_{mid}} \cong \mathbf{f}^{in}|_{r=r_{in}} + \mathbf{f}^{ex}|_{r=1} + (p^{in}|_{r=r_{in}} - p^{ex}|_{r=1})\mathbf{e}_r \quad (1.33)$$

defines a general inviscid, fluid-shell interaction. Such interactions are implemented through Equation 1.33 by including the contributions of both the internal and external fluid pressures at each surface of the shell, i.e. $r = r_{in}$ and r_{ex} . This approach is contrary to the common practice of taking pressure values at an imaginary midsurface, $r = r_m$. See, for example, [Huang, 1969, Huang, 1979, Geers and Zhang, 1994, Bahari and Popplewell, 2015]. This apparently small modification

noticeably improves the performance of the shell theories especially for higher wall thicknesses. The fluid-solid interaction, however, is described more rigorously in the GL approach by applying continuities at all the traction boundaries. Moreover, approximate shell models always assume that the fluid particle velocities at adjacent traction surfaces are identical to that at the imaginary midsurface. Kinematic compatibility and continuity conditions at the inner and outer surfaces of the shell require that

$$s u_r|_{r=r_{mid}} \cong v_r^{in}|_{r=r_{in}}, \quad s u_r|_{r=r_{in}} \cong v_r^{ex}|_{r=r_{ex}} \quad (1.34)$$

in which the modal components of v^{ex} and v^{in} are given in Equation 1.18 and Equation 1.19a for the adjacent, inviscid fluid media. The unknown coefficients A_n and B_n in the previous equations are determined from Equation 1.34. Moreover, Equation 1.32 infers that $\mathbf{f}$ acts only at the shell's imaginary midsurface so that time delays are ignored for those elastic waves which travel repeatedly between the two surfaces of the shell. Consequently shell theories do not distinguish whether a traction force is applied at the shell's inner or outer surface.

The two most commonly used approximations of the GL are known as the *four mode* (4M) and *five mode* (5M) shell theories. The 4M model, proposed by [Federhofer, 1937], accounts for membrane and bending effects as a *thin* shell approximation. It has been the most popular choice for elastoacoustic problems. See, for instance, [Huang, 1969, Lou and Klosner, 1973, Zhang and Geers, 1993], [Chen and Stepanishen, 1994] as well as [Zakout, 2001] and [Bahari and Popplewell, 2015]. On the other hand, the 5M theory for a spherical shell was developed first by [Prasad, 1964] and was then improved by [Wilkinson and Kalnins, 1965]. It lies within the category of so called *thick* shell theories as it includes shear deformation, rotary inertia terms as well as membrane and bending effects. For instance, [Tang and Yen, 1970] used the variation of 5M proposed by [Prasad, 1964], whereas [Menton and Magrab, 1974] utilized the one of [Wilkinson and Kalnins, 1965]. The terms *four-* and *five-* modes also refers to maximum number of required independent kinematic variables. These names, also, corresponds to the maximum number of branches can be predicted by these shell theories in their associated dispersion curves. The latter 5M model is the one considered currently.

The displacement field, $\mathbf{u}$, is defined as a function of the midsurface's displacement, $\bar{\mathbf{u}}$, and the radial coordinate, r , in a shell theory. Consequently

$$\mathbf{u}(r, \theta, \varphi, s) \equiv \mathbf{u}(\bar{\mathbf{u}}(\theta, \varphi, s), z(r)), \quad -h/2 < z < h/2 \quad (1.35)$$

where $z \equiv r - r_{min}$. Equation 1.35 in conjunction with Equation 1.32, implies unrealistically that any variation in displacement vector $\mathbf{u}$ across a shell's thickness happens spontaneously. The foregoing approximation of Equation 1.35 is obtained from a Taylor's series expansion of displacements elements in terms of a thickness coordinate, z , by neglecting terms of order z^2 and

higher, such that

$$\mathbf{u} \equiv \begin{Bmatrix} u_r(r, \theta, \varphi, s) \\ u_\theta(r, \theta, \varphi, s) \\ u_\varphi(r, \theta, \varphi, s) \\ \beta_\theta(r, \theta, \varphi, s) \\ \beta_\varphi(r, \theta, \varphi, s) \end{Bmatrix} = \underbrace{\begin{Bmatrix} \bar{u}_r(\theta, \varphi, s) \\ \bar{u}_\theta(\theta, \varphi, s) \\ \bar{u}_\varphi(\theta, \varphi, s) \\ \bar{\beta}_\theta(\theta, \varphi, s) \\ \bar{\beta}_\varphi(\theta, \varphi, s) \end{Bmatrix}}_{\bar{\mathbf{u}}} + z \begin{Bmatrix} 0 \\ \bar{\beta}_\theta(\theta, \varphi, s) \\ \bar{\beta}_\varphi(\theta, \varphi, s) \\ 0 \\ 0 \end{Bmatrix}, \quad -\frac{h}{2} < z < \frac{h}{2} \quad (1.36)$$

where $\bar{\beta}_\theta$ and $\bar{\beta}_\varphi$ are the midsurface rotations. The $\bar{\beta}_\theta$ is assumed to be zero in the 4M theory. It is inferred from Equation 1.36 and zero radial strain assumption that frequency spectra of 4M and 5M have only four and five branch for each n , respectively. This is in contrary to GL which predicts infinite number of branches for each mode n . In an axisymmetric, shear-free loading situation, $\bar{\beta}_\varphi$ and $\bar{u}_\varphi$ are zero and response is independent of azimuthal angel φ . Furthermore the traction force load on shell simplifies to

$$\mathbf{f} = [\varrho_r(\theta, s), 0, 0]^T \quad (1.37)$$

where

$$\varrho(\theta, s) = f_r^{(in)}(\theta, s) - f_r^{(ex)}(\theta, s) + p^{(in)}(r, \theta, s)|_{r=r_{in}} - p^{(ex)}(r, \theta, s)|_{r=r_{ex}}. \quad (1.38)$$

Consequently Equation 1.36 is reduced to

$$\begin{Bmatrix} u_r(r, \theta, s) \\ u_\theta(r, \theta, s) \\ \beta_\theta(r, \theta, s) \end{Bmatrix} = \begin{Bmatrix} \bar{u}_r(\theta, s) \\ \bar{u}_\theta(\theta, s) \\ \bar{\beta}_\theta(\theta, s) \end{Bmatrix} + z \begin{Bmatrix} 0 \\ \bar{\beta}_\theta(\theta, s) \\ 0 \end{Bmatrix}, \quad -\frac{h}{2} < z < \frac{h}{2}. \quad (1.39)$$

Equation 1.39 indicates that while the radial displacement remains constant across the thickness in both shell theories, the 5M theory conversely allows the meridional component to vary linearly in radial direction z . Irrespective of the adopted shell theory, the radial displacement can be written in terms of a Fourier-Legendre expansion as

$$\bar{u}_r(\theta, s) = \sum_{n=0}^{\infty} \bar{u}_n^r(s) P_n(\cos \theta). \quad (1.40)$$

By using the expansion given in Equation 1.40 and the compatibility conditions of Equation 1.34, the unknown coefficients $A_n(s)$ and $B_n(s)$ in Equation 1.16 become

$$A_n(s) = \frac{-s \bar{u}_n^r(s)}{\rho_{in} \alpha_{in} [i_{n+1}(\alpha_{in} r_{in} s) + \frac{n}{r_{in} \alpha_{in} s} i_n(\alpha_{in} r_{in} s)]} \quad (1.41a)$$

and

$$B_n(s) = \frac{s \bar{u}_n^r(s) + (2n+1)g(s)e^s[i_{n+1}(s) + \frac{n}{s}i_n(s)]}{-k_{n+1}(s) + \frac{n}{s}k_n(s)} \quad (1.41b)$$

regardless of the shell theory. Subsequently the closed form expressions of the midsurface's radial

displacement, $\bar{u}_n^r(s)$, can be deduced from Equation 1.32 and Equation 1.41, such that

$$\bar{u}_n^r(s) = \frac{\mathbb{A}_n(s)}{\mathbb{B}_n(s)}, \quad (1.42)$$

where

$$\begin{aligned} \mathbb{A}_n(s) = \hat{a}_n^{in} \left[\hat{a}_n^{sca} (\xi_n^{net} + \hat{p}_n^{inc}) - \hat{a}_n^{inc} \hat{p}_n^{sca} \right] & \left[c_3(\nu - 1) + \lambda_n(b_3 + b_4 - 2k_s b_5) \right. \\ & \left. + 2k_s \lambda_n) - (b_3 + b_4 - 2k_s b_5)\nu \right] \end{aligned} \quad (1.43)$$

and

$$\begin{aligned} \mathbb{B}_n(s) = \hat{a}_n^{sca} \hat{p}_n^{in} s^2 & \left[c_3 \lambda_n (b_3 + b_4 - 2b_5 k_s + 2k_s \lambda_n) - c_3 \nu + (b_3 + b_4 - 2b_5 k_s) \lambda_n \nu \right] \\ & + \hat{a}_n^{in} \left[a_n^{sca} h(1 - \nu) (c_3 (2 + k_1 s^2 \alpha_{Lsh}^2 + 2\nu) - \lambda_n (c_2 - c_1 \lambda_n + \lambda_n^2)) \right] \\ & \hat{p}_n^{sca} s^2 \left[c_3 (\nu - 1) + \lambda_n (b_3 + b_4 - 2b_5 k_s) \nu \right] \end{aligned} \quad (1.44)$$

in which $\lambda_n = n(n + 1)$ and $k_s = 6/5$. Expressions for $\{\hat{p}_n^{sca}, \hat{p}_n^{in}, \hat{p}_n^{inc}, \hat{a}_n^{sca}, \hat{a}_n^{in}, \hat{a}_n^{inc}, \xi_n^{net}\}$ in Equation 1.43 and Equation 1.44 are

$$\begin{aligned} \hat{p}_n^{sca} &= k_n(s), & \hat{a}_n^{in} &= -\rho_{in} \alpha_{in} \left[i_{n+1}(\alpha_{in} s) + \frac{n}{\alpha_{in} s} i_n(\alpha_{in} s) \right], \\ \hat{a}_n^{sca} &= k_n(s), & \hat{a}_n^{inc} &= (2n + 1)g(s) \left[i_{n+1}(s) + \frac{n}{s} i_n(s) \right], \\ \hat{p}_n^{in} &= i_n(\alpha_{in} s), & \hat{p}_n^{inc} &= (2n + 1)g(s) i_n(s), \quad \xi_n^{net} = \xi_n^{in} - \xi_n^{ex}. \end{aligned} \quad (1.45)$$

The remaining of parameters and constants in Equation 1.43 and Equation 1.44 are given in Equation C.1 and Equation C.2 of Appendix C. However Equation 1.42 is developed for 5M, if $k_s = k_r = k_2 = c_r = 0$, the same equation represents $\bar{u}_n^r(s)$ in the case of 4M. Some computational details and validations of the current approach is discussed next.

1.3 Validations

The current solutions are implemented in MATHEMATICA[©] environment. The code is made parallel by using WOLFRAM'S KERNEL TECHNOLOGY with a high level of computational efficiency. The objective here is to examine the validity of these solutions and their associated computer codes under a general loading situation. This goal is fulfilled by performing numerical comparisons against the multiple cases of published data to ensure the validity and numerical robustness of the current computer codes. The results from the existing cases are reproduced first by using the original methodologies. Then the outputs of these codes are compared with those of the current work. However only three representative cases are presented next.

(i) The scattering of a plane incident wave from an empty spherical shell

[Huang, 1969] studied the scattering of an incident transient plane wave from an empty steel shell submerged in water. The 4M theory was used with $h = 0.01$ and $\bar{g}(t) = \mathcal{H}(t)$. Figure 1.2c is a reproduction of Figures 2 and 3 of [Huang, 1969] in which the modal components of the radial displacements, u_r^n , were given for $n = 0$ and $2 \leq n \leq 7$.

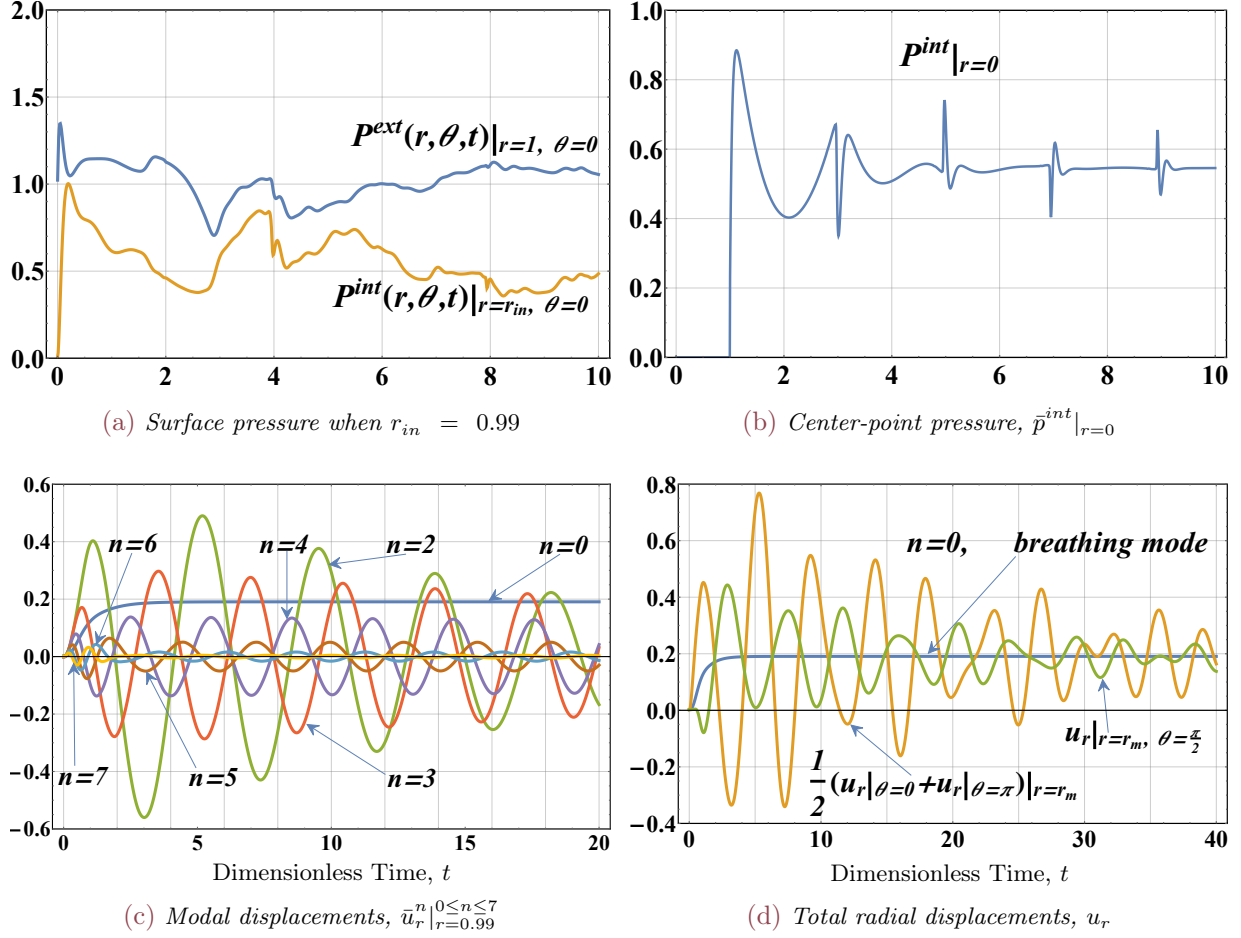


Figure 1.2: Time histories of the (a) shell's modal displacements, $\bar{u}_r^n, 0 \leq n \leq 7$ at $r = 0.99$; (b) shell's total displacement at $\theta = \pi/2$ as well as the average value for the apex and tail in comparison to the breathing mode displacement (the late time static asymptotic deflection). The water submerged, empty steel spherical shell has a nondimensionalized thickness of $h = 0.02$. It is subjected to a plane incident pressure wave with Heaviside time dependency. (c) inner and outer surface pressure time histories at a spherical shell's apex ($\theta = 0$) and (d) its center point ($r = 0$).

(ii) The scattering of an incident plane wave from a fluid filled spherical shell

[Zhang and Geers, 1993] studied the scattering of an incident transient plane wave from a water filled and water submerged, steel spherical shell. A *four-mode* shell theory was considered for $h = 0.01$ and $\bar{g}(t) = \mathcal{H}(t)$. Surface pressure histories of the internal and external fluid media are shown in Figure 1.2a as well as the pressure at the sphere's center, $r = 0$, as given in Figure 1.2b. These figures agree very well with Figures 5 and 16 of [Zhang and Geers, 1993], respectively. Barely visible differences, however, are observed by comparing these results with those of the 5M and the GL solutions.

(iii) The transient acoustical radiation from an empty spherical shell

The transient acoustical radiation from an underwater, empty steel shell caused by a surface excitation was studied by [Chen and Stepanishen, 1994], [Zakout, 2001] and [Bahari and Poplewell, 2015] who all used the 4M theory. [Chen and Stepanishen, 1994] considered the Dirac impulse ($\bar{g}(t) = \delta(t)$) as the time dependency of the force located at the outer surface of the shell. The transient response was retrieved by using in vacuo eigenvectors. On the other hand, [Zak-

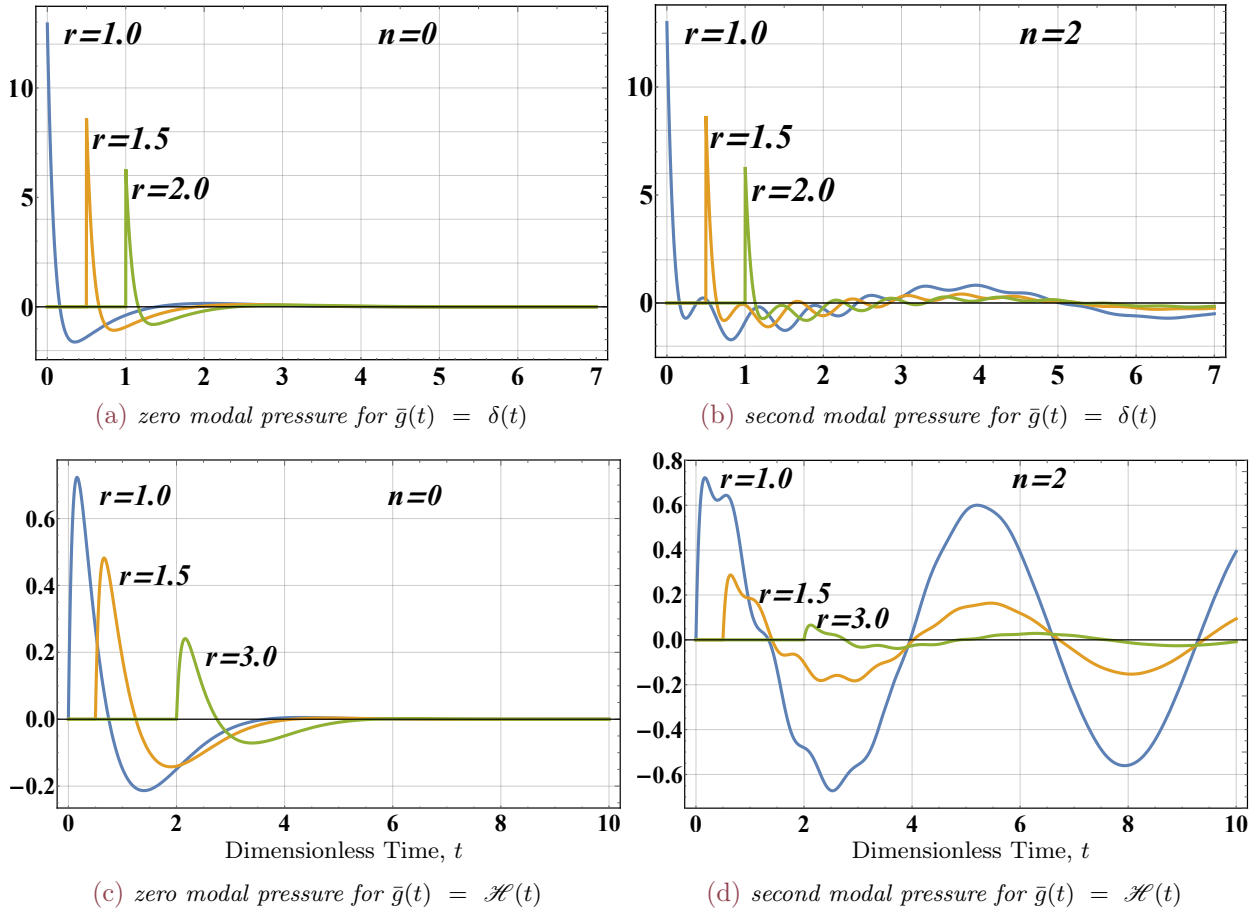


Figure 1.3: Time histories of the modal pressure at different nondimensionalized distances from the sphere's center ($r = 1.0, 1.5$, and 2) for (a) breathing mode $n = 0$ and (b) second vibratory mode $n = 2$ where an Dirac impulse $\bar{g}(t) = \delta(t)$ applied at the shell's outer surface. Time histories of the modal pressure at different nondimensionalized distances from the sphere's center ($r = 1.0, 1.5$, and 3) for (a) breathing mode $n = 0$ and (b) second vibratory mode $n = 2$ where an Heaviside pulse $\bar{g}(t) = \mathcal{H}(t)$ applied at the shell's inner surface.

out, 2001] used the Heaviside excitation whereas [Bahari and Popplewell, 2015] considered both a Heaviside and a Dirac impulse acting on the inner surface of a shell for which a direct Laplace transform approach was adopted. It should be remembered that shell theories do not distinguish an inner or outer loads except for a sign difference (Equation 1.38). The pressure histories of the zero and second modes are given at different distances from the shell's center for both the impulse and Heaviside loads considered in Figure 3.8. A fairly good agreement between Figure 1.3a and Figure 1.3b is observed with the corresponding Figure 10 and Figure 12 of [Chen and Stepanishen, 1994], respectively. Furthermore, Figure 1.3c and Figure 1.3d agree well with Figure 7 and Figure 2c given respectively in [Zakout, 2001] and [Bahari and Popplewell, 2015]. An extensive set of newly conducted numerical tests are presented next.

1.4 Numerical Experiments

The main objective of this section is to examine thoroughly the range of validity of two shell theories in comparison to the data from the GL model for a shell which is subjected to a general transient excitation. This objective is achieved by performing a truly exhaustive and diverse set of carefully selected numerical experiments. The main computations of the modal components are performed in the MATHEMATICA[©] environment as noted in 2.2. An extensive

set of MATLAB codes are developed for the post processing and automatic plotting of the outputs of the MATHEMATICA[©] code. These MATLAB codes automatically plot several thousands of comparative cases to produce high quality figures. The subplots are arranged in a way to maximize the data in the least possible space. Moreover the limits of a vertical axis is set exactly by using the global maximum and minimum values among all the 4M, 5M and GL data. A typical run-time using this hardware for each set of 40 modes takes around 7 hours for each shell theory and an hour extra for the the GL code. Such a difference is almost inconsequential. The results presented here are a tiny fraction of thousands of cases.

1.4.1 Configurations of The Case Studies

air	$c_{in} = 340 \text{ m/s}$ and $\rho_{in} = 1.225 \text{ kg/m}^3$
sea water	$c_{ex} = 1500 \text{ m/s}$ and $\rho_{ex} = 1000 \text{ kg/m}^3$
steel	$\rho_s = 7800 \text{ m/s}$, $\nu = 0.3$ and $E = 207 \text{ Gpa}$
aluminum	$\rho_s = 2700 \text{ m/s}$, $\nu = 0.32$ and $E = 69 \text{ Gpa}$
pyroceram glass	$\rho_s = 2700 \text{ m/s}$, $\nu = 0.3$ and $E = 69 \text{ Gpa}$
shell's thickness	$h = \{0.001, 0.04, 0.1, 0.2\}$

Table 1.1: *Material properties and shell's thicknesses.*

The case studies are selected so that the most effective factors are incorporated in the determination of the accuracy of the two shell approximations. The following investigated factors include: the relative shell wall thickness (h), mode number (n), time dependency of an excitation ($f(t)$), the source of excitation (an incident wave and a surface traction force) and the relative acoustical properties of the interacting media. A steel shell is submerged invariably in water and it is filled with either water or air. Four relative shell wall thicknesses are used to encompass very thin and thick shells. Aluminum and pyroceram glass are chosen to be compared with steel for the material of the shell. All the material properties as well as the shell's thicknesses are given in [Table 1.1](#).

Individual modes represent a specific physical motion. The zero ($n = 0$) and first ($n = 1$) modes give rise to a breathing deformation and a rigid body motion, respectively. All other higher mode numbers, $n > 1$, on the other hand, are known as labour nontranslational modes relate to different symmetric and asymmetric vibratory modes. See, for example, [Zakout, 2001]. The breathing mode refers to a symmetric vibratory motion in which that any given circular cross section of the shell remains circular as the shell's radius changes uniformly. This motion corresponds to the long term asymptotic response is the static deflections when the rigid body motion is omitted. These three categories of modes are considered in the current numerical experiments.

Different excitation functions $\tilde{g}(t)$ are chosen to represent various types of frequency spectra. The time dependencies of these functions and the corresponding closed form expressions in the Laplace and Fourier $\check{g}(\omega)$ domains are given in [Table 1.2](#). The Dirac delta $\delta(t)$ and Heaviside $\mathcal{H}(t)$ pulses represent ideal excitations which have been previously used most extensively. A response to a general excitation can be found by simply employing the convolution or Duhamel's integral of responses to Dirac or Heaviside functions, respectively, because of the complete system's linearity. See, for example, [Tang and Yen, 1970, Huang, 1969, Zhang and Geers, 1993, Zakout, 2001]. The $\delta(t)$ provides a uniform frequency spectrum whereas the Heaviside function gives more weight

Excitation Pulses		
Ricker	$\tilde{g}(t) = \left[1 - 2 \left(\frac{t - t_{r_0}}{t_{r_\omega}} \right)^2 \right] e^{-\left(\frac{t - t_{r_0}}{t_{r_\omega}} \right)^2}$ $\check{g}(\omega) = t_{r_\omega}^3 \frac{\omega^2 \mathbf{e}^{\left(i t_{r_0} \omega - \frac{t_{r_\omega}^2 \omega^2}{4} \right)}}{2\sqrt{2}}$ $g(s) = \mu_6 s + \mu_7 - \mu_3 \mathbf{e}^{(-s\mu_4 - \mu_2)^2 s^2} \operatorname{erfc}(s\mu_4 - \mu_2)$	
Gaussian	$\tilde{g}(t) = \mathbf{e}^{-\left(\frac{t - t_{g_0}}{\sqrt{2}t_{g_\omega}} \right)^2}, \quad \check{g}(\omega) = t_{g_\omega} \mathbf{e}^{i t_{g_0} \omega - \frac{t_{g_\omega}^2 \omega^2}{2}}$ $g(s) = \mu_{17} \mathbf{e}^{s^2 \mu_{20} - s t_{g_0}} \operatorname{erfc} \left(\frac{2t_{g_\omega}^2 - t_{g_0}}{\mu_{19}} \right)$	
Modulated Gaussian	$\tilde{g}(t) = \mathbf{e}^{-\left(\frac{t - t_{m_0}}{\sqrt{2}t_{m_\omega}} \right)^2} \cos \omega_g (t - t_{m_0})$ $\check{g}(\omega) = \mu_{10} \mathbf{e}^{(i t_{m_\omega} \omega)} \left(\mathbf{e}^{-\frac{1}{2} t_{m_\omega}^2 (\omega - \omega_m)^2} + \mathbf{e}^{-\frac{1}{2} t_{m_\omega}^2 (\omega + \omega_m)^2} \right)$ $g(s) = \mu_{10} \mathbf{e}^{(s^2 \mu_{12} - s t_0 - i t_{m_\omega}^2 s \omega_m - \mu_{13})} \dots$ $\left[\operatorname{erfc} \left(\frac{\mu_{14} + s t_{m_\omega}^2}{\mu_{11}} \right) + \mathbf{e}^{(s \mu_{16})} \operatorname{erfc} \left(\frac{\mu_{15} + s t_{m_\omega}^2}{\mu_{11}} \right) \right]$	
Delta Dirac	$\tilde{g}(t) = \delta(t), \quad \check{g}(\omega) = \frac{1}{\sqrt{2\pi}}, \quad g(s) = 1$	
Heaviside	$\tilde{g}(t) = \mathcal{H}(t), \quad \check{g}(\omega) = \frac{1}{2} \left(\delta(\omega) - \frac{\mathbf{i}}{\pi \omega} \right), \quad g(s) = \frac{1}{s}$	
Exponential	$\tilde{g}(t) = \mathbf{e}^{-\frac{t}{t_e}}, \quad \check{g}(\omega) = \sqrt{2\pi} \delta \left(\frac{\mathbf{i}}{t_e} + \omega \right), \quad g(s) = \frac{t_e}{s t_e + 1}$	
Explosion	$\tilde{g}(t) = \frac{t_e + t_d - t}{t_e + t_d} \mathbf{e}^{-\frac{t}{t_e}}, \quad g(s) = t_e \frac{t_d + s t_e t_d + s t_e^2}{(t_e + t_d)(1 + s t_e)^2}$ $\check{g}(\omega) = \frac{\sqrt{2\pi}}{t_e + t_d} \left((t_d + t_e) \delta \left(\frac{\mathbf{i}}{t_e} + \omega \right) + \mathbf{i} \mathcal{H} \left(\frac{\mathbf{i}}{t_e} + \omega \right) \right)$	
$\mu_1 = \mathbf{e}^{-(t_{r_0}/t_{r_\omega})^2}, \mu_2 = t_{r_0}/t_{r_\omega}, \mu_3 = \sqrt{\pi} \mu_1 t_{r_\omega}^3/4, \mu_4 = t_{r_\omega}/2, \mu_6 = \mu_1 t_{r_\omega}^2/2$		
$\mu_7 = \mu_1 t_{r_0}, \mu_{10} = \sqrt{\pi}/8 t_{m_\omega}, \mu_{11} = \sqrt{2} t_{m_\omega}, \mu_{12} = t_{m_\omega}^2/2, \mu_{13} = t_{m_\omega}^2 \omega_g^2/2$		
$\mu_{14} = -t_{m_0} - \mathbf{i} t_{m_0}^2 \omega_m, \mu_{15} = -t_{m_0} + \mathbf{i} t_{m_0}^2 \omega_m$		
$\mu_{19} = 2 \mathbf{i} t_{m_\omega}^2 \omega_g, \mu_{17} = \sqrt{\pi}/2 t_{g_\omega}, \mu_{19} = \sqrt{2} t_{g_\omega}, \mu_{20} = t_{g_\omega}^2/2$		
Narrow-Band: $t_{r_\omega} = t_{g_\omega} = t_{m_\omega} = t_e = 1/5, t_{r_0} = t_{g_0} = t_{m_0} = 1$ and $\omega_m = 25$		
Broad-Band: $t_{r_\omega} = t_{g_\omega} = t_{m_\omega} = t_e = 1/100, t_{r_0} = t_{g_0} = t_{m_0} = 1/10$ and $\omega_m = 200$		
UT range: $t_{r_\omega} = t_{g_\omega} = t_{m_\omega} = t_e = 1/1000, t_{r_0} = t_{g_0} = t_{m_0} = 1/100$ and $\omega_m = 2000$		

Table 1.2: Mathematical Expressions of excitation in time $\tilde{g}(t)$, frequency $\check{g}(\omega)$ and Laplace $g(s)$ domains where $\operatorname{erfc}(x)$ denotes the complementary error function.

to the lower modes. The mathematical expressions of these two functions are very simple in the Fourier and Laplace spaces. Consequently they are better suited for analytical developments which explains their popularity. The pulse created by an explosion is considered for its obvious practical application. Furthermore Ricker, Gaussian as well as the modulated Gaussian pulses are employed. As an example of their practical usefulness, such pulses are used widely in conventional ultrasonic testing (UT). Three frequency ranges, a *narrow-* and *broad-* broad as well as a band in the UT range, are chosen. A broader frequency band corresponds to a sharper temporal behaviour. Their associated non-dimensional parameters are given in the last two rows of [Table 1.2](#). The time histories of these functions are given in [Figure 1.4](#) along with their corresponding frequency

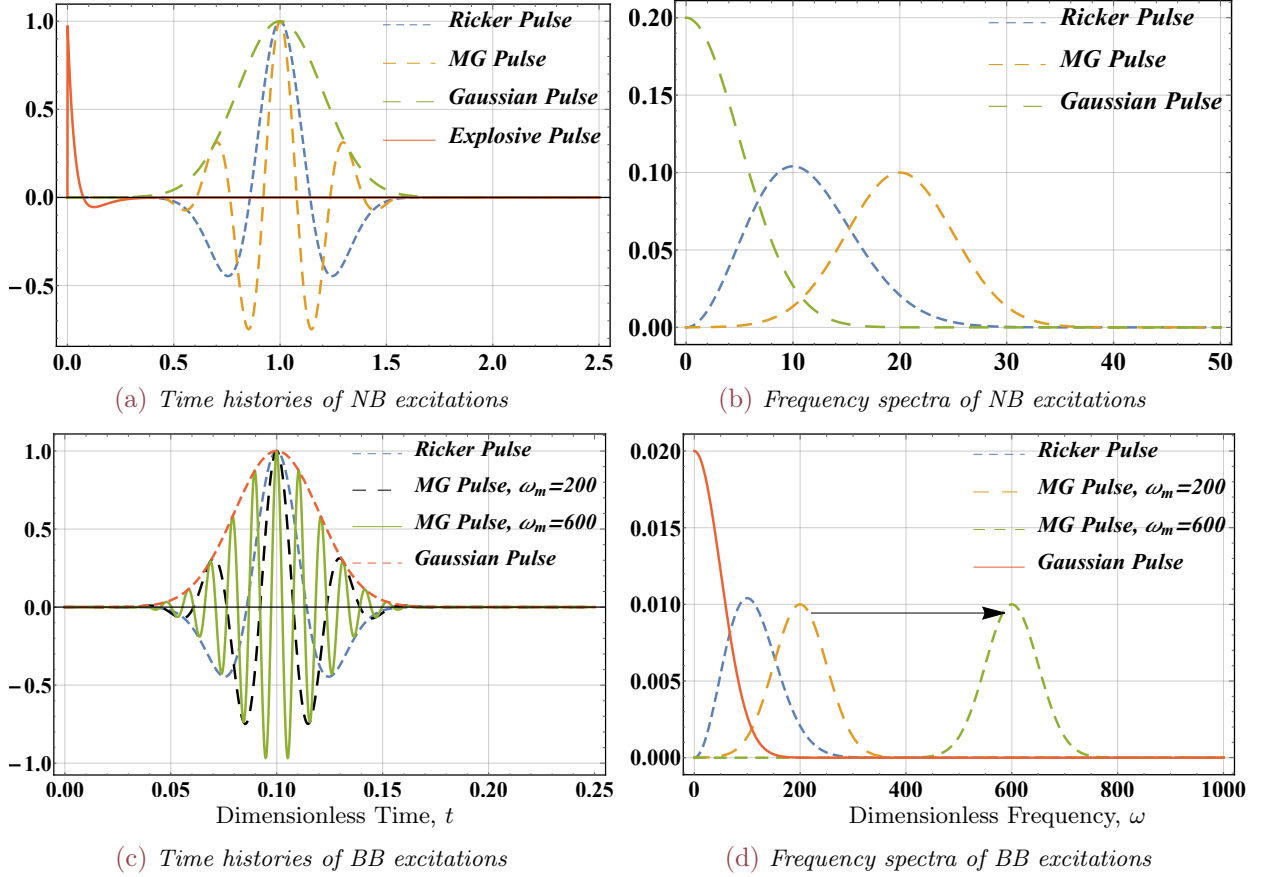


Figure 1.4: Shows the normalized time histories and corresponding frequency spectra of (a) and (b) the Narrowband (NB) as well as (c) and (d) the broadband (BB) excitations, respectively.

spectra. The frequency spectra illustrated in [Figure 1.4d](#) have a ten times broader frequency ranges in comparison that of [Figure 1.4b](#). See also the corresponding time histories given in [Figure 1.4a](#) and [Figure 1.4c](#). Both the Gaussian pulse and its modulated version, unlike the Ricker pulse, have symmetrical frequency spectra. Moreover a larger modulation frequency ω_m leads to a more oscillatory behaviour in time as well as a spectral shift towards higher frequencies. This situation is illustrated in [Figure 1.4c](#) and [Figure 1.4d](#). The Gaussian pulse, on the other hand, allocates the highest weight to the lowest modes. See, for example, [Figure 1.4d](#) and [Figure 1.4b](#).

All plots include the time histories for the 4M, 5M and GL approaches. The majority of the results presented here are in the form of figures which contain 12 subplots for a given mode number. These figures are composed for four shell thicknesses and three different excitations.

1.4.2 The Mode Number Effect

[Figure 1.5](#) gives the pressure time histories of different modes at the outer surface of a water filled shell submerged in sea water where $h = 0.10$. The shell is excited invariably by the NB Ricker pulse in the form of a traction force at the outer surface of the shell. A higher mode number leads to a greater error in results of the shell theories. Compare, for example, $n = 0$ and $n = 1$ with $n = 10$. Two other important observation can be made from this figure which are unexpected. First, the 5M approach does not give better results in comparison to the 4M theory. See, for example, $n = 6$ and $n = 8$ in [Figure 1.5](#). Consequently there is no universal certainty that the 5M produce better than 4M. Moreover noncausal perturbations may appear for the 5M

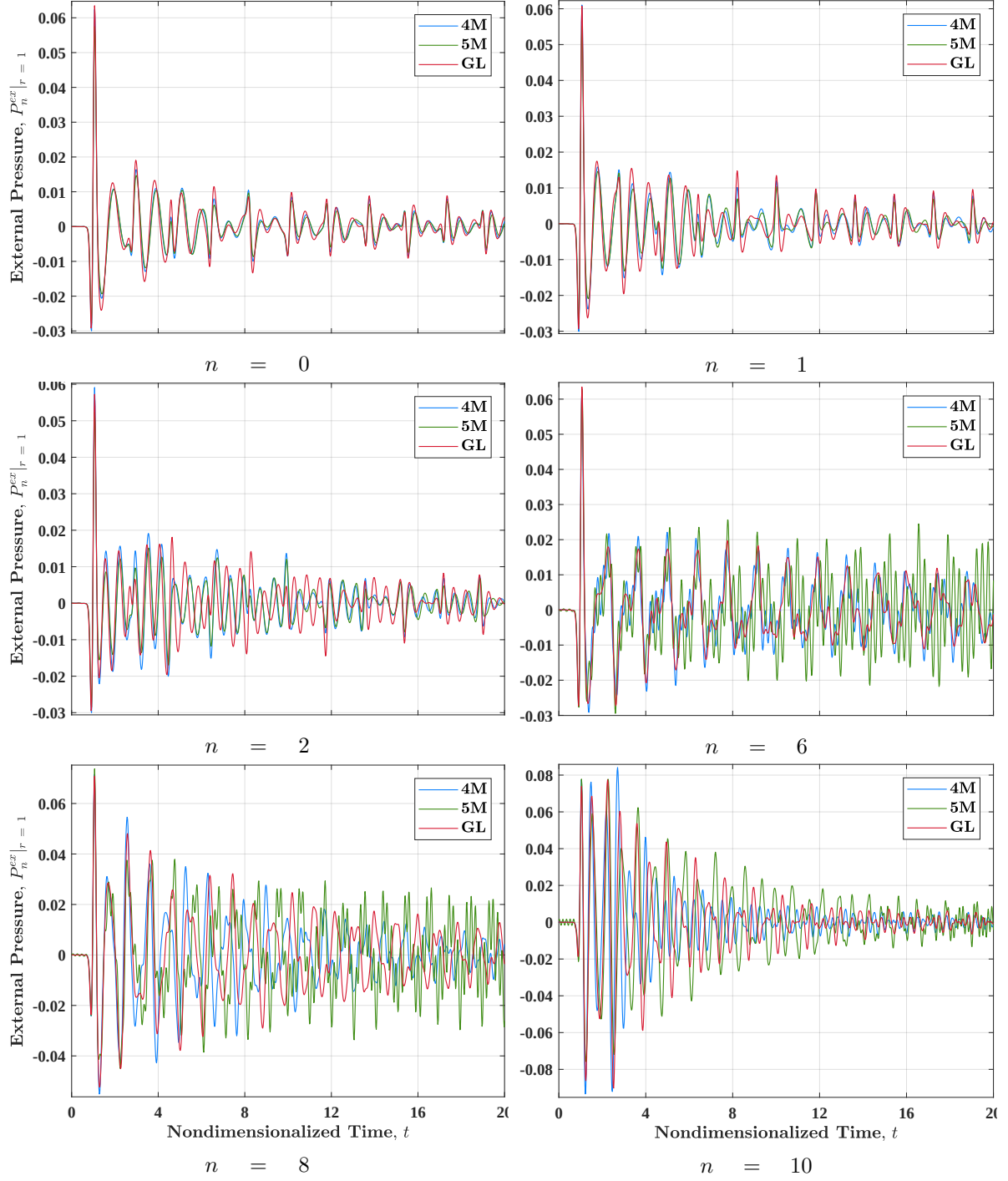


Figure 1.5: Modal ($n = 0, 1, 2, 6, 8, 10$) time histories of the external pressure field at the outer surface of a water filled, steel shell submerged in water when it is subjected to a traction force at its outer surface. The shell wall thickness and the excitation's time-dependency are invariably $h = 0.1$ and the NB Ricker pulse, respectively.

theory ($n = 8$ and $n = 10$). The shell theories underestimate the GL results for the lower modes ($n = 0, 1$ and $n = 2$). The 4M and 5M overestimate the GL results in the higher mode numbers ($n = 6$ and $n = 8$).

Pressure histories at external surface of a water filled, steel shell are given in [Figure 1.6](#) and [Figure 1.7](#) for the mode numbers $n = 10$ and $n = 20$, respectively. Three excitations are applied including Dirac delta, Heaviside and explosive pulses along with four shell wall thicknesses. Both of the shell theories demonstrate significant errors in the thinner shells when mode number increases,

invariably. Furthermore the amplitudes of the results remain a significant level at these higher mode numbers, especially, in the case of the Dirac delta. Combination of the last two observations has an important practical implication: superposition of considerably higher mode numbers are required to accurately predict the complete response. Consequently deviations of the shell theories from GL in total responses are expected to be even more. The other crucial observation is regarding resonance like behaviour in shell theories, which becomes more significant by increasing the mode number. The shell theories and GL for $n = 10$ and $h = 0.06$ stay at almost the same amplitude with a slight phase difference, whereas they start diverging significantly in both the trend and the amplitude for $n = 20$.

1.4.3 The Sources of Excitation

The next objective is to investigate the role of an excitation source. Two different configurations are selected to meet this objective. Each configuration is subjected to both of an incident acoustical wave and a surface traction force on the external surface of the shell. Then the response to these two sources are compared. Four shell wall thicknesses, six excitation functions, two types of the contained fluids and two different modes are considered. The pressure at the external surface of a steel shell whose interior is filled with air is the first configuration considered. The Dirac delta, Heaviside and explosive pulses are applied separately. On the other hand, the pressure at the inner surface of a steel shell which is filled with water is selected as the second configuration. This shell is subjected the Narrowband Gaussian, modulated Gaussian and Ricker pulses. The results of the first configuration are given in [Figure 1.8](#) for the $n = 2$ and [Figure 1.9](#) whereas those of the second configuration are illustrated in [Figure 1.10](#) and [Figure 1.11](#) for $n = 5$.

1.4.3.1 Traction forces versus plane incident acoustical wave

[Figure 1.8](#) and [Figure 1.9](#) demonstrate the results for the first configuration. The sources of excitation are an incident plane wave and a direct mechanical traction force acting on the shell's outer surface, respectively. Clearly the deviations of the shell theories from the GL model are significantly higher when the surface traction force rather than an incident wave is applied invariably. These deviations are more noticeable in the case of the Dirac delta and explosion. A phase difference can be seen at $h = 0.2$ when an incident wave is applied. The discrepancies become significant at $h = 0.1$ and $h = 0.2$ for both of the explosion and delta Dirac pulse in [Figure 1.9](#). The initial peaks are underestimated and high frequency contents are missing at early time. Conversely, the shell theories' responses become overestimated as well as completely out of phase at later times. The encapsulated fluid field, however, behaves noticeably differently when the source of excitation is an impinging incident wave.

The second configuration is illustrated in [Figure 1.10](#) and [Figure 1.11](#) for the fifth mode of the pressure histories at the internal surface of a water filled, steel shell. The shell is subjected to an incident acoustical wave and a traction force at its outer surface, respectively. The following general and important observations are made by comparing the first and second configurations. These behaviours are not unique to these two configurations, similar situations have been observed in the several hundreds of other cases. A considerably higher level of discrepancies is observed within the encapsulated fluid field compared to the external fluid when an incident acoustical wave is applied. The pressure histories of the contained fluid, however, show lower level of erroneous

results when an incident wave rather than a traction force is applied. Besides the 4M approach produces slightly more accurate results than the 5M in both configurations.

Figure 1.12 and Figure 1.13 give the time histories of the radial displacement of an air filled, steel shell at its external surface where $n = 5$ and 10 , respectively. The shell is excited by an incident plane acoustical wave having the Narrowband frequency content. Interestingly, unlike the external pressure field, the accuracy of the shell theories is quite sensitive to an increase in the mode number and wall thickness. In other words, errors grow for the shell theories as the n or h increase. The 4M approach performs better in $n = 5$ whereas the 5M model gives better accuracy for $n = 10$.

Deviations from the GL approach generally increase, regardless of the approximate model, as the wall thickness or mode number grows. However very few exceptions have been observed but only within the external fluid medium when an incident acoustical wave is applied. See, for example, Figure 1.14 which gives the tenth modal pressure component at the external surface of an air filled, steel shell. When this shell is subjected to an incident plane wave, errors in the resulting responses increase as the thickness grows from $h = 0.01$ to $h = 0.10$. However the error diminishes when the thickness increases further to $h = 0.20$. This discrepant behaviour occurs invariably for all three excitations considered. Following the previous trend, the Heaviside function produces the highest errors. Figure 1.15 illustrates an even more interesting behaviour. Figure 1.15 has the same configuration as Figure 1.14 whereas for $n = 20$. The discrepancies almost disappear significantly for two highest thicknesses whilst slight deviations are introduced for $h = 0.01$ and $h = 0.06$. Surprisingly it can be concluded that an increase in mode number as well as the thickness can lead to higher level of accuracy.

Generally, it would be a fair conclusion to say: the shell theories demonstrate notably higher level of accuracy in the external fluid field in comparison with the contained fluid field or the shell surface displacements.

1.4.3.2 Traction forces at the inner- versus outer- surface

The shell theories, regardless of their type, do not distinguish whether a force is applied at a shell's inner or outer surface. (See Equation 1.32 and the subsequent discussion.) The objective here is to examine the performance of the shell theories when an identical force is applied on either surfaces of the shell. A water filled, steel shell is excited by the BB modulated Gaussian pulse in the UT range where $h = 0.10$. Figure 1.16 gives the time histories of the shell when the traction force is applied on the shell's inner (left column) and outer (right column) surfaces for $n = 0$. The inner and outer surface pressures are given in the first and second rows, whereas the displacement at the shell's outer surface is given in the lowest row. First the results of shell theories stay the same regardless of whether the loading at the inner or the outer loading surfaces. Interestingly, all the responses from shell theories approach become almost completely out of phase with respect to their GL counterparts when a traction force applied at the inner rather than outer surface of the shell. This type of deviation is enhanced by the increasing shell wall thickness. This statement stays correct based on similar comparisons for several other cases.

1.4.4 The Breathing Mode, $n = 0$

The radial displacement's time-histories of the shell's midsurface is given in Figure 1.17 when excited by a traction force at the shell's external surface for $n = 0$. Both shell theories demonstrate

very comparable results in all three excitation functions and thicknesses. The discrepancies are mostly from the phase difference kind. Heaviside shows the highest level of errors. The total static displacement constitutes solely by this breathing mode when the excitation's time-dependency is the Heaviside.

Figure 1.18 and Figure 1.19 show the pressure time histories of water filled steel shell at the outer surface of the shell in response to an incident wave and a surface traction force, respectively, for $n = 0$. Both shell theories demonstrate an acceptable performance when the incident wave is the source of excitation. The traction force, conversely, leads to significantly higher level of discrepancies even in thin shells. In terms of the excitation functions, Gaussian pulse leads to the least level of error whereas the modulated Gaussian pulse depicts the highest level of discrepancies. Shell theories underestimate the pressure peaks and jumps except for very thin shells.

The pressure at center of the shell is caused solely by the breathing motion. The corresponding time histories are given in Figure 1.20 when the shell is excited by a surface traction force at the shell's outer surface. The instant arrival time of the wavefront is predicted well in all cases in both figures. The time-intervals between two successive deeps and peaks is about $\Delta t \approx 2$ in $h = 0.01$ and $h = 0.06$ which is as expected. Both of the shell theories show a good performance in the two lowest thicknesses, however, they deviate from GL significantly in $h = 0.10$ and $h = 0.20$. The discrepancies are higher in the case of surface traction force. The errors are the least where a Heaviside excitation is applied. The 5M time histories overestimates or underestimates the GL results from case to case. Furthermore, the 5M completely fails at $h = 0.1$ when the shell is excited by a surface traction load.

1.4.5 The Rigid Body Motion, $n = 1$

Figures Figure 1.21 and Figure 1.22 depicts the rigid body motion of an air filled, steel shell due to an incident wave and a surface traction force, respectively. Common observations from the two figures are given next. Trends are very similar in all cases as expected because there are no shell deformations in a rigid body motion. Both shell theories always underestimate the GL approach. The differences between corresponding displacement profiles caused by an incident wave and a traction force are explained next.

A Heaviside traction load is essentially a constant force. This generates a second order (parabolic) displacement profile history which leads to a constant acceleration of the shell Figure 1.21. The incident wave, on the other hand, produces an initial parabolic and subsequent linear motion. The shell moves initially with a constant acceleration which leads to a parabolic profile of the motion until the wavefront reaches the middle of the shell at $t = 1$. Then a force starts to progressively counteract the shell's movement as the wavefront moves further so that the shell's acceleration decreases. The net force and the shell's subsequent acceleration eventually become zero after the wavefront completely engulfs the shell at $t = 2$. Consequently the profile of the shell's motion becomes linear which corresponds to a constant velocity over time. The case of for both the explosion and Dirac delta pulse are discussed next.

The shell's motion initially increases sharply due to the rapidly rising force acting over the shell. Subsequently the motion's profile becomes linear in the case of the traction force after the excitation pulses die. Similarly, the shell accelerates initially when an incident wave is applied until the wavefront reaches the middle of the shell at $t = 1$. then the shell starts to decelerate after $t = 1$ and subsequently essentially comes to a complete halt after the wavefront passes the

shell and applies a strong decelerating pulse. It should be remembered, however, that in the real world all rigid body motions eventually come to rest due to the viscosity of the external fluid.

1.4.6 The Labour Modes, $n > 1$

Figure 1.23 and Figure 1.24 depict the radial displacement of the shell's external surface when a traction force is applied at its outer surface for $n = 5$ and $n = 15$, respectively. The amplitudes are pretty comparable and they stay almost in phase for the $n = 5$. However an interesting situation occurs in $h = 0.2$ where there are a vertical shifts in long term asymptotic of values of GL, 5M and 4M. The same issue appears at a relatively thinner shell's wall-thickness of $h = 0.1$ as depicted in Figure 1.24. Furthermore the amplitudes of radial displacements stay at the same level even for higher mode. Overall the 5M shows a better performance compared to the 4M in this case. This situation, however, occurs only when a traction force on the shell's surface is applied. If the same configuration is subjected to an incident acoustical wave these offset displacements do not appear in the Heaviside excitation. See, for example, the plots given in Figure 1.25. The error of shell theories in results associated with all the excitations increases in higher thicknesses, whereas the 5M demonstrate considerably better accuracy. It almost consistently produces slightly overestimated results compared to the GL case.

1.4.7 Narrowband versus Boardband Excitations, $\bar{f}(t)$

The aim here is to investigate the effect of the frequency shift of an identical functions but having different frequency ranges. Three excitation suctions are considered having three different level of frequency content, ranging from relatively narrowband (NB), broadband (BB) and even a higher broadband case at Ultrasonic Testing level (UT). It is reminded that the levels are changing by factor of ten. Consequently the results are plotted in different time spans in order to actually being able to see the curves. The pressure histories corresponding to the breathing mode at the external surface of a water filled, steel shell is considered. This shell is subjected to a traction force at its outer surface whose having three time dependencies of Ricker, Gaussian and modulated Gaussian pulses.

Figure 1.26, Figure 1.27 and Figure 1.28 are time histories associated with excitations at BB, NB and UT levels, respectively. Generally the level of deviations from the GL approach increases significantly as the bandwidth increases in all cases. The results for the first three lower shell wall thicknesses remain pretty comparable with the GL responses in the NB range. The 5M completely diverges at $h = 0.20$ for the Ricker and Modulated Gaussian pulses. It fails to capture the GL trend and shows some noncausal results as well. The 4M in case of this NB excitation shows a better performance than the 5M model. The level of the erroneous results at $h = 0.06$ for Gaussian pulse in the UT range becomes much higher than the BB range. Furthermore the magnitudes of the deviations even at $h = 0.01$ becomes noticeable when the UT range excitation is applied. Basically the nature of these high frequency peaks and deeps at the BB and the UT ranges are due to the multiple reflections within the shell wall thickness that can not be captured in the shell theories. Generally a broader range of excitation associated with higher level of deviations where it becomes more significant at a thinner shell wall thicknesses. The modulated Gaussian and Gaussian pulses lead to the the most and the least level of discrepancies, respectively. These observations remain valid throughout for hundreds other cases of comparative case studies.

1.4.8 Shell's Material

The aluminum and pyroceram glass are chosen as the shell material in comparison with the steel shell. The rationales of the these choices are explained as follows. Both of the aluminum and pyroceram glass have comparable wave propagation velocities with respect to the steel. For example, the speed of compressional wave in steel and aluminum are $5700m/s$ and $6400m/s$, respectively. However the density of glass and aluminum is almost a third of steel. Consequently, their effective impedance is significantly smaller than steel. Lower impedance leads to lower capacity in holding acoustic energy within the shell. Glass, however, has a higher Poisson's ratio than aluminum whereas its Young modulus and density are almost identical with the aluminum.

Inner and outer surface modal pressures of the breathing mode are given in [Figure 1.30](#). In these comparisons the effect of these properties are intended to be investigated. The shell theories introduce a higher level of error at an earlier time in the case aluminum and glass compared to steel. The first set of these sharp peaks and deeps are associated with the initial interaction. The subsequent peaks and deeps correspond to the strong triple points' superposition and the arrival of the primary focal wavefronts. The magnitude of these sharp deeps/peaks in the aluminum and glass shells are almost three time of those occurring in the steel shell. The order of error in estimating these significant maximum values in shell theories are higher in aluminum and pyroceram glass compared to steel. On the other hand they have better performance when the wave is damped. In general it can be concluded that the lower impedance values when the wave celerity are at the same level leads to higher errors in the predicting the absolute maximum values. The Poisson's ratio, however, seems not to be an important factor in the performance of the shell theories. The conclusions and closing remarks are given next.

1.5 Conclusions

Transient elasto-dynamics of a fluid filled, spherical shell are investigated when the shell is submerged in an acoustical medium. The shell is subjected to a general excitation source which include the surface traction forces as well as incident acoustical waves. Universal solutions are found for two of the most popular shell theories and one solution is based on the exact local theory of linear elasticity. Theoretical comparisons are made between the approximate and exact approaches in which fundamental differences are highlighted. Corresponding computer codes are validated meticulously for their accuracy and robustness by using comparisons with published numerical results. Moreover an extensive set of numerical experiments have been conducted to determine the most effective parameters which affect the approximations' accuracies. They include the effect of a shell's relative wall thickness, mode number, time dependency and frequency of the excitations, the source and form of an excitation, and each shell's acoustical properties. Numerical experiments are performed for four shell wall thicknesses, surface pressures and displacements of the first fourty modes, six excitation functions in the form of traction forces acting at either one of a shell's surfaces, the type of incident wave, three frequency bandwidths (the NB, BB and UT ranges), two encapsulated fluids (water and air) and three shell materials (steel, aluminum and a pyroceram glass). The main observations are summarized next.

- Performances of the shell theories generally declines as:
 - the mode number or a shell's wall thickness increases;

- an excitation's bandwidth is broadened;
- a traction force rather than an incident acoustical wave is applied;
- the impedance of either a shell's material or a contained fluid is lowered leading to, specifically, a larger level of error in predicting the peaks and deeps;
- Exhaustive comparisons between the 4M and 5M approximations result into the following observations;
 - These shell approximations under- or over- estimate results from the exact model and, more importantly, such discrepancies are unpredictable;
 - However results from both of the approximations agree well when a shell is thin and an applied excitation has a Narrower bandwidth;
 - Deviations between the results from the two approximate theories stem commonly from phase differences. They arise because the 5M model involves two different wave speeds rather than the single one of the the 4M model;
 - Interestingly 4M approximation produce more accurate results compared to the 5M in multiple cases;
 - The major source of inaccuracy in the shell approximations is likely rooted in their indifference to wave motions within a shell's wall and their associated mode conversions at both shell's surfaces;
 - The 5M theory occasionally produces noncausal responses as well as unrealistic resonances and beating like behaviours. These surprising anomalies arise most likely from the assumption that every wave travels instantaneously across a shell's wall;
- The excitations are an important factor in determining the performance of the approximate shell theories. For example:
 - Shell approximations are more accurate for an incident plane wave in the external fluid. However errors in the responses which occur elsewhere are most likely much higher.
 - Shell theories do not distinguish whether a force is applied at the inner or outer surface of the shell.
 - A traction force applied on a shell's inner surface produces the most inaccurate results.
 - The amplitudes of the modal responses remain comparable even for very high mode numbers when they are given rise to by a broadband excitation, especially, in the common industrial frequency range of ultrasonic testing.
 - The responses given rise to by the explosion or Dirac delta produce very similar trends. The amplitude of responses, however, are significantly different.
 - Unlike the Dirac delta, the magnitudes of responses decline considerably with the increasing mode numbers when the explosion is applied.
 - The amplitudes of individual modal components produced in response to the the Dirac delta and Heaviside functions have been shown to be comparable well beyond those of the first ten lowest modes. This observation suggests that significantly higher mode numbers need to be considered when constructing a corresponding total response.

It has been demonstrated clearly that a shell's wall thickness, the mode number and the excitation's configuration are the main factors in determining the accuracy of the two shell theories considered here. These shell theories may noticeably over- or under- estimate the modal responses given by the exact linear theory. The 5M approximation does not necessarily produce more accurate results in comparison to those of the 4M model. The principal advantage of the evaluated shell theories is their twenty or so percent lower computational demand. Overall, however, the exact local theory of linear elasticity is recommended strongly, henceforth.

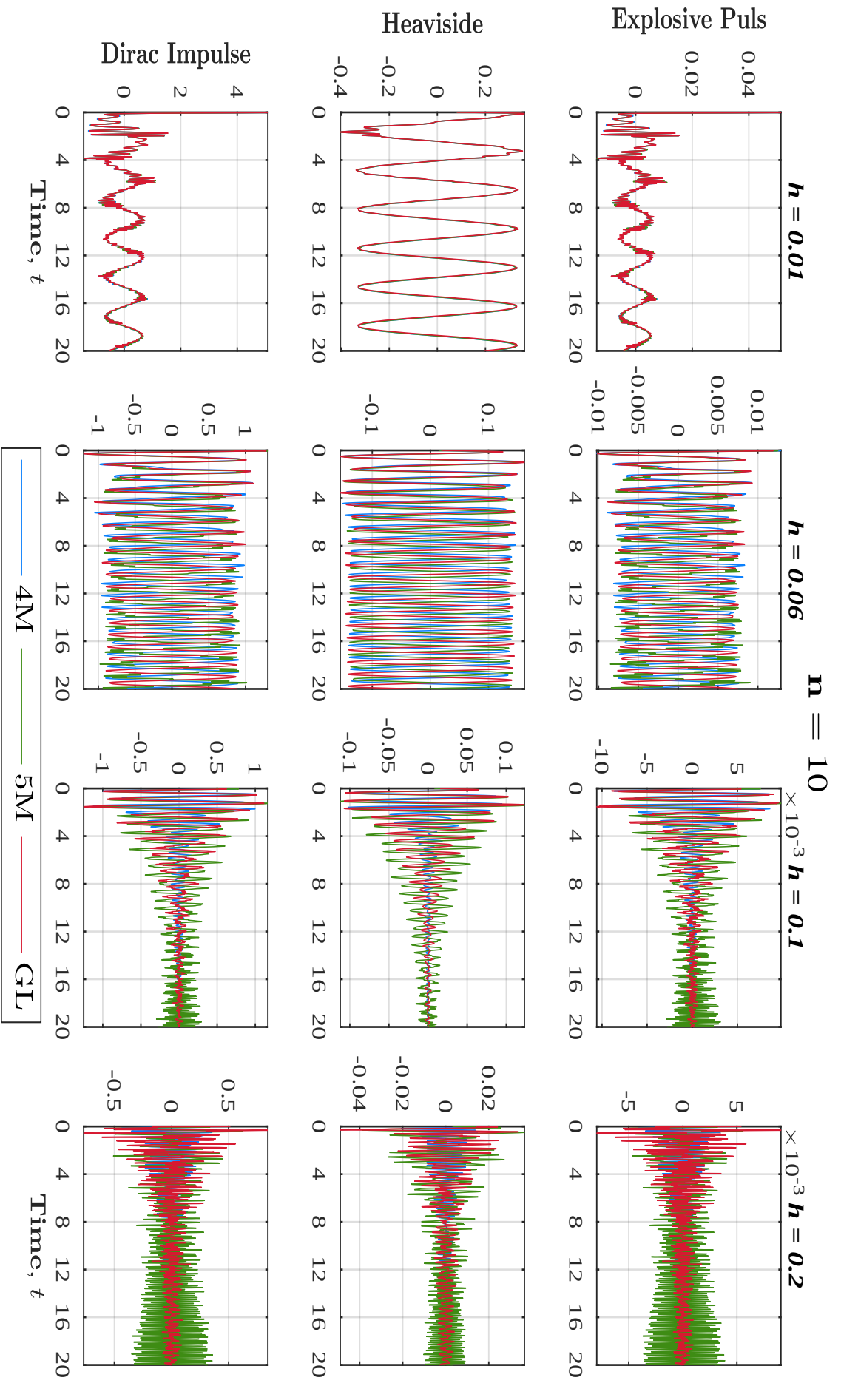


Figure 1.6: Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at the outer surface of the shell. Three different excitations and four relative wall thicknesses are considered for $n = 10$.

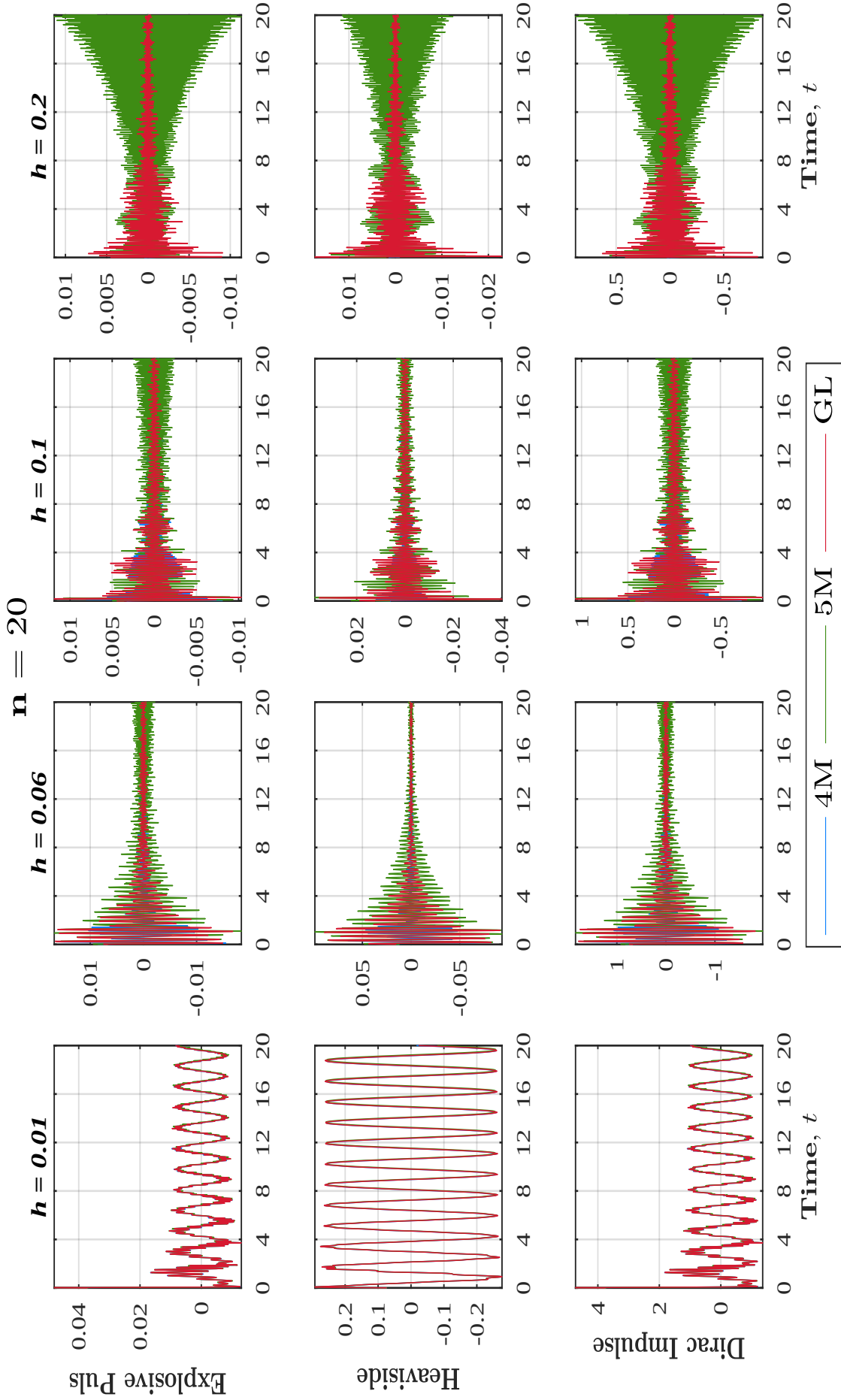


Figure 1.7: Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at the outer surface of the shell. Three different excitations and four relative wall thicknesses are considered for $n = 20$.

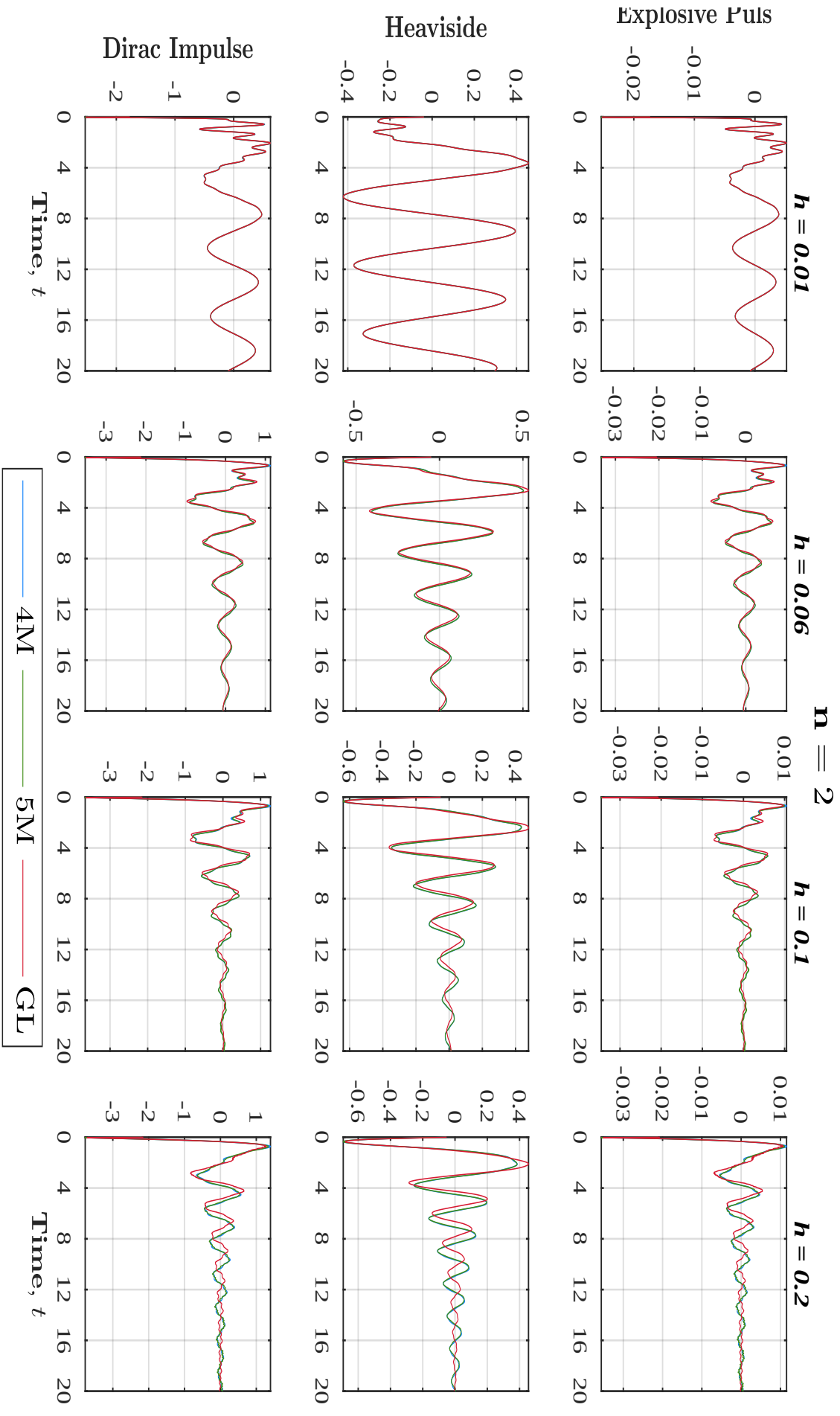


Figure 1.8: Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to an incident plane wave.

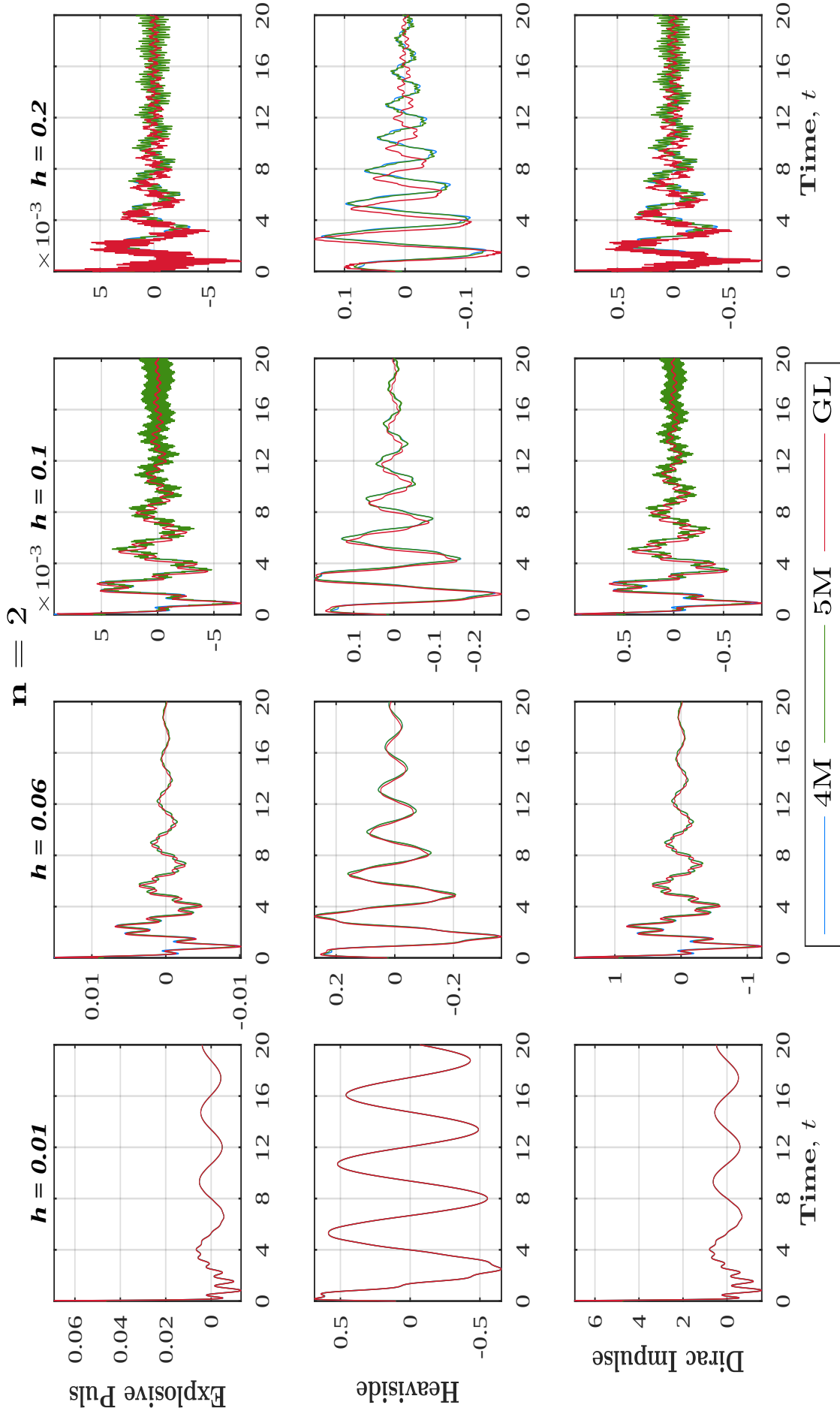


Figure 1.9: Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at its outer surface.

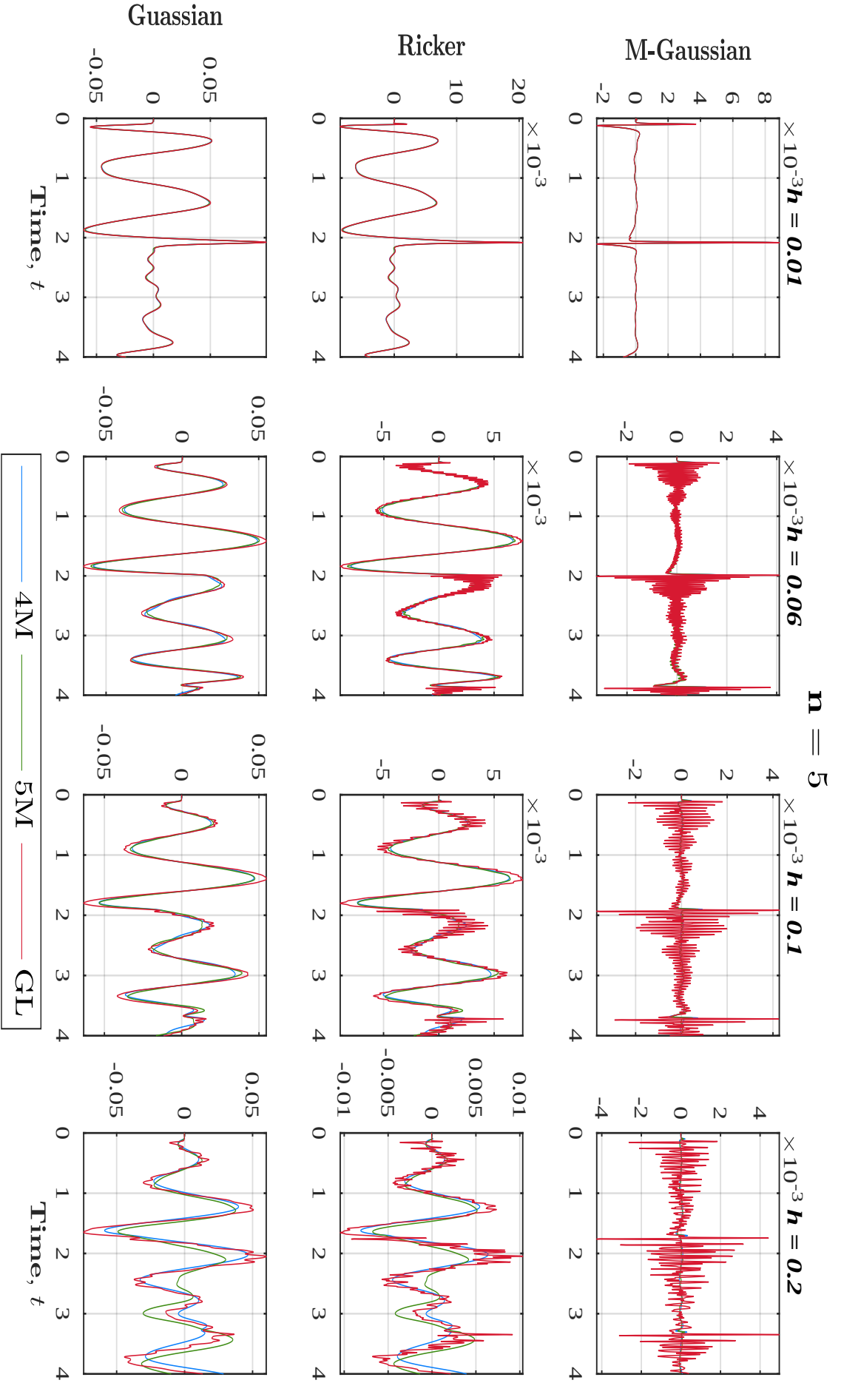


Figure 1.10: Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at its outer surface.

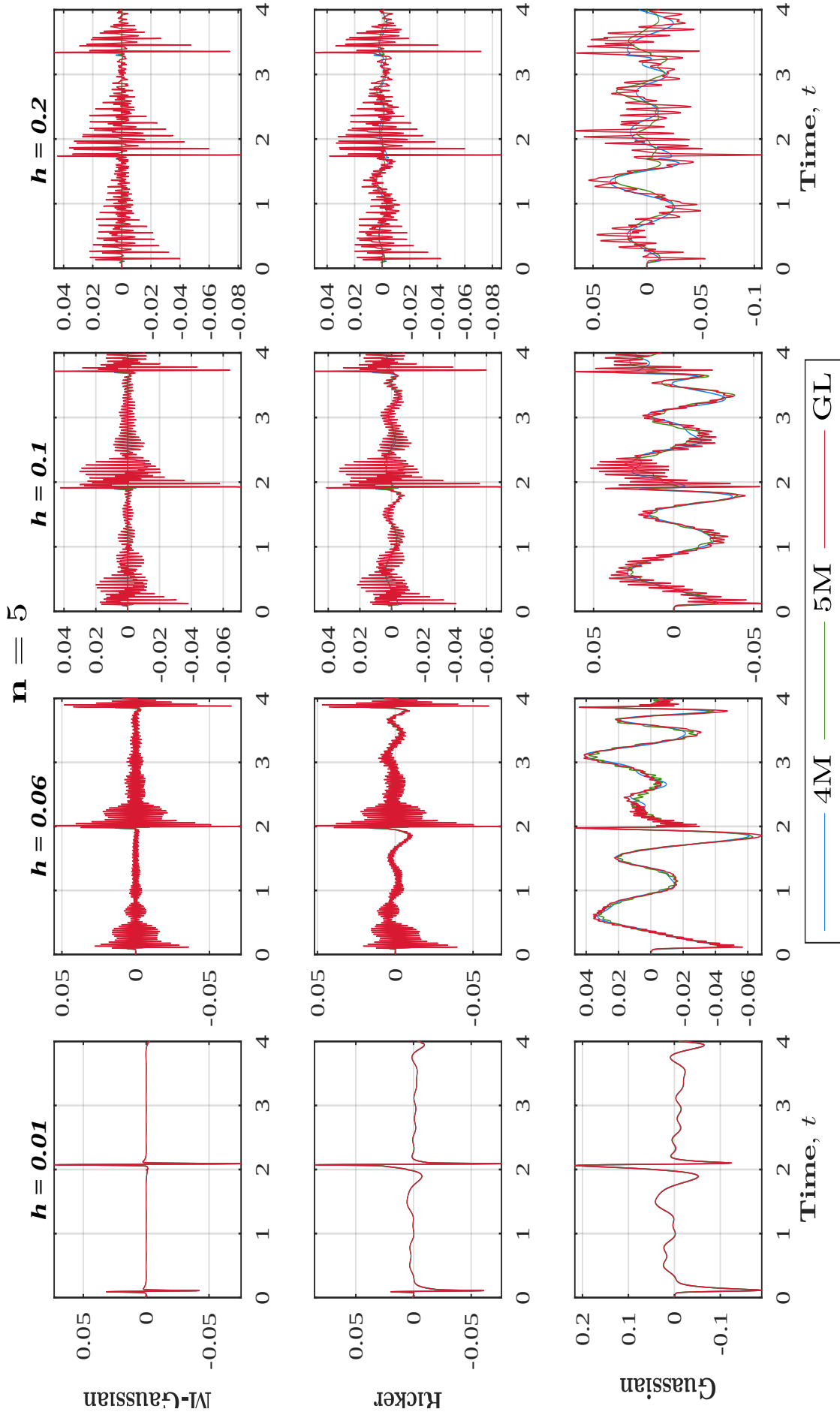


Figure 1.11: Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at its outer surface.

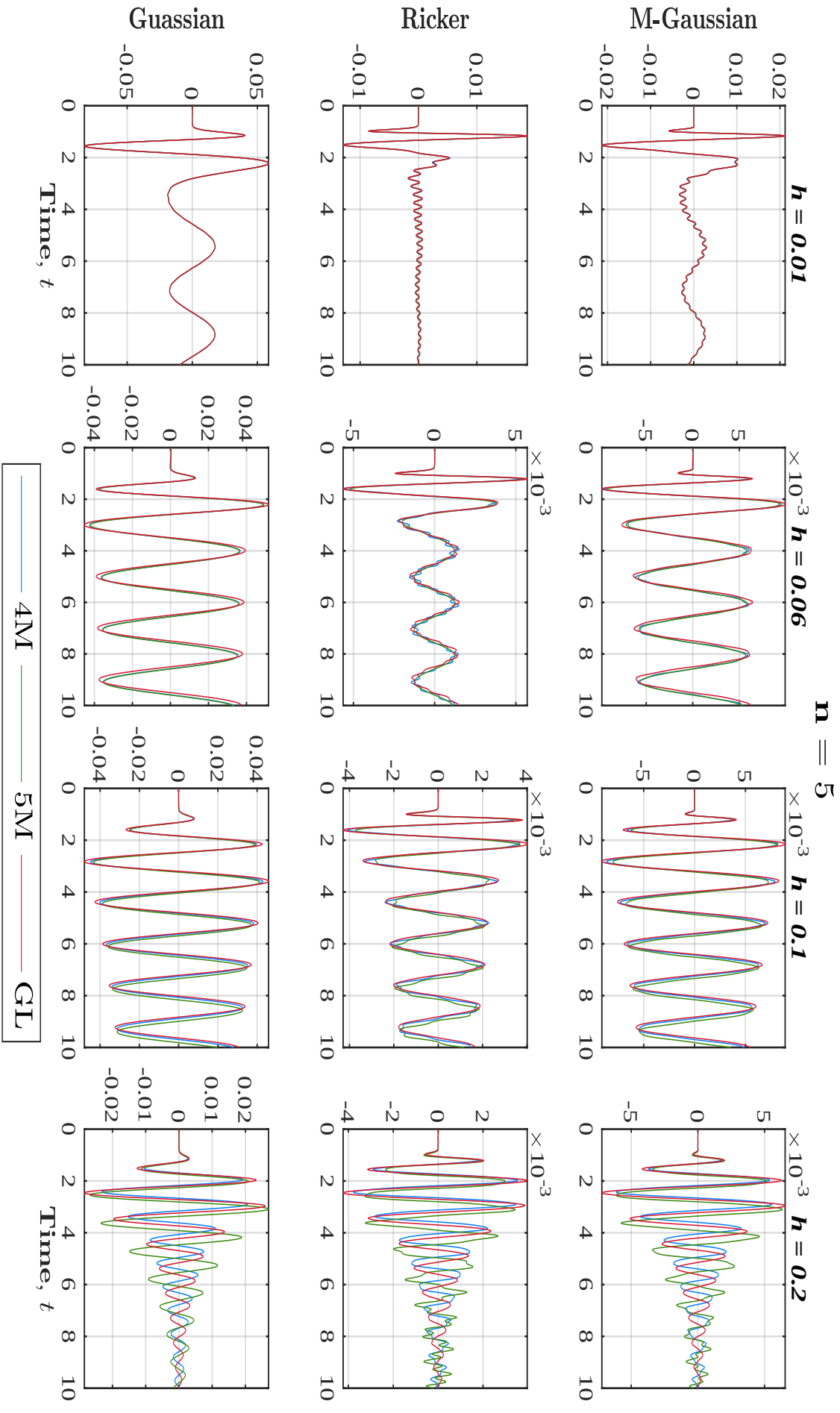


Figure 1.12: Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at its outer surface.

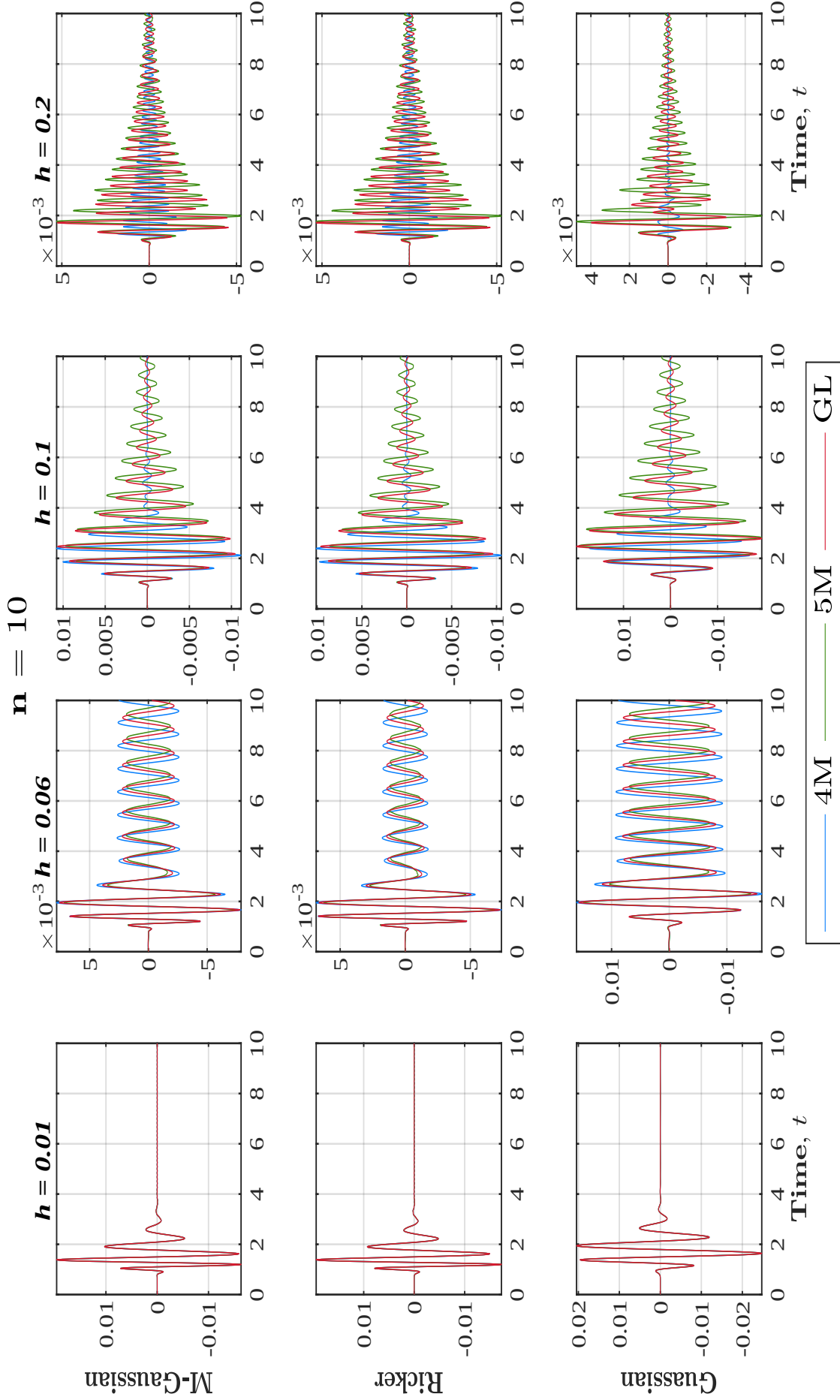


Figure 1.13: Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at its outer surface.

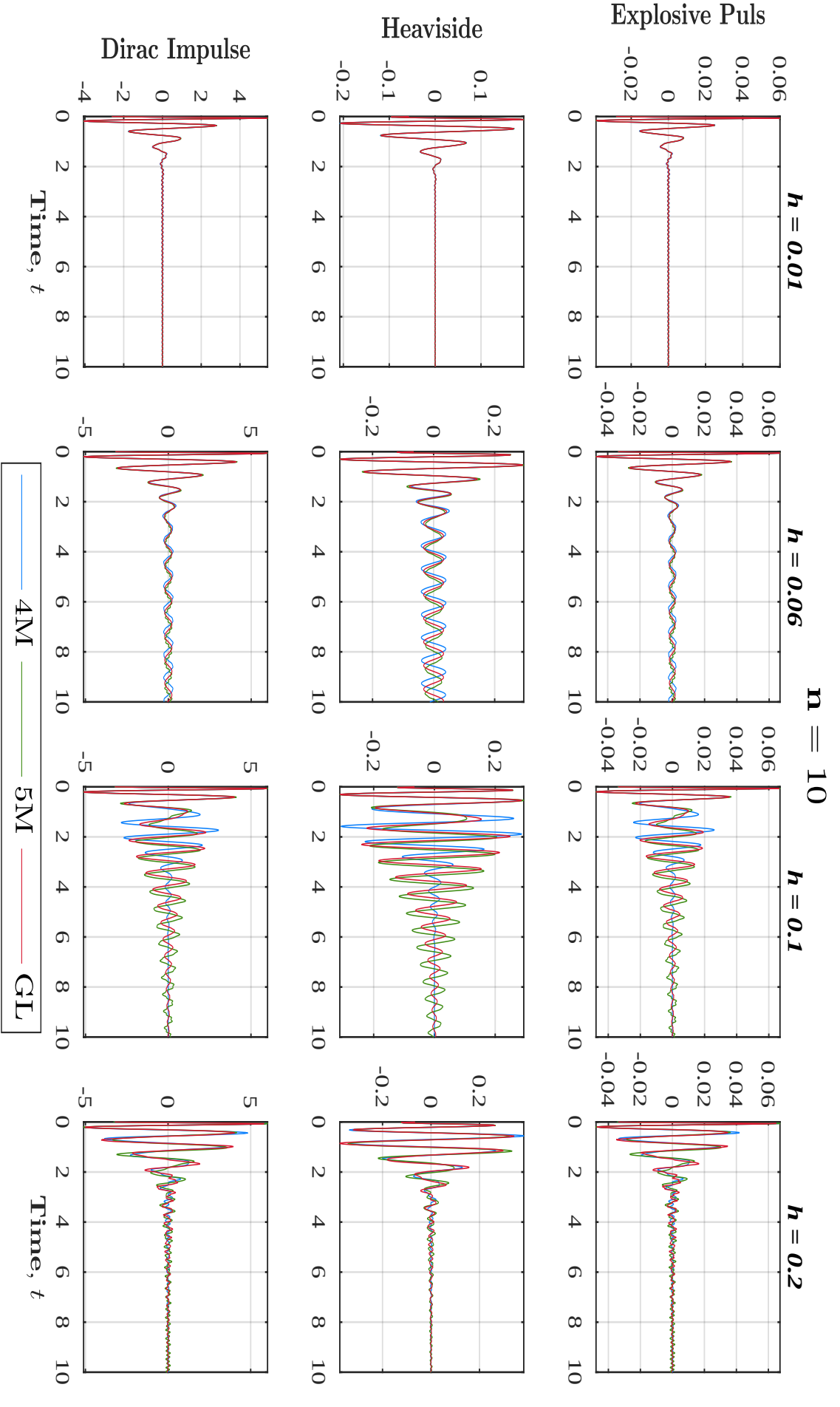


Figure 1.14: Modal time histories of the external pressure field at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at the outer surface of the shell. Three different excitations and four relative wall thicknesses are considered for $n = 20$.

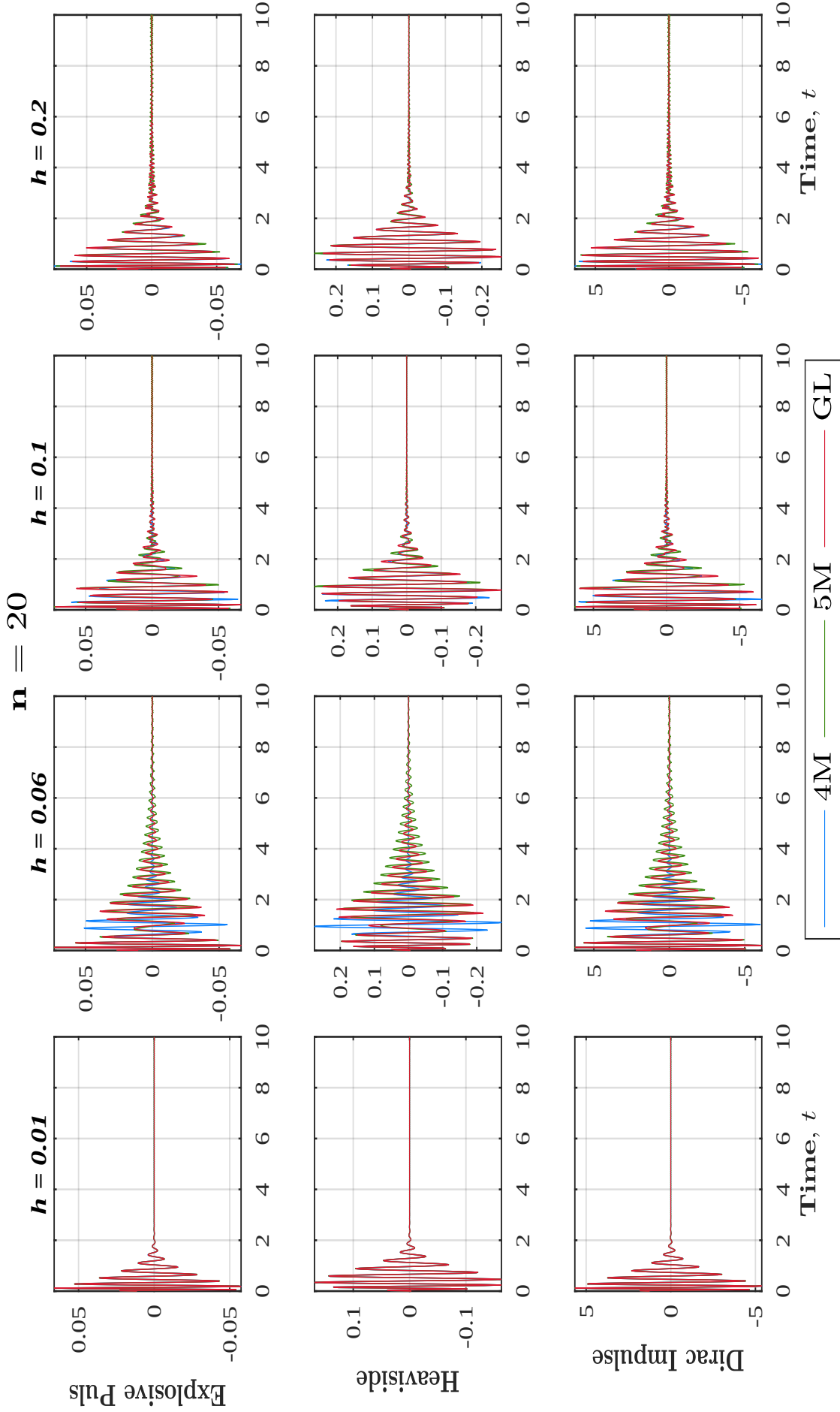


Figure 1.15: Modal time histories of the external pressure field at the outer surface of an air-filled steel shell submerged in water when subjected to a traction force at the outer surface of the shell. Three different excitations and four relative wall thicknesses are considered for $n = 20$.

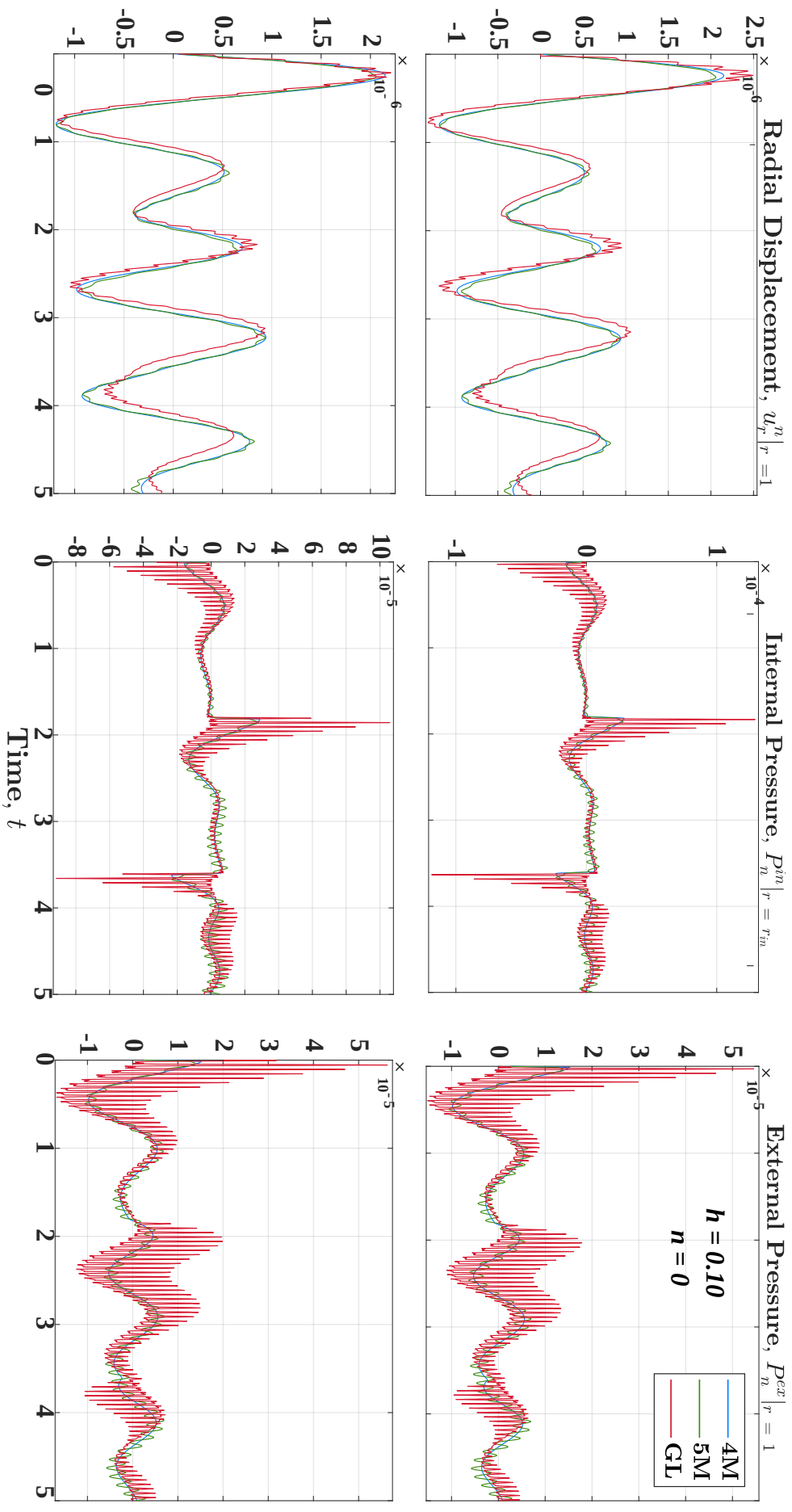


Figure 1.16: Showing the breathing mode's time histories in response of a traction force applied at (first row) outer and (second row) inner surfaces of a steel shell where $h = 0.1$ and the excitation is the broadband modulated Gaussian pulse.

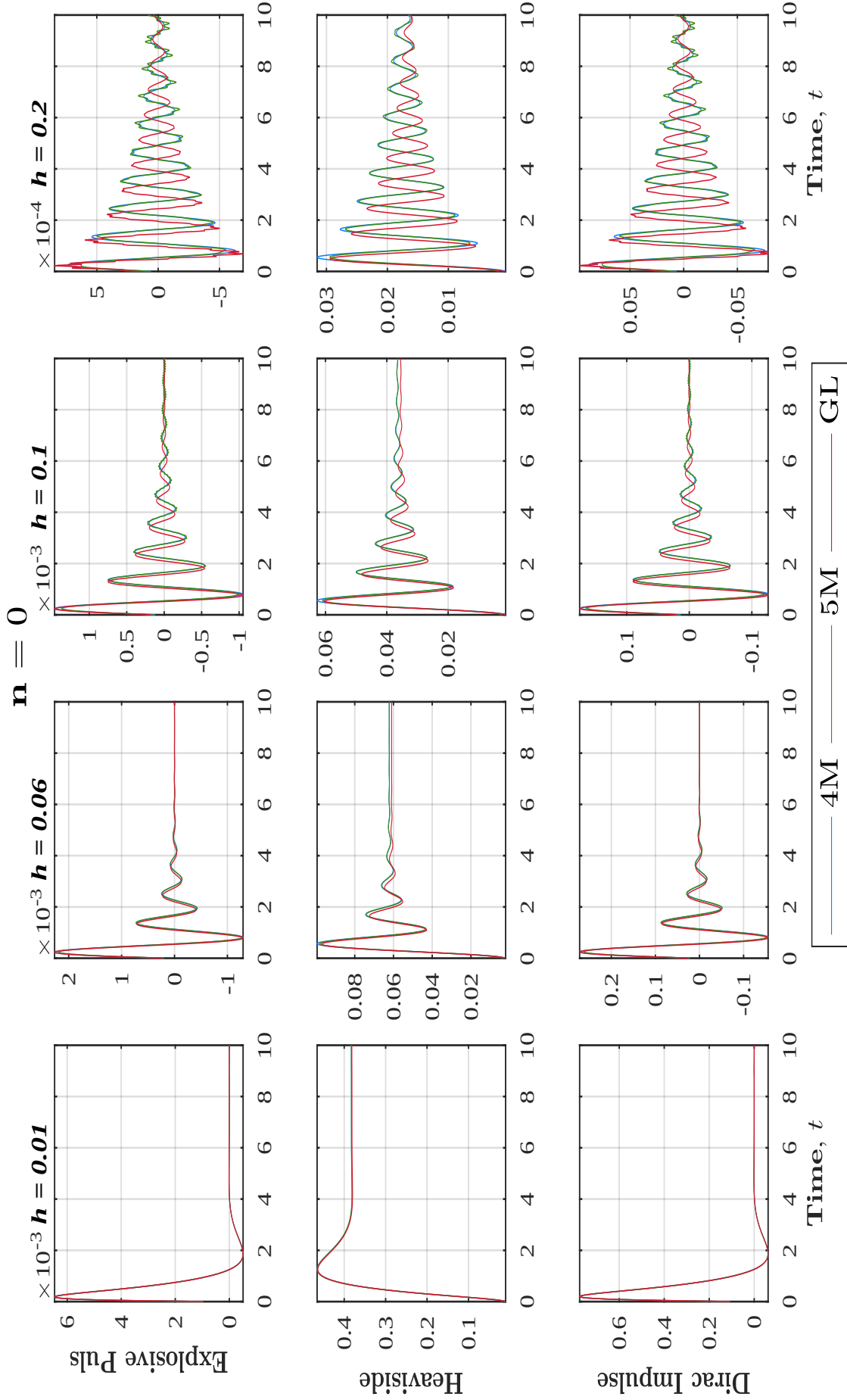


Figure 1.17: Modal time histories of the breathing mode's radial displacement at the outer surface of an air filled steel shell submerged in water when subjected to a surface traction force at the shell's external surface.

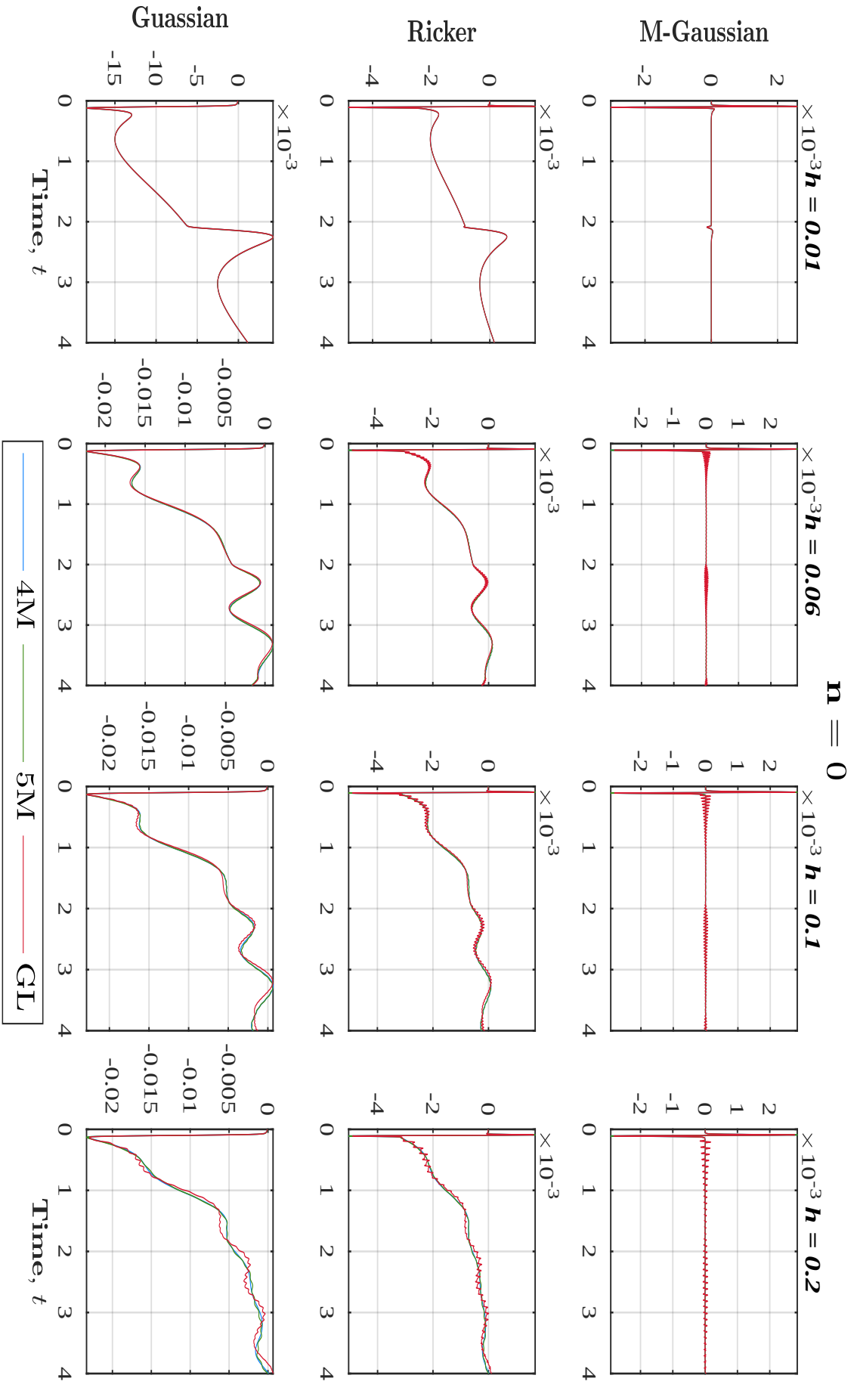


Figure 1.18: Modal time histories of the radial displacement at the outer surface of a water filled steel shell submerged in water when subjected to an plane incident wave.

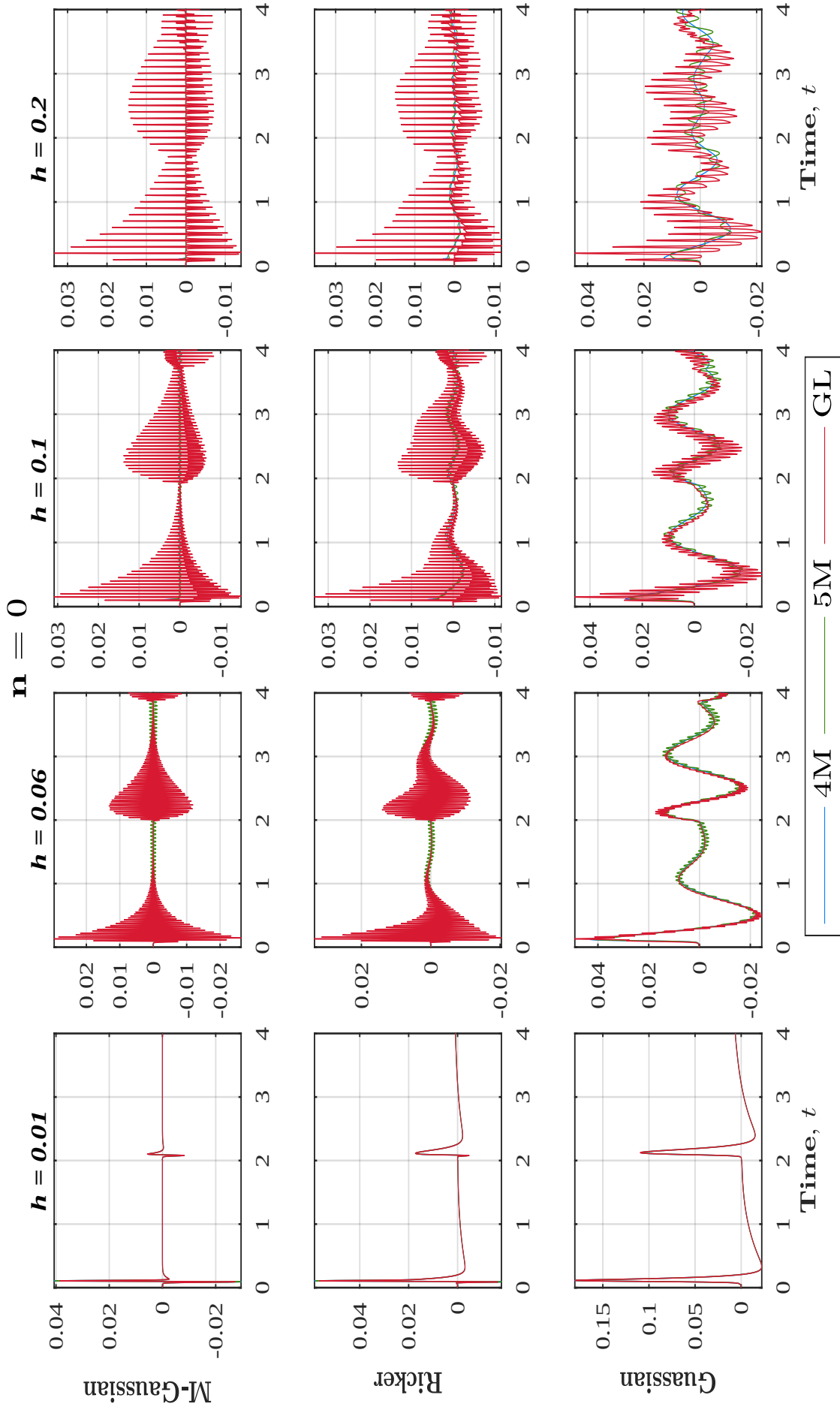


Figure 1.19: Modal time histories of the radial displacement at the outer surface of a water filled steel shell submerged in water when subjected to a traction surface load at the shell's outer surface.

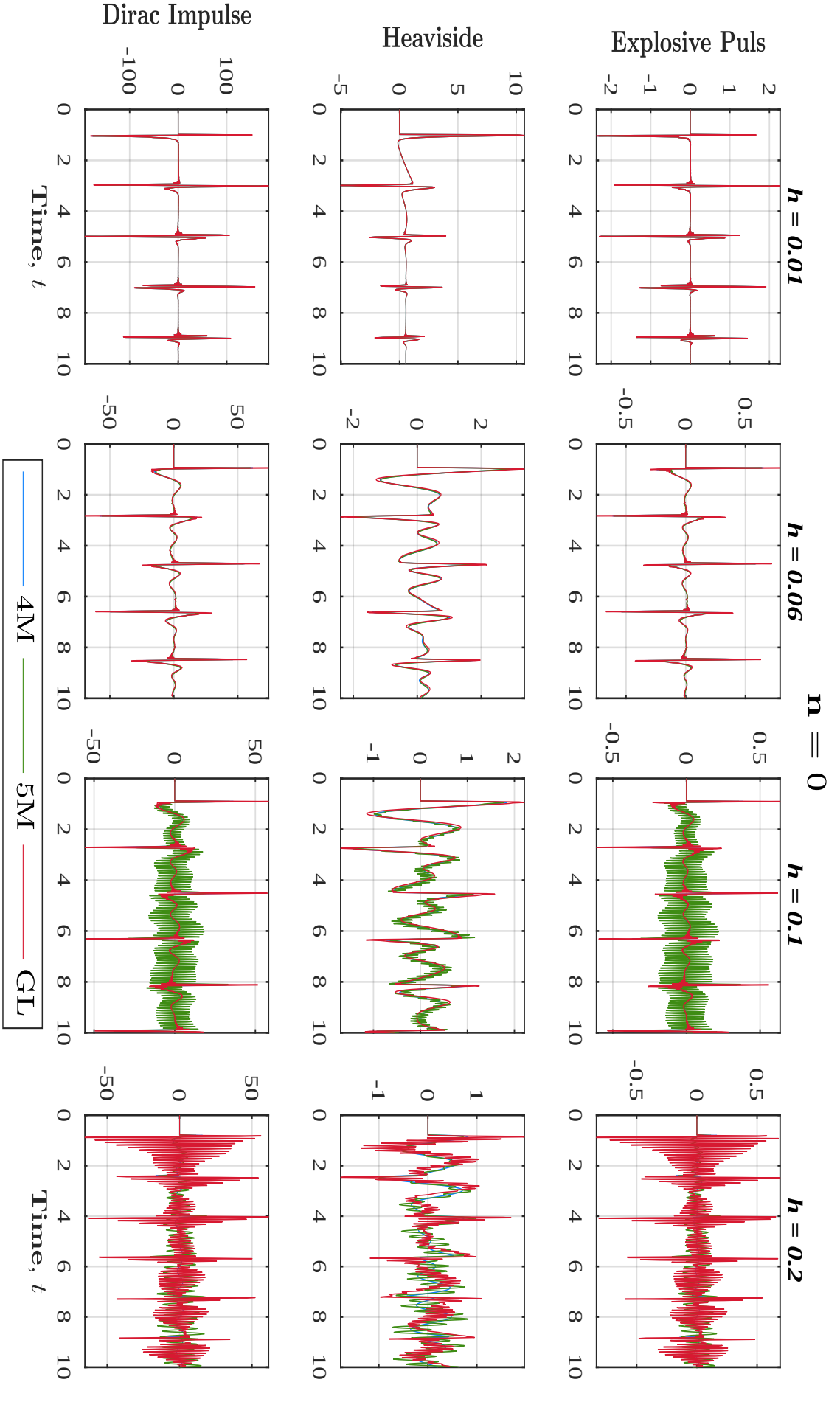


Figure 1.20: Modal time histories of the radial displacement at the outer surface of a water filled steel shell submerged in water when subjected to a traction surface load at the shell's outer surface.

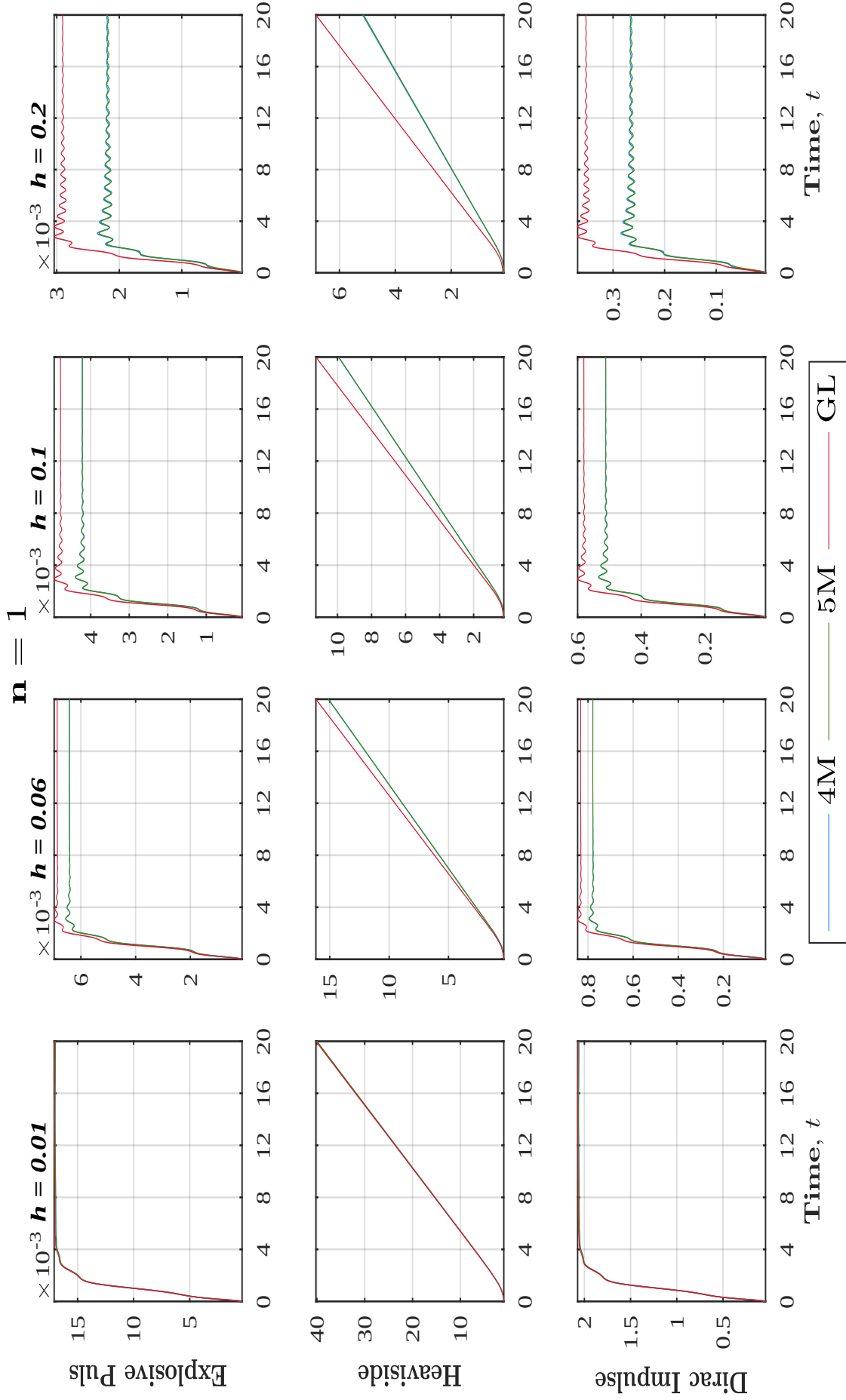


Figure 1.21: Modal time histories of the radial displacement at the outer surface of an air filled steel shell submerged in water when subjected to an plane incident wave.

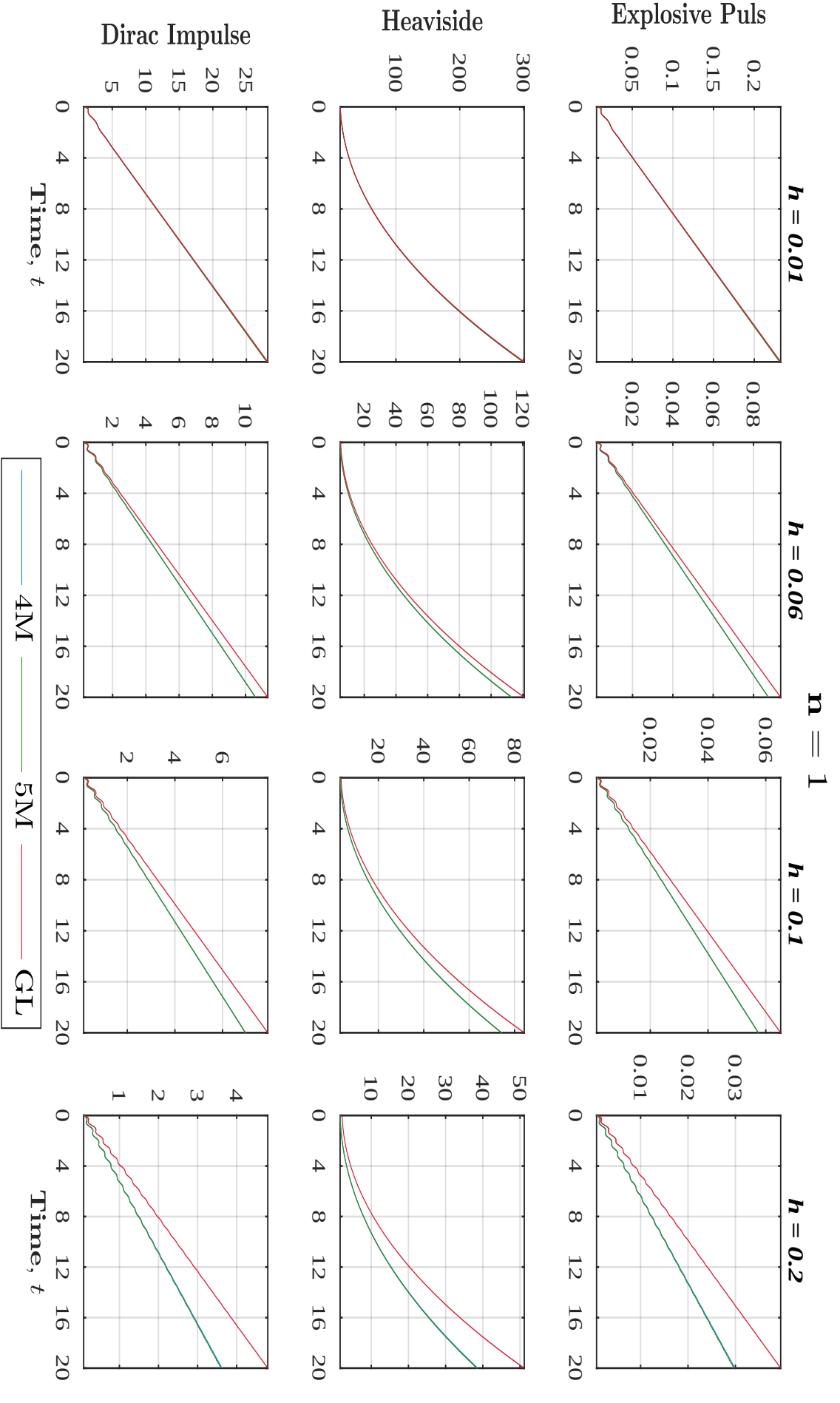


Figure 1.22: Showing modal time histories of the radial displacement at the outer surface of an air-filled steel shell submerged in water when subjected to a traction surface load at the shell's outer surface.

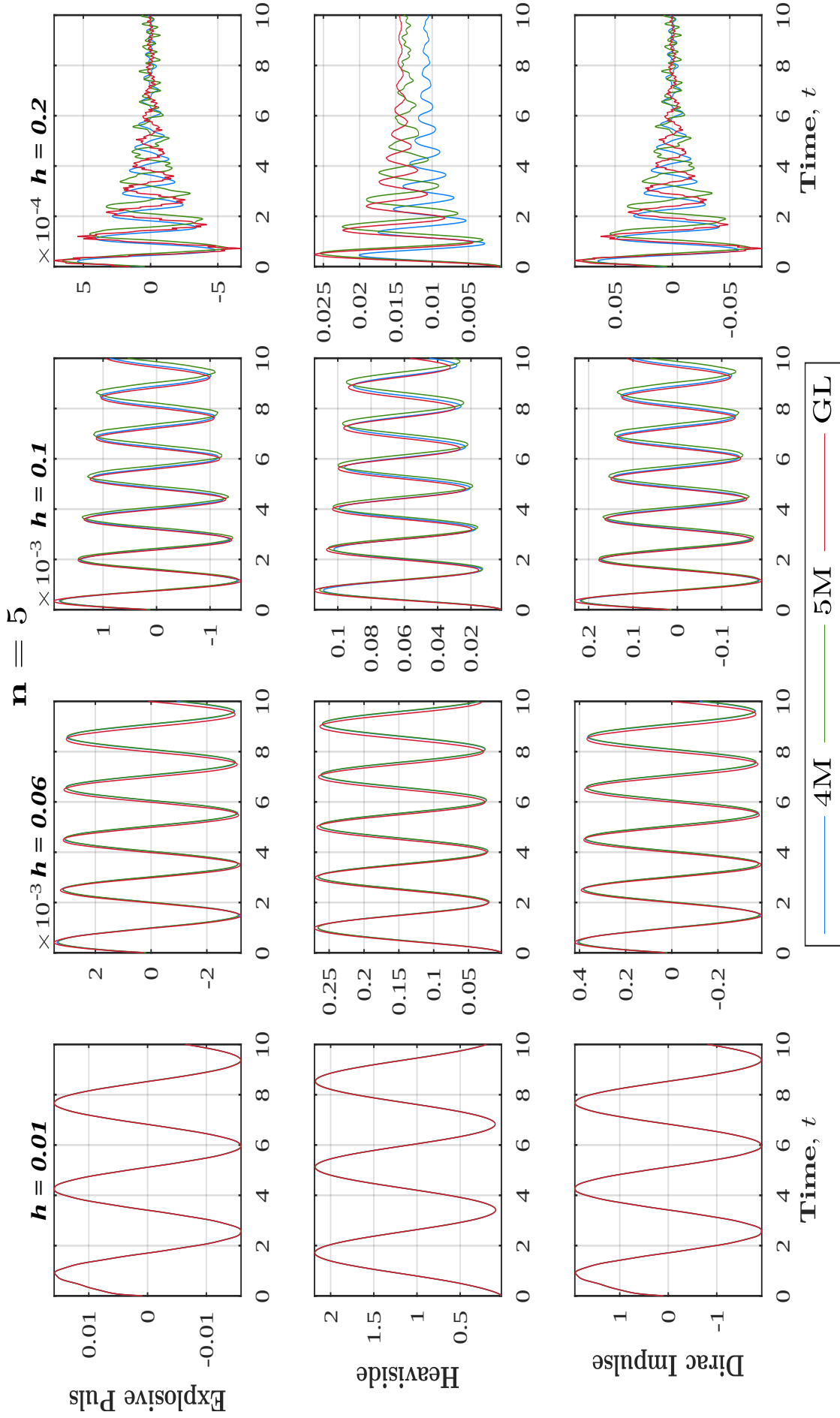


Figure 1.23: Showing modal time histories of the radial displacement at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at the shell's outer surface.

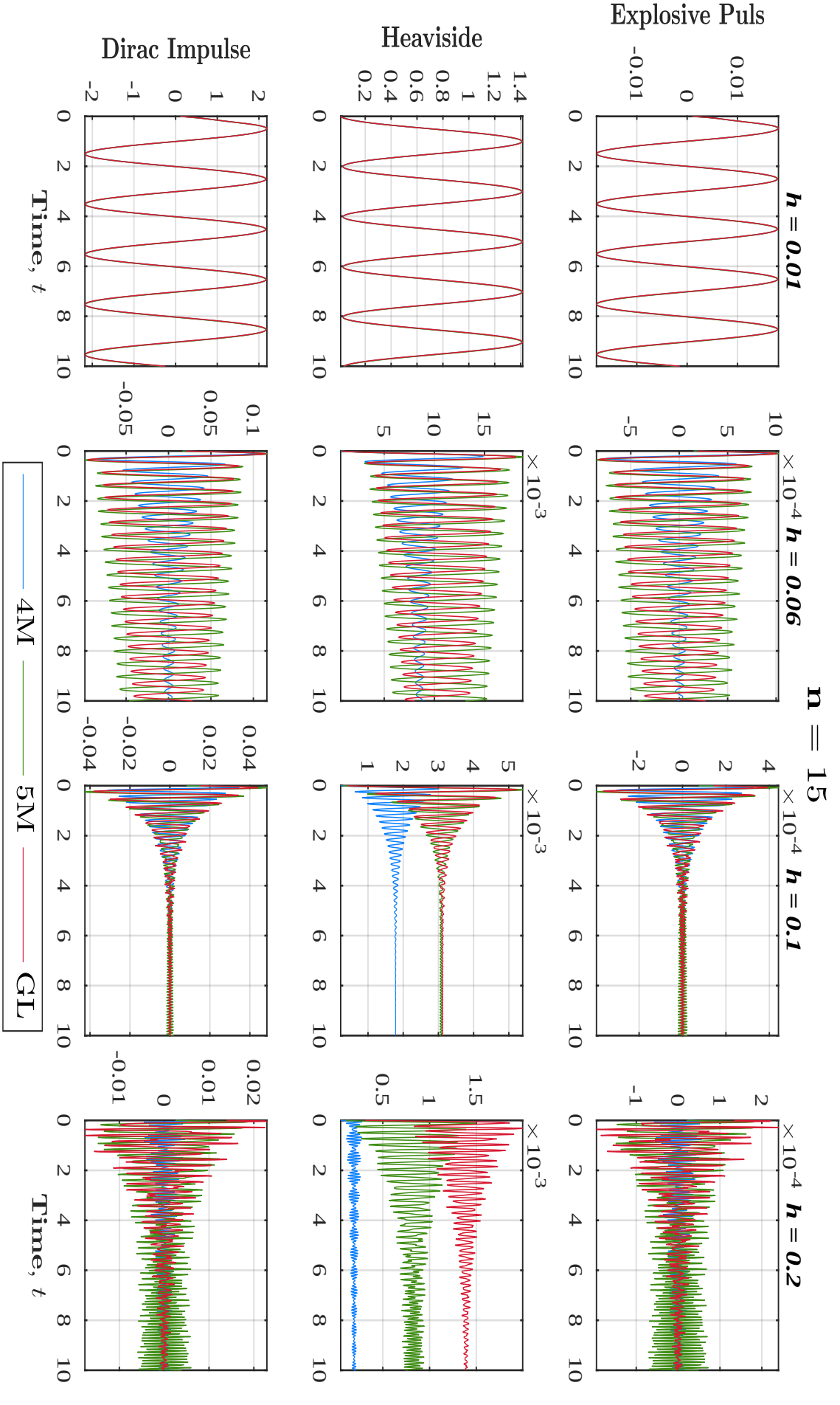


Figure 1.24: Showing modal time histories of the radial displacement at the outer surface of an air filled steel shell submerged in water when subjected to a traction force at the shell's outer surface.

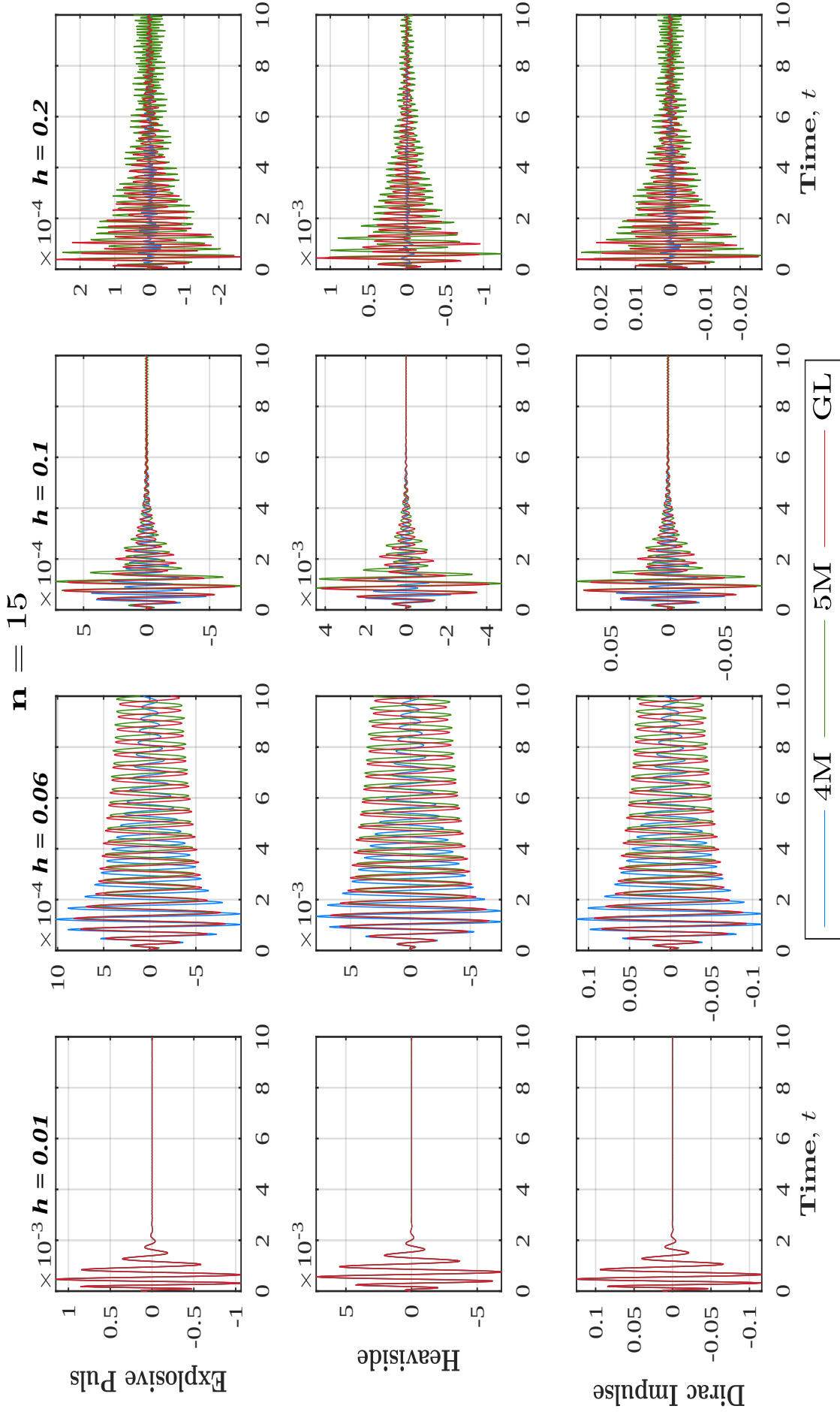


Figure 1.25: Showing modal time histories of the radial displacement at the outer surface of a water filled steel shell submerged in water when subjected to an plane incident wave.

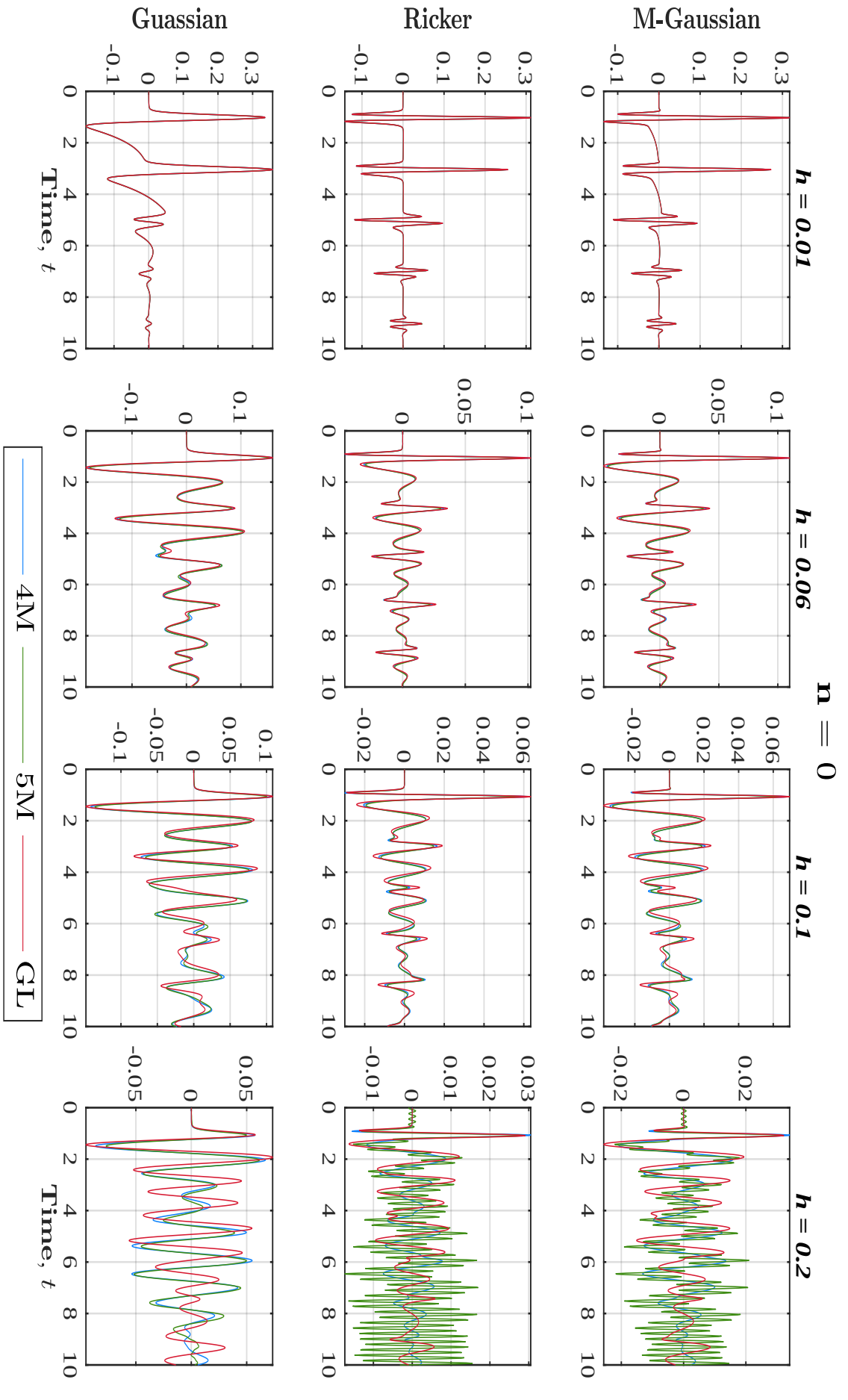


Figure 1.26: Showing modal time histories of the surface pressures at the outer surface of a water filled steel shell submerged in water when subjected to a traction surface load for $n = 0$ where the excitation's time-dependencies are in the narrowband range.

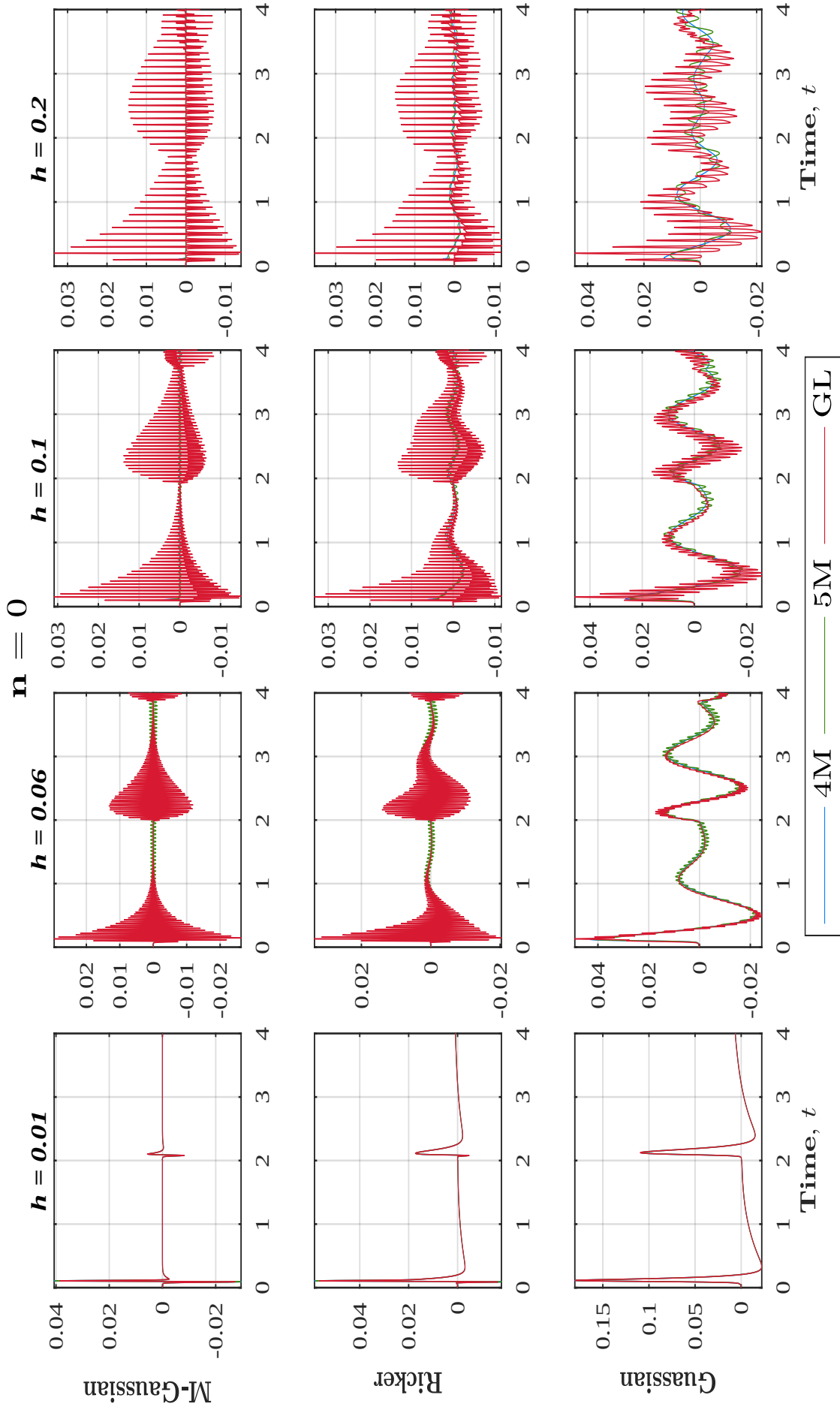


Figure 1.27: Showing modal time histories of the surface pressures at the outer surface of a water filled steel shell submerged in water when subjected to a traction surface load for $n = 0$ where the excitation's time-dependencies are in the broadband range.

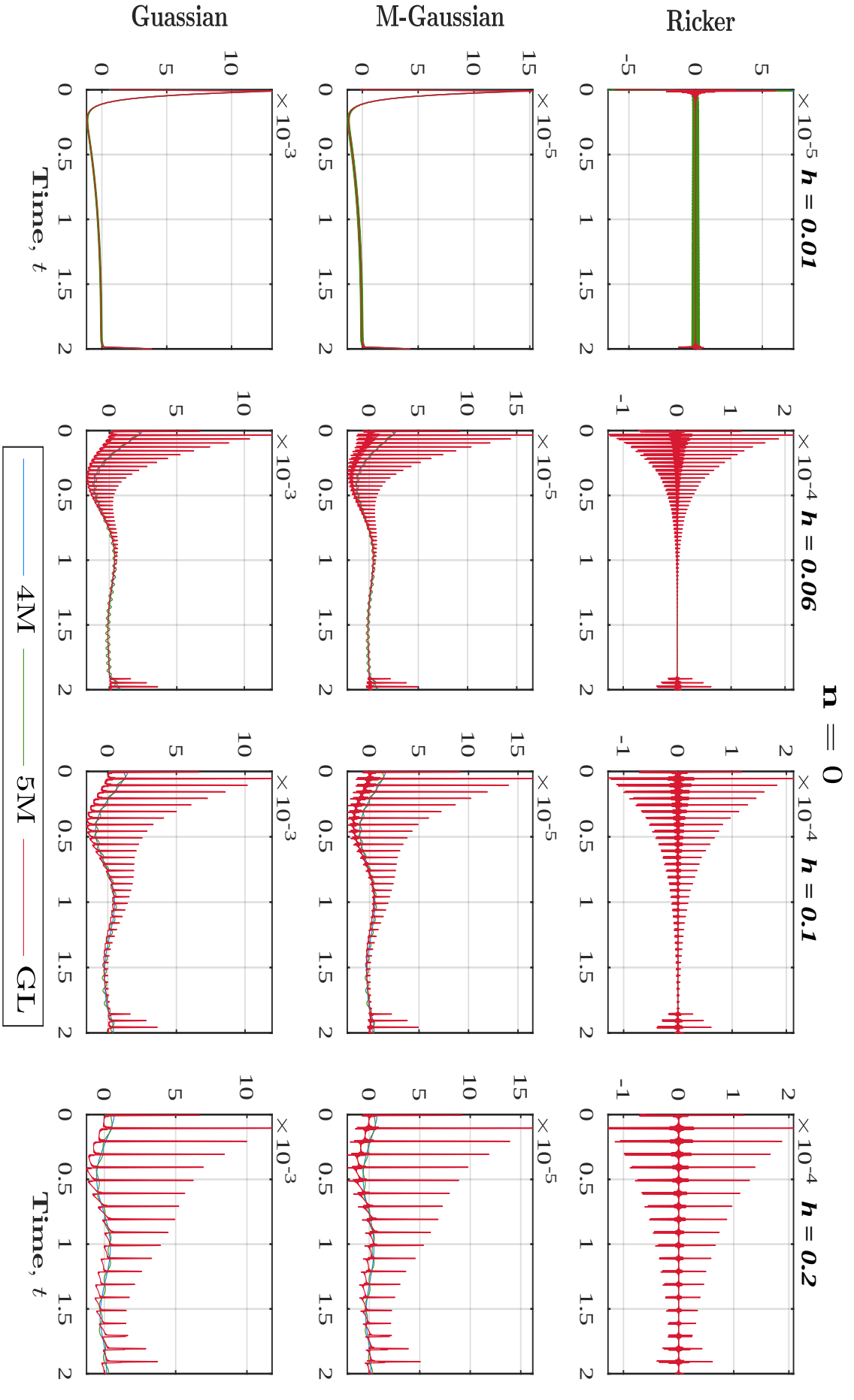


Figure 1.28: Showing modal time histories of the surface pressures at the outer surface of a water filled steel shell submerged in water when subjected to a traction surface load for $n = 0$ where the excitation's time-dependencies are in the UT-broadband range.

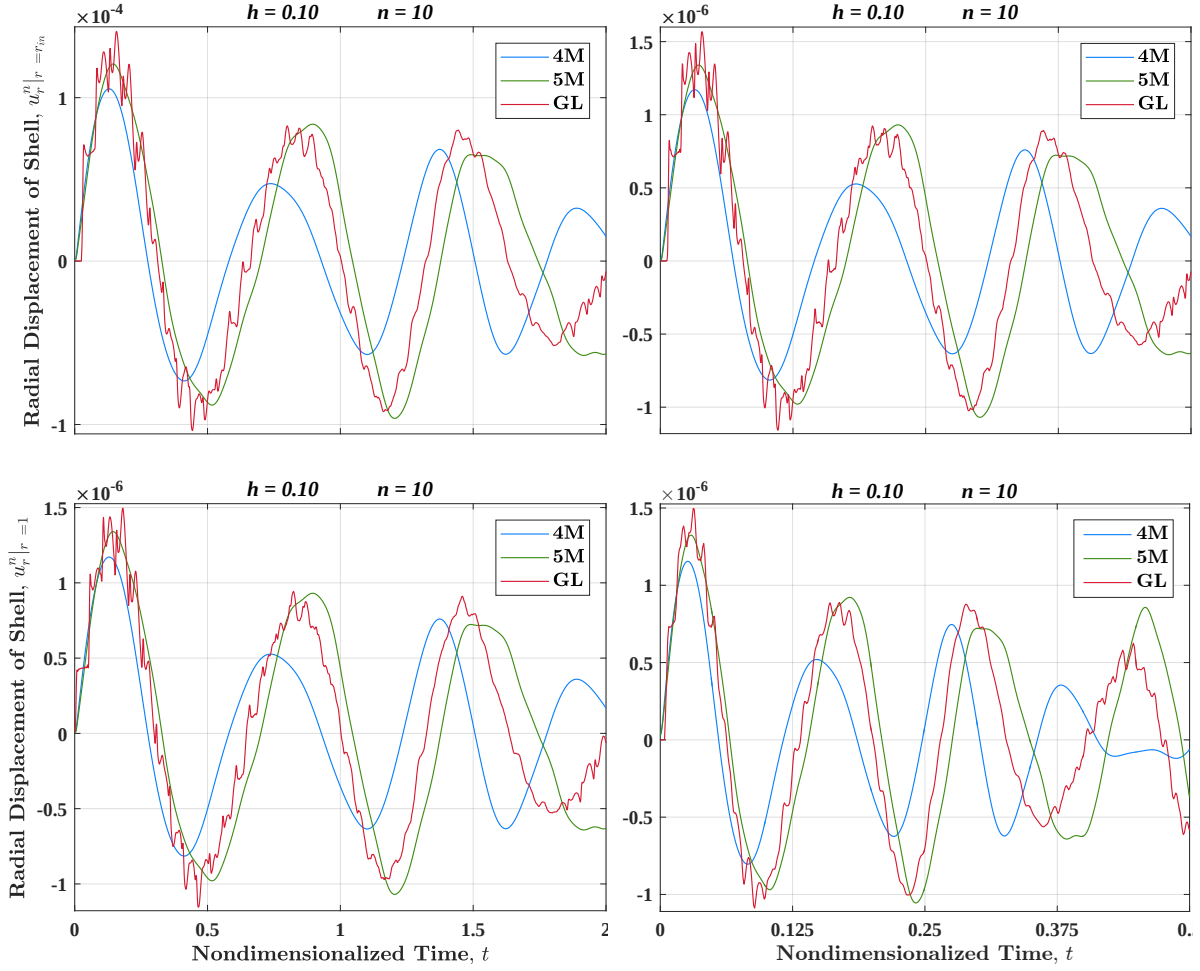


Figure 1.29: Showing modal time histories of the internal (first row) and external (second row) radial displacements of a water filled steel shell submerged in water when subjected to a traction force at its outer surface for $n = 10$. The shell wall-thickness is $h = 0.1$ and the excitation's time-dependency is a broadband Ricker pulse at UT testing range with central frequency of (first column) $\omega_r = 2000$ and (second column) $\omega_r = 6000$.

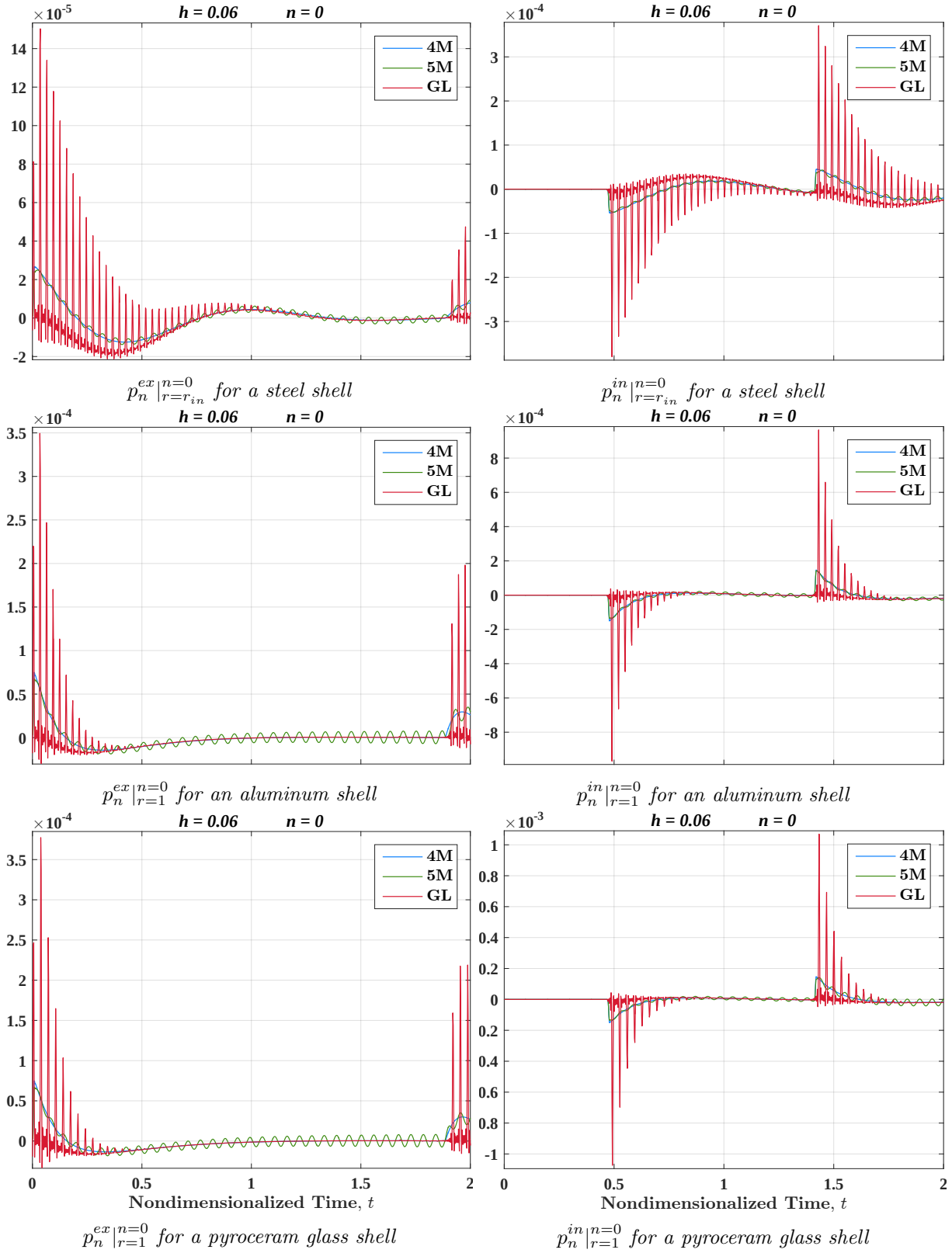


Figure 1.30: Showing the modal pressure histories associated with the breathing mode at outer (left column) and inner (right column) surfaces for a steel (first row), aluminum (second row) and pyroceram glass (third row) when filled with and submerged in water subjected to a traction force at its outer surface. The shell wall thickness and the excitation's time-dependency are invariably $h = 0.1$ and the NB Ricker pulse, respectively.

Chapter 2

Transient acoustic radiation from a spherical shell filled with and submerged in compressible fluids subjected to a centrally symmetric impact

Transient response of a spherical shell subjected to a centrally symmetric impact on its either surfaces is treated analytically. A *Hertzian* distribution profile of the loading is assumed, in addition to the commonly used uniform assumption. The shell is submerged in and filled with compressible, isothermal and inviscid fluids. Exact linear theory is applied to isentropic and isothermal acoustical perturbations. The exact local theory of linear elasticity, rather than approximate shell theories, is further applied to the shell leading to *Navier* equation which is reduced to a set of wave equations by using the *Helmholtz decomposition*. Modal expansions of all field variables are carried out in terms of *Fourier-Legendre* series. Modal responses are realized in closed-form in terms of the *modified spherical Bessel* functions in the Laplace domain. A *Cesáro summation* approach is taken into account to alleviate the slow convergence rate of *Fourier-Legendre* series. Special visualization techniques are subsequently employed to further illustrate the wave propagation and its associated physics. A number of remarkable acoustical phenomena are observed having significant practical importance. The likelihood of cavitation occurrence, for instance, is investigated. This situation is associated with the strong focusing of acoustical energy, in absence of anti-symmetric modes, within the contained fluid which is capable of amplifying the initial pulse by several folds.

2.1 Introduction

Over the last century, a great deal of interest has been shown in better understanding the coupled interaction of a transiently excited elastic structure within an acoustic medium. This is due to the wide range of applications from ultrasound imaging and nondestructive testing to underwater acoustics. However, a computationally efficient approach for modeling of the underlying acoustical interactions remains an impediment, see for example, [Perlman, 1941, Huang, 1969, Huang and Mair, 1996, Leblond and Sigrist, 2010, Hasheminejad et al., 2011, Bahari and Popplewell, 2015]. Applications of such modeling approach are quite wide and include the followings.

- *Head injury* is approximated by a doubly impacted fluid filled shell ([Engin, 1969a]).
- A fluid filled spherical shell has been utilized as an approximation of human *eyeball* ([Navarro et al., 1985, Von Ramm et al., 1991, Shih and Guo, 2016]).
- *Diagnostic ultrasound* uses such solution as a highly accurate forward kernel in a wave field inversion scheme for imaging of cysts and other anomalies ([Grant and West, 1965, Grandjean and Leparoux, 2004, Wirgin, 2016]).

- *Nondestructive testing* in ultrasonic immersion testing of submerged structures such as offshore oil and gas containers ([Krautkrämer and Krautkrämer, 2013]).
- *Underwater target identifications* in active and passive sonars ([Gaunaud et al., 1995]).
- Shell structure under *impact and shock* to be accounted in their safe design ([Sprague and Geers, 1999]).

Theoretical investigations can be categorized broadly into numerical studies and analytical solutions. Some specific numerical methods have been developed for this type of coupled elasto-acoustic problems in lieu of conventional FEM and BEM. For example, Geers developed Doubly Asymptotic Approximation (DAA) in [Geers, 1978, Geers and Zhang, 1994] or [Leblond and Sigrüst, 2010] applied a boundary integral approach *in-vacuo* eigenvectors were used. The current review of literature is, however, restricted to the analytical or semi-analytical solutions that have been carried out almost exclusively by using integral (Laplace or Fourier) transform(s). For example, Ref. [Engin, 1969a] investigated the dynamic response of a fluid-filled spherical shell subjected to a localized radial impulse in order to model a human head exposed to an external impact. Since numerical accuracy of external and internal fluid fields' responses highly depend on the accuracy of the shell model, a special attention is given to this aspect of the problem. An idyllic *four-mode* shell theory was used in conjunction with the linearized acoustic wave equation to which a Laplace transform was introduced to obtain the head's axisymmetric, transient response. (It is worth recalling that the conventional terminology of *four-* and later *five-mode* shell theories should not be confused with the number of the modes used in the summation of the *modal expansion*. *Four-mode* shell theory refers to the classical bending model whereas, if rotary inertia and shear effects are also included at the shell's mid-surface, it is termed the *five-mode* shell theory. Details may be found in the work of Leissa [Leissa, 1973]). Lou and Klosner [Lou and Klosner, 1973] studied the radiated acoustic pressure from a transiently excited spherical shell and compared both the plane and spherical wave approximations with the exact solution. They employed the same shell theory by Ref. [Engin, 1969a] to determine the transient response produced by a concentrated load having a Heaviside or Delta Dirac impulse time variation. Later, Menton and Magrab [Menton and Magrab, 1974] applied a Fourier transform, in conjunction with a *five-mode* shell theory, to find the transient, far-field pressure radiation from a submerged, fluid-filled spherical shell. A variable frequency increments was utilized in their numerical inversion procedure of the Fourier transform as an important feature of their inversion procedure. Ref. [Chen and Stepanishen, 1994] proposed a method by using an *in-vacuo* modal eigenvector expansion of field variables having time dependent coefficients. Based on the *four-mode* shell theory, the transient acoustic radiation was investigated for a fluid-loaded, empty spherical shell excited by an axisymmetric, impulsive load. Later, Zakout [Zakout, 2001] gave an interesting closed-form reformulation of the work of [Lou and Klosner, 1973] for a distributed Heaviside load. More recently, some discrepancies in the numerical results of Zakout [Zakout, 2001] were rectified by Bahari and Popplewell [Bahari and Popplewell, 2015], who also computed the impulse response of the radiated field as the Green's function in the time domain. [Hasheminejad et al., 2011] investigated the scattering of a transient, incident acoustic wave by a fluid-filled, spherical shell using the Laplace transform. An exact 3D elasticity model was utilized for the shell and the inversion of the Laplace transform was performed numerically. Ref. [Hasheminejad and Mousavi-Akbarzadeh, 2015] recently analyzed the

transient acoustic radiation from an eccentric sphere and a spherical shell in an acoustic wedge, respectively by employing the numerical inversion procedure detailed in Ref. [Hasheminejad et al., 2011]. However, the previously slow convergence rate was improved here by using the σ -factor filter, which is due to *Lanzcos* ([Laverty and Gazonas, 2006]).

This review of literature indicates that so far there has not been any study on a doubly impacted fluid filled and submerged spherical shell in time domain. The closest work is [Engin, 1969a] where a fluid-filled but in-vacuo thin shell was used to model a head injury. An approximate thin shell theory known as the *four-mode* was employed in which the bending and membrane effects were included. In the current work, however, an exact local theory of linear elasticity is applied to the shell leading to significant improvement in the accuracy of the fluid field responses. The effects of external fluid on the physics of the problem are investigated as well. Further, *Hertzian* distributions of the surface traction forces are considered in addition to the commonly-used uniform profile.

The order of the organization of this chapter is as follows. The mathematical formulation section starts by applying the exact isothermal, inviscid linear wave equation both fluid media to describe acoustical perturbations. The *Navier* equation is employed to elastic shell as the consequence of exact linear theory. Fields' modal responses are realized for elastic and fluid media in closed-form in terms of the *modified spherical Bessel* functions. After evaluating the inversion of the Laplace transform, the total responses are obtained by superposing the modal components. The current solutions and the associated computer codes are compared against the available data obtained from the pertinent literature. An extensive set of numerical experimentation is carried out by considering different types of excitation and geometrical parameters. Special visualization techniques as well as quantitative presentations of the field responses are then used to provide an in-depth description of the physics of interactions.

2.2 Mathematical Modelling

The wave propagation in shell's elastic medium is described by shear and compressional wave equations after the reduction of Navier equation by using the Helmholtz decomposition. The coupling of internal and external fluid fields is obtained by continuity of Cauchy traction stress vectors and particle velocities at the shell's either surfaces. A *Fourier-Legendre* modal expansion is applied to all field variables and corresponding unknown modal coefficients are settled by satisfying the boundary conditions at traction surfaces. The total solution is achieved through the modal summation, where a *matrix summability method* is applied to enhance the convergence rate of the associated *Fourier Legendre* series.

2.2.1 Preliminaries

Let's assume that a spherical shell (later labelled with subscript s) is filled internally (*in*) with a fluid and is submerged externally (*ex*) inside the same or a different fluid. The fluids are considered to be respectively inviscid and compressible. The shell itself is subject to a surface traction force or an incident plane wave both of which are transient. This configuration is illustrated in Figure 2.1. These excitation amplitudes are sufficiently weak that the assumption of linearity is reasonable. Consequently, wave phenomena in either fluid medium is characterized by an unperturbed mass density and sound speed denoted, respectively, by ρ_f^* and c_f^* where $f = in, ex$. The elastic shell has an inner (outer) radius of $r_{in}^*(r_{ex}^*)$ and it is assumed to be homogeneous and isotropic. The

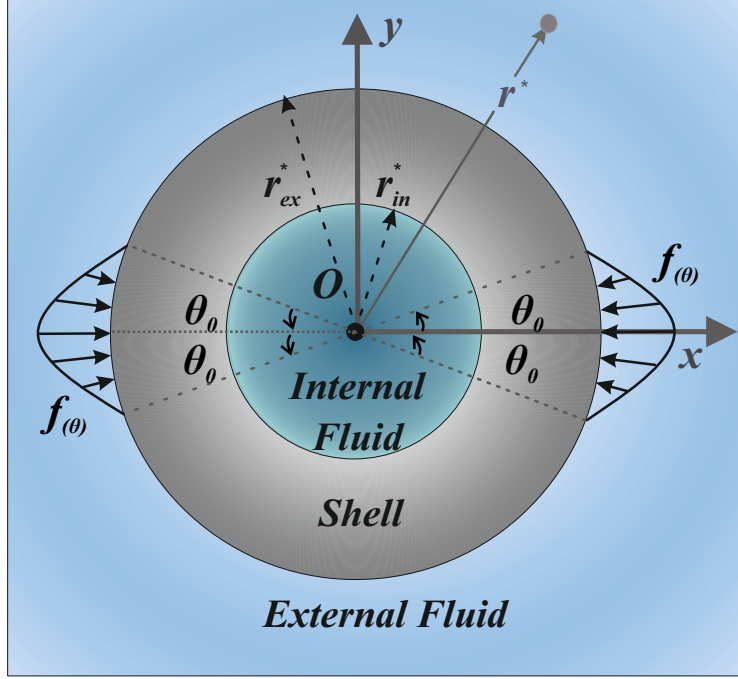


Figure 2.1: depicting the problem's geometry and the double impact excitation. Azimuthal angle φ is perpendicular to the illustrated plane.

mass density and Lamé constants of the shell are ρ_s^* and μ^*, λ^* . These and other parameters are nondimensionalized. Hence, length parameters including the displacements and distances from the sphere's center, are taken with respect to r_{ex}^* . Further, stresses and pressures are given relative to $\{\sigma_0^* = \rho_{ex}^* c_{ex}^{*2}\}$ whereas time t^* is nondimensionalized with respect to $\{t_0^* = r_{ex}^*/c_{ex}^*\}$. The following dimensionless quantities, with no star superscripts, are used throughout the current study. They are the same as those given by Equation 1.1 in § 1.2.1.

A Laplace transform is applied to all time dependent field variables. The Laplace transform for an arbitrary function, $\tilde{\Lambda}(t)$, exists if $|\tilde{\Lambda}(t)| < +\infty$ as $t \rightarrow +\infty$. This transform and its complex inversion are defined respectively as

$$\Lambda(s) = \int_0^\infty \tilde{\Lambda}(t) e^{-s t} dt \quad \text{and} \quad \tilde{\Lambda}(t) = \frac{1}{2\pi i} \lim_{\eta \rightarrow \infty} \int_{\gamma - i\eta}^{\gamma + i\eta} \Lambda(s) e^{s t} ds. \quad (2.1a)$$

where η is an arbitrary real number greater than the real parts of all the singularities of $\Lambda(s)$. An algorithm based on a Fourier series and trapezoidal integration rule is utilized for the inverse Laplace transform. Subsequently, the estimated inversion integration in Equation 2.1a becomes

$$\tilde{\Lambda}_{N_d} \approx \frac{e^{at}}{T_0} \left(\frac{\Lambda(a)}{2} + \sum_{j=1}^{N_d} \Xi_j(t) \right), \quad t \in [0, 2T_0] \quad (2.2)$$

that

$$\Xi_j(t) = \varrho_j \left[\Re\{\tilde{\Theta}_j(t)\} \cos b_j(t) + \Im\{\tilde{\Theta}_j(t)\} \sin b_j(t) \right], \quad (2.3)$$

where

$$\tilde{\vartheta}_j(t) = \Lambda [a + i b_j(t)], \quad b_j(t) = \frac{j\pi}{T_0}t,$$

in which ϱ_j , a filter due to *Lanzcos*, is $\varrho_j = \sin\left(\frac{j\pi}{N_d}\right)/\left(\frac{j\pi}{N_d}\right)$.

2.2.2 Excitation Source

The traction force, $\tilde{\mathbf{f}}^i \equiv \tilde{\mathbf{f}}^i(\theta, \varphi, t)$, which is normal to either the shell's inner or outer surface, generally takes the vector form of

$$\tilde{\mathbf{f}}^i = \tilde{f}_r^i(\theta, \varphi, t)\mathbf{e}_r + \tilde{f}_\theta^i(\theta, \varphi, t)\mathbf{e}_\theta + \tilde{f}_\varphi^i(\theta, \varphi, t)\mathbf{e}_\varphi. \quad (2.4)$$

Assuming that the traction force is time and spatially separable, the radial traction force can be expressed as

$$\tilde{f}_r^i(\theta, t) = f_r^i(\theta)\tilde{f}_t^i(t), \quad i = in, ex. \quad (2.5)$$

The $f_r^i(\theta)$ in the last equation can be expanded in terms of the orthogonal Legendre polynomials $P_n(\cos \theta)$ as

$$f_r^i(\theta) = \sum_{n=0}^{\infty} f_n^i P_n(\cos \theta), \quad (2.6)$$

where f_n^i is the n^{th} modal component of $f^i(\theta)$ that is given by

$$f_n^i = \frac{2n+1}{2} \int_0^\pi f_r^i(\theta) P_n(\cos \theta) \sin \theta d\theta. \quad (2.7)$$

By combining Equation 2.5 with Equation 2.7, the Laplace transform of $\tilde{f}_r^i(\theta, t)$ becomes

$$f_r^i(\theta, s) = \sum_{n=0}^{\infty} \xi_n^i P_n(\cos \theta), \quad (2.8)$$

where $\xi_n^i = f_n^i f_t^i(s)$. The spatial distribution of an axisymmetric, double normal impact on a shell's either surfaces, as illustrated in Figure 2.1, according to the *Hertz* elastic contact theory reduces to

$$f_r^i(\theta) = \frac{3}{2\pi} \sqrt{1 - \left(\frac{r_i}{r_c} \sin \theta\right)^2} \begin{cases} 1 & -\theta_c \leq \theta \leq \theta_c \\ 0 & \theta_c \leq \theta \leq \pi - \theta_c \\ 1 & \pi - \theta_c \leq \theta \leq \pi + \theta_c \end{cases} \quad (2.9)$$

where the contact angle, θ_c , and the radius of the contact circle, r_c , are

$$r_c = \sqrt[3]{\frac{3Fr_i(1-\nu^2)}{4E}}, \quad \theta_c = \sin^{-1}\left(\frac{r_c}{r_i}\right),$$

in which F , E and ν are the resultant contact force, shell's Young modulus and Poisson's ratio, respectively [Sneddon, 1965]. Only even modes stay nonzero due to a central symmetry of loading. Consequently, Equation 2.8 becomes

$$f_r^i(\theta, s) = \sum_{n=0}^{\infty} \xi_{2n}^i P_{2n}(\cos \theta). \quad (2.10)$$

An assumption of constant distribution of contact forces over contact surfaces,

$$f_r^i(\theta) = \begin{cases} 1 & -\theta_c \leq \theta \leq \theta_c \\ 0 & \theta_c \leq \theta \leq \pi - \theta_c \\ 1 & \pi - \theta_c \leq \theta \leq \pi + \theta_c \end{cases}, \quad (2.11)$$

is very common due to its consequent considerable simplifications in calculating the modal components of a load. [Zakout, 2001, Bahari and Popplewell, 2015]. The modal component of this uniform loading by using the integration given in Equation 2.7 becomes

$$f_n^i = \frac{1}{2} \left(P_{n-1}(-1) - P_{n-1}(1) - P_{n-1}(-\beta) + P_{n-1}(\beta) + P_{n+1}(-\beta) - P_{n+1}(\beta) \right), \quad (2.12)$$

where $\beta = \cos \theta_c$. The integration given in Equation 2.7, however, is required to be taken numerically when a Hertzian distribution is assumed. A quadrature integration of Gaussian-Legendre kind is utilized such that

$$f_n \equiv \int_a^b g_n(\theta) d\theta \approx \frac{b-a}{2} \sum_{j=1}^m w_j^{\mathbb{G}} g_n \left(\frac{b-a}{2} t_j^{\mathbb{G}} + \frac{b+a}{2} \right) \quad (2.13)$$

for $g_n(\theta) = (2n+1)f_r(\theta) \sin \theta P_n(\cos \theta)/2$ where

$$P_{m+1}(t_j^{\mathbb{G}}) = 0 \quad \text{and} \quad w_j^{\mathbb{G}} = \frac{2(1-t_j^{\mathbb{G}})}{(1+m)^2 (P_{m+2}(t_j^{\mathbb{G}}))^2}. \quad (2.14)$$

The $w_j^{\mathbb{G}}$ and $t_j^{\mathbb{G}}$ are the corresponding weights and nodes. It provides an exactness degrees of $2m-1$. Satisfactory results are obtained by using $m=5$.

2.2.3 Fluid Media

Let's assume $\bar{p}^*$, $\bar{\mathbf{v}}^*$ and $\bar{\rho}'$ are fluid's acoustic perturbations of pressure, particle velocity and density, respectively, from the equilibrium condition in a stagnant uniform fluid in the absence of a body force. By omitting $\bar{\rho}'$ and $\bar{\mathbf{v}}^*$ in Equation A.1a by using Equation A.1b and Equation A.1c, the nondimensional form of wave equation (Equation A.2) for an acoustic phenomenon emerges as [Rienstra and Hirschberg, 2017]

$$\nabla^2 p = \alpha_f^2 s^2 p, \quad (2.15)$$

in the Laplace domain with zero initial conditions. The acoustical pressure perturbation, $p(r, \theta, s,)$, generated within the internal or external fluid is governed by Equation A.3. The solution is found,

by using a Fourier-Legendre expansion, to be

$$p(r, \theta, s) = \sum_{n=0}^{\infty} p_n(r, s) P_n(\cos \theta), \quad (2.16)$$

where $p_n \equiv p_n(r, s)$ is the n^{th} modal component of the acoustical pressure. Substitution of Equation 2.16 into Equation A.3 yields

$$\frac{d^2 p_n}{dr^2} + \frac{2}{r} \frac{dp_n}{dr} - \frac{n(n+1)}{r^2} p_n = s^2 \alpha^2 p_n. \quad (2.17)$$

Solutions of $p_n \equiv p_n(r, s)$ must be bounded as (i) $r \rightarrow \infty$ for the scattered/radiated pressure field in the fluid surrounding the shell (p^{sct}) and (ii) as $r \rightarrow 0$ for the radiated pressure in the contained fluid (p^{int}). Consequently

$$p_n^{in}(r, s) = A_n(s) i_n(\alpha_{in} r s) \quad \text{and} \quad p_n^{sct}(r, s) = B_n(s) k_n(r s) \quad (2.18a)$$

in which $i_n(x)$ and $k_n(x)$ are the modified spherical Bessel functions of the first and second kind, respectively. A rigorous treatment of the singularity at the sphere's center is given by Equation 1.20a through Equation 1.21b. The $A_n(s)$ and $B_n(s)$ are the unknown n^{th} modal coefficients determined in where the required boundary conditions are satisfied.

2.2.4 Elastic Medium

The Generalized Hooke's law as the constitutive law with the kinematic assumption of the small displacement and conservation of linear momentum, in the absence of a body force leads to the well-known Navier equation such that

$$\left(-\mu \nabla \times \nabla \times + (\lambda + 2\mu) \nabla \nabla \cdot - \rho_s s^2 \right) \mathbf{u} = \emptyset \quad (2.19)$$

in nondimensionalized form with zero initial conditions in the Laplace domain. The displacement vector can be decomposed into two elements as

$$\mathbf{u} = \nabla \phi + \nabla \times \boldsymbol{\psi} \quad \text{iff} \quad \nabla \cdot \boldsymbol{\psi} = 0. \quad (2.20)$$

$\nabla \phi$ is longitudinal, irrotational displacement (P -wave). However, $\nabla \times \boldsymbol{\psi}$ corresponds to *equivolume*, transverse shear displacement (S -wave) which is in a plane normal to longitudinal displacement vector. This procedure is known as the classical *Helmholtz* decomposition in which the solution of Equation 2.19 is reduced to finding three dimensionless displacement potentials, $\phi(r, \theta, \varphi, s)$, $\psi(r, \theta, \varphi, s)$ and $\chi(r, \theta, \varphi, s)$, which are governed by

$$\nabla^2 \phi = s^s \alpha_p^2 \phi, \quad \nabla^2 \psi = s^s \alpha_s^2 \psi \quad \text{and} \quad \nabla^2 \chi = s^s \alpha_s^2 \chi \quad (2.21)$$

The $\alpha_p = c_{ex}/c_p$ and $\alpha_s = c_{ex}/c_s$ in which $c_p = \sqrt{(\lambda + 2\mu)/\rho_s}$ and $c_s = \sqrt{\mu/\rho_s}$ are the compressional and shear wave velocities, respectively. In an ax-symmetric, non-torsional loading case, χ vanishes. Then two remaining wave potentials, similar to Equation 5.24, are expressed in terms

of Fourier-Legendre series, such that

$$\phi(r, \theta, s) = \sum_{n=0}^{\infty} \phi_n(r, s) P_n(\cos \theta) \quad \text{and} \quad \psi(r, \theta, s) = \sum_{n=0}^{\infty} \psi_n(r, s) P_n(\cos \theta). \quad (2.22a)$$

Substituting ϕ and ψ from Equation 2.21 into Equation 2.22 leads to

$$\frac{d^2 \varpi_n}{dr^2} + \frac{2}{r} \frac{d\varpi_n}{dr} - \frac{n(n+1)}{r^2} \varpi_n = s^2 \alpha^2 \varpi_n \quad (2.23)$$

where ϖ_n and α are the abbreviated versions of ϕ_n and α_l as well as ψ_n and α_s , respectively. The solution of Equation 2.23 is

$$\begin{Bmatrix} \phi_n(r, s) \\ \psi_n(r, s) \end{Bmatrix} = \begin{Bmatrix} C_n(s) i_n(\alpha_l r s) + D_n(s) k_n(\alpha_l r s) \\ E_n(s) i_n(\alpha_s r s) + F_n(s) k_n(\alpha_s r s) \end{Bmatrix}. \quad (2.24)$$

By substituting the last equations into Equation 2.20, displacement vector is obtained. Then the stress tensor is determined by inserting the displacement vector into ???. Six independent boundary conditions (BCs) are required to find the six unknown coefficients $A_n(s)$ through $F_n(s)$ as discussed next.

2.2.5 Fluid-Solid Interaction

Continuity of two sets of vectors at the shell's surfaces are needed to be satisfied. One vector relates to the *traction stress*, $\mathbf{t}$, and the other vector to the velocity $\mathbf{v} \equiv \mathbf{s}\mathbf{u}$ in the Laplace domain. Vector $\mathbf{t}$ is determined by projecting the Cauchy-Green stress *tensor*, $\boldsymbol{\sigma}$, in the direction of $\mathbf{e}_n$ which is the unit vector normal to a given traction surface. Consequently,

$$\mathbf{t}^{(i)} \equiv \boldsymbol{\sigma} \cdot \mathbf{e}_n, \quad i = in, ex. \quad (2.25)$$

As the centres of the spherical coordinate frame and shell coincide, $\mathbf{e}_n \equiv \mathbf{e}_r$, hence,

$$\mathbf{t}^{(i)} = \sigma_{rr} \mathbf{e}_r + \sigma_{r\theta} \mathbf{e}_\theta + \sigma_{r\phi} \mathbf{e}_\phi. \quad (2.26)$$

In an axisymmetric configuration, however, the normal component of $\mathbf{t}^{(i)}$ (i.e. σ_{rr}) and only one shear component, $\sigma_{r\theta}$ or $\sigma_{r\phi}$, in addition to the radial component of the surface velocity, v_r , are needed to satisfy the BCs. The stress components σ_{rr} and $\sigma_{r\theta}$ as well as radial velocity $v_r \equiv \mathbf{s}\mathbf{u}_r$ are needed to enforce the BCs at shell's traction surfaces. The implementation of these BCs are described in § 1.2.5.1 (Equation 1.29 through Equation 1.31).

2.2.6 Construction of Total Response

A linear superposition of the modal time histories are required to produce the total temporal responses of fields' variables through a *Fourier-Legendre* series. However, this series shows relatively slow convergence and even in high mode numbers leads to a superfluous oscillations known as *Gibbs phenomenon* [Zhang and Geers, 1993, Hasheminejad et al., 2011]. This deficiency

is addressed by using a procedure known as *modified Cesàro summation*, such that

$$\bar{\Lambda}(r, \theta, t) \approx \sum_{n=0}^{N_{CS}} \Xi_n \bar{\Lambda}_n(r, t) P_n(\cos \theta) \quad (2.27)$$

where

$$\Xi_n = \begin{cases} 0 & \text{if } 0 \leq n \leq 2 \\ \frac{N_{CS} + 1 - n}{N_{CS} + 1} & \text{if } 2 \leq n \leq N_{CS} \end{cases} \quad (2.28)$$

for $0^\circ \leq \theta \leq 10^\circ$ and $170^\circ \leq \theta \leq 180^\circ$, whereas

$$\Xi_n = \frac{N_{CS} + 1 - n}{N_{CS} + 1} \quad 0 \leq n \leq N_{CS} \quad (2.29)$$

in which $10^\circ \leq \theta \leq 170^\circ$ and N_{CS} is a chosen truncation number.

2.3 Results and Discussions

Validity of the current approach and the associated computer codes were examined through an extensive comparison of the numerical results against the pertinent published literature. However, for the sake of conciseness, only three cases are presented here. A selected set of case studies are considered next. Two steel spherical shell is selected having two thickness. The shell assumed to be fully filled and immersed in water. The mechanical properties of the steel, air and water are given in [Table 2.1](#). Temporal, Laplace and Fourier expressions of various excitation functions used in the validations and the subsequent case studies are given in [Table 1.2](#). Effect of different frequency contents of the excitation are investigated on the dynamic response of the system. Truncation numbers are chosen as $N_d = 400$ and $N_{CS} = 50$ based on a similar error analysis given in [[Hasheminejad et al., 2011](#)] and the fact that the temporal response vanishes as $t \rightarrow \infty$.

water	$c_f = 1.5 \text{ km/s}, \quad \rho_f = 1000 \text{ kg/m}^3$
air	$c_f = 0.34 \text{ km/s}, \quad \rho_f = 1 \text{ kg/m}^3$
steel	$\rho_s = 7.8 \text{ kg/s}, \nu = 0.3, E = 207 \text{ Gpa}$
h	0.01, 0.03, 0.06, 0.10, 0.20
θ_c	$\pi/3, \pi/2$

Table 2.1: Material properties and geometrical specifications.

Special visualization techniques are employed to better understand the dynamics of wave propagation through this coupled and complex system of fluid and elastic fields. It is advantageously carried out through depiction of total fields' response by using a series of consecutive snapshots illustrating pressure and stress responses. To have a better physical sense of these, all data are normalized to $[-1, 1]$ such that a constant color remains as the representation of zero value. Subsequently, brighter or darker colored regions correspond to positive or negative values, respectively. Further, to study the surface waves, a number of time-angle contour-plots are considered with the same normalization concept at a constant radius.

Darker colors corresponds to negative pressures and compressional stress whereas the brighter regions represent the positive gauge pressures in fluids and tensile stress within the shell's material.

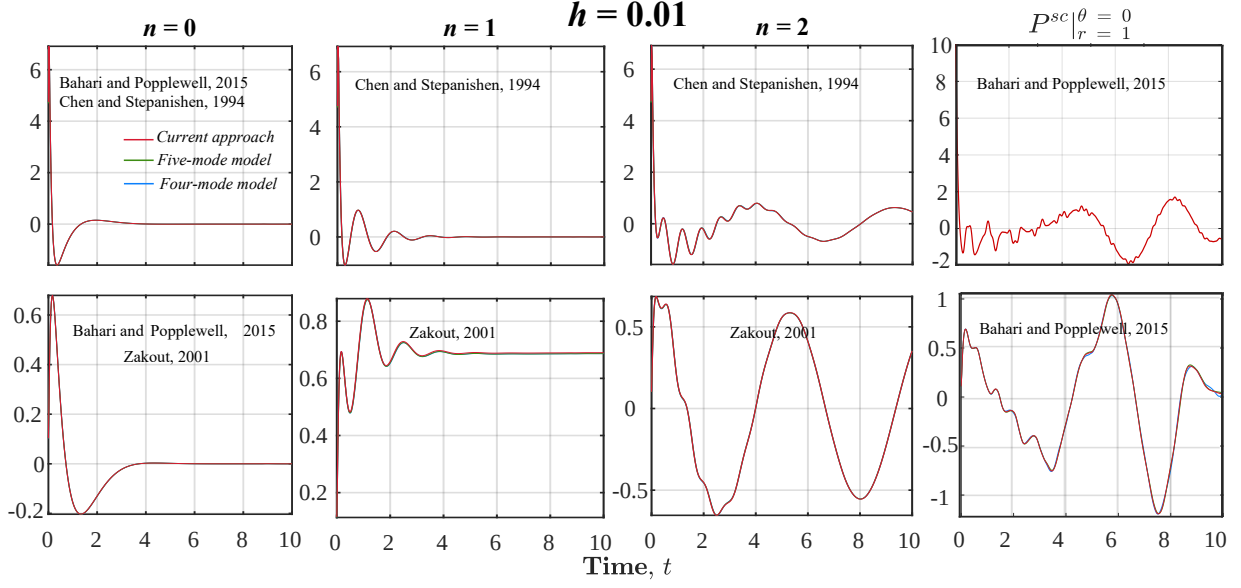


Figure 2.2: External acoustical pressure histories for an empty (air-filled), steel spherical shell with $h = 0.01$ at its apex when subjected to (first row) a Dirac delta pulse, $\delta(t)$ (see Figures 10 through 12 in [Chen and Stepanishen, 1994] and Figure 3b in [Bahari and Popplewell, 2015]) and (second) Heaviside pulse, $\mathcal{H}(t)$ (see Figures 7 and 8 in [Zakout, 2001] as well as Figures 1c and 3b). The first three columns from left gives modal components, whereas the last column illustrates the total responses when the spatial distribution is described by $\theta_c = \pi/3$.

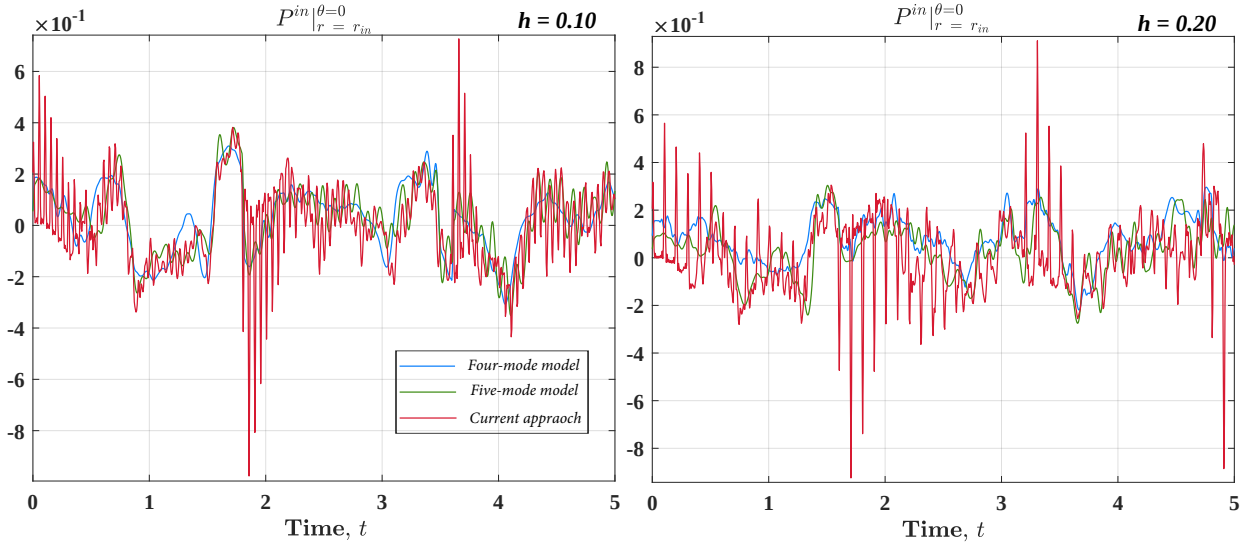


Figure 2.3: Acoustical pressure history at the inner surface of water-filled, steel spherical shell with $h = 0.10$ and 0.20 when subjected to a broadband modulated Gaussian pulse with $\theta_c = \pi/3$.

The order of snapshots is from left to right and top to bottom.

2.3.1 Validations

Transient acoustic radiation from an empty steel shell was studied in Ref. [Chen and Stepanishen, 1994, Zakout, 2001, Bahari and Popplewell, 2015] where the shell's wall thickness was $h = 0.01$. A thin shell theory known as *four-mode* was applied invariably to approximate the coupled elastic medium. Ref. [Chen and Stepanishen, 1994] considered a Dirac delta excitation, $\bar{f}_t(t) = \delta(t)$ at shell's outer surface. A in-vacuo eigenvector expansion with time-dependent coefficients were utilized to retrieve a transient response of shells of revolution submerged in a fluid medium. The method was originally given in [Stepanishen, 1997] for a general thin shell of rev-

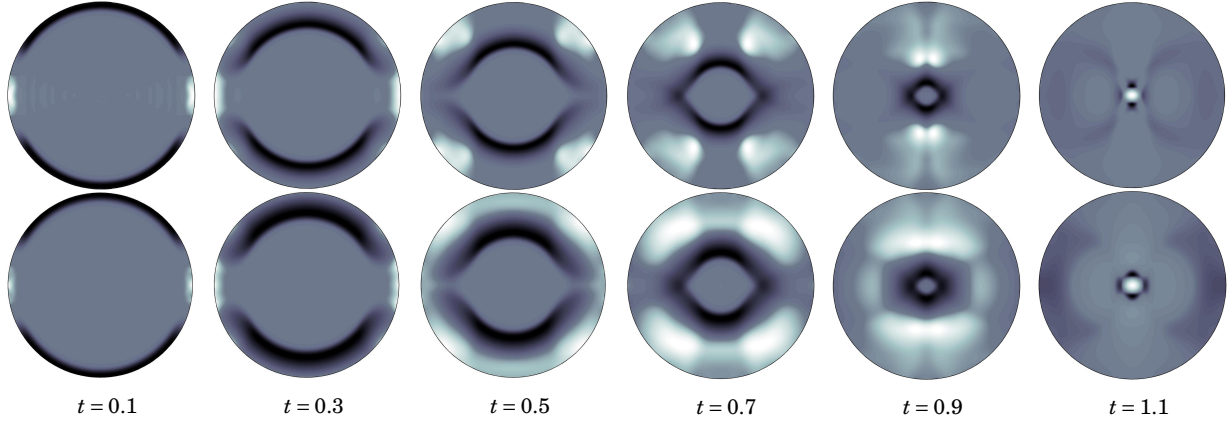


Figure 2.4a: A series of internal pressure's successive snapshots ($0.1 \leq t \leq 1.1$) in response to an exponentially decaying excitation with $\theta_c = \pi/3$ where (first row) $h = 0.01$ and (second row) $h = 0.03$ in which the Hertzian distribution profile is assumed.

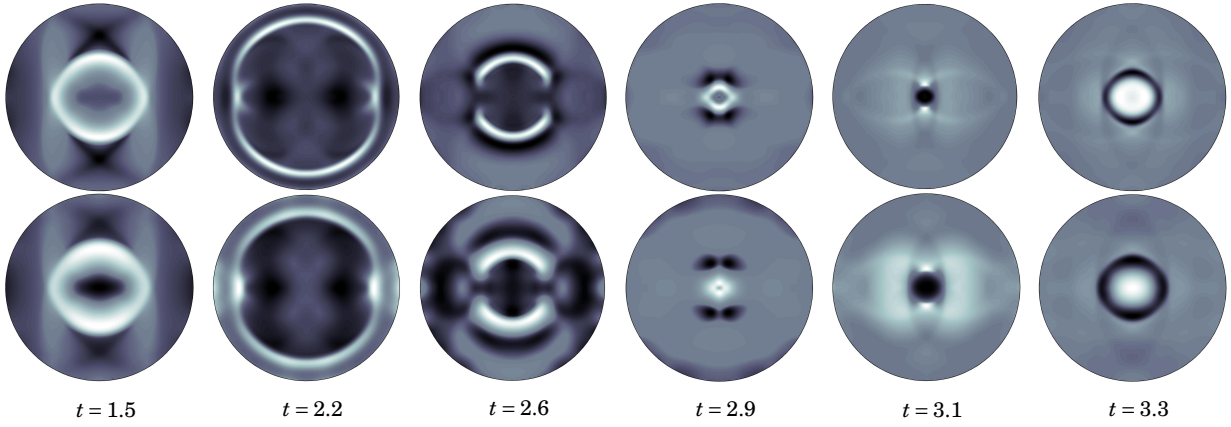


Figure 2.4b: A series of internal pressure's successive snapshots ($1.5 \leq t \leq 3.3$) in response to an exponentially decaying excitation with $\theta_c = \pi/3$ where (first row) $h = 0.01$ and (second row) $h = 0.03$ in which the Hertzian distribution profile is assumed.

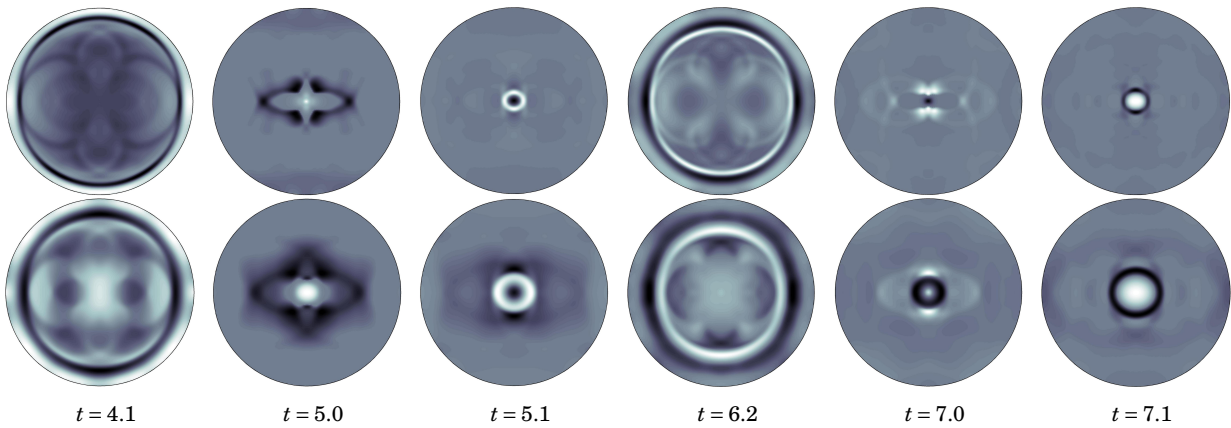


Figure 2.4c: A series of internal pressure's successive snapshots ($4.1 \leq t \leq 7.1$) in response to an exponentially decaying excitation with $\theta_c = \pi/3$ where (first row) $h = 0.01$ and (second row) $h = 0.03$ in which the Hertzian distribution profile is assumed.

olution. Ref [Zakout, 2001] studied a similar situation as [Chen and Stepanishen, 1994] where a Heaviside function, $\bar{f}_t(t) = \mathcal{H}(t)$. Both of these excitations were considered in [Bahari and Popplewell, 2015]. A direct Laplace transform approach was utilized in [Zakout, 2001, Bahari and

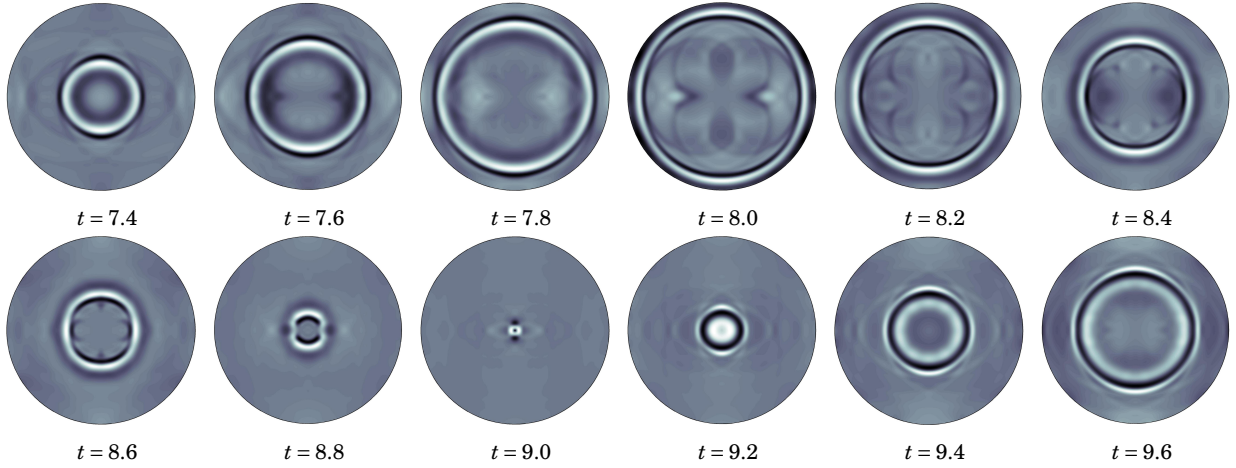


Figure 2.5: A series of successive snapshots of internal pressure which are evenly spaced in time, $\Delta t = 0.2$, from $t = 7.4$ to $t = 9.6$ in response to an exponentially decaying excitation with $\theta_c = \pi/3$ when $h = 0.01$ in which the Hertzian distribution profile is assumed.

Popplewell, 2015]. A *five-mode* shell theory are considered in addition to the *four-mode* model. Multiple results from these references are reproduced in [Figure 2.2](#) where a high level of agreement is observed regardless.

The total responses given in the last column of [Figure 2.2](#) are due to a centrally symmetrical load at the shell's inner surface was considered which is similar to the loading configuration described in [Equation 2.11](#). [Equation 2.12](#) provides the corresponding modal components of this loading in closed form. Rigid body motion is not possible due to central symmetry of loading. Furthermore, the odd modes' contribution to a total response remains zero when a centrally symmetric excitation applies. This level of agreement is achieved by considering a similar number of modes, i.e. $N_{CS} = 12$, without using the *Cesáro summation*.

Hence a significant level of discrepancies, however, appears when higher relative thicknesses are considered. [Figure 2.3](#), for example, gives the pressure history at the inner surface of shells with $h = 0.10$ and 0.20 when subjected to a broadband modulated Gaussian pulse. The approximate shell theories demonstrate considerably high level of error in comparison to the exact theory of linear elasticity. Consequently this suggest the necessity of using the current approach versus the commonly used shell theories. A selected set of case studies are presented and discussed next.

2.3.2 Case Studies

A more qualitative presentations of numerical results is taken initially through the visualization techniques, which was described earlier at the beginning of this section. Quantitative discussions follow afterwards. various choices of material, geometrical and loading configurations are given in [Table 2.1](#). The shells are subjected to a centrally symmetric double impact at shells' either surfaces.

A series of pressures snapshots for the contained fluid field is given in [Figure 2.4a](#) through [Figure 2.4c](#) for two wall thicknesses of $h = 0.01$ and $h = 0.03$. [Figure 2.5](#), however, is dedicated only to $h = 0.01$ with constant time steps of $\Delta = 0.2$. These shells are excited by an exponentially decaying excitation as described in [Table 1.2](#). Its spatial distribution is specified by $\theta_c = \pi/3$ according to [Equation 2.11](#). At the beginning of the interaction, such as $t = 0.1, 0.3$ in [Figure 2.4a](#), positive pressure regions appear right below the impact zones while a negative pressure starts to

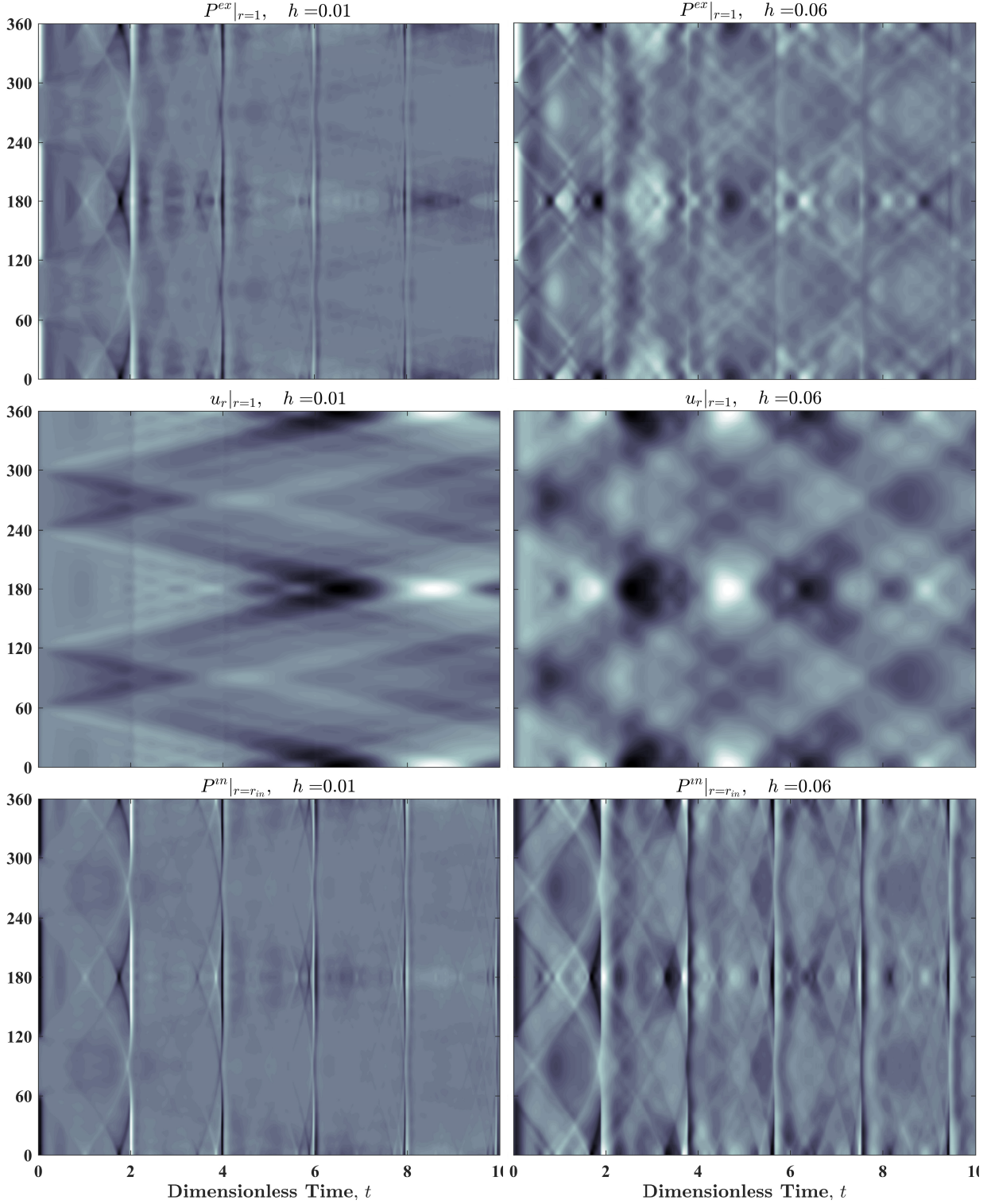


Figure 2.6: Contour plots of total field response to an explosive pulse (Table 1.2) applied over the region specified by $\theta_0 = \pi/6$ for two shell's wall thicknesses of $h = 0.01$ and $h = 0.06$. (1st row) acoustical energy's density; (2nd row) radial displacement and (3rd row) acoustical energy's density at the shell's entire outer, mid- and inner surfaces, respectively.

propagate from the vicinity of regions not to the impact. This observation has two implications. First, it highlights the higher speed of compression waves, which is more than four times larger than the speed of sound in internal fluid. Further, the negative pressure is a vacuum like due to the shell's positive (outward) displacement, u_r , whereas right below the impact zone, an initial

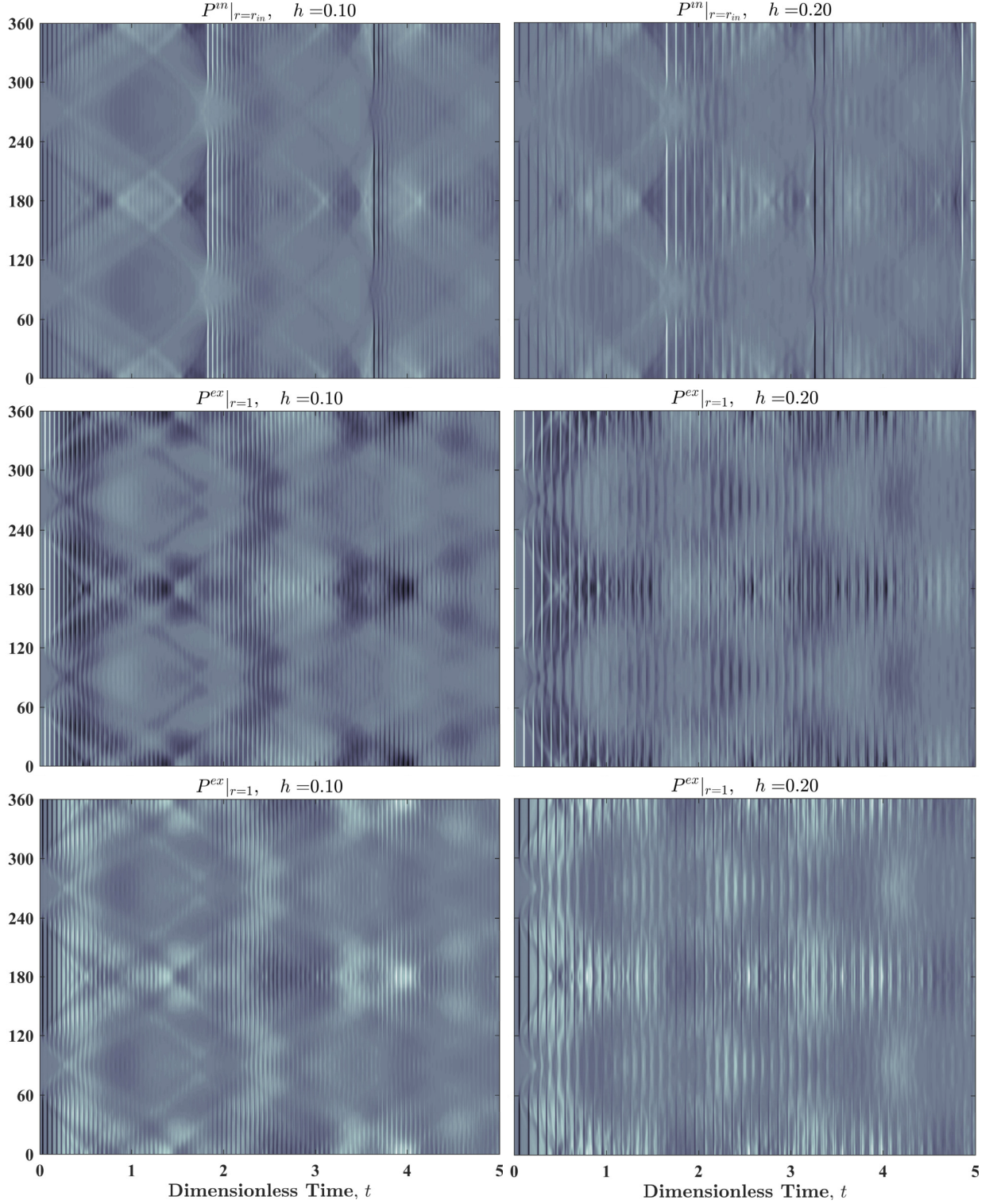


Figure 2.7: Contour plots of density of acoustical energy in response to a broadband modulated Gaussian pulse ($t_0 = 1/200, t_G = 1/2000$ and $\omega_G = 600$) applied over the region specified by $\theta_0 = \pi/6$ for two shell's wall thicknesses of $h = 0.1$ and $h = 0.2$. Load is applied on the shell's external surface in the (1st row) and (2nd row) whereas the shell's inner surface is excited in (3rd row).

negative (inward) displacement compresses the adjacent fluid. Once an acoustic perturbation initiates permeation, it keeps propagating normal to its wavefront with constant speed until it reaches the shell's inner surface. This simple point is very useful in better understanding the wave propagation mechanics in all subsequent snapshots.

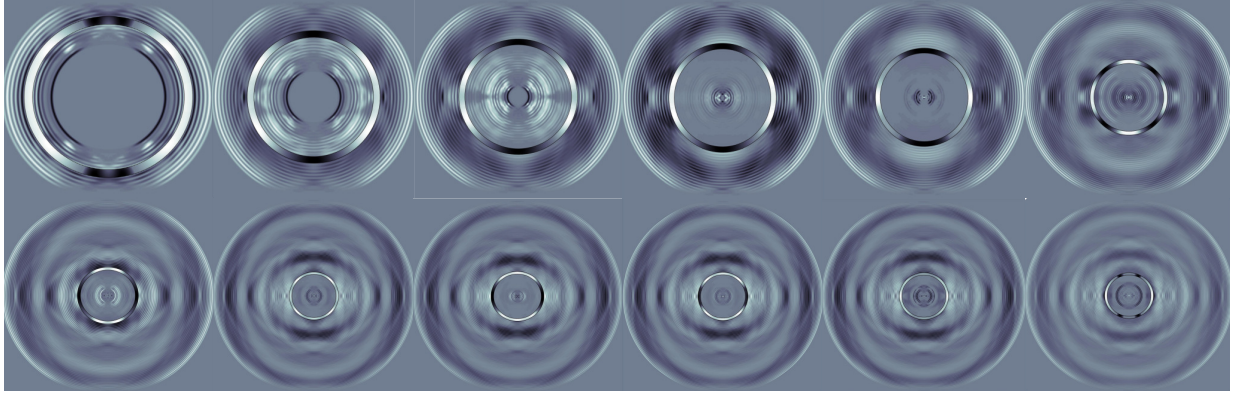


Figure 2.8: Successive snapshots of total field response from left to right in response to a broadband modulated Gaussian pulse with $t_0 = 1/200$ and $t_\omega = 1/2000$ when $\theta_c = \pi/3$ and $h = 0.1$. Each snapshot contains internal and external fluid fields' acoustical pressures as well as the shell's radial displacement u_r .

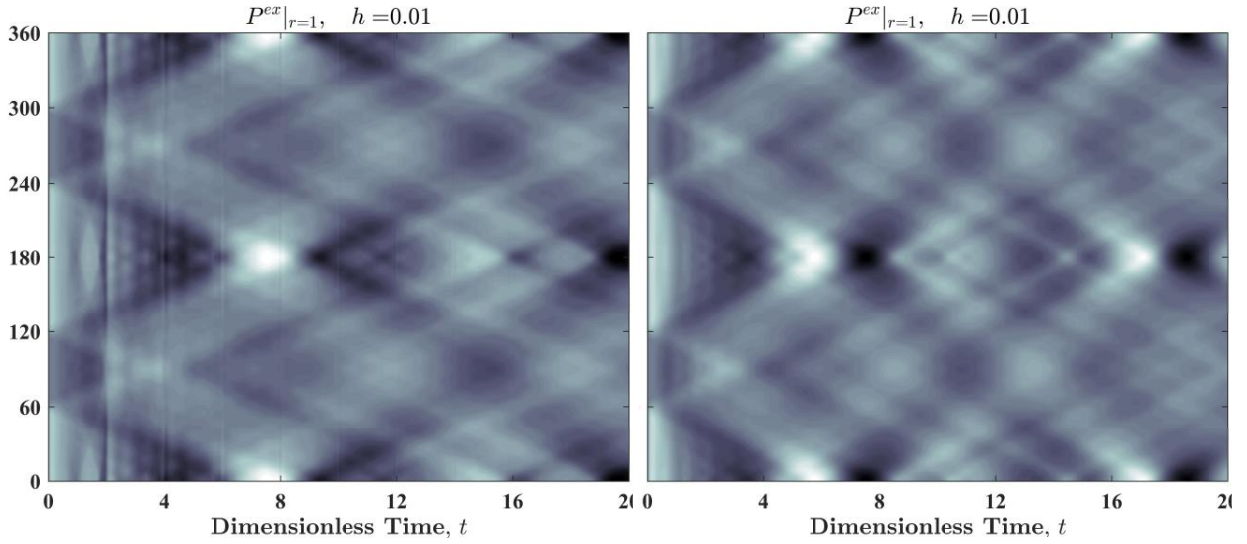


Figure 2.9: Contour plots of total field response to a Heaviside excitation, $\bar{f}_t(t) = \mathcal{H}(t)$, applied over the region specified by $\theta_0 = \pi/6$ for a shell filled with (a) water (b) air where $h = 0.06$.

A strong primary focusing of the initial wavefronts starts forming around $t \approx 1$, see, for example, $t = 0.9$ and $t = 1.1$ in **Figure 2.4a**. This focusing results in both positive and negative peaks in gauge pressures. The negative pressure, specifically, are of interest because they provide the information about the cavitation likelihood. If the resulting absolute pressure becomes close to the thermodynamic saturation pressure of water in its current temperature, that region indicates a high probability of cavitation to occur. There is a delay in this process in the case of $h = 0.01$ compared to $h = 0.03$. This is basically because of the difference in the wave's required time to travel the shells' inner space. Hint that the inner radius is $\Delta r = 0.02$ smaller in the thicker shell.

Initial wave fronts pass each other after superposing at the primary focusing and keep propagating towards the shell's inner walls. See, for instance, the snapshots at $t = 1.5$ and $t = 2.2$ given in **Figure 2.4b**. Once starts interacting with the shell's surface, part of it transmits to the outer fluid through the elastic shell. The reflected waves from the walls keep moving towards each others to form a secondary focusing around $t \approx 3$. See the corresponding snapshots at $t = 2.9$ and $t = 3.1$ in **Figure 2.4b**. As it can be seen, the effect of shell's thickness on the fluid field becomes more conspicuous as time goes on. Wave fronts, for instance, are sharper and more focused in the

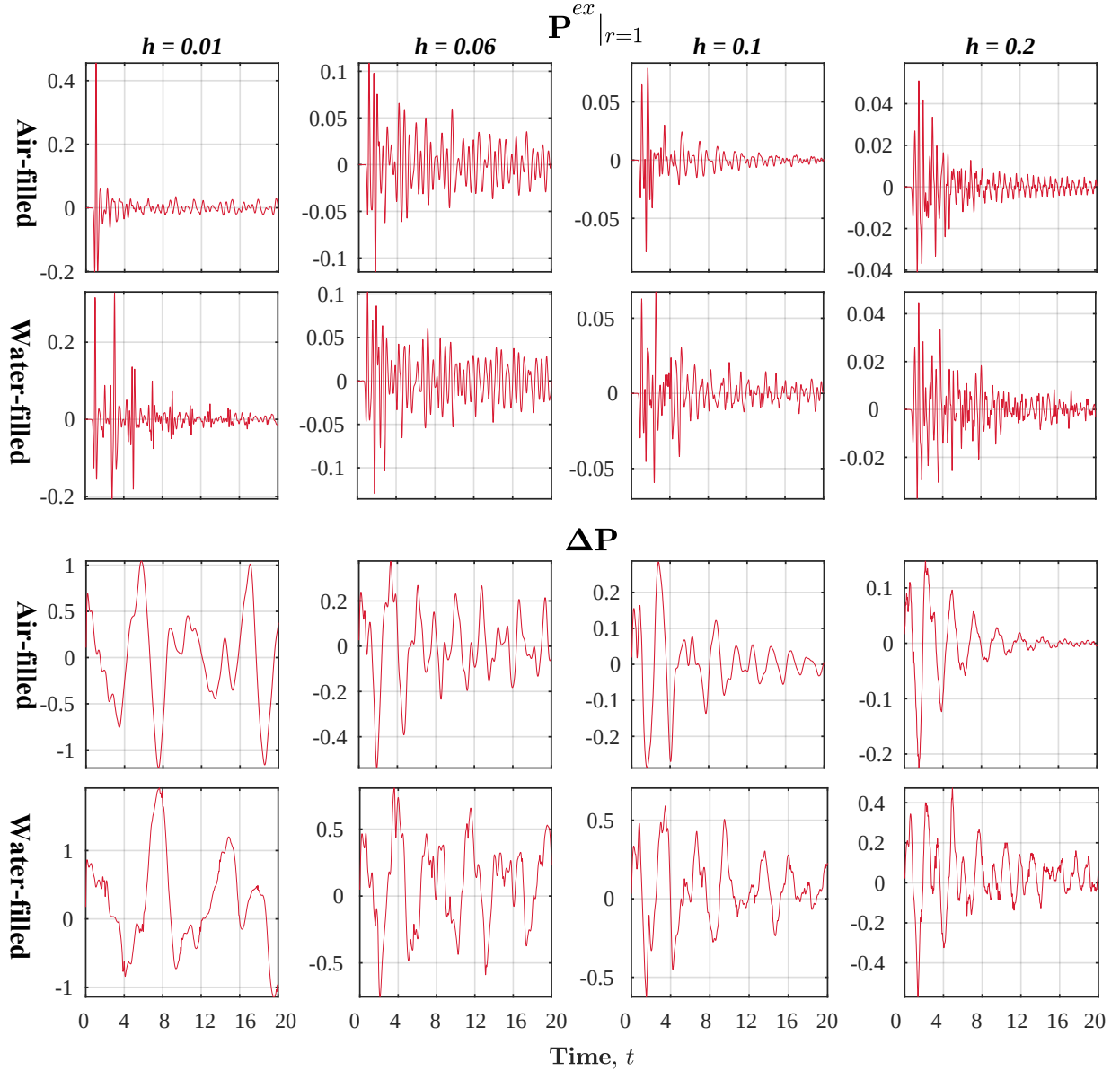


Figure 2.10: Pressure histories at the shell's center when $\theta_c = \pi/3$. The excitation are the narrowband Ricker pulse and Heaviside for the external surface pressure and the pressure gradient across the shell's wall, respectively.

shell with lower thickness and, consequently, easier to distinguish. This is basically due to larger amount of *shell-born* waves forming in the elastic field in a higher wall-thickness. Figure 2.4c shows the next two major focusing mechanisms around time $t \approx 5$ and 7. However, as it can be seen in, for example, $t = 4.1$ and $t = 6.1$ they are secondary, tertiary and so on wave fronts leading to other focusing mechanisms but with smaller amplitudes. The formation of those corresponding triple points in $t = 4.1$ and $t = 6.1$ are distinguishable easily.

Figure 2.5 is dedicated to only one shell thickness, $h = 0.01$, so that, some subsequent triple point formations and the associated focusings can be followed easier. Another major focusing can be observed around time $t \approx 9$. Multiple triple point formation can be detected effortlessly at $t = 8.0, 8.2, 8.4$ and 8.36 . In general, the intervals between successive primary focusings stays about constant value of $\Delta t \approx 2$.

Another visualization technique is applied in the form of time-space contour plots as in Fig-

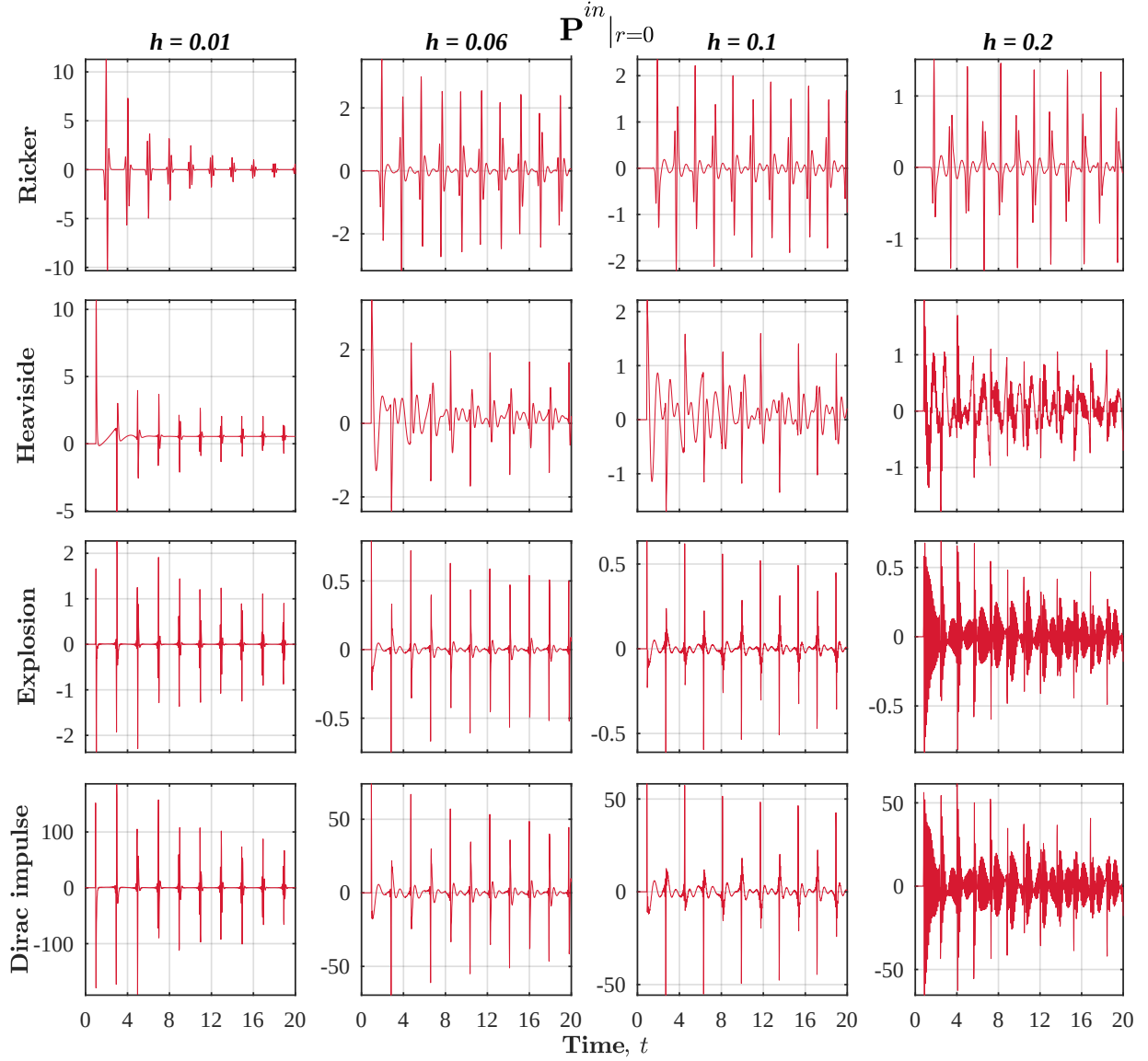


Figure 2.11: Internal pressure histories at the shell's center when $\theta_c = \pi/2$ for four excitations and four shell wall thicknesses.

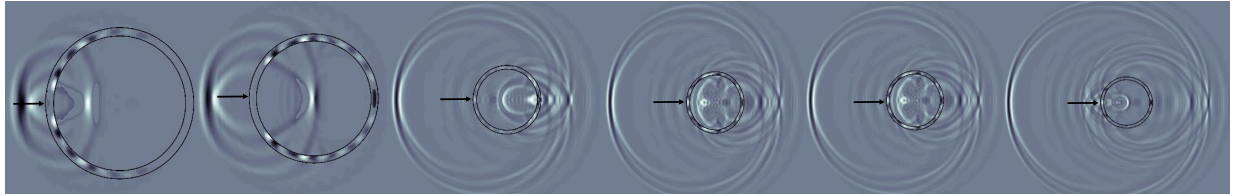


Figure 2.12: Successive snapshots of total field response to a concentrated ($f_\theta = \delta(\theta)$) narrowband Ricker pulse with $t_0 = 1/5$ and $t_\omega = 1/20$. Each snapshot contains internal and external fluid field's acoustical pressures. The radial stress, σ_{rr} , is selected as the representation of the elastic shell's response.

Figure 2.6. A careful arrangement is made so that for each shell's wall thicknesses, external fluid medium comes first at the top, which is followed by shell's radial displacement and then the internal fluid field at the bottom. Acoustical energy density is considered at both shell's surfaces to have a better understanding of energy communication rate from the shell and its encapsulated

fluid to the infinite fluid medium. The rate of acoustic energy transfer from the inner fluid largely depends on the shell's thickness where both fluids are the same. One should be reminded that total acoustic energy density is the sum of its corresponding potential, $\mathcal{I}_p$ and kinetic, $\mathcal{I}_v$ energies as

$$\mathcal{I} \equiv \mathcal{I}_p + \mathcal{I}_v = \frac{1}{2}\rho_f v_f^2 + \frac{p^2}{2\rho_f c_f^2} = \frac{p^2}{\rho_f c_f^2}. \quad (2.30)$$

Indeed the potential energy and kinetic acoustic energies are converted to each others and they have a $\pi/2$ phase difference because of the first time derivative from p to v . A quick comparison between first and second columns of [Figure 2.6](#) clearly demonstrates that the acoustical energy density becomes far more concentrated and localized in smaller shell's wall thicknesses. Furthermore the surface acoustical energy dissipates faster to infinite fluid medium in a thinner shell. This is due to the lower capacity of a thinner shell structure in maintaining acoustical energy. The focusings are clearly tractable in all contour plots. The intervals between successive focusings are $\Delta t \approx 1.98$ and $\Delta \approx 1.94$ for $h = 0.01$ and $h = 0.03$, respectively. A major negative (positive) peak forms in each focusing which is followed immediately by a positive (negative) peak. This order is inverted in the subsequent focusing. These wave trends are very well distinguished they are readily traceable at the shell's inner surface (the third row of [Figure 2.6](#)), whilst they are not easily differentiated at the shell's outer surface (the second row of [Figure 2.6](#)).

It can be noted easily that the initial pressure response ($t = 0^+$) right below the applied force, alternates between positive and negative from the shell's inner to outer surface pressure. This can be explained straightforwardly as follows. The inward (outward) radial displacement of shell leads positive (negative) gauge pressure at the shell's inner (external) surface. Interestingly, shell's maximum inward or outward deflections does not happen at early time. For example, they occur around $t \approx 7, 9$ when $h = 0.01$. These moments, however, are shifted to earlier times as the shell's wall thickness increases, as illustrated in [Figure 2.6](#). Intensity and number of head waves within the shell's material increases considerably as h increases. See the second row of [Figure 2.6](#). Distinguishing these waves, however, becomes more difficult in displacement field within thicker shells. This difficulty can be managed by referring to the internal and external pressure fields because the shell born waves eventually are transmitted to the adjacent fluid medium through the traction boundaries. Different waves are distinguished most clearly within the internal fluid field. The time and angle of the primary focusings can be seen in all the contour plots of [Figure 2.6](#). The maximum pressures and deflections occur around the center of the applied loads.

[Figure 2.7](#) demonstrates surface pressure contour plots for two higher thicknesses of $h = 0.10$ and 0.20 when the shell is excited by a broadband modulated Gaussian ([Table 1.2](#)) with $\theta_c = \pi/3$. The top two rows are responses when the shell is excited at its outer surface whereas the bottom row is due to an excitation at the shell's inner surface. The most conspicuous difference between this figure and [Figure 2.6](#) is in the presence of multiple focal wavefronts in higher thicknesses and broader frequency content of excitation. These wavefronts are formed because the excited wavelengths are partially smaller than shells' thicknesses. The interval of these wavefronts increases as higher thicknesses are considered. The primary focusings occur at $t \approx 1.8$ and $t \approx 1.6$ corresponding to $h = 0.1$ and $h = 0.2$, respectively. Higher thicknesses results in a way slower decaying of the acoustical energy in the contained fluid because the shell can sustain higher amount of acoustical

energy. Excitation at the shell's inner surface results into dominantly positive gauge pressure in the external fluid (third row) which is in contrast with the results for the excitation on the outer surface (second row).

Successive snapshots of the total field response are given in [Figure 2.8](#) when a shell with $h = 0.1$ is excited by a broadband excitation which produces wavelengths comparable to the shell's thickness. This figure helps to better understand the wave's mechanics of the contour plots given in [Figure 2.7](#). The first row shows the superposition process of multiple radiated wavefronts at the shell's center. These multiple wavefronts are the results of successive mode conversions at the shell's either surfaces which gradually transmits its energy into the contained fluid. After superposing at the shell's center these wavefronts keep travelling back and forth from a shell's inner surface and again towards the center of the sphere. Strong negative gauge pressures are formed within the external acoustical field occurs later after completion of the first and second primary focusings corresponding to the snapshots in the second row.

[Figure 2.9](#) aims to briefly investigate the effect of the enclosed fluid on the external pressure's wave dynamics. The contour plots are the system's responses when shell is filled with water and air, respectively, from left to right when the shell is excited by Heaviside pulse invariably. Arrival of the successive focusing wavefronts from center to the shell's inner surface with $\Delta t \approx 2$ are detectable when the shell is filled with water. They, however, are missing in the contour plot associated with the air-filled shell. This is due to the insignificant value of air's impedance compared to water. Furthermore, formation of major higher pressure regions on the shell's external surface is delayed in the shell filled with water in comparison to the air-filled shell.

[Figure 2.10](#) has a similar objective as [Figure 2.9](#) but with a quantitative rather than a qualitative approach. Pressures at the shell's external surface are slightly larger than when the shell is filled with air rather than water. The response dissipates quickly in the case of air-filled situation specially for very thin shells. This observation can be explained by the energy capacity of the overall structure, shell and its contained fluid. Significantly higher impedance of the water makes the shell filled with it more capable of holding larger acoustical energy in comparison to the air-filled shell. The difference between the shell's inner and outer surfaces at apex is given when a Heaviside excitation is applied. In complete contrast to the $P^{ex}|_{r=r_i n}$, the Δp dissipates significantly faster at higher thicknesses. It is more conspicuous when the shell is filled with air. The initial peak values of Δp are considerably less sensitive to the increase in the shell's thickness when filled with water. Compare, for example, $h = 0.01$ with $h = 0.1$ and $h = 0.2$ in [Figure 2.10](#).

From all previous snapshots and contour-plots, we have a qualitative feeling that the magnitudes of the focused pressures at the shell's center are significantly higher than other regions. [Figure 2.11](#) is to quantitatively study the center point pressure. The effects of the shell's relative wall thickness and various excitation are investigated. The breathing mode, $n = 0$, is the only contributing mode at $r = 0$ due to its geometrical symmetry. The modal expression of the loading, f_n , for a uniform distribution is given in [Equation 2.12](#) which reduced into $f_0 = -2 \cos \theta_c$ for $n = 0$. Obviously this modal coefficient is maximized ($f_0 = -2$) when $\theta_c = \pi/2$. In other words, the maximum values of focused pressures occurs when the load is applied to the entire shell's surface. One should be reminded that center-point's pressure for a general value of θ_c is obtained simply by multiplying the values due to $\pi/2$ by $2 \cos \theta_c$.

General observations from [Figure 2.11](#) are as follows regardless of the excitation. The peak

values decrease by the increasing shell's wall thickness and as time goes by. The successive peaks alternate between positive and negative values. The intervals between successive peaks are reduced by increasing h . A better energy preservation behaviour is observed in consecutive pressure peaks and troughs for the thicker shells. This can be physically explained by noting that energy can be better sustained in a more rigid container. These observations are in complete agreement with observations on the previous snapshot series and contour plots.

Specific observations to each excitation are given next. The narrowband Ricker pulse remains with single focal peak at regular intervals of $\Delta t \approx 2r_{in}$ for all thicknesses through out. This time is required by a surface wave to travel between shell's inner surfaces to reach the center with a sonic speed. However, multiple secondary focused peaks appears at $h = 0.2$. They become very significant in cases of the Dirac impulse and explosion. These secondary peaks are due to multiple reflections within the shell's thickness when the wavelengths become comparable to the shell's thickness. See, for example, the snapshots given in [Figure 2.8](#) and contour-plots of [Figure 2.7](#). The Heaviside pulse converges to its long term asymptotic (static values) quickly when $h = 0.01$. Interestingly the first peak values in Ricker (Heaviside) pulse reaches 12 (12) and -10 (-5) times of the original excitation. The Dirac impulse, on the other hand, leads to the highest amplitudes by several folds compared to other excitations. The explosion leads to a very similar trend but about 100 times lower in value. Hence other three pulses convey a higher practical importance whereas the Dirac pulse has theoretical significance because of its frequency content. Especially the negative peak values are a useful indication in predicting the likelihood of cavitation. Finally an important concluding remarks can be made by comparing [Figure 2.11](#) and [Figure 2.10](#): the values of the surface pressures are significantly lower than those peaks forming at the center of the sphere.

At last, a case of concentrated load, $f_\theta = \delta(\theta)$, on the apex of the shell is considered in [Figure 2.12](#), which is not centrally symmetric. The corresponding modal coefficients become $f_n = \frac{2n+1}{2}$. In such situation, unlike former case studies, the rigid body motion, $n = 1$, and other anti-symmetric modes, $n = 2i + 1$, persist and have nonzero contribution. A narrowband [Figure 2.12](#) pulse is applied. This case was chosen as it is in contrast with of double impact. A very different focusing mechanism happens in the internal fluid medium, which leads to the development of smaller amplitudes in comparison to a centrally symmetrical loading situation. Triple points and focusing of the internal fluid are formed through a different mechanisms rather than those discussed in [Figure 2.4a](#) through [Figure 2.5](#) as well as [Figure 2.8](#). This is basically due to the contribution of odd modes. The formation of the symmetric and asymmetric *Lamb* and head waves in both external and internal fluid media can be effortlessly followed and distinguished. Formation of the triple points and the associated primary and secondary focusings are detectable most readily. The further analysis of this kind of loading, however, is beyond the main objective of this work.

2.4 Conclusions

Transient acoustic radiation from a fluid filled elastic spherical shell due to a double impact on its either surfaces is investigated analytically. The shell is fully submerged in a compressible fluid medium. Internal and external fluid fields were governed by the exact linear theory of an acoustic phenomenon. An exact local theory of linear elasticity is employed to describe the elastic shell.

A Helmholtz decomposition was applied to elastic shell to resolve the displacement field vector. The complete closed-form modal solutions are obtained for elastic and fluid media in terms of the *modified spherical Bessel* functions in Laplace domain, which are then numerically inverted to time. The coupling between fluid fields and elastic shell is enforced through the continuity boundary conditions. Finally, total responses were achieved by a linear superposition of modal components through the associated spherical harmonics after numerically inverting the Laplace field responses back into the time domain. A *Cesáro summation* approach was taken to alleviate a relatively slow convergence rate of *Fourier-Legendre* series.

After validating the current approach and associated computer codes against related literature, several numerical case-studies are considered. Special visualization techniques were applied to better understand the wave propagation phenomena of a coupled nature. A variety of wave phenomena were observed and discussed. More specifically, different focusing behaviour is observed within the internal fluid medium. Strong amplification of the internal pressure as high as twelve and ten folds of the loading's amplitude is observed for positive and negative gauge pressures, respectively, when a Heaviside excitation or a Ricker pulse is considered. This is an important indication of cavitation likelihood which, for example, can be taken into considerations in a safe design of the underwater shell structures.

Chapter 3

A Comprehensive, Transient Elasto-Acoustics Study of A Fluid Filled Spherical Shell Embedded in An Elastic Medium I:

Analytical Formulations and Validations

An exhaustive study is carried out on the mechanics of waves travelling through a coupled elasto-acoustic system of a fluid-filled, thick-walled spherical shell embedded in an elastic formation. Important but previously untouched aspects of this classical problem are addressed rigorously. An exact local theory of linear elasticity is applied to the elastic formation and the shell, whereas the linear theory of acoustics for compressible fluids is utilized. Two sets of excitation sources are considered: (i) either incident compressional or shear waves, and (ii) a general traction force acting on either one of the shell's surfaces. The origin of the incident waves can be point sources of stress waves positioned within the infinite elastic medium, shell's material or an eccentric pressure source located within the encapsulated fluid. These point sources give rise to spherical wavefronts impinging on shell's either surfaces. A plane wavefront is the limiting case of a spherical wave if a point source within the infinite elastic formation is located far from the shell. Translational addition theorems are modified accordingly in the Laplace domain. They are employed advantageously to unify the description of various incident waves in terms of a common set of spherical harmonics having their origin coinciding with the shell's center. A rigorous cross check of the current approach is performed meticulously against multiple existing limiting cases taken from the pertinent literature where the relevant mathematical and physical aspects are discussed accordingly. The comparisons lend credence to the current approach and the associated computer code developments. Then a brief overview is given of the unifying physical aspects. An extensive set of carefully selected numerical case studies are presented in a companion chapter where more general configurations are discussed.

3.1 Introduction

Investigations of spherical wave scatterers have practical relevance to electromagnetism, optics, acoustics, bio-medical imaging, seismology and ultrasonics. The particular problem considered here is the elastodynamics of an embedded, possibly fluid-filled spherical shell or an elastic inclusion. It has been a subject of interest since the mid-twentieth century [Knopoff, 1959]). In the particular case of mechanically generated waves, a spherical scatterer has applications in diverse fields such as geology, civil and nuclear engineering as well as the petroleum industry. For instance a spherical geometry reasonably approximates a range of actual objects and anomalies like oil trapped in an underground cavity. Moreover, analytical solutions for a complex system of

multiple spherical scatterers can be built from the response of a single spherical obstacle. See, for example, [Ávila-Carrera and Sánchez-Sesma, 2006, Grandjean and Leparoux, 2004, Martin, 2006]. Another example is in nuclear waste management. Underground spherical waste containers are usually preferred to those above ground for reasons of safety, [Reddy and Kierans, 1976]. A more intricate design however requires a better understanding of the dynamic behavior caused by a seismic or shock load. Furthermore, a computationally efficient analysis of wave scattering from a buried spherical obstacle having known properties may help to inversely define an unknown scatterer. The additional complexity likely requires a greater computational efficiency however which leads, in turn, to the need for a more precise forward solver. Practical examples requiring similar forward or corresponding inverse solutions are listed next.

- *Diagnostic medical ultrasound* (i) to detect a tumor or cyst as well as to monitor a cancer growth in lymph nodes or a thyroid gland [Norton and Linzer, 1981, Wirgin, 2016]; (ii) to enhance ultrasound imaging through double-scattering in a nanoparticle such as a rattle-type Mesoporous Silica Nanostructure or MSN [Zhang et al., 2015];
- *Geophysics* related to oil and gas exploration in which underground anomalies and cavities need to be discovered. Then mechanical wave techniques are preferred in comparison to electromagnetic conduction and induction based methods because elastic waves are more likely to produce direct and less ambiguous results [Grant and West, 1965, Grandjean and Leparoux, 2004]; Oscillations of earth have been approximated by a spherical shell with a fluid core [Satô et al., 1963].
- *Ultrasound in non-destructive testing (NDT)* used to detect welding cavities in steel structures or voids in the base material [Darmon et al., 2003, Krautkrämer and Krautkrämer, 2013].
- *Safe design of Underground Structures* such as nuclear disposal containers against natural and human induced seismic activities [Reddy and Kierans, 1976].

The focus of this particular chronological review is restricted to purely analytical or semi-analytical studies. Among the earliest publications, [Ying and Truell, 1956] investigated the steady-state scattering of longitudinal plane waves by a spherical obstacle such as a cavity, rigid or elastic sphere within an isotropic, elastic medium by using the classical Helmholtz decomposition. By employing the Fourier Transform, [Eringen, 1957] analyzed the dynamic behavior of an infinite elastic medium containing a spherical cavity when exposed to a general surface traction force. Later, [Einspruch and Truell, 1960] determined the steady-state scattering of an incident plane wave impinging upon a spherical obstacle filled with an ideal, inviscid fluid. They compared the responses caused by separate incident transverse and longitudinal waves and noted significant differences. Three years later, [Pao and Mow, 1963] revisited and extended the work of [Ying and Truell, 1956]. They incorporated the rigid body motion of the spherical obstacle by appropriately modifying the boundary conditions. Moreover they also showed that a spherical obstacle whose inside is empty, rigid or fluid-filled are different limiting cases of an elastic inclusion. These results were also reported in a later monograph ([Mow and Pao, 1971], 6.2). [Satô et al., 1963] considered transient wave propagation due to an impulsive torsional disturbance at the surface of an elastic sphere in which the sphere had a fluid core. The corresponding wave mechanics is a basic

model for seismology applications and earth dynamics. A complete normal mode approach was utilized based on the Helmholtz decomposition. Subsequently [Yeh, 1964] developed an interesting analytical solution by differently treating an incident plane wave as a load moving over a spherical cavity's surface. [Murthy and Reddy, 1967] applied the latter technique to a similar geometrical configuration but with an additional spherical shell lining. However a thin shell theory rather than a more universally applicable kinematic assumptions of Green-Lagrange strain tensor was utilized to approximate an underground, spherical contaminant container exposed to a seismic or shock wave. A closed-form solution was derived for the dilatation mode. The approach adopted by [Yeh, 1964] was used in a subsequent short note of [McLeary, 1969]. Reasonable numerical convergence was achieved with the lowest 25 modes describing the scattering by a spherical cavity of a slowly (as opposed to sharply) rising incident wave. In two companion chapters, [Chadwick and Trowbridge, 1967b, Chadwick and Trowbridge, 1967a] studied the vibrations of a rigid sphere embedded in an infinite elastic medium due to harmonic and impulsive forces by using a Laplace transform approach. First, a pure troupe was applied so that merely torsional modes were excited. These waves travel at the speed of a shear wave. Both dilatational and transverse modes were excited to study rectilinear oscillations by applying a normal traction. These waves, however, travel at two different speeds. [Moon and Pao, 1967] and subsequently [Moon, 1972] showed that the spatial distribution of both the steady-state stress and scattered energy caused by plane and spherical incident waves are very different.

In a benchmark paper, [Nussenzveig, 1965] investigated the high frequency, steady-state behavior of a totally reflecting spherical inclusion in response to an incident wave. A modified Watson transformation was used, in conjunction with appropriate near- and far- field asymptotes, to study the formation of various wave phenomena formed in different spatial regions. Afterwards, [Norwood and Miklowitz, 1967] presented analytical solutions for (1) elastic waves radiated from a cavity caused by the impingement of a transient point source excitation located on the cavity's internal surface, and (2) diffracted elastic waves due to a nonstationary plane dilatational incident stress pulse. A Watson transform successfully mitigated the slow convergence associated with the higher frequency components in the early time response. Their method, however, is valid only within the so called shadow zone - a region which cannot be connected to the source of disturbance by straight line rays. About two decades later, [Wijeyewickrema and Keer, 1989] gainfully applied the same procedure to a similar configuration except that an empty cavity was replaced by a rigid spherical inclusion. By using the same approach as [Nussenzveig, 1965] in the frequency domain, [Datta and Shah, 1970] investigated the diffraction of SH waves originating from a point source that acts on a sphere having a rigid or fluid core. The medium outside the cavity was assumed to be spherically isotropic so that the motion stayed independent of the longitudinal wave and no radial displacement occurred. [Iwashimizu, 1972] considered the scattering of shear or longitudinal incident waves in an elastic medium from a rigid sphere. He studied the differences between the scattering by a movable rigid sphere in comparison to an immovable one, especially in the Rayleigh limit.

The Watson transform was shown to be equivalent to the *Poisson summation* which was popular until the early 1970s. See, for example, [Nussenzveig, 1965]. This popularity can be attributed to the slow convergence of a Fourier series type summation leading to a relatively high computational cost in the context of a limited computing technology of the time. This transform,

conversely, enabled a temporal response to be estimated by using few algebraic terms (e.g. [Norwood and Miklowitz, 1967]). However, the latter extreme computational efficiency came at the expense of a very limited range of applicability. An analytical evaluation of a Watson transform (integration) usually requires several asymptotic limit cases such as a very high frequency evaluation of the one corresponding to early time response and specific spaces within, for example, so-called shadow zone. On the other hand, the recent dramatic rise in computational capabilities, apparently, has been the major factor in the significant discontinuation of using the Watson transform in favour of Fourier series like approaches.

[Huang and Wang, 1972] employed a Laplace transform to study the transient, stress wave diffraction from a spherical cavity due to an incident longitudinal, plane step wave travelling within an elastic medium. An analytical expression was derived for the elastic wave potentials by using a truncated Legendre Fourier series. They claimed that a sufficient convergence was achieved generally in a temporal response by employing only the first two modes. However, up to the first six modes were required in the immediate region of the scattered field near the cavity. It was conjectured plausibly that this still fast convergence happened because the cavity's smooth (free) boundary did not produce a sharp stress jump immediately after the incident wave's arrival. [Jain and Kanwal, 1980] required few terms to approximate the harmonic scattering of plane waves by an empty or fluid filled cavity by utilizing an asymptotic expansion for low frequencies.

[Zhao and Harkrider, 1992] considered the interaction of an explosively generated wave with a solid sphere embedded in an elastic medium. A numerically evaluated inverse Fourier transform was employed. Later [Lebedeva, 1994] determined the displacement of a rigid sphere embedded in an elastic medium that was caused by a transient longitudinal, plane wave. A Laplace rather than a Fourier transform was used. [Akkas et al., 2000] investigated the transient radiation emanating from a spherical cavity, with and without a shell lining, that was initiated by an axisymmetric surface load. They essentially used a Laplace transform approach to solve the linear spherical wave equation with a procedure known as the Residual Variable Method (RVM). However a *four-mode*, thin shell theory approximation modelled the embedded body. [Zakout et al., 2000] applied a RVM to obtain the transient response of an infinite elastic medium containing a spherical cavity due to a torsional surface loading. Only the torsional (second class vibratory) modes were excited because this traction force does not lead to mode conversion in a purely uniform torsional traction. A similar geometrical configuration to that examined by [Akkas et al., 2000], but with a completely symmetrical (1D) surface load, was investigated by [Gan-bin and Kang-he, 2005], [Dwivedi et al., 2013] and [Xu et al., 2015] in which different constitutive relations, such as viscoelastic and porous, represented the surrounding soil. More recently, [Schwartz et al., 2016] developed a solution for the steady state, axisymmetric scattering of an incident cylindrical wave by a spherical inclusion located within a viscoelastic, isotropic medium. A multipole expansion technique was employed so that the incident and scattered wave fields could be defined in either cylindrical or spherical coordinates. This duality permitted the boundary conditions to be enforced exactly. However the procedure involved an infinite number of integrals which were evaluated numerically.

This aforementioned review indicates so far that there have been numerous studies of the transient scattering of elastic waves from *spherical cavities* and *inclusions* but without a shell lining ([Yeh, 1964, Norwood and Miklowitz, 1967, McLeary, 1969, Wijeyewickrema and Keer, 1989, Huang and Wang, 1972, Zhao and Harkrider, 1992, Lebedeva, 1994, Schwartz et al., 2016]).

On the other hand, there have been far fewer cases involving the transient elastic wave radiation from an embedded shell in which a surface load rather than an incident wave has been considered (e.g. [Eringen, 1957, Rice et al., 1978, Duffey and Johnson, 1981, Akkas et al., 2000, Zakout et al., 2000, Gan-bin and Kang-he, 2005]). To the authors' knowledge, there have been very few studies of the non-stationary scattering of an incident elastic wave by an embedded spherical shell. [Murthy and Reddy, 1967] considered one example of an empty shell but a simple, thin shell theory was employed. Indeed, a transient elastoacoustic analysis of a thick-walled, fluid-filled spherical shell embedded in an elastic medium remains unaddressed. Consequently a comprehensive and detailed formulation of this problem is given here based on the exact theory of linear elasticity. Different excitation sources are considered in the time domain. These include plane and spherical waves arising from various origins as well as surface traction forces. The following companion papers give comprehensive numerical case studies from which general themes may be observed. Specific highlights and novelties of the solutions given in the first companion paper are summarized next:

- **(i)** transient scattering of spherical incident, elastic waves (shear or longitudinal) which originate from within the shell's material are investigated for the first time;
- **(ii)** transient diffraction and refraction of a pressure wave stemming from an eccentric point source located inside the enclosed fluid medium are considered;
- **(iii)** exact theory of linear elasticity rather than a simplified shell theory represents a shell which transiently scatters or refracts plane or spherical stress waves originating from a surrounding infinite elastic medium;
- **(iv)** effect of an encapsulated fluid is included in the non-stationary elasto-acoustics of an embedded, thick-walled spherical shell;
- **(v)** transient radiation from a thick walled spherical shell is presented by using exact linear theory when the shell is subjected to the most generalized form of traction force(s).

Number of key observations within the numerical experimentation of the second companion paper having significant practical implications are listed next:

- **(i)** different focusing mechanisms are noted within the contained fluid, the shell's material and its surrounding elastic formation;
- **(ii)** various configurations of the geometrical and relative material properties are identified being able of concentration the acoustical energy into a very small region;
- **(iii)** specific material arrangements are tested which lead to a significant damping of the impinging shocks.
- **(iv)** situations prone to a damaging cavitation within the contained fluid are investigated.

The order adopted for the following presentation starts with a generalized mathematical formulation (§ 3.2) whose results are compared with a wide variety of previously published data (§ 3.3). The mathematical developments begin with preliminaries, terminologies and the problem's definition. It is followed by a detailed description of the excitation sources. Wave propagation in

elastic media (shell and external elastic formation) and the internal fluid medium is formulated subsequently. This is followed by implementation of appropriate boundary conditions. This section ends by constructing total solutions from corresponding modal components. Finally, the current solutions are compared meticulously with multiple existing published limit cases to validate the current approach and associated computer codes (§ 3.3). A comprehensive set of general circumstances are considered in a companion chapter.

3.2 Mathematical Formulations

The choice of an accurate and numerically efficient Laplace inversion procedure is considered in § 3.2.1. The problems' overall geometry and relevant nondimensionalized parameters as well as terminologies and conventions are given in § 3.2.2. Then detailed descriptions of different excitation sources are considered in § 3.2.3. A generalized description of a traction force acting at either surface of the shell is given first. Next, a unified description is outlined for various incident waves which include spherical and plane stress waves generated by a longitudinal or shear origin located at a point in the infinite elastic medium and the shell's material or an eccentric pressure initiated at a point within the internal fluid. When translational addition theorems are used, their derivations are summarized briefly and consistently in Appendix E in a novel way to gleam their relevance. The 3D description of an elastic wave propagating through an elastic medium and a linear acoustical phenomenon occurring within a fluid medium are discussed in § 3.2.4 and § 3.2.5, respectively. Continuity boundary conditions are applied subsequently at a shell's traction surfaces before a solution is completed in § 3.2.6. Details are provided later in the Appendices when difficulties require more explanation.

3.2.1 Laplace Transform

The earlier literature review indicated that the Laplace transform is a versatile tool for solving dynamic problems of the Newman-Dirichlet type. Particular examples of interest here include the elastic and acoustic wave equations. See, for example, [Huang and Wang, 1972, Akkas et al., 2000]. Therefore the one-sided Laplace transform operator is applied invariably to all the time dependent functions. Then the Laplace transform and its inversion (to time) are respectively

$$\Lambda \equiv \Lambda(s) = \int_0^\infty \bar{\Lambda}(t) e^{-st} dt \quad \text{and} \quad \bar{\Lambda} \equiv \bar{\Lambda}(t) = \frac{1}{2\pi i} \lim_{\eta \rightarrow \infty} \int_{\gamma-i\eta}^{\gamma+i\eta} \Lambda(s) e^{st} ds. \quad (3.1)$$

However an analytical inversion is often challenging. Consequently, it has been limited largely to simple expressions of $\Lambda(s)$. See, for example, [Akkas et al., 2000, Bahari and Popplewell, 2015]. Consequently an efficient numerical inversion method has been found necessary for more physically realistic models such as [Hasheminejad et al., 2011]. Different numerical algorithms have been proposed but few have been shown to be reliable ([Hsu and Dranoff, 1987, D'amore et al., 1999, Abate and Valkó, 2004, Cohen, 2007]). [Naeeni et al., 2015] recently compared the performance of several numerical inversion techniques for typical engineering problems which included elastic wave propagation examples. They concluded that methods based upon a Fourier series, such as the one proposed by [Durbin, 1974], were generally the most reliable and numerically stable. Our preliminary numerical investigations supported this contention. The major drawback, however, is that the convergence is slow. Therefore [Crump, 1976] applied the ϵ -algorithm to mitigate this

deficiency. More recently, the *Lanzcos* filter was utilized in conjunction with the *Durbin* approach, by [Laverty and Gazonas, 2006] and later [Keles, 2016] to enhance the convergence rate. Just as importantly, superfluous Gibbs oscillations especially in the neighbourhood of a discontinuous wavefront were alleviated. This modified form of the *Durbin* approach is used here. It is outlined next.

The trapezoidal rule is applied, in combination with a trigonometric Fourier series expansion, to the inversion integration of Equation 3.1. Then the exact value of this integration is restored as

$$\bar{\Lambda}(t) = \lim_{N_d \rightarrow \infty} \Lambda_{N_d}(t, a, N_d, T_0) + \underbrace{\sum_{j=1}^{\infty} e^{-2j(a T_0)} \bar{\Lambda}(2jT_0 + t)}_{**}, \quad t \in [0, 2T_0] \quad (3.2)$$

where $**$ is an error arising from the discretization. Here $\bar{\Lambda}_{N_d}$, N_d , a , and T_0 are respectively the truncated estimate of the inversion, the truncation number, a free parameter (to be assigned), and an arbitrary instant. Then the modified *Durbin* approximation of $\bar{\Lambda}_{N_d} \equiv \bar{\Lambda}_{N_d}(t, a, N_d, T_0)$ for $\bar{\Lambda} \equiv \bar{\Lambda}(t)$ is expressed in the closed form

$$\bar{\Lambda}_{N_d} = \frac{e^{at}}{T_0} \left(\frac{\Lambda(a)}{2} + \sum_{j=1}^{N_d} \varrho(j) \left[\Re\{\bar{\eth}_j(t) \cdot \cos b_j(t)\} + \Im\{\bar{\eth}_j(t) \cdot \sin b_j(t)\} \right] \right) \quad (3.3)$$

where

$$\bar{\eth}_j(t) = \Lambda[a + i b_j(t)], \quad b_j(t) = \frac{j\pi}{T_0} t, \quad \varrho(j) = \sin\left(\frac{j\pi}{N_d}\right) / \left(\frac{j\pi}{N_d}\right) \quad \text{and} \quad t \in [0, 2T_0].$$

The $\{\varrho(j)\}$ is termed the *Lanzcos* filter. [Durbin, 1974] as well as [Honig and Hirdes, 1984] suggested that the relative discretization error tolerance level of 10^{-er} can be met by choosing $\{aT_0 = (er \ln 10)/2\}$. [Honig and Hirdes, 1984] argued that the choice of the free parameter, $\{aT_0\}$, is optimal if the absolute values of the discretization and truncation errors are equal. From the examinations by [Durbin, 1974, Honig and Hirdes, 1984, Valkó and Abate, 2004], $5 \leq a T_0 \leq 10$ has been suggested. Accordingly the choice of $5 \leq er \leq 6$ is adopted here so that reasonable results are expected ([Laverty and Gazonas, 2006]).

3.2.2 Definition of The Problem

Consider the elastic spherical shell surrounded by the infinite elastic formation illustrated in Figure 3.1. The shell and formation are distinguished invariably by subscripts sh and ∞ , respectively, or more succinctly by the subscript $k = \{sh \text{ or } \infty\}$. By following this convention $\{\mu_k, \lambda_k\}$ and ρ_k are the Lamé constants and densities, respectively. Furthermore the corresponding longitudinal (p) and shear (s) wave speeds are respectively

$$c_{p,k} = \sqrt{\frac{\lambda_k + 2\mu_k}{\rho_k}} \quad \text{and} \quad c_{s,k} = \sqrt{\frac{\mu_k}{\rho_k}} \quad (3.4)$$

The sphere's interior, on the other hand, is filled completely with an inviscid, compressible fluid (denoted by subscript f) having density ρ_f and sound speed c_f .

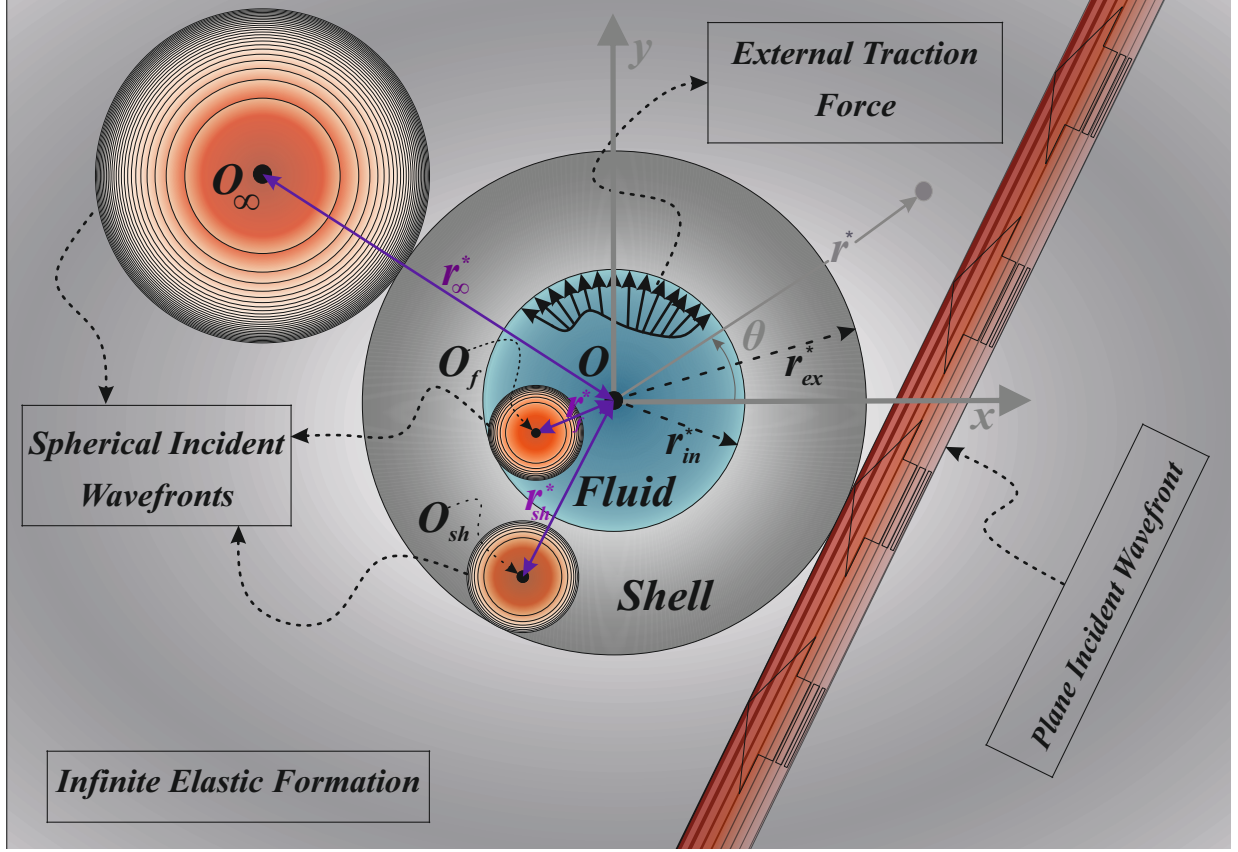


Figure 3.1: Depicting the problem's geometry and different excitation sources including incident waves and traction forces. Azimuthal angle φ is perpendicular to the illustrated plane.

The fixed origin, O , of the principal spherical coordinate frame (r^*, θ, φ) shown in Figure 3.1 coincides with the center of the spherical shell when at rest. The different excitation sources considered are located at points O_∞ , O_{sh} or O_f which are in the infinite elastic medium, the shell's material or internal fluid, respectively, as shown in Figure 3.1. Corresponding distances from the global origin, O , of these excitation positions are r_∞^* , r_{sh}^* and r_f^* . Moreover this figure indicates that r_{in}^* and r_{ex}^* are the shell's internal and external radii, respectively. The following nondimensionalized parameters (which, contrary to convention, have no $*$ superscript) are always used. They are defined by

$$\begin{aligned} t &= t^*/t_0, \quad r = r^*/r_{ex}^*, \quad \bar{u}_i = \bar{u}_i^*/r_{ex}^*, \quad \bar{v}_i = \frac{\bar{v}_i^*}{c_p}, \quad \bar{\sigma}_{ij} = \frac{\bar{\sigma}_{ij}^*}{\sigma_0^*}, \\ \bar{p} &= \bar{p}^*/\sigma_0^*, \quad \alpha_{m,k} = c_p/c_m^{(k)}, \quad \alpha_f = c_p/c_f \quad \text{and} \quad \gamma_f = \rho_\infty/\rho_f \end{aligned} \quad (3.5)$$

in which $i, j = \{r, \theta, \varphi\}$, $m = \{s, p\}$, $c_p = c_p^{(\infty)}$, $t_0 = c_p^{(\infty)} r_{ex}^*$ and $\sigma_0^* = \lambda_\infty + 2\mu_\infty$. In addition, $t, \sigma_{ij} \equiv \sigma_{ij}(r, \theta, \varphi, s)$, $p \equiv p(r, \theta, \varphi, s)$, $u_i \equiv u_i(r, \theta, \varphi, s)$, and $v_i \equiv v_i(r, \theta, \varphi, s)$ are the dimensionless time, Cauchy stress tensor, acoustic pressure perturbations, displacement and velocity of material particles in a Lagrangian description, respectively. All the excitation amplitudes are considered sufficiently small that the superposition principle and reciprocity theorem apply.

3.2.3 Excitation Sources

A unified description is formulated first for different incident waves originating from various excitation sources. A source can be located at a given point within the encapsulated fluid, the shell's material or its surrounding elastic medium. Regardless, the temporal origin, $t = 0$, is always taken to coincide with the first instant when an incident wavefront or a traction force impinges upon the shell's inner or outer surfaces. Solutions prior to this instant are usually uninformative. Not only radial but generally oriented traction forces are considered next. The magnitudes of the stress and pressure arising at a specified position from an incident wave or a traction force are normalized with respect to the σ_0^* defined immediately after Equation 3.5. The superposition principle can be applied, of course, when multiple excitation sources exist simultaneously. Moreover, $\bar{f}(t)$ is always taken to represent the time dependency of an excitation.

3.2.3.1 Incident Waves

The incident waves considered are categorized and described mathematically next. The fixed main coordinate frame centered at O is chosen, without loss of generality, so that the azimuth angle, φ_i where $i = \{f, sh, \infty\}$, of a point source located at O_i is zero. A disturbance at O_i at a given instant produces a propagating spherical wavefront which eventually collides with the sphere after time t_i . This incident wave is described by employing the *convolution* of the full-space *Green's function*, $\bar{\mathbf{G}}(t|t_i, \mathbf{r}|\mathbf{r}_i)$, and a specified excitation whose time variation is $\bar{f}(t)$. The t_i is the initial arrival time of the wavefront at one of the shell's surfaces. The derivation of $\bar{\mathbf{G}}$ is given explicitly in Appendix D. Subsequently a general temporal description for an incident spherical wave becomes

$$\bar{\Lambda}^{(inc)} = \frac{r''}{r'} \bar{f}(\tau) \mathcal{H}(\tau) \quad \text{and} \quad \tau = t + \tau', \quad (3.6)$$

where $\bar{\Lambda}^{(inc)} \equiv \bar{\Lambda}^{(inc)}(r, \theta, \varphi, t)$ may correspond to either the displacement (potentials), stress, pressure or particle velocity. Furthermore $\mathcal{H}(t)$ is the Heaviside function whereas

$$\tau' = \alpha(r'' - r') \quad \text{with} \quad \alpha \equiv \alpha_f, \alpha_{s,k}, \alpha_{p,k} \quad \text{and} \quad k = \infty, sh. \quad (3.7)$$

As depicted in Figure 3.3, r' is the distance between A (a representative point on either a plane or spherical wavefront) and an excitation source located at point O_i . Furthermore the r'' is the normal distance between a source and either surfaces of the shell which is closer. These situations are depicted in Figure 3.2. From the geometry given in the aforementioned figure

$$r' \equiv |\mathbf{r} - \mathbf{r}_i| = \sqrt{r^2 + r_i^2 - 2r_i r \cos \gamma} \quad \text{and} \quad i = \infty, sh, f. \quad (3.8)$$

The corresponding expression for r'' depends whether the source of the excitation is located within (i) the shell's interior, (ii) the shell itself or (iii) its exterior, as illustrated in Figure 3.1 and Figure 3.3. Equations are given next for each of these possibilities.

- (i) A source positioned at point O_f within the encapsulated fluid leads to

$$r'' = r_{in} - r_f. \quad (3.9)$$

This situation may represent an internal explosion or a a probe pulse for an ultrasonic immersion test.

- (ii) The source's location at O_{sh} within the shell determines if the elastic wavefront of the generated stress produced by a single excitation first touches the shell's inner or outer surface. The demarcation between these two possibilities is given by

$$r'' = \begin{cases} r_{sh} - r_{in}, & r_{sh} - r_{in} \leq h' \\ 1 - r_{sh}, & r_{sh} - r_{in} > h' \end{cases} \quad (3.10)$$

where $h' = (1 - r_{in})/2$. This situation may arise, for example, from a deep seismic activity where the earth is approximated as a mantle and a liquid core. See, for instance, [Satô et al., 1963]. Another example for this configuration is the initiation of a hydrogen induced crack in a carbon steel pipe or pressure vessel containing natural gas. This configuration, also, can be an approximation of the earth and its liquid core where a deep seismic activity occurs.

- (iii) A s - or p - wave may be initiated in the surrounding infinite elastic medium by a source located at O_∞ where

$$r'' = r_\infty - 1. \quad (3.11)$$

An underground explosion or an earthquake are particular examples of this situation.

An appropriate value of r'' in conjunction with r' found from Equation 3.8 are substituted into Equation 3.6. Then the Laplace transform is applied to produce

$$\Lambda^{(inc)} = r'' f(s) e^{\alpha s r''} \underbrace{\left(\frac{e^{-\alpha s \sqrt{r^2 + r_i^2 - 2 r_i r \cos \gamma}}}{\sqrt{r^2 + r_i^2 - 2 r_i r \cos \gamma}} \right)}_* \quad (3.12)$$

in which the $*$ expression is considered by itself later. The spherical wavefront becomes plane as $r_\infty \rightarrow \infty$, so that, $r' \rightarrow r_\infty$ and, by using Equation 3.11, $r'' \rightarrow 1$. As a consequence, Equation 3.6 turns into

$$\bar{\Lambda}^{(inc)} = \bar{f}(\tau) \mathcal{H}(\tau) \quad \text{where} \quad \tau = t + \alpha(z - 1) \quad \text{and} \quad z = r \cos \gamma. \quad (3.13)$$

Substituting $\cos \gamma$ obtained from Equation E.1 of E into Equation 3.13 produces

$$\Lambda^{(inc)} = f(s) e^{-\alpha s} \underbrace{e^{-\alpha s r (\cos \theta \cos \theta_i + \sin \theta \sin \theta_i \cos \varphi)}}_{**} \quad (3.14)$$

in the Laplace domain. The θ_i (γ) is the angle between the incident wave's direction (vector $\vec{OA}$ in Figure 3.3 which defines the receiver's position at point A) and the z axis. Alternatively, Equation 3.12 reverts to Equation 3.14 after applying L'Hospital's rule to the term marked by the $*$ in Equation 3.12.

The next goal is to express the incident wave $\Lambda^{(inc)}$ from Equations 3.12 and 3.14 in terms of the spherical harmonics, Y_n^m , and of Legendre polynomial $P_n(\cos \theta)$ for 3D and 2D cases,

respectively. The $Y_n^m \equiv Y_n^m(\theta, \varphi)$ signifies the normalized spherical harmonics, details of which are given more conveniently in [B](#). Then these equations take the respective forms

$$\Lambda^{(inc)} = \sum_{n,m} \Lambda_{n,m}^{(inc)} Y_n^m \quad \text{and} \quad \Lambda^{(inc)} = \sum_{n=0}^{\infty} \Lambda_n^{(inc)} P_n(\cos \theta) \quad (3.15)$$

in which $\sum_{n,m}^{\infty} \equiv \sum_{n=0}^{\infty} \sum_{m=-n}^{+n}$ whereas $\Lambda_{n,m}^{(inc)} \equiv \Lambda_{n,m}^{(inc)}(r, s)$ or $\Lambda_n^{(inc)} \equiv \Lambda_n^{(inc)}(r, s)$ are the modal components of the incident wave. These modal components are determined by expanding the $*$ and $**$ expression indicated in [Equations 3.12](#) and [3.14](#), respectively. This task is performed by using two translational and Legendre *addition theorems* which are detailed in [Appendix E](#). The expansion of the $*$ expression in [Equation 3.12](#) is determined by applying the *addition theorem I* through [Equation E.8](#), such that

$$\Lambda^{(inc)} = \frac{2 \alpha s r''}{\pi} e^{\alpha s r''} f(s) \sum_{n=0}^{\infty} \begin{cases} (2n+1) i_n(\alpha r_i s) k_n(\alpha r s) P_n(\cos \gamma), & r < r_i \\ (2n+1) i_n(\alpha r s) k_n(\alpha r_i s) P_n(\cos \gamma), & r \geq r_i \end{cases} \quad (3.16)$$

Furthermore the Legendre polynomial $P_n(\cos \gamma)$ is expanded by applying the *Legendre addition theorem* given in [Equation E.2](#). Then [Equation 3.16](#) becomes

$$\Lambda^{(inc)} = f(s) \frac{2 \alpha s r''}{\pi} e^{\alpha s r''} \sum_{n,m}^{\infty} \begin{cases} \mathcal{S}_{nm} i_n(\alpha r_i s) k_n(\alpha r s) P_n^m(\cos \theta) e^{im\varphi}, & r < r_i \\ \mathcal{S}_{nm} i_n(\alpha r s) k_n(\alpha r_i s) P_n^m(\cos \theta) e^{im\varphi}, & r \geq r_i \end{cases} \quad (3.17)$$

where $\mathcal{S}_{nm} = (-1)^m (2n+1) P_n^{-m}(\cos \theta_i)$. Subsequently modal components of the spherical incident wave in [Equation 3.17](#), according to the definition given in [Equation E.4](#), become

$$\Lambda_{n,m}^{(inc)} = 8f(s) \alpha s r'' e^{\alpha s r''} \begin{cases} i_n(\alpha r_i s) k_n(\alpha r s) \check{Y}_n^m(\theta_i, 0), & r < r_i \\ i_n(\alpha r s) k_n(\alpha r_i s) \check{Y}_n^m(\theta_i, 0), & r \geq r_i \end{cases} \quad (3.18)$$

For an asymmetric configuration $\cos \gamma = \cos \theta$ because φ and θ_i in [Equation 3.8](#) are zero. The modal components of a spherical incident wave are determined by using the 2D version of the *addition theorem I* given in [Equation E.8](#). Then

$$\Lambda_n^{(inc)} = f(s) \frac{2 \alpha s r''}{\pi} e^{\alpha s r''} (2n+1) \begin{cases} i_n(\alpha r_i s) k_n(\alpha r s), & r < r_i \\ i_n(\alpha r s) k_n(\alpha r_i s), & r \geq r_i \end{cases} \quad (3.19)$$

On the other hand, the plane incident wave expansion used in [Equation 3.14](#) becomes

$$\bar{\Lambda}^{(inc)} = f(s) e^{-\alpha s} \sum_{n,m}^{\infty} \mathcal{X}_{nm} i_n(\alpha r s) P_n^m(\cos \theta) e^{im\varphi} \quad (3.20)$$

where

$$\mathcal{X}_{nm} = (2n+1) \frac{(n-|m|)!}{(n+|m|)!} P_n^m(\cos \theta_i)$$

by applying the *addition theorem II* given in Equation E.9. Modal components of an incident plane wave, by incorporating Equations E.4 and E.3 into 3.20, become

$$\Lambda_{n,m}^{(inc)} = 4 \pi f(s) e^{-\alpha s} i_n(\alpha r s) \check{Y}_n^m(\theta_i, 0). \quad (3.21)$$

The last equation is simplified for an asymmetric configuration when θ_i and φ in Equation 3.14 are zero. Then, $**$ is expanded by using 2D *addition theorem II* as given in Equation E.10. As a result, the asymmetric (2D) modal components of a plane incident wave in an asymmetric configuration become

$$\bar{\Lambda}_n^{(inc)} = f(s) e^{-\alpha s} (2n+1) i_n(\alpha r s). \quad (3.22)$$

A general description of a mechanical traction force applied on the spherical shell is given next.

3.2.3.2 Surface Traction Forces

The next objective is to give a succinct, universal description of externally exerted traction forces (per unit area) which may act on either one or both of the shell's two surfaces. For instance, these excitations can be a representation for an impact load or an implanted ultrasonic probe on either of the shell's surfaces. Externally employed surface traction forces, $\hat{\mathbf{t}} \equiv \hat{\mathbf{t}}(\theta, \varphi, s) = (\hat{t}_r, \hat{t}_\theta, \hat{t}_\varphi)^T$, correspond to the *Cauchy stress vector*, $\mathbf{t} \equiv \mathbf{t}(\theta, \varphi, s)$, which is determined by projecting the *Cauchy stress tensor*, $\boldsymbol{\sigma}$, in the direction of $\mathbf{e}_n$ which is the unit vector normal and outwards to a given traction surface. Then

$$\mathbf{t}^{(i)} \equiv \boldsymbol{\sigma} \cdot \mathbf{e}_n^{(i)} \quad \text{on} \quad \partial\Omega \quad (3.23)$$

where $\Omega = \{(r, \theta, \varphi) | r_{in} \leq r \leq 1\}$ and $i = in, ex$. From the shell's perspective, if its centres center coincidences with that of the spherical coordinate frame, O , (as illustrated in Figure 3.1 $\mathbf{e}_n^{(ex)} \equiv \mathbf{e}_r$ and $\mathbf{e}_n^{(in)} \equiv -\mathbf{e}_r$. Hence,

$${}^{(k)}t_j^{(i)}(\theta, \varphi, s) = \sigma_{rj}^{(k)}(r, \theta, \varphi, s)|_{r=r_i}, \quad (3.24)$$

in which

$$k = inc, sc \quad \text{and} \quad j = r, \theta, \varphi, \quad \text{whereas} \quad i = in, ex.$$

Expansions of the $\mathbf{t}$ components in terms of spherical harmonics reveal that $t_r^{(i)}$ depends only upon Y_n^m . On the other hand if a nonzero shear component $t_\theta^{(i)}$ or $t_\varphi^{(i)}$ exists, both Y_{n-1}^m and Y_n^m are present. However, a generalized traction force vector, $\mathcal{F}^{(i)}$, can be defined by following the procedure suggested by [Eringen, 1957] whose general *Fourier-Legendre* expansion was rewritten

solely in terms of Y_n^m . Then

$$\mathcal{F}^{(i)}(\theta, \varphi, s) = \left\{ \begin{array}{c} t_r^{(i)} \\ \frac{\partial(t_\theta^{(i)} \sin \theta)/\partial \theta + \partial t_\varphi^{(i)}/\partial \varphi}{-n(n+1) \sin \theta} \\ \frac{\partial t_\theta^{(i)}/\partial \varphi - \partial(t_\varphi^{(i)} \sin \theta)/\partial \theta}{-n(n+1) \sin \theta} \end{array} \right\} = \sum_{n,m}^\infty \left\{ \begin{array}{c} a_n^m(s) \\ b_n^m(s) \\ c_n^m(s) \end{array} \right\}^{(i)} Y_n^m(\theta, \varphi) \quad (3.25)$$

in which $(a_n^m, b_n^m, c_n^m) \equiv (a_n^m(s), b_n^m(s), c_n^m(s))$ are the modal coefficients of $\mathcal{F}$'s expansion in terms of normalized spherical harmonics. These coefficients can be found by using the orthogonality properties of Y_n^m given in [Equation E.3](#) of [Appendix E](#). Subsequently

$$\left\{ \begin{array}{c} a_n^m(s) \\ b_n^m(s) \\ c_n^m(s) \end{array} \right\}^{(i)} = \oint_{\partial \Omega} \mathcal{F}^{(i)}(\theta, \varphi, s) \check{Y}_n^m(\theta, \varphi) d\Omega \quad (3.26)$$

where Y_n^{*m} is the complex conjugate of Y_n^m and $d\Omega = \sin \theta \, d\theta d\varphi$ in which $\partial \Omega = \{(\theta, \varphi) | \theta \in [0, \pi] \wedge \varphi \in [-\pi, \pi]\}$. The particular representation given in [Equation 3.25](#) is useful when a shear or torsional traction force exists. It is needed later in [§ 3.2.6](#) where the boundary conditions are defined and applied.

3.2.4 Elastic Media

The next aim is to determine the displacement field vector of the elastic media. This vector is governed by the *Navier* equation which is given in [Equation B.2](#). By superimposing the *Hansen* vectors of [Equations B.5](#) into [B.6](#), the displacement field becomes

$$\bar{\mathbf{u}} = \nabla \bar{\phi} + \nabla(r\bar{\psi}) \times \mathbf{e}_r + \nabla \left(\frac{\partial(r\bar{\chi})}{\partial r} \right) - r \nabla^2 \bar{\chi} \mathbf{e}_r \quad (3.27)$$

where $\mathbf{e}_r$ and ∇ are the radial unity vector and the *Nabla* operator, respectively. The $\bar{\phi} \equiv \bar{\phi}(r, \theta, \varphi, t)$ (longitudinal) as well as $\bar{\psi} \equiv \bar{\psi}(r, \theta, \varphi, t)$ and $\bar{\chi} \equiv \bar{\chi}(r, \theta, \varphi, t)$ (shear) are displacement potentials. Consequently the solution of the *Navier* [Equation B.2](#) is reduced to finding $\bar{\phi}$, $\bar{\psi}$ and $\bar{\chi}$. In the Laplace domain these potentials must satisfy

$$\nabla^2 \phi = s^2 \alpha_{p,k}^2 \phi, \quad \nabla^2 \psi = s^2 \alpha_{s,k}^2 \psi, \quad \nabla^2 \chi = s^2 \alpha_{s,k}^2 \chi \quad (3.28)$$

for zero initial conditions. Three dimensional solutions of [Equation 3.28](#) for the shell and the encompassing infinite elastic medium are found by using a standard separation of variables approach. The response, however, must satisfy *Sommerfeld's* radiation condition i.e. the solution

must be bounded as $r \rightarrow \infty$. Consequently

$$\begin{pmatrix} \phi^{(\infty)} \\ \psi^{(\infty)} \\ \chi^{(\infty)} \\ \vdots \\ \phi^{(sh)} \\ \psi^{(sh)} \\ \chi^{(sh)} \end{pmatrix} = \sum_{n,m}^{\infty} \begin{pmatrix} A_n^m k_n(r s \alpha_{p,\infty}) \\ B_n^m k_n(r s \alpha_{s,\infty}) \\ C_n^m k_n(r s \alpha_{s,\infty}) \\ \vdots \\ D_n^m i_n(r s \alpha_{p,sh}) + E_n^m k_n(r s \alpha_{p,sh}) \\ F_n^m i_n(r s \alpha_{s,sh}) + G_n^m k_n(r s \alpha_{s,sh}) \\ H_n^m i_n(r s \alpha_{s,sh}) + I_n^m k_n(r s \alpha_{s,sh}) \end{pmatrix} Y_n^m \quad (3.29)$$

where $\{i_n(z), k_n(z)\}$ are *modified spherical Bessel* functions whose character can be found in [Olver, 2010, § 14.30(iv); 10.47(ii)]. Furthermore $A_n^m \equiv A_n^m(s)$ through $I_n^m \equiv I_n^m(s)$ are unknown coefficients. They are determined by applying boundary conditions which are formulated and detailed later in § 3.2.6. By incorporating the given solution in Equation 3.29 into the *Hansen* vectors described in Equation B.5, the displacement field turns into the compact vector form

$$\begin{pmatrix} \mathbf{u}_{\infty} \\ \vdots \\ \mathbf{u}_{sh} \end{pmatrix} = \sum_{n,m}^{\infty} \begin{pmatrix} A_n^m \mathbf{\infty}_k \mathbf{L}_n^m + B_n^m \mathbf{\infty}_k \mathbf{M}_n^m + C_n^m \mathbf{\infty}_k \mathbf{N}_n^m \\ \vdots \\ D_n^m \mathbf{\infty}_i \mathbf{L}_n^m + E_n^m \mathbf{\infty}_k \mathbf{L}_n^m + F_n^m \mathbf{\infty}_i \mathbf{M}_n^m \\ + G_n^m \mathbf{\infty}_k \mathbf{M}_n^m + H_n^m \mathbf{\infty}_i \mathbf{N}_n^m + I_n^m \mathbf{\infty}_k \mathbf{N}_n^m \end{pmatrix} \quad (3.30)$$

To expand Equation 3.30 into a more tractable form, another set of orthogonal vectors $\{\mathbb{L}, \mathbb{M}, \mathbb{N}\}$ are required. Introduced originally by [Morse and Ingard, 1968, § 7.2.32] for the spherical harmonics Y_n^m , these vectors are

$$\mathbb{L}(\theta, \varphi) = \mathbf{e}_r Y_n^m, \quad \mathbb{M}(\theta, \varphi) = r \nabla Y_n^m, \quad \mathbb{N}(\theta, \varphi) = \nabla \times \{\mathbf{e}_r r Y_n^m\}. \quad (3.31)$$

By inserting the vectors of Equation 3.31 into Equation 3.30, the displacement field becomes

$$\begin{pmatrix} \mathbf{u}_{\infty} \\ \vdots \\ \mathbf{u}_{sh} \end{pmatrix} = \sum_{n,m}^{\infty} \begin{pmatrix} A_n^m [\mathbf{\infty}_k \vartheta_r^a \mathbb{L}_n^m + \mathbf{\infty}_k \vartheta_{\theta}^a \mathbb{M}_n^m] + B_n^m [\mathbf{\infty}_k \vartheta_{\varphi}^b \mathbb{N}_n^m] \\ + C_n^m [\mathbf{\infty}_k \vartheta_r^c \mathbb{L}_n^m + \mathbf{\infty}_k \vartheta_{\theta}^c \mathbb{M}_n^m] \\ \vdots \\ D_n^m [\mathbf{\infty}_i \vartheta_r^d \mathbb{L}_n^m + \mathbf{\infty}_i \vartheta_{\theta}^d \mathbb{M}_n^m] + E_n^m [\mathbf{\infty}_k \vartheta_r^e \mathbb{L}_n^m + \mathbf{\infty}_k \vartheta_{\theta}^e \mathbb{M}_n^m] \\ + F_n^m [\mathbf{\infty}_i \vartheta_{\varphi}^f \mathbb{N}_n^m] + G_n^m [\mathbf{\infty}_k \vartheta_{\varphi}^g \mathbb{N}_n^m] + \\ H_n^m [\mathbf{\infty}_i \vartheta_r^g \mathbb{L}_n^m + \mathbf{\infty}_i \vartheta_{\theta}^h \mathbb{M}_n^m] + I_n^m [\mathbf{\infty}_k \vartheta_r^i \mathbb{L}_n^m + \mathbf{\infty}_k \vartheta_{\theta}^i \mathbb{M}_n^m] \end{pmatrix}. \quad (3.32)$$

The coefficients $j_k \vartheta_r^1 \equiv j_k \vartheta_r^i(r, s, n)$ with $\{i = a, b, \dots, i\}$ and $\{j = \infty, sh\}$ are given in F. After algebraically manipulating Equation 3.32, the components of the displacement vector u_i become

$$\begin{pmatrix} u_r^{(\infty)} \\ \vdots \\ u_r^{(sh)} \end{pmatrix} = \sum_{n,m}^{\infty} \begin{pmatrix} A_n^m \mathbf{\infty}_k \vartheta_r^a + B_n^m \mathbf{\infty}_k \vartheta_r^b + C_n^m \mathbf{\infty}_k \vartheta_r^c \\ \vdots \\ D_n^m \mathbf{\infty}_i \vartheta_r^d + E_n^m \mathbf{\infty}_k \vartheta_r^e + F_n^m \mathbf{\infty}_i \vartheta_r^f \\ + G_n^m \mathbf{\infty}_k \vartheta_r^g + H_n^m \mathbf{\infty}_i \vartheta_r^h + I_n^m \mathbf{\infty}_k \vartheta_r^i \end{pmatrix} Y_n^m \quad (3.33a)$$

as well as

$$\begin{Bmatrix} u_{\theta}^{(\infty)} \\ u_{\theta}^{(sh)} \end{Bmatrix} = \sum_{n,m} \begin{Bmatrix} A_n^{m\infty} \vartheta_{\theta}^a + C_n^{m\infty} \vartheta_{\theta}^c \\ D_n^{msh} \vartheta_{\theta}^d + E_n^{msh} \vartheta_{\theta}^e \\ + H_n^{msh} \vartheta_{\theta}^h + I_n^{msh} \vartheta_{\theta}^i \end{Bmatrix} \frac{\partial Y_n^m}{\partial \theta} + \begin{Bmatrix} B_n^{m\infty} \vartheta_{\theta}^b \\ F_n^{msh} \vartheta_{\theta}^f \\ + G_n^{msh} \vartheta_{\theta}^g \end{Bmatrix} \frac{imY_n^m}{\sin \theta} \quad (3.33b)$$

and

$$\begin{Bmatrix} u_{\varphi}^{(\infty)} \\ u_{\varphi}^{(sh)} \end{Bmatrix} = \sum_{n,m} \begin{Bmatrix} A_n^{m\infty} \vartheta_{\varphi}^a + C_n^{m\infty} \vartheta_{\varphi}^c \\ D_n^{msh} \vartheta_{\varphi}^d + E_n^{msh} \vartheta_{\varphi}^e \\ + H_n^{msh} \vartheta_{\varphi}^h + I_n^{msh} \vartheta_{\varphi}^i \end{Bmatrix} \frac{imY_n^m}{\sin \theta} + \begin{Bmatrix} B_n^{m\infty} \vartheta_{\varphi}^b \\ F_n^{msh} \vartheta_{\varphi}^f \\ + G_n^{msh} \vartheta_{\varphi}^g \end{Bmatrix} \frac{\partial Y_n^m}{\partial \theta}. \quad (3.33c)$$

Corresponding stress elements can be found by using Equations B.1b and 3.33. However only the radial σ_{rr} and shear stress components $\{\sigma_{r\theta}, \sigma_{r\varphi}\}$ are required to enforce the boundary conditions. These components are

$$\begin{Bmatrix} \sigma_{rr}^{(\infty)} \\ \sigma_{rr}^{(sh)} \end{Bmatrix} = \sum_{n,m} \begin{Bmatrix} A_n^{m\infty} \varsigma_r^a + B_n^{m\infty} \varsigma_r^b + C_n^{m\infty} \varsigma_r^c \\ D_n^{msh} \varsigma_r^d + E_n^{msh} \varsigma_r^e + F_n^{msh} \varsigma_r^f \\ + G_n^{msh} \varsigma_r^g + H_n^{msh} \varsigma_r^h + I_n^{msh} \varsigma_r^i \end{Bmatrix} Y_n^m \quad (3.34a)$$

and

$$\begin{Bmatrix} \sigma_{r\theta}^{(\infty)} \\ \sigma_{r\theta}^{(sh)} \end{Bmatrix} = \sum_{n,m} \begin{Bmatrix} B_n^{m\infty} \varsigma_{\theta}^b \\ F_n^{msh} \varsigma_{\theta}^f \\ + G_n^{msh} \varsigma_{\theta}^g \end{Bmatrix} \frac{im}{\sin \theta} Y_n^m + \sum_{n,m} \begin{Bmatrix} A_n^{m\infty} \varsigma_{\theta}^a + C_n^{m\infty} \varsigma_{\theta}^c \\ D_n^{msh} \varsigma_{\theta}^d + E_n^{msh} \varsigma_{\theta}^e \\ + H_n^{msh} \varsigma_{\theta}^h + I_n^{msh} \varsigma_{\theta}^i \end{Bmatrix} \frac{\partial Y_n^m}{\partial \theta} \quad (3.34b)$$

whereas

$$\begin{Bmatrix} \sigma_{r\varphi}^{(\infty)} \\ \sigma_{r\varphi}^{(sh)} \end{Bmatrix} = \sum_{n,m} \begin{Bmatrix} A_n^{m\infty} \varsigma_{\varphi}^a + C_n^{m\infty} \varsigma_{\varphi}^c \\ D_n^{msh} \varsigma_{\varphi}^d + E_n^{msh} \varsigma_{\varphi}^e \\ + H_n^{msh} \varsigma_{\varphi}^h + I_n^{msh} \varsigma_{\varphi}^i \end{Bmatrix} \frac{im}{\sin \theta} Y_n^m + \sum_{n,m} \begin{Bmatrix} B_n^{m\infty} \varsigma_{\varphi}^b \\ F_n^{msh} \varsigma_{\varphi}^f \\ + G_n^{msh} \varsigma_{\varphi}^g \end{Bmatrix} \frac{\partial Y_n^m}{\partial \theta}. \quad (3.34c)$$

Hence the modal terms corresponding to $n = 0$ in shear stress components of Equations 3.34b and 3.34c are zero. The $j_{\varsigma_r}^1 \equiv j_{\varsigma_r}^i(r, s, n)$ are given in Appendix F. Super- and sub- scripts in these coefficients' notation are: upper-left specifies the shell (*sh*) or the infinite medium(∞); the lower-left denotes whether the coefficient is a multiplier to $i_n(z)$ or $k_n(z)$; the upper-right (m) and the lower-left (n) state the mode numbers; the upper-right $\{a, b, \dots, i\}$ indicates the multiplier of $\{A, B, \dots, I\}$; and the lower-right gives the $\{r, \theta, \varphi\}$ directions. Wave propagation in the fluid contained in the sphere is addressed next.

3.2.5 Fluid Field

A linear acoustical phenomenon happens when a density fluctuation, $\Delta\rho_f/\rho_f$, and its associated change in the fluid's particle velocity ($\Delta v/v$), are very small. See, for example, [Rienstra and Hirschberg, 2003, § 2.1]. Then the conservation of mass and linear momentum for a stagnant fluid (i.e., one having zero mean velocity) leads to

$$\frac{1}{\gamma_f} \frac{\partial \bar{v}(r, \theta, \varphi, t)}{\partial t} + \nabla \bar{p}(r, \theta, \varphi, t) = 0 \quad (3.35)$$

in the absence of body force where the $\bar{p} \equiv \bar{p}(r, \theta, \varphi, t)$ represents the acoustical fluctuations from the static pressure. The fluid particle's radial displacement, u_r , is derived straightforwardly in the Laplace domain from Equation 3.35. Thus the radial component of the fluid's particle displacement in a *Lagrangian* description becomes

$$u_r(r, \theta, \varphi, s) = \frac{-\gamma_f}{s^2} \frac{\partial p(r, \theta, \varphi, s)}{\partial r} \quad (3.36)$$

for zero initial conditions. Combining the acoustical constitutive relation $\bar{p} = \Delta\rho_f/\alpha_f$ with the conservation law of Equation 3.35 leads to the acoustical wave equation

$$\nabla^2 p(r, \theta, \varphi, s) = \alpha_f^2 s^2 p(r, \theta, \varphi, s). \quad (3.37)$$

The solution for p under the constraint that it is bounded at vicinity of the sphere's center, as $r \rightarrow 0$, is

$$p(r, \theta, \varphi, s) = \sum_{n,m}^{\infty} J_n^m(s) i_n(\alpha_f r s) Y_n^m(\theta, \varphi). \quad (3.38)$$

The corresponding radial component of the fluid's particle velocity is obtained simply by inserting the latter equation into Equation 3.36 so that

$$v_r(r, \theta, \varphi, s) = \gamma_f \alpha_f \sum_{n,m}^{\infty} J_n^m(s) \left(i_{n+1}(\alpha_f r s) + \frac{n}{\alpha_f r s} i_{n+1}(\alpha_f r s) \right) Y_n^m(\theta, \varphi). \quad (3.39)$$

However Equation 3.37 suggests that the corresponding modal pressure is indeterminate. This singularity has to be treated separately for (i) $n = 0$ and (ii) $n > 0$. First, the well-known *L'Hôpital's rule* is applied once as $r \rightarrow 0$ for $n = 0$ which leads to

$$p_0^0(r, s)|_{r=0} = \lim_{r \rightarrow 0} J_0^0(s) \frac{\sinh(\alpha_{in} r s)}{\alpha_{in} r s} \equiv J_0^0(s) \quad (3.40)$$

Then by using the limit form of $i_n(z)$ given in [Olver, 2010, § 10.52.1] when $n > 0$, the pressure at the shell's centre becomes

$$p_n^m(r, s)|_{r=0}^{n>1} = \lim_{r \rightarrow 0} J_n^m(s) i_n(\alpha_{in} r s) \equiv J_n^m(s) \lim_{r \rightarrow 0} \frac{(\alpha_{in} r s)^n}{(2n+1)!!} = 0 \quad (3.41)$$

A similar procedure can be used to show that the fluid's particle velocity is zero at the center. An immediate physical interpretation of Equation 3.40 is that the breathing mode, $n = 0$, is

the only non-zero modal contribution at the center of the shell. The $J_n^m = J_n^m(s, n, m)$ used in Equation 3.38 is an unknown coefficient, which together with the unknown coefficients of the elastic media $\{A_n^m, \dots, I_n^m\}$ are determined next by applying the appropriate boundary conditions.

3.2.6 Boundary Conditions

The boundary conditions are formulated most readily by employing the continuity of two sets of vectors at both surfaces of the shell. One relates to *Cauchy traction stress vector*, $\mathbf{t}$, and the other to the particle displacement vector, $\mathbf{u}$, in a *Lagrangian* description. Complete continuity of all traction stress vector components is required on the shell's external surface. Consequently enforcing equilibrium of the traction forces, $\mathbf{t} \equiv 0$ leads to

$$t_i^{(ex)} = \sigma_{ri}^{(\infty)} + \hat{t}_r^{(ex)} - \sigma_{ri}^{(sh)} = 0, \quad \text{on } \partial\Omega_{ex} \quad \text{where } i = r, \theta, \varphi \quad (3.42a)$$

by using Equation 3.23. The $\Omega = \{(r, \theta, \varphi) | r_{in} \leq r \leq 1\}$ and $\partial\Omega_{ex} = \{(\theta, \varphi) | r = 1\}$ specify the shell's domain and its boundary at the shell's external surface, respectively. In addition, the continuity of all surface displacement components is required to be satisfied so that

$$u_i^{(\infty)} - u_i^{(sh)} = 0, \quad \text{on } \partial\Omega_{ex} \quad \text{where } i = r, \theta, \varphi. \quad (3.42b)$$

On the other hand, the inner traction surface has to be treated differently because the contained fluid, unlike the external elastic medium, cannot sustain any shear stress. Continuity of only the radial component of the displacement vector, u_r , meets the required kinematic compatibility conditions in an inviscid interaction. Consequently,

$$\mathbf{t}^{(in)} \equiv \left(p + \hat{t}_r^{(in)} - \sigma_{rr}^{(sh)}\right)\mathbf{e}_r + \left(\hat{t}_\theta^{(in)} - \sigma_{r\theta}^{(sh)}\right)\mathbf{e}_\theta + \left(\hat{t}_\varphi^{(in)} - \sigma_{r\varphi}^{(sh)}\right)\mathbf{e}_\varphi = 0 \quad (3.43a)$$

and

$$u_r^{(f)} - u_r^{(sh)} = 0, \quad \text{on } \partial\Omega_{in} \quad (3.43b)$$

where $\partial\Omega_{in} = \{(\theta, \varphi) | r = r_{in}\}$ denotes the boundary at the shell's inner surface. The definitions of stress, displacement and fluid's pressure components used in Equation 3.42 and 3.43 can be found in Equations 3.34, 3.33, 3.36 and 3.38, respectively. Hence these fields' variables (i.e. σ_{ij} , u_i and p) are composed implicitly of two parts. One part corresponds to a wave which is incident on the shell, the other part relates to waves scattered by or refracted through the shell. These parts are denoted by

$$\Lambda = \Lambda^{inc} + \Lambda^{sct} \quad (3.44)$$

where superscripts *inc* and *sct* denote the incident and scattered components, respectively, whereas Λ can be any one of the p , σ_{ij} , u_i , ϕ , χ or ψ . The incident component is detailed in § 3.2.3 whereas the scattered or refracted parts have been described in § 3.2.4 and § 3.2.5. Equation 3.42 and Equation 3.43 provide ten independent equations. Addition theorems required in the expansion of spherical incident terms lead to two various expansions corresponding to the two spatially different regions. See, for example, Equation 3.17. A similar structured formulae shall be considered for

the scattered and transmitted (*sc*) terms as well. This important point, however, is commonly overlooked. By separating the scattered (or radiated) terms (*sc*) which are unknown on one side, from the incident terms (*inc*) and external surface forces ($\hat{t}$) which are known on the other side, the resulting equation system of equations become

$$\underbrace{\begin{pmatrix} \infty \sigma_{rr}^{sc}|_{r=1} - sh \sigma_{rr}^{sc}|_{r=1} \\ sh \sigma_{rr}^{sc}|_{r=r_{in}} - p|_{r=r_{in}} \\ \infty \sigma_{r\theta}^{sc}|_{r=1} - sh \sigma_{r\theta}^{sc}|_{r=1} \\ sh \sigma_{r\theta}^{sc}|_{r=r_{in}} \\ \infty \sigma_{r\varphi}^{sc}|_{r=1} - sh \sigma_{r\varphi}^{sc}|_{r=1} \\ sh \sigma_{r\varphi}^{sc}|_{r=r_{in}} \\ \infty u_r^{sc}|_{r=1} - sh u_r^{sc}|_{r=1} \\ \infty u_\theta^{sc}|_{r=1} - sh u_\theta^{sc}|_{r=1} \\ \infty u_\varphi^{sc}|_{r=1} - sh u_\varphi^{sc}|_{r=1} \\ sh u_r^{sc}|_{r=r_{in}} - u_r^{sc}|_{r=r_{in}} \end{pmatrix}}_{\boldsymbol{\delta}} = \underbrace{\begin{pmatrix} sh \sigma_{rr}^{inc}|_{r=1} - \infty \sigma_{rr}^{inc}|_{r=1} - \hat{t}_r^{(ex)} \\ p|_{r=r_{in}} - \infty \sigma_{rr}^{inc}|_{r=r_{in}} + \hat{t}_r^{(in)} \\ sh \sigma_{r\theta}^{inc}|_{r=1} - \infty \sigma_{r\theta}^{inc}|_{r=1} + \hat{t}_\theta^{(ex)} \\ - \infty \sigma_{r\theta}^{inc}|_{r=r_{in}} - \hat{t}_\theta^{(in)} \\ sh \sigma_{r\varphi}^{inc}|_{r=1} - \infty \sigma_{r\varphi}^{inc}|_{r=1} + \hat{t}_\varphi^{(ex)} \\ - \infty \sigma_{r\varphi}^{inc}|_{r=r_{in}} - \hat{t}_\varphi^{(in)} \\ sh u_r^{inc}|_{r=1} - \infty u_r^{inc}|_{r=1} \\ sh u_\theta^{inc}|_{r=1} - \infty u_\theta^{inc}|_{r=1} \\ sh u_\varphi^{inc}|_{r=1} - \infty u_\varphi^{inc}|_{r=1} \\ u_r^{inc}|_{r=r_{in}} - sh u_r^{inc}|_{r=r_{in}} \end{pmatrix}}_{\boldsymbol{\xi}}. \quad (3.45)$$

The $\boldsymbol{\delta}$ is a vector containing all unknown terms whereas $\boldsymbol{\xi}$ is a vector which represents excitation with known values. Constraints arising from Equation 3.45 provide ten independent modal equations for each set of (n, m) . They correspond to the ten unknown coefficients of $A_n^m(s)$ through $J_n^m(s)$ in the Laplace domain. This system of linear equations can be written succinctly for each (n, m) as

$$\boldsymbol{\delta}_n^m \equiv \boldsymbol{\Gamma}_n^m \boldsymbol{\omega}_n^m = \boldsymbol{\xi}_n^m \quad (3.46)$$

where $\boldsymbol{\Gamma}_n^m$, $\boldsymbol{\xi}_n^m$ and $\boldsymbol{\omega}_n^m = \{A_n^m(s), \dots, J_n^m(s)\}^T$ are respectively a 10×10 matrix of known modal values, a 10×1 vector containing the unknown modal coefficients and the modal components of the excitation vector. However there are only seven independent equations in an axisymmetric, non-torsional situation because one of either the $\sigma_{r\theta}$ or $\sigma_{r\varphi}$ requirements as well as both u_θ and u_φ conditions become redundant.

It can be inferred straightforwardly from the boundary conditions given in Equation 3.42a and Equation 3.43a that a shell's surface may act as a wave conversion mechanism. If, for example, a plane *p*-wave is incident on the outer traction surface, the scattered field consists not only of *p*-waves but also *s*-waves. (It would be remembered, however, that the procedure given in §3.2.3.2 may be followed when a combination of shear and normal traction forces exists simultaneously.) Next step is to retrieve the transient modal responses by using the inversion algorithm detailed in §3.2.1. Construction of total temporal responses from the computed time-domain modal components are discussed next.

3.2.7 Construction of Total Responses

A linear superposition of the modal time histories are required to produce the total temporal responses by using an infinite *Fourier-Legendre* series. However not only the convergence of such a series is known to be slow, but also superfluous oscillations known as *Gibbs phenomenon* may appear when higher modes are included. See, for example, [Zhang and Geers, 1993, Hasheminejad et al., 2011]. This deficiency is effectively overcome here by employing the *modified Cesáro summation* in the form of

$$\Lambda(r, \theta, \varphi, t) \approx \sum_{n=0}^{N_{CS}} \Xi_n \sum_{m=-n}^{+n} \Lambda_n^m(r, s) Y_n^m(\theta, \varphi). \quad (3.47)$$

The N_{CS} is a chosen truncation number and

$$\Xi_n = \begin{cases} 0 & \text{if } 0 \leq n \leq 2 \\ \frac{N_{CS} + 1 - n}{N_{CS} + 1} & \text{if } 2 \leq n \leq N_{CS} \end{cases} \quad (3.48)$$

for $0^\circ \leq \theta \leq 10^\circ$ and $170^\circ \leq \theta \leq 180^\circ$, whereas

$$\Xi_n = \frac{N_{CS} + 1 - n}{N_{CS} + 1} \quad 0 \leq n \leq N_{CS} \quad (3.49)$$

in which $10^\circ \leq \theta \leq 170^\circ$. Hence the breathing mode, $n = 0$, contributes only to the radial displacement, $u_{r,n}^m$, whereas the summations for $u_{\theta,n}^m$ and $u_{\varphi,n}^m$ start from $n = 1$. Computational details and numerical validations are presented next.

3.3 Numerical Procedure and Validations

Calculations are divided between two programming environments. Computations of the transient modal responses are performed MATHEMATICA[©] environment; whereas the total solutions and post-processing are carried out with MATLAB[©] 2015b. The truncation number N_d in Equation 3.2 is set to 400 by trial and error. This gives a smooth and reliable convergence in extensive numerical experiments. A higher numerical efficiency is achieved by using *Lanzcos* summation in comparison to the classical summation which its numerical convergence is discussed in [Durbin, 1974, Honig and Hirdes, 1984, Naeeni et al., 2015]. This technique is implemented in Equation 3.3. Truncation number N_{CS} in Equation 3.47, generally, is set to 120 modes for a smooth convergence in all the case studies given in the second part. Hence that the choice of N_{CS} varies in this part depending on the limiting case. Computational cost rises as higher modes are incorporated into the calculations. This rise is associated primarily with the corresponding increase in the order of the Bessel functions.

The associated MATHEMATICA code is parallelized by using the Wolfram's Kernel technology. These temporal modal components are imported into MATLAB environment and, then, integrated into the total solutions in time-domain by using the modified Cesáro Mean summation as applied in Equation 3.47. A typical run time for 2000 data points in the time domain reaches the order of 10^0 s where 120 modes are incorporated. Limiting cases of the general configuration illustrated in Figure 3.1 are discussed next.

3.3.1 Descriptions of The Limiting Situations

The current mathematical developments and its corresponding computer cross checked rigorously against numbers of previous publications. These comparisons are the limiting cases of the configuration given in [Figure 3.1](#). These situations pertain essentially to limit cases of an elastic medium which can be fluid, rigid or vacuous. More general situations are considered in a companion chapter. A succinct conceptual discussion of some general limiting situations is given next.

3.3.1.1 An Elastic Medium Transformation to a Fluid Field

An elastic medium effectively behaves as an inviscid fluid if only compressional waves (ϕ) are sustained. Shear waves cannot propagate, i.e. $\chi = \psi = 0$, and, subsequently shear stresses cannot be produced so that $\sigma_{r\theta} = \sigma_{r\varphi} = 0$. In other words, the shear modulus of elasticity vanishes as $\mu \rightarrow 0$. Hence, by using generalized Hooke's law, [Equation B.1b](#), and the Helmholtz decomposition, [Equation B.3](#), the *hydrostatic* stress tensor and the associated displacement field vector respectively become

$$\bar{\sigma}_{ij} = -\bar{p} \delta_{ij} \quad \text{or} \quad \bar{\sigma}_{ii} = -\bar{p} \equiv \lambda \bar{\phi}_{,ii} \quad (3.50a)$$

and

$$\bar{u}_i = \bar{\phi}_{,i}. \quad (3.50b)$$

The δ_{ij} and ϕ are an identity tensor and a compressional wave potential, respectively. Conservation of linear momentum, [Equation B.1c](#), by using the latter stress tensor turns into

$$\bar{\sigma}_{ij,j} \equiv -\bar{p}_{,i} = \rho_f \frac{\partial \bar{v}_i}{\partial t}. \quad (3.51)$$

This last equation is identical to [Equation 3.35](#) where $\bar{v}_i = \partial \bar{u}_i / \partial t$ represents a fluid particle velocity with a *Lagrangian* description, which is identical to [Equation 3.35](#). The wave equation for an acoustical phenomenon emerges ([Equation 3.37](#)) by combining [Equation 3.50](#) and [Equation 3.51](#) where $c_f \equiv (\lambda / \rho_f)^{1/2}$.

The radial stress is the only nonzero element of *Cauchy stress vector* and shear components are relaxed at the traction surfaces,

$$t_i = \sigma_{ir} e_r, \quad \text{so that} \quad \sigma_{rr} = -p \quad \text{and} \quad \sigma_{r\theta} = \sigma_{r\varphi} = 0. \quad (3.52)$$

Furthermore, solely the continuity of the radial component of the displacement vector, u_r , is enforced although the circumferential displacements (u_θ, u_φ) may become discontinuous at the boundary due to the zero viscosity assumption. This analysis agrees with [[Einspruch and Truell, 1960](#), [Pao and Mow, 1963](#)].

3.3.1.2 An Elastic Medium Transformation to a Rigid Body

Necessary kinematic assumption for a rigid body situation is

$$\varepsilon_{ij} = 0, \quad \text{or} \quad u_{i,j} = u_{j,i} \quad (3.53)$$

where ε_{ij} is the Green-Lagrange strain tensor for infinitesimal displacement field u_i which is given in Equation B.1a. In other words the relative displacement of any given two points of the medium must be zero. Furthermore, the kinematics of a rigid material can be also expressed in terms of *Green-Lagrange tensor* $\mathbb{C}_{ij}$ as

$$\mathbb{C}_{ij} = \delta_{ij} \quad (3.54)$$

where the Kronecker delta δ_{ij} is the manifestation of a rigid body motion. From the material properties' point of view, the Lamé constants of a rigid body must correspond to $\mu \rightarrow \infty$ and $\lambda \rightarrow \infty$. These assumptions leads to an infinite rigidity or Young's modulus $E \rightarrow \infty$ because

$$E = \mu \frac{3\lambda + 2\mu}{\mu + \lambda}. \quad (3.55)$$

For instance consider a rigid inclusion within an elastic medium. The inclusion has a unit radius and it is subjected to an incident stress wave travelling in x direction. (See, for example, [Mow, 1965] or [Mow and Pao, 1971, § 6.2]). The translational displacement of the inclusion u_x^{in} must satisfy the conservation of linear momentum, so that,

$$\frac{\partial^2 \bar{u}_x^{in}}{\partial t^2} = \frac{3}{4\pi\rho} \oint_{\partial\Omega} (\bar{\sigma}_{rr}|_{r=1} \cos \theta - \bar{\sigma}_{r\theta}|_{r=1} \sin \theta) ds, \quad (3.56)$$

where $\partial\Omega = \{(\theta, \varphi) | r = 1\}$ and $ds = \sin \theta d\theta d\varphi$. It can be shown straightforwardly that only $n = 1$ is the nonzero contributing mode in the latter integration. This observation is a consequence of the orthogonality of Legendre polynomial $P_n^m(\cos \theta)$ in the expansion of the stress components. See, for example, Equation 3.34a and 3.34b. Consequently the compatibility boundary conditions between the rigid sphere and the elastic medium at $r = 1$ become

$$u_r^\infty \equiv u_x^{in} P_1(\cos \theta) = u_x^{in} \cos \theta \quad \text{and} \quad u_\theta^\infty \equiv u_x^{in} P_1^1(\cos \theta) = -u_x^{in} \sin \theta \quad (3.57)$$

in which the u_x is given in Equation 3.56. Furthermore it can be inferred straightforwardly from Equation 3.56 that the rigid body motion vanishes $u_x^{in} \rightarrow 0$ if $\rho \rightarrow \infty$, so that

$$u_r^\infty = u_\theta^\infty = 0. \quad (3.58)$$

However it should be noted that the traction stresses can be nonzero in such a situation.

3.3.1.3 Cavity

A medium behaves effectively as an empty space if the density $\rho \rightarrow 0$. This condition applies to both a fluid or an elastic medium. In this situation traction boundary conditions includes complete relaxation of Cauchy stress vector while there is no constraints on surface the displacement vector. hence

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{r\varphi} = 0. \quad (3.59)$$

The following numerical comparisons have been selected from various results published previously.

3.3.2 Numerical Comparisons

3.3.2.1 LIMIT CASE I an empty spherical shell embedded in an infinite elastic medium

[Reddy and Kierans, 1976] studied the response of an empty spherical shell having thickness $h = 0.019$ and embedded in an infinite elastic medium. A plane, compressional incident wave with $\bar{g}(t) = \mathcal{H}(t)$ impinged on the shell. An analytical expression was given for the radial displacement of the shell's mid-surface. It is illustrated in their Figure 3 but only for the dilatational (breathing) mode, $n = 0$. A *three-mode*, thin shell theory was employed so that only membrane effects were considered. The aforementioned figure is reproduced here in Figure 3.4a with the same identifications. It predicts correctly that the total displacement converges quickly to the late time asymptotic, ≈ 2.8 static response. This behaviour is not surprising because $n = 0$ breathing mode is the sole contributor. It is worth mentioning however that a considerable difference was observed in the higher modes $n > 1$ for a *three-mode* shell model.

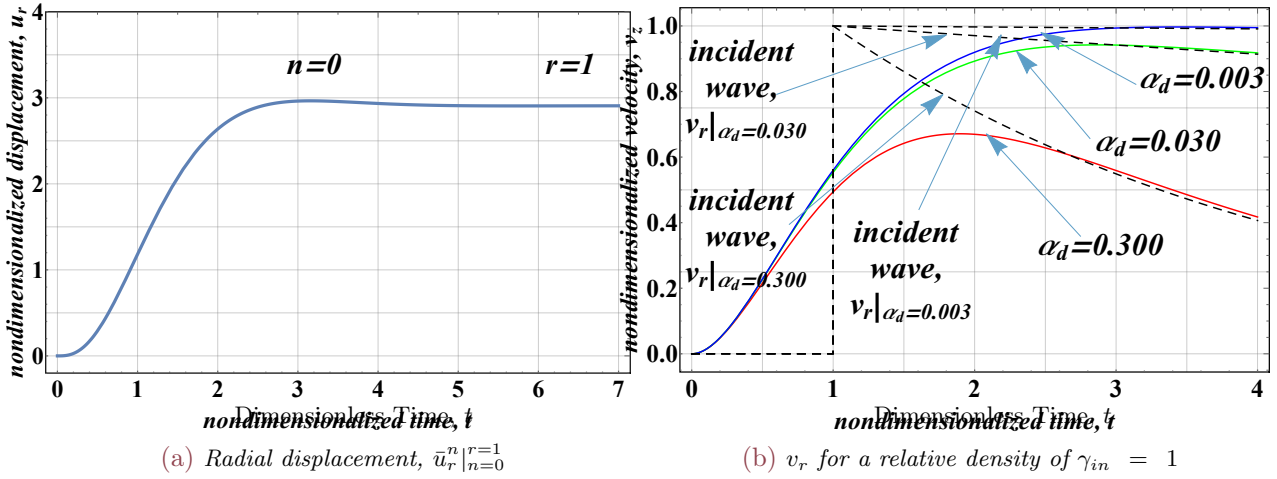


Figure 3.4: (a) illustrates the radial displacement of a buried, empty steel spherical shell's mid-surface due to a Heaviside incident compressional wave where the shell's thickness is $h = 0.19$ ([Reddy and Kierans, 1976]) (b) rigid body radial velocity of a spherical rigid inclusion subjected to an explosive compressional incident wave for different decaying constant where the density of inclusion and elastic formation is the same ([Mow, 1965]).

3.3.2.2 LIMIT CASE II a rigid spherical inclusion in an infinite elastic medium

The configuration of Figure 3.1 effectively becomes an elastic inclusion in an infinite elastic medium if the internal radius becomes very small, $r_{in} \rightarrow 0$. Then a similar arrangement as the one considered by [Mow, 1965] for a plane compressional wave impingement and elastic properties of the inclusion chosen according to Equation 3.55. The rigid body motion, u_x^∞ , can be deduced from Equation 3.56 or, equivalently, by considering only the $n = 1$ translational mode such that

$$u_x^{in} = u_\theta^{in} \sin \theta + u_r^{in} \cos \theta \quad \text{at } r = 1 \quad (3.60)$$

where $\theta = 0$ or π . Afterwards, Figure 3.4b illustrates the normalized radial particle velocity of the elastic formation at the inclusion's surface created by an exponentially decaying excitation having a general temporal form $\bar{g}(t) = \mathcal{H}(t)e^{-\alpha_d t}$. Different values of the decaying rates α_d are tested where the inclusion and elastic formation have an identical density. Incident particle velocities due to various values of α_d are shown. There is very good agreement initially between

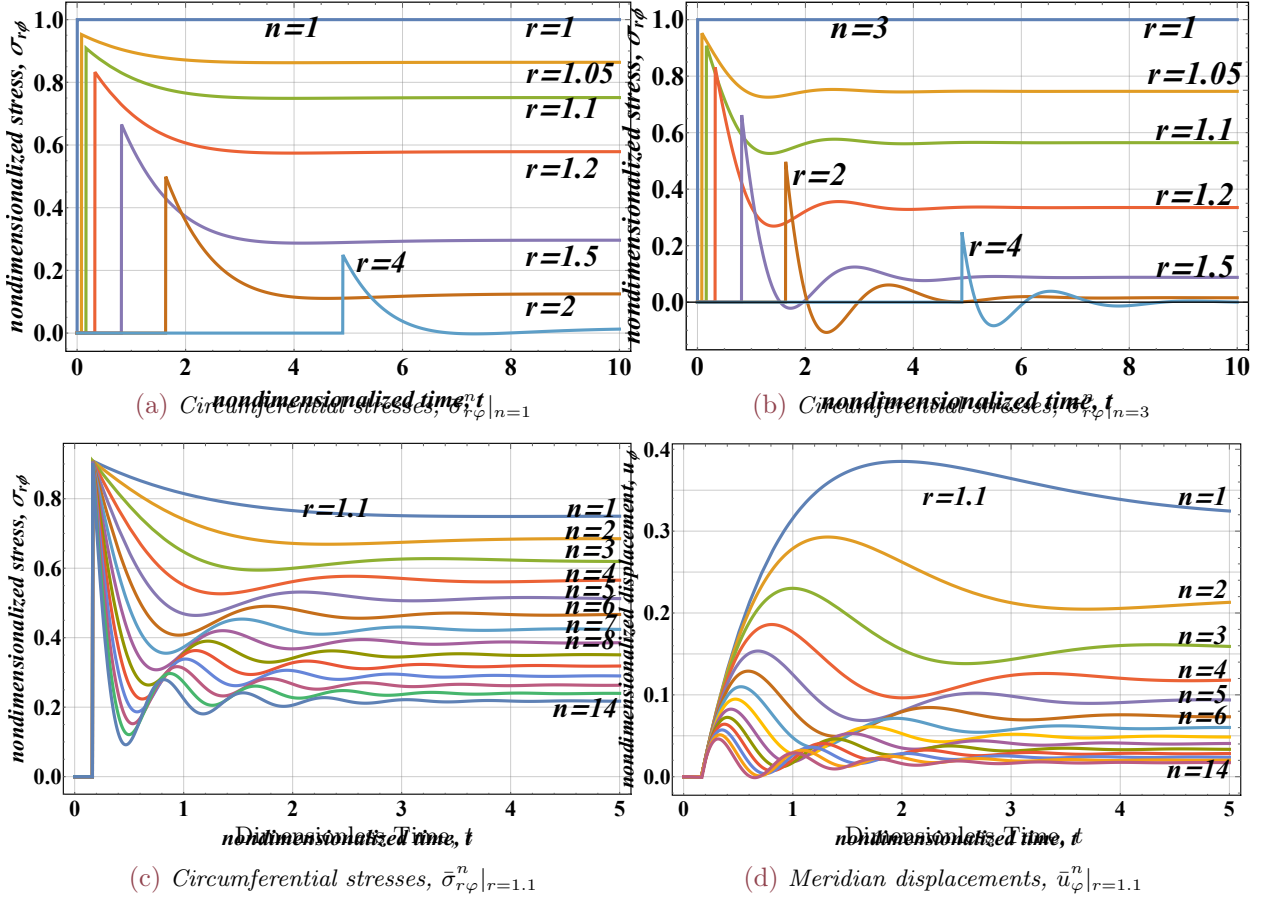


Figure 3.5: A spherical cavity subjected to an asymmetric Heaviside torsional loading is considered as the one of [Zakout et al., 2000]. Modal circumferential stress components of $\bar{\sigma}_{r\phi}^n$ for (a) $n = 1$ (b) $n = 3$ (c) $n = 1 - 14$ and $r = 1.1$ (d) modal circumferential displacement components for $n = 1 - 14$ and $r = 1.1$. Hence, the stress at $r = 1$ is the applied actual stress.

Figure 3.4b and Figure 5 of [Mow, 1965]. a slight but increasingly noticeable disagreement can be observed after the full engulfment of the inclusion, i.e. $t > 2$. The inclusion velocity catches the incident wavefront and slightly passes the current figure whereas in the original one they seems to be completely simultaneously.

3.3.2.3 LIMIT CASE III an empty spherical cavity in an infinite elastic medium

If both the shell's thickness and the relative density of the internal fluid become very small, i.e. $h \rightarrow 0$ and $\gamma_{in} \rightarrow 0$, the conditions for §3.3.1.3 are met for a spherical cavity. Furthermore if the shell and surrounding elastic medium have identical material properties and the contained fluid has zero density, the situation effectively reverts to that of the cavity. The two examples considered by [Zakout et al., 2000] and [Blake Jr, 1952, Goldsmith and Allen, 1955] are chosen for this limiting case situation.

[Zakout et al., 2000] developed a closed-form solution for an asymmetric response of an elastic cavity to a purely torsional, transient excitation applied at the cavity's surface. The excitation's time variation is $\bar{g}(t) = \mathcal{H}(t)$ again. This loading is treated according to §3.2.3.2. A torsion alone excites only the ψ shear potential. Moreover there are no mode conversions at the inner surface and the $n = 0$ breathing mode does not come into play. The ensuing circumferential stresses of the first and third modes are plotted in Figure 3.5a and Figure 3.5b at various radial distances.

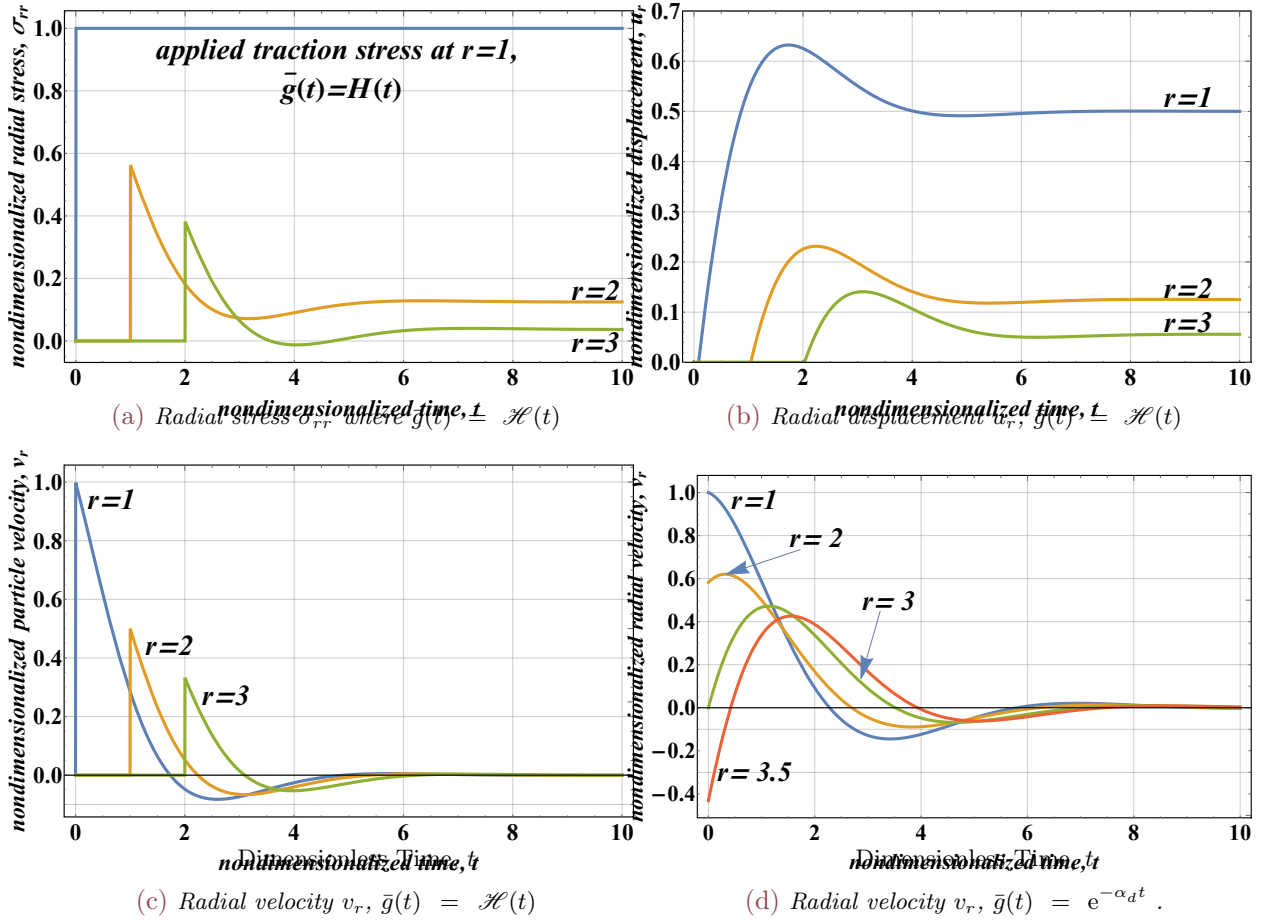


Figure 3.6: A spherical cavity's response histories to a symmetric pressure which is known a priori as a traction force, at different distances from cavity's surface, r , (a) radial stresses σ_{rr} (b) radial displacements u_r and (c) radial particle velocities v_r time histories are due to a Heaviside $\mathcal{H}(t)$ excitation while (d) depicts particle velocities due to an explosive excitation $e^{-\alpha_d t}$ with a decaying constant of $\alpha_d = 2 \times 10^{-6} \text{ s}^{-1}$. See [Goldsmith and Allen, 1955].

They correspond to Figures 2 and 4 of [Zakout et al., 2000], respectively.

It should be noted that the response at $r = 1$ is the actual traction force and it is useful as a simple reference. The agreement in the $n = 1$ mode is seen to be excellent. However more noticeable differences can be observed for the $n = 3$ mode, especially around the peak responses and as the distance from the cavity, r , increases. Figure 3.5c and Figure 3.5d which correspond to Figures 5 and 8 of [Zakout et al., 2000] respectively, illustrates circumferential displacements and stresses at $r = 1.1$ for $n = 1 - 14$. Excellent agreement is seen again, particular at the earliest times and for the lowest modes. Even better agreement occurred when Figure 3.5 was reproduced by applying a higher numerical precision in the RVM approach used originally by [Zakout et al., 2000]. The dynamic response of a spherical cavity induced by a uniform transient pressure was considered as early as [Jeffreys, 1931]. Closed form expression were derived for a spherical cavity's radial displacement and stress produced by a uniform pressure having a uniform Heaviside pressure in Equation 21 where the Lamé elastic constants were assumed to be equal in value, $\lambda = \mu$. Later on [Blake Jr, 1952] provide a similar solution for an exponentially decaying excitation and general Lamé constants. Next, [Goldsmith and Allen, 1955] performed a rigorous set of numerical case studies on the Blake's solutions. Only dilation (breathing) mode $n = 0$ is excited by this symmetric

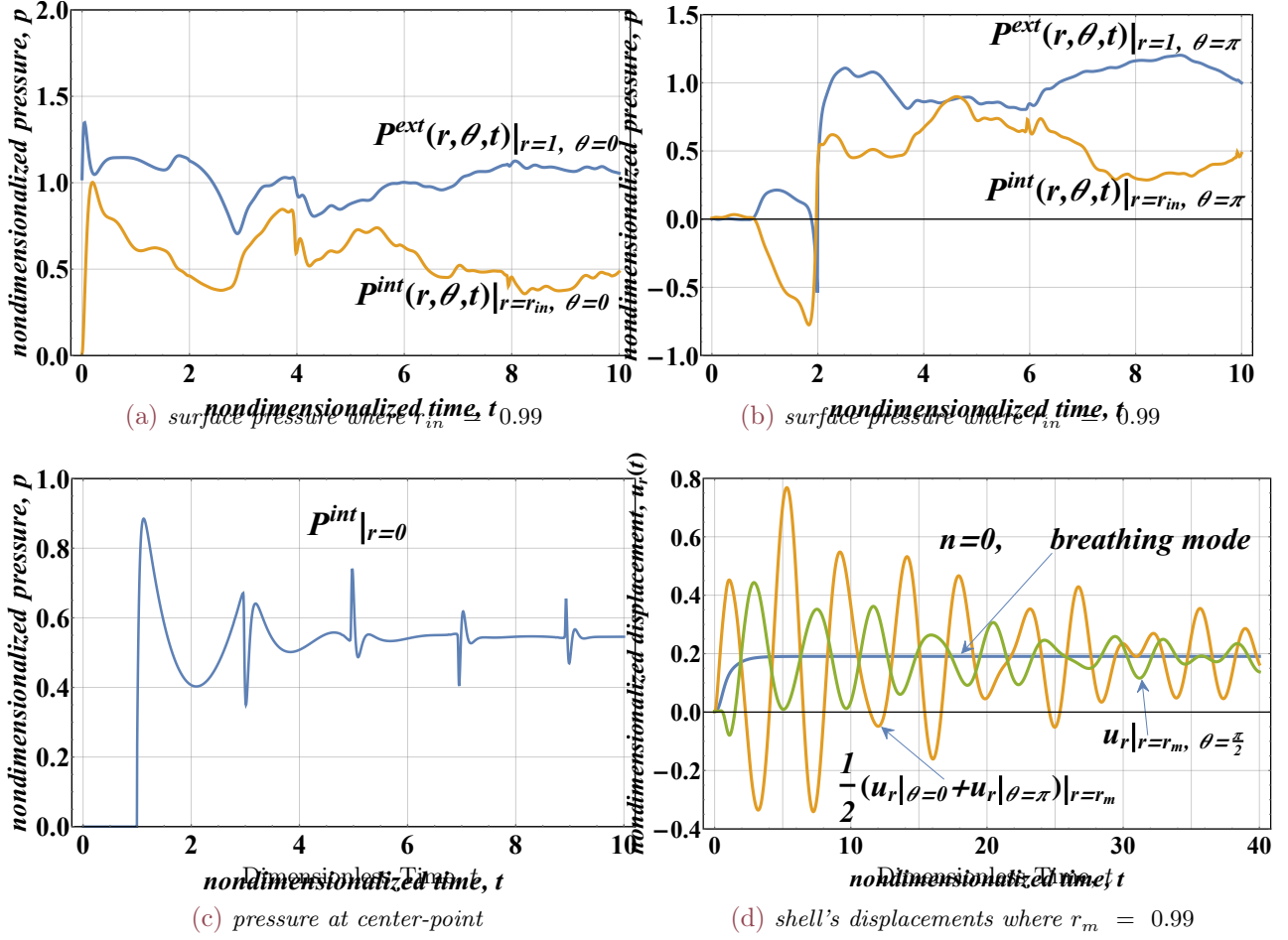


Figure 3.7: (a) inner and outer surface pressures at a spherical shell's (b) apex ($\theta = 0$), (c) tail ($\theta = \pi$) and (c) its center-point ($r = 0$). This spherical steel shell is filled and submerged in water which is subjected to an incident plane pressure wave. (d) shell's displacement at $\theta = \pi/2$ as well as the average value between apex and tail in comparison with breathing mode displacement as the late time static asymptotic deflection where $h = 0.01$ [Huang, 1969].

excitation. Consequently, Equation B.4 and Equation B.1b turns into

$$\nabla^2 \phi \equiv \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) \phi = s^2 \phi, \quad 1 < r < \infty \quad (3.61a)$$

$$\sigma_{rr} \equiv \frac{\partial^2 \phi}{\partial r^2} + \frac{2\lambda}{\lambda + 2\mu} \frac{\partial \phi}{r \partial r} = g(s), \quad r = 1 \quad (3.61b)$$

where $\bar{g}(t) = \mathcal{H}(t)e^{-\alpha_d t}$ is an exponentially decaying function. The closed-form temporal response of Equation 3.61 to a Heaviside excitation, $\alpha_d = 0$, becomes

$$\bar{\phi}_{\mathcal{H}}(t) = \left(\omega_1 e^{-\frac{2\mu}{2\mu+\lambda}(t-r+1)} (\cos \omega_0 t + \omega_2 \sin \omega_0 t) + \frac{1}{2} \right) \mathcal{H}(t - r + 1) \quad (3.62)$$

in which $\omega_0 = 2\sqrt{\mu^2 + \mu\lambda/(\lambda + 2\mu)}$, $\omega_2 = \mu/\sqrt{\mu^2 + \lambda\mu}$ and $\omega_1 = (\lambda + 2\mu)/(4\mu)$. Response to a general excitation is simply retrieved by using the Duhamel's theorem, such that

$$\bar{\phi}(t) = \bar{\phi}_{\mathcal{H}}(t)\bar{g}(t)|_{t=0^+} + \int_0^t \frac{d\bar{g}(\tau)}{d\tau} \big|_{t=(t-\tau)} \bar{\phi}_{\mathcal{H}}(\tau) d\tau \quad (3.63)$$

where $\bar{\phi}_{\mathcal{H}}(t)$ is given in Equation 3.62. Figure 3.6a illustrates the time history of the radial stress at $r = 2$ and $r = 3$ within the surrounding elastic medium caused by Heaviside force applied at the cavity's surface, $r = 1$. Corresponding radial displacements and particle velocities are depicted in Figure 3.6b and Figure 3.6c, respectively. Moreover particle velocities at various distances from cavity's surface in responses of an explosive excitation are depicted in Figure 3.6c. This corresponds to Figure 3 of [Goldsmith and Allen, 1955] where an excellent agreement is met. Furthermore, the late asymptotic stresses and displacements in Figure 3.6a and Figure 3.6b, respectively, agree completely with the corresponding static values.

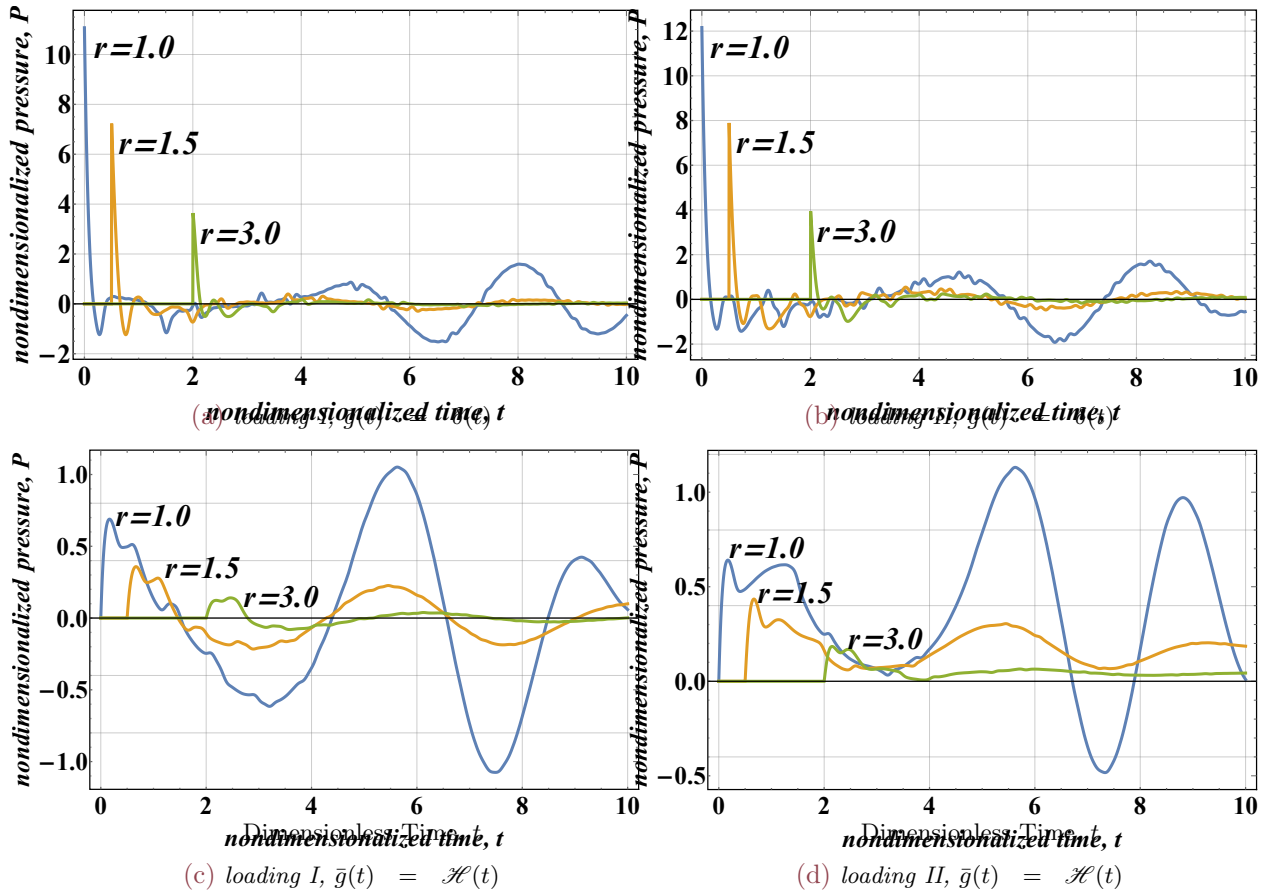


Figure 3.8: Time histories of the total pressure at the shell's apex ($\theta = 0$) and different nondimensionalized distances from the sphere's center ($r = 1.0, 1.5$, and 3) for (a) the loading case I and Heaviside excitation $\bar{g}(t) = \mathcal{H}(t)$ (b) the loading case II and Heaviside excitation $\bar{g}(t) = \mathcal{H}(t)$ (c) the loading case I and Impulse excitation $\bar{g}(t) = \delta(t)$ and (d) the loading case II and Impulse excitation $\bar{g}(t) = \delta(t)$. The loading case I allows the rigid body motion while the loading case II does not. These results correspond to Figures 2(b,c) and Figures 3(b,c) of [Bahari and Popplewell, 2015] for loading cases I and II, respectively.

LIMIT CASE IV a spherical shell filled with and submerged in compressible fluids

The configuration of [Zhang and Geers, 1993] is reproduced if the procedure outlined in § 3.3.1.1 is applied to the external elastic medium and the acoustical properties of water are adjusted appropriately. A plane, Heaviside compressional wave impinges on the water filled shell having a thickness of $h = 0.01$. Figure 3.7a and Figure 3.7b illustrate the internal and external pressure histories at the head and tail of the shell. These figures demonstrate a fairly good agreement with figures 6 and 7 of [Zhang and Geers, 1993], respectively, where a truncation of $N_{CS} = 9$ is made.

Cf §3.2.7. Moreover, center-point pressure in Figure 3.7c is perfectly consistent with Figure 16 of [Zhang and Geers, 1993].

More obvious differences, however, can be observed at the tail of the shell when (i) $t \ll 1$ and (ii) $t \approx 2$ around the shell's full engulfment by the incident wavefront. Besides, a stronger negative external pressure deep at the tail point is observed in Figure 3.7b. These discrepancies seem to be caused by the errors introduced by the application of the approximate *four-mode* shell theory in [Zhang and Geers, 1993] versus the exact Green-Lagrange kinematic assumptions used here.

LIMIT CASE V an empty spherical shell submerged in a compressible fluid

The configuration of [Huang, 1969] is replicated if the internal fluid is treated as explained in §3.3.1.3 in the previous limit case and $h = 0.02$. Figure 3.7d illustrates the radial displacement time histories which are fair good reproduction of Figures 2 and 4 of [Huang, 1969]. The rigid body motion due to $n = 1$ in the Figure 3.7d is taken out. Consequently the long term asymptotic response of radial displacements are expected to be the same as static response in which zero's mode is expected to be the only contributing mode. Accordingly both total displacements oscillate around the zero's mode $(u_r^n(r, s)|_{r=r_m})^{n=0}$ in Figure 3.7d.

If the source of excitation is an axisymmetric surface load at the shell's inner surface, the configuration matches the one of [Zakout, 2001, Bahari and Popplewell, 2015]. Two types of excitation function as delta Dirac impulse and Heaviside are considered along with two spatial distributions of the surface loading. The rigid body motion in the loading situation I exists while it is suppressed in the loading case II. Figure 3.8a, Figure 3.8b, Figure 3.8c and Figure 3.8d are fairly accurate reproductions of Figures 2(b), 2(c), 3(b) and 3(c) of [Bahari and Popplewell, 2015], respectively.

LIMIT CASE VI buried elastic inclusion

An elastic shell buried in an elastic medium is a limiting situation of the configuration presented in Figure 3.1 when the internal radius vanishes as $r_{in}^* \rightarrow 0$. This case has many potential applications such as ultrasonic imaging of the lesions within human tissues. This situation is discussed and studied by using some numerical experiments in the following companion chapter.

The aforementioned limiting case studies demonstrate the correctness and robustness of the proposed approach. Significant wave phenomena, however, are disregarded in these limiting situations. Multiple reflections and refractions of different types of waves at shell's surfaces occur along with the associated mode conversions. Elastic-elastic or elastic-fluid interfaces are associated with some specific features of wave propagation. A more accurate prediction of different body or surface waves' formation and propagation requires the consideration of the general configuration depicted in Figure 3.1. Some specific examples of these waves are asymmetric p -/ s - waves, symmetric and asymmetric *Lamb* types and head waves. Numerically capturing these physical features of waves is vital, for example, in ultrasonic A-scan testing of structure having such an internal anomaly. A set of meticulously selected situations are investigated numerically in the follow up where special visualizations techniques are used to better comprehend the physics involved in the wave propagation's associated phenomena.

3.4 Conclusions

A comprehensive, detailed study on a elastodynamics of an embedded elastic shell, filled with a compressible fluid is performed with a semi-analytical approach. Two general categories of excitation sources are considered: incident waves impinging on the shell's either surfaces as well as directly applied traction forces. A unified mathematical description is given for incident waves of various origins by using an associated time-domain, full-space Green's function. Incident sources stemming from either elastic media, the shell or infinite formation, may have an origin of a purely equivoluminal shear nature, an irrotational dilational motion or a combination of both. Furthermore, an eccentric pressure pulse within the encapsulated fluid, also, is included. Appropriate *translational addition theorems* are developed and applied to all incident terms to express them in terms of *spherical harmonics* of a frame whose its center coincides with the sphere's center. An exact linear theory of local elasticity in conjunction with *Helmholtz* decomposition is employed to describe elastic fields in terms of orthogonal *Hansen* vectors. A linearized acoustic wave model is utilized to describe the acoustic phenomenon within the contained fluid medium where a singularity at the sphere's center, $r = 0$, is handled accordingly. The boundary conditions at either shell surfaces are applied including: (i) kinematic continuity with a Lagrangian description; and (ii) continuity of Cauchy traction stress vector. Finally, the total fields' responses in time-domain are retrieved by superposing the transient modal components after performing a Laplace transform inversion algorithm. A relatively slow convergence of the associated Fourier-Legendre series is improved by employing the *modified Cesàro summation*. It should be reminded that any combined loading situation can be retrieved simply through the application of the superposition principle.

Validating the current approach and associated computer codes are performed by a rigorously cross-checking against multiple sources from the relevant literature as different limiting cases. Discussions on the mathematical and physical interpretations of these limiting cases are proposed, accordingly. The comparisons have confirmed the accuracy and the reliability of the current approach. Important physical features of the interaction, however, are missing in the limiting cases. Specific examples of such physical characteristics are simultaneous mode conversions at an elastic-elastic interface occurring for both reflected and refracted waves. These conversions affect the formation and physics of symmetric and asymmetric *Lamb's* waves, head waves and creeping waves. A fluid and elastic interface, on the other hand, do not convert modes. Limiting case of thin shell theories, for instance, completely overlook the body waves' propagation and reflections within a shell's wall. A completely rigid inclusion, as another example, simply does not exist in the nature and every materials can sustain and transmit mechanical wave propagation. An incident wave with a shear origin compared to compressional source leads to significantly different stress and pressure concentration patterns. The companion chapter is focused on the study of features of this kind through a number of carefully selected case studies.

Chapter 4

A Comprehensive, Transient Elasto-Acoustics Study of A Fluid Filled Spherical Shell Embedded in An Elastic Medium II: Numerical Results

An analytical approach was taken principally in the previous companion chapter to describe the transient elasto-acoustics of a spherical shell buried in an elastic formation. Validations were performed successfully for existing but relatively simple, limit cases taken from the literature. More general situations that emphasize important physical features are considered here, which were overlooked previously. Accordingly, case studies are chosen to represent different geometrical and material configurations when subjected to excitation sources having various origins. The effects of the relative acoustical properties and shell wall thicknesses are shown to be very essential in predicting the physics of interaction. Focusing within an encapsulated fluid or an elastic inclusion, for instance, is determined mainly by the relative celerity of the interacting media, to a lesser extend, by a shell's relative thickness. This has practical significance in predicting the likelihood of cavitation in the fluid field as well as the determination of the dynamic stress concentration. Several other important observations are made which have significant practical implications, for instance, in design of shock resistance configurations, producing highly focused ultrasonic pulses for the non-destructive testing or ultrasonic cleaning purposes and structural integrity considerations. The numerical experiments are centred around an incident plane wave of a compressional type. Brief comparisons are made when the shell is excited by other sources. Detailed numerical investigations of the other sources of excitations, however, require to be addressed later.

4.1 Preliminaries

Mathematical developments are being detailed and validated for the existing simple limit situations in the first companion chapter. More complicated situations are investigated numerically here whose organization is described next for a fluid filled, spherical shell embedded within an infinite elastic medium. A brief comment is made on the treatment of incident terms of an excitation by demonstrating the corresponding procedure in three specific examples. Configurations of the case studies are specified and the rationale of their choice are given next. An extensive set of numerical experiments are performed in the next section to investigate the dynamics of the interactions between the physical components of interest. The most important observations and their associated practical implications as well as concluding remarks are given finally.

4.1.1 Material and Geometrical Configurations

Extensive preliminary computer experiments suggested that the relative acoustical properties of interacting elastic media and an enclosed fluid field are the main factors which govern the physics of the interactions. Granite and steel are taken as principle choices for the surrounding elastic formations and shell, respectively. The choice of granite and steel are based on essentially two reasons. They have a very comparable elastic wave celerity and they are encountered widely in different industries. See, for instance, [Reddy and Kierans, 1976] who justify the use of steel for nuclear waste containers or in natural environments. Another example can be a very Low noise underground experimental complex known as LSBB (Noise Inter-Disciplinary Underground Science and Technology) in southern France whose its test facilities are buried deeply within a mostly granite formation ¹. However, other sets of materials are considered in order to examine the influences of their relative mechanical properties on the dynamics of the interaction. Hard clay, for example, as an external elastic formation with a steel shell is tested for its practical importance. Furthermore this configuration provides significantly high level of contrast in the acoustical properties. Pyroceram glass, for example, is taken as an external elastic formation in which steel shell resides due to its significantly lower acoustical impedance (density) in comparison to granite whilst their wave velocities are very similar. Moreover an aluminum shell being embedded within a pyroceram glass formation is also interesting as their Young's modules and density are identical whereas the Poisson ratios are 30% different. Water is taken invariably as the contained fluid. Relative shell thicknesses range widely from $h = 0.02$ to $h = 0.98$. The $h = 0.98$, essentially, represents a limiting situation of an elastic inclusion which is mentioned in §3.1 as the LIMIT CASE VI. All material properties and shell's relative wall thicknesses are given in Table 4.1. Comments on the description of the variously chosen incident excitation sources are given next.

sea water	$c_{ex} = 1500 \text{ m/s}$ and $\rho_{ex} = 1000 \text{ kg/m}^3$
steel	$\rho_s = 7800 \text{ kg/m}^3$, $\nu = 0.3$ and $E = 207 \text{ Gpa}$
granite	$\rho_s = 2700 \text{ kg/m}^3$, $\nu = 0.26$ and $E = 78.8 \text{ Gpa}$
hard clay	$\rho_s = 2000 \text{ kg/m}^3$, $\nu = 0.15$ and $E = 0.1 \text{ Gpa}$
aluminum	$\rho_s = 2700 \text{ kg/m}^3$, $\nu = 0.32$ and $E = 69 \text{ Gpa}$
pyroceram glass	$\rho_s = 2700 \text{ kg/m}^3$, $\nu = 0.2$ and $E = 69 \text{ Gpa}$
shell's thickness	$h = \{0.02, 0.04, 0.1, 0.2, 0.4, 0.8, 0.98\}$

Table 4.1: Material properties and nondimensionalized shell's relative thicknesses, h .

4.1.2 Excitation Sources

The excitation's time dependency, $\bar{f}(t)$, is chosen mainly as the Ricker pulse as it describes real signals such as seismic pulses and usually leads to a stable numerical behaviour. See, for example, [Gholamy and Kreinovich, 2014]. An explosion is considered occasionally but only in one case as its practical importance is limited essentially to a blast wave. Closed form expressions for these excitation functions in time ($\bar{f}(t)$), frequency ($\check{f}(\omega)$) and Laplace ($f(s)$) domains are given in Table 4.2. Pulses' parameters are chosen so that their dominant wavelengths are comparable to the thinnest shell's relative thickness.

¹<http://lsbb.eu/>

Excitation Pulses	
Ricker	$\bar{f}(t) = \left[1 - 2 \left(\frac{t - t_{r_0}}{t_{r_\omega}} \right)^2 \right] - \left(\frac{t - t_{r_0}}{t_{r_\omega}} \right)^2$ $f(s) = \mu_6 s + \mu_1 t_{r_0} - \mu_3 e^{(-s\mu_4 - \mu_2)^2 s^2} \operatorname{erfc}(s\mu_4 - \mu_2)$ $\tilde{f}(\omega) = t_{r_\omega}^3 \frac{\omega^2 e^{\left(i t_{r_0} \omega - \frac{t_{r_\omega}^2 \omega^2}{4} \right)}}{2\sqrt{2}}$
Explosion	$\bar{f}(t) = \frac{t_e + t_d - t}{t_e + t_d} e^{-\frac{t}{t_e}}$ $f(s) = t_e \frac{t_d + s t_e t_d + s t_e^2}{(t_e + t_d)(1 + s t_e)^2}$ $\tilde{f}(\omega) = \frac{\sqrt{2\pi}}{t_e + t_d} \left((t_d + t_e) \delta\left(\frac{\mathbf{i}}{t_e} + \omega\right) + \mathbf{i} \mathcal{H}\left(\frac{\mathbf{i}}{t_e} + \omega\right) \right)$
$\mu_1 = e^{-(t_{r_0}/t_{r_\omega})^2}, \mu_2 = t_{r_0}/t_{r_\omega}, \mu_3 = \sqrt{\pi} \mu_1 t_{r_\omega}^3/4, \mu_4 = t_{r_\omega}/2, \mu_6 = \mu_1 t_{r_\omega}^2/2$ $t_{r_\omega} = t_e = 1/100, t_{r_0} = t_d = 1/10$	

Table 4.2: Mathematical expressions of the Ricker pulse and explosion in time, $\bar{f}(t)$, frequency, $\tilde{f}(\omega)$, and Laplace, $f(s)$, domains where $\operatorname{erfc}(x)$ denotes the complementary error function.

Different excitation sources which include plane and spherical incident waves having various origins as well as a traction force are formulated in the first part. Then incident terms and their corresponding modal expansions are given in §3.2.3 of the first companion chapter where the required addition theorems are applied. These incident terms are derived directly from the pressure in the contained fluid or displacement potentials within elastic media before the modal expansions. Examples of these procedures are given next for the sources originating from internal fluid, within the shell material and infinite elastic formation.

A general, unified formula for a spherical incident stress wave and eccentric pressure pulse are given in Equation 6 of the previous part whereas the corresponding formula for a plane stress incident wave is presented in Equation 13. The eccentric incident pressure pulse is expressed as

$$\bar{p}^{inc}(r, \theta, t) = \frac{r''}{r'} f \mathcal{H} \quad (4.1)$$

then the corresponding particle acceleration of the wavefront becomes

$$\bar{a}_r^{inc}(r, \theta, t) = \frac{\gamma_f r'' (r - r_f \cos \theta) (\alpha_{in} r' \underline{f} + \underline{f})}{r'^3} \mathcal{H} \quad (4.2)$$

where $\mathcal{H} \equiv \mathcal{H}(\tau(t, \theta))$ and $f \equiv f(\tau(t, \theta))$ are the Heaviside function and the excitation's time dependency, respectively. Furthermore $\tau \equiv \tau(r, \theta)$, $r' \equiv r'(r, \theta)$ and r'' are given in Equations Equation 3.6 through Equation 3.11 whereas $\underline{f} \equiv df(z)/dz$. Similarly, elements of displacement and Cauchy stress vector at the wavefront due to a compressional plane incident wave originated

within the infinite elastic formation turns into, respectively,

$$\bar{\mathbf{u}}_p^{inc}(r, \theta, t) \equiv \begin{Bmatrix} \bar{u}_{r,p}^{inc} \\ \bar{u}_{\theta,p}^{inc} \end{Bmatrix} = \alpha_{p,\infty} \underline{f} \begin{Bmatrix} \cos \theta \\ -\sin \theta \end{Bmatrix} \mathcal{H} \quad (4.3)$$

and

$$\bar{\mathbf{t}}_p^{inc}(r, \theta, t) \equiv \begin{Bmatrix} \bar{\sigma}_{rr,p}^{inc} \\ \bar{\sigma}_{r\theta,p}^{inc} \end{Bmatrix} = \alpha_{p,\infty}^2 \begin{Bmatrix} (\lambda + \mu + \mu \cos 2\theta) \underline{f} \\ -2\mu \left(\underline{f} \cos \theta + 2\underline{f}/(r\alpha_{p,\infty}) \right) \sin \theta \end{Bmatrix} \mathcal{H} \quad (4.4)$$

where $\underline{f} = d^2 f(z)/dz^2$. On the other hand, if this compressional incident stress wave happens to have its origin at $r = r_\infty$, the elements of displacement and the Cauchy stress vector at the spherical wavefront becomes, respectively,

$$\begin{Bmatrix} \bar{u}_{r,p}^{inc} \\ \bar{u}_{\theta,p}^{inc} \end{Bmatrix} = \begin{Bmatrix} r'' (r_\infty \cos \theta - r) (r' \underline{f} \alpha_{p,\infty} + f) \\ -r_\infty r'' (r' \underline{f} \alpha_{p,\infty} + f) \sin \theta \end{Bmatrix} \frac{\mathcal{H}}{r'^3} \quad (4.5)$$

and

$$\begin{Bmatrix} \bar{\sigma}_{rr,p}^{inc} \\ \bar{\sigma}_{r\theta,p}^{inc} \end{Bmatrix} = \begin{Bmatrix} rr'' \left[r' \alpha_{p,\infty} \left\{ r' \underline{f} \alpha_{p,\infty} \left[r_\infty^2 (\mu \cos 2\theta + \lambda + \mu) + r(\lambda + 2\mu) \right. \right. \right. \\ \left. \left. (r - 2r_\infty \cos \theta) \right] + \mu \underline{f} (4r^2 - 8rr_\infty \cos \theta + 3r_\infty^2 \cos 2\theta + r_\infty^2) \right. \right. \\ \left. \left. + \mu f (4r^2 - 8rr_\infty \cos \theta + r_\infty^2 (3 \cos 2\theta + 1)) \right] \right. \\ \left. 2r_\infty r'' \mu \sin \theta \left[\alpha_{p,\infty} \left(r \alpha_{p,\infty} \underline{f} \left\{ r (r^2 + r_\infty^2 (\cos 2\theta + 2)) \right. \right. \right. \right. \right. \\ \left. \left. \left. - r_\infty \cos \theta (3r^2 + r_\infty^2) \right\} + r' \underline{f} (r_\infty r^2 + r \cos \theta - 2r_\infty^2) \right) \right. \right. \\ \left. \left. + f (r^2 + r_\infty r \cos \theta - 2r_\infty^2) \right] \right. \end{Bmatrix} \frac{\mathcal{H}}{r r'^5} \quad (4.6)$$

where $r'' = r_\infty - 1$. A spherical incident wave generated from a shear source originating within the shell's material leads to the following radial component of displacement at its wavefront as

$$\bar{u}_{r,s}^{inc} = \frac{1}{2r'^5} \left\{ \begin{array}{l} r_{sh} (r_{sh} - 1) \left(\alpha_{s,sh} \left[-r' \underline{f} \left[4 \cos \theta \left(r_{sh}^2 + r^2 \right) \right. \right. \right. \\ \left. \left. \left. - r(\cos 2\theta + 7)r_{sh} \right] + 2rr_{sh}\alpha_{s,sh} - \sin^2 \theta r'^2 \underline{f} \right] \right. \\ \left. \left. + f \left(r(\cos 2\theta + 7)r_{sh} - 4 \cos \theta \left(r_{sh}^2 + r^2 \right) \right) \right) \right. \\ \left. \begin{array}{l} r_{sh} (r_{in} - r_{sh}) \left(\alpha_{s,sh} \left[2r \sin^2 \theta r'^2 \underline{f} r_{sh} \alpha_{s,sh} \right. \right. \\ \left. \left. - r' \underline{f} \left\{ 4 \cos \theta \left(r_{sh}^2 + r^2 \right) - r(\cos 2\theta + 7)r_{sh} \right\} \right] + \right. \\ \left. \left. f \left(r(\cos 2\theta + 7)r_{sh} - 4 \cos \theta \left(r_{sh}^2 + r^2 \right) \right) \right) \right. \end{array} \right. \end{array} \right. \quad \begin{array}{l} r_{sh} - r_{in} \leq \frac{h}{2} \\ \\ r_{sh} - r_{in} > \frac{h}{2}. \end{array} \quad (4.7)$$

Shall note that various excitation configurations gives rise to different dynamics of interactions. For instance, dynamics stress concentration pattern created by an eccentric pressure source is essentially different from the one given rise to by a point source within the elastic formation. The focus in this paper, however, is on compressional plane stress wave as the main source of excitation. The rationale for the choice of materials is given next.

4.2 Numerical Experimentation

A series of extensive numerical experiments are undertaken to determine and investigate some important physical aspects of the interactions. An incident plane stress wave is taken as the principal source of excitation which has the Ricker pulse time dependency. Two different approaches are taken in illustrating the dynamics of the system: qualitative and quantitative. Successive 2D snapshots of the total field's response along with the 2D time-space contour-plots are used in the wave motions' qualitative descriptions. Note that the term *total field's* response refers to the stress and pressure states within the entire elastic shell and its encompassing formation as well as the contained fluid. Distributions of maximum absolute values of the pressure and stresses, temporally as well as spatially, are considered as quantitative descriptions of the interactions.

Complete field response snapshots are depicted in a series of consecutive 2D snapshots which includes internal fluid pressure and radial stresses within the shell's material and elastic formation. The incident term is omitted in the snapshots so that the scattered wave components are more visible and details have a higher discernibly. Illustrations are normalized between $[-1, 1]$ in gray level color to insure that any given color represents a fixed normalized value throughout. Each positive gauge pressure and tensile stress is divided by its corresponding maximum positive value whilst the negative gauge pressure and compressional stresses are normalized with respect to their corresponding maximum absolute values. A brighter color corresponds to a higher positive gauge pressure or tensile stress value whereas a darker region signifies a negative gauge pressure or compressional stress value. Consequently the gray level color assigned to the zero value remains the

same throughout. Each snapshot comprises 216000 data points including 300 radii and 720 angles in r and θ directions, respectively. These visualizations are performed in MATLAB[®] environment. The 2D space-time contour-plots are normalized in the same way as stated before unless mentioned otherwise. There are several general comments on the dynamics of wave motion which remains valid throughout. Some of presented 2D time-space contour-plots are used to depict the maximum values of the pressure and stresses. They are colored rather than the gray scale. These contour-plots are normalized between either $[0, 1]$ or $[-1, 0]$ which are mainly colored in blue or red.

Effect of relative material properties of three interacting media is studied after which the impact of the geometrical parameter of structure is investigated thoroughly. Subsequently effects of different excitations in terms of the time dependency as well as their origin are briefly addressed briefly. Two different excitation functions as well as two incident sources rather than plane longitudinal type are compared in their effect on dynamic stress and pressure concentrations. They are mentioned briefly next as they are essential to the understanding of the nature interactions.

4.2.1 General Comments

An incident p -wave is susceptible to a wave mode conversion upon each contact with a shell's surface. In other words, an incident p -wave is converted at boundaries to reflected as well as refracted shear and longitudinal waves. This behaviour is due to the Cauchy traction boundary conditions and it is explained in §2.6. However the shell's inner surface behaves differently. Once any wave arrives in the contained fluid, it is transmitted only in the form of pressure waves as the fluid is assumed to be inviscid. The easiest way to distinguish a shear wave from a longitudinal motion is based upon their different speeds of propagation. For instance, the speed of longitudinal wave is almost twice that of the shear deformation in a carbon steel. This considerable difference helps to visually distinguish p - (primary) and s - (secondary) wavefronts. Furthermore, *head waves* arise from the tangent transition between shear and compressional wave fronts. Different types of surface waves, such as creeping, Lamb, Love and Franz waves are specific examples of superposition of these body waves. The next subsection is dedicated to analyzing the effect of a shell's relative thickness on these dynamics of the interactions.

4.2.2 Effect of a Shell's Relative Thickness

Figure 4.1a through Figure 4.2b give successive snapshots of a water filled, steel shell embedded in a granite formation and impinged upon by a plane compressional wave having a Ricker pulse's time history. The first to fourth rows are the results for relative shell's wall thicknesses of $h = 0.02, 0.04, 0.20$ and 0.40 , respectively. Incident waves are omitted in these figures for a better visibility and differentiation of various waves. Specular reflections are generally the dominant scattered waves, initially, as expected. See, for example, $t = 0.3$ given in Figure 4.1a. The sharpness and contrast of these specular reflections become stronger as the relative thickness increase. This is simply because of the overall rise in the collective acoustical impedance of the shell as its thickness increases. A specularly reflected shear wave in addition to the compressional one is produced initially through the mode conversion at the shell's external surface. Generally these two waves are more distinctively visible as the shell gets thicker. See, for example, the snapshot at $t = 0.7$ when $h = 0.4$ where the longitudinal and transverse wave speeds of granite can be estimated simply through the normal separation distance of these two reflected wavefronts.

Smaller and larger curvatures correspond to the specularly reflected longitudinal and transverse waves, respectively.

Waves other than those reflected specularly start to appear in the external medium at very early time for $h = 0.02$ and $h = 0.04$, the thinnest shells. See, for example, $t = 0.3$, $t = 0.4$ and $t = 0.5$. These are given rise to by multiple reflections which occur between two surfaces of the shell. They are mostly Lamb wave in the thinner shells. Symmetric Lamb waves are more dominant for $h = 0.02$, whereas asymmetric Lamb wave are stronger and hence more distinctly visible when $h = 0.04$. Antisymmetric Lamb waves appear significantly stronger by a slight increase in the shell's thickness. Compare, for example, $h = 0.02$ with $h = 0.04$ in [Figure 4.1b](#). This is expected as the Lamb waves are highly sensitive to the overall impedance of the structure. See, for example, Figure 7-d given in [[Hasheminejad et al., 2011](#)] and its corresponding discussion. Antisymmetric waves become significantly more dominant in a higher acoustical impedance through an increase in the shell's thickness.

Further significant increases in the shell's relative wall thickness to $h = 0.20$ and $h = 0.40$, however, change the nature of the scattered field from the Lamb wave types to more pure body waves and their corresponding head waves. See the snapshots given in [Figure 4.1a](#) and [Figure 4.1b](#). A higher mechanical energy can be sustained in the shell as its wall thickness increases. Consequently the shell is able to re-transmit stronger waves back into the external elastic formation. See, for instance, $t = 0.5$ to $t = 0.8$ in [Figure 4.1a](#) as well as the entire snapshots given in [Figure 4.1b](#). The waves rather those specularly reflected do not appear effectively in the elastic formation until $t \approx 0.50$ and $t \approx 0.90$ when $h = 0.20$ and $h = 0.40$, respectively. See the corresponding snapshots given in [Figure 4.1a](#) and [Figure 4.1b](#). This is simply the approximate time required for the transmitted compressional wavefronts to travel from the shell's outer surface to the inner surface and the way back.

Overall, the multiple reflections and secondary refracted waves into the external formation are significantly smaller in amplitude in comparison to those of specularly reflected initially for the thinnest shell, $h = 0.02$. Initial compressional and shear specular reflections, for instance, remain highly distinct and relatively strong within the granite formation when $h = 0.02$ for the snapshots given in [Figure 4.1a](#) through [Figure 4.1c](#). These amplitudes, however, become more comparable for higher shell's wall thicknesses. See, for example, the snapshots given in [Figure 4.1b](#) for $h = 0.04, 0.20$ and $h = 0.40$.

In the meantime, a clearly visible straight (plane) wavefront in the shell's wall appears in the thickest shells arising directly from the impinging incident plane wavefront in the snapshots given in [Figure 4.1a](#) and [Figure 4.1b](#) at $h = 0.20$ and $h = 0.40$. The plane geometry of the transmitted wavefront is due to almost identical longitudinal waves' velocities within granite and steel. These transmitted plane wavefronts behave analogous to plane incident waves impinging upon the shell's inner surface. See, for instance, snapshots $t = 0.3$ to $t = 0.7$ when $h = 0.20$ and $h = 0.40$. Specular reflections occur at the shell's inner surface similar to those happening at the outer surface. The reflected waves split into compressional and shear wavefronts. This can be observed the best by following the snapshots at $t = 0.70$ through $t = 1.06$ which are given in [Figure 4.1a](#) to [Figure 4.1b](#) for $h = 0.40$. The wave dynamics occurs in the response of this transmitted flat wavefront which incident at the shell's inner surface is analogous to those happening at the shell's external surface. See, for instance, all the snapshots given in [Figure 4.1b](#) for $h = 0.40$.

Formations of multiple triple points and their subsequent mode conversions are easily detectable at the shell's surfaces through the snapshots given in [Figure 4.1a](#) to [Figure 4.1c](#) in both of the thickest shells. A clear, comprehensive discussion over the triple points on circular reflectors can be found, for example, in [\[Sun and Takayama, 1996\]](#). Developments of multiple triple points within the shell's wall at both inner and outer surfaces are clearly tractable by the snapshots presented in [Figure 4.1a](#) through [Figure 4.1c](#) when $h = 0.20$ and $h = 0.40$. For instance, triple points' formation are clearly detectable at the shell's inner surface for the snapshots given in [Figure 4.1a](#) at $t = 0.50$ and $t = 0.80$ when $h = 0.20$ and $h = 0.40$, respectively. These triple points keep superposing with each others while new triple points are created as time passes. These can be traced more readily in [Figure 4.1a](#) through [Figure 4.1c](#) for $h = 0.20$ and $h = 0.40$. A set of Franz waves of compressional and shear types appears in thinner shell, $h = 0.02$ and $h = 0.04$. See, for example, $t = 2.30$ through $t = 3.60$. For $h = 0.2$, on the other hand, there are multiple secondary Franz type waves along with other creeping and surfaces waves created as depicted in [Figure 4.1a](#). See, for example, [\[Nussenzveig, 1965\]](#).

Waves reflected from the shell's inner surface form spherical wavefronts act as spherical incident waves impinging upon the shell's outer surface. This procedure is depicted distinctly through the same snapshots in addition to those at $t = 1.05$ and $t = 1.25$ which are given in [Figure 4.1b](#) for $h = 0.20$ and $h = 0.40$, respectively. The interactions of this secondary spherical wavefront with the shell's outer surface lead to full mode conversion. See, for example, snapshots $t = 1.05$ through $t = 1.55$ given in [Figure 4.1b](#) for $h = 0.40$. Both sides of the boundary can sustain shear waves in addition to compressional deformations unlike the inner surface. Formation of the head waves as well as the associated secondary s - and p - wavefronts are easily traceable within granite formation. See, for example, the snapshots $t = 0.6$ through $t = 0.8$ given in [Figure 4.1a](#) along with all the snapshots given in [Figure 4.1b](#) for $h = 0.20$ and especially $h = 0.40$.

The incident wavefront faces the shell's concave surface until it travels halfway in the shell at time $t = 1.0$. Main components of the scattered field are the specular reflections up to this instance. See the snapshots given in [Figure 4.1a](#). However, the nature of the scattered field changes to *creeping waves* within the shadow region, the second half of the shell ($1 < t < 2$). See the snapshots given in [Figure 4.1b](#). These creeping waves start to appear at $(\theta = \pi/2, r = 1)$ and continue to travel along the shell's circumference while they radiate into the granite formation in a tangential direction. This continues until the incident wavefront passes the shell as illustrated in [Figure 4.1b](#) at the interval $1 < t < 2$. These creeping waves, however, keep travelling in the opposite direction afterwards. [Figure 4.1c](#) starts with $t = 2.0$ which is exactly when the incident wave completely engulfs the shell. Strong transmitted creeping waves into the elastic formation are observed in the vicinity of the tail region at time $t = 2.20$. These wavefronts are the result of superimposing of the surface creeping waves. The features of these strong, flat wavefronts are quite similar regardless of the shell's thickness because they mainly depend on the shape of the reflector's geometry.

Waves transmitted into the internal fluid have to pass through the shell's inner surface which permits the passage of longitudinal waves because an inviscid fluid is able to sustain only the compressional waves and shear stresses vanish at the shell's inner surface. These transmitted pressure waves propagate normal to their original wavefront with a constant speed which is the speed of sound in the water. This simple but important point is the key in understanding the

formation of the subsequent wave geometries within the encapsulated fluid. All transmitted pressure wavefronts into the contained fluid are highly concave regardless of the shell's thickness. This can be observed easily within all snapshots given in [Figure 4.1a](#) through [Figure 4.1c](#). This curved geometry of the wavefront is simply because of significantly lower speed of sound in water compared to the compressional wave's celerity (propagation velocity of the *incident* wavefront) in the granite (and steel) by more than twice. This concave shape leads to a very strong focusing mechanism that we talk about more in the following paragraphs. These initial wave fronts from $h = 0.02$ to $h = 0.04$ gets stronger. Once the shell's relative wall thickness increase to $h = 0.2$ and $h = 0.4$, the internal space of the shell decreases significantly. Consequently the pressure wavefronts require less time to travel through the inner space of the shell. Consequently these wavefronts' curvatures increase as the shell becomes thicker. This leads to faster focusing process in the thicker shells. See, for instance, the corresponding snapshots given for $h = 0.40$ in [Figure 4.1b](#) through [Figure 4.1c](#).

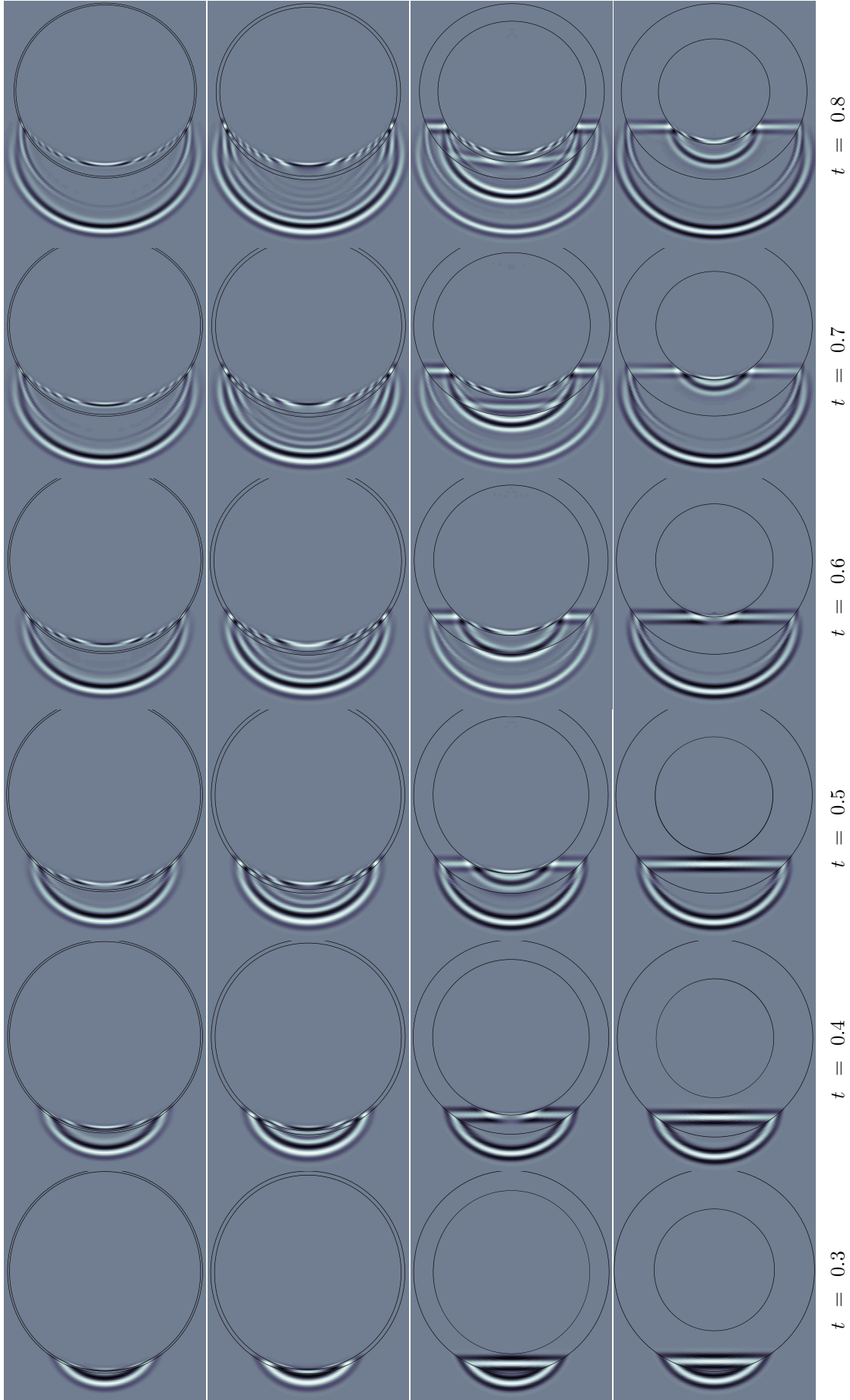


Figure 4.1a: A series of total field successive snapshots of a water filled steel shell embedded in granite in response to a plane compressional incident wave. Relative shell wall thicknesses from top to bottom, respectively, are $h = 0.02, 0.04, 0.20$ and 0.40 . The time dependency of the excitation, $f(t)$, is the Ricker pulse. The virtual camera zooms out as time passes. Initial specular reflections and head waves are depicted.

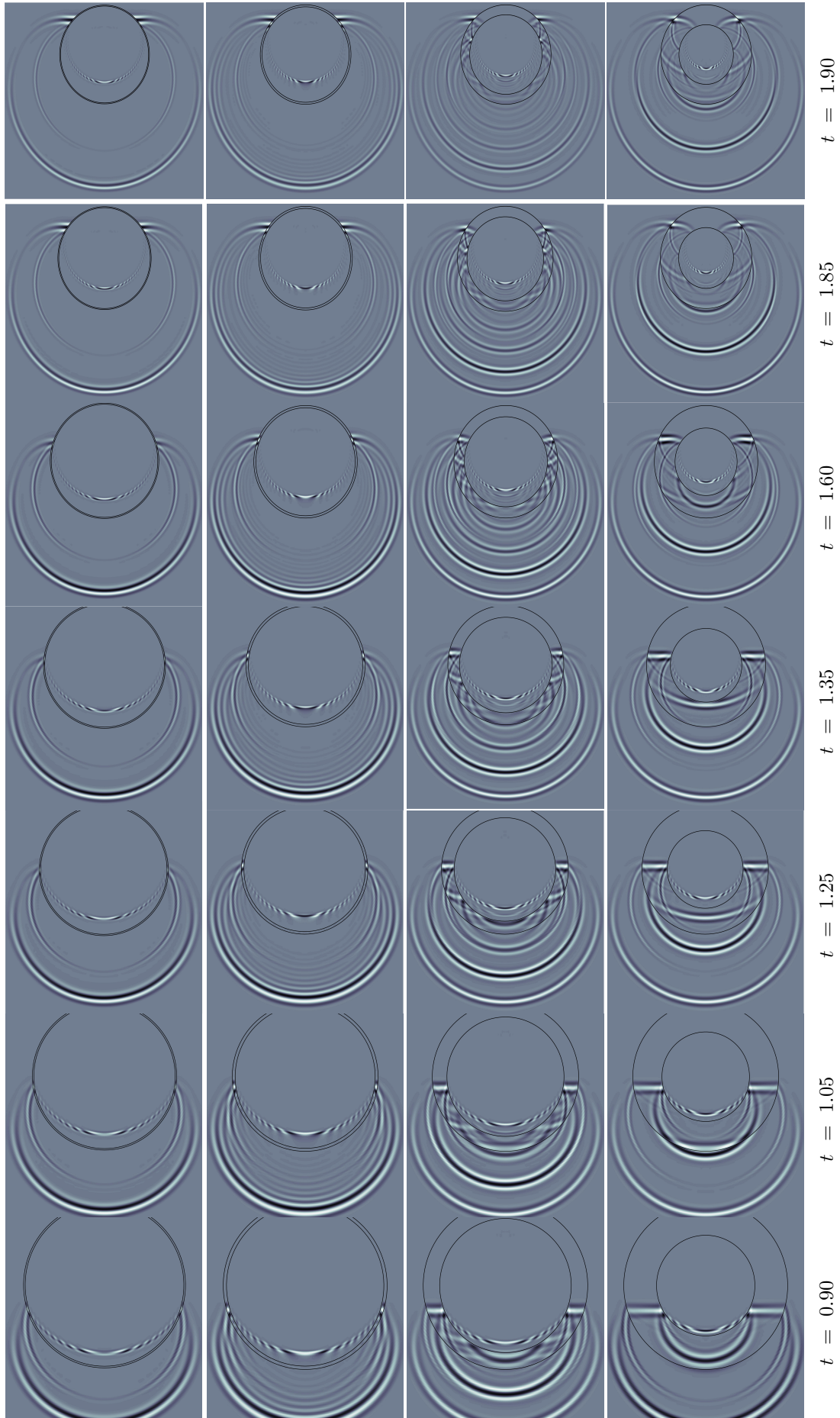


Figure 4.1b: A series of total field successive snapshots of a water filled steel shell embedded in granite in response to a plane compressional incident wave. Relative shell wall thicknesses from top to bottom, respectively, are $h = 0.02, 0.04, 0.20$ and 0.40 . The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse. The virtual camera zooms out as time passes. The formation of creeping waves in shadow region is illustrated. (continued)

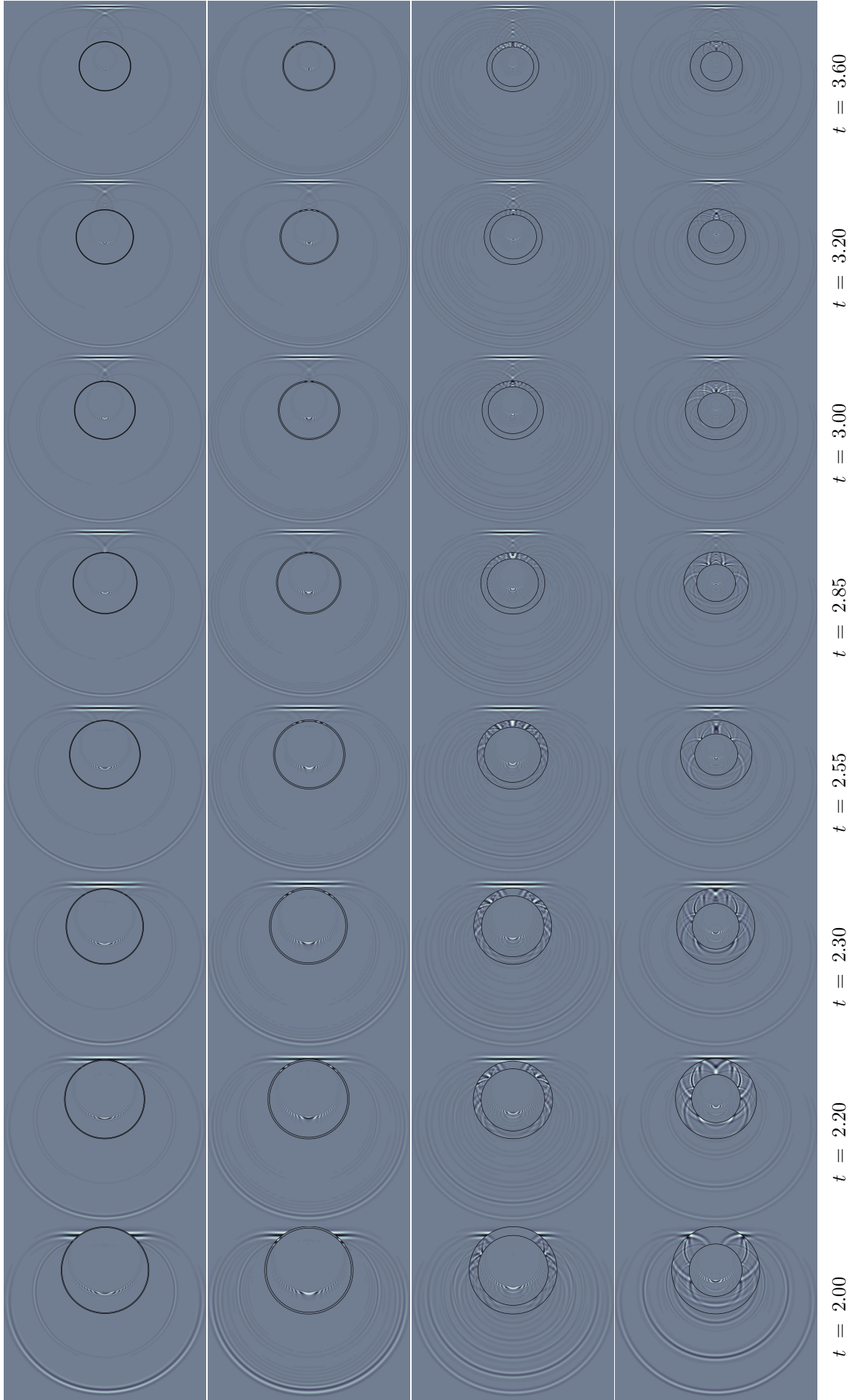


Figure 4.1c: A series of total field successive snapshots of a water filled steel shell embedded in granite in response to a plane compressional incident wave. Relative shell wall thicknesses from top to bottom, respectively, are $h = 0.02, 0.04, 0.20$ and 0.40 . The time dependency of the excitation, $f(t)$, is Ricker pulse. The virtual camera zooms out as time passes. Creeping waves' superposition and their movement towards the head region (continued).

The 2D time-space contour plots presented in [Figure 4.3](#) are used in conjunction with the snapshots given [Figure 4.1a](#) and [Figure 4.2b](#) to understand better the focusing mechanism within the internal fluid field. As seen by the consecutive snapshots in [Figure 4.1a](#) through [Figure 4.1c](#) the pressure wavefronts move towards each other and subsequently become superimposed as time passes as illustrated in [Figure 4.1a](#) and [Figure 4.2b](#). The first row of [Figure 4.3](#) shows the temporal gauge pressure at the entire shell's internal surface. The transmitted surface pressure waves start to appear in the internal fluid at the sphere's head area, $\theta = 0$, and start to fade more quickly in the thinner shells. A general observation is that the surface acoustic energy becomes more distributed as the shell's thickness increases. Stronger head waves appear in the internal shell's surface as the shell gets thicker. This behaviour can be seen more clearly in the first row of [Figure 4.3](#) between $0 < t < 5$. Furthermore, multiple focal points appear at the tail area, $\theta = \pi$, as the shell gets thicker at $2 < t < 5$. These high pressure spots are the ones in vicinity of the triple points forming within shell material at inner surface of the shell. See, for example, $t = 2.20$, $t = 3.00$, $t = 3.60$ and $t = 4.50$ when $h = 0.40$.

The second and third rows of [Figure 4.3](#), on the other hand, present the maximum absolute values of all the positive and negative gauge pressures, not only at the shell's surface, but throughout the entire contained water. Their negative values, for example, has a significant practical importance because they can indicate the likelihood of possible cavitations. Indeed once a corresponding absolute (summation of the acoustical and stagnant pressures) pressure at a given temperature reaches a saturation pressure of the fluid, possible trigger point for cavitation becomes feasible. Maximum positive pressure is also important in structural integrity's considerations. The second and third rows in [Figure 4.3](#) suggest that the primary focusing of the internal pressure field happens around $t \approx 5$ when $h = 0.02$. However this behaviour occurs sooner when the shell gets thicker. For instance, it occurs as early as $t \approx 3.5$ when $h = 0.4$. Also stronger secondary focusing points happen at larger thicknesses. In general as the shell's thickness increases, higher distribution of high acoustical energy also happens. High intensity pressure wavefronts keep travelling towards the shell's tail region to meet its inner surface after which the primary focusing is completed. Arrivals of these strongly focused wavefronts to the shell's inner surface at tail region occur at $t \approx 8, 7.75, 6.5$ and 5.0 corresponding to $h = 0.02, 0.04, 0.2$ and 0.4 , respectively. Following the snapshots after these times reveals an interesting common phenomenon: shells' prior stress state become insignificant compared to those given rise through these high intensity pressure wavefronts. These newly induced stresses keep travelling circumferentially through the shell material whilst they give rise to create new set of high intensity of body wave and their associated head waves as well as Lamb waves. Only one pair of easily distinguishable shear and compressional waves are created when $h = 0.02$. See the corresponding snapshots at $t > 8$ for $h = 0.02$. However the transmitted waves to the elastic formation become multiple strong wave fronts of body and Lamb types when $h = 0.04$. The Lamb waves become insignificant in comparison to the body and head waves where $h = 0.20$. See the corresponding snapshots at $t > 6.5$ given in [Figure 4.1a](#) and [Figure 4.2b](#) when $h = 0.20$. The rate of elastic energy emission into the infinite elastic formation, on the other hand, decreases significantly when $h = 0.40$. In other words, the strain energy is trapped within the steel and its contained water for a longer time. Besides considerably fewer waves are transmitted to the external medium when $h = 0.4$ compared to $h = 0.20$ because of longer times of flight through the shell's wall. Formation of the secondary focusing processes are

initiated after which the primary focusing wavefronts meet shells' inner surfaces at the tail region. These secondary focusing sequences are higher in intensity for thicker shell because the shell has higher capability in trapping the energy within its contained fluid.

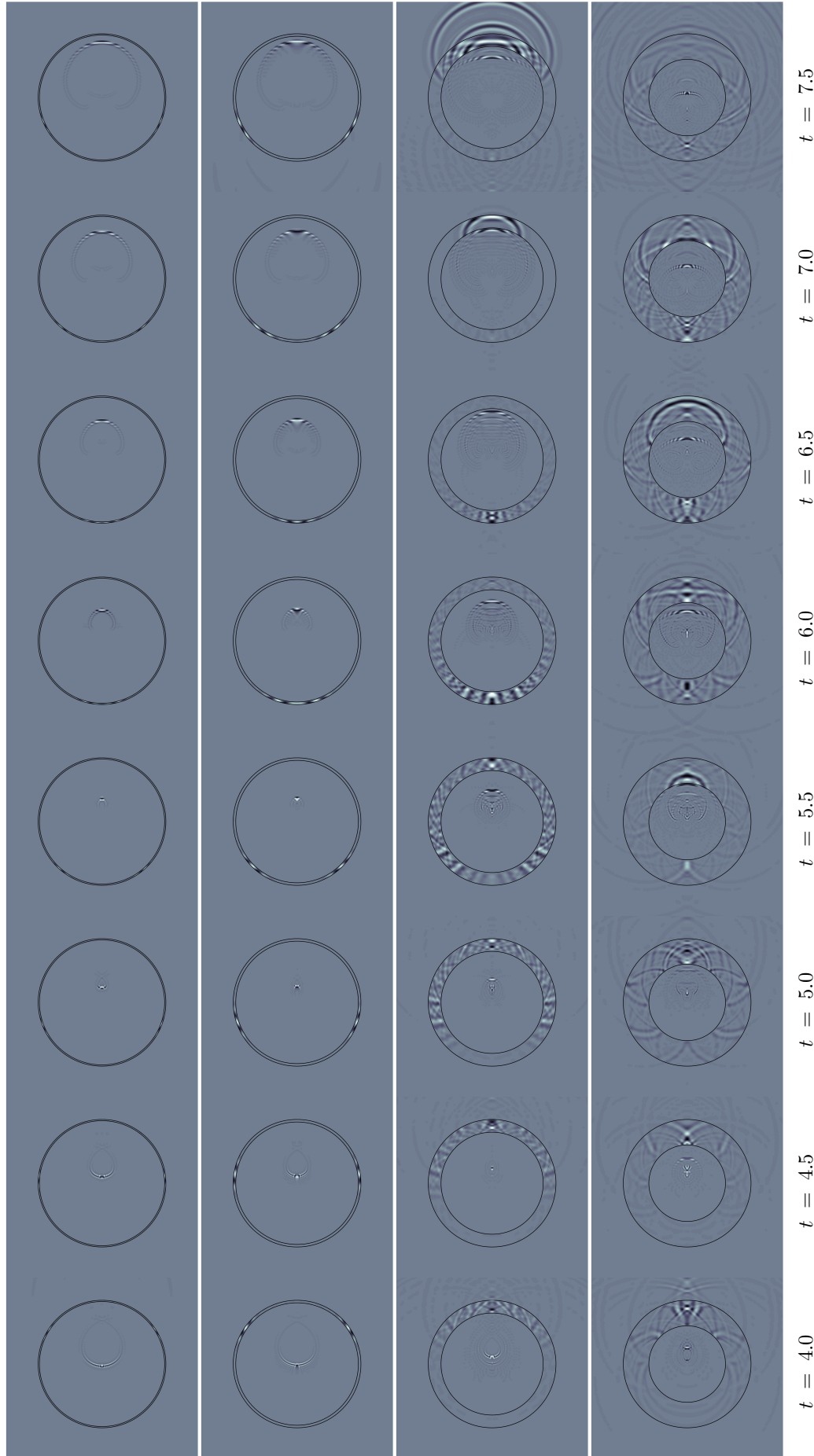


Figure 4.2a: A series of total field successive snapshots of a water filled steel shell embedded in granite in response to a plane compressional incident wave. Relative shell wall thicknesses from top to bottom, respectively, are $h = 0.02, 0.04, 0.20$ and 0.40 . The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse. Strong focusing within the internal pressure field.

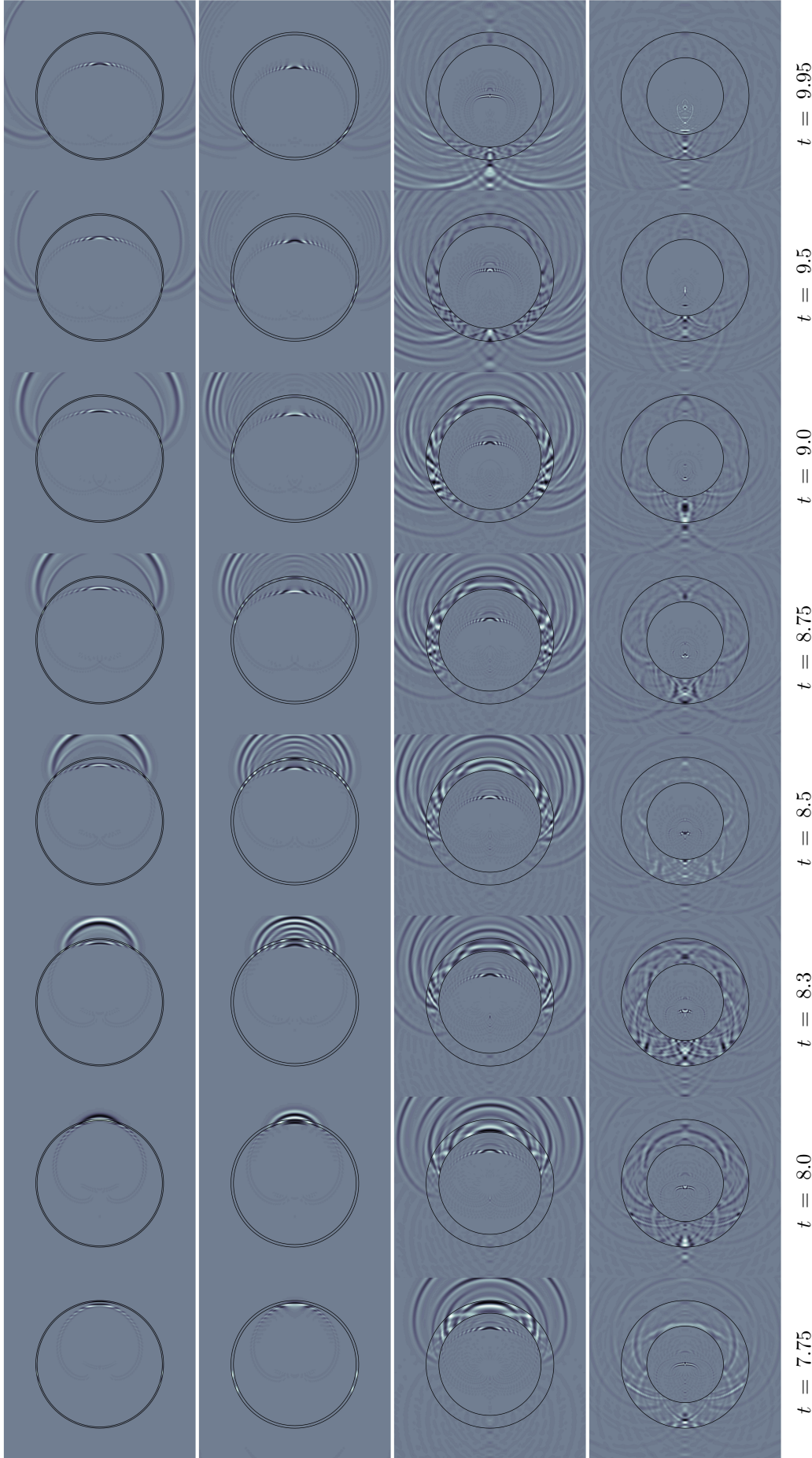


Figure 4.2b: A series of total field successive snapshots of a water filled steel shell embedded in granite in response to a plane compressional incident wave. Relative shell wall thicknesses from top to bottom, respectively, are $h = 0.02, 0.04, 0.20$ and 0.40 . The time dependency of the excitation, $f(t)$, is Ricker pulse. Strong focusing within the internal pressure field. (continue)

Figure 4.4 provides a very helpful qualitative overview of the entire interaction in a single figure for the radial stress concentration. The maximum values of tensile (top row) and compressional (bottom row) stresses correlate although the compression shows slightly higher distribution. Furthermore, the stress concentration due to radial stress keep being more distributed within the entire shell's material as the shell's thickness increases. The early high intensity stress regions at the head region (around $\theta \approx 0$) is very focused in a short period and a small space. The stress concentration, however, grows as the shell's thickness increases. The first focal region at the tail area happens around $t \approx 3$ is a single spot when $h = 0.02$ but its intensity is not comparable to the initial one occurring just after $t = 0^+$. The first focal spot at the tail region is a consequence of superposition in the stress waves which travel the shell's wall towards the tail region. Its relatively weak amplitude for $h = 0.02$ is due to its relative smaller energy carrying capacity of shell as larger dominant wavelengths in comparison to the shell's wall thicknesses are involved. However these focal spots created by the stress waves' superposition increase in both number and intensity as the relative thickness increases. In the later time, the stress jump due to internal focal pressure wavefronts at tail region is visible in all thicknesses and increases in higher thicknesses. Furthermore the initial occurrence of these spots is shifted to earlier times in the larger thicknesses. For instance, these highly focused stress spots occur around $2 < t < 4.5$. The next high intensity region relates to the arrival of the focused pressure wavefront to the shell's inner surface at the tail region. The arrival time ranges from $t \approx 8$ to $t \approx 4.5$ when $h = 0.02$ and $h = 0.40$, respectively. Only one major high intensity fringe is formed when $h = 0.02$, whereas multiple ones are created in higher thicknesses corresponding to consecutive reflections between two surfaces of the shell. The time differences between these fringes become larger as h increases simply because more time is required for a wave to travel the shell's thickness. These observations are better understood if Figure 4.4 is considered along with the corresponding snapshots given in Figure 4.1a through Figure 4.2b.

The first row of Figure 4.5 illustrates the development of stress history at the entire shell's external surface. It should be recollected that a wave's group velocity is simply the tangent to the corresponding fringe at any given point in an angle-time contour plots. Dynamics of the external surface pressure, for example, mostly constitute of the initial interaction when $h = 0.02$. The interaction starts with a high group velocity (compressional) and as time passes it slows down ($0 < t < 3$). Other fringes appear as the h increases which correspond to different group velocities at early time around head region. They have different ranges of the group wave velocities relating to various wave types including the body waves and their corresponding head waves. This feature can be used to categorize and specify the kind of the wave motion which occur. Furthermore significantly stronger creeping waves and their corresponding focal regions appear in the tail region for higher thicknesses.

The second and third rows of Figure 4.5 present the time histories of the maximum absolute tensile and compressional stresses. Observations which are common to all thicknesses are given next. Stress concentration area are mainly the head and tail regions. The stress concentration in the head region decreases dramatically compared to the tail area as the shell's thickness increases. Furthermore the head region associated with high stress concentration becomes narrower. Highly focused stress amplitudes at the tail region corresponds to the strong creeping waves around $t \approx 2$. This can be best observed in conjunction with the snapshots correspond to $t > 1.60$

in [Figure 4.1b](#) and [Figure 4.1c](#) regardless of the shell's thickness. These strong creeping waves occupy larger surface before meeting each other at $t = 2.20$. This keeps to travel with a flat wave front geometry afterwards as presented in [Figure 4.1c](#). The sudden discontinuity in this highly focused stress wavefronts at tail region after $t \approx 7$ is simply because of the spatial limitation in our selected radial domain of computations to $r \leq 6$. No new phenomenon or significant observation, however, can be made if a larger domain is selected anyways.

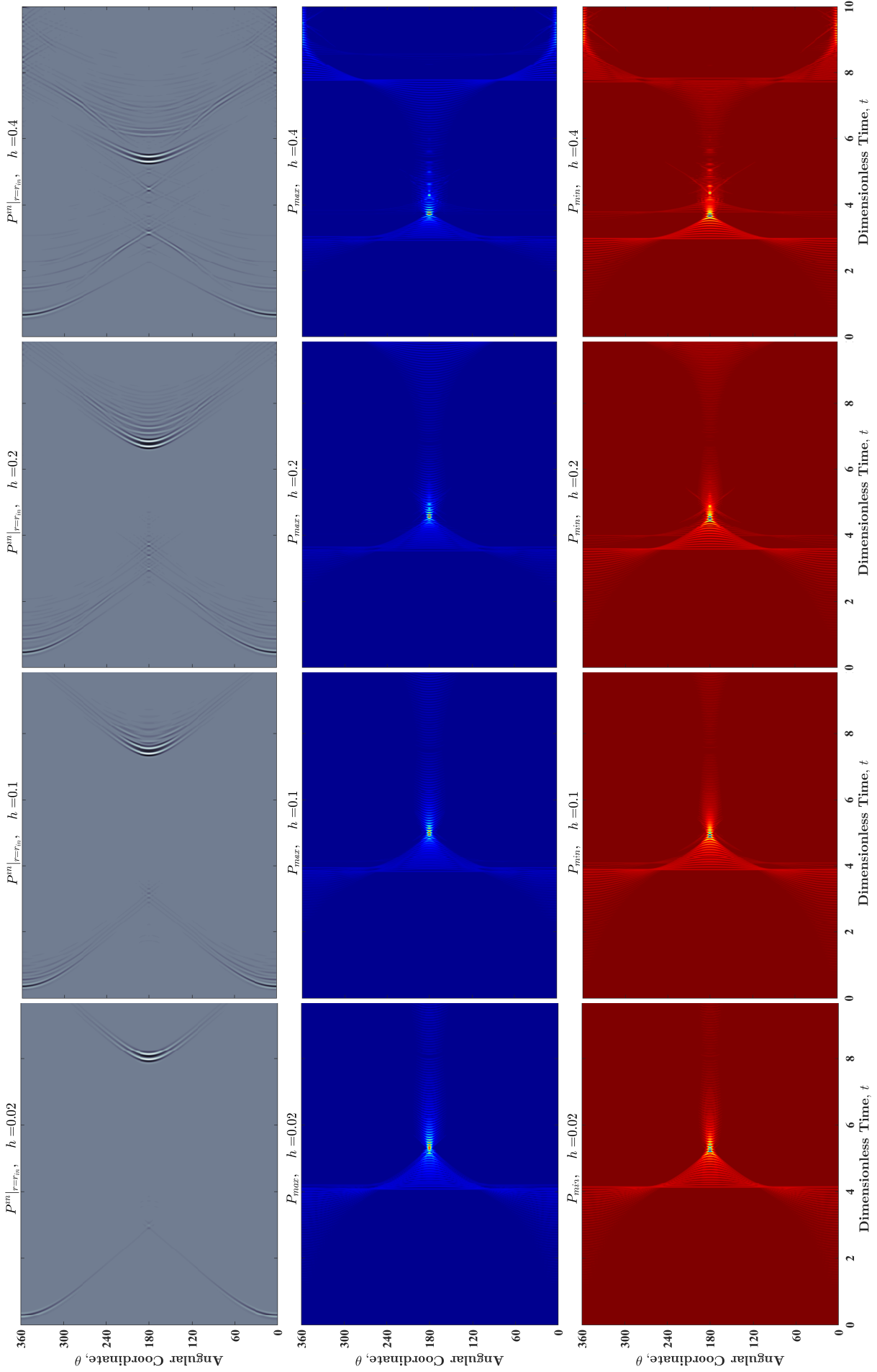


Figure 4.3: Pressure contour plots of a water filled steel shell embedded in granite in response to a plane compressional incident wave for different thicknesses: (1st row) gauge pressure at the shell's entire inner surface (normalized); (2nd row) maximum of positive gauge pressure of all radii at each angle; (3rd row) minimum of negative gauge pressure of all radii at each angle. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.

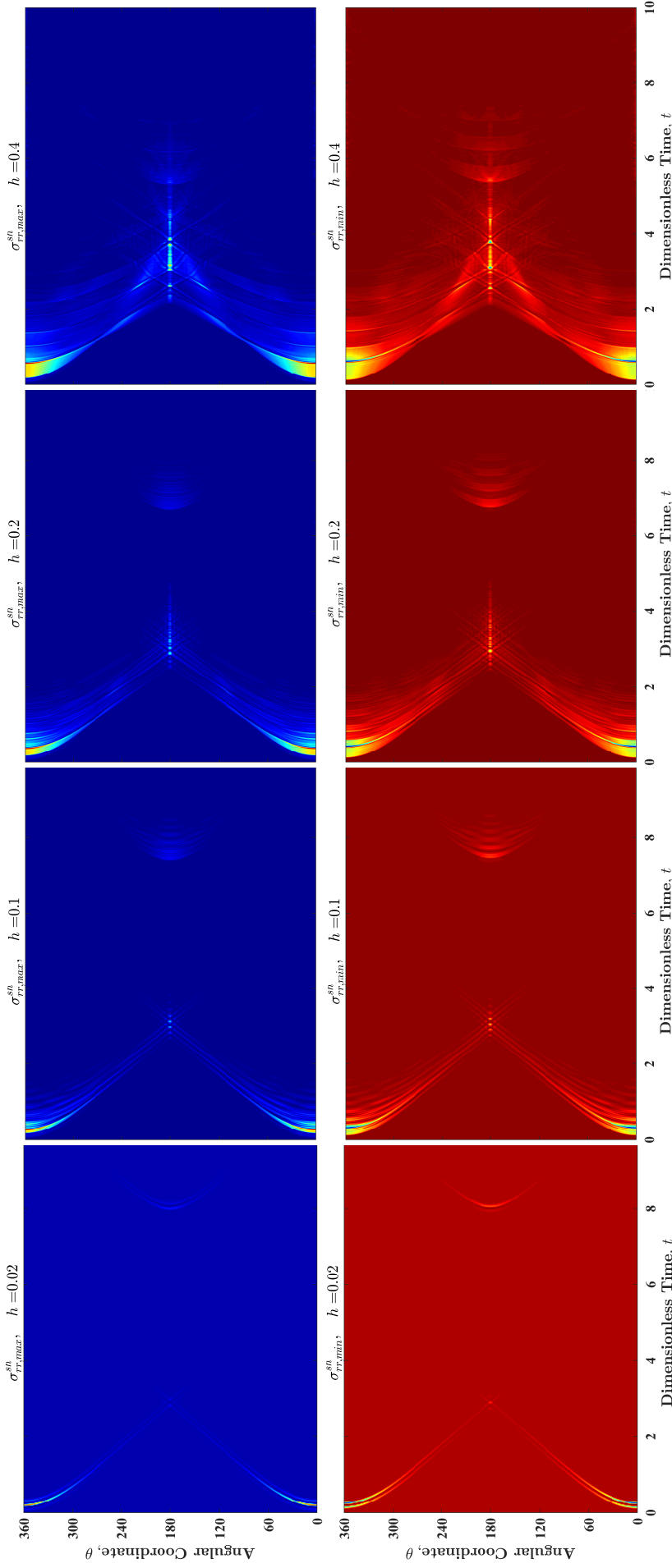


Figure 4.4: Maximum absolute values of radial stress within shell material in response to a plane compressional incident wave impinging on a water filled steel shell embedded in granite: maximum (1^{st} row) tensile stress; (2^{nd} row) compressional stress. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.

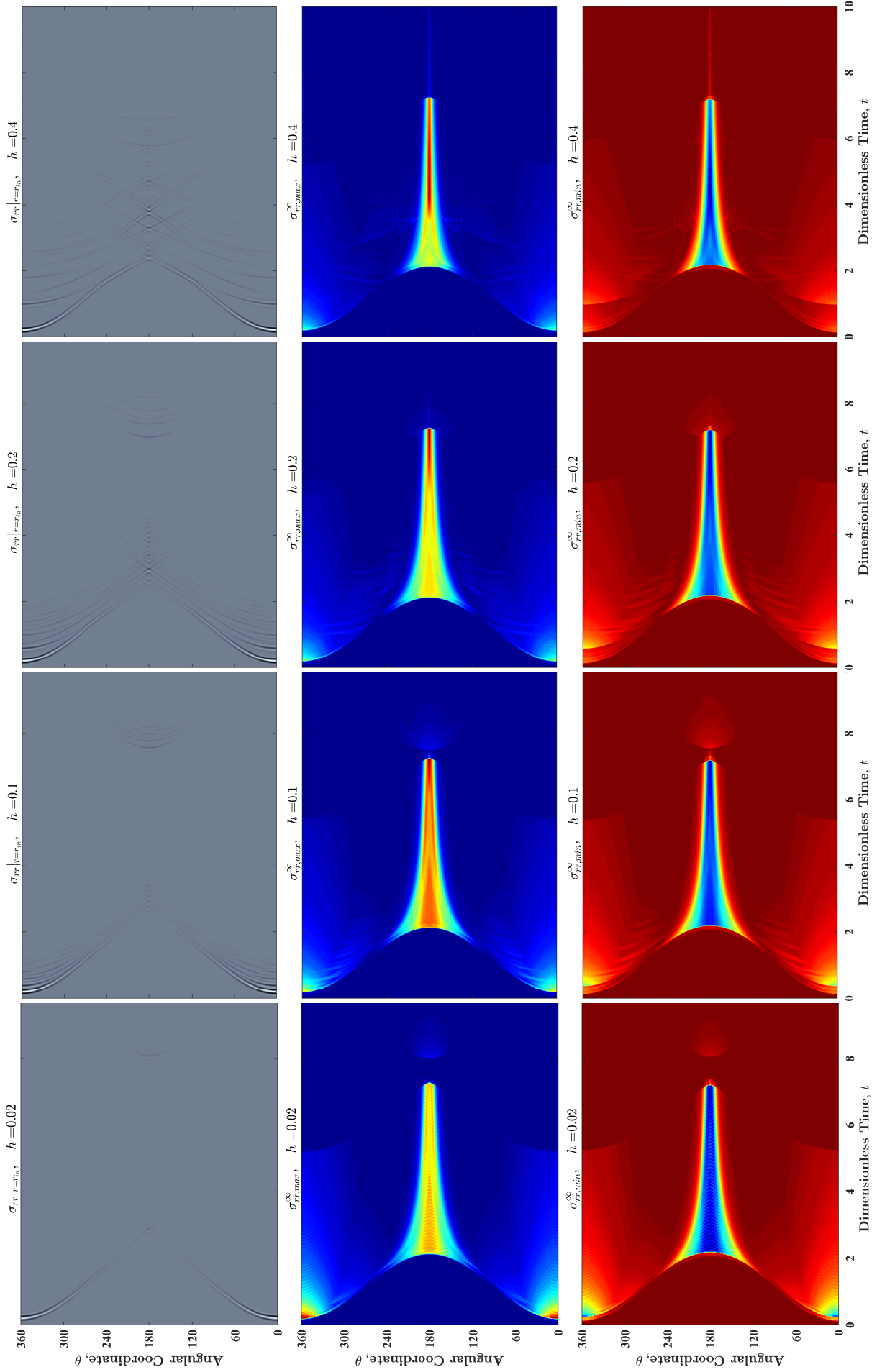


Figure 4.5: Radial stress contour plots of a water filled steel shell embedded in granite in response to a plane compressional incident wave for different thicknesses: (1st row) stress at the shell's entire inner surface (normalized); maximum of absolute (2nd row) tensile and (3rd row) compressing stresses of all radii at each angle within the infinite granite formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.

Figure 4.6 and Figure 4.7 give a quantitative presentation of the pressures and radial stresses' maximum absolute values in time and space, respectively. Figure 4.6 gives the maximum absolute values through the entire spatial space for any given moment in time. Figure 4.7, on the other hand, gives the polar distribution of the absolute maximum magnitudes by searching through entire time and radial domains for each angle. Both figures have one thing in common: the trend in absolute maximum values of negative and positive pressure and stresses closely correlate in time as well as in their spatial angle location.

The first row of Figure 4.6 illustrates the time-history of maximum absolute values of positive and negative gauge pressures. Obviously the maximum peaks corresponds to each primary focal point within the internal fluid regardless of the shell's thickness. A primary focal point migrates in time from $t \approx 5$ when $h = 0.02$ to $t \approx 3.5$ for $h = 0.4$. The reason for this change has been discussed previously. See the snapshots given in Figure §4.2.2 and Figure 4.2b along with their corresponding discussions. The peak magnitudes increase generally for higher thicknesses when the internal radii is the same. There is a slight decrease, however, from $h = 0.2$ to $t = 0.4$. This is basically due to considerably smaller mass of water in the shell by $h = 0.4$ compare to $h = 0.2$ when the external rather than internal radius is fixed. A smaller fluid mass has obviously less capacity in absorbing acoustic energy. There are some secondary focusing peaks around $t \approx 9$ and afterwards when $h = 0.4$. The pressure peaks around $4 < t < 6$ are related to triple points' superposition at the shell's head region. See the corresponding snapshots given in Figure 4.2b. The polar plots given in Figure 4.7 correspond to those temporal curves given at the first row of Figure 4.6. These figures reveal an interesting common point: the absolute maximum pressures are very directional and they occur at the shell's tail region regardless of the thickness. The following differences, however, are observed when the shell's wall thickness increases. First, the absolute maximum value of the negative gauge pressure is higher than the corresponding positive gauge pressures when $h = 0.02$. This difference decreases, however, when $h = 0.04$ and even the order is reversed for $h = 0.2$ and $h = 0.4$. The stronger peak gauge pressure at the shell's head region relates to the superposition of the triple points meeting at the shell's inner surface around $\theta = 0$ when $h = 0.4$. These quantitative observations complement those of the corresponding snapshots.

The second row of Figure 4.6 gives the absolute maximum amplitudes of the stress generated within the shell's wall. The initial peak increases slightly from $h = 0.02$ to $h = 0.04$, and then remains constant at next two higher thicknesses. However the peaks get delayed in travel time as the shell becomes thicker. This delay is related to a longer time required for a wave to through the shell's thickness. Furthermore there is a smaller jump before the initial main peak. Maximum positive values are higher than negative ones except for $h = 0.02$. The initial strong peak relates to the formation of triple points at the shell's inner surface after which the transmitted wavefronts travels the shell's wall thickness and meets the inner surface. There is also a slight increase in the amplitudes as the shell gets thicker. Secondary, but weaker, peaks relate mostly to the superposition of Franz waves at the shell's tail region when $h = 0.02$ and $h = 0.04$. See the corresponding snapshots at $t = 2.86$ given in Figure 4.1c. The secondary peaks when $h = 0.20$ and $h = 0.40$, on the other hand, are given rise to by the multiple superposition of those triple points formed on both surfaces of the shell. See the corresponding snapshots after $t > 2$ when $h = 0.20$ and $h = 0.40$. Number and magnitudes of these secondary peaks increase as the shell

gets thicker. This trend is excepted due to higher capacity of the shell's greater mass to trap mechanical energy. The next set of, even smaller peaks, are created by the interaction of the internal focused wavefronts with the shell's inner surface at the tail region. These peaks are separated distinctly from the previous ones in thinner shells.

The third row of [Figure 4.6](#) gives the absolute maximum amplitudes of the stresses generated within the entire granite formation but without the incident term. The compressional stress peaks remain superior after the initial interaction throughout. The pattern for all the thicknesses are very similar. However the initial positive and negative peaks decreases with the increasing wall thickness of the shell. These changes in amplitudes relate to the increase in structural overall impedance as the shell gets thicker. Initial phase difference between compressional and tensile peaks increases for the higher thicknesses. This phenomenon relates mostly to the different required travel time through the shell's wall as well as the wave mode conversions at the interface with the shell.

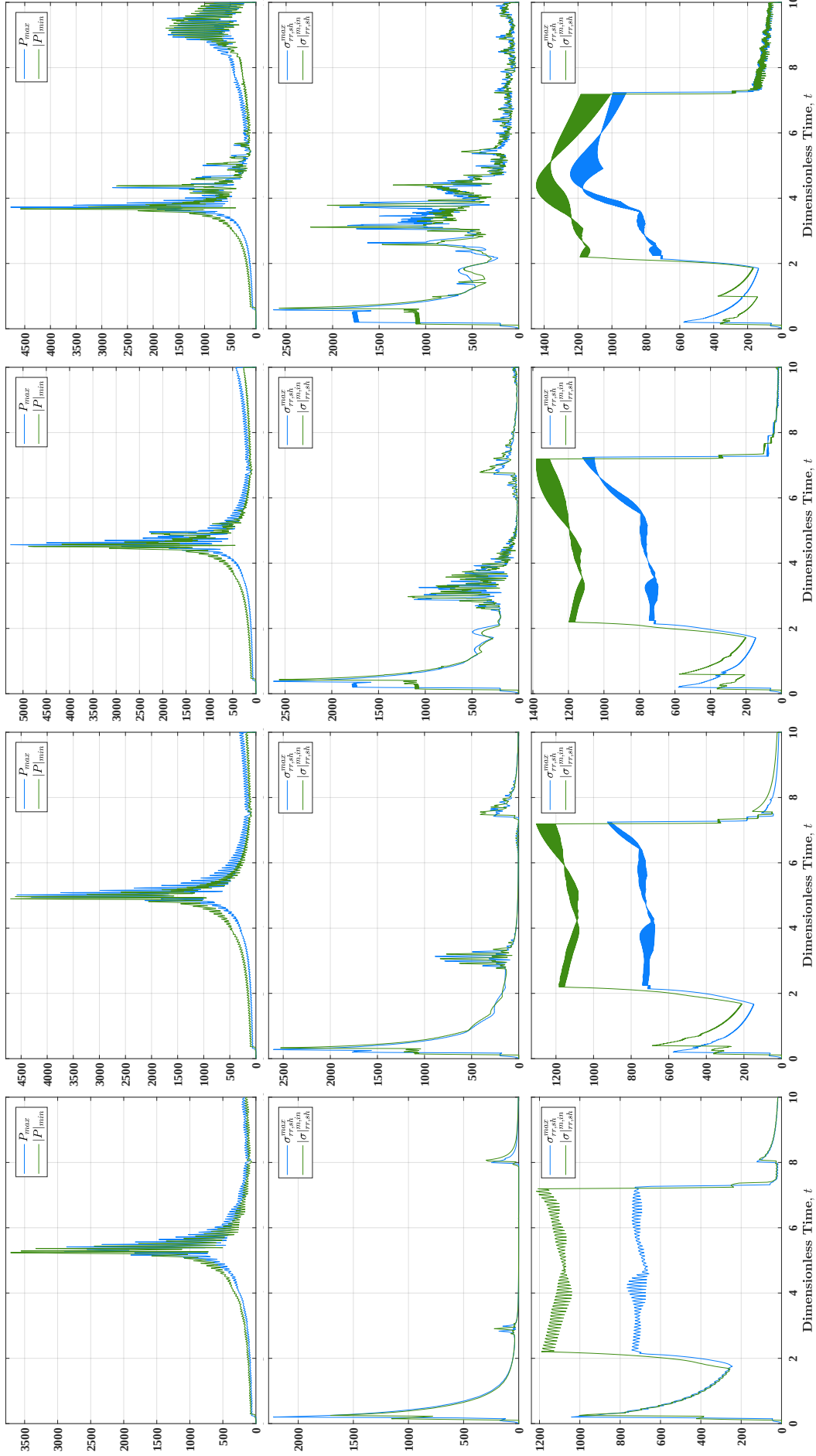


Figure 4.6: Absolute values of the positive and negative pressures and stresses occurrence through the entire spatial domain of a water filled steel shell embedded in granite. The responses are created by a plane compressional incident wave for different shell wall thicknesses. The time dependency of the excitation, $f(t)$, is the Ricker pulse. (1st row) internal pressure; radial stresses within (2nd row) steel shell and (3rd row) infinite granite formation. Columns correspond to $h = 0.02, 0.1, 0.2$ and 0.4 from right to left.

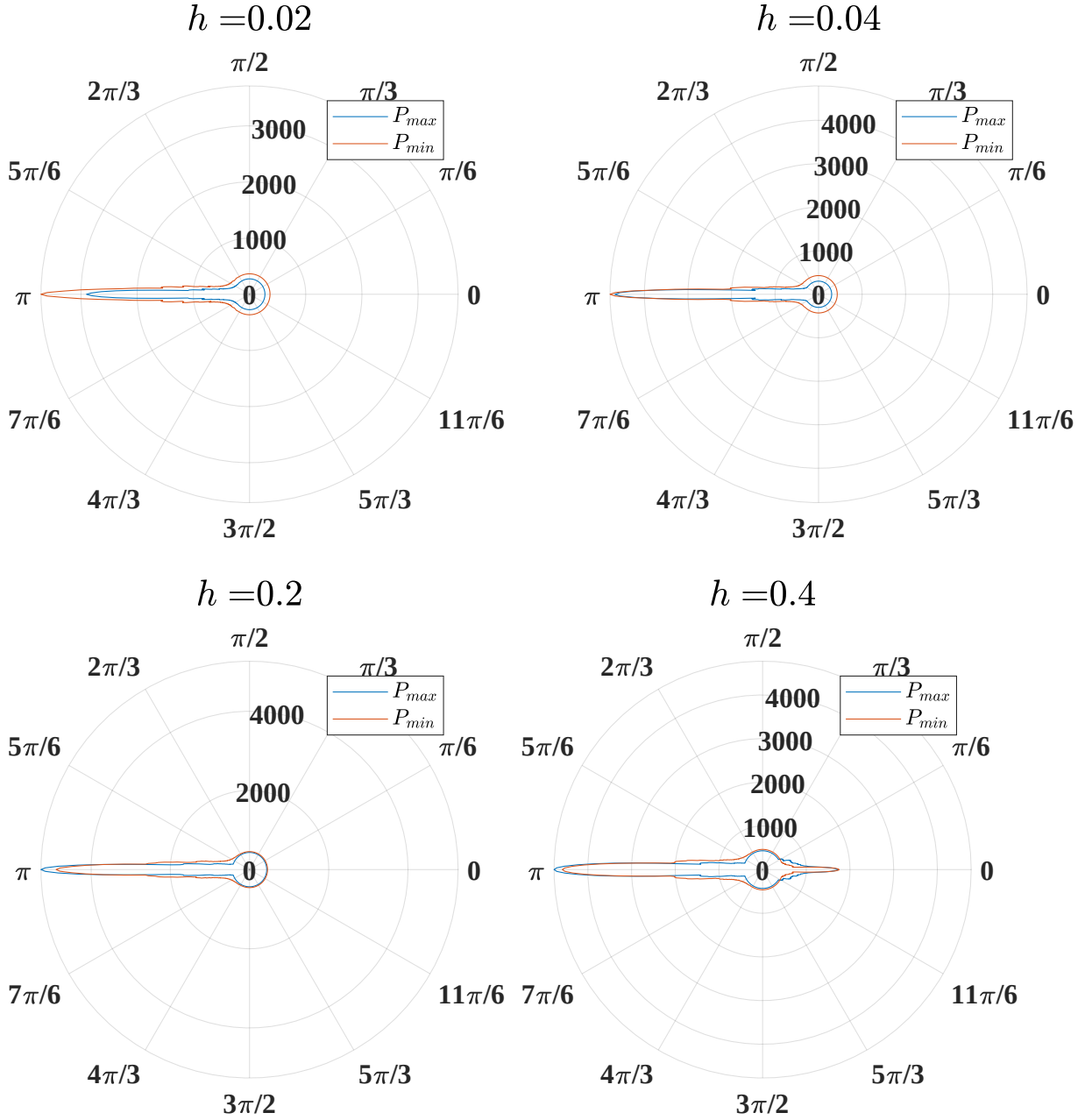


Figure 4.7: Polar plots of the absolute positive and negative gauge pressures throughout the entire spatial domain of a water filled steel shell embedded in granite. Responses are created by a plane compressional incident wave for different thicknesses. The time dependency of the excitation, $\bar{f}(t)$, is the Ricker pulse.

Figure 4.8a through Figure 4.8d give full size (i.e. $0 \leq r < 6$) snapshots for configurations as before but with two significantly higher thicknesses of $h = 0.80$ and $h = 0.98$. Furthermore the first row of Figure 4.9 shows the radial stresses' history of the entire granite-steel interface whereas the second and third rows of Figure 4.9 illustrate the maximum tensile stresses generated over the entire granite and steel material, respectively. The water filled steel shell is embedded in the same granite formation. It is subjected to a plane incident compressional stress wave with a Ricker pulse time dependency again. These values of h are close to limit case of an inclusion within an elastic formation. The thick hollow sphere ($h = 0.80$) and the spherical inclusion ($h = 0.98$) are simply called “hollow sphere” and “inclusion”, respectively. This limiting situation has not been discussed in the first companion chapter. Such a configuration represents, for example, a cysts in

a human tissue.

The scattered fields are identical in trend and amplitude upto $t \approx 1.60$ when the waves scattered from the inner surface reaches the sphere's outer surface of the hollow sphere. This includes the formations of easily distinguishable, specular shear and compressional reflections upto $t = 1.0$. See the corresponding snapshots given in [Figure 4.8a](#). The nature of the waves alternates, however, to creeping waves afterwards until the incident wave engulfs both spheres completely by $t = 2.00$. See the corresponding snapshots given in [Figure 4.8a](#) and [Figure 4.8b](#). The strong creeping wavefront has a more straight and broader geometry for the inclusion. Interestingly this wavefront, however, is noticeably stronger when the hollow sphere is the scatterer. See the corresponding snapshots at $t = 2.50$ given in [Figure 4.8b](#). This phenomenon can be observed with an overall overview through the contour plots given in the first and second rows of [Figure 4.9](#). Dynamic stress concentrations within the elastic formation occurs at head and tail regions for the inclusion and hollow sphere, respectively. These high intensity stress regions relates to the initial specular reflections from the inclusion whereas it corresponds to creeping waves from the hollow sphere. Consequently a hollow sphere, unlike an inclusion, acts as an acoustic lens in focusing the incident wave's energy. This observation has a significant practical implication. Scattering of the transmitted waves from the hollow sphere's inner surface causes these differences in scattered fields within the granite formation, an aspect which is discussed next.

The initially refracted wavefront into the inclusion is as straight as the original plane incident wavefront as the propagation velocity of the compressional waves are almost identical. This transmitted wavefront itself acts like an incident wave upon encountering the inner surface for the hollow sphere. Scattering from the hollow sphere's inner surface initiates around $t \approx 0.90$ and this refracted wavefront passes the inner fluid inclusion by $t \approx 1.35$. This processes is similar to those from the sphere's external surface. Longitudinal and transverse specular reflections constitutes the scattered field upto $t \approx 1.15$, whereas the creeping waves start forming after this moment until $t \approx 1.35$. These creeping waves travel towards the sphere's tail region after which they superimpose into a strong wavefront. See, for example, the snapshot at $t = 2.00$ given in [Figure 4.8b](#). A part of this wavefront is transmitted into the external elastic formation in the case of the hollow sphere. This effect is so strong that the stress intensity around tail region becomes noticeably stronger than the initial specular reflections for the hollow sphere. Compare two snapshots at $t = 2.5$ given in [Figure 4.8b](#).

The flat geometry of the refracted wavefront into the steel leads to a different focusing mechanism within the internal medium in comparison to those explained previously. The focusing initiates with formation of triple points over the steel-granite interface after which the transmitted wavefront passes both sphere's center around $t \approx 1.15$. These points grow stronger as they approach the tail region. The triple points on both sides of the refracted wavefront start forming exactly after $t = 1.0$, once the reflector become convex. They meet and superimpose in magnitude afterwards. See the corresponding snapshots between $1.25 < t < 2.5$ given in [Figure 4.8b](#) and [Figure 4.8c](#). These triple points after superimposing each other, continue to travel in the opposite direction. These lead to formation of a convex wavefront travelling toward the head region. This wavefront goes through a similar focusing procedure in the head region which leads to a secondary focusing. Subsequent multiple secondary focusings are formed likewise. See, for example, the entire snapshots after $t = 2.50$ given in [Figure 4.8b](#) through [Figure 4.8d](#). This

mechanism occurs when a flat or convex wavefront meets a circular concave reflector. A detailed discussion on this subject can be found in [Sun and Takayama, 1996] regarding the focusing of the compressional (pressure waves). The first focusing of the compressional wave, however, is followed by another focusing created by the shear waves. See the contour plots given in the third row of Figure 4.9. The primary focusings of the compressional and shear waves approximately are completed at $t = 2.65$ and $t = 2.35$ for both of the sphere and inclusion. Interestingly focusing the shear waves is stronger compared to the longitudinal one. The formation of the triple points for the first secondary focusing can be observed in the snapshots at $t = 4.0$ and $t = 4.5$. The following secondary completed focusings occur around $t = 5.00, 6.50, 7.00$ and $t = 7.5$ given in Figure 4.8d. The amplitude of the secondary focal points are weaker for the hollow sphere compared to the inclusion. Both of the hollow sphere and inclusion demonstrate a similar pattern of the maximum compressional stress within steel as it is shown in third row of Figure 4.9. The inclusion, however, shows slightly higher distribution. Those two high intensity stress points at the tail region at approximately around $t \approx 2.65$ and $t \approx 3.35$ are associated with the compressional and shear focal points, respectively. A short comment on the observed Franz waves are given next.

The Franz waves start forming at the tail region within the external medium. See, for example, both the snapshots at $t = 2.50$ given in Figure 4.8b. These waves keep travelling on opposite directions over the surface. See the corresponding snapshots in at $t \leq 2.50$ given in Figure 4.8b and Figure 4.8c. The Franz waves forming around the hollow sphere's inner surface, on the other hand, are more visible compared to those on the sphere's external surface. This is because of the presence of the associated *quasi-incident* (transmitted) wavefront, unlike the external medium. Compare, for example, the snapshot at $t = 1.5$ for the hollow sphere given in Figure 4.8a with both snapshots at $t = 2.50$ given in Figure 4.8b.

Figure 4.10 gives polar distributions of the absolute maximum tensile and compressional stresses. The first row relates to the stresses within the steel in which both of the hollow sphere and inclusion show a comparable trend, especially, at the tail region. The distribution patterns, however, are somewhat different at the head region. This difference is due to the specular reflections from the hollow sphere's inner surface. Both of the tensile and compressional stresses have quite similar patterns whereas the amplitudes are higher about 10% for the steel inclusion. The dynamic stress concentrations within the granite formation, however, are quite different in both trend and amplitudes at the tail region. The hollow sphere creates higher level stresses at its tail region by four folds compared to the inclusion. Furthermore the pattern is more focused to a narrower region in the hollow sphere. The reasons for such behaviours are explained earlier. The inclusion, on the other hand, demonstrates a higher stress concentration at the head region in comparison with its tail area. This pattern at the head region is identical with those of the hollow sphere exactly. Because the peak stresses are given rise to the initial specular reflections which are the same in both cases. See the snapshots upto $t = 1$ given in Figure 4.8a as well as the contour plots given at the second row of Figure 4.9. Effects of the relative material properties are considered next.

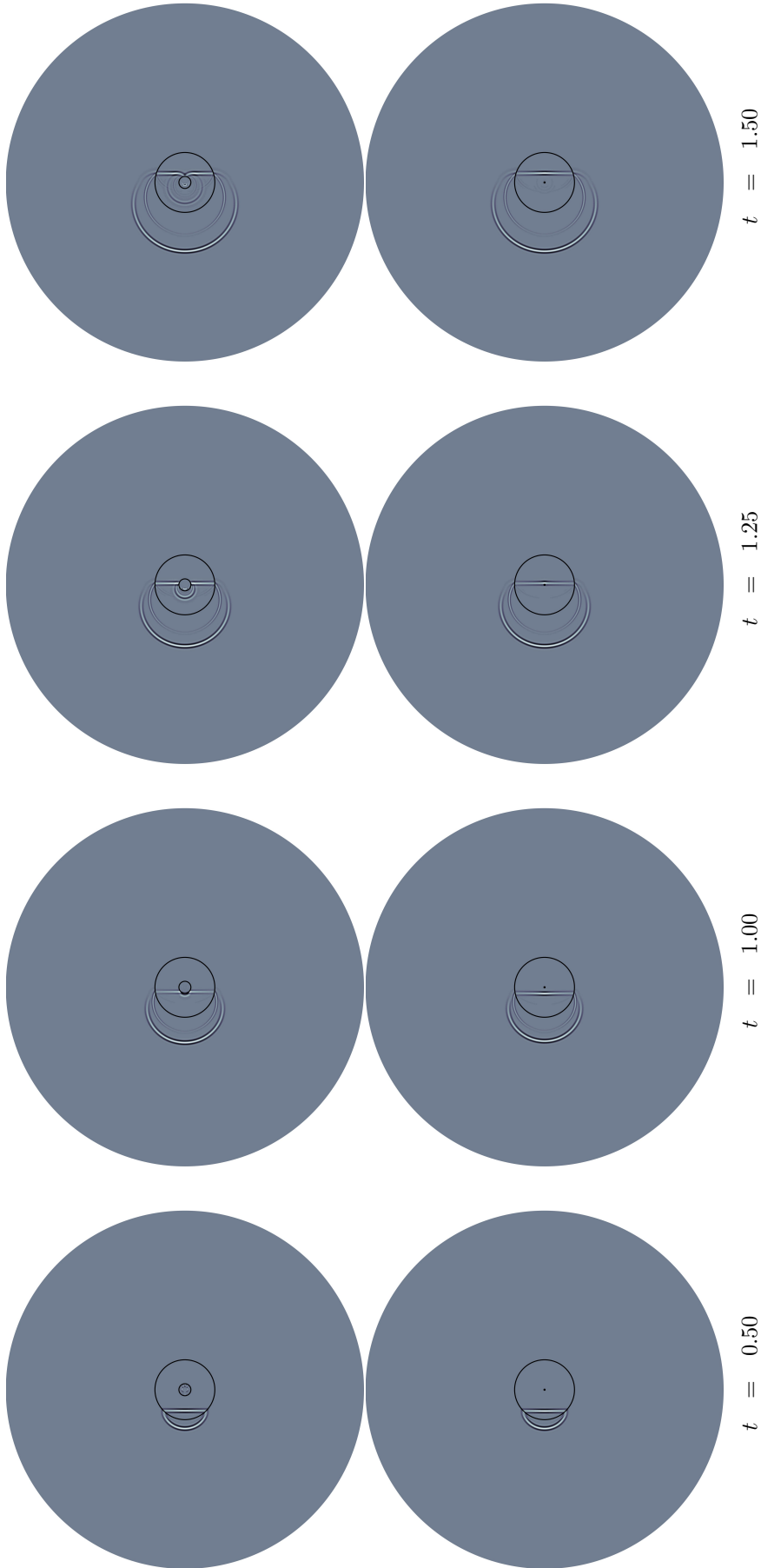


Figure 4.8a: A series of total field successive snapshots of a water filled, hollow steel sphere (the top row for $h = 0.80$) and a steel inclusion (bottom row) both embedded in granite in response to a plane compressional incident wave. The time dependency of the excitation, $f(t)$, is Ricker pulse.

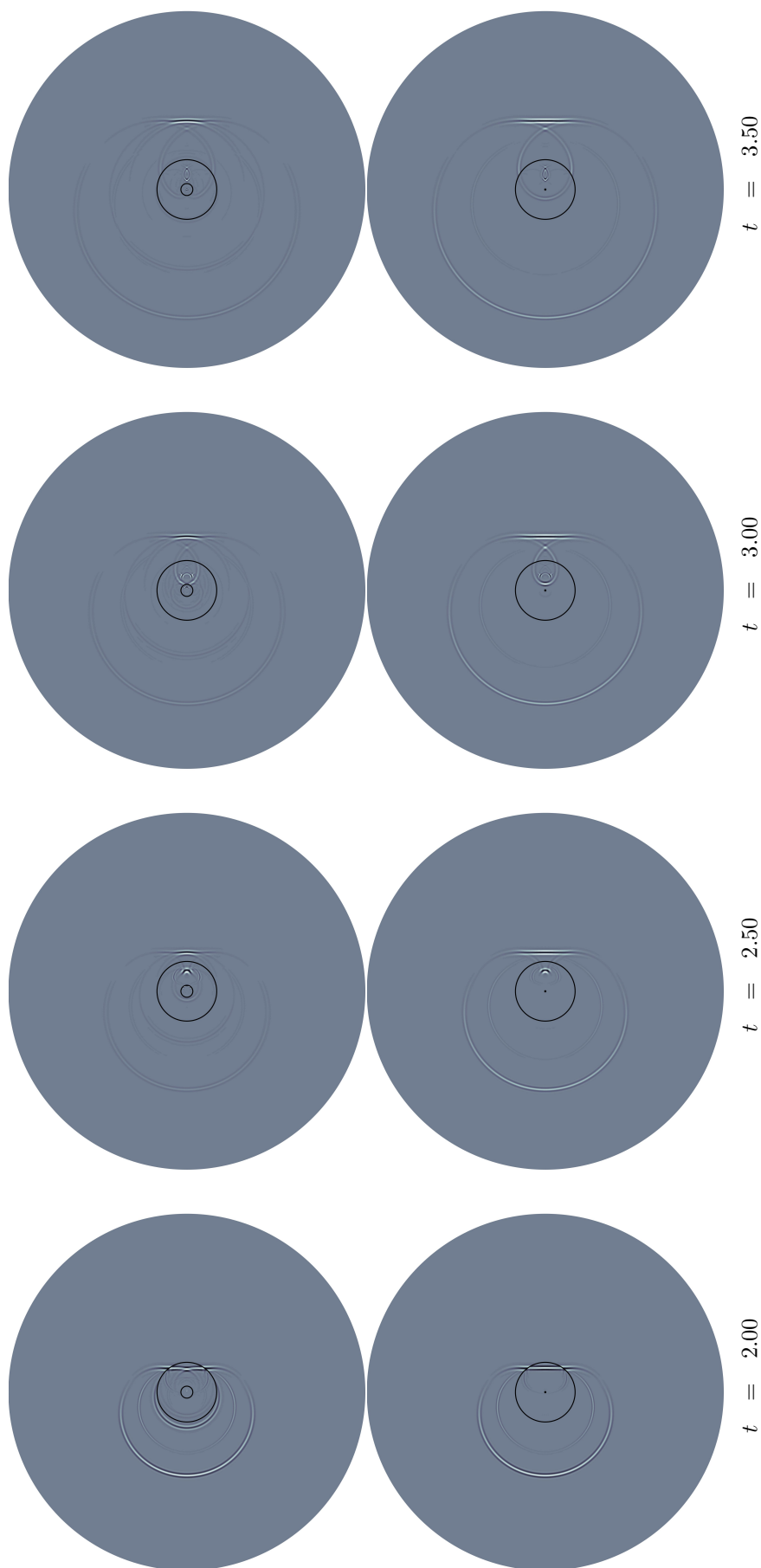


Figure 4.8b: A series of total field successive snapshots of a water filled, hollow steel sphere (the top row for $h = 0.80$) and a steel inclusion (bottom row) both embedded in granite in response to a plane compressional incident wave. The time dependency of the excitation, $f(t)$, is Ricker pulse. (continued)

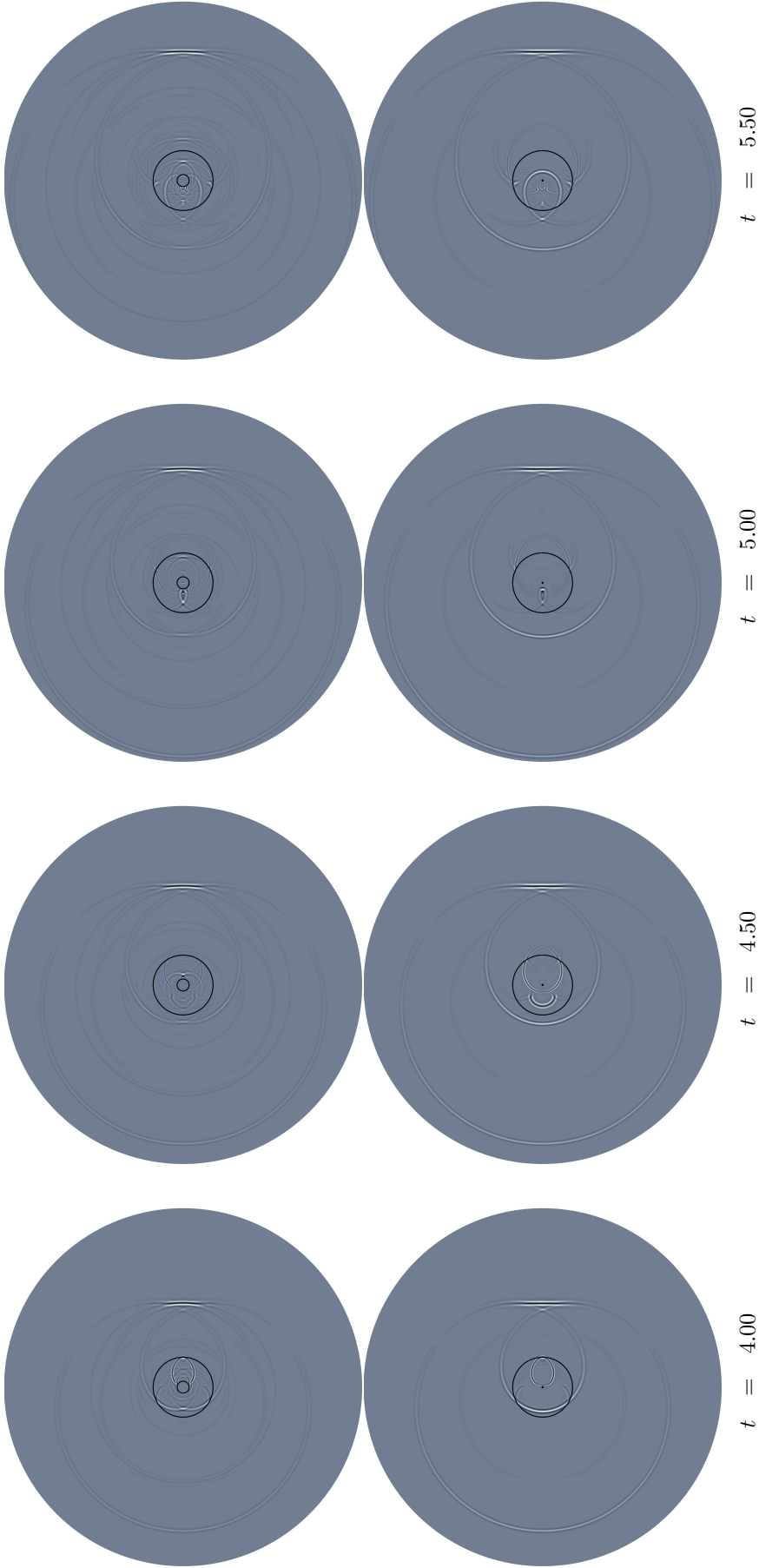


Figure 4.8c: A series of total field successive snapshots of a water filled, hollow steel sphere (the top row for $h = 0.80$) and a steel inclusion (bottom row) both embedded in granite in response to a plane compressional incident wave. The time dependency of the excitation, $f(t)$, is Ricker pulse. (continued)

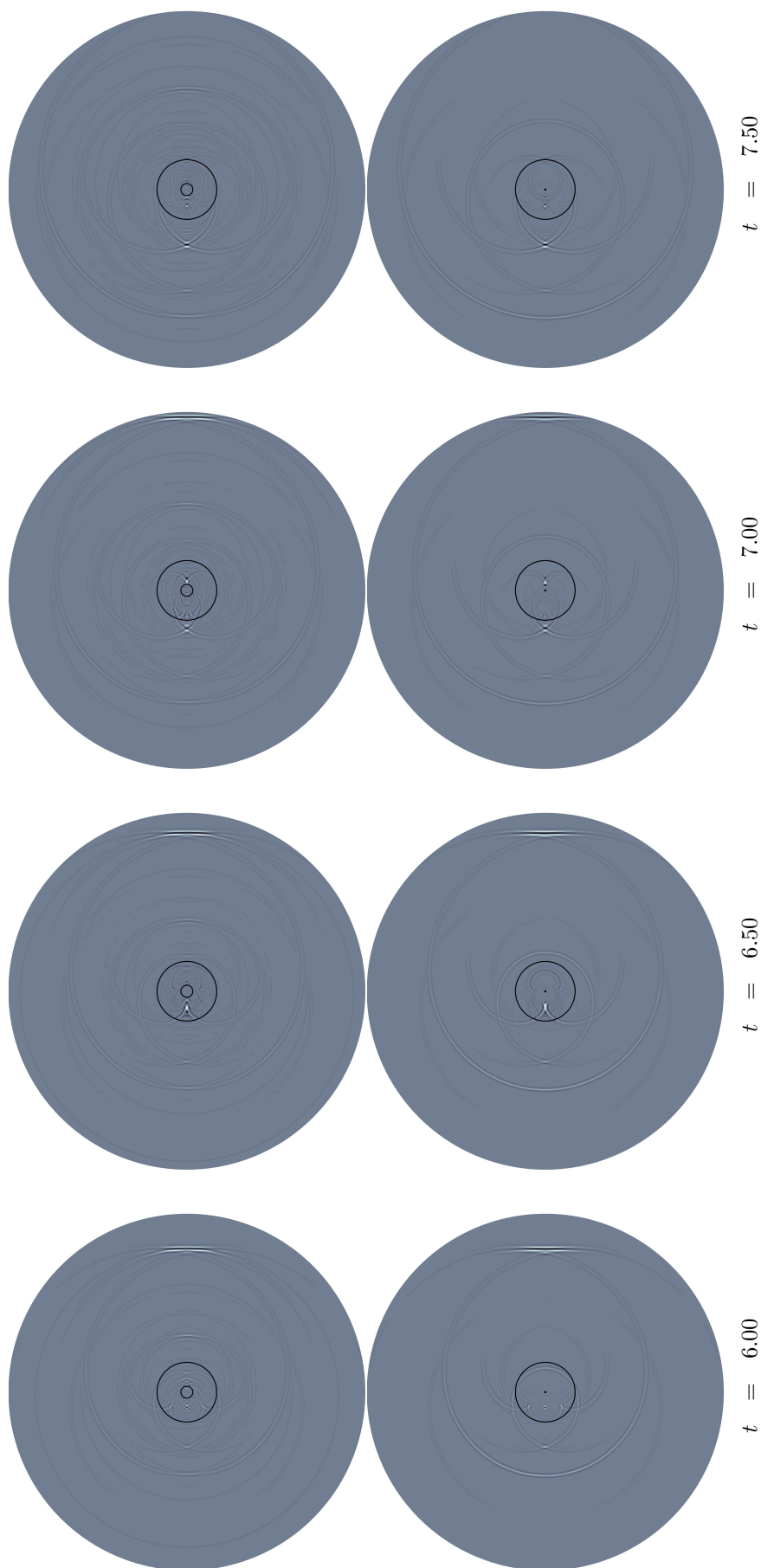


Figure 4.8d: A series of total field successive snapshots of a water filled, hollow steel sphere (the top row for $h = 0.80$) and a steel inclusion (bottom row) both embedded in granite in response to a plane compressional incident wave. The time dependency of the excitation, $f(t)$, is Ricker pulse. (continued)

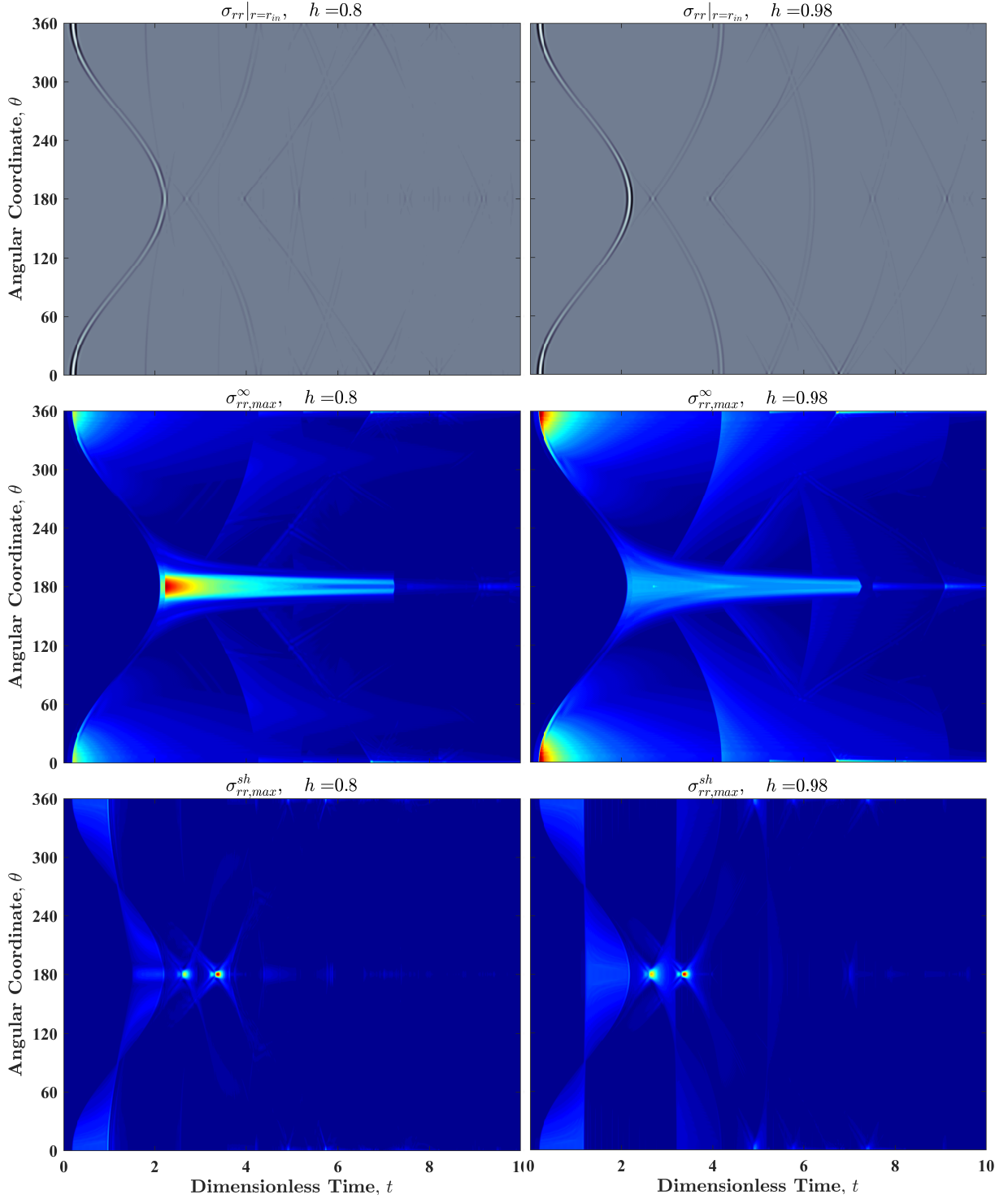


Figure 4.9: Radial stress contours of a water filled steel, hollow sphere ($h = 0.80$) and a steel inclusion both embedded in granite in response to a plane compressional incident wave: (1st row) radial stress at the shell's entire inner surface (normalized); maximum of (2nd row) tensile and (3rd row) compressing stresses of all radii at each angle within the infinite granite formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.

4.2.3 Effect of Relative Material Properties

The amplitude of the stresses and pressures are proportional to the acoustic impedance of a medium whereas the wave motion's dynamics is mostly a function of the relative wave celerity. Three different material arrangements of the shell and elastic formation are considered next.

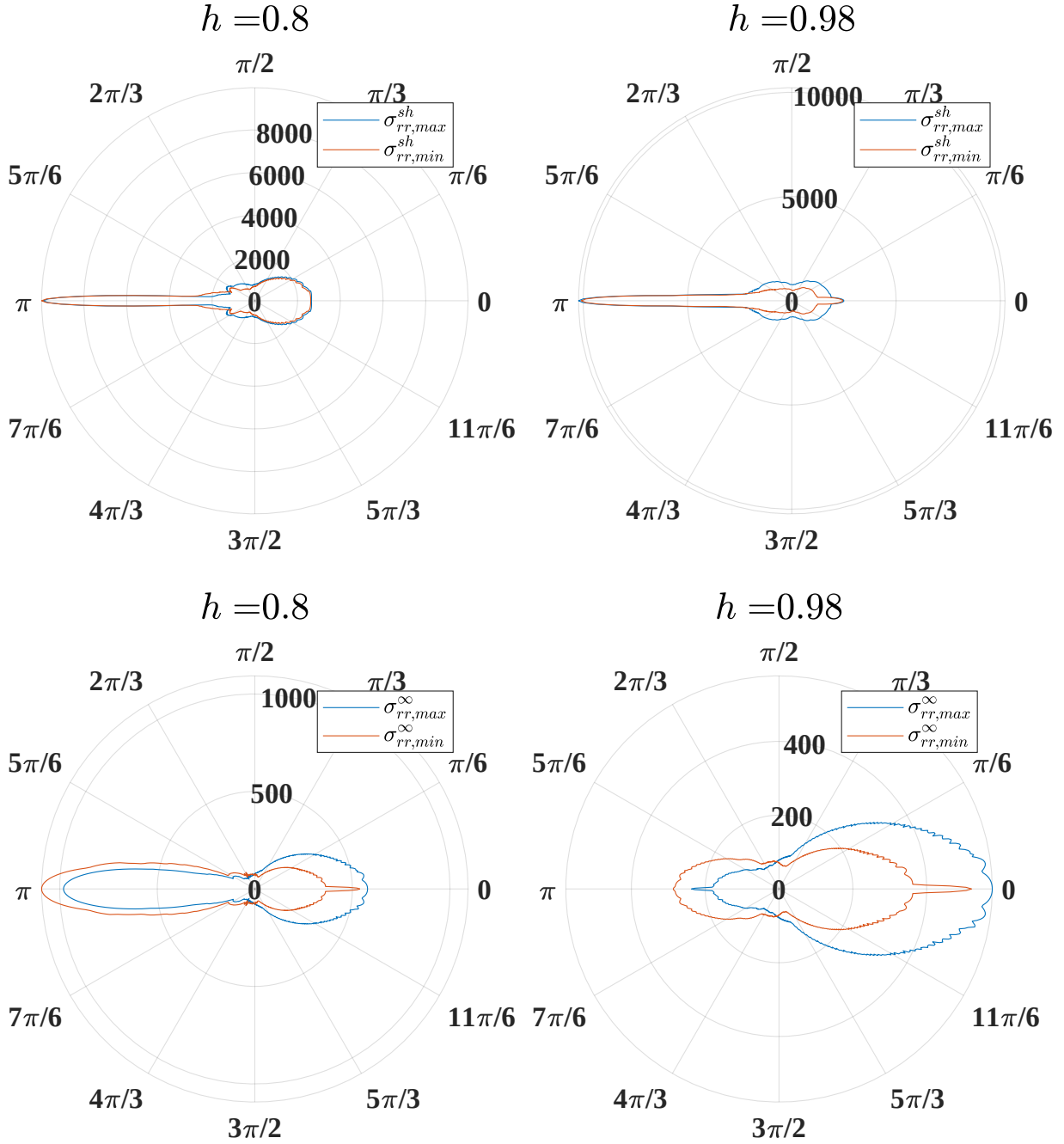


Figure 4.10: Polar plots of the absolute maximum values of tensile and compressional and negative gauge pressures through the entire spatial domain within (first row) steel and (second row) granite formation in response of a compressional plane, incident wave having the Ricker pulse as its time dependency.

Figure 4.11a and Figure 4.11b give snapshots of a water filled shell in response to a plane compressional incident wave with the Ricker pulse time dependency. The first and second rows correspond respectively to steel and aluminum shells embedded within an exterior pyroceram medium whereas the third row shows the results for a steel shell buried in a hard clay formation. These three cases are addressed simply as configurations “T”, “II” and “III”. The rationale behind these choices have been discussed in §4.1.1. Major differences appear in the wave’s dynamics between the steel shell embedded in the hard clay and two other configurations even from the beginning of the interaction. Significantly lower body waves’ celerity within the hard clay compared

to the corresponding ones in the steel shell and contained water is the main contributing factor. The compressional and shear wave velocities within steel are higher than those of the hard clay by approximately 26 and 13 folds. The ratio of the sound propagation's speed within water with respect to the hard clay's compressional wave is more than 6. The wave celerities within steel and aluminum, on the other hand, are pretty comparable to the corresponding ones in pyroceram. This leads to a completely different dynamics of the interaction in configuration III compared to the other two cases. Refracted wavefront into the shell has not reached to the shell's inner surface by $t = 0.20$ in the first two configurations whilst the transmitted wavefront has covered already around three quarters of the contained water when configuration III is considered. Compare the corresponding snapshots given in [Figure 4.11a](#).

The transmitted waves into the contained water for configuration III at $t = 0.20$ given in [Figure 4.11a](#) have two major components. A wavefront refracted directly from the head region having a positive gauge pressure (brighter color) with a convex wavefront. Such a geometry of the wavefront leads to an internal focusing mechanism within the diffracted pressure waves which is similar to the one explained previously for the inclusion. Weaker wavefronts can be observed which have a negative gauge pressure (the darker colored region). These head waves are created ahead the main convex refracted wavefront when the speed of the sound in the contained fluid is equal or higher the celerity of external medium. The compressional wave velocity in the shell material must be higher than the enclosed fluid's celerity. See, for example, [\[Sun and Takayama, 1996\]](#) and [\[Hasheminejad et al., 2011\]](#).

Formation of relatively weak multiple focusing mechanisms are identifiable throughout the successive snapshots given in the third row of [Figure 4.11a](#) and [Figure 4.11b](#) for the configuration III. Such a significant level of the contrast in the relative wave celerities, however, leads to a highly dispersive response. Amplitudes of focal pressures, for example, become considerably weaker compared to other two configurations. This observation has important implications, for instance, in the design of shock resisting shell structures because acoustical energy is distributed as being focused into small regions in other two configurations.

The first two rows of the snapshots given in [Figure 4.11a](#) are associated with configuration I and II, respectively. The refracted wavefronts in these two configurations are concave in contrast to the plane wavefront refracted into the steel shell when embedded in granite. This difference is due to the higher compressional wave speeds in steel or aluminum relative to that in pyroceram whereas granite and steel have almost identical compressional wave speeds. Specular reflections are stronger and sharper in the steel shell because its impedance is almost three times of the one of aluminum. Furthermore the shear wavefront of the specular reflection is stronger in the steel shell compared to that in the aluminum shell. See, for example, the snapshots at $t = 0.5$ given in [Figure 4.11a](#). Shear and compressional specularly reflected wavefronts are easily distinguishable in case of steel shell embedded in clay. The specularly reflected compressional wavefront closely followed by the shear one because the propagation speeds of s - and p - waves are closer in the hard clay in comparison to those of the pyroceram formation. See, for instance, the snapshots given in [Figure 4.11a](#) at $t = 1.0$.

[Figure 4.13](#) depicts contour plot time-histories of the inner and outer surfaces of the shell for the aforementioned three material configurations. The surface stresses for steel and aluminum shells buried in pyroceram formations are given in the first two rows, respectively, whereas the

case of steel shell in a hard clay formation is illustrated at the last row. Time history of the inner surface stress (surface pressure) clearly depicts multiple, succeeding focusing processes in which the stress concentrations occur around head and tail regions where they are expected. Multiple mode conversions and evolution of the associated head waves are easily distinguishable.

Moreover the steel shell (first row) shows multiple triple points at both of the shell's surfaces which have relatively comparable strengths. The aluminum shell, on the other hand, gives rise to only one major, secondary triple point on its external surface. These observations can be confirmed by considering [Figure 4.13](#) in which wave conversions associated with the triple points are depicted clearly at both shell surfaces. Compare, for instance, the wave dynamics at the head region at both traction surfaces of the steel and aluminum shells during $0 < t < 2$ through the contour-plots given at the first and second rows [Figure 4.13](#), respectively.

The steel shell has an obviously greater capacity to sustain strain energy because of its higher density. This factor is the reason for stronger secondary waves emitting from the shell into the external elastic formation in steel shell (first row) in comparison to the aluminum shell (second row) in a pyroceram formation. See, for instance, $t = 1.0$ through 4.0 of the first two rows in [Figure 4.11a](#) and [Figure 4.11b](#).

The subsequent multiple, high intensity stress regions appear at approximately $2.5 < t < 5$ around the tail region in the case of the steel shell embedded in a pyroceram formation. These points are associated with the triple points generated at the inner and outer surfaces of a shell which meet eventually and are superimposed at the tail region. These high intensity pressure points associated with triple points do not appear in the aluminum shell. See, for example, snapshots given in [Figure 4.12a](#).

The contour plots of the absolute maximum amplitude of the pressure and stresses time-history are useful in this case. Multiple focal regions are visible around the tail area ($\theta \approx \pi$) in third row of [Figure 4.15](#). At the same time, high negative pressure concentration spots are detectable around head regions ($\theta \approx 0$) are visible which are corresponding to secondary focal regions as well as the triple points developments. Corresponding quantitative plots in [Figure 4.16](#) and [Figure 4.17](#) show clearly way higher distribution over time and as well as space. Accordingly [Figure 4.16](#) and [Figure 4.17](#) demonstrate much smaller amplitudes up to ten folds for the clay embedded steel shell as expected. The main conclusion from these comparisons is that a much higher distribution acoustical energy forms when the shell is subjected to an incident plane stress wave if a high contrast of wave celerities exists. This has an immediate, obvious practical implication for shock resistant designs of the buried shells so the transmitted energy to the contained fluid does not get focused and amplified but rather distributed and damped more safely. Similar observation can be made in [Figure 4.14](#) about occurrence of a higher distribution of the tensile stress within the steel shell buried in the hard clay.

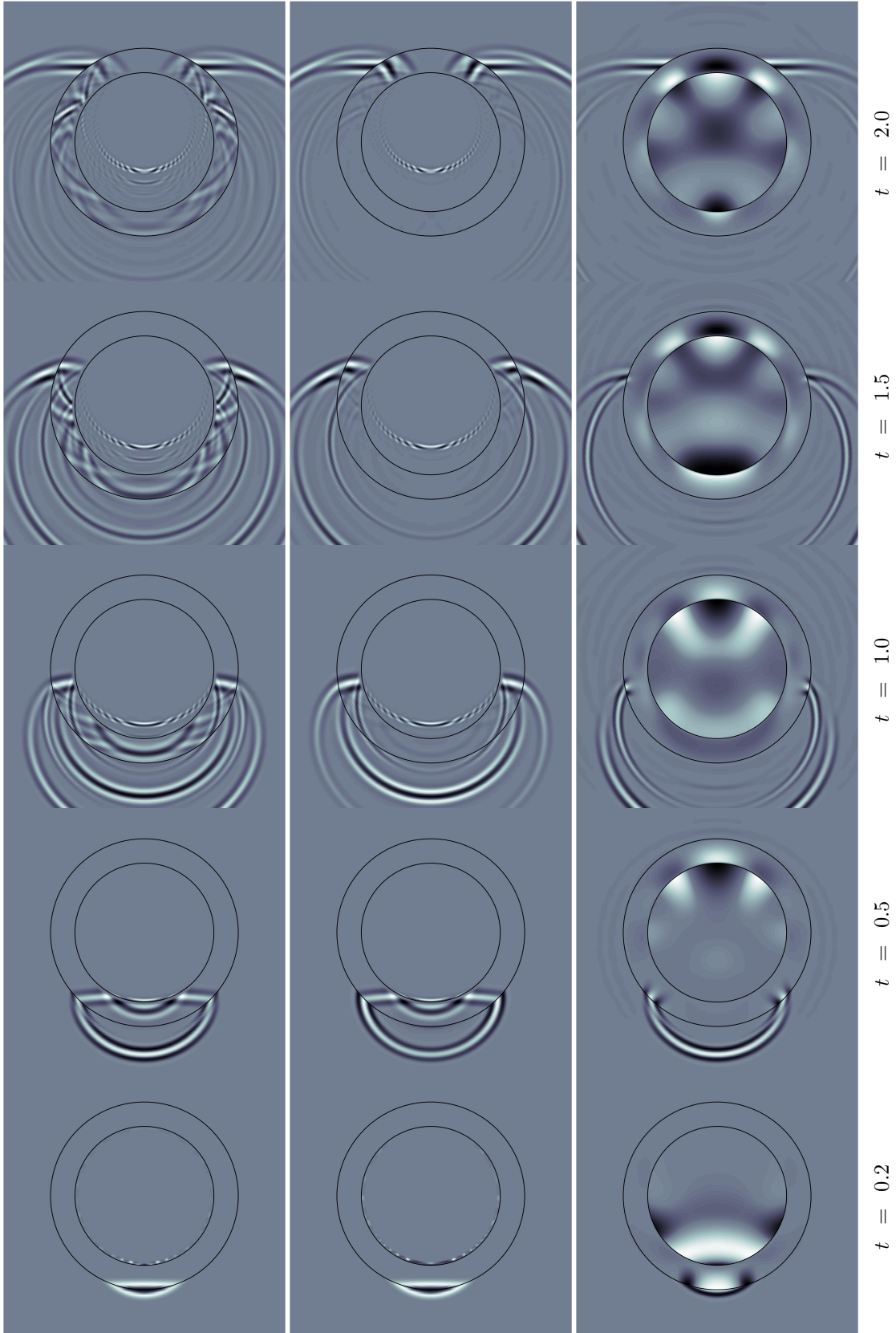


Figure 4.11a: A series of total field successive snapshots of a water filled shell with shell wall thickness of $h = 0.25$ in response to a plane compressional incident wave. (1st row) steel shell in an pyroceram formation; (2nd row) aluminum shell in an pyroceram formation and (3rd row) steel shell embedded in hard clay. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.

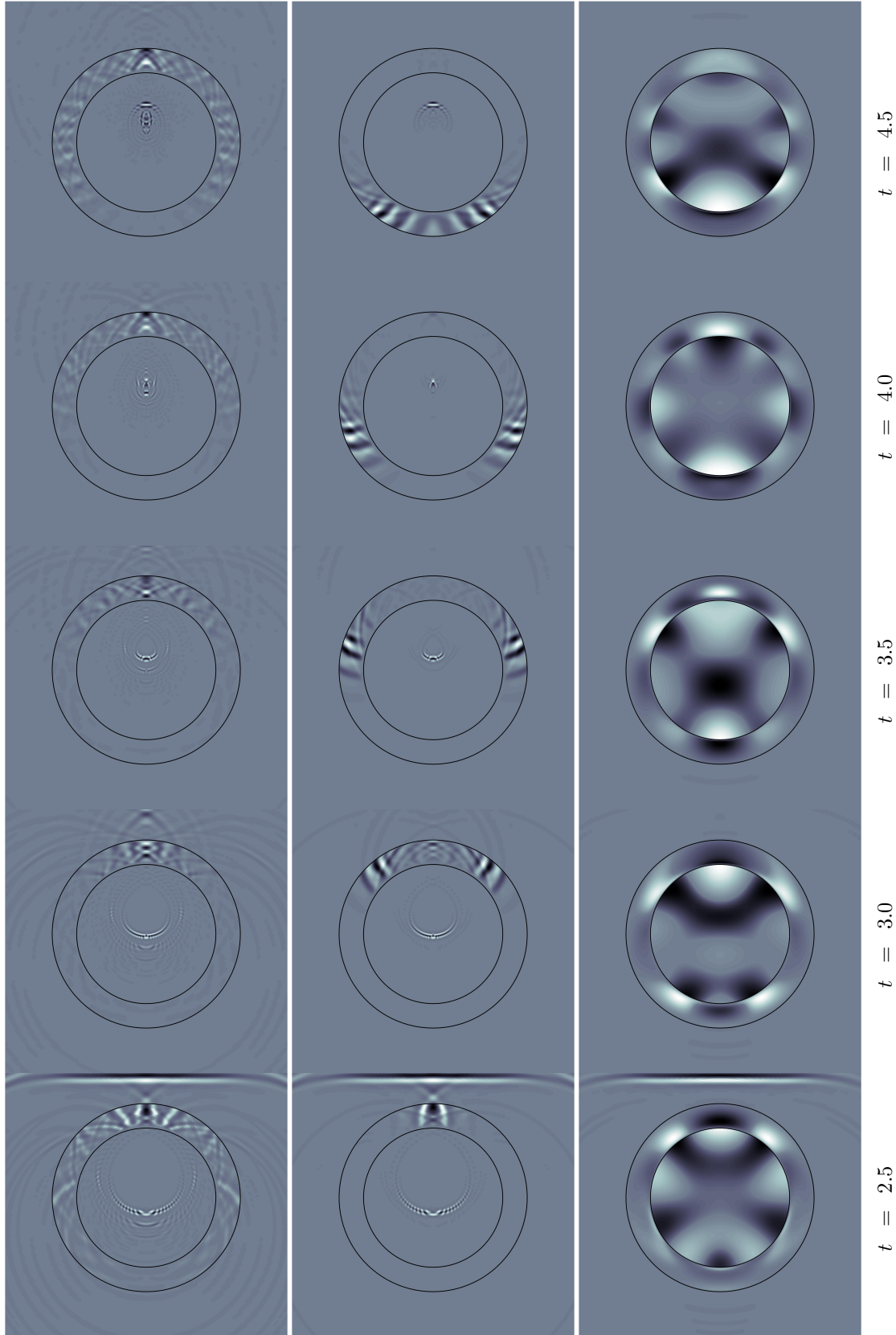


Figure 4.11b: A series of total field successive snapshots of a water filled shell with shell wall thickness of $h = 0.25$ in response to a plane compressional incident wave. (1st row) steel shell is embedded in an pyroceram shell is embedded in an pyroceram formation and (3rd row) steel shell embedded in hard clay. The time dependency of the excitation, $f(t)$, is Ricker pulse.

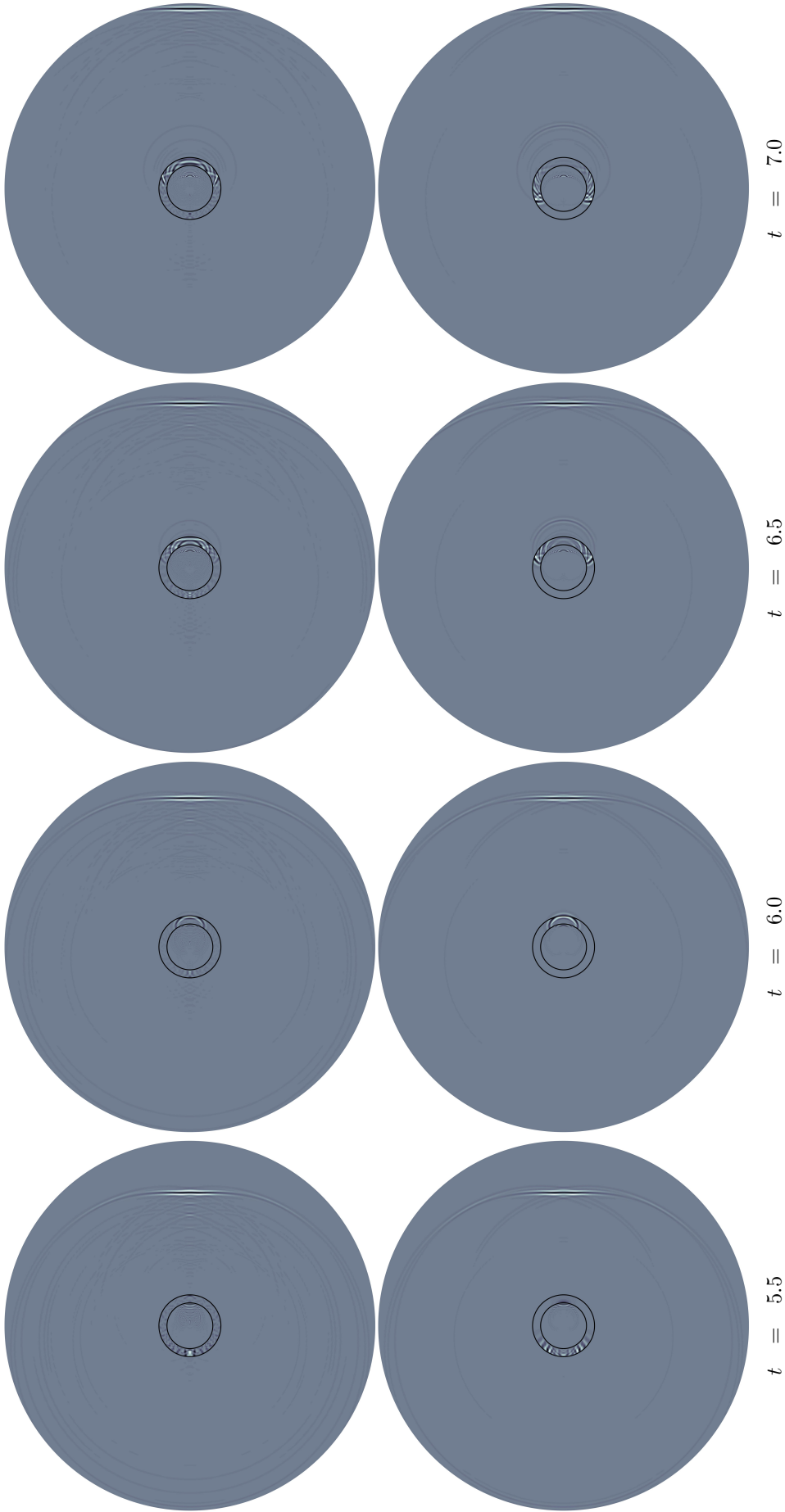


Figure 4.12a: A series of total field successive snapshots of a water filled shell with shell wall thickness of $h = 0.25$ in response to a plane compressional incident wave. (1^{st} row) steel shell is embedded in an pyroceram formation and (2^{nd} row) aluminum shell is embedded in an pyroceram formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.

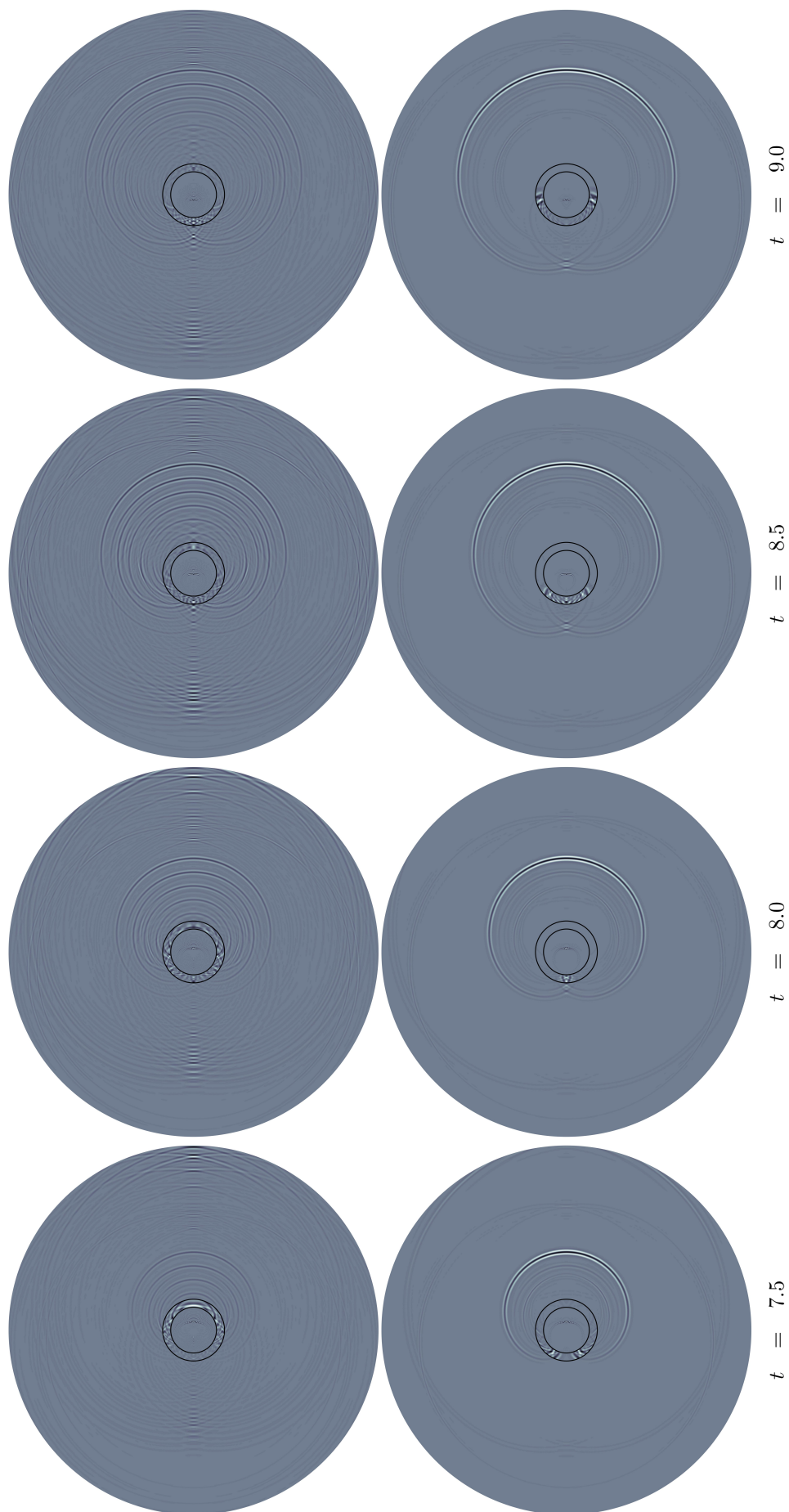


Figure 4.12b: A series of total field successive snapshots of a water filled shell with shell wall thickness of $h = 0.25$ in response to a plane compressional incident wave. (1^{st} row) steel shell is embedded in an pyroceram formation and (2^{nd} row) aluminum shell is embedded in an pyroceram formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.

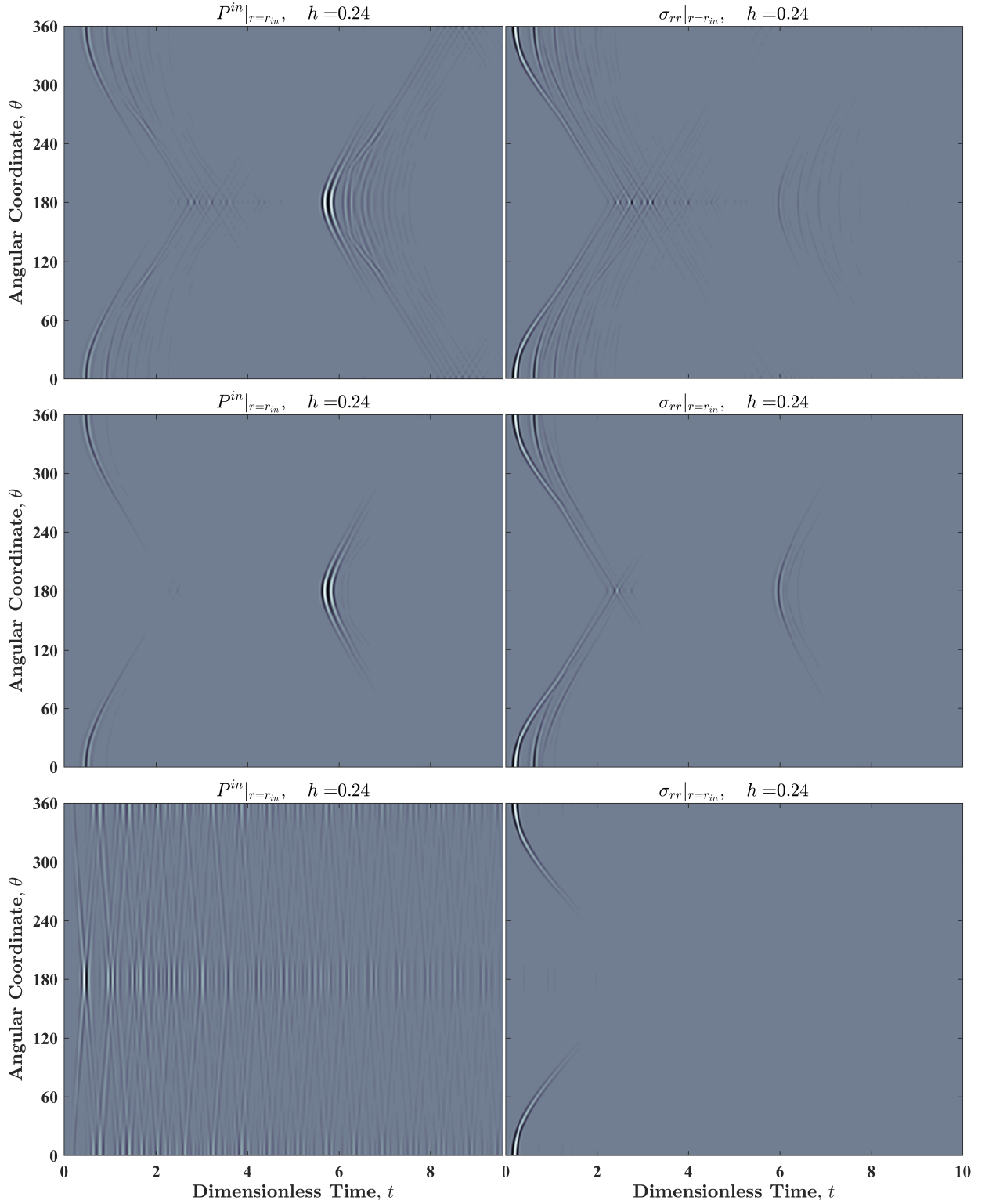


Figure 4.13: Contour plots of the surface pressures (first column) and radial stresses (second column) of a water filled shell with the wall thickness of $h = 0.24$ due to a plane compressional incident wave: (1st row) a steel shell in an pyroceram formation; (2nd row) an aluminum shell is embedded in an pyroceram formation and (3rd row) steel shell embedded in hard clay formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.

Strong focused pressure wave front of the internal fluid reach the shell's inner surface slightly after $t \approx 5.5$. This then creates a high pressure wavefront which is transmitted into the material of the shell repeated reflections occur between the two surfaces of the shell leads to the formation

of multiple triple points. See [Figure 4.12a](#) and [Figure 4.12b](#) when $t > 5.5$ in conjunction with the first two rows of [Figure 4.13](#). Two main points can be made by comparing the wave dynamics of a steel and an aluminum shell after which the focused pressure wavefront meets the inner surface of the shell by following snapshots given in [Figure 4.12a](#). First, the aluminum shell can store considerably less energy than steel due to its much lower density and, hence, lower acoustical impedance. The stronger wavefront is transmitted from the aluminum shell into the pyroceram compared to the steel shell. This observation is even more conspicuous in the snapshots given in [Figure 4.12b](#) compared to those given in [Figure 4.12a](#). The primary specular reflections and creeping waves are out of the snapshots' rendering domain. The steel shell, on the other hand, releases the acoustical energy into the external medium in a more comparable steps. It should be noted that the triple points and their associated surface waves move slightly faster in the aluminum shell because of having a higher longitudinal wave speed.

[Figure 4.17](#) shows that the material configuration considerably alters not only the intensity but also the distribution of the maximum tensile and compressional stresses in the shell's thickness. The maximum tensile stresses at the head region is almost twice of the corresponding compressional ones in the configuration I, however, they are very comparable in other two configurations. The peak values happening around $\theta \approx \pi/2$ and $\theta \approx 3\pi/2$ in only the configuration II. These peaks are basically due to the formation of strong triple points at both shell surfaces. The significant effect of different excitations is considered next.

4.2.4 Effect of different excitations

The objective here is to demonstrate briefly the significant effect of various excitation sources on the wave's dynamics described. These sources are formulated in §2.3 of the first companion chapter. The effect of the time-dependency of an excitation, $\bar{f}(t)$, is tested first by comparing two functions. Various sources of excitation are compared to the plane, compressional incident wave next. They are incident waves of a compressional spherical, a plane shear as well as an eccentric internal pressure types. A water-filled, steel shell with $h = 0.2$ which is embedded in granite is considered regardless.

The system's response to an incident plane compressional wave to an explosion is compared to that of Ricker pulse whose characteristics are given in [Table 4.2](#). These characteristics are chosen so that they have very similar temporal shapes and effective duration. [Figure 4.18](#) illustrates the spatial and temporal distributions of the resulting negative and positive gauge pressures.

The time and angle of occurrence of the extreme values are similar in the tail region. Ricker pulse, however, incites amplitudes which are almost twice of those produced by the explosion. Besides, the difference between negative and positive pressure extrema increase when an explosion impinges the shell. The timing of the focal pressure formations are identical in [Figure 4.18](#) as it depends solely on the material and geometrical configuration. [Figure 4.19](#) depicts spatial and time-history distributions of the maximum absolute values of tensile and compressional radial stresses happening within the shell. The dynamic radial stress concentration occurring from the Ricker pulse are compared to those from the explosion pulse is approximately 65% higher. Furthermore the maximum tensile and compressional stresses are not in phase for the explosion unlike those for the Ricker pulse as illustrated in the second row of [Figure 4.19](#).

These extrema are the result of primary and secondary triple points being individually superimposed as well as those given rise to by the focal pressure wavefront at the inner surface. The

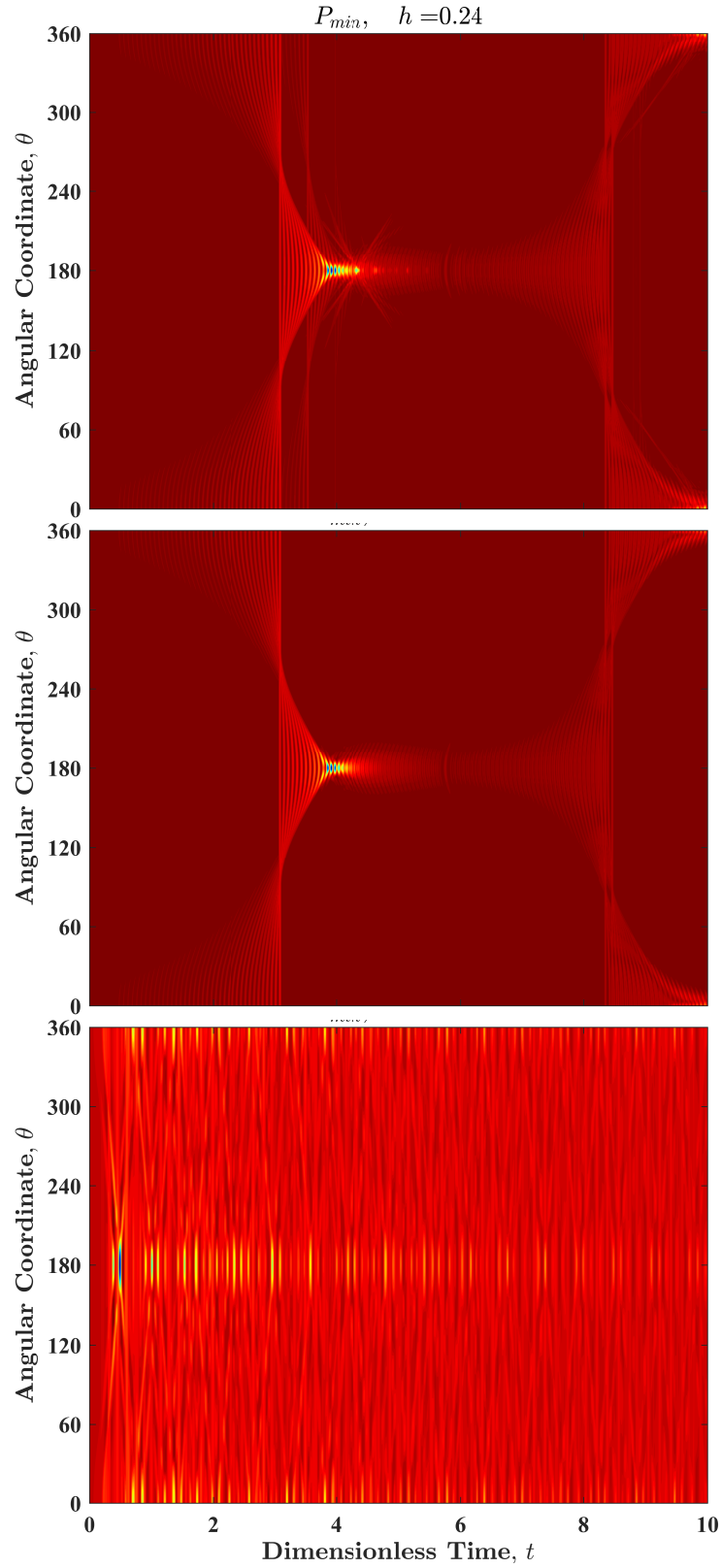


Figure 4.14: Contour plots of the maximum absolute value of negative gauge pressure of a water filled shell with the wall thickness of $h = 0.24$ due to a plane compressional incident wave: (1st row) a steel shell in an pyroceram formation; (2nd row) an aluminum shell is embedded in an pyroceram formation and (3rd row) steel shell embedded in hard clay formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.

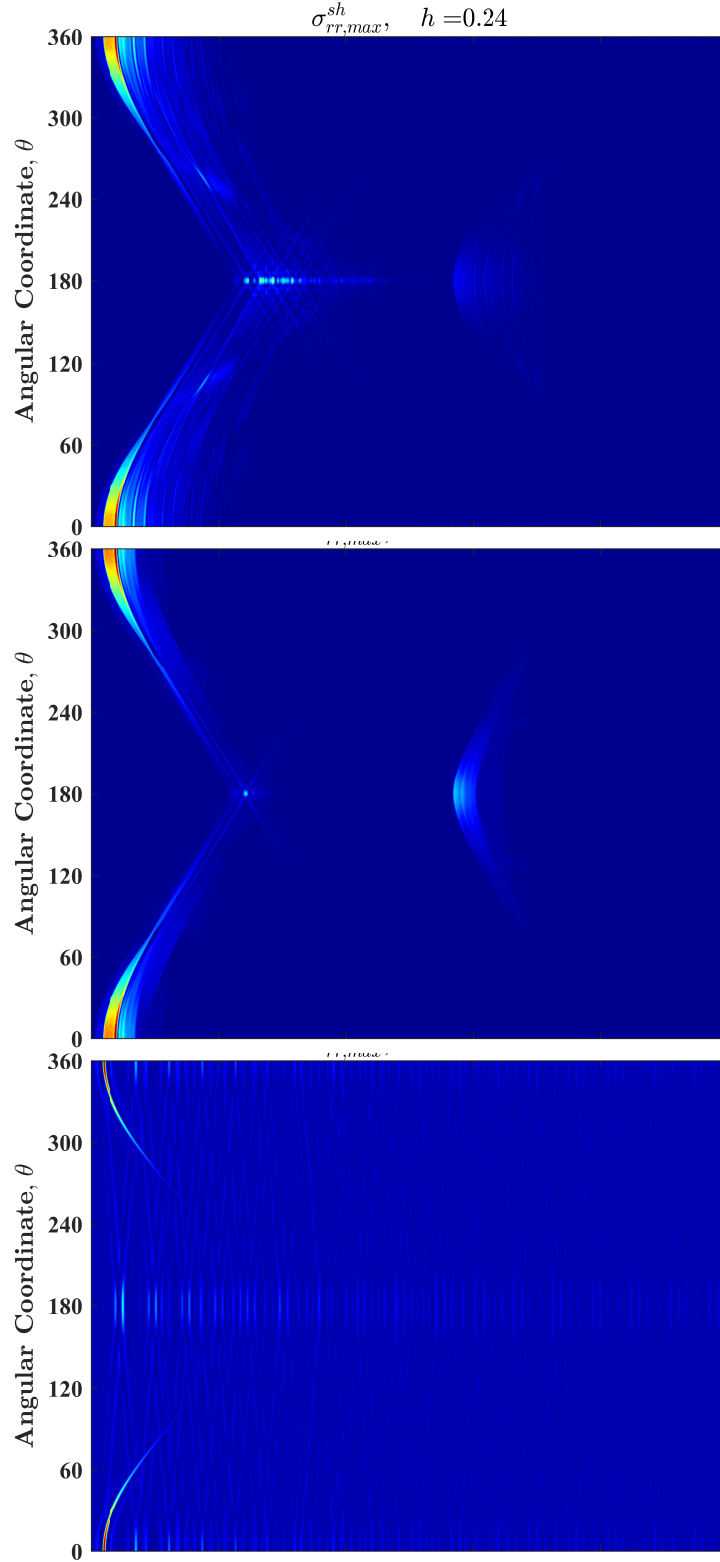


Figure 4.15: Contour plots of the maximum stresses within the shell material of a water filled shell with the wall thickness of $h = 0.24$ due to a plane compressional incident wave: (1st row) a steel shell in an pyroceram formation; (2nd row) an aluminum shell is embedded in an pyroceram formation and (3rd row) steel shell embedded in hard clay formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.

amplitudes, however, are at different levels.

Figure 4.20 compares the results for incident compressional waves whose fronts are plane or

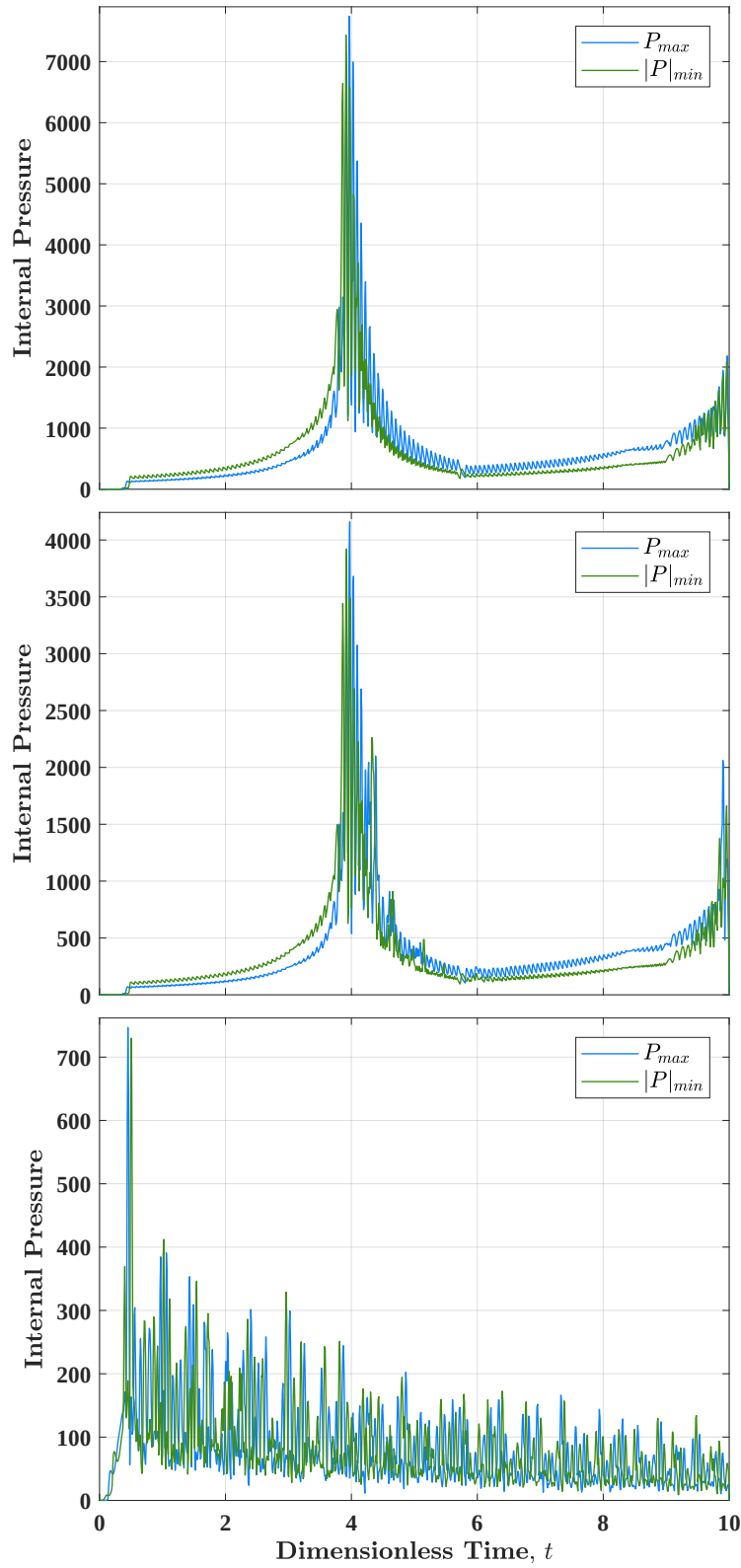


Figure 4.16: Absolute values of positive and negative gauge pressures through the entire spatial domain for a water filled shell ($h = 0.24$) in response to a plane compressional incident wave: (1st row) a steel shell embedded in an pyroceram formation; (2nd row) an aluminum shell is embedded in an pyroceram formation and (3rd row) steel shell embedded in hard clay formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.

spherical. The origin of spherical wave is located at $r_\infty = 1.5$ measuring from the shell's center. The first and second columns from the left are the responses to the plane and spherical wavefronts, respectively. The maximum absolute positive and negative gauge pressures in the first and second rows show a completely different patterns and intensities. The plane incident wave creates a much more directional and focused pattern within the internal pressure whereas the spherical wave shows larger distribution of the acoustical energy. Conversely, the extrema of tensile and compressional stresses within the shells material becomes more directional for the case of spherical incident. See the second row of [Figure 4.20](#). Furthermore the maximum tensile stress at the head region of the shell is about twice that of its compressional counterpart.

The effect of shear rather than a compressional incident waves are presented in [Figure 4.21](#). The first and second columns in this figure correspond, respectively, to the compressional and shear incident waves with a plane geometry. The shear wave clearly leads a more focused acoustical energy in a narrower region. Its wavefront intensifies the pressure extrema much higher in the tail region where the negative gauge pressure dominates. Conversely the maximum positive gauge pressure is higher than the negative one when a compressional wave is impinged upon the shell. This observation has a significant practical implication: the likelihood of a cavitation occurrence is considerably greater when a shear wave is applied. Furthermore, it is more focused spatially with a higher intensity so that shell's integrity may also be of concern. The duration of the focused region on the internal pressure field is delayed in the case of shear incident due to a lower speed of a shear wave in a granite formation compared with the compressional wave.

The traction stresses of the inner and outer surfaces of the shell are illustrated in the first row of [Figure 4.22](#) in response to a plane incident wave (the first row) in comparison to an eccentric pressure source within the encapsulated fluid at $r_f = 0.4$ from the center of the shell (the second row). These two sources lead to very different wave dynamics at the shell's surface. The associated focusing mechanism of the internal pressure is completely different for the eccentric pressure source. See the contour plots in the first column of [Figure 4.22](#). For example, no significant head wave formation occurs at the initial interaction unlike the plane compressional incident wave. Furthermore the associated mode conversions follow different paths in these two cases. The stress concentration associated with the focusing mechanism within the encapsulated fluid occurs at the tail region has a much higher intensity for the eccentric pressure source around $t \approx 7$. A much higher stresses concentration, however, happens in the head region when the plane, compressional wave is the incident. See the contour plots in the second column of [Figure 4.22](#). The concluding remarks are given next.

4.3 Conclusions

An extensive set of numerical experiments are performed in the this part. A plane incident wave of longitudinal type is chosen as the main excitation source with a Ricker pulse time dependency. Especial visualization techniques are used through 2D snapshots as well as time-angle contour plots to better understand the complex nature of the interaction. These qualitative presentations are followed by a specific set of quantitative arrangement of field's responses. The effect of the geometrical parameter, the relative shell wall-thickness, is investigated first. Comprehensive comparisons are performed for $h = 0.02$ through $h = 0.4$. Juxtapositions between $h = 0.8$ and $h = 0.98$ are considered next which are basically a migration to the limiting case of elastic inclusion

that is not considered at the first part. The significant effect of the relative material properties of the interacting media is studied subsequently by choosing three different configurations. The impact of the relative celerity and acoustical impedance are investigated accordingly. Dynamics of the system in response to an explosive load charge is compared with a Ricker pulse with similar parameters. Finally brief comparisons are made between different sources of excitation on one hand and the plane longitudinal incident wavefront on the other hand. These cases includes a spherical wave of the compressional type, a plane shear wave and an eccentric internal pressure pulse. The main observations and the associated concluding remarks are listed next.

- Generally absolute maximum value of either negative or positive internal pressure occurs at the tail region.
- Generally maximum absolute values of the negative gauge pressure is higher than its corresponding positive ones.
- An increase in relative shell wall-thickness leads to a higher intensity of the focal wavefront in the internal fluid medium if a similar mass of the contained fluid is considered.
- As the shell wall-thickness increases multiple focusing processes occur in the contained fluid. This is basically due to stronger triple point formation at the inner surface of the shell transmitting stronger wavefronts into the enclosed fluid.
- Primary focusings of a diffracted wavefront into an inclusion with a flat geometry leads to two focal points: one with a compressional origin which is followed by a stronger shear type.
- A small fluid or vacuum inclusion within an elastic inclusion acts as an acoustic lens and focuses the incident waves at the tail region.
- Highly contrasting acoustical properties of the interacting media produces lower level of the maximum amplitudes of pressures and stresses and, hence, increases the shock resistance properties of a buried, fluid-filled, spherical shell.
- Presence of small fluid inclusion within a buried hollow sphere leads to a considerably higher concentration of the refracted acoustical energy in comparison with the solid sphere. This point can be used in the design of the acoustical lenses within an elastic matrix.
- Shell materials, having higher impedance, release energy more slowly into the contained fluid, which makes it safer environment for an enclosed instrumentation.
- Similar compressional and shear wave speeds of a shell and its encompassing elastic formation which are higher than the speed of sound in the shell's contained fluid can lead to very highly focused pressures. This phenomena maybe used, in a controlled fashion, to remotely clean the inner surface of of the shell or for non-destructive testing purposes.
- Large amplitude, negative gauge pressure may be used to forecast the likelihood of damaging cavitations.
- Incident waves of shear type, generally, lead to a much higher level of acoustical energy concentration compared with the longitudinal incident waves.

The high intensity focused pressure regions have obvious destructive potentials, especially due to likelihood of cavitation occurrence. The consequences, however, can be constructive if it is used in a controlled way. A focused wavefront can be used, for example, as a remote cleaning tool for the inner surface of a shell without direct access to the shell's inside. Furthermore, this phenomena can be used as a focusing mechanism of ultrasonic beams for the nondestructive testing applications. Higher impedance of the structure leads to a higher capacity in absorbing acoustical energy. See, for example, the steel shell in comparison with the aluminum one has considerably higher strain energy. The best choice in distributing the internal energy happens when the wave propagation speed within the internal fluid medium is higher than the wave celerity of the external elastic formation. This situation results in no significant focusing in the wavefronts transmitted into the external formation as well.

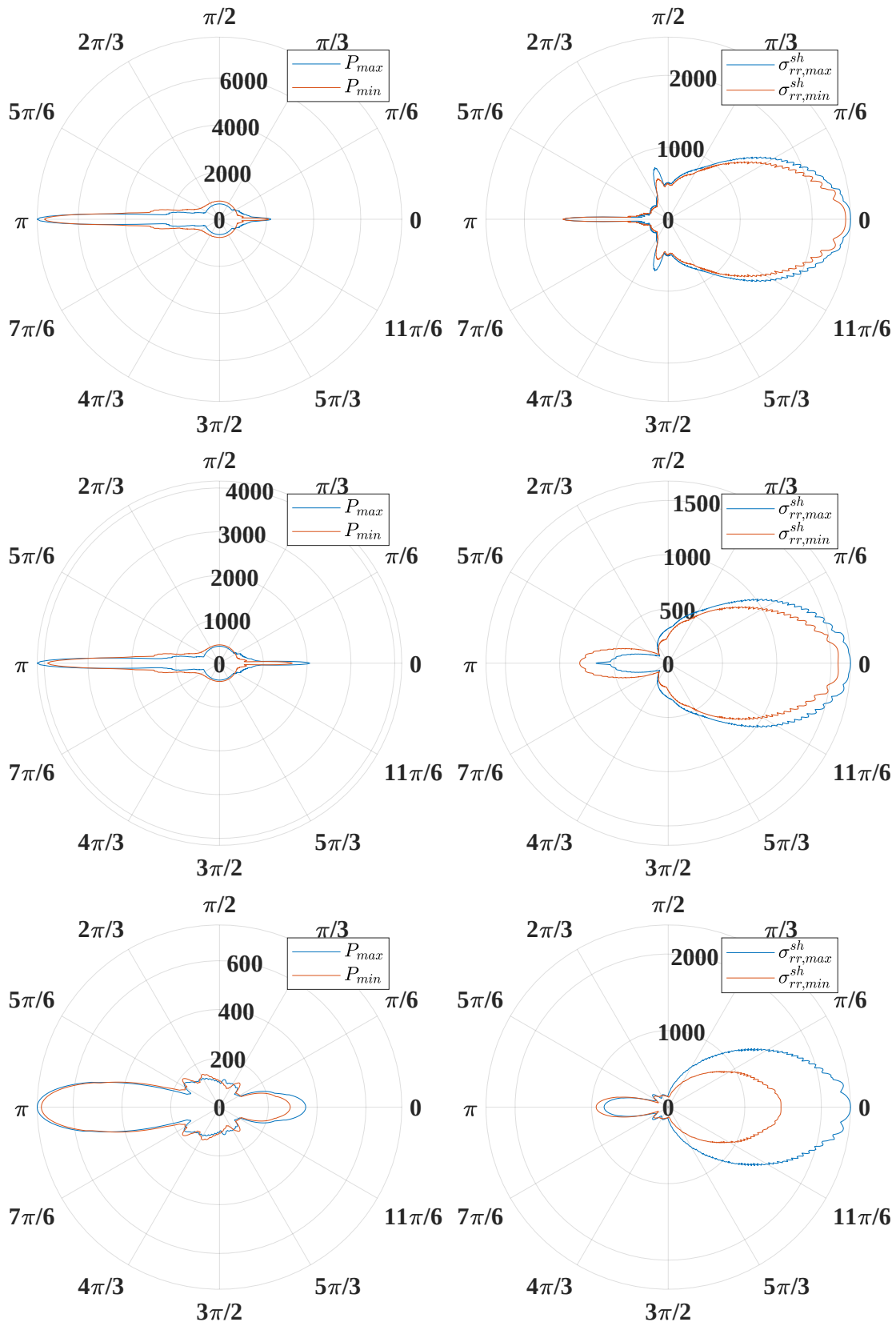


Figure 4.17: Absolute values of positive and negative gauge pressures (first column) and radial stresses (second column) within the shell material through the entire spatial domain for a water filled shell ($h = 0.24$) in response to a plane compressional incident wave: (1st row) a steel shell in an epoxy formation; (2nd row) an aluminum shell is embedded in an epoxy formation and (3rd row) steel shell embedded in hard clay formation. The time dependency of the excitation, $\hat{f}(t)$, is Ricker pulse.

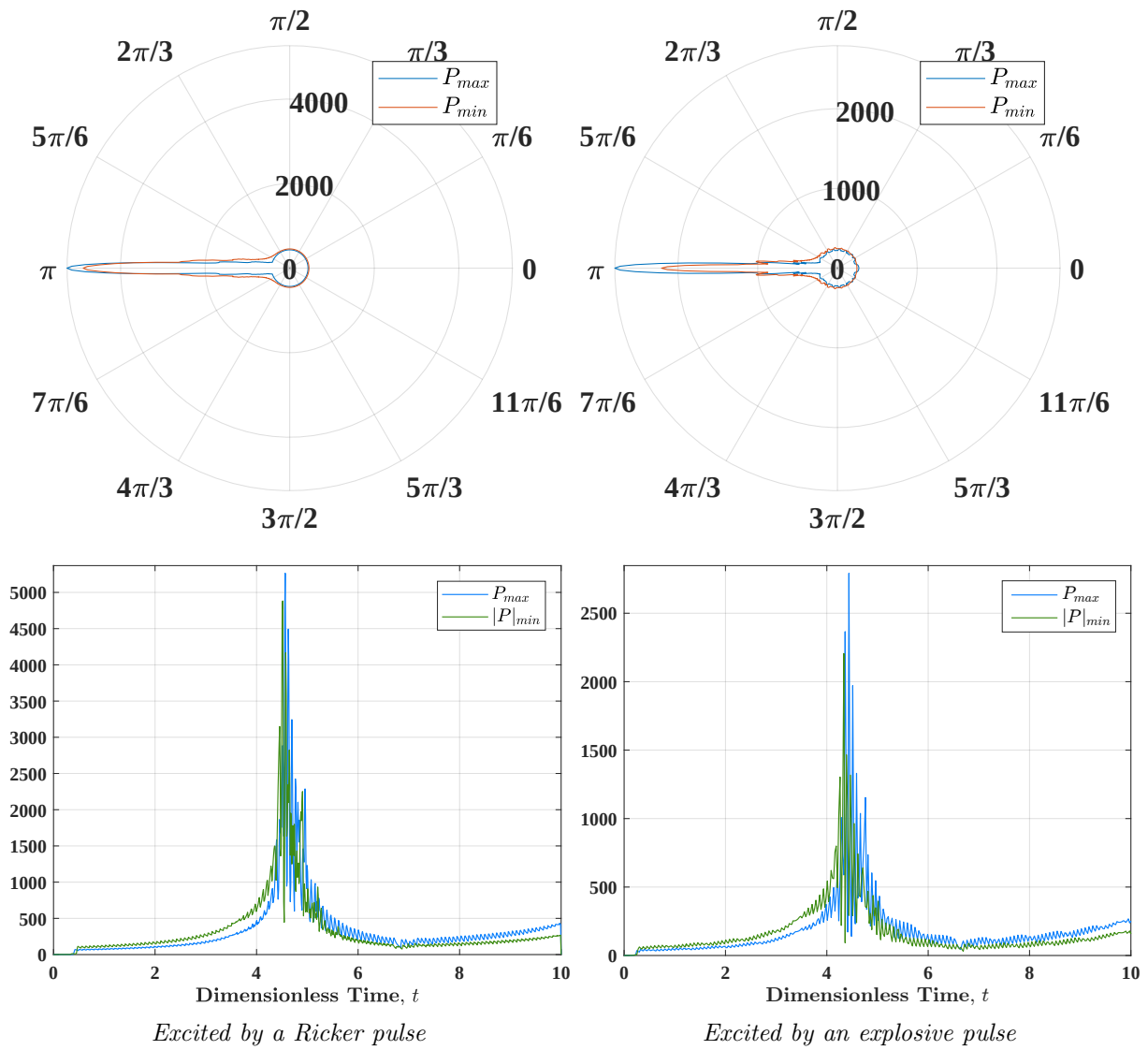


Figure 4.18: Absolute values of positive and negative gauge pressures through the entire spatial domain for a water filled steel shell ($h = 0.2$) embedded in granite in response to a plane compressional incident wave where the time dependency of the excitation, $\bar{f}(t)$, are (1st column) a Ricker pulse and (2nd column) an explosive load.

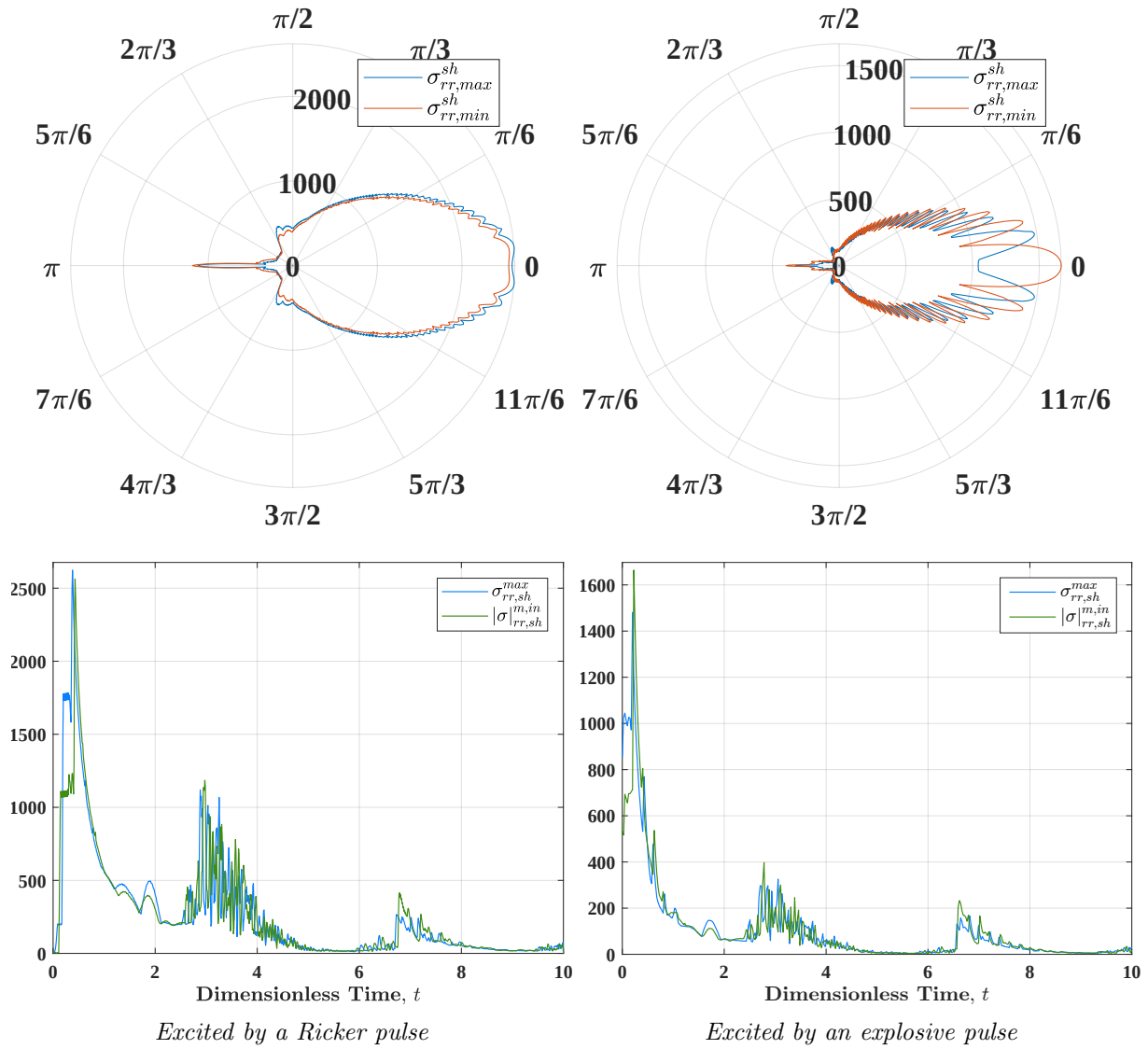


Figure 4.19: Absolute values of compressional and tensile stresses through the entire spatial domain for a water filled steel shell ($h = 0.2$) embedded in granite in response to a plane compressional incident wave where the time dependency of the excitation, $\bar{f}(t)$, are (1st column) a Ricker pulse and (2nd column) an explosive load.

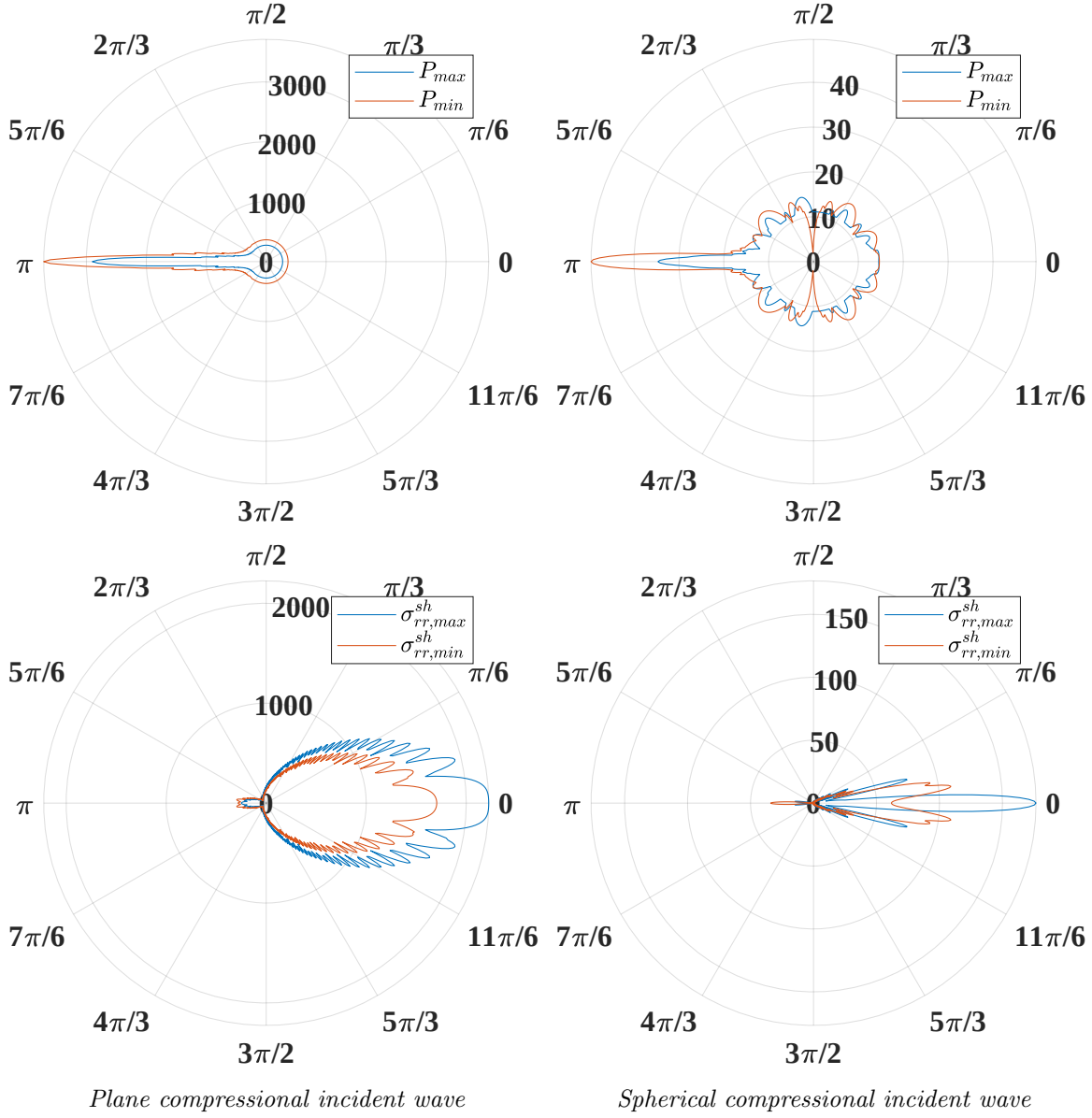


Figure 4.20: Absolute values of positive and negative gauge pressures through the entire spatial domain for a water filled steel shell ($h = 0.2$) embedded in granite in response to a (1st column) plane compressional and (2nd column) spherical compressional incident wave with $r_{\infty} = 1.5$ where the time dependency of the excitation, $\bar{f}(t)$, is a Ricker pulse.

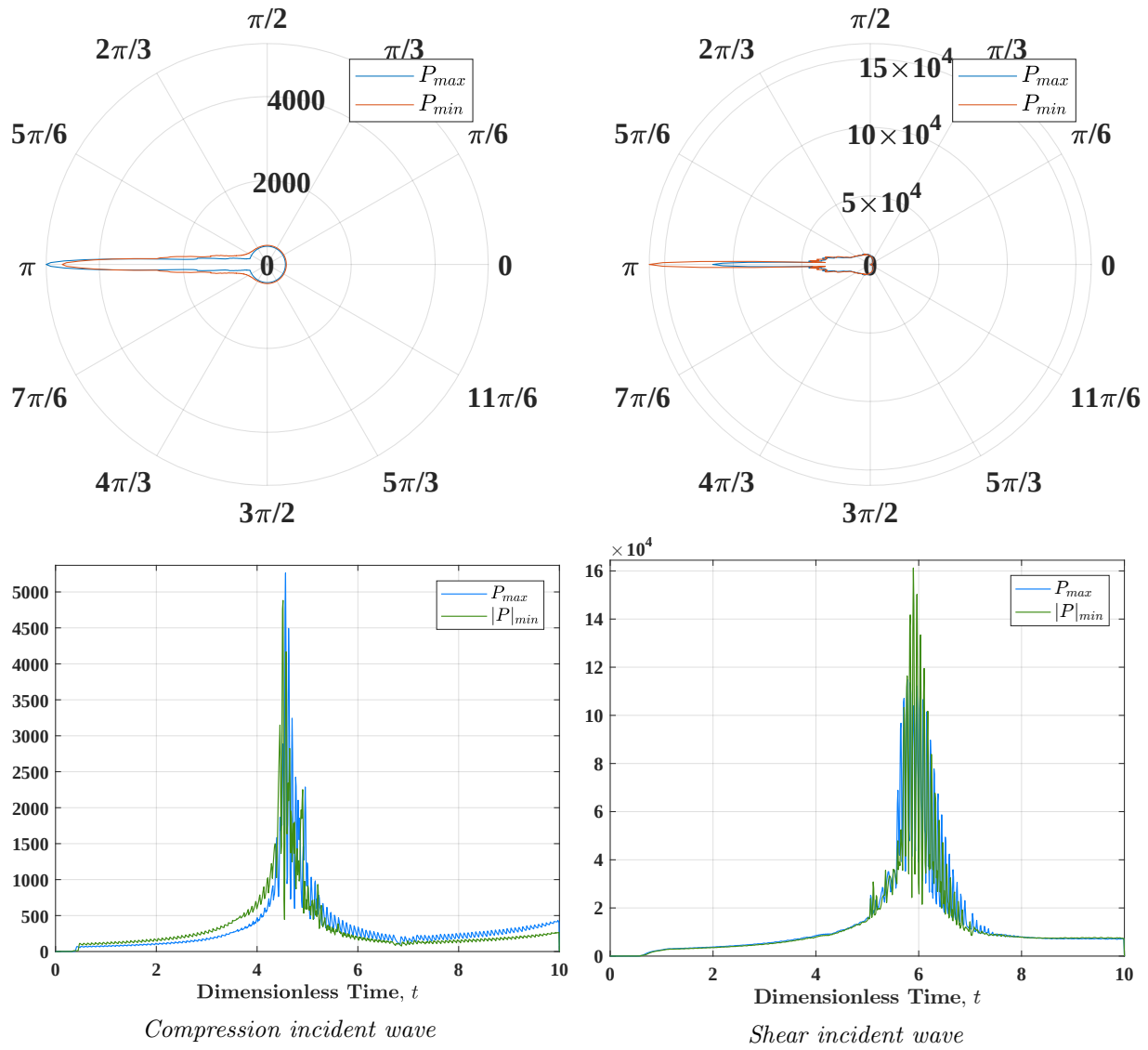


Figure 4.21: Absolute values of positive and negative gauge pressures through the entire spatial domain for a water filled steel shell ($h = 0.2$) embedded in granite in response to plane (1st column) compressional and (2nd column) shear incident waves where the time dependency of the excitation, $\bar{f}(t)$, is a Ricker pulse.

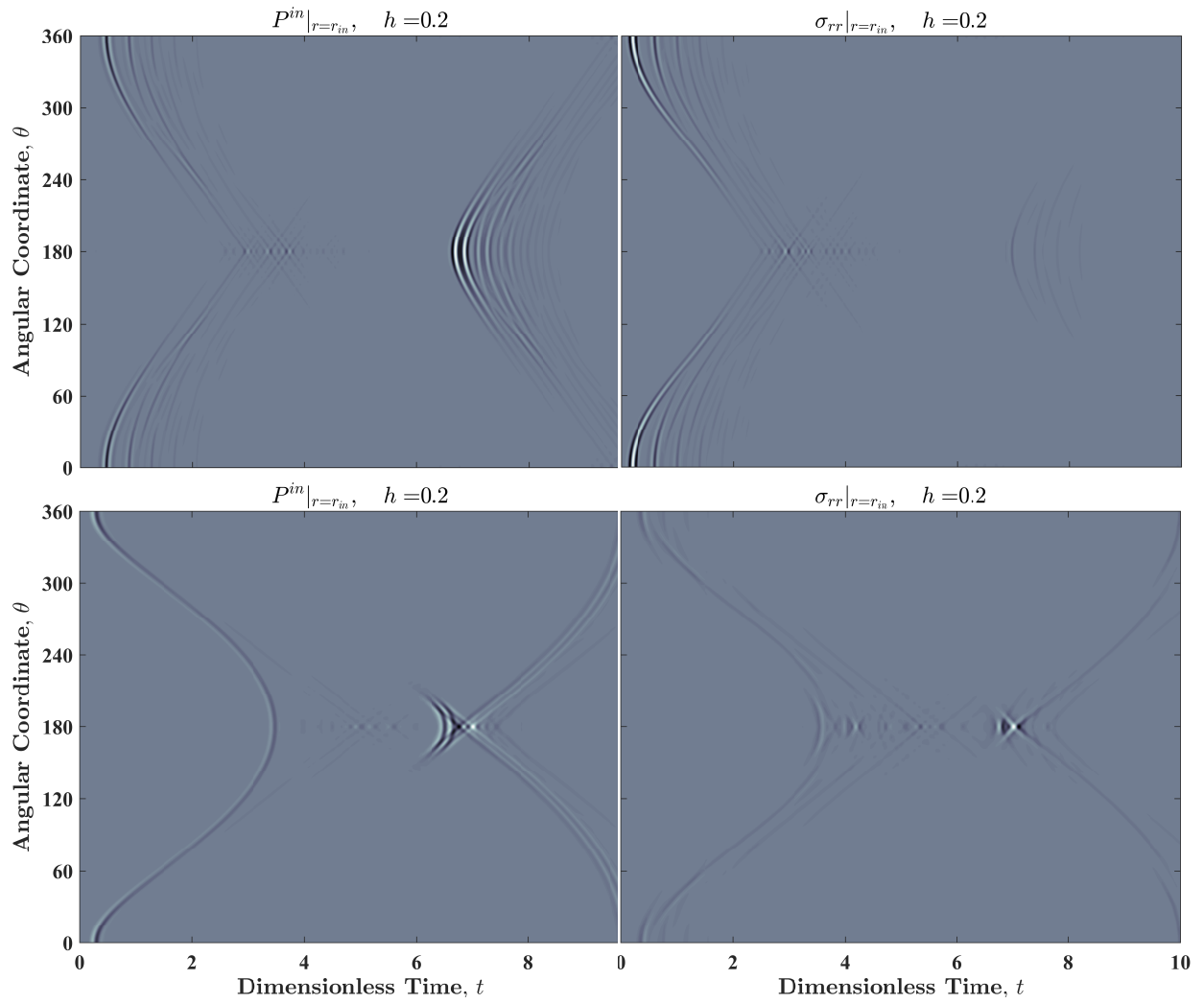


Figure 4.22: Contour plots of the surface pressures (first column) and radial stresses (second column) of a water filled shell with the wall thickness of $h = 0.2$ due to a (1^{st} row) plane compressional incident wave and (2^{nd} row) eccentric pressure pulse in granite formation. The time dependency of the excitation, $\bar{f}(t)$, is Ricker pulse.

On elasto-acoustic of a fluid-filled anisotropic FGM cylindrical shell in an acoustical wedge having anisotropic piezo-FGM coatings I:

Formulations

Fully coupled elasto-acoustic responses of a thick-walled, multi-layered piezo-composite cylindrical shell is treated analytically where it is filled with fluid and freely submerged in an acoustical wedge medium. Plane incident acoustical waves as well as traction forces on either surfaces of shell are considered as the sources of excitation. The exact theories of linear elasticity and piezoelectricity are applied to the shell's material. Acoustic perturbations within internal and external fluid media are modeled for a compressible fluid within linear domain. A classical *T-Matrix* (Transfer Matrix) technique applied first by enforcing the continuity traction boundary conditions between adjacent assumed sub-layers in a full-space configuration. Subsequently the global *T-Matrix* is constructed. The solution to the resulting state-space form are retrieved by applying a Taylor's series expansion. Then the classical method of images is utilized to satisfy exactly the boundary conditions in an acoustical wedge domain. In this process, a translational addition theorem due to *Neumann* is required to be applied to cylindrical wave harmonic expansions in terms of Hankel and Bessel functions of the first kind. Two sources of excitation are considered: an external traction force of either surfaces of the shell as well as an incident plane acoustical wave. Finally a general solution procedure is used to assemble the global solution. It is important to note that the current approach is applicable to any fully anisotropic material as well as radially, axially and circumferentially polarized piezoelectric configurations. The validations and numerical case studies are given in the companion paper.

5.1 Introduction

Diffraction of waves by cylindrical and spherical objects have been topic of interest of scholars from the late nineteen century within different branches of fundamental and applied physics such as electromagnetism, quantum mechanics, geology and acoustics. See, for example, [Rayleigh, 1896, Morse, 1936, Beranek, 1940, Junger, 1952, Knopoff, 1959, De Branges and Rovnyak, 1966, Balanis, 1999, Faddeev, 1995, Chadani and Sabatier, 2012, Kahnert, 2003, Colton and Kress, 2012]. Applications of acoustic and ultrasonic based methods, however, have been receiving a growing interest due to their non-ionizing radiations along with the capability of working with significantly short wavelengths. Practical examples requiring such a forward solver in an acoustical field are listed below.

- *Medical ultrasound imaging* is frequently used in monitoring the growth of a tumor and soft-tissue organ containing a cyst ([Foiret et al., 2012, Potsika et al., 2014, Wirgin, 2016]). This requires a solution to serve as a forward solver kernel of an inversion algorithm, such as Born approximation. See, for example, ([Robinson and Greenleaf, 1986]).
- *Sonars and underwater acoustics* have many applications in fishery. It requires a scattering analysis of plane waves from a cylindrical body as a forward solver for the corresponding inversion scheme ([Clay, 1991]). Design of *hydrophones* is an area that continues to be using analytical methods in, for example, configuring effects of nearby baffles on the directional characteristics of arrays of projectors ([Gustafson III and Stepanishen, 1981]).
- *NDT* (Nondestructive Testing) is applied extensively in inspecting pressure vessels and pipelines through variety of UT (Ultrasonic Testing) techniques such as A-scan, phased array's C-scan and guided waves ([Amir, 2013, Krautkrämer and Krautkrämer, 2013]).

There are many situations which require considering acoustical propagation rather than a full-space configuration. Exterior acoustic problems are frequently referring to propagation of sources in half-spaces. Such problems are, for instance, encountered in environmental acoustics design of noise barriers and require the handling of particular boundary conditions along the related half-plane. If this plane is either a rigid boundary which enforces a zero normal velocity or a soft boundary which requires a zero pressure (normal stress) condition ([Coyette and Van den Nieuwenhof, 2000]). Large number of studies have been reported on acoustical scattering from a submerged cylindrical object, however, the current chronological literature review is mostly focused on analytical or semi-analytical approaches. A brief review is given next on the most pertinent studies.

[Skidan et al., 1974] utilized the Helmholtz integral approach in conjunction with Fourier transform to study the harmonic response of an infinite cylindrical shell submerged in an acoustical medium bonded by an elastic half space. Following, [Shaw, 1975] utilized the boundary integral equation technique to investigate the scattering of transient acoustical plane waves from a cylinder with a Gaussian distribution of boundary impedance. Next, [Gustafson III and Stepanishen, 1981] presented a solution for the acoustical multiple scattering of two infinitely long cylindrical hydrophone bounded by a planar baffle, excited by a harmonic incident plane wave. Later, [Huang, 1981] employed the *method of images* in the Laplace domain to study the transient scattering of an incident shock from an elastic shell near a hard surface under a plain strain condition. It was concluded that the effect of the hard surface is insignificant when the incident wave is of long duration (narrow-band). Shortly afterwards, [Varadan et al., 1982] used Half-space Green's function along with Helmholtz integral to study the effect of acoustical multiple scattering of submerged objects close to a rigid wall. Furthermore [Aleamar et al., 1986b] investigated the acoustical scattering from a cylindrical fluid obstacle in a denser and a lighter fluid such as water and air to estimate the attenuation and obtain dispersion curves. Subsequently they considered the inverse scattering problem of the same configuration to obtain the orientation, dimensions and material properties of the scatterer in [Aleamar et al., 1986a]. [Doolittle et al., 1988] used the experimental data of the east Australian continental shelf to check the reliability of analytical method of three dimensional refraction from a wedge shaped acoustical medium in shallow waters when subjected to a line source parallel to the wedge.

Boundary Integral Equation (BIE) technique by using a modified Green's function was applied by [Seybert and Soenarko, 1988] to investigate the three dimensional radiation and sound diffraction from arbitrary shape bodies immersed in an acoustical half space. Integrals on the infinite plane were removed by using a modified Green's function associated with the Helmholtz integral. Thus, all discretizations and numerical approximations are confined to the body or obstacle. [Chandra and Thompson, 1992] studied the acoustical scattering from a fluid cylinder with high compressibility contrast by utilizing the *Padé* approximates method to speed up the Neumann series' convergence. Boundary element method (BEM) was employed by [Li et al., 1994] to study the sound radiation and scattering from an arbitrary shape body in vicinity of infinite plane with global boundary conditions by using a half-space Green's function.

An exact solution is proposed by [Buckingham and Tolstoy, 1990] to determine the transmission loss due to a point source of excitation within an ideal acoustical wedge with pressure release boundaries. later [Bishop and Smith, 1997] proposed a null-field, T-matrix method to investigate the acoustic scattering from a motionless elastic cylindrical shell submerged in an isotropic, homogeneous acoustical half-space in the vicinity of rough elastic-fluid interface. Concurrently, [Norris and Osipov, 1997] utilized *Sommerfeld* integral transform to study interactions between two dimensional acoustical waves and semi-infinite wedge shaped elastic plates under the weight of an ideally compressible fluid. Next [Osipov and Norris, 1999] proposed a solution for the wave scattering within an acoustical wedge when it is subjected to an incident plane wave with arbitrary impedance boundary conditions. A Sommerfeld integral and the crucial nullification theorem based on the Mal'uzhinets approach was used. Later, The conjugated infinite element and its associated Green's function are utilized by [Coyette and Van den Nieuwenhof, 2000] to study the acoustical phenomena in half-space domains.

[Costa et al., 2011] used BEM and the Method of Fundamental Solution (MFS) to study the acoustical wave propagation due to a point source within different boundary conditions. The Green function is defined by utilizing the method of images and normal mode approach to simulate different seabed and free surface interface configurations. The 2.5 and 3 dimensional Green's functions are used by [Tadeu et al., 2013] to initiate acoustical wave propagation near two dimensional acoustical wedges. The applicability of the image source Green's functions was tested through numerical case studies involving three-dimensional wave propagation situations. Later, an exact analytical solution has been presented by [Wen-Yu and Ren-He, 2015] to determine the three dimensional pressure field excited by a point source in an acoustical wedge shaped medium. Rigid wall boundary conditions were utilized to approximate the sea floor. [Fawcett, 2014] used Boundary Integral Equation Method (BIEM) to study the acoustical scattering from a finite, rigid cylinder near an interface.

The aforementioned literature review, however, suggests that the acoustic scattering of a fully anisotropic piezo-composite cylindrical shell which is freely submerged in an acoustical wedge is yet to be addressed. Subsequently, specific highlights and novelties of the current work are listed next.

- (i) scattering and radiation of acoustical incident waves from a fluid-filled and submerged shell which is located within an acoustical wedge is considered for the first time with the exact theory of linear elasticity;

- (ii) Fully coupled piezoelectric coatings with exact theory of linear piezoelectricity are included in this configuration.
- (iii) The proposed method is capable of handling full anisotropy of the shell's elastic core and its piezoelectric coatings.
- (iv) Any radially functionally graded profile of the shell's core materials can be treated accordingly.
- (v) Different piezoelectric orientations including radially, axially and circumferentially polarized are within the scope of the proposed approach.

The main objective of the current work is to investigate an analytical solution procedure for three dimensional sound radiation and diffraction from an arbitrary thick-walled, fluid-filled piezo-composite layered cylindrical shell submerged in an acoustical wedge-shaped medium. Two sources of excitation are considered which are an oblique incident plane wave and a normal traction force on the shell's either surfaces. The solution is obtained by using a *transfer matrix* technique along with the *method of images*. The developed procedures can be utilized as a reliable benchmark to validate the accuracy of other semi-analytical or numerical techniques.

The specific organization of this paper is outlined as follows. The mathematical modelling section is developed based on the theory of linear piezoelectricity and acoustical phenomena. This includes the constitutive relations, kinematic assumptions, conservation laws, traction boundaries, solution assembly and ultimate transfer matrix in solid media. The method of images is employed so the shell's traction boundary conditions can be satisfied exactly after which the excitation source are described. Assembly of the ultimate solution are given lastly. The validations of the current work and numerical results for different configurations are given in a companion paper.

5.2 Mathematical Developments

A detailed description of the problem for the associated general configuration of interest is given first. The coupled electro-mechanical domain solutions is performed first in a full-space configuration. The full-space solution, then, is extended to a generalized wedge domain by using the appropriate translational addition theorems.

5.2.1 Problem Description

A three dimensions schematic of a three-layered, hollow FGM cylindrical shell is depicted in [Figure 5.1](#) subjected to traction forces and oblique incident waves. The shell composed of an anisotropic FGM core layer with overall thickness of $h_{el} = r_a - r_s$ which is perfectly connected to its inner and outer FGPM (functionally graded piezoelectric material) sensor/actuator layers with total thickness of $h_s = r_s - r_{in}$ and $h_a = r_{ex} - r_a$, respectively. This composite shell is assumed to be plunged into a perfectly compressible wedge-shaped acoustical medium with the density and sound velocity of ρ_{ex} and c_{ex} , respectively. The shell is filled with another compressible fluid having density and celerity of ρ_{in} and c_{in} , respectively. This structure is excited by an arbitrary non-axisymmetric, radial mechanical forces at its internal f_{in} and/or external f_{ex} surfaces. Furthermore this system can be subjected to an oblique incidence plane wave with an

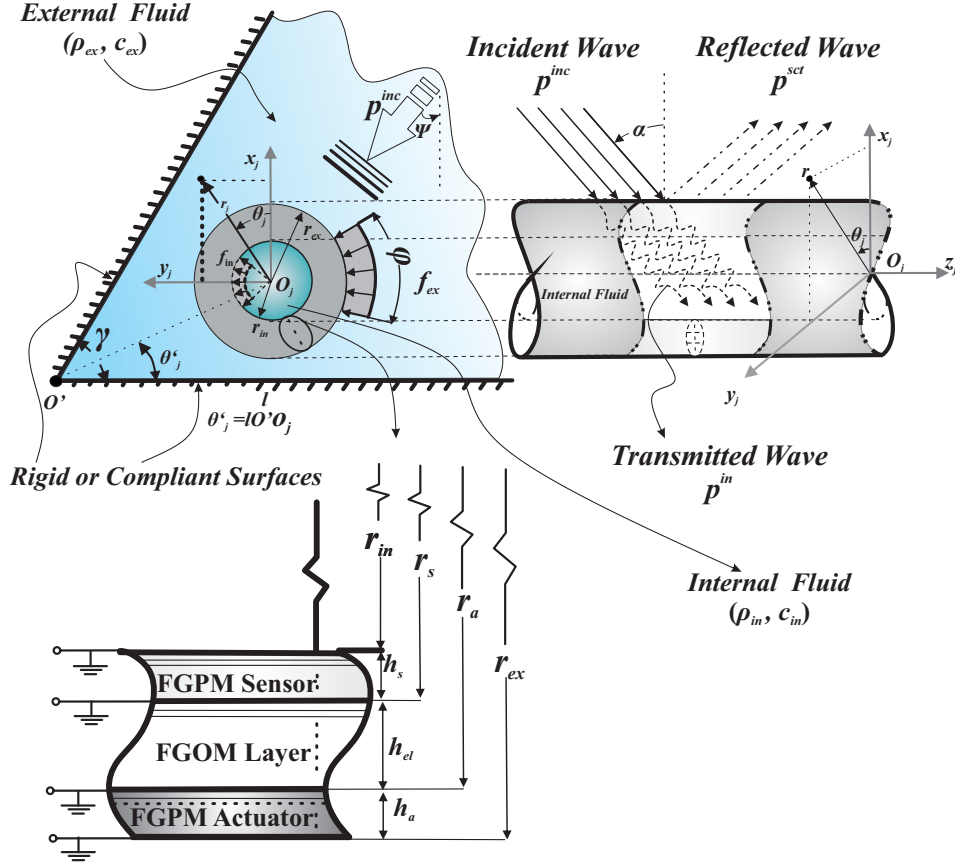


Figure 5.1: Problem Configuration.

arbitrary angle of α as illustrated in Figure 5.1. A Fourier transform is applied invariantly to all time variables $\tilde{\Lambda}(t)$ throughout, such that

$$\Lambda(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\Lambda}(t) e^{i\omega t} dt$$

where the Λ and ω denotes the Fourier transformed of $\tilde{\Lambda}$ and circular frequency, respectively. Furthermore $i = \sqrt{-1}$.

5.2.2 Structural Modelling: Elasticity and Piezoelectricity

5.2.2.1 Constitutive Relations

Anisotropic constitutive relations for piezoelectricity within the linear domain is ([Chopra and Sirohi, 2013])

$$\sigma_{ij}^{pz} = \mathbb{C}_{ijkl}^{pz} \epsilon_{kl}^{pz} - e_{ijk} E_k, \quad (5.1a)$$

$$D_i = e_{ikl} \epsilon_{kl}^{pz} + \kappa_{ij} E_j \quad (5.1b)$$

where e_{ijk} is third order dielectric permittivity tensor having upto 18 independent components. Moreover $\mathbb{C}_{ijkl}^{pz}$ and κ_{ij} are the fourth order stiffness and second order dielectric permittivity tensors, respectively when the permittivity is measured under a zero strain and electric fields condition. They can have maximum of 21 and 6 independent parameters in a triclinic orientation. The number of the associated independent constants, however, are reduced to 9, 5 and 3 correspond-

ing to the fourth order elastic, third order piezoelectric and second order dielectric permittivity tensors, respectively for an orthotropic arrangement. The number of independent constants of an orthotropic piezoelectric material are reduced further to 4, 3 and 2, respectively, in a plain strain situation. The expanded matrix form of Equation 5.1 for an orthotropic orientation are given in Equation G.1 and Equation G.2. Furthermore, the σ_{ij}^{pz} and ϵ_{ij}^{pz} denote the Green-Cauchy stress and strain tensors whereas the E_i and D_i specify the electrical field and electrical displacement vectors, respectively. The electrical field E_i can be defined by using an electrical potential, ϕ , so that

$$E_i = -\phi_{,i} \quad (5.2)$$

where $_{,i} \equiv \vec{\nabla}$ is the forward gradient operator in cylindrical coordinates which is given in Equation H.1. The constitutive relation of a purely elastic material, on the other hand, can be deduced from Equation 5.1a, such that

$$\sigma_{ij}^{el} = \mathbb{C}_{ijkl}^{el} \epsilon_{kl}^{el} \quad (5.3)$$

where $\mathbb{C}_{ijkl}^{el}$ is the fourth order elastic tensor with 21 independent parameters in a triclinic crystal. An orthotropic material, however, is specified by only 9 independent constants. This number is reduced to 4 for a plain strain situation. The expanded matrix form of Equation 5.3 for an orthotropic elastic orientation is given in Equation G.3. Kinematic assumptions are defined next.

5.2.2.2 Kinematic Assumptions

The Green-Cauchy strain tensor ϵ_{ij} is

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{i,j} u_{j,i}) \quad (5.4)$$

where $_{,i} \equiv \vec{\nabla}$ and $_{,j} \equiv \overleftarrow{\nabla}$ are the forward and backward Nabla operators. This tensor is linearized if the third term in Equation 5.4 is neglected for infinitesimal deformations u_i , such that

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \quad (5.5)$$

Hence, the ϵ_{ij} represents both of ϵ_{ij}^{pz} and ϵ_{ij}^{el} . The expanded matrix form of Equation 5.5 is given in Equation H.2. The necessary conservation laws are employed next.

5.2.2.3 Conservation Laws

Conservation of linear momentum in the differential form is

$$\sigma_{ij,j} + f_i - \rho \ddot{u}_i = 0 \quad (5.6)$$

where f_i , ρ and $\ddot{u}_i$ are a body force, unperturbed material density and the Lagrangian particle acceleration, respectively. The first term, divergence of stress tensor which is determined by inner product of the second order stress tensor with the cylindrical gradient operator. This is given in Equation H.1. This equation remains valid for both purely linear elastic and piezo-elastic

materials. In other words, σ_{ij} denotes σ_{ij}^{el} as well as σ_{ij}^{pz} and similarly ρ specifies ρ^{el} and ρ^{pz} . The expanded form of Equation 5.6 can be found in Equation I.1.

The electrostatic charge equilibrium of a piezoelectric material can be expressed as

$$D_{i,i} = Q_f \quad (5.7)$$

where Q_f is the free charge density. Hence the equilibrium conditions for a piezoelectric material are satisfied by implementation of both of Equation 5.6 and Equation 5.7. The expanded form of Equation 5.7 is given in Equation I.2.

5.2.2.4 Traction Boundaries

Continuities of traction force (t_i) and displacement (u_i) vectors are required by mechanical boundary conditions. The traction force can be defined as

$$t_i = \sigma_{ij} n_j \quad (5.8)$$

where t_i and n_i are the Cauchy stress vector at a traction surface and unit-length direction vector, respectively.

The mixed electrical and mechanical boundary conditions, when perfect bounding is assumed, require the continuities of displacement u_i , traction force t_i , electric potential ϕ_i and the electrical displacement D_i . The $e_n \equiv e_r$ if the z -axis and the shell's centre-line coincide. So that

$$\mathbf{t} = \sigma_{rr} \mathbf{e}_r + \sigma_{r\theta} \mathbf{e}_\theta + \sigma_{rz} \mathbf{e}_z \quad (5.9)$$

where e_n is the unit vector in the normal direction to a traction surface.

5.2.3 Solution Assembly in Solid Media

The field's variables can be expanded in terms of trigonometric functions in the circumferential θ and axial z directions in a cylindrical coordinate, such that

$$\mathbf{V}^i = \sum_{n=-\infty}^{\infty} \mathbf{V}_n^i e^{i(\omega t + n\theta - k_z z)}, \quad i = pz, el \quad (5.10)$$

in which $\mathbf{V}^{el} \equiv \mathbf{V}^{el}(r, \theta, z, \omega)$ and $\mathbf{V}^{pz} \equiv \mathbf{V}^{pz}(r, \theta, z, \omega)$ are spatial *state vectors* associated with the purely elastic and piezoelectric layers, respectively. Furthermore ω , n and k_z are angular frequency, circumferential and axial wave number, respectively. In the plain strain case, however, $k_z = 0$. The components of modal state-space vectors of $\mathbf{V}_n^{el}(r, \omega)$ and $\mathbf{V}_n^{pz}(r, \omega)$, are

$$\mathbf{V}_n^{el}(r, \omega) = \begin{Bmatrix} u_{r,n}^{el}(r, \omega) \\ u_{\theta,n}^{el}(r, \omega) \\ u_{z,n}^{el}(r, \omega) \\ \sigma_{rr,n}^{el}(r, \omega) \\ \sigma_{r\theta,n}^{el}(r, \omega) \\ \sigma_{rz,n}^{el}(r, \omega) \end{Bmatrix} \quad \text{and} \quad \mathbf{V}_n^{pz}(r, \omega) = \begin{Bmatrix} u_{r,n}^{pz}(r, \omega) \\ u_{\theta,n}^{pz}(r, \omega) \\ u_{z,n}^{pz}(r, \omega) \\ \sigma_{rr,n}^{pz}(r, \omega) \\ \sigma_{r\theta,n}^{pz}(r, \omega) \\ \sigma_{rz,n}^{pz}(r, \omega) \\ D_{r,n}(r, \omega) \\ \phi_n(r, \omega) \end{Bmatrix}. \quad (5.11)$$

Hence the variables in the z direction vanish in the case of a plain strain situation. After performing monotonous algebraic manipulations associated with substituting modal expanded forms provided in Equation 5.11 into the Equation 5.1, Equation 5.2, Equation 5.3, Equation 5.5, Equation 5.6 and Equation 5.7 along with the employing of the trigonometric functions' orthogonality in the absence of any electrical charge density or body force, $f_i = Q_f = 0$, modal components of the state space become

$$\frac{d\mathbf{V}_n^{el}(r, \omega)}{dr} = \mathbf{G}_n^{el}(r, \omega) \mathbf{V}_n^{el}(r, \omega), \quad n = -\infty, \dots, -1, 0, 1, \dots, +\infty \quad (5.12a)$$

$$\frac{d\mathbf{V}_n^{pz}(r, \omega)}{dr} = \mathbf{G}_n^{pz}(r, \omega) \mathbf{V}_n^{pz}(r, \omega), \quad n = -\infty, \dots, -1, 0, 1, \dots, +\infty. \quad (5.12b)$$

The $\mathbf{G}_n^{el}(r, \omega)$ and $\mathbf{G}_n^{pz}(r, \omega)$ are 6×6 and 8×8 modal coefficient matrices whose elements are given in, respectively, Equation J.1 and Equation J.2. In the plain strain situation, however, $\mathbf{G}_n^{el}(r, \omega)$ and $\mathbf{G}_n^{pz}(r, \omega)$ are 4×4 and 6×6 matrices.

A closed form solution of Equation 5.12 is intricate as the material property varies radially with a general profile. Alternatively an assumed laminated model is used to approximate the continuous variations with small step changes. See, for example, [Ding and Chen, 2001]. Accordingly each elastic and piezoelectric actual layers are assumed to be composed of n_{el} , n_s and n_a perfectly bonded sub-layers with equal thicknesses of $\hat{h}_{el} = (r_a - r_s)/n_{el}$, $\hat{h}_s = (r_s - r_{in})/n_s$ and $\hat{h}_a = (r_{out} - r_a)/n_a$, respectively. Each sub-layer is assumed to have constant values at its mid-surfaces $\bar{r}_k = (r_k + r_{k-1})/2$. This is a reasonable approximation when sub-layers are taken to be very thin, $\hat{h} \ll 1$. Consequently the solution of Equation 5.12 for each sub-layer, simply, turns into

$$\begin{aligned} \mathbf{V}_n^{el}(r, \omega) &= e^{((r-r_{k-1})\mathbf{G}_n^{el}(\bar{r}_k, \omega))} \mathbf{V}_n^{el}(r_{k-1}, \omega), & r_{k-1} \leq r \leq r_k \\ r_{k-1} &= r_s + (k-1)\hat{h}_{el}, \quad r_k = r_s + k\hat{h}_{el} & k = 1, 2, \dots, n_{el}, \end{aligned} \quad (5.13a)$$

$$\begin{aligned} \mathbf{V}_n^s(r, \omega) &= e^{((r-r_{k-1})\mathbf{G}_n^s(\bar{r}_k, \omega))} \mathbf{V}_n^s(r_{k-1}, \omega), & r_{k-1} \leq r \leq r_k \\ r_{k-1} &= r_{in} + (k-1)\hat{h}_s, \quad r_k = r_{in} + k\hat{h}_s, & k = 1, 2, \dots, n_s, \end{aligned} \quad (5.13b)$$

and

$$\begin{aligned} \mathbf{V}_n^a(r, \omega) &= e^{((r-r_{k-1})\mathbf{G}_n^a(\bar{r}_k, \omega))} \mathbf{V}_n^a(r_{k-1}, \omega), & r_{k-1} \leq r \leq r_k \\ r_{k-1} &= r_a + (k-1)\hat{h}_a, \quad r_k = r_a + k\hat{h}_a, & k = 1, 2, \dots, n_a. \end{aligned} \quad (5.13c)$$

The last equations for the k^{th} sub-layer becomes

$$\mathbf{V}_n^{el}(r_k, \omega) = e^{(\mathbf{G}_n^{el}(\bar{r}_k, \omega)\hat{h}_{el})} \mathbf{V}_n^{el}(r_{k-1}, \omega), \quad (5.14a)$$

$$\mathbf{V}_n^s(r_k, \omega) = e^{(\mathbf{G}_n^s(\bar{r}_k, \omega)\hat{h}_s)} \mathbf{V}_n^s(r_{k-1}, \omega) \quad \text{and} \quad (5.14b)$$

$$\mathbf{V}_n^a(r_k, \omega) = e^{(\mathbf{G}_n^a(\bar{r}_k, \omega)\hat{h}_a)} \mathbf{V}_n^a(r_{k-1}, \omega). \quad (5.14c)$$

Global T -matrices are achieved, subsequently, when the continuity of the state-space vector $\mathbf{V}_n^i$ is satisfied for all actual and imaginary interfaces. Then, global transfer matrices relate the $\mathbf{V}_n^i$ at

the outer and inner surfaces, $\mathbf{T}_n$. Thus

$$\mathbf{V}_n^{el}(r_a, \omega) = \mathbf{T}_n^{el}(\omega) \mathbf{V}_n^{el}(r_s, \omega), \quad (5.15a)$$

$$\mathbf{T}_n^{el}(\omega) = \prod_{k=1}^{n_{el}} e^{(\mathbf{G}_n^{el}(\bar{r}_k, \omega) \hat{h}_{el})}, \quad (5.15b)$$

in the core elastic layer, whereas

$$\mathbf{V}_n^s(r_s, \omega) = \mathbf{T}_n^s(\omega) \mathbf{V}_n^s(r_{in}, \omega), \quad (5.15c)$$

$$\mathbf{T}_n^s(\omega) = \prod_{k=1}^{n_s} e^{(\mathbf{G}_n^s(\bar{r}_k, \omega) \hat{h}_s)}, \quad (5.15d)$$

$$\mathbf{V}_n^a(r_{ex}, \omega) = \mathbf{T}_n^a(\omega) \mathbf{V}_n^a(r_a, \omega), \quad (5.15e)$$

$$\mathbf{T}_n^a(\omega) = \prod_{k=1}^{n_a} e^{(\mathbf{G}_n^a(\bar{r}_k, \omega) \hat{h}_a)}, \quad (5.15f)$$

for the piezoelectric coatings in which $\mathbf{T}_n^{el}(\omega)$, $\mathbf{T}_n^s(\omega)$ and $\mathbf{T}_n^a(\omega)$ are 6×6 , 8×8 and 8×8 matrices, respectively. However, $\mathbf{T}_n^{el}(\omega)$, $\mathbf{T}_n^s(\omega)$ and $\mathbf{T}_n^a(\omega)$ are 4×4 , 6×6 and 6×6 matrices, respectively, when a plain strain assumption is made. In the next section the assembly of the ultimate transfer matrix will be discussed.

5.2.4 Assembly of The Ultimate Transfer Matrix

The elastic core and the piezoelectric layer are assumed to be perfectly bounded such that

$$\left\{ \begin{array}{l} u_{r,n}^{el}(r_k, \omega) \\ u_{\theta,n}^{el}(r_k, \omega) \\ u_{z,n}^{el}(r_k, \omega) \\ \sigma_{rr,n}^{el}(r_k, \omega) \\ \sigma_{r\theta,n}^{el}(r_k, \omega) \\ \sigma_{rz,n}^{el}(r_k, \omega) \end{array} \right\} = \left\{ \begin{array}{l} u_{r,n}^k(r_k, \omega) \\ u_{\theta,n}^k(r_k, \omega) \\ u_{z,n}^k(r_k, \omega) \\ \sigma_{rr,n}^k(r_k, \omega) \\ \sigma_{r\theta,n}^k(r_k, \omega) \\ \sigma_{rz,n}^k(r_k, \omega) \end{array} \right\} \quad \text{where } k = s, a \quad (5.16)$$

Field's variables in the z direction can be taken out for an axisymmetric situation. A modal 8×8 transfer matrix, $\mathbf{S}^n$, is emerged by applying the perfect bonding conditions at both core's interfaces with FGPM coatings, $r = r_s, r_a$, such that

$$\mathbf{V}_n^a(r_{ex}, \omega) = \mathbf{S}^n(\omega) \mathbf{X}_n(\omega), \quad (5.17)$$

in which

$$\mathbf{S}_n(\omega) = [\mathbf{T}_n^a(\omega)(:, 1 : 6) \mathbf{T}_n^{el}(\omega) \quad \mathbf{T}_n^a(\omega)(:, 7 : 8)],$$

and

$$\mathbf{X}_n(\omega) = \begin{Bmatrix} \mathbf{V}_n^{el}(r_s, \omega) \\ D_{r,n}^a(r_a, \omega) \\ \phi_n^a(r_a, \omega) \end{Bmatrix}$$

The $\mathbf{T}_n^a(\omega)(:, 1 : 6)$ contains only the first six columns of $\mathbf{T}_n^a(\omega)$ whereas $\mathbf{T}_n^a(\omega)(:, 7 : 8)$ includes only the last two columns of $\mathbf{T}_n^a(\omega)$. Similarly, the ultimate transfer matrix, $\mathbf{W}_n(8 \times 8)$, is obtained by enforcing the traction boundaries at the inner and outer radii ($r = r_{in}, r_{ex}$) with the contained and containing fluids, such that

$$\mathbf{Y}_n(w) = \mathbf{W}_n(\omega) \mathbf{R}_n(\omega), \quad n = -\infty, \dots, 0, \dots, +\infty \quad (5.18)$$

where

$$\mathbf{W}_n(\omega) = \begin{pmatrix} \mathbf{S}_n(\omega)(:, 1 : 6) \mathbf{T}_n^s(\omega)(1 : 6, :) & \mathbf{S}_n(\omega)(:, 7 : 8) \\ \mathbf{T}_n^s(\omega)(7 : 8, :) & \mathbf{0}_{2 \times 2} \end{pmatrix},$$

$$\mathbf{Y}_n(w) = \begin{Bmatrix} \mathbf{V}_n^a(r_{ex}, \omega) \\ D_{r,n}^s(r_s, \omega) \\ \phi_n^s(r_s, \omega) \end{Bmatrix}$$

and

$$\mathbf{R}_n(w) = \begin{Bmatrix} \mathbf{V}_n^s(r_{in}, \omega) \\ D_{r,n}^a(r_a, \omega) \\ \phi_n^a(r_a, \omega) \end{Bmatrix}.$$

Hence, the $\mathbf{T}_n^s(\omega)(7 : 8, :)$ contains only the last two rows of $\mathbf{T}_n^s(\omega)$.

Distribution of material properties through the thickness of a FGM is given by so called *Combination Law*. See, for example, [Jiangong et al., 2009]. Accordingly an effective material property, ${}^k\varrho^{st}$, at k -th layer can be defined with respect to the layer's mid-surface at $r = \bar{r}_k$. Such that

$${}^k\varrho^{st} = {}^kV^{st} \left({}^{in}\varrho^{st} \right) + \left[1 - {}^kV^{st} \right] \left({}^{ex}\varrho^{st} \right), \quad st = \{el, s \text{ and } a\} \quad (5.19)$$

where

$${}^k\varrho^{st} \equiv \varrho^{st}|_{r=\bar{r}_k}, \quad {}^{in}\varrho^{st} \equiv \varrho^{st}|_{r=r_{in}}, \quad {}^{ex}\varrho^{st} \equiv \varrho^{st}|_{r=r_{ex}},$$

in which ${}^kV^{st}$ is a volume fraction whereas ϱ can be any of $C_{ij}^{st}, \rho^{st}, e_{ij}$ or ε_{ij} . Mathematical descriptions of the various loading sources are given next.

5.2.5 Excitation Sources

The excitation sources are divided to the oblique incident plane waves and the traction forces. The plane incident wave is discussed first.

5.2.5.1 Oblique Incident Plane Wave

An acoustical incident plane wave with an arbitrary angle α impinges on the infinite cylinder depicted in [Figure 5.1](#). All derivations, hence forth, are carried out in the cylindrical coordinate (r, θ, z) . An arbitrary incident plane wave is expressed as ([\[Huang, 1970\]](#))

$$\tilde{p}^{inc}(r_j, \theta_j, z_j, t) = \tilde{g} \left(\frac{r_j \cos \theta_j - r_{ex}}{c_{ex}} + t \right) \mathcal{H} \left(\frac{r_j \cos \theta_j - r_{ex}}{c_{ex}} + t \right) \quad (5.20)$$

if it strikes the vertex of the shell at $(r = r_{ex}, \theta = 0)$ first. The $j = 1, 2, \dots, k$, and k is the number of imaginary sinks and sources. Furthermore $\tilde{g}(t)$, $\mathcal{H}(x)$ and c_{ex} are a harmonic function of excitation, Heaviside function and the speed of sound in the external domain, respectively. The Fourier expansion of this incident wave is

$$p^{inc}(r_j, \theta_j, z_j, \omega) = \sum_{n=0}^{\infty} p_{j,n}^{inc}(r_j, \omega) e^{i(\omega t + n\theta_j - k_z^{ex} z_j)} \quad (5.21)$$

in which its modal components, $p_{j,n}^{inc}$, are

$$p_{j,n}^{inc}(r_j, \omega) = p_0 i^n J_n(k_r^{ex} r_j) \quad (5.22)$$

that

$$k_r^{ex} = k_{ex} \cos(\alpha), \quad k_z^{ex} = k_{ex} \sin(\alpha) \quad \text{and} \quad k_{ex} = \frac{\omega}{c_{ex}}$$

where p_0 and α are the incident wave's amplitude and the angle of the incidence with respect to z_j axis. The traction forces are discussed next as the source of excitation.

5.2.5.2 Traction Forces

The normal loads applied at the inner or outer surfaces of the shell are expanded as

$$f_k^j(\theta, \omega) = \sum_{n=-\infty}^{\infty} f_n^{j,k}(\omega) e^{in\theta}, \quad \text{at } r = r_k \quad (5.23)$$

where

$$f_n^{j,k}(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_k(\theta, \omega) e^{-in\theta} d\theta$$

in which $k = in, ex$. Acoustical phenomena within the fluid media is addressed as follows.

5.2.6 Fluid Media

The acoustical pressure perturbation, $p(r, \theta, z, \omega)$, generated within the internal or external fluid is governed by [Equation A.2](#) of [Appendix A](#). Its solution by using a Fourier expansion becomes

$$p(r, \theta, z, \omega) = \sum_{n=-\infty}^{\infty} p_n(r, \omega) e^{i(\omega t + n\theta - k_z^i z)}. \quad (5.24)$$

Inserting Equation 5.24 into Equation A.2 yields

$$\frac{d^2 p_n}{dr^2} + \frac{1}{r} \frac{dp_n}{dr} - \frac{n^2}{r^2} p_n = \alpha_i^2 p_n, \quad i = in, sct, inc, ex \quad (5.25)$$

where $\alpha_i = \sqrt{k_z^2 - (\omega/c_i)^2}$ and $p_n \equiv p_n(r, \omega)$ is the n^{th} modal component of the acoustical pressure. Hence, the wave number in the z direction in both internal and external fluid domains shall be matched, $k_z = k_z^{in} = k_z^{ex}$, because all travelling waves are driven by the plane incident wave. See, for example, [Yuan et al., 2012]. Moreover *in*, *sct*, *inc* and *ex* refer to internal, scattered, incident and total external pressures, respectively. Solutions of Equation 5.25 must be bounded as (i) $r \rightarrow \infty$ for the scattered/radiated pressure field in the fluid surrounding the shell (p^{sct}) and (ii) as $r \rightarrow 0$ for the radiated pressure in the contained fluid (p^{in}). See, for example, [Junger and Feit, 1986]. Consequently

$$p_n^{in}(r_j, \omega) = P_{j,n}^{in}(\omega) J_n(k_r^{in} r_j) \quad \text{and} \quad (5.26a)$$

$$p_n^{sct}(r_j, \omega) = P_{j,n}^{sct}(\omega) H_n(k_r^{ex} r_j) \quad (5.26b)$$

in which $k_r^{in} = \sqrt{(\omega/c_{in})^2 - k_z^2}$ and c_{in} is the speed of sound in the encapsulated fluid. Moreover $J_n(r)$ and $H_n(r)$ are Bessel and Hankel functions of the first kind, respectively. The $P_{j,n}^{in}$ and $P_{j,n}^{sct}$ are n^{th} unknown modal coefficients determined by applying the appropriate boundary conditions. The total acoustical pressure within the containing fluid, p^{ex} , is the sum of the incident term by generated scattered waves. They are given by Equation 5.22 and Equation 5.26b, respectively. Then the external pressure is

$$p_{j,n}^{ex}(r_j, \omega) = p_0 i^n J_n(k_r^{ex} r_j) + P_{j,n}^{sct}(\omega) H_n(k_r^{ex} r_j). \quad (5.27)$$

The corresponding fluid's particle displacement, $\mathbf{u}$, are determined by substituting the acoustical pressure found from Equation 5.26a and Equation 5.27 into Equation A.1b in the frequency domain. Then, the radial component of $\mathbf{u}$ within internal and external fluids becomes

$$u_{j,n}^i(r_j, \omega) = \frac{1}{\omega^2 \rho_i} \frac{\partial p_{j,n}^i(r_j, \omega)}{\partial r_j}, \quad i = in, ex. \quad (5.28)$$

Subsequently the modal components of the internal fluid's particle displacement by substituting Equation 5.26a into Equation 5.28 turns into

$$u_{j,n}^{in}(r_j, \omega) = \frac{P_{j,n}^{in}(\omega) J'_n(k_r^{in} r_j)}{\omega^2 \rho_{in}} \quad (5.29)$$

where “ $\prime$ ” indicates differentiation with respect to the radial coordinate system. Hence these solutions are valid only for a full space configuration with the coordinate centered at O_j . However, the surrounding fluid field in presence of wedge's walls shall be treated differently. A *translational addition theorem* is required to be applied within a *method of images* to take into account for the exact effects of the walls.

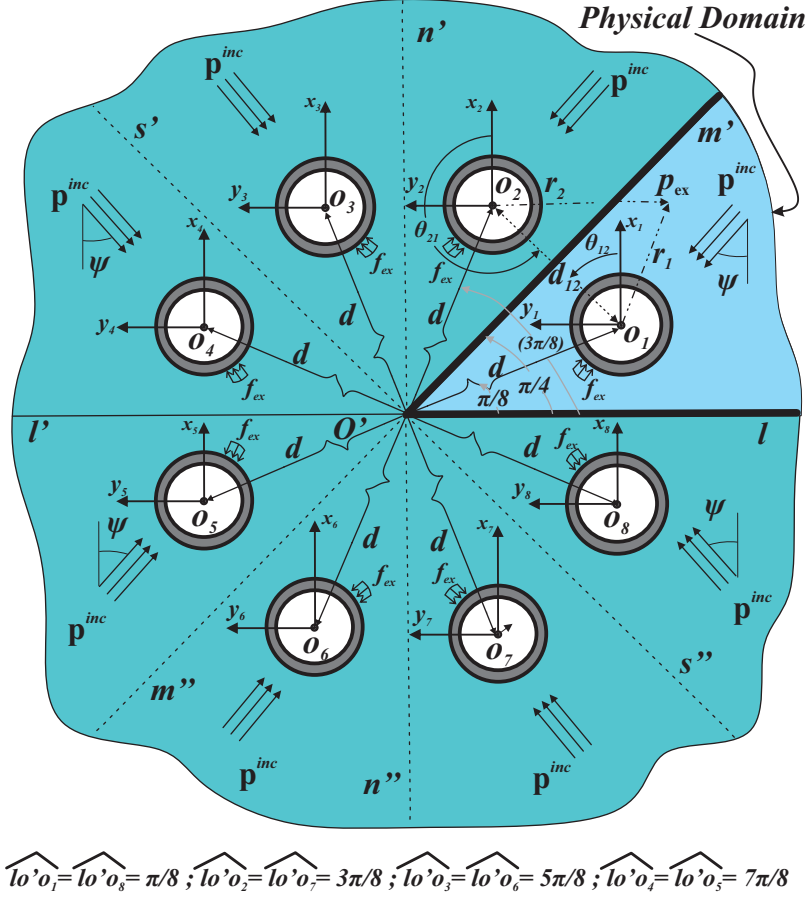


Figure 5.2: Shell position, loading configuration, and measurement points.

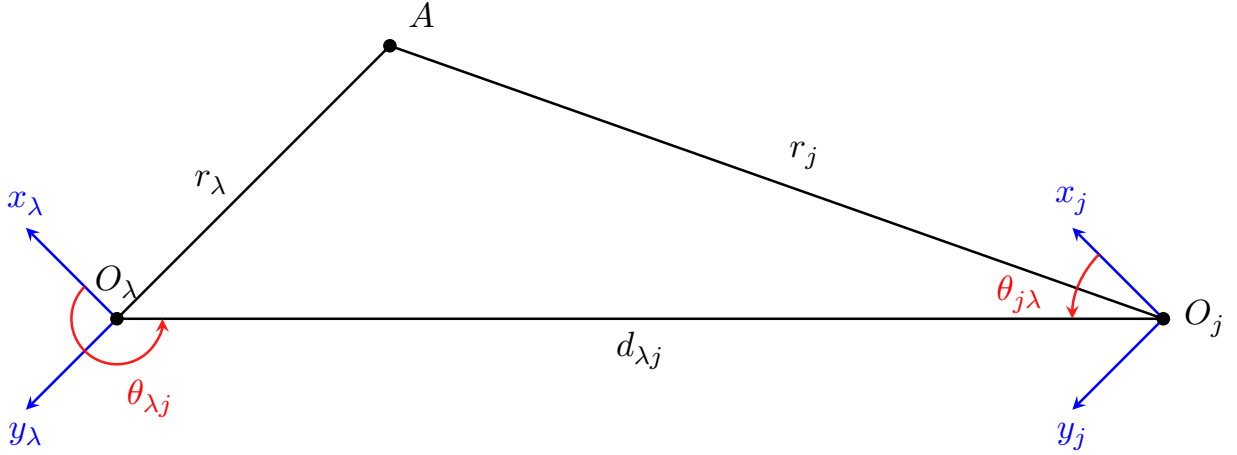


Figure 5.3: Illustrating description of an arbitrary point A with respect to a coordinate origin at O_j and O_λ where $d_{\lambda j}$ is the distance between to centers. The z axis coordinate distinguished by j is translated to the one of λ by using the vector $\mathbf{r}_{\lambda j} \equiv \mathbf{O}_j \mathbf{O}_\lambda (|\mathbf{r}_{\lambda j}| = d_{\lambda j})$, while all the axes remain parallel. z axes are perpendicular and outward with respect to the current plane.

5.2.7 Wedge Domain: Method of Images

An analytical description of total field responses requires an appropriate treatment of exact boundary conditions at the wedge's surfaces through the classical *method of images*. This technique uses imaginary sinks and sources within the imaginary acoustical field to satisfy boundary conditions at the wedge surfaces. Accordingly an imaginary out-of-plane source is assumed at

an equal distance from a wall as a real source. Such a situation is illustrated in [Figure 5.2](#) for a wedge angle of $\pi/4$. First an appropriate configuration of images shall be chosen. A coordinate frame is allocated to each imaginary shell at its center in a way that the axes of all corresponding Cartesian frames remain parallel. Then the Fourier expansion of each incident and scattered values in terms of cylindrical harmonics of λ^{th} frame are expressed in terms of the harmonics of the frame centered at O_j by using a translational addition theorem. The corresponding geometry of this translational addition theorem is depicted in [Figure 5.3](#).

Hence all compensatory imaginary sources must be placed outside the real domain of $(lO'm')$ as as specified in [Figure 5.2](#). A mirror image is considered at a same distance from a wall as its real source. One should note that a sink or a source are chosen when the wall can be assumed as compliant or rigid, respectively.

Cylindrical wave functions must be expressed in terms of multi-cylindrical coordinate systems of $(r_\lambda, \theta_\lambda)$, $\lambda = 1, 2, 3, \dots, k$ before applying boundary conditions at wall surfaces. In other words, any expansions of a field variable at A shall be expressed in terms of cylindrical harmonics with its origin at O_λ . This situation is illustrated schematically in [Figure 5.3](#). This task is possible through the application of the *Numman's* translational addition theorem in the frequency domain for Hankel function of first kind, such that

$$H_n(kr_\lambda)e^{in\theta_\lambda} = \sum_{m=-\infty}^{\infty} J_m(kr_j)H_{n-m}(kd_{\lambda j})e^{i[(n-m)\theta_{\lambda j}+m\theta_j]} \quad (\lambda \neq j). \quad (5.30)$$

The $\theta_{\lambda j}$ is the angle between $\overline{O_\lambda O_j}$ and the x_λ which is the axis of Cartesian coordinate with the origin at O_λ . Furthermore $d_{\lambda j}$ is the absolute distance between origins O_λ and O_j . These geometrical parameters are illustrated in [Figure 5.3](#). See, for example, [[Olver, 2010](#), § 10.23.2] and [[Ivanov, 1970](#)]. Subsequently the cylindrical wave harmonics of outgoing reflected pressure waves in the coordinate $(r_\lambda, \theta_\lambda, z_\lambda)$ can be expressed in terms of those in (r_j, θ_j, z_j) , $(\lambda, j = 1, 2, 3, \dots, k; \lambda \neq j)$ coordinates. Consequently [Equation 5.26b](#) turns into

$$p_{\lambda,j}^{sct} = e^{i(\omega t - k_z z)} \sum_{n,m} P_{\lambda,m}^{sct}(\omega) \begin{cases} H_{m-n}(k_r^{ex} d_{\lambda j}) J_n(k_r^{ex} r_j) e^{i[(m-n)\theta_{\lambda j} + n\theta_j]}, & d_{\lambda j} > r_j \\ H_{m-n}(k_r^{ex} r_j) J_n(k_r^{ex} d_{\lambda j}) e^{i[(m-n)\theta_j + n\theta_{\lambda j}]}, & d_{\lambda j} < r_j \end{cases} \quad (5.31)$$

with application of [Equation 5.30](#) in which $p_{\lambda,j}^i \equiv p_\lambda^i(r_j, \theta_j, z_j, \omega)$ where $i = sct, inc, ex, in$ and $\sum_{n,m} \equiv \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty}$. Furthermore $P_{\lambda,m}^{sct}(\omega)$ is the modal coordinate associated with cylindrical coordinate “ λ ”. A similar procedure can be found in [[Ivanov, 1970](#)] and [[Hasheminejad and Alibakhshi, 2006](#)].

Moreover the λ -th incident wave corresponding to the λ coordinate system is required to be expressed in terms of the cylindrical harmonics at j -th coordinate system. Consequently, by *Neumann's* addition theorem for Bessel functions of the first kind as given in [[Olver, 2010](#),

§ 10.23.2], any real or imaginary incident wave turns into

$$p_{\lambda,j}^{inc} = \sum_{n=-\infty}^{\infty} i^n J_n(k_r^{ex} r_j) e^{-in\beta_\lambda} e^{-ik_r^{ex} d_{\lambda j} \cos(\beta_j - \theta_{\lambda j})} e^{i(\omega t + n\theta_j - k_z^{in} z)} \quad (5.32)$$

in which

$$P_{j,n}^{inc} = i^n J_n(k_r^{ex} r_j) e^{-in\beta_\lambda} e^{-ik_r^{ex} d_{\lambda j} \cos(\beta_j - \theta_{\lambda j})} \quad (5.33)$$

where $j, \lambda=1,2$, and $\beta_1 = \psi + \pi$, $\beta_2 = -\psi$, ψ is an incident angle as shown in [Figure 5.1](#). The $\theta_{\lambda j}$ is the angle between $\overline{O_\lambda O_j}$ and the x_λ whereas $d_{\lambda j}$ is the absolute distance between origins O_λ and O_j . Obviously $d_{\lambda j} = 0$ if $\lambda = j$. See the schematic presentation given in [Figure 5.3](#).

Hence the total acoustical pressure at any given point within the acoustical wedge medium is the summation of all scattered and incident terms due to all real and imaginary sources and sinks. See, for example, [Figure 5.2](#). Thereby, the total external acoustical pressure, in terms of (r_j, θ_j, z_j) , becomes

$$p_j^{ex} = p_{j,j}^{sct} + \sum_{\lambda=1, \lambda \neq j}^k p_{\lambda,j}^{sct} + \sum_{\lambda=1}^k p_{\lambda,j}^{inc} \quad (5.34)$$

in which

$$p_{j,j}^{sct} = \sum_{n=-\infty}^{\infty} P_{j,n}^{sct}(\omega) H_n(k_r^{ex} r_j) e^{i(\omega t + n\theta_j - k_z^{ex} z_j)}. \quad (5.35)$$

The corresponding modal components of the total external pressure given in [Equation 5.34](#) turns into

$$\begin{aligned} p_{j,n}^{ex}(r_j, \omega) &= P_{j,n}^{sct}(\omega) H_n(k_r^{ex} r_j) + \\ &\sum_{\lambda=1}^k i^n J_n(k_r^{ex} r_j) e^{-in\beta_\lambda} e^{-ik_r^{ex} d_{\lambda j} \cos(\beta_j - \theta_{\lambda j})} + \sum_{\lambda=1, \lambda \neq j}^k \sum_{m=-\infty}^{\infty} \\ &P_{\lambda,m}^{sct}(\omega) \begin{cases} H_{m-n}(k_r^{ex} d_{\lambda j}) J_n(k_r^{ex} r_j) e^{i[(m-n)\theta_{\lambda j} + n\theta_j]}, & d_{\lambda j} > r_j \\ H_{m-n}(k_r^{ex} r_j) J_n(k_r^{ex} d_{\lambda j}) e^{i[(m-n)\theta_j + n\theta_{\lambda j}]}, & d_{\lambda j} < r_j \end{cases} \end{aligned} \quad (5.36)$$

by using [Equation 5.33](#) and [Equation 5.31](#). Fluid particle displacements are deduced from the corresponding acoustical pressures simply by using [Equation 5.28](#). So that

$$\begin{aligned} u_{j,n}^{ex}(r_j, \omega) &= \frac{1}{\omega^2 \rho_{ex}} (P_{j,n}^{sct}(\omega) H'_n(k_r^{ex} r_j) + \\ &\sum_{\lambda=1}^k i^n J'_n(k_r^{ex} r_j) e^{-in\beta_\lambda} e^{-ik_r^{ex} d_{\lambda j} \cos(\beta_j - \theta_{\lambda j})}) + \frac{1}{\omega^2 \rho_{ex}} \sum_{\lambda=1, \lambda \neq j}^k \sum_{m=-\infty}^{\infty} \\ &P_{\lambda,m}^{sct}(\omega) \begin{cases} H_{m-n}(k_r^{ex} d_{\lambda j}) J'_n(k_r^{ex} r_j) e^{i[(m-n)\theta_{\lambda j} + n\theta_j]}, & d_{\lambda j} > r_j \\ H'_{m-n}(k_r^{ex} r_j) J_n(k_r^{ex} d_{\lambda j}) e^{i[(m-n)\theta_j + n\theta_{\lambda j}]}, & d_{\lambda j} < r_j \end{cases} \end{aligned} \quad (5.37)$$

Application of the pertinent boundary conditions at the shell's inner and outer surfaces is presented

next.

5.2.8 Shell's Traction Boundary Conditions

Continuities at traction surfaces are applied to the Cauchy stress vector $\mathbf{t}_r = (\sigma_{rr}, \sigma_{rz}, \sigma_{r\theta})^T$ and only the radial component of displacement vector u_r , such that

$$\begin{Bmatrix} u_{r,n}^{k,(j)}(r_k, \omega) \\ \sigma_{rr,n}^{k,(j)}(r_k, \omega) - \beta f_n^{j,k}(\omega) \\ \sigma_{r\theta,n}^{k,(j)}(r_k, \omega) \\ \sigma_{rz,n}^{k,(j)}(r_k, \omega) \end{Bmatrix} = \begin{Bmatrix} u_{j,n}^k(r_k, \omega) \\ \beta p_{j,n}^k(r_k, \omega) \\ 0 \\ 0 \end{Bmatrix} \quad (5.38)$$

where $k = in, ex$ and $\beta = 1$ for $k = in$ whereas $\beta = -1$ once $k = ex$. Electrical boundary conditions of being *short-circuited* in the FGPM sensor and actuator layers are

$$\phi_n(r_j, \omega) = 0 \quad \text{at} \quad r_j = r_{in}, r_s, r_a \text{ and } r_{ex}. \quad (5.39)$$

See, for example, [Bian et al., 2006].

Finally, by direct implementation of mechanical and electrical boundary conditions Equation 5.38 and Equation 5.39 into the ultimate transfer matrix Equation 5.18, after some tedious rearrangement, one can obtain the following matrix form for the transformed modal coefficients, such that

$$\mathbf{Ax} = \mathbf{B} \quad (5.40)$$

where

$$\begin{aligned} \mathbf{x} = [& \dots, u_{r,-1}^{in,(j)}, u_{r,0}^{in,(j)}, u_{r,+1}^{in,(j)}, \dots, u_{\theta,-1}^{in,(j)}, u_{\theta,0}^{in,(j)}, u_{\theta,+1}^{in,(j)}, \dots, u_{z,-1}^{in,(j)}, u_{z,0}^{in,(j)}, u_{z,+1}^{in,(j)}, \\ & \dots, \sigma_{rr,-1}^{in,(j)}, \sigma_{rr,0}^{in,(j)}, \sigma_{rr,+1}^{in,(j)}, \dots, \sigma_{r\theta,-1}^{in,(j)}, \sigma_{r\theta,0}^{in,(j)}, \sigma_{r\theta,+1}^{in,(j)}, \dots, \sigma_{rz,-1}^{in,(j)}, \sigma_{rz,0}^{in,(j)}, \sigma_{rz,+1}^{in,(j)}, \dots, p_{j,-1}^{in} \\ & \dots, \underbrace{\sigma_{rr,-1}^{in,(j)}, \sigma_{rr,0}^{in,(j)}, \sigma_{rr,+1}^{in,(j)}}_{r=r_{in}}, \underbrace{\sigma_{r\theta,-1}^{in,(j)}, \sigma_{r\theta,0}^{in,(j)}, \sigma_{r\theta,+1}^{in,(j)}}_{r=r_s}, \underbrace{\sigma_{rz,-1}^{in,(j)}, \sigma_{rz,0}^{in,(j)}, \sigma_{rz,+1}^{in,(j)}}_{r=r_s}, \dots, u_{r,-1}^{out,(j)}, u_{r,0}^{out,(j)}, u_{r,+1}^{out,(j)}, \\ & \dots, u_{\theta,-1}^{out,(j)}, u_{\theta,0}^{out,(j)}, u_{\theta,+1}^{out,(j)}, \dots, u_{z,-1}^{out,(j)}, u_{z,0}^{out,(j)}, u_{z,+1}^{out,(j)}, \dots, \sigma_{rr,-1}^{out,(j)}, \sigma_{rr,0}^{out,(j)}, \sigma_{rr,+1}^{out,(j)}, \\ & \dots, \sigma_{r\theta,-1}^{out,(j)}, \sigma_{r\theta,0}^{out,(j)}, \sigma_{r\theta,+1}^{out,(j)}, \dots, \sigma_{rz,-1}^{out,(j)}, \sigma_{rz,0}^{out,(j)}, \sigma_{rz,+1}^{out,(j)}, \dots, p_{j,-1}^{sct}, p_{j,0}^{sct}, p_{j,1}^{sct}, \dots, \\ & \dots, \underbrace{D_{r,-1}^j, D_{r,0}^j, D_{r,+1}^j}_{r=r_a}, \underbrace{D_{r,-1}^j, D_{r,0}^j, D_{r,+1}^j}_{r=r_{ex}}, \dots]^T \end{aligned}$$

and

$$\mathbf{B} = [\dots, p_{j,-1}^{inc} + f_{-1}^{(j),ex}, p_{j,0}^{inc} + f_0^{(j),ex}, p_{j,1}^{inc} + f_1^{(j),ex}, \dots, f_{-1}^{(j),in}, f_0^{(j),in}, f_1^{(j),in}, \dots]^T.$$

However $\mathbf{x}$ turns into

$$\mathbf{x} = [\dots, u_{r,-1}^{in,(j)}, u_{r,0}^{in,(j)}, u_{r,+1}^{in,(j)}, \dots, u_{\theta,-1}^{in,(j)}, u_{\theta,0}^{in,(j)}, u_{\theta,+1}^{in,(j)}, \dots, \sigma_{rr,-1}^{in,(j)}, \sigma_{rr,0}^{in,(j)}, \sigma_{rr,+1}^{in,(j)}, \dots, \sigma_{r\theta,-1}^{in,(j)}, \sigma_{r\theta,0}^{in,(j)}, \sigma_{r\theta,+1}^{in,(j)}, \dots, u_{r,-1}^{out,(j)}, u_{r,0}^{out,(j)}, u_{r,+1}^{out,(j)}, \dots, u_{\theta,-1}^{out,(j)}, u_{\theta,0}^{out,(j)}, u_{\theta,+1}^{out,(j)}, \dots, p_{j,-1}^{in}, p_{j,0}^{in}, p_{j,1}^{in}, \dots, \overbrace{D_{r,-1}^j, D_{r,0}^j, D_{r,+1}^j}^{r=r_{in}}, \dots, \overbrace{D_{r,-1}^j, D_{r,0}^j, D_{r,+1}^j}^{r=r_s}, \dots, \overbrace{D_{r,-1}^j, D_{r,0}^j, D_{r,+1}^j}^{r=r_a}, \dots, \overbrace{D_{r,-1}^j, D_{r,0}^j, D_{r,+1}^j}^{r=r_{ex}}, \dots, p_{j,-1}^{sct}, p_{j,0}^{sct}, p_{j,1}^{sct}, \dots, \sigma_{rr,-1}^{out,(j)}, \sigma_{rr,0}^{out,(j)}, \sigma_{rr,+1}^{out,(j)}, \dots, \sigma_{r\theta,-1}^{out,(j)}, \sigma_{r\theta,0}^{out,(j)}, \sigma_{r\theta,+1}^{out,(j)}, \dots]^T$$

in a plain strain situation. The solution is completed here. Concluding remarks are given next.

5.3 Conclusions

A fluid-filled, anisotropic cylindrical shell is considered which is submerged in an acoustical wedge. The shell composed of three graded triclinic layers in which the inner and outer layers are piezoelectric. The system can be excited with a plane acoustical wave or a surface traction force(s) in the frequency domain. The solution to the system's response is investigated analytically. A classical transfer matrix approach is applied to the shell in the full-space to assemble the global *T-Matrix*. A Taylor's series approximation is applied to find a closed form solution in full-space. The classical method of images through a translational addition theorem is utilized to enforce the pertinent boundary conditions at the wedge's surfaces. The proposed model is capable of handling any wedge angle of $\alpha = 2\pi/n$ where n is a natural number simply by assuming a proper configuration of imaginary sinks and sources. Furthermore the solution is applicable to any level of anisotropy in the shell's core material as well as different piezoelectric polarization of its inner and outer layers. Numerical verification of the current solutions against FEM simulations along with the number of numerical experiments are given in a companion paper.

Chapter 6

On elasto-acoustic of a fluid-filled anisotropic FGM cylindrical shell in an acoustical wedge having anisotropic piezo-FGM coatings II:

Numerical Results

Theoretical developments associated with elasto-acoustic of an anisotropic cylindrical shell in an acoustical wedge is given in the first companion paper. The reliability of the proposed solutions and associated computer codes are cross-checked rigorously with the results of a multi-physics, commercial FE (Finite Element) package. Comparisons and validations are performed when the shell is excited by a concentrated traction force or a plane acoustical incident wave. Numerical case studies are carried out for an oil-filled shell which is submerged in seawater for various FGM (Functionally Graded Material) and FGPM (Functionally Graded Piezoelectric Material) shell's material profiles. Effects of material properties' profile, wedge angle as well as the excitation's sources and frequencies are investigated on the system's harmonic response. The selected excitation sources are a plane incident wave and mechanical traction forces with three different configurations in distribution of the surface forces. Three acoustical wedge angles of π (half-space), $\pi/2$ (quarter-space) and $\pi/4$ are examined. Specific emphasis is given to quarter- and half- space configurations due to their wider range of applications. Moreover two arrangements of shell's layers are considered which includes a two layered FGPM-FGM and a three layered FGPM-FGM-FGPM shells. Furthermore, three different graded profiles for each of the FGM and FGPM layers are chosen. The corresponding radial variations have metal-rich, ceramic-rich and linear profiles. Effects of this cylindrical obstacle and wedge's walls on the spatial distribution of acoustical pressure field are investigated. Physical phenomena of practical importance are observed such as the focusing of acoustical energy when a wave is incident on a shell.

Introduction

The mathematical formulations for elasto-acoustic response of a fluid-filled anisotropic cylindrical shell submerged in an acoustical wedge is detailed in the first part of this paper. This part, however, has two specific objectives. The first one is to validate the proposed solutions and associated computer codes with a commercial FE package. The second objective is about conducting a number of numerical experiments for a variety of material, excitation and geometrical configurations. Specific highlights of this part are listed next.

- High level of agreement between the results of FEM with the outputs of the current codes are observed especially at relatively low frequencies.

- FEM results at higher frequencies becomes very expensive and numerically unstable whereas the proposed approach remains stable.
- Computational challenges with the infinite domains are non-existent in current analytical approach
- The shell structure acts as a focusing lens when impinged by an acoustical wave at higher ultrasonic frequency range.
- The spatial distribution patterns of the stress concentration mainly is a function of wedge angle when rigid body motion is not suppressed.
- Generally a metal rich radially graded profile leads to a higher level of the stress concentration.
- The absolute maximum stress concentration accrues in direction of the wedge angle's bisector.

An extensive series of numerical comparisons are performed first against Finite Element Method (FEM) simulations to ensure the accuracy and robustness of the current approach (§ 6.1). Limitations of the classical FEM approach are discussed briefly. Various arrangements of materials are considered. The corresponding material properties of the shell's elastic core and its piezoelectric coatings as well as the acoustical properties of the containing and encapsulated fluid media are given first (§ 6.2). Then different geometrical configurations of the shell's relative arrangement with respect to a wedge's boundaries are described also in the same section. Numerical experiments are performed for different cases and relevant physical implications are discussed subsequently in § 6.3. Acoustical radiation from the shell due to surface traction forces are discussed and considered after which scattering of the incident wave from the cylindrical shell is investigated. The main concluding remarks are listed finally.

6.1 Validations

The objective of this section is to rigorously cross-check and validate the stability and accuracy of the current 3D computer codes for different loading and geometrical configurations by making comparisons with the corresponding COMSOL MULTIPHYSICS[®] results. COMSOL is chosen due to its well-known capacity in handling multiphysics problems. More specifically, defining an incident acoustic perturbation in the frequency domain is straightforward by using COMSOL.

A paralleled MATLAB code is developed to construct the ultimate transfer matrix W_n ($n = -N, \dots, -1, 0, 1, \dots, +N$) and the coefficient matrices, $\mathbf{A}$ and $\mathbf{B}$, given in Equations 18 and 40, respectively. This code solves Equation 40 for $n = -N, \dots, -1, 0, 1, \dots, +N$. The convergence of numerical results is assessed through a simple trial and error procedure to determine the minimum number of required modes. The choice of $N = 50$ leads to a consistently stable numerical behaviour for a given range of frequencies of interest.

Computations are performed on a cluster of Intel[®] Xeon[®] processor E7-8893 v4 desktop computer which benefits from paralleled core technology for multi-threaded applications. This code can exploit the full processing power of multi-core desktops through executing on 12 MATLAB

workers running locally by utilization of MATLAB's Parallel Computing Toolbox. Before addressing the numerical results we shall validate the accuracy of the proposed solution. Validations of the current code's results when traction force(s) are the source of excitation are given after which the comparisons are performed for the scattering due to a plane incident wave.

There is, however, a numerical stability issue inherent with the use of FEM associated with the mesh size relative to the smallest wave length. In other words the degrees of freedom are required to be increased as the frequency are targeted. The subsequent increase in the computational cost is significant. The other major problem associated with the infinite physical domain which is required to be truncated in FEM simulations. To address this issue, COMSOL uses a perfectly matched layer (PML) which is not a boundary condition, but rather a domain that absorbs most of the outgoing waves' energy. See, for more details, [Comsol, 2005]. However, always a partial reflection of waves occurs from PML layer which is mostly due to the discretization of the domain.

A PML boundary is employed in the FEM modeling of the truncated acoustic domain in order to maximize the energy absorption of outgoing wave energy at the PML boundary, without any impedance mismatch causing spurious reflections at the far-field boundaries. The PML is positioned 10 meters away from the center of the cylinder. Around 3792 mapped mesh points are used to model the PZT and elastic layers. Moreover, 22828, 261632 and 258342 free triangular elements were employed to the internal fluid medium, external acoustic half space medium and PML, respectively. A mesh study was conducted to achieve a reasonable accuracy in the results.

Level of errors in FEM results rises as frequency increases if the degrees of freedom remains the same. The current analytical approach, on the other hand, does not have any difficulty in dealing with high frequency or infinite physical domains.

Oil-filled, piezo-composite hollow cylinders submerged in seawater is considered regardless. Acoustical properties of these fluids are given in Table 6.1.

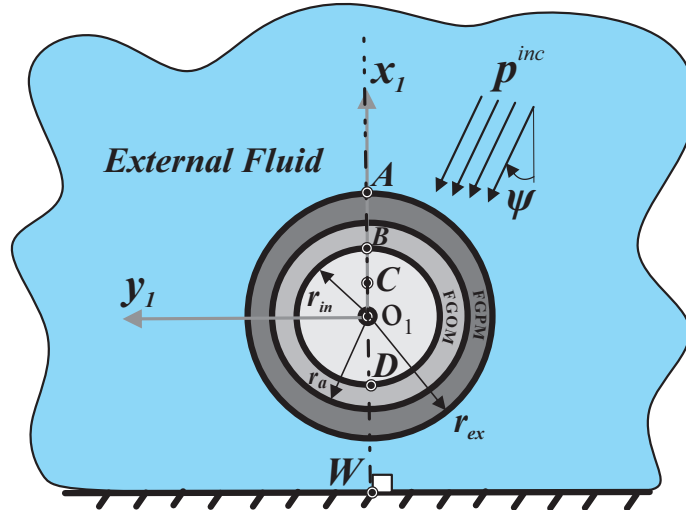


Figure 6.1: Shell position in the external fluid and measurement points A through D and W for the three dimensional acoustical scattering angles are given in rads.

6.1.1 Scattering of Incident Waves

A cylindrical shell is positioned in an acoustical half space, $\gamma = \pi$, with rigid boundaries $d = 2.1 \text{ m}$ subjected to an incident plane wave with $\psi = \alpha = 0; p_0 = 1 \text{ N}$. See Figure 6.1

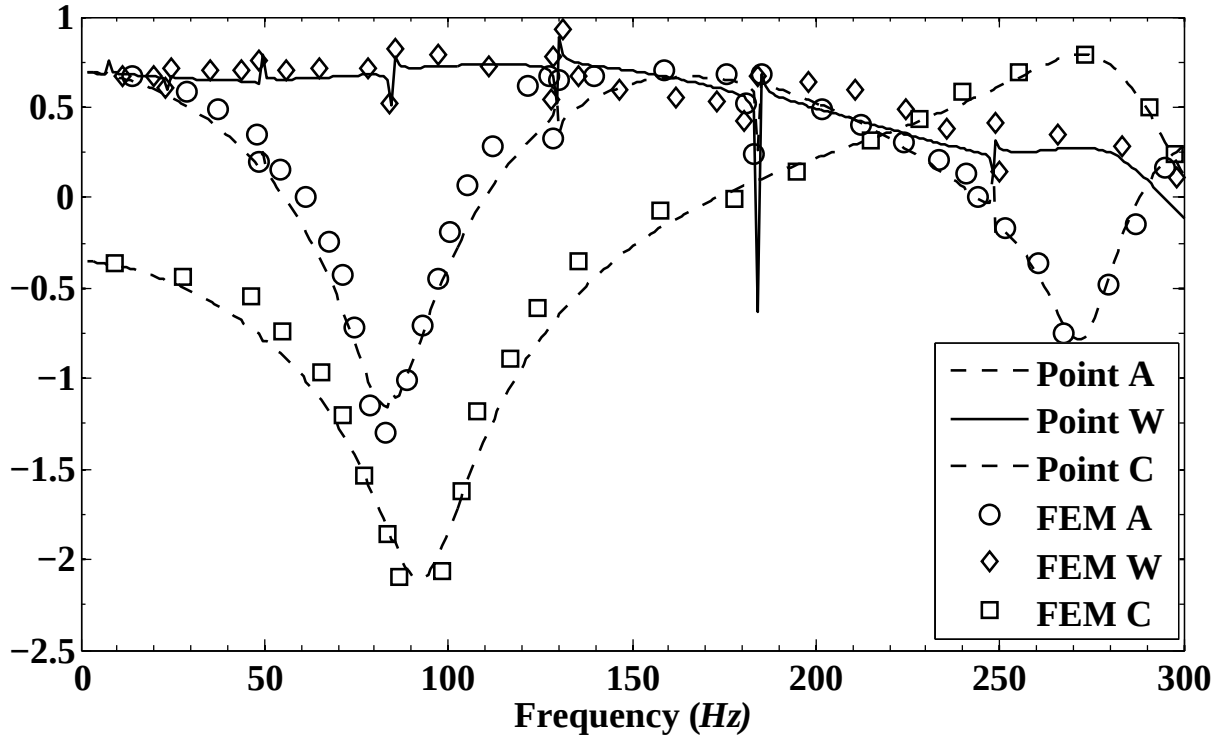


Figure 6.2: Showing the acoustically transmitted pressure (pa) at point, C, the radiated pressure on the wall at point, W, and the total on-surface acoustical pressure at the apex of the cylinder. The cylinder is the acoustic half space illustrated in Figure 6.1 and it is subjected to an external plane wave.

and Figure 6.1 given in the first companion and current papers, respectively. The $n_{el} = n_s = n_a = 10$ sub-layers are used to construct sub-T-matrices of each sub-layer to achieve a satisfactory convergence stability in the shell's ultimate transfer matrix. Subsequently the quantities of interest are calculated at a number of given positions which are specified in Figure 6.1. They are at the observation points

- C: transmitted acoustic pressure, $p^{\text{in}}(r = 1 \text{ m}, \theta = 0, z = 0, \omega) \text{ pa}$,
- W: the radiated pressure attached to the wall, $p^{\text{ex}}(r = 2.1 \text{ m}, \theta = \pi, z = 0, \omega) \text{ pa}$,
- A: the total on-surface acoustic pressure located at the apex of the cylinder, $p^{\text{ex}}(r = 2 \text{ m}, \theta = 0, z = 0, \omega) \text{ pa}$
- B and D: the radial structural displacements at the inner surfaces of the cylinder, $u_r(r = 1.9 \text{ m}, \theta = 0, z = 0, \omega) \text{ m}$ and $u_r(r = 1.9 \text{ m}, \theta = \pi, z = 0, \omega) \text{ m}$.

Comparisons between results from the current approach and the FEM code given in Figure 6.2 and Figure 6.3 in which a reasonable agreement is observed.

6.1.2 Traction Forces

In plane traction forces are applied leading to a plain strain (2D) situation. The corresponding dimensions are $r_{in} = 0.98 \text{ m}$, $r_s = 0.99 \text{ m}$, $r_a = 1 \text{ m}$, $r_{ex} = 1.01 \text{ m}$. The piezoelectric sensor and actuator are assumed to be fabricated from non-graded PZT4, $V_s = V_a = 1$, with a core layer

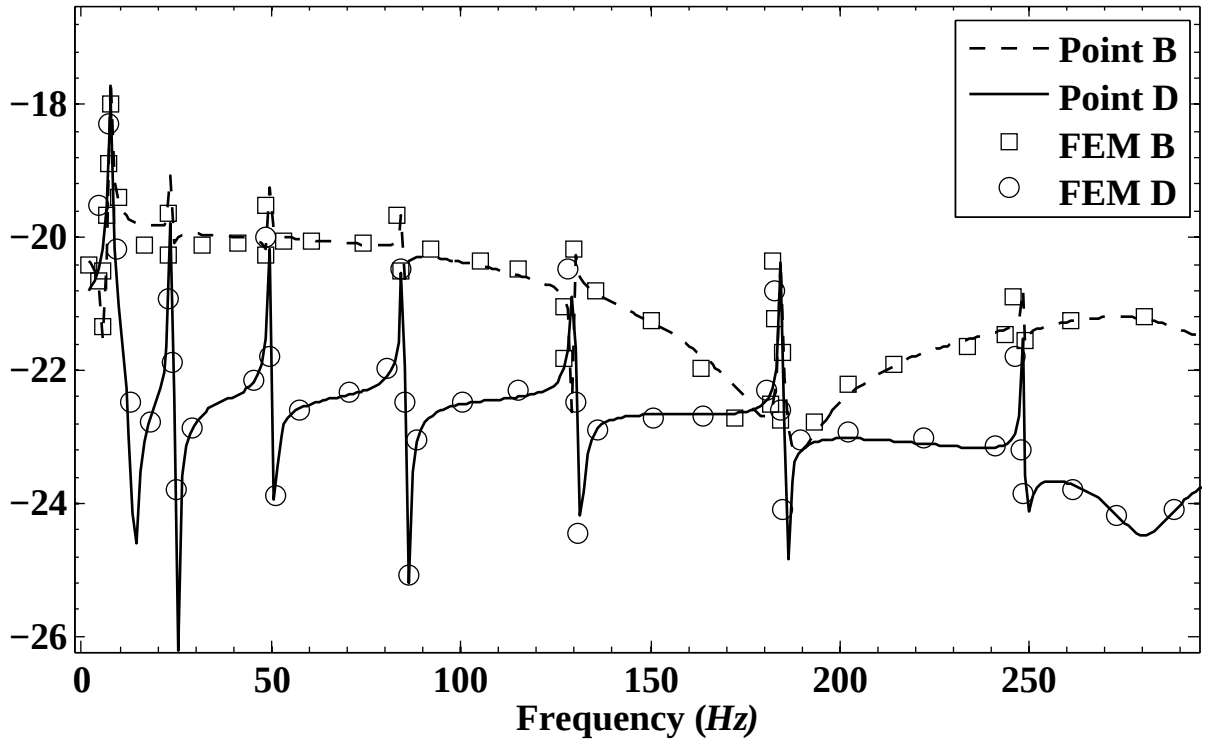


Figure 6.3: Frequency response of shell's radial displacements (μm) at the inner surfaces of the cylinder, points B and D when the shell positioned in an acoustical half space when the shell is acted upon by an incident plane wave.

made of homogeneous metal-based material, $V_{el} = 1$. The material properties of the host layer is given in [Table 6.4](#). The cylinder is positioned in an acoustical half space with $d = 1.1 \text{ m}$, $\gamma = \pi$, which is subjected to a concentrated load applied radially at the shell's outer surface where $f_{ex}(\theta, \omega) = F_0 / [\exp(in\theta_1)(2\pi r_{ex})]$ with $F_0 = 1 \text{ N}$ applied at $\theta_1 = 0$. This situation is illustrated in [Figure 6.4](#). The convergence requirements in constructing the ultimate transfer matrix is satisfied by having $n_a = n_{el} = n_s = 10$ sub-layers. The system's responses in the frequency domain are given at

- O_1 : cylinder's center-point pressure $p_{in}(r = 0 \text{ m}, \theta = 0, \omega) \text{ pa}$,
- W' : the radiated pressure attached to the wall $p_{ex}(r = 1.1 \text{ m}, \theta = 3\pi/2, \omega) \text{ pa}$,
- F' : the radiated acoustic pressure at a certain distance from the shell $p_{ex}(r = 3 \text{ m}, \theta = \pi/2, \omega) \text{ pa}$
- D' : the radial structural displacement at the outer surface of the PZT actuator layer $u_r(r = 1.01 \text{ m}, \theta = \pi/2, \omega) \text{ m}$

which are illustrated in [Figure 6.4](#). A satisfactory agreement is observed between the results from the current approach and the FEM simulations. Slight deviations, however, start to appear as higher frequencies are targeted.

A shell with the same material arrangement and identical loading conditions are considered but in an acoustical quarter space wedge ($O' : d = 1.555 \text{ m}, \gamma = \pi/2$). See the corresponding schematic given in [Figure 6.4](#). The pressures are calculated at the shell's center-point $p_{in}(r = 0 \text{ m}, \theta =$

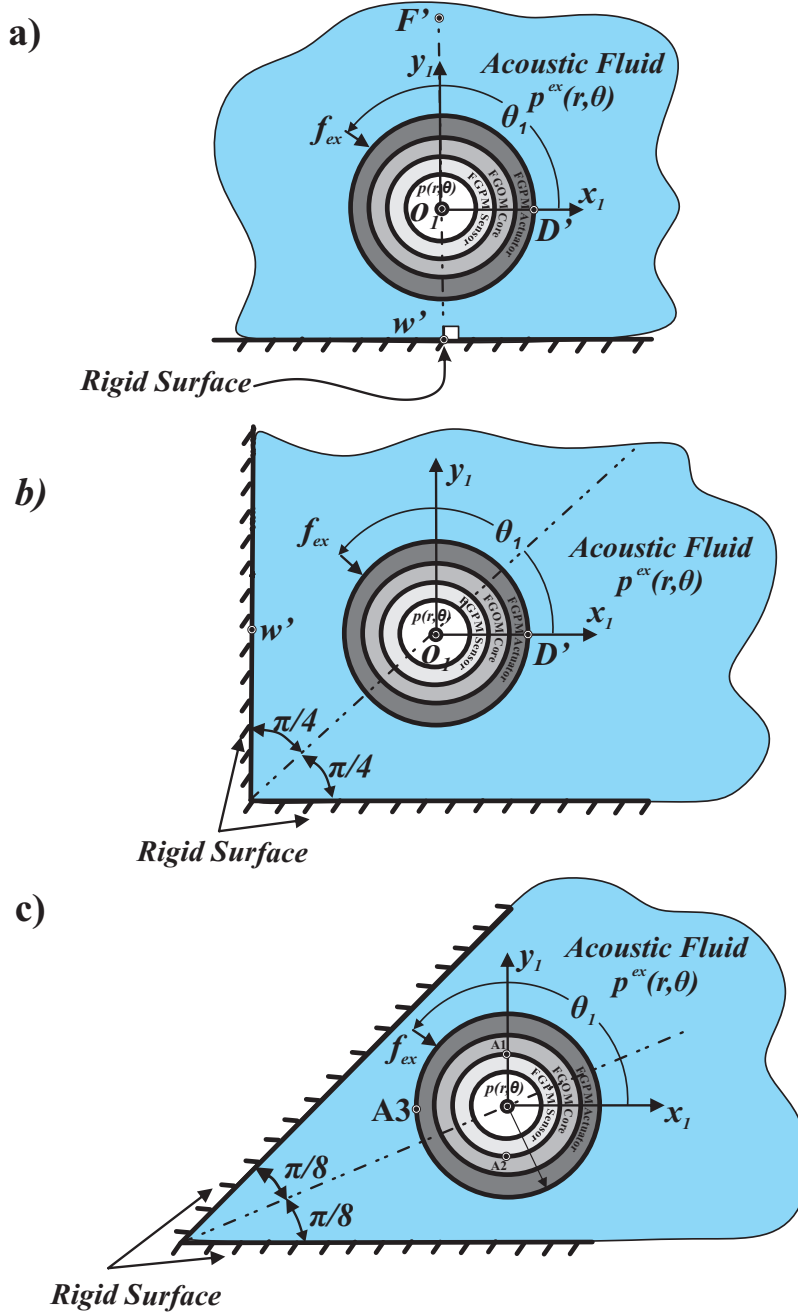


Figure 6.4: Shell position, loading configuration, and measurement points. (a) Half space (180-degree acoustical wedge) (b) quarter space (90-degree acoustical wedge) (c) 45-degree acoustical wedge.

$0, \omega)pa$ and on the rigid wall's surface $p_{ex}(r = 1.1 \text{ m}, \theta = \pi, \omega)pa$ corresponding to the observation points O_1 and W' , respectively. Furthermore, the surface pressure $p_{ex}(r = 1.01 \text{ m}, \theta = 0, \omega)pa$ and the radial displacement at the outer surface of the PZT actuator layer $U_r(r = 1.01 \text{ m}, \theta = 0, \omega) \text{ m}$ are estimated at the observation point D' . The corresponding results are given in Figure 6.6. A good agreement between the current and FEM simulations is observed, especially, at lower frequencies.

The same arrangement is considered again but this time in a 45° acoustical wedge ($O' : d = 2.874 \text{ m}, \gamma = \pi/4$) located at $(\theta_1 = \pi/8)$. See the schematic given in Figure 6.4. The

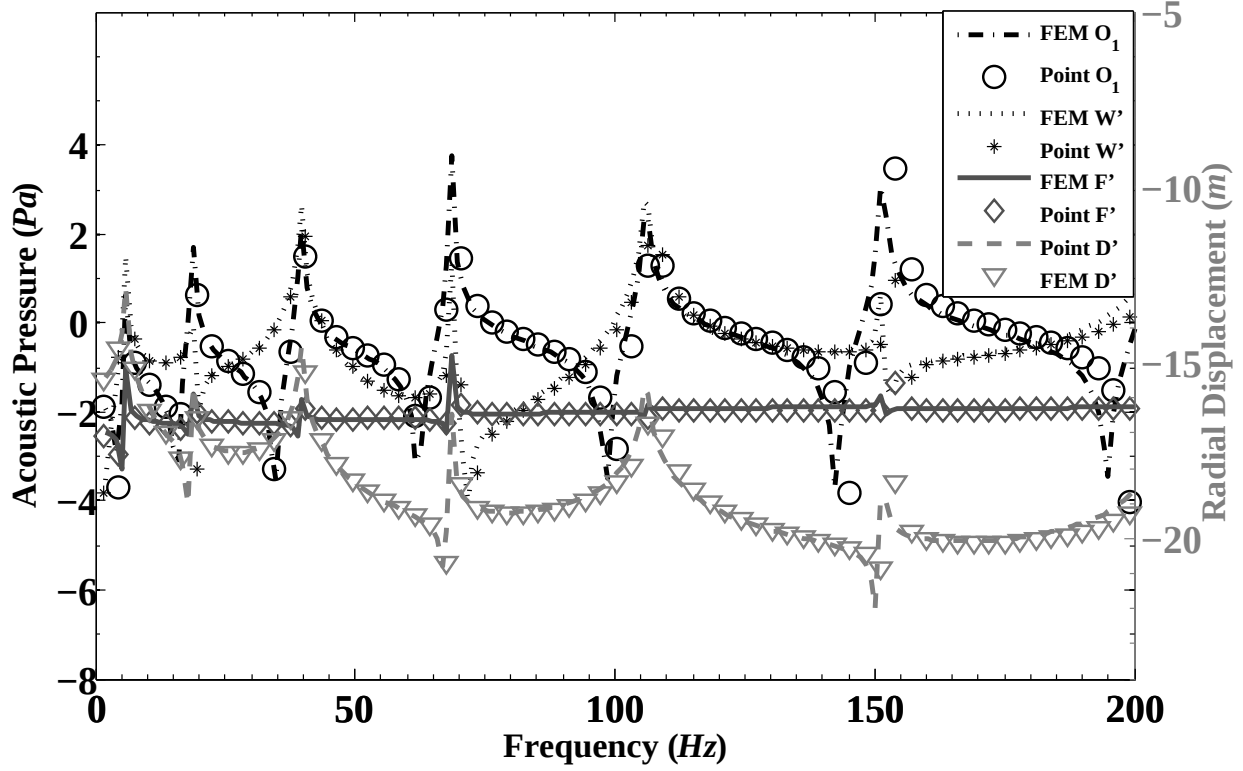


Figure 6.5: Pressure's frequency responses at the shell's center-point O_1 , W' and farther from cylinder at F' as well as the radial displacement at D' when the shell is positioned in an acoustical half space and excited by a point load applied radially at the shell's outer surface.

frequency response of the center-point pressure, $p_{in}(r = 0, \theta = 0, \omega)pa$ and the corner pressure, $p_{ex}(r = 2.874 m, \theta = 9\pi/8, \omega)pa$ are considered corresponding to the observation points at O_1 and W , respectively. Furthermore the surface acoustical pressure, $p_{ex}(r = 1.01 m, \theta = \pi/8, \omega)pa$ and the radial structural displacement, $U_r(r = 1.01 m, \theta = 0, \omega) m$ are calculated at the observation point D' . The corresponding results are shown in **Figure 6.7** where an excellent agreement between the current approach and FEM results are observed. More general situations are given next.

6.2 Material Arrangements

Oil-filled, piezo-composite hollow cylinder submerged in the seawater is considered regardless. Acoustical properties of these fluids are given in **Table 6.1**. Free vibration of a triclinic crystal for a single layer piezo-composite was considered in a recent communication, [Rabbani et al., 2018], but for a single-layer, in-vacuo shell where no fluid coupling is accounted for. An orthotropic material orientation is used in the current work because of its wider range of applications. The solution procedure, however, remains the same for higher level of anisotropy. Two chosen arrangements of the layers and their material compositions are given next.

- *Material arrangements when surface traction forces are applied.* The shell's core layer is made of FGM which its properties changes in the radial direction through its thickness ($h_{el} = 0.1 m$). The material properties' profile varies as linear, metal-rich and ceramic-rich patterns between Zirconia (ZrO_2) and Aluminum (Al). The material properties of Zirconia (ZrO_2) and Aluminum (Al) are given in **Table 6.3**. Furthermore interfaces between layers are located at $r_{in} = 0.8 m$, $r_s = 0.9 m$, $r_a = 1 m$, $r_{ex} = 1.1 m$ for a FGPM-FGM-FGPM

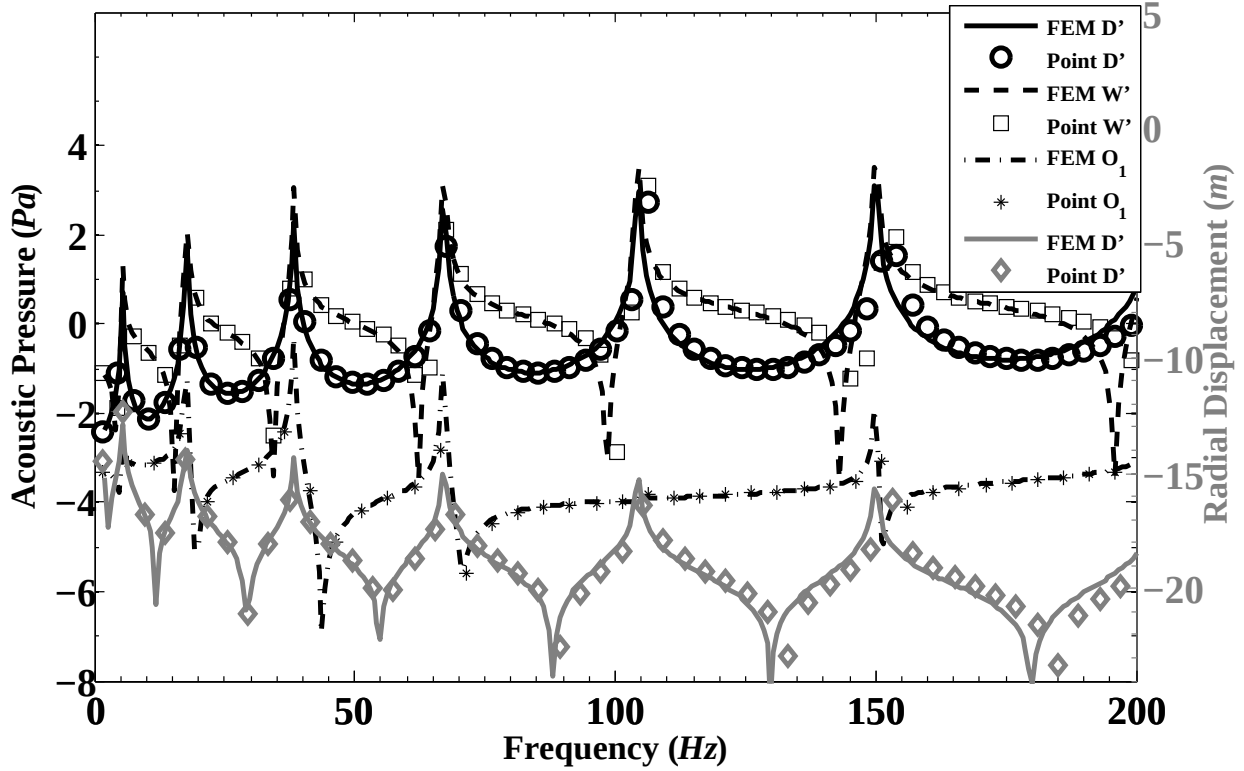


Figure 6.6: Pressure's frequency responses at the shell's center-point O_1 , W' and farther from cylinder at F' as well as the radial displacement at D' when the shell is positioned in an acoustical quarter space and excited by a point load applied radially at the shell's outer surface.

arrangement.

Volume fraction of the internal core layer varies from fully ceramic at interface $r = r_s$ to fully metallic at $r = r_a$. See Equation 19 of the first companion paper. The volume fraction of the FGM layers is

$$V_{elr} = [1 - (r - r_s)/h_{el}]^\kappa.$$

Three distinct FGM materials are considered whose κ values are 0.1, 1 and 10 corresponding to, respectively, metal rich (MR), linear and ceramic rich (CR) profiles. The inner/outer piezoelectric sensor/actuator is assumed to be fabricated from a non-graded, radially polarized PZT4 for which $V_s = V_a = 1$.

The vibration modes of the shell are assumed to be axisymmetric when excited with traction force(s) in which axial modes are not excited. Required material properties of PZT4, subsequently, are provided in Table 6.4 under a plain strain situation.

- *Material arrangements when a plane incident wave impinges on the shell.* The layers' interfaces are located at $r_{in} = 1.899 \text{ m}$, $r_s = 1.9 \text{ m}$, $r_a = 1.95 \text{ m}$, $r_{ex} = 2.00 \text{ m}$. The $r_{in} \rightarrow r_s$ essentially is equivalent with the removal of the internal piezoelectric layer. Consequently it represents basically a FGM-FGPM arrangement. The inner elastic layer is made of Steel with thickness of $h_{el} = 0.05 \text{ m}$). The outer axially polarized piezoelectric layer is assumed to be fabricated from non-graded PZT4 with $V^s = V^a = 1$.

The incident angle of the impinging wave excites vibratory modes asymmetrically. Conse-

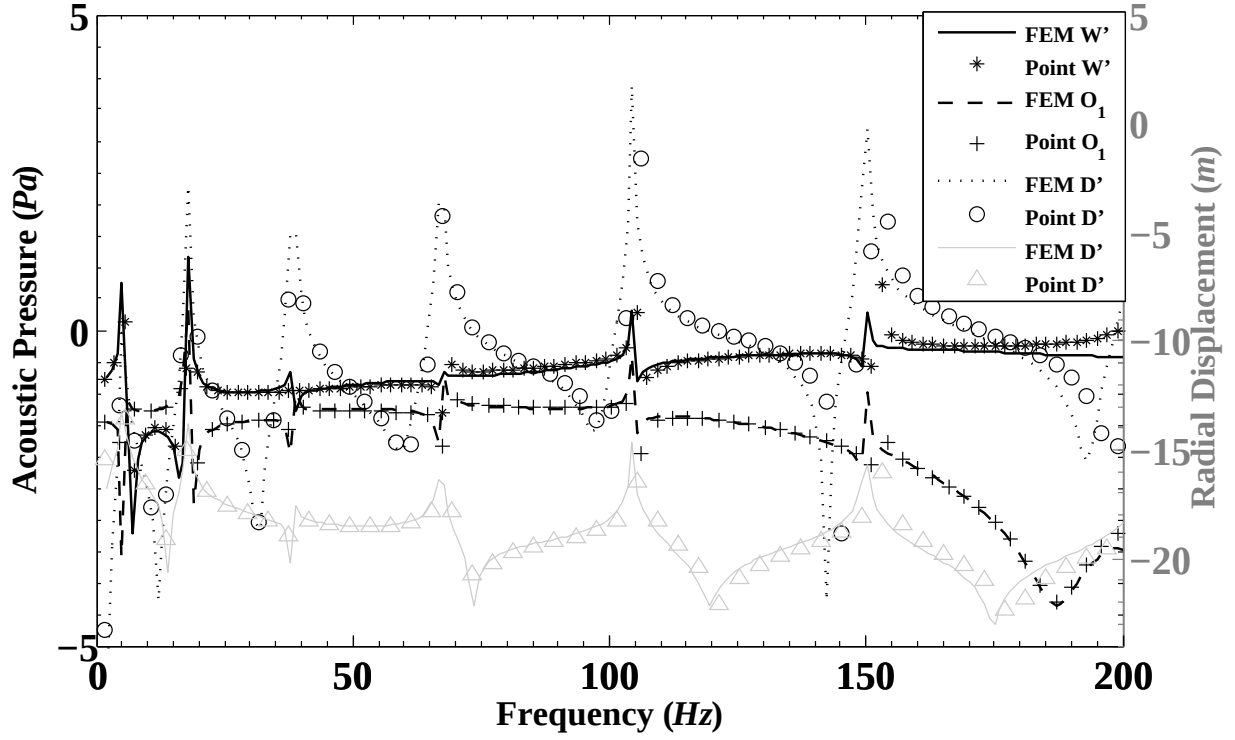


Figure 6.7: Pressure's frequency responses at the shell's center-point O_1 , W' and farther from cylinder at F' as well as the radial displacement at D' when the shell is positioned in a 45-degree wedge-shaped acoustical medium and excited by a point load applied radially at the shell's outer surface.

quently material and electrical properties of the constituent materials in a three-dimensional (asymmetric) vibration condition are provided in Table 6.2.

	Seawater	Oil
$c \text{ (m/s}^2\text{)}$	1500	1490
$\rho \text{ (Kg/m}^3\text{)}$	1010	960

Table 6.1: Acoustical properties of the constituent materials.

6.3 Case Studies

The chosen numerical experiments when a plane wave is incident upon the shell is given first. These are followed by the system's response to surface traction forces as the source of excitation.

6.3.1 Incident Wave

The selected material arrangements are given in § 6.2 when the source of excitation is an incident wave. Figure 6.8 shows the total spatial distribution of acoustical pressure magnitude, $|p(r_1, \theta_1, z_1, t)|$ at selected plane wave frequencies ($F = \omega/2\pi$) for an acoustical half-space ($\gamma = \pi$) at the neighborhood of a piezo-composite cylinder. Interestingly as the incident wave's frequency increases the resulting acoustical pressure field becomes highly directional. For example the distribution of the pressure field at the lowest frequency $5kHz$ barely shows any signs of directionality. Both internal and external pressure fields, however, become quickly highly directional as frequency increases to $10kHz$ and $50kHz$. Further increase in frequency leads to much higher directionality

	PZT Layer		Steel Layer
C_{11}	13.9×10^{10}	c_{11}	28.269×10^{10}
C_{12}	7.8×10^{10}	c_{12}	12.115×10^{10}
C_{13}	7.43×10^{10}	c_{13}	12.115×10^{10}
C_{22}	13.9×10^{10}	c_{22}	28.269×10^{10}
C_{23}	7.43×10^{10}	c_{23}	12.115×10^{10}
C_{33}	11.5×10^{10}	c_{33}	28.269×10^{10}
C_{44}	51.2×10^{10}	c_{44}	8.077×10^{10}
C_{55}	51.2×10^{10}	c_{55}	8.077×10^{10}
C_{66}	61.2×10^{10}	c_{66}	8.077×10^{10}
e_{31}	-5.2		
e_{32}	-5.2		
e_{33}	15.1		
e_{15}	12.7		
e_{24}	12.7		
ε_{11}	650×10^{-11}		
ε_{22}	650×10^{-11}		
ε_{33}	560×10^{-11}		

Table 6.2: Mechanical and electrical properties of the constituent materials in a non-symmetric (3D) loading situation when the shell is subjected to an oblique incident wave.

		Zirconia(ZrO_2)		Aluminum
$C_{ij}^{el}(\text{N/m}^2)$	C_{11}^{el}	31.86×10^{10}	C_{11}^{el}	11.047×10^{10}
	C_{12}^{el}	12.9×10^{10}	C_{12}^{el}	5.7×10^{10}
	C_{22}^{el}	31.86×10^{10}	C_{22}^{el}	11.047×10^{10}
	C_{33}^{el}	18.96×10^{10}	C_{33}^{el}	5.338×10^{10}
$\rho_{el}(\text{kg/m}^3)$	ρ_{el}	5700	ρ_{el}	2706

Table 6.3: Mechanical properties of the cylinder's constituent materials for the plane strain case.

but only in the external pressure field. Note a highly focused pressure region between the wall and shell's surface in the higher frequencies.

The previous figure gives a qualitative measure of the pressure field. On the other hand, Figure 6.9 provides a quantitative presentation of the pressure field. This figure gives polar plots of the maximum external acoustical pressure. The pressure field demonstrates a highly depressive behaviour at the lowest excitation frequency 5kHz . Patterns quickly start changing to a highly direction behaviour at 10kHz where the maximum value happens at the head region. Further increase in the frequency to 50kHz leads to a complete shift in the maximum from head region to the tail region. Indeed the presence of the rigid wall nearest the cylinder dominate until their effects counterbalance. Distribution pattern interchanges between the tail and head region by a further increase to 100kHz , however, the absolute maximum remains at the tail region. A further increase to 500kHz and 1MHz leads to narrower but nearly equal peaks at both head and tail regions. This figure shows that the distribution of the acoustical energy is highly frequency dependent. Energy is more focused in a smaller region as higher frequencies are involved regardless. The focal regions are located in head and tail regions. The numerical results for the traction forces case are

$\rho(Kg/m^3)$	ρ_{pz}	Actuator/Sensor Layer	ρ_{el}	Host Layer
		7500		7000
$\left(\frac{N}{m}\right)$	C_{11}^{pz}	13.9×10^{10}	C_{11}^{el}	1.3462×10^{10}
	C_{12}^{pz}	7.8×10^{10}	C_{12}^{el}	5.7692×10^{10}
	C_{22}^{pz}	13.9×10^{10}	C_{22}^{el}	1.3462×10^{10}
	C_{33}^{pz}	6.12×10^{10}	C_{33}^{el}	7.6923×10^{10}
$\left(\frac{C}{m}\right)$	e_{11}	15.1		
	e_{21}	-5.2		
	e_{32}	12.7		
$\left(\frac{F}{m}\right)$	ϵ_{11}	560×10^{-11}		
	ϵ_{22}	560×10^{-11}		
ϵ_{ij}				

Table 6.4: Mechanical and electrical properties of the constituent materials when the loading conditions leads to a plane strain situation.

discussed next.

6.3.2 Surface Traction Force

Configurations and material selections for the traction force case are given in [section § 6.2](#). Three acoustical wedge angle are considered as illustrated in the same figure. The shell is subjected to three different loading conditions over its external surface. They are listed as

- **Case I:** A uniformly distributed harmonic force with the subtending angle $\varphi = \pi/36$ centred at the cylinder's apex ($35\pi/72 \leq \theta \leq 37\pi/72, r = r_{ex}$) which can be expressed as

$$f_{ex}^j(\theta, \omega) = \begin{cases} F_0 & 35\pi/72 \leq \theta \leq 37\pi/72 \\ 0 & \text{otherwise} \end{cases},$$

where F_0 is the load's amplitude. The corresponding modal force coefficients, by using Equation 23, turn into

$$f_n^{(j),ex}(\omega) = F_0 \sin(\varphi n/2) / [e^{(in\theta_1)}(n\pi)],$$

where $\theta_1 = \pi/2$.

- **Case II:** Similar to the configuration in case I but the subtended angle is $\varphi = \pi/6$. Then

$$f_{ex}^{(j)}(\theta, \omega) = \begin{cases} F_0 & 5\pi/12 \leq \theta \leq 7\pi/12 \\ 0 & \text{otherwise} \end{cases},$$

where the corresponding modal force coefficients become

$$f_n^{(j),ex}(\omega) = F_0 \sin(\varphi n/2) / [e^{(in\theta_1)}(n\pi)],$$

in which $\theta_1 = \pi/2$.

- **Case III:** A pair of equal and opposite radially distributed harmonic load are applied at the shell's apogee and perigee with the subtended angle $\varphi = \pi/6$. This situation can be

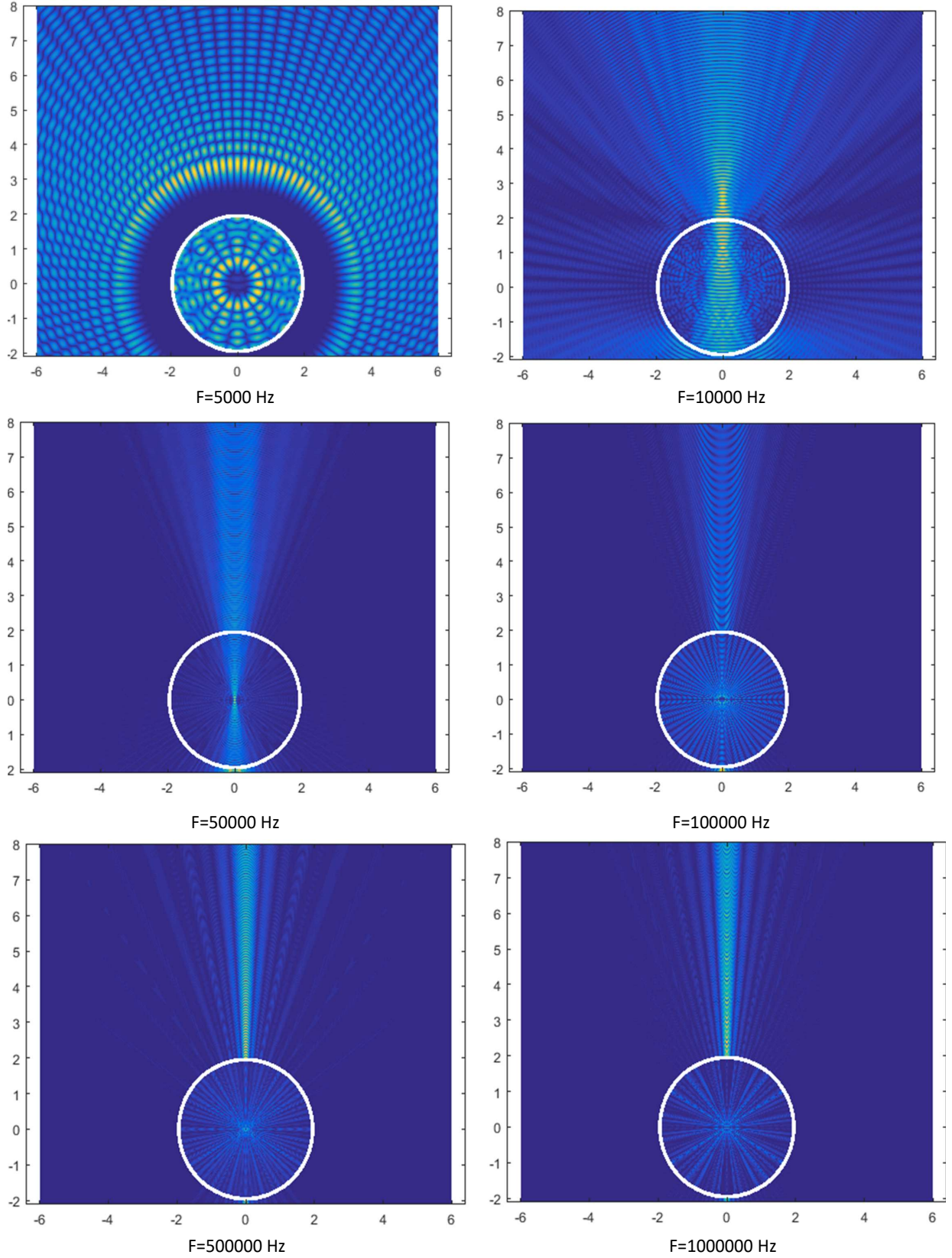


Figure 6.8: Spatial distributions of Total acoustical pressure, $|p(r_1, \theta_1, z_1, t)|$, at selected plane wave frequencies for a half space ($\gamma = \pi$) acoustical medium when the shell is impinged by a plane acoustical incident wave.

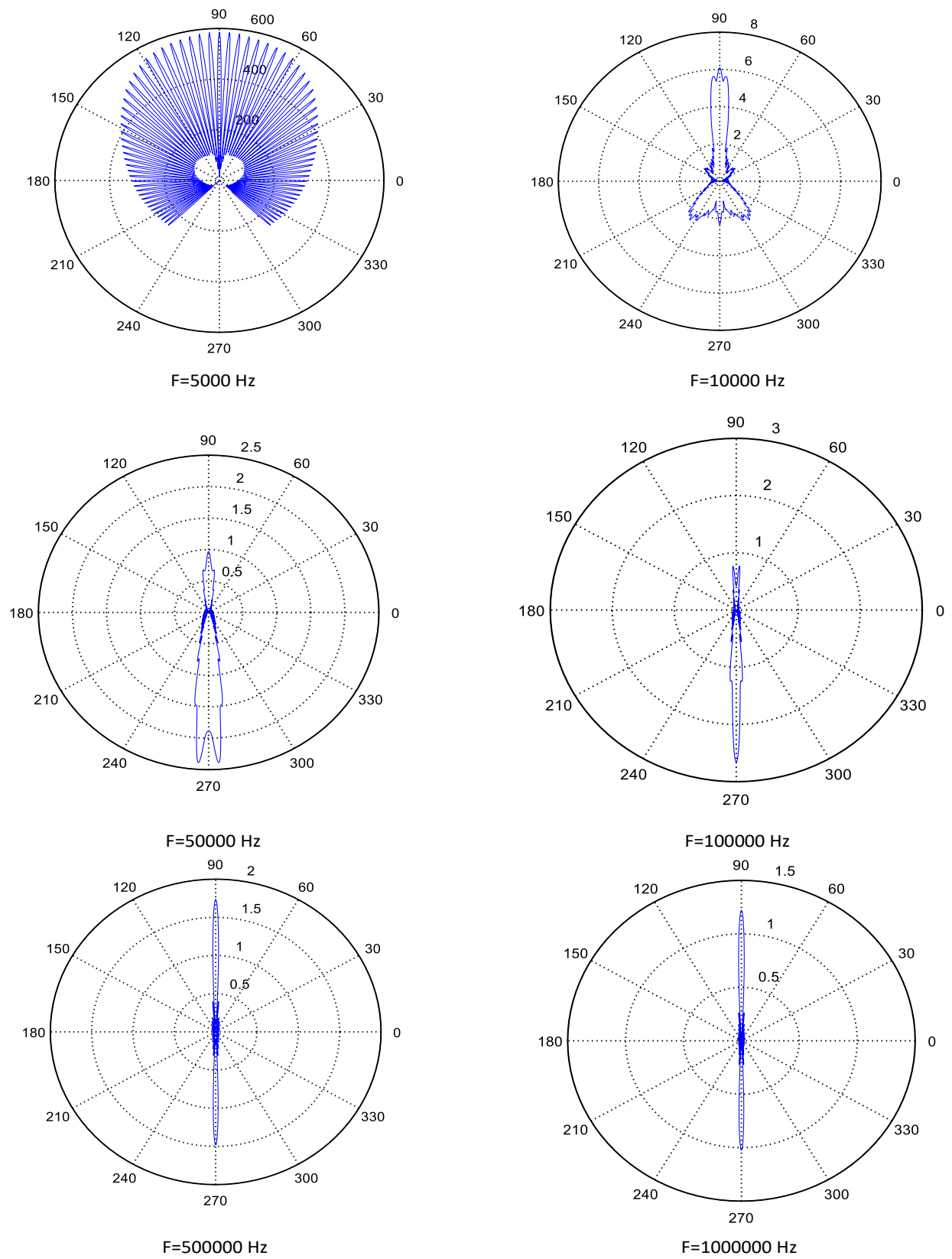


Figure 6.9: Maximum acoustical pressure within the submerging fluid, $|p_{ex}(r_1, \theta_1, z_1, t)|$, when the shell is located in an acoustical half space ($\alpha = \pi$) and it is subjected to a plane acoustical incident wave with the selected frequencies.

expressed as

$$f_{ex}^{(j)}(\theta, \omega) = \begin{cases} F_0 & 5\pi/12 \leq \theta \leq 7\pi/12 \\ F_0 & 17\pi/12 \leq \theta \leq 19\pi/12 \\ 0 & \text{otherwise} \end{cases},$$

where F_0 is the amplitude of radial harmonic load. The corresponding modal force coefficients are

$$f_n^{(j),ex}(\omega) = \frac{F_0 \sin(\varphi n/2)}{n\pi} \left(e^{-in\theta_1} + e^{-in\theta_2} \right),$$

in which $\theta_1 = \pi/2$, $\theta_2 = 3\pi/2$.

The loading configuration of case III does not lead to rigid body motion whereas other two configurations give rise to rigid body motion.

Figure 6.10 shows the polar plots of the maximum hoop stress, $\text{Max}|\sigma_{\theta\theta}|$ along with the shell's thickness. The results are plotted for shells with three graded profiles including a metal-rich (MR), ceramic-rich (CR) and linear (L) compositions which are subjected to three surface loading configurations as detailed previously. Three wedge angles are considered as $\alpha = \pi$, $\pi/2$ and $\pi/4$. The distribution pattern in the double sided loading (case III) shows the least sensitivity to the wedge angle. The four loop pattern in double sided loading is inclined according to the wedge angle of $\pi/4$. Distribution patterns, however, are significantly affected by the wedge angle and it approaches more uniform patterns as the wedge angle decreases in other two cases of one-sided loading configurations. Double sided loading also leads to the larger stress concentration compared to the one-sided one. Increasing the region under the distributed loading configuration I ($\pi/6$) and loading case II ($\pi/6$) leads to higher level of stresses whereas, interestingly, the corresponding patterns remain the same.

It is clear that the maximum hoop stress for the acoustical half-space occurs at $\theta = 3\pi/4$ regardless of loading configurations or cylinder's material profile when the shell is located in the half-space. This independence happens because the hoop stress is affected principally by the acoustical pressure field near the rigid wall. The grading profile of the shell's core material has significant effect on the level of stresses regardless. Interestingly the order of the amplitudes corresponding to each material profile is a function of wedge angle. The MR and L profiles show the same level of stresses and almost twice of the one of CR in the half-space, nevertheless. Unlike the half space, distribution patterns differ significantly for various material profiles when the wedge angle is decreased. This is valid, however, only for one-sided loading configurations. The 90- and 45- degree acoustical wedges produce more directional maximum hoop stresses only in the case of CR. The MR and L profiles, on the other hand, have a tendency towards more uniform stress distributions as wedge angle decreases. Overall, the CR core lowers the hoop stress almost invariably. Consequently it looks as a better choice from structural integrity's point of view.

Figure 6.11 demonstrates the same arrangement of polar plots but for radial stresses $\text{Max}|\sigma_{rr}|$ rather than Hoop stresses in the previous figure. Two figures show very similar patterns, however, the level of the stress concentrations are significantly lower in **Figure 6.11**. An observation common to **Figure 6.10** and **Figure 6.11** is that the distribution pattern of the maximum stress has a tendency to adjust with the bisector of the acoustical wedge. This observation is more visible

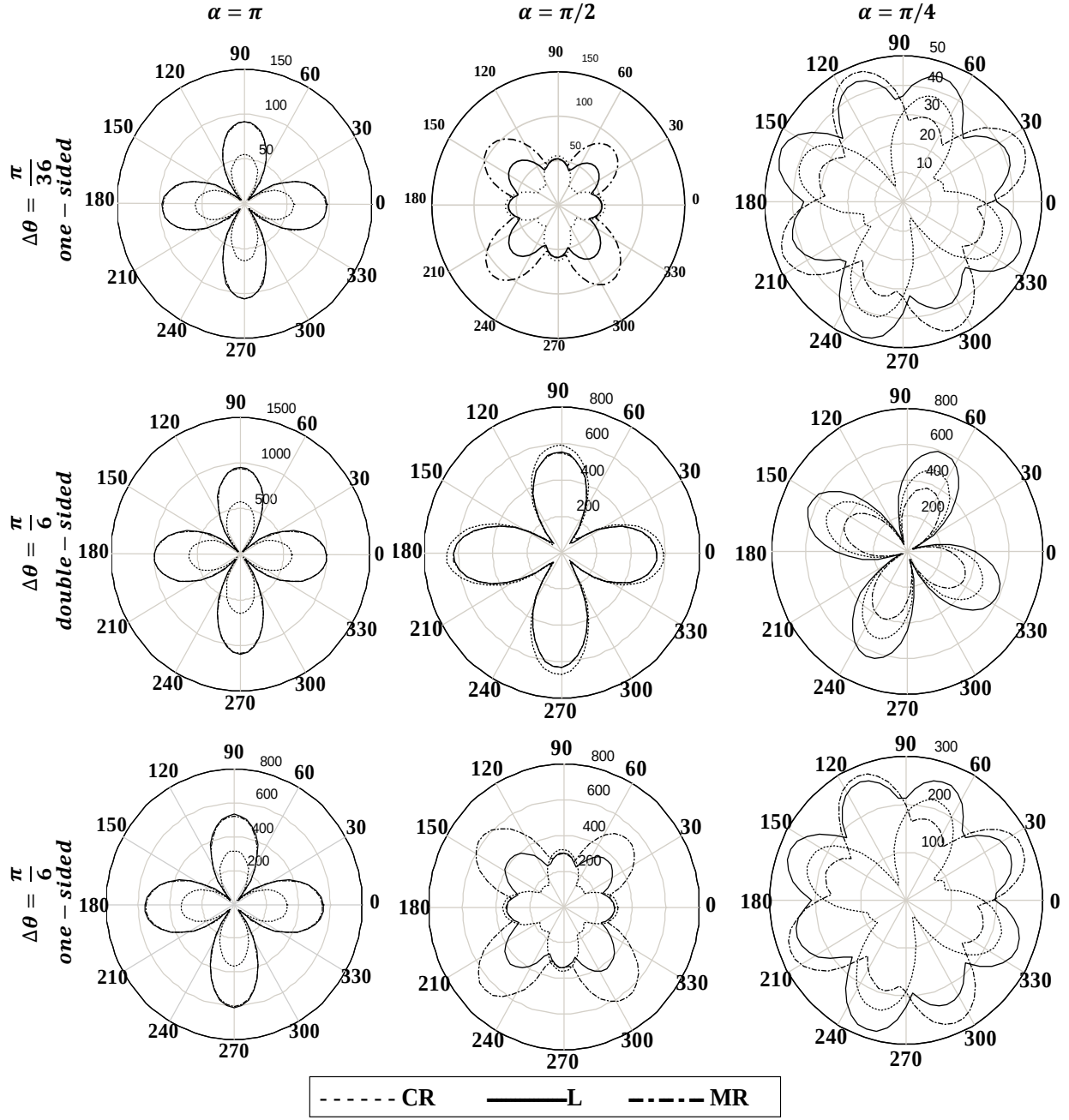


Figure 6.10: Maximum hoop stress, $\sigma_{\theta\theta}$, when the shell is positioned in an acoustical half space, a quarter space and 45-degree acoustical wedge. Three surface loading configurations are considered for three graded profile of the shell's material.

when $\gamma = \pi/4$. The ceramic rich profile as the core material produces lower stress magnitudes. Unlike hoop stresses, the absolute maximum radial stresses occur at head and tail region ($\theta = 0$ and π). Furthermore two one-sided loading configurations produced more even distribution of radial stresses near the wedge corner compared to the corresponding distribution of hoop stresses when the shell is located in the quarter space, acoustical wedge.

Figure 6.12 displays the marginally different center point pressures for the three wedge media when each one is excited by the three different surface loads. The center-point pressure is affected only by the cylinder's breathing mode ($n = 0$) so that it is not directional. First, the order of

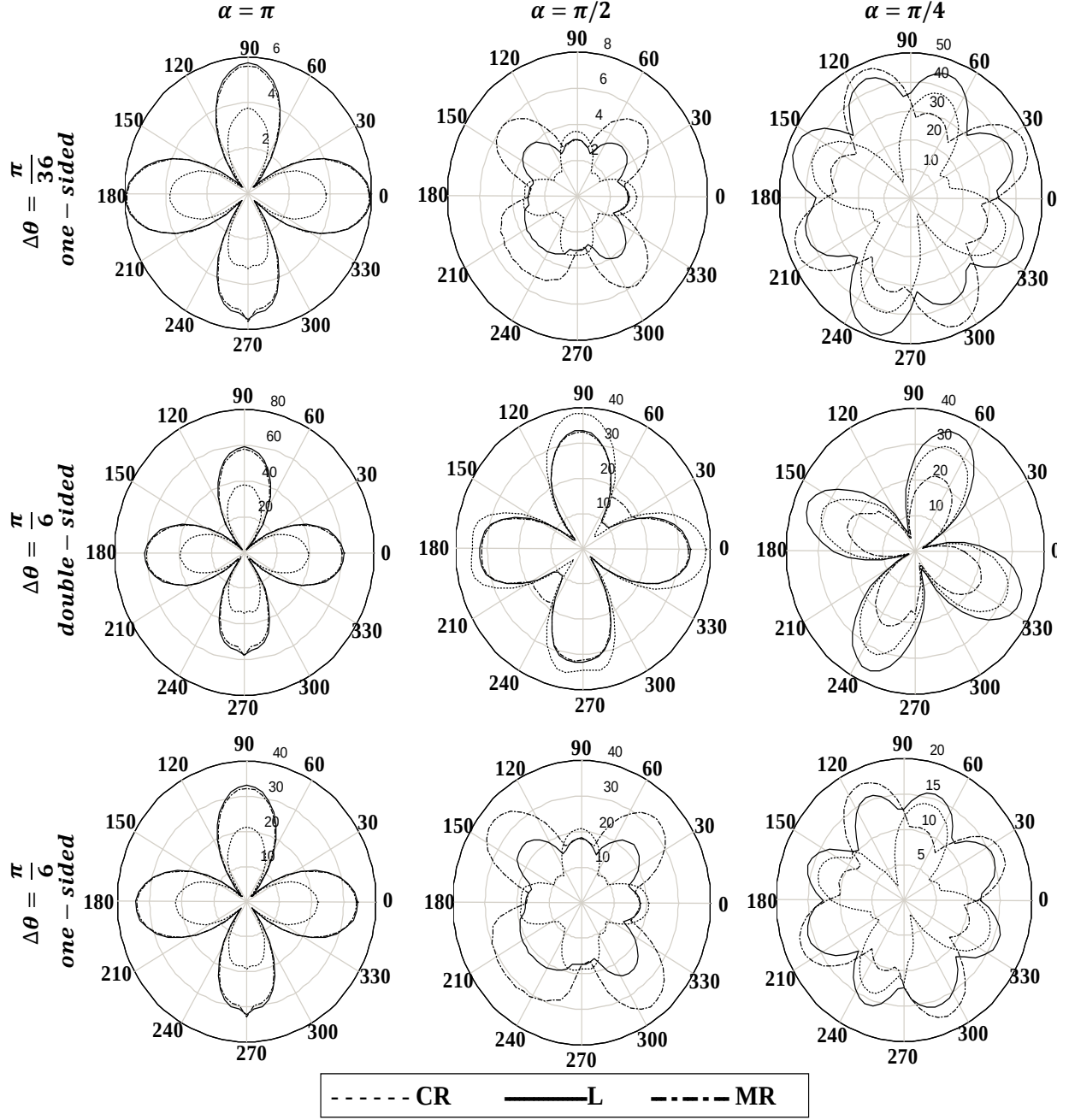


Figure 6.11: Maximum radial stress, σ_{rr} , when the shell is positioned in an acoustical half space, a quarter space and 45-degree acoustical wedge. Three surface loading configurations are considered for three graded profile of the shell's material.

the center point pressure's magnitudes in a descending order is, generally, as metal-rich, linear and ceramic rich core. The lower resonant frequencies coincide, however, a phase shift start to appear as the frequency increases. Furthermore, the resonant frequencies start by metal-rich and it is followed by linear and ceramic rich core, respectively. As the wedge angle decreases, progressively new resonant frequencies start showing up regardless of the core material's profile or loading configuration. Having an axisymmetric double-sided loading situation cancels out some of the resonant frequencies. This may happen due to the cancellation in some of the out of phase modes. The concluding remarks are given next.

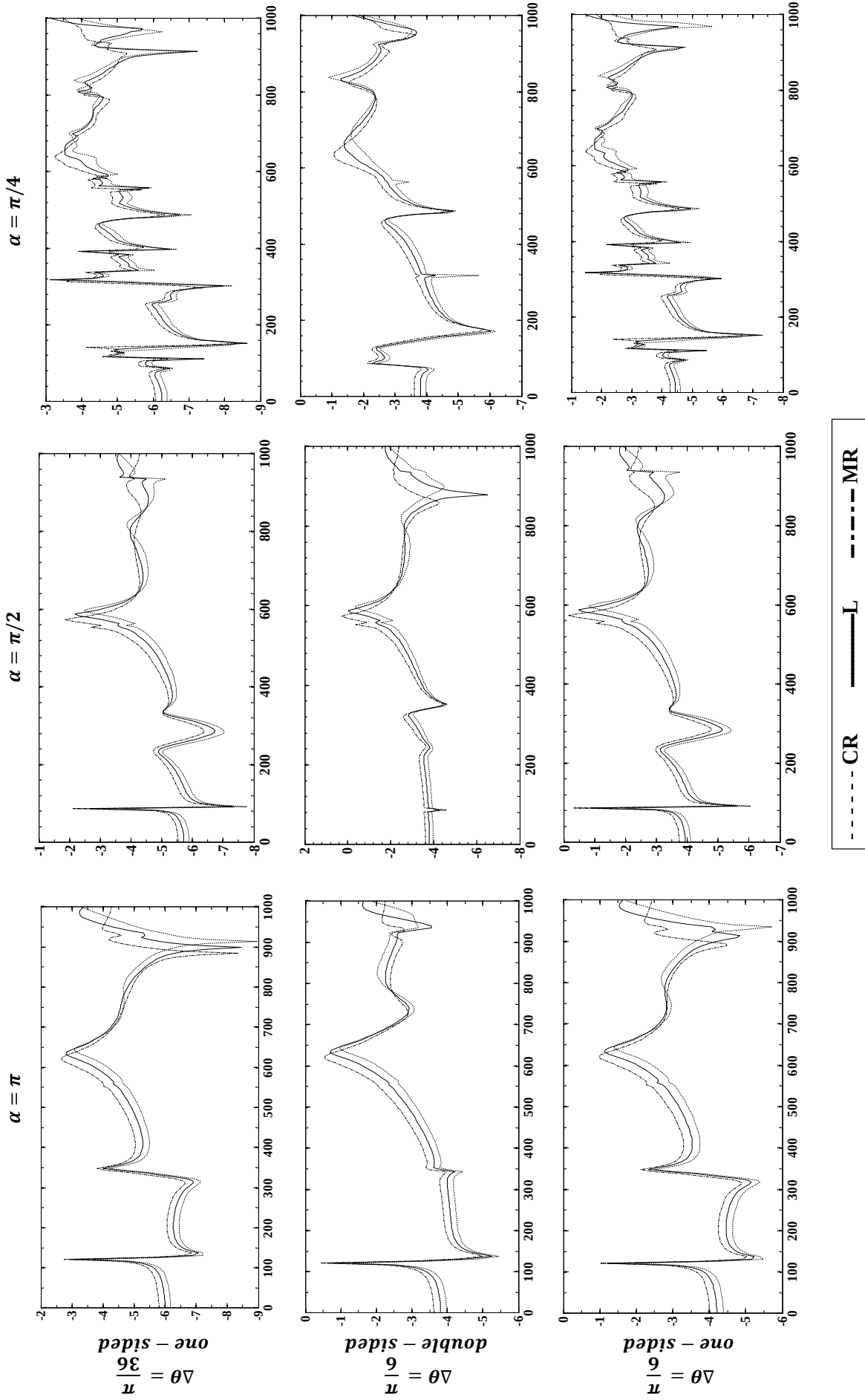


Figure 6.12: Center point pressure in frequency range of 1 – 1000Hz for the half space, quarter space, and 45 degree acoustical wedge configurations with three loading situations and three graded material profile.

6.4 Conclusions

A general solution was given for elasto-acoustic response of a fluid-filled piezo-composite cylindrical shell submerged in an acoustical wedge in the first companion paper. Extensive numerical comparisons are performed between the current approach against FEM simulation results to ensure the validity and robustness of the current approach. These validations are carried out for two sources of excitations as incident waves and surface traction forces. Important remarks, beyond the validations purposes, can be made on the limitations of the conventional FEM approach. Hence,

- Higher frequencies produce shorter wavelengths which requires smaller mesh sizes to maintain a prescribed accuracy. This leads to a significantly higher computational cost associated with the increase in system's degrees of freedom.
- Truncation of infinite domains are usually carried out by applying absorbing boundary conditions. This approach leads to a considerable additional computational cost. Furthermore, this approach is not able to produce perfectly absorbing boundaries and there will be always some reflections.
- The computational costs associated with modelling at frequency ranges of industrial ultrasonic testing methods (typically between 1 to 10 MHz) becomes extremely high so that make it almost impractical.
- High level of contrast in acoustical properties of the interacting media makes the numerical stability problems.

The current analytical approach, however, is not associated with these problems. The proposed solution can be used also as a benchmark for other semi-numerical and numerical solutions.

Three different acoustical wedge angles are considered including half-space, quarter-space and 45-degree configurations. Furthermore three different profiles of FGM (functionally graded material) host materials are investigated. In addition to the incident wave, three configurations of the surface traction forces are tested as well. Effects of these parameters on the pressure and stress fields are studied. The main observations from the conducted numerical experiments are listed next.

- The shell structure can act effectively as a lens in focusing an incident acoustical wave to a very narrow region as the excitation frequency increases.
- The stress concentration patterns in an one-sided surface loading depends highly on wedge angle. The pattern, generally, becomes more evenly distributed in smaller wedge angles.
- Generally the stress concentration's intensity for different FGM profiles in order from highest to lowest is metal rich, linear and ceramic rich.
- The absolute maximum stress concentration accrues in direction of the wedge angle's bisector.
- The resonance frequencies of the system are increased in number in smaller wedge angles when a one-sided surface load is applied regardless of the core material's profile.

Numerical case studies here are carried out for orthotropic shell material, however, it should be noted that the current solution can be applied to any fully triclinic material orientation. The current solution can easily be used to determine the acoustical transmission loss when the shell is located in an acoustical wedge such as a pipe near an oceanic slope. The proposed model can be further developed to be applicable to active or passive vibration control for energy harvesting purposes.

Quadrature Formulae For a Circular Domain

The present paper focuses on quadrature formulae for integration on circular domains. The focus is on the construction of quadrature rules with a prescribed exactness, a quasi-uniform distribution of the points within the integration domain and a given number of nodes on the boundary. Five prominent families of quadrature rules are carefully selected based on an extensive literature search as well as an exhaustive set of numerical experiments. Explicit formulae are provided in each quadrature rule to calculate weights and associated nodes. Numerical experiments are conducted next on continuous and discontinuous functions to assess their performance.

These quadrature rules are well suited in the framework of various computational tools such as the mixed finite element method and meshless technique. These formulae are applicable to a wide range of problems within the realm of computational mechanics when circular cross sections are involved and boundary conditions has a significant influence. Examples range from linear local theory to the generalized nonlocal and micropolar theories of continua. Elasto-plastic or hyper-elastic behaviour of material, thermoacoustic instabilities in combustors, solids with strong Cosserat effects and fluid flows with slip boundary conditions are some specific applications.

7.1 Motivations

Many applications require the solution of complex problems with tubular or spherical geometries in which boundary conditions are imposed on cylinders' or spheres' traction surface(s). Examples of possible situations are listed next.

- Strong shock waves interacting with structures having circular cross section may involve discontinuity at the shock front which may lead to the structure's nonlinear response [Whitham, 2011].
- Seismic, elastic and acoustical wave propagation in cylindrical or spherical domains require integration over circular cross sections [Iwashimizu, 1972, Leblond and Sigrist, 2010, Hasheminejad et al., 2011]. An efficient numerical integration scheme, however, still remains problematic [Faccioli et al., 1997, Atkins and Shu, 1998, Wilcox et al., 2010].
- Fiber-based structural models in terms of resultant forces often require an accurate integration scheme, especially in the presence of a non-linear response [Aurichio and Sacco, 1999, Soydas and Saritas, 2013, Spacone et al., 1996].
- Thermoacoustic instability in turbulent combustors are shown to be inherently nonlinear in nature [Balasubramanian and Sujith, 2008, Nair et al., 2014]. Two and three dimensional analysis of this behaviour requires a highly accurate numerical integration scheme.

- Numerical investigations on Navier-Stokes equations inside nanochannels with slip boundary conditions led to unexpectedly contradictory experimental observations [Calabrò et al., 2013, Mattia and Calabrò, 2012, Ritos et al., 2014] when usual numerical schemes are incorporated which are meant for no slip boundary conditions [Rajani et al., 2009]. As a result, numerical integration techniques are required having a higher degree of accuracy.

Some recent computational tools are settled more naturally within circular domains. Specific examples are the method of finite spheres and the mixed finite element approach.

- Meshless methods in 2D problems are often based on local basis functions [Asprone et al., 2011, Sladek et al., 2013] in circular domains [Atluri and Zhu, 1998, Wendland, 1999]. In particular, the method of finite spheres as a truly natural meshless technique, requires numerical integration to be performed without a mesh [De and Bathe, 2000, De and Bathe, 2001]. Obviously the quadrature rules discussed here are of utmost interest.
- An effective mixed finite element method for structural elements such as beams and frames requires an accurate integration scheme in 2D and 3D [Taylor et al., 2003].

Consequently, the purpose of the present work is to numerically compute the following integral:

$$\iint_{\Omega} f(\zeta) d\zeta, \quad \zeta \in \mathbb{R}^2, \quad \Omega = \{(x, y) \mid x^2 + y^2 \leq 1\}. \quad (7.1)$$

The goal is to solve the aforementioned integration by using a quadrature rule which requires N nodes ζ_i with the corresponding relative weights w_i , such that

$$\iint_{\Omega} f(\zeta) d\zeta \approx \sum_{i=1}^N w_i f(\zeta_i). \quad (7.2)$$

In particular *global quadrature rules* are considered, i.e. based on a construction involving the whole domain. An alternative strategy, called a *composite quadrature rule*, is to subdivide the domain and apply copies of a given quadrature in each sub-domain. Therefore, an assessment is made to determine whether the introduced rules are well-suited to be used also in a composite manner.

Quadrature rules are grouped into *families*. Rules within the same family are constructed by following the same integration scheme and differ only in the number of points used. Examples in 1D are the interpolant quadrature rules based on equispaced nodes which are referred as the family of Newton-Cotes rules. Accordingly particular attention is given to number of parameters in the development of the quadrature formulae. These parameters are listed subsequently.

1. *Quadrature point localization property.* Quadrature points are required to be localized inside the integration domain, i.e. $\zeta_i \in \Omega$. This condition may seem trivial but in the literature, rules have been proposed that evaluate the integration function outside the domain. Distributions of nodes only within the circle, however, are targeted here. A quasi-uniform distribution is preferred. Moreover, there is an interest in placing a given number of points, $N_{\partial\Omega}$, on the boundary so that

$$\zeta_j \in \partial\Omega \quad \forall j = 1, \dots, N_{\partial\Omega}.$$

2. *Explicit weight computation.* Weights have to be computed after which the node's localization is defined. The literature presents many quadrature rules with tabulated points and weights since their calculations are not trivial to perform. Consequently, the focus is placed on rules having explicit formulae for the computation of the nodes and weights. As an example, formulae constructed by optimization (approximate) procedures are not considered in this work.
3. *Convergence property.* A numerical study is undertaken on the performance of quadrature rules when the number of nodes increases. This property determines the rate of convergence with an increasing number of points when the formulae are applied to regular functions.
4. *Requirement of exactness.* Degree of exactness is determined according to the highest degree of the polynomial that can be exactly integrated. Exactness is assessed in the space of polynomials. In particular, the space of bivariate polynomials¹ of degree k which is denoted by

$$\mathbb{P}^k = \left\{ \sum_{0 \leq i+j \leq k} a_{ij} x^i y^j \right\} \\ \equiv \text{span}\{1, x, y, x^2, xy, y^2, \dots, x^k, x^{k-1}y, \dots, xy^{k-1}, y^k\}.$$

The quadrature rule given in [Equation 7.1](#) is defined to have *degree of exactness* $k \in \mathbf{N}$ if

$$\iint_{\Omega} \pi(\zeta) d\zeta = \sum_{i=1}^N w_i \pi(\zeta_i) \quad \forall \pi \in \mathbb{P}^k.$$

A quadrature rule is described as *Optimal* if it achieves the maximum degree of exactness with a minimum number of nodes. Henceforth the number of nodes required to achieve a prescribed exactness k is specified by $N_{(k)}$.

Optimality is a challenging issue for the multidimensional cases in contrast to 1D situations where the optimal rules are, simply, Gaussian quadratures. See, for example, [\[Cools, 1997\]](#). In particular, the minimum required number of nodes, $N_{(k)}^{OPT}$, in a quadrature rule over a circular domain with a prescribed degree of exactness, k , becomes

$$N_{(k)}^{OPT} \geq \frac{(k+1)(k+3)}{8}$$

so that $\dim(\mathbb{P}^k) = \frac{(k+2)(k+1)}{2}$. See, for example, [\[Stroud, 1971, Cools and Kim, 2000\]](#). Optimal formulae are known only for few degrees in [\[Cools and Kim, 2000\]](#) whilst some of these rules require the nodes outside the circle. See, for instance, the table given in [\[Cools and Kim, 2000\]](#). The current study, on the other hand, is restricted to those rules in which their nodes are located only inside the disk or on its boundary.

¹Notice that in several cases the tensor product space is considered alternatively.

7.2 Literature review and preliminaries

Much literature is available regarding quadrature formulae in 2D as well as for circular domains. Little attention, however, has been paid to the properties listed in §7.1. See, for example, the Stroud’s monograph [Stroud, 1971] and the review articles by Cools and co-authors [Cools, 1997, Cools, 2003, Cools and Kim, 2000] which focus on determining optimal rules.

Five prominent families of quadrature rules are carefully selected based on an extensive literature search and numerical experiments on various quadrature rules for circular domains. The first two selected families are due to *Peirce* [Peirce, 1957] and *Stroud* [Stroud, 1971] which are probably the best known quadrature rules for circular domains. These quadrature families have been used extensively although they cannot encompass points on the boundary. They, for example, are employed in meshless methods [Babuška et al., 2003, Babuška et al., 2008, De and Bathe, 2000, De and Bathe, 2001, Mazzia et al., 2007, Mazzia and Pini, 2010, Zhang and Banerjee, 2012]. The third and forth selections are constructed following recent developments by Vianello and co-authors [Bos and Vianello, 2012, Da Fies and Vianello, 2012] as well as Bojanov [Bojanov, 2005, Bojanov and Petrova, 1998]. Peirce’s rules are modified to develop the third family allowing simple computations in more general situations. Consequently, quadratures with points on a boundary or quadratures on subdomains of a circle can be constructed. However the quadrature rules from this family are not optimal with respect to the number of nodes for a prescribed degree of exactness.

On the other hand, Bojanov [Bojanov, 2005, Bojanov and Petrova, 1998] proposed the so-called line methods which is in a different category of multidimensional methods. This family, straightforwardly, leads to an explicit formula for the nodes and their weights where points on boundary can be utilized. This leads to a quasi-optimal number of points required for a given degree of accuracy. Conversely, the formulation is not well suited for circular subdomains thus for the use of composite formulae.

The last quadrature rule family selected is based on multivariate interpolation however it is not well suited for high degrees of exactness. In particular, the quadrature constructed on the star-like point distribution developed by Hakopian [Hakopian, 1982] is considered. The particular pattern of the nodes, on the other hand, may be of interest for some specific physical problems.

For the sake of completeness, it should be noted that an automatic cubature (a quadrature in higher dimensions) software is available [Cools and Haegemans, 2003]. This software is based on an adaptive composite strategy in which nodes and weights are computed in an iterative process. Such an approach is not considered as nodes are not fixed a priori and thus, it is not well suited for the discretization of a PDE.

Necessary notations on Gaussian quadrature in 1D are given at the beginning as it provides the ingredients for the calculation of 2D formulae. The 1D μ -Gaussian quadrature rule denoted by $\mathbb{G}_\mu$ is a unique rule for $\mu + 1$ nodes and provides an exactness $2\mu + 1$. Symbols $\mathbf{t}^\mathbb{G}$ and $\mathbf{w}^\mathbb{G}$ refer to the nodes and weights of the quadrature rule $\mathbb{G}_\mu$, they have the generic components $t_j^\mathbb{G} = \mathbf{t}^\mathbb{G}|_j$ and $w_j^\mathbb{G} = \mathbf{w}^\mathbb{G}|_j \ \forall j = 1, \dots, \mu + 1$. The calculation of the Gaussian nodes follows from evaluating the zeros of the Legendre polynomial $P_{\mu+1}(t)$. This polynomial is constructed to be orthogonal with respect to the Lebesgue measure in the finite interval of interest to the polynomial of lower indices, $P_{m \leq \mu}(t)$. Moreover, the weights $\mathbf{w}^\mathbb{G}$ can be computed by using the same polynomials

such that

$$P_{\mu+1}(t_j^{\mathbb{G}}) = 0 \quad \text{and} \quad w_j^{\mathbb{G}} = \frac{2(1-t_j^{\mathbb{G}})}{(1+\mu)^2(P_{\mu+2}(t_j^{\mathbb{G}}))^2} \quad \text{for } j = 1, 2, \dots, \mu+1. \quad (7.3)$$

The so-called weighted-Gaussian rules are obtained by changing the Lebesgue measure in the orthogonality construction of $P_{\mu+1}$. Furthermore, optimal quadrature rules with several preassigned nodes can be constructed. The μ -Gauss-Lobatto rule, however, is the only one which takes into account two boundary (end) points. The degree of exactness is $2\mu - 1$ if $\mu + 1$ nodes are used. The nodes and weights of this rule are indicated, respectively, by the symbols $\mathbf{t}^{\mathbb{GL}}$ and $\mathbf{w}^{\mathbb{GL}}$. The corresponding nodes and weights are given by

$$\begin{cases} P'_{\mu+1}(t_j^{\mathbb{GL}}) = 0 & \text{and} & w_j^{\mathbb{GL}} = \frac{2}{\mu(\mu+1)P_{\mu}^2(t_j^{\mathbb{GL}})} & \text{for } j = 2, \dots, \mu \\ t_1^{\mathbb{GL}} = -1, t_{\mu+1}^{\mathbb{GL}} = 1 & \text{and} & w_j^{\mathbb{GL}} = \frac{2}{\mu(\mu+1)} & \text{for } j = 1, \mu+1 \end{cases} \quad (7.4)$$

Refer, for example, to [Gautschi, 1981] for a complete survey on the theory concerning Gaussian quadrature in 1D and various extensions.

7.3 Quadrature methods

The selected quadrature rules are detailed after which being categorized into two groups as follows.

1. *Product rules.* These rules are constructed in a rectangular domain, Ψ , by taking into account the tensor product of two 1D quadratures along the two perpendicular directions. It is well known, that polar coordinates can describe a circle in terms of variables taken relative to a rectangle. One of the two variables in the rectangular domain always represents the angular length. In the other direction, however, two choices are available. One is taken along a circle's diameters whereas the other is chosen along radii. The first choice leads to quadrature rules which use a number of nodes close to optimal whereas the second one, on the other hand, is well suited for composite calculations. However, all the rules use the symmetries of a circle to generate a set of nodes which lie on concentric circles. Thus a transfer is needed between circular (Ω) and rectangular (Ψ) co-ordinates as $[0, 1] \times [0, 2\pi] \rightarrow [-1, 1] \times [-\pi/2, \pi/2]$. Consequently, the quadrature in the rectangular domain is applied to the function of concern and then multiplied by the determinant of the Jacobian, $|\partial(x, y)/\partial(\rho, \theta)|$. This is associated with transformation from circular to the rectangular coordinates, such that

$$\iint_{\Omega} f(\zeta) d\zeta = \iint_{\Psi} F(\rho, \theta) d\xi \quad \text{with} \quad F(\rho, \theta) = |\rho| f(\rho \cos(\theta), \rho \sin(\theta)).$$

A quadrature formula is constructed to integrate exactly the initial trigonometric polynomials in the rectangular domain to recover the polynomial exactness in the final domain.

2. *Line rules.* These rules consider an integral in a 2D circular domain to be a linear combination of integrals made on the chords² of the circle. These quadrature rules, however, are

²Notice that if all the chords are taken as diameters we obtain a rule that is also a product rule.

not well suited for composite calculations. The most commonly used line rules are derived by using a reduction formula. See, for example, [Quarteroni et al., 2007, 9.9.1]. These reduction formulae involve parallel chords. Subsequently the integral is written as

$$\iint_{\Omega} f(\zeta) d\zeta = \int_{-1}^1 \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y) dy \right] dx \approx \sum_{l=1}^{M_L} \alpha_l \mathbb{I}_l \quad (7.5)$$

where

$$\mathbb{I}_l := \int_{-\sqrt{1-x_l^2}}^{\sqrt{1-x_l^2}} f(x_l, y) dy$$

after which nodes $\{x_l\}$ are chosen for $l = 1, \dots, M_L$.

Three families of quadrature rules are presented in the category of the product rules in § 7.3.1. The differences are in their choice of the formulae in the radial, ρ , and the angular, θ , directions. Two families of quadrature rules in the category of line rules are presented later in § 7.3.2. The first of these two is an optimal choice of reduction formulae. Notice that all these rules are well suited for the application of sum factorization techniques that are useful for the reduction of the numerical cost of the quadratures. See, for example, [Antolin et al., 2015] for more detailed discussion.

7.3.1 Product rules

Three different cubature families within the class of product rules are presented next. These product rules are due to *Stroud*, *Peirce* and *Vianello*.

7.3.1.1 Stroud's cubature formula: the spherical product rule

The spherical product quadrature rule used here was developed originally by Stroud [Stroud, 1971]. Its construction is performed in the rectangular domain $\Psi \equiv [-1, 1] \times [-\pi/2, \pi/2]$, along a circle's diameters. Exactness is recovered by employing two weighted Gaussian quadratures with respect to $|\rho|$ and $1/\sqrt{1-t^2}$ in the radial ρ and angular directions θ , respectively, where $t = \sin\theta$. An orthogonal polynomial was utilized where its recursive presentation is

$$Q_n(t) = \begin{cases} 1, & \text{for } n = 0 \\ t, & \text{for } n = 1 \\ Q_{n-1}(t)t - \varsigma_n Q_{n-2}(t), & \text{for } 0 \leq n \leq 2 \end{cases} \quad \text{where} \quad (7.6)$$

in which

$$\varsigma_n = \begin{cases} \frac{n-1}{4n}, & \text{if } \{n = 2m+1 | m \in \mathbb{N}\} \\ \frac{n}{4(n-1)}, & \text{if } \{n = 2m | m \in \mathbb{N}\}. \end{cases}$$

The radial coordinates of the nodes are determined through the zeros $(t_i^{\mathbb{Q}})$ of the polynomial $(Q_n(t))$ given in Equation 7.6. Consequently the angular and radial coordinates of nodes, respectively, are

$$\theta_i = \frac{2i-1}{2M} \pi \quad \text{and} \quad \rho_j = t_j^{\mathbb{Q}}. \quad (7.7)$$

The closed form formula for this quadrature family turns into

$$\iint_{\Omega} f(\zeta) d\zeta \approx \sum_{i=1}^M w_i^{\mathbb{S}} \sum_{j=1}^M f(\rho_i \cos \theta_j, \rho_i \sin \theta_j) \quad (7.8)$$

where

$$w_i^{\mathbb{S}} = \vartheta_i^{(M)}(-1) + \vartheta_i^{(M)}(1) - 2\vartheta_i^{(M)}(0) \quad (7.9)$$

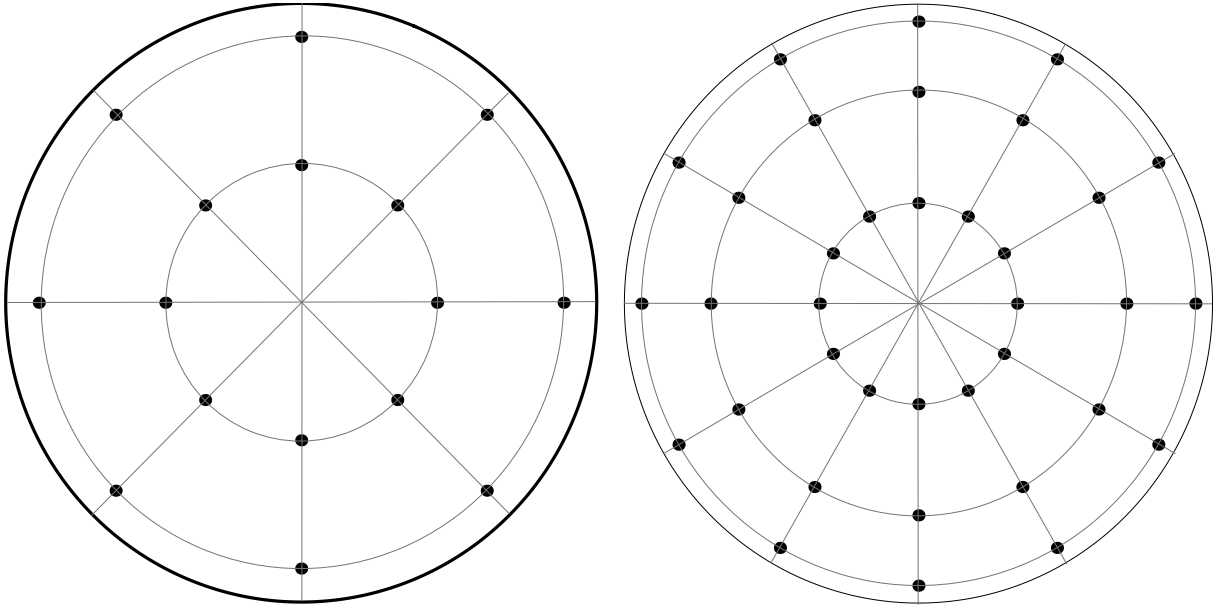
in which

$$\vartheta_i^{(n)}(r) = \int_0^r \frac{|\rho| Q_n(t)}{(t - t_i^{\mathbb{Q}}) \frac{dQ_n(t)}{dt} \big|_{t=t_i^{\mathbb{Q}}}} dt .$$

Analytical evaluation of this integral for individual orders of n results into a significant reduction in overall computational complexity of the quadrature scheme. The weights are positive because the construction employs Gaussian rules. The overall formula has an optimal exactness degree of $2M - 1$. This is obtained in the class of product rule for M points in each of the ρ and θ directions. This cubature rule is denoted by superscript $\mathbb{S}$. If k is odd³, the number of nodes needed to achieve exactness k is

$$N_{(k)}^{\mathbb{S}} = \frac{(k+1)^2}{4} .$$

This rule, however, does not provides nodes on the boundary.



(a) The quadrature with degree of exactness $k = 7$, thus with $N = 16$ nodes. (b) The quadrature with degree of exactness $k = 11$, thus with $N = 36$ nodes.

Figure 7.1: Stroud's quadrature rule: a product rule based on weighted Gaussian quadratures, both along diameters and angular distances. The nodes are at the vertices of regular polygons.

The localization of the nodes⁴ are plotted in Figure 7.1 relative to the degree of exactness $k = 7$

³The even cases are slightly different in all the family of formulae that we consider. In order to enable rapid comparison and maintain simple notations we always refer to the odd cases.

⁴The plotted nodes are rotated accordingly in order to have some points on the x -axis for rapid comparison.

(with 16 points, see the left panel) and $k = 11$ (with 36 points, on the right). The calculations of weights are products of the weighted Gaussian 1D quadratures. They are well known but laborious. The corresponding detailed derivation procedures can be found in [Stroud, 1971]. The final explicit formulae, however, are given here more conveniently in Equation 7.9.

7.3.1.2 Peirce's cubature formula: equispaced angles

The second product rule is adopted from the cubature formula proposed in [Peirce, 1957]. This approach has been used extensively. See, for example, [De and Bathe, 2000, De and Bathe, 2001]. Its construction is made, as in the previous family, in the rectangular domain $\Psi \equiv [-1, 1] \times [-\pi/2, \pi/2]$. This quadrature is denoted by superscript $\mathbb{P}$. Inside this domain there are M Gaussian nodes in both the radial ρ and θ directions. Thus the integration formulae over a disk with an inner radius r_{in} , $\Omega = \{(x, y) \text{ s.t. } r_{in} \leq x^2 + y^2 \leq 1\}$, becomes

$$\iint_{\Omega} f(\zeta) d\zeta \approx \sum_{i=1}^M w_i^{\mathbb{P}} \sum_{j=1}^{4(M+1)} f(\rho_i \cos \theta_j, \rho_i \sin \theta_j) \quad (7.10)$$

where the angular and radial coordinates of nodes, respectively, are

$$\theta_i = \frac{i\pi}{2(M+1)} \quad \text{and} \quad \rho_j = \sqrt{\frac{1-r_{in}^2}{2} t_j^{\mathbb{G}} + \frac{1-r_{in}^2}{2}}. \quad (7.11)$$

The corresponding weights, $w_i^{\mathbb{P}}$, turns into

$$w_i^{\mathbb{P}} = \frac{\pi(r_{in}^2 - \rho_i^2)(1 - \rho_i^2) \left(\phi_i^{(M)}(1) - \phi_i^{(M)}(r_{in}) \right)}{2(1+M)(2+M) \left((1 - r_{in}^2) P_{M+2} \left(\frac{\varrho_i}{r_{in}^2 - 1} \right) + \varrho_i P_{M+1} \left(\frac{\varrho_i}{r_{in}^2 - 1} \right) \right)}$$

in which $\varrho_i = 1 + r_{in}^2 - 2\rho_i^2$ and

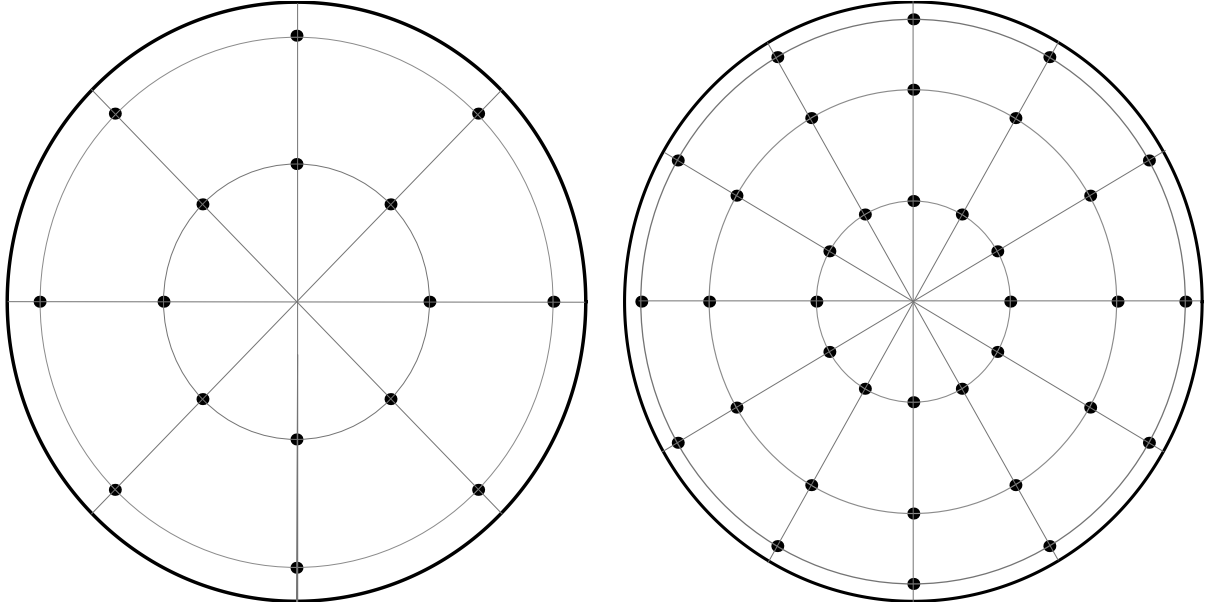
$$\phi_i^{(m)}(r) = \int_0^r \frac{\left(\frac{r_{in}^2 + 1 - 2t}{r_{in}^2 - 1} \right) P_{m+1}}{t - \rho_i^2} dt \quad \text{where} \quad r = 1, r_{in}.$$

Analytical evaluation of the last integration for individual M s leads to a significant reduction in the overall computational costs. The weights can be calculated so that the degree of exactness is $k = 2M - 1$. Subsequently the minimum number of nodes required to achieve the degree of exactness, k , is

$$N_{(k)}^{\mathbb{P}} = \frac{(k+1)^2}{4}.$$

Figure 7.2 indicates that the circular boundary lacks quadrature points again. Moreover, the overall quadrature is very similar to that of Stroud as weights are invariably positive. The number of points for the exactness requirement coincide by comparing Figure 7.1 and Figure 7.2. Furthermore the distribution patterns of points are seen to be very similar.

Peirce's quadrature has been used widely. This method has been extended to treat only portions of the circle such as circular sectors. See, for more information, [De and Bathe, 2001, Mazzia et al., 2007, Mazzia and Pini, 2010].



(a) The quadrature with $k = 7$, and $N = 16$ nodes.

(b) The quadrature with $k = 11$, and $N = 36$ nodes.

Figure 7.2: Peirce's quadrature rule which is a product rule based on Gauss nodes positioned along the diameters and equispaced angles.

7.3.1.3 Vianello and co-authors' product cubature formulae

More recently, Vianello and co-authors from the Padua group have derived an elegant reformulation of Peirce's quadrature. This is to accommodate for more complicated geometries arising from the intersections of circles and semi-planes such as lunes and lenses. See, for example, [Bos and Vianello, 2012] for more details. Their resulting formulation is summarized here in order to construct a product rule in the domain $\Psi \equiv [0, 1] \times [-\pi, \pi]$. Subsequently the overall number of nodes are doubled when diameters is replaced by radii. This approach, however, leads a simple way of treating subsets of the circle. We, specifically, choose the Gauss-Lobatto rule along the radial ρ direction, $\rho_l = 1/2t_l^{\text{GL}} + 1/2$, to obtain a rule which includes nodes on the boundary. Consequently

$$\begin{aligned} \iint_{\Omega} f(x, y) \, dx \, dy &= \int_0^1 \rho \int_{-\pi}^{\pi} f(\rho \cos \theta, \rho \sin \theta) \, d\theta \, d\rho \\ &\approx \int_0^1 \rho \sum_j w_j^{\text{V}} f(\rho \cos \theta_j, \rho \sin \theta_j) \, d\rho \\ &\approx \frac{1}{2} \sum_{jl} w_j^{\text{V}} w_l^{\text{GL}} \rho_l f(\rho_l \cos \theta_j, \rho_l \sin \theta_j) \end{aligned} \quad (7.12)$$

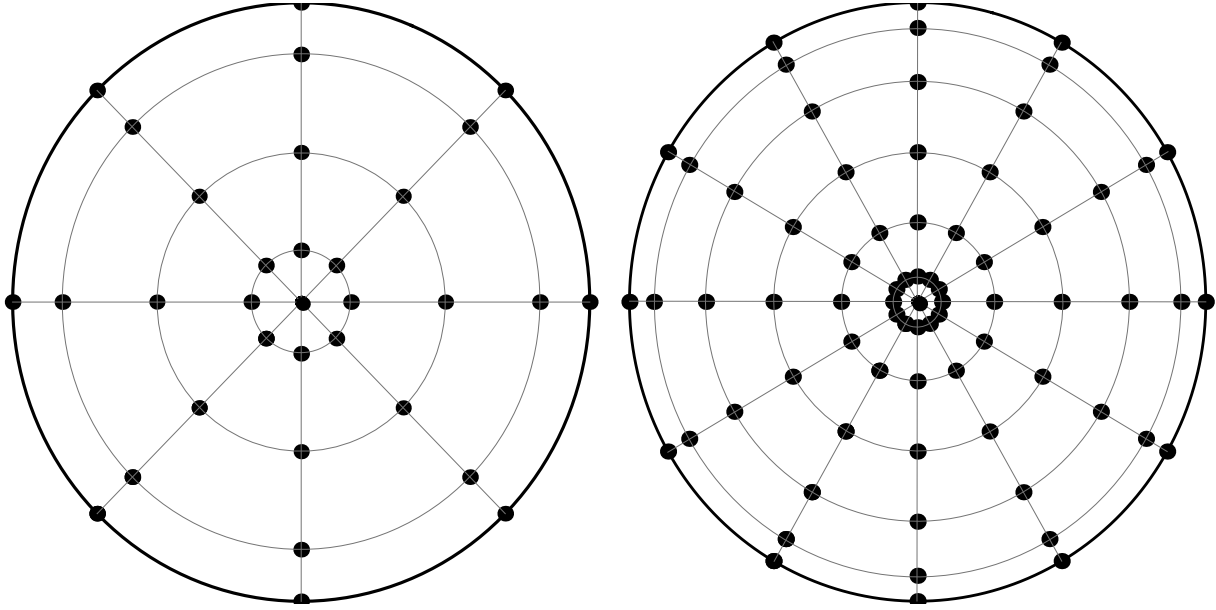
where

$$w_j^{\text{V}} = \frac{2}{M+1} \left(v_0 + 2 \sum_{k=1}^M v_{2k} T_{2k} \left[\cos \left(\frac{2j-1}{2(M+1)} \pi \right) \right] \right) \quad (7.13)$$

in which

$$v_k = \int_{-1}^1 \frac{T_k(x)}{\sqrt{1-x^2}} \, dx \quad (7.14)$$

is the so-called momentum associated with Chebyshev of the first kind $T_k(x)$ of degree k . The t^{GL} and w_l^{GL} can be obtained by using Equation 7.4. The integral given in Equation 7.14 is evaluated analytically for individual orders of k . A detailed derivation of Equation 7.13 is given in [Bos and Vianello, 2012] for the recursive calculation of w_j when the nodes are equispaced in angular direction, θ_j . Then the overall quadrature has the degree of exactness $k = 2M - 1$ where the $(M + 1)$ -degree Gauss-Lobatto rule is taken in the radial direction along with $2M$ equispaced angles. It should be noted that there are $N = 2M^2 + 1$ nodes if the center of the circle is counted only once.



(a) The quadrature with degree of exactness $k = 7$, thus with $N = 33$ nodes. (b) The quadrature with degree of exactness $k = 11$, thus with $N = 73$ nodes.

Figure 7.3: Vianello's quadrature rule which is a product rule based on Gauss-Lobatto radial abscissae and equispaced angles.

This quadrature is denoted by superscript $\mathbb{V}$. Number of nodes used to achieve exactness degree of k and the number of nodes placed on the boundary are, respectively,

$$N_{(k)}^{\mathbb{V}} = \frac{(k+1)^2}{2} + 1 \quad \text{and} \quad N_{\partial\Omega}^{\mathbb{V}} = k + 1 .$$

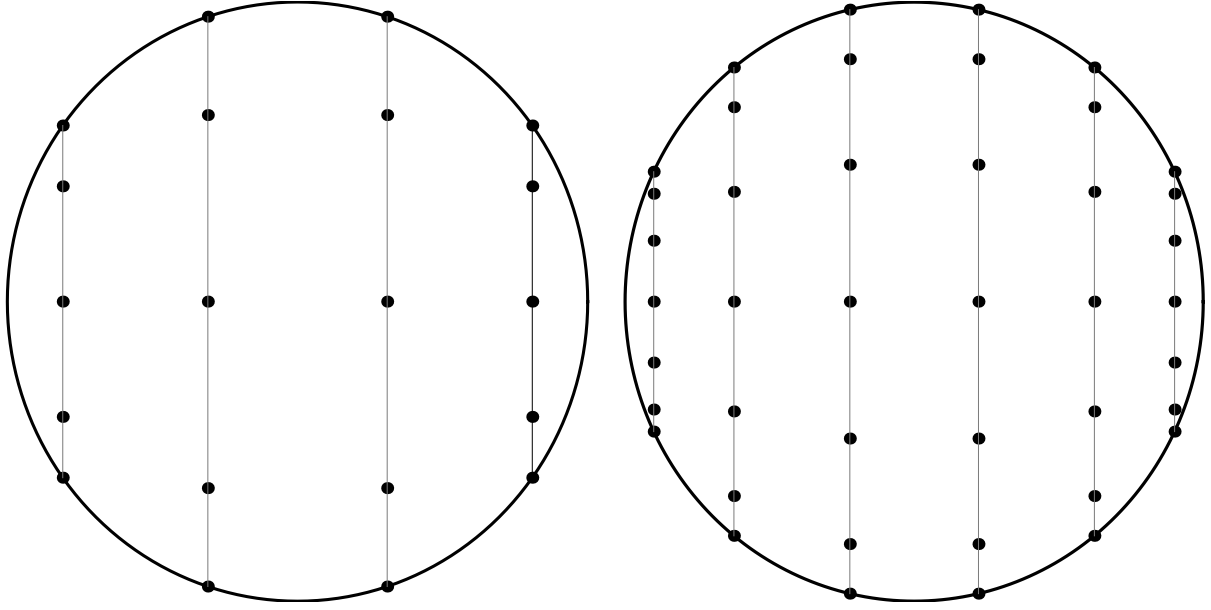
Figure 7.3 illustrates the distribution of nodes for degrees of exactness $k = 7$ (left side) and $k = 11$ (right side).

7.3.2 Line rules

Two different quadrature families, due to Bojanov and Hakopian, within the class of line rules are investigated next.

7.3.2.1 Bojanov's cubature formula: the optimal reduction rule

The method proposed by Bojanov has been described previously and culminates in the class of reduction formula Equation 7.5. The following results are deduced from [Bojanov and Petrova, 1998].



(a) The quadrature with degree of exactness $k = 7$, thus with $N = 20$ nodes. (b) The quadrature with degree of exactness $k = 11$, thus with $N = 42$ nodes.

Figure 7.4: Bojanov's quadrature rule: a line quadrature where the lines are chosen on the zeros of Chebychev polynomials and on the lines a Gauss-Lobatto rule is chosen.

Lemma 7.1. If x_l are the zeros of a Tchebycheff polynomial of the second kind and degree n so that

$$x_l = \cos \frac{l\pi}{n+1}, \quad i = 1, 2, \dots, n,$$

and

$$\alpha_l = \frac{\pi}{n+1} \sin \frac{l\pi}{n+1},$$

then formulae Equation 7.5 are exact for polynomials having degree $2n - 1$. Moreover, if $\mathbb{I}_l$ in this formula are calculated with a rule having an exactness degree of $2n - 1$, then the overall bivariate quadrature also has the degree of exactness $2n - 1$.

The $(M_L + 1)$ -Gauss-Lobatto rule⁵ is used to achieve the maximum exactness of $k = 2M_L - 1$. This cubature formula can be expressed explicitly as

$$\iint_{\Omega} f(\zeta) d\zeta \approx \sum_{l=1}^{M_L} \alpha_l q_{rt1} - x_l^2 \sum_{j=1}^{M_L+1} w_j^{\text{GL}} f(x_l, t_j^{\text{GL}} \sqrt{1 - x_l^2}) \quad (7.15)$$

where $N = M_L(M_L + 1)$ nodes are used. Consequently the number of nodes used to achieve exactness k and the number of points located on the boundary are, respectively,

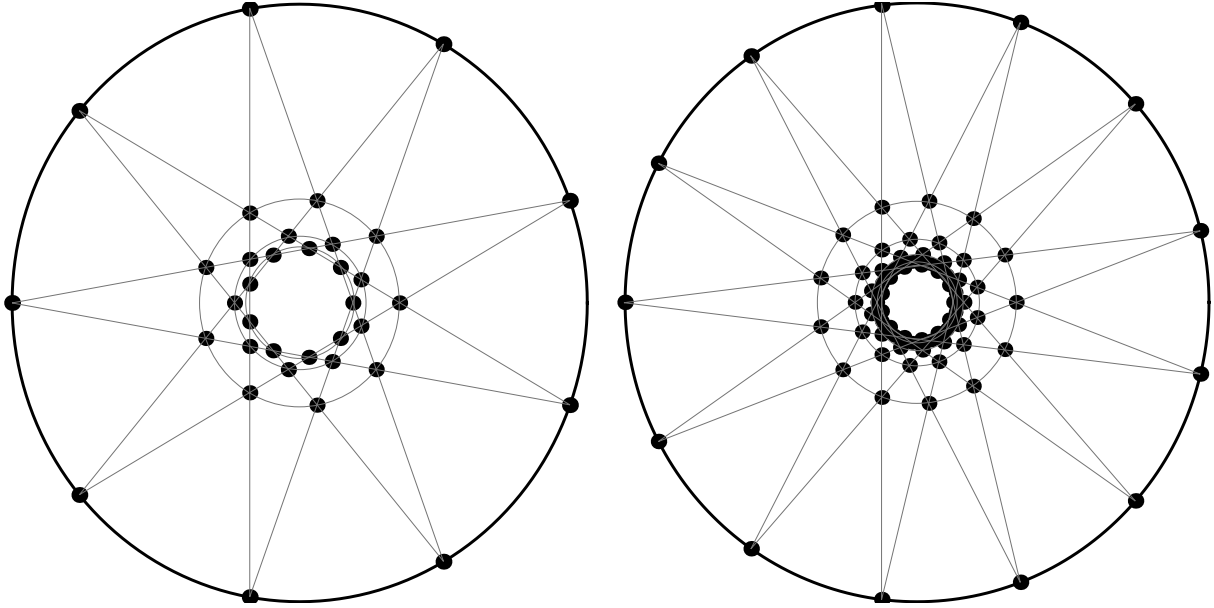
$$N_{(k)}^{\mathbb{B}} = \frac{(k+1)(k+3)}{4} \quad \text{and} \quad N_{\partial\Omega}^{\mathbb{B}} = k + 1.$$

The number of points on the boundary are identical in this cubature compared to the one of Vianello for a prescribed degree of exactness however the overall number is almost half. Bojanov's rule, on the other hand, is not well suited for composite computations because there is no

⁵Alternatively, applying Gaussian quadrature, for the same degree of exactness we need M_L nodes at each line. Nevertheless in this case, none of the quadrature points lies on the boundary.

straightforward generalization for subdomains of a circle. Localization of points are illustrated in [Figure 7.4](#) for degree of exactness $k = 7$ (left side) and $k = 11$ (right side). The corresponding number of points are 20 (along 4 lines) and 42 (along 6 lines), respectively.

7.3.2.2 Hakopian's cubature formula: star-like points



(a) The quadrature with degree of exactness $k = 7$, thus with $N = 39$ nodes. (b) The quadrature with degree of exactness $k = 11$, thus with $N = 78$ nodes.

Figure 7.5: Hakopian's quadrature rule: a line quadrature based on interpolation in star-like points.

The last family of quadrature rules presented here was proposed originally by Hakopian in [[Hakopian, 1984](#)] and then was developed further in [[Sauer and Xu, 1996](#)]. This quadrature is denoted here by $\mathbb{H}$. Similar to the product rules, the nodes are placed on concentric circles. There are $2M_C + 1$ equispaced nodes of

$$\theta_i = \begin{cases} \frac{2i+1}{2M_C+1}\pi, & \text{if } \{M_C + i = 2m+1 | m \in \mathbb{N}\} \\ \frac{2i}{2M_C+1}\pi, & \text{if } \{M_C + i = 2m | m \in \mathbb{N}\} \end{cases}$$

which are located on M_C circles with radii of

$$\rho_i = \frac{\sin \frac{\pi}{4M_C+2}}{\sin \frac{(2i-1)\pi}{4M_C+2}}.$$

The closed form formula for this quadrature family turns into

$$\iint_{\Omega} f(\zeta) d\zeta \approx \sum_{i=1}^M w_i^{\mathbb{H}} \sum_{j=0}^{2M} f(\rho_i \cos \theta_j, \rho_i \sin \theta_j) \quad (7.16)$$

where

$$w_i^{\mathbb{H}}(r) = \frac{\pi}{2m+1} \int_0^1 \left(\prod_{k=1}^{i-1} \frac{x - r_k^2}{r_i^2 - r_k^2} \prod_{k=i+1}^m \frac{x - r_k^2}{r_i^2 - r_k^2} \right) dx .$$

Analytical evaluation of the latter integral for individual orders of m results into a significant reduction in overall computational complexity of the quadrature scheme.

Integrations are made along chords as in Bojanov's quadrature. The nodes' distribution, however, forms a star-like pattern such as those demonstrated in [Figure 7.5](#). The degree of exactness is $k = 2M_C - 1$ when nodes are on M_C concentric circles. The number of nodes used to achieve exactness k and the number of points placed on the boundary are, respectively,

$$N_{(k)}^{\mathbb{H}} = \frac{(k+1)(k+2)}{2} \quad \text{and} \quad N_{\partial\Omega}^{\mathbb{H}} = k+2 .$$

[Figure 7.5](#) illustrates the distributions of the nodes for degrees of exactness $k = 7$ (with 39 points lying on 4 circles, see the left panel) and $k = 11$ (with 78 points lying on 6 circles, on the left and right sides). This quadrature has a number of unfavourable features despite of having a comparable number of nodes required for a prescribed exactness to those of previous formulae. First, nodes are not “well-distributed” as the number of nodes are increased. Indeed the circles where nodes are taken on have radius that are far from equispaced as the center is approached. Moreover the resulting weights are not always positive. Thus convergence is not guaranteed in functions, $f(\xi)$, having irregularities. These situations are discussed in the next section.

7.4 Numerical evaluations

All computations including the calculations of nodes, weights, reference solutions and cubature integrals are made in MATHEMATICA[©] environment. Relative errors are always plotted on a semi-log scale as either (a) the number of nodes or (b) the degree of exactness increases.

Convergence plots are presented to compare the performance of the previously introduced quadrature rules as the number of nodes increases. Examples are chosen so that the number of nodes is related to an odd degree of exactness. Then the numerical tests are divided into the following two classes.

1. *Continuous functions.* Quadrature rules invariably converge when they are applied to continuous functions which are also differentiable over the integration domain. The quadrature rules not always having positive weights, however, do not converge when applied to continuous but non-differentiable functions. generally the convergence speed (i.e the slopes in the later plots) depend on the function's regularity and the exactness properties of each family of quadrature formulae.
2. *Discontinuous functions.* The quadrature family diverges as nodes are increased if negative weights are involved. The cubature formulae with only positive weights achieve some degrees of convergence when they are used for discontinuous functions. The convergence rate, however, is considerably slower when compared to results of continuous test functions. This situation happens because the convergence is not always improved near a jump discontinuity

if interpolating polynomials of increasing degrees are taken. In other words, an increasing degree of exactness is not a sufficient condition to recover a prescribed convergence.

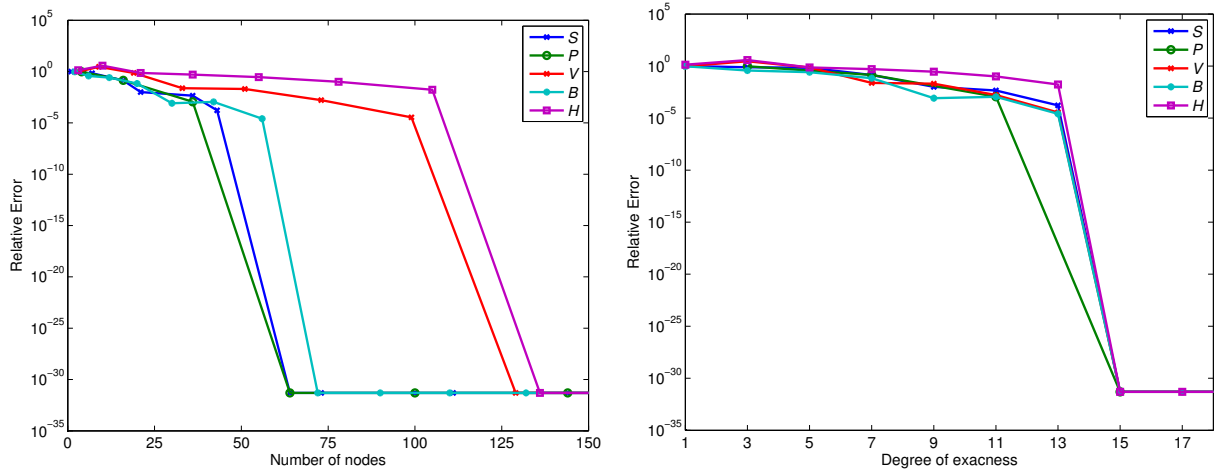
7.4.1 Continuous functions

Consider first the following polynomial

$$f(x, y) = (3x - 2y + 1)^{15} . \quad (7.17)$$

As expected, all the quadrature families obtain the exact value for this function up to machine precision for a sufficiently high number of interpolation points as presented in Figure 7.6. Moreover, there is no apparent instabilities as the number of nodes increases.

7.4.2 Continuous integrand functions



(a) Convergence history: relative error against number of nodes. (b) Convergence history: relative error against degree of exactness.

Figure 7.6: Convergence histories when applying the quadrature formulae to the polynomial test function Equation 7.17 in semi-log scale. It is reminded that S, P, V, B and H denote quadratures due to Stroud, Peirce, Vianello, Bojanov and Hakopian, respectively.

The second test concerns a regular function rather than a polynomial such as

$$f(x, y) = e^{-x^2} \cos^2 y . \quad (7.18)$$

The same comparisons are made in Figure 7.7 in the same way as the previous test functions. Cubature rules of Stroud, Peirce and Bojanov demonstrate a very comparable behaviour. These three families converge rapidly to a relative error of less than 10^{-16} and with approximately 125, 130 and 140 nodes, respectively. The other two families, however, converge with a slower pace. Overall the Hakopian's approach shows the slowest convergence rate.

The third test involves a continuous, but non-differentiable function such as

$$f(x, y) = |y| . \quad (7.19)$$

convergence test histories of this function are presented in Figure 7.8. Hakopian's quadrature does not converge which is basically because of the presence of negative weights. This demonstrates

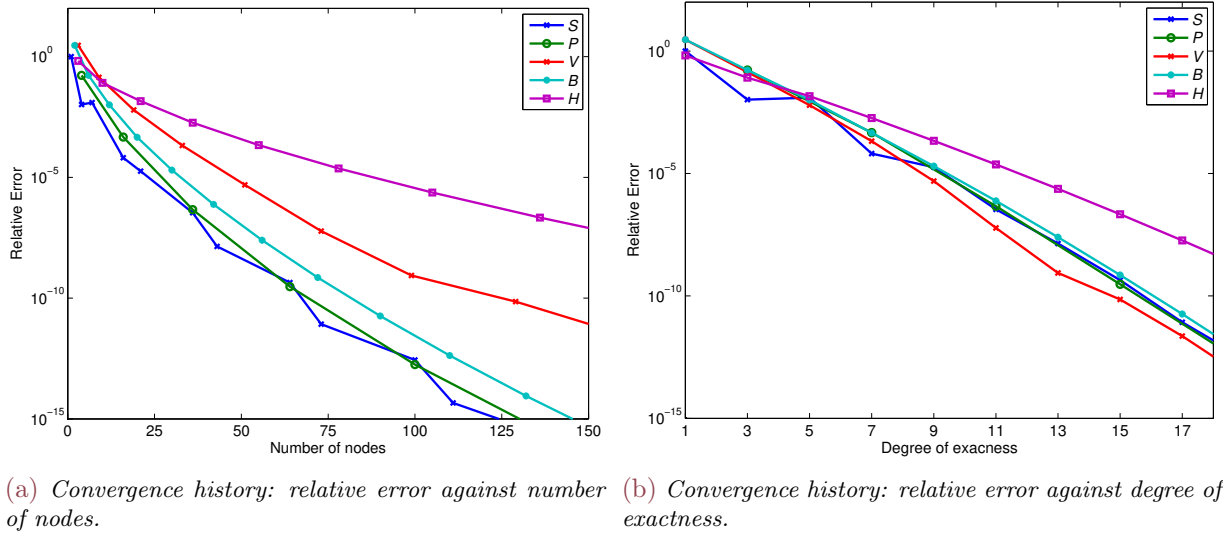


Figure 7.7: Convergence histories when the quadrature formulae are applied to the regular test function Equation 7.18 in semi-log scale. It is reminded that S, P, V, B and H denote quadratures due to Stroud, Peirce, Vianello, Bojanov and Hakopian, respectively.

well the importance of having only positive weight in the stability of a given quadrature rule. See, for example, [Keshavarzzadeh et al., 2018]. All the other families of quadrature rules converge similarly, but relative to previous test functions, slowly due to the function's reduced regularity.

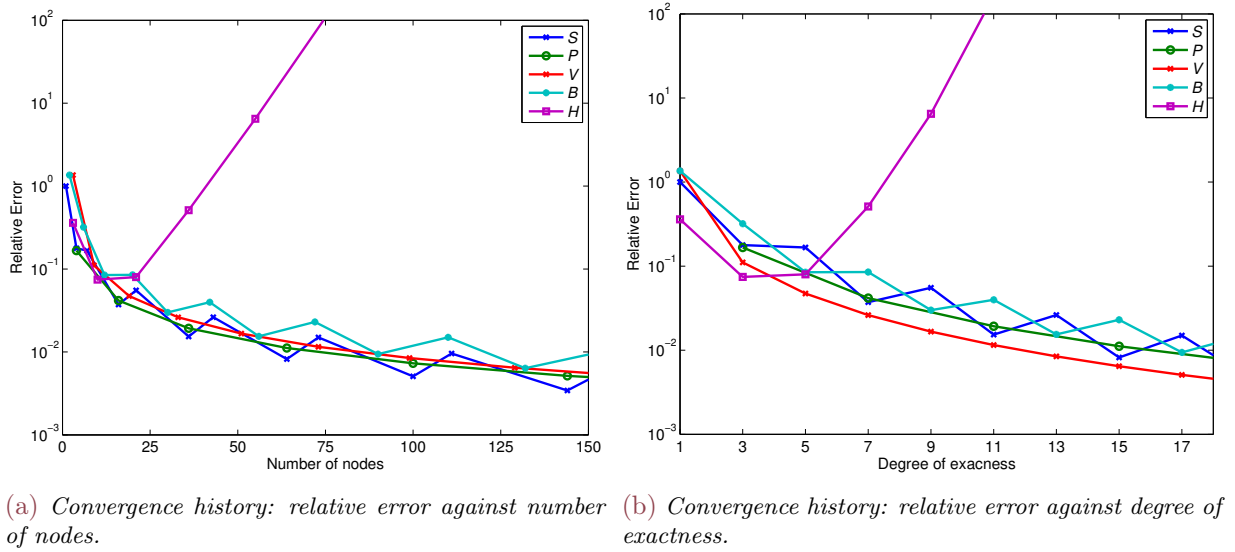


Figure 7.8: Convergence histories when applying the quadrature formulae to the continuous, non-differentiable test function Equation 7.19 in semi-log scale. It is reminded that S, P, V, B and H denote quadratures due to Stroud, Peirce, Vianello, Bojanov and Hakopian, respectively.

7.4.3 Discontinuous functions

Two cases of discontinuous functions are considered in this part which can be a model for heterogeneous domains such as shock wave on structures with circular cross sections, elastoplastic deformation, fiber structures, fouled channels in micro-fluids or particles near a boundary in meshless methods. Although a prescribed exactness is not necessarily guaranteed, tests are undertaken to determine also if a distribution of points is “good enough”.

Piecewise-constant functions are considered in both examples. This can be considered as the worst-case-scenario for a test function. A unity value and 0.1 are assumed in the two subdomains of white and black, respectively, as it is shown in Figure 7.9. Each of these subdomains has a different shape within an outer circle as demonstrated in the aforementioned figure.

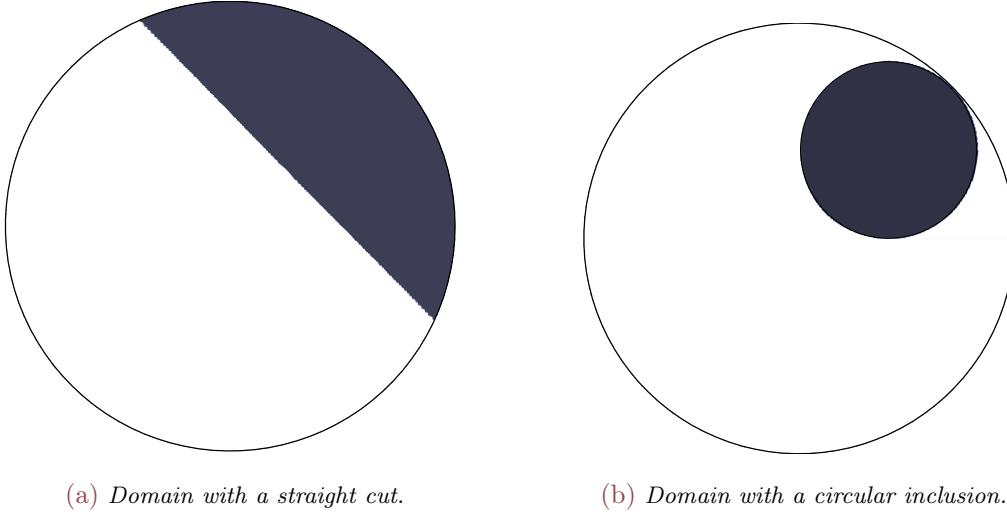
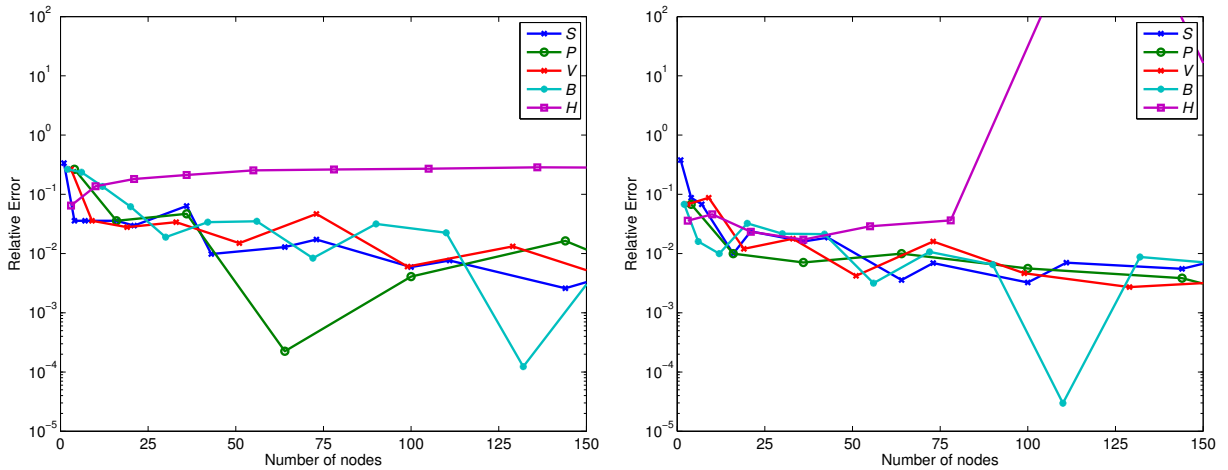


Figure 7.9: Two discontinuous domains are chosen for convergence history tests having a very sharp step-wise jumps.

Convergence tests for these two discontinuous functions are given in Figure 7.10. Hakopian's quadrature shows a rapid convergence at the beginning however it completely fails later on in both cases. Other cubature formulae demonstrate generally a slower convergence rate compared to regular functions. The convergence history can behave somewhat erratically. Relative errors less than 1% are achieved by using around 150 nodes, almost regardless.



(a) Convergence history in the domain with cut in a semi-log scale. (b) Convergence history in the domain with inclusion in a semi-log scale.

Figure 7.10: Convergence histories when applying the quadrature formulae to discontinuous function. In both cases we plot relative error against number of nodes in semi-log scale.

Clearly composite strategies are needed to possibly improve this situation if the location of discontinuities are known a priori.

Conclusions

Different families of quadrature formulae suited for integration over a circular domain are presented and compared. It is found that recent developments can be used profitably to obtain simple formulae which give a quick and accurate convergence providing (1) calculation nodes which are reasonably distributed and ordered, and (2) negative weights are avoided. All cubature presented formulae show a very rapid and smooth convergence for regular functions which are continuous and differentiable over the circular domain. The Hakopian's rule, however, fails completely when the data is not continuous or non-differentiable. Other quadrature rules show slower but reasonably smoother convergence history. A summary of the main points observed regarding cubature rules presented here are given next.

- Bojanov quadrature rule can provide nodes on the boundary. It shows a very stable behaviour and high convergence rate invariably. However it can not be used in a composite approach.
- Peirce and Stroud's demonstrate a comparably stable behaviour and high convergence rates. Both of these quadrature rules do not provide nodes on the boundary. Peirce formula, however, can be modified to be utilized in a composite approach.
- Padova rule demonstrates a very stable behaviour; can provide nodes on the boundary and can be used in a composite approach. Its convergence rate, however, is lower than Peirce, Stroud and Bojanov's approaches.
- Associated weights in Hakopian quadrature can be negative and consequently it does not converge for irregular functions. It has nodes on the boundary, however, this cubature formula does not provide an even distribution of nodes, especially in higher degrees of exactness.

A "composite" approach, such as those in Padova class, are suggested for irregular functions to improve the convergence and accuracy.

Conclusions

Main achievements of each chapter are given concisely in few sentences followed by a brief description of the works in progress and suggested researches as possible continuations of the current works.

A comparative study is carried out on the performance of the most popular shell theories in elasto-acoustic modelling of a fluid-filled, submerged shell structure subjected to transient excitations of various sources. Generalized unified, closed form solutions are given here for different models considered. A discontinuation in application of shell theories under a general nonstationary excitation is strongly recommended, henceforth. The presented study is done for the individual modal components. Another paper with a similar objective is in progress but on the total response rather than individual modal components. This will complement the current work. Inclusion of viscosity in the fluid media is the next step which is also in progress. This is an important factor in making a more realistic model, especially, when dealing with the ultrasonics frequency ranges. This part, also, is extended to a multi-layered and FGM shell structure which will appear in the future communications.

Subsequently, the exact linear theory is utilized to find the transient response of a shell filled with and submerged in compressible fluids under a general centrally symmetric excitation. A *Hertzian* distribution of contact forces is considered as well as commonly used assumption of a uniform profile. Such a situation can be a reasonably idealized model for a number of significant practical configurations such as head injury and high intensity ultrasound surgery. This loading configuration leads to omission of asymmetric modes which consequently leads to a very strong focusing of acoustical energy at the center of the shell. It is demonstrated quantitatively as well as qualitatively that under special arrangements, the focal pressure can be magnified, for example, by more than ten folds of an input load. Such findings convey practically significant implications.

Truly comprehensive investigations are performed on buried shells within elastic formations rather than submerged ones in fluid medium where the most general form of various excitation sources are considered and closed form solutions are developed. Numerical experiments are performed when the shell excited by a general surface traction, eccentric pressure source within the internal fluid as well as various combinations of spherical incident waves of different origin. Exact solutions, also, for laminated spheres are developed where the production of the corresponding numerical results are in progress. Furthermore, the effect of damping in the form of viscoelasticity also is under investigation. A modified Biot's constitutive model for the fluid saturated porous formations is a work in progress. A half-space situation, rather than the current full space configuration will be the subject of future researches. Multiple scattering by an array of spherical shells also will be the subject of upcoming communications.

The generalized Hooke's law along with generalized linear piezoelectricity for anisotropic materials for shells submerged in an acoustical wedge, rather than isotropic shells in an acoustical full space, is studied subsequently. Plane incident acoustical waves as well as traction forces on either surfaces of the shell are considered as the sources of excitation in the frequency domain.

Transient analysis of the same configuration is an immediate extension. Diffraction of incident waves by multiple cylindrical scatterers is, also, a straightforward extension to an acoustical wedge solution. Active and passive control of the pressure fields, however, can be an interesting topic for the future research.

Scattering of elastic and acoustical waves by spherical obstacles within materials having microstructure is an area with many thrilling applications from nano- and bio- mechanics to hypersonic aerospace industry. An appropriate theory of micropolars continua is required to be applied according to the specific intended application. Number of micropolars theories are established since the Cosserat brothers proposed their theory at the beginning of the last century [Cosserat et al., 1909]. Even though at early stages of such an elegant and exciting research front, it looks definitely a promising path with enormous opportunities that yet need to be explored.

Quadrature formulae are presented for circular domains being applicable, for example, to cylindrical and spherical shells. These integration schemes can be used within a mixed finite element framework or a meshless technique. A study on elasto-plastic response of cylinders is at its final stages where a mixed finite element framework is employed by using of these cubature formulae. These approach, however, can be extended to other problems with different sources of nonlinearity such as large deformations kinematics, hyperelastic constitutive models and discontinuities occurring due to the interaction of strong shock waves and structures.

Bibliography

- [Abate and Valkó, 2004] Abate, J. and Valkó, P. (2004). Multi-precision laplace transform inversion. International Journal for Numerical Methods in Engineering, 60(5):979–993.
- [Akkas et al., 2000] Akkas, N., Zakout, U., and Tupholme, G. (2000). Propagation of waves from a spherical cavity with and without a shell embedment. Acta Mechanica, 142(1-4):1–11.
- [Aleamar et al., 1986a] Aleamar, J., Delsanto, P., Rosario, E., Nagl, A., and Überall, H. (1986a). Spectral analysis of the scattering of acoustic waves from a fluid cylinder. iii. solution of the inverse scattering problem. Acta Acustica united with Acustica, 61(1):14–20.
- [Aleamar et al., 1986b] Aleamar, J. D., Delsanto, P. P., Rosario, E., Nagl, A., and Überall, H. (1986b). Spectral analysis of the scattering of acoustic waves from a fluid cylinder, part i & part ii: Denser fluid loading and denser fluid inside. Acta Acustica united with Acustica, 61(1):1–13.
- [Amir, 2013] Amir, N. (2013). Advances in non-invasive tube inspection using pulse reflectometry. In Proceedings of the 13th International Symposium on Nondestructive Characterization of Materials (NDCM-XIII). LAUM Le Mans.
- [Anderson, 1955] Anderson, R. E. (1955). Scattering of acoustic pulses by a solid sphere in air. The Journal of the Acoustical Society of America, 27(5):1006–1006.
- [Antolin et al., 2015] Antolin, P., Buffa, A., Calabrò, F., Martinelli, M., and Sangalli, G. (2015). Efficient matrix computation for tensor-product isogeometric analysis: The use of sum factorization. Computer Methods in Applied Mechanics and Engineering, 285:817–828.
- [Arons and Yennie, 1948] Arons, A. and Yennie, D. (1948). Energy partition in underwater explosion phenomena. Reviews of modern Physics, 20(3):519.
- [Asprone et al., 2011] Asprone, D., Auricchio, F., and Reali, A. (2011). Novel finite particle formulations based on projection methodologies. International Journal for Numerical Methods in Fluids, 65(11-12):1376–1388.
- [Atkins and Shu, 1998] Atkins, H. L. and Shu, C. (1998). Quadrature-free implementation of discontinuous galerkin method for hyperbolic equations. AIAA journal, 36(5):775–782.
- [Atluri and Zhu, 1998] Atluri, S. and Zhu, T. (1998). A new meshless local Petrov-Galerkin (MLPG) approach in computational mechanics. Computational mechanics, 22(2):117–127.
- [Auricchio and Sacco, 1999] Auricchio, F. and Sacco, E. (1999). A temperature-dependent beam for shape-memory alloys: constitutive modelling, finite-element implementation and numerical simulations. Computer Methods in Applied Mechanics and Engineering, 174(1):171–190.

- [Ávila-Carrera and Sánchez-Sesma, 2006] Ávila-Carrera, R. and Sánchez-Sesma, F. (2006). Scattering and diffraction of elastic p-and s-waves by a spherical obstacle: A review of the classical solution. Geofísica internacional, 45(1):3–21.
- [Babuška et al., 2003] Babuška, I., Banerjee, U., and Osborn, J. E. (2003). Survey of meshless and generalized finite element methods: a unified approach. Acta Numerica, 12(1):1–125.
- [Babuška et al., 2008] Babuška, I., Banerjee, U., Osborn, J. E., and Li, Q. (2008). Quadrature for meshless methods. International journal for numerical methods in engineering, 76(9):1434–1470.
- [Bahari and Popplewell, 2015] Bahari, A. and Popplewell, N. (2015). Comment on “transient response of an acoustic medium by an excited submerged spherical shell”[j. acoust. soc. am. 109 (6), 2789–2796 (2001)]. The Journal of the Acoustical Society of America, 137(5):2966–2969.
- [Balanis, 1999] Balanis, C. A. (1999). Advanced engineering electromagnetics. John Wiley & Sons.
- [Balasubramanian and Sujith, 2008] Balasubramanian, K. and Sujith, R. (2008). Non-normality and nonlinearity in combustion–acoustic interaction in diffusion flames. Journal of Fluid Mechanics, 594:29–57.
- [Bandak, 1997] Bandak, F. A. (1997). Biomechanics of impact traumatic brain injury. In Crashworthiness of transportation systems: Structural impact and occupant protection, pages 53–93. Springer.
- [Beranek, 1940] Beranek, L. L. (1940). Precision measurement of acoustic impedance. The Journal of the Acoustical Society of America, 12(1):3–13.
- [Bian et al., 2006] Bian, Z., Lim, C., and Chen, W. (2006). On functionally graded beams with integrated surface piezoelectric layers. Composite structures, 72(3):339–351.
- [Bishop and Smith, 1997] Bishop, G. C. and Smith, J. (1997). Scattering from an elastic shell and a rough fluid–elastic interface: Theory. The Journal of the Acoustical Society of America, 101(2):767–788.
- [Blake Jr, 1952] Blake Jr, F. (1952). Spherical wave propagation in solid media. The Journal of the Acoustical Society of America, 24(2):211–215.
- [Bojanov, 2005] Bojanov, B. (2005). Quadrature formulae of non-standard type. Rend. Mat. Appl, 25:195–221.
- [Bojanov and Petrova, 1998] Bojanov, B. and Petrova, G. (1998). Numerical integration over a disc. A new Gaussian quadrature formula. Numerische Mathematik, 80(1):39–59.
- [Bos and Vianello, 2012] Bos, L. and Vianello, M. (2012). Subperiodic trigonometric interpolation and quadrature. Applied Mathematics and Computation, 218(21):10630–10638.
- [Bucaro et al., 2008] Bucaro, J., Houston, B., Saniga, M., Dragonette, L., Yoder, T., Dey, S., Kraus, L., and Carin, L. (2008). Broadband acoustic scattering measurements of underwater unexploded ordnance (uxo). The Journal of the Acoustical Society of America, 123(2):738–746.

- [Buckingham and Tolstoy, 1990] Buckingham, M. J. and Tolstoy, A. (1990). An analytical solution for benchmark problem 1: the “ideal”wedge. The Journal of the Acoustical Society of America, 87(4):1511–1513.
- [Calabrò et al., 2013] Calabrò, F., Lee, K., and Mattia, D. (2013). Modelling flow enhancement in nanochannels: Viscosity and slippage. Applied Mathematics Letters, 26(10):991–994.
- [Chadan and Sabatier, 2012] Chadan, K. and Sabatier, P. C. (2012). Inverse problems in quantum scattering theory. Springer Science & Business Media.
- [Chadwick and Trowbridge, 1967a] Chadwick, P. and Trowbridge, E. (1967a). Oscillations of a rigid sphere embedded in an infinite elastic solid: part II. rectilinear oscillations. Mathematical Proceedings of the Cambridge Philosophical Society, 63(4):1207–1227.
- [Chadwick and Trowbridge, 1967b] Chadwick, P. and Trowbridge, E. A. (1967b). Oscillations of a rigid sphere embedded in an infinite elastic solid: part I. torsional oscillations. Mathematical Proceedings of the Cambridge Philosophical Society, 63(4):1189–1205.
- [Chandra and Thompson, 1992] Chandra, K. and Thompson, C. (1992). Improved perturbation method for scattering from a fluid cylinder. The Journal of the acoustical Society of america, 92(2):1047–1055.
- [Chen and Stepanishen, 1994] Chen, H.-W. and Stepanishen, P. (1994). Acoustic transient radiation from fluid-loaded shells of revolution using time-dependent in vacuo eigenvector expansions. The Journal of the Acoustical Society of America, 95(2):601–616.
- [Chopra and Sirohi, 2013] Chopra, I. and Sirohi, J. (2013). Smart structures theory, volume 35. Cambridge University Press.
- [Clay, 1991] Clay, C. (1991). Low-resolution acoustic scattering models: fluid-filled cylinders and fish with swim bladders. The Journal of the Acoustical Society of America, 89(5):2168–2179.
- [Cohen, 2007] Cohen, A. M. (2007). Numerical Methods for Laplace Transform Inversion, volume 5. Springer Science & Business Media.
- [Colton and Kress, 2012] Colton, D. and Kress, R. (2012). Inverse acoustic and electromagnetic scattering theory, volume 93. Springer Science & Business Media.
- [Comsol, 2005] Comsol, A. (2005). Comsol multiphysics user’s guide. Version: September, 10:333.
- [Cools, 1997] Cools, R. (1997). Constructing cubature formulae: the science behind the art. Acta numerica, 6(1):1–54.
- [Cools, 2003] Cools, R. (2003). An encyclopaedia of cubature formulas. Journal of complexity, 19(3):445–453.
- [Cools and Haegemans, 2003] Cools, R. and Haegemans, A. (2003). Algorithm 824: CUBPACK: A package for automatic cubature; framework description. ACM Transactions on Mathematical Software (TOMS), 29(3):287–296.

- [Cools and Kim, 2000] Cools, R. and Kim, K. J. (2000). A survey of known and new cubature formulas for the unit disk. Korean Journal of Computational & Applied Mathematics, 7(3):477–485.
- [Cosserat et al., 1909] Cosserat, E., Cosserat, F., et al. (1909). Théorie des corps déformables.
- [Costa et al., 2011] Costa, E., Godinho, L., Santiago, J., Pereira, A., and Dors, C. (2011). Efficient numerical models for the prediction of acoustic wave propagation in the vicinity of a wedge coastal region. Engineering Analysis with Boundary Elements, 35(6):855–867.
- [Coyette and Van den Nieuwenhof, 2000] Coyette, J. P. and Van den Nieuwenhof, B. (2000). A conjugated infinite element method for half-space acoustic problems. The Journal of the Acoustical Society of America, 108(4):1464–1473.
- [Crump, 1976] Crump, K. S. (1976). Numerical inversion of Laplace transforms using a Fourier series approximation. Journal of the ACM (JACM), 23(1):89–96.
- [Da Fies and Vianello, 2012] Da Fies, G. and Vianello, M. (2012). Trigonometric Gaussian quadrature on subintervals of the period. Electronic Transactions on Numerical Analysis, 39:102–112.
- [D’amore et al., 1999] D’amore, L., Laccetti, G., and Murli, A. (1999). An implementation of a Fourier series method for the numerical inversion of the Laplace transform. ACM Transactions on Mathematical Software (TOMS), 25(3):279–305.
- [Darmon et al., 2003] Darmon, M., Calmon, P., and Bele, B. (2003). Modelling of the ultrasonic response of inclusions in steels. In AIP Conference Proceedings, number A, pages 101–108.
- [Datta and Shah, 1970] Datta, S. K. and Shah, A. H. (1970). Diffraction of sh-waves in a spherically isotropic medium by a rigid or fluid spherical core. Geophysical Journal International, 21(1):33–46.
- [De and Bathe, 2000] De, S. and Bathe, K. (2000). The method of finite spheres. Computational Mechanics, 25(4):329–345.
- [De and Bathe, 2001] De, S. and Bathe, K.-J. (2001). The method of finite spheres with improved numerical integration. Computers & Structures, 79(22-25):2183–2196.
- [De Branges and Rovnyak, 1966] De Branges, L. and Rovnyak, J. (1966). Canonical models in quantum scattering theory. Perturbation theory and its applications in quantum mechanics, pages 295–392.
- [Diercks and Hickling, 1967] Diercks, K. J. and Hickling, R. (1967). Echoes from hollow aluminum spheres in water. The Journal of the Acoustical Society of America, 41(2):380–393.
- [Ding and Chen, 2001] Ding, H. and Chen, W. (2001). Three dimensional problems of piezoelectricity. Nova Science Publishers.
- [Doolittle et al., 1988] Doolittle, R., Tolstoy, A., and Buckingham, M. (1988). Experimental confirmation of horizontal refraction of cw acoustic radiation from a point source in a wedge-shaped ocean environment. The Journal of the Acoustical Society of America, 83(6):2117–2125.

- [Drinkwater and Wilcox, 2006] Drinkwater, B. W. and Wilcox, P. D. (2006). Ultrasonic arrays for non-destructive evaluation: A review. NDT & E International, 39(7):525–541.
- [Duffey and Johnson, 1981] Duffey, T. and Johnson, J. (1981). Transient response of a pulsed spherical shell surrounded by an infinite elastic medium. International Journal of Mechanical Sciences, 23(10):589–593.
- [DuMond et al., 1946] DuMond, J. W., Cohen, E. R., Panofsky, W., and Deeds, E. (1946). A determination of the wave forms and laws of propagation and dissipation of ballistic shock waves. The Journal of the Acoustical Society of America, 18(1):97–118.
- [Durbin, 1974] Durbin, F. (1974). Numerical inversion of laplace transforms: an efficient improvement to *dubner* and *abate*’s method. The Computer Journal, 17(4):371–376.
- [Dwivedi et al., 2013] Dwivedi, J., Singh, V., and Lal, R. K. (2013). Analysis of transient dynamic behaviour of spherical cavity in viscoelastic soil medium. ISRN Mechanical Engineering, 2013.
- [Einspruch and Truell, 1960] Einspruch, N. G. and Truell, R. (1960). Scattering of a plane longitudinal wave by a spherical fluid obstacle in an elastic medium. The Journal of the Acoustical Society of America, 32(2):214–220.
- [Engin, 1969a] Engin, A. E. (1969a). The axisymmetric response of a fluid-filled spherical shell to a local radial impulse—a model for head injury. Journal of Biomechanics, 2(3):325–341.
- [Engin, 1969b] Engin, A. E. (1969b). Vibrations of fluid-filled spherical shells. The Journal of the Acoustical Society of America, 46(1B):186–190.
- [Eringen, 1957] Eringen, A. C. (1957). Elasto-dynamic problem concerning the spherical cavity. The Quarterly Journal of Mechanics and Applied Mathematics, 10(3):257–270.
- [Faccioli et al., 1997] Faccioli, E., Maggio, F., Paolucci, R., and Quarteroni, A. (1997). 2d and 3d elastic wave propagation by a pseudo-spectral domain decomposition method. Journal of seismology, 1(3):237–251.
- [Faddeev, 1995] Faddeev, L. (1995). Inverse problem of quantum scattering theory ii. In 40 Years In Mathematical Physics, pages 121–183. World Scientific.
- [Fawcett, 2014] Fawcett, J. A. (2014). Scattering from a finite cylinder near an interface. The Journal of the Acoustical Society of America, 136(2):485–493.
- [Federhofer, 1937] Federhofer, K. (1937). Zur Berechnung der Eigenschwingungen der Kugelschale. Hölder-Pichler-Tempsky.
- [Firoozbakhsh and DeSilva, 1975] Firoozbakhsh, K. and DeSilva, C. (1975). A model of brain shear under impulsive torsional loads. Journal of Biomechanics, 8(1):65–73.
- [Fischell and Schmidt, 2015] Fischell, E. M. and Schmidt, H. (2015). Classification of underwater targets from autonomous underwater vehicle sampled bistatic acoustic scattered fields. The Journal of the Acoustical Society of America, 138(6):3773–3784.

- [Foiret et al., 2012] Foiret, J., Minonzio, J.-G., Talmant, M., and Laugier, P. (2012). Cortical bone quality assessment using quantitative ultrasound on long bones. In Engineering in Medicine and Biology Society (EMBC), 2012 Annual International Conference of the IEEE, pages 1121–1124. IEEE.
- [Gan-bin and Kang-he, 2005] Gan-bin, L. and Kang-he, X. (2005). Transient response of a spherical cavity with a partially sealed shell embedded in viscoelastic saturated soil. Journal of Zhejiang University Science A, 6(3):194–201.
- [Gaunaurd et al., 1995] Gaunaurd, G., Huang, H., and Strifors, H. (1995). Acoustic scattering by a pair of spheres. The Journal of the Acoustical Society of America, 98(1):495–507.
- [Gaunaurd and Wertman, 1991] Gaunaurd, G. and Wertman, W. (1991). Transient acoustic scattering by fluid-loaded elastic shells. International Journal of Solids and Structures, 27(6):699–711.
- [Gautschi, 1981] Gautschi, W. (1981). A survey of Gauss-Christoffel quadrature formulae. In E. B. Christoffel (Aachen/Monschau, 1979), pages 72–147. Birkhäuser, Basel.
- [Geers and Zhang, 1994] Geers, T. and Zhang, P. (1994). Doubly asymptotic approximations for submerged structures with internal fluid volumes: formulation. Journal of Applied Mechanics, 61(4):893–899.
- [Geers, 1972] Geers, T. L. (1972). Scattering of a transient acoustic wave by an elastic cylindrical shell. The Journal of the Acoustical Society of America, 51(5B):1640–1651.
- [Geers, 1978] Geers, T. L. (1978). Doubly asymptotic approximations for transient motions of submerged structures. The Journal of the Acoustical Society of America, 64(5):1500–1508.
- [Gholamy and Kreinovich, 2014] Gholamy, A. and Kreinovich, V. (2014). Why ricker wavelets are successful in processing seismic data: Towards a theoretical explanation. In Computational Intelligence for Engineering Solutions (CIES), 2014 IEEE Symposium on, pages 11–16. IEEE.
- [Goldsmith and Allen, 1955] Goldsmith, W. and Allen, W. A. (1955). Graphical representation of the spherical propagation of explosive pulses in elastic media. The Journal of the Acoustical Society of America, 27(1):47–55.
- [Grandjean and Leparoux, 2004] Grandjean, G. and Leparoux, D. (2004). The potential of seismic methods for detecting cavities and buried objects: experimentation at a test site. Journal of Applied Geophysics, 56(2):93–106.
- [Grant and West, 1965] Grant, F. S. and West, G. F. (1965). Interpretation theory in applied geophysics. McGraw-Hill Book.
- [Gustafson III and Stepanishen, 1981] Gustafson III, H. L. and Stepanishen, P. R. (1981). Effects of multiple scattering and acoustic interactions among two parallel hydrophones and a planar baffle. The Journal of the Acoustical Society of America, 69(2):380–389.
- [Hakopian, 1982] Hakopian, H. (1982). Multivariate divided differences and multivariate interpolation of Lagrange and Hermite type. Journal of Approximation Theory, 34(3):286–305.

- [Hakopian, 1984] Hakopian, H. (1984). Multivariate interpolation ii of langrange and hermite type. Studia Mathematica, 80:77–88.
- [Hasheminejad and Alibakhshi, 2006] Hasheminejad, S. and Alibakhshi, M. (2006). Two-dimensional scattering from an impenetrable cylindrical obstacle in an acoustic quarterspace. Forschung im Ingenieurwesen, 70(3):179–186.
- [Hasheminejad et al., 2011] Hasheminejad, S. M., Bahari, A., and Abbasian, S. (2011). Modelling and simulation of acoustic pulse interaction with a fluid-filled hollow elastic sphere through numerical laplace inversion. Applied Mathematical Modelling, 35(1):22–49.
- [Hasheminejad and Mojahed, 2016] Hasheminejad, S. M. and Mojahed, A. (2016). Transient sound radiation from an impulsively excited piezoelectric composite hollow sphere in a wedge-shaped acoustic domain. Wave Motion, 62:1–19.
- [Hasheminejad and Mousavi-Akbarzadeh, 2015] Hasheminejad, S. M. and Mousavi-Akbarzadeh, H. (2015). Transient acoustic radiation from an eccentric sphere. Applied Mathematical Modelling, 39(17):5033–5046.
- [Heaviside, 1893] Heaviside, O. (1893). A gravitational and electromagnetic analogy. The Electrician, 31(18):5125–5134.
- [Honig and Hirdes, 1984] Honig, G. and Hirdes, U. (1984). A method for the numerical inversion of laplace transforms. Journal of Computational and Applied Mathematics, 10(1):113–132.
- [Hsu and Dranoff, 1987] Hsu, J. and Dranoff, J. (1987). Numerical inversion of certain laplace transforms by the direct application of fast fourier transform (fft) algorithm. Computers & chemical engineering, 11(2):101–110.
- [Huang, 1969] Huang, H. (1969). Transient interaction of plane acoustic waves with a spherical elastic shell. The Journal of the Acoustical Society of America, 45(3):661–670.
- [Huang, 1970] Huang, H. (1970). An exact analysis of the transient interaction of acoustic plane waves with a cylindrical elastic shell. Journal of Applied mechanics, 37(4):1091–1099.
- [Huang, 1979] Huang, H. (1979). Transient response of two fluid-coupled spherical elastic shells to an incident pressure pulse. The Journal of the Acoustical Society of America, 65(4):881–887.
- [Huang, 1981] Huang, H. (1981). Interaction of acoustic shock waves with a cylindrical elastic shell immersed near a hard surface. Wave motion, 3(3):269–278.
- [Huang et al., 1971] Huang, H., Lu, Y.-P., and Wang, Y.-F. (1971). Transient interaction of spherical acoustic waves and a spherical elastic shell. Journal of Applied Mechanics, 38:71.
- [Huang and Mair, 1996] Huang, H. and Mair, H. U. (1996). Neoclassical solution of transient interaction of plane acoustic waves with a spherical elastic shell. Shock and Vibration, 3(2):85–98.
- [Huang and Wang, 1972] Huang, H. and Wang, Y. (1972). Transient stress concentration by a spherical cavity in an elastic medium. Journal of Applied Mechanics, 39(4):1002–1004.

- [Ivanov, 1970] Ivanov, E. A. (1970). Diffraction of electromagnetic waves on two bodies, volume 597. National Aeronautics and Space Administration, Springfield, Va.
- [Iwashimizu, 1972] Iwashimizu, Y. (1972). Scattering of elastic waves by a movable rigid sphere embedded in an infinite elastic solid. Journal of Sound and Vibration, 21(4):463–469.
- [Jain and Kanwal, 1980] Jain, D. and Kanwal, R. (1980). Scattering of elastic waves by an elastic sphere. International Journal of Engineering Science, 18(6):829–839.
- [Jeffreys, 1931] Jeffreys, H. (1931). On the cause of oscillatory movement in seismograms. Geophysical Journal International, 2(s8):407–416.
- [Jiangong et al., 2009] Jiangong, Y., Bin, W., and Guoqiang, C. (2009). Wave characteristics in functionally graded piezoelectric hollow cylinders. Archive of Applied Mechanics, 79(9):807–824.
- [Junger, 1952] Junger, M. C. (1952). Sound scattering by thin elastic shells. The Journal of the Acoustical Society of America, 24(4):366–373.
- [Junger and Feit, 1986] Junger, M. C. and Feit, D. (1986). Sound, structures, and their interaction, volume 225. MIT press Cambridge, MA.
- [Kahnert, 2003] Kahnert, F. M. (2003). Numerical methods in electromagnetic scattering theory. Journal of Quantitative Spectroscopy and Radiative Transfer, 79:775–824.
- [Keles, 2016] Keles, I. (2016). Novel approach to forced vibration behavior of anisotropic thick-walled spheres. AIAA Journal, pages 1438–1442.
- [Keshavarzzadeh et al., 2018] Keshavarzzadeh, V., Kirby, R. M., and Narayan, A. (2018). Numerical integration in multiple dimensions with designed quadrature. arXiv preprint arXiv:1804.06501.
- [Knopoff, 1959] Knopoff, L. (1959). Scattering of compression waves by spherical obstacles. Geophysics, 24(1):30–39.
- [Krautkrämer and Krautkrämer, 2013] Krautkrämer, J. and Krautkrämer, H. (2013). Ultrasonic testing of materials. Springer Science & Business Media.
- [Lamb, 1882] Lamb, H. (1882). On the vibrations of a spherical shell. Proceedings of the London Mathematical Society, 1(1):50–56.
- [Laverty and Gazonas, 2006] Laverty, R. R. and Gazonas, G. A. (2006). An improvement to the fourier series method for inversion of laplace transforms applied to elastic and viscoelastic waves. International Journal of Computational Methods, 3(01):57–69.
- [Lebedeva, 1994] Lebedeva, I. (1994). The motion of a rigid spherical inclusion in an elastic medium under the action of plane waves. Journal of Mathematical Sciences, 70(5):1978–1984.
- [Leblond and Sigrist, 2010] Leblond, C. and Sigrist, J.-F. (2010). A versatile approach to the study of the transient response of a submerged thin shell. Journal of Sound and Vibration, 329(1):56–71.

- [Leissa, 1973] Leissa, A. W. (1973). Vibration of shells, volume 288. Scientific and Technical Information Office, National Aeronautics and Space Administration Washington.
- [Li et al., 1994] Li, W., Wu, T., and Seybert, A. (1994). A half-space boundary element method for acoustic problems with a reflecting plane of arbitrary impedance. Journal of Sound and Vibration, 171(2):173–184.
- [Lou and Klosner, 1973] Lou, Y. K. and Klosner, J. (1973). Transient response of a point-excited submerged spherical shell. Journal of Applied Mechanics, 40(4):1078–1084.
- [Love, 1920] Love, A. E. H. (1920). A treatise on the mathematical theory of elasticity. at the University Press.
- [Martin, 2006] Martin, P. A. (2006). Multiple Scattering: Interaction of Time-Harmonic Waves with N Obstacles. Number 107. Cambridge University Press.
- [Mattia and Calabrò, 2012] Mattia, D. and Calabrò, F. (2012). Explaining high flow rate of water in carbon nanotubes via solid–liquid molecular interactions. Microfluidics and nanofluidics, 13(1):125–130.
- [Mazzia et al., 2007] Mazzia, A., Ferronato, M., Pini, G., and Gambolati, G. (2007). A comparison of numerical integration rules for the meshless local Petrov–Galerkin method. Numerical Algorithms, 45(1-4):61–74.
- [Mazzia and Pini, 2010] Mazzia, A. and Pini, G. (2010). Product gauss quadrature rules vs. cubature rules in the meshless local Petrov–Galerkin method. Journal of Complexity, 26(1):82–101.
- [McLeary, 1969] McLeary, R. (1969). The interaction of a plane wave with a spherical cavity. Journal of Applied Mechanics, 36(3):644–646.
- [Menton and Magrab, 1974] Menton, R. T. and Magrab, E. B. (1974). Farfield radiation of internally generated transient acoustic pulses transmitted through a spherical shell. The Journal of the Acoustical Society of America, 55(2):237–246.
- [Milenkovic and Raynor, 1966] Milenkovic, V. and Raynor, S. (1966). Reflection of a plane acoustic step wave from an elastic spherical membrane. The Journal of the Acoustical Society of America, 39(3):556–563.
- [Moon, 1972] Moon, F. (1972). On the scattering of spherical elastic waves by a spherical cavity. Journal of Applied Mechanics, 39(2):591–593.
- [Moon and Pao, 1967] Moon, F. and Pao, Y. (1967). The influence of the curvature of spherical waves on dynamic stress concentration. Journal of Applied Mechanics, 34(2):373–379.
- [Morse, 1936] Morse, P. M. (1936). Vibration and sound, volume 3. McGraw-Hill New York.
- [Morse and Ingard, 1968] Morse, P. M. and Ingard, K. U. (1968). Theoretical Acoustics. Princeton university press.

- [Mow, 1965] Mow, C. (1965). Transient response of a rigid spherical inclusion in an elastic medium. Journal of Applied Mechanics, 32(3):637–642.
- [Mow and Pao, 1971] Mow, C.-C. and Pao, Y.-H. (1971). The Diffraction of Elastic Waves and Dynamic Stress Concentrations. Rand Corporation.
- [Murthy and Reddy, 1967] Murthy, G. N. and Reddy, D. (1967). Response of an embedded spherical shell to a plane wave. Nuclear Engineering and Design, 5(4):426–432.
- [Naeeni et al., 2015] Naeeni, M. R., Campagna, R., Eskandari-Ghadi, M., and Ardalan, A. A. (2015). Performance comparison of numerical inversion methods for laplace and hankel integral transforms in engineering problems. Applied Mathematics and Computation, 250:759–775.
- [Naghdi and Cooper, 1956] Naghdi, P. and Cooper, R. (1956). Propagation of elastic waves in cylindrical shells, including the effects of transverse shear and rotatory inertia. The Journal of the Acoustical Society of America, 28(1):56–63.
- [Naghdi, 1962] Naghdi, P. M. (1962). Foundations of elastic shell theory. Technical report, California Univ Berkeley Inst of Engineering Research.
- [Nair et al., 2014] Nair, V., Thampi, G., and Sujith, R. (2014). Intermittency route to thermoacoustic instability in turbulent combustors. Journal of Fluid Mechanics, 756:470–487.
- [Navarro et al., 1985] Navarro, R., Santamaría, J., and Bescós, J. (1985). Accommodation-dependent model of the human eye with aspherics. JOSA A, 2(8):1273–1280.
- [Norris and Osipov, 1997] Norris, A. and Osipov, A. (1997). Structural and acoustical wave interaction at a wedge-shaped junction of fluid-loaded plates. The Journal of the Acoustical Society of America, 101(2):867–876.
- [Norton and Linzer, 1981] Norton, S. J. and Linzer, M. (1981). Ultrasonic reflectivity imaging in three dimensions: exact inverse scattering solutions for plane, cylindrical, and spherical apertures. IEEE Transactions on biomedical engineering, (2):202–220.
- [Norwood and Miklowitz, 1967] Norwood, F. R. and Miklowitz, J. (1967). Diffraction of transient elastic waves by a spherical cavity. Journal of Applied Mechanics, 34(3):735–744.
- [Nussenzveig, 1965] Nussenzveig, H. M. (1965). High-frequency scattering by an impenetrable sphere. Annals of Physics, 34(1):23–95.
- [Olver, 2010] Olver, F. W. (2010). NIST handbook of mathematical functions hardback and CD-ROM. Cambridge University Press.
- [Osipov and Norris, 1999] Osipov, A. and Norris, A. (1999). The malyuzhinets theory for scattering from wedge boundaries: a review. Wave motion, 29(4):313–340.
- [Pao and Mow, 1963] Pao, Y.-H. and Mow, C. (1963). Scattering of plane compressional waves by a spherical obstacle. Journal of Applied Physics, 34(3):493–499.
- [Payton, 1960] Payton, R. G. (1960). Transient interaction of an acoustic wave with a circular cylindrical elastic shell. The Journal of the Acoustical Society of America, 32(6):722–729.

- [Peirce, 1957] Peirce, W. H. (1957). Numerical integration over the planar annulus. Journal of the Society for Industrial & Applied Mathematics, 5(2):66–73.
- [Perlman, 1941] Perlman, H. (1941). Acoustic trauma in man: Clinical and experimental studies. Archives of Otolaryngology, 34(3):429–452.
- [Potsika et al., 2014] Potsika, V. T., Grivas, K. N., Protopappas, V. C., Vavva, M. G., Raum, K., Rohrbach, D., Polyzos, D., and Fotiadis, D. I. (2014). Application of an effective medium theory for modeling ultrasound wave propagation in healing long bones. Ultrasonics, 54(5):1219–1230.
- [Prasad, 1964] Prasad, C. (1964). On vibrations of spherical shells. The Journal of the Acoustical Society of America, 36(3):489–494.
- [Primakoff and Keller, 1947] Primakoff, H. and Keller, J. B. (1947). Reflection and transmission of sound by thin curved shells. The Journal of the Acoustical Society of America, 19(5):820–832.
- [Pujol, 2003] Pujol, J. (2003). Elastic Wave Propagation and Generation in Seismology. Cambridge University Press.
- [Qatu, 2004] Qatu, M. S. (2004). Vibration of laminated shells and plates. Elsevier.
- [Quarteroni et al., 2007] Quarteroni, A., Sacco, R., and Saleri, F. (2007). Numerical mathematics, volume 37. Springer.
- [Rabbani et al., 2018] Rabbani, V., Bahari, A., Hodaei, M., Maghoul, P., and Wu, N. (2018). Three-dimensional free vibration analysis of triclinic piezoelectric hollow cylinder. Composites Part B: Engineering.
- [Rajani et al., 2009] Rajani, B., Kandasamy, A., and Majumdar, S. (2009). Numerical simulation of laminar flow past a circular cylinder. Applied Mathematical Modelling, 33(3):1228–1247.
- [Rayleigh, 1896] Rayleigh, J. W. S. B. (1896). The theory of sound, volume 2. Macmillan.
- [Reddy and Kierans, 1976] Reddy, D. and Kierans, T. (1976). Dynamic analysis for design criteria for underground nuclear reactor containments. Nuclear Engineering and Design, 38(2):177–205.
- [Rice et al., 1978] Rice, J., Rudnicki, J., and Simons, D. A. (1978). Deformation of spherical cavities and inclusions in fluid-infiltrated elastic materials. International Journal of Solids and Structures, 14(4):289–303.
- [Rienstra and Hirschberg, 2017] Rienstra, S. and Hirschberg, A. (2017). An introduction to acoustics.
- [Rienstra and Hirschberg, 2003] Rienstra, S. W. and Hirschberg, A. (2003). An Introduction to Acoustics, volume 18.
- [Ritos et al., 2014] Ritos, K., Mattia, D., Calabrò, F., and Reese, J. (2014). Flow enhancement in nanotubes of different materials and lengths. The Journal of Chemical Physics, 140(1):014702.

- [Robinson and Greenleaf, 1986] Robinson, B. S. and Greenleaf, J. F. (1986). The scattering of ultrasound by cylinders: Implications for diffraction tomography. The Journal of the Acoustical Society of America, 80(1):40–49.
- [Satô et al., 1963] Satô, Y., Usami, T., Landisman, M., and Ewing, M. (1963). Basic study on the oscillation of a sphere. Geophysical Journal International, 8(1):44–63.
- [Sauer and Xu, 1996] Sauer, T. and Xu, Y. (1996). Regular points for Lagrange interpolation on the unit disk. Numerical Algorithms, 12(2):287–296.
- [Schwartz et al., 2016] Schwartz, B. L., Liu, Y., Royston, T. J., and Magin, R. L. (2016). Axisymmetric diffraction of a cylindrical transverse wave by a viscoelastic spherical inclusion. Journal of Sound and Vibration, 364:222–233.
- [Seybert and Soenarko, 1988] Seybert, A. and Soenarko, B. (1988). Radiation and scattering of acoustic waves from bodies of arbitrary shape in a three-dimensional half space. ASME, Transactions, Journal of Vibration, Acoustics, Stress, and Reliability in Design, 110:112–117.
- [Shaw, 1975] Shaw, R. (1975). Diffraction of waves by circular obstacles with variable impedance boundary conditions. Journal of sound and vibration, 38(4):417–424.
- [Shih and Guo, 2016] Shih, P.-J. and Guo, Y.-R. (2016). Resonance frequency of fluid-filled and prestressed spherical shell—a model of the human eyeball. The Journal of the Acoustical Society of America, 139(4):1784–1792.
- [Skidan et al., 1974] Skidan, O., Klosner, J. M., and Baron, M. L. (1974). Sound radiation from a cylinder immersed in an acoustic fluid bounded by an elastic half-space. The Journal of the Acoustical Society of America, 56(2):427–439.
- [Sladek et al., 2013] Sladek, J., Stanak, P., Han, Z., Sladek, V., and Atluri, S. (2013). Applications of the MLPG Method in Engineering & Sciences: A Review. CMES: Computer Modeling in Engineering & Sciences, 92(5):423–475.
- [Sneddon, 1965] Sneddon, I. N. (1965). The relation between load and penetration in the axisymmetric boussinesq problem for a punch of arbitrary profile. International journal of engineering science, 3(1):47–57.
- [Soydas and Saritas, 2013] Soydas, O. and Saritas, A. (2013). An accurate nonlinear 3d Timoshenko beam element based on Hu-Washizu functional. International Journal of Mechanical Sciences, 74:1–14.
- [Spacone et al., 1996] Spacone, E., Filippou, F. C., and Taucer, F. F. (1996). Fibre beam-column model for non-linear analysis of R/C frames: Part I. Formulation. Earthquake Engineering and Structural Dynamics, 25(7):711–726.
- [Sprague and Geers, 1999] Sprague, M. A. and Geers, T. L. (1999). Response of empty and fluid-filled, submerged spherical shells to plane and spherical, step-exponential acoustic waves. Shock and Vibration, 6(3):147–157.

- [Stepanishen, 1997] Stepanishen, P. R. (1997). Acoustic transient radiation and scattering from fluid-loaded elastic shells using convolution methods. The Journal of the Acoustical Society of America, 102(1):110–119.
- [Stroud, 1971] Stroud, A. H. (1971). Approximate calculation of multiple integrals, volume 431. Prentice-Hall Englewood Cliffs, NJ.
- [Sun and Takayama, 1996] Sun, M. and Takayama, K. (1996). A holographic interferometric study of shock wave focusing in a circular reflector. Shock Waves, 6(6):323–336.
- [Tadeu et al., 2013] Tadeu, A., Costa, E., Antonio, J., and Amado-Mendes, P. (2013). 2.5 d and 3d green’s functions for acoustic wedges: Image source technique versus a normal mode approach. Journal of Computational Acoustics, 21(01):1250025.
- [Tang and Yen, 1970] Tang, S.-C. and Yen, D. H. (1970). Interaction of a plane acoustic wave with an elastic spherical shell. The Journal of the Acoustical Society of America, 47(5B):1325–1333.
- [Taylor et al., 2003] Taylor, R., Filippou, F., Saritas, A., and Auricchio, F. (2003). A mixed finite element method for beam and frame problems. Computational Mechanics, 31(1-2):192–203.
- [Valkó and Abate, 2004] Valkó, P. P. and Abate, J. (2004). Comparison of sequence accelerators for the gaver method of numerical laplace transform inversion. Computers & Mathematics with Applications, 48(3-4):629–636.
- [Varadan et al., 1982] Varadan, V., Varadan, V., and Flax, L. (1982). Scattering of acoustic waves by submerged objects in the presence of a nearby boundary. The Journal of the Acoustical Society of America, 72(S1):S99–S100.
- [Von Ramm et al., 1991] Von Ramm, O. T., Smith, S. W., and Pavy, H. G. (1991). High-speed ultrasound volumetric imaging system. ii. parallel processing and image display. IEEE transactions on ultrasonics, ferroelectrics, and frequency control, 38(2):109–115.
- [Watson, 1995] Watson, G. N. (1995). A Treatise on The Theory of Bessel Functions. Cambridge University Press.
- [Wen-Yu and Ren-He, 2015] Wen-Yu, L. and Ren-He, Z. (2015). A benchmark model for three-dimensional sound propagation in an ideal wedge-shaped waveguide. Chinese Physics Letters, 32(2):024301.
- [Wendland, 1999] Wendland, H. (1999). Meshless Galerkin methods using radial basis functions. Mathematics of Computation of the American Mathematical Society, 68(228):1521–1531.
- [Weston, 1959] Weston, V. (1959). Pulse return from a sphere. IRE Transactions on Antennas and Propagation, 7(5):43–51.
- [Whitham, 2011] Whitham, G. B. (2011). Linear and nonlinear waves, volume 42. John Wiley & Sons.
- [Wijeyewickrema and Keer, 1989] Wijeyewickrema, A. C. and Keer, L. M. (1989). Transient elastic wave scattering by a rigid spherical inclusion. The Journal of the Acoustical Society of America, 86(2):802–809.

- [Wilcox et al., 2010] Wilcox, L. C., Stadler, G., Burstedde, C., and Ghattas, O. (2010). A high-order discontinuous galerkin method for wave propagation through coupled elastic–acoustic media. Journal of Computational Physics, 229(24):9373–9396.
- [Wilkinson and Kalnins, 1965] Wilkinson, J. and Kalnins, A. (1965). On nonsymmetric dynamic problems of elastic spherical shells. Journal of Applied Mechanics, 32(3):525–532.
- [Wirgin, 2016] Wirgin, A. (2016). Retrieval of the equivalent acoustic constitutive parameters of an inhomogeneous fluid-like object by nonlinear full waveform inversion. Ultrasonics, 65:353–369.
- [Xu et al., 2015] Xu, C., Chen, Q., Zhou, J., and Cai, Y. (2015). Analytical solution of transient dynamic response of spherical cavity in viscoelastic and saturated soils. KSCE Journal of Civil Engineering, 19(7):2035–2040.
- [Yeh, 1964] Yeh, G. C. (1964). Diffraction of a plane compressional wave by a spherical cavity in an elastic medium. The Journal of the Acoustical Society of America, 36(5):1009–1009.
- [Ying and Truell, 1956] Ying, C. and Truell, R. (1956). Scattering of a plane longitudinal wave by a spherical obstacle in an isotropically elastic solid. Journal of Applied Physics, 27(9):1086–1097.
- [Yuan et al., 2012] Yuan, C., Bergsma, O., and Beukers, A. (2012). Sound transmission loss prediction of the composite fuselage with different methods. Applied Composite Materials, 19(6):865–883.
- [Zakout, 2001] Zakout, U. (2001). Transient response of an acoustic medium by an excited submerged spherical shell. The Journal of the Acoustical Society of America, 109(6):2789–2796.
- [Zakout et al., 2000] Zakout, U., Akkas, Z., and Tupholme, G. (2000). Transient response of an infinite elastic medium containing a spherical cavity subjected to torsion. Journal of applied mechanics, 67(2):282–287.
- [Zhang et al., 2015] Zhang, K., Chen, H., Guo, X., Zhang, D., Zheng, Y., Zheng, H., and Shi, J. (2015). Double-scattering/reflection in a single nanoparticle for intensified ultrasound imaging. Scientific reports, 5.
- [Zhang and Geers, 1993] Zhang, P. and Geers, T. L. (1993). Excitation of a fluid-filled, submerged spherical shell by a transient acoustic wave. The Journal of the Acoustical Society of America, 93(2):696–705.
- [Zhang and Banerjee, 2012] Zhang, Q. and Banerjee, U. (2012). Numerical integration in Galerkin meshless methods, applied to elliptic Neumann problem with non-constant coefficients. Advances in Computational Mathematics, 37(4):453–492.
- [Zhao and Harkrider, 1992] Zhao, L.-S. and Harkrider, D. G. (1992). Wave fields from an off-center explosion in an embedded solid sphere. Bulletin of the Seismological Society of America, 82(4):1927–1955.

BIBLIOGRAPHY

- [Zou and Schmiedeler, 2008] Zou, H. and Schmiedeler, J. (2008). Predicting brain injury under impact with a strain measure from analytical models. International journal of crashworthiness, 13(3):337–348.

Appendices

Appendix A

Acoustical Phenomena

The wave equation for a fluid medium is derived from the pertinent fundamental assumptions. Let's assume $\bar{p}^*$, $\bar{\mathbf{v}}^*$ and $\bar{\rho}'$ are fluid's acoustic perturbations of pressure, particle velocity and density, respectively, from the equilibrium situation in a stagnant uniform fluid in absence of a body force. The conservation equations of the mass and linear momentum as well as the constitutive rule in a quiescent, *isentropic* acoustic phenomenon are, respectively,

$$\bar{\rho}'_{,t} + \rho_f \nabla \cdot \bar{\mathbf{v}}^* = 0, \quad (\text{A.1a})$$

$$\nabla \bar{p}^* = -\rho_f \bar{\mathbf{v}}^*_{,t} \quad \text{and} \quad (\text{A.1b})$$

$$\bar{p}^* = c_f^2 \bar{\rho}' . \quad (\text{A.1c})$$

By omitting $\bar{\rho}'$ and $\bar{\mathbf{v}}^*$ in [Equation A.1a](#) and using [Equation A.1b](#) as well as [Equation A.1c](#), the linear wave equation for an acoustic phenomenon emerges as

$$c_f^2 \nabla^2 \bar{p} = \bar{p}^*_{,tt} . \quad (\text{A.2})$$

See, for example, [[Rienstra and Hirschberg, 2017](#)]. The nondimensional form of [Equation A.2](#) becomes

$$\nabla^2 p = \alpha_f^2 s^2 p \quad (\text{A.3})$$

in the Laplace domain for zero initial conditions.

Appendix **B**

Elastic Waves

The *Helmholtz* decomposition of the displacement field of an elastic medium and its associated *Hansen* displacement vectors are explained here concisely. The kinematic assumption for an infinitesimal deformation $\bar{\mathbf{u}}^*$ in the form of the *Green-Cauchy* strain tensor, the generalized Hooke's law as the assumed linear constitutive relation and conservation of linear momentum in the absence of a body force are, respectively,

$$\bar{\boldsymbol{\varepsilon}} = \frac{1}{2} (\nabla \bar{\mathbf{u}}^* + \bar{\mathbf{u}}^* \nabla), \quad (\text{B.1a})$$

$$\bar{\boldsymbol{\sigma}}^* = \mathbb{C} : \bar{\boldsymbol{\varepsilon}} \quad \text{and} \quad (\text{B.1b})$$

$$\nabla \cdot \bar{\boldsymbol{\sigma}}^* = \rho_s^* \bar{\mathbf{u}}^*_{,tt}. \quad (\text{B.1c})$$

The $\bar{\boldsymbol{\varepsilon}}$ and $\mathbb{C}$ are a symmetric *Green-Lagrange* strain tensor and the fourth order stiffness tensor, respectively. The well-known *Navier* equation emerges by combining the latter three equations for an isotropic elastic medium along with the following identity of $\nabla^2 \equiv \nabla \nabla \cdot - \nabla \times \nabla \times$, the well-known *Navier* equation emerges as

$$(\lambda + 2\mu) \nabla \nabla \cdot \bar{\mathbf{u}}^* - \mu \nabla \times \nabla \times \bar{\mathbf{u}}^* = \rho_s^* \bar{\mathbf{u}}^*_{,tt}. \quad (\text{B.2})$$

The standard displacement solution of Equation B.2 is found by superimposing the longitudinal and shear wave motions. This technique is known as the *Helmholtz* decomposition. Its mathematical form is

$$\bar{\mathbf{u}} = \nabla \bar{\phi} + \nabla \times \bar{\boldsymbol{\Psi}} \quad \text{iff} \quad \nabla \cdot \bar{\boldsymbol{\Psi}} = 0 \quad (\text{B.3})$$

when it is nondimensionalized. The $\bar{\phi}$ and $\bar{\boldsymbol{\Psi}}$ are the scalar longitudinal (dilatational, primary or the so called *p*-wave) wave potential and the solenoidal (shear, secondary or the *s*-wave) vector potential, respectively. Obviously $\nabla \bar{\phi}$ is an *irrotational* displacement vector whereas the $\nabla \times \bar{\boldsymbol{\Psi}}$ represents a *equivoluminal* deformation if the *gauge* condition $\nabla \cdot \bar{\boldsymbol{\Psi}} = 0$ is satisfied. Substituting Equation B.3 into Equation B.2 and taking, first a divergence and then a curl from both sides of the resulting equation along with the application of the aforementioned identity, leads to a scalar and a vector wave equations, respectively, as $c_p^2 \nabla^2 \bar{\phi}^* = \bar{\phi}^*_{,tt}$ and $c_s^2 \nabla^2 \bar{\boldsymbol{\Psi}}^* = \bar{\boldsymbol{\Psi}}^*_{,tt}$ in which $c_p = \sqrt{(\lambda + 2\mu)/\rho_s}$ and $c_s = \sqrt{\mu/\rho_s}$ are the compressional and shear wave speeds, respectively. The latter wave equations become

$$\nabla^2 \bar{\phi} = \alpha_p^2 \bar{\phi}_{,tt} \quad \text{and} \quad \nabla^2 \bar{\boldsymbol{\Psi}} = \alpha_s^2 \bar{\boldsymbol{\Psi}}_{,tt}. \quad (\text{B.4})$$

when nondimensionalized. It is obvious that only two independent scalar potentials, $\bar{\chi}^*$ and $\bar{\psi}^*$ arise from Equation B.4 due to the gauge condition. Motion associated with the *solenoidal* component can be decomposed further into two orthogonal vectors $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$ which are associated with two scalar shear wave potentials $\bar{\psi}$ and $\bar{\chi}$, respectively. Both these vectors are located in a plane normal to the *irrotational* displacement $\bar{\mathbf{L}}$. Then in a spherical coordinate frame

$$\bar{\mathbf{L}} = \nabla \bar{\phi}, \quad \bar{\mathbf{M}} = \nabla(r\bar{\psi}) \times \mathbf{e}_r \quad \text{and} \quad \bar{\mathbf{N}} = \nabla \left(\frac{\partial(r\bar{\chi})}{\partial r} \right) - r \nabla^2 \bar{\chi} \mathbf{e}_r. \quad (\text{B.5})$$

These three orthogonal vectors are due to Hansen's as in [Pujol, 2003, § 5.6.1] which their superposition constitutes the total displacement field as

$$\bar{\mathbf{u}} = \bar{\mathbf{L}} + \bar{\mathbf{M}} + \bar{\mathbf{N}}. \quad (\text{B.6})$$

Components of $\bar{\mathbf{u}}$ in a spherical coordinate frame become

$$\bar{\mathbf{u}} \equiv \begin{Bmatrix} \bar{u}_r \\ \bar{u}_\theta \\ \bar{u}_\varphi \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \bar{\phi}}{\partial r} + \frac{\partial^2(r\bar{\chi})}{\partial r^2} - r \nabla^2 \bar{\chi} \\ \frac{1}{r} \frac{\partial \bar{\phi}}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial \bar{\psi}}{\partial \varphi} + \frac{\partial^2(r\bar{\chi})}{r \partial \theta \partial r} \\ \frac{1}{r \sin \theta} \frac{\partial \bar{\phi}}{\partial \varphi} - \frac{\partial \bar{\psi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial^2(r\bar{\chi})}{\partial \varphi \partial r} \end{Bmatrix}. \quad (\text{B.7})$$

The stress components σ_{rr} and $\sigma_{r\theta}$ as well as the radial velocity $v_r \equiv su_r$ are needed to enforce the boundary conditions at shell's traction surfaces by using GL in § 1.2.5.1 for an axisymmetric situation. These components may be found in terms of the displacement potentials by using Equation B.1b, Equation B.3 and Equation B.7 in spherical coordinate. Then

$$\begin{Bmatrix} v_r \\ \sigma_{rr} \\ \sigma_{r\theta} \end{Bmatrix} = \begin{Bmatrix} s \left(\frac{\partial^2(r\psi)}{\partial r^2} - r \nabla^2 \psi \right) \\ 2\mu \frac{\partial}{\partial r} \left(\frac{\partial^2(r\psi)}{\partial r^2} - r \nabla^2 \psi \right) \\ \frac{\mu}{r} \left(\frac{\partial}{\partial \theta} \left(\frac{\partial^2(\psi)}{\partial r^2} - r \nabla^2 \psi \right) - \frac{1}{r} \frac{\partial^2(r\psi)}{\partial \theta \partial r} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial^2(r\psi)}{\partial \theta \partial r} \right) \right) \end{Bmatrix} + \begin{Bmatrix} \frac{s \partial \phi}{\partial r} \\ \lambda \nabla^2 \phi + 2\mu \frac{\partial^2 \phi}{\partial r^2} \\ \frac{2\mu}{r} \left(\frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) \end{Bmatrix}. \quad (\text{B.8})$$

Corresponding modal components for u_r , σ_{rr} and $\sigma_{r\theta}$ are derived by substituting Equation 2.24 into Equation B.8 so their n^{th} modal components are, respectively,

$$\begin{aligned} u_n^r(r, s) &= C_n(s) \{n i_n(\alpha_p r s) + \alpha_p r s i_{n+1}(\alpha_p r s)\} + D_n(s) \{n k_n(\alpha_p r s) - \alpha_p r s k_{n+1}(\alpha_p r s)\} \\ &+ E_n(s) \{n(n+1) i_n(\alpha_s r s)\} + F_n(s) \{n(n+1) k_n(\alpha_s r s)\}, \end{aligned}$$

and

$$\begin{aligned}
\sigma_n^{rr}(r, s) = & C_n(s) \left\{ \left(\frac{(r s \alpha_s)^2}{2} - n + n^2 \right) i_n(\alpha_p r s) - 2 r s \alpha_p i_{n+1}(\alpha_p r s) \right\} \\
& + D_n(s) \left\{ \left(\frac{(r s \alpha_s)^2}{2} - n + n^2 \right) k_n(\alpha_p r s) + 2 r s \alpha_p k_{n+1}(\alpha_p r s) \right\} \\
& + E_n(s) \left\{ n(n+1) \left((n-1) i_n(\alpha_p r s) + r s \alpha_p i_{n+1}(\alpha_p r s) \right) \right\} \\
& + F_n(s) \left\{ n(n+1) \left((n-1) k_n(\alpha_s r s) - r s \alpha_p k_{n+1}(\alpha_s r s) \right) \right\},
\end{aligned}$$

whereas

$$\begin{aligned}
\sigma_n^{r\theta}(r, s) = & C_n(s) \left\{ (n-1) i_n(\alpha_p \theta s) + r s \alpha_p i_{n+1}(\alpha_p r s) \right\} \\
& + D_n(s) \left\{ (n-1) k_n(\alpha_p \theta s) - r s \alpha_p k_{n+1}(\alpha_p r s) \right\} \\
& + E_n(s) \left\{ \frac{1}{2} \left(n^2 - 1 + \frac{(\alpha_s r s)^2}{2} \right) i_n(\alpha_p \theta s) - r s \alpha_s i_{n+1}(\alpha_s r s) \right\} \\
& + F_n(s) \left\{ \frac{1}{2} \left(n^2 - 1 + \frac{(\alpha_s r s)^2}{2} \right) k_n(\alpha_p \theta s) + r s \alpha_s k_{n+1}(\alpha_s r s) \right\}.
\end{aligned} \tag{B.9}$$

Equations B.9 are used to particularize the Cauchy traction boundary conditions.

Appendix **C**

Shell Theories' Operators

The linear operator $\mathbf{L} \equiv \mathbf{L}_{2 \times 2}$ for th 4M theory is

$$\begin{pmatrix} - \left\{ \begin{array}{c} (1 + \beta) \left[(\nu + \cot^2 \theta) + \nabla_\theta^2 \right] \\ + \alpha_{L_{sh}}^2 \frac{\partial^2}{\partial t^2} \end{array} \right\} & \left\{ \begin{array}{c} \frac{\partial}{\partial \theta} [\beta(1 - \nu) - (1 + \nu)] \\ + \beta \nabla_\theta^2 \end{array} \right\} \\ \left\{ \begin{array}{c} \left[\frac{\partial}{\partial \theta} + \cot \theta \right] \\ \left((1 + \nu) + \beta(\nu + \cot^2 \theta) - \beta \nabla_\theta^2 \right) \end{array} \right\} & - \left\{ \begin{array}{c} 2(1 + \nu) + \beta(1 - \nu) \nabla_\theta^2 \\ + \beta \nabla_\theta^4 + \alpha_{L_{sh}}^2 \frac{\partial^2}{\partial t^2} \end{array} \right\} \end{pmatrix}$$

where $\nabla_\theta^2 = \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta}$. In the case of 5M theory, however, the enlarged operator $\mathbf{L} \equiv \mathbf{L}_{3 \times 3}$ becomes

$$\begin{pmatrix} \left\{ \begin{array}{c} \frac{\partial^2}{\partial \theta^2} + \nu \cot \theta \frac{\partial}{\partial \theta} \\ - \csc^2 \theta + (1 - \nu) \times \\ \left(\cot \theta \frac{\partial}{\partial \theta} - \cot^2 \theta \right) \\ - \alpha_{sh}^2 - \alpha_{L_{sh}}^2 \frac{\partial^2}{\partial t^2} \end{array} \right\} & \left\{ -\alpha_{sh}^2 \right\} & \left\{ \begin{array}{c} (1 + \nu) \\ + \alpha_{sh}^2 \frac{\partial}{\partial \theta} \end{array} \right\} \\ \{1\} & \left\{ \begin{array}{c} \left(\nu \cot \theta + \frac{\beta}{\alpha_{sh}^2} \times \right. \\ \left. [\nu + \cot \theta (1 - \nu)] \right) \\ \times \frac{\partial}{\partial \theta} + \frac{\beta}{\alpha_{sh}^2} \frac{\partial^2}{\partial \theta^2} \\ - \nu \frac{\beta}{\alpha_{sh}^2} \csc^2 \theta \\ + (1 - \nu) \cot \theta \\ \left. - 1 - \beta \alpha_{sh}^2 \frac{\partial^2}{\partial t^2} \right\} & \left\{ -\frac{\partial}{\partial \theta} \right\} \\ \left\{ \begin{array}{c} - \left(\alpha_{sh}^2 + (1 + \nu) \right) \frac{\partial}{\partial \theta} \\ - \cot \theta \left(\alpha_{sh}^2 + (1 + \nu) \right) \end{array} \right\} & \left\{ \alpha_{sh}^2 \frac{\partial}{\partial \theta} + \alpha_{sh}^2 \cot \theta \right\} & \left\{ \begin{array}{c} \alpha_{sh}^2 \frac{\partial^2}{\partial \theta^2} + \\ \alpha_{sh}^2 \cot \theta \frac{\partial}{\partial \theta} \\ - 2(1 + \nu) \\ - \alpha_{L_{sh}}^2 \frac{\partial^2}{\partial t^2} \end{array} \right\} \end{pmatrix}$$

in which

$$\begin{aligned}
c_{L_{sh}} &= \sqrt{\frac{E}{\rho}(1-\nu^2)}, \quad c_{S_{sh}} = \sqrt{\frac{\kappa G}{\rho}(1-\nu^2)}, \quad \beta = \frac{(1-r_{in})^2}{12}, \\
\alpha_{L_{sh}} &= \frac{c_{ex}}{c_{L_{sh}}}, \quad \alpha_{S_{sh}} = \frac{c_{ex}}{c_{S_{sh}}}, \quad \alpha_{sh} = \frac{c_{S_{sh}}}{c_{L_{sh}}}.
\end{aligned} \tag{C.1}$$

The $c_{L_{sh}}$ and $c_{S_{sh}}$ are the dilatational and modified shear wave speeds, respectively in a thin shell. Moreover κ is a shear correction factor and G is the shear modulus of the shell's material. Then parameters used in [Equation 1.43](#) and [Equation 1.44](#) of the main text are defined as

$$\begin{aligned}
b_1 &= 1 + \nu + k_2 \alpha_{L_{sh}}, & b_2 &= 2 + k_1 \frac{d_s}{1 + \nu}, \\
b_3 &= \epsilon - 2k_s + 2k_s k_r \frac{d_s}{1 - \nu}, & b_4 &= 1 - 2 \frac{k_s k_2 d_s}{1 - \nu}, \\
b_5 &= 1 - (k_1 + k_2) \frac{d_s}{1 - \nu}, & b_6 &= 1 - \nu - k_r d_s, \\
b_7 &= 1 - (c_r + k_r) \frac{d_s}{1 - \nu}, & \epsilon &= \frac{12}{h^2}, \\
c_1 &= 4 - 2k_s(1 + \nu) - [k_1 + k_r + \frac{2k_s}{1 - \nu}]d_s, & k_1 &= 1 + \frac{1}{\epsilon}, \\
c_2 &= (1 - \nu)(b_5 b_6 + b_1 b_7) + (1 + \nu)c_4, & k_2 &= \frac{2}{\epsilon}, \\
c_3 &= (1 - \nu)(b_4 b_7 - b_3 b_7), & k_r &= 1 + \frac{9}{5\epsilon}, \\
c_4 &= (b_4 b_2 - b_1 b_3 + b_2 b_3 + b_4 b_6 - 2k_s b_5 b_2), & d_s &= \alpha_{L_{sh}}^2 s^2.
\end{aligned} \tag{C.2}$$

Appendix **D**

Green's Function

The transient full-space Green's function $\bar{\mathbf{G}} \equiv \bar{\mathbf{G}}(t|t_i, \mathbf{r}|\mathbf{r}_i)$ for the wave equation is the solution to

$$\mathcal{L}\bar{\mathbf{G}} = \delta(t')\delta(\mathbf{r}') \quad (\text{D.1})$$

where $\mathcal{L} \equiv \nabla^2 + \alpha^2(\partial^2/\partial t^2)$, $\mathbf{r}' = \mathbf{r} - \mathbf{r}_i$ and $t' = t - t_i$. The $\mathbf{r}$, $\mathbf{r}_i$ and $\mathbf{r}'$ are the vectors illustrated in [Figure 3.3](#). The t_i is arrival time of a wavefront to one of the shell's surfaces. In a spherical coordinate $\delta(\mathbf{r}') = \frac{1}{r^2 \sin \theta} \delta(r - r_i) \delta(\theta - \theta_i) \delta(\varphi - \varphi_i)$. The solution of [Equation D.1](#) is

$$\mathbf{G} = e^{-s t_i} \frac{e^{-s \alpha |\mathbf{r}'|}}{4\pi |\mathbf{r}'|} \quad (\text{D.2})$$

in the Laplace domain where $|\mathbf{r}'|$ is given in [Equation 3.8](#). The time domain Green's function is found by taking an inverse Laplace transform of the last equation, such that

$$\bar{\mathbf{G}} = \frac{\delta(\alpha |\mathbf{r}'| - t')}{4\pi |\mathbf{r}'|}. \quad (\text{D.3})$$

Subsequently the unified expression for an incident wave is obtained, simply, by performing *convolution* integration on $\bar{\mathbf{G}}$ and an arbitrary excitation $\bar{f}(t)$ which leads to [Equation 3.6](#). The corresponding expression of this incident term in Laplace domain is deduced by multiplication of $f(s)$ with $\mathbf{G}$ given in [Equation D.2](#). This is identical to the one of [Equation 3.12](#). Hence, the corresponding harmonic Green's function may be found in [[Ivanov, 1970](#), § 3.6] or [[Gaunaud et al., 1995](#), Equation 5].

Appendix **E**

Spherical Addition Theorems

The *Legendre* addition theorem is explained first after which two *translational* addition theorems are newly derived. Suppose $\mathbf{r} = r \hat{\mathbf{r}}$ is an arbitrary vector in which r is its magnitude and $\hat{\mathbf{r}}$ represents the associated unit vector. The Cartesian components of $\hat{\mathbf{r}}$ are $(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. Thus the inner product of two arbitrary unit vectors becomes

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}_i \equiv \cos \gamma = \cos \theta \cos \theta_i + \sin \theta \sin \theta_i \cos(\varphi - \varphi_i) \quad (\text{E.1})$$

where γ is the angle between these two unit vectors. Applying Legendre polynomial symbolized by P_n to both sides of Equation E.1 leads to

$$P_n(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}_i) \equiv P_n(\cos \gamma) = \frac{4\pi}{2n+1} \sum_{m=-n}^n Y_n^m(\hat{\mathbf{r}}) \check{Y}_n^m(\hat{\mathbf{r}}_i), \quad (\text{E.2})$$

where $\check{Y}_n^m(\hat{\mathbf{r}}) = \check{Y}_n^m(\theta, \varphi)$ is the complex conjugate of the spherical harmonics Y_n^m . This equation is known as the *Legendre addition theorem*. The spherical harmonics, Cf [Martin, 2006, Equation 3.20] or [Olver, 2010, § 14.18.2]. The spherical harmonics are orthonormal functions, i.e. $\langle Y_n^m, Y_p^q \rangle = \delta_{np} \delta_{mq}$, where $\langle a, b \rangle = \oint a b \, d\Omega$ in which $d\Omega = \sin \theta d\theta d\varphi$. If $\tilde{Y} \equiv \tilde{Y}_n^m(\theta, \varphi) = P_n^m(\cos \theta) e^{im\varphi}$ and $\mathcal{A}_n^m \equiv \langle \tilde{Y}_n^m, \tilde{Y}_p^q \rangle$ then $Y_n^m = \mathcal{A}_n^m \tilde{Y}_n^m$ where

$$\mathcal{A}_n^m = \sqrt{\frac{(2n+1)(n-|m|)!}{4\pi(n+|m|)!}}. \quad (\text{E.3})$$

It also can be shown straightforwardly that

$$Y_n^{-m} = (-1)^m \check{Y}_n^m \quad \text{and} \quad P_n^{-m} = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m, \quad (\text{E.4})$$

Cf [Olver, 2010, § 10.30.6; 14.30.8]. A *translational* addition theorem is required if a coordinate frame centered at O_i is moved to O by using the translation vector $\mathbf{r}_i = \overrightarrow{OO_i}$ but with the original frame's axes maintaining their direction. The result of combining [Olver, 2010, Equations 10.60.1; 10.60.2], $h_n^{(1)}(ix) = j_n(x) + iy_n(x)$ and the Euler's formula, $e^{ix} = \cos x + i \sin x$ becomes

$$\frac{e^{i z |\mathbf{r} - \mathbf{r}_i|}}{|\mathbf{r} - \mathbf{r}_i|} = \sum_{n=0}^{\infty} \begin{cases} i j_n(z r) h_n^{(1)}(z r_i) P_n(\cos \gamma) & r < r_i \\ i j_n(z r_i) h_n^{(1)}(z r) P_n(\cos \gamma) & r \geq r_i \end{cases} \quad (\text{E.5})$$

for the geometry given in Figure 3.3. The $\{j_n(x), y_n(x)\}$, $\{i_n(x), k_n(x)\}$ are the spherical Bessel and modified spherical Bessel functions of the first and second kinds, respectively, whereas $h_n^{(1)}(x)$

is the spherical Hankel function of the first kind. Equation E.5, subsequently, turns into

$$\frac{e^{-s \alpha |\mathbf{r} - \mathbf{r}_i|}}{|\mathbf{r} - \mathbf{r}_i|} = \frac{2}{\pi} \sum_{n=0}^{\infty} \begin{cases} (2n+1) k_n(\alpha r_i) i_n(\alpha r) P_n(\cos \gamma) & r < r_i \\ (2n+1) i_n(\alpha r) k_n(\alpha r_i) P_n(\cos \gamma) & r \geq r_i \end{cases} \quad (\text{E.6})$$

by using $j_n(ix) = i^{-n} i_n(x)$ and $h_n^{(1)}(ix) = -2i^{-n} k_n(x)/\pi$ [Olver, 2010, § 10.47(vi)] and substituting $z = i\alpha s$. An alternative derivation of Equation E.6 can be given by starting from [Martin, 2006, theorem 3.15] for the configuration illustrated in Figure 3.3, such that

$$h_0(z |\mathbf{r} - \mathbf{r}_i|) = \sum_{n=0}^{\infty} \begin{cases} (2n+1) h_n(z r_i) j_n(z r) P_n(\cos \gamma) & r < r_i \\ (2n+1) h_n(z r) j_n(z r_i) P_n(\cos \gamma) & r \geq r_i \end{cases}. \quad (\text{E.7})$$

Then simply by using $k_0(x) = \pi e^{-x}/(2x)$ and substituting $z = i\alpha s$, the latter equation turns into Equation E.6. This equation is a special case of the modified *Gegenbauer's* addition theorem. See, for example, [Watson, 1995, § 11.4.1, Equation 11] or [Olver, 2010, § 10.60.3]. Finally the nonaxisymmetric (3D) *addition theorem I* emerges as

$$\frac{e^{-s \alpha |\mathbf{r} - \mathbf{r}_i|}}{|\mathbf{r} - \mathbf{r}_i|} = 8 \sum_{n,m}^{\infty} \begin{cases} i_n(\alpha r_i s) k_n(\alpha r s) \check{Y}_n^m(\theta_i, \varphi_i) Y_n^m(\theta, \varphi), & r < r_i \\ i_n(\alpha r s) k_n(\alpha r_i s) \check{Y}_n^m(\theta_i, \varphi_i) Y_n^m(\theta, \varphi), & r \geq r_i \end{cases} \quad (\text{E.8})$$

by substituting Equation E.2 into E.6. This addition theorem is used for expanding a spherical incident wave. The substitution of $\cos \gamma$ with $\cos \theta$ in Equation E.6 gives the asymmetric (2D) version of this theorem. The addition theorem required to express a plane incident wave is deduced next.

The derivation of the *addition theorem II* may start from the theorem due to *Bauer* which is

$$e^{i x \cos \gamma} = \sum_{n=0}^{\infty} (2n+1) i^n j_n(x) P_n(\cos \gamma).$$

See Equation 1 of [Watson, 1995, § 11.5]. The *Bauer* formula turns into the nonaxisymmetric (3D) *addition theorem II* as

$$\begin{aligned} e^{\alpha [\cos \theta \cos \theta_i + \sin \theta \sin \theta_i \cos(\varphi - \varphi_i)] r s} &= 4\pi \sum_{n,m}^{\infty} i_n(\alpha r s) \check{Y}_n^m(\theta_i, \varphi_i) Y_n^m(\theta, \varphi) \\ &= \sum_{n,m}^{\infty} (2n+1) \frac{(n-m)!}{(n+m)!} i_n(\alpha r s) P_n^m(\cos \theta_i) P_n^m(\cos \theta) e^{im(\varphi - \varphi_i)} \end{aligned} \quad (\text{E.9})$$

by utilizing *Legendre* addition theorem given in Equation E.2, substituting γ from Equation E.1, $j_n(ix) = i^{-n} i_n(x)$ and having $x = i\alpha s z$. Cf [Olver, 2010, § 10.60.9]. Equation E.9 reduces into

$$e^{\alpha s r \cos \theta} = \sum_{n=0}^{\infty} (2n+1) i_n(\alpha r s) P_n(\cos \theta) \quad (\text{E.10})$$

by using $\{x = i\alpha s r \cos \gamma\}$ and substituting γ with θ in an asymmetric (2D) configuration. Cf [Olver, 2010, § 10.60.8].

Appendix **F**

$\{^j_k \vartheta_{\theta,\varphi}^i\}$ and $\{^j_k \varsigma_{\theta,\varphi}^i\}$

Mathematical expressions are given here for $\{^j_k \vartheta_{\theta,\varphi}^i\}$ and $\{^j_k \varsigma_{\theta,\varphi}^i\}$. These are the relevant elements of the displacement and stress of the shell and its surrounding as they appear in [Equation 3.33](#) and [Equation 3.34](#), respectively. For the radial displacement, u_r , they are

$$\begin{aligned}
 {}^\infty_k \vartheta_r^a &= n k_n (\alpha_{p,\infty} r s) - \alpha_{p,\infty} r s k_{n+1} (\alpha_{p,\infty} r s), \\
 {}^\infty_k \vartheta_r^b &= 0, \\
 {}^\infty_k \vartheta_r^c &= n(n+1) k_n (\alpha_{s,\infty} r s), \\
 {}^{sh}_i \vartheta_r^d &= n i_n (\alpha_{p,sh} r s) + \alpha_{p,sh} r s i_{n+1} (\alpha_{p,sh} r s), \\
 {}^{sh}_k \vartheta_r^e &= n k_n (\alpha_{p,sh} r s) - \alpha_{p,sh} r s k_{n+1} (\alpha_{p,sh} r s), \\
 {}^{sh}_i \vartheta_r^f &= 0, \\
 {}^{sh}_k \vartheta_r^g &= 0, \\
 {}^{sh}_i \vartheta_r^h &= n(n+1) i_n (\alpha_{s,sh} r s) \text{ and} \\
 {}^{sh}_k \vartheta_r^i &= n(n+1) k_n (\alpha_{s,sh} r s).
 \end{aligned}$$

Corresponding elements for u_θ become

$$\begin{aligned}
 {}^\infty_k \vartheta_\theta^a &= k_n (\alpha_{p,\infty} r s), \\
 {}^\infty_k \vartheta_\theta^b &= k_n (\alpha_{s,\infty} r s), \\
 {}^\infty_k \vartheta_\theta^c &= (n+1) k_n (\alpha_{s,\infty} r s) - \alpha_{s,\infty} r s k_{n+1} (\alpha_{s,\infty} r s), \\
 {}^{sh}_k \vartheta_\theta^d &= i_n (\alpha_{p,sh} r s), \\
 {}^{sh}_k \vartheta_\theta^e &= k_n (\alpha_{p,sh} r s), \\
 {}^{sh}_k \vartheta_\theta^f &= i_n (\alpha_{s,sh} r s), \\
 {}^{sh}_k \vartheta_\theta^g &= k_n (\alpha_{s,sh} r s), \\
 {}^{sh}_k \vartheta_\theta^h &= (n+1) i_n (\alpha_{s,sh} r s) + \alpha_{s,sh} r s i_{n+1} (\alpha_{s,sh} r s) \text{ and} \\
 {}^{sh}_k \vartheta_\theta^i &= (n+1) k_n (\alpha_{s,sh} r s) - \alpha_{s,sh} r s k_{n+1} (\alpha_{s,sh} r s),
 \end{aligned}$$

whereas for u_φ

$$\begin{aligned}
{}^\infty_k \vartheta_\varphi^a &= k_n(\alpha_{p,\infty} r s), \\
{}^\infty_k \vartheta_\varphi^b &= -k_n(\alpha_{s,\infty} r s), \\
{}^\infty_k \vartheta_\varphi^c &= (n+1)k_n(\alpha_{s,\infty} r s) - \alpha_{s,\infty} r s k_{n+1}(\alpha_{s,\infty} r s), \\
{}^{sh}_k \vartheta_\varphi^d &= i_n(\alpha_{p,sh} r s), \\
{}^{sh}_k \vartheta_\varphi^e &= k_n(\alpha_{p,sh} r s), \\
{}^{sh}_k \vartheta_\varphi^f &= -i_n(\alpha_{s,sh} r s), \\
{}^{sh}_k \vartheta_\varphi^g &= -k_n(\alpha_{s,sh} r s), \\
{}^{sh}_k \vartheta_\varphi^h &= (n+1)i_n(\alpha_{s,sh} r s) + \alpha_{s,sh} r s i_{n+1}(\alpha_{s,sh} r s) \text{ and} \\
{}^{sh}_k \vartheta_\varphi^i &= (n+1)k_n(\alpha_{s,sh} r s) - \alpha_{s,sh} r s k_{n+1}(\alpha_{s,sh} r s).
\end{aligned}$$

Similarly the elements of the radial stress, σ_{rr} , are

$$\begin{aligned}
{}^\infty_k \varsigma_r^a &= \left(\frac{(r s \alpha_{s,\infty})^2}{2} - n + n^2 \right) k_n(\alpha_{p,\infty} r s) + 2 r s \alpha_{p,\infty} k_{n+1}(\alpha_{p,\infty} r s), \\
{}^\infty_k \varsigma_r^b &= 0, \\
{}^\infty_k \varsigma_r^c &= n(n+1) \left((n-1) k_n(\alpha_{s,\infty} r s) - r s \alpha_{s,\infty} k_{n+1}(\alpha_{s,\infty} r s) \right), \\
{}^{sh}_k \varsigma_r^d &= \left(\frac{(r s \alpha_{s,sh})^2}{2} - n + n^2 \right) i_n(\alpha_{p,sh} r s) - 2 r s \alpha_{p,sh} i_{n+1}(\alpha_{p,sh} r s), \\
{}^{sh}_k \varsigma_r^e &= \left(\frac{(r s \alpha_{s,sh})^2}{2} - n + n^2 \right) k_n(\alpha_{p,sh} r s) + 2 r s \alpha_{p,sh} k_{n+1}(\alpha_{p,sh} r s), \\
{}^{sh}_k \varsigma_r^f &= 0, \\
{}^{sh}_k \varsigma_r^g &= 0, \\
{}^{sh}_k \varsigma_r^h &= n(n+1) \left((n-1) i_n(\alpha_{p,sh} r s) + r s \alpha_{p,sh} i_{n+1}(\alpha_{p,sh} r s) \right) \text{ and} \\
{}^{sh}_k \varsigma_r^i &= n(n+1) \left((n-1) k_n(\alpha_{s,sh} r s) - r s \alpha_{p,sh} k_{n+1}(\alpha_{s,sh} r s) \right).
\end{aligned}$$

For the tangential stress, $\sigma_{r\theta}$, they are

$$\begin{aligned}
\infty_k \zeta_\theta^a &= (n-1)k_n(\alpha_{p,\infty}\theta s) - r s \alpha_{p,\infty}k_{n+1}(\alpha_{p,\infty}r s), \\
\infty_k \zeta_\theta^b &= (n-1)k_n(\alpha_{s,\infty}r s) + r s \alpha_{s,\infty}k_{n+1}(\alpha_{s,\infty}r s), \\
\infty_k \zeta_\theta^c &= \frac{1}{2} \left(n^2 - 1 + \frac{(\alpha_{s,\infty}r s)^2}{2} \right) k_n(\alpha_{p,\infty}\theta s) + r s \alpha_{s,sh}k_{n+1}(\alpha_{s,sh}r s), \\
sh_k \zeta_\theta^d &= (n-1)i_n(\alpha_{p,sh}\theta s) + r s \alpha_{p,sh}i_{n+1}(\alpha_{p,sh}r s), \\
sh_k \zeta_\theta^e &= (n-1)k_n(\alpha_{p,sh}\theta s) - r s \alpha_{p,sh}k_{n+1}(\alpha_{p,sh}r s), \\
sh_k \zeta_\theta^f &= (n-1)i_n(\alpha_{s,sh}r s) - r s \alpha_{s,sh}i_{n+1}(\alpha_{s,sh}r s), \\
sh_k \zeta_\theta^g &= (n-1)k_n(\alpha_{s,sh}r s) + r s \alpha_{s,sh}k_{n+1}(\alpha_{s,sh}r s), \\
sh_k \zeta_\theta^h &= \frac{1}{2} \left(n^2 - 1 + \frac{(\alpha_{s,sh}r s)^2}{2} \right) i_n(\alpha_{p,sh}\theta s) - r s \alpha_{s,sh}i_{n+1}(\alpha_{s,sh}r s) \text{ and} \\
sh_k \zeta_\theta^i &= \frac{1}{2} \left(n^2 - 1 + \frac{(\alpha_{s,sh}r s)^2}{2} \right) k_n(\alpha_{p,sh}\theta s) + r s \alpha_{s,sh}k_{n+1}(\alpha_{s,sh}r s).
\end{aligned}$$

Terms for the tangential stress of $\sigma_{r\varphi}$ are

$$\begin{aligned}
\infty_k \zeta_\varphi^a &= (n-1)k_n(\alpha_{p,\infty}r s) - r s \alpha_{p,\infty}k_{n+1}(\alpha_{p,\infty}r s), \\
\infty_k \zeta_\varphi^b &= - \left((n-1)k_n(\alpha_{s,\infty}r s) + r s \alpha_{s,\infty}k_{n+1}(\alpha_{s,\infty}r s) \right), \\
\infty_k \zeta_\varphi^c &= \frac{1}{2} \left(n^2 - 1 + \frac{(\alpha_{s,\infty}r s)^2}{2} \right) k_n(\alpha_{p,\infty}r s) + r s \alpha_{s,sh}k_{n+1}(\alpha_{s,sh}r s), \\
sh_k \zeta_\varphi^d &= (n-1)i_n(\alpha_{p,sh}\varphi s) + r s \alpha_{p,sh}i_{n+1}(\alpha_{p,sh}r s), \\
sh_k \zeta_\varphi^e &= (n-1)k_n(\alpha_{p,sh}r s) - r s \alpha_{p,sh}k_{n+1}(\alpha_{p,sh}r s), \\
sh_k \zeta_\varphi^f &= - \left((n-1)i_n(\alpha_{s,sh}r s) - r s \alpha_{s,sh}i_{n+1}(\alpha_{s,sh}r s) \right), \\
sh_k \zeta_\varphi^g &= - \left((n-1)k_n(\alpha_{s,sh}r s) + r s \alpha_{s,sh}k_{n+1}(\alpha_{s,sh}r s) \right), \\
sh_k \zeta_\varphi^h &= \frac{1}{2} \left(n^2 - 1 + \frac{(\alpha_{s,sh}r s)^2}{2} \right) i_n(\alpha_{p,sh}r s) - r s \alpha_{s,sh}i_{n+1}(\alpha_{s,sh}r s) \text{ and} \\
sh_k \zeta_\varphi^i &= \frac{1}{2} \left(n^2 - 1 + \frac{(\alpha_{s,sh}r s)^2}{2} \right) k_n(\alpha_{p,sh}r s) + r s \alpha_{s,sh}k_{n+1}(\alpha_{s,sh}r s).
\end{aligned}$$

Appendix **G**

Axially Polarized Orthotropic Constitutive Model

As a specific case of fully anisotropic triclinic crystal, the Piezoelectricity constitutive equation for an axially polarized orthotropic material in an expanded matrix form becomes

$$\begin{Bmatrix} \sigma_{rr}^{pz} \\ \sigma_{\theta\theta}^{pz} \\ \sigma_{zz}^{pz} \\ \sigma_{\theta z}^{pz} \\ \sigma_{rz}^{pz} \\ \sigma_{r\theta}^{pz} \end{Bmatrix} = \begin{pmatrix} C_{11}^{pz} & C_{12}^{pz} & C_{13}^{pz} & 0 & 0 & 0 \\ \cdot & C_{22}^{pz} & C_{23}^{pz} & 0 & 0 & 0 \\ \cdot & \cdot & C_{33}^{pz} & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & C_{44}^{pz} & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & C_{55}^{pz} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & C_{66}^{pz} \end{pmatrix} \begin{Bmatrix} \epsilon_{rr}^{pz} \\ \epsilon_{\theta\theta}^{pz} \\ \epsilon_{zz}^{pz} \\ \epsilon_{\theta z}^{pz} \\ \epsilon_{rz}^{pz} \\ \epsilon_{r\theta}^{pz} \end{Bmatrix} - \begin{pmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & 0 & e_{33} \\ 0 & e_{24} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} E_r \\ E_\theta \\ E_z \end{Bmatrix} \quad (\text{G.1})$$

and

$$\begin{Bmatrix} D_r \\ D_\theta \\ D_z \end{Bmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & e_{51} & 0 \\ 0 & 0 & 0 & e_{42} & 0 & 0 \\ e_{13} & e_{23} & e_{33} & 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \epsilon_{rr}^{pz} \\ \epsilon_{\theta\theta}^{pz} \\ \epsilon_{zz}^{pz} \\ \epsilon_{\theta z}^{pz} \\ \epsilon_{rz}^{pz} \\ \epsilon_{r\theta}^{pz} \end{Bmatrix} + \begin{pmatrix} \kappa_{11} & 0 & 0 \\ \cdot & \kappa_{22} & 0 \\ \cdot & \cdot & \kappa_{33} \end{pmatrix} \begin{Bmatrix} E_r \\ E_\theta \\ E_z \end{Bmatrix}. \quad (\text{G.2})$$

Equation G.1 and Equation G.2 are associated with those of Equation 5.1a and Equation 5.1b in the main text, respectively. The similar procedure for a purely orthotropic elastic material results into

$$\begin{Bmatrix} \sigma_{rr}^{el} \\ \sigma_{\theta\theta}^{el} \\ \sigma_{zz}^{el} \\ \sigma_{\theta z}^{el} \\ \sigma_{rz}^{el} \\ \sigma_{r\theta}^{el} \end{Bmatrix} = \begin{pmatrix} C_{11}^{el} & C_{12}^{el} & C_{13}^{el} & 0 & 0 & 0 \\ \cdot & C_{22}^{el} & C_{23}^{el} & 0 & 0 & 0 \\ \cdot & \cdot & C_{33}^{el} & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & C_{44}^{el} & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & C_{55}^{el} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & C_{66}^{el} \end{pmatrix} \begin{Bmatrix} \epsilon_{rr}^{el} \\ \epsilon_{\theta\theta}^{el} \\ \epsilon_{zz}^{el} \\ \epsilon_{\theta z}^{el} \\ \epsilon_{rz}^{el} \\ \epsilon_{r\theta}^{el} \end{Bmatrix}. \quad (\text{G.3})$$

Equation G.3 corresponds to the one of Equation 5.3. However, for a radially polarized piezoelectric material in a plain strain situation Equations G.1, G.2 and G.3, respectively, turn into

$$\begin{Bmatrix} \sigma_{rr}^{pz} \\ \sigma_{\theta\theta}^{pz} \\ \sigma_{r\theta}^{pz} \end{Bmatrix} = \begin{pmatrix} C_{11}^{pz} & C_{12}^{pz} & 0 \\ \cdot & C_{22}^{pz} & 0 \\ \cdot & \cdot & C_{33}^{pz} \end{pmatrix} \begin{Bmatrix} \epsilon_{rr}^{pz} \\ \epsilon_{\theta\theta}^{pz} \\ \epsilon_{r\theta}^{pz} \end{Bmatrix} - \begin{pmatrix} e_{11} & 0 \\ e_{21} & 0 \\ 0 & e_{32} \end{pmatrix} \begin{Bmatrix} E_r \\ E_\theta \end{Bmatrix}, \quad (\text{G.4})$$

$$\begin{Bmatrix} D_r \\ D_\theta \end{Bmatrix} = \begin{pmatrix} e_{11} & e_{21} & 0 \\ 0 & 0 & e_{32} \end{pmatrix} \begin{Bmatrix} \epsilon_{rr}^{pz} \\ \epsilon_{\theta\theta}^{pz} \\ \epsilon_{r\theta}^{pz} \end{Bmatrix} + \begin{pmatrix} \varepsilon_{11} & 0 \\ \cdot & \varepsilon_{22} \end{pmatrix} \begin{Bmatrix} E_r \\ E_\theta \end{Bmatrix}. \quad (\text{G.5})$$

Applying a similar procedure to a purely orthotropic elastic material results into

$$\begin{Bmatrix} \sigma_{rr}^{el} \\ \sigma_{\theta\theta}^{el} \\ \sigma_{r\theta}^{el} \end{Bmatrix} = \begin{pmatrix} C_{11}^{el} & C_{12}^{el} & 0 \\ \cdot & C_{22}^{el} & 0 \\ \cdot & \cdot & C_{33}^{el} \end{pmatrix} \begin{Bmatrix} \epsilon_{rr}^{el} \\ \epsilon_{\theta\theta}^{el} \\ \epsilon_{r\theta}^{el} \end{Bmatrix}. \quad (\text{G.6})$$

Appendix H

Kinematic Relations

By using the cylindrical gradient operator,

$$\nabla = \frac{\partial}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \mathbf{e}_\theta + \frac{\partial}{\partial z} \mathbf{e}_z, \quad (\text{H.1})$$

the Lagrangian Cauchy-Green infinitesimal deformation tensor in an expanded matrix-form becomes

$$\begin{pmatrix} \epsilon_{rr} & \epsilon_{r\theta} & \epsilon_{rz} \\ \cdot & \epsilon_{\theta\theta} & \epsilon_{\theta z} \\ \cdot & \cdot & \epsilon_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} \right) & \frac{1}{2} \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) \\ \cdot & \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} & \frac{1}{2} \left(\frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{r \partial \theta} \right) \\ \cdot & \cdot & \frac{\partial u_z}{\partial z} \end{pmatrix}. \quad (\text{H.2})$$

Equation H.2 corresponds to Equation 5.5 in the main text. The latter equations, in a plain strain situation, are reduced to

$$\begin{pmatrix} \epsilon_{rr} & \epsilon_{r\theta} \\ \cdot & \epsilon_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} \right) \\ \cdot & \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \end{pmatrix}. \quad (\text{H.3})$$

Appendix I

Conservation Laws

The expanded matrix form of conservation of linear momentum in the absence of body force is

$$\begin{pmatrix} \frac{\partial}{\partial r} + \frac{1}{r} & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{1}{r} \frac{\partial}{\partial \theta} - \frac{1}{r} \\ 0 & \frac{1}{r} \frac{\partial}{\partial \theta} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} + \frac{2}{r} \\ 0 & 0 & \frac{\partial}{\partial z} & \frac{1}{r} \frac{\partial}{\partial \theta} & \frac{\partial}{\partial r} + \frac{1}{r} & 0 \end{pmatrix} \begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{\theta z} \\ \sigma_{rz} \\ \sigma_{r\theta} \end{Bmatrix} = \rho \begin{Bmatrix} \ddot{u}_r \\ \ddot{u}_\theta \\ \ddot{u}_z \end{Bmatrix}. \quad (\text{I.1})$$

Equation I.1 corresponds to the one of Equation 5.6. The electrostatic charge equilibrium, on the other hand, for a piezoelectric material in the absence of free charge density is

$$\frac{1}{r} \frac{\partial (r \partial D_r)}{\partial r} + \frac{1}{r} \frac{\partial D_\theta}{\partial \theta} + \frac{\partial D_z}{\partial z} = 0. \quad (\text{I.2})$$

Equation I.2 corresponds to the one of Equation 5.7. In the last two equations for an axially symmetric (2D) configuration, all variables in z direction as well as differentiations with respect z become zero.

Appendix J

Modal Coefficient Matrices

Modal coefficient elements of elastic medium are as follows

$$\mathbf{G}_n^{el}(r, \omega) = \begin{pmatrix} S_n^{(1,1)}(r, \omega) & S_n^{(1,2)}(r, \omega) & S_n^{(1,3)}(r, \omega) \\ S_n^{(2,1)}(r, \omega) & S_n^{(2,2)}(r, \omega) & S_n^{(2,3)}(r, \omega) \end{pmatrix} \quad (\text{J.1})$$

where $S_n^{(i,j)} \equiv S_n^{(i,j)}(r, \omega)$ are

$$S_n^{(1,1)} = \begin{pmatrix} -\frac{C_{12}^{el}}{rC_{11}^{el}} & -\frac{inC_{12}^{el}}{rC_{11}^{el}} \\ -\frac{in}{r} & \frac{1}{r} \\ -ik_z & 0 \end{pmatrix}, \quad S_n^{(1,2)} = \begin{pmatrix} -\frac{ik_z C_{13}^{el}}{C_{11}^{el}} & \frac{1}{C_{11}^{el}} \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$S_n^{(2,1)} = \begin{pmatrix} \frac{C_{22}^{el} - \frac{(C_{12}^{el})^2}{C_{11}^{el}}}{r^2} - \omega^2 \rho_{el} & \frac{in \left(C_{11}^{el} C_{22}^{el} - (C_{12}^{el})^2 \right)}{r^2 C_{11}^{el}} \\ \frac{in \left((C_{12}^{el})^2 - C_{11}^{el} C_{22}^{el} \right)}{r^2 C_{11}^{el}} & C_{44}^{el} k_z^2 + \frac{n^2 \left(C_{11}^{el} C_{22}^{el} - (C_{12}^{el})^2 \right)}{r^2 C_{11}^{el}} - \omega^2 \rho_{el} \\ \frac{ik_z \left(C_{12}^{el} C_{13}^{el} - C_{11}^{el} C_{23}^{el} \right)}{r C_{11}^{el}} & \frac{k_z n \left(C_{11}^{el} \left(C_{23}^{el} + C_{44}^{el} \right) - C_{12}^{el} C_{13}^{el} \right)}{r C_{11}^{el}} \end{pmatrix},$$

$$S_n^{(2,2)} = \begin{pmatrix} \frac{ik_z \left(C_{11}^{el} C_{23}^{el} - C_{12}^{el} C_{13}^{el} \right)}{r C_{11}^{el}} & \frac{\frac{C_{12}^{el}}{C_{11}^{el}} - 1}{r} \\ \frac{k_z n \left(C_{11}^{el} \left(C_{23}^{el} + C_{44}^{el} \right) - C_{12}^{el} C_{13}^{el} \right)}{r C_{11}^{el}} & -\frac{in C_{12}^{el}}{r C_{11}^{el}} \\ \left(C_{33}^{el} - \frac{(C_{13}^{el})^2}{C_{11}^{el}} \right) k_z^2 + \frac{n^2 C_{44}^{el}}{r^2} - \omega^2 \rho_{el} & -\frac{ik_z C_{13}^{el}}{C_{11}^{el}} \end{pmatrix} \text{ and}$$

$$S_n^{(1,3)} = \begin{pmatrix} 0 & 0 \\ \frac{1}{C_{66}^{el}} & 0 \\ 0 & \frac{1}{C_{55}^{el}} \end{pmatrix}, \quad S_n^{(2,3)} = \begin{pmatrix} -\frac{in}{r} & -ik_z \\ -\frac{1}{r} & 0 \\ 0 & -\frac{1}{r} \end{pmatrix}.$$

Equation J.1 corresponds to the one of Equation 5.12a in the main text. Furthermore, the modal coefficient elements of piezoelectric materials can be written as

$$\mathbf{G}_n^{pz}(r, \omega) = \begin{pmatrix} W_n^{(1,1)}(r, \omega) & W_n^{(1,2)}(r, \omega) & W_n^{(1,3)}(r, \omega) & W_n^{(1,4)}(r, \omega) \\ W_n^{(2,1)}(r, \omega) & W_n^{(2,2)}(r, \omega) & W_n^{(2,3)}(r, \omega) & W_n^{(2,4)}(r, \omega) \end{pmatrix} \quad (\text{J.2})$$

in which $W_n^{(i,j)} \equiv W_n^{(i,j)}(r, \omega)$ are

$$\begin{aligned} W_n^{(1,1)} &= \begin{pmatrix} -\frac{C_{12}^{pz}}{rC_{11}^{pz}} & -\frac{inC_{12}^{pz}}{rC_{11}^{pz}} \\ -\frac{in}{r} & \frac{1}{r} \\ -ik_z & 0 \\ \frac{C_{22}^{pz} - \frac{(C_{12}^{pz})^2}{C_{11}^{pz}}}{r^2} - \omega^2 \rho_{pz} & \frac{in(C_{11}^{pz}C_{22}^{pz} - (C_{12}^{pz})^2)}{r^2C_{11}^{pz}} \end{pmatrix}, \\ W_n^{(1,2)} &= \begin{pmatrix} -\frac{ik_z C_{13}^{pz}}{C_{11}^{pz}} & \frac{1}{C_{11}^{pz}} \\ 0 & 0 \\ 0 & 0 \\ \frac{ik_z(C_{11}^{pz}C_{23}^{pz} - C_{12}^{pz}C_{13}^{pz})}{rC_{11}^{pz}} & \frac{\frac{C_{12}^{pz}}{C_{11}^{pz}} - 1}{r} \end{pmatrix}, \quad W_n^{(1,3)} = \begin{pmatrix} 0 & 0 \\ \frac{1}{C_{66}^{pz}} & 0 \\ 0 & \frac{\epsilon_{11}}{e_{15}^2 + C_{55}^{pz}\epsilon_{11}} \\ -\frac{in}{r} & -ik_z \end{pmatrix}, \\ W_n^{(1,4)} &= \begin{pmatrix} 0 & -\frac{ik_z e_{31}}{C_{11}^{pz}} \\ 0 & 0 \\ \frac{e_{15}}{e_{15}^2 + C_{55}^{pz}\epsilon_{11}} & 0 \\ 0 & \frac{ik_z(C_{11}^{pz}e_{32} - C_{12}^{pz}e_{31})}{rC_{11}^{pz}} \end{pmatrix}, \\ W_n^{(2,1)} &= \begin{pmatrix} \frac{in((C_{12}^{pz})^2 - C_{11}^{pz}C_{22}^{pz})}{r^2C_{11}^{pz}} & C_{44}^{pz}k_z^2 + \frac{n^2(C_{11}^{pz}C_{22}^{pz} - (C_{12}^{pz})^2)}{r^2C_{11}^{pz}} - \omega^2 \rho_{pz} \\ \frac{ik_z(C_{12}^{pz}C_{13}^{pz} - C_{11}^{pz}C_{23}^{pz})}{rC_{11}^{pz}} & \frac{k_z n(C_{11}^{pz}(C_{23}^{pz} + C_{44}^{pz}) - C_{12}^{pz}C_{13}^{pz})}{rC_{11}^{pz}} \\ \frac{ik_z(C_{12}^{pz}e_{31} - C_{11}^{pz}e_{32})}{rC_{11}^{pz}} & \frac{k_z n(C_{11}^{pz}(e_{24} + e_{32}) - C_{12}^{pz}e_{31})}{rC_{11}^{pz}} \\ 0 & 0 \end{pmatrix}, \\ W_n^{(2,2)} &= \begin{pmatrix} \frac{k_z n(C_{11}^{pz}(C_{23}^{pz} + C_{44}^{pz}) - C_{12}^{pz}C_{13}^{pz})}{rC_{11}^{pz}} & -\frac{inC_{12}^{pz}}{rC_{11}^{pz}} \\ \left(C_{33}^{pz} - \frac{(C_{13}^{pz})^2}{C_{11}^{pz}}\right)k_z^2 + \frac{n^2C_{44}^{pz}}{r^2} - \omega^2 \rho_{pz} & -\frac{ik_z C_{13}^{pz}}{C_{11}^{pz}} \\ \left(e_{33} - \frac{C_{13}^{pz}e_{31}}{C_{11}^{pz}}\right)k_z^2 + \frac{n^2e_{24}}{r^2} & -\frac{ik_z e_{31}}{C_{11}^{pz}} \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

$$W_n^{(2,3)} = \begin{pmatrix} -\frac{2}{r} & 0 \\ 0 & -\frac{1}{r} \\ 0 & 0 \\ 0 & \frac{e_{15}}{e_{15}^2 + C_{55}^{pz} \epsilon_{11}} \end{pmatrix},$$

$$W_n^{(2,4)} = \begin{pmatrix} 0 & \frac{k_z n (C_{11}^{pz} (e_{24} + e_{32}) - C_{12}^{pz} e_{31})}{r C_{11}^{pz}} \\ 0 & \left(e_{33} - \frac{C_{13}^{pz} e_{31}}{C_{11}^{pz}} \right) k_z^2 + \frac{n^2 e_{24}}{r^2} \\ -\frac{1}{r} & -\epsilon_{33} k_z^2 - \frac{e_{31}^2 k_z^2}{C_{11}^{pz}} - \frac{n^2 \epsilon_{22}}{r^2} \\ -\frac{C_{55}^{pz}}{e_{15}^2 + C_{55}^{pz} \epsilon_{11}} & 0 \end{pmatrix} \text{ and}$$

Equation J.2 corresponds to the one of Equation 5.12b in the main text.

