Optimal Mixed-Level Robust Parameter Designs

by

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Abstract

Fractional factorial designs have been proven useful for efficient data collection and are widely used in many areas of scientific investigation and technology. The study of these designs' structure paves a solid ground for the development of robust parameter designs.

Robust parameter design is commonly used as an effective tool for variation reduction by appropriating selection of control factors to make the product less sensitive to noise factor in industrial investigation. A mixed-level robust parameter design is an experimental design whose factors have at least two different level settings. One of the most important consideration is how to select an optimal robust parameter design. However, most experimenters paid their attention on two-level robust parameter designs. It is highly desirable to develop a new method for selecting optimal mixed-level robust parameter designs.

In this thesis, we propose a methodology for choosing optimal mixed-level fractional factorial robust parameter designs when experiments involve both qualitative factors and quantitative factors. At the beginning, a brief review of fractional factorial designs and two-level robust parameter designs is given to help understanding our method. The minimum aberration criterion, one of the most commonly used criterion for design selection, is introduced. We modify this criterion and develop two generalized minimum aberration criteria for selecting optimal mixed-level fractional factorial robust parameter designs. Finally, we implement an effective computer

program. A catalogue of 18-run optimal designs is constructed and some results are given.

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Dedication Page

With love and thanks to my parents and favorite teacher, Po Yang.

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Chapter 1

Introduction

Design of experiment is the process of planning studies and investigating efficient methods for collecting data for scientific investigation. It has a long history of successful applications in scientific research and engineering development. It is a basic tool to identify the key factors among a large number of factors and test the effects among the involved factors on the process.

Particularly, factorial designs are widely employed when the treatments have factorial structure. R. A. Fisher (1971) was a pioneer of factorial design and contributed many basic ideas in agricultural experiments. Box, Hunter and Hunter (2005) were early advocators of using experimental design methods in the industrial setting. In the last several decades, researchers' interest in factorial designs was increased. They developed various theories to further understand factorial designs and their applications in industrial, agricultural, biological, pharmaceutical, and manufacturing sciences, etc.

Full factorial designs, which include all the possible runs, allow us to estimate all the factor effects. However, they are time consuming and costly. Therefore, it is not suitable when an experiment involves a large number of factors, which is often the case in practice.

To save time and cost of running experiments, fractional factorial designs (FFDs), whose runs are chosen from full factorial designs, were presented because of their small run sizes and flexibility. Clearly, there are many ways to choose the runs from full factorial designs. A *regular* fractional factorial design (Box and Hunter, 1961, and Fries and Hunter, 1981) can be obtained when its runs are chosen according to some rule. Otherwise, the design is called a *non-regular* fractional factorial design (Plackett and Burman, 1946).

However, there exists a disadvantage when fractional factorial designs are used, that is, the effects of factors can be *aliased* with each other and can not be estimated. Therefore, it is hard to separate the influence of effects on responses.

Usually, investigators consider not only the estimation of the effects of interest but also the limitation of the cost of performing the experiment. Since when the number of factors and run size are fixed, we have many possible FFDs to choose, how to select optimal designs, which allow us to estimate the most effects with the least cost, among these designs becomes a primary problem for us. The problem led to the development and use of various optimality criteria. The most common criterion is the minimum aberration. It was proposed by Fries and Hunter (1980) for regular designs and was further extended to general designs by Tang and Deng (1999), Cheng and Ye (2004), and Lin (2014), etc.

The study of detailed structure of FFDs paves a solid ground for the development of robust parameter design (RPD), which was pioneered by Taguchi (1986). RPD is used as a statistical methodology for variation reduction of a process through changing the setting of controllable factors in order to reduce process sensitivity to uncontrollable factors. There are two types of factors included in RPDs: control factors and noise factors. Control factors have adjustable values during experiments as opposed to noise factors whose values are difficult to control under normal operating conditions.

Robust parameter designs have been studied extensively in the last several decades (See Wu and Hamada (2000), Montgomery (2005), Wu and Zhu (2003), and Bingham and Li (2002), etc). In particular, Zhu (2000) and Bingham and Sitter (2003) studied minimum aberration regular robust parameter designs and Loeppky (2006) investigated the minimum aberration non-regular robust parameter designs and provided a catalogue of optimal designs.

In practical experiments, factors can be classified into two broad categories: quantitative factors and qualitative factors. A quantitative factor, such as temperature or reactant concentration, is a variable whose levels can be connected with a numerical scale. A qualitative or categorical factor, such as gender, color or batch of material, is a variable whose level cannot be ordered.

Although a lot of research has been done for robust parameter designs, almost all the work focused on two-level designs, in which all the factors have two levels. However, in practice, many experiments can involve factors with three or more levels, such designs are called mixed-level designs. Up to now, none of the existing work studies optimal mixed-level robust parameter designs.

The objective of this thesis is to develop an approach to the selection of optimal mixed-level RPDs with the consideration of both qualitative factors and quantitative factors. Optimal RPDs will be searched and tabulated for 18-run mixed-level designs.

In Chapter 2, we start with an overview of factorial designs, including regular and non-regular two-level fractional factorial designs and mixed-level fractional factorial designs. The tool of indicator function and minimum aberration optimality criteria are introduced to study these designs. In addition, two-level RPDs are introduced. In Chapter 3, we propose a general framework that can be used to express mixed-level general RPDs. A new ordering principle of RPDs that include both qualitative and quantitative factors is introduced and the minimum aberration criterion is extended to mixed-level fractional factorial RPDs. Examples are given to further illustrate the method. In Chapter 4, we construct minimum aberration designs from a 18-run mixed-level design. A catalogue of 18-run optimal mixed-level fractional factorial RPDs is provided. We also give two illustrative examples.

Chapter 2

Factorial Designs

In practice, many experiments involve factors. Factorial designs are commonly used in agricultural and industrial experiments (see Box, Hunter and Hunter, 1978 and Kempthorne, 1952) as an important systematic way to test the effect of multiple factors on a response. An important consideration is how to select optimal factorial designs so that experimenters can balance their need for as much precision of parameters' estimators and their desire to reduce as much costs of experimentation as possible. Some criteria such as *resolution* (See Box and Hunter, 1961) and *minimum aberration* (see Fries and Hunter, 1980), have been used to select good designs.

This chapter presents factorial designs. In section 2.1, we will discuss regular and non-regular two-level factorial designs. The difference between full factorial designs and fractional factorial designs will be given. In section 2.2, we will introduce more general designs, their indicator functions and some optimality criteria, such as minimum aberration. Finally, in section 2.3, two-level factorial designs for robust parameter experiments will be presented.

2.1 Two-level Factorial Designs

2.1.1 Two-level Full Factorial Designs

In many industrial experiments, investigators are interested in studying the effects of factors, as well as interactions between factors on a process. A two-level factorial design is a design with each factor at two level settings. Generally speaking, if there are k factors, each at 2 levels, then an experimental design with all possible combinations of all the k factors would have 2^k runs in total, which is called a *two-level full factorial design*. These designs are used to decide which factors contribute to important effects on the process.

In order to illustrate the estimation of effects, we consider a 2^3 full factorial design which is used to investigate the influence of three factors, X_1: initial pH (7.4 or 8.2), X_2: temperature (30^oC or 37^oC) and X_3: the number of baffles in the flask (0 or 2), on the amount of nuclease (Max Morris, 1950)

Each factor has two level settings, denoted by -1 (low) and +1 (high). For example, two temperatures are tested in the experiment, 30^oC and 37^oC, which are denoted by -1 and +1, respectively. We want to try various combinations of these settings so as to establish the best way for estimating the effects. There are $2^3 = 8$ runs in total. These eight combinations are shown in the *design matrix* X:

$$X = \begin{pmatrix} X_1 & X_2 & X_3 \\ -1 & -1 & -1 \\ +1 & -1 & -1 \\ -1 & +1 & -1 \\ +1 & +1 & -1 \\ -1 & -1 & +1 \\ +1 & -1 & +1 \\ -1 & +1 & +1 \\ +1 & +1 & +1 \end{pmatrix} \tag{2.1}$$

Matrix 2.1 helps us to test all the main effects. For Example, the effect of X_1 is $E(X_1) = \bar{y}(X_1+) - \bar{y}(X_1-)$, where $E(X_1)$ represents the effect of X_1, $\bar{y}(X_1+)$ is the mean of the response at the high level of X_1 and $\bar{y}(X_1-)$ is the mean of the response at the low level of X_1.

Running the full complement of all possible factor combinations means that we can estimate all the main and interaction effects. There are three main effects, X_1, X_2 and X_3, three two-factor interactions, X_1X_2, X_1X_3, X_2X_3, and one three-factor interaction, $X_1X_2X_3$, all of which appear in the full model as follows:

$$\begin{aligned} Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_{12} X_1 X_2 + \\ \beta_{13} X_1 X_3 + \beta_{23} X_2 X_3 + \beta_{123} X_1 X_2 X_3 + \epsilon, \end{aligned} \tag{2.2}$$

where $\epsilon \sim N(0, \sigma^2)$. A full factorial design allows us to estimate all eight coefficients

$\{\beta_0, ..., \beta_{123}\}$. The 2^3 full factorial design matrix is

$$X = \begin{array}{c} \begin{array}{cccccccc} (1) & X_1 & X_2 & X_3 & X_1X_2 & X_1X_3 & X_2X_3 & X_1X_2X_3 \end{array} \\ \begin{pmatrix} +1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 \\ +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 \\ +1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 \\ +1 & +1 & +1 & -1 & +1 & -1 & -1 & -1 \\ +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 \\ +1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 \\ +1 & -1 & +1 & +1 & -1 & -1 & +1 & -1 \\ +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \end{pmatrix} \end{array} \tag{2.3}$$

2.1.2 Regular Two-level Fractional Factorial Designs

In the previous subsection, some methodologies and concepts about full factorial designs have been introduced. This subsection will give a brief and formal introduction of the basics of two-level regular *fractional factorial designs* (FFD) so as to build a foundation for the later extension and development.

In practice, many experiments involve plenty of factors, full factorial designs are not feasible for experiments. For instance, when there are 10 factors in an experiment, the total runs needed is $2^{10} = 1024$ which is too expensive and time consuming. Thus, *fractional factorial designs*, which consist of carefully chosen fractions of full factorial designs, were proposed.

The following is an Example to illustrate how to construct a two-level regular FFD with five factors. We start with the 2^3 full factorial design (2.1) with three factors X_1, X_2, and X_3. Add a fourth column to the design matrix as factor X_4, using $X_4 = X_1X_2$ to manufacture it, i.e., create new column by multiplying the indicated column 1 and column 2 together. Do like wise for factor $X_5 = X_1X_3$. The resultant matrix is the design matrix of a 8-run FFD with 5 factors. We call X_4 and X_5 as *generators*.

The design matrix is

$$X = \begin{pmatrix} (1) & X_1 & X_2 & X_3 & X_4 = X_1X_2 & X_5 = X_1X_3 \\ +1 & -1 & -1 & -1 & +1 & +1 \\ +1 & +1 & -1 & -1 & -1 & -1 \\ +1 & -1 & +1 & -1 & -1 & +1 \\ +1 & +1 & +1 & -1 & +1 & -1 \\ +1 & -1 & -1 & +1 & +1 & -1 \\ +1 & +1 & -1 & +1 & -1 & +1 \\ +1 & -1 & +1 & +1 & -1 & -1 \\ +1 & +1 & +1 & +1 & +1 & +1 \end{pmatrix} \tag{2.4}$$

As can be seen, $X_1X_2X_4$ will yield a column of +1's. Similarly $X_1X_3X_5$ and $(X_1X_2X_4)(X_1X_3X_5) = X_2X_3X_4X_5$ will yield two columns of +1's, respectively. These relations can be written as

$$I = X_1X_2X_4 = X_1X_3X_5 = X_2X_3X_4X_5, \tag{2.5}$$

which is called the *defining relation* of this 2^{5-2} design.

Usually, a two-level regular FFD with k factors and p generators is denoted by a 2^{k-p} *FFD*. The generators can completely determine the design. Each term in the defining relation is called a *word*. The *word length* for a regular design is defined as the number of factors in a word, for instance, the length of the word $X_1X_2X_4$ is 3.

Although we can use a 2^{5-2} FFD to explore the effect of each factor without performing $2^5 = 32$ runs, the reduction in run-size come with a cost. Some factorial effects among the involved factors are *aliased* with each other. For Example, it is hard to distinguish the effect of X_1 from the effect of X_2X_4 interaction. This can be found from the defining relation (2.5) by multiplication:

$$X_1 = X_1I = X_1(X_1X_2X_4) = X_1^2X_2X_4 = X_2X_4.$$

In this case, we can say X_1 is *aliased* with X_2X_4. Similarly, we can find all the alias relations of the design. For Example, X_2X_3 is aliased with X_4X_5, since

$$X_2X_3 = X_2X_3I = X_2X_3(X_2X_3X_4X_5) = X_4X_5.$$

An important property of an FFD is its *resolution* (Box and Hunter, 1961) which describe how badly the effects of a design are aliased (or confounded). Formally, the *resolution* is defined as the length of the shortest word in the defining relation. Generally, designs with higher-resolution are considered better, and the aim of a design is usually to find an FFD with the highest-resolution for a given number of factors when the run size is fixed. However, designs with the same resolution do not mean they have same properties, which is illustrated through the use of Example 2.1.2.

Example 2.1.2 Consider two 2^{7-2} FFDs $\mathcal{A}_1$ and $\mathcal{A}_2$ generated by

$$X_6 = X_1X_2X_3, \quad X_7 = X_3X_4X_5,$$

and

$$X_6 = X_1X_2X_3, \quad X_7 = X_2X_3X_4X_5,$$

respectively. Their defining relations are

$$I = X_1X_2X_3X_6 = X_3X_4X_5X_7 = X_1X_2X_4X_5X_6X_7$$

and

$$I = X_1X_2X_3X_6 = X_2X_3X_4X_5X_7 = X_1X_4X_5X_6X_7,$$

respectively. Both $\mathcal{A}_1$ and $\mathcal{A}_2$ has Resolution IV, since the shortest word in both defining relations is 4. Although they have the same resolution, $\mathcal{A}_1$ has two four-letter words and $\mathcal{A}_2$ has only one four-letter word, which means there are more aliased two-factor interactions in design $\mathcal{A}_1$. Since main effects and two-factor interactions are usually important and needed to be estimated, $\mathcal{A}_2$ is better than $\mathcal{A}_1$.

Fries and Hunter (1980) introduced *word length pattern (WLP)* and *minimum aberration* as an additional index to distinguish among designs with the same resolution. It is based on the following *hierarchical ordering principle* (Wu and Hamada, 2000)

(1) lower order effects are more important than higher order effects

(2) effects of the same order are equally important.

For example, X_1 is more significant than X_1X_2 and X_1X_2 is equally important as X_2X_3.

Now we present the definitions of word length pattern and minimum aberration of a design.

Definition 2.1.1 *For a regular 2^{k-p} FFD $\mathcal{A}$, let $W_i(\mathcal{A})$ be the number of words of length i in the defining relation of $\mathcal{A}$. Define*

$$W(\mathcal{A}) = (W_1(\mathcal{A}), W_2(\mathcal{A}), ..., W_k(\mathcal{A})),$$

to record the frequencies of different lengths of the words. $W(\mathcal{A})$ is called the Word Length Pattern (WLP) of the design $\mathcal{A}$.

Definition 2.1.2 *(Minimum Aberration) Given two 2^{k-p} fractional factorial designs $\mathcal{A}_1$ and $\mathcal{A}_2$. $\mathcal{A}_1$ is said to have less aberration if there exists s such that*

$$W_1(\mathcal{A}_1) = W_1(\mathcal{A}_2), W_2(\mathcal{A}_1) = W_2(\mathcal{A}_2), ..., W_{s-1}(\mathcal{A}_1) = W_{s-1}(\mathcal{A}_2), W_s(\mathcal{A}_1) < W_s(\mathcal{A}_2).$$

We say $\mathcal{A}_1$ is the minimum aberration design if no other design has less aberration than $\mathcal{A}_1$.

In Example 2.1.2, $\mathcal{A}_2$ has only one four-factor interaction and two five-factor interaction. So by Definition 2.1.1, the wordlength pattern of $\mathcal{A}_2$ is (0, 0, 0, 1, 2, 0, 0). Meanwhile, $\mathcal{A}_1$ has two four-factor interaction and one six-interaction. So $\mathcal{A}_1$'s wordlength pattern is (0, 0, 0, 2, 0, 1, 0). The number of four-factor interaction of $\mathcal{A}_2$ is less than that of $\mathcal{A}_1$. Therefore, $\mathcal{A}_2$ has *less aberration.* The resolution and minimum aberration criterion can be combined by looking at an ordered list of the number of words of each length for $\mathcal{A}_1$ and $\mathcal{A}_2$ as shown in the following table:

Design	Generators	Wordlength Pattern	Resolution
$\mathcal{A}_1$	$X_6 = X_1X_2X_3$, $X_7 = X_3X_4X_5$	(0, 0, 0, 2, 0, 1, 0)	IV
$\mathcal{A}_2$	$X_6 = X_1X_2X_3$, $X_7 = X_2X_3X_4X_5$	(0, 0, 0, 1, 2, 0, 0)	IV, less aberration

We can see that a minimum aberration design is the one for which the first unequal entry is the smallest.

Minimum aberration designs are generally deemed optimal since they have the smallest number of alias relationships between low-order effects. In Example 2.1.2,

$\mathcal{A}_1$ leads to 6 pairs of aliased 2-factor interactions, which are

$$(X_1X_2, X_3X_6), (X_1X_3, X_2X_6), (X_1X_6, X_2X_3) \text{ because } I = X_1X_2X_3X_6;$$

$$(X_3X_4, X_5X_7), (X_3X_5, X_4X_7), (X_3X_7, X_4X_5) \text{ because } I = X_3X_4X_5X_7.$$

But $\mathcal{A}_2$ aliases only the first 3 pairs of two-factor interactions, and therefore we think $\mathcal{A}_2$ is better.

As such, it is popular to use minimum aberration criterion to decide how to assign additional factors to interaction columns in a design matrix. Except for FFDs, the minimum aberration criterion can be adapted for more complex experimental situations. The generalized minimum aberration, introduced by Tang and Deng (1999), is a generalization of the minimum aberration, and applied to estimate the goodness of general FFDs. Zhu (2000) and Bingham and Sitter (2003) developed the minimum aberration criterion for Robust Parameter Designs. We will discuss these developments in the following sections. For more work on minimum aberration designs, see Bingham and Sitter (1999), Li and Lin (2003), Cheng and Tang (2005), and Ai, Xu, and Wu (2010), etc.

2.1.3 Non-regular Two-level Fractional Factorial Designs

A major constraint attached to the use of regular two-level FFDs is the requirement that run sizes must be powers of two. When this restriction is not practical, *non-regular* fractions, which do not preserve all of the structure of regular plans, may be used. In the past decade or so, a great deal of work has been done on the class of non-regular designs and its various extensions. Except for the flexibility of run size, non-regular designs has one more advantage that some factorial effects are partially aliased, unlike regular FFDs, in which effects are either fully aliased or independent. Wu and Hamada (2000) discussed some methods for analyzing non-regular designs.

Unlike regular designs, which can be studied through their generators or defining relations, non-regular designs do not have generators, thus, it is hard to explore these designs. Recently, Fontana *et al.* (2000) and Ye (2003) showed that any FFD can be represented by an *indicator polynomial function*, or *indicator function*.

Definition 2.1.3 *(Fontana, Pistone and Rogantin, 2000 and Ye, 2003)*

Denoted by $\mathcal{D}$ a 2^k full factorial design. Each run is denoted by the vector $\boldsymbol{x} = (x_1, x_2, ..., x_k)$, where $x_i^2 = 1$, $i = 1, 2, ..., k$. The indicator function of a two-level fractional factorial design $\mathcal{A}$ with n runs from $\mathcal{D}$ is defined as

$$F_{\mathcal{A}}(\boldsymbol{x}) = \begin{cases} N_{\boldsymbol{x}} & \text{if } \boldsymbol{x} \in \mathcal{A}, \\ 0 & \text{if } \boldsymbol{x} \notin \mathcal{A} \text{ and } \boldsymbol{x} \in \mathcal{D}, \end{cases}$$

where N_x indicates how many times the given run $\boldsymbol{x}$ appears in the design $\mathcal{A}$.

Suppose $\mathcal{M}$ is the set of all subsets of $\{1, 2, ..., k\}$ and $M \in \mathcal{M}$. Let $X_M(\mathbf{x}) = \prod_{m \in M} x_m$. Note that $\varnothing$ is an element of $\mathcal{M}$ and $X_\varnothing(\mathbf{x}) = 1$. Fontana *et al.* (2000) and Ye (2003) pointed out that $F_{\mathcal{A}}(\mathbf{x})$ can be represented by a polynomial function, that is

$$F_{\mathcal{A}}(\mathbf{x}) = \sum_{M \in \mathcal{M}} b_M X_M(\mathbf{x}), \tag{2.6}$$

where the coefficient of X_M can be determined by

$$b_M = \frac{1}{2^k} \sum_{\mathbf{x} \in \mathcal{F}} X_M(\mathbf{x}). \tag{2.7}$$

Obviously, the constant $b_\varnothing = \frac{n}{2^k}$. Fontana *et al.* (2000) showed that $|b_M|$ is always less than or equal to $|b_\varnothing|$ and a design is a regular design if and only if $|b_M| = |b_\varnothing|$. The indicator function of a regular design can be easily obtained from its generators.

Example 2.1.3 Consider the regular 2^{5-2} FFD (2.5). The indicator function can be easily determined by its generators

$$\begin{aligned} F(\mathbf{x}) &= \frac{2^3}{2^5}(1 + X_1X_2X_4)(1 + X_1X_3X_5) \\ &= \frac{1}{4}(1 + X_1X_3X_5 + X_1X_2X_4 + X_1^2X_2X_3X_4X_5) \\ &= \frac{1}{4}(1 + X_1X_3X_5 + X_1X_2X_4 + X_2X_3X_4X_5), \end{aligned}$$

However, for non-regular factorial designs, their indicator functions are not as simple as those of regular designs. They can be obtained by (2.6) and (2.7).

Example 2.1.4 We consider a non-regular 12-run factorial design listed in Table 2.1.

Table 2.1: A projection of 12-run non-regular factorial design

X_1	X_2	X_3	X_4	X_5
1	1	1	1	1
-1	1	-1	1	1
-1	-1	1	-1	1
1	-1	-1	1	1
-1	1	-1	-1	-1
-1	-1	1	-1	-1
-1	-1	-1	1	-1
1	-1	-1	-1	-1
1	1	-1	-1	1
1	1	1	-1	-1
-1	1	1	1	-1
1	-1	1	1	-1

The indicator function of the design is

$$\begin{aligned}
F(\mathbf{x}) = {} & \frac{1}{2^5}(12 + 4X_1X_2X_3 - 4X_1X_2X_4 + 4X_1X_2X_5 + 4X_1X_3X_4 - 4X_1X_3X_5 \\
& + 4X_1X_4X_5 + 4X_2X_3X_4 - 4X_2X_3X_5 + 4X_2X_4X_5 - 4X_3X_4X_5 \\
& - 4X_1X_2X_3X_4 + 4X_1X_2X_3X_5 - 4X_1X_2X_4X_5 + 4X_1X_3X_4X_5 \\
& + 4X_2X_3X_4X_5 + 8X_1X_2X_3X_4X_5) \\
= {} & \frac{1}{8}(3 + X_1X_2X_3 - X_1X_2X_4 + X_1X_2X_5 + X_1X_3X_4 - X_1X_3X_5 + X_1X_4X_5 \\
& + X_2X_3X_4 - X_2X_3X_5 + X_2X_4X_5 - X_3X_4X_5 - X_1X_2X_3X_4 + X_1X_2X_3X_5 \\
& - X_1X_2X_4X_5 + X_1X_3X_4X_5 + X_2X_3X_4X_5 + 2X_1X_2X_3X_4X_5)
\end{aligned} \tag{2.8}$$

$b_{\mathbf{t}}$ can be used as a measurement that reflects the degree of aliasing between effects. In Example 2.1.3, the coefficient of $X_1X_3X_5$ is equal to $1/4$ which is the same as the constant $b_\varnothing$ of the indicator function. It indicates that X_1 is fully aliased with X_3X_5, or X_3 is fully aliased with X_1X_5, or X_5 is fully aliased with X_1X_3. In Example 2.1.4, the coefficient of $X_1X_2X_3$ is 1/8 which is smaller than the constant 3/8. This indicates that X_1 is partially aliased with X_2X_3, or X_2 is partially aliased with X_1X_3, or X_3 is partially aliased with X_1X_2.

Indicator function is a powerful tool for studying factorial designs, especially non-regular designs. For example, using indicator functions, Li, Lin, and Ye (2003) studied optimal non-regular foldover designs, Balakrishnan and Yang (2006) investigated the relation between the designs with different resolutions, Lokeppy (2006) studied optimal non-regular two-level RPDs, and Balakrishnan and Yang (2009) discussed various non-regular semifoldover designs. For more work, see Edwards (2011) and Ou, Qin, and Li (2011), etc.

In section 2.1.2, we briefly introduced wordlength, wordlength pattern, and minimum aberration criterion for regular designs. Deng and Tang (1999) extended these concepts to general two-level factorial designs. Li, Lin and Ye (2002) redefined them using indicator functions as follows.

Definition 2.1.4 *Let $\mathcal{A}$ be a two-level FFD and (2.6) be its indicator function. The* generalized word length *of $X_{\boldsymbol{M}}$ is $||\boldsymbol{M}|| + (1 - |b_{\boldsymbol{M}}/b_\varnothing|)$, where $||\boldsymbol{M}||$ is the number of letters in the word $X_{\boldsymbol{M}}$.*

For instance, in Example 2.1.3, the length of the word $X_1X_3X_5$ is $||\mathbf{M}|| + (1 - |b_{\mathbf{M}}/b_\varnothing|) = 3 + 1 - |\frac{1}{4}/\frac{1}{4}| = 3$. In Example 2.1.4, the length of the word $X_1X_2X_3X_4X_5$ is $||\mathbf{M}|| + (1 - |b_{\mathbf{M}}/b_\varnothing|) = 5 + (1 - |\frac{8}{2^5}/\frac{12}{2^5}|) = 5 + 1 - 2/3 = 5\frac{1}{3}$.

Definition 2.1.5 *Let $\mathcal{A}$ be a two-level FFD. Define the extended word length pattern of $\mathcal{A}$ to be*

$$(f_1, ..., f_{1+(n-1)/n}, f_2, ..., f_{2+(n-1)/n}, ..., f_k., ..., f_{k+(n-1)/n}),$$

where $f_{i+j/n}$ is the number of words of length $i + j/n$.

Tang and Deng (1999) proposed the *generalized resolution* of two-level FFD $\mathcal{A}$ based on extended WLP, which is the smallest $i + j/n$ such that $f_{i+j/n} \neq 0$.

In Example 2.1.4, there are 10 length $3\frac{2}{3}$ words, 5 length $4\frac{2}{3}$ words and 1 length $5\frac{1}{3}$. So $f_{3\frac{2}{3}} = 10$, $f_{4\frac{2}{3}} = 5$ and $f_{5\frac{1}{3}} = 1$. The extended WLP is [(0,10,0), (0,5,0), (1,0,0)] and its generalized resolution is $3\frac{2}{3}$. The *minimum aberration design* can be obtained by sequentially minimizing extended WLP.

2.2 Mixed-level Fractional Factorial Designs

In practical experiments, many factors have more than two levels. For instance, quite often, experimenters would like to choose several different relative humidities to increase the analyzing accuracy. If three relative humidities 70%, 80%, 90% are chosen, then the factor "relative humidity" would have three levels: 70%, 80% and 90%. Considering these situations, *general factorial designs* or *mixed-level factorial designs* which include factors with different level settings, are usually used. For example, the design in Table 2.2 is a mixed-level design with 8 factors: one has two levels and the others have three levels.

Table 2.2: L18: A 18-Run $2^1 3^7$ Mixed-Level Design

Two-level factor	Three-level factor						
X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	2	2	2	2	2	2
0	1	0	0	1	1	2	2
0	1	1	1	2	2	0	0
0	1	2	2	0	0	1	1
0	2	0	1	0	2	1	2
0	2	1	2	1	0	2	0
0	2	2	0	2	1	0	1
1	0	0	2	2	1	1	0
1	0	1	0	0	2	2	1
1	0	2	1	1	0	0	2
1	1	0	1	2	0	2	1
1	1	1	2	0	1	0	2
1	1	2	0	1	2	1	0
1	2	0	2	1	2	0	1
1	2	1	0	2	0	1	2
1	2	2	1	0	1	2	0

Note that each row corresponds to a run, such as treatment combination, and each column to a factor. Label the eight columns as X_1, X_2, X_3, X_4, X_5, X_6, X_7, and X_8 and let x_1, x_2,..., x_8 denote the levels of the eight columns.

A lot of work has been done for mixed-level fractional factorial designs. For example, Wu and Zhang (1993) discussed minimum aberration factorial designs with two-level and four-level factors; Xu and Wu (2001) studied a generalized minimum aberration criterion for comparing general fractional factorial designs and Tang and Xu (2014) establish some theoretical results for three-level fractional factorial designs and obtain the best level permutations for regular designs with 27 and 81 runs. For more work, see Cheng and Wu (2002), Liu, Fang, and Hichernell (2006), and Guo, Simpson, and Pignatiello (2007), etc.

2.2.1 Indicator Functions of General Factorial Designs

In section 2.1.3, indicator function of a two-level FFD is briefly discussed. Cheng and Ye (2004) extended this definition further to general factorial designs, where factors can have different levels.

Definition 2.2.1 *(Cheng and Ye, 2004) Denoted by $\mathcal{D}$ a full factorial design and $\mathcal{A}$ a general FFD in $\mathcal{D}$, that is, for any $\boldsymbol{x} \in \mathcal{A}$, $\boldsymbol{x} \in \mathcal{D}$. The indicator function $F_{\mathcal{A}}(\boldsymbol{x})$ of $\mathcal{A}$ is a function defined on $\mathcal{D}$, such that for any $\boldsymbol{x} \in \mathcal{D}$, it counts the number of appearances of the run $\boldsymbol{x}$ in $\mathcal{A}$.*

Assume that $\mathcal{D}$ contains k factors X_1,X_2,...,X_k. The levels of ith factor X_i are elements in the set $S_i = \{0, 1, 2, ..., s_i - 1\}$. Clearly, the number of runs in $\mathcal{D}$ is $N = s_1 s_2 ... s_k$. For each factor X_i, define $c_0^i(x)$, $c_1^i(x)$,..., $c_{s_i-1}^i(x)$ such that

$$\sum_{x \in \{0,1,...,s_i-1\}} c_u^i(x) c_v^i(x) = \begin{cases} 0 & \text{if} \quad u \neq v \\ s_i & \text{if} \quad u = v. \end{cases}$$

$c_0^i(x), c_1^i(x), ..., c_{s_i-1}^i(x)$ are called *orthogonal contrasts*. Obviously, if X_i is a two-level factor, then

$$(c_0^i(0), c_0^i(1)) = (1, 1), \qquad (c_1^i(0), c_1^i(1)) = (-1, 1).$$

Wu and Hamada (2000) pointed out that the linear and quadratic contrasts for a three-level factor are

$$(c_0^i(0), c_0^i(1), c_0^i(2)) = (1, 1, 1),$$

$$(c_1^i(0), c_1^i(1), c_1^i(2)) = (-\sqrt{3/2}, 0, \sqrt{3/2}),$$

$$(c_2^i(0), c_2^i(1), c_2^i(2)) = (1/\sqrt{2}, -\sqrt{2}, 1/\sqrt{2}).$$

For a given design and a run $\mathbf{x} = (x_1, x_2, ..., x_k) \in \mathcal{D}$, a polynomial term is defined as

$$C_{\mathbf{t}}(\mathbf{x}) = \prod_{i=1}^{k} c_{t_i}^i(x_i), \tag{2.9}$$

where $\mathbf{t} = (t_1, t_2, ..., t_k) \in \mathcal{T}$ and $\mathcal{T} = S_1 \times S_2 \times ...S_k$. In general, $C_{\mathbf{0}}(\mathbf{x}) = 1$.

Cheng and Ye (2004) showed that $F_{\mathcal{A}}(\mathbf{x})$ has a polynomial form and a general FFD can be represented by an indicator function.

Theorem 2.2.2 *(Cheng and Ye, 2004)*

Denoted by $\mathcal{D}$ a full factorial design with N runs and $\mathcal{A}$ a general FFD in $\mathcal{D}$. $F_{\mathcal{A}}(\boldsymbol{x})$ has the form

$$F_{\mathcal{A}}(\boldsymbol{x}) = \sum_{\boldsymbol{t} \in \mathcal{T}} b_{\boldsymbol{t}} C_{\boldsymbol{t}}(\boldsymbol{x}), \tag{2.10}$$

for all $\boldsymbol{x} \in \mathcal{D}$. The coefficients $b_{\boldsymbol{t}}$ can be calculated by the formula

$$b_{\boldsymbol{t}} = \frac{1}{N} \sum_{\boldsymbol{x} \in \mathcal{A}} C_{\boldsymbol{t}}(\boldsymbol{x}). \tag{2.11}$$

To illustrate the use of mixed-level factorial designs, we consider an example. Suppose a m-run mixed-level factorial design $\mathcal{A}$ with k_1 two-level factors and k_2 three-level factors. The design space $\mathcal{D}$ is a collection of $2^{k_1} \times 3^{k_2}$ runs: $\{(x_1, x_2, ..., x_{k_1}, ..., x_{k_1+k_2}), x_i = 0, 1, i = 1, 2, ..., k_1; x_j = 0, 1, 2; j = k_1+1, ..., k_1+k_2\}$. From Theorem 2.2.2, any mixed-level factorial design can be expressed through an indicator function $F_{\mathcal{A}}(\mathbf{x})$. By (2.11), the coefficients of the indicator function of $\mathcal{A}$ can be calculated by the formula

$$b_{\mathbf{t}} = \frac{1}{2^{k_1} \times 3^{k_2}} \sum_{x \in \mathcal{A}} C_{\mathbf{t}}(\mathbf{x})$$

and $b_{\mathbf{0}} = \frac{m}{2^{k_1} \times 3^{k_2}}$. According to the notation in Cheng and Ye (2004), we denote $\mathbf{t} = (t_1, t_2, ..., t_k)$ by $\mathbf{t} = t_1 t_2 ... t_k$ in this thesis. Thus, $C_{\mathbf{t}}(\mathbf{x}) = C_{t_1 t_2 \ldots t_k}(\mathbf{x})$ and $b_{\mathbf{t}} = b_{t_1 t_2 \ldots t_k}$. The following example provides the detail calculation on finding an indicator function.

Example 2.3 Consider a mixed-level factorial design $\mathcal{O}$ with four factors X_1, X_2, X_3 and X_5. X_1 is a two-level factor chosen from the first column of Table 2.2 and X_2, X_3, X_5 are three 3-level factors chosen from columns 2, 3, and 5 of Table 2.2. By (2.11), we can find the coefficient of each term in the indicator function of design $\mathcal{O}$.

For instance, for the first run $\mathbf{x} = (0, 0, 0, 0)$ in $\mathcal{O}$,

$$C_{1211}(\mathbf{x}) = c_1^1(0) c_2^2(0) c_1^3(0) c_1^5(0) = (-1)(1/\sqrt{2})(-\sqrt{3/2})(-\sqrt{3/2}) = -\frac{3}{2\sqrt{2}};$$

for the second run $\mathbf{x} = (0, 0, 1, 1)$ in $\mathcal{O}$,

$$C_{1211}(\mathbf{x}) = c_1^1(0) c_2^2(0) c_1^3(1) c_1^5(1) = (-1)(1/\sqrt{2})(0)(0) = 0;$$

for the third run $\mathbf{x} = (0, 0, 2, 2)$ in $\mathcal{O}$,

$$C_{1211}(\mathbf{x}) = c_1^1(0) c_2^2(0) c_1^3(2) c_1^5(2) = (-1)(1/\sqrt{2})(\sqrt{3/2})(\sqrt{3/2}) = -\frac{3}{2\sqrt{2}}$$

Do like wise for the rest of runs and we can get the values of C_{1211} for all of the runs in $\mathcal{O}$. Hence, by (2.11) we can obtain b_{1211}. Other coefficients can be found similarly. Note that $C_{0000}(\mathbf{x}) = 1$, we obtain the indicator function of $\mathcal{O}$:

$$\begin{aligned} F_{\mathcal{O}}(\mathbf{x}) = \frac{1}{3} &- 0.167C_{1011}(\mathbf{x}) + 0.117C_{0211}(\mathbf{x}) - 0.117C_{1211}(\mathbf{x}) + 0.096C_{1021}(\mathbf{x}) \\ &- 0.117C_{0121}(\mathbf{x}) - 0.117C_{1121}(\mathbf{x}) - 0.136C_{1112}(\mathbf{x}) - 0.096C_{1012}(\mathbf{x}) \\ &+ 0.117C_{0112}(\mathbf{x}) + 0.117C_{1112}(\mathbf{x}) + 0.136C_{1212}(\mathbf{x}) - 0.166C_{1022}(\mathbf{x}) \\ &+ 0.117C_{0222}(\mathbf{x}) - 0.117C_{1222}(\mathbf{x}). \end{aligned} \tag{2.12}$$

$F_{\mathcal{O}}(\mathbf{x})$ counts the number of times that the run $\mathbf{x}$ appears in the design. For example, one can check that when the run $\mathbf{x} = (0, 0, 2, 2)$, $F_{\mathcal{O}}(\mathbf{x}) = 1$, which implies that $\mathbf{x} = (0, 0, 2, 2)$ appears one time in $\mathcal{O}$; when $\mathbf{x} = (1, 1, 1, 1)$, $F(\mathbf{x}) = 0$, which indicates that this run is not in $\mathcal{O}$.

In an indicator function, $C_{\mathbf{t}}(\mathbf{x})$ is called a *word* which represents $X_1^{t_1} X_2^{t_2} ... X_k^{t_k}$. If $b_{\mathbf{t}} \neq 0$, where $\mathbf{t} = (t_1, t_2, ..., t_k)$, then $t_i = 0$ means the effect of X_i is not active. For a quantitative factor X_i, $t_i = 1$ indicates the linear effect of X_i is active, and $t_i = 2$ indicates the quadratic effect of X_i is active, and so forth. For a qualitative factor X_i, $t_i \neq 0$ implies treatment effects are active.

For example, consider $b_{1012} = -0.096$ in (2.12). Since $t_2 = 0$, the effect of X_2 is not included in the corresponding term $C_{1012}(\mathbf{x})$, which is $X_1 X_3 X_5^2$. Since $b_{1012} \neq 0$, it involves with three possible explanations: the treatment effect of X_1 is aliased with the interaction of the linear effect of X_3 and the quadratic effect of X_5; or the linear effect of X_3 is aliased with the interaction of treatment effect X_1 and the quadratic effect of X_5; or the quadratic effect of X_5 is aliased with the interaction of the treatment effect of X_1 and the linear effect of X_3.

Similarly, $C_{0222}(\mathbf{x})$ represents the quadratic-by-quadratic interaction $X_1^0 X_2^2 X_3^2 X_5^2 = X_2^2 X_3^2 X_5^2$. The coefficient $b_{0122} = 0$ implies that the linear-by-quadratic interaction $X_2 X_3^2$ is orthogonal to X_5^2 in the design, or $X_3^2 X_5^2$ is orthogonal to X_2, or $X_2 X_5^2$ is orthogonal to X_3^2.

In experiments, factors can be classified into two types: qualitative factors and quantitative factors. The former deal with descriptions and focus on collecting information that is not numerical. Thus, there is no ordering among levels for this type of factors. For example, for a qualitative factor, contrast $(c_1(0), c_1(1), c_1(2)) = \sqrt{\frac{3}{2}}(-1, 0, 1)$ represents the difference of treatment effects between levels 0 and 2; contrast $(c_2(0), c_2(1), c_2(2)) = \sqrt{\frac{1}{2}}(-1, 2, 1)$ represents the difference of treatment effects between level 1 and the average of levels 0 and 2. On the other hand, quantitative factors can take only numbers, which are usually used to fit a regression model for prediction. Thus, there is an ordering among levels of these factors.

In a mixed-level fractional factorial design, there are three types of words: 1) consists of only qualitative factors; 2) consists of only quantitative factors; and 3) consists of both qualitative and quantitative factors. A word is named as a *mixed-type word* if both quantitative factors and qualitative factors are in the word.

Cheng and Ye (2004) defined two norms for $\mathbf{t}$. The norm of $\mathbf{t}$, denoted by $||\mathbf{t}||_0$ is defined as the number of factors it involves. When all the factors are quantitative, the norm of $\mathbf{t}$ is defined as the sum of its polynomial degree, denoted by $||\mathbf{t}||_1 = \sum_{i=1}^{k} t_i$. Cheng and Ye (2004) defined generalized wordlength patterns for the designs with only qualitative factors and the designs with only quantitative factors, respectively, as shown in Definition 2.2.3.

Definition 2.2.3 *(Cheng and Ye, 2004) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two factorial designs with k qualitative factors and k quantitative factors, respectively. The generalized wordlength patterns of $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined as $(\alpha_1(\mathcal{A}_1), ..., \alpha_k(\mathcal{A}_1))$, and $(\beta_1(\mathcal{A}_2), ..., \beta_K(\mathcal{A}_2))$, respectively, where*

$$\alpha_i(\mathcal{A}_1) = \sum_{||\boldsymbol{t}||_0 = i} (\frac{b_{\boldsymbol{t}}}{b_{\boldsymbol{0}}})^2,$$

$$\beta_i(\mathcal{A}_2) = \sum_{||\boldsymbol{t}||_1 = i} (\frac{b_{\boldsymbol{t}}}{b_{\boldsymbol{0}}})^2,$$

and $K = \sum_{i=1}^{k}(s_i - 1)$. The resolution of $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined as the smallest integer j such that $\alpha_j > 0$ and $\beta_j > 0$, respectively.

When a design $\mathcal{A}$ has mixed-type words, Lin (2014) extended the idea of Cheng and Ye (2004) and defined the norm of $\mathbf{t}$ as $||\mathbf{t}|| = \sum_{i=1}^{k}[I(t_i \neq 0)I(g_i = 1) + t_i I(g_i = 2)]$, where $I(\cdot)$ is an indicator function, and

$$g_i = \begin{cases} 1 & \text{if } X_i \text{ is a qualitative factor} \\ 2 & \text{if } X_i \text{ is a quantitative factor} \end{cases}.$$

For example, assume X_1 and X_2 are qualitative factors, and X_3 and X_4 are quantitative factors, the length of $X_1 X_2^2 X_3 X_4^2$ is $1 + 1 + 1 + 2 = 5$.

Lin (2014) also extended the *generalized wordlength pattern* of $\mathcal{A}$ as $(\gamma_1(\mathcal{A}), \gamma_2(\mathcal{A}), ..., \gamma_k(\mathcal{A}))$, where

$$\gamma_i(\mathcal{A}) = \sum_{||\mathbf{t}||=i} (\frac{b_{\mathbf{t}}}{b_{\mathbf{0}}})^2,$$

The *resolution* of $\mathcal{A}$ is defined as the smallest j such that $\gamma_j(\mathcal{A}) > 0$. Meanwhile, sequentially minimize $\gamma_i(\mathcal{A})$ for $i = 1, 2, ..., k$ can get minimum aberration designs.

2.2.2 Geometric isomorphism

In this section, we review the effect of level permutation of a factor on the properties of a design.

For qualitative factors, level permutations will not change the structure of the design since there is no ordering among levels of qualitative factors. However, for quantitative factors, Cheng and Ye (2004) pointed out that permuting levels for one or more factors will generate designs with different geometric structures and properties, since there is an ordering among levels of quantitative factors.

Geometric isomorphism of two designs is defined as follows.

Definition 2.2.4 *Denoted by $\mathcal{D}$ a full factorial design. Let $\mathcal{A}$ and $\mathcal{B}$ be two FFDs in $\mathcal{D}$. We say designs $\mathcal{A}$ and $\mathcal{B}$ are geometrically isomorphic if $\mathcal{A}$ can be obtained from $\mathcal{B}$ by reversing the level order of factors and/or variable exchange.*

Lin and Po (2014) pointed out that for two geometrically isomorphic designs, $\mathcal{A}$ and $\mathcal{B}$, with both quantitative and qualitative factors in design space $\mathcal{D}$, their generalized WLPs are identical.

For three-level factors, there are six permutations among the levels:

$$\{0,1,2\} \to \{0,1,2\}, \qquad \{0,1,2\} \to \{0,2,1\}, \qquad \{0,1,2\} \to \{1,0,2\},$$

$$\{0,1,2\} \to \{1,2,0\}, \qquad \{0,1,2\} \to \{2,0,1\}, \qquad \{0,1,2\} \to \{2,1,0\}.$$

These permutations can be classified into three pairs as shown in Table 2.3. Within each pair, two permutations generate two designs that are geometrically isomorphic. That is, we only need to consider one permutation in each pair.

Table 2.3: Geometric Isomorphism Groups

Rule	Notation	Permutation
I	X_i	{0,1,2}→{0,1,2} or {2,1,0}
II	$\grave{X}_i$	{0,1,2}→{1,2,0} or {1,0,2}
III	$\hat{X}_i$	{0,1,2}→{2,0,1} or {0,2,1}

$\grave{X}_i$ indicates the level permutation rule II is chosen for factor X_i, while $\hat{X}_i$ indicates the level permutation rule III is chosen for factor X_i. Moreover, level permutations of a design may change the indicator function of the design.

Example 2.4 Consider the factorial design $\mathcal{O}$ in Example 2.3. Assume X_1 is a two-level qualitative factor and X_2, X_3, and X_5 are three-level quantitative factors. If we change the levels of X_2, X_3 and X_5 by level permutation rules I, II, and III, respectively. Then, the indicator function of the resulting factorial design $\mathcal{O}_1$ is

$$\begin{aligned} F_{\mathcal{O}_1}(\mathbf{x}) = \frac{1}{3}C_{0000}(\mathbf{x}) &+ 0.167C_{1011}(\mathbf{x}) - 0.102C_{0111}(\mathbf{x}) - 0.102C_{1111}(\mathbf{x}) \\ &- 0.058C_{1211}(\mathbf{x}) + 0.09C_{0121}(\mathbf{x}) + 0.058C_{1121}(\mathbf{x}) + 0.058C_{0221}(\mathbf{x}) \\ &- 0.102C_{1221}(\mathbf{x}) - 0.09C_{0112}(\mathbf{x}) - 0.058C_{1112}(\mathbf{x}) + 0.102C_{0212}(\mathbf{x}) \\ &- 0.17C_{1212}(\mathbf{x}) + 0.167C_{1022}(\mathbf{x}) + 0.17C_{1012}(\mathbf{x}) - 0.102C_{0122}(\mathbf{x}) \\ &- 0.102C_{1122}(\mathbf{x}) - 0.058C_{0222}(\mathbf{x}) - 0.058C_{1222}(\mathbf{x}) - 0.058C_{0211}(\mathbf{x}). \end{aligned} \tag{2.13}$$

Obviously, $F_{\mathcal{O}_1}(\mathbf{x})$ is different from $F_{\mathcal{O}}(\mathbf{x})$ listed in Example 2.3.

2.3 Two-Level Robust Parameter Designs

Robust parameter designs (RPD), initially proposed by Genichi Taguchi (1986), are usually used to minimize variation in industrial processes and products. Typical example of variation in operating levels of manufacturing processes include raw material variation and temperature variation. The quality of a process is mainly affected by two types of factors: control factors and noise factors.

Control factors are the variables whose values can be controlled in an experiment and also under normal operating conditions.

Noise factors are the variables that are hard to control in a real application of the process. Noise factors can be humidity, light and the temperature of environment, etc.

Consider, the cake baking example in Montgomery (2005). An experimenter is interested in improving the taste of cake. Some factors can be controlled by the cake manufacturer, such as amounts of flour, amount of egg powder, and amount of sugar. These are control factors. However, bake temperature and time are noise factors since they can not be controlled by the manufacturer and may vary among the users.

The manufacturer wants to reduce the effects in variation of oven temperature and time on cake quality under the best settings for the control factors. This is also the goal of an RPD.

Because RPD has two types of factors, the factorial effects of RPD have more complicated interpretation. We call the effects involving both control factors and noise factors as *control-by-noise effects or interactions*.

In this thesis, we denote a control factor by "C" and a noise factor by "N". Zhu (2000) discussed the importance of control-by-noise interaction and considered C, N and CN as equally important. To help define the ranking, Zhu (2000) proposed a definition of word length for two-level regular RPDs:

$$W(k_1, k_2) = \begin{cases} 1 & \text{if} \quad max(k_1, k_2) = 1, \\ k_1 & \text{if} \quad k_1 > k_2 \quad \text{and} \quad k_1 > 1, \\ k_2 + 1/2 & \text{if} \quad k_1 \leq k_2 \quad \text{and} \quad k_2 \geq 2, \end{cases}$$

where k_1 is the number of control factors and k_2 is the number of noise factors. Zhu (2000) provides the words with lengths smaller than 4.5. We list these words in Table 2.4.

Table 2.4: Word Lengths for RPDs defined by Zhu (2000)

Word length	Words
1.0	C, N, CN
2.0	CC, CCN
2.5	$CCNN, CNN, NN$
3.0	$CCC, CCCN, CCCNN$
3.5	$CCCNNN, CCNNN, CNNN, NNN$
4.0	$CCCC, CCCCN, CCCCNN, CCCCNNN$
4.5	$CCCCNNNN, CCCNNNN, CCNNNN, CNNNN, NNNN$

However, Bingham and Sitter (2003) increased the ranking of "CN" to reflect their increased interest in the "CN" and suggested different word lengths for two-level RPDs, which are shown in Table 2.5.

Table 2.5: Word Lengths for RPDs defined by Bingham and Sitter (2003)

Word length	Words
1.0	C, N
1.5	CN
2.0	CC, NN
2.5	CCN, CNN
3.0	$CCC, CCNN$
3.5	$CCCN, CNNN$
4.0	$CCCC, NNN, CCCNN, CCNNN$
4.5	$CCCCN, CNNNN$
5.0	$CCCCC, NNNN, CCCNNN, CCNNNN, CCCCNN$

For a two-level fractional factorial RPD $\mathcal{A}$, the *word length pattern* can be defined as:

$$W(\mathcal{A}) = (W_{1.0}, W_{1.5}, W_{2.0}, ..., W_k),$$

where W_l counts the number of words of length l.

The *minimum aberration designs* for fractional factorial RPDs can be obtained by sequentially minimizing the composite $W(\mathcal{A})$.

To rank non-regular two-level FF RPDs, Loeppky et al.(2006) defined the word length of $X_M(\mathbf{x})$ as $r + (1 - |b_M/b_\varnothing|)/2$, where r is the word length defined by Zhu (2000) or Bingham and Stter (2003), and the extended word length pattern as

$$(g_{2.0}, ..., g_{2.0+(t-1)/2t}, ..., g_{m-1}, ..., g_{m+(t-1)/2t}),$$

where $g_{r+l/2t}$ is the number of words of length $r + l/2t$. Loeppky et al.(2006) also extended the notions of generalized aberration and generalized resolution based on the extended wordlength pattern of RPDs. Particularly, they provided a catalog of optimal robust parameter designs with 12, 16 and 20 runs.

Chapter 3

Mixed-Level General Robust Parameter Designs

In Chapter 2, some notations and concepts of FFDs and two-level RPDs have been introduced. We also discussed the properties and structure of two-level fractional factorial RPD's and shown how to rank them using the minimum aberration criterion. However, in practice, many RPDs involve factors with three or more levels, and factors can be qualitative or quantitative. In this chapter, we will develop a new methodology for ranking mixed-level fractional factorial RPDs with both qualitative factors and quantitative factors..

In section 3.1, we will give the format of indicator functions for a mixed-level RPD. This will provide a framework for the later development. In section 3.2, we will define the wordlength for a mixed-level fractional factorial RPD by distinguishing qualitative factors and quantitative factors. Then, we will extend the generalized WLP and the minimum aberration criterion to mixed-level fractional factorial RPDs.

3.1 Indicator Functions of Robust Parameter Designs

In this section, to study the properties of a general robust parameter design, we redefine (2.10) and take both control and noise factors into account. Denoted by $\mathcal{D}$ a mixed-level full factorial RPD with k_1 control factors, k_2 noise factors and M runs. A design point $\mathbf{x}$ of $\mathcal{D}$ is defined as $\mathbf{x} = (x_1, ..., x_{k_1}, x_{k_1+1}, ..., x_k)$, where $k = k_1 + k_2$.

Note that the design matrix of a factorial design is the same as that of an RPD. The difference between them is that there exist two types of factors in the RPD. So, the results from FFDs can be adapted to fractional factorial RPDs. That is, the design matrix can be considered as a normal factional fractional design if we do not distinguish control factors and noise factors.

Suppose $\mathcal{A}$ is a fractional factorial mixed-level RPD with k_1 noise factors, k_2 control factors and n runs. $\mathcal{D}$ is the full factorial mixed-level RPD with N runs and $\mathcal{A} \in \mathcal{D}$. The levels of ith control factor X_{C_i} are elements in the set $S_{C_i} = \{0, 1, 2, ..., s_{C_i} - 1\}$. The levels of ith noise factor X_{N_i} are elements in the set $S_{N_i} = \{0, 1, 2, ..., s_{N_i} - 1\}$. Further, define $\mathcal{T}_C = S_{C_1} \times S_{C_2} \times \cdots \times S_{C_{k_1}}$ and $\mathcal{T}_N = S_{N_1} \times S_{N_2} \times \cdots S_{N_{k_2}}$. So $\mathcal{T}$ can be rewritten as $\mathcal{T} = \mathcal{T}_C \times \mathcal{T}_N$. Let $\mathbf{t} \equiv A \cup B$, where $\mathbf{t} \in \mathcal{T}$, $A \in \mathcal{T}_C$ and $B \in \mathcal{T}_N$. Then, by (2.10), the polynomial form of the indicator function of $\mathcal{A} \in \mathcal{D}$ can be rewritten as

$$F_{\mathcal{A}}(\mathbf{x}) = \sum_{t \in \mathcal{T}} b_{\mathbf{t}} C_{\mathbf{t}}(\mathbf{x}) = \sum_{A \in \mathcal{T}_C} \sum_{B \in \mathcal{T}_N} b_{A \cup B} C_{A \cup B}(\mathbf{x}), \tag{3.1}$$

where

$$b_{A \cup B} = \frac{1}{N} \sum_{\mathbf{x} \in \mathcal{A}} C_{A \cup B}(\mathbf{x}).$$

In particular, $b_{\mathbf{0}} = n/N$, where $\mathbf{0} = (0, 0, ..., 0)$, and $C_{\mathbf{0}}$ represents a column of ones.

Example 3.1 Consider a fractional factorial RPD $\mathcal{D}_1$ with 3 control factors and 1 noise factor consisting of columns 1-4 in Table 2.2. Denoted by X_1, X_2, and X_3 the control factors that represent columns 1, 2, and 3, respectively, and X_4 the noise factor which represents column 4.

The indicator function of $\mathcal{D}_1$ is

$$\begin{aligned} F_{\mathcal{D}_1}(\mathbf{x}) = \frac{1}{3} &- 0.167C_{1011}(\mathbf{x}) - 0.102C_{0111}(\mathbf{x}) + 0.102C_{1111}(\mathbf{x}) - 0.059C_{0211}(\mathbf{x}) \\ &+ 0.059C_{1211}(\mathbf{x}) + 0.096C_{1021}(\mathbf{x}) - 0.059C_{0121}(\mathbf{x}) + 0.059C_{1121}(\mathbf{x}) \\ &+ 0.102C_{0221}(\mathbf{x}) + 0.1701C_{1221}(\mathbf{x}) - 0.096C_{1012}(\mathbf{x}) + 0.059C_{0112}(\mathbf{x}) \\ &- 0.059C_{1112}(\mathbf{x}) - 0.102C_{0212}(\mathbf{x}) - 0.1701C_{1212}(\mathbf{x}) - 0.167C_{1022}(\mathbf{x}) \\ &- 0.102C_{0122}(\mathbf{x}) + 0.102C_{1122}(\mathbf{x}) - 0.059C_{0222}(\mathbf{x}) + 0.059C_{1222}(\mathbf{x}), \end{aligned} \tag{3.2}$$

Note that $b_{1021} \neq 0$ which implies that the main control effect X_1 is aliased with the control-by-noise interaction ${X_3}^2X_4$, the quadratic control effect ${X_3}^2$ is aliased with the control-by-noise interaction X_1X_4, the main noise effect X_4 is aliased with the control-by-control interaction $X_1{X_3}^2$.

The concepts and definitions mentioned above build a foundation for ranking and ordering designs. In chapter 2, we discussed two-level RPDs without distinguishing qualitative factors and quantitative factors. In the next section, we will discuss how to rank mixed-level RPDs with considering quantitative factors and quantitative factors. and will extend the definition of wordlength of two-level RPDs to mixed-level RPDs based on Table 2.4 and formula 3.3.

3.2 Word Length

In chapter 2, we introduced the wordlengths defined by Zhu (2000) (see Table 2.4) and Bingham and Sitter (2003) (see Table 2.5) for two-level RPDs. Zhu (2000) also provides a formula to find the wordlengths, which is very helpful for writing a computer program and searching optimal designs. However, Bingham and Sitter (2003) did not provide such formula. To determine the wordlength, they consider the shortest length of each word. For example, consider a word $CCCNN$ which can be viewed as $C + CCNN$, $CC + CNN$, $CN + CCN$, $CCC + NN$ and $N + CCCN$. Then, Bingham and Sitter (2003) add the aliased words' ranks together. $Rank(C) + rank(CCNN) = 1 + 3 = 4$; $Rank(CN) + rank(CCN) = 1.5 + 2.5 = 4$; $rank(CC) + rank(CNN) = 2 + 2.5 = 4.5$; $rank(CCC) + rank(NN) = 3 + 2 = 5$ and $Rank(N) + rank(CCCN) = 1 + 3.5 = 4.5$. Therefore, the length of $CCCNN$ is declared to be 4, the smallest of these three combinations.

Based on their definition, we find the following formula as shown in (3.3):

$$W(k_1,k_2)=\begin{cases} k_1 & \text{if } k_2=0 \\ k_2 & \text{if } k_1=0 \text{ and } k_2\leq 2, \\ k_2+1 & \text{if } k_1=0 \text{ and } k_2\geq 3, \\ k_2+0.5 & \text{if } k_1=1 \text{ and } k_2\geq 1, \\ k_1+0.5 & \text{if } k_1\geq 1 \text{ and } k_2=1, \\ k_1+k_2-1 & \text{if } min\{k_1,k_2\}\geq 2, \end{cases} \tag{3.3}$$

where k_1 is the number of control factors and k_2 is the number of noise factors. Notice that the case for $min\{k_1, k_2\} \leq 2$ is defined based on their definition, but for $min\{k_1, k_2\} \geq 3$, we define the wordlength to be k_1+k_2-1 to decrease the difference between the number of factors in the word and the wordlength. This alternative numerical rule will be very useful for searching optimal designs in computer program. Now we show that when $min\{k_1, k_2\} \leq 2$, the wordlength defined in Bingham and Sitter (2003) follows the formula 3.3.

Proof:

A.For $k_2 = 0$, $W(k_1, k_2) = k_1$. It is obvious.

B.For $k_1 = 0$ and $k_2 \leq 2$, $W(k_1, k_2) = k_2$. It is listed in Table 2.5 .

C.For $k_1 = 0$ and $k_2 \geq 3$, $W(k_1, k_2) = k_2 + 1$.

The statement holds for $k_2 = 3$ as listed in Table 2.5. We assume it holds for $k_2 = m$, where $m \geq 3$, that is $rank(N_1...N_m) = m + 1$. Then, show that $rank(N_1...N_{m+1}) = m + 2$ for $k_2 = m + 1$.

The word of type $N_1N_2...N_mN_{m+1}$ can be viewed as creating alias structures of the form $N_1 + N_2...N_{m+1}$, $N_1N_2 + N_3...N_{m+1}$ and $N_1N_2..N_i + N_{i+1}...N_{m+1}$, where

$i = 3, 4, ...$ In the first case, the ranks sum as $rank(N_1) + rank(N_2...N_{m+1}) = 1+m+1 = m+2$; in the second case as $rank(N_1N_2)+rank(N_3...N_{m+1}) = 2+m$; and in the third case as $rank(N_1N_2..N_i)+rank(N_{i+1}...N_{m+1}) = i+1+m-i+1 = m+2$. Therefore, the word has length $m + 2$ and the statement is true.

D.For $k_1 = 1$ and $k_2 \geq 1$, $W(k_1, k_2) = k_2 + 0.5$. The statement holds for $k_2 = 3$ as listed in Table 2.5. We assume it holds for $k_2 = m$, where $m \geq 3$, that is $rank(CN_1...N_m) = m+0.5$. Then, show that $rank(CN_1...N_mN_{m+1}) = m+1+0.5 = m + 1.5$ for $k_2 = m + 1$.

The word of type $CN_1...N_mN_{m+1}$ can be viewed as creating alias structures of the form $C+N_1...N_{m+1}$, $CN_1...N_i+N_{i+1}...N_{m+1}$ and $CN_1...N_m+N_{m+1}$ where $i = 1, 2, ...$ In the first case, the ranks sum as $rank(C)+rank(N_1...N_{m+1}) = 1+m+2 = m+3$; in the seconde case as $rank(CN_1...N_i)+rank(N_{i+1}...N_{m+1}) = i+0.5+m-i+1 = m+1.5$ and in the third case as $rank(CN_1...N_m) + rank(N_{m+1}) = m + 0.5 + 1 = m + 1.5$. Thus, the word has length $m + 1.5$ and the statement is true.

E. For $k_2 = 1$ and $k_1 \geq 1$, $W(k_1, k_2) = k_1 + 0.5$. It is similar with part D.

F. For $min\{k_1, k_2\} = 2$, $W(k_1, k_2) = k_1 + k_2 - 1$. The statement holds for $k_1 = 3$ and $k_2 = 2$ as listed in Table 2.5. We assume the statement holds for $k_1 = m_1$ and $k_2 = 2$, where $m_1 \geq 2$that is $rank(C_1...C_{m_1}N_1N_2) = m_1 + 2 - 1 = m_1 + 1$. Then, we fix k_2 and show that $rank(C_1...C_{m_1+1}N_1N_2) = m_1 + 2$ for $k_1 = m_1 + 1$ and $k_2 = 2$.

Rank(N)+$Rank(C_1...C_{m_1}C_{m_1+1}N) = 1 + m_1 + 1.5 > m_1 + 2$;

rank(NN)+$rank(C_1C_2...C_{m_1}C_{m_1+1}) > m_1 + 2$;

$rank(C_1NN) + rank(C_2...C_{m_1+1}) = 2.5 + m_1 + 1 > m_1 + 2$;

$rank(C_1...C_iNN) + rank(C_{i+1}...C_{m_1+1}) = (i+2-1) + m_1 + 1 - i = m_1 + 2;$

$rank(C_1) + rank(C_2...C_{m_1+1}NN) = m_1 + 2.$

We can also proof it when we fix k_1 and $k_2 = m_2 + 1$ by using the similar way.

Based on the formula above, in this section, we will define wordlengths for mixed-level fractional factorial RPDs.

First, we consider a word which contains only qualitative factors. Since the interest of using qualitative factors is to test whether the difference of treatments exists, Cheng and Ye (2004) pointed out changing the polynomial degree of a word not influence its wordlength. Let $\mathbf{t} = (t_1, t_2, ..., t_k)$, $S = \{j | j \in \{1, 2, ..., k\}$ and $t_j \neq 0\}$, and $\mathbf{W} = \prod_{j \in S} X_j$. Based on the definitions of Zhu (2000) and the above formula, we define the length of a word as follows:

Definition 3.2.1 *When the word $C_{\boldsymbol{t}}(\boldsymbol{x}) = X_1^{t_1} X_2^{t_2} \cdots X_k^{t_k}$ contains only qualitative factors, the length of $C_{\boldsymbol{t}}(\boldsymbol{x})$, denoted by $||C_{\boldsymbol{t}}(\boldsymbol{x})||_0$ is defined as the length of the word $\boldsymbol{W}$.*

Consider, for example, the word "C^2NN^2" which contains only qualitative factors. Since the length of the word $\mathbf{W} = CNN$ is 2.5 as listed in Table 2.4 or Table 2.5, so the length of C^2NN^2 is also 2.5.

Table 3.1 and Table 3.2 illustrate the lengths of the words with only qualitative factors for three-level RPDs in accordance with the definition of Zhu (2000) and Bingham and Sitter (2003), respectively. To save space, we list the words with lengths less than or equal to 4.5.

Table 3.1: Word Lengths for three-level RPDs based on Zhu's definition of wordlength when all the factors are qualitative

Word Length	Words
1.0	$C, N, C^2, N^2, CN, C^2N, CN^2, C^2N^2$
1.5	
2.0	$CCN, C^2CN, C^2C^2N, C^2C^2N^2, CCN^2, C^2CN^2, CC, C^2C, C^2C^2$
2.5	$NN, N^2N, N^2N^2, CNN, C^2NN, CN^2N, C^2N^2N, CN^2N^2,$ $CCNN, C^2CNN, CCN^2N, C^2C^2NN, CCN^2N^2, C^2C^2N^2N,$ $CC^2N^2N^2, C^2C^2N^2N^2, C^2N^2N^2, C^2CN^2N$
3.0	$CCC, C^2CC, C^2C^2C, C^2C^2C^2, CCCN, C^2CCN, CCCN^2,$ $C^2CCN^2, C^2C^2C^2N, C^2C^2CN^2, C^2C^2C^2N^2, CCCNN,$ $C^2C^2CNN, C^2CCN^2N, CCCN^2N^2, C^2C^2C^2NN,$ $C^2C^2C^2N^2N, C^2C^2CN^2N^2, C^2C^2C^2N^2N^2, CCCN^2N,$ $C^2C^2CN, C^2CCNN, CC^2C^2N^2N, CCC^2N^2N^2$
3.5	$NNN, N^2NN, N^2N^2N, N^2N^2N^2, CNNN, C^2NNN,$ $CN^2N^2, C^2N^2N^2N, CN^2N^2N^2, C^2N^2N^2N^2, CCNNN,$ $CC^2N^2NN, CCN^2N^2N, C^2C^2N^2NN, CC^2N^2N^2N,$ $CCN^2NN, C^2C^2NNN, CC^2N^2N^2N^2, C^2C^2N^2N^2N^2,$ $CCC^2N^2NN, CCCN^2N^2N, C^2C^2C^2NNN, CC^2C^2N^2NN,$ $CCCN^2N^2N^2, C^2C^2C^2N^2NN, CC^2C^2N^2N^2N,$ $C^2C^2C^2N^2N^2N, CC^2C^2N^2N^2N^2, C^2C^2C^2N^2N^2N^2,$ $C^2N^2NN, C^2CNNN, C^2C^2N^2N^2N, C^2CCNNN,$ $C^2C^2CNNN, CCCN^2NN, CN^2NN, CCN^2N^2N^2, CCCNNN,$ $CCC^2N^2N^2N, CCC^2N^2N^2N^2$
4.0	$CCCC, C^2CCC, C^2C^2CC, C^2C^2CC^2, C^2C^2C^2C^2,$ $C^2C^2CCN, CCCC^2N^2, C^2C^2C^2CN, C^2C^2CCN^2, C^2C^2C^2C^2N,$ $CCCCN^2, C^2C^2C^2C^2N^2, CCCCNN, C^2CCCNN, CCCCNN^2,$ $C^2C^2C^2CNN, C^2C^2CCNN^2, C^2CCCN^2N^2, C^2C^2C^2C^2NN,$ $C^2CCCNN^2, CCCCN^2N^2, C^2C^2CCN^2N^2, C^2C^2C^2C^2N^2N,$ $C^2CCCNNN, CCCCN^2NN, C^2C^2CCNNN, CCCC^2NNN^2,$ $C^2CCCN^2N^2N, C^2C^2CCN^2NN, CCCCN^2N^2N^2,$ $C^2C^2CCNN^2N^2, CCCC^2N^2N^2N^2, C^2C^2C^2C^2N^2NN,$ $CCC^2CN^2N^2N^2, CC^2C^2CN^2N^2N^2, C^2C^2C^2CN^2N^2N,$ $C^2C^2C^2C^2N^2N^2, CCCCNNN, C^2C^2C^2CNNN,$ $C^2C^2C^2CN^2, C^2C^2CCNN, CC^2C^2C^2NN^2, C^2C^2C^2CN^2N^2,$

Table 3.1 (Continued)

Word Length	Words
	$C^2C^2C^2CN^2N^2N, C^2C^2C^2CN^2N^2N^2, CCCCNN^2N^2$
	$CCCCN, C^2CCCN, C^2C^2C^2C^2NNN, C^2C^2C^2CN^2NN,$
4.5	$NNNN, NNNN^2, NNN^2N^2, NN^2N^2N^2, N^2N^2N^2N^2,$
	$C^2NNNN^2, CNNN^2N^2, C^2N^2N^2NN, CN^2N^2N^2N,$
	$CN^2N^2N^2N^2, C^2N^2N^2N^2N^2, CCNNNN, CC^2NNNN,$
	$C^2CN^2NNN, CCNNN^2N^2, C^2C^2N^2NNN, C^2CN^2N^2NN,$
	$C^2C^2N^2N^2NN, CC^2N^2N^2N^2N, CCN^2N^2N^2N^2,$
	$C^2C^2N^2N^2N^2N, CC^2N^2N^2N^2N^2, C^2C^2N^2N^2N^2N^2,$
	$C^2CCNNNN, CCCNNNN^2, C^2C^2CNNNN, C^2CCN^2NNN,$
	$C^2C^2C^2NNNN, CCCN^2N^2N^2N, C^2CCN^2N^2NN,$
	$C^2C^2C^2N^2NNN, CC^2C^2N^2N^2NN, CCC^2N^2N^2N^2N,$
	$CCC^2N^2N^2N^2N^2, CC^2C^2N^2N^2N^2N, C^2C^2C^2N^2N^2NN,$
	$CNNNN^2, CC^2C^2N^2N^2N^2N^2, C^2C^2C^2N^2N^2N^2N^2,$
	$C^2NNNN, CCNN^2N^2N^2, CCCN^2N^2NN, CCCN^2N^2N^2N^2,$
	$C^2C^2C^2N^2N^2N^2N, CCNNNN^2, C^2N^2N^2N^2N,$
	$CCCNNNN, C^2C^2CN^2NNN, C^2C^2NNNN, CNNNN,$
	$CC^2N^2N^2N^2N^2$

Table 3.2: Word Lengths for three-level RPDs based on Bingham and Sitter's definition of wordlength when all the factors are qualitative

Word length	Words
1.0	C, N, C^2, N^2
1.5	CN, C^2N, CN^2, C^2N^2
2.0	$CC, NN, C^2C, C^2C^2, N^2N^2, N^2N$
2.5	$CCN, CNN, C^2CN, CCN^2, C^2C^2N, C^2CN^2$ $C^2C^2N^2, C^2NN, CN^2N, C^2N^2N, CN^2N^2, C^2N^2N^2$
3.0	$CCC, CCNN, C^2CC, C^2C^2C, C^2C^2C^2, C^2CNN, C^2C^2NN$ $C^2C^2N^2N, C^2C^2N^2N^2, CCN^2N, CCN^2N^2, C^2CN^2N, CC^2N^2N^2$
3.5	$CCCN, CNNN, C^2CCN, CCCN^2, C^2C^2CN, C^2CCN^2,$ $C^2C^2C^2N^2, C^2NNN, CN^2NN, C^2N^2NN, CN^2N^2N,$ $C^2N^2N^2N^2, C^2C^2C^2N, C^2C^2CN^2, C^2N^2N^2N, CN^2N^2N^2$
4.0	$CCCC, NNN, CCCNN, CCNNN, C^2CCC, C^2C^2CC, C^2C^2C^2C,$ $N^2NN, N^2N^2N, N^2N^2N^2, C^2CCNN, CCCN^2N, C^2C^2CNN,$ $CCCN^2N^2, C^2C^2C^2NN, C^2CCN^2N^2, C^2C^2C^2N^2N, C^2C^2CN^2N^2$ $C^2C^2C^2N^2N^2, C^2CNNN, CCN^2NN, C^2C^2NNN, CC^2N^2NN,$ $CCN^2N^2N^2, C^2C^2N^2N^2N, CC^2N^2N^2N^2, C^2C^2N^2N^2N^2,$ $C^2C^2N^2NN, C^2C^2C^2C^2, CCN^2N^2N, C^2CN^2N^2N, C^2CCN^2N$
4.5	$CCCCN, CNNNN, C^2CCCN, CCCCN^2, C^2C^2CCN, C^2CCCN^2,$ $CCC^2C^2N^2, C^2C^2C^2C^2N, CC^2C^2C^2N^2, C^2C^2C^2C^2N^2, C^2NNNN$ $CN^2NNN, C^2N^2NNN, CN^2N^2NN, C^2N^2N^2NN, CN^2N^2N^2N,$ $CN^2N^2N^2N^2, C^2N^2N^2N^2N^2, C^2C^2C^2CN, C^2N^2N^2N^2N$

Second, we consider the situation when all the factors involved in a word are quantitative. Recall that Cheng and Ye (2004) defined two norms $||\mathbf{t}||_0$ and $||\mathbf{t}||_1$, which were reviewed in section 2.2.1, for factorial desgins, where $||\mathbf{t}||_0$ counts the number of letters in the corresponding word and $||\mathbf{t}||_1 = \sum_{i=1}^{k} t_i$ calculates the polynomial degree of the corresponding word. In fact, we find there is a relationship between $||\mathbf{t}||_0$ and $||\mathbf{t}||_1$, that is $||\mathbf{t}||_1 = ||\mathbf{t}||_0 + \sum_{j \in S}(t_j - 1)$.

Definition 3.2.2 *When the word $C_{\boldsymbol{t}}(\boldsymbol{x}) = X_1^{t_1} X_2^{t_2} \cdots X_k^{t_k}$ contains only quantitative* *factors, the length of* $C_{\boldsymbol{t}}(\boldsymbol{x})$ *is defined as* $||C_{\boldsymbol{t}}(\boldsymbol{x})||_1 =$ *the length of* $\boldsymbol{W} + \sum_{j \in S}(t_j - 1) = ||C_{\boldsymbol{t}}(\boldsymbol{x})||_0 + \sum_{j \in S}(t_j - 1)$.

For example, consider the word C^2NN^2, which contains only quantitative factors. Because the length of the word $\mathbf{W} = CNN$ is 2.5 by the definition of Zhu (2000) (see Table 2.4), we have $||C_{\mathbf{t}}(\mathbf{x})||_0 = 2.5$. Thus, the length of C^2NN^2 is $||C_{\mathbf{t}}(\mathbf{x})||_1 = ||C_{\mathbf{t}}(\mathbf{x})||_0 + \sum_{j \in S}(t_j - 1) = 2.5 + (2-1) + (1-1) + (2-1) = 4.5$.

Table 3.3 and Table 3.4 list the words with lengths at most 5 for three-level RPDs when the words contain only quantitative factors. The wordlengths defined in Table 3.3 and Table 3.4 are based on the definition of wordlengths in Zhu (2000) and Bingham and Sitter (2003), respectively.

Table 3.3: Word Lengths for three-level RPDs based on Zhu's definition of wordlength when all the factors are quantitative

Word length	Words
1.0	C, N, CN
1.5	
2.0	$CCN, CC, C^2, N^2, C^2N, CN^2$
2.5	$NN, CNN, CCNN$
3.0	$CCC, CCCN, CCCNN, C^2N^2, C^2C, C^2CN, CCN^2$
3.5	$NNN, CNNN, CCNNN, CCCNNN, C^2NN, CN^2N,$ C^2CNN, CCN^2N, N^2N
4.0	$CCCC, CCCCN, CCCCNN, CCCCNNN, C^2C^2N, C^2CN^2,$ $C^2CCNN, CCCN^2N, CCCN^2, C^2C^2, C^2CC, C^2CCN$
4.5	$NNNN, CNNNN, CCNNNN, CCCNNNN, CCCCNNNN,$ $CCN^2N^2, N^2NN, C^2NNN, CN^2NN, N^2N^2, C^2CN^2N,$ $CN^2N^2, C^2N^2N, C^2C^2NN$
5.0	$CCCCC, CCCCCN, CCCCCNN, CCCCCNNN,$ $C^2CCN^2, C^2CCC, C^2CCCN, CCCCN^2, C^2CCCNN,$ $C^2C^2CNN, C^2CCN^2N, CCCN^2N^2, C^2C^2CN, CCCCN^2NN,$ $C^2CCCNNN, C^2C^2N^2, CCCCN^2N, C^2C^2C, CCCCCNNNN,$

Table 3.4: Word Lengths for three-level RPDs based on Bingham and Sitter's definition of wordlength when all the factors are quantitative

Word length	Words
1.0	C, N
1.5	CN
2.0	CC, NN, C^2, N^2
2.5	CCN, CNN, C^2N, CN^2
3.0	$CCC, CCNN, C^2C, N^2N$
3.5	$CCCN, CNNN, C^2N^2, C^2CN, CCN^2, C^2NN, CN^2N$
4.0	$CCCC, NNN, CCCNN, CCNNN, C^2CC, C^2CNN, CCN^2N$
4.5	$CCCCN, CNNNN, C^2C^2N, C^2CN^2, C^2N^2N, CN^2N^2$
	$C^2NNN, CN^2NN, C^2CCN, CCCN^2$
5.0	$CCCCC, NNNN, CCCNNN, CCNNNN, CCCCNN, C^2C^2C$
	$C^2C^2NN, C^2CN^2N, CCN^2N^2, C^2CCC$

However, in many practical experiments, the words involved in designs are more complicated. They can be mixed-type words which contain both qualitative and quantitative factors. For calculating these type of words in RPDs, we define the the length of a mixed-type word in following:

Definition 3.2.3 *The length of a mixed-type word $C_{\boldsymbol{t}}(\boldsymbol{x}) = X_1^{t_1}X_2^{t_2}\cdots X_k^{t_k}$ is defined as $||C_{\boldsymbol{t}}(\boldsymbol{x})|| =$ the length of $\boldsymbol{W}+\sum_{j\in S} I(g_i=1)(t_j-1)$, that is*

$$||C_{\boldsymbol{t}}(\boldsymbol{x})|| = ||C_{\boldsymbol{t}}(\boldsymbol{x})||_0 + \sum_{j\in S} I(g_i=1)(t_j-1), \tag{3.4}$$

where

$$\mathrm{I}(\cdot) = \begin{cases} 1 & \textit{if ``}\cdot\textit{'' is true} \\ 0 & \textit{otherwise;} \end{cases}$$

and

$$g_i = \begin{cases} 1 & \textit{if } X_i \textit{ is a quantitative factor} \\ 0 & \textit{if } X_i \textit{ is a qualitative factor.} \end{cases}$$

One can see that $||C_{\mathbf{t}}(\mathbf{x})||$ reduces to $||C_{\mathbf{t}}(\mathbf{x})||_0$ when all the factors in $C_{\mathbf{t}}(\mathbf{x})$ are qualitative and to $||C_{\mathbf{t}}(\mathbf{x})||_1$ when all the factors in $C_{\mathbf{t}}(\mathbf{x})$ are quantitative.

For example, consider a word CC^2NN^4. Suppose all the control factors are qualitative and all the noise factors are quantitative. The length of the word $\mathbf{W} = CCNN$ is 2.5 in Table 2.4 or 3.0 in Table 2.5. Therefore, the length of $C_{\mathbf{t}}(\mathbf{x})$ is $||C_{\mathbf{t}}(\mathbf{x})|| = 2.5 + (0 + 0 + 1 \times (2-1) + 1 \times (4-1)) = 6.5$ on the basis of Zhu's definition, or $||C_{\mathbf{t}}(\mathbf{x})|| = 3.0 + 1 + 3 = 7$ on the basis of formula 3.3.

3.3 Generalized Wordlength Pattern And Minimum Aberration

By modifying Definition 2.2.3, we define the *generalized word length pattern* of a mixed-level RPD as follows:

Definition 3.3.1 *Let $\mathcal{A}$ be a mixed-level fractional factorial RPD with indicator function $F_{\mathcal{A}}(\boldsymbol{x})$ in (3.1). Let $\beta_i(\mathcal{A}) = \sum_{L_i}(b_{J\cup K}/b_{\boldsymbol{0}})^2$, where L_i is the set of all $J \cup K$ such that the length of $C_{J\cup K}(\boldsymbol{x})$ equals to i in (3.4). Then the generalized wordlength pattern of $\mathcal{A}$ is $\mathcal{B}(\mathcal{A}) = (\beta_{1.0}(\mathcal{A}), \beta_{1.5}(\mathcal{A}), ...,)$.*

With this new definition of wordlength pattern, the usual minimum aberration criterion can be simply utilized for mixed-level fractional factorial RPDs.

Definition 3.3.2 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two different mixed-level fractional factorial RPDs with generalized WLPs $\mathcal{B}(\mathcal{A}_1)$ and $\mathcal{B}(\mathcal{A}_2)$, respectively. $\mathcal{A}_1$ is said to have less aberration if $\beta_s(\mathcal{A}_1) < \beta_s(\mathcal{A}_2)$ and s is the smallest number such that $\beta_s(\mathcal{A}_1) \neq \beta_s(\mathcal{A}_2)$. We say $\mathcal{A}_1$ is the minimum aberration mixed-level fractional factorial RPD if no other RPD has less aberration than $\mathcal{A}_1$.*

For better illustration, in the rest of this thesis, we denote $\mathcal{B}_1(\mathcal{A})$ and $\mathcal{B}_2(\mathcal{A})$ the generalized WLPs based on the lengths of words defined by Zhu (2000) and formula 3.3, respectively. The corresponding generalized minimum aberration criteria are called G_1 *minimum aberration* criterion and G_2 *minimum aberration* criterion, respectively.

3.4 An Illustrative Example

Consider a non-isomorphic mixed-level fractional factorial RPD $\mathcal{D}_2$ with its columns 1-3, and 7 selected from Table 2.2. Assume the first three columns X_1, X_2, and X_3 of $\mathcal{D}_2$ are control factors and the fourth column X_7 represents a noise factor. Moreover, we assume only X_1 is qualitative and other factors are quantitative in Table 2.2.

The indicator function of $\mathcal{D}_2$ is

$$\begin{aligned} F_{\mathcal{D}_2}(\mathbf{x}) = \frac{1}{3} &+ 0.204C_{1111}(\mathbf{x}) + 0.118C_{0211}(\mathbf{x}) + 0.118C_{1121}(\mathbf{x}) - 0.204C_{0221}(\mathbf{x}) \\ &- 0.118c_{1112}(\mathbf{x}) + 0.204C_{0212}(\mathbf{x}) + 0.204C_{1122}(\mathbf{x}) + 0.118C_{0222}(\mathbf{x}). \end{aligned} \tag{3.5}$$

Assume the wordlength defined based on Bingham and Sitter (2003) is used. To illustrate how to obtain the generalized WLP, we list all the coefficients of the indicator function and the wordlengths of all the corresponding words in Table 3.5.

Table 3.5: The results of $\mathcal{D}_2$ in Example 3.2

54 coefficients	($\frac{1}{3}$ 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0.204 0.118 0 0 0 0 0.118 -0.204 0 0 0 0 0 0 0 0 0 0 -0.118 0.204 0 0 0 0 0.204 0.118 0)
The length of each words	(0 1.0 1.0 2.0 2.0 3.0 1.0 2.0 2.0 3.0 3.0 4.0 2.0 3.0 3.0 4.0 4.0 5.0 1.0 1.5 1.5 2.5 2.5 3.5 1.5 2.5 2.5 3.5 3.5 4.5 2.5 3.5 3.5 4.5 4.5 5.5 2.0 2.5 2.5 3.5 3.5 4.5 2.5 3.5 3.5 4.5 4.5 5.5 3.5 4.5 4.5 5.5 5.5 6.5)

The first wordlength, "0", implies the length of C_{0000} is 0. Now we consider calculating $\beta_{1.0}(\mathcal{D}_2)$. Totally, there are 4 words whose length is "1". The corresponding words are the 2nd term, the 3rd term, the 7th term and the 19th term. Since their coefficients are "0" as shown in Table 3.5, by Definition 3.3.1, $\beta_{1.0}(\mathcal{D}_2) = (\frac{0}{1/3})^2 + (\frac{0}{1/3})^2 + (\frac{0}{1/3})^2 + (\frac{0}{1/3})^2 = 0$. Do like wise for the rest $\beta_i(\mathcal{D}_2)$, where $i = 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0, 6.5$, we can get the generalized WLP $\mathcal{B}_2(\mathcal{D}_2)$ =(0, 0, 0, 0, 0, 0.5 0, 1, 0, 0.5, 0).

Now, consider the $\mathcal{D}_1$ in Example 3.1. The wordlength defined based on Bingham and Sitter (2003) is used. Under the same assumption that only X_1 is qualitative and other factors are quantitative, we then obtain its generalized WLP: $\mathcal{B}_2(\mathcal{D}_1)$ =(0, 0, 0, 0.34375, 0, 0.35416, 0, 0.625, 0, 0.64583, 0.03125). As can be seen, $\mathcal{D}_2$ has less generalized aberration than $\mathcal{D}_1$.

To further illustrate, consider assigning a control factor to the column 2 and the noise factors to other three columns. Then the generalized WLP for $\mathcal{D}_1$ is (0, 0, 0, 0.09375, 0, 0.09375, 0, 0.375, 0.25, 0.125, 0.1667, 0.61458, 0.25, 0.03125), whereas the generalized WLP for $\mathcal{D}_2$ becomes (0, 0, 0, 0, 0, 0.125, 0, 1.125, 0, 0.375, 0, 0.375, 0, 0). Obviously, different way to assign the columns to control factors and noise factors can generate designs with different structures.

Chapter 4

Optimal Robust Parameter Designs

In this chapter, we apply the construction method and the generalized G_1 and G_2 minimum aberration optimality criteria introduced in Chapter 3 to obtain a catalogue of optimal mixed-level robust parameter designs with 18 runs.

We use a computer program to calculate the wordlengths and generalized wordlength patterns for each choice of the robust parameter designs with the given number of control factors and noise factors. Then search over all the class of non-isomorphism designs constructed from Table 2.2 and select the best one that has generalized minimum aberration. We assume that in Table 2.2, the factor with two levels, X_1, is qualitative and the rest seven factors with three levels are quantitative.

In section 4.1 and section 4.2, we list optimal 18-run fractional factorial RPDs according to the corresponding G_1 and G_2 minimum aberration criteria as defined in Chapter 3, respectively. In section 4.3, two illustrative examples are employed to help understand the catalogue and some results are presented.

4.1 Optimal G_1 Minimum Aberration 18-run Robust Parameter Designs

Recall that in section 3.2, we defined the wordlength for mixed-level RPDs based on the wordlength defined by Zhu (2000) and Bingham and Sitter (2003), respectively. Then we defined the corresponding wordlength patterns, $\mathcal{B}_1(\mathcal{A})$ and $\mathcal{B}_2(\mathcal{A})$, and generalized G_1 and G_2 minimum aberration. In this section, we search optimal mixed-level robust parameter designs using G_1 minimum aberration criterion.

Table 4.1 presents a complete catalogue of optimal RPDs selected from Table 2.2 with only three-level factors. Table 4.2 and Table 4.3 list optimal RPDs chosen from Table 2.2 with the inclusion of the two-level factor X_1. It is assumed that X_1 is a control factor in Table 4.2 and a noise factor in Table 4.3.

In each table, columns 1-3 list the number of factors k, the number of control factors k_1, and the number of noise factors k_2, respectively. For each choice of k_1 and k_2, there are usually more than one geometrically isomorphic optimal RPDs. To save space, only one optimal design is shown as an example in column 4, where the bold numbers represent the columns of control factors. In column 5, we present the generalized wordlength patterns of the corresponding optimal designs.

For example, in Table 4.1, when $k = 4$, $k_1 = 1$ and $k_2 = 3$, $\{2, \grave{4}, 5, \mathbf{8}\}$ in column 4 means that the optimal design is constructed from columns 2, 4, 5 and 8 with column 8 as a control factor and columns 2, 4 and 5 as noise factors. Among these, column 4 is changed by the permutation rule II. The generalized wordlength pattern of the optimal design is $\mathcal{B}_1(\mathcal{A}) = (0, 1.125, 1.5, 0.375, 0.5, 0)$, which is listed in column 5.

Table 4.1: Optimal G_1 minimum aberration 18-Run RPDs selected from Table 2.2 with only three-level factors

k	k_1,k_2	**Columns**	$\beta_1(2.0)$, $\beta_1(2.5)$,..., $\beta_1(5.0)$
3	1,2	$\{\mathbf{2},3,7\}$	(0.01852,0,0,0.1713,0,0.75926,0)
	2,1	$\{\mathbf{2},\mathbf{3},7\}$	(0.01852,0,0.1713,0,0.75926,0,0.125)
4	1,3	$\{3,\grave{4},5,\mathbf{6}\}$	(0.02778,0.0625,0.02083,1.10243,0,0.84375,0)
	2,2	$\{\mathbf{2},\grave{3},4,\mathbf{5}\}$	(0.05208,0.0434,0.80671,2.02315,0.01273,0.05671,0.29)
	3,1	$\{\mathbf{2},\grave{\mathbf{4}},\mathbf{5},6\}$	(0.04282,0,1.20023,0,1.54282,0,0.64468)
5	1,4	$\{\grave{3},\grave{4},\grave{5},\mathbf{6},7\}$	(0.03241,0.15625,0.02778,2.50694,0,3.54948,0)
	2,3	$\{\hat{2},\hat{3},\mathbf{4},5,\mathbf{8}\}$	(0.08333,0.427,0.81,3.5,0.778,2.51,0.666)
	3,2	$\{\grave{2},\grave{\mathbf{4}},\grave{5},\mathbf{6},\mathbf{7}\}$	(0,0.25,3.1875,2.25,0.625,0.75,2.25)
	4,1	$\{\grave{\mathbf{2}},\grave{\mathbf{4}},\grave{\mathbf{5}},\grave{\mathbf{6}},7\}$	(0.09606,0,3.16146,0,2.78472,0,4.05556)
6	1,5	$\{\hat{2},\hat{3},\hat{\mathbf{4}},5,6,7\}$	(0.18751,6.09375,5.90625,8.375,7.625,6.28125,3.28125)
	2,4	$\{\hat{2},\hat{3},\hat{\mathbf{4}},\mathbf{5},6,7\}$	(0,0.9375,2.625,6.1875,0,6,0.875)
	3,3	$\{\hat{\mathbf{2}},\hat{3},\hat{\mathbf{4}},\mathbf{5},6,7\}$	(0,2.25,4.125,3.375,1.875,9,2.25)
	4,2	$\{\grave{2},\grave{\mathbf{4}},\grave{\mathbf{5}},\mathbf{6},\mathbf{7},8\}$	(0.375,0.53125,5.5,2.125,5.9375,2.4375,8.73437)
	5,1	$\{\hat{\mathbf{2}},\hat{\mathbf{3}},\hat{\mathbf{4}},5,\mathbf{6},\mathbf{7}\}$	(0.1875,6.1,5.9,8.375,7.625,6.28125,3.28125)
7	1,6	$\{2,3,\grave{\mathbf{4}},5,\grave{6},\grave{7},8\}$	(0.375,9.1875,13.125,19.5,20.6875,22.4,15.1875)
	2,5	$\{2,3,\grave{\mathbf{4}},\mathbf{5},\grave{6},\grave{7},8\}$	(0,1.875,3,10.875,0,15.375,1)
	3,4	$\{\grave{\mathbf{2}},\grave{\mathbf{3}},4,5,\mathbf{6},7,8\}$	(0.5625,2.1,4.211,11.98,5.367,16.539,5.766)
	4,3	$\{\mathbf{2},\grave{\mathbf{3}},\grave{\mathbf{4}},5,6,7,\mathbf{8}\}$	(0.84375,2.046875,7.375,7.394,10.756,15.25,13.8)
	5,2	$\{\grave{\mathbf{2}},\grave{\mathbf{3}},\grave{4},\grave{5},\mathbf{6},\mathbf{7},\mathbf{8}\}$	(0.75,0.75,8.625,4.3125,13.5,4.6875,21.625)
	6,1	$\{\mathbf{2},\mathbf{3},\grave{\mathbf{4}},5,\grave{\mathbf{6}},\grave{\mathbf{7}},\mathbf{8}\}$	(0.375,9.1875,13.125,19.5,20.6875,22.40625,15.1875)

Table 4.2: Optimal G_1 minimum aberration 18-Run RPDs selected from Table 2.2 with X_1 as a control factor

k	k_1,k_2	**Columns**	$\beta_1(2.0)$, $\beta_1(2.5)$,..., $\beta_1(5.0)$
3	1,2	$\{\mathbf{1},3,6\}$	(0,0,0,0.03704,0,0)
	2,1	$\{\mathbf{1},\mathbf{2},3\}$	(0.02778,0,0.05093,0,0.02778)
4	1,3	$\{\grave{\mathbf{1}},2,4,5\}$	(0,0,0,0.18519,0,1.5,0)
	2,2	$\{\mathbf{1},\grave{2},3,\mathbf{6}\}$	(0.03935,0.1875,0.0625,0.95602,0.03472,0.58102,0)
	3,4	$\{\mathbf{1},2,\mathbf{3},\mathbf{8}\}$	(0.05556,0,0.49537,0,1.15278,0,0.49537)
5	1,4	$\{\grave{\mathbf{1}},2,3,6,7\}$	(0,0,0,0.97917,0,4.3125,0)
	2,3	$\{\mathbf{1},\grave{2},\mathbf{4},\grave{5},8\}$	(0.02315,0.12,0.5765,0.088,2.8,0.02778,3.37731)
	3,5	$\{\mathbf{1},\hat{\mathbf{2}},\mathbf{3},6,8\}$	(0.09606,0.75,1.75463,0.7338,0.83333,1.6412,2.17)

Table 4.2 (Continued)

k	k_1,k_2	**Columns**	$\beta_1(2.0)$, $\beta_1(2.5)$,..., $\beta_1(5.0)$
	4,1	$\{\mathbf{1},\hat{2},\mathbf{3},\mathbf{4},\mathbf{8}\}$	(0.09028,0,2.29861,0,1.90972,0,3.84028)
6	1,5	$\{\mathbf{1},2,\hat{3},\grave{5},6,7\}$	(0,4,6.0625,5.75,4.125,3.5,2.3125)
	2,4	$\{\mathbf{1},\grave{3},\mathbf{5},\grave{6},7,8\}$	(0,0.875,2,4.5625,0,5,6.625)
	3,3	$\{\mathbf{1},\grave{2},\grave{\mathbf{3}},\mathbf{4},5,6\}$	(0,1.2708,3.2292,4.7292,1.2292,4.9895,1.5208)
	4,2	$\{\mathbf{1},\mathbf{2},\grave{4},5,\grave{\mathbf{6}},\mathbf{8}\}$	(0,0.97917,5.6666,2.25,2,1.625,5.8333)
	5,1	$\{\grave{\mathbf{1}},2,\hat{\mathbf{4}},\mathbf{5},\mathbf{6},\mathbf{7}\}$	(0,5,5.89583,5.91667,3.70833,3.91667,1.4)
7	1,6	$\{\mathbf{1},\hat{2},3,4,5,6,7\}$	(0.5,5.81,12.094,14.19,15.1875,13.875,9.344)
	2,5	$\{\mathbf{1},\grave{2},\grave{3},4,\mathbf{5},\grave{6},7\}$	(0,1.625,2,9.4375,0,11.84375,17.57812)
	3,4	$\{\mathbf{1},\grave{2},\grave{\mathbf{3}},4,\mathbf{5},\grave{6},7\}$	(0,3.14583,4.625,9,1.70833)
	4,3	$\{\mathbf{1},\mathbf{2},\grave{4},\mathbf{5},\grave{\mathbf{6}},7,8\}$	(0.1875,2.93,8.78,4.5625,6.14,13.3177,8.7968)
	5,2	$\{\mathbf{1},\grave{\mathbf{2}},\grave{\mathbf{3}},\mathbf{4},5,6,\mathbf{7}\}$	(0.5625,1.10938,8.79427,2.79,12.065,4.427,13.852)
	6,1	$\{\mathbf{1},\mathbf{2},\hat{\mathbf{4}},\grave{5},\mathbf{6},\mathbf{7},\mathbf{8}\}$	(0.28125,8.589,10.4,15.516,14.6875,16.11,7.1875)
8	1,7	$\{\grave{\mathbf{1}},\hat{2},\grave{3},\hat{4},\grave{5},6,7,8\}$	(0.75,9.375,20.91,30.09375,37.633,43.453,39.2)
	2,6	$\{\mathbf{1},\grave{2},\grave{3},4,\mathbf{5},\grave{6},7,8\}$	(0,2.75,2,15.5625,0,24.14,39.2)
	3,5	$\{\mathbf{1},2,3,4,5,6,7,8\}$	(0,5.66676.20833,13.9583,2.45833,35.1875,2.958)
	4,4	$\{\mathbf{1},\hat{2},3,\hat{\mathbf{4}},5,6,\mathbf{7},\mathbf{8}\}$	(1.1,4.3,7.2448,17.3515,13.3411,29.426,14.95)
	5,3	$\{\mathbf{1},\hat{2},\mathbf{3},\hat{\mathbf{4}},\mathbf{5},\mathbf{6},7,8\}$	(1.188,3.88,11.145,10.467,20.66,20.1,30.1)
	6,2	$\{\mathbf{1},\grave{2},\grave{3},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7},\mathbf{8}\}$	(1.468,1.41,12.8281,4.75,24.29687,4.5625,38.28)
	7,1	$\{\mathbf{1},\mathbf{2},\grave{\mathbf{3}},\grave{\mathbf{4}},\mathbf{5},\grave{6},\mathbf{7},\mathbf{8}\}$	(0.56,12.34,20.4375,33.52,37.172,47.95,32.67)

Table 4.3: Optimal G_1 minimum aberration 18-Run RPDs selected from Table 2.2 with X_1 as a noise factor

k	k_1,k_2	**Columns**	$\beta_1(2.0)$, $\beta_1(2.5)$,..., $\beta_1(5.0)$
3	1,2	$\{1,\mathbf{2},3\}$	(0.00926,0,0,0.0463,0,0.00926,0)
	2,1	$\{\mathbf{1},2,\mathbf{3}\}$	(0.01389,0.00463,0.00926,0.02315,0.02778,0.02778)
4	1,3	$\{1,\mathbf{2},6,8\}$	(0.01852,0,0,0.26389,0,1.02778,0)
	2,2	$\{1,\mathbf{2},\mathbf{3},7\}$	(0.0185,0.037,0.1713,0.4491,0.76,0.2963,0.125)
	3,4	$\{1,\grave{\mathbf{2}},\mathbf{3},\mathbf{6}\}$	(0.028,0.00926,0.225,0.0648,0.993,0.055,0.63657)
5	1,4	$\{1,4,\grave{6},7,\mathbf{8}\}$	(0.02778,0.06713,0.02083,1.19271,0,2.58681,0)
	2,3	$\{1,\mathbf{2},\grave{4},5,\mathbf{8}\}$	(0.0521,0.08,0.81,3.00116,0.01273,0.71,0.295)
	3,5	$\{1,\hat{\mathbf{2}},\hat{\mathbf{4}},\mathbf{5},6\}$	(0.04282,0.037,1.2,0.87153,1.54282,1.2,0.68171)
	4,1	$\{1,\grave{\mathbf{2}},\grave{\mathbf{3}},\mathbf{6},\mathbf{7}\}$	(0.05556,0,0.05556,0.8125,0.16667,4.3125,0)
6	1,5	$\{1,2,\grave{4},5,\grave{6},\mathbf{8}\}$	(0,3.4375,4.5,5.375,5.25,3.6875,2.5)
	2,4	$\{1,\hat{2},\hat{\mathbf{4}},\mathbf{5},6,7\}$	(0,0.1875,2.25,3.42708,0,4.5625,0.75)

Table 4.3 (Continued)

k	k_1,k_2	**Columns**	$\beta_1(2.0)$, $\beta_1(2.5)$,..., $\beta_1(5.0)$
	3,3	$\{1,2,\grave{\mathbf{4}},5,\grave{\mathbf{6}},\mathbf{8}\}$	(0,0.25,3.22916,4.7916,0.625,2.5,2.375)
	4,2	$\{1,\mathbf{2},\grave{\mathbf{4}},\mathbf{5},\grave{\mathbf{6}},8\}$	(0,0,4.2291,2.54166,2.625,1.0833,4.625)
	5,1	$\{1,\grave{\mathbf{2}},\grave{\mathbf{4}},\grave{\mathbf{5}},\mathbf{6},\mathbf{7}\}$	(1,0.1666,0.6666,9.0625,1,2.25,7.375)
7	1,6	$\{1,2,3,\grave{\mathbf{4}},5,\text{‘}6,8\}$	(0.1875,6.344,8.26,12.74,14.78125,13.72,12.4)
	2,5	$\{1,2,3,\grave{\mathbf{4}},\mathbf{5},\grave{6},8\}$	(0,0.9375,2.625,7.14583,0,9.5416,0.875)
	3,4	$\{1,\grave{\mathbf{2}},\hat{3},\grave{\mathbf{4}},\grave{\mathbf{5}},6,7\}$	(0,2.25,4.125,5.75,1.875,13.1875,2.25)
	4,3	$\{1,2,\grave{\mathbf{4}},\mathbf{5},\grave{\mathbf{6}},7,\mathbf{8}\}$	(0.375,0.53125,6,6.75,5.9375,6.10937,10.734375)
	5,2	$\{1,\mathbf{2},\mathbf{3},\grave{\mathbf{4}},5,\grave{\mathbf{6}},\mathbf{8}\}$	(0.1875,0.25,6.94792,4.21875,7.5,3.47917,13.9375)
	6,1	$\{1,\grave{\mathbf{2}},\grave{\mathbf{4}},\grave{\mathbf{5}},\mathbf{6},\mathbf{7},\mathbf{8}\}$	(1.5,1.5,1,15.125,1.5,10.09375,19.40625)
8	1,7	$\{1,2,3,\grave{\mathbf{4}},5,\grave{6},\grave{7},8\}$	(0.375,9.6875,16,26.5625,34.75,40.65,38.0625)
	2,6	$\{1,\grave{2},\grave{3},\grave{\mathbf{4}},\grave{5},6,\mathbf{7},8\}$	(0,2.875,1.875,10.895,0,22.927,0.625,35.6458)
	3,5	$\{1,\grave{\mathbf{2}},\grave{\mathbf{3}},\grave{4},5,6,\mathbf{7},8\}$	(0.46875,2.9375,4.82,11.6289,5.2421)
	4,4	$\{1,\hat{\mathbf{2}},3,\hat{\mathbf{4}},5,\mathbf{6},\mathbf{7},8\}$	(0.84375,2.484375,7.73698,10.6549,12.123)
	5,3	$\{1,\grave{\mathbf{2}},\grave{\mathbf{3}},\grave{4},\grave{5},\mathbf{6},\mathbf{7},\mathbf{8}\}$	(0.4687,2.9375,4.82,11.6289,5.2422,27.21,5.6718)
	6,2	$\{1,\mathbf{2},\mathbf{3},\grave{\mathbf{4}},5,\grave{\mathbf{6}},\grave{\mathbf{7}},\mathbf{8}\}$	(0.375,0.5,10.5625,5.9375,18.8125,6.375,32.5,9.5)
	7,1	$\{1,\grave{\mathbf{2}},\grave{\mathbf{3}},\grave{\mathbf{4}},\grave{\mathbf{5}},\mathbf{6},\mathbf{7},\mathbf{8}\}$	(2.25,3,1.5,25.3125,2.25,22.3125,50.64)

4.2 Optimal G_2 Minimum Aberration 18-run Robust Parameter Designs

In this section, G_2 minimum Aberration mixed-level RPDs in 18 runs are obtained and tabulated.

Table 4.4 lists the G_2 minimum aberration designs selected from Table 2.2 with only three-level factors. Table 4.5 and 4.6 present the G_2 optimal minimum aberration RPDs constructed from Table 2.2 with the inclusion of the two-level factor X_1. In particular, in Table 4.5, X_1 is assumed to be a control factor and in Table 4.6, X_1 is considered as a noise factor. Similar to the tables in section 4.1, Tables 4.4-4.6, columns 1-5 show the number of factors (k), the number of control factors (k_1), the number of noise factors (k_2), an example of optimal designs, and the corresponding generalized wordlength pattern, respectively.

Table 4.4: Optimal G_2 minimum aberration 18-Run RPDs selected from Table 2.2 with only three-level factors

k	k_1,k_2	**Columns**	$\beta_2(2.5)$**,** $\beta_2(3.0)$**,...,** $\beta_2(5.0)$
3	1,2	$\{2,\mathbf{3},8\}$	(0.01389,0.00926,0.38194,0.02778,0.00926,0)
	2,1	$\{\mathbf{2},\mathbf{3},8\}$	(0.01389,0.00926,0.38194,0.02778,0.00926,0)
4	1,3	$\{2,\grave{3},4,\mathbf{8}\}$	(0.02778,0.04167,1.43056,0.00694,0.08333,0.30556)
	2,2	$\{\grave{\mathbf{2}},\grave{\mathbf{3}},4,5\}$	(0.03819,0.04167,2.37153,0.02778,0.05903,0.02083)
	3,1	$\{\mathbf{2},\grave{\mathbf{3}},\mathbf{4},8\}$	(0.02778,0.04167,1.43056,0.3125,0.08333,0.02778)
5	1,4	$\{\grave{2},\grave{4},\grave{5},\mathbf{6},7\}$	(0.03704,0.08333,4.13889,0.01389,0.16667,2.11806)
	2,3	$\{\grave{\mathbf{2}},\grave{\mathbf{4}},\grave{5},6,7\}$	(0.06481,0.86806,5.02778,0.05556,0.24074,1.86111)
	3,2	$\{\grave{2},\grave{\mathbf{4}},\grave{5},\mathbf{6},\mathbf{7}\}$	(0.06481,0.86806,5.02778,0.43056,0.24074,1.52315)
	4,1	$\{\grave{\mathbf{2}},\grave{\mathbf{4}},\grave{\mathbf{5}},\grave{\mathbf{6}},7\}$	(0.03704,0.08333,4.13889,2.13194,0.16667,0.11111)
6	1,5	$\{\hat{2},\hat{3},\hat{\mathbf{4}},5,6,7\}$	(0.1875, 5.15625, 0.9375, 2.8125, 3.09375, 6.312)
	2,4	$\{\hat{2},\hat{3},\hat{\mathbf{4}},\mathbf{5},6,7\}$	(0.375,0.5625,6.5625,2.25,3.375,7.25,3)
	3,3	$\{\grave{\mathbf{3}},\grave{\mathbf{4}},\grave{\mathbf{5}},6,7,8\}$	(0.5625,1.78125,5.0625,2.4375,3.9375,6.5625,6.1875)
	4,2	$\{\hat{\mathbf{2}},\hat{\mathbf{3}},\hat{4},5,\mathbf{6},\mathbf{7}\}$	(0.375,1.3125,6.5625,2.625,2.625,3,7.25,3.125)
	5,1	$\{\hat{\mathbf{2}},\hat{\mathbf{3}},\hat{\mathbf{4}},5,\mathbf{6},\mathbf{7}\}$	(0.1875, 0.9375, 5.15625, 3.09375, 2.8125, 2.0625)
7	1,6	$\{2,3,\grave{\mathbf{4}},5,\grave{6},\grave{7},8\}$	(0.375,7.3125,1.875,6.375,6.75,14.625,4.875,9.75)
	2,5	$\{2,\grave{\mathbf{3}},\grave{4},5,6,7,\mathbf{8}\}$	(1.3125,0.78125,7.73438,3.98,10.664,9.37,12.1875)
	3,4	$\{2,\grave{\mathbf{3}},\grave{\mathbf{4}},5,6,7,\mathbf{8}\}$	(1.6875,0.82813,9.765625,6.4,10.148,12.043,8.38281)
	4,3	$\{\mathbf{2},\grave{3},\grave{4},\mathbf{5},\mathbf{6},\mathbf{7},8\}$	(1.6875,1.48437,9.7656,5.789,10.148,13.4,8.383)
	5,2	$\{\mathbf{2},\grave{3},\grave{\mathbf{4}},\mathbf{5},\mathbf{6},\mathbf{7},8\}$	(1.3125,1.8125,7.734375,5.8281,10.664,9.8,12.1875)
	6,1	$\{\mathbf{2},\mathbf{3},\grave{\mathbf{4}},5,\grave{\mathbf{6}},\grave{\mathbf{7}},\mathbf{8}\}$	(0.375,1.875,7.3125,6.75,6.375,4.875,14.625,10.9375)

Table 4.5: Optimal G_2 minimum aberration 18-Run RPDs selected from Table 2.2 with X_1 as a control factor

k	k_1,k_2	**Columns**	$\beta_2(2.5)$**,** $\beta_2(3.0)$**,...,** $\beta_2(5.0)$
3	1,2	$\{\mathbf{1},3,6\}$	(0,0.01852,0.01852,0,0)
	2,1	$\{\mathbf{1},\mathbf{2},3\}$	(0.00926,0.01389,0.03704,0,0.02778)
4	1,3	$\{\mathbf{1},\grave{3},6,7\}$	(0,0.055,0.066,0.26,0.84375,0.09375)
	2,2	$\{\mathbf{1},\grave{\mathbf{2}},6,7\}$	(0.02083,0.42593,0.1412,0.26389,0.794,0.384)
	3,1	$\{\mathbf{1},\mathbf{2},\mathbf{6},7\}$	(0.00926,0,0.03704,0.01852,0.0463,0.57407)

Table 4.5 (Continued)

k	k_1,k_2	**Columns**	$\beta_2(2.5)$, $\beta_2(3.0)$,..., $\beta_2(5.0)$
5	1,4	$\{\mathbf{1},\grave{2},3,6,7\}$	(0.0417,0.0833,0.6458,0.6319,1.6875,1.118)
	2,3	$\{\mathbf{1},\grave{2},\mathbf{3},\grave{4},8\}$	(0.04282,0.8669,2.80208,0.06597,0.08449,2.29282)
	3,2	$\{\mathbf{1},\mathbf{2},4,\hat{5},\mathbf{8}\}$	(0.01389,0.00463,0.05787,0.06944,0.79861,3.39815)
	4,1	$\{\mathbf{1},\mathbf{2},\grave{\mathbf{3}},\mathbf{4},8\}$	(0.02778,0.09028,2.20833,1.72917,0.18056,0.048)
6	1,5	$\{\mathbf{1},2,3,\grave{6},\grave{7},8\}$	(0,2.5,1.5,2.5,3.5625,4.25)
	2,4	$\{\mathbf{1},\grave{2},\grave{3},\mathbf{4},5,6\}$	(0.46875,2.9375,4.82,11.6289,5.2421,27.208)
	3,3	$\{\mathbf{1},\mathbf{2},\grave{4},5,\grave{\mathbf{6}},8\}$	(1.375,8.3125,0,0,5.5,4.625)
	4,2	$\{\mathbf{1},\mathbf{2},\grave{4},5,\grave{\mathbf{6}},\mathbf{8}\}$	(0,0.97916,5.6666,2.25,2,1.625)
	5,1	$\{\mathbf{1},\mathbf{2},\grave{\mathbf{4}},\mathbf{5},\grave{\mathbf{6}},8\}$	(0,0,5.3125,4.75,0,0)
7	1,6	$\{\mathbf{1},\hat{2},3,\hat{4},5,\grave{6},8\}$	(0.5,3.84375,1.96875,6.79687,5.29687,5.79687)
	2,5	$\{\mathbf{1},2,3,\grave{\mathbf{4}},5,\grave{6},8\}$	(0.6875,1.27,9.01,2.0313,4.91,8.4)
	3,4	$\{\mathbf{1},2,3,\grave{\mathbf{4}},\mathbf{5},\grave{6},8\}$	(0.875,2.1875,10.0625,4.1875,3.875,10,10)
	4,3	$\{\mathbf{1},2,\grave{\mathbf{4}},5,\grave{6},\mathbf{7},\mathbf{8}\}$	(0.75,2.40625,11.1875,3.65625,4,11.2708)
	5,2	$\{\mathbf{1},2,\grave{\mathbf{4}},5,\grave{\mathbf{6}},\mathbf{7},\mathbf{8}\}$	(0.5625,1.85938,10.33854,3.84896,4.64,7.2)
	6,1	$\{\mathbf{1},\mathbf{2},\grave{\mathbf{4}},5,\grave{\mathbf{6}},\mathbf{7},\mathbf{8}\}$	(0.28125,0.96875,7.62,6.6,3.796875,3.61)
8	1,7	$\{\mathbf{1},\hat{2},3,\hat{4},5,\grave{6},7,8\}$	(0.75,6.5625,2.8125,10.64,10.266,11.95312)
	2,6	$\{\mathbf{1},\hat{2},3,\hat{\mathbf{4}},5,6,7,8\}$	(1.91,0.91,10.625,6,15.289,11.13)
	3,5	$\{\mathbf{1},2,\grave{\mathbf{3}},4,5,6,7,\mathbf{8}\}$	(2.5938,2.308,11.12,11.4323,15.07,18.155)
	4,4	$\{\mathbf{1},\grave{2},\grave{3},\mathbf{4},5,6,\mathbf{7},\mathbf{8}\}$	(2.72,2.6146,13.0573,12.148,15.1016,21.423)
	5,3	$\{\mathbf{1},\grave{2},\grave{3},\mathbf{4},5,\mathbf{6},\mathbf{7},\mathbf{8}\}$	(2.468,2.73,12.984,10.778,16.8,18.587)
	6,2	$\{\mathbf{1},2,\grave{\mathbf{3}},\grave{\mathbf{4}},\mathbf{5},\grave{6},\mathbf{7},\mathbf{8}\}$	(1.125,2.96875,15.09375,8.09375,11.03125,14.797)
	7,1	$\{\mathbf{1},\mathbf{2},\grave{\mathbf{3}},\grave{\mathbf{4}},\mathbf{5},\grave{6},\mathbf{7},\mathbf{8}\}$	(0.5625,1.9375,10.40625,12.09375,8.34,8.97)

Table 4.6: Optimal G_2 minimum aberration 18-Run RPDs selected from Table 2.2 with X_1 as a noise factor

k	k_1,k_2	**Columns**	$\beta_2(2.5)$, $\beta_2(3.0)$,..., $\beta_2(5.0)$
3	1,2	$\{1,\mathbf{3},6\}$	(0.0093,0,0.028,0.0046,0.00463)
	2,1	$\{1,\mathbf{2},\mathbf{3}\}$	(0.00463,0.00926,0.01389,0.03704,0,0.02778)
4	1,3	$\{1,\mathbf{2},6,7\}$	(0.01852,0.00926,0.403,0.042,0.02315,1.05)
	2,2	$\{1,\mathbf{2},\mathbf{3},7\}$	(0,0,0.125,0.375,0.75,0.25)
	3,1	$\{1,\mathbf{2},\mathbf{6},\mathbf{7}\}$	(0.375,0.125,0.625,0.875,0)
5	1,4	$\{1,\grave{2},\grave{3},4,\mathbf{8}\}$	(0.02778,0.04167,1.43,0.021,0.86111,0.35417)
	2,3	$\{\mathbf{1},\mathbf{2},\grave{4},5,6\}$	(0.0463,0.02778,2.73611,0.0463,0.361,0.972)
	3,2	$\{\mathbf{1},\hat{\mathbf{2}},3,4,\mathbf{8}\}$	(0.04167,0.03704,1.34722,1.26389,1.43981,0.13426)

Table 4.6 (Continued)

k	k_1,k_2	**Columns**	$\beta_2(2.5)$**,** $\beta_2(3.0)$**,...,** $\beta_2(5.0)$
	4,1	$\{1,\grave{\mathbf{2}},\grave{\mathbf{3}},\mathbf{4},\mathbf{8}\}$	(0.01389,0.07639,0.04167,3.9375,0,0.22917)
6	1,5	$\{1,2,\grave{4},5,\grave{6},\mathbf{8}\}$	(0,3.4375,0,1.875,2.625,3.25,2.125,4.375,0.875)
	2,4	$\{1,\mathbf{2},\grave{\mathbf{4}},5,\grave{6},8\}$	(0,0.25,5.4375,0.75,1.875,1.375,3,3.375,3.625)
	3,3	$\{1,2,\grave{\mathbf{4}},5,\grave{\mathbf{6}},\mathbf{8}\}$	(0,0.25,5.4375,2.9583,1.41667,1,3,4.2916,1.5833)
	4,2	$\{1,\mathbf{2},\hat{\mathbf{4}},\grave{5},\mathbf{6},\mathbf{8}\}$	(0,0,3.4375,4.625,2,0,3.25,6.625,0,0,2.3125,3.5)
	5,1	$\{1,\mathbf{2},\hat{\mathbf{4}},\grave{\mathbf{5}},\mathbf{6},\mathbf{8}\}$	(0,10.0625,0,9.875,0,5.8125,0,0.25,0)
7	1,6	$\{1,2,3,\grave{\mathbf{4}},5,\grave{6},8\}$	(0.1875,5.40625,0.9375,4.92,3.3438,7.656)
	2,5	$\{1,\mathbf{2},3,\grave{\mathbf{4}},5,\grave{6},8\}$	(0.6875,1.2708,9.0104,2.0312,4.9062,8.4062)
	3,4	$\{1,\grave{\mathbf{3}},\grave{4},\grave{5},\mathbf{6},\mathbf{7},8\}$	(0.5625,1.78125,7.3125,2.8125,5.71875,8.25)
	4,3	$\{1,\grave{\mathbf{2}},\grave{\mathbf{4}},\grave{\mathbf{5}},6,7,\mathbf{8}\}$	(0.4688,1.3438,7.375,3.375,4.1875,10.15625)
	5,2	$\{1,\mathbf{2},\mathbf{3},\grave{\mathbf{4}},5,\grave{\mathbf{6}},\mathbf{8}\}$	(0.1875,1.1875,5.40625,7.38541,3.645833,5)
	6,1	$\{1,\grave{\mathbf{2}},\grave{\mathbf{3}},\grave{\mathbf{4}},\mathbf{5},\mathbf{7},\mathbf{8}\}$	(2.3125,10.07812,13.42187,15.2539,14.543)
8	1,7	$\{1,\grave{2},\grave{\mathbf{3}},4,5,6,7,8\}$	(0.65625,5.64,1.6875,12.72,6.578,13.69,10.9)
	2,6	$\{1,\grave{2},\grave{3},\grave{\mathbf{4}},\grave{\mathbf{5}},6,7,8\}$	(0.938,0.5625,11.625,3,8.688,13.75,23.1)
	3,5	$\{1,\grave{2},\grave{\mathbf{3}},\grave{4},\grave{5},\mathbf{6},\mathbf{7},8\}$	(1.125,2.4375,12.65625,4.5,9.46875,20.0625)
	4,4	$\{1,\grave{2},\grave{3},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7},8\}$	(2.71875,2.6145,13.05729,12.1484,15.1015)
	5,3	$\{1,\grave{\mathbf{2}},\grave{\mathbf{3}},\grave{4},\grave{5},\mathbf{6},\mathbf{7},\mathbf{8}\}$	(0.9375,1.875,11.625,8,8.125,20.625,18.5937)
	6,2	$\{1,\mathbf{2},\mathbf{3},\grave{\mathbf{4}},5,\grave{\mathbf{6}},\grave{\mathbf{7}},\mathbf{8}\}$	(0.375,0.5,10.5625,5.9375,18.8125,6.375,32.5)
	7,1	$\{1,\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7},\mathbf{8}\}$	(5.0625,13.36,28.75781,36.484,35.34375,41.1)

4.3 Two Illustrative Examples and Results

When we plan an experiment which involves several control factors and noise factors that are either qualitative or quantitative with each at two or three levels, we may choose an optimal design directly from Tables 4.1-4.6 to satisfy our needs. Now two examples is applied to gain further insight of these optimal design tables.

Example 4.1

Consider a mixed-level fractional factorial RPD $\mathcal{A}$ with one factor assigned to column 1 of Table 2.2 and other four factors assigned to the four columns selected from columns 2-8 of Table 2.2. Assume $\mathcal{A}$ has two control factors and two noise factors, that is $k_1 = 2$ and $k_2 = 2$. Totally, we obtain 4 optimal RPDs using G_1 minimum aberration criterion by searching among the entire class of non-isomorphic RPDs with X_1 as one of the control factors, and 18 optimal RPDs for the case that X_1 is a noise factor. All the optimal RPDs are presented in Table 4.7 and Table 4.8, respectively.

Table 4.7: Optimal G_1 Minimum Aberration RPDs when $k_1 = 2$, $k_2 = 2$ and X_1 is a control factor

Columns	Generalized Word Length Pattern $\mathcal{B}_1(\mathcal{A})$
$\{\mathbf{1},\grave{2},3,\mathbf{6}\}$	(0.03935, 0.1875, 0.0625, 0.95602, 0.03472, 0.58102, 0)
$\{\mathbf{1},\grave{2},3,\mathbf{7}\}$	(0.03935, 0.1875, 0.0625, 0.95602, 0.03472, 0.58102, 0)
$\{\mathbf{1},2,4,\grave{\mathbf{8}}\}$	(0.03935, 0.1875, 0.0625, 0.95602, 0.03472, 0.58102, 0)
$\{\mathbf{1},2,5,\grave{\mathbf{8}}\}$	(0.03935, 0.1875, 0.0625, 0.95602, 0.03472, 0.58102, 0)

Table 4.8: Optimal G_1 Minimum Aberration RPDs when $k_1 = 2$, $k_2 = 2$ and X_1 is a noise factor

Columns	Generalized Word Length Pattern $\mathcal{B}_1(\mathcal{A})$
$\{\mathbf{1},\hat{2},\hat{3},\mathbf{7}\}$	(0.01852,0.03704,0.1713,0.44907,0.75926,0.2963,0.125)
$\{\mathbf{1},2,3,\grave{\mathbf{6}}\}$	(0.01852,0.03704,0.1713,0.44907,0.75926,0.2963,0.125)

In column 1 of Tables 4.7 and 4.8, the optimal designs are listed, where the control factors are written in bold numbers. For example, $\{\mathbf{1},\grave{2},3,\mathbf{6}\}$ in Table 4.7 implies this optimal design is constructed by columns 1, 2, 3, and 6 in Table 2.2 with two control factors X_1 and X_6 and two noise factors X_2 and X_3. Among these factors, X_2 is changed by the level permutation rule II. The second column list the generalized wordlength pattern $\mathcal{B}_1(\mathcal{A})$ of the corresponding optimal designs.

Example 4.2

Consider the design $\mathcal{A}$ in Example 4.1. Now G_2 minimum aberration criterion is used instead of G_1 minimum aberration criterion. For the case that X_1 is one of the control factor, we obtain 2 optimal RPDs by searching among the entire class of non-isomorphic RPDs. For the case that X_1 is one of the noise factor, there are 2 optimal RPDs totally. All the optimal RPDs are presented in Table 4.9 and Table 4.10, respectively.

Table 4.9: Optimal G_2 Minimum Aberration RPDs when $k_1 = 2$, $k_2 = 2$ and X_1 is a control factor

Columns	Generalized Word Length Pattern $\mathcal{B}_2(\mathcal{A})$
$\{\mathbf{1},\grave{2},\grave{6},\mathbf{7}\}$	(0.02083, 0.42593, 0.1412, 0.26389, 0.79398, 0.38426)
$\{\mathbf{1},\hat{2},\hat{\mathbf{6}},7\}$	(0.02083, 0.42593, 0.1412, 0.26389, 0.79398, 0.38426)

Table 4.10: Optimal G_2 Minimum Aberration RPDs when $k_1 = 2$, $k_2 = 2$ and X_1 is a noise factor

Columns	Generalized Word Length Pattern $\mathcal{B}_2(\mathcal{A})$
$\{1,\grave{\mathbf{2}},\grave{\mathbf{6}},7\}$	(0.01852, 0.00926, 0.40278, 0.04167, 0.02315, 1.05556)
$\{1,\hat{\mathbf{2}},\hat{6},\mathbf{7}\}$	(0.01852, 0.00926, 0.40278, 0.04167, 0.02315, 1.05556)

The optimal designs are listed in the first column, where the control factors are written in bold numbers. In column 2, the generalized wordlength pattern $\mathcal{B}_2(\mathcal{A})$ of the corresponding optimal designs is presented.

Chapter 5

Conclusion

Fractional factorial designs are popular in areas of science and technology and used to identify the effect of factors on a process because they are more flexible and have smaller run sizes than full factorial designs. Some investigators proposed new designs build upon the work on detailed structure of fractional factorial designs. Robust parameter design is one of the representative examples. The main purpose of this project is to study mixed-level fractional factorial robust parameter designs and to introduce a new ranking principle of lengths of words that is helpful to select optimal mixed-level robust parameter designs.

In Chapter 2, we reviewed two-level full factorial designs, two-level regular and non-regular fractional factorial designs, mixed-level fractional factorial designs, and two-level robust parameter designs. In regular designs, generators in the defining relation play key roles in determining optimal designs. However, they are not available to choose good designs by generators in non-regular designs. With the development of indicator function, the concept of defining relation can be extended to non-regular designs, mixed-level fractional factorial designs and robust parameter designs. Furthermore, we introduced minimum aberration as an optimality criterion for ordering fractional factorial designs and two-level robust parameter designs.

Chapter 3 discussed the use of indicator function for a mixed-level robust parameter design and developed a new methodology for selecting optimal designs. We proposed a new definition for lengths of words when the difference of qualitative factors and quantitative factors and the difference of control factors and noise factors are taken into consideration. Moreover, we defined generalized wordlength patterns $\mathcal{B}_1(\mathcal{A})$ and $\mathcal{B}_2(\mathcal{A})$, and generalized G_1 and G_2 minimum aberration optimality criteria to mixed-level robust parameter designs.

Chapter 4 investigates optimal robust parameter designs using the generalized G_1 and G_2 minimum aberration criteria developed in Chapter 3. An optimal robust parameter design should have the minimum aberration among all the possible designs. We focused our efforts on searching for optimal mixed-level robust parameter designs of 18 runs from Table 2.2. We considered every possible combination of control and noise factors and every possible permutation assigned to the columns of Table 2.2. G_1 and G_2 minimum aberration robust parameter designs are searched among the set of all non-isomorphic fractional factorial mixed-level robust parameter designs. A catalogue of generalized minimum aberration fractional factorial robust parameter designs with 18 runs is tabulated. Lastly, we used two examples to illustrative this catalogue and find some rules between these tables.

Overall, while dealing with the important problem of selecting optimal mixed-level fractional factorial robust parameter designs, the thesis attempts to a framework for the study of robust parameter designs which involves two groups of factors.

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