METHODS OF SOLVING HEAT CONDUCTION PROBLEMS, WITH PARTICULAR REFERENCE TO FROZEN DAMS

by

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ABSTRACT

The various methods of solving heat conduction problems are thoroughly reviewed, followed by the application of several of these methods to solve a series of thermal problems associated with the design and construction of a potential Nelson River frozen dike. The solutions to these problems are presented in detail. The main results were obtained using a finite-difference approach with the numerical iterations carried out on a digital computer. General observations and conclusions on the frozen dam heat transfer study are given. The methods of solving heat conduction problems are summarized, and recommendations are made regarding selection and best use of the various methods. The important matter of formulating the actual problem is discussed.

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NOMENCLATURE

All symbols appearing in this thesis have been defined as used. The main symbols are listed below.

A	-	area, sq.ft.
C	-	heat capacity, Btu/^{o}F
C_p	-	specific heat, Btu/lb_m-^{o}F
e	-	natural logarithmic base
f	-	function
h	-	convective heat transfer coefficient, Btu/hr-ft^2-^{o}F
k	-	thermal conductuvity, Btu/hr-ft-^{o}F
K	-	conductance, Btu/hr-^{o}F
ℓ	-	length, ft.
ln	-	natural logarithm
ly	-	langley (gm cal/cm^2)
M	-	dimensionless modulus
m	-	meters
Q	-	heat, Btu
q	-	heat flow, Btu/hr
S	-	shape factor
t	-	temperature, ^{o}F
W	-	internal heat generation per unit time and volume

Greek Symbols

α	-	thermal diffusivity, ft^2/hr
Δ	-	a finite increment
δ	-	thickness of thermal layer, ft
ϵ	-	absorptivity (dimensionless)
θ	-	angle
τ	-	time, hr
Σ	-	summation
ϕ	-	angle
ω	-	angular velocity, rad/hr

Subscripts

a	-	air
av	-	average
eff	-	effective
F	-	frozen
m	-	mean
UF	-	unfrozen
w	-	water

METHODS OF SOLVING HEAT CONDUCTION PROBLEMS, WITH PARTICULAR REFERENCE TO FROZEN DAMS

I INTRODUCTION

This thesis primarily concerns methods of solving heat conduction problems, and is the result of an extensive study of heat conduction in a frozen dam. The presentation consists essentially of a review of methods of solving heat conduction problems, followed by the application of several of these methods to determine the thermal regime of a frozen dike.

The mathematical theory of heat conduction is outlined in Section II. This is prerequisite to the resume of the various methods of solving heat conduction problems which follows in Section III. For the purpose of presentation, the methods considered have been categorized as analytical, finite-difference, analog and experimental. Although basically academic in nature, this survey has considerable practical value since the emphasis in dealing with the various methods was not only to indicate the physical basis of each method, but also to discuss the practical aspects of application and utility.

The frozen dam thermal problem is introduced early in Section IV. The solution of the thermal regime was obtained through a simplified heat conduction model. The solutions to six specific problems associated with the design and construction of frozen dikes, are presented. The solutions to these problems have been used in a feasibility study of the hydro-electric power development of the lower Nelson River. Both analytical and analog methods were used to determine the steady state solutions. The finite-difference approach was used exclusively to solve the cyclic problems involving phase change. The nature and size of these periodic heat conduction problems required the use of a digital computer to carry out the numerical iterations. General observations and conclusions on the frozen dam heat transfer study are given.

The methods of solving heat conduction problems are summarized in Section V. Recommendations are included regarding selection and best use of the various methods. The important matter of suitable formulation of an actual heat conduction problem based on theoretical and practical considerations, is discussed.

II CONDUCTION HEAT TRANSFER THEORY

Heat transfer by conduction takes place in both solids and fluids providing temperature differences exist. For the case of fluids, heat transfer by convection and/or radiation will frequently occur simultaneous with conduction. The treatment of heat conduction as the sole means of heat transfer therefore implies the medium under consideration is or acts as a solid. Accordingly, heat conduction may be considered as a process of redistribution of internal energy in a medium without measurable displacement of mass. It is conventional to use the term 'heat flow' when referring to the actual physical process of transferring internal energy. This term used qualitatively is often accompanied by a statement as to the general direction of the 'flow'.

The mathematical theory of heat conduction treats matter as being continuous, and is based on a generalized macroscopic analysis of the transfer of energy in a solid. The fundamental law of heat conduction was deduced from a study of the results of experiments on the linear flow of heat through a slab perpendicular to the faces. The quantity of heat (Q) per unit area (A) transferred from one face to the other during an arbitrary period of time under conditions of steady temperature was found to be directly proportional to the thermal conductivity (k) of the slab material, the temperature difference (Δt) between the two faces, and the duration of the time interval ($\Delta\tau$), and inversely proportional to the distance between the two faces (Δx). This law was described by Biot in 1804, and was first used as a fundamental mathematical equation by Fourier in 1822 in his analytical theory of heat. In differential form, the fundamental equation of heat conduction is:

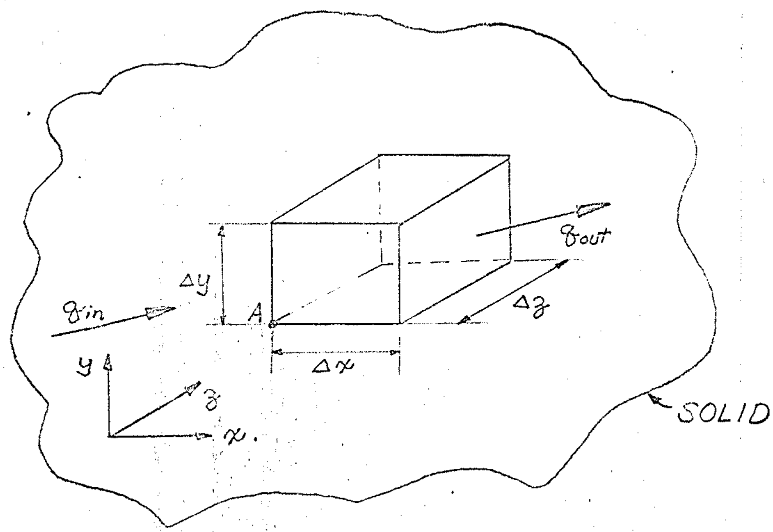
$$\frac{q}{A} = -k \frac{dt}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $q = dQ/d\tau$ is the heat flow (or rate of heat flow) corresponding to the instantaneous rate of transferring energy across a defined surface. The minus sign is required in keeping with a convention that dx be positive in the direction of heat flow, whereby dt must be negative since heat must flow in the direction of decreasing temperature. The derivative dt/dx is defined as the temperature gradient in the x-direction, or more generally as the gradient of the temperature field. The quantity q/A is called the heat flux.

Equation (1) is completely general for unidirectional heat flow, and simply states that the heat flux at any point at any instant is proportional to the temperature gradient at the point at the instant under consideration. The constant of proportionality is the thermal conductivity which is a property of the material. The equation applies to any point in the continuum, and at any point, dt/dx (or more specifically, dt) may vary with time. The heat flux at a point in a region is a vector whose direction indicates the direction of heat flow and whose magnitude corresponds to the quantity of heat per unit time crossing a unit area normal to the vector at the point.

2.1 General Heat Conduction Equation

The general heat conduction equation together with equation (1) constitute the basic mathematical relations involved in the theory of heat conduction. Whereas equation (1) gives the heat flux in terms of the thermal conductivity of the medium and the temperature gradient, the general heat conduction equation describes the dependence of temperature on the spatial coordinates and time in the presence of heat sources/sinks. In view of the importance of the general heat conduction equation, the derivation for a Cartesian coordinate system is given in detail.



The physical model shown is a parallelepiped having dimensions Δx, Δy and Δz, subject to transient heat conduction and internal heat generation. For the immediate purpose, the medium is taken to be homogeneous and isotropic, the thermal conductivity is independent of temperature, and the rate of heat generation is uniform.

$t = f(x, y, z \;\&\; \tau)$

Heat in is the summation of the heat flows into the element through the x, y, & z faces. Heat out is the summation of the heat flows out of the element through the x + Δx, y + Δy, and z + Δz faces.

By the law of conservation of energy:

Heat in + heat generation - heat out = time rate of change of internal energy.

The rate given by (q in - q out) may be positive or negative. Internal heat generation is positive for a heat source or negative for a heat sink. For a heat source, using W to denote the heat generation per unit time and volume, the heat generated within the element per unit time is $W(\Delta x.\Delta y.\Delta z)$. The time rate of change of internal energy of the element is the rate of heat storage if positive or the rate of heat release if negative. Expressed in terms of the time rate of change of the average temperature of the element, the time rate of change of internal energy is $\rho\, C_p(\Delta x.\Delta y.\Delta z)\frac{\partial t_{av}}{\partial \tau}$, where density ($\rho$) and specific heat ($C_p$) are assumed constant.

For the heat flow into the element, from the fundamental law of heat conduction ($q = -kA\frac{dt}{dx}$), the heat flow into the element through the x-face (area $\Delta y \cdot \Delta z$) is:

$$q_{in,x} = -k(\Delta y \cdot \Delta z)\,\frac{\partial t}{\partial x}\Big|_x ,$$

where $\frac{\partial t}{\partial x}\big|_x$ denotes the temperature gradient with respect to x at the x-face.

Similarly, the heat flow out of the element through the x + Δx face is:

$$q_{out,x+\Delta x} = -k(\Delta y \cdot \Delta z)\frac{\partial t}{\partial x}\Big|_{x+\Delta x} .$$

An expression for $\frac{\partial t}{\partial x}\big|_{x+\Delta x}$ in terms of $\frac{\partial t}{\partial x}\big|_x$ can be obtained by expanding the temperature gradient at the x-face in a Taylor's series as follows:

$$\frac{\partial t}{\partial x}\Big|_{x+\Delta x} = \frac{\partial t}{\partial x}\Big|_x + \frac{\partial}{\partial x}\left(\frac{\partial t}{\partial x}\Big|_x\right)\Delta x + \frac{\partial^2}{\partial x^2}\left(\frac{\partial t}{\partial x}\Big|_x\right)\frac{\Delta x^2}{2!} + \cdots\cdots$$

$$\therefore q_{out,x+\Delta x} = -k(\Delta y\cdot\Delta z)\left[\frac{\partial t}{\partial x}\Big|_x + \frac{\partial}{\partial x}\left(\frac{\partial t}{\partial x}\Big|_x\right)\Delta x + \frac{\partial^2}{\partial x^2}\left(\frac{\partial t}{\partial x}\Big|_x\right)\frac{\Delta x^2}{2!} + \cdots\right]$$

The difference between the two heat flow rates is:

$$q_{in,x} - q_{out,x+\Delta x} = -\left\{-k(\Delta y\cdot\Delta z)\left[\frac{\partial}{\partial x}\left(\frac{\partial t}{\partial x}\Big|_x\right)\Delta x + \frac{\partial^2}{\partial x^2}\left(\frac{\partial t}{\partial x}\Big|_x\right)\frac{\Delta x^2}{2!} + \cdots\right]\right\}$$

$$= +\,k(\Delta x\cdot\Delta y\cdot\Delta z)\left[\frac{\partial}{\partial x}\left(\frac{\partial t}{\partial x}\Big|_x\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial t}{\partial x}\Big|_x\right)\frac{\Delta x}{2!} + \cdots\right]$$

Parallel expressions may be developed for the y and z pair of faces. These terms, divided by the volume, are designated as Y & Z. Now applying the heat rate balance equation as previously written and dividing by the volume ($\Delta x.\Delta y.\Delta z$), gives:

$$k\left[\frac{\partial}{\partial x}\left(\frac{\partial t}{\partial x}\Big|_x\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial t}{\partial x}\Big|_x\right)\frac{\Delta x}{2!} + \cdots\right] + Y + Z + W = \rho\, C_p\frac{\partial t_{av}}{\partial \tau}\ .$$

Now let Δx, Δy and $\Delta z \rightarrow 0$, i.e. approach point A.

$\frac{\partial t}{\partial x}\Big|_x \rightarrow \frac{\partial t}{\partial x}\Big|_A$ or simply $\frac{\partial t}{\partial x}$ at a point.

Also $t_{av} \rightarrow t_A$ or t at a point.

The general equation becomes

$$\frac{k}{\rho C_p}\left[\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}\right] + \frac{W}{\rho C_p} = \frac{\partial t}{\partial \tau}\ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ . \quad (2)$$

This linear second order partial differential equation is the general heat conduction equation written in Cartesian coordinates for a homogeneous and isotropic solid whose thermal conductivity is independent of temperature. If the medium is heterogeneous or homogeneous but anisotropic, equation (2) becomes

$$\frac{1}{\rho C_p}\left[k_x \frac{\partial^2 t}{\partial x^2} + k_y \frac{\partial^2 t}{\partial y^2} + k_z \frac{\partial^2 t}{\partial z^2}\right] + \frac{W}{\rho C_p} = \frac{\partial t}{\partial \tau}$$

If k varies with temperature, equation (2) written quite generally becomes

$$\frac{1}{\rho C_p}\left[\frac{\partial}{\partial x}\left(k \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial t}{\partial z}\right)\right] + \frac{W}{\rho C_p} = \frac{\partial t}{\partial \tau}$$

Writing the equation to incorporate the relationship $k = f(t)$ makes the equation non-linear. Carslaw and Jaeger[1] give the equation for the case of k varying with temperature but independent of position (for which an exact solution is possible) as:

$$\frac{k}{\rho C_p}\left[\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}\right] + \frac{W}{\rho C_p} + \frac{\partial k}{\partial t}\left[\left(\frac{\partial t}{\partial x}\right)^2 + \left(\frac{\partial t}{\partial y}\right)^2 + \left(\frac{\partial t}{\partial z}\right)^2\right]\frac{1}{\rho C_p} = \frac{\partial t}{\partial \tau}$$

Although the thermal conductivities of solids generally vary with temperature, equation (2) is frequently used in engineering practice using k equal to the average k for the temperature range encountered in a problem. This procedure will give results which are usually within the required degree of accuracy.

Equation (2) may be derived in a similar manner for cylindrical or spherical polar coordinates. The equation in cylindrical coordinates (r,θ,z) corresponding to equation (2) will be used later and is given as follows for reference.

$$\frac{k}{\rho C_p}\left[\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \theta^2} + \frac{\partial^2 t}{\partial z^2}\right] + \frac{W}{\rho C_p} = \frac{\partial t}{\partial \tau} \quad . \quad . \quad . \quad . \quad (3)$$

2.2 Type of Problems

Equation (2) or its equivalent in other coordinate systems will generally be the form of the general heat conduction equation used in this thesis, i.e. k, ρ, and C_p are constant and independent of temperature and position. In solving the partial differential equation corresponding to a particular problem, in order to determine the exact solution, certain information associated with the problem must be known which can be applied to evaluate the constants of integration. Accordingly, problems in heat conduction fall into a class of problems termed initial and boundary value problems. Boundary conditions are the values of the required solution (usually temperature) at the bounding surfaces of the geometry. Boundary condition temperature values may be constant or vary with time in a prescribed manner. Occasionally, the heat flux at a surface provides a boundary condition. For unidirectional heat flow, given the heat flux at a surface and the thermal conductivity of the medium, the temperature gradient at the surface may be calculated using equation (1).

Similarly, as the name implies, initial conditions give the values of temperature for the problem at time, $\tau = 0$. Hence, initial conditions are required for transient heat conduction problems where it is required to solve for temperature for particular values of $\tau > 0$. Initial conditions are the base values of temperature from which a transient departs. For many practical problems, the initial conditions are that the material is isothermal.

Regarding equation (2), if the temperature at a given point in the medium varies with time (or alternatively, if temperature varies with both time and position) the process is called conduction in the unsteady state or transient heat conduction. On the other hand, if the temperature at each point in the continuum does not vary with time, i.e. $\partial t/\partial\tau = 0$, the process is called conduction in the steady state. For this case, the temperature field is constant; hence for constant k, the heat fluxes are also constant and independent of time.

The general heat conduction equation applies to all heat conduction problems --- heat flow in one, two or three dimensions, in steady or unsteady states, with or without internal heat generation. The equation is linear, and in writing the equation appropriate to any particular situation, certain terms in the equation may equal zero. For example, for two-dimensional transient heat flow without internal heat generation:

$$\frac{k}{\rho C_p}\left[\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2}\right] = \frac{\partial t}{\partial \tau}$$

The two dimensions x and y are required to describe the temperature distribution at any time, τ. For any given value of τ, all sections through the three dimensional geometry in the x,y plane have identical temperature fields, hence $\frac{\partial^2 t}{\partial z^2} = 0$, and in fact $\frac{\partial t}{\partial z}$ = const = 0, i.e. there are no temperature gradients in the z direction.

2.2.1 Steady State Problems

For steady state situations, the problem is generally that of determining the temperature distribution and/or the heat flow. The two special cases, written using all three coordinates, are as follows:

Steady state temperature fields without heat sourses

$$\frac{k}{\rho C_p}\left[\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}\right] = \frac{\partial t}{\partial \tau} = 0$$

Since $\frac{k}{\rho C_p} \neq 0$,

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = 0$$

Using the symbol $\nabla^2 t$ (the Laplacian) for the second-order differential parameter of t, the equation is

$$\nabla^2 t = 0 \quad \text{(Laplace equation)}$$

From the Laplace equation, since the temperature distribution is independent of any thermal property of the medium, it is apparent that for a given geometry and set of boundary conditions, different materials will have the same temperature distribution. The actual heat flows for different materials will of course vary depending on the thermal conductivity of the material. In addition, for this type of problem, it should be realized that certain heat sources/sinks must be physically involved in the situation in order to maintain steady temperature conditions. These heat sources/sinks exist at or beyond the boundaries of the problem, and their effects are included in the specifications of the boundary conditions.

Steady state temperature fields with uniform heat sources

$$\frac{k}{\rho C_p}\left[\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}\right] + \frac{W}{\rho C_p} = \frac{\partial t}{\partial \tau} = 0$$

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = -\frac{W}{k} = \text{constant}$$

or

$$\nabla^2 t = \text{const} \quad \text{(Poisson equation)}$$

2.2.2 Transient Problems

For ordinary transient problems, it is usually required to determine the temperature distributions for various values of time. In some cases the instantaneous heat fluxes or the integrated heat flows over certain time intervals may be of importance. A special case involving transient heat conduction is that for which the temperature boundary conditions change with time in a periodic manner. The required solution for this type of problem is usually the steady periodic solution for which the temperature fields for the medium vary in a repetitious cyclic manner with the boundary conditions. Periodic heat conduction is discussed in detail in section 3.1.2 b) since it is the main type of problem involved in the frozen dam study.

From equation (2), the equation for unsteady state heat conduction without internal heat generation is:

$$\frac{k}{\rho C_p}\left[\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}\right] = \frac{\partial t}{\partial \tau}$$

The quantity $(k/\rho C_p) \equiv \alpha$, the thermal diffusivity, is contained in the equation. The actual transient behaviour depends on the thermal diffusivity which may be considered as a form of reciprocal time constant. This concept may be illustrated by considering the case of a small pellet of undefined geometry having a temperature t_2 which is suddenly immersed in a large bath of temperature t_1. It is assumed that the convective heat transfer coefficient is infinite, and the surface of the pellet attains a temperature t_1 instantaneously upon immersion. Using the differential operator $p \equiv \frac{d(\ )}{d\tau}$, as a result of equation (1), at the instant of immersion:

$pQ = \frac{t_1 - t_2}{R}$, for $t_1 > t_2$,

where R, the thermal resistance, is of the form $\frac{x}{kA}$ with units of

$ft \times \frac{hr\text{-}ft\text{-}^{\circ}F}{Btu} \times \frac{1}{ft^2} = \frac{hr\text{-}^{\circ}F}{Btu}$.

By the law of conservation of energy;

Heat input = time rate of heat storage in the pellet.

$\therefore pQ = Cpt_2$,

where C is the thermal capacity of the pellet, i.e.

$C = f(V, \rho, C_p)$ with units $ft^3 \times \frac{lbm}{ft^3} \times \frac{Btu}{lbm\,^{\circ}F} = \frac{Btu}{^{\circ}F}$.

Equating the two expressions for pQ:-

$\frac{t_1 - t_2}{R} = Cpt_2$

$t_1 - t_2 = RCpt_2$

$t_1 = (1 + Tp)\, t_2$ where $T = RC$ is the time constant (hrs.).

The time constant of a system is a measure of the speed of response. If T is small, the pellet temperature $t_2 \rightarrow t_1$ relatively quickly. Conversely, if T is large, more time is required for the change to occur. Now $T = RC = \frac{x}{kA} \cdot V\rho C_p$ where x, V and A are components of the geometry. Hence for a given geometry, a large value of $\alpha = \frac{k}{\rho C_p}$ gives a small T and hence a short transient period. Accordingly, thermal diffusivity is a measure of the thermal sensitivity of a material - the time response of a material to a temperature disturbance. The units of thermal diffusivity are:

$\alpha = \frac{\partial t}{\partial \tau} / \frac{\partial^2 t}{\partial x^2} + \ldots = \frac{^{\circ}F}{hr} \times \frac{ft^2}{^{\circ}F} = \frac{ft^2}{hr}$

III METHODS OF SOLVING HEAT CONDUCTION PROBLEMS

Of the numerous methods available for solving heat conduction problems, in general, each method has a fairly definite range of application which is either inherent in the method or due to mathematical complexities. For example, free-hand flux plotting may be used only for two dimensional steady state problems. For any given problem, usually more than one method can be employed, and any one of these methods will tend to have certain advantages and disadvantages compared to the others. The selection of a method to solve a particular problem within the required degree of accuracy with a minimum expenditure of time and effort is largely a matter of engineering judgment. The lack of suitable facilities/apparatus may, in many cases, rule out the use of certain methods (particularly analog methods). A finite-difference approach to a large-scale problem by numerical iteration may be impractical by hand computation, whereas the solution may be readily obtainable through the use of an electronic digital computer after only a few hours of programming and computer operation.

For the purpose of presentation, the various methods of solving heat conduction problems were arbitrarily classified as <u>analytical</u>, <u>finite-difference</u> (both graphical and numerical), <u>analog</u> and <u>experimental</u>. The objectives in dealing with each method were to present its physical basis and to discuss the practical aspects of its application and utility. The survey was by no means exhaustive; new methods/techniques within a method are being developed continually. In this respect, the emphasis was in dealing with the older and better known methods which have been established through years of use. Certain solution methods were treated primarily by illustrative example. In most of these cases, the examples presented have a bearing on the frozen dam problem.

3.1. Analytical Methods

Analytical methods of solving heat conduction problems consist essentially of straight solutions of the partial differential equation of heat conduction, i.e. mathematical solutions to equation (2) written to correspond to a given problem, with the constants of integration evaluated through the use of boundary condition data. Following selection of a suitable coordinate system, the problem is first formulated by writing the appropriate differential equation and the initial and boundary conditions, and by specifying the nature of the required solution. The problem then becomes strictly mathematical, and may be subjected to any mathematical technique to obtain a particular solution. In theory, any heat conduction problem can be solved analytically. In practise, even if the problem can be successfully formulated, the solution may be indeterminate due to the intricate mathematics involved.

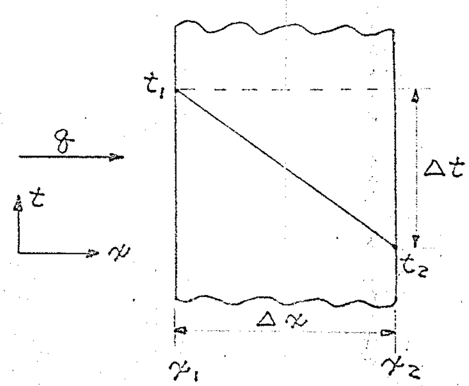
3.1.1 Heat Conduction in a Single Independent Variable

This class of problem is one-dimensional steady state heat conduction with or without internal heat generation. (The case of $t = f(\tau)$ alone is no longer a heat conduction problem since zero temperature gradients preclude conduction of heat). The three basic geometries are the plane wall, hollow cylinder (tube) and hollow sphere. The plane wall or plate has finite thickness, however its parallel bounding surfaces are infinite in extent. The hollow cylinder is infinitely long. The bounding surfaces for all three geometries are isothermal and at different temperatures. Heat flows from one surface through the medium to the other surface in a direction which is normal to the surfaces.

Many heat transfer situations encountered in engineering practice approach unidirectional steady state heat conduction. Examples without internal heat generation are the walls of a refrigerator, a circular insulated steam pipe,etc. Applications involving internal heat generation may include electric heating elements, nuclear fuels, curing of concrete, etc. For the refrigerator wall for example, heat flow will be predominately unidirectional on a surface area basis. Two or three coordinates will be required to specify the temperature distribution at the edges and corners. If the surface area is relatively large compared to the thickness of the wall, edge effects will tend to be negligible, and the entire wall may be treated to a very good approximation as a case of one-dimensional heat flow.

In view of the importance of unidirectional heat flow relations in heat conduction analysis, two cases are considered in detail - the plane wall without heat sources and the cylinder with uniform internal heat generation.

a) Plane Wall (W = 0)



From equation (2),

$$\frac{k}{\rho C_p}\left(\frac{\partial^2 t}{\partial x^2}\right) = 0 \text{ and since } t = f(x)$$

alone, $$\frac{d^2 t}{dx^2} = 0$$

For $t_1 > t_2$, q is as shown and x increases in the same direction as q.

Since $\frac{d^2t}{dx^2} = 0$, $\frac{dt}{dx} = C$, a constant.

$\therefore \int dt = C \int dx$

$t = Cx + D$

Let the boundary conditions be

$t = t_1$ at $x = x_1$, and

$t = t_2$ at $x = x_2$.

Solving for C and D from these boundary conditions gives an equation for the temperature distribution (linear). Heat flow is unidirectional, and for equation (1), an expression for dt/dx can be obtained by differentiating the temperature distribution equation with respect to x. The resulting heat flow equation is:

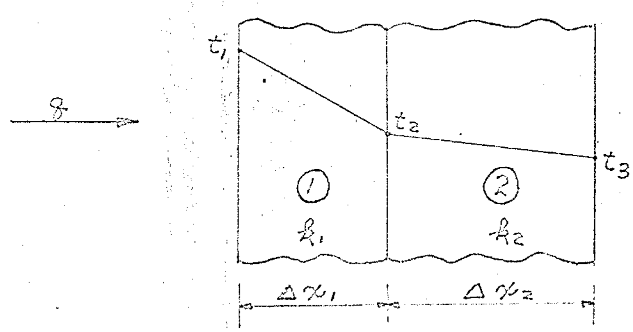
$$\frac{q}{A} = \frac{+k(t_1 - t_2)}{x_2 - x_1} = \frac{k\,\Delta t}{\Delta x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence the heat flux (for k independent of t) is simply the product of the thermal conductivity and the ratio of temperature difference to wall thickness.

(Equation (4) might have been derived more readily by integrating equation (1) over the limits given by the boundary conditions. This 'shortcut' method is possible only because unidirectional heat flow is involved for which equation (1) is applicable).

If thermal conductivity varies linearly with temperature, for a temperature range bounded by t_1 and t_2, it may be shown[2] that the mean thermal conductivity value suitable for use in the heat flow equation is k at $t = \frac{1}{2}(t_1 + t_2)$, i.e. k at the average temperature of the range. The actual temperature distribution will deviate from a straight line relationship. The position of maximum temperature deviation from a linear temperature distribution occurs at the value of x for which $t = \frac{1}{2}(t_1 + t_2)$. The general expression for mean thermal conductivity (including k non-linear in temperature) is $k_m = \frac{1}{t_1 - t_2} \int_{t_1}^{t_2} k \cdot dt$, $k=f(t)$

The multilayer wall is an obvious extension to the case of steady state unidirectional heat flow through a plane wall. The multilayer wall consists of uniform layers of different materials of either equal or unequal widths. A two layer wall is first considered.



The usual problem is to calculate the heat flow through the composite wall. Temperature t_2, being an interior temperature, is usually not known. Under steady temperature conditions, there can be no heat storage or release for the medium, hence the heat flow through each layer must be the same. Assuming no contact resistance between the two layers, i.e. no temperature discontinuity at the interface, from equation (4), for unit area:

$$q = \frac{k_1(t_1 - t_2)}{\Delta x_1} = \frac{k_2(t_2 - t_3)}{\Delta x_2}$$

Solving for t_2 in terms of t_1, t_3 and the thermal resistances, and substituting for t_2 in the heat flow equation, leads to $q = \dfrac{t_1 - t_3}{\dfrac{\Delta x_1}{k_1} + \dfrac{\Delta x_2}{k_2}}$

per unit surface area.

By induction, for a wall of 'n' layers:

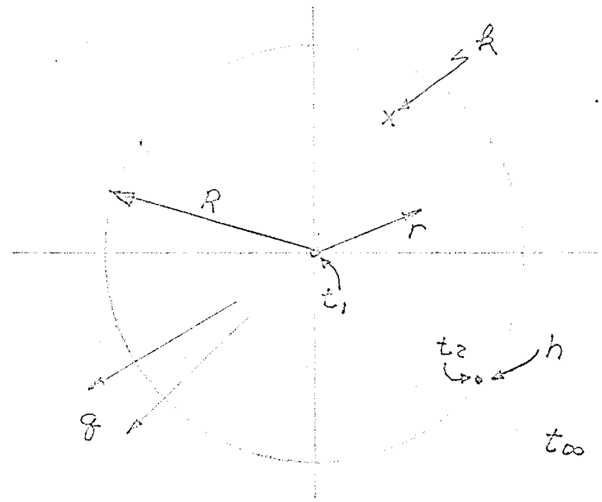
$$q = \frac{t_1 - t_{n+1}}{\frac{\Delta x_1}{k_1} + \frac{\Delta x_2}{k_2} + \dots + \frac{\Delta x_n}{k_n}} \text{ per unit surface area.} \quad . \quad . \quad . \quad (5)$$

The condition of no contact resistance implies a perfect thermal bond between the materials. In practice, some temperature drop will occur at an interface due to a finite contact resistance. If the thermal resistances of the materials are large and the materials are well fitted, neglecting contact resistances may not cause significant error in the computed heat flow. On the other hand, if the materials are good thermal conductors and the layers are relatively thin, the contact resistances may very well govern the heat flow.

An estimation of the magnitude of contact resistance for a given problem tends to be a rather complicated matter even if parameters such as surface roughness and interfacial pressure are accurately known. Theories of real contact area between plane surfaces in contact and of constriction resistance have been established[3], however there is still relatively little data published on the subject.

b) <u>Cylinder (W = const.)</u>

A solid generating heat uniformly on a volume basis and of long straight cylindrical geometry, is considered. The situation might, for example, represent the ideal case of a wire carrying electric current with no skin effects and electrical resistance independent of temperature.



Since radial heat flow is involved, equation (3) in cylindrical coordinates is applied, i.e.

$$\frac{k}{\rho C_p}\left[\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r}\right] + \frac{W}{\rho C_p} = 0.$$

Since $t = f(r)$ only,

$$\frac{d^2 t}{dr^2} + \frac{1}{r}\frac{dt}{dr} + \frac{W}{k} = 0.$$

Substituting $S = \frac{dt}{dr}$ (gradient) and multiplying through by $r\,dr$ results in

$$r dS + S dr = -\frac{W}{k} r\,dr \quad \text{where L.H.S.} = d(rS).$$

Two successive integrations leads to

$$t = -\frac{W}{2k} \cdot \frac{r^2}{2} + C_1 \ln r + C_2$$

The constants of integration may be evaluated by applying known boundary conditions. The cylinder dissipates heat by convection to its surroundings, hence

$$\frac{q}{A} = -k \frac{dt}{dr}\Big|_{r=R} = h(t_2 - t_\infty).$$

Also $\frac{dt}{dr}\Big|_{r=0} = 0$ by symmetry which requires $C_1 = 0$.

Using $t_2 = -\frac{W}{2k} \cdot \frac{R^2}{2} + C_2$ and the first B.C. results in

$$C_2 = t_\infty + \frac{WR^2}{4k} \left(1 + \frac{2k}{hR}\right)$$

Finally, the expression for temperature distribution becomes

$$t = t_\infty + \frac{WR^2}{4k}\left(1 + \frac{2k}{hR}\right) - \frac{W}{4k} \cdot r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

Temp. at axis ($r=0$) : $t_1 = C_2$

Temp. at surface ($r=R$) : $t_2 = t_\infty + \frac{WR}{2h}$

Also $t_1 - t_2 = \frac{WR^2}{4k}$

As expected, the temperature levels in the solid depend on the ambient temperature (t_∞) and the convective heat transfer coefficient (h). The temperature difference ($t_1 - t_2$), however, depends only on internal factors; being directly proportional to W and R^2, and inversely proportional to the thermal conductivity of the medium. Note, as long as steady temperature conditions prevail, the heat flow for the problem simply depends on W, i.e. $q = \pi R^2 W$ Btu/hr. per ft. of cylinder length. The surface heat flux, $\frac{q}{A} = \frac{1}{2}RW$ Btu/hr-ft^2.

One particular analytical approach to one-dimensional steady state problems warrants special mention. This approach is to write out a heat balance equation on an incremental element of the geometry, thereby capitalizing on the fact that heat flow is unidirectional. This approach was used to specify one of the boundary conditions of the preceding cylinder problem. It is also commonly employed in analyzing extended surfaces which are used in many applications to increase the heat dissipation capability of a structure. If the fin diameter or width is small compared to its length, convection tends to control the heat flow - there are large axial temperature gradients, however sections normal to the longitudinal axis are essentially isothermal - hence one-dimensional steady state heat conduction. The heat rate balance equation is:-

Heat in by cond. = Heat out by cond. + conv. loss at the perimeter.

For a fin element of length dx (x in the axial direction), applying equation (1) of the form $q = -k\,A(x)\,\frac{dt}{dx}$, and expanding the 'heat out' term in a Taylor series using the first two terms as a reasonable approximation, leads to:

$$-k\,A(x)\frac{dt}{dx} = -k\,A(x)\frac{dt}{dx} + \frac{d}{dx}\left[-k\,A(x)\frac{dt}{dx}\right]\,dx + h(Cdx)(t - t_\infty),$$

where C = wetted perimeter and h and t_∞ are as defined earlier.

$$\therefore\ k\,\frac{d}{dx}\left[A(x)\,\frac{dt}{dx}\right]\,\cancel{dx} = h\,C\,\cancel{dx}\,(t - t_\infty)$$

$$\text{and}\quad \frac{d}{dx}\left[A(x)\,\frac{dt}{dx}\right] = \frac{hC}{k}(t - t_\infty)\quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

For A independent of x,

$$\frac{d^2t}{dx^2} - \frac{hC}{kA}(t - t_\infty) = 0,$$

which has a general solution of the form

$$t - t_\infty = C_1\,e^{mx} + C_2\,e^{-mx} \quad \text{where } m = \sqrt{\frac{hC}{kA}}.$$

A large amount of theoretical information and data is available to the designer/analyst involved in a fin problem; both in heat transfer texts[4,5] and the general literature. Gardner[6] gives curves for fin efficiencies of several varieties of straight fins, annular fins and spines.

The preceding illustrates the general analytical approach to heat conduction in a single independent variable. Mathematically, these problems tend to be elementary in nature since formulation involves an ordinary differential equation. Due to ease of handling, analytical methods are used almost exclusively in solving problems of this category.

3.1.2 Heat Conduction in Two Independent Variables

Heat conduction in two independent variables may be either two-dimensional steady state or one-dimensional transient. These two cases are considered separately in the above mentioned sequence. The presence of heat sources/sinks within the temperature fields is excluded throughout this section.

a) Two-Dimensional Steady State Problems

From equation (2) since $t = f(x,y)$, the temperature field is described by:

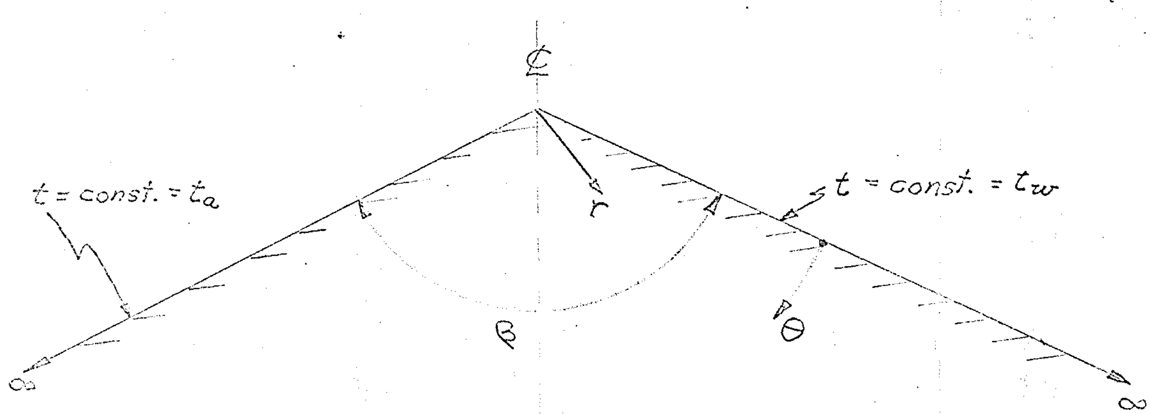
$$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

As usual, for a given geometry and set of boundary conditions, the required solution to the problem is either the heat flow at a particular surface or the temperature field. If the former is the requirement, it is necessary usually to solve first for the spatial distribution of temperature. Several analytical methods are available; the main ones are reviewed as follows:

i) Method of Separation of Variables (Product Method)

The solution of partial differential equations by separation of variables consists basically of reducing the partial differential equation to one involving the product of two (or more) ordinary differential equations. Success in obtaining the final solution justifies the assumption that the dependent variable can be expressed as a product of functions of different independent variables. A separation constant is involved (see example following), and occasionally the boundary conditions require this parameter to assume certain discrete characteristic values (eigenvalues) for the problem. In this case, unless the solution can be expressed in closed form or tabular data is available, numerical evaluation of temperature as a function of a sum of eigenfunctions may be somewhat tedious. In general, however, the product method is an extremely powerful and useful mathematical tool. Fortunately, many partial differential equations describing heat conduction belong to the class of separable domains, and the method is frequently used with success.

A secondary advantage associated with the method involves the fact that boundary conditions are often specified for constant values of a geometric coordinate. The method of separation of variables permits direct application of the boundary conditions, thereby facilitating evaluation of the constants. The method is illustrated by the following example.



The geometry shown is a cross-section through a long wedge whose bounding surfaces are isothermal and at different temperatures. For practical purposes, the temperature at the apex may be considered as the average of the two boundary temperatures. The situation might represent a dike with its sides exposed to water and atmosphere respectively, where the interest is in temperature for values of radius sufficiently small that the length of the slope appears to go to infinity before breaking horizontally.

A cylindrical coordinate system is used with references as indicated. From the boundary conditions (t constant for all values of r), it is suspected that an equation for t might be written in terms of θ alone. This is in fact true, and a short derivation would be:

$$t = f(\theta), \text{ and } \frac{1}{r^2}\frac{d^2t}{d\theta^2} = 0$$

However, to illustrate the product method, temperature is considered as being dependent on both radius and angle, i.e. $t = f(r,\theta)$. From equation (3),

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \theta^2} = 0 \quad ; \quad \frac{\partial^2 t}{\partial z^2} = 0.$$

Assume $t = R\phi$ where $R = f(r)$ alone,

and $\phi = f(\theta)$ alone.

Substitution into the Laplace equation gives

$$\phi\frac{d^2R}{dr^2} + \frac{\phi}{r}\frac{dR}{dr} + \frac{R}{r^2}\frac{d^2\phi}{d\theta^2} = 0, \text{ and therefore}$$

$$\frac{-r^2}{R}\left[\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right] = \frac{1}{\phi}\cdot\frac{d^2\phi}{d\theta^2} = S, \text{ the separation constant.}$$

The two sides are functions of different independent variables and can be equal only if each is equal to a constant. This separation constant is an integer which may be positive or negative. Since the problem has been over-formulated; try S = 0.

$$\therefore \frac{d^2\phi}{d\theta^2} = 0, \text{ hence } \frac{d\phi}{d\theta} = A$$

$$\int d\phi = A\int d\theta$$

$$\phi = A\theta + B$$

$$\text{Also } \frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} = 0$$

$$\frac{d^2R}{dr^2} = -\frac{1}{r}\frac{dR}{dr}$$

$$\int \frac{d\left(\frac{dR}{dr}\right)}{(dR/dr)} = -\int \frac{dr}{r}$$

$$\ln \frac{dR}{dr} = -\ln r + \ln C \text{ or } \frac{dR}{dr} = \frac{C}{r}$$

$$\int dR = C\int \frac{dr}{r}$$

$$R = C\ln r + D$$

$$\therefore t = R\phi = (C\ln r + D)(A\theta + B)$$

Now C = 0, since in order to match the boundary conditions, t must be constant for a given θ for all values of r.

$$\therefore t = (AD)\theta + (BD)$$

$$= E\theta + F$$

Now $t = t_w$ at $\theta = 0$, hence $F = t_w$

and the equation becomes $t = E\theta + t_w$

Also $t = t_a$ at $\theta = \beta$, and substitution yields

$$t_a = E\beta + t_w \text{ where } E = \frac{t_a - t_w}{\beta}$$

$$\therefore t = \left(\frac{t_a - t_w}{\beta}\right)\theta + t_w$$

or for $t_w > t_a$

$$t = t_w - \left(\frac{t_w - t_a}{\beta}\right)\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Only one coordinate is required to specify temperature in the final expression which makes the problem one-dimensional steady state. If the problem were extended to include a dike plus foundation, a Cartesian co-ordinate system might very well be used due to foundation considerations. For this case, two space coordinates would be required to specify temperature within the dike proper. The point then is that the coordinate system has a bearing on the dimensionality of the problem. Failure to choose the 'best' coordinate system may complicate a problem needlessly by requiring an extra independent variable.

ii) <u>Conformal Mapping</u>

In the theory of functions of a complex variable, the term 'mapping' refers to the transformation of points in the z-plane ($z = x + iy$) into corresponding points of the w-plane ($w = u + iv$) according to their functional relationship $w = f(z)$, where w(a second complex variable) is a function of the complex variable z within a defined domain. If the function is analytic and has non-vanishing derivatives, the mapping is conformal, i.e. the transformation preserves angles, hence in particular orthogonal curves are mapped into orthogonal curves.

Any function that has continuous partial derivatives of the second order and that satisfies the Laplace equation in two independent variables is called a harmonic function. Both the real and imaginary parts of an analytic function of a complex variable are harmonic functions. This may be shown as follows:

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{-\partial v}{dx}$ by the Cauchy - Riemann conditions.

Hence $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y^2} = \frac{-\partial^2 v}{\partial y \partial x}$

For well behaved functions u and v, the order of partial is immaterial;

and in particular $\frac{\partial(\frac{\partial v}{\partial y})}{\partial x} = \frac{\partial(\frac{\partial v}{\partial x})}{\partial y}$.

$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = \frac{-\partial^2 u}{\partial y^2}$;

from which

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

The functions u and v are called conjugate harmonic functions. In general, if a harmonic function is given, it can be shown that its harmonic conjugate exists, and the latter can be calculated to within an arbitrary constant. In potential theory, the harmonic conjugates are referred to generally as the 'potential function' (ϕ) and the 'stream function' (ψ).

An important characteristic of conjugate harmonic functions is that the curves $u(x,y) = C_1$, a constant, and $v(x,y) = C_2$ are mutually orthogonal. In order for the curves to be perpendicular at the point of intersection, the slope of one curve must be the negative reciprocal of the slope of the other curve at the point of intersection.

For $u(x,y) = C_1$,

$$du = 0 = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

whence $\left.\frac{dy}{dx}\right|_u = -\left(\frac{\partial u/\partial x}{\partial u/\partial y}\right)$ providing $\frac{\partial u}{\partial y} \neq 0$.

Similarly, for $v(x,y) = C_2$,

$$\frac{dy}{dx}\Big|_v = -\left(\frac{\partial v/\partial x}{\partial v/\partial y}\right) .$$

$$\therefore \frac{dy}{dx}\Big|_v = \frac{\partial u/\partial y}{\partial u/\partial x}$$, by substitution from the Cauchy-Riemann conditions.

$$= \frac{-1}{\dfrac{\partial u/\partial x}{\partial u/\partial y}} = \frac{-1}{\dfrac{dy}{dx}\Big|_u}$$ at the same point (x,y).

The temperature function, $t = f(x,y)$, and the temperature gradients for steady state heat conduction are continuous throughout a domain consisting of the interior of the solid under consideration. Hence since equation (8) is a Laplace equation in two dimensions, temperature (t) is a harmonic function of x, y. It follows, lines $t(x,y) = C_1$ are the isotherms. Furthermore, from equation (1), lines orthogonal to the isotherms (the conjugate harmonic, say $S(x,y) = C_2$) must be heat flow lines since the heat flux vector is in the direction of the temperature gradient which is perpendicular to the isotherm at each point.

The preceding coupled with one important theorem associated with conformal mapping, form the basis for the use of conformal mapping in solving two-dimensional steady state heat conduction problems. This theorem[7] states - "Every harmonic function of x and y transforms into a harmonic function of u and v under the change of variables $x + iy = f(u + iv)$ where f is an analytic function". Accordingly, under conformal transformations, equation (8) applies (with appropriate independent variables) to the transformed geometry as well as the original geometry.

The general technique in solving two-dimensional steady state heat conduction problems via conformal mapping is to locate the geometry in the x,y plane, and seek a transformation which will provide a geometry in the u, v plane which is amenable to solution. Problems of boundary conditions of either prescribed temperature (Dirichlet) or specified normal temperature gradient (Neumann) can be handled. Tables of transformations are available[8]. For a given problem, an explicit equation for temperature can generally be obtained by writing the temperature relation in the u, v plane (usually by inspection), and substituting the transformation function to obtain temperature in terms of the coordinates (x,y) of the actual geometry. (Note, locating the geometry in the u,v plane and working in reverse is permissible providing the inverse function (analytic) is single-valued.) The application of this mathematical tool is illustrated by the following simple example.

The problem is a long pipe with an eccentric bore of geometry; O.D. = 2.0", I.D. = 1.25" and eccentricity, e = 0.25". The temperature field is steady, and the boundary conditions are that the constant temperatures of the inner and outer surfaces are $t_i = 160^\circ F$ and $t_o = 60^\circ F$. It is required to compute the heat loss from the pipe. The average thermal conductivity of the material for the temperature range involved is $k_{av} = 1.0$ Btu/hr-ft-$^\circ$F.

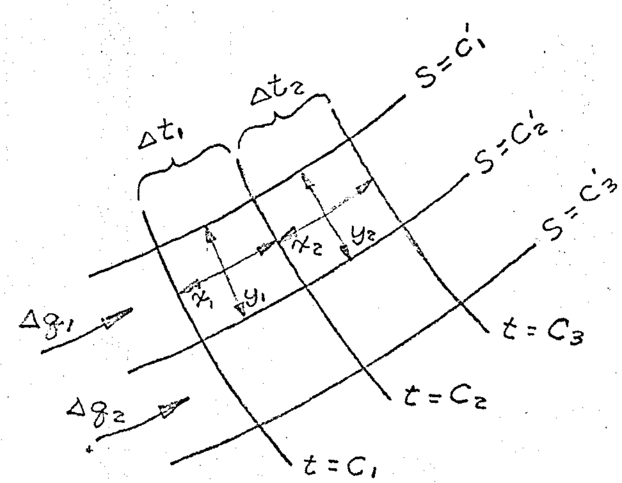
A transformation was found[9] (Fig. 1) which maps the region into a concentric cylinder geometry for which the heat flow equation is well known. No scale change is involved. The inner radius of this new cylinder is 1.0 inches, and an equation is given for the outer radius (R_o) which was calculated to be 1.42 inches. Using a cylinder length of 1 ft., and solving directly in the transformed plane, results in

$$q = \frac{2\pi \ell k \Delta t}{\ell n \dfrac{r_o}{r_i}} = \frac{2\pi(100)}{\ell n \dfrac{1.42}{1.0}} = 1790 \text{ Btu/hr. per ft. of length.}$$

An explicit equation for heat flow exists[10] for this problem in terms of the actual geometry. The time required to obtain a numerical result is about the same for both methods given the formula or the transform.

iii) Flux Plot

Freehand flux-plotting may be used to solve two-dimensional (Cartesian) steady state heat conduction problems in a rapid though approximate manner. Although graphical in nature, this method is included in this section for the sake of continuity since its basis is related to the principles of conformal mapping. As previously outlined, in two-dimensional steady state heat flow, the isotherms and heat flow lines intersect at right angles. Heat flow lines are adiabatic in the sense that heat will not flow across such a line. If a grid of isotherms and heat flow lines is constructed so as to provide orthogonality, the region bounded by two adjacent adiabatic lines constitutes a heat flow lane. If in addition, the lines of the grid form curvilinear squares (average length of the sides opposite each other are equal), the temperature difference between a given isotherm and its two neighbouring itotherms is the same.



In finite-difference form, for a length 'ℓ' of the cross-section (see sketch);

$$\Delta q_1 = \frac{k(y_1 \ell)\ \Delta t_1}{x_1} = \frac{k(y_2 \ell)\ \Delta t_2}{x_2}$$

since the same heat flows through each element under steady temperature conditions.

Now, $y_1 = x_1$ and $y_2 = x_2$ from the construction, hence $\Delta t_1 = \Delta t_2 = \Delta t_i$.

To solve a two-dimensional steady state problem, the geometry is first examined for symmetry. This generally leads to isolation of a component region which is bounded by two isotherms (at different temperatures) connected by two adiabatics. The procedure now is to sketch a flux plot for this region by forming a grid of curvilinear squares. When completed, the number of temperature intervals (N) and the number of heat flow lanes (M) are counted. The heat flow for the region is the sum of the incremental heat flows of each heat flow lane between the boundary isotherms, i.e.

$$q = \sum_1^M \Delta q_i$$

$$= k\ell \sum_1^M \Delta t_1 \quad \text{since } \Delta q_2 = k\ \ell\ \Delta t_1 = \Delta q_i$$

$$= k\ \ell\ M\ \Delta t_1$$

Now $\Delta t_1 = \Delta t_2 = \Delta t_i = \Delta t$ in general.

Also $\Delta t = \dfrac{t_A - t_B}{N}$ where t_A and t_B are the boundary condition temperatures.

Hence by substitution for $\Delta t_1 = \Delta t$

$$q = k\ell\left(\frac{M}{N}\right)(t_A - t_B) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The ratio $M/N = S$ is called the shape factor.

The total heat flow and over-all shape factor for the geometry are q or S times the number of symmetrical sections respectively.

Accordingly, flux-plotting both allows an estimate of the heat flow and provides information as to the location and value of the isotherms. Accuracy depends on the degree of refinement of the network. Increased accuracy can be achieved by decreasing the size of the grid. This technique is illustrated by solving the previous problem of the pipe with an eccentric bore (Fig. 2). The geometry is symmetrical in two halves. From the solution by conformal mapping, the exact value of the shape factor is $S = \dfrac{2\pi}{\ell n \dfrac{r_o}{r_i}} = \dfrac{2\pi}{0.351} = 17.9$. The value obtained by flux-plotting ($S = 17.5$) is only approximately 2% in error.

Values for the shape factors (or equivalence) of variations of several geometries have been published; in particular for insulation designs[11] and buried heat sources,[12,13] the latter including three dimensional effects.

iv) Method of Fictitious Sources and Images

The method of fictitious sources and images, originated mainly by Lord Kelvin (1880), has proven extremely useful in solving certain heat conduction problems having non-linear boundary conditions. Probably the best known solution in heat conduction by this method is that for the two-dimensional steady state problem of a buried cable dissipating heat to its surroundings. The boundary conditions for this problem are:

1) a linear isotherm bounding the medium (semi-infinitely extended) surrounding the buried cable, and
2) a circular isotherm located in the medium, representing the constant surface temperature of the buried cable.

Even formulating the boundary conditions for any one coordinate system will tend to disuade attempts to solve the problem using the usual analytical approach. The procedure then is to look for a combination of fictitious heat sources/sinks which will produce the boundary temperatures. This approach may be used because the general heat conduction equation is linear, and the net effect on a point in the geometry due to either fictitious or actual heat sources/sinks, may be determined by the principle of superposition. The combination which gives the boundary conditions for the buried cable is a heat sink which is a mirror image of the heat source, and in doing so, the surroundings become infinitely extended and the actual heat source (buried cable) becomes 'line'. The details of the solution are given in several heat transfer texts including Eckert and Drake[14].

The method of fictitious sources and images may be used for either steady or unsteady state problems. Jakob[15] deals with both instantaneous and continuous point sources, and with instantaneous line and plane sources. The case of a point source (sphere) located in an infinite medium, all at uniform initial temperature, instantaneously brought to a fixed higher temperature at $\tau = 0$, becomes a transient heat conduction problem whose analysis leads to the fundamental solution for temperature for heat conduction in three dimensions. Carslaw and Jaeger[16] give several cases of heat conduction solved by this method.

Incidentally, buried cable theory may be used to solve the problem of the pipe with eccentric bore - the surface temperatures become isotherms surrounding the buried line heat source. The solution, however, involves a trial and error approach which is time consuming. A further analytical solution is given by El-Saden[17].

b) <u>One-Dimensional Transient Problems</u>

From equation (2) since $t = f(x,\tau)$, the temperature field is described by:

$$\alpha\left(\frac{\partial^2 t}{\partial x^2}\right) = \frac{\partial t}{\partial \tau} \quad , \quad \alpha = \frac{k}{\rho C_p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Since only one space coordinate is required to specify temperature at any given time, the geometries are the infinite slab, the infinitely long cylinder and the sphere.

Problems of this type may be classified as either <u>pure transient</u> or <u>steady periodic</u>. Pure transients basically entail moving from one steady state condition to another due to a change of a boundary condition temperature. The boundary temperature may change from one temperature level to another either instantaneously or over a period of time. Heat will be stored or released from the medium depending on whether the surface temperature change causes the body to be heated or cooled.

On the other hand, the boundary condition temperature may vary continuously in a regular periodic manner. After a suitable period of time with the boundary conditions having been applied for several cycles, the transient involved in moving from the initial temperature conditions will pass, and the temperatures of points in the medium will vary in a repetitious cyclic manner in response to the periodic boundary conditions. The heat conduction process at this stage is called 'steady periodic' or 'quasi-steady'. Alternate heat storage and release takes place. Although the initial conditions together with the thermal diffusivity of the material, determine the duration of the transient to when a steady periodic condition is attained, the steady periodic pattern depends only on the boundary conditions. Hence initial conditions are not required to solve for the steady periodic temperature regime. Actually, the usual problem is to solve for temperature under steady periodic conditions. The requirement here is to solve for the temperature history over the time period required to complete one cycle. The temperature of any interior point of the geometry which is influenced by the boundary conditions, will lag the surface temperature by a certain time interval. The time intervals for various points may be of interest particularly for cases having long cycle times.

i) Pure Transient Solutions

Various metallurgical and manufacturing processes approach unidirectional transient heat conduction. A typical example is a heated plate (length and breadth >> thickness) suddenly immersed in a quenching bath. Heat leaves from both sides of the plate by convection. If the initial condition is that the temperature is uniform or the distribution is symmetrical about the midplane, the temperature history will be the same for each half of the plate, i.e. the midplane is adiabatic. Hence the half geometry is effectively infinite slab - one face insulated and the other losing heat by convection.

The general classical solution to this type of problem is reviewed first, followed by a discussion of the use of charts. Application of the method of separation of variables (described earlier) to equation (11) or its equivalent in other coordinate systems, reduces the problem to ordinary differential equations in the space and time domains.

Cartesian Coordinates

Assuming $t = XT$ where $X = f(x)$ alone and $T = f(\tau)$ alone, and substituting into equation (11) results in:

$$\frac{1}{X} \cdot \frac{d^2X}{dx^2} = \frac{1}{\alpha T} \cdot \frac{dT}{d\tau} = K, \text{ the separation constant.}$$

$$\therefore \int \frac{dT}{T} = \alpha K \int d\tau$$

$$\ln T = \alpha K\tau + \ln A$$

$$T = A e^{\alpha K\tau}$$

As $\tau \to \infty$, $T \to \infty$ for +ve K, hence $t = XT \to \infty$ since x is bounded. This is of course not possible as t must remain finite. Therefore $K < 0$; and let $K = -\lambda^2$.

$$\therefore T = A e^{-\alpha \lambda^2 \tau}.$$

Now $\frac{1}{X} \cdot \frac{d^2X}{dx^2} = -\lambda^2$,

hence $\frac{d^2X}{dx^2} + \lambda^2 X = 0$ (Helmholtz's equation).

The solution is

$$X = B \cos \lambda x + C \sin \lambda x.$$

$$t = XT = A e^{-\alpha \lambda^2 \tau} (B \cos \lambda x + C \sin \lambda x). \quad \ldots \ldots \quad (12)$$

The constants are evaluated by applying the boundary conditions. Note, the solution to the Helmholtz equation for the rectangular coordinate system involves sine and cosine terms - oscillating periodic functions of constant amplitude.

Cylindrical Coordinates

From equation (3), the equation equivalent to equation (11) is:

$$\alpha \left[\frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \frac{\partial t}{\partial r} \right] = \frac{\partial t}{\partial \tau}$$

Assuming $t = RT$ leads to

$$\frac{1}{R}\left[\frac{d^2R}{dr^2} + \frac{1}{r}\,\frac{dR}{dr}\right] = \frac{1}{\alpha T}\cdot\frac{dT}{d\tau} = K, \text{ the separation constant}$$

Again K must be negative, and letting $K = -\lambda^2$ results in $T = A\,e^{-\alpha\lambda^2\tau}$.

Now $\frac{d^2R}{dr^2} + \frac{1}{r}\,\frac{dR}{dr} + R\lambda^2 = 0$ (Helmholtz)

or

$$\frac{d^2R}{d(\lambda r)^2} + \frac{1}{(\lambda r)}\,\frac{dR}{d(\lambda r)} + R = 0$$

This is Bessel's equation of zero order ; Bessel's equation of order 'n' being in general :-

$$\frac{d^2y}{dx^2} + \frac{1}{x}\,\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0\ , \quad n = \text{const.}$$

The solution, a Bessel (cylinder) function, to the Helmholtz equation is

$$R = BJ_o(\lambda r) + C\,Y_o(\lambda r),$$

where J_o = Bessel function of the first kind, zero order, and

Y_o = Bessel function of the second kind, zero order.

$$\therefore t = RT = A\,e^{-\alpha\lambda^2\tau}\left[B\,J_o(\lambda r) + C\,Y_o(\lambda r)\right] \quad . \quad . \quad . \quad . \quad (13)$$

Again the constants must be determined by the boundary conditions. The solution to the Helmholtz equation this time involves two oscillating functions whose amplitudes decrease as r increases. The solution[18] to the Helmholtz equation in a spherical coordinate system contains $\frac{1}{\lambda r}$ $\sin(\lambda r)$ and $\frac{1}{\lambda r}\cos(\lambda r)$ terms which are again oscillating functions whose amplitudes decrease as r increases at a rate even faster than the Bessel functions.

A common situation for unidirectional transient heat conduction is sudden change of either the boundary temperature of the body or the temperature of a fluid surrounding the body. The temperature and heat flow solutions for these and other similar problems frequently involve the sum of a convergent series or the sum of the roots of a transcendal equation. It follows, obtaining a numerical result may be a time-consuming proposition. For this reason, various authors have prepared charts for use in solving transient problems. Heisler[19], for example, gives data for the sudden temperature change of the environment of bodies of initial uniform temperature. His charts may be used to determine the temperature histories at the surface and center of a semi-infinite plate, and at the surface, center and half radius of an infinitely long cylinder or sphere. In a recent publication, Schneider[20] presents approximately fifty graphical solutions for a variety of one-dimensional constant-property transient problems for various geometries and boundary conditions which include both changes in surface temperature and surface heat flux. In general, temperature history charts are plotted in terms of dimensionless groups making up the theoretical relations, thereby making the data universal in application. The usual parameters used are:

1) $N_{Bi} = h\ell/k$ (Biot modulus), an index of the relative resistance of the medium to that of the convective surroundings. (The symbol 'ℓ' designates the thickness of the infinite plate or radius of the cylinder or sphere)
2) $N_{Fo} = \alpha\,\tau/\ell^2$ (Fourier modulus), a dimensionless time and measure of the response.
3) Various dimensionless position and temperature ratios.

ii) Steady Periodic Solutions

The boundary condition temperature is periodic, and since any periodic function can be expressed in terms of a Fourier series, the surface temperature ($t_o = f(\tau)$) can be written as

$$t_o = C_o + \sum_{n=1}^{n=\infty} A_n \cos n\,\omega\,\tau + \sum_{n=1}^{n=\infty} B_n \sin n\,\omega\tau,$$

where the various constants are determined by the usual procedures of harmonic analysis. By working from equation (11) via the product method, and taking the separation constant to be an imaginary number, the general solution[21] corresponding to equation (12) has the form:

$$t = A\, e^{-\lambda x/\sqrt{2}} \left[B \cos (\lambda^2\alpha\,\tau - \lambda x/\sqrt{2}) + C \sin (\lambda^2 \alpha\,\tau - \lambda x/\sqrt{2}) \right].$$

At $x = 0$ (at the surface);

$t = t_o = D \cos \lambda^2\alpha\,\tau + E \sin \lambda^2\alpha\,\tau$, where D and E are arbitrary constants, hence matching the boundary condition is straightforward since the equation compares closely to the Fourier series expression.

The case of a slab (infinite in extent) extending from the surface $x = 0$ to $x = \infty$, subjected to one-dimensional unsteady state heat conduction by a periodic surface temperature, is now considered. If t_o(boundary temperature at $x = 0$) varies in a simple harmonic manner between $-t_{om}$ and $+t_{om}$ such that $t_o = t_{om} \cos \omega\tau$, then it can be shown that

$$t = t_{om}\, e^{-\lambda x} \cos (\omega\tau - \lambda x) \text{ where } \lambda = \sqrt{\frac{\omega}{2\alpha}} ,$$

is the solution to equation (11).

$$\frac{\partial t}{\partial x} = t_{om} \left[-\lambda e^{-\lambda x} \cos(\omega\tau - \lambda x) + \lambda e^{-\lambda x} \sin(\omega\tau - \lambda x) \right].$$

$$\frac{\partial^2 t}{\partial x^2} = t_{om} \Big[\lambda^2\, e^{-\lambda x} \cos(\omega\tau - \lambda x) - \lambda^2\, e^{-\lambda x} \sin(\omega\tau - \lambda x)$$
$$- \lambda^2\, e^{-\lambda x} \sin(\omega\tau - \lambda x) - \lambda^2\, e^{-\lambda x} \cos(\omega\tau - \lambda x) \Big]$$
$$= -\, 2\,\lambda^2\, t_{om}\, e^{-\lambda x} \sin(\omega\tau - \lambda x).$$

$$\frac{\partial t}{\partial \tau} = -\omega\, t_{om}\, e^{-\lambda x} \sin(\omega\tau - \lambda x).$$

Substituting into equation (11) results in

$$2\,\alpha\,\lambda^2 = \omega \quad \text{or} \quad \lambda = \sqrt{\frac{\omega}{2\alpha}} \qquad \text{Q.E.D.}$$

Note also, at $x = 0$,

$t = t_o = t_{om} \cos \omega\tau$ as required.

Several important observations can be made by inspection of the temperature equation

$$t = t_{om}\, e^{-\sqrt{\frac{\omega x^2}{2\alpha}}} \cos\left[\omega\tau - \sqrt{\frac{\omega x^2}{2\alpha}}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

1) For a particular depth x, the maximum temperature (amplitude) occurs when the cosine term has a value of unity, i.e. when $\omega\tau - \sqrt{\frac{\omega x^2}{2\alpha}} = 0$. Hence the time lag is $\Delta\tau = x\sqrt{\frac{1}{2\alpha\omega}}$. The lag increases with depth; the temperature variations at certain depths will be in phase with the surface temperature variations.

2) The temperature variation at any depth has the same periodicity as the surface temperature variation, however the amplitude decreases exponentially with depth. The depth of zero amplitude occurs where the temperature variations are considered (specified) negligibly small. At still greater depths, the surface temperature has no influence other than that of establishing the steady temperature level.

3) The damping factor contains ω - the higher the frequency of the thermal oscillation, the less the penetration. On the other hand the greater the thermal diffusivity, the deeper the penetration.

Heat conduction under steady periodic conditions occurs in machine operations and various industrial processes involving thermal cycling. In addition, quasi-steady heat conduction takes place on a very large scale in nature with the surface of the earth being heated and cooled by the atmosphere both on a daily and annual cycle basis. The preceding slab analysis applies (with appropriate surface temperature cycle) to the temperature variations in the ground, as well as to several other situations such as the penetration of temperature into the walls of a cylinder of an internal combustion engine. In general, Fourier analysis is widely used in solving problems of this type.

3.1.3 Heat Conduction in Three or Four Independent Variables

Heat conduction in three and four independent variables encompasses three-dimensional steady state and two- and three-dimensional transient. These constitute the complex cases of heat conduction. Unfortunately, an analytical solution to any given problem of this type may not be practical or even possible due to mathematical difficulties. This is somewhat ironical since solids undergoing heat conduction are always finite in size, hence an exact solution for temperature requires its specification using three coordinates. For many practical cases, however, end effects are sufficiently localized that a solution suitable for the bulk of the geometry can be obtained by considering temperature to depend on only one or two space variables. McAdams[22] gives empirical equations to solve for three - dimensional steady heat flow for rectangular boxes of uniform wall thickness with isothermal inner and outer surfaces. Similar equations have been published[23] for rectangular blocks, square plates, etc. Morse and Feshbach[24] give solutions to several cases of the Laplace equation in three dimensions, however, they point out that the Laplace equation separates in only a few coordinate systems, and the three-dimensional solutions nearly always turn out to be infinite series or integrals which generally do not converge rapidly. Regarding transient problems, the product superposition principle due to Newman[25] can be applied to solve certain two - and three-dimensional transient cases (finite cylinder, brick-shaped objects,etc) using unidirectional transient solutions (charts) - the solutions for these cases being the product of two or three solutions each in one-dimension.

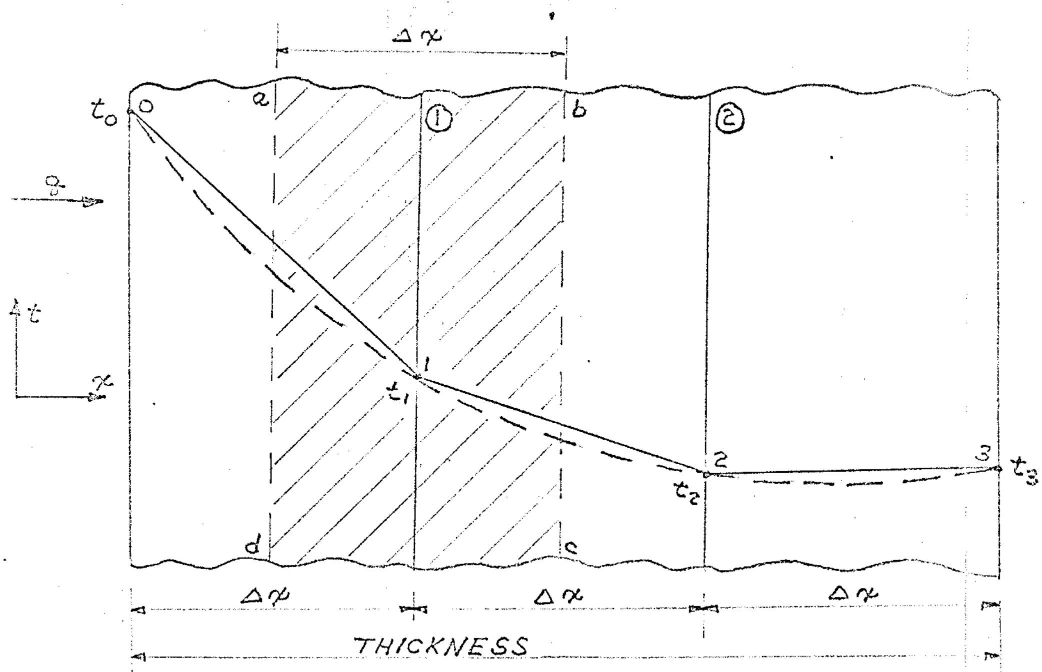
A number of solutions to specific problems of the types under consideration have been published over the years. For example, Dicker and Friedman[26] give the solution to the transient heat conduction equation for two-dimensional convex quadrilateral and three-dimensional convex hexahedral domains (non-separable) using Galerkin's method in conjunction with the Laplace transform. The examples given in their paper are for uniform initial temperature with the boundaries subjected to sudden change of temperature. Steady state solutions as $\tau \rightarrow \infty$ are included. Although these various solutions are available in the literature, their values are largely academic since they are intended basically to demonstrate particular analytical techniques. The final results generally have no direct application, i.e. the chance of having the boundary conditions of a real problem match those of a published solution is quite remote. Any real problem will tend to be unique in at least certain aspects, hence an analytical solution must be approached from basic fundamentals.

To summarize, solutions to heat conduction problems in three or four independent variables by purely mathematical methods, even if theoretically possible, are subject to limitations which depend both on the complexity of the problem and on the mathematical ability of the analyst. The current restrictions on analytical methods of solution, however, are being continually removed by advances in mathematics. In this respect, the more modern mathematical techniques - operational calculus (in particular, the Laplace transform to take care of the time variable), integral transforms, matrix formulation, etc - are meeting with increasing popularity and success.

3.2 Finite - Difference Methods

In the finite-difference approach to solving heat conduction problems, the geometry is subdivided into a number of regions; the thermal relations for each region (zone) being represented by its central point (node). Temperature is computed for each node, i.e. at discrete points in the geometry rather than at any point as in analytical solutions. The basis and procedure for the method is given for the case of one-dimensional transient heat conduction - this case is relatively simple but general in that it involves both space and time increments.

The geometry is as shown in the following diagram. The initial conditions ($\tau = 0$) give the true initial temperature distribution as indicated by chain line. The geometry has been subdivided, and attention is focused on the shaded zone abcd. The temperatures at points 0 and 1 are joined by a straight line, from which the temperature gradient at plane ad is approximated by $(\frac{t_o - t_1}{\Delta x})$. Similarly, the temperature gradient at plane bc is approximated by $(\frac{t_1 - t_2}{\Delta x})$. In addition, the temperature at plane 1, i.e. t_1, approximates the average temperature of the zone.



From the general heat conduction equation for a homogeneous isotropic medium with constant thermal properties and $W = 0$;

$$\alpha\left[\frac{\partial^2 t}{\partial x^2}\right] = \frac{\partial t}{\partial \tau} .$$

The term in brackets $\dfrac{\partial(\frac{\partial t}{\partial x})}{\partial x}$ is the slope of the temperature gradient.

Written in finite-difference form, the equation becomes

$$\alpha\left[\frac{\Delta^2 t}{(\Delta x)^2}\right] = \frac{\Delta t}{\Delta \tau}$$

Now $\dfrac{\partial(\frac{\partial t}{\partial x})}{\partial x} \rightarrow \dfrac{\Delta^2 t}{(\Delta x)^2} = \dfrac{1}{\Delta x}\left[\dfrac{(t_o - t_1)}{\Delta x} - \dfrac{(t_1 - t_2)}{\Delta x}\right]$ for plane 1 (zone 1);

$$= \frac{t_o - 2\,t_1 + t_2}{(\Delta x)^2}$$

Also $\dfrac{\partial t}{\partial \tau} \rightarrow \dfrac{\Delta t}{\Delta \tau} = \dfrac{t_1' - t_1}{\Delta \tau}$,

where t_1' is the future temperature at point 1 at the end of the time increment $\Delta\tau$.

$$\therefore \quad \alpha \left[\frac{t_o - 2t_1 + t_2}{(\Delta x)^2} \right] = \frac{t_1' - t_1}{\Delta\tau}$$

$$t_o - 2t_1 + t_2 = \left[\frac{(\Delta x)^2}{\alpha\Delta\tau} \right] (t_1' - t_1)$$

Defining a modulus, $M = (\Delta x)^2/\alpha\Delta\tau$, gives

$$t_o - 2t_1 + t_2 = Mt_1' - Mt_1$$

Hence an explicit equation for t_1' becomes

$$t_1' = \frac{t_0 + (M - 2)\, t_1 + t_2}{M} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (15)$$

Similarly,

$$t_2' = \frac{t_1 + (M - 2)\, t_2 + t_3}{M}$$

From the preceding, if the true temperature distribution is non-linear, it is obvious that the finer the geometric subdivision (i.e. the smaller Δx), the more closely the finite-difference formulation approaches the partial differential equation.

Equation (15) can be derived in another manner which sheds more light on the nature of the approximations inherent in the method. For the alternative approach, a heat balance is set up on the element abcd which has a volume $(A \cdot \Delta x)$, where A is the cross-sectional area in plane 1.

Heat in - Heat out = Heat stored or released

$$\Delta\tau \left[\frac{kA(t_o - t_1)}{\Delta x} - \frac{kA(t_1 - t_2)}{\Delta x} \right] = (A\Delta x)\, \rho C_p\, (t_1' - t_1)$$

Simplification and substitution for M leads to equation (15). It is noted that the initial temperature gradients are used to compute the heat flows during the time interval. This indicates - the smaller $\Delta\tau$, the better the accuracy.

3.2.1 Transient Numerical Solutions

The general procedure can be amply illustrated by considering the previous development to represent an actual problem. Temperatures t_o and t_3 are given by the boundary conditions. Knowing Δx from the construction, a suitable $\Delta\tau$ could be selected based on convergence criterion (to be covered later), and hence M could be calculated. Now knowing t_1 and t_2 from the initial conditions ($\tau = 0$), values for t_1' and t_2' can be calculated from the derived equations, thereby solving for the temperature distribution at $\tau = 0 + \Delta\tau$. Using t_1' and t_2' as the new base values and the appropriate specified boundary temperatures, the procedure is repeated to solve for temperature at $\tau = 0 + \Delta\tau + \Delta\tau = 2\Delta\tau$. Etc.

The numerical method for transient heat conduction can accommodate practically any type of surface condition, including the case of a variable convective surface which is of considerable practical importance. If a boundary condition varies with time, the relationship is handled numerically by a series of small step changes. Dusinberre[27] gives a number of rules and suggestions regarding subdivision of the geometry to suit the boundary conditions, and handling of the various surface conditions.

The matter of convergence is considered next. For unidirectional transient heat conduction, the convergence criterion is $M \geq 2$. No formal justification will be given, however several facts are noted. From the definition of M, for $M < 2$, $\Delta\tau$ must be relatively large which implies poor accuracy. Regarding equation (12), if $M < 2$, temperature t_1'

depends on t_1 in a negative sense. This suggests convergence will be oscillatory if at all possible. Establishment of a convergence criterion was initially found necessary by experience - using a value of M too small resulted in the calculated numerical answers oscillating and finally diverging as time became large. The convergence criterion was subsequently set by a combination of empiricism and intuitive argument. At least one mathematical investigation of convergence has been made. Fowler[28] recognizes and discusses two distinct types of convergence; firstly, the convergence of numerical solutions as time becomes large, and secondly, the convergence as time and space increments go to zero of the numerical solution becoming identical with the corresponding analytical solution. His investigation using a one-dimensional slab of initially uniform temperature includes several types of boundary conditions. The various solutions are expressed in terms of a set of polynomials or finite Fourier series. The methods used for establishing convergence of the second kind can be extended to two and three dimensions with arbitrary initial temperature distribution.

The convergence criterion can be put into a more useful form. This is done following the procedure of Dusinberre[27,29]. Several equations useful for setting up a calculation result.

Equation (15) may be written as

$$t_1' = \left(\frac{1}{M}\right) t_o + \left(\frac{M-2}{M}\right) t_1 + \left(\frac{1}{M}\right) t_2, \text{ or as}$$

$$t_1' = (F_{01})t_o + (F_{11})\, t_1 + (F_{21})\, t_2 \;;$$

a 'weighting' procedure where the values of the F factors are defined by the correspondence.

Now $\Sigma F = 1$ since $\frac{1}{M} + \left(\frac{M-2}{M}\right) + \frac{1}{M} = 1.$

Also $F_{01} = F_{21} = \frac{1}{M} = \frac{\alpha\Delta\tau}{(\Delta x)^2} = \frac{k\Delta\tau}{\rho C_p(\Delta x)^2} \cdot \frac{A}{A} = \frac{K\Delta\tau}{C}$,

where $K = \frac{kA}{\Delta x}$, say in $\frac{Btu}{hr-{}^oF}$,

and $C = (A\cdot\Delta x)\,\rho C_p$, say in Btu/^{o}F.

$$\text{Hence in general, } F = \frac{K\Delta\tau}{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

Now $F_{11} = 1 - (F_{01} + F_{21})$

$= 1 - \Sigma\frac{K\Delta\tau}{C}$ in general.

Convergence requires F_{11} to be zero or positive, i.e. negative coefficients must be avoided.

Hence $\Sigma\frac{K\Delta\tau}{C} \leq 1.$

This relationship can be rewritten to form the general convergence criterion which is given by

$$\Delta\tau \leq \frac{C}{\Sigma K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

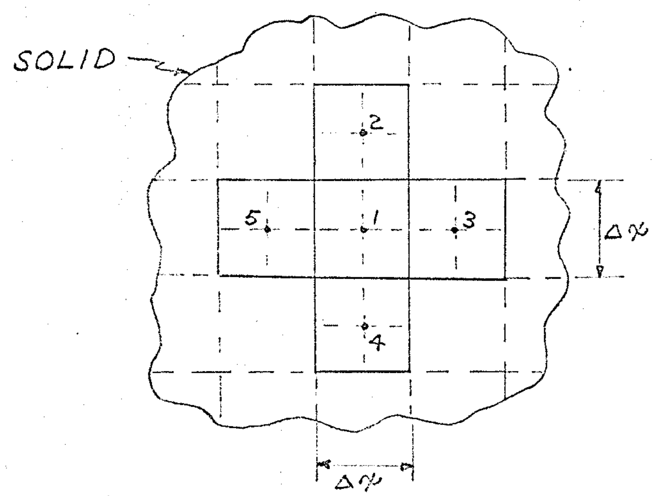
The preceding may be readily extended to two and three dimensions. For the two-dimensional transient case, the simplest subdivision of geometry is a network of squares as indicated in the following diagram. Setting up a heat rate balance on Zone/node 1; for unit area:-

$$\frac{k}{\Delta x}\left[(t_2 - t_1)+(t_3 - t_1)+(t_4 - t_1)+(t_5 - t_1)\right] = \frac{\Delta x\rho C_p(t_1' - t_1)}{\Delta\tau} .$$

Again letting $M = \frac{(\Delta x)^2}{\alpha\Delta\tau}$, leads to

$t_2 + t_3 + t_4 + t_5 - 4t_1 = M(t_1' - t_1)$ or

$$t_1 = \frac{t_2 + t_3 + t_4 + t_5 + (M - 4)t_1}{M}$$



The convergence criterion is M ≥ 4. The general equations (16) and (17) apply. Dusinberre[27] discusses various ways of subdividing a cylinder, and provides useful data on squares, rectangles, annular segments and triangles as elements of an area.

The basic relations for a three-dimensional transient system can be derived in a similar fashion. The convergence criterion for a solid sectioned into cubical regions is M ≥ 6.

The basic ideas of the finite-difference numerical approach are summarized at this point. As outlined, the procedure is essentially that of a heat balance. The future temperature of a node depends on the temperatures of the surrounding nodes (heat flows) and its own present temperature (heat storage or release). For heat capacity purposes, the material in a zone is effectively lumped at the node. The lines connecting adjacent nodes may be considered as heat flow paths. Internal heat generation can be handled by including the heat generation term in the initial heat balance.

In general, although the basic principles of the method are elementary, a certain amount of experience is required in order for the method to be applied most effectively. The main concern in using the method is the question of accuracy. Some truncation errors in distance and time are always inherent in the method. In general, the smaller the space increments (with accompanying smaller $\Delta\tau$), the better the accuracy. A stepped subdivision particularly near boundaries is often advantageous. The degree of refinement of the subdivision depends on the accuracy sought, which in turn has a bearing on the solution use and the accuracy of the problem data. Accuracy for its own sake is at the expense of time and effort. There are no simple 'thumbrules' for estimating the accuracy obtained in solving a given problem using a given subdivision of geometry. If a check on all or part of the calculation is not possible by an analytical method, the inaccuracies can usually only be discussed qualitatively. Accordingly, preparing a problem for solution requires a fair amount of engineering judgment. In order to ensure suitable accuracy, it is generally necessary to use either a relatively fine subdivision of geometry with the corresponding near maximum $\Delta\tau$, or a time increment which is substantially less than the maximum permissible $\Delta\tau$ as given by the convergence criterion. In any event, the computation usually involves a large amount of arithmetic which tends to make the method tedious and time consuming. This disadvantage can be overcome by using a digital computer. This method of computation is very practical since the calculation procedure consists of a repetitious fixed sequence of arithmetic operations which can be readily programmed.

The finite-difference numerical approach coupled with modern computational facilities, leads to an extremely versatile and powerful method of solving transient heat conduction problems. The method is most advantageously used for solving problems with geometries such as an assembly consisting of several different materials. Here with judicious subdivision, the different material properties can be readily transformed into appropriate F factors. The time increment is set by the governing node(s). Instantaneous heat flows between any two nodes for any case at any time can be obtained by applying equation (1) in finite difference form i.e. $q = \frac{kA}{\Delta x}(t_n - t_{n \pm 1})$. Cases of variable thermal conductivity or other material properties can be handled by applying suitably altered F factors as the computation progresses. The method can be used for the more complex transient heat conduction problems such as with moving heat sources or phase change. The latter will be discussed later in this thesis.

The method under consideration can be used for both pure transient and steady periodic problems. For the steady periodic case, the computation procedes from assumed initial temperature conditions to the final repetitive state by a process of iteration. The cycle time is divided into a number of equal time steps ($\Delta\tau$) in keeping with the convergence criterion. Iteration may similarly be used to obtain a steady state solution, i.e. if a problem having constant boundary conditions is set up as a transient, given sufficient time, the process will move from the initial conditions to steady temperature conditions. The restriction here is that the heat conduction medium must be homogeneous.

The finite-difference numerical method described is called the 'explicit' method due to the fact the equations give future temperature in terms of the present (known) temperature. An alternative numerical method uses the temperature gradients at the end of the time increment in conjunction with the heat stored or released during the time interval. This formulation is called the Liebmann[30] method, and has no apparent restriction on the value of the modulus, M. Nevertheless, even though the calculation is always stable regarding convergence, using a large $\Delta\tau$ indiscriminately can result in a solution which may be substantially in error. This method is 'implicit' in that the equations contain all future (unknown) temperatures except for one present temperature. Hence solving for temperature at each time step involves the solution of a set of simultaneous equations. Compared to the explicit method, the advantage achieved by removing the restriction on the magnitude of $\Delta\tau$ tends to be offset by the more complicated calculation.

3.2.2 Transient Graphical Solutions

From equation (15), for $M = 2$, the expression for future temperature is:

$$t'_1 = \tfrac{1}{2}(t_o + t_2).$$

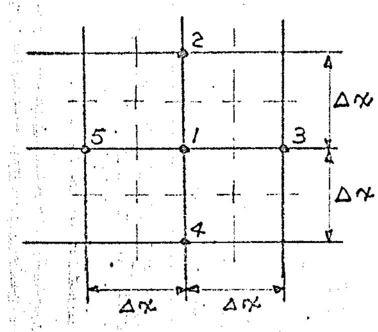
This equation for one-dimensional transient heat conduction can be easily solved graphically. Equation (15) is based on a geometry such as a plane wall subdivided into several layers of uniform thickness (Δx). Hence, if temperature versus position is plotted to scale, since t'_1 is the average of t_o and t_2, the graphical solution for t'_1 is the temperature corresponding to the point of intersection of a straight line joining t_o and t_2 and plane 1. This method of solving one-dimensional transient problems is generally referred to as the Schmidt plot. Since $M = 2$, from the definition of M, the time interval is given by $\Delta\tau = (\Delta x)^2/2\alpha$. This equation shows the fixed relationship between subdivision of the geometry and the time step. The selection of a suitable Δx for a given problem depends on the nature of the required solution. The choice of the space increment, however, need not remain fixed for the complete graphical construction, but rather can be varied as desired. The Schmidt plot can be used for constant or variable boundary temperatures or heat flux, and for convective surfaces. This method can also be used for multi-layer wall problems. For this case, the space increments for the different materials are proportional such that the same $\Delta\tau$ applies throughout.

Jakob[31] describes other graphical methods for solving transient problems involving cylinders and spheres. As in the case of the Schmidt plot, these methods are limited by their nature to one-dimensional heat flow. The main feature of a graphical solution is that it provides a picture of temperature change with time.

3.2.3 Steady State Numerical Solutions

The two main methods of solving steady state heat conduction problems by the finite-difference approach are the iteration and relaxation methods. Both methods can be used to solve problems in one, two or three dimensions with or without internal heat generation. The bases for both methods are essentially identical; the methods differ only in procedure. As in the transient numerical method, the geometry is subdivided into a number of similar regions. Actually, this subdivision can be made using practically any type of element, however the usual (and simplest) procedure for two and three-dimensional cases is to use a network of squares or cubes respectively. By virtue of steady temperature conditions, there can be no heat storage or release, hence the emphasis is on the nodes (points of intersection of the grid) and the lines connecting the nodes (heat flow paths).

The basic equation is now derived for the two-dimensional steady state case with no internal heat generation using a square grid.



For a homogeneous isotropic material, by a heat rate balance on node 1:

$$\frac{kA}{\Delta x}\left[(t_2 - t_1) + (t_3 - t_1) + (t_4 - t_1) + (t_5 - t_1)\right] = 0$$

Since $\frac{kA}{\Delta x} \neq 0$,

$$t_2 + t_3 + t_4 + t_5 - 4t_1 = 0 \quad . \; . \; . \; . \; . \; . \; . \qquad (18)$$

It can be shown[32] that this equation is a solution to the Laplace equation in two dimensions to within good accuracy.

a) Iteration (Gauss-Seidel) Method

Continuing with the preceding development using the Dusinberre nomenclature, the basic equation can be kept general by writing as:

$$K_{21}(t_2-t_1)+K_{31}(t_3-t_1)+K_{41}(t_4-t_1)+K_{51}(t_5-t_1) = 0$$

or

$$K_{21}t_2 + K_{31}t_3 + K_{41}t_4 + K_{51}t_5 - \Sigma Kt_1 = 0.$$

Dividing through by ΣK and assigning F' factors such as $F'_{21} = K_{21}/\Sigma K$, etc, leads to:

$$F'_{21} t_2 + F'_{31} t_3 + F'_{41} t_4 + F'_{51} t_5 = t_1 \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

Here again $\Sigma F = 1$.

Similar equations can be written for all the nodes, and the first step is to assume a temperature at each node based on all available information. The procedure is now by iteration whereby a new temperature at each node in turn is computed based on the tentative temperature of surrounding nodes, i.e. using equation (19) explicitly. The procedure is convergent since the fixed boundary temperatures are continually averaged in on the provisional unknown temperatures. At the final stage, the same temperatures repeat throughout the field which is equivalent to all equations being solved simultaneously. The better the initial guess on temperature distribution, the fewer the number of iterations required. Dusinberre suggests if the final temperature distribution is difficult to predict, it is usually worthwhile to solve the problem first using a course network. The initial temperature distribution for a finer network is then based on this solution. Upon solving the problem, if desired, the isothermal field can be plotted by interpolation.

Since the iteration procedure entails a fixed repetitious sequence of operations, the method can be readily adapted to digital computer computation. Internal heat generation can be handled by the iteration method by including the heat generation term in the heat balance used to derive the working equation.

The method of squaring[33] can also be used to solve the Laplace and Poisson equations in two dimensions. The basis of this method is identical to the iteration method; the procedure differs in that the final temperatures are obtained by a process involving differences which is used to modify the initial guesses of temperature. The advantage of the method is speed of solution. This method can also be programmed for a digital computer.

b) Relaxation Method

For the purpose of the method, equation (18) is written as $t_2 + t_3 + t_4 + t_5 - 4t_1 = R$, a residual which of course must be zero at the solution. As for iteration, the first step for the relaxation method is to assume a temperature for each node. The initial residuals are calculated. The procedure then is to adjust the temperatures of the various nodes so as to make the residuals zero everywhere. The sequence of the relaxation is arbitrary; as a general rule, the larger residuals are relaxed first. The amount of temperature adjustment at any stage is based on the relaxation pattern, i.e. for the Laplace equation in two dimensions using a square network, from equation (18) it can be seen that a change of one unit of temperature at t_1 changes the residual by ∓ 4, whereas the same change in temperature for any one of the adjacent nodes causes a residual change of only ± 1. In general, the relaxation pattern for the nodes near the boundaries will be different, and depend on the boundary conditions. As with iteration, the better the initial guess, the sooner the solution is reached. Experience is necessary in order for the method to be used efficiently. Over-relaxation early in the procedure is often advantageous in reducing the time for solution.

The relaxation method is best known as a method for solving the Laplace and Poisson equations in two dimensions. The method can be extended to three-dimensional problems, however the procedure becomes complicated due to the additional two(or four) relaxation possibilities. It is generally easier to use the iteration method for this case.

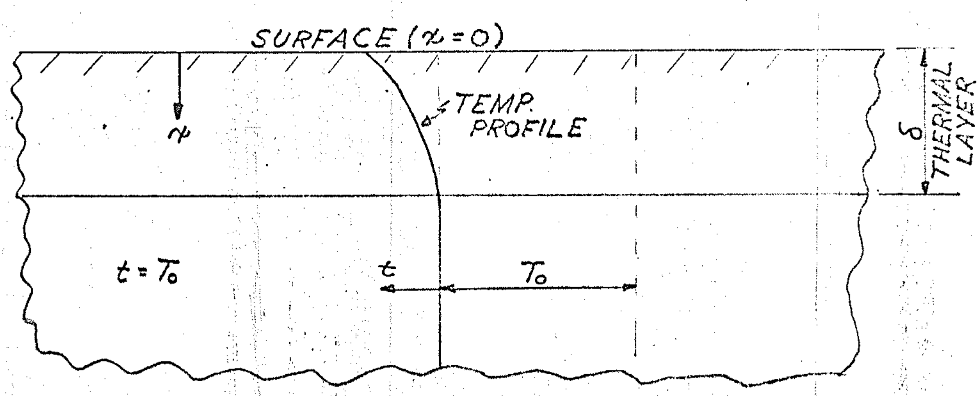
3.2.4 Steady State Graphical Solutions

The only graphical method of any importance is the freehand flux plot used to solve two-dimensional steady state problems. This method was described and discussed in Section 3.1.2. a) iii).

3.2.5 Heat-Balance Integral Method

The heat-balance integral method for solving one-dimensional transient problems is included in this section for two reasons - it is semi-numerical and approximate; the latter due both to a finite-difference approximation and an assumption regarding the temperature profile.

The basic method is illustrated using an example given by Goodman[34] in which a semi-infinite slab at uniform temperature (T_0), is subjected at the surface $x = 0$ to a constant heat flux (F) starting at $\tau = 0$.



The boundary condition is

$$k \frac{\partial t}{\partial x}\Big|_{x=0} = -F \quad \text{for } \tau > 0$$

From equation (2), for W = 0 and $\alpha = \frac{k}{\rho C_p}$, the governing equation is

$$\alpha\left(\frac{\partial^2 t}{\partial x^2}\right) = \frac{\partial t}{\partial \tau} \quad \text{for } \tau > 0 \text{ and } \delta > x > 0,$$

where δ is the thermal layer and a function of time. The concept of a thermal layer is analogous to the laminar boundary layer thickness in convection heat transfer, except that the thermal layer thickness increases with time. At any time, for $x > \delta$ the slab remains at its original temperature since there is no heat conduction beyond $x = \delta$. The heat-balance integral is obtained by multiplying both sides of the heat conduction equation by dx and integrating over the thermal layer, i.e.

$$\alpha \int_0^\delta \frac{\partial^2 t}{\partial x^2}\, dx = \int_0^\delta \frac{\partial t}{\partial \tau}\, dx \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; (20)$$

In order to solve this equation, the shape of the temperature profile within the thermal layer is assumed to be represented by a second-degree polynomial in x, i.e.

$$t = a + bx + cx^2, \text{ hence}$$

$$\frac{\partial t}{\partial x} = b + 2cx.$$

From the boundary condition, $\frac{\partial t}{\partial x}\Big|_{x=0} = -\frac{F}{k} = b.$

Also $\frac{\partial t}{\partial x}\Big|_{x=\delta} = 0$ since there is no heat transfer beyond $x = \delta$.

$$\therefore\; b + 2c\delta = 0 = -\frac{F}{k} + 2c\delta, \text{ whence } c = \frac{F}{2k\delta}.$$

Now $t = T_o$ at $x = \delta$, hence

$$T_o = a + b\delta + c\delta^2 \quad \text{or } a = T_o - b\delta - c\delta^2 .$$

$$= T_o + \frac{F\delta}{k} - \frac{F\delta}{2k} = T_o + \frac{F\delta}{2k} .$$

$$\therefore t = \left(T_o + \frac{F\delta}{2k}\right) - \frac{F}{k}x + \frac{F}{2k\delta} x^2$$

(Note, the coefficients a and c depend on δ and hence on time).

Using this expression for t,

$$\int_o^\delta t\,dx = T_o\delta + \frac{F\delta^2}{6k}$$

Also, using a finite-difference approximation for the change of slope of the temperature gradient, the average $\frac{\partial^2 t}{\partial x^2}$ over the boundary layer is:

$$\frac{\partial^2 t}{\partial x^2}\bigg|_{av} = \frac{1}{\delta}\left[\frac{\partial t}{\partial x}\bigg|_{x=\delta} - \frac{\partial t}{\partial x}\bigg|_{x=0}\right]$$

$$\therefore \alpha\int_o^\delta \frac{\partial^2 t}{\partial x^2}\,dx \approx \alpha\left[\frac{\partial t}{\partial x}\bigg|_{x=\delta} - \frac{\partial t}{\partial x}\bigg|_{x=0}\right] = -\alpha\frac{\partial t}{\partial x}\bigg|_{x=0} = \alpha\frac{F}{k}$$

since $\frac{\partial t}{\partial x}\big|_{x=\delta} = 0$.

Substituting into the heat-balance integral (equation (20)) gives

$$\alpha\frac{F}{k} = \frac{\partial}{\partial\tau}\left[\int_o^\delta t\,dx - T_o\delta\right]$$

$$= \frac{\partial}{\partial\tau}\left(\frac{F\delta^2}{6k}\right)$$

$$\alpha\frac{F}{k}\int_o^\tau \partial\tau = \int_o^\tau \partial\left(\frac{F\delta^2}{6k}\right)$$

$$\alpha\frac{F}{k}\cdot\tau = \frac{F\delta^2}{6k}\bigg|_o^\tau = \frac{F\delta^2}{6k} ,$$

whence $\delta = \sqrt{6\alpha\tau}$.

The surface temperature at any time is given by

$$T_s = T_o + \frac{F\delta}{2k}, \quad x = 0$$

$$= T_o + \frac{F}{2k}(6\ \alpha\ \tau)^{\frac{1}{2}}$$

Goodman points out this solution for surface temperature is 9% in error compared to the corresponding analytical solution. This error can be reduced to 2% by using a third-degree polynomial in x to approximate the temperature profile.

The heat-balance integral method appears to offer good accuracy coupled with solution times comparable to analytical methods. The method is relatively new and still under development. Yang[35] outlines an improved integral procedure.

3.3 Analog Methods

The solution to a heat conduction problem by an analog method involves the measurement of variables in a system which obeys an equation indentical in form to that which governs the heat conduction. Although several non-electric analogs (hydraulic, soap film, etc) have been used to solve the heat conduction equation, only electric analogs will be considered since these are by far most extensively used due to their speed, accuracy and ease of construction and operation. The general technique, following selection and programming of an appropriate analog system, is to impose analogous boundary/initial conditions and measure or record the pertinent electrical quantities. An important feature of the electric analog method is the ease at which boundary/initial conditions and/or the equivalent thermal diffusivity can be varied in the handling of transient heat conduction problem. Hence if desired for the purpose of design optimization or other reason, the solution to a problem can be readily investigated over a range.

3.3.1 Steady State Problems

Unidirectional steady state heat conduction problems are best solved by analytical methods; however, a simple thermal-electric analog exists which on occasions is useful for heat flow computations. The equations

$q = \left(\frac{kA}{x_2-x_1}\right)(t_1-t_2)$ and Ohm's law, $I = \left(\frac{1}{R}\right)E$, are similar; the analogous quantities being:

1) $(t_1 - t_2)$ and E (potential)
2) q and I (flow)
3) $\left(\frac{x_2 - x_1}{kA}\right)$ and R (resistance)

Two and three-dimensional steady state heat conduction problems with or without internal heat generation, under conditions already mentioned, form respectively Poisson and Laplace equations. In a much broader sense, the Laplace equation governs fields having no internal excitation, whereas the Poisson equation pertains to fields with distributed internal excitation. The Laplace equation in two dimensions, for example, describes the velocity potential in two-dimensional irrotational flow of an ideal fluid, and the two-dimensional distribution of electric potential in a region of constant resistivity. The latter of course forms the basis for the electrical continuous type geometric analog.

The dry paper type of field plotter, commonly referred to as the analog field plotter, is a very useful tool for solving two-dimensional Laplace problems having intricate geometries. The required apparatus consists essentially of a DC power supply, voltage divider, high sensitivity null detector and marking stylus. The general procedure is to draw the geometry to scale on Teledeltos paper (a conducting sheet having uniform resistance and thickness characteristics), paint (silver, copper, etc) in the boundary lines, and apply an electrical potential across the opposing boundary strips. If temperature is the main interest, specific isotherms can be plotted using the stylus-voltage divider-detector circuit. Alternatively, it is more usual to plot the isothermal field using equal increments of temperature (potential) drop. The flux plot described in Section 3.1.2. a) iii) can be completed by sketching in the heat flow lanes, thereby permitting evaluation of the shape factor and hence the heat flow for the problem. Shape factors may also be determined directly[36] by comparing the voltage drop across an unknown shape to that for one of known shape factor.

The Poisson equation in two dimensions can also be solved by the analog field plotter by employing a change of variables which transforms[37] the Poisson equation $\nabla^2 t = c$ into a Laplace equation $\nabla^2 t' = 0$. The transformed boundary conditions are fairly difficult to apply but are manageable.

McDonald and Nikiforuk[38] estimate the limit of accuracy for the conducting sheet analog to be about 2 per cent. The wet type of continuous medium, i.e. the electrolytic tank, is capable of even better accuracies. Using the same basic procedure as for the analog field plotter, the electrolytic tank can be used to solve two-and certain three-dimensional steady state problems.

3.2.2. Transient Problems

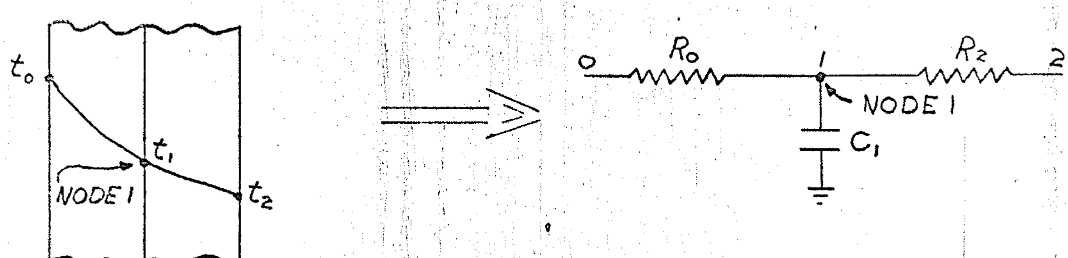
The two main electric analog methods for solving transient heat conduction problems are the resistance - capacitance (R-C) analog computer and the electronic analog computer. Both systems perform by replacing the continuous medium by lumped networks which is equivalent to using a finite-difference approximation.

a) R-C Analog (Beuken Model)

The analog can be shown by considering the one-dimensional transient equation (diffusion equation) replaced by the following partial finite-difference equation:

$$t_o - 2t_1 + t_2 = \frac{(\Delta x)^2}{\alpha} \frac{dt_1}{d\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The geometric subdivision together with the analogous R-C network is illustrated in the following sketch.



According to Kirchhoff's current law, the electric potential in the vicinity of node 1 is:

$$\frac{V_o - V_1}{R_o} + \frac{V_2 - V_1}{R_2} = C_1 \frac{dV_1}{d\tau}$$

Hence for $R_o = R_2 = R$,

$$V_o - 2V_1 + V_2 = RC_1 \frac{dV_1}{d\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

which is analogous to the temperature equation. The capacitor is the analog of the heat storage capacity of the element of material, while the resistors represent the thermal resistances;

i.e. $RC_1 \longrightarrow \frac{(\Delta x)^2}{\alpha} = \left(\frac{\Delta x}{kA}\right)\left[(\Delta x \cdot A)\ \rho C_p\right]$

Similar analogs for two-and three-dimensional transient heat conduction can be devised using R-C networks[39].

A transient heat conduction problem can be programmed for an R-C network analyzer by choosing scale factors which permit suitable capacitor and resistor settings (or selection) and establish the relation between network time and problem time. The initial and boundary conditions establishing the voltage range, are applied; and the transient voltages at the various nodes are recorded. In addition to truncation errors due to the finite-difference approximation, R-C networks are subject to errors due to capacitor leakage and input-output equipment.

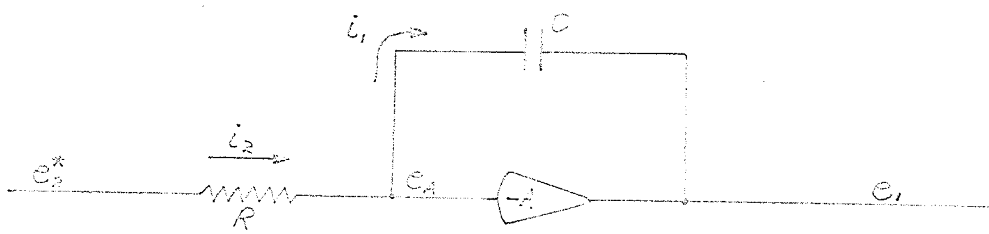
b) Analog Computer (Differential Analyzer)

The finite-difference approach for analog computer simulation is necessary due to the fact the analog computer can continuously integrate with respect to only one independent variable at any given time. Equation (21) rewritten as

$$t_1 = \frac{\alpha}{(\Delta x)^2} \int_0^{\tau} (t_o - 2t_1 + t_2)\, d\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

forms the basic operating equation.

The direct-coupled electronic analog computer consists essentially of operational amplifiers in groups, together with precision resistors, capacitors and potentiometers. Operations such as addition, subtraction, multiplication by a constant, and integration with respect to time, can be performed by making appropriate electrical connections. Integration, for example, is obtained by placing a capacitor in parallel with the amplifier as indicated in the following schematic.



The amplification (-A) is negative due to the fact the amplifier reverses the phase of the input.

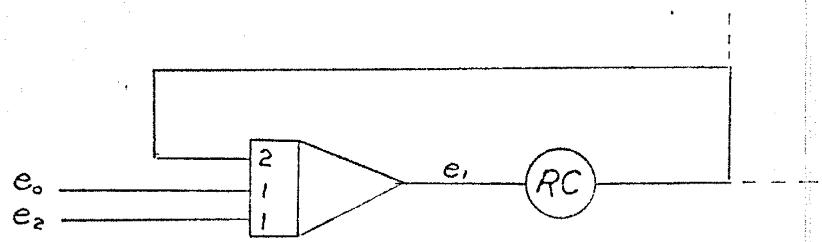
Using the differential operator, $p \equiv \frac{d(\ )}{d\tau}$:-

$i_2 = \frac{e_2^* - e_A}{R} = Cp(e_A - e_1)$ since $i_1 \approx i_2$, the reason being that the amplifier draws negligible current. Now $e_A = \frac{-e_1}{A} \approx 0$ since $A \approx 10^6$ while $e_1 < 100$ volts in general.

$$\therefore \frac{e_2^*}{R} = -Cpe_1,$$

$$\text{or } e_1 = -\frac{1}{RC}\left(\frac{e_2^*}{p}\right) = -\frac{1}{RC}\int_0^\tau e_2^* \, d\tau .$$

This equation parallels equation (23) in the same manner as equations (22) and (21). An integrator having three inputs is required since e_2^* represents $(t_o - 2t_1 + t_2)$. Using analog computer symbolism, the general circuit employing a coefficient setter is as follows:



As for the R-C analog, programming a transient heat conduction problem involves the use of appropriate time scales, and the selection of scale factors relating computer and problem variables. A wide variety of transient problems can be handled, including periodic temperature boundary conditions. The size of the problem that can be handled by any particular analog computer depends on its number of amplifiers. Other than errors due to the discretized model, the accuracy of an analog computer depends on the quality of its components. Amplifier drift is a typical source of error. Freed and Rallis[40] estimate the accuracy of typical heat conduction solutions to be approximately 2 to 3 per cent.

To summarize, there is little doubt that analog methods are capable of solving heat conduction problems in an efficient manner with accuracies within normal engineering requirements. Although usually reserved for the more complex heat conduction situations, analog methods also compare favourable with analytical and numerical methods for the solution of routine problems. Both analog field plotters and electronic analog computers are available commercially. These devices may of course be used to solve problems in other fields. Under certain circumstances, the design and construction of a special purpose analog system may be warranted. A number of schemes, both electric and non-electric have been reported in the literature. For example, Paschkis[41] describes a combined geometric and network method for solving three-dimensional transient heat flow problems (continuous resistance with lumped capacitors).

3.4 Experimental Methods

Heat conduction situations arise occasionally for which the required solution cannot be obtained with sufficient accuracy by the methods already outlined. For these cases, it is necessary to use experimental methods in order to determine the required information. In general, the solution to a heat conduction problem is obtained experimentally only as a last resort due to reasons of cost and time.

Two methods are distinguished for the experimental investigation of a heat conduction problem - the direct approach characterized by field testing, or the more indirect laboratory type method involving the use of models tested under controlled conditions. In design work, the former generally consists of building an instrumented mock-up of a tentative design followed by a series of 'worst condition', etc, tests. Although boundary conditions may be simulated, actual service environmental conditions are usually used. If these conditions tend to vary at random, relatively lengthy test periods are necessary in order that interpretation of results be conclusive. Prototype testing is generally fairly expensive. In fact, the nature of the problem may be such that the cost of a full scale replica is prohibitive.

Size reduction alone through the use of models tends to reduce the cost of an experimental solution. Other features of the model approach can be best shown through a review of the theory of models as applied to heat conduction. The model must only be geometrically similar to the prototype.

The parameters involved in designing a model for heat conduction purposes are summarized as follows:

For steady state heat flow:

$$\left(\frac{q}{kL\Delta t}\right)_M = \left(\frac{q}{kL\Delta t}\right)_P ; \quad \ldots \quad (24)$$

while for the boundary conditions,

$$\left(\frac{hL}{k}\right)_M = \left(\frac{hL}{k}\right)_P \quad \ldots \quad (25)$$

For transient heat conduction:

$$\left(\frac{\alpha\tau}{L^2}\right)_M = \left(\frac{\alpha\tau}{L^2}\right)_P \quad \ldots \quad (26)$$

The subscripts M and P denote model and prototype respectively. L is a characteristic dimension, and h is the surface convective/radiative heat transfer coefficient. All groups are dimensionless. Equations (25) and (26) have been derived by Jakob[42] by the application of the principle of similarity to unidirectional heat conduction. Using his procedure, equation (24) may be derived as follows:

For steady state heat flow, the equations for two systems 1 and 2 are

$$F_1 = \frac{k_1 \, \Delta t_1}{\Delta x_1} , \quad \ldots \quad (a)$$

and $$F_2 = \frac{k_2 \, \Delta t_2}{\Delta x_2} , \quad \ldots \quad (b)$$

where F denotes the heat flux = q/A.

In order for the heat flows for the two systems to be similar, a number of similarity factors must apply which can be written as

$$F_1 = C_F F_2$$

$$k_1 = C_k k_2$$

$$\Delta t_1 = C_t \Delta t_2$$

$$\Delta x_1 = C_x \Delta x_2$$

Substituting into equation (a) gives

$$C_F F_2 = \frac{(C_k k_2)(C_t \Delta t_2)}{(C_x \Delta x_2)} .$$

By comparing with equation (b),

$$C_F = \frac{C_k \cdot C_t}{C_x}$$

Substituting for the C's in this equation gives

$$\frac{F_1}{F_2} = \frac{k_1}{k_2} \cdot \frac{\Delta t_1}{\Delta t_2} \cdot \frac{\Delta x_2}{\Delta x_1}$$

or

$$\left(\frac{F\Delta x}{k\Delta t} \right)_1 = \left(\frac{F\Delta x}{k\Delta t} \right)_2 .$$

Now $F_1 = \frac{q_1}{A_1}$, hence $F_1 \Delta x_1 = \frac{q_1 \Delta x_1}{A_1} = \frac{q_1}{L_1}$,

where $L_1 = A_1/\Delta x_1$ is a characteristic length.

Similarly $F_2 \Delta x_2 = q_2/L_2$, and equation (24) follows.

The model theory as outlined provides a means of extrapolating data from a calculation or prototype/model test, to new temperature conditions. Brown[43] in his paper describing model tests on a buried pair of insulated pipes and on a basementless building, demonstrates the calculation of heat flows for different steady state boundary conditions. The equations for both steady state and transient models also show the material of construction is optional.

A main advantage of the use of models for transient heat conduction investigations lies in the choice of a suitable time scale through appropriate selection of model scale and materials. For the frozen dam problem, for example, assuming the actual material is required due to the phase change aspect, a simple calculation shows that a 1/30 scale model will compress the annual cycle into a test cycle of 9.7 hours. Depending on the size of the dike, even this scale may be too large for practical purposes, however the cycle time would appear to be fairly convenient even though it would be necessary to operate the model continuously for several days in order to achieve steady periodic conditions. This is not to imply that model testing of a frozen dam is feasible. A cursory examination indicates a number of problems other than boundary condition simulation. A main purpose of such a study would be to examine aspects such as moisture migration and its associated heat transfer - a phenomena which involves physical parameters other than those used for the heat conduction theory of models. The point nevertheless remains that model tests would permit the required data to be obtained in a relatively short period of time compared to field tests.

IV FROZEN DAM HEAT TRANSFER STUDY

Feasibility studies recently carried out by Manitoba Hydro on the hydro-electric power development of the lower Nelson River, included the possibility of using frozen earth materials for the construction of dams in areas where permafrost exists. The Department of Mechanical Engineering of the University of Manitoba with assistance from Manitoba Hydro undertook a heat transfer study concerning the thermal aspects of such structures. The main part of this study consisted of solving a number of heat conduction problems associated with the design and construction of frozen dikes. The author was involved in the calculations for these problems; the solutions and the methods used form the body of this part of the thesis.

4.1 General Background

Permafrost research over a period of many years has produced a vast literature containing its own set of terminology[44]. Permanently frozen ground or permafrost is any earth material in which a temperature below freezing has existed continually for many years. The surface material which thaws annually is called the active layer. This depth of thaw locates the permafrost table. The thickness of the permafrost belt depends on many factors including the geothermal gradient, i.e. heat flow from the interior of the earth. The condition of temperature equilibrium between the permafrost and the surface, is referred to generally as the thermal regime. This term implies steady periodic temperature conditions. Other terms used in connection with the frozen dam are self-explanatory. The term "zone of alternate freezing and thawing' has been borrowed from the

Russian literature[45].

Frozen earth materials have much greater mechanical strengths than their thawed counterparts due to the cementing action of ice. The actual strength of a given material depends largely on the ice content and the nature of the ice formations. The necessary structural stability of a frozen dike can be achieved, in principle at least, given a sufficient quantity of a suitable frozen material. Frozen soil is also impervious to water flow, hence a natural grout curtain tends to be automatically provided for protection against seepage through the dike foundation. A number of serious problems arise in construction in permafrost areas due to the destructive action of freezing and thawing, etc, however these were of no immediate concern in the frozen dam heat transfer study.

Approximately 45 per cent of the total land area of both Canada and the U.S.S. R. is underlain by permafrost[46] which ranges from continuous in the north to discontinuous (sporadic) in the more southern regions. The Nelson River, flowing northeasterly between the northern end of Lake Winnipeg and Hudson Bay, is situated within a band of discontinuous permafrost (Fig. 3)[46]. The total power potential of the Nelson River is about 5000 megawatts of which only 160 megawatts (Kelsey) has been developed. A number of feasible power sites exist all along its 400 mile length.

4.2 The Frozen Dam Problem

A theoretical solution of the thermal regime of a potential Nelson River frozen dam was not possible without simplifying assumptions. A review of literature relevant to the subject indicated the thermal regime would be influenced by a large number of complex inter-related factors.

Externally, several processes of energy and moisture exchange would take place between the exposed downstream slope of the dike and its environment. The main factors involved here are air temperature, wind, surface cover (type and colour), insolation, precipitation, condensation and evaporation. Similar processes taking place at the interface of the dike and the water reservoir would depend on water temperature and seepage. Collectively, these processes defined the boundary conditions.

Internally, heat transfer would take place by conduction and through mass movement of water (and air) in the voids of the dike material(s). For a given type of earth material, the thermal diffusivity depends largely on the moisture content and its phase. The effective (measured) thermal conductivity reflects heat conduction through the mineral skeleton and mixed mode heat transfer through pores of the base material. From a broad macroscopic point of view, the frozen or thawed material of the dike could be considered as homogeneous and isotropic, hence equations (1) and (2) would describe the heat conduction for either phase.

On the other hand, the convective type of heat transfer accomplished by moisture movement (air contribution is usually negligible[47]) was more complex. Apparently moisture may move as either a liquid or a vapour. Vapour migration depends on temperature differences[48] since it entails evaporation and subsequent condensation. Martynov[47] points out that heat transfer through vapour migration is negligible in soils with temperatures below 50°C. This then left liquid-phase moisture which could be transferred by either filtration or migration. Both phenomena depend on pressure gradients. Filtration is basically due to gravitational force, whereas migration depends on internal forces (capillary effects, etc.).

The internal pressure is a function of the moisture content and temperature, i.e. migration would be influenced by both the moisture gradient and the temperature gradient. Since these gradients are not necessarily in the same direction, the conductive and convective heat fluxes could have any orientation with respect to each other.

An important factor influencing the thermal regime remained - namely, phase change. A zone of alternate freezing and thawing (active layer) would exist at the downstream slope. In addition, the permanently frozen and thawed zones would be separated by a moving phase change boundary. The heat associated with phase change would depend on the moisture content. It is a fairly well known fact that at temperatures below normal water freezing temperature, earth materials generally contain some moisture that is not frozen. Most of the moisture freezes at 32°F (or a temperature slightly less), followed by some of the remaining moisture freezing as the temperature decreases. Thereafter, despite temperature, a small quantity of moisture remains liquid. Martynov calls this the 'firmly bound' water. It is a characteristic of the material, ranging from zero for sand to 10-20 per cent or higher for clays.

Based on the preceding, the initial task was that of formulating the frozen dam (thermal) problem. Although there is no close relationship between permafrost distribution and air temperature[46], soil temperatures in general have been closely correlated with air temperature and summer sunshine[49]. Pearce and Gold[50] measured temperatures and heat flows in clay soils in a non-permafrost area, and using Fourier analysis, showed that the annual temperature variation in the soil at various depths is remarkably consistent with the theory of heat flow in a homogeneous infinitely thick slab [outlined in Section 3.1.2 b) ii]. Deviations occurred at shallow depths presumably due to property changes as the result of phase change. Their investigation certainly indicated a periodic surface temperature. Compared to air temperature, other factors might have negligible effects or be mutually compensating. Condensation on the surface of the downstream slope of the dike, would tend to offset evaporation. Percolation as the result of rainfall and melting snow, has the effect of equalizing temperatures of the soil at varying depths[49]. This, however, would be localized in the first few feet of the active layer. Regarding the upstream slope, seepage from the water reservoir would be minimized through design. Accordingly, it appeared not unrealistic to assume the boundary conditions consisted of air temperature (with possible allowance for solar radiation) and reservoir water temperature.

Investigations concerning the convective mechanism of heat transfer in soils are still in their initial stages[47]. Even a typical ratio of the heat transferred by water movement to that transferred by conduction was not known. The work of Pearce and Gold[50] suggested this ratio might be small. In any event, since moisture migration cannot occur in frozen earth materials, pure conduction alone would take place in the frozen core of the dike. Hence, since most of the dike was to be kept frozen (either naturally or artificially) for structural reasons, heat transfer by water movement would be restricted to the active layer and to a region of unknown size adjacent to the water reservoir. Accordingly, the thermal regime of the frozen dam might be determined to within some unknown error (hopefully small) by heat conduction theory alone.

The resulting model for the frozen dam problem was fairly straightforward. Annual air and reservoir water temperatures formed the boundary conditions. Heat conduction only took place throughout. Considering a section through a long dike, the problem was two-dimensional transient with the complication of phase change. The steady periodic solution gave the thermal regime corresponding to a dike after several years of service.

The phase change problem ('problem of Stefan') is non-linear since the boundary between the two phases moves at a velocity which is dependent on temperature. As this boundary moves a distance dx', a heat quantity $L\rho dx'$ per unit area is absorbed or released by the material which must be provided or removed by conduction. Hence, for the dike, the velocity of the phase change boundary was given by:

$$\frac{dx'}{d\tau} = \frac{1}{\rho L}\left| k_{UF}\frac{\partial t}{\partial x}\Big|_{x=x'} - k_F\frac{\partial t}{\partial x}\Big|_{x=x'}\right|, \quad \ldots \quad (27)$$

where L = latent heat of fusion of ice,

and ρ = weight of moisture per ft^3 of soil (constant).

General equations for the phase change problem have been formulated and several analytical solutions exist[51], however, these solutions are of no great practical importance. Numerical[52,53,34] and analog[54] methods have been used successfully.

4.3 Program and Methods of Solution

The problems analyzed as part of the frozen dam heat transfer study, were those outlined[55] by G.E. Crippen and Associates Ltd. on behalf of Manitoba Hydro. All problems are described in detail in Section 4.5

where the calculations are summarized and the results presented. The first problem consisted of determining the thermal regime of a flat portion of ground, i.e. the cyclic condition before construction. As for the frozen dam problem, this problem was simplified to heat conduction in a semi-infinite medium which was homogeneous and isotropic in both the thawed and frozen phases. The problem was solved first using ambient air temperature as the boundary condition, and later, using modified air temperature to include insolation. The affect of the geothermal gradient on the temperature regime is discussed. The results substantiate the qualitative discussion of Lachenbruch et al[56].

The second problem was to determine the isotherm field for a homogeneous fill dike on a steady state basis. The problem was first solved for a dike of constant thermal conductivity using an analog field plotter. The basic solution for temperature for this case is given by equation (9). The analog method proved an efficient means of determining the isotherm field in the dike foundation region. The temperature superposition principle[57] was used to include geothermal gradient in obtaining the final solution. The results were checked by the conformal mapping method. The Schwarz-Christoffel transform was not manageable; the partial solution obtained added little other than to indicate the shape of the adiabatic lines. The problem was finally solved using two thermal conductivity values, one for each phase.

The third problem required the solution of the thermal regime of a homogeneous fill dike under natural conditions. The problem was solved for two sets of thermal properties which were sufficiently similar to permit certain conclusions to be made based on a detailed comparison of results. The solution to this problem was of prime importance. If the natural thermal regime was not suitable from the structural point of view (this turned out to be the case), sufficient improvement might be possible through design or cooling. Problems 3 to 6 inclusive were concerned with methods of increasing the size of the frozen core. Problem 4 (steady state) and Problem 5 (cyclic) through a design incorporating a layer of insulation-type material used as a liner for the water reservoir, and Problem 6 by operating air cooling ducts during the winter.

Conditions of 'no surface cover' were used throughout the calculations. All steady periodic solutions were obtained using the finite-difference numerical method. The iterations were carried out using the University of Manitoba Bendix G-15D digital computer. The computation was set up in the usual manner as outlined in Section 3.2.1. The frozen state governed convergence. Phase change was handled by the method of 'excess degrees'[58,27]. Subdivision of the geometry was a matter of compromise. On the one hand, it was necessary to select a geometric network sufficiently refined so that reasonable results could be obtained, however, on the other hand, the number of nodes and hence the amount of arithmetic involved for a solution, had to be kept within practical limits. The main considerations were accuracy and computer space and time.

4.4. Data

4.4.1.

Reference Appendix 'A' and Fig. 4, data obtained for the boundary conditions used in the computations are fairly representative for the lower Nelson River region.

a) Air temperature (B.C. # 1) values were obtained from the Winnipeg DOT Meteorological Record Office for Gillam, Manitoba. Gillam is situated near the Nelson River approximately mid-way between Kelsey and Hudson Bay. The mean monthly air temperatures are average values for a 16 year period (1946 to 1961 inclusive).

b) Reservoir water temperatures (B.C.# 2) were obtained from a graphical record of daily mean forebay water temperature at Kelsey for the year 1960. Variations in reservoir water temperature are quite uniform from year to year. Since Nelson River water further north will tend to be somewhat cooler, this boundary condition was conservative.

c) The accuracy of the average daily insolation (includes cloudy days) data[59] was approximately ±10%.

d) The geothermal gradient was taken to be 3°C per 100 meters[55]. This figure was 'order of magnitude' only.

4.4.2.

The following thermal properties of an earth material containing 10% moisture by weight were used for all calculations except where otherwise noted. These properties may be considered as roughly corresponding to a saturated glacial till.

	ρ[60] (lbm/ft³)	k[61] (Btu/hr-ft°F)	C_p[60] (Btu/lbm - °F)
Unfrozen	120	0.7	0.20
Frozen	120	1.0	0.15

$(\alpha_F/\alpha_{UF} \approx 2)$

a) For the purpose of simplification, density (ρ) was taken to be constant, i.e. independent of temperature and not affected by phase change.

b) The thermal conductivity (k) values should be considered as mean values for the temperature ranges encountered in each specific case. In general, the thermal conductivity of soil increases with temperature, however the variation is relatively small particularly over the temperature range from -20 to +25°F[62]. Since the thermal conductivity of a soil depends primarily on the water content and whether the soil is frozen or unfrozen, the relationship between k_{UF} and k_F was established knowing the moisture content (12 lbs water/ft^3 of material) and the thermal conductivity values for ice and water of 1.3 and 0.343 Btu/hr-ft-°F respectively. The ratio of $k_F : k_{UF}$ is 1.43. This may be compared to a value of 1.54 used in one Russian paper[45] for frozen dam calculations (soil not described). At high moisture contents the thermal conductivity of frozen soil is generally about 50% greater than when in a thawed state[49].

c) The specific heat (C_p) value for the frozen state was determined based on C_p = 0.2 Btu/lbm-°F in the thawed state and $C_{p(ice)}/C_{p(water)} = 0.5$.

4.4.3.

The heat of fusion of ice is 144 Btu/lbm. This gave a value for 'excess degrees' applied at phase change of:

$$L = \frac{\text{Latent heat}}{\text{Av. specific heat}} = \frac{144 \times 0.1 \times 120}{0.175 \times 120} = 82F^{o} \text{ to freeze or thaw.}$$

4.4.4.

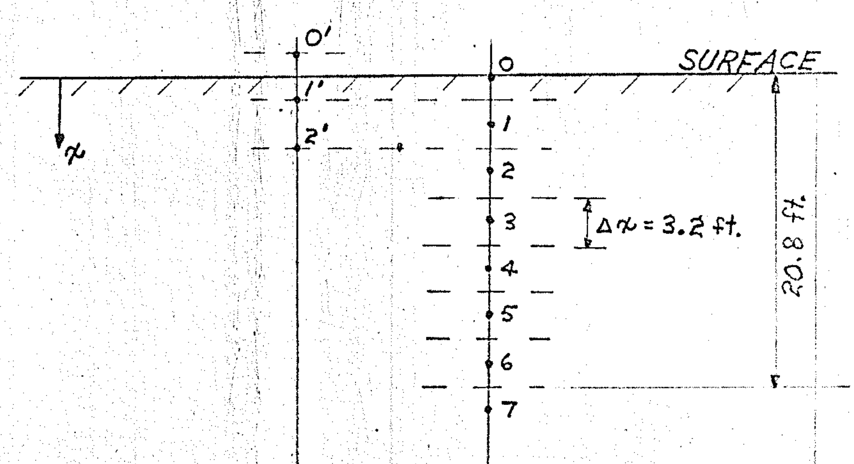
For all calculations, phase change was taken to occur at 32°F. In general, the average freezing point of a number of soils is close to 32°F[49]. For steady state calculations in particular, the affect of a freezing point less than 32°F (dissolved salts in the soil, fine grained soils, etc) is readily apparent.

4.5 Calculations and Results

This section consists of reporting on six problems. For each problem, a statement of the problem is given, the basic finite-difference data is listed, and the results are presented.

4.5.1 Cyclic Condition Before Construction (Problem 1) (1-D Slab-Transient - No cover)

The problem was to examine a flat portion of ground having no surface cover (vegetation, snow, etc), with its surface exposed to atmosphere (air temp. varies, B.C. # 1), and to determine the variations of ground temperature with depth and time. The soil was assumed to be homogeneous and isotropic in each phase. The geometry used for the calculations was as follows:



$t = f(x,\tau)$

$\left(\frac{k}{\rho C_p}\right) \frac{\partial^2 t}{\partial x^2} = \frac{\partial t}{\partial \tau}$ except at locations where phase change is taking place.

For the finite difference calculations, $\Delta x = 3.2$ ft.

$\Delta\tau = 91.25$ hrs ($\sim \frac{1}{2}$ week)

$t_7 = t_6$

The total number of $\Delta\tau$ periods in the annual cycle was 96 (one computer channel).

a) The first calculation neglected solar radiation and the geothermal gradient. Temperature t_o equaled ambient air temperature at all times. The results are shown in Fig. 5 (curve 'A') and Fig. 6.

i) The permafrost table is located at a depth of 7.6 ft. Temperature t_o is the average temperature of a zone which includes the upper 1.6 ft of material. Phase change associated with this material was neglected, however the thermal resistance of the path 0 - 1 was fully recognized. The assumption t_o = B.C. # 1 generates a minor error in the calculated temperatures; node temperature values are slightly high. The error in the location of the permafrost table, however, is relatively large. The calculated temperature values apply equally well to a geometry consisting of nodes 0', 1',etc. In this case the average convective heat transfer coefficient (h) for both phases is 0.53 Btu/hr-ft^2-^{o}F. This value is considered somewhat low for a truly exposed surface subject to natural convective with some wind. Nevertheless the value is of reasonable order of magnitude considering the heat flow magnitude and the fact that the geometry during part of the cycle is effectively a cooled flat plate facing up. The net result is the permafrost table is raised 1.6 ft. It follows, the most probable location of the permafrost table is about 6.5 ft. for this calculation.

ii) Fig. 6 illustrates temperature lag with depth; for example, warmest in January, coolest in May at 16 ft. As outlined in Section 3.1.2 b) ii), under idealized conditions (isotropic material, no phase change), a periodic surface temperature having a cosine wave form will penetrate an infinitely thick slab in a manner such that the advancing wave is decreasing in amplitude with increasing depth; the extremes on a diagram having the co-ordinates of Fig. 6 are enclosed in a bell-shaped envelope. A theoretical equation is available[63] to solve for the depth to which the temperature osciallation penetrates the medium. The yearly fluctuations of B.C. # 1 approximate a cosine wave (Fig. 4). Applying the equation directly using average thermal properties for the soil indicated this depth to be approximately 53 feet. The location of zero amplitude as well as the characteristic of the theoretical temperature oscillation are affected by phase change.

b) The above results apply also for the case of geothermal gradient. The location of the 32°F isotherm and hence the permafrost table is not affected by the geothermal gradient.

i) During the months of February, March, April and part of May, the portion of ground under consideration is completely frozen - in general temperatures increase with depth and presumably a temperature of 32°F is reached somewhere well within the permafrost belt. Part (and possibly all) of $Q_{(geothermal)}$ would be transmitted to this region to influence the temperature regime in the upper 20 ft of ground. The affect, however, is negligible. The geothermal gradient of $3^{\circ}C/100$ m may be considered equivalent to a heat flux in frozen soil of 0.0165 Btu/hr-ft^2 along the x-axis originating from a source located beneath the slab. Comparison of this value with typical instantaneous heat flows involved in the problem, showed $Q_{(geothermal)}$ to be only a very small fraction ($\sim$ 1%) of the normal heat flows, hence its affect is negligible in most areas. Note, the $Q_{(geothermal)}$ influence will of course increase with depth.

ii) During the months of October to January inclusive, the location of the permafrost table and the temperatures of the soil above the permafrost table are not affected by $Q_{(geothermal)}$ due to the temperature profiles involved. $Q_{(geothermal)}$ affects only the temperature of the permafrost belt.

iii) During the remaining months of the year not covered above, $Q_{(geothermal)}$ may affect the temperature regime, but only slightly. The temperature gradients reverse relatively quickly in the spring of the year, hence $Q_{(geothermal)}$ is required to do considerable warming at depth before it can be transmitted to influence the temperature of frozen ground in the upper 20 ft.

c) The second calculation was the same as la) except solar heating was included.

$$t_o = t_{(ambient\ air)} + \frac{\varepsilon Q_{(solar)}}{hA}$$, sol-air temperature.

The values used were $h = 4$ Btu/hr-ft^2-$^{\circ}$F, the absorptivity $\varepsilon = 0.7$ in summer and 0.1 in winter (existence of snow or at least a frosted surface was allowed), and $Q_{(solar)}$ as per Appendix 'A' data. The values were selected so as to represent the net radiative heat exahange - absorbed insolation less surface emission. The net effect of the above was to increase the mean annual air temperature from 23.4°F to 28.8°F. The result is shown in Fig. 5 (curve 'B'). The permafrost table is located at a depth of 10.4 ft. This level is 2.8 ft. lower than the equivalent geometry for the case of radiation neglected. Actually, the lowering of the permafrost table due to solar heating by 37% of the original depth should be considered as order of magnitude only. The result may be considerably in error since, for example, a relatively large portion of the absorbed solar energy tends to be dissipated by surface evaporation. Note, the nature of Curve 'B' in covering the entire cycle makes the very existence of permafrost questionable.

d) The condition of no surface cover made the above described calculations unique. The location of the permafrost table depends very definitely on the surface cover, both as to type and colour. At Kelsey, for instance, the depth of the permafrost table has been reported[64] to vary from 2 to 3 ft in areas of swamp and organic cover to 5 to 6 ft on the more exposed higher ground.

e) The lower Nelson River region lies within the band of discontinuous permafrost where the distribution of permafrost is sporadic. The condition of no surface cover made the problem equivalent to determining the maximum depth to the permafrost table. Considering solar heating and moisture migration as applicable in the thawed phase, for the type of soil considered, this depth is estimated to be about 8 ft. Additionally, as a rough 'thumb-rule' for an exposed surface and a material with properties similar to those used in the problem, it appears that permafrost _may_ not exist if it does not appear within the first 8 ft of soil.

4.5.2 Homogeneous Fill Dike - Steady State (Problem 2)

The problem was to examine a dike cross-section on a steady state basis to determine the isotherm field. The geometry used for the dike was a symmetrical cross-section having a 1 in 3 slope. The temperature solution to this problem for the case of a single isotropic material depends only on the constant boundary condition temperature values and the geometry.

The heat flow rate depends in addition on the thermal conductivity of the dike material. For the actual case, since thermal conductivity depended on the phase, the dike was considered for the purpose of steady state analysis as consisting of two materials with different k values, having a common boundary defined by the isotherm $t = 0^{\circ}C$. For this case the isotherm field depends also on the ratio of the k values.

a) The problem was solved first by an analog method for the case of a constant k value for the dike material. The results obtained using an analog field plotter are shown in Figs. 7 and 8 for a constant t_a and for $t_w = 10^{\circ}C$ (Case 1 - extreme) and $t_w = 6.1^{\circ}C$ (Case 2) respectively. Approximately 42% of the volume of the dike proper is frozen for Case 1 compared to 47% for Case 2. Taking bedrock as typically occurring at about 30 meters ($\sim$ 100 ft), the percentage of frozen dike structure above bedrock contained in a 60 meter wide section symmetrical about the center line, falls to 30% for Case 1 compared to 46% for Case 2. The above figures are based on the final isotherms resulting from 'normal' heat flow and the geothermal gradient.

b) Conformal mapping (Section 3.1.2 a) ii) was used to check the results of Case 2 above. The method and results are outlined in Appendix 'B'. The complete solution involves a Schwarz-Christoffel transformation, however the method used (biased temperature) is valid, and a complete solution can be obtained by this method coupled with free-hand flux plotting in the dike foundation region. A simplified flux plot neglecting foundation distortions is shown in Fig. 9. The frost line ($t = 0^{o}C$) is located at $\emptyset = 98.5^{o}$.

c) The boundary conditions used for Case 2 are the actual mean annual temperatures for this problem. The problem was solved next using these boundary conditions and k = 0.7 for thawed soil and k = 1.0 for frozen soil. The location of the frost line of Fig. 9 shifts towards the water side since the lower conductivity material is situated on this side. The same heat flows through each material, and for a flux plot this applies for each heat flow lane. By writing equation (1) for each phase and using a common flux plot, the location of the frost line was readily established. The isotherm positions for each phase were then determined by the analytical method. The solution is shown in Fig. 10. The frost line is located at $\emptyset = 85.9^{o}$ - a shift of 12.6^{o} towards the water side.

4.5.3 <u>Homogeneous Fill Dike - Cyclic (Problem 3)</u>

The problem consisted of solving for the natural thermal regime of a dike having the same cross-section as that examined in the steady state analysis. For a long dike, due to symmetry, the heat conduction is two-dimensional transient with phase change. The steady periodic solution was required.

$t = f(x,y,\tau)$.

$\frac{k}{\rho C_p}\left[\frac{\partial^2 t}{\partial x^2} + \frac{\partial t}{\partial y^2}\right] = \frac{\partial t}{\partial \tau}$ except at locations where phase change is taking place.

A finite-difference numerical calculation was set up and programmed using the geometry shown in Fig. 11. The downstream slope had no surface cover, and solar heating was neglected. The dike surfaces followed air and water temperatures as applicable. The node zones were 2.5 m by 7.5 m rectangles. The time increment ($\Delta\tau$) was 365 hours (~ 2 weeks) - hence 24 $\Delta\tau$ periods in the annual cycle.

a) The problem was solved using the usual thermal properties for the dike material (Case 1). The results are given in Appendix 'C' and Fig. 12.

b) The problem was solved using the usual thermal properties except the k values (hence thermal diffusivities) were halved (Case 2) i.e. effectively ρ, C_p and moisture content were held constant while k was varied by a factor of two. The α and k ratios were still the same. The results are presented in Appendix 'D' and Fig. 13.

c) The solutions were compared and the following points noted:

i) The general heat transfer pattern is the same for both cases. Temperature extremes are greater for Case 1 than for Case 2 due to the higher thermal diffusivity of Case 1, i.e. the material is more sensitive to the boundary conditions. The general heat transfer pattern is illustrated in Fig. 14. The heat flow process is rather complex compared to the steady state concept of a uniform flow of heat from the water side to the air side. Heat transfer to and from the soil adjacent to the downstream slope may be considered as being similar to the Problem 1 slab calculation. This same basic pattern also holds for the region beneath the water side surface. (Due to thermal lag, the temperature gradients continued to change over the period of approximately 6 months during which B.C. # 2 remained constant at 32°F). The actual pattern for either one is influenced by the existence (and nature) of the other. For fixed boundary conditions and materials, the interaction will be a function of the geometry. The third heat flux vector shown in Fig. 14 reflects the basic flow of heat from water side to air side. Note from Appendices 'C' and 'D' the locations of the 36°F and 34°F isotherms (also the adjoining 32°F) remain fairly constant throughout the annual cycle, indicating an almost steady flow of heat into the frozen core. The central region essentially floats in equilibrium between the two sub-surface phenomena.

ii) The depth of the zone of alternate thawing and freezing beneath the downstream slope averaged about $6\frac{1}{2}$ ft for Case 1 compared to about 5 ft for Case 2. This difference is due again to the higher thermal diffusivity of Case 1, i.e. deeper penetration. The above figures for zone depth may <u>not</u> be taken as absolute values. Accuracy in this respect, as for Problem 1, required exact application of the boundary conditions. (A better approximation for the actual depth is given in Problem 6). Furthermore, there was some error in the relative depths for this zone since the inherent assumption of more or less instantaneous thawing/freezing of the triangular zones of Nodes 56, 64, etc, affected the location of the depth of maximum thaw in slightly different ways. Both solutions indicated the depth of thaw increased going up the downstream slope. Additionally, Node 89 zone thawed more than Node 81 zone as should be the case since the heat transfer would approach the Problem 1 slab calculation with increasing lateral distance from the dike.

iii) The boundary between permanently frozen and thawed soils for Case 1 was closer to the dike cross-section centerline than for Case 2 by about 2 ft. The difference is considered essentially due once again to the higher thermal diffusivity of Case 1. Following the summer thaw, the active zone adjacent to the downstream slope refreezes and the frozen core is sub-cooled. The degree of sub-cooling depends on the thermal sensitivity of the material. (The average temperature of the frozen core in early April was 21.8°F for Case 1 versus 24.3°F for Case 2). Since the temperature gradient is in the same direction across the frozen core during the period of sub-cooling, the lower boundary tends to be pushed out as the amount of heat extracted increases.

iv) The location of the boundary between permanently frozen and thawed soils was practically fixed at depths below about 25 ft from the peak of the dike, i.e. there was almost negligible shift over the annual cycle. For Case 1, the total swing at Node 51 was about 7 inches decreasing to about 2 inches at Node 54. There was some evidence of underthawing of the frozen core, however this point could not be established definitely. Further calculations using a deeper cross-section are required to check this aspect.

d) The preceding solutions generally apply also for the case of geothermal gradient for reasons similar to those outlined in Problem 1. To iterate, the geothermal influence again increases with depth. This time the affect has a new characteristic. The geothermal flux vector is vertically up whereas the normal flux vector for the bulk of the dike is generally horizontal from water to air side. The net result is that not only do the temperatures of points remote from the surface tend to increase due to the geothermal influence, but the points are effectively moved similar to the shift indicated in Figures 7 and 8. For a newly constructed dike with permafrost beneath the water reservoir, the geothermal gradient cannot directly influence the thermal regime of the thawed soil since the temperatures above and below the permafrost belt exceed 32°F. However the permafrost would tend to vanish with time due to both the existence of the water reservoir and the geothermal action at the lower boundary of the permafrost belt.

e) Due to the size of the problem, the finite-difference network had to be kept relatively 'coarse' for practical reasons. The main significant error in the solutions due to subdivision of geometry involved the use of the method of 'excess degrees' to account for phase change. Since the heat associated with phase change is comparatively large, the boundary between frozen and thawed soils is in effect a narrow band of material having a very low thermal diffusivity, providing of course phase change is taking place. In using the 'excess degrees' method, there is a very definite suppression of temperature (which alters the heat flow) when the temperature of a node tends to move through 32°F from above or below. In general, the true temperature of a node is 32°F when the node zone is about half frozen/thawed. If the node is less than half frozen, the true temperature exceeds 32°F, and vice versa. If a zone undergoes complete phase change, i.e. either freezes or thaws completely, the net affect on the temperatures of surrounding nodes as a result of holding the temperature of the phase changing node constant at 32°F, is more or less negligible due to the compensating characteristic. By the same argument, if a zone is about half frozen on the average throughout the full cycle, the error is negligible since the variations are such that thaw equals freeze during the cycle. Unfortunately, with a coarse geometry these idealized situations occurred rarely, and the error present in the final solutions is now discussed in some detail:

i) Nodes 57,65, etc.

These nodes thaw partially and then refreeze. Since thaw equals freeze, from point of view of phase change this region contains alternate heat sinks and sources where sink equals source in magnitude. Actually for Case 1 (maximum thaw of node zones about 34%) the $(t_a - 32).\tau$ integrals for thawing and freezing (τ is the period for each) are very nearly equal despite the fact that the true average temperature of a node during freezing will always be less than 32°F, whereas the true average node temperature during thawing will be approximately 32°F when about 40% of the node zone is thawed (allowing for steeper temperature gradients above the frost line than below). Accordingly, the thawing and then freezing action retards heating and cooling respectively in more or less equal degrees. Due to this action the actual application of B.C. # 1 loses some of its significance since an overestimate of thawing power is offset by an equal overestimate of freezing ability. The total cycle for the nodes under consideration consisted of partial thawing and refreezing, and included a period during which the region was completely frozen. Hence conditions were biased in favour of an overestimate of the size of the frozen core.

ii) Nodes 50 to 54 inclusive.

Since the frost line is on the air side of the center line, holding a node at $32^{o}F$ in order to account for latent heat of fusion results in a node temperature which is less than the true average temperature for the zone. The affect is that opposite nodes appear colder than they should be, hence a slightly distorted isotherm picture resulted. In addition, a latent heat balance was not possible since the heat inflow to the zone was overestimated while the heat outflow was underestimated (thaw exceeded freeze). In view of the fact that the frost line through the subject area for Case 1 shifted only a very small amount over the annual cycle, phase change was neglected for the Case 2 calculation. This procedure simplified locating the frost line. The amount of freezing /thawing was estimated by the changes of node temperature over the cycle.

4.5.4 Zoned Fill Dike - Steady State (Problem 4)

The problem was to investigate the affect on the temperature regime of a dike due to a layer of insulation-type material used as a liner for the water reservoir. For comparison purposes, the dike geometry used was the same as for Problems 2 and 3, i.e. a 1 in 3 slope. The main dike material was a homogeneous fill having k values in the two phases of $k_{UF} = 0.7$ and $k_F = 1.0$ Btu/hr-ft-^{o}F. The liner for the reservoir was arbitrarily chosen to be a 3 ft thick slab consisting of a hypothetical insulating material of $k = 0.1$ Btu/hr-ft-^{o}F.

a) For the steady-state calculation (surfaces at - 4.8^{o}C and +6.1^{o}C,) a wedge was substituted for the uniform 3 ft thick layer of insulating material; the wedge having the same cross-sectional area as the rectangular slab. This approximation simplified the solution since the temperature drop across the wedge is constant.

b) The solution is the frost line is located at $\emptyset$ = 75.1^{o} - compared to Figure 10, a shift of 10.8^{o} towards the water side. The temperature drop across the wedge of insulating material is 1.89C^{o}.

4.5.5 <u>Zoned Fill Dike - Cyclic (Problem 5)</u>

The steady periodic solution for the temperature regime based on the geometry and materials of Problem 4, was obtained by the finite-difference method using the geometric sub-division shown in Figure 11. Since the insulating material is located immediately beneath the water reservoir, there is no phase change involved for nodes whose zones contain the insulating material. In addition, the zones of nodes 41, 33 etc, contain only a small quantity of the insulating material ($k = 0.1$), hence for calculation purposes, the ρ and C_p values for the insulating material were taken to be the same as for thawed glacial till.

a) The main results of the computation are as follows:

i) The boundary between permanently frozen and thawed soils almost coincided with the centerline of the dike - the mean frost line over the annual cycle was located just to the left of the centerline as indicated in Figure 12. There is very little geometric error of the type discussed in Section 4.5.3 e) ii) in locating this line since the nodes involved are nearly half frozen at the solution.

ii) The depth of the active zone adjacent to the downstream slope was almost identical to Problem 3a). Maximum thaw increased by less than ½%.

b) The calculation demonstrated the feasibility of using an insulating material adjacent to the water reservoir to expand the frozen core in the lateral direction. The major problem in incorporating this method in the design of a dike is likely to be that of finding an insulating material which is suitable for the application and available at reasonable cost.

c) Regarding this technique for increasing the size of the frozen core, note, although the insulating material constitutes a barrier to the flow of heat from the reservoir to the soil during the summer months, it also impedes the flow of heat from the soil to the reservoir during the winter months. The success of the method appears to depend largely on the periodic nature of B.C. # 2 (Figure 4). For example, the average temperature of Node 25 over the annual cycle was 41.6°F compared to 42.7°F for Problem 3 a) - a difference of only 1.1F°. The maximum temperature for Node 25 was 44.9°F versus 52.1°F for Problem 3a), whereas the minimum temperature for Node 25 was 39.0°F versus 36.2°F for Problem 3a) - differences of -7.2F° and +2.8F° on the high and low respectively.

4.5.6 Homogeneous Fill Dike - Cyclic - Cooling Ducts (Problem 6)

During construction of the dike, a network of horizontal pipes is incorporated in the dike section. During the winter, cold air is drawn mechanically through the ducts, thereby extracting heat. The duct cooling system is sealed during the summer months. Using natural cold winter air makes the method analogous to the technique of extended surfaces as contrasted to pure refrigeration. In principle, the piping layout can be designed to provide the necessary frozen core size and the degree of sub-cooling.

The main objectives for the analysis concerning the use of air cooling ducts, were to determine the feasibility of the method and to provide data useful for the cooling system design. The cooling ducts used were 3 ft. diameter steel pipes. The period of cooling duct operation was 5 months, November to March inclusive. Theoretically, an optimum cooling period would exist for each particular duct depending on location. In practise, actual weather conditions would deviate considerably from the mean values of B.C. # 1, hence a certain amount of day-to-day adjustment of the cooling operation would be required either manually or automatically. Hence this type of optimization was not considered since it is best carried out on the job.

a) The design for winter cooling as to number and location of ducts (especially design optimization) would obviously require at least some 'trial and error' procedure. This aspect led to considerable preliminary work in developing a computer program which could be readily altered to accommodate changes in duct locations and addition/deletion of ducts. The required flexibility was achieved by using the concept of a duct zone which is patched into a basic network similar to that used for Problem 3. Details of the program are given in Appendix 'F'; further discussion concerning the duct zone follows.

b) The feasibility of using a symmetrical model for the duct zone was investigated (duct in an unlimited mass of soil with no external heat flows). A relatively high heat flux was anticipated at locations immediately adjacent to the duct during cooling duct operation. This would be characterized by steep temperature gradients and isotherms which would be essentially concentric circles. The heat transfer area for heat flow to the duct increases with the distance from the duct, hence the heat flux and temperature gradient would decrease with radial distance from the duct. The accuracy achieved in using a symmetrical model for the duct zone would therefore depend on the size of the duct zone. The procedure is equivalent to assuming the future temperature of the duct zone during cooling operations is independent of the temperatures of the surrounding nodes. The temperatures of surrounding nodes are influenced at all times by the temperature of the duct zone. During the period in which the duct is blocked-off, the zone containing the duct is treated the same as any other node.

c) Several calculations were made at this stage to determine the magnitude of heat transfer in the vicinity of the duct during duct operation; this information to be used to select a suitable zone size for the geometry used for the final calculation. These calculations dealt mostly with a duct located in an unlimited mass of soil using the geometry of Figure 16.(Limiting temperature at the boundary was $32^{o}F$ - the phase change temperature at the thawed front, and the maximum temperature for the frozen core). No purpose can be served in describing the calculations in detail, hence only the main results are noted as follows:

i) The heat flux in the immediate vicinity of the duct is apparently very much greater than the normal heat flux. The calculation from which the data used later was extracted, indicated the instantaneous heat flow rates half way through the period of duct operation (mid-January) were such that the normal heat flux in the vicinity of Node 43 of Figure 11 (Problem 3, Case 1) was only 2% of the duct heat flux at the surface of the duct and about $6\frac{1}{2}$% of the duct heat flux at a radius of $4\frac{1}{2}$ ft from the center of the duct. This comparison gives at least some order of relative magnitude - the true comparison for the purpose at hand should of course have been between the heat flux due to the heat extraction power of the duct and the heat flux due to the water reservoir with the duct in position and operating.

ii) There is every indication that the duct zone enters a new period of cooling operation at a temperature very nearly $32^{o}F$. Following 5 months of cooling duct operation, calculations showed the duct zone temperature would relax itself to $32^{o}F$ within a period of about 3 months - this without the benefit of the warming influence of the water reservoir.

d) The value for h_{eff} (a lumped thermal resistance composed of the convective heat transfer coefficient and all miscellaneous thermal resistances associated with the duct proper) used in the duct calculations was by no means negligible and had to be given due consideration. In arriving at a realistic value for h_{eff}, the following factors were considered:

i) Numerous theoretical equations and empirical correlations are available to predict the convective heat transfer coefficient for the inner surface of the duct. These equations indicated an 'h' value of very nearly 4 Btu/hr-ft^2-$^{\circ}$F based on an air flow of 1500 ft/min and using properties at -3.8°F average.

ii) The calculations for the duct zone assumed the cooling medium (air) to be at the actual ambient air temperature. In practise, with a suction system, the average temperature of the cooling medium would be approximately actual air temperature less half the air temperature rise. The allowable air temperature rise is a variable; the value for Δt_{air} is likely to be established as a compromise between desired duct length and the ρAVC_p (Btu/hr-$^{\circ}$F) product based on a knowledge of the average heat extraction rate per unit length of duct, i.e.

$$\Delta t_{air} = \frac{(\text{Btu/hr-ft of duct})_{av} \times \text{length of duct (ft)}}{\rho AVC_p}$$

Since V is likely to have a value of approximately 1000 to 2000 ft/min, the ρAVC_p value is relatively large. Accordingly, even with frictional heating, Δt_{air} is likely to be of the order of a few degrees only. Nevertheless, these considerations indicated compensation in the form of a suitable fictitious thermal resistance was in order to effectively reduce the cooling air temperature by a small amount.

iii) A large contact resistance could exist between the duct and the surrounding dike material. The thermal conductivity value for air at the temperatures of this problem is about 0.013 Btu/hr-ft-^{o}F, hence the ratio $k_{(\text{frozen soil})}$ to $k_{(\text{air})}$ is 77. There is no literature readily available regarding the contact resistance between a buried pipe and soil. (The situation is usually reversed in buried pipe applications, i.e heat loss to be minimized, and use of insulation yields an insulation resistance which is large compared to the combined soil and contact resistance). At any rate, the design of the cooling system would have to be such so as to ensure a reasonably good thermal bond between the ducts and the soil, since otherwise, in the limit, a high contact resistance would tend to undermine the whole purpose of duct cooling. The use of extended surfaces (corrugated pipe, fins, etc) appears advantageous in this respect. In view of the preceding, the value for h_{eff} was somewhat arbitrarily taken as h_{eff} = 2.0 Btu/hr-ft^2-^{o}F.

e) A calculation was carried out for a single duct located centrally in the Node 74 zone. The geometries were as shown in Figures 15 and 16. The time interval corresponding to Figure 15 was 365 hours, the same as for Problems 3 and 5. For Figure 16, Δx was 3 ft and $\Delta\tau$ was 60.8 hrs (1/6 of 365 hrs.). The duct zone temperature was the average temperature of the material within a radius of 6.17 ft. of the duct center. The values for duct zone temperature are listed in Appendix 'E'. This appendix also contains the boundary condition temperatures used in this problem and Problems 3 and 5. The main results are shown in

Figure 17. The isotherm field at the end of January is given in Figure 18.

f) In order to fully assess the affect of the cooling duct operation, it was necessary to solve for the initial conditions. This solution is equivalent to resolving Problem 3, Case 1, using a new geometry. The results are also indicated in Figure 17. The thickness of the zone of alternate thawing and freezing near the downstream slope was about 4 ft. This value is a better approximation of the actual depth of this active zone since the geometry permitted accounting for phase change of more of the actual material near the surface.

g) For the final solution, the temperatures of nodes beyond the neighbourhood of the duct were very nearly identical with the initial condition temperatures. The thickness of the active zone at the downstream slope remained essentially the same, however Node 86 froze an additional 11%. Nodes 76, 83 and 90 remained virtually constant throughout the cycle at 33.0°F, 32.4°F and 32.1°F respectively. The thaw/freeze variations for Nodes 67, 73 and 75 were surprisingly small, hence Figure 17 shows only the mean frost line. There is some geometry error of the type mentioned in Section 4.5.3 e) ii) which makes the calculation conservative in that the extent of frozen material surrounding the duct on the thawed front tends to be underestimated.

h) Regarding the cooling duct zone calculation, based on $h_{eff} = 2.0$ Btu/hr-ft²-°F and $k_F = 1.0$ Btu/hr-ft-°F, the average heat extraction for the 5 month cooling period was about 10 Btu/hr per square foot of duct surface. This rate produces an air temperature rise of about

3.6 F° in 500 feet of duct based on a flow velocity of 1500 ft/min. The maximum heat extraction rate was about 140% of the average and occurred in early January. This data on heat extraction rates is considered fairly representative - inspection of the isotherm field of Fig. 18 in the neighbourhood of the duct together with other information indicated the values will tend to be slightly high due to the error resulting from the use of the symmetrical model.

i) In general, the use of this method of winter cooling to increase the size of the frozen core, appears to be definitely feasible. The single duct calculation both demonstrates the effectiveness of the method and provides information as to duct spacing. The results show that the duct (purposely positioned well into the original thawed zone) has the effect of blocking the reservoir heat flow, and is able to stand up to the summer thawing action. There is every indication that a string of ducts could be used to hold almost any frost line within limits. Auxiliary ducts located in the frozen core could also be used for sub-cooling purposes if desired. Further calculations based on specific objectives could be carried out using the computer program contained in Appendix 'F'.

4.6 General Observations and Conclusions

Regarding the first problem, the thickness of the active layer is known to have a strong dependence on surface cover. For the soil-moisture combination considered, the thickest active layer for the lower Nelson River region (corresponding to no surface cover) is estimated to average about 8 ft over a period of several years. This result cannot be extrapolated to other conditions. The heat flow pattern for the calculations were such that an increase in the water content of the soil would decrease the depth to the permafrost table due to the phase change influence. On the other hand, the depth to the permafrost table tends to increase with the thermal diffusivity of the soil. The thermal diffusivity of a given soil depends largely on the moisture content; for most soils, thermal diffusivity increases with moisture content up to a maximum value and then decreases[49]. Similarly, moisture migration increases with increasing water content to a certain level and then decreases[49]. The actual moisture content, therefore, appears to be a very important parameter in the location of the permafrost table.

The geothermal gradient has negligible influence on the thermal regime of the upper 20 ft of soil. However, the geothermal gradient would definitely have to be included in a calculation on a deep cross-section such as for determining the location of zero amplitude of the periodic surface temperature. Such a computation would also determine the thickness of the permafrost belt since the distance from the point of zero amplitude to the lower boundary of the permafrost belt, depends simply on the value of the geothermal gradient.

Concerning the dike calculations, it is first emphasized that the steady state solutions (Problems 2 and 4) have no real significance in so far as detailed information relating to the actual problem is concerned. The main reason for this is that the boundary conditions are distinctly periodic in nature and the temperature field is time dependent.

Steady state anlysis of a problem of this kind should be used only for general comparison purposes such as progressively varying a boundary temperature, geometry or thermal conductivity ratio, and noting the changes in the temperature solution. The actual solution for any particular configuration will provide qualitative data at best. The discrepancy between the location of the boundary between frozen and thawed material of Figures 10 and 12 supports these conclusions.

The primary solutions (Figures 12 and 13) of the two cases of the natural thermal regime of a homogeneous fill dike are fairly similar. This suggests the solutions obtained may be used as a first approximation for a Nelson River dike of any saturated homogeneous material having an α ratio of about 2. The thermal regime solutions, however, differ more in detail. In this respect, the use of a high thermal diffusivity material on the downstream slope of the dike to increase the size of the frozen core has considerable merit. The possible lateral increase in the size of the frozen core is fairly limited due to the influence of the water reservoir, however, the vertical penetration was indicated to be quite substantial.

Phase change appears to be a very important factor in establishing the temperature regime of a frozen dike. It is probably subordinated only by the boundary conditions. To simplify the situation for the purpose of illustrating a point, as far as the bulk of the dike is concerned, the boundary condition at (or near) the downstream slope consists of B.C. # 1 and $32^{\circ}F$ (the phase change temperature), each effective over approximately half the cycle. For a given soil moisture content, the $32^{\circ}F$ portion is a function of B.C. # 1 in that the latter determines the length and position of the $32^{\circ}F$ period. The phase change temperature therefore enters as a common denominator in the solution of the thermal regime of any dike.

The condition of no surface cover on the downstream slope of the dike is fairly idealized. The only improvement would be to place an insulating layer on this surface during the summer months (possibly vegetation which is burned off in the autumn). The importance of fully utilizing the downstream slope for cooling purposes warrants emphasis. The large area involved is an indication of the available potential for natural cooling. In order to capitalize on this potential, it is necessary to reduce the summer thaw (plant growth, sun shades, etc) and provide an exposed surface during the winter months (in particular, keep insulating snow off the surface). Such conditions could be approximated through the use of snow sheds[65] which have been used successfully in certain Russian designs. Under normal circumstances, the high insulating value of snow compared to vegetation could foreseeable seriously affect the frozen core of the dike by impeding refreezing of the active layer and sub-cooling.

The criterion for structural stability of a frozen dam as given by Bogoslovsky[45] is that a minimum of half the dam section must be kept permanently frozen. The requirement for a Nelson River dike is still to be established. Using the Russian rule as a tentative criterion, the results of Problem 3 (Case 1) indicate the natural thermal regime provides conditions which are below the minimum requirement. Artificial means of increasing the size of the frozen core are therefore necessary.

As already mentioned in conjunction with Problem 5, designing a dike with an insulating liner on the upstream slope has fair possibility for this purpose, however the main problem would be locating or developing an insulating material which is suitable and available at reasonable cost. The use of air cooling ducts appear to be a more practical method. There is every indication that a frozen core comprising 60% or more of the dike section could be achieved without difficulty through internal cooling using natural cold winter air. One possible source of trouble might well be deterioration of the quality of the thermal bond between a duct and its surrounding frozen soil due to cracking as the result of thermal cycling.

A conceptual design for an internally cooled dike for the lower Nelson River is shown in Figure 19. Available information for the Nelson River region indicates a maximum depth to bedrock of about 100 ft. Extrapolated calculations indicated four cooling ducts positioned approximately as shown, would provide permafrost conditions to bedrock.

The term 'accuracy' as used in Section III refers to the deviation of a solution to a given formulated problem from the exact analytical solution to the problem. This usage of the term was generally continued in Section 4.5. The main question then concerns the formulation of the frozen dam problem. Just how closely the predicted thermal regime based on simplified models, will approach the actual thermal regime remains to be seen. Due to the complexity of the real problem, model or field tests appear to be necessary in order to permit even an approximate answer to the question of the true accuracy of solution.

V SUMMARY AND RECOMMENDATIONS

A purely mathematical approach is undoubtedly the most efficient method of solving unidirectional steady state heat conduction problems. The expressions for heat flow and temperature together with the numerical answers can almost always be determined easily and quickly. In certain cases, such as a fin with a variable convective heat transfer coefficient, a finite-difference approach may be superior.

For two-dimensional steady state problems involving single homogeneous materials and no heat sources, the analog field plotter provides an excellent means of obtaining a good engineering solution. This method is especially recommended for irregular geometries. The concept of a shape factor for evaluating heat flow is particularly useful. Free-hand flux plotting may be used to obtain a rough solution as a first approximation, however investing any great length of time to obtain an accurate solution defeats the main purpose of this method. The possible use of conformal mapping should not be over-looked. This method can be extremely powerful. As demonstrated by the simple example included in Section 3.1.2 a)ii), heat flow solutions for the transformed geometry may well be one-dimensional. A mathematical solution for temperature using the product method is likely to be practical only for regular shapes. This method is best used to spot check the accuracy of a flux plot.

Either numerical iteration or relaxation is suggested for two-dimensional steady state problems involving heat sources and/or heterogeneous materials. There is little to choose between these two methods on theoretical grounds. The choice in practice depends on the nature of the problem and personal experience.

For three-dimensional steady state problems, if the shape is fairly regular, the possibility of using existing published data[22,23] should be investigated for at least an approximate solution for heat flow. A solution for temperature distribution may be possible through the principle of superposition. For composite structures , the problem is best tackled by numerical iteration. The electrolytic tank or prototype/model tests constitute the only other practical alternatives.

The choice of method for a one-dimensional transient problem depends mostly on the nature of the required solution. If the depth of penetration of the temperature disturbance versus time is of primary interest, the heat-balance integral method offers fair accuracy and speed. In general, if only specific temperature information is required, the analytical method is preferred since it is capable of yielding exact results quickly. Similarly, charts[19,20] can be used for quick solutions but less accurate results. On the other hand, if general temperature field variations are required, a solution by the finite-difference numerical method automatically contains a complete set of temperature values for discrete points at uniform time intervals. Interpolation in space and time is relatively simple. If graphical data is desired, the Schmidt plot is the obvious choice. The qualitative features of this method, however, are partially offset by the necessity of having to translate the graphical results to numerical data. This also applies to an analog computer solution which would provide continuous temperature variations with time for discrete points of the geometry.

The finite-difference numerical method is exceptionally versatile in accommodating non-uniform initial conditions, unusual boundary conditions, variable thermal properties, phase change, etc. Combined with a digital computer, this method is capable of solving a wide range of two- and three-dimensional transient problems. As illustrated in the frozen dam heat transfer study, a steady periodic solution can be obtained by iteration. In general, if accuracy is especially critical or if the problem cannot be formulated adequately, model experiments or field tests are necessary. The outstanding feature of the model method is the choice of a suitable time scale. The analog computer also permits selection of a convenient time scale. Since the number of amplifiers available limits the number of nodes, the analog computer is best used for qualitative analysis of transient problems involving more than one independent space variable. As already mentioned, the main advantage of electrical analog simulation lies in the ease at which problem parameters can be varied in order to investigate the solution to a heat diffusion problem over a range for design work or performance studies.

For engineering purposes, the method of separation of variables to obtain an analytical solution is more or less limited to two independent variables. For steady state problems, simple expressions can often be derived for heat flow. The equations for temperature for both steady and unsteady state problems are usually more difficult to evaluate. A considerable amount of qualitative information can often be deduced by examining the equations themselves. Solving the equations to provide numerical results may however be tedious and time consuming. In some cases the use of a digital computer for this purpose may be warranted. For such cases, the finite-difference numerical method is probably just as economical.

The key to successful application of the finite-difference numerical method lies in the use of a digital computer. The method is easy to program. The program can be checked by spot checking the computer results using a desk calculator. Type-out can be arranged to suit. For example, in the dike computation, a 'profile' type-out was used to facilitate interpretation of results. For an iteration procedure, convergence can be assisted by the computer operator if desired.

Heat conduction problems encountered in engineering work may range from a simple calculation of the steady state heat loss through a plane wall to a problem concerning re-entry kinetic heating of an ablative nose cone of a space vehicle. For any heat conduction problem of even moderate complexity, probably the most important part of solving the actual problem consists of formulating the problem. The accuracy of the data available for solving the problem has an immediate direct bearing on the accuracy of the final solution. Boundary conditions are usually difficult to apply exactly. Approximations may be necessary regarding contact resistance, internal heat generation, etc - for example, consider heating due to exponential attenuation of gamma rays. No purpose can be served in further generalizing on the matter of formulation since every problem is unique. The main point is that approximations and simplifying assumptions are invariably necessary in formulating a problem, and these in turn generally have more affect on the true solution accuracy than the method used to solve the heat conduction model.

BIBLIOGRAPHY

1. Carslaw, H.S. and J.C. Jaeger : Conduction of Heat in Solids Second Edition, Oxford-Clarendon Press, 1959, p.10.

2. McAdams, W.H. : Heat Transmission, Third Edition, McGraw-Hill, 1954, p.11.

3. Ross A.& R.L. Stoute : Heat Transfer Coefficient Between UO2 and Zircoloy-2, Appendix 'A', AECL - 1552, June, 1962,(CRFD-1075)

4. Eckert, E.R.G, and R.M. Drake : Heat and Mass Transfer, Second Edition, McGraw-Hill, 1959, Chapt. 3.

5. Chapman, A.J. : Heat Transfer, MacMillan, 1960, Chapt. 4.

6. Gardner, K.A. : Efficiency of Extended Surfaces, Trans ASME, Vol 67, No. 8, Nov. 1945, pp. 621-631.

7. Churchill, R.V. : Complex Variables and Applications, Second Edition, McGraw-Hill, 1960, p.184.

8. Kober, H. : Dictionary of Conformal Representations, Dover Publications.

9. Churchill : op.cit. App.2 (Table of Transformations of Regions), p.287.

10. Giedt, W.H. : Principles of Engineering Heat Transfer, D.Van Nostrand, 1957, p.63.

11. Jakob, M., and G.A. Hawkins : Elements of Heat Transfer, Third Edition, Wiley, p.40.

12. McAdams, : op.cit., p.25.

13. Grober, H., S. Erk and V. Grigull : Fundamentals of Heat Transfer, Third Edition, McGraw-Hill, 1961, p.120.

14. Eckert and Drake : op.cit., pp. 60-64.

15. Jakob, M. : Heat Transfer, Wiley, Vol. 1, pp 336-342.

16. Carslaw and Jaeger : op.cit., Chapt. 10.

17. El-Saden, M.R. : Heat Conduction in an Eccentrically Hollow, Infinitely Long Cylinder with Internal Heat Generation, Trans. A.S.M.E. - Journal of Heat Transfer, Nov. 1961, pp 510-512.

18. Grober et al : op.cit. pp 28-30.

19. Heisler, M.P. : Temperature Charts for Induction and Constant-Temperature Heating, Trans. A.S.M.E. Vol. 69, pp.227-236.

20. Schneider, P.J. : Temperature Response Charts, Wiley.

21. Eckert and Drake, op.cit. p.100.

22. McAdams, : op.cit., p.26.

23. Glazebrook (Editor) : Dictionary of Applied Physics, MacMillan, Vol. 1, p.463 (Langmuir, Adams and Meikle).

24. Morse, P.M., and H. Feshback : Methods of Theoretical Physics, McGraw-Hill (1953), Part II, Chapter 10.

25. Newman, A.B. : Trans. Am. Inst.Chem. Engrs, Vol. 24(1930), p.44.

26. Dicker, D & M.B. Friedman : Solutions of Heat-Conduction Problems with Nonseparable Domains, A.S.M.E. Trans. Journal of Applied Mechanics, Dec., 1963, pp 493-499.

27. Dusinberre, G.M. : Heat-Transfer Calculations by Finite Differences, International Textbook Co. (1961).

28. Fowler, C.M. : Analysis of Numerical Solutions of Transient Heat-Flow Problems, Quart. Appl. Math., Vol. 3, No. 4, 1946, pp. 361-376.

29. Dusinberre, G.M. : Numerical Methods for Transient Heat Flow, A.S.M.E. Trans. Vol. 67, 1945, pp. 703-712.

30. Liebmann, G. : A New Electrical Analog Method for the Solution of Transient Heat-Conduction Problems, A.S.M.E. Trans. Vol. 78, No. 3, 1956, p.655.

31. Jakob, M.: Heat Transfer, Wiley, Vol. 1, Chap. 19.

32. Eckert, and Drake, : op.cit. p.70

33. Thom, A. and C.J. Apelt : Field Computations in Engineering and Physics, D. VanNostrand Co. 1961.

34. Goodman, T.R. : The Heat-Balance Integral and Its Application to Problems Involving a Change of Phase., A.S.M.E. Trans, Vol. 80 (1958), pp 335-342.

35. Yang , K.T. : Calculation of Unsteady Heat Conduction in Single-layer and Composite Finite Slabs with and without Property Variations by an Improved Integral Procedure. A.S.M.E. 1961 International Heat Transfer Conference -'International Developments in Heat Transfer', Part 1.

36. Giedt : op.cit. p.64.

37. Karplus, W.J. : Analog Simulation, McGraw-Hill, (1958), p.296.

38. McDonald T.W., and P.N. Nikiforuk, : Engineering Applications of Analog Computers, EIC Engineering Journal, Vol. 45, No. 4, April 1962, pp. 72-83.

39. Karplus, : op.cit. p.207.

40. Freed, N.H. and C.J. Kallis, : Analogue Computer Solutions of the Heat Conduction Equation. Journal Mechanical Engineering Science, Vol. 5, No. 2 (1963), pp 157-162.

41. Paschkis, V. : Combined Geometric and Network Analog Computer for Transient Heat Flow, Trans. A.S.M.E. Journal of Heat Transfer, May 1959, pp. 144-149.

42. Jakob, M. : Heat Transfer, Wiley, Vol. 1, pp 322-325.

43. Brown, W.G. : Ground Heat Exchange Problems - The Use of Models and Model Theory, NRC 4688, June 1958.

44. Muller, S.W. : Permafrost or Permanently Frozen Ground and Related Engineering Problems, J.W. Edwards, Inc. 1947.

45. Bogoslovsky, P.A. : Temperature Conditions in an Impervious Earth Dam Constructed on Permafrost, After Long Use, (Translation)

46. Brown, R.J.E. : The Distribution of Permafrost and Its Relation to Air Temperature in Canada and the U.S.S.R. NRC 5941.

47. Martynov, G.A. : Heat and Moisture Transfer in Freezing and Thawing Soils, NRC TT-1065 (1963) from Principles of Geocryology - Academy of Sciences of the U.S.S.R., V.A. Obruchev Institute of Permafrost Studies (1959).

48. Crawford, C.B. and R.F. Legget, : Soil Temperatures in Water Works Practice, NRC 2910.

49. Crawford, C.B. : Soil Temperatures, A Review of Published Records, NRC 2785.

50. Pearce, D.C. and L.W. Gold, : Observations of Ground Temperature and Heat Flow at Ottawa, Canada, NRC 5304.

51. Carslaw and Jaeger, op.cit. Chapt. 11.

52. Murray, W.D. and F. Landis : Numerical and Machine Solutions of Transient Heat-Conduction Problems Involving Melting or Freezing, A.S.M.E. Trans, Journal of Heat Transfer, May, 1959, pp. 106-112.

53. Sunderland, J.E. and R.J. Grosh, : Transient Temperature in a Melting Solid, A.S.M.E. Trans. Journal of Heat Transfer, Nov. 1961, pp 409-414.

54. Otis, D.R. : Solving the Melting Problem Using an Electric Analog to Heat Conduction, Trans. A.S.M.E. Vol. 79 (1957), pp 759-764.

55. Crippen, G.E. and Associates Ltd. : Manitoba Hydro Program for a Frozen Dam Heat Transfer Study, April, 1963.

56. Lachenbruch, A.H. et al, : Temperatures in Permafrost Temperature, Its Measurement and Control in Science and Industry, Vol. 3, Part 1, pp. 791-803.

57. Brown, W.G. : Simple Graphical Methods for Estimating the Location of Permafrost Under Shallow Lakes and Rivers, Proceedings of the First Canadian Conference on Permafrost NRC Technical Memorandum No. 76, pp 226-231.

58. Dusinberre, G.M. : A Note on Latent Heat in Digital Computer Calculations, A.S.M.E. Paper No. 58-HT-7 (1958).

59. Mateer, C.L. : A Preliminary Estimate of the Average Insolation in Canada, Canadian Journal of Agricultural Science, Vol. 35, Nov./Dec. 1955, pp. 579-594.

60. USSR Ministry for Construction (Heavy Industry) : Thawing Frozen Ground by Means of Water-Heated Metal Stakes, Scientific Information Centre, DRB.Canada, T51R, 15 Nov. '51, Table 6.

61. Kersten, M.S. : University of Minnesota Eng. Ex. Sta. Bull. 28, June 1949.

62. District Heating Handbook, Third Edition, NDHA, p.180,

63. Eckert and Drake, : op.cit. p.104.

64. MacDonald, D.H., R.A. Pillman and H.R. Hopper, : Kelsey Generating Station Dam and Dykes, The Engineering Journal, Vol. 43, No. 10, Oct. 1960, pp 87-98.

65. Borisov, G.A. and G.Y. Shamshura, : Experience in the Design Construction and Use of Earth Dams at Norilsk, (Translation).

$w = \dfrac{z-a}{az-1}$ where $a = \dfrac{1+\alpha_1\alpha_2+\sqrt{(1-\alpha_1^2)(1-\alpha_2^2)}}{\alpha_1+\alpha_2}$

$R_0 = \dfrac{1-\alpha_1\alpha_2+\sqrt{(1-\alpha_1^2)(1-\alpha_2^2)}}{\alpha_1-\alpha_2}$, ($a>1$ & $R_0>1$ when $-1<\alpha_2<\alpha_1<1$)

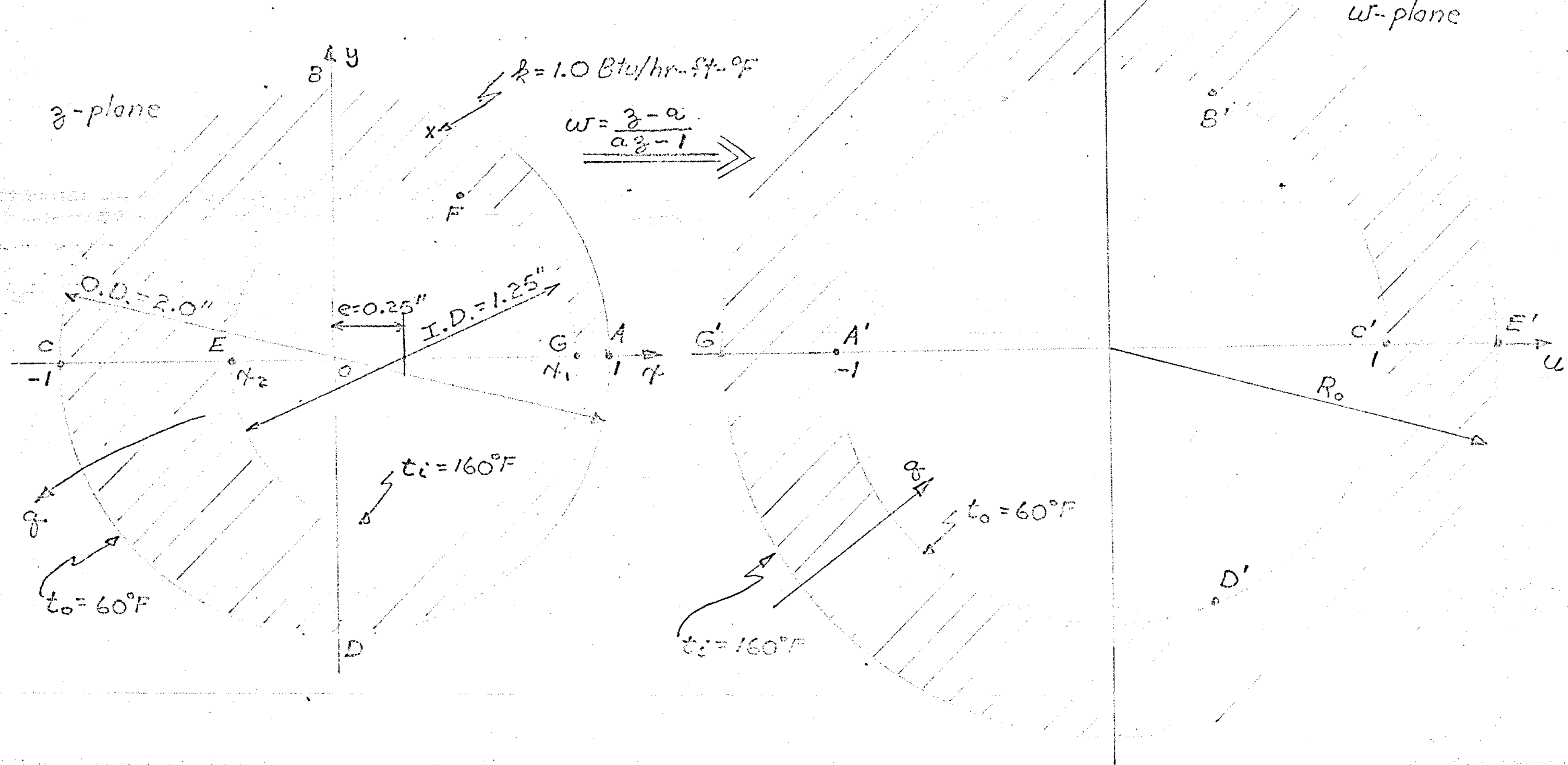


FIG.1 : Conformal Transformation for Solution of Heat Loss from a Pipe with Eccentric Bore in 2-D Steady State Heat Conduction

Shape Factor, $S = 2\left(\frac{M}{N}\right) = 2\left(\frac{35}{4}\right) = 17.5$

Heat Flow, $q = k l S (t_i - t_o)$

$= 1750$ Btu/hr per foot of pipe length

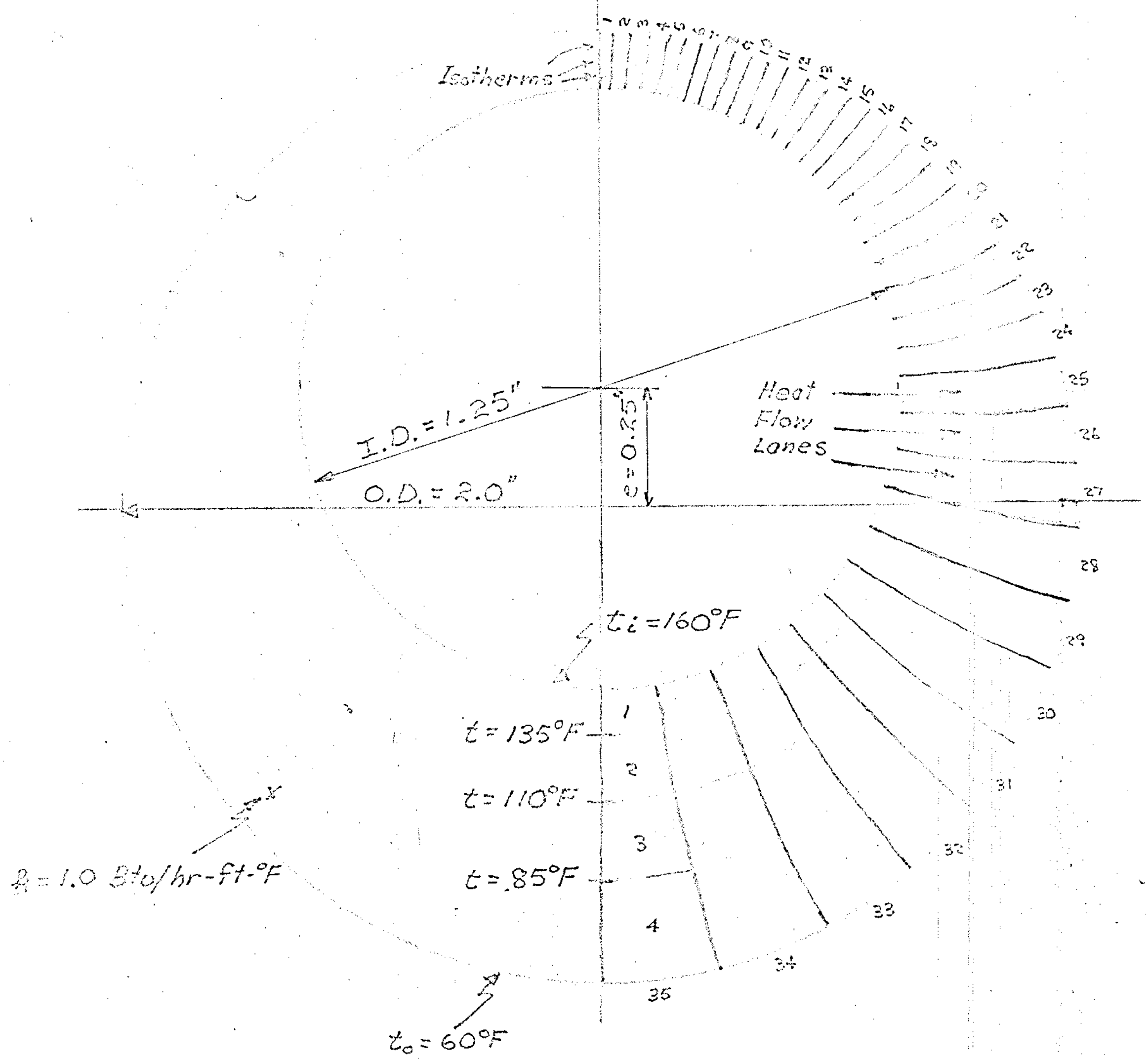
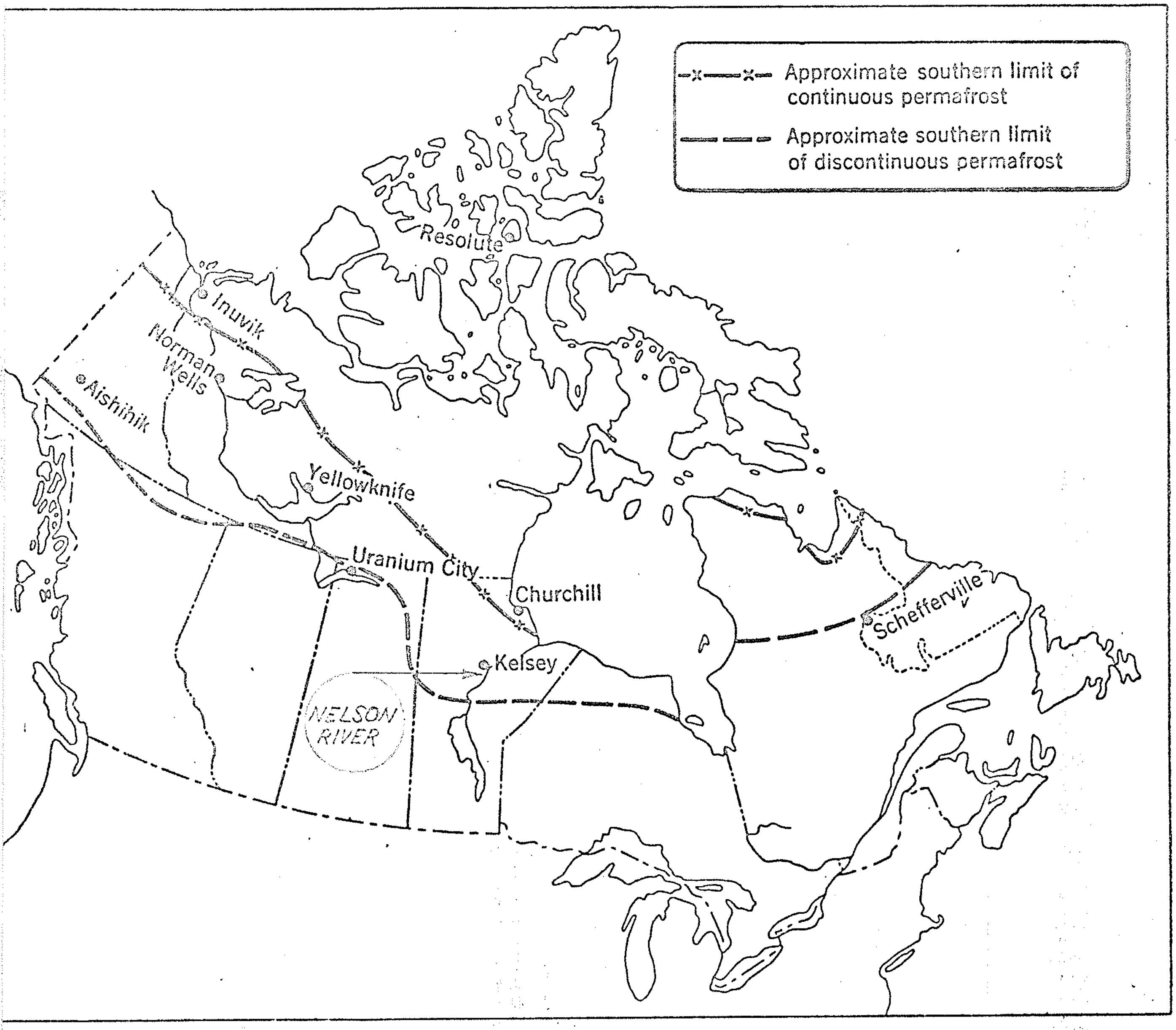


FIG. 2 : Flux Plot for Pipe with Eccentric Bore in 2-D Steady State Heat Conduction



Locations of ground temperature installations.

FIG. 3 : PERMAFROST MAP OF CANADA

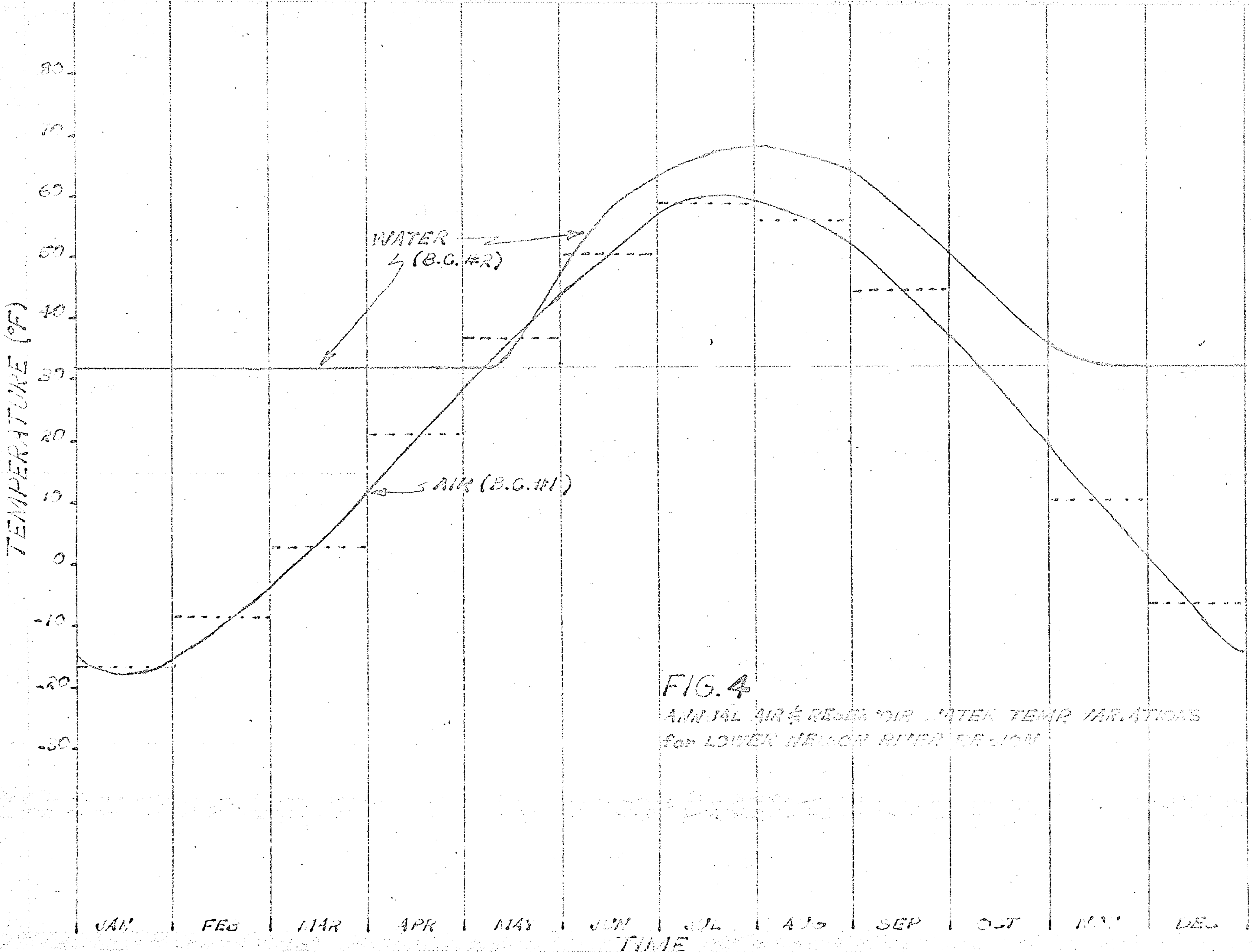
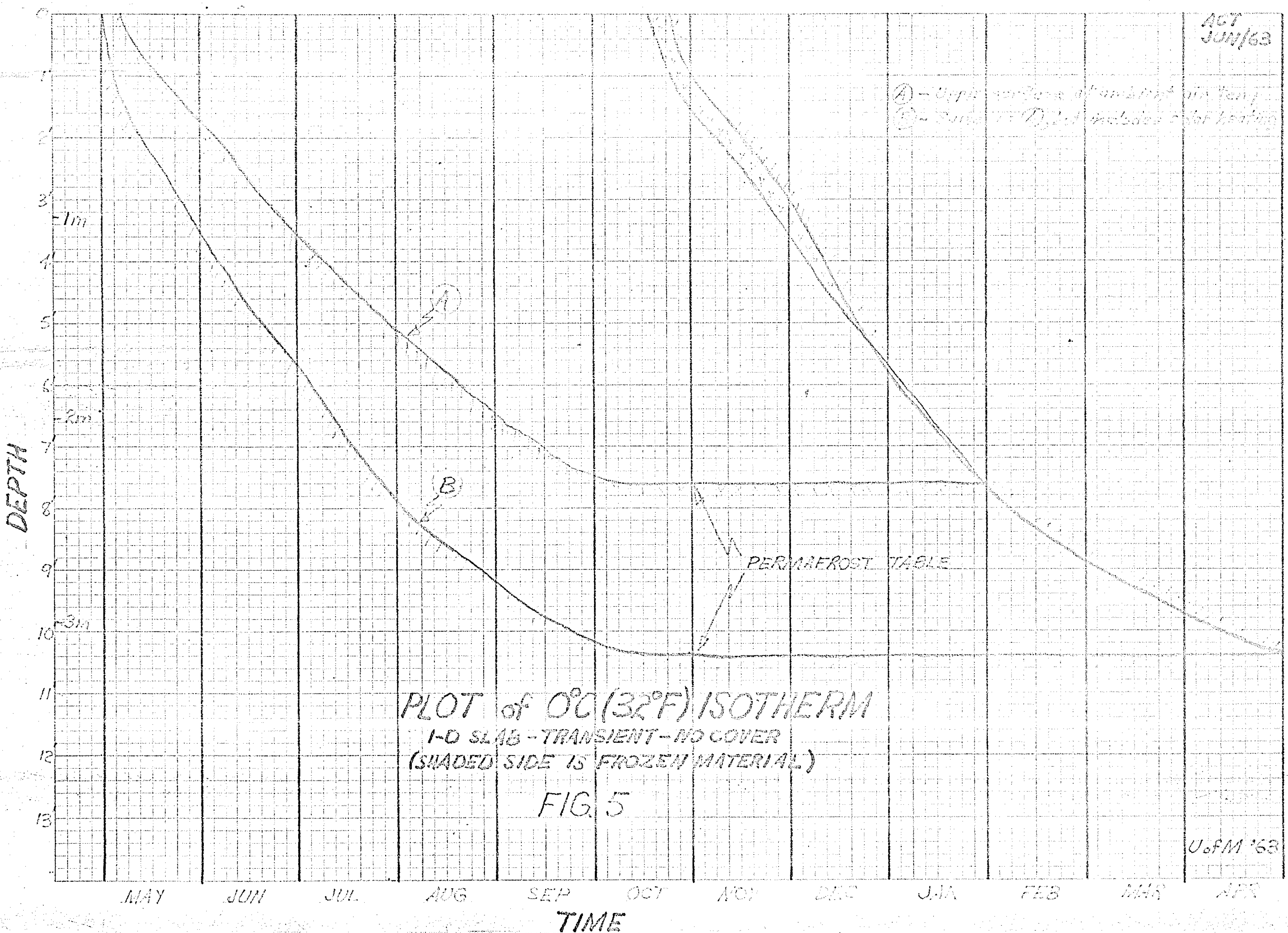


FIG. 4
ANNUAL AIR & RESERVOIR WATER TEMP VARIATIONS
for LOWER NELSON RIVER REGION



PLOT of 0°C (32°F) ISOTHERM
1-D SLAB - TRANSIENT - NO COVER
(SHADED SIDE IS FROZEN MATERIAL)
FIG. 5

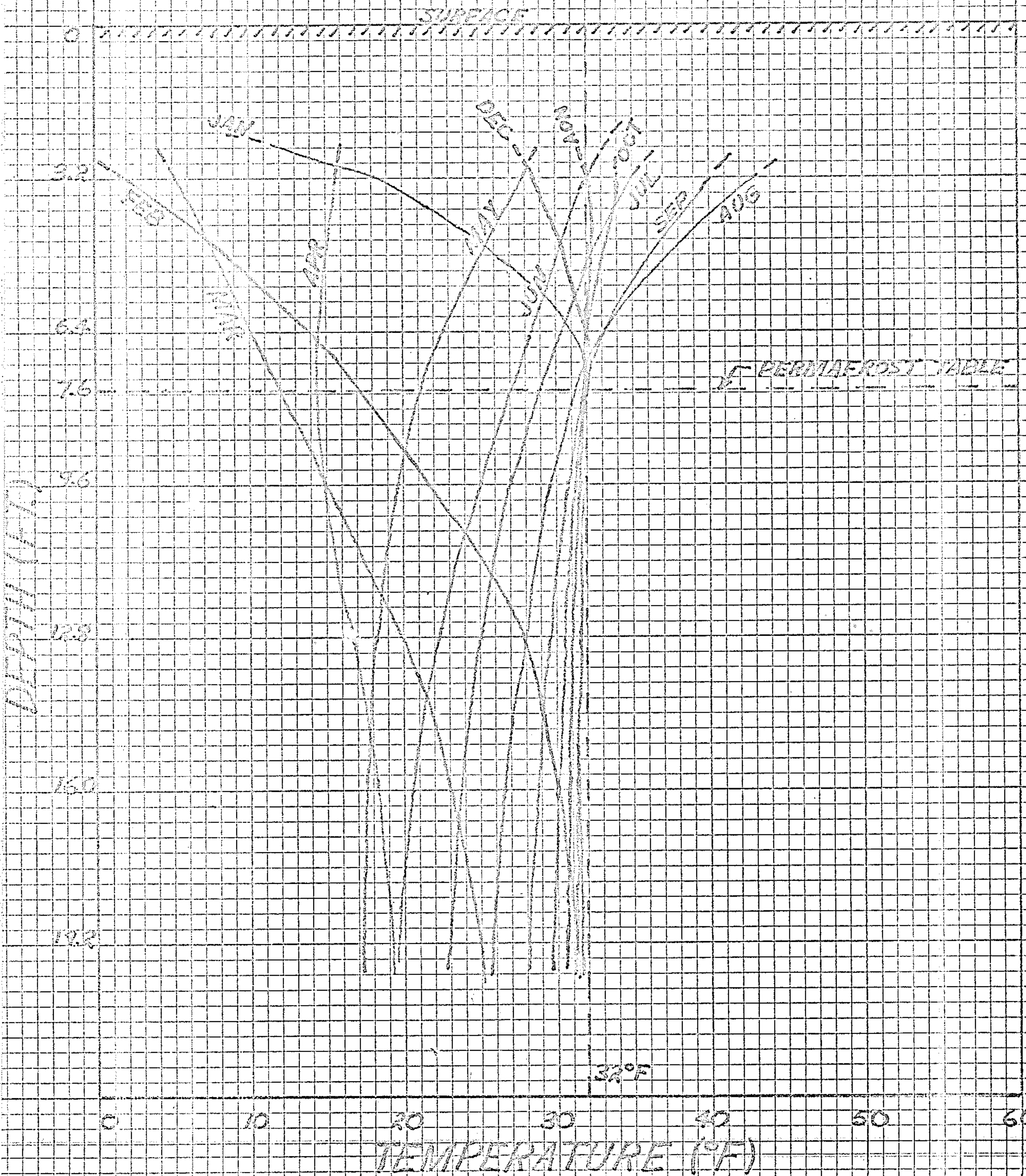


FIG. 6

— MID-MONTH TEMPS FOR CALC. (9) —

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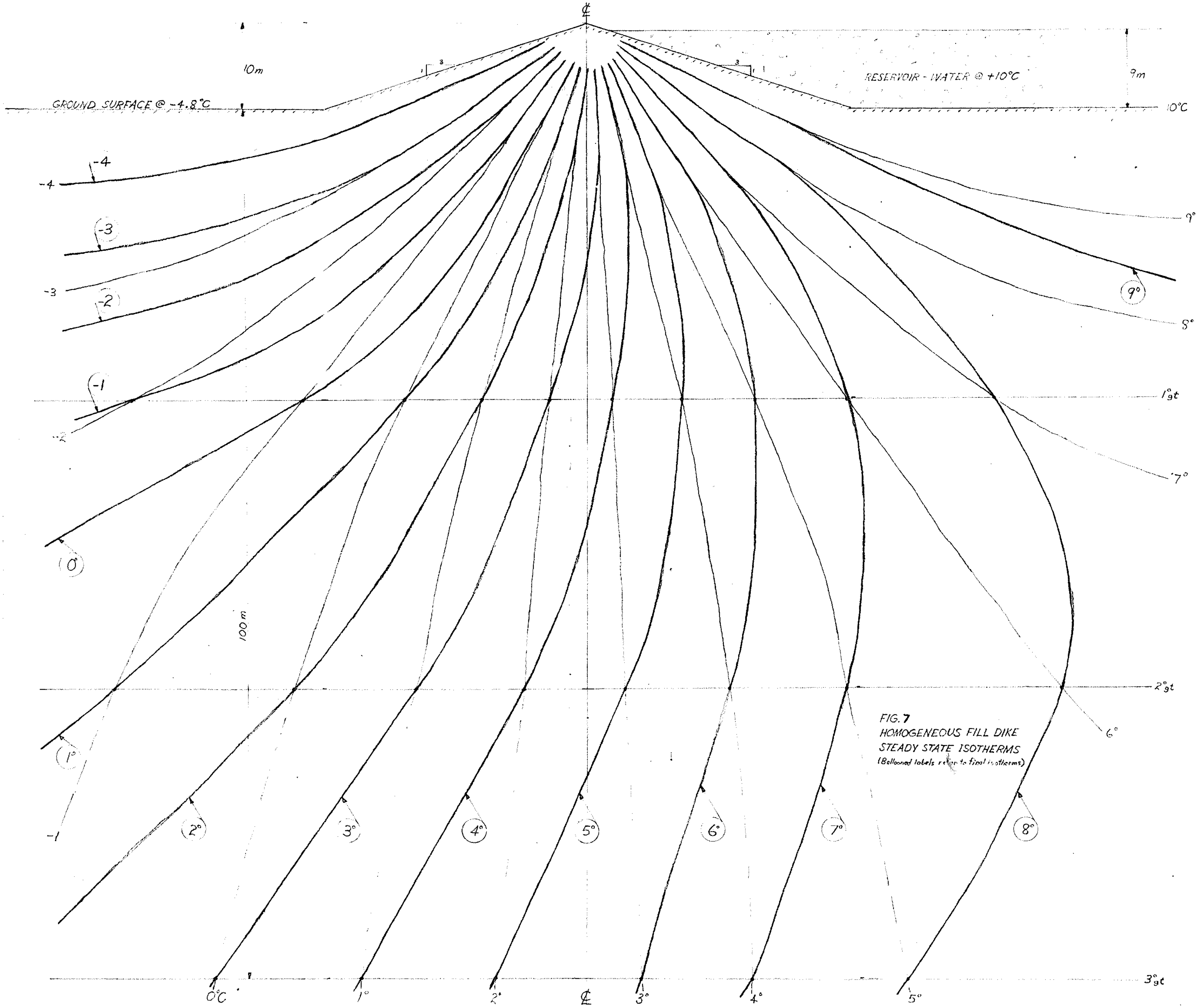

FIG. 7
HOMOGENEOUS FILL DIKE
STEADY STATE ISOTHERMS
(Ballooned labels refer to final isotherms)

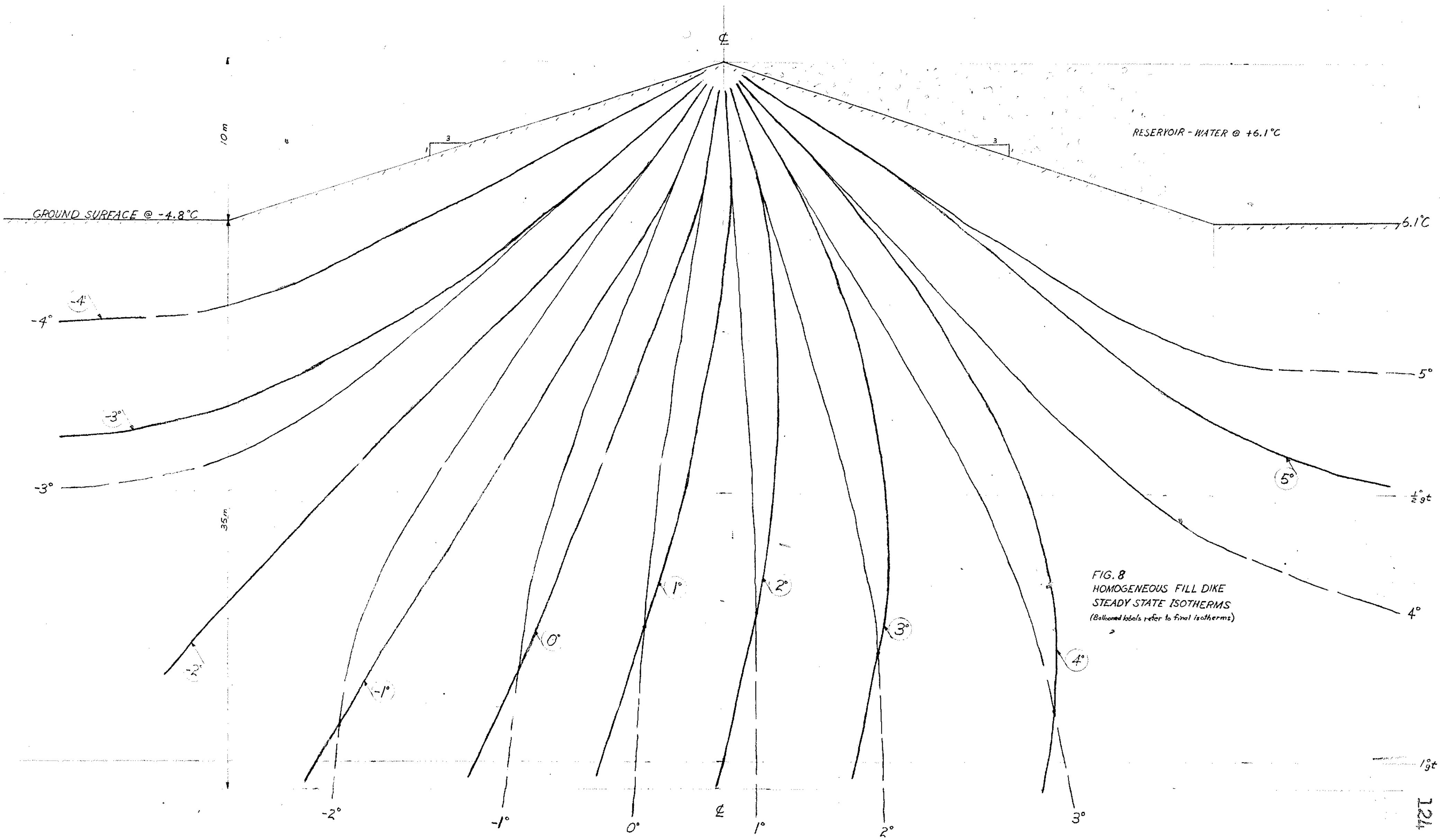

FIG. 8
HOMOGENEOUS FILL DIKE
STEADY STATE ISOTHERMS
(Ballooned labels refer to final isotherms)

FIG. 9

HOMOGENEOUS FILL DIKE - STEADY STATE
SIMPLIFIED FLUX PLOT for
SINGLE ISOTROPIC DIKE MATERIAL

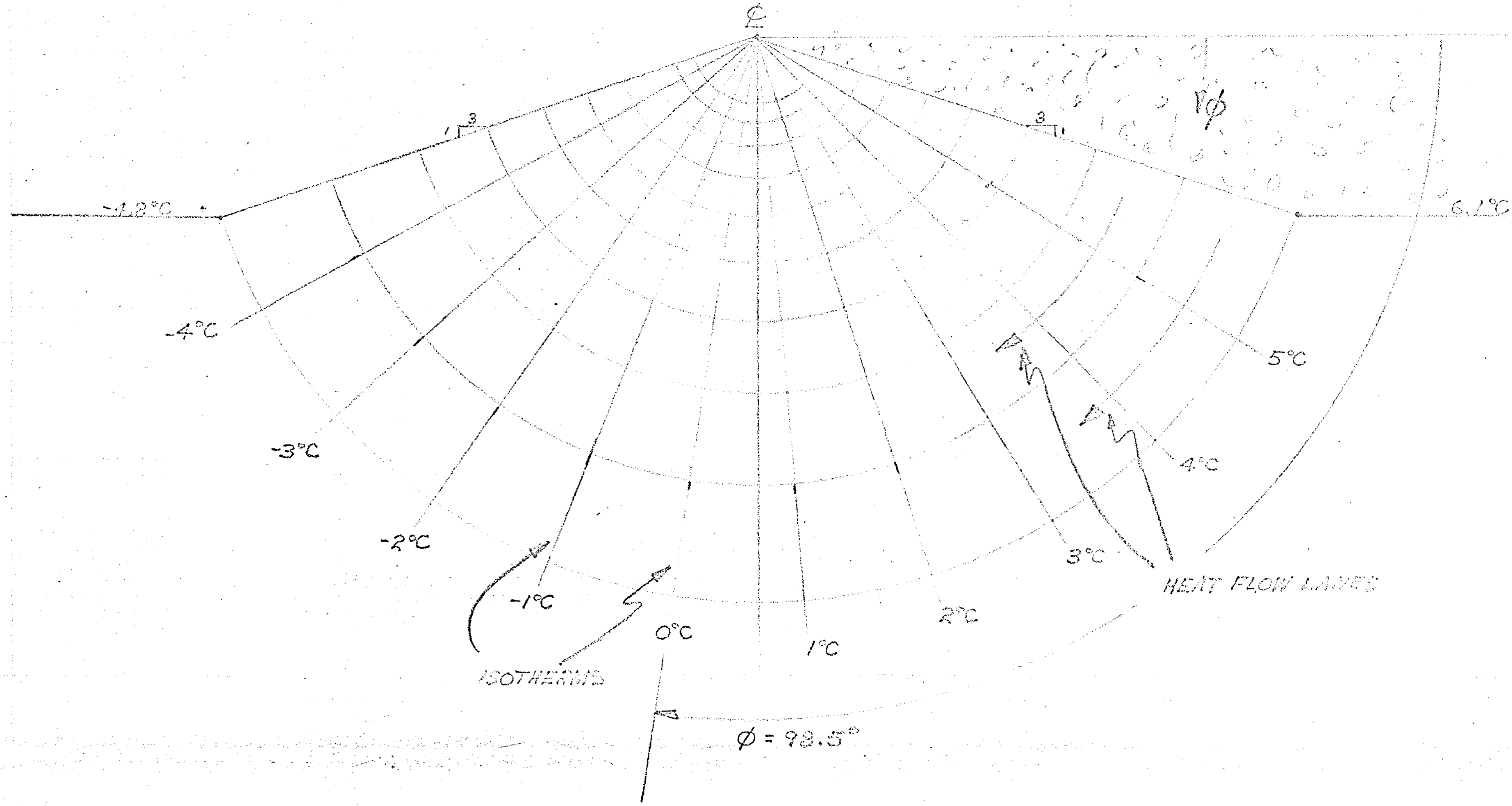

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FIG. 10

HOMOGENEOUS FILL DIKE - STEADY STATE

ISOTHERMS for $k_{uf}/k_f = 0.7$

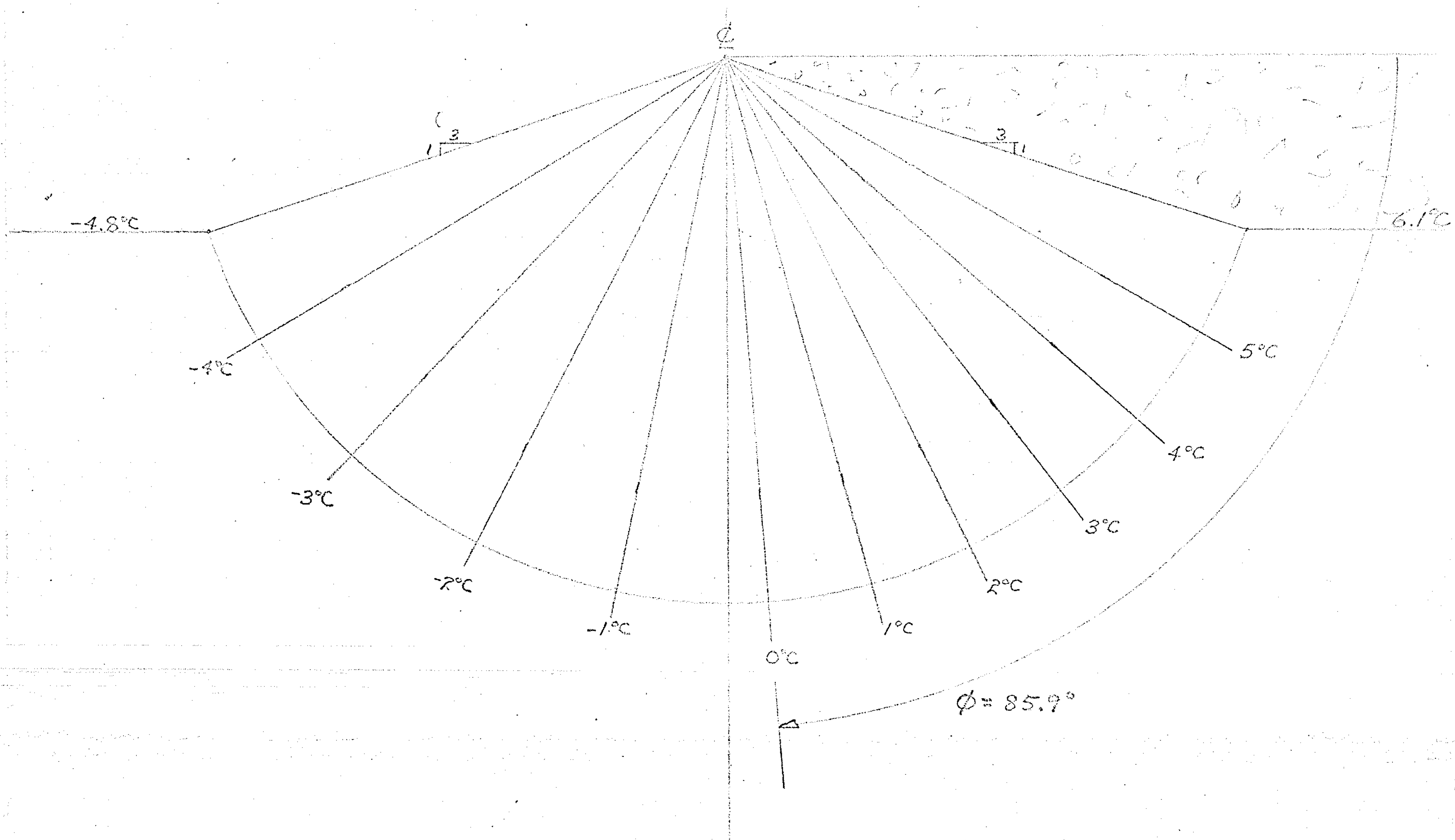

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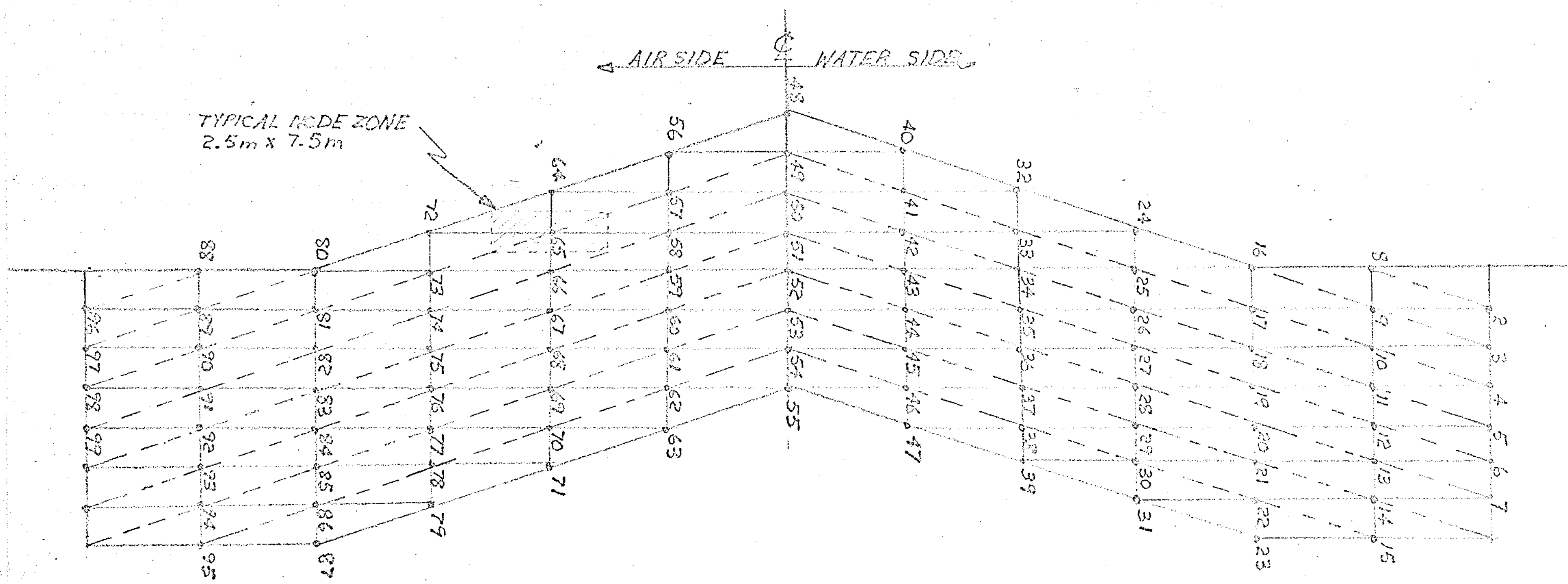
FIG. 11

HOMOGENEOUS FILL DAM

TRIANGULAR/RECTANGULAR NETWORK

FOR CYCLIC CASE CALCULATIONS



SCALE: 1" = 32.8' (10m)

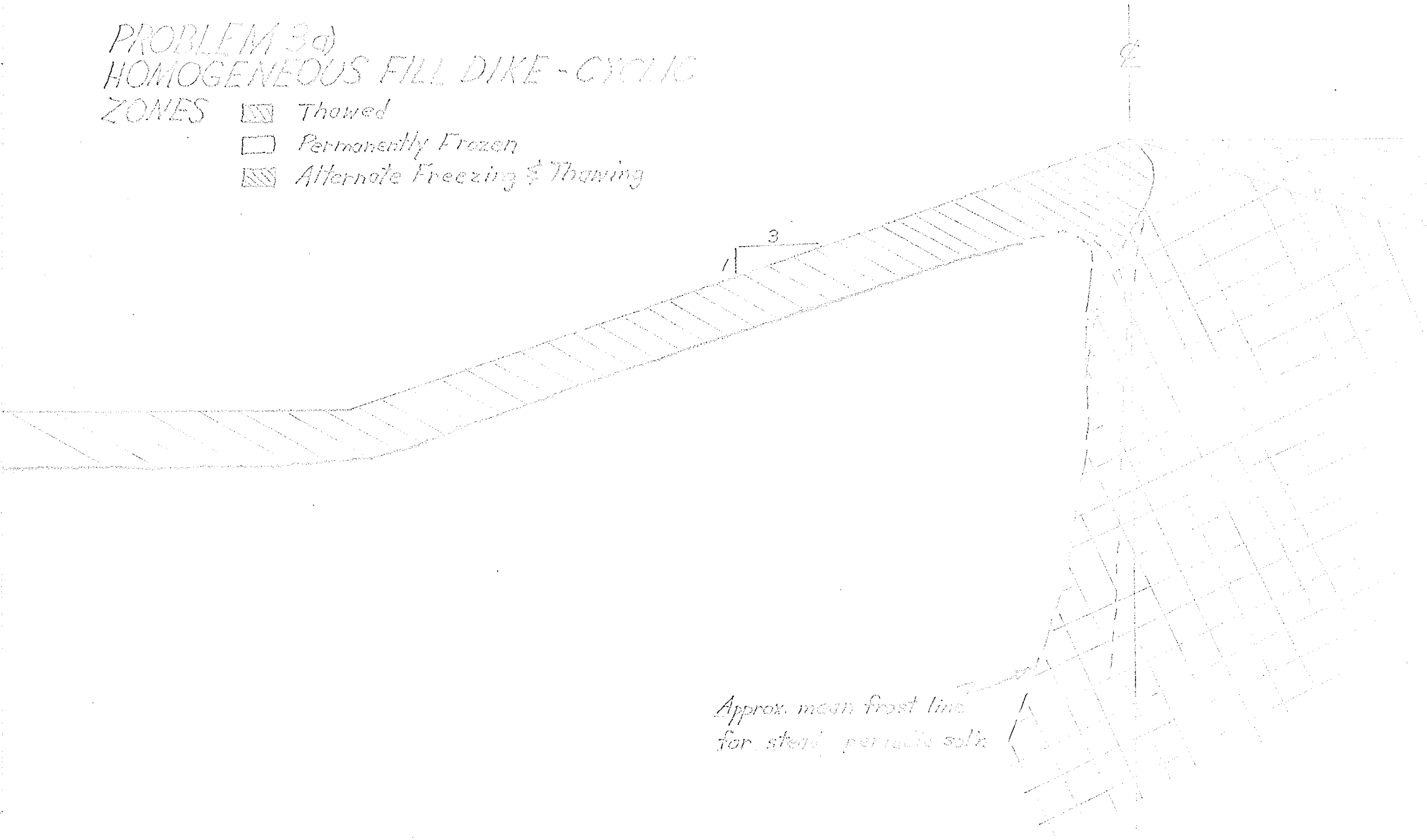


FIG. 12
SCALE : 1" = 16.4' (5m)

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PROBLEM 3b)
HOMOGENEOUS FILL DIKE - CYCLIC

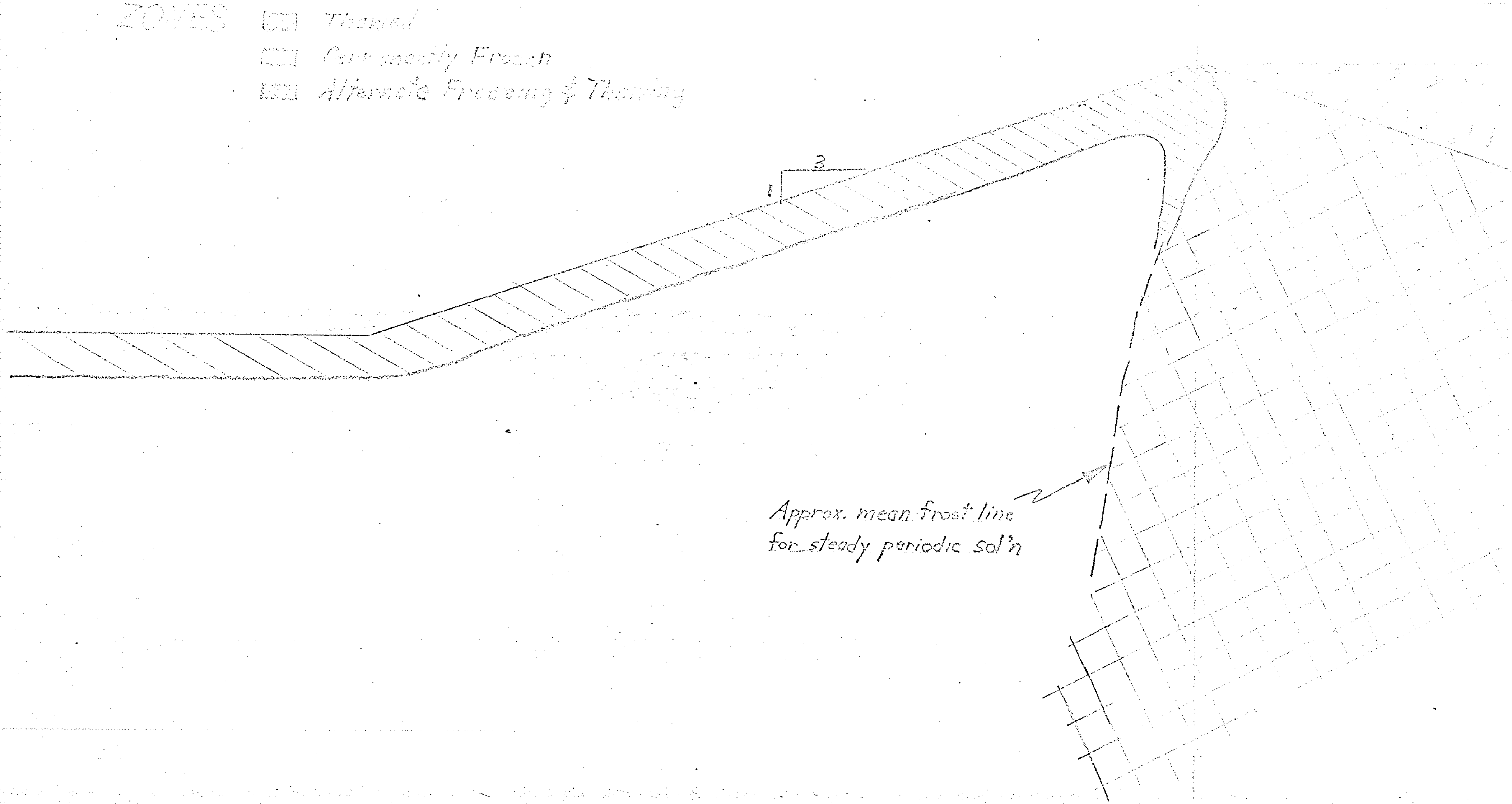

FIG. 13
SCALE : 1" = 16.4' (5m)

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FIG. 14
HOMOGENEOUS FILL DIKE - CYCLIC
GENERAL HEAT FLOW PATTERN

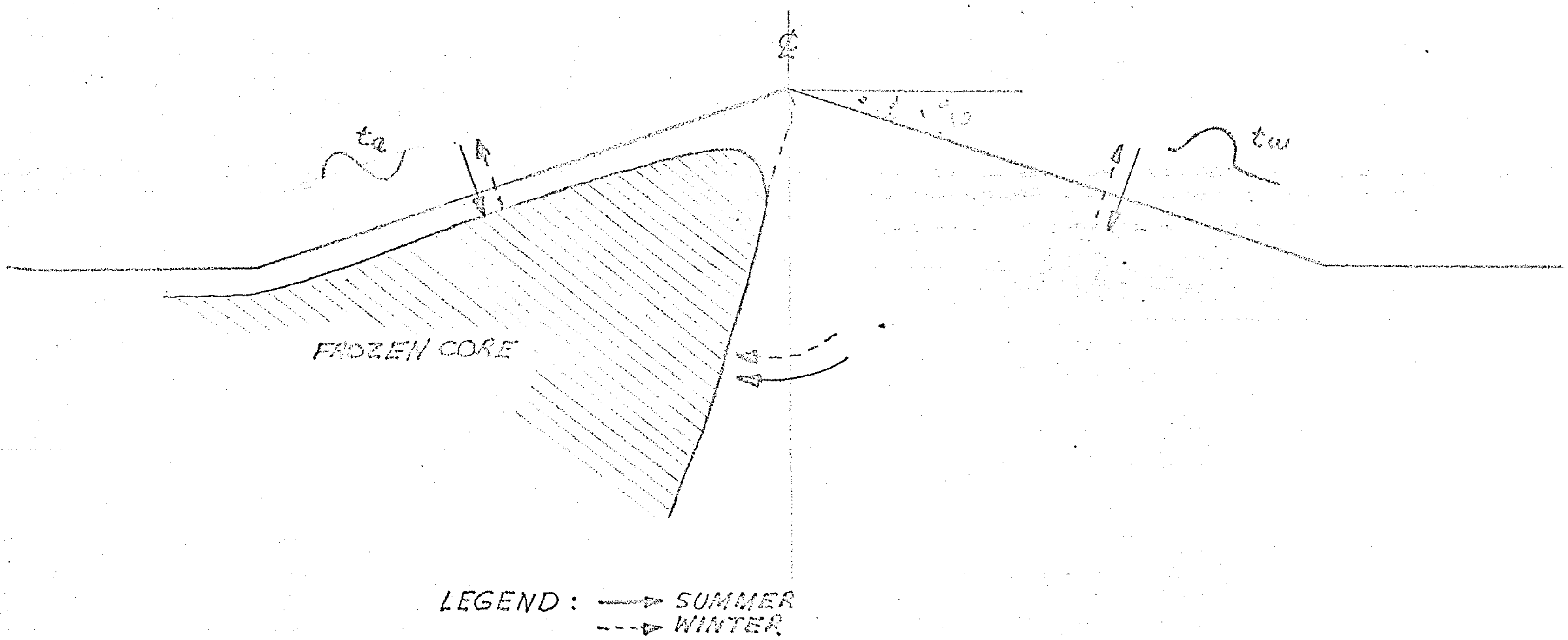

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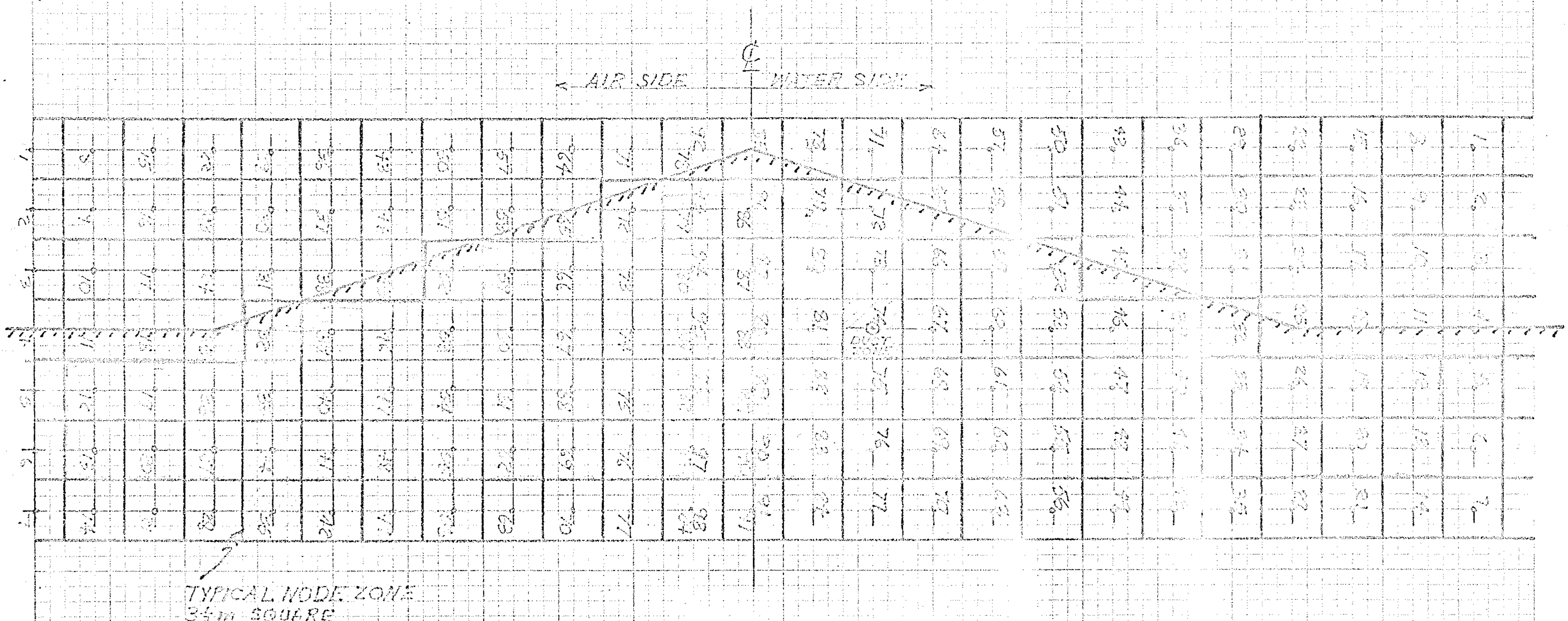
FIG. 15

HOMOGENEOUS FILL DIKE - CYCLIC

SQUARE NETWORK FOR COOLING DUCT CALCULATIONS



SCALE : 1" = 27.4' (8⅓m)

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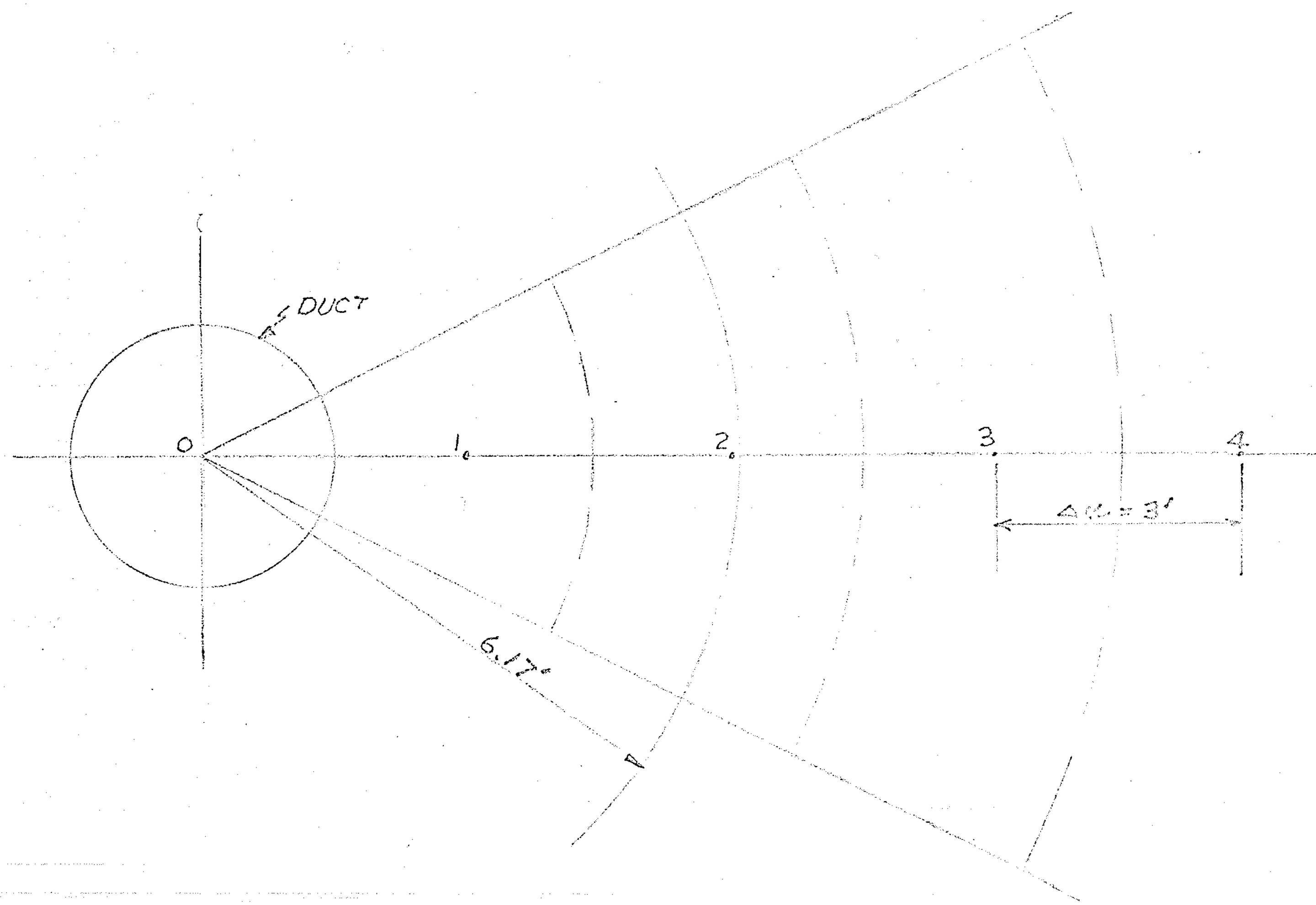
FIG. 16
GEOMETRY FOR DUCT ZONE
TEMPERATURE CALCULATION



SCALE: 1" = 2'

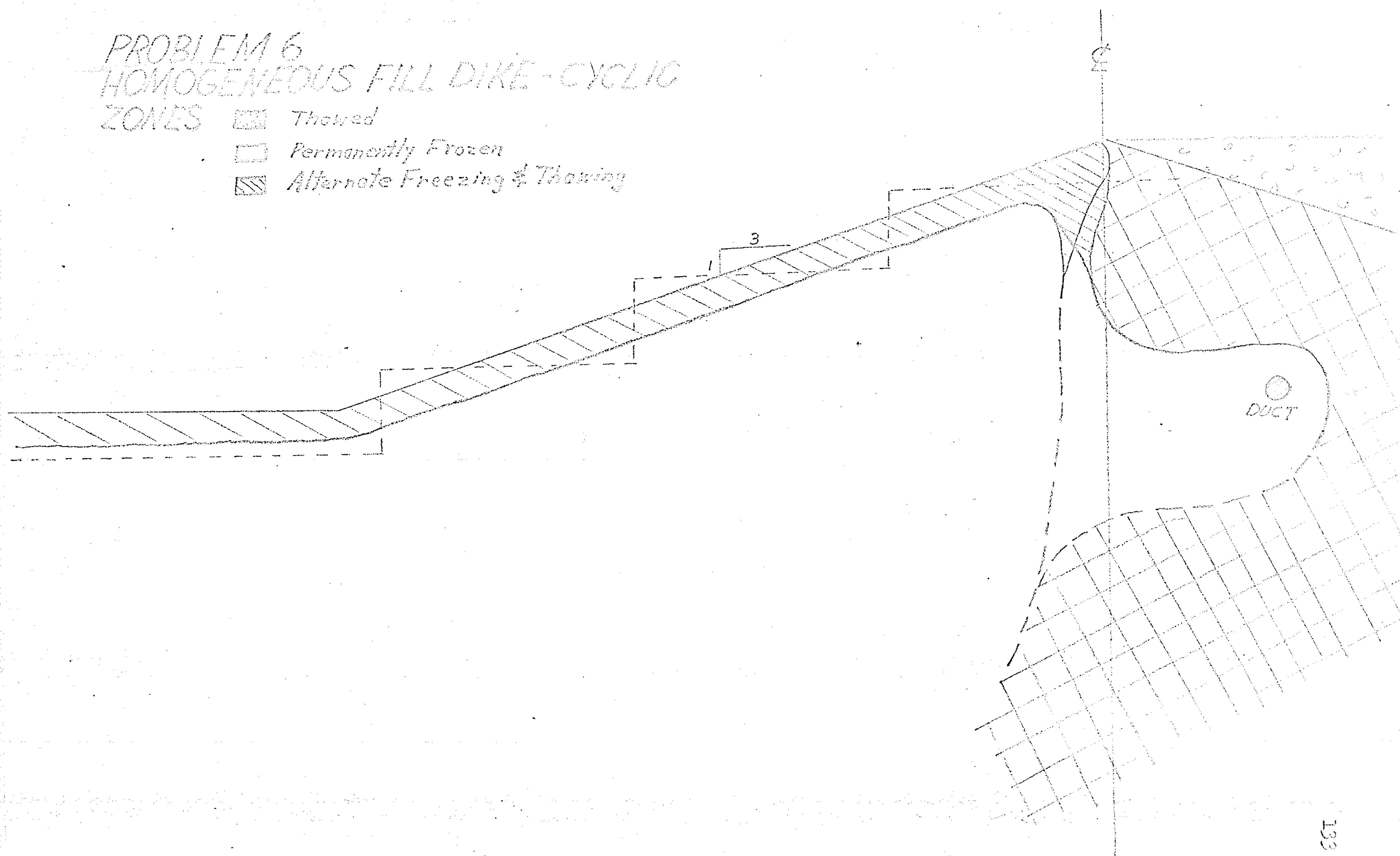
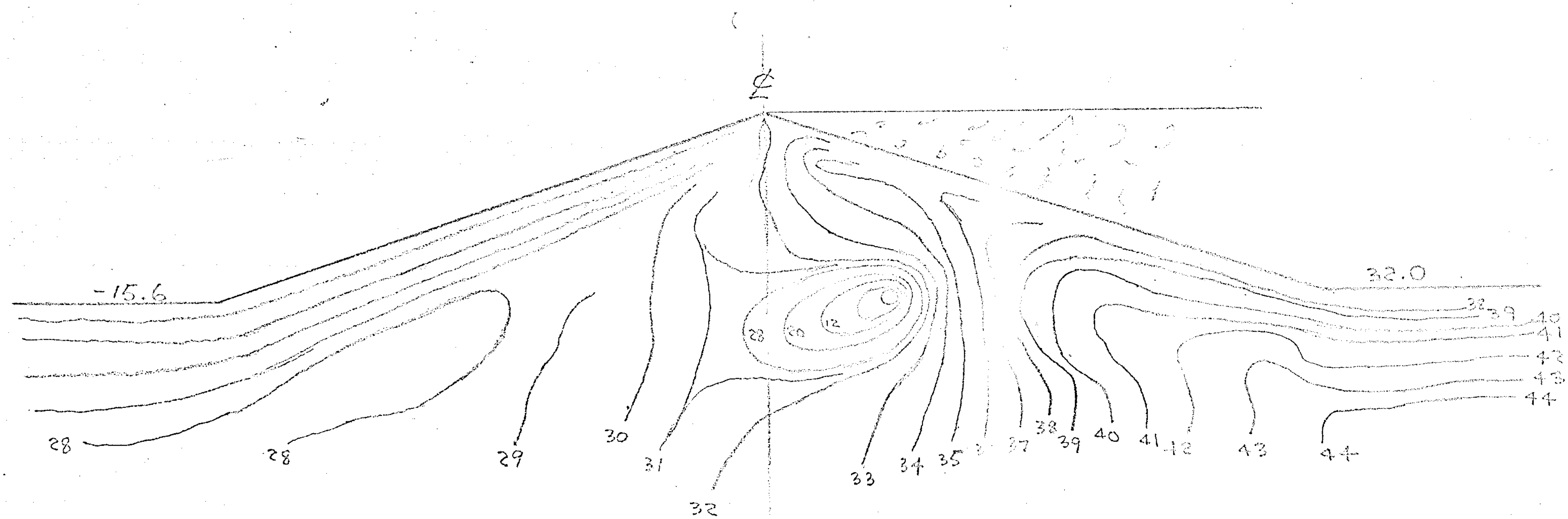


FIG. 17
SCALE : 1" = 16.4' (5m)

U of M '63

FIG. 18

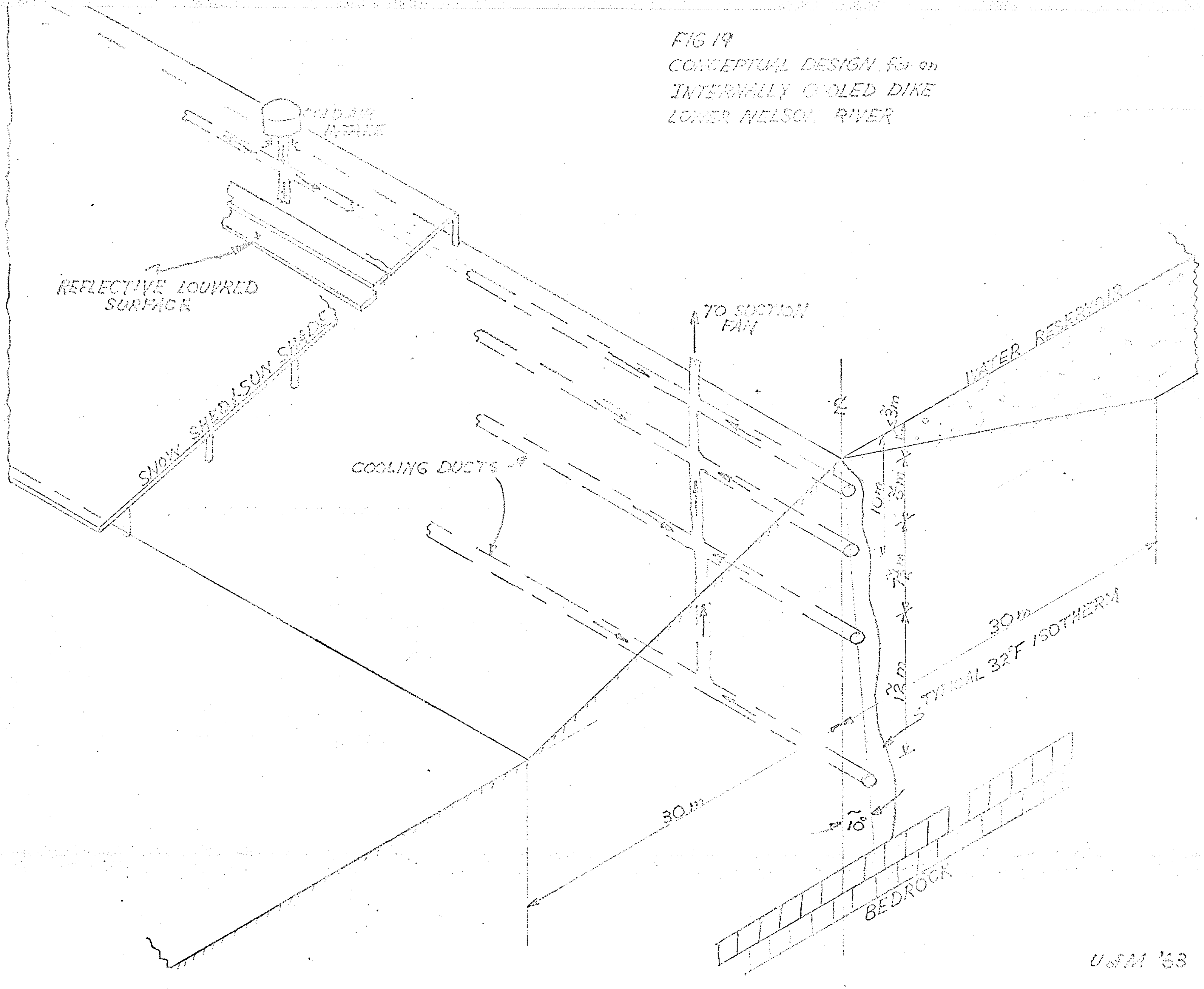
HOMOGENEOUS FILL DIKE - CYCLIC
WINTER COOLING - SINGLE DUCT
ISOTHERMS @ END OF JANUARY



SCALE: 1" = 27.4' (8⅓ m)

UofM '63

FIG 19
CONCEPTUAL DESIGN for an
INTERNALLY COOLED DIKE
LOWER NELSON RIVER



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A P P E N D I C E S

APPENDIX 'A'

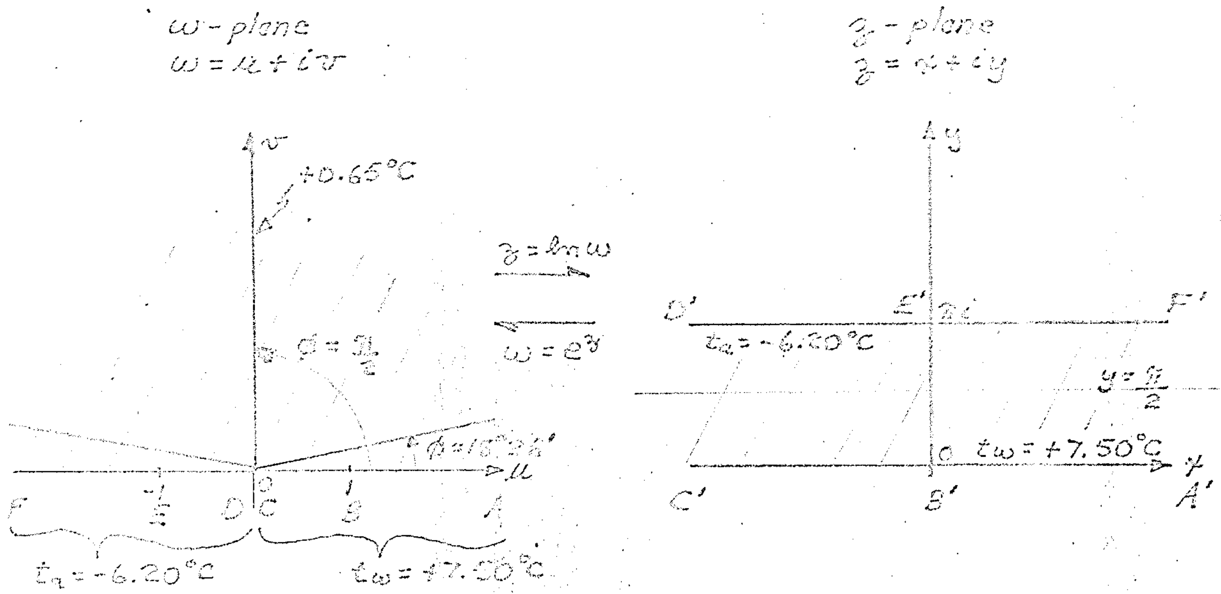
TABLE 1

BOUNDARY CONDITION DATA

	Mean Monthly Air Temperature (°F)	Mean Monthly Water Temperature (°F)	Approx. Average Insolation (ly/day)*
Jan	- 16.6	32	75
Feb	- 8.6	32	180
Mar.	+ 2.8	32	320
Apr	+ 20.9	32	450
May	+ 36.9	36	500
Jun	+ 50.4	57	515
Jul.	+ 59.0	66	500
Aug.	+ 56.0	66	360
Sep.	+ 44.2	57	225
Oct.	+ 32.0	44	140
Nov.	+ 10.0	33	75
Dec.	- 6.7	32	50
Year	+ 23.4 (-4.8°C)	43 (6.1°)	

*1 ly = 3.68 Btu/ft^2

HOMOGENEOUS FILL DIKE-STEADY STATE
SOLUTION BY CONFORMAL MAPPING



The transformation maps the upper half of the w-plane into the infinite strip $0 \leq y \leq \pi$ of the z-plane.

$$w = e^z \quad \text{or} \quad \rho e^{i\phi} = e^x e^{iy}$$

$$\therefore \rho = e^x \quad \& \quad \phi = y$$

Lines $x = c_1$ $\qquad \rho = e^{c_1}$, a circle centered at the origin

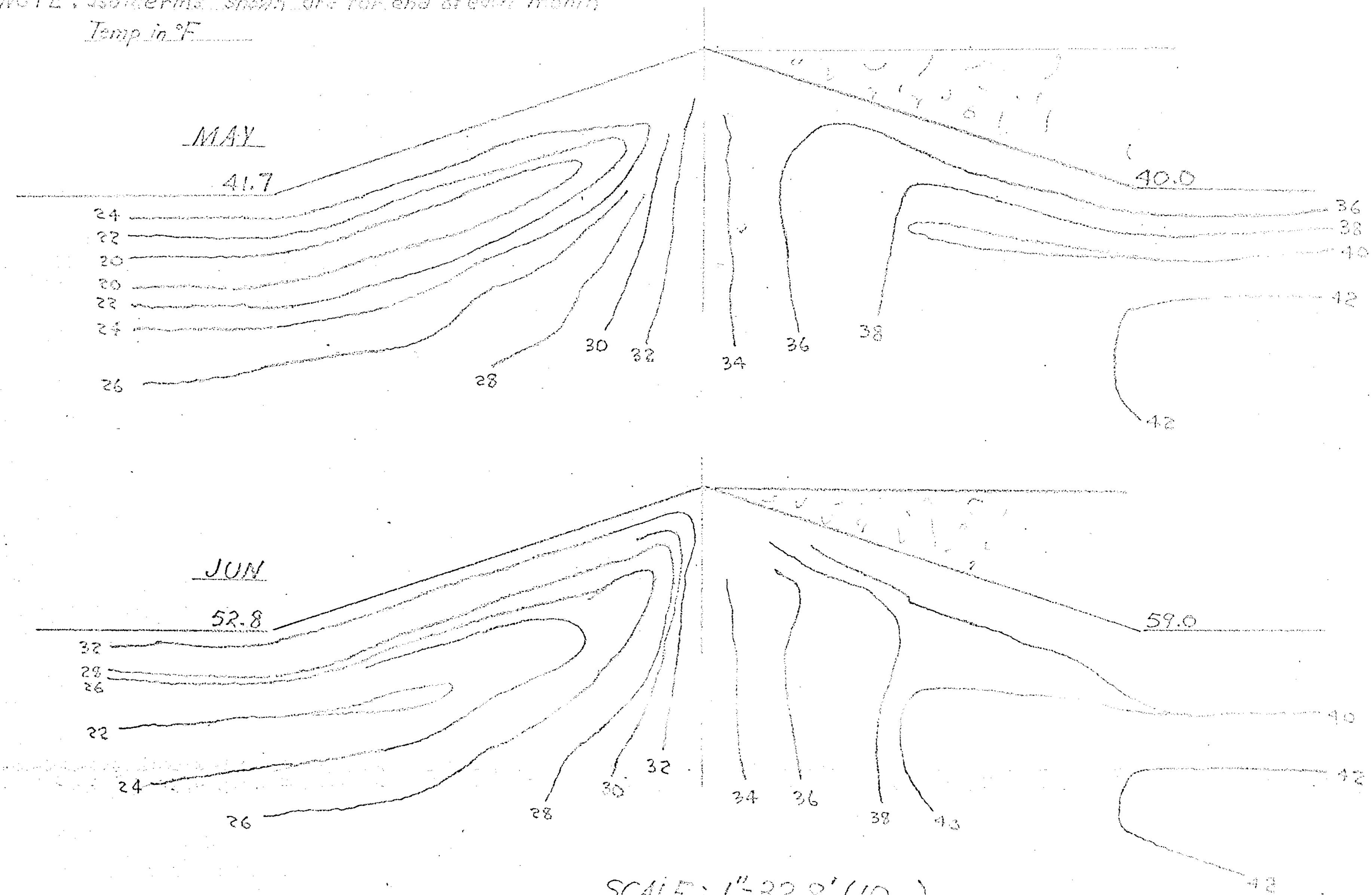
Lines $y = c_2$ $\qquad \phi = c_2$, radial lines

Isotherm	ø	Corresponding value of ø from Analog Field Plotter Results - Fig. 8
6.1°C	18.4°	18.4°
5	32.9	33½
4	46.0	47½
3	59.1	59½
2	72.3	73½
1	85.4	86
0	98.5	98½
-1	111.7	111½
-2	124.8	125½
-3	137.9	138
-4	151.1	152
-4.8	161.6	161.6

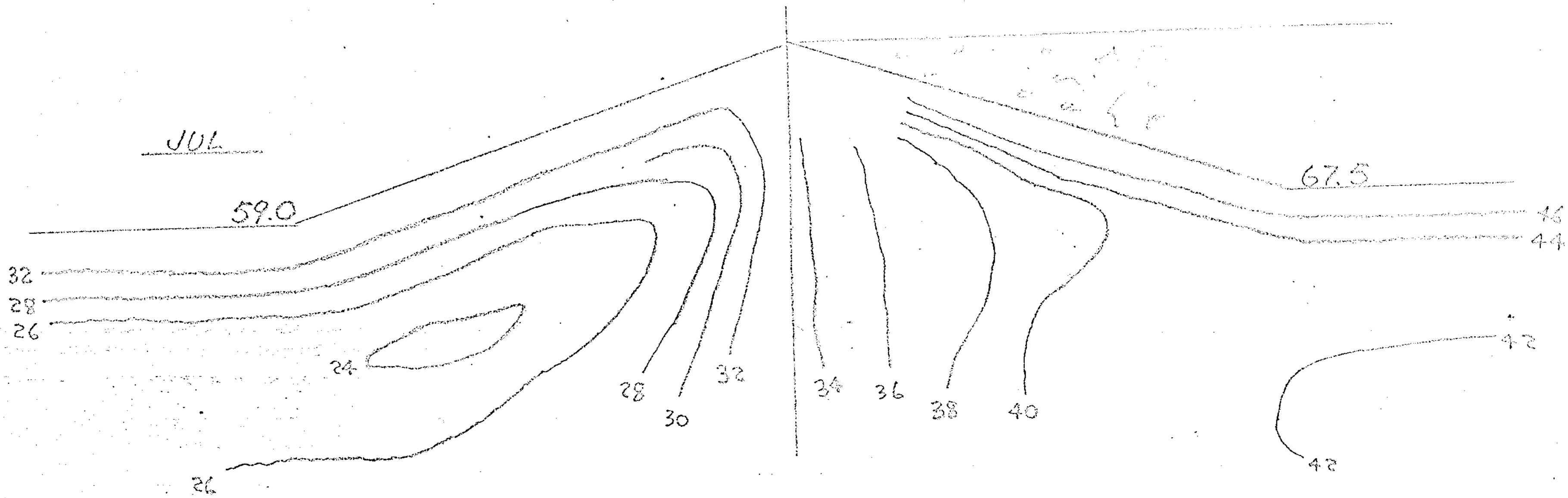
PROBLEM 3 a)
HOMOGENEOUS FILL DIKE - CYCLIC
MONTHLY ISOTHERMS

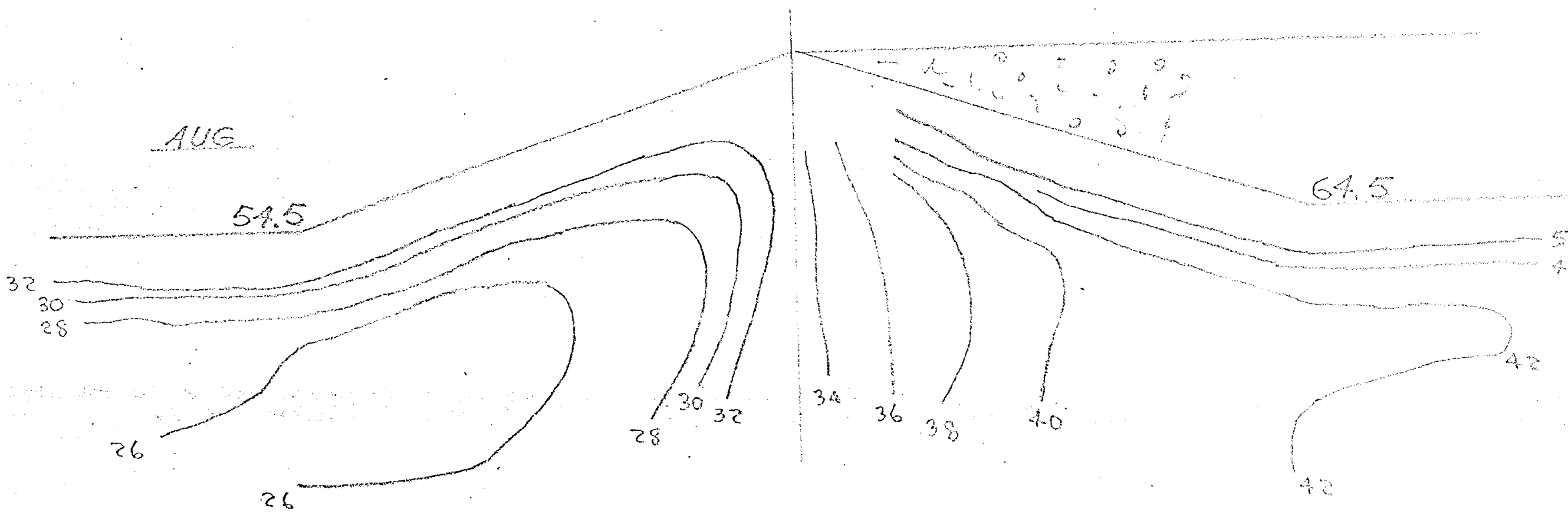
APPENDIX 'C'
Sheet 1 of 4

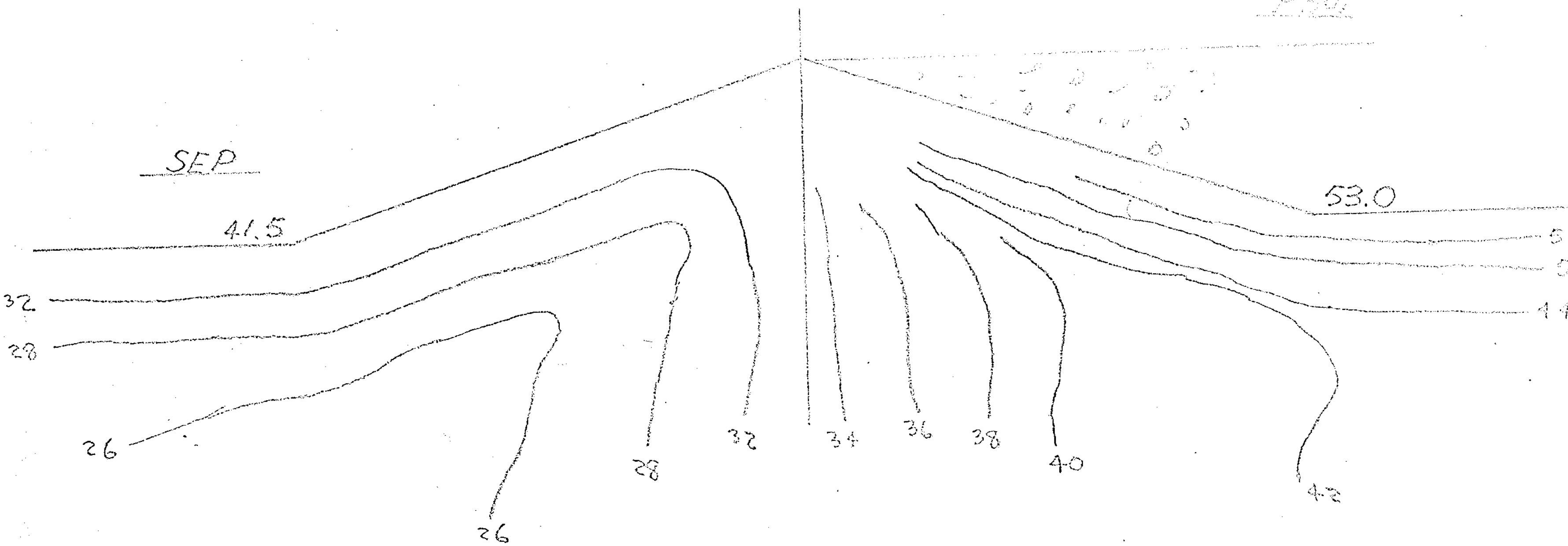
NOTE: Isotherms shown are for end of each month
Temp in °F

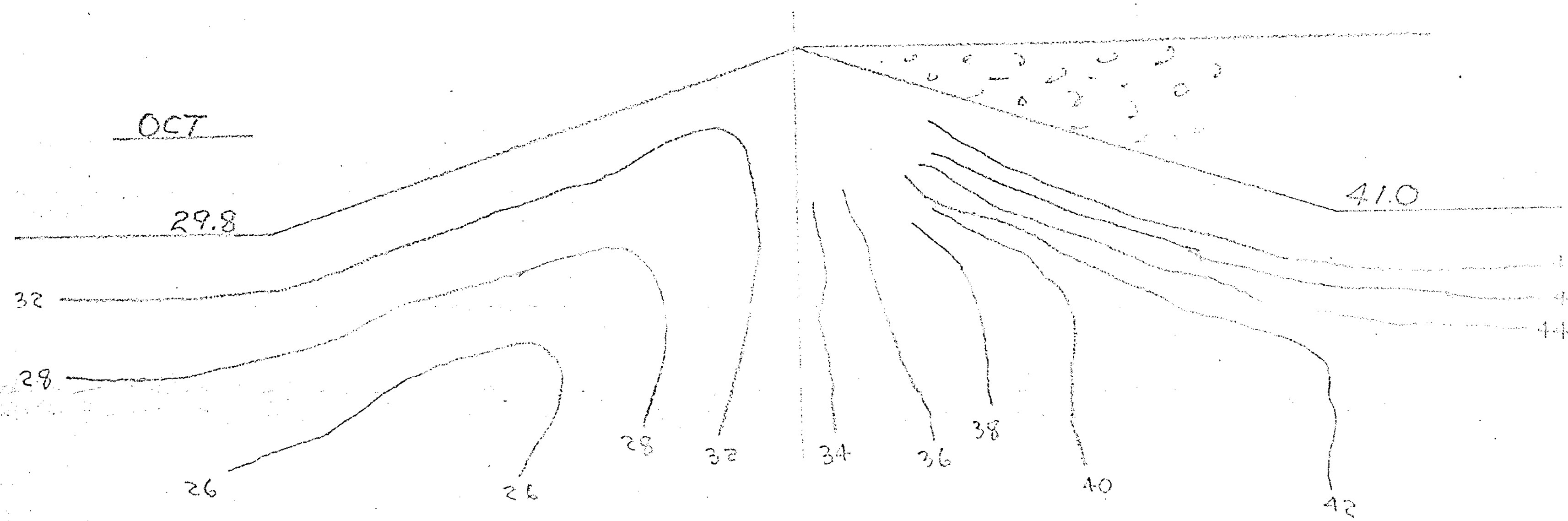


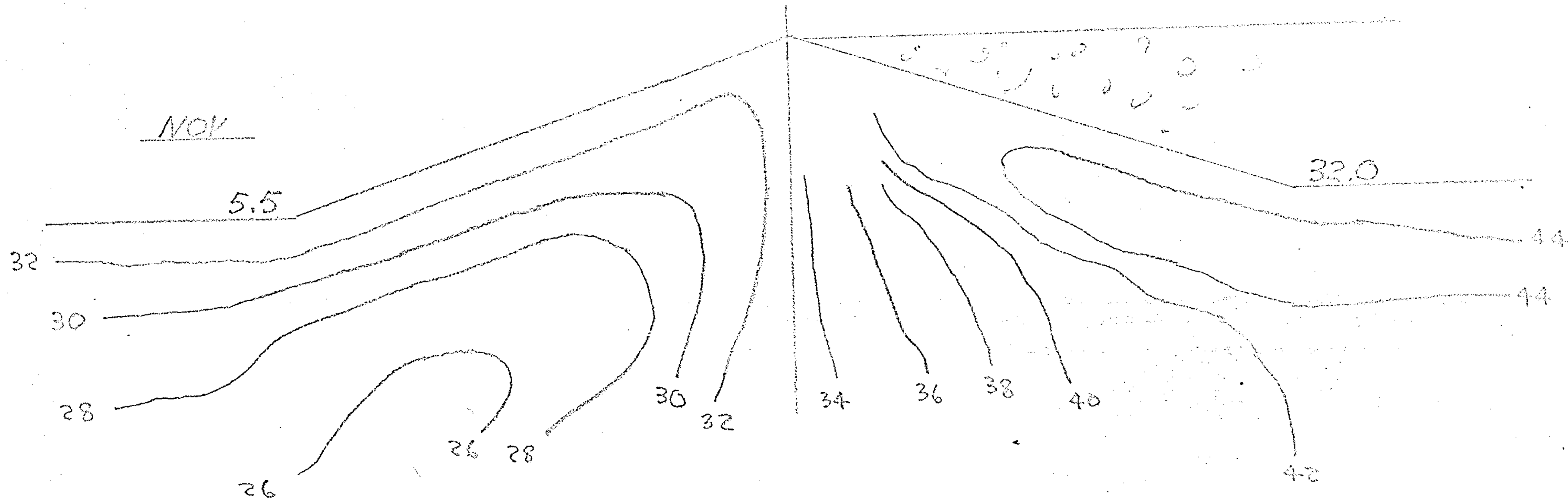
SCALE: 1" = 32.8' (10m)

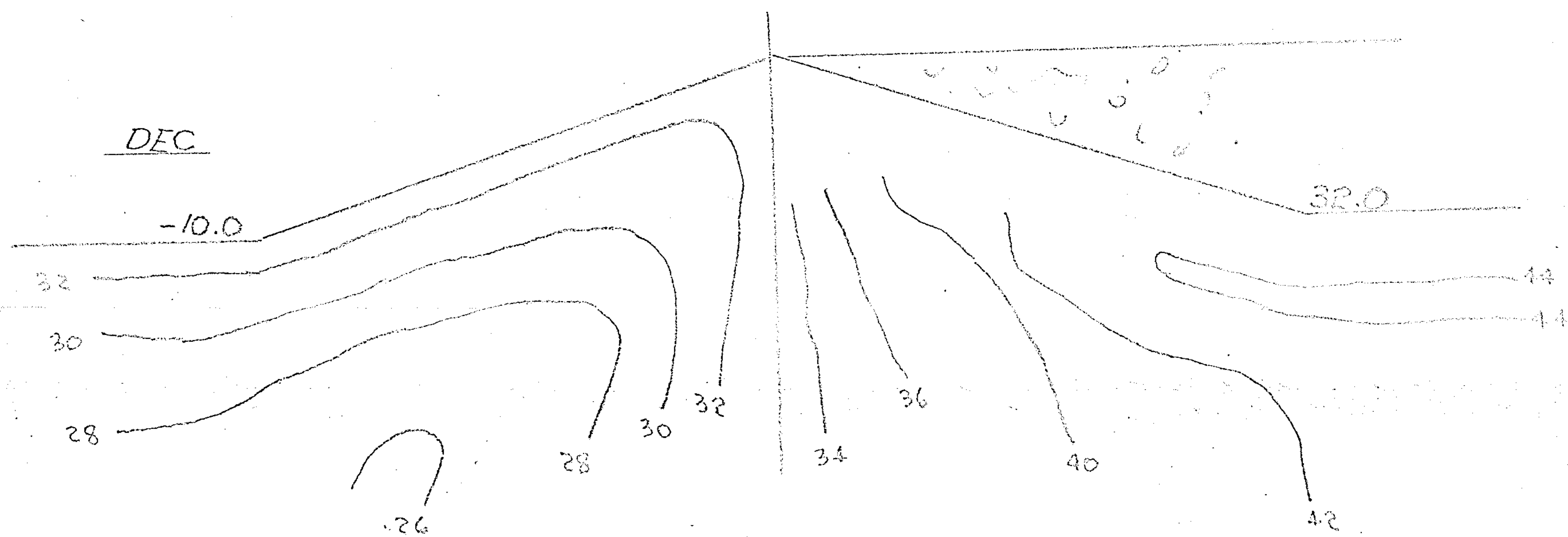

JUL
59.0
67.5
32
28
26
24
26
28
30
32
34
36
38
40
42
42
46
44


AUG
54.5
64.5
32
30
28
26
26
28
30
32
34
36
38
40
42
42
50
46

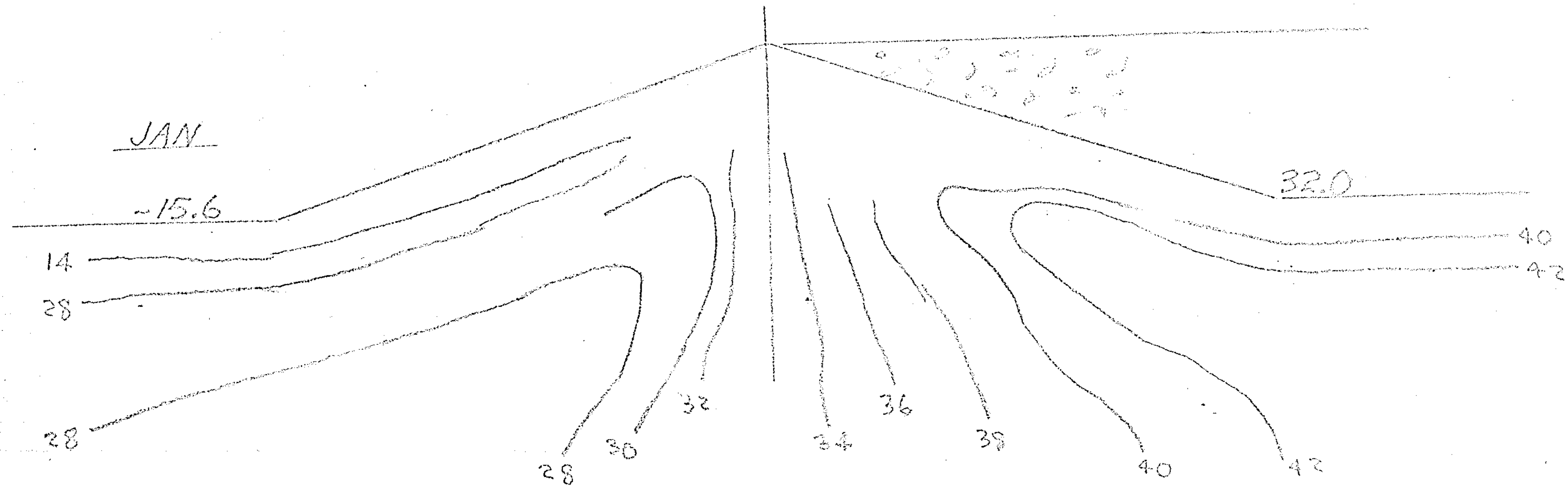

SEP
41.5
53.0
32
28
26
52
50
44
26
28
32
34
36
38
40
42

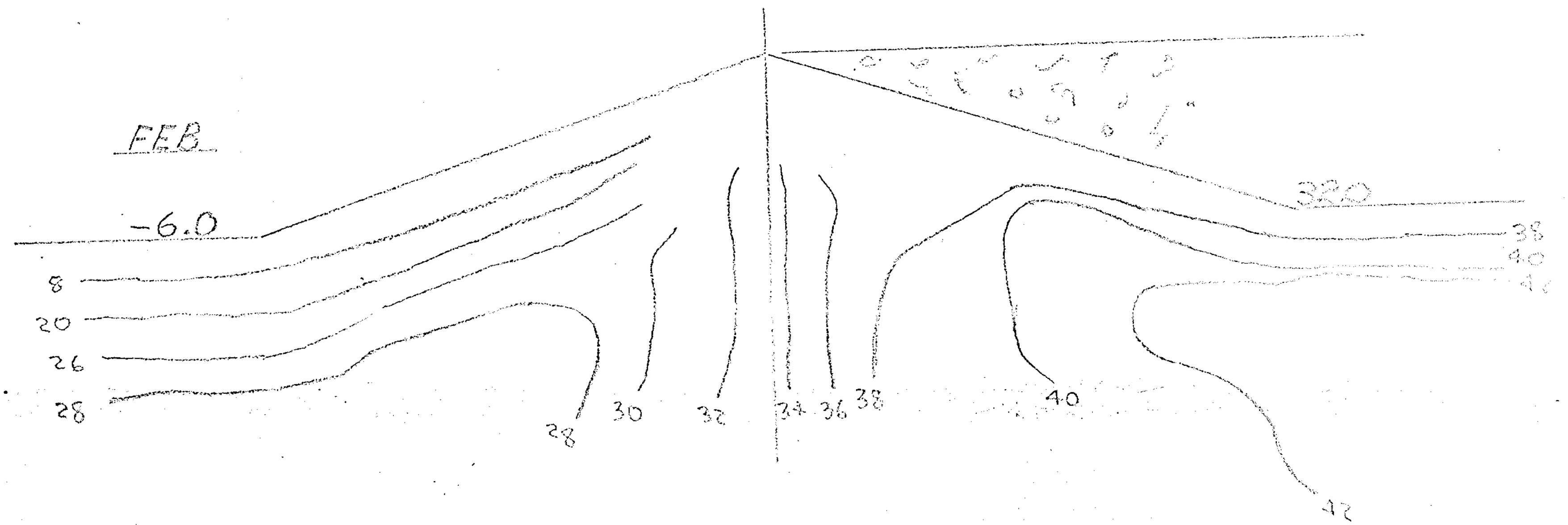

OCT
29.8
41.0
32
28
26
26
28
32
34
36
38
40
42
46
45
44

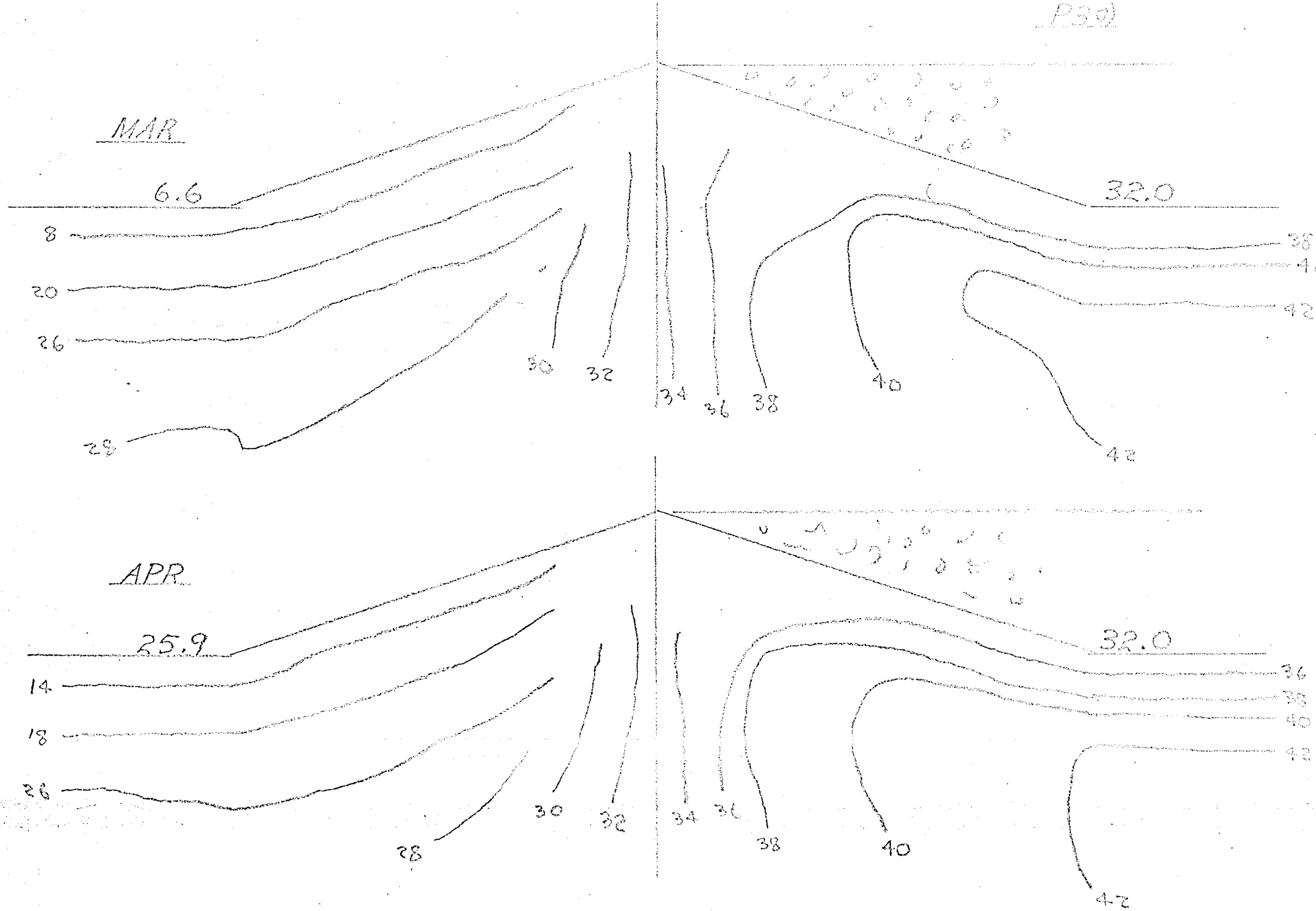

NOV
5.5
32.0
32
30
28
26
26
28
30
32
34
36
38
40
42
44
44


DEC
-10.0
32.0
32
30
28
26
28
30
32
34
36
40
42
44
44

Sheet 5 of 6
P30)



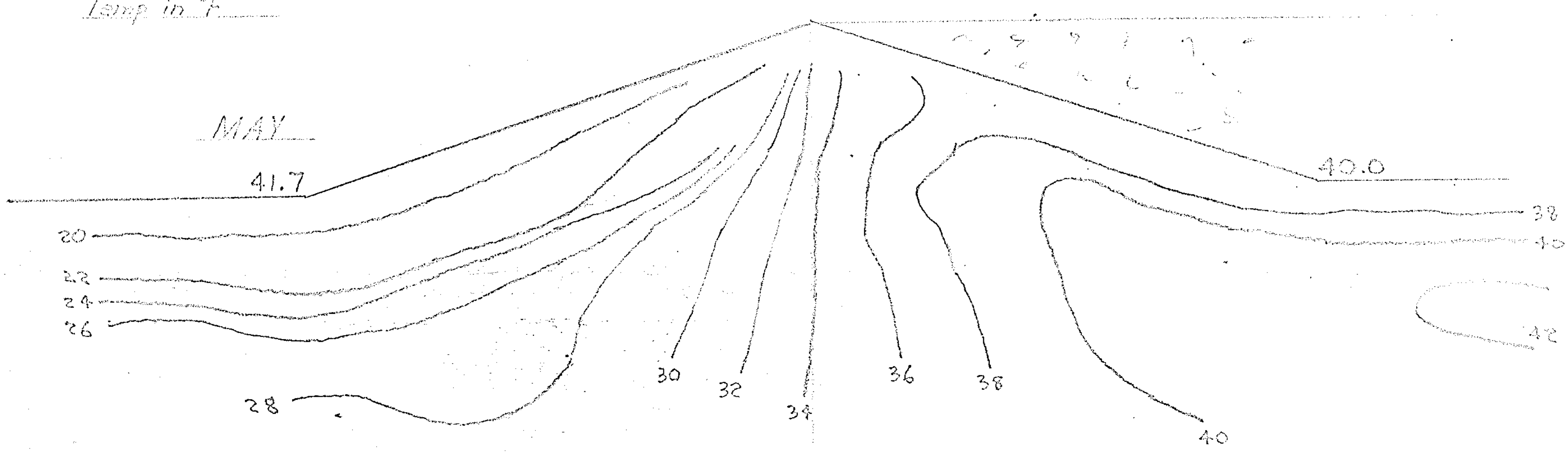


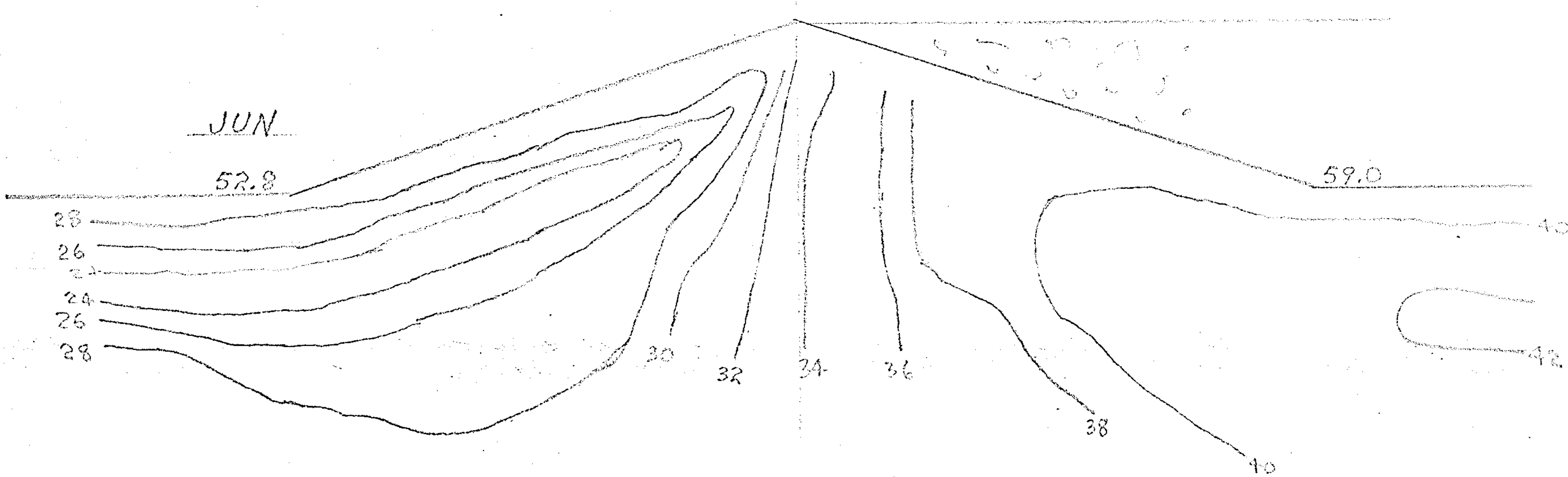

MAR
6.6
32.0
8
20
26
28
30
32
34
36
38
40
42
APR
25.9
32.0
14
18
26
28
30
32
34
36
38
40
42

PROBLEM 3 b)
HOMOGENEOUS FILL DIKE - CYCLIC
MONTHLY ISOTHERMS

NOTE: Isotherms shown are for end of each month
Temp in °F

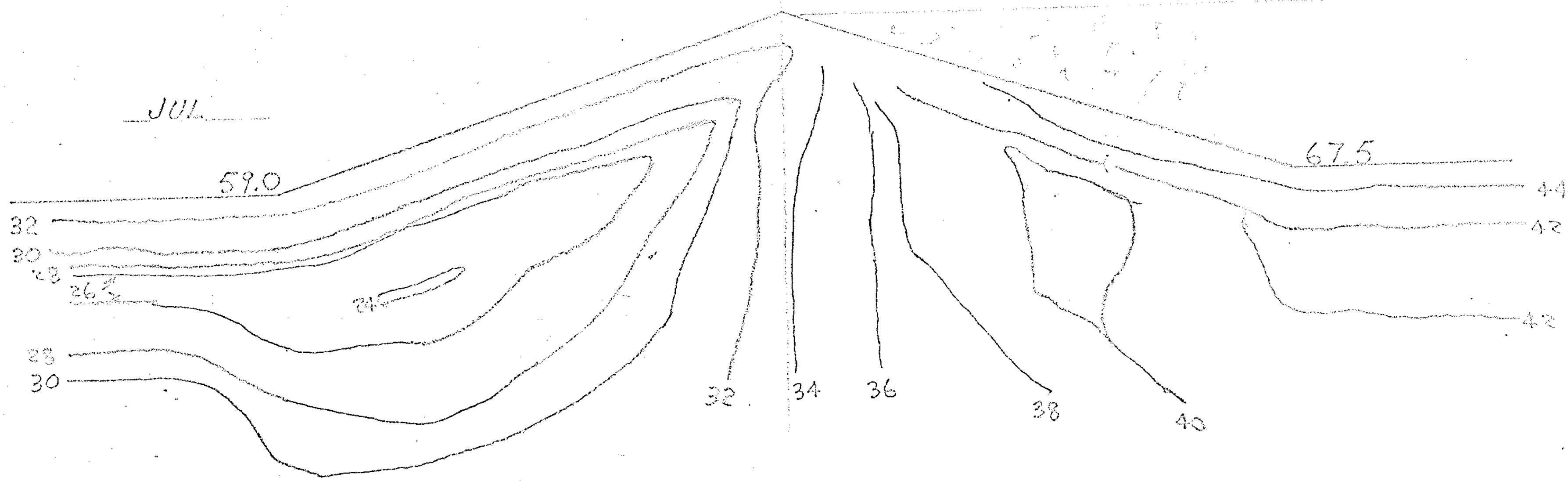
APPENDIX 'D'
Sheet 1 of 6

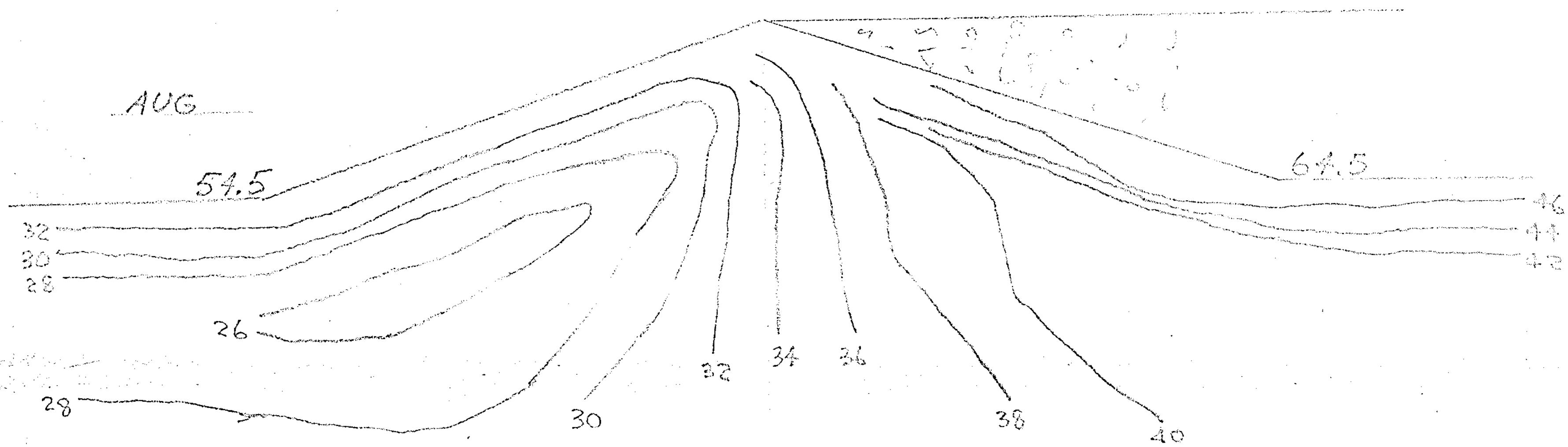




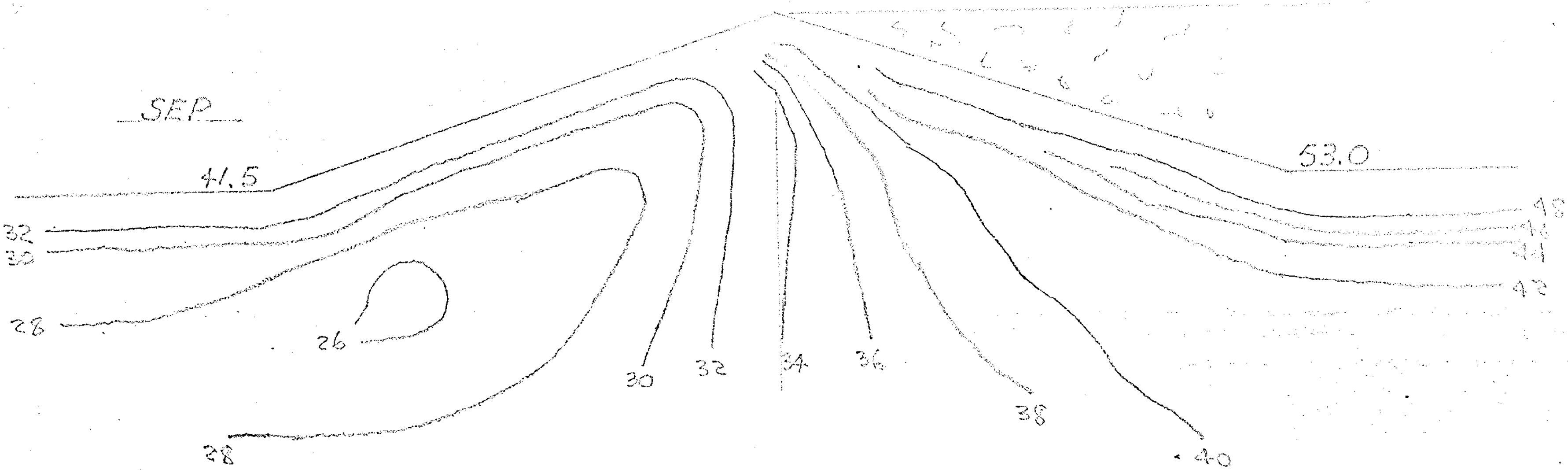
SCALE: 1" = 32.8' (10m)

Sheet 2 of [illegible]
P381

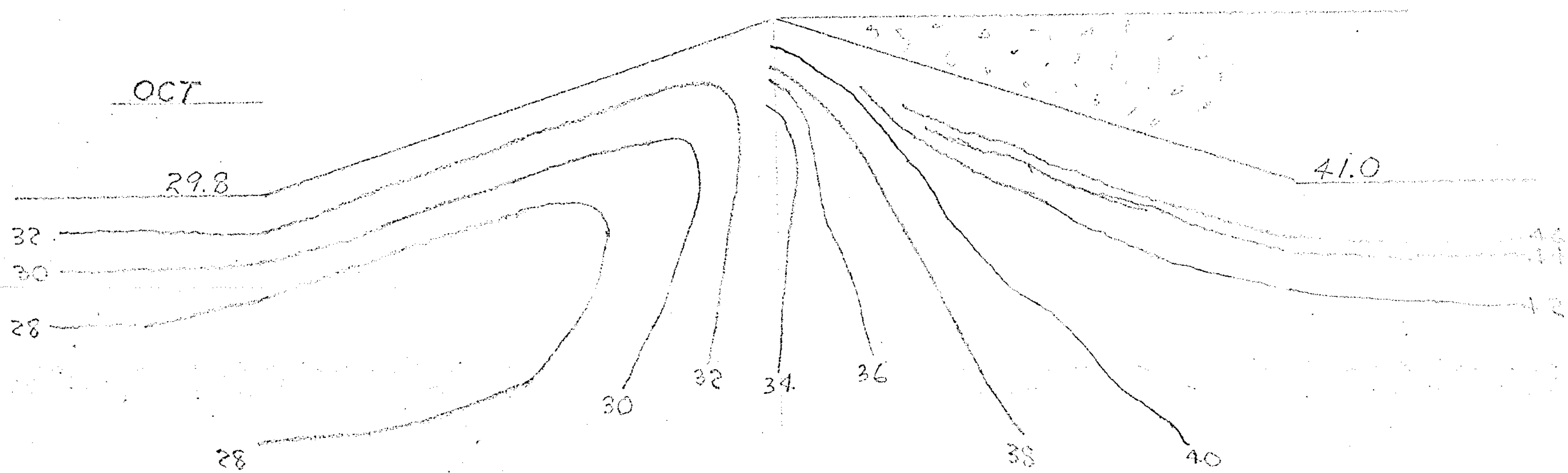


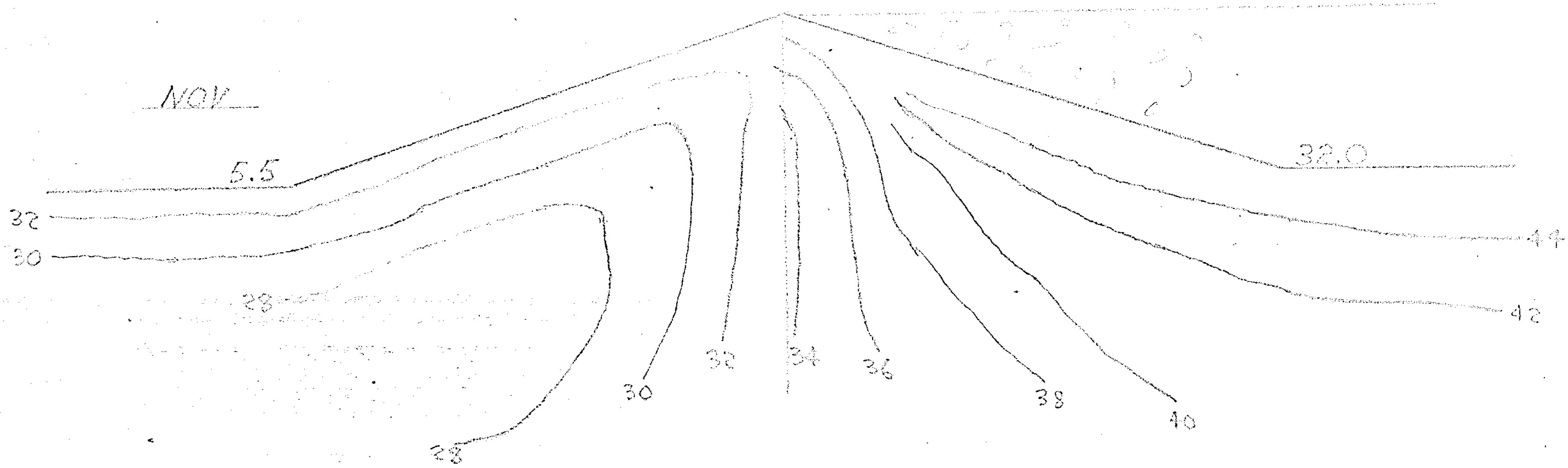


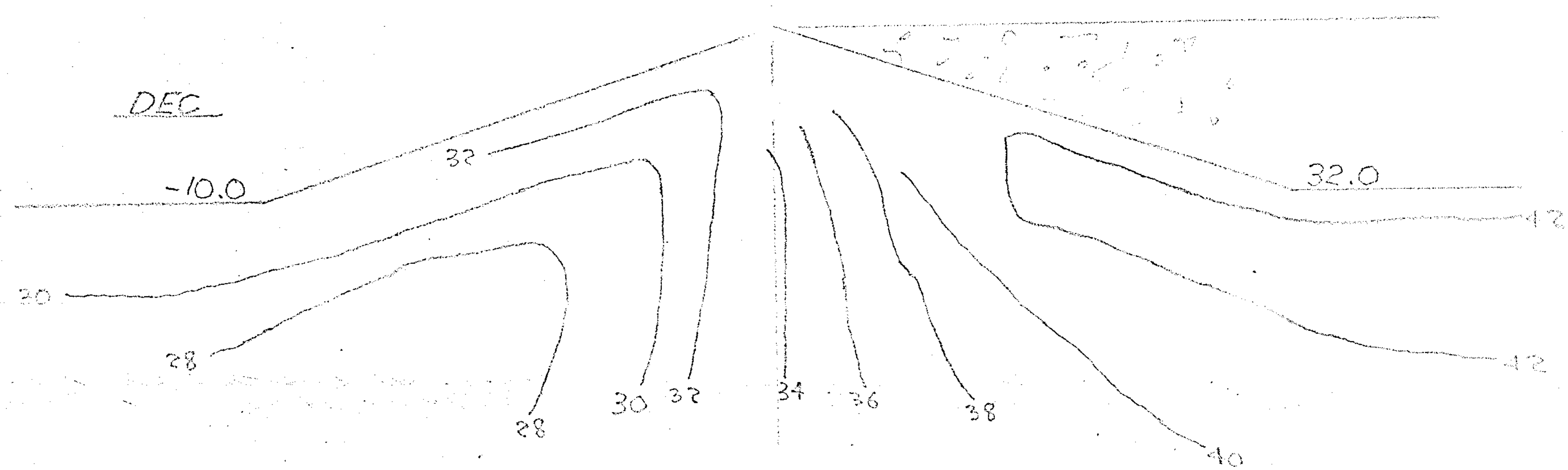
SEP


41.5
53.0
32
30
28
26
28
30
32
34
36
38
40
48
46
44
42

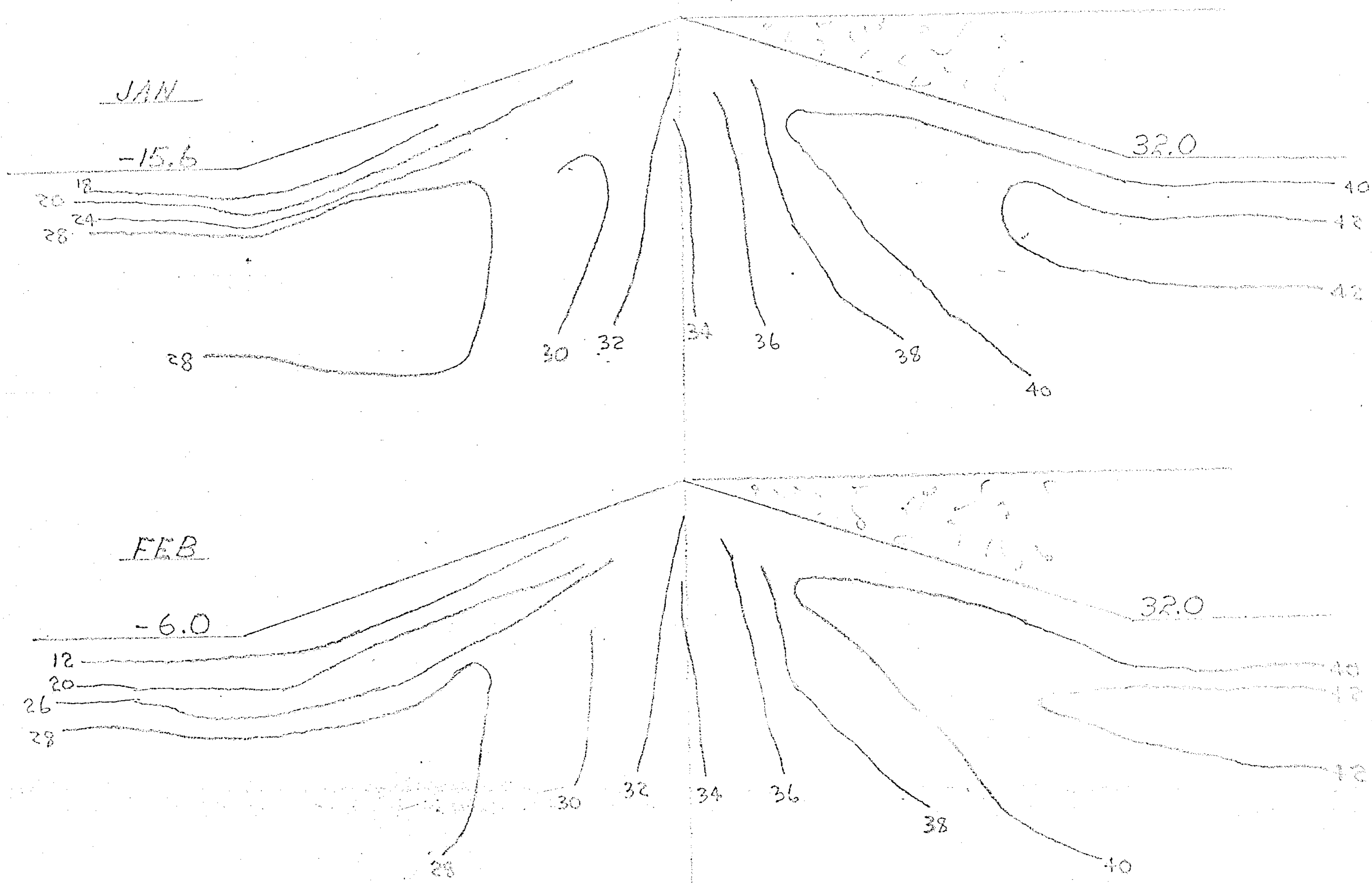
OCT

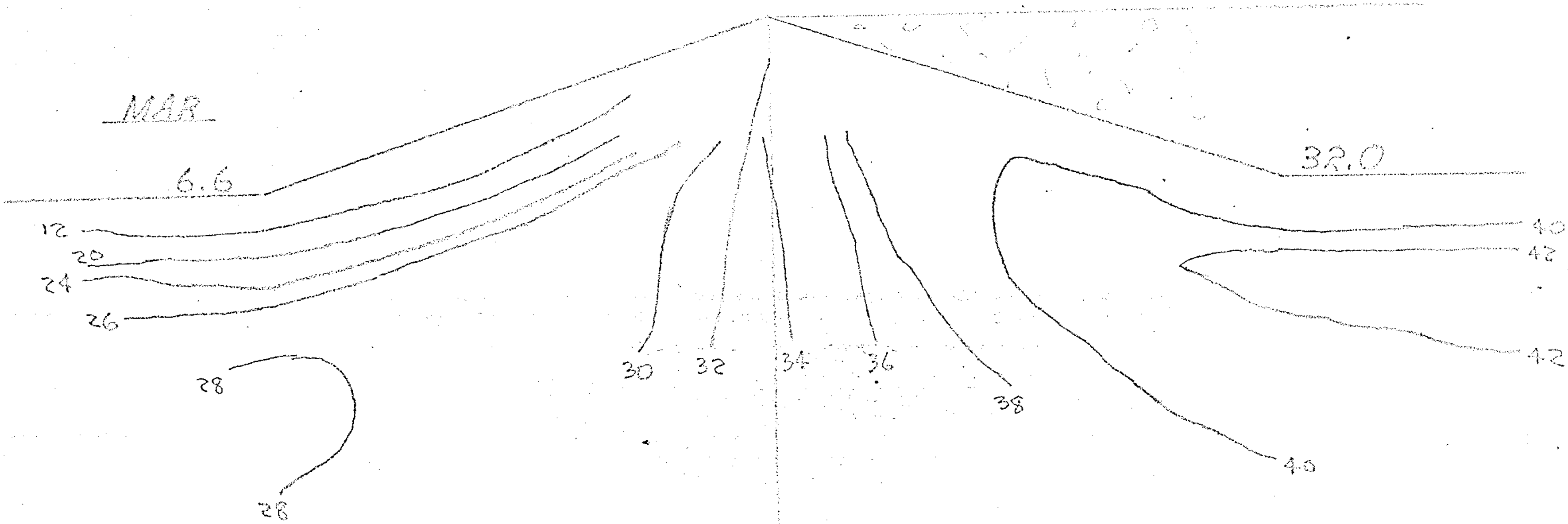

29.8
41.0
32
30
28
28
30
32
34
36
38
40
46
44
42

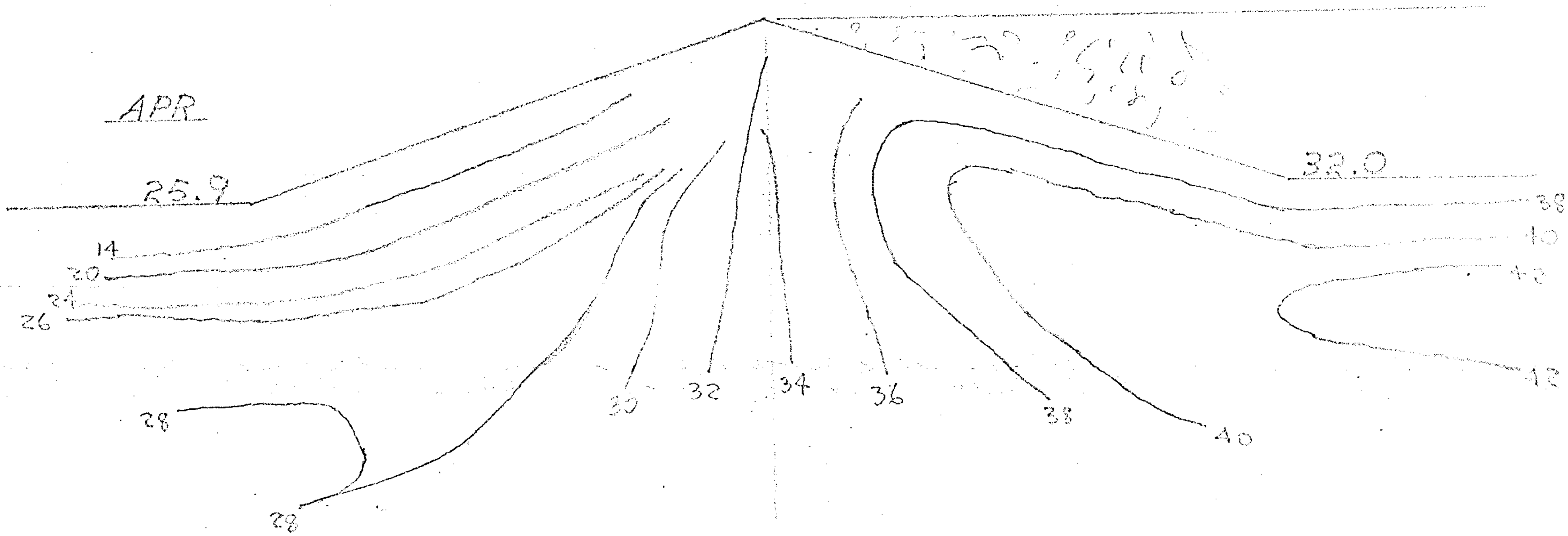

NOV
5.5
32.0
32
30
28
44
42
30
32
34
36
38
40
28


DEC
-10.0
32.0
32
30
28
42
42
28
30
32
34
36
38
40

Sheet 5 of 6
P361




MAR
6.6
32.0
12
20
24
26
28
28
30
32
34
36
38
40
40
42
42


APR
25.9
32.0
14
20
24
26
28
28
30
32
34
36
38
38
40
40
42
42

APPENDIX 'E'

TABLE 2

BOUNDARY CONDITION AND DUCT ZONE TEMPERATURE DATA

Note: For actual temperature (^{o}F) subtract 100.

Period Ending		Water	Air	Duct
Jan.	15	132.0	82.5	107.6
	30	"	84.4	107.9
Feb.	15	"	88.9	110.7
	30	"	94.0	113.1
Mar.	15	"	99.0	116.2
	30	"	106.6	119.3
Apr.	15	"	116.0	127.9
	30	"	125.9	130.7
May.	15	"	132.0	131.6
	30	140.0	141.7	131.8
Jun.	15	155.0	148.0	131.9
	30	159.0	152.8	132.0
Jul.	15	164.5	159.0	"
	30	167.5	159.0	"
Aug.	15	167.5	157.5	"
	30	164.5	154.5	"
Sep.	15	161.0	147.0	"
	30	153.0	141.5	"
Oct.	15	147.0	134.2	"
	30	141.0	129.8	"
Nov.	15	134.0	114.5	122.2
	30	132.0	105.5	118.9
Dec.	15	"	96.5	113.6
	30	"	90.0	110.4

APPENDIX 'F'

BENDIX G-15D COMPUTER PROGRAM FOR HOMO-
GENEOUS FILL DIKE-CYCLIC-COOLING DUCTS

General

1) The geometry for the program which follows is as shown in Figure 15. This program is for a single duct at 1774. Other ducts can be handled by the program as indicated.

2) To avoid negative values of B.C. # 1, the computer operates on temperature values equal to true temperature plus 100, i.e. freezing/thawing at 132°F.

3) Type-out is profile, i.e. temperature values are printed according to node position in the dike cross-section. Type-out can be eliminated without difficulty.

4) Three sets of 'F' factors are used in the computation. These correspond to thawed, frozen or phase change nodes. The computer has a choice of three paths. The choice is made by examining the original node temperature value and proceding on the basis of $t > 132$ ($L=0$), $t < 132$ ($L=82$) or $t = 132$ ($0 \leq L \leq 82$).

5) Phase change is handled by the method of 'excess degrees'. Separate accounts of the accumulated excess degrees (L) are maintained for each node. The method is applied according to the following flow diagram.

$$\begin{array}{ccccc} & & \text{freezing} \longrightarrow & & \\ UF \longrightarrow & 0 & \text{to} & +82 & \\ & \uparrow & & \downarrow & \\ & 0 & \text{to} & +82 & \longleftarrow F \\ & & \longleftarrow \text{thawing} & & \end{array}$$

The fraction of the volume of a node zone which is frozen at any time is $(\frac{1}{82})$ times the L value at the time under consideration.

Data

0700 to 0724 inclusive - t_w data (Appendix 'E')

0725 to 0749 inclusive - t_a data

0750 to 0774 inclusive - t_d data

$\left.\begin{array}{l} 0775 = 0.0890 \\ 0776 = 0.6440 \end{array}\right\}$ UF , $F_{n-\ n\pm m}$ F_{n-n}

$\left.\begin{array}{l} 0777 = 0.1695 \\ 0778 = 0.3220 \end{array}\right\}$ F

$\left.\begin{array}{l} 0779 = 0.1292 \\ 0780 = 0.4832 \end{array}\right\}$ F/T

0781 = 82.0 to freeze/thaw

0782 = 132.0

0783 = 2.0

0790 = N_{sides}

0791 = Miscellaneous

0792 = $S_{F/T}$

0800 = S_w

0900 = S_a

1500 = L_w

1600 = L_a

1700 = N_w

1800 = N_a

UNIVERSITY OF MANITOBA

INTERCOM 1000S

PROBLEM HOMOGENEOUS FILL DIKE WINTER COOLING - CYCLIC	Prepared by: A.C. TRUPP	
	Date: AUG/63	Page 1 of 14

LOC	K	OP CODE	AD	DR	(A)	Notes
1000		30	00	02		pos paper
01	2	70	00	00		$WB_2 = 0$
02	2	71	00	01		$WD_2 = 1$
03	2	72	00	23		$WL_2 = 23$
04	1	73	00	00		$CB_1 = 0$ 11
05	1	74	01	00		$CD_1 = 1$
06	1	75	01	00		$CL_1 = 1$
07	3	73	00	00		$CB_3 = 0$
08	3	74	01	00		$CD_3 = 1$
09	3	75	01	00		$CL_3 = 1$
1010	2	42	07	00		$0700 \xrightarrow{+} Ac$
11		49	17	11		store in 1711
12		49	17	18		1718
13		49	17	25		1725
14		49	17	31		1731
15		49	17	38		1738
16		49	17	45		1745
17		49	17	51		1751
18		49	17	58		1758
19		49	17	65		1765
1020		49	17	71		1771
21		49	17	78		1778
22		49	18	92		1892
23	2	43	07	25		$0725 \xrightarrow{+} A$
1024		48	07	83		$0783 \xrightarrow{\div} A$

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INTERCOM 1000

PROBLEM	Prepared by:	
	Date:	Page 2 of 14

LOC	K	OP CODE	AD	DR	(A)	Notes
1025		49	17	85		store in 1785
26		49	18	85		1885
27	2	42	07	25		0725 $\xrightarrow{+}$ Ac
28		49	18	11		store in 1811
29		49	18	18		1818
1030		49	18	25		1825
31		49	18	31		1831
32		49	18	38		1838
33		49	18	45		1845
34		49	18	51		1851
35		49	18	58		1858
36		49	18	65		1865
37		49	18	71		1871
38		49	18	78		1878
39		49	17	92		1792
1040		29	11	72		Go to → 1
41	3	70	00	00		$WB_3 = 0$ ← 9
42	3	71	00	05		$WD_3 = 5$
43	3	72	00	70		$WL_3 = 70$
44	3	29	11	00		Go to → 6
45						
46						
47	2	33	07	01		type 0701 & tab ← 7
48	1	42	17	12		1712 $\xrightarrow{+}$ Ac ← 5
1049		41	07	82		0782 $\xrightarrow{-}$ A

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PROBLEM

Prepared by:

Date: | Page 3 of 14

LOC	K	OP CODE	AD	DR	(A)	Notes
1050		23	13	55	If A = 0 → freeze/thaw	If A = 0 go to → (2)
51		22	10	75	If A < 0 → frozen	If A < 0 go to → (3)
52	1	42	17	05	A > 0 → UF	1705 $\xrightarrow{+}$ Ac
53	1	43	17	19		1719 $\xrightarrow{+}$ A
54	1	43	17	11		1711 $\xrightarrow{+}$ A
55	1	43	17	13		1713 $\xrightarrow{+}$ A
56		44	07	75		0775 $\xrightarrow{\times}$ A
57		49	07	90	UF	store in 0790
58	1	42	17	12		1712 $\xrightarrow{+}$ Ac
59		44	07	76		0776 $\xrightarrow{\times}$ A
1060		43	07	90		0790 $\xrightarrow{+}$ A
61	1	49	08	12		store in 0812
62		41	07	82		0782 $\xrightarrow{-}$ A
63		22	13	50	If A < 0 → start to freeze	If A < 0 go to → (4)
64	1	33	08	12		type 0812 & tab ← (12)
65	1	76	10	48		Inc WB1 & go to → (5)
66	3	29	11	05		Go to → (6')
67						
68	2	42	07	01		0701 $\xrightarrow{+}$ Ac ← (8)
69	2	43	07	26		0726 $\xrightarrow{+}$ A
1070		48	07	83		0783 $\xrightarrow{\div}$ A
71		33	21	01		type A & tab
72		29	10	48		Go to → (5)
73	2	33	07	26		type 0726 & tab ← (10)
1074		29	10	48		Go to → (5)

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PROBLEM

Prepared by:

Date:

LOC	K	OP CODE	AD	DR	(A)	Notes
1075	1	42	17	05		1705 $\xrightarrow{+}$ Ac (3)
76	1	43	17	19		1719 $\xrightarrow{+}$ A
77	1	43	17	11		1711 $\xrightarrow{+}$ A
78	1	43	17	13		1713 $\xrightarrow{+}$ A
79		44	07	77		0777 $\xrightarrow{x}$ A
1080		49	07	90	F	store in 0790
81	1	42	17	12		1712 $\xrightarrow{+}$ Ac
82		44	07	78		0778 $\xrightarrow{x}$ A
83		43	07	90		0790 $\xrightarrow{+}$ A
84	1	49	08	12		store in 0812
85		42	07	82		0782 $\xrightarrow{+}$ Ac
86	1	41	08	12		0812 $\xrightarrow{-}$ A
87		22	10	89	If A < 0 → start to thaw	If A < 0 go to
88		29	10	64	If +ve or 0 → still frozen	Go to → (12)
89		49	07	91	-ve quantity	store in 0791
1090	1	42	15	12	+82	1512 $\xrightarrow{+}$ Ac
91		43	07	91	+82 + (-ve quantity)	0791 $\xrightarrow{+}$ A
92	1	49	15	12		store in 1512
93		42	07	82		0782 $\xrightarrow{+}$ Ac (13)
94	1	49	08	12		store in 0812
1095		29	10	64		Go to → (12)

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PROBLEM	Prepared by:	
	Date:	Page 5 of 14

LOC	K	OP CODE	AD	DR	(A)	Notes
1100	1	70	00	00	12 & 13	
01	1	71	00	01		
02	1	72	00	01		
03		30	03	00		
04		29	10	47		
05	1	70	00	07	19 & 20	
06	1	71	00	01		
07	1	72	00	08		
08		30	03	01		
09	3	76	10	47		
1110	1	70	00	14	26 & 27	
11	1	71	00	01		
12	1	72	00	15		
13		30	03	01		
14	3	76	10	47		
15	1	70	00	20	32, 33 & 34	
16	1	71	00	01		
17	1	72	00	22		
18		30	02	01		
19	3	76	10	47		
1120	1	70	00	27	39, 40 & 41	
21	1	71	00	01		
22	1	72	00	29		
23		30	02	01		
1124	3	76	10	47		

6

7

6'

7

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PROBLEM	Prepared by:	
	Date:	Page 6 of 14

LOC	K	OP CODE	AD	DR	(A)	Notes
1125	1	70	00	34		
26	1	71	00	01	46, 47 & 48	
27	1	72	00	36		
28		30	02	01		
29	3	76	10	47		
1130	1	70	00	40		
31	1	71	00	01	52, 53, 54 & 55	
32	1	72	00	43		
33		30	01	01		
34	3	76	10	47		
35	1	70	00	47		
36	1	71	00	01	59, 60, 61 & 62	
37	1	72	00	50		
38		30	01	01		
39	3	76	10	47		
1140	1	70	00	54		
41	1	71	00	01	66, 67, 68 & 69	
42	1	72	00	57		
43		30	01	01		
44	3	76	10	37		
45	1	70	00	60		
46	1	71	00	01	72 & 73	
47	1	72	00	61		
48		30	00	01		
1149	3	76	10	47		

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LOC	K	OP CODE	ADDR		(A)	Notes
1150		33	17	74	} 74	
51	3	76	11	55		
52						
53						
54						
55	1	70	00	63		
56	1	71	00	01	} 75 & 76	
57	1	72	00	64		
58	3	76	10	48		
59						
1160	1	70	00	67		
61	1	71	00	01	} 79, 80, 81, 82 & 83	
62	1	72	00	71		
63		30	00	01		
64	3	76	10	47		
65	1	70	00	74		
66	1	71	00	01	} 86, 87, 88, 89 & 90	
67	1	72	00	78		
68		30	00	01		
69	3	76	10	68		→ 8
1170	1	77	11	71		
71	3	77	10	41		→ 9
72	2	42	07	50		$0750 \xrightarrow{+} Ac$ ← 1
73		49	17	74		store in 1774
1174		29	10	41		Go to → 9

Remainder 1100 for duct entries as required.

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LOC	K	OP CODE	ADDR	(A)	Notes
1200	1	70	00 67		6
01	1	71	00 01	79, 80, 81, 82 & 83	
02	1	72	00 71		
03		30	00 01		
04		29	10 73		10
05	1	70	00 60		6'
06	1	71	00 01	72, 73, 74, 75 & 76	
07	1	72	00 64		
08		30	00 01		
09	3	76	10 73		10'
1210	1	70	00 54		
11	1	71	00 01	66, 67, 68 & 69	
12	1	72	00 57		
13		30	01 01		
14	3	76	10 73		
15	1	70	00 47		
16	1	71	00 01	59, 60, 61 & 62	
17	1	72	00 50		
18		30	01 01		
19	3	76	10 73		
1220	1	70	00 40		
21	1	71	00 01	52, 53, 54 & 55	
22	1	72	00 43		
23		30	01 01		
1224	3	76	10 73		

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PROBLEM	Prepared by:	
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LOC	K	OP CODE	AD	DR	(A)	Notes
1225	1	70	00	34		
26	1	71	00	01	46, 47 ≢ 48	
27	1	72	00	36		
28		30	02	01		
29	3	76	10	73		
1230	1	70	00	27		
31	1	71	00	01	39, 40 ≢ 41	
32	1	72	00	29		
33		30	02	01		
34	3	76	10	73		
35	1	70	00	20		
36	1	71	00	01	32, 33 ≢ 34	
37	1	72	00	22		
38		30	02	01		
39	3	76	10	73		
1240	1	70	00	14		
41	1	71	00	01	26 ≢ 27	
42	1	72	00	15		
43		30	03	01		
44	3	76	10	73		
45	1	70	00	07		
46	1	71	00	01	19 ≢ 20	
47	1	72	00	08		
48		30	03	01		
1249	3	76	10	73		

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PROBLEM	Prepared by:	
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LOC	K	OP CODE	AD	DR	(A)	Notes
1250	1	70	00	00		
51	1	71	00	01	12 & 13	
52	1	72	00	01		
53		30	03	01		
54	3	76	10	73		
55		30	00	01		
56	1	73	00	00		$CB_1 = 0$
57	1	74	01	00		$CD_1 = 1$
58	1	75	01	00	Storage	$CL_1 = 1$
59	1	72	00	78	0812 → 1712	$WL_1 = 78$
1260	1	71	00	01	0890 → 1790	$WD_1 = 1$
61	1	70	00	00	and	$WB_1 = 0$
62	1	42	08	12	0912 → 1812	0812 $\xrightarrow{+}$ Ac
63	1	49	17	12	0983 → 1883	store in 1712
64	1	76	12	62		Inc WB_1 & go to
65	1	72	00	71		$WL_1 = 71$
66	1	77	12	60		Inc CB_1 & go to
67	1	73	00	00		$CB_1 = 0$
68		42	08	12	12 in 5	0812 $\xrightarrow{+}$ Ac
69		49	17	05		store in 1705
1270		42	08	13		0813 $\xrightarrow{+}$ Ac
71		49	17	06	13 in 6 & 14	store in 1706
72		49	17	14		1714
73	1	70	00	00		$WB_1 = 0$
1274	1	71	00	07		$WD_1 = 7$

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LOC	K	OP CODE	ADDR		(A)	Notes
1275	1	72	00	70	Storage	$WL_1 = 70$
76	1	42	08	20	0820 → 1721 0827 → 1728	0820 $\xrightarrow{+}$ Ac
77	1	49	17	21	0890 → 1791	store in 1721
78	1	76	12	76		Inc WB_1 & go to
79		42	09	12	12 in 5	0912 $\xrightarrow{+}$ Ac
1280		49	18	05		store in 1805
81		42	09	13		0913 $\xrightarrow{+}$ Ac
82		49	18	06	13 in 6 & 14	store in 1806
83		49	18	14		1814
84	1	70	00	00		$WB_1 = 0$
85	1	71	00	07	Storage	$WD_1 = 7$
86	1	72	00	63	0920 → 1821	$WL_1 = 63$
87	1	42	09	20	0983 → 1884	0920 $\xrightarrow{+}$ Ac
88	1	49	18	21		store in 1821
89	1	76	12	87		Inc WB_1 & go to
1290	1	70	00	00		$WB_1 = 0$
91	1	71	00	01	Storage	$WD_1 = 1$
92	1	72	00	05	1879 → 1793	$WL_1 = 5$
93	1	42	18	79	1884 → 1798	1879 $\xrightarrow{+}$ Ac
94	1	49	17	93		store in 1793
95	1	76	12	93		Inc WB_1 & go to
96	1	70	00	00		$WB_1 = 0$
97	1	71	00	01		$WD_1 = 1$
98	1	72	00	05		$WL_1 = 5$
1299	1	42	17	86		1786 $\xrightarrow{+}$ Ac

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PROBLEM	Prepared by:	
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LOC	K	OP CODE	AD	DR	(A)	Notes
1300	1	49	18	86		store in 1886
01	1	76	12	99		Inc WB1 & go to
02		33	16	12	⌐ L variations	
03		33	16	19		
04		33	16	26		
05		33	16	32		
06		33	16	39		
07		33	16	46		
08		33	16	52		
09		33	16	59		
1310		33	16	66		
11		33	16	72		
12		33	16	79		
13		33	15	86		
14		33	15	80		
15		33	15	73		
16		33	15	67		
17		33	15	75		
18		33	15	82		
19		33	15	90		
1320		30	00	01		pos paper
21	2	76	10	04		Inc WB2 & go to → 11
1322		67	00	00		halt
		29	OR 10	01	for continuous operation	

1322 - 1349 for L values as required.

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PROBLEM	Prepared by:	
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LOC	K	OP CODE	AD	DR	(A)	Notes
1350		49	07	91	-ve quantity	store in 0791 ← (4)
51	1	42	15	12	L should be 0	1512 $\xrightarrow{+}$ Ac
52		41	07	91	-(-ve) = +ve	0791 $\xrightarrow{-}$ A
53	1	49	15	12		store in 1512
54		29	10	93		Go to → (13)
55	1	42	17	05		1705 $\xrightarrow{+}$ Ac ← (2)
56	1	43	17	19		1719 $\xrightarrow{+}$ A
57	1	43	17	11		1711 $\xrightarrow{+}$ A
58	1	43	17	13	F/T	1713 $\xrightarrow{+}$ A
59		44	07	79		0779 $\xrightarrow{\times}$ A
1360		49	07	90		store in 0790
61	1	42	17	12		1712 $\xrightarrow{+}$ Ac
62		44	07	80		0780 $\xrightarrow{\times}$ A
63		43	07	90		0790 $\xrightarrow{+}$ A
64		49	07	92	$S_{F/T}$	store in 0792
65	1	42	15	12		1512 $\xrightarrow{+}$ Ac
66		23	13	82		If A = 0 go to → (14)
67		42	07	82	132	0782 $\xrightarrow{+}$ Ac
68		41	07	92	132 - ans = +, -, or 0 freezing thawing	0792 $\xrightarrow{-}$ A
69		49	07	91		store in 0791
1370	1	43	15	12	1512 is +ve	1512 $\xrightarrow{+}$ A
71	1	49	15	12	+ve, -ve or 0	store in 1512
72		22	13	92	If A < 0 → completely thawed	If A < 0 go to → (15)
73		42	07	81	+82	0781 $\xrightarrow{+}$ Ac
1374	1	41	15	12	+ve	1512 $\xrightarrow{-}$ A

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PROBLEM	Prepared by:	
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LOC	K	OP CODE	AD	DR	(A)	Notes	
1375		22	13	77	%A<0 → completely frozen	%A<0 go to →	
76		29	10	93		Go to →	13
77		43	07	82	132+(-ve) → temp falls	0782 —+→ A ←	
78	1	49	08	12		store in 0812	
79		42	07	81	Reset L = +82 on completion of freezing	0781 —+→ Ac	
1380	1	49	15	12		store in 1512	
81		29	10	64		Go to →	12
82		42	07	92	$S_{F/T}$	0792 —+→ Ac ←	14
83		41	07	82		0782 —−→ A	
84		30	13	89		%A ≥ 0 go to →	
85		49	07	91	A<0 → freezing	store in 0791	
86		40	07	91	-(-ve) = +ve	0791 —−→ Ac	
87	1	49	15	12		store in 1512	
88		29	10	93		Go to →	13
89		42	07	92		0792 —+→ Ac ←	
1390	1	49	08	12		store in 0812	
91		29	10	64		Go to →	12
92		42	07	82		0782 —+→ Ac ←	15
93	1	41	15	12	-ve quantity	1512 —−→ A	
94	1	49	08	12	132-(-ve) → temp rises	store in 0812	
95		41	21	01	Reset L=0 on completion of thaw	A —−→ A	
96	1	49	15	12		store in 1512	
1397		29	10	64		Go to →	12