"NON-LINEAR ELECTRONIC EFFECTS IN SOLIDS"

A thesis
submitted in partial fulfilment of
the requirements for the degree of
Master of Science
at
the University of Manitoba

by

Hing Cheung Law

June, 1967

CONTENTS

CONTENTS

ACKNOWLEDGEMENT

The author wishes to thank Dr. V. V. Paranjape for suggesting the problem and for his guidance throughout the development of this thesis.

The author would also like to take this opportunity to thank Mr. A. B. Berkal for his help in proof-reading and Mrs. Claudia Lin for typing this thesis.

Finally, the author is indebted to the Canadian National Research Council for the financial support given to him.

ABSTRACT

Non-linear effects in solids due to interaction of mobile carriers with an external electromagnetic field are investigated, using the hydrodynamical approach. In the case when the conduction or valence band is non-parabolic, non-linear effects are also observed. It is shown that in non-linear effects due to non-parabolic bands in indium antimonide and germanium, the generation of even harmonic components in the current density can be achieved by imposing an external dc electric field.

Some of the results in this thesis are in the process of publication.

CHAPTER I

INTRODUCTION

When electrons and ions in solids are subjected to an external electromagnetic field, their states of motion are changed, resulting in two phenomena. One is that the ions and bound electrons are relatively displaced, producing a net dipole moment per unit volume. The net dipole moment per unit volume is called the polarisation. The other is that the mobile electrons are affected to produce a net current density. The current density have a component oscillating with the frequency w of the incident field, but there are also present components which oscillate with frequencies 2w, 3w .... These components are the harmonics of w. If the incident field contains more than one frequency component, say w_1 and w_2, there would be present in the polarisation and current density mixed frequency components like $w_1 \pm w_2$, $2w_1 \pm w_2$, $w_1 \pm 2w_2$ .... The generation of harmonic and mixed frequency components as a result of electrons in a solid interacting with the incident field will be called non-linear electronic effects in solids in this thesis. Non-linear effects due to the interaction of the incident with ions and bound electrons have been reviewed by P. A. Franken (1963). In the band theory of solids, a mobile electron is not the only kind of current carrier;

current is also carried by holes. Thus non-linear effects can also be observed in the interaction of electromagnetic field with holes. It is the purpose of this thesis to investigate the mechanism of non-linear effects due to the interaction of the incident field with mobile electrons and holes, and the non-linear effects produced when the solid is under forces such as dc electric and magnetic fields.

Non-linear effects are described by non-linear equations. Non-linear equations have the following property: if S_1 and S_2 are separately solutions of the equation, $S_1 + S_2$ is not a solution of the equation. The equation

$$\frac{dy}{dx} + y = 0 \qquad (1.1)$$

is linear while

$$\frac{dy}{dx} + y^2 = 0 \qquad (1.2)$$

is non-linear. If y_1 and y_2 are two solutions of these two equations, $y_1 + y_2$ is a solution of Eq.(1.1) but not a solution of Eq.(1.2). In general, non-linear equations contain terms quadratic or of higher power in the unknown, or if there are two equations in two unknowns x and y, the equations contain product of powers of x and y, i.e. $x^{l_1} \; y^{l_2}$ where l_1, l_2 are integers.

The hydrodynamical equations will be used to describe the interaction between the external electromagnetic field and the current carriers. Using the perturbation technique to solve these equations, explicit expressions for the harmonics and mixed frequency components are obtained. For current carriers whose velocity is not proportional to the momentum, which is true when the conduction or valence band of the solid is non-parabolic, non-linearity is also introduced.

It is characteristic of non-linear phenomena that they have steady state solutions only for a finite interval of time. This thesis is only concerned with the steady state solutions.

All calculations are carried out in the Gaussian system of units.

CHAPTER II.

THE HYDRODYNAMICAL APPROACH

The behaviour of an ionised medium under an applied electromagnetic field can often be described by a fluid model, rather than by the trajectories of individual particles. This fluid is made up from two electrically charged components, one being the gas of electrons and the other the gas of positive ions. Each component is described by the following set of equations:

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad (2.1)$$

$$\nabla \cdot \vec{B} = 0 \qquad (2.2)$$

$$\nabla \times \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c} ne\vec{v} \qquad (2.3)$$

$$\nabla \cdot \vec{D} = 4\pi e(n-n_0) \qquad (2.4)$$

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{v}) = 0 \qquad (2.5)$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = \frac{e}{m}\left[\vec{E}-\nabla\phi\right] + \frac{e}{mc}\left[\vec{v}\times\vec{B}\right] - \frac{\vec{v}}{\tau} - \frac{1}{n}\nabla\left[\frac{3nkT}{m}\right] \qquad (2.6)$$

where $\vec{D} = \varepsilon_0 \vec{E}$, $\vec{B} = \mu_0 \vec{H}$, $\vec{E}$ and $\vec{H}$ being the incident electric and magnetic field intensity respectively;

ε_0 and μ_0 are the dielectric constant and permeability of free space and are therefore both numerically equal to unity.

n is the density of electrons (ions)

$\vec{v}$ is the average velocity of electrons (ions)

m is the mass of electrons (ions)

n_0 is the density of electrons at equilibrium which is equal to the density of ions at equilibrium (at equilibrium means in the absence of applied field and temperature gradient)

c is the velocity of light

τ is the relaxation time of a particular collision process

k is the Boltzmann constant

T is the temperature in degrees Kelvin.

ϕ is an effective potential determined via poisson's equation $\nabla^2\phi = -4\pi e(n-n_0)$, by the deviation of the electron (ion) density from its equilibrium value.

Equations(2. 1)-(2. 6)are called hydrodynamical equations.

In the case of solids, the ions are much less mobile than the electrons and their motion can be neglected. Another situation where the motion of the ions can be neglected is when they are heavy.

The first four equations are just the Maxwell electrodynamic equations, where it is recognised that the current density $\vec{J} = ne\vec{v}$.

Equations(2. 5) and (2. 6) can be derived from the Boltzmann transport equation.(Appendix A) Eq. (2.5), called the continuity equation, expresses the conservation of the number of electrons (ions) Eq. (2.6) express the conservation of momentum of electrons (ions). Eq. (2.5) and (2.6) can be obtained by taking first and second velocity moments of the Boltzmann equation, if certain approximations and assumptions are made. $\vec{v}$ then is the velocity of electrons (ions) averaged over the steady state solution of the Boltzmann equation. In the case of electrons in solids, m must be replaced by m*, the conductivity effective mass of the electrons.

In equation (I2. 6), $\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}$ is just the average acceleration $\frac{d\vec{v}}{dt}$ of the charged particle. $\vec{v}$ is a function of position and time i.e. $\vec{v} = \vec{v}(x,y,z,t)$

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \vec{v}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \vec{v}}{\partial z}\frac{\partial z}{\partial t}$$

But $\frac{\partial x}{\partial t} = v_x$, $\frac{\partial y}{\partial t} = v_y$, $\frac{\partial z}{\partial t} = v_z$

So $\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}$

$\frac{e}{m}\left[\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right]$ is the force on the charged particle due to the external applied field; $e(-\nabla\phi)$ is the force on the charged particle due to the internal field $-\nabla\phi$.

$\frac{\vec{v}}{\tau}$ describes the effect of collision and scattering processes. There are electron-electron collisions and electron-ion collisions. Electron-ion collisions may be elastic or inelastic. Inelastic collisions result in ionisation and recombination processes. Electrons in solids may also be scattered by impurities, crystal defects and lattice vibrations. To this day, there exists no rigorous theory to account for the effect of collision processes on motion of charged particles. The introduction of a relaxation time τ is just an attempt in this endeavour and represents no more than an approximation. Furthermore, it is not possible to define a universal τ for all collision processes. The total effect may be given by a τ where

$$\frac{1}{\tau} = \frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_3} + \cdots\cdots\cdots\cdots$$

$\tau_1, \tau_2, \tau_3, \ldots\ldots$ being the relaxation time of different collision processes under consideration. Depending on the particular collision process concerned, may or may not depend on the velocity $\vec{v}$.

$\nabla \left(\frac{3nkT}{m} \right)$ is the pressure force arising from the thermal motion of the charged particles.

The terms in Eq. (2.5) and (2.6) responsible for production of non-linear effects are $n\vec{v}$, $(\vec{v}.\nabla)\ \vec{v}$, $\vec{v} \times \vec{B}$ and possibly $\frac{\vec{v}}{\tau}$ if τ is a function of $\vec{v}$. To illustrate how quadratic terms like $n\vec{v}$ give rise to harmonic componentsof input frequency, consider an applied field in the form $\vec{E}_1 e^{iw_1 t}$. As a first approximation, n and $\vec{v}$ will separatelyhave a time dependence of $e^{iw_1 t}$

$n\vec{v}$ therefore contains terms of time dependence e^{2iw_1t}. It follows that n and $\vec{v}$ should also separately contain terms of time dependence of e^{2iw_1t}. Repeating the argument, higher harmonics of w_1 are also present in n and $\vec{v}$.

Consider a non-uniform applied electric field $\vec{E}$. If the field variation is along the direction of the electric field*, the charged particles are exposed to a field of varying amplitude. This effect is included in the term $(\vec{v}.\nabla)\,\vec{v}$, which generates harmonics of the input frequency, w_1. $\vec{B}$ in this case is not time dependent explicitly, for $\nabla \times \vec{E} = 0$ so that $-\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ On the other hand, if the electric field varies in amplitude in a direction perpendicular to the field, $\nabla \times \vec{E} \neq 0$ and there is an explicitly time dependent magnetic field associated with it. This magnetic field interacts with the moving electrons via the Lorentz force $\frac{\vec{v} \times \vec{B}}{c}$, which produces harmonics of the input frequency. Thus any non-uniformity of the applied electric field produces harmonic components of input frequency in the electron velocity.

Even if the applied electric field is uniform, harmonic generation can still be produced by the spatial variation of the static charged particle density. The movement

---

* It is to be noted that the hydro-dynamical equations apply to any external electromagnetic field, not necessarily transverse fields.

of electrons from a region of high density to a region of lower density under an oscillating electric field gives rise to a net oscillating charge density, for $\nabla \cdot (n\vec{v}) \neq 0$ so that $\frac{\partial n}{\partial t} \neq 0$. The product $n\vec{v}$ of this charge density with the velocity therefore generates harmonic components in the current density $\vec{J} = ne\vec{v}$.

There are numerous reports, both theoretical and experimental, on research done on non-linear effects due to the mechanisms discussed above. Many of them start from the Boltzmann equation, which is theoretically more rigorous but mathematically more involved. In the next Chapter, a review of theoretical calculations of non-linear effects due to these mechanisms will be presented, using the less rigorous but simpler hydrodynamical equations (2 . 5) and (2.. 6). But before these calculations are presented, something must be said about the range of validity of the hydrodynamical equations.

The hydrodynamical equations are derived from the Boltzmann transport equations. So the first question is under what conditions is the Boltzmann transport equation valid. To answer this question, one would have to start from quantum statistics and follow the arguments which lead to the Boltzmann transport equation. It is unfortunate that no review pertaining to thses arguments is available in the literature and the author does not intend to give such a review here. However, if one adopts the one-electron approximation (Reitz,1955) for describing the electrons in solids, then provided that theexternal field is not too strong and not too high in frequency, the Boltzmann equation is applicable to regions larger than one within which there may be deviation from charge neutrality in the absence of external field(Ziman,1960; Luttinger,1951; Adams, 1952,1953; Feuer, 1952; Kittel & Mitchell, 1954; Luttinger& Kohn, 1955; Adams & Argyres, 1956; Linhart, 1960)The dimension of a region within which such a deviation may occur is the Debye length (Appendix B). So the smallest length of interest L must be greater than the Debye length λ_D. The smallest length of interest is the wavelength of the applied field. Eqs. (2.5) and (2.6) are therefore applicable when the wavelength λ of the applied field is greater than the Debye length. The criteria $\lambda > \lambda_D$ was discussed by Rostoker(1960), Balescu (1960) and some others.

Assuming the validity of the Boltzmann equations, there is still one restriction on the applicability of Eqs. (2.5) and (2.6). Since Eqs. (2.5)and (2.6) are non-relativistic equations, the electron velocity must be very much smaller than the velocity of light.

CHAPTER III

CALCULATION OF NON-LINEAR EFFECTS IN THE HYDRODYNAMICAL EQUATIONS

In this chapter, the continuity equation (2. 5) and the equation of motion (2 . 6) will be solved explicitly with the aim of obtaining an expression for the current density $\vec{J} = ne\vec{v}$, where n is the electron density, e is the electronic charge and $\vec{v}$ is the average velocity of electron defined in chapter II. The equations will be solved first in a simple case in which the incident electric field $\vec{E}$ is sinusoidal and the electron density is assumed constant. Later in the chapter, the following more complicated situations will be considered.

1) $\vec{E}$ with more than one frequency component
2) $\vec{E}$ in the presence of an external d.c. magnetic field
3) $\vec{E}$ in the presence of an external d.c. electric field
4) variation of n is taken into account
5) the relaxation time τ depends on the magnitude of the average electron velocity.

Except in case 4), n is assumed to be equal to n_0 , the value of the electron density at equilibrium. This assumption is made in order to avoid mathematical intricacies, so that the features of the solution to the hydrodynamical equations in the various cases can be brought out in a simple way.

3.1 $\vec{E}$ sinusoidal , $n = n_0$

Let $\vec{E} = \vec{E}_1 e^{iw_1 t} + \vec{E}_1^* e^{iw_1 t}$ (3.1)

$$\vec{B} = \vec{B}_1 e^{iw_1 t} + \vec{B}_1^* e^{iw_1 t} \qquad (3.2)$$

where w_1 is the frequency of the incident field and E and B are connected by

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

The equation of motion becomes

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} = \frac{e}{m^*}\left[\vec{E} - \nabla\phi_0\right] + \frac{e}{m^*}\frac{\vec{v}\times\vec{B}}{c} - \frac{\vec{v}}{\tau} - \frac{1}{n}\nabla\left(\frac{3nkT}{m^*}\right) \qquad (3.3)$$

where $-\nabla\phi_0$ is an internal static electric field which may be present.

Throughout the calculations in this chapter, the term $\frac{1}{n}\nabla\left(\frac{3nkT}{m^*}\right)$ will be neglected. This is equivalent to assuming that the ordered motion of the electrons caused by the electromagnetic forces predominates over the random motion caused by the thermal and pressure effects. This assumption is valid at a sufficiently low temperature.

A method of obtaining a general solution to Eq.(3.3) is not available. Usually a perturbation technique is used. In this technique, it is assumed that the incident field $\vec{E}$ is of sufficiently low amplitude such that $\vec{v}$ can be expanded in a rapidly converging power series.

$$\vec{v} = \vec{v}_0 + \sum_{j=1}^{\infty} \vec{v}_j e^{ijw_1t} \quad + \text{complex conjugate} \qquad (3.4)$$

where $v_1 \gg v_2 \gg v_3 \gg \quad \ldots\ldots\ldots\ldots, \quad v_j$ denoting the magnitude of $\vec{v}_j$.

Putting Eqs. (3.1), (3.2) and (3.4) into (3.3), and equating coefficients of corresponding powers of e^{iw_1t}, the following set of equations is obtained:

$$iw_1\vec{v}_1 \simeq \frac{e}{m^*}\vec{E} - \frac{\vec{v}}{\tau} \qquad (3.5)$$

$$2iw_1\vec{v}_2 \simeq -(\vec{v}_1 . \nabla)\ \vec{v}_1 + \frac{e}{m^*}\frac{\vec{v}_1 \times \vec{B}_1}{c} - \frac{\vec{v}_2}{\tau} \qquad (3.6)$$

In the above set of equations, terms of order higher than j are neglected in an equation for $\vec{v}_j$. For example, in the equation for $\vec{v}_1$, terms like $(\vec{v}_1^* . \nabla)\ \vec{v}_2$, $(\vec{v}_2^* . \nabla)\ \vec{v}_3$ .... and $\frac{e}{m^*c}(\vec{v}_1 \times \vec{B}_2^*)$ are neglected. Terms containing $\vec{v}_0$ are also neglected because $\vec{v}_0$ is the electron velocity in the absence of applied field and is very small under the assumption that thermal motion is negligible.

Substituting Eq.(3.5) into Eq.(3.6), and using the vector identity

$$(\vec{A}.\vec{B}) = (\vec{A}.\nabla)\vec{B} + (\vec{B}.\nabla)\vec{A} + \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) \quad (3.7)$$

and using the relation ($-\frac{1}{c}\frac{\partial\vec{B}}{\partial t} = \nabla\times\vec{E}$)

$$\vec{B}_1 = \frac{ic}{w_1}\nabla\times\vec{E}_1 \quad (3.8)$$

$$\vec{v}_2 = \frac{1}{2}\frac{e^2}{m^*}\frac{1}{(2iw_1+\frac{1}{\tau})(iw_1+\frac{1}{\tau})^2}\nabla\left(\vec{E}_1.\vec{E}_1\right) \quad (3.9)$$

v_2 is of second order in E_1, and in general v_j is of j^{th} order in E_1.

In order that $v_2 << v_1$, E_1 and W_1, should satisfy the inequality

$$\frac{e}{m^* w_1}E_1 >> \frac{e^2}{m^{*2}w_1^3}\left|\nabla(\vec{E}_1.\vec{E}_1)\right| \quad (3.10)$$

or

$$E_1 >> \frac{e^2}{m^* w_1^2}E_1\left|\nabla\vec{E}_1\right|$$

If $\vec{E}_1$ is a plane wave, the magnitude of $\nabla\vec{E}_1$ is $E_1 k$ where $\vec{k}$ is the propagation vector and is equal to w_1/c. Thereforethe condition for $v_2<<v_1$ becomes

$$\frac{m^* c}{e} >> \frac{E_1}{w_1} \quad (3.11)$$

Consider now the time independent equation obtained from Eq.(3.3)

$$(\vec{v}_1.\nabla)\vec{v}_1^* \ (\vec{v}_1^*.\nabla)\vec{v}_1 = -\frac{e}{m^*}\nabla\phi_o + \frac{e}{m^*c}\left[\vec{v}_1\times\vec{B}_1^* + \vec{v}_1^*\times\vec{B}_1\right] \quad (3.12)$$

using Eqs.. (3.5),(3.7)and (3.8), one gets

$$\frac{e}{m^{*2}}\,\frac{1}{w_1^2+\frac{1}{\tau^2}}\,\nabla(\vec{E}_1.\vec{E}_1^*) = -\frac{e}{m^*} \quad (3.13)$$

Thus if $\vec{E}_1$ is a plane wave, $\nabla(\vec{E}_1.\vec{E}_1^*)=0$ and $\nabla\phi_o = 0$, provided the assumption that $\nabla\left(\frac{3nkT}{m^*}\right)$ is negligible holds. This simple conclusion seems to have been overlooked in the literature.

3.2 $\vec{E}$ with more than one frequency component, $n = n_o$

Suppose $\vec{E}$ has two frequency components:

$$\vec{E} = \vec{E}_1 e^{iw_1t} + \vec{E}_2 e^{iw_2t} + \text{complex conjugate} \quad (3.14)$$

and $$\vec{B} = \vec{B}_1 e^{iw_1t} + \vec{B}_2 e^{iw_2t} + \text{complex conjugate} \quad (3.15)$$

The calculation can be carried out in exactly the same manner as in the previous case, but there is one very important difference: besides harmonics of w_1 and w_2, there are present in the solution for $\vec{v}$ components at mixed frequencies $j_1 w_1 \pm j_2 w_2$, where j_1, j_2 can take positive or negative integral values. The mixed frequency components arise because of the non-linear interaction of the components $\vec{E}_1 e^{iw_1t}$ and $\vec{E}_2 e^{iw_2t}$.

The first four time dependent terms of $\vec{v}$ are

$$\vec{v}_1 e^{iw_1 t} + \vec{v}_2 e^{iw_2 t} + \vec{v}_{12} e^{i(w_1+w_2)t} + \vec{v}_{21} e^{i(w_1-w_2)t} \quad (3.16)$$

$\vec{v}_2$ in this expression is not the same as $\vec{v}_2$ in Eq.(3.6) $\vec{v}_{12}$ and $\vec{v}_{21}$ are obtained by substituting Eq.(3.14) and Eq. (3.15)into the equation of motion

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = \frac{e}{m^*}\vec{E} + \frac{\vec{v}\times\vec{B}}{c} - \frac{\vec{v}}{\tau}$$

with the result that

$$iw_1\vec{v}_1 \simeq \frac{e}{m^*}\vec{E}_1 - \frac{\vec{v}_1}{\tau} \quad (3.17)$$

$$iw_2\vec{v}_2 \simeq \frac{e}{m^*}\vec{E}_2 - \frac{\vec{v}_2}{\tau} \quad (3.18)$$

$$i(w_1+w_2)\vec{v}_{12}+(\vec{v}_1\cdot\nabla)\vec{v}_2+(\vec{v}_2\cdot\nabla)\vec{v}_1 \simeq \frac{e}{m^*}\left[\frac{\vec{v}_1\times\vec{B}_2}{c}+\frac{\vec{v}_2\times\vec{B}_1}{c}\right] - \frac{\vec{v}_{12}}{\tau} \quad (3.19)$$

$$i(w_1-w_2)\vec{v}_{21}+(\vec{v}_1\cdot\nabla)\vec{v}_2^* \simeq \frac{e}{m^*}\left[\frac{\vec{v}_1\times\vec{B}_2^*}{c}\right] - \frac{\vec{v}_{21}}{\tau} \quad (3.20)$$

Substituting Eqs. (3.17)and (3.18) into Eq. (3.19) and Eq. (3.20)and using $-\frac{1}{c}\frac{\partial\vec{B}}{\partial t} = \nabla\times\vec{E}$, one gets

$$\vec{v}_{12} \simeq \frac{1}{i(w_1+w_2)+\frac{1}{\tau}} \frac{e^2}{m^{*2}(iw_1+\frac{1}{\tau})(iw_2+\frac{1}{\tau})} \nabla(\vec{E}_1\cdot\vec{E}_2) \quad (3.21)$$

$$\vec{v}_{21} \simeq \frac{1}{i(w_1-w_2)+\frac{1}{\tau}} \frac{e^2}{2m^{*2}} \frac{1}{(iw_1+\frac{1}{\tau})(iw_2+\frac{1}{\tau})} \nabla(\vec{E}_1\cdot\vec{E}_2^*) \quad (3.22)$$

It is to be noted that when $\vec{E}_1$ is perpendicular to $\vec{E}_2$, $\vec{v}_{12}$ and $\vec{v}_{21}$ vanish.

3.3 $\vec{E}$ in the presence of a d.c. magnetic field, $n=n_0$

Consider a d.c. magnetic field $\vec{B}_o$ being applied in the z-direction, in addition to $\vec{E}_1$ and $\vec{B}_1$. (There is no loss of generality in assuming that $\vec{B}_o$ is pointing in the positive z-direction) One then has:

$$\vec{E} = \vec{E}_1 e^{iw_1 t} + \vec{E}_1^* e^{-iw_1 t} \tag{3.1}$$

$$\vec{B} = \vec{B}_0 + \vec{B}_1 e^{iw_1 t} + \vec{B}_1^* e^{-iw_1 t} \tag{3.23}$$

$$\vec{v} = \vec{v}_0 + \vec{v}_1 e^{iw_1 t} + \vec{v}_2 e^{2iw_1 t} + \ldots\ldots + \text{complex conjugate} \tag{3.4}$$

After substituting Eqs. (3.1), (3.23) and (3.4) into the equation of motion, the following equations are obtained:

$$iw_1 \vec{v}_1 \simeq \frac{e}{m^*}\left[\vec{E}_1 + \frac{\vec{v}_1 \times \vec{B}_0}{c}\right] - \frac{\vec{v}_1}{\tau} \tag{3.24}$$

$$2iw_1 \vec{v}_2 + (\vec{v}_1 \cdot \nabla)\vec{v}_1 \simeq \frac{e}{m^*}\left[\frac{\vec{v}_1 \times \vec{B}_1}{c} + \frac{\vec{v}_2 \times \vec{B}_0}{c}\right] - \frac{\vec{v}_2}{\tau} \tag{3.25}$$

Eq.(3.24) is equivalent to the following three equations:

$$iw_1 v_{1x} - \frac{e}{m^*} E_{1x} = v_{1y} w_c - \frac{v_{1x}}{\tau} \tag{3.26}$$

$$iw_1 v_{1y} - \frac{e}{m^*} E_{1y} = -v_{1x} w_c - \frac{v_{1y}}{\tau} \tag{3.27}$$

$$iw_1 v_{1z} - \frac{e}{m^*} E_{1z} = -\frac{v_{1z}}{\tau} \tag{3.28}$$

where $w_c = \dfrac{e}{m^* c} B_0$. These three equations have the solutions:

$$v_{1x} = \frac{e}{m^*} \frac{(iw_1 + \frac{1}{\tau})E_{1x} - w_c E_{1y}}{w_c^2 - w_1^2 + \frac{2iw_1}{\tau} + \frac{1}{\tau^2}} \qquad (3.29)$$

$$v_{1y} = \frac{e}{m^*} \frac{(iw_1 + \frac{1}{\tau})E_{1y} + w_c E_{1x}}{w_c^2 - w_1^2 + \frac{2iw_1}{\tau} + \frac{1}{\tau^2}} \qquad (3.30)$$

$$v_{1z} = \frac{e}{m^*} \frac{1}{iw_1 + \frac{1}{\tau}} E_{1z} \qquad (3.31)$$

Usually -- is very much smaller than w_1. So that

$$\vec{v}_1 \simeq \frac{e}{m^*} \frac{iw_1}{w_c^2 - w_1^2} \left[\vec{E} - \frac{w_c^2}{w_1^2} E_{1z} \vec{i}_3 \right] + \frac{e}{m^*} \frac{\vec{E} \times \vec{w}_c}{w_c^2 - w_1^2} \qquad (3.32)$$

where $\vec{i}_3$ is the unit vector in the positive z-direction and $\vec{w}_c = \frac{e}{m^* c} \vec{B}_0$

Thus v_1 attains its maximum when the applied field has a frequency $w_1 = w_c$. ($\vec{v}_1$, however, will not become infinitely large because of the presence of the terms $\frac{2iw_1}{\tau} + \frac{1}{\tau^2}$ in Eqs. (3.29) and (3.30) , which have been neglected in Eq. (3.32)). This phenomenon is called cyclotron resonance, and W_c is called the cyclotron frequency.

Consider now an equation in $\vec{v}_j$. The terms containing $\vec{v}_j$ in this equation are $ijw_1\vec{v}_j$ and $\frac{e}{m^* c}\left[\vec{v}_j \times \vec{B}_0\right]$ if $\frac{\vec{v}_j}{\tau}$ is neglected. Writing out this equation in cartesian co-ordinates, the following three equations are obtained:

$$ijw_1 v_{jx} + w_c v_{jy} = F \tag{3.33}$$

$$ijw_1 v_{jy} - w_c v_{jx} = G \tag{3.34}$$

$$ijw_1 v_{jz} = H \tag{3.35}$$

where F, G, H are terms not containing v_j. The solution for $\vec{v}_j$ therefore contains the factor $\frac{1}{w_c^2 - (jw_1)^2}$.
Moreover, to get the right time dependence, F, G, H must contain terms which are functions of $\vec{v}_{j-1}$, $\vec{v}_{j-2}$, ... $\vec{v}_1$. It can therefore be concluded that cyclotron resonance for $\vec{v}_j$ occurs at $jw_1 = w_c$, $(j-1)\, w_2 = w_2$, $(j-2)\, w_1 = w_c$, ... $w_1 = w_c$.

3.3 $\vec{E}$ in the presence of a d.c. electric field, $n=n_0$

Consider an external dc electric field $\vec{E}_o$ being applied in addition to $\vec{E}_1$ and $\vec{B}_1$, i.e.

$$\vec{E} = \vec{E}_0 + \vec{E}_1 e^{iw_1 t} + \vec{E}_1^* e^{-iw_1 t} \tag{3.36}$$

$$\vec{B} = \vec{B}_1 e^{iw_1 t} + \vec{B}_1^* e^{-iw_1 t} \tag{3.2}$$

The important point in this case is that $\vec{v}_o$ depends on E_o. The equation in $\vec{v}_0$ is

$$(\vec{v}_0 . \nabla)\vec{v}_0 + \frac{\vec{v}_0}{\tau} \simeq \frac{e}{m^*} \vec{E}_0 \tag{3.37}$$

The larger $\vec{E}_o$ is, the larger $\vec{v}_0$ becomes. When $\vec{E}_0$

is sufficiently large, terms containing $\vec{v}_0$ will predominate over those which do not.

3.5 $\vec{E}$ sinusoidal , n deviates from n_0

The equations in n and $\vec{v}$ are

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} + \frac{\vec{v}}{\tau} - \frac{e}{m^*}\left[\vec{E} - \nabla\phi\right] - \frac{e}{m^* c}\left[\vec{v} \times \vec{B}\right] = 0 \quad (3.38)$$

$$\frac{\partial n}{\partial t} + \nabla.(n\vec{v}) = 0 \quad (2.4)$$

$$\nabla^2 \phi = -4\pi e(n - n_0) \quad (3.39)$$

$$-\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} \quad (2.1)$$

These are in fact just the hydrodynamical and Maxwell equations in Chapter II. Also $\vec{E}, \vec{B}, \vec{v}$ are given by:

$$\vec{E} = \vec{E}_1 e^{iw_1 t} + \vec{E}_1^* e^{-iw_1 t} \quad (3.1)$$

$$\vec{B} = \vec{B}_1 e^{iw_1 t} + \vec{B}_1^* e^{-iw_1 t} \quad (3.2)$$

$$\vec{v} = \vec{v}_0 + \vec{v}_1 e^{iw_1 t} + \vec{v}_2 e^{2iw_1 t} + \ldots\ldots + \text{complex conjugate} \quad (3.4)$$

For the same arguments by which $\vec{v}$ can be expanded in a power series, n can be expressed as the power series.

$$n = n_0 + n_1 e^{iw_1 t} + n_2 e^{2iw_1 t} + \ldots. + \text{complex conjugate} \quad (3.40)$$

An equation in n_1 can be obtained in the following way: from Eq. (2.4), one has

$$\frac{\partial n_1}{\partial t} + n_0 \nabla . \vec{v}_1 = 0 \quad (3.41)$$

Differentiate Eq.(3.41) with respect to time t.

$$\frac{\partial^2 n}{\partial t^2} + n_0 \nabla \cdot \frac{\partial \vec{v}_1}{\partial t} = 0 \qquad (3.42)$$

From Eq. (3.38), one gets

$$\frac{\partial \vec{v}_1}{\partial t} + \frac{\vec{v}_1}{\tau} - \frac{e}{m^*} \vec{E}_1 + \frac{e}{m^*} \text{ (coefficient of } e^{iw_1 t} \text{ in} \nabla \phi) = 0 \qquad (3.43)$$

Take divergence of Eq. (3.43)

$$\nabla \cdot \frac{\partial \vec{v}_1}{\partial t} + \frac{\nabla \cdot \vec{v}_1}{\tau} - \frac{e}{m^*} \nabla \cdot \vec{E}_1 + \frac{e}{m^*} \text{ (coefficient of } e^{iw_1 t} \text{in } \nabla^2 \phi) = 0 \qquad (3.44)$$

From Eq. (3.39), one obtains

$$\text{(coefficient of } e^{iw_1 t} \text{ in } \nabla^2 \phi) = -4\pi e n_1 \qquad (3.45)$$

Substituting Eqs. (3.41), (3.44) and (3.45) into Eq. (3.42), the result is:

$$\frac{\partial^2 n_1}{\partial t^2} + \frac{1}{\tau} \frac{\partial n_1}{\partial t} + \frac{4\pi e^2}{m^*} n_0 n_1 = -\frac{e}{m^*} \nabla \cdot \vec{E}_1 \qquad (3.46)$$

For a transverse incident field, $(\nabla \cdot \vec{E} = 0)$ this is simplified to

$$\frac{\partial^2 n_1}{\partial t^2} + \frac{1}{\tau} \frac{\partial n_1}{\partial t} + w_p n_1 = 0 \qquad (3.47)$$

where

$$w_p^2 = \frac{4\pi e^2}{m^*} n_0$$

Eq. (3.47) in general describes a system oscillating with frequency w_p in the absence of a driving force and damped by friction. In this case, the equation describes the fluctuation to first order of the electron density from equilibrium value, damped by collisions, w_p is called the plasma frequency. In the steady state, the fluctuation is reduced to zero by damping. Therefore $n_2 = o$ in the steady state.

$\vec{v}_1$ can now be solved from Eq.(3.43), giving

$$i w_1 \vec{v}_1 + \frac{\vec{v}_1}{\tau} = \frac{e}{m^*} \vec{E}_1$$

$$\vec{v}_1 = \frac{e}{m^*} \frac{\vec{E}_1}{i w_1 + \frac{1}{\tau}} \qquad (3.47)$$

In exactly the same way, an equation for n_2 can be set up. It turns out to be

$$\frac{\partial^2 n_2}{\partial t^2} + \frac{1}{\tau}\frac{\partial n_2}{\partial t} + w_p^2 n_2 = n_0 \nabla \cdot \vec{A} \qquad (3.48)$$

where $A = (\vec{v}_1 \cdot \nabla)\vec{v}_1 - \frac{e}{m^* c} (\vec{v}_1 \times \vec{B}_1)$

In the steady state, the solution for n_2 is

$$n_2 = \frac{n_0}{w_p^2 - 4w_1^2 + \frac{2 i w_1}{\tau}} \nabla \cdot \vec{A} \qquad (3.49)$$

A resonance occurs at $2w_1 = w_p$ for n_2.

It should be noted that $\vec{J}$ being equal to $ne\vec{v}$, its frequency components have contributions both from n and $\vec{v}$. The third order current density, for example, is $n_0 e\vec{v}_3 + n_1 e\vec{v}_2 + n_2 e\vec{v}_1$. ($n_3 e\vec{v}_0$ is small compared with these three terms).

3.6 When τ depending on magnitude of velocity $\vec{v}$

Suppose $\tau = \alpha v^2$, where α is a constant having dimension $\frac{\text{time}}{\text{velocity}^2}$. Since

$$\vec{v} = \vec{v}_0 + \vec{v}_1 e^{iw_1t} + \vec{v}_2 e^{2iw_1t} + \dots\dots\dots$$

(In this case, it is more convenient to write $\vec{v}$ without adding the complex conjugate). v^2 is given by:

$$v^2 = \vec{v}\cdot\vec{v}^*$$

$$= v_0^2 + 2\vec{v}_0\cdot\vec{v}_1\cos(w_1t) + v_1^2 + 2\vec{v}_1\cdot\vec{v}_2\cos(2w_1t) + \dots\dots (3.50)$$

where v_0, v_1, v_2 .... have been taken to be real for mathematical convenience. So one has

$$\frac{\vec{v}}{\tau} = \frac{1}{\alpha}\,\frac{\vec{v}}{v_0^2 + 2\vec{v}_0\cdot\vec{v}_1\cos(w_1t) + v_1^2 + 2\vec{v}_1\cdot\vec{v}_2\cos(2w_1t) + \dots\dots} \quad (3.51)$$

In the absence of a strong external d.c. electric field, v_0 is small compared with v_1, and $v_1 \gg v_2 \gg v_3 \gg \dots\dots$. Eq. (3.51) can therefore be expanded in a binomial series:

$$\frac{\vec{v}}{\tau} \simeq \frac{1}{\alpha}\vec{v}\left[1 - \frac{2\vec{v}_0\cdot\vec{v}_1}{v_1^2}\cos(w_1t) - \frac{2\vec{v}_1\cdot\vec{v}_2}{v_1^2}\cos(2w_1t) - \dots\dots\right]$$

$$= \frac{1}{\alpha}\left[\vec{v}_0 + \vec{v}_1 e^{iw_1t} + \vec{v}_2 e^{2iw_1t} + \dots\dots\right]\left[1 - \frac{2\vec{v}_0\cdot\vec{v}_1}{v_1^2}\cos(w_1t) - \frac{2\vec{v}_1\cdot\vec{v}_2}{v_1^2}\cos(2iw_1t) - \dots\dots\right] \quad (3.52)$$

The factor $\frac{2\vec{v}_0\cdot\vec{v}_1}{v_1^2}\cos(w_1t)\,v_1 e^{iw_1t}$ in the product gives the second harmonic of w_1 and the factor

$2\,\frac{\vec{v}_1\cdot\vec{v}_2}{v_1^2}\vec{v}_1 e^{iw_1t}\cos(2w_1t)$ gives the third harmonic of W_1. Other factors in the product give higher harmonics of W_1.

As a specific example, harmonic generation due to the presence of an electron density gradient and ionising collisions has been discussed by Chiyoda. (1967)

All calculations in this chapter also apply to the interaction of electromagnetic field with holes, with two modifications:

(i) holes carry a positive charge which has the same magnitude as the electronic charge.

(ii) holes in general have a different effective mass from that of mobile electrons.

Experiments on non-linear effects arising from the mechanisms discussed in this chapter have been performed and the results studied by many authors (Whitmer and Barrette 1961; Proceedings of the Conference on the Physics of Quantum Electronics, 1965) Extensive references can also be found in the paper by Bloembergen, (1966). The experimental results and the theoretical predictions agree within the uncertainties due to the experimental errors.

CHAPTER IV.

NON-PARABOLIC CONDUCTION AND VALENCE BANDS

In the hydrodynamical equations, it was assumed that the average velocity $\vec{v}$ is proportional to the average momentum $\vec{Q}$ (see appendix A) Whether this is true depends on the structure of the conduction and valence bands of the solid. (Interaction of conduction and valence band with higher or lower bands can usually be neglected.) If electrons are the current carriers, the dependence is on the structure of the conduction band. If holes are the current carriers, the dependence is on the structure of the valence band.

For solids with an incompletely filled band, the dependence of velocity on momentum is determined mainly by the band structure up to the level to which the band is filled. Sodium, for example, except near the edge of the Brillouin zones, has energy bands of the form (Smith, 1959)

$$\mathcal{E} = \frac{p^2}{2m^*} \qquad (4.1)$$

where $\vec{p}$ is the momentum of an electron and $\mathcal{E}$ is the energy with the zero of the energy taken at the bottom of the partially filled band. The velocity $\vec{u}$ averaged over a unit cell of the solid is given by:

$$\vec{u} = \nabla_p \mathcal{E} = \frac{\vec{p}}{m^*} \qquad (4.2)$$

It is important to note that $\vec{u}$ is not the same as $\vec{v}$, which is the velocity averaged over the solution of the Boltzmann equation.

The velocity $\vec{u}$ is proportional to the momentum $\vec{p}$ (and therefore as shown in the appendix, the average velocity $\vec{v}$ is proportional to the average momentum $\vec{Q}$) for mobile electrons in sodium. Energy bands which give rise to a linear relationship between velocity and momentum said to be parabolic. Energy bands which give rise to nonlinear relationship between velocity and momentum are said to be non-parabolic.

Sodium crystals have cubic symmetry, and that is why the constant-energy surfaces are spherical in momentum space. There are energy bands which are not spherical but still parabolic. Energy bands with ellipsoidal constant-energy surfaces of the form

$$\mathcal{E} = \frac{p_x^2}{2m_1^*} + \frac{p_y^2}{2m_2^*} + \frac{p_z^2}{2m_3^*} \qquad (4.3)$$

where m_1^*, m_2^* and m_3^* are effective masses in x, y and z directions respectively, so that

$$u_x = \frac{p_x}{m_1^*} \quad , \quad u_y = \frac{p_y}{m_2^*} \quad , \quad u_z = \frac{p_z}{m_3^*} \qquad (4.4)$$

are parabolic. Energy bands with ellipsoidal constant-energy surfaces but displaced in the momentum space of the form:

$$\mathcal{E} = \frac{(p_x-p_c)^2}{2m_1^*} + \frac{p_y^2}{2m_2^*} + \frac{p_z^2}{2m_3^*} \qquad (4.5)$$

where p_c is a constant, so that

$$u_x = \frac{p_x-p_c}{m_1^*}, \quad u_y = \frac{p_y}{m_2^*}, \quad u_z = \frac{p_z}{m_3^*} \qquad (4.6)$$

are also parabolic. In general, an energy band is parabolic if the energy $\mathcal{E}$ is a quadratic function of the momentum p.

For semiconductors which have filled valence band and empty conduction band at zero temperature, the relationship between velocity and momentum is determined mainly by the energy-momentum relation at the highest maximum of the valence band and the lowest minimum of the conduction band. (There may be more than one maximum or minimum.)

Many semiconductors have the highest maximum of the valence band or the lowest minimum of the conduction band at $\vec{K} = 0$ (where $\vec{K}$ is the propagation vector of the current carrier.) If in addition the corresponding bands are non-degenerate, the energy will be a quadratic function near $\vec{K} = 0$ and the bands are therefore parabolic.

Indium antimonide has an outstanding band structure.

in that both the valence band highest maximum and the conduction band lowest minimum occur at $\vec{k} = 0$. Both the conduction and valence bands, however, are degenerate. By taking into account the mutual interaction of the valence and conduction bands, E. O. Kane (1957) showed that for energies above $\vec{k} = 0$ of the order of a fraction of the band gap between the conduction and valence bands, the conduction band becomes non-parabolic:

$$\mathcal{E} = \left[\left(\frac{E_G}{2} \right)^2 + \left(\frac{p^2}{2m^*} \right) E_G \right]^{\frac{1}{2}} \qquad (4.7)$$

where E_G is the band gap. $\vec{u}$ is therefore given by

$$\vec{u} = \nabla_p \mathcal{E} = \frac{\vec{p}}{m^*} \left[1 - \frac{2\, p^2}{m^* E_G} \right]^{-\frac{1}{2}} \qquad (4.8)$$

Because of the small band gap, the non-parabolic nature of the conduction band will be important at high temperature or high electron concentration.

Calculations can be simplified somewhat while keeping the non-parabolic nature of the conduction band by not considering explicitly the terms due to the filling of the bands up to the Fermi level. Then for current carriers near the edge of a nonparabolic band, the energy expanded to the fourth order in momentum is:

$$\mathcal{E} \simeq \frac{p^2}{2m^*} - \frac{p^4}{4m^*E_G} \tag{4.9}$$

and
$$\vec{u} \simeq \frac{\vec{p}}{m^*} - \frac{p^2}{m^{*2}E_G}\vec{p} \tag{4.10}$$

An example of non-parabolic valence band is the valence band of germanium.

In germanium, the highest valence band maximum occurs at $\vec{K} = 0$, but the valence band is degenerate. The constant energy surfaces near $\vec{K} = 0$ are warped and hence non-parabolic (Smith,1959). In particular, for heavy holes, the energy-momentum relation of the centre section $p_z = 0$ in the [100] direction can be well approximated by

$$\mathcal{E} = \frac{p^2}{2m'} + 4\alpha \frac{p_x^2\, p_y^2}{p^2} \tag{4.11}$$

where

$$\frac{1}{2m'} = \frac{1}{2m}\left[A - \left(B^2 + \frac{1}{4}C^2\right)^{1/2}\right]$$

$$\alpha = C^2 \Big/ \left[16m\left(B^2 + \frac{1}{4}C^2\right)^{1/2}\right]$$

A, B, C are standard energy band notations (Smith,1959) and α is a measure of the warping. The velocity is

$$u_x = \frac{p_x}{m} + 8\frac{p_x p_y^4}{p^4} \tag{4.12}$$

$$u_y = \frac{p_y}{m} + 8\frac{p_x^4 p_y}{p^4} \tag{4.13}$$

$$u_z = 0 \tag{4.14}$$

The foregoing discussion is not intended to be an exhaustive discussion on non-parabolic conduction and valence bands of solids. The important point is that for solids with non-parabolic conduction or valence bands, the velocity of current carriers is no longer proportional to the momentum. This will be shown in the next chapter to be an additional source of non-linear effects. Since the explicit non-parabolic terms in the energy-momentum relations depends on the actual band structure of the solid, it is not possible to perform calculations for current carriers in a general solid. The specific cases, Eqs. (4.7)-(4.14), are therefore used to illustrate non-linear effects due to non-parabolic conduction and valence bands.

CHAPTER V

NON-LINEAR EFFECTS DUE TO NON-PARABOLIC BANDS

Let $\vec{E} = \vec{E}_1 e^{iw_1 t} + \vec{E}_1 e^{-iw_1 t}$ be the incident electromagnetic field. The equation of motion is

$$\frac{d\vec{p}}{dt} = e\,\vec{E} \qquad (5.1)$$

neglecting the comparatively weak magnetic forces. The solution of Eq. (5.1) is

$$\vec{p} = \vec{p}_0 + \frac{e\,\vec{E}_1}{iw_1} e^{iw_1 t} - \frac{e\vec{E}_1^*}{iw_1} e^{-iw_1 t} \qquad (5.2)$$

Substituting Eq. (5.2) into the velocity-momentum relations Eqs. (5.10),(5.12) and (5.13) gives the corresponding expressions for velocity $\vec{u}$. The current density $\vec{J}$ is then obtained by averaging $\vec{u}$ over the Fermi distribution function and multiplying by the electronic charge e (positive for holes, negative for electrons). When $\vec{P}_o$ is neglegible, this is equivalent to multiplying $\vec{u}$ by the carrier density n and e.

5.1 NON-PARABOLIC CONDUCTION BAND IN INDIUM ANTIMONIDE

Case (i) $\vec{E}$ sinusoidal, $\vec{p}_0$ negligible

Setting $\vec{p}_0=0$, substitution of Eq.(5.2) into Eq.(5.10) gives the velocity

$$\vec{u} \simeq \frac{1}{m^*}\frac{e\vec{E}_1}{iw_1} e^{iw_1 t} + \frac{1}{m^* E_G}\frac{e^3}{iw_1^3}\left[E_1^2\vec{E}_1^* + 2(\vec{E}_1.\vec{E}_1^*)\vec{E}_1\right] e^{iw_1 t}$$

$$- \frac{1}{m^* E_G}\frac{e^3}{iw_1^3} E_1^2\vec{E}_1 e^{3iw_1 t} + \text{their complex conjugates} \quad \ldots\ldots\ldots\ldots\ldots(5.3)$$

It is seen from Eq.(5.3) that apart from the fundamental frequency, the third harmonic of w_1 in velocity is generated. The ratio of the magnitude of this third harmonic to that of the first harmonic is

$$\frac{e^2}{w_1^2}\frac{|E_1^2|}{m^* E_G} \qquad (5.4)$$

For $w_1 \sim 10^{14}$, $m \sim 0.1\, m_e$, $E_G \sim$ lev, $E_1 \sim 10^4$ volt/cm, this ratio is of the order of 10^{-8}. If the incident wave is circularly polarised i.e. $E_{1x} = i E_{1y}$, $E_1{}^2 = 0$ so that essentially no non-linear effects can be observed.

For the purpose of this thesis, it should be noted from Eq.(5.3) that no second harmonic component of w_1 is generated.

CASE (ii) $\vec{E}$ CONTAINS MORE THAN ONE FREQUENCY COMPONENT, $\vec{P}_G$ NEGLIGIBLE

Let $\vec{E}$ be given by $\vec{E}_1 e^{iw_1 t} + \vec{E}_1^* e^{iw_1 t} + \vec{E}_2 e^{iw_2 t} + \vec{E}_2^* e^{-iw_2 t}$, then one has:

$$\vec{p} = \frac{e}{iw_1}\left[\vec{E}_1 e^{iw_1 t} - \vec{E}_1^* e^{-iw_1 t}\right] + \frac{e}{iw_2}\left[\vec{E}_2 e^{iw_2 t} - \vec{E}_2^* e^{-iw_2 t}\right] \quad \ldots\ldots\ldots\ldots(5.5)$$

The velocity $\vec{u}$ is given by:

$$\vec{u} = \frac{\vec{p}}{m^*} - \frac{p^2}{m^{*2} E_G}\vec{p}$$

$$\simeq \frac{1}{m^*}\left[\frac{e}{iw_1}\vec{E}_1 e^{iw_1 t} + \frac{e}{iw_2}\vec{E}_2 e^{iw_2 t}\right]$$

$$- \frac{1}{m^{*2} E_G}\left[- \frac{e^3}{iw_1^3} E_1^2 \vec{E}_1 e^{3iw_1 t} - \frac{e^3}{iw_2^3} E_2^2 \vec{E}_2 e^{3iw_2 t}\right.$$

$$- \frac{e^3}{iw_1^2 w_2}(E_1^2 \vec{E}_2 + 2\vec{E}_1 . \vec{E}_2 \vec{E}_1) e^{i(2w_1 + w_2)t}$$

$$+ \frac{e^3}{iw_1^2 w_2}(E_1^2 \vec{E}_2^* + 2\vec{E}_1 . \vec{E}_2^* \vec{E}_1) e^{i(2w_1 - w_2)t}$$

$$- \frac{e^3}{iw_1 w_2^2}(E_2^2 \vec{E}_1 + 2\vec{E}_1 . \vec{E}_2 \vec{E}_1) e^{i(w_1 + 2w_2)t}$$

$$\left. - \frac{e^3}{iw_1 w_2^2}(E_2^{*2} \vec{E}_2 + 2\vec{E}_1 . \vec{E}_2^* \vec{E}_2^*) e^{i(w_1 - 2w_2)t}\right]$$

+ their complex conjugates (5.6)

where all first harmonic components in w_1 and w_2 in the product $P^2 P$ are neglected in comparison with

$$\frac{1}{m^*}\left[\frac{\vec{E}_1}{iw_1} e^{iw_1 t} + \frac{\vec{E}_2}{iw_2} e^{iw_2 t}\right]$$

Eq. (5.6) reveals that in addition to third harmonics of w_1 and w_2, there are mixed frequency components at $2w_1 + w_2$, $w_1 + 2w_2$, $2w_1 - w_2$, $w_1 - 2w_2$. If the incident waves are circularly polarised, one has

$$E_1^2 = E_2^2 = 0$$

$$\begin{aligned} \vec{E}_1 \cdot \vec{E}_2 &= E_{1x}E_{2x} + E_{1y}E_{2y} \\ &= - E_{1y}E_{2y} + E_{1y}E_{2y} \\ &= 0 \end{aligned} \tag{5.7}$$

and the components at frequencies $2w_1 + w_2$ and $w_1 + 2w_2$ will be absent.

No components at frequencies $2w_1$, $2w_2$, $w_1 + w_2$ and $w_1 - w_2$ are generated.

CASE(iii) $\vec{E}$ in the presence of a d.c. magnetic field, $\vec{P}_o$ negligible

Suppose a d.c. magnetic field $\vec{B}_o$ is applied in the positive z-direction in addition to $\vec{E}$. The equation of motion becomes

$$\frac{dp}{dt} = e\left(\vec{E}_1 e^{iw_1t} + \vec{E}_1^* e^{-iw_1t}\right) + \vec{u} \times \frac{e\vec{B}_0}{c}$$

$$= e\left(\vec{E}_1 e^{iw_1t} + \vec{E}_1^* e^{-iw_1t}\right) + \left(1 - \frac{p^2}{m^* E_G}\right) \vec{p} \times \vec{w}_c \qquad \text{..................(5.8)}$$

where $\vec{w}_c = \dfrac{e\,\vec{B}_0}{m^* c}$

Eq.(5.8) is a non-linear equation. Using the perturbation technique outlined in Chapter III, $\vec{P}$ is assumed to be expandable in a converging series

$$\vec{p} = \vec{p}_1 e^{iw_1t} + \vec{p}_2 e^{2iw_1t} + \ldots\ldots + \text{complex conjugate} \qquad \text{...............(5.9)}$$

where $P_1 \gg P_2 \gg P_3 \gg$ ....

$\vec{P}_1$ is given by the equation:

$$iw_1\vec{p}_1 \simeq e\vec{E}_1 + \vec{p}_1 \times \vec{w}_c \qquad (5.10)$$

terms in $\dfrac{p^2}{m^* E_G}\vec{P}$ being negligible. Solving this equation, one gets

$$p_{1x} = \frac{e(w_c E_y - iw_1 E_x)}{w_c^2 - w_1^2}$$

$$p_{1y} = \frac{-e(iw_1 E + w_c E_x)}{w_c^2 - w_1^2} \qquad (5.11)$$

$$p_{1z} = 0$$

Resonance occurs at $w_1 = w_c$.

$\vec{P}_2$ is given by the equation

$$2iw_1\vec{p}_2 \simeq \vec{p}_2 \times \vec{w}_c \qquad (5.12)$$

Writing out in cartesion co-ordinates: one gets

$$2iw_1 p_{2x} = p_{2y} w_c$$

$$2iw_1 p_{2y} = -p_{2x} w_c \qquad (5.13)$$

$$p_{2z} = 0$$

This is a system of homogeneous equations in P_{2x} and P_{2y}. A non-trivial solution of these equations determines the magnitude of $\vec{p}_2$ only up to an arbitary constant which is independent of $\vec{E}$. This is physically inadmissible. $\vec{P}_2$ must therefore be equal to zero. Similar arguements show that all even harmonic components $\vec{P}_4$, $\vec{P}_6$, $\vec{P}_8$ ...... must vanish.

$\vec{P}_3$ is given by the equation

$$3iw_1\vec{p}_3 \simeq \vec{p}_3 \times \vec{w}_c - \frac{p_1^2}{m^* E_G} \vec{p}_1 \times \vec{w}_c \qquad (5.14)$$

Solving this equation, the result is

$$p_{3x} = \frac{p_1^2}{m^*E_G} \frac{w_c}{w_c^2 - 9w_1^2} \left[w_c p_{1x} + 3iw_1 p_{1y} \right]$$

$$p_{3y} = \frac{p_1^2}{m^*E_G} \frac{w_c}{w_c^2 - 9w_1^2} \left[-3iw_1 p_{1x} + w_c p_{1y} \right] \qquad (5.15)$$

$$p_{3z} = 0$$

Resonance occurs at $W_1 = W_c$ and $W_c = 3W_1$.

In general any odd harmonic component $\vec{P}_j$ is given by an equation of the form:

$$ji w_1 \vec{p}_j = \vec{p}_j\, w_c - \frac{1}{m^*E_G} \vec{X} \times \vec{w}_c \qquad (5.16)$$

where $\vec{X}$ are the terms in the product $P^2\,\vec{P}$ which have a time dependence of e^{ijw_1t}. $\vec{X}$ therefore is a function of $\vec{p}_{j-2}$, $\vec{p}_{j-4}$ . ..... $\vec{P}_3$, $\vec{P}_1$, so that resonance is exhibited at $jw_1 = w_c$, $(j-2)\, w_1 = w_c$, $(j-4)\, w_1 = w_c$ .... $3w_1 = w_c$, $w_1 = w_c$.

Consider now the velocity:

$$\vec{u} = \frac{\vec{p}}{m^*} - \frac{p^2}{m^{*2}E_G} \vec{p}$$

Since P contains only odd harmonic components, $P^2 \vec{P}$ will also contain only odd harmonic components. Hence it can be concluded that $\vec{u}$ contains only odd harmonic components and the odd harmonic component $\vec{u}_j$ exhibits resonance at frequencies $jw_1 = w_c$, $(j-2)w_1 = w_c$....... $3w_1 = w_c$, $w_1 = w_c$. The expression for the third velocity harmonic component calculated by Lax and Zawadzki (1967) is in agreement with this conclusion.

Case (iv) P_o large compared with eE_1/w_1

Solution of equation of motion Eq. (5.1) is

$$\vec{p} = \vec{p}_0 + \frac{e\vec{E}_1}{iw_1} e^{iw_1 t} - \frac{e\vec{E}_1^*}{iw_1} e^{-iw_1 t}$$

Substituting into the velocity-momentum relation

$$\vec{u} = \frac{\vec{p}}{m^*} - \frac{p^2}{m^{*2} E_G} \vec{p}$$

it is seen that the second harmonic components in $p^2 \vec{P}$ is given by

$$\frac{1}{m^{*2} E_G} \frac{e^2}{w_1^2} \vec{p}_0 \cdot \vec{E}_1 \qquad (5.17)$$

However, if this is integrated over the equilibrium distribution function to obtain the current density, the integral vanishes because in the absence of an applied field, there is no preferred direction of electron motion. Thus only odd harmonic components are generated in the current density.

5.2. Non-parabolic valence band in germanium

The velocity-momentum relation is

$$u_x = \frac{p_x}{m} + 8\alpha \frac{p_x p_y^4}{p^4}$$

$$u_y = \frac{p_y}{m} + 8\alpha \frac{p_x^4 p_y}{p^4}$$

Case (i) $\vec{E}$ sinusoidal, $\vec{P}_o$ negligible

The solution of the equation of motion is:

$$p_x = \frac{eE_x}{iw_1} e^{iw_1 t} - \frac{eE_x^*}{iw_1} e^{iw_1 t}$$

$$p_y = \frac{eE_y}{iw_1} e^{iw_1 t} - \frac{eE_y^*}{iw_1} e^{-iw_1 t}$$

where E_x and E_y are the cartesian components of $\vec{E}_1$ in the x and y direction respectively.

Consider incident waves which are plane polarised. If $E_x=E_y=0$, $u_x=u_y=0$. If $E_y=E_z=0, u_x=p_x/m$, $u_y=0$. If $E_x=E_z=0$, $u_x=0$, $u_y=p_y/m$.

No non-linear effects can be observed.

Consider incident waves which are circularly polarised. Without loss of generality, let E_y be real and for circularly polarised wave $E_x = i\,E_y$. Thus

$$p_x = \frac{eE_y}{w_1} e^{iw_1 t} + \frac{eE_y}{w_1} e^{-iw_1 t}$$

$$= 2\mathcal{P}\cos(w_1 t) \qquad (5.18)$$

$$p = \frac{eE_y}{iw_1} e^{iw_1 t} - \frac{eE_y}{iw_1} e^{-iw_1 t}$$

$$= 2\mathcal{P}\sin(w_1 t) \qquad (5.19)$$

where $\mathcal{P}$ is defined to be the real quantity eE_y/w_1 .

From Eqs.(5.18) and.(5.19), one obtains

$$p^4 = (\, p_x^2 + p_y^2 \,)^2 = 16\,\mathcal{P}^4 \qquad (5.20)$$

$$p_x^4 = 16\mathcal{P}^4\cos^4(w_1 t)$$
$$= 2\mathcal{P}^4\left[\cos(4w_1 t) + 4\cos(2w_1 t) + 3\right] \qquad (5.21)$$

$$p_y^4 = 16\mathcal{P}^4\sin^4(w_1 t)$$
$$= 2\,\mathcal{P}^4\left[\cos(4w_1 t) - 4\cos(2w_1 t) + 3\right] \qquad (5.22)$$

Substituting Eqs. (5.20),(5.21),(5.22) into the velocity momentum relation (4.10), the velocity is given by:

$$u_x = (\frac{2}{m} + 2\alpha)\rho\cos(w_1 t) - 3\alpha\rho\cos(3w_1 t) + \alpha\rho\cos(5w_1 t) \quad \text{.........(5.23)}$$

$$u_y = (\frac{2}{m} + 2\alpha)\rho\sin(w_1 t) + 3\alpha\rho\sin(3w_1 t) + \alpha\rho\sin(5w_1 t) \quad \text{........(5.24)}$$

The velocity contains third and fifth harmonic components. No even harmonic components are present.

Case (ii) P_o large compared with eE_1/w_1

Solution of the equation of motion gives:

$$p_x = p_{0x} + 2\rho\cos(w_1 t)$$
$$p_y = p_{0y} + 2\rho\sin(w_1 t) \quad (5.25)$$

$$p_x^4 = p_{0x}^4 + 8p_{0x}^3\rho\cos(w_1 t) + 24p_{0x}^2\rho^2\cos^2(w_1 t) + 16p_{0x}\rho^3\cos^3(w_1 t) + 16\rho^4\cos^4(w_1 t) \quad (5.26)$$

$$p_y^4 = p_{0y}^4 + 8p_{0y}^3\rho\sin(w_1 t) + 24p_{0y}^2\rho^2\sin^2(w_1 t) + 16p_{0y}\rho^3\sin\ (w_1 t) + 16\rho^4\sin^4(w_1 t) \quad (5.27)$$

$$p^4 = (p_x^2 + p_y^2)^2$$
$$= (p_0^2+4\rho^2)^2 + 8(p_0^2+4\rho^2)p_{0x}\rho\cos(w_1 t)$$
$$+8(p_0^2+4\rho^2)p_{0y}\rho\sin(w_1 t) + 24p_{0x}p_{0y}\rho^2\sin(w_1 t)\cos(w_1 t)$$
$$+16p_{0x}^2\rho^2\cos^2(w_1 t) + 16p_{oy}^2\rho^2\sin^2(w_1 t) \quad (5.28)$$

where $p_0^2 = p_{0x}^2 + p_{0y}^2$

Since P_o is large, $1/p^4$ can be approximated by:

$$\frac{1}{p^4} \simeq \frac{1}{p_0^4}\left[\left(1+\frac{4\rho^2}{p_0^2}\right)^2 + 8\left(\frac{1}{p_0^2}+\frac{4\rho^2}{p_0^4}\right)p_{0x}\rho\cos(w_1t)\right.$$

$$+8\left(\frac{1}{p_0^2}+\frac{4\rho^2}{p_0^2}\right)p_{0y}\rho\sin(w_1t) + 24\frac{p_{0x}p_{0y}}{p_0^4}\rho\sin(w_1t)\cos(w_1t)$$

$$\left. +16\frac{p_{0x}^2}{p_0^4}\rho^2\cos^2(w_1t) + 16\frac{p_{0y}^2}{p_0^4}\rho^2\sin^2(w_1t)\right] \qquad (5.28)$$

Consider now the even harmonic components of the velocity. For u_x they are produced by the term $\frac{p_x p_y^4}{p^4}$. Even harmonic terms are terms whose time dependence has the form $\sin^a(w_1t)\cos^b(w_1t)$ where a and b are positive integers such that a + b is an even number. When a + b is an odd integer, $\sin^a(w_1t)\cos^b(w_1t)$ is the time dependence of odd harmonic terms. An inspection of Eqs. (5.26), (5.27), (5.28) shows that p_x^4, p_y^4, $1/p^4$ all have the following property.

Harmonic terms	Dependence on p_{0x} p_{0y} is $p_{0x}^g p_{0y}^h$ where g,h, are positive intergers
Even	g+h is even
Odd	g+h is odd

Denoting the dependence on P_{ox}, P_{oy} of harmonic terms in p_y^4 and $1/p^4$ by $p_{0x}^{g_1} p_{0y}^{h_1}$ and $p_{0x}^{g_2} p_{0y}^{h_2}$ respectively, dependence of even harmonic terms on $p_{0x} p_{0y}$ in $p_x p_y^4/p^4$ can be obtained by:

Term T_3 from p_x is $p_{0x}^{g_3} p_{0y}^{h_3}$	Term T_1 from p_y is $p_{0x}^{g_1} p_{0y}^{h_1}$	Term T_2 from $1/p^4$ is $p_{0x}^{g_2} p_{0y}^{h_2}$
$g_3=1, h_3=0$	g_1+h_1 is even	g_2+h_2 is even
	g_1+h_1 is odd	g_2+h_2 is odd
$g_3=0, h_3=0$	g_1+h_1 is even	g_2+h_2 is odd
	g_1+h_1 is odd	g_2+h_2 is even
Dependence of even harmonic terms on $p_{0x} p_{0y}$ in u_x = dependence on $p_{0x} p_{0y}$ of $T_1 T_2 T_3$		

The above table is set up using the fact that even harmonic terms can only be obtained by multiplying together two even harmonic terms or two odd harmonic terms, and odd harmonic terms can only be obtained by multiplying together one even harmonic term and one odd harmonic term. The dependence of even harmonic terms on p_{ox}, p_{oy} in $p_x p_y^4/p^4$ is thus given by $p_{ox}^g p_{oy}^h$, where $g + h$ is an odd integar. It will vanish on averaging over the equilibrium distribution function, being an odd function in p_{0x} p_{0y}. They do not contribute to the final expression for current density.

Again there are no even harmonic components in the current density.

5.3 Effect of an external d.c. electric field

Calculations in sections 5.1 and 5.2 show that under the various conditions specified, non-linearity due to non-parabolic conduction band in indium antimonide and non-parabolic valence band in germanium does not produce even harmonic components in the current density. It will now be shown that if a strong external d.c. electric field is imposed, such production can be achieved.

To discuss the effect of an external d.c. electric field, collision processes must be taken into account. The presence of collision processes is necessary for the mobile current carriers to attain a steady state. Using the hydrodynamical approach presented in Chapter II, the equation of motion is, neglecting the magnetic forces.

$$\frac{d\vec{Q}}{dt} = e\vec{E}_0 + e\vec{E}_1 e^{iw_1 t} + e\vec{E}_1^* e^{-iw_1 t} - \frac{\vec{Q}}{\tau} \qquad (5.29)$$

where $\vec{Q}$ is the average momentum, $\vec{E}_0$ is the d.c. electric field.

Eq.(5.29) can be solved exactly to give

$$\vec{Q} = e\vec{E}_0\tau + \frac{e\vec{E}_1 e^{iw_1 t}}{iw_1 + \frac{1}{\tau}} - \frac{e\vec{E}_1^{*} e^{-iw_1 t}}{iw_1 - \frac{1}{\tau}} \qquad (5.30)$$

Neglecting $1/\tau$ compared with w_1 and letting $\vec{Q}_0 = e\vec{E}_0\tau$, one obtains

$$\vec{Q} \simeq \vec{Q}_0 + \frac{e\vec{E}_1}{iw_1}e^{iw_1 t} - \frac{e\vec{E}_1^{*}}{iw_1}e^{-iw_1 t} \qquad (5.31)$$

The average velocity - average momentum relation for non-parabolic conduction band in indium antimonide and non-parabolic valence band in germanium are respectively.

$$\vec{v} = \frac{\vec{Q}}{m^*} - \frac{Q^2}{m^{*2}E_G}\vec{Q} \qquad (5.32)$$

$$v_x = \frac{Q_x}{m} + 8\alpha\frac{Q_x Q_y^4}{Q^4}$$

$$v_y = \frac{Q_y}{m} + 8\alpha\frac{Q_x^4 Q_y}{Q^4} \qquad (5.33)$$

The current density is given by $ne\vec{v}$, where n, the carrier density, is assumed to be constant.

Three simple cases will be discussed to illustrate the role played by the d. c. electric field.

<u>Case (i)</u> <u>Non-linearity due to non-parabolic conduction band in indium antimonide. $\vec{E}$ sinusoidal</u>

The velocity $\vec{v}$ is obtained by substituting

$$\vec{Q} = \vec{Q}_0 + \frac{e\vec{E}_1}{iw_1} e^{iw_1 t} - \frac{e\vec{E}_1^*}{iw_1} e^{-iw_1 t}$$

into

$$\vec{v} = \frac{\vec{Q}}{m^*} - \frac{Q^2}{m^{*2} E_G} \vec{Q}$$

The mathematics is exactly the same as in case (iv) of section 5.1, with $\vec{Q}$ replacing $\vec{p}$ and $\vec{Q}_0$ replacing $\vec{P}_0$. Hence the second harmonic velocity component in $\vec{v}$ is

$$\frac{1}{m^{*2} E_G} \frac{e^2}{w_1^2} \vec{Q}_0 \cdot \vec{E}_1$$

The difference is that the second harmonic component in the current density is just

$$\frac{n}{m^{*2} E_G} \frac{e^3}{w_1^2} \vec{Q}_0 \cdot \vec{E}_1 \qquad (5.34)$$

which is obviously non-vanishing except when $\vec{Q}_0 \cdot \vec{E}_1 = 0$ i.e. when $\vec{E}_0$ is perpendicular to $\vec{E}_1$.

Case (ii) Non-linearity due to non-parabolic conduction band in indium antimonide, $\vec{E}$ has two frequency components

Let $\vec{E} = \vec{E}_1 e^{iw_1 t} + \vec{E}_2 e^{iw_2 t}$ +complex conjugate

Then $\vec{Q}$ becomes

$$\vec{Q} = \vec{Q}_0 + \vec{Q}_1 e^{iw_1 t} + \vec{Q}_2 e^{iw_2 t} \text{ +complex conjugate} \qquad \ldots\ldots\ldots\ldots (5.35)$$

where $\vec{Q}_1 = \dfrac{e\vec{E}_1}{iw_1}$, $\vec{Q}_2 = \dfrac{e\vec{E}_2}{iw_2}$

Substituting Eq. (5.35) into

$$\vec{v} = \frac{\vec{Q}}{m^*} - \frac{Q^2}{m^{*2}E_G}\vec{Q}$$

gives the various harmonic components in $\vec{v}$.

Comparing with a similar case but in the absence of $\vec{E}_o$, i.e. case (ii) of section 5.1, one finds that in the presence of $\vec{E}_o$, there are components at the mixed frequencies $w_1 + w_2$ and $w_1 - w_2$ in the current density. The spatial parts of these frequency components are:

$$-\frac{ne}{m^{*2}E_G}\left[(\vec{Q}_1\cdot\vec{Q}_2)\vec{Q}_0 + (\vec{Q}_0\cdot\vec{Q}_1)\vec{Q}_2 + (\vec{Q}_2\cdot\vec{Q}_1)\vec{Q}_0\right] \quad (5.36)$$

and

$$-\frac{ne}{m^{*2}E_G}\left[(\vec{Q}_1\cdot\vec{Q}_2^*)\vec{Q}_0 + (\vec{Q}_0\cdot\vec{Q}_1^*)\vec{Q}_2 + (\vec{Q}_0\cdot\vec{Q}_1)\vec{Q}_2^*\right] \quad (5.37)$$

These two components, however, will vanish if $\vec{E}_0$, $\vec{E}_1$ and $\vec{E}_2$ are mutually perpendicular, as can be seen from Eqs. (5.36) and (5.37).

Case (iii) Non-linearity due to non-parabolic valence band in germanium, $\vec{E}$ sinusoidal and circularly polarised.

The mathematics is the same as in case (ii) of section 5.2 with $\vec{Q}_0$ replacing $\vec{p}_0$. It is seen there that in general there are even harmonic components in the velocity. These harmonic components will therefore be the even harmonic components in the current density after multiplying by n and e.

Finally, it is easy to see that the harmonic components can be made large by applying a sufficiently strong d.c. electric field.

Non-linear effects due to nOn-parabolic bands was first observed only very recently (Patel, Slusher, and Fleury, 1966) The experimental results agree with the theoretical calculations within the limitations imposed by the experimental error, which is about thirty percents.

CHAPTER VI

CONCLUSIONS

Non-linear effects in solids can be made large by applying, simultaneously with the electric fields of the incident electromagnetic waves, a d.c. magnetic field of appropiate strength to stimulate cyclotron resonance, or a sufficiently strong d.c. electric field. For non-linear effects in indium antimonide and germanium, the application of a d.c. electric field also results in the generation of even harmonic components. No attempts have yet been made to observe these even harmonic components.

REFERENCES

ADAMS E.N. J. Chem Phys. 21 2013 (1953)

ADAMS E.N. & ARGYRES P. N. Phys. Rev. 102 605 (1956)

BALESCU R. Phys. Fluids 52 (1960)

BLOEMBERGEN N. and SHEN Y.R. Phys. Rev. 141 298 (1966)

CHANG H. and LEE Y.C. Phys. Rev. Lets. 14 426 (1965)

CHANG H. and MILLER P. B. Phys. Rev. 134 A 683 (1964)

CHIOYDA K. J. Phys. Soc. Japan 20 290 (1965)

DUBOIS D. F. Phys. Rev. Lets. 14 818 (1965)

DUBOIS D. F. and GILINSHY V. Phys. Rev. 135 A 995 (1964)

FEUER P. Phys. Rev. 88 92 (1952)

FRANKEN F. A. Rev. of Mod. Phys. 35 23 (1963)

KANE E. O. J. Phys. Chem. Solids 1 249 (1957)

KITTEL C. and MITCHELL A. H. Phys. Rev. 96 1488 (1954)

LAX B., ZAWADZKI W. and WEILER M. H. Phys. rev. Lets. 18 462 (1967)

LINHART J. G. Plasma Physics North-Holland Publishing Co. Amsterdam (1960)

LUTTINGER J. M. and KOHN W. Phys. Rev. 97 869 (1955)

PATEL C. K.N., SLUSHER R. E. and FLEURY P. A. Phys. Rev. Lets. 17 1011 (1966)

Proceedings of the Conference on the Physics of Quqntum Electronics Puerto Rico. McGraw Hill Book Co. Inc. New York (1965)

REITZ J. R. Solid State Phys. 1 1 (1955)

ROSTOKER N. and ROSENBLUTH M.N. Phys. Fluids 3 1 (1960)

SMITH R. A. Wave Mechanics of Crystalline Solids Chapman and Hall Ltd., London (1961)

WHITMER R.F. and BARRETTE E. E. Phys Rev. 121 661 (1961)

WOLFF P. A. and PEARSON G. A. Phys Rev. Lets. 17 1015 (1966)

ZIMAN J. M. Electrons and Phonons Oxford, Clarendon Press (1960)

APPENDIX A

DERIVATION OF HYDRODYNAMICAL EQUATIONS FROM BOLTZMANN EQUATION

The distribution function of a system of identical particles $f(\vec{r}, \vec{p}, t)$ is defined such that $f(\vec{r}, \vec{p}, t)$ $dx\,dy\,dz\,dp_x\,dp_y\,dp_z$ is the number of particles which lie within the spatial volume dxdydz, centred at $\vec{r}$, and whose momenta lie within the intervals dp_x, dp_y and dp_z centred at $\vec{p}$. The equation describing the change of f under external forces is the Boltzmann equation:

$$\frac{\partial f}{\partial t} + \sum_i \frac{p_i}{m} \frac{\partial f}{\partial x_i} + \sum_i F_i \frac{\partial f}{\partial p_i} = \left(\frac{\partial f}{\partial t}\right)_{coll.} \qquad (A.1)$$

where i takes the values 1, 2, and 3, with x_1, x_2 and x_3 representing the x, y, z co-ordinates; p_i and F_i represent the ith components of the particle momentum and external force respectively $\left(\frac{\partial f}{\partial t}\right)_{coll.}$ represents the change in f due to collisions between particles at a fixed point in space and time. m is the mass of the particle.

It is assumed that the velocity $\vec{u}$ is proportional to $\vec{p}$. This is true for classical particles and for conduction electrons in solids with a parabolic conduction band.

The particla density $n(\vec{r},t)$ and the average momentum Q are defined by

$$n(\vec{r},t) = \iiint_{-\infty}^{\infty} f(\vec{r},\vec{p},t)\, dp_x dp_y dp_z \qquad (A.2)$$

$$\vec{Q} = \iiint_{-\infty}^{+\infty} \vec{p}\, f(\vec{r},\vec{p},t)\, dp_x dp_y dp_z \,/\, n(\vec{r},t) \qquad (A.3)$$

In general, the mean value $\psi(\vec{r}, t)$ for any quantity $\psi(\vec{p})$ is defined by

$$\bar{\psi}(\vec{r},t) = \iiint_{-\infty}^{\infty} \psi(\vec{p}) f(\vec{r},\vec{p},t) dp_x dp_y dp_z / n(\vec{r},t) \quad (A.4)$$

Multiplying. Eq.(A.1) by $\psi(\vec{p}) dp_x dp_y dp_z$ where $\psi(\vec{p})$ is an arbitary function of $\vec{p}$ and integrating over all p, one obtains the following equations:

$$\iiint_{-\infty}^{\infty} \psi(\vec{p}) \frac{\partial f}{\partial t} dp_x dp_y dp_z = \frac{\partial}{\partial t} \iiint_{-\infty}^{\infty} \psi(\vec{p}) f dp_x dp_y dp_z = \frac{\partial}{\partial t}(n\bar{\psi}) \quad (A.5)$$

$$\iiint_{-\infty}^{\infty} \psi(p) p_i \frac{\partial f}{\partial x_i} dp_x dp_y dp_z = \frac{\partial}{\partial x_i} \iiint_{-\infty}^{\infty} \psi(\vec{p}) p_i f dp_x dp_y dp_z = \frac{\partial}{\partial x_i}(n\overline{p_i \psi}) \quad (A.6)$$

$$\iiint_{-\infty}^{\infty} \psi(\vec{p}) F_i(\vec{r},\vec{p}) \frac{\partial f}{\partial p_i} dp_x dp_y dp_z$$

$$= - \iiint_{-\infty}^{\infty} f \frac{\partial}{\partial p_i} \left[F_i(\vec{r},\vec{p}) \psi(p)\right] dp_x dp_y dp_z$$

$$= - n \overline{\frac{\partial}{\partial p_i}(F_i \psi)} \qquad (A.7)$$

Putting $\psi = 1$, and noting that $\frac{\partial F_i}{\partial p_i} = 0$ for electric and magnetic forces, and the integral of $\left(\frac{\partial f}{\partial t}\right)_{coll.}$ over momentum space vanishes since collisions cannot change the total number of particles per unit volume, Eqs.(A.5)&(A.6) yield the equation:

$$\frac{\partial n}{\partial t} + \nabla.(n\frac{\vec{Q}}{m}) = 0 \qquad (A.8)$$

The equation obtained by putting $\psi(\vec{p}) = \vec{p}$ is

$$\frac{\partial}{\partial t}(n\vec{Q}) + \nabla.(n\frac{\overline{\vec{p}\,\vec{p}}}{m}) - n\overline{\vec{F}} = \iiint_{-\infty}^{\infty} \vec{p}(\frac{\partial f}{\partial t})_{coll.} dp_x dp_y dp_z \qquad (A.9)$$

The various terms can be rewritten as:

$$\frac{\partial}{\partial t}(n\vec{Q}) = n\frac{\partial \vec{Q}}{\partial t} + \vec{Q}\frac{\partial n}{\partial t} \qquad (A.10)$$

Defining the random momentum $\vec{\beta}$ as

$$\vec{\beta} = \vec{p} - \vec{Q} \qquad (A.11)$$

one gets

$$\nabla.(n\frac{\overline{\vec{p}\,\vec{p}}}{m}) = \nabla.(n\frac{\overline{\vec{Q}\,\vec{Q}}}{m}) + \nabla.(n\frac{\overline{\vec{\beta}\,\vec{\beta}}}{m}) \qquad (A.12)$$

Since $\overline{\vec{\beta}} = 0$

Assuming appropriate symmetry conditions such that the tensor $\vec{\beta}\vec{\beta}$ reduces to the scalar β^2, where $\frac{\beta^2}{2m}$ is just the energy due to thermal motion, $\nabla.(n\frac{\overline{\vec{\beta}\,\vec{\beta}}}{m})$ becomes $\nabla(\frac{3nkT}{m})$

Expanding $\nabla\cdot(n\frac{\overline{\vec{Q}\,\vec{Q}}}{m})$, one has

$$\nabla\cdot(n\frac{\overline{\vec{p}\,\vec{p}}}{m}) = (\frac{n}{m}\vec{Q}\cdot\nabla)\vec{Q} + \frac{\vec{Q}}{m}\nabla\cdot(n\vec{Q}) + \nabla(\frac{3nkT}{m}) \qquad (A.14)$$

For the external force term, neglecting the gravitational force, $\vec{F}$ is given by

$$\vec{F} = e\left[\vec{E} + \frac{\vec{p}}{m}\times\frac{\vec{B}}{c}\right] \qquad (A.15)$$

where E and B are electric and magnetic field intensities.

For the collision term, it is assumed that a relaxation time τ exists such that

$$\left(\frac{\partial f}{\partial t}\right)_{coll.} = -\frac{f}{\tau} \qquad (A.16)$$

Substituting Eqs.(A.10),(A.14),(A.15)and(A.16) into Eq.(A.9)and using the continuity equation, one obtains

$$n\frac{\partial\vec{Q}}{\partial t} + \frac{n}{m}(\vec{Q}\cdot\nabla)\vec{Q} = ne\left[\vec{E} + \frac{\vec{Q}}{m}\times\frac{\vec{B}}{c}\right] - \nabla(\frac{3nkT}{m}) - \frac{\vec{Q}}{\tau} \qquad (A.17)$$

Relating the average momentum $\vec{Q}$ to the average velocity $\vec{v}$ by $\vec{Q} = m\vec{v}$, the equation of motion (2 . 6) is obtained.

APPENDIX B

THE DEBYE LENGTH

Consider a positive ion placed at the origin of the co-ordinate system. This positive ion will attract neighbouring electrons while at the same time repelling other positive ions. The result is that, in a region surrounding the positive ion, an excess electronic charge will exist. Thus within this region, there will not be charge neutrality, and a space charge will be set up which is given by $e(N_+ - N_-)$ where N_+ and N_- are the density of ion and electrons respectively. This space charge will give rise to a static electric field whose potential ϕ satisfies Poisson's equation.

$$\nabla^2 \phi = -4\pi e(N_+ - N_-) \tag{B.1}$$

Now assume the ions are at rest and that N_+ is equal to the equilibrium charged particle density N_0, while within the region under consideration N_- is a slowly variging function of position, Eq.(B.1) becomes

$$\nabla^2 \phi = -4\pi e \left[N_0 - N_-(r)\right] \tag{B.2}$$

$N_-(r)$ is given by:

$$N_-(r) = \iiint_{-\infty}^{\infty} f dp_x dp_y dp_z \tag{B.3}$$

where f is the solution of the Boltzmann transport equation for electrons at steady state. This solution is

$$f = N_0\left(\frac{m}{2\ kT}\right)^{3/2} e^{\frac{-\beta^2}{2mkT}} \exp\left(\frac{e\phi}{kT}\right) \tag{B.4}$$

where β is defined in Eq.(A.11).

Substituting Eq.(B.4) into (B.3), one gets

$$N_-(r) = N_0 \exp\left(\frac{e\phi}{kT}\right) \tag{B.5}$$

and assuming that $e\phi \ll KT$, one has

$$N_-(r) \simeq N_0\left(1 + \frac{e\phi}{kT}\right) \tag{B.6}$$

Finally Eq. (b.6) is subtituted into Eq. (B.2) to give

$$\nabla^2\phi = \frac{4\pi e^2 N_0}{kT}\phi$$

$$= \frac{1}{\lambda_D^2}\phi \tag{B.7}$$

where $\lambda_D = \left(\frac{4\pi e^2 N_0}{kT}\right)^{1/2}$ is the Debye Length.

is spherically symmetric about the origin and has the solution $\phi = \frac{e}{r}\exp\left(\frac{r}{\lambda_D}\right)$ (B.8)

Thus in a distance from the origin large compared with λ_D, ϕ is small so that in a region whose dimension is large compared with λ_D, charge neutrality is observed.