ON SINGULAR TIME- AND FUEL-OPTIMAL SOLUTIONS

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ABSTRACT

The time-optimal control processes are considered for a class of linear time-invariant systems which are singular. It is shown that singularity occurs when there is a cancellation in the system transfer function. For single-input systems, a unique solution exists which is bang-bang if the controllable states are constrained. A generalized non-unique bang-bang control law is shown to exist for multi-input systems.

For fuel-optimal control processes, it is proved that for linear time-invariant systems, singular controls cannot be optimal. In the case of non-linear systems, optimal singular controls may occur quite often, in particular, when the problems considered are subjected to non-linear friction forces.

Conditions characterizing singular problems are derived and several examples are presented.

TABLE OF CONTENTS

CHAPTER PAGE

LIST OF FIGURES

FIGURE PAGE

CHAPTER 1

GENERAL INTRODUCTION

The structure and synthesis of optimal control processes is the subject of much current research. The Pontryagin's Maximum (Minimum) Principle (PMP) [28] has been one of the main tools in these studies. This principle makes it possible to find optimal control steering functions for a wide class of optimal processes in terms of a linear differential equation called the adjoint system of the original system equation.

A difficulty arises when the control variable enters linearly in the system equation or the index of performance. In such problems, situations may occur when PMP fails to provide effective optimality conditions. The corresponding Hamiltonian function ceases to be an explicit function of the control variables and thus yields no information about the desired optimal control. This class of problems is referred to as "singular" problems or "non-normal" problems, in the sense of LaSalle.

In this thesis, the necessary and sufficient (if possible) conditions are established for the existence of singular problems. The optimization under study is time-optimal, fuel-optimal and time-weighted fuel-optimal problems.

REVIEW OF LITERATURE

In the earlier literature of the calculus of variations, curves with singular subarcs received little analytical attention except to be explicitly excluded from the hypothesis of most of the established theorems. Evidently the concept "singular control" is of a very recent origin.

LaSalle[20] (1960) observes the computational difficulties in constructing optimal solutions to a class of time-optimal problems, for which there exist many control laws, each equally optimal. He establishes a set of theorems which characterize a normal and a proper system.

At this stage, a carefully expressed definition of the term "singular" extremals does not appear to exist in the literature. Haynes and Hermes[13] (1963) put forward a general definition of the singular problem using the associated Pfaffian system approach. They are concerned only with problems which are mildly nonlinear.

Following their works, a number of contributions (Kelly [16 , 17], Snow [30], Kopp and Moyer [18], Johnson et al. [14,15], Athans et al. [4, 5], Thau [31], and Goh [8, 9]) have appeared which throw considerable light on the difficulties of these problems.

Kelly [17] produces a set of necessary conditions for singular arcs via the second variation of the function to be minimized. This set of necessary conditions is the generalized Clebsch (Legendre) necessary condition:

$$\frac{\partial}{\partial u}\left[\frac{d^k}{dt^k}\left(\frac{\partial H}{\partial u}\right)\right] \geqq 0 \tag{1.1}$$

$$k=2,4,\ldots\ldots\ldots\ldots$$

Kopp and Moyer [18] extended this result by considering a specially chosen class of perturbations on the singular control u_s. They have shown that, for a special class of piecewise continuous perturbations the second variation of the index of performance (IP) along singular arcs is strictly positive if and only if the inequality in (1.1) holds along the singular arc.

Johnson et al work out a class of singular problems to illustrate the significance of singular solutions.
Goh extends the test of singular extremals for conventional Bolza problems. In a series of papers he indicates the procedures for the derivation of singular extremals.

Most of the works mentioned above consider problems in which the IP is quadratic .

The works of Thau, Snow and Athans et al. are closest to the material presented in this thesis . Some of their theorems are discussed, amplified, and extended to cover the problems considered here.

CHAPTER 2

SINGULAR EXTREMALS IN THE PROBLEMS OF OPTIMAL CONTROL

The main objective of this chapter is to review and discuss the fundamental relationships existing between the extremal of PMP, the solution of the adjoint system equation induced by PMP and the max or min Hamiltonian scalar function. During the course of discussion a scheme is suggested for classifying extremals of PMP.

It is now necessary to formulate the control process to establish a quantitative basis for discussion of the material to follow.

The dynamical system which is controlled is assumed to satify the following vector differential equation

(2.1) $$\dot{\underline{x}}(t) = \underline{f}\,[\underline{x}(t);\underline{u}(t)]$$

where $\underline{x}(t)$ is a vector with n components representing the state of the system at time t, and $\underline{u}(t)$ is a vector with r components representing the control input to the system at time t and $\underline{f}[\underline{x}(t);\underline{u}(t)]$ is a function of $\underline{x}(t)$ and $\underline{u}(t)$. For the remainder of this thesis, $\underline{x}$ and $\underline{u}$ or other symbols are functions of t explicitly unless otherwise stated.

For reasons of mathematical expediency, it is assumed that

(a) the function $\underline{f}$ is continuous,

(b) for each fixed $(t,\underline{u})$ the function $\underline{f}$ is differentiable and its derivatives are continuous.

The precise statement of optimization is as follows:

Given the system (2.1), the boundary conditions $\underline{x}(t_o)$, the constraint set Ω(where Ω is a set of r-dimensional space of the control variables $\underline{u}$, usually closed, bounded and convex), the target set S and the IP

(2.2) $$J(\underline{u}) = \int_{t_o}^{T} L(\underline{x},\underline{u})\, dt$$

then find the control $\underline{u}$ that

(a) satisfies the constraint $\underline{u} \in \Omega$

(b) transfers the state of the system from $\underline{x}(t_o)$ of (2.1) to $\underline{x}(T)$ so that $\underline{x}(T) \in S$ and in so doing

(c) minimizes the IP (2.2)

The search for the optimal control u* is facilitated via the PMP which will be stated.

PONTRYAGIN'S MAXIMUM (MINIMUM) PRINCIPLE

If $\underline{u}^*$ is the optimal control and $\underline{x}^*$ is the generated optimal trajectory, then corresponding to $\underline{u}^*$ and $\underline{x}^*$ there exists a co-state (adjoint) vector p* such that the following relationships hold:

(a) Canonical equations

(2.3) $$\dot{x}_i = \left.\frac{\partial H}{\partial p_i}\right|_* \qquad \dot{p}_i = -\left.\frac{\partial H}{\partial x_i}\right|_*$$

where H is the Hamiltonian function given by

$$H(\underline{x},\underline{p},\underline{u}) = L(\underline{x},\underline{u}) + \langle \underline{p}, \underline{f}(\underline{x},\underline{u})\rangle$$

and $|_*$ means the partial derivatives must be evaluated at the optimal values.

(b) Boundary conditions

$$\underline{x}^*(t_o) = \underline{\xi}$$

$$\underline{x}^*(T) \in S$$

$$\underline{p}^*(T) \perp S$$

(c) Minimization of the Hamiltonian

(2.4) $$H^\circ = \min_{u\in\Omega} H(\underline{x}^*,\underline{p}^*,\underline{u}) = H(\underline{x}^*,\underline{p}^*,\underline{u}^*) \leq H(\underline{x}^*,\underline{p}^*,\underline{u})$$

for every t, $t_o \leq t \leq T$ and $\forall$ $\underline{u}$.

The proof of the PMP can be found in [28].

The conditions provided by PMP are local in nature and in general not sufficient.

Athans and Falb [4] have shown for $u \leq 1$ and $\underline{x}^*(t_o) = \underline{\xi}$ and $\underline{x}^*(T) = \underline{0}$ the optimal control law is given by

(2.5) $$\underline{u}^* = -\mathrm{SGN}^\dagger\{B^T\underline{p}^*\}$$

$$\equiv -\mathrm{SGN}\ \{\underline{q}^*\} \qquad \text{for time-optimal}$$

(2.6) $$\underline{u}^* = -\mathrm{DEZ}^\dagger\{B^T\underline{p}^*\}$$

$$\equiv -\mathrm{DEZ}\ \{\underline{q}^*\} \qquad \text{for fuel-optimal}$$

These equations provide a unique solution to time- or fuel-optimal problems if $q_j^* \neq 0$ in time-optimal and $|q_j^*| \neq 1$ in fuel-optimal problems over any finite interval of positive length. Such problems are called normal. If, however, $\underline{q}^* \equiv 0$ in

† The signum and deadzone functions are defined in pp. 380 and 438 of Athans and Falb [4].

time-optimal or $|q_j^*| \equiv 1$ in fuel-optimal problems over some finite interval of positive length, the signum and the deadzone functions are not defined, and accordingly an arbitrary $|u| \leq 1$ will satisfy PMP. The control problems are called non-normal in the sense of LaSalle, or totally singular.

With these preliminary facts in mind, a clear-cut definition of singularity for a class of control processes is advanced.

Before attempting this, the following definition and theorem of Lebesgue measure are reviewed.

DEFINITION 1

Suppose $M \varepsilon E_1$. By a Lebesgue covering of a set M, it is meant a countable sequence $I = \{I_1, I_2, \ldots\ldots\}$ of open intervals which covers M. If $L(I_k)$ is the length of I_k, covering L(I) is defined to be the number

$$L(I) = \sum_{k=1}^{\infty} L(I_k)$$

whenever the series on the right converges. The number

$$m(M) = \inf\{ L(I) \mid I \text{ is a Lebesgue covering of } M\}$$

is called the Lebesgue measure of M.

When M is bounded, say $M \subset [a, b]$, then

$$0 \leq m(M) \leq b-a$$

If $m(M) = 0$, M is said to be a set of measure zero.

THEOREM 1.

If F is a countable collection of sets in E_1 say

$$F = \{F_1, F_2, \ldots\ldots\ldots F_n\}$$

such that $m(F_k) = 0$ for each k, let

$$M = \bigcup_{k=1}^{\infty} F_k$$

then $m(M) = 0$.

DEFINITION 2.

For each value j, j=1,2,.......r, designate

$$q_j = \sum_{i=1}^{n} p_i \; b_{ij}$$

to define a switching function and denote the set

$$\Gamma(k) = \{\underline{p},\underline{x} \mid q_k = 0 \text{ (time-optimal) or } |q_k| - 1 = 0 \text{ (fuel-optimal)}\}$$

and denote by Γ the set

$$\Gamma = \bigcup_{k=1}^{r} \Gamma(k)$$

then an extremal $(\underline{p},\underline{x})$ given on an interval I is called "totally singular" if the set

$$\beta = \{ t \mid t \in I \text{ and } \underline{p},\underline{x} \in \Gamma \}$$

have positive Lebesgue measure in I, where I denotes the time interval $[t_o, T]$

In other words, a totally singular arc or subarc occurs whenever the inequality in (2.4) becomes an equality over a finite interval of time and H thus becomes independent of $\underline{u}$. If for example H is linear in u_k, $\frac{\partial H}{\partial \bar{u}_k}$ vanishes over a finite interval; the PMP fails to select the optimal control in this case. It is clear that if a $\underline{u}$* exists ,it must satisfy the requirement that the system remains on a path such that $\frac{\partial H}{\partial u_k} = 0$.

In general, the determination of whether or not a totally singular control satisfies the necessary conditions of PMP is not a difficult task; the difficulty is to prove that this extremal is optimal.

In this thesis, a distinction is made between a singular and a totally singular problem." By total singularity" it is meant that the PMP fails to provide any information about all the control variables. If one or more control variables are defined, then the problem is said to be singular.

CHAPTER 3

TIME-OPTIMAL SINGULAR PROBLEMS

§1. Introduction

It is well known that if a control system is to be operated from a limited source of power, and if the transition time from some initial state to some final state is to be minimum, then the control strategy is to utilize all available power. This hypothesis [21] is called "The Bang-Bang Principle". For a normal time-optimal problem the control law is uniquely defined by this hypothesis. However, if the problem is singular, the validity of this hypothesis is open to conjecture. Moreover, the time-optimal solution(s) is non-unique even if the solution(s) exists. It is found that under certain conditions and restrictions, a generalized "Bang-Bang Principle" may be established with regard to the singular problem.

§2. The Singular Problem

The problem posed is to determine the control law which steers the system from some initial state to some final state $\underline{x}_F$ (in particular, the origin $\underline{0}$) in minimum time satisfying the dynamical system of the form

$$(3.1) \qquad \dot{\underline{x}} = A\underline{x} + B\underline{u}$$

$$\underline{y} = C\underline{x}$$

where A is an nxn and B is an nxr constant matrix, with $\underline{u}$ constrained in magnitude by the relation

$$(3.2) \qquad |u_j| \leq 1$$

Treating the problem via the PMP method of optimization, the Hamiltonian is

$$(3.3) \qquad H(\underline{x},\underline{p},\underline{u}) = 1 + \langle \underline{p}, A\underline{x} \rangle + \langle \underline{p}, B\underline{u} \rangle$$

where $\underline{p}$ is the co-state vector satisfying the canonical equation

$$(3.4) \qquad \underline{\dot{p}} = -\frac{\partial H}{\partial \underline{x}} = -A^T \underline{p}$$

and the control law is given by

$$(3.5) \qquad \underline{u} = -\mathrm{SGN}\{B^T \underline{p}\} = -\mathrm{SGN}\{\underline{q}\}$$

which can be written in component form as

$$(3.6) \qquad u_j = -\mathrm{sgn}\{\sum_{i=1}^{n} b_{ij} p_i\} = -\mathrm{sgn}\{ q_j \} \qquad j = 1,2,\ldots\ldots\ldots r$$

The condition for totally singular control occurs when $B^T\underline{p} \equiv 0 \ \forall \ t \varepsilon \ [t_1, t_2] \ \varepsilon [0, T]$; then $\underline{u}$ is arbitrary and may be optimal if $|\underline{u}| \leq 1$. If for some j, j=1,2,.....r, $q_j \equiv 0 \ \forall t \ \varepsilon \ [t_1, t_2] \ \varepsilon \ [0,T]$ then the problem is said to be singular. With these preliminaries the following definitions are made.

<u>DEFINITION 3.</u>Normal time-optimal problem

If for all j, j=1,2,.....r, $q_j \neq 0$ except at $t = t_{\gamma j}$ where $t_{\gamma j}$ is a countable set of times t_{1j}, t_{2j}, t_{3j} ....

$t_{\gamma j} \; \varepsilon \; [0,T]$ $\qquad \gamma=1,2,\ldots\ldots m$

$\qquad j=1,2,\ldots\ldots r$

then the problem is said to be normal time-optimal.

DEFINITION 4. Singular time-optimal problem

Suppose in the interval $I=[0,T]$ there is one or more proper subinterval $[t_1,t_2]_j$, $[t_1,t_2]_j \varepsilon\ [0,T]$ such that $q_j \equiv 0$ for all $t\ \varepsilon\ [t_1,t_2]$ and for some j, then the problem is said to be singular time-optimal . If the condition holds for all j then the problem is totally singular.

§ 3. Controllability and Existence of Optimal Control

It has been shown [4] that a necessary and sufficient condition that a linear system defined by (3.1) be completely controllable at time $t > t_o$ is that the matrix

(3.7) $\qquad G= [\ B,\ AB,\ A^2B,\ldots\ldots\ldots\ldots A^{n-1}B]$ has a rank n for all $t > t_o$.

This condition can be shown to be equivalent to the statement that the matrix [28]

(3.8) $\qquad W(t_o,t_1) = \int_{t_o}^{t_1} \Phi(t_o,t)BB^T\Phi(t_o,\ t)^T\ dt$

be positive definite for some time $t_1 \geq t_o$, where the superscript T denotes the transpose of the matrix. The matrix $\Phi(t,\ t_o)$ is the fundamental transition matrix of (3.1) with the following properties:

(3.9) $\dot{\Phi}(t,t_o) = A\Phi(t,t_o)$

$\Phi(t_o,t_o) = I$ the identity matrix

$\Phi(t,t_o)\Phi(t_o,t) = I$

and consequently $\Phi(t_o,t)$ satisfies the equation

(3.10) $\dot{\Phi}(t_o,t) = -\Phi(t_o,t)A$

From linear algebra, in order that the matrix $W(t_o,t_1)$ be positive definite at time t_1, the quadratic form

(3.11) $\underline{z}^T W(t_o,t_1)\underline{z}$ (where $\underline{z}$ is a constant vector) must be greater that zero for all $\underline{z} \neq 0$.

Substitute (3.11) into (3.8) this condition can be written as

(3.12) $$\int_{t_o}^{t_1} \underline{z}^T \Phi(t_o,t) BB^T \Phi(t_o,t)^T \underline{z} \; dt > 0 \quad \forall \; \underline{z} \neq 0$$

Now partition the B matrix into column vectors, i.e.

$$B = [\underline{b}_1 \; \underline{b}_2 \; \ldots\ldots\ldots\ldots \underline{b}_r]$$

or

$$B^T = \begin{bmatrix} \underline{b}_1^T \\ \underline{b}_2^T \\ \cdot \\ \cdot \\ \cdot \\ \underline{b}_r^T \end{bmatrix}$$

and

$$BB^T = \sum_{i=1}^{r} \underline{b}_i \underline{b}_i^T$$

and (3.12) reduces to

(3.13) $$\sum_{i=1}^{r} \left[\int_{t_o}^{t_1} \underline{z}^T \Phi(t \;\; ,t)\underline{b}_i \right]^2 dt > 0 \qquad \forall \; \underline{z} \neq 0$$

Note this condition implies that

(3.14) $\underline{z}^T\Phi(t_o,t)\underline{b}_i$ $i=1,2,\ldots\ldots\ldots r$

are not identically zero over the interval $[t_o,t_1]$. For linear time invariant system there is no loss of generality if $t_o=0$ and $t_1=T$.

LaSalle [20] in his treatment of time-optimal control systems discusses the concept of normal system and proper system. In the notation of the present formulation the following definitions are made.

DEFINITION 5.

A system is called proper if $\underline{z}^T\Phi(0,T)B=0$ on an interval of positive length implies $\underline{z}=0$, otherwise it is improper.

DEFINITION 6.

A system is called normal if for each i, $\underline{z}^T\Phi(0,T)\underline{b}_i=0$ on an interval of non-zero length implies $\underline{z}=0$ on that interval, otherwise it is non-normal or singular.

From these definitions it is obvious that every normal system is proper, but not every proper system is normal. Thus a proper system is equivalent to complete controllability of the system [20].

To relate the definition of normality of a linear system to the definition of a normal time-optimal control problem, take the transpose of (3.14) to obtain

(3.15) $[\underline{z}^T\Phi(0,T)\underline{b}_i]=0$

Since z is any arbitrary vector, $\underline{z}$ may be viewed as the initial co-state vector $\underline{\pi}$ which is also arbitrary as will be shown later in this chapter.

Thus the system being normal is equivalent to the normality of time-optimal problem.

A criterion for testing a linear system described by (3.1) to be singular or totally singular is given by the following theorems.

THEOREM 1.

The time-optimal problem is totally singular if and only if the system is not completely controllable.

Proof: Assume the system is not completely controllable; then the matrix defined by (3.7) has a rank less than n. It follows that there exists a non-zero vector $\underline{z}$ such that

(3.16) $\underline{z}^T G = 0^T$

$\underline{z}^T [B, AB, \ldots\ldots\ldots\ldots A^{n-1}B] = 0^T$

or $\underline{z}^T B = \underline{z}^T AB = \ldots\ldots\ldots\ldots = \underline{z}^T A^{n-1} B = 0^T$

By the Cayley-Hamilton Theorem matrix A satisfies its own characteristic equation, i.e.,

$$A^n = \sum_{i=0}^{n-1} c_i A^i$$

for certain real numbers c_i, $i = 0, 1, \ldots\ldots\ldots n-1$.

Thus

$$\underline{z}^T A^n B = \underline{z}^T \sum_{i=0}^{n-1} c_i A^i B = 0^T$$

By induction

$$\underset{\sim}{z}^T A^{n+k} B = 0 \qquad \forall\ k = 0, 1, 2, \ldots\ldots\ldots\ldots\ldots$$

Therefore

$$(3.17) \qquad \underset{\sim}{z}^T e^{-At} B = \underset{\sim}{z}^T [I - At + \frac{A^2 t^2}{2!} - \ldots\ldots\ldots\ldots] B = 0^T$$

for all real t.

Since $\underset{\sim}{z}$ is non-zero (3.17) implies that

$$B^T e^{-At} \underset{\sim}{z} = 0 \quad \text{or}$$

$$B^T \underset{\sim}{p} = 0.$$

Hence the time-optimal problem is totally singular.

Assume the time-optimal problem to be totally singular.

By definition

$$(3.18) \qquad \underset{\sim}{q} = B^T \underset{\sim}{p} = 0 \qquad \forall\ t \in [t_1, t_2]$$

It follows that $\underset{\sim}{q}$ and its first n-1 time derivatives are zero, i.e.

$$(3.19) \qquad \begin{aligned} B^T \underset{\sim}{p} &= 0 \\ B^T A^T \underset{\sim}{p} &= 0 \\ B^T {A^T}^2 \underset{\sim}{p} &= 0 \\ \cdots\cdot &\ \cdot\cdot \\ B^T {A^T}^{n-2} \underset{\sim}{p} &= 0 \\ B^T {A^T}^{n-1} \underset{\sim}{p} &= 0 \end{aligned} \qquad \forall\ t \in [t_1, t_2]$$

Denoting G as the matrix defined (3.7), (3.19) can be written as

$$(3.18) \qquad G^T \underset{\sim}{p} = 0$$

Since $\underset{\sim}{p} \neq 0$, a singular G^T => the system is uncontrollable.

A conjecture that one might be tempted to make is that if the system is completely controllable, then it admits no totally singular extremals. This is not true as shown in Example 2. However, it may be asserted that totally singular extremals cannot be optimal. This is because G has a rank= n, in which case $\underline{z}$ has to be zero. But by PMP, the extremals are not optimal.

In order to prove the next theorem, it is assumed that the matrix is of simple structure [33]. In other words, the eigenvectors associated with each eigenvalue of A are linearly independent. If A has distinct eigenvalues or if A has indistinct eigenvalues which can be linked in a Jordan chain, then their eigenvectors are linearly independent, and they constitute a basis for the state space.

THEOREM 2.

Assume A is of simple structure . The time-optimal problem is singular if and only if for some j, j=1,2,.....r the matrix given by

(3.21) $$G_j = [\underline{b}_j \; A\underline{b}_j \; \ldots\ldots\ldots\ldots\ldots A^{n-1}\underline{b}_j]$$

is singular.

Proof: Necessity-Mimic the proof of Theorem 1 using (3.18) and re-writing it as

(3.22) $$\underline{q}_j = \langle \underline{b}_j, \underline{p} \rangle = 0$$

to show G_j has a rank less than n.

Sufficiency: Since A is of simple structure, G_j can be written as [26]

(3.23) $[Ib_{kj} \quad (A-\lambda_k I)b_{kj} \ldots\ldots\ldots\ldots (A-\lambda_k I)^{n-1} b_{kj}]$

where λ_k is the kth eigenvalue and $k = 1,2,\ldots\ldots\ldots n$ and b_{kj} is the row component of the column vector b_j.

Since G_j has a rank less than n, it implies that there exist certain real numbers c_i, $i=0,1\ldots\ldots\ldots,n-1$ such that

(3.24) $$c_o I b_{kj} + c_1(A-\lambda_k I)b_{kj} + \ldots\ldots\ldots\ldots + c_{n-1}(A-\lambda_k I)^{n-1} b_{kj} = 0$$

i.e.
$$\begin{aligned}
&[c_o I + c_1(A-\lambda_1 I) + \ldots\ldots\ldots\ldots + c_{n-1}(A-\lambda_1 I)^{n-1}]b_{1j} = 0\\
&[c_o I + c_1(A-\lambda_2 I) + \ldots\ldots\ldots\ldots + c_{n-1}(A-\lambda_2 I)^{n-1}]b_{2j} = 0\\
&\ldots \quad \ldots \quad \ldots\ldots\ldots\ldots \quad \ldots \quad \ldots \quad \ldots\\
&[c_o I + c_1(A-\lambda_n I) + \ldots\ldots\ldots\ldots + c_{n-1}(A-\lambda_n I)^{n-1}]b_{nj} = 0
\end{aligned}$$

By assumption, the matrices $(A-\lambda_i I)^i$, $i=0,1, \ldots\ldots n-1$ are linearly independent, the coefficients are uniquely determined: unique in the sense that n-1 of them can be solved in terms of the remaining one.

The n equations [27] which determine the coefficients c_i can be written in matrix form as

(3.25) $$\begin{bmatrix} F(\lambda_1) \\ F(\lambda_2) \\ \cdot \\ F(\lambda_n) \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots\ldots\ldots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \ldots\ldots\ldots & \lambda_2^{n-1} \\ \cdot & & & \ldots\ldots\ldots & \\ 1 & \lambda_n & \lambda_n^2 & \ldots\ldots\ldots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} c_o \\ c_1 \\ \cdot \\ c_{n-1} \end{bmatrix} = 0$$

or $\underline{F}(\lambda) = \underline{\Lambda}\, \underline{c}$

Therefore

(3.26) $\underline{c} = \Lambda^{-1} \underline{F}(\lambda)$

Since $e^{-\Lambda} = \sum_{i=0}^{n-1} \alpha_i \Lambda^i$, the coefficients α_i are precisely those c_i when one computes $e^{-\Lambda}$ by Sylvester's Theorem [26].

Thus $e^{-\Lambda t}\underline{b}_j = 0$ and for any non-zero $\underline{z}$, $\underline{z}^T e^{-\Lambda t}\underline{b}_j = 0 \; \forall \; t \in [t_1, t_2]$.

Therefore the time-optimal problem is singular.

Note that the above proof uses the fact that the eigenvaules are distinct. The same result can be obtained if the eigenvalues are indistinct, provided that they can be linked in a Jordan chain.

For properties of matrix A other than that A is of simple structure, (3.21) is, in general, insufficient to guarantee that the time-optimal problem is singular. In such problems(e.g. A is singular), there exist some initial states for which the time-optimal is normal, while for other initial states, it is singular.

The fact that G or G_j is independent of $\underline{x}$, $\underline{p}$, $\underline{u}$, and t implies that a time-optimal problem is singular or totally singular $\forall$ t, tϵ $[t_1,t_2]$ ϵ $[0,T]$, then it is singular or totally singular for all t, t >0. The verification of this statement may be made by invoking the Principle of Optimality [4], which states that any portion of the optimal trajectory is also optimal.

Regarding the existence of singular optimal control, a few facts and the terminology of linear algebra are used to provide a framework to discuss the results which are to follow.

Appendix A is devoted to basic definitions and derivation of the fundamental results to be used in this chapter.

The definition of domain of controllability [24] is given below to facilitate discussion on the existence of singular controls.

DEFINITION 7

The domain of null controllability T for the differential equation (3.1) and for the given constraint set Ω is defined to consist of all points $\underset{\sim}{x}_o \in R^n$ for which there exists a measurable $\underset{\sim}{u}(t) \in \Omega$ (defined on some finite interval) which steers $\underset{\sim}{x}_o$ to the origin $\underset{\sim}{0}$.

Lee and Markus [24] prove that if the system of (3.1) is completely controllable and the eigenvalues of A have negative real parts with the assumption that u = 0 lies in the interior of Ω, then the domain of null controllability T is the whole state space R^n.

Controllability of a system as formulated above is sufficient to ensure the existence of an input $\underline{u}$ on the interval [0,T] to regulate the output to the origin. However, this property certainly does not represent a necessary condition. It is observed that in order to regulate the output, it suffices to find a control $\underline{u}$ for t > T such that the solution of (3.1) denoted by $\underset{\sim}{x}$ for t > T satisfies the equation

(3.27) $\quad \underline{y} = C\underline{x} = \underline{0} \quad \forall\ t > T$

Denote $\hat{R}$ the set of all $\underline{x}$ for which a control can be found to drive the system to $\underline{0}$. If the system is controllable then a $\underline{u}$ certainly can be found. However, the system may not be controllable, yet it is conceivable that a control can be found that will force $\underline{x}_o$ to $\underline{0}$. For this the following definition is essential.

DEFINITION 8

A state $\underline{x}_o$ is controllable if there exists a $\underline{u}$ such that $\underline{x}(T) = 0$ for all $t > T$.

THERORFM 3

A state is controllable if and only if it is in the range of G.

The proof of this theorem is presented in Appendix A.

Theorem 3 states that the range of the matrix G denoted by R(G) is the subspace of all controllable states. If the dimension of R(G) is r' such that $r' < n$, one can pick a new basis for R^n. The first r' vectors constitute a basis for R(G) and the (n-r') remaining ones constitute a basis for the null space of G denoted by N(G).

In the new basis chosen, denote matrices by a prime. It is asserted that

$$A' = \begin{bmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} B'_1 \\ 0 \end{bmatrix} \quad \text{or} \tag{3.28a}$$

(3.28b)
$$A' = \begin{bmatrix} A'_{11} & 0 \\ A'_{21} & A'_{22} \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} 0 \\ B'_2 \end{bmatrix}$$

where in (3.28a) A'_{11} is $r'xr'$, A'_{12} is $r'x(n-r')$, A'_{22} is $(n-r')x(n-r')$, B'_1 is $r'xr$ and in (3.28b) A'_{11} is $(n-r')x(n-r')$, A'_{21} is $r'x(n-r')$, A'_{22} is $r'xr'$, B'_2 is $r'xr$ constant matrices and r is the number of components in $\underline{u}$. Eq. (3.1) can be partitioned into two vectors if it is written as

(3.29) $\underline{x}' = (\underline{x}_1'^T \; \underline{x}_2'^T)^T$ where $\underline{x}_1' \in R(G)$ and $\underline{x}_2' \in N(G)$.

The lower matrix of B' in equation (3. 28a) is zero because the columns are neces sary in R(G) by definition of G. The lower left hand submatrix of A' is zero because R(G) is an invariant subspace under A. i.e. $\underline{x}' \in R(G)$ implies $A'\underline{x}' \in R(G)$

Thus if $\underline{x}' = (\underline{x}_1'^T \; \underline{0}^T)^T$, then for all possible $\underline{x}'$, $A'\underline{x}' = \{(A'_{11}\underline{x}_1' + A'_{12}\,\underline{x}_2')^T, (A'_{22}\underline{x}_2')^T\}^T = \{(A'_{11}\underline{x}_1')^T, \underline{0}^T\}^T$ and therefore $A'_{12} = \underline{0}$.

The equation shows that whenever $r' < n$, any state vector $\underline{x}'$ can be decomposed into a form $[\underline{x}_1'^T \underline{x}_2'^T]^T$ where $\underline{x}_2'$ unaffected by the input $\underline{u}$, whatever it may be. Futhermore $\underline{x}'$ is controllable if and only if $\underline{x}_2'$ at t=0 is equal to zero. It is of interest to note that $\underline{x}'$ and $\underline{x}'_1$ depend for all $t > 0$ on $\underline{x}_2'(0)$. The same argument can be made for the second of (3.28) except $\underline{x}'_2$ is in the range of G and $\underline{x}_1'$ is on the null space of G.

In the following corollary ,an explicit criterion for the existence of an optimal control $\underline{u}$ for singular problem is established.

COROLLARY 1

Assume A is stable i.e Re $\lambda < 0$, $\underline{x}_2'(0)=0$ and $\underline{x}_1' \varepsilon R(G)$ and $\underline{u} = 0$ is in the interior of Ω, then there exists a control $|u_j| \leqq 1$ which steers $\underline{x}_0$ to the origin. If $\underline{u}$ satisfies the PMP then $\underline{u}$ is a minimal time controller.

Proof: By Theorem 3 and Definition 7, the domain of null controllability Υ is the subspace $\underline{x}_2' = 0$. Thus there exists a $|u_j| \leqq 1$ which steers $\underline{x}'$ to $\underline{0}$ in minimum time. If $\underline{u}$ satisfies PMP then there exists an optimal controller $\underline{u}$ on the interval $[0 \leqq t \leqq t^* \leqq t_1]$ steering $\underline{x}$ to the origin.

The existence of an optimal control for completely controllable systems has been proved by Pontryagin et al [28].

§ 4. The Synthesis of Singular Optimal Controls

The synthesis of optimal controls for singular time-optimal problems is now considered.

In some cases, it is possible to use the transversality condition to rule out the existence of singular arcs in the optimal strategy. In time-optimal problems however, additional difficulties spring from the fact that along an extremal the Hamiltonian scalar function is constant for both normal [4] and singular extremals. The constant is zero by the transversality condition.

THEOREM 4

For linear time-invariant systems, the Hamiltonian for a totally singular time-optimal problem is constant along any totally singular arc. The constant is zero.

Proof: $H^* = 1 + \langle \underline{p}^*, A\underline{x}^* \rangle$ (totally singular condition)

Differentiate H^* with respect to time to obtain

$$\dot{H}^* = \langle \dot{\underline{p}}^*, A\underline{x}^* \rangle + \langle \underline{p}^*, A\dot{\underline{x}}^* \rangle$$

Since $A\underline{x}^* = \dot{\underline{x}}^* - B\underline{u}$ and

$$\dot{\underline{p}}^* = - A^T p^*$$

Substituting these equations into $\dot{H}^*$, one would obtain [28]†

$$\dot{H}^* = \frac{d}{dt} \langle \underline{p}^*, B \rangle \; \underline{u}$$

The totally singular condition $\Rightarrow \langle \underline{p}^*, B \rangle = 0$. Therefore $H^* = \text{constant} \geq 0$. However, at $t=T^*$, $H^*(T^*) = 0 \Rightarrow H^* = 0$ $\forall\, t \in [0,T^*]$.

§ 4.1 Single-input Systems

For single-input systems, matrix B is a column vector with n rows. Note u is a scalar function. The condition for singularity may be re-expressed in the next theorem.

THEOREM 5

The time-optimal problem is singular if and only if $\langle \underline{p}, \underline{x} \rangle = 0$.

Proof: Assume the problem is singular. The solution to (3.1) is

† The proof holds if $\underline{u}$ is piecewise constant and reference [28] proves that H is continuous and constant for the general case.

(3.30) $$\underline{x}(t) = \Phi(t,\tau)\underline{x}(\tau) + \int_{\tau}^{t} \Phi(t,\lambda)B\underline{u}(\lambda)\, d\lambda$$

For some T and for an admissible $u(t') \neq 0 \quad t \leq t' \leq T$ with $\underline{x}(T) = 0$, (3.30) reduces to

(3.31) $$\underline{x}(\tau) = -\int_{\tau}^{T} \Phi(\tau,\lambda)Bu(\lambda)\, d\lambda$$

where τ is some initial time.

Therefore

(3.32) $$\begin{aligned} \langle \underline{p}(\tau), \underline{x}(\tau)\rangle &= -\int_{\tau}^{T} \langle \Phi(\tau,\lambda)\underline{p}(\tau), B\rangle\, u(\lambda)\, d\lambda \\ &= -\int_{\tau}^{T} \langle \underline{p}(\lambda), B\rangle u(\lambda)\, d\lambda \quad \text{since } \underline{p} = -A^T\underline{p} \\ &= 0 \qquad \text{singular condition} \end{aligned}$$

The proof of 'if' part is trivial since $u(t) \neq 0$, (3.32) necessarily implies the problem to be singular if $\langle \underline{p}(\tau), \underline{x}(\tau)\rangle = 0$.

Note that (3.32) implies that

(3.33) $$\langle \underline{\pi}, \underline{\xi}\rangle = 0$$

By Lemma 1 in Appendix A, there exists a real, skew-symmetric M such that

(3.34) $$\underline{\pi} = M\underline{\xi} \qquad \text{where M is non-unique.}$$

For n=2, M is of the form:

(3.35) $$M = \pm k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad k \text{ is a constant} > 0$$

It can be shown that the hyperplane defined by (3.32) is in the range of G and hence at least one of the states is controllable.

THEOREM 6

For single-input systems, if A is stable i.e. Re $\lambda < 0$ and (3.32) holds, then the singular time-optimal control is unique.

Proof :Partition the state equation to obtain (3.28) and constrain the controllable states to the hyperplane defined by (3.32).Therefore $n-r'$ components of the vectors may be eliminated from the state equation to obtain

(3.36) $\qquad \dot{\underline{x}}_1' = A_{11}' \underline{x}_1' + B_1' u$

The matrix G' associated with this equation is of rank r' which is precisely the dimension of the state equation (3.36). Thus G' is non-singular in the new basis and an optimal control is uniquely given by

(3.37) $\qquad u^* = - \operatorname{sgn}\{ \langle B_1' , \underline{p}' \rangle \}$

It is the intent here to show that the singular problem arises because the systems have the property that their transfer function contain zeros, with at least one zero cancelling a pole.

Consider again the system equation(3.1). Taking the Laplace transform of (3.1) and solving for $X(s)$ one obtains

(3.38) $\qquad X(s) = (sI-A)^{-1} (X_o + BU(s))$

$\qquad\qquad Y(s) = CX(s)$

Since the transfer function is the Laplace transform of the impulse response of the system. i.e. the zero state response of (3.1) to $\delta(t)$, and $L\{\delta(t)\} = 1$, it follows that the transfer function $Z(s)$ of the system is

(3.39) $\quad Z(s) = C\,(sI-A)^{-1}\,B.$

Without loss of generality assume $Y(s)=X(s)$, then

$$Z(s) = (sI-A)^{-1}B$$

$$= \frac{1}{(sI-A)} \begin{bmatrix} n_1(s) \\ n_2(s) \\ \cdot\cdot \\ n_n(s) \end{bmatrix}$$

where n_i, $i=1,2,.......n$, are polynomials in s.

Assuming the problem is singular implies G has a rank less than n. If the rank of G is less than n, then there exist constants c_i, $i= 0,1.....n-1$ such that

$$[c_oB + c_1AB + \;.............+c_{n-1}A^{n-1}B] = 0$$

Define $J = (sI-A)^{-1}B \Rightarrow B = (sI-A)J$

Thus

(3.40) $$\sum_{i=0}^{n-1} c_iA^i(sI-A)J = (sI-A)\sum_{i=0}^{n-1} c_iA^iJ = 0$$

For s such that $|sI-A| \neq 0$, (3.40) implies that

$$\sum_{i=0}^{n-1} c_iA^iJ = 0$$

Using the identities

$$J = J$$

$$AJ = sJ-B$$

$$\cdot\cdot \qquad \cdot\cdot$$

$$A^{n-1}J = s^{n-1}J - A^{n-2}B - sA^{n-3}B - ....-s^{n-2}B$$

one obtains

$$\sum_{i=0}^{n-1} c_i A^i J = \sum_{i=0}^{n-1} c_i s^i J - \sum_{i=1}^{n-1} c_i A^{i-1} B - s \sum_{i=2}^{n-1} c_i A^{i-2} B$$

$$- \ldots\ldots\ldots - s^{n-3} \sum_{i=n-3}^{n-1} c_i A^{i-n+2} B$$

$$- s^{n-2} c_{n-1} B$$

$$= 0$$

Hence, $\sum_{i=0}^{n-1} c_i s^i J = \sum_{j=0}^{n-2} s^j \sum_{i=j+1}^{n-1} c_i A^{i-j-1} B$

or

$$J = \frac{\sum_{j=0}^{n-2} s^j \sum_{i=j+1}^{n-1} c_i A^{i-j-1} B}{\sum_{i=0}^{n-1} c_i s^i} \tag{3.41}$$

The denominator of (3.41) indicates that a cancellation **occurred,** because it is of degree one less than n.

Now assume J has a cancellation. By expressing J as follows:

$$J = \frac{\sum_{j=0}^{n-1} s^j \sum_{i=1+j}^{n} \alpha_i A^{i-j-1} B}{\sum_{i=0}^{n} \alpha_i s^i} \qquad (\alpha_n = 1)$$

and defining $v_j = \sum_{i=1+j}^{n} \alpha_i A^{i-j-1} B$

where v_j is an $n \times r$ matrix, and one can write J as follows:

$$J = \frac{\sum_{j=0}^{n-1} s^j v_j}{\sum_{i=0}^{n} \alpha_i s^i} \tag{3.42}$$

Suppose that J has a cancellation. Then the numerator of the right hand side of (3.42) must have the following form:

$$\sum_{j=0}^{n-1} s^j v_j = (s-s_k) \sum_{j=0}^{n-2} s^j w_j \tag{3.43}$$

where s_k is an eigenvalue of A. By equating the coefficients of s^i ,i=0,1,2.....n-1 of both sides of (3.43) one obtains

$$\begin{aligned} v_0 &= -s_k w_0 \\ v_1 &= w_0 - s_k w_1 \\ v_2 &= w_1 - s_k w_2 \\ \cdot\cdot \quad & \cdot\cdot \quad \cdot\cdot \\ v_{n-1} &= w_{n-2} \end{aligned} \tag{3.44}$$

Then

$$\begin{aligned} & v_0 + s_k v_1 + \dots\dots\dots + s_k^{n-1} v_{n-1} \\ &= -(s_k w_0) + (s_k w_0 - s_k^2 w_1) + (s_k^2 w_1 - s_k^3 w_2) + \dots \\ &\quad + (s_k^{n-2} w_{n-3} - s_k^{n-1} w_{n-2}) + s_k^{n-1} w_{n-2} \\ &= 0 \end{aligned} \tag{3.45}$$

which implies that

$$\sum_{i=1}^{n} \alpha_i A^{i-1} B + s_k \sum_{i=2}^{n} \alpha_i A^{i-2} B + \dots\dots + s^{n-1} \alpha_n B = 0 \tag{3.46}$$

which can be written as

$$\sum_{i=0}^{n-1} \kappa_i A^i B = 0 \tag{3.47}$$

where,

$$\kappa_i = \sum_{j=0}^{n-i-1} \alpha_{i+j+1} s_k^j$$

Since the coefficient of $A^{n-1}B$ is $\kappa_{n-1} = \alpha_n = 1$, [28], (3.47) implies that the vectors B, AB,.......$A^{n-1}B$ are linearly dependent. Hence the rank of G is less than n and the problem is singular.

Example 1 is chosen to illustrate the techniques discussed above.

§4.2. An Example

EXAMPLE 1

Consider the control system shown in Fig. 1. Assume there is no input to the system and the system is subjected to only initial conditions.

The problem to be solved is to determine the control u(t) which will steer the system from $\underline{x}_o$ to $\underline{0}$ in minimum time, subjected to the magnitude constraint $|u| \leqq 1$.

The equivalent equation for the normal case (the zero(s-1) is absent) , is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 2x_1 - x_2 + u$$

The Hamiltonian is

$$H = 1 + p_1x_2 + p_2 2x_1 - p_2x_2 + p_2u$$

The co-state equation is

$$\dot{p}_1 = -2p_2$$
$$\dot{p}_2 = -p_1 + p_2$$

Therefore

$$\underline{p} = \begin{bmatrix} 1/3\ e^{2t} + 2/3\ e^{-t} & -2/3\ e^{2t} + 2/3\ e^{-t} \\ -1/3 e^{2t} + 1/3\ e^{-t} & 2/3\ e^{2t} + 1/3\ e^{-t} \end{bmatrix} \underline{\pi}$$

The optimal control is given by

$$u^* = -\mathrm{sgn}\{\ p_2\ \}$$

and is unique.

Fig. 2A is a state plane diagram showing the saddle point, separatrices and several trajectories correspond to u=1.For u= -1 the state plane is identical to the foregoing except it is shifted to the right by 1 unit. The domain of null controllability T is the region bounded by separatrices I and II as shown in Fig. 2B. The optimal paths are shown for initial conditions in T . For initial conditions exterior to T, no solutions exist.

Now consider the non-normal case in which the zero (s-1) is present.

The equivalent system equation is

$$\dot{y}_1 = y_2 + u$$

$$\dot{y}_2 = 2y_1 - y_2 - 2u$$

where $y_1 = x$ and $y_2 = \dot{x} - u$

The problem is singular because G is singular.

The Hamiltonian is

$$H = 1 + p_1 y_2 - p_2 y_2 + 2p_2 y_1 + (p_1 - 2p_2)u$$

The optimal control is

$$u^* = -\mathrm{sgn}\,\{p_1 - 2p_2\} = -\,\mathrm{sgn}\{[\pi_1 - 2\pi_2]e^{2t}\}$$

If $\pi_1 = 2\pi_2$, then the control is undefined. From (3. 34)

$$\pm\,\xi_2 = \mp 2\xi_1$$

which is just a straight line passing through the origin with slope = -2. Graphically it may be seen that the two separatrices I and II approach one another resulting in a singular problem. Because the zero is non-minimum phase the target set is the origin in the y-plane [4].

Investigations into the existence of optimal solutions with $|u| < 1$ show that the target will never be reached. The only optimal solution is found for initial states constrained in the hyperplane $<\underline{p},\underline{x}> = 0$ (which is the line $y_2 = -2y_1$) with $|u| = 1$. The optimal strategy is depicted in Fig. 3.

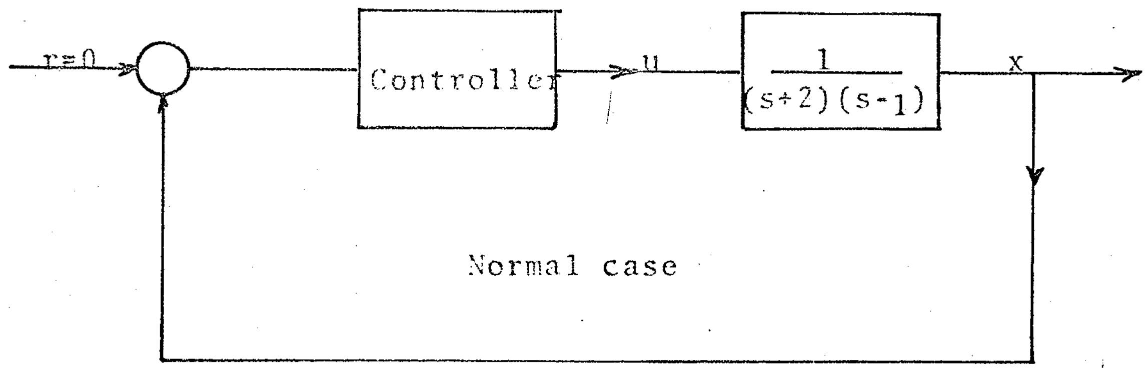
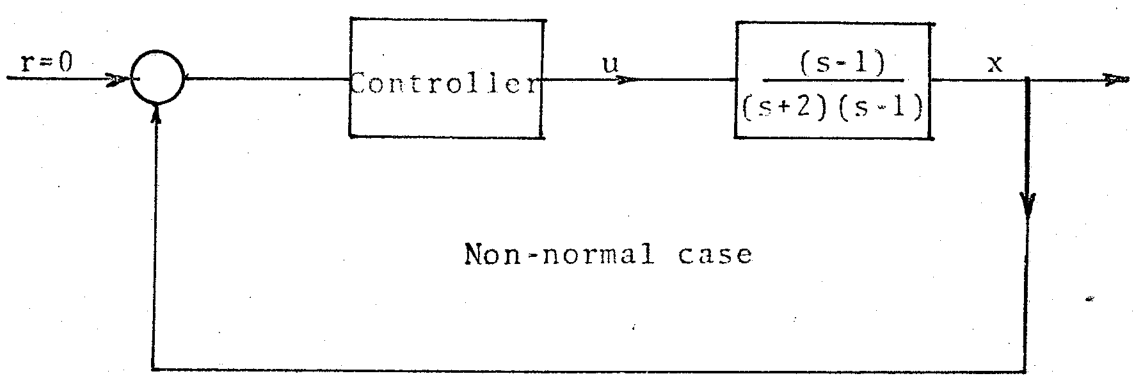


Fig. 1A Block diagram from Example 1



Fi . 1B Block diagram for Example 1.

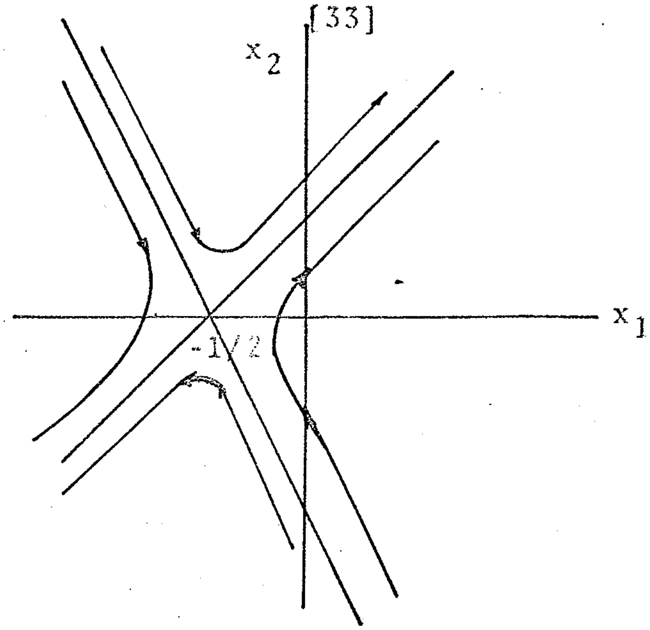


Fig. 2A Trajectories for u = +1

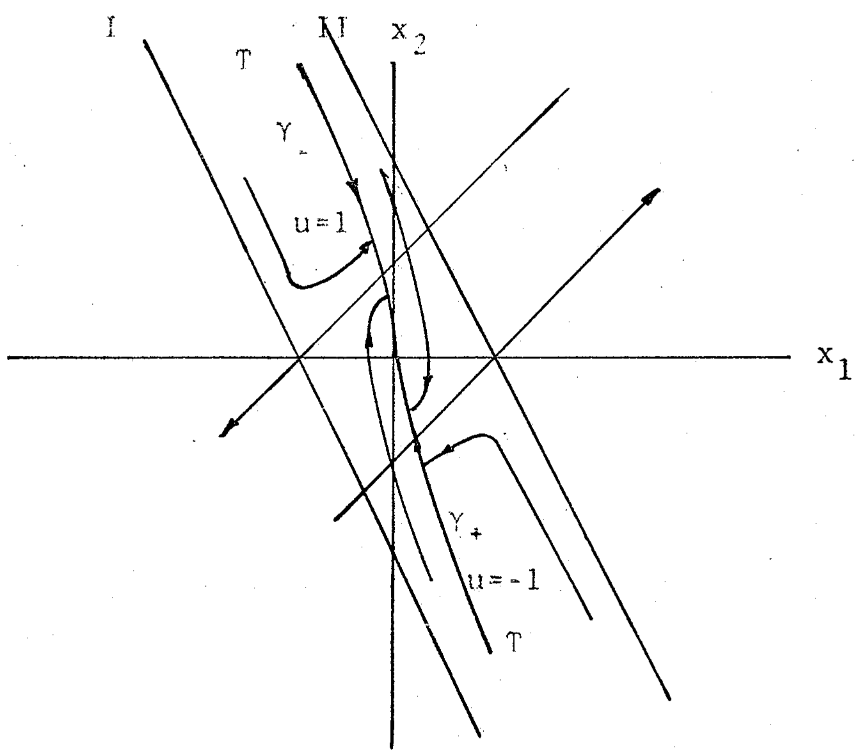


Fig 2B Optimal trajectories for normal time-optimal problem.

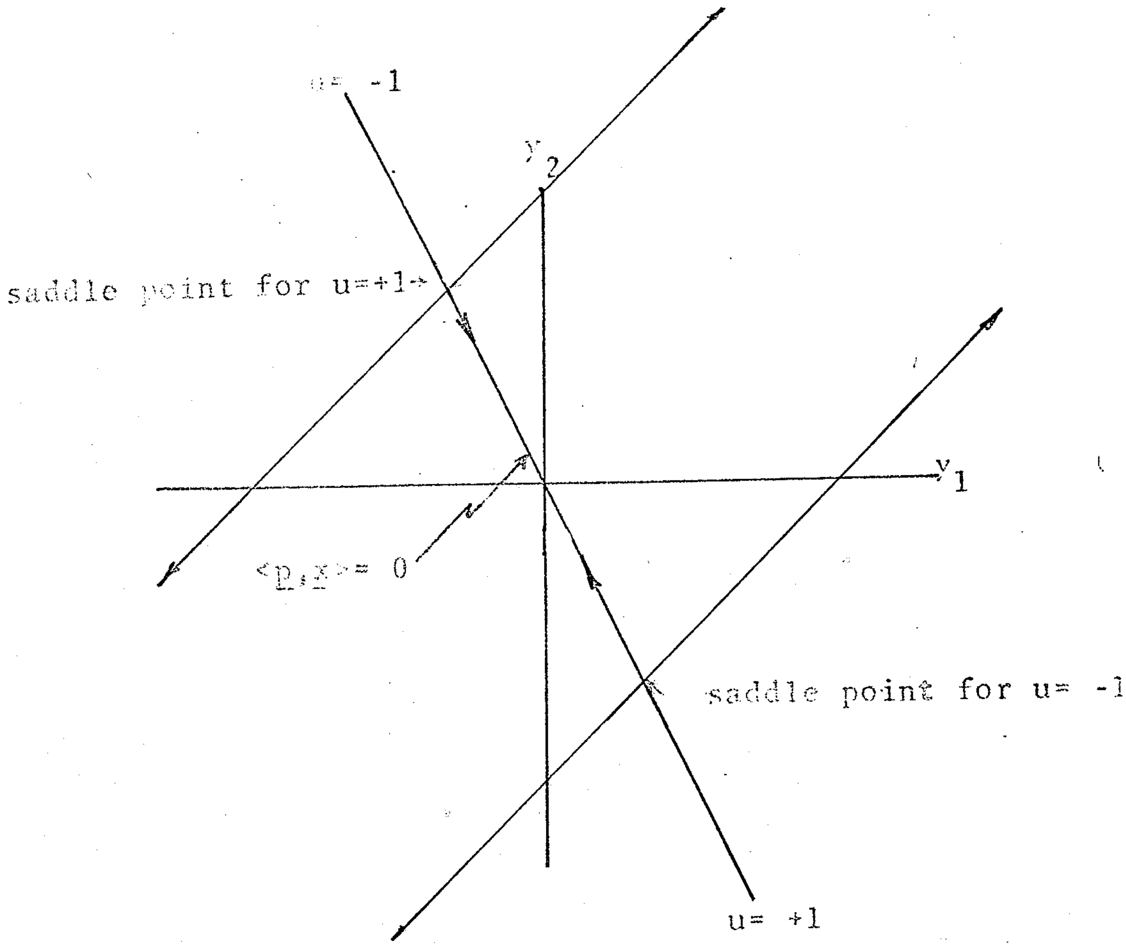


Fig.3 Singular optimal controls for Example 1

§4.3 Multiple-input systems

In general, the optimal solution to singular multiple-input systems is non-unique. Thau [31] has reported a technique by which a unique time-optimal may be obtained. He reformulates the problems using the reserve force approach. He assumes that the system (3.1) is non-normal with $<\underline{b}_j, \underline{p}> = 0$ and u_i , $i \neq j$ are then uniquely determined by (3.6). The control u is found such that

(a) x(0) is brought to $\underline{0}$ in minimum time.

(b) $u(t'), 0 \leqq t' \leqq T$, is chosen so that $u_j(t') \leqq c$, for $t > t'$ and c is the smallest positive real number possible for the given x(0)

Condition (b) requires that at each instant, the jth component be chosen so that in the remaining control interval the maximum magnitude is as small as possible. The result of this supposition is that a unique control law can be derived. The problems he considered are systems with $Re\lambda < 0$, (stable). The smallest possible value for c in this case is 0. Thus in the singular region an analogous minimum time-minimum fuel solution exists which will drive the system to the origin. This approach must be used with caution because in some cases it fails to account for optimal solution in certain region, as will be demonstrated in Example 2.

It will be shown that a generalized non-unique "Bang-Bang Principle" may be derived which is applicable to singular problems.

The set R(t) is defined by

(3.48) $$R(t)=\{x(t): x(t)= \Phi(t)\{\underline{x}_0 + \int_0^t \Phi^{-1}(s)B\underline{u}(s)\ ds\} \quad |u_j|\leq 1,\ j=1,2,\ldots,r\}$$

where R(t) is the set of all points to which $\underline{x}_0$ can be transferred in t second using an admissible control.

Thau [31] has shown that for a non-normal problem, if $\underline{x}(0)$ can reach $\underline{0}$ in time t but not at any time prior to t, then $\underline{0}$ is on the boundary of R(t). The optimal control law is not unique.

Consider now the system equation (3.1) and assume Λ is of simple structure. The solution of the adjoint system is $\underline{p}=e^{-\Lambda^T t}\underline{\pi}$. If the problem is non-normal, namely for some initial conditions, $\langle \underline{b}_j, e^{-\Lambda^T t}\underline{\pi}\rangle \equiv 0$ on proper interval $[t_1,t_2]$, then the optimal control u_j^* is not unique on $t_1 \leq t \leq t_2$, where the PMP fails to select an optimal u_j. The optimal control u_j^*, outside $[t_1,t_2]$ is given by

(3.49) $$u_j^* = -\mathrm{sgn}\{\sum_{i=0}^{n} b_{ij}p_i\} \quad , j=1,2,\ldots,r$$

It is clear that the singularity conditions occur when all b_{ij} or π_i is zero. Define $e^{-\Lambda t}\underline{b}_j = \underline{h}(t)$. If $\underline{h}(t)\equiv 0$, then $\underline{0}$ on R(t) can be reached with an arbitrary u_j, $|u_j| \leq 1$. Assume $\underline{h}(t) \neq 0$ and consider two controls u_j^o and u_j^1 such that on $[t_1,t_2]$; $u_j^o=0$ and $|u_j^1|\leq 1$. These two controls are uniquely defined by (3.49) outside $[t_1,t_2]$, and they lead to two points on R(t), $\underline{x}_1(t)$ and $\underline{x}_2(t)$ respectively in the interval $t_1 \leq t \leq t_2$. Clearly

$$\Delta\underline{x}(t) = \underline{x}_1(t) - \underline{x}_2(t) = \int_{t_1}^{t} \underline{h}(t,s)u_j(s)\ ds \quad \text{and}$$

$$|u_j|\leq 1 \text{ on } [t_1,t_2].$$

For all $|u_j| \leq 1$, $\underline{x}(t)$ is not zero. The assumption that $\underline{h}(t) \neq 0$ implies that $\pi_j = 0$. Then an orthogonal matrix P exists such that the first r' components of the column vector $P\underline{h}(t)$ are non-zero, and the remaining n-r' components are zero. Denote the first r' components of $P\underline{h}(t)$ by $\underline{h}_r(t)$, then the control u_j^* is given by

(3.50) $$u_j^* = -\text{sgn}\,\{<\underline{b}_{jr'}, e^{-\Lambda^T t}\underline{\pi}_{r'}>\} \quad \text{on } [t_1, t_2].$$

Since a problem is singular for all $t \varepsilon [0,T]$, then a non-unique Bang-Bang Principle exists for all $t \varepsilon [0,T]$.

LaSalle's Bang-Bang Principle [21] is generalized to extend to singular problems by Halkin [10] with the additional assumptions that a bang-bang steering function be piecewise continuous, i.e. continuous at all but a finite number of points. Halkin proves:

THEOREM 7 . If of all piecewise continuous bang-bang steering functions, there is an optimal relative to Ω'(where Ω' is the set of all piecewise constant bang-bang steering functions), then it is optimal relative to Ω.

THEOREM 8. If there is an optimal steering function there is always a piecewise continuous bang-bang control that is optimal.

Halkin's results can be summarized as follows:

Anything that can be done with an arbitrary control can be done with a relay control with a finite number of switching times.

While LaSalle [21] advocates that an analogous bang-

bang principle holds for a proper system, he can only assert that:

THEOREM 9

If a control system is proper and if Ω is a bounded convex set, then the optimal steering function has values on the boundary of Ω, and more precisely if $\underline{u}^*$ is optimal, then $\underline{u}^*$ cannot be in the interior of Ω for a positive interval of time.

THEOREM 10

If the system is proper, then the optimal steering function for the special problem (i.e. the target set is the origin) has values on the boundary of Ω.

To show that for some systems whose eigenvectors do not span the state space, (3.21) is only a necessary condition, denote K to consist of all initial states which are singular and L to consist of all initial states which are non-singular. It is clear that

(a) $K \cap L = \phi$ (the empty set)

(b) $K \cup L = S$ the entire space.

(c) L is a closed subset of S (possibly the empty set)

(d) K is an open subset of S (possibly the empty set)

Consider further the set T(k) such that $q_k \neq 0$. Note that T(k) is the complement of set $\Gamma(k)$ of Definition 2, i.e.

$$T(k) = \{(p,x) \mid q_k \neq 0\} = C(\Gamma(k))$$

then

$$T = \bigcap_{k=1}^{r} T(k) = C(\Gamma)$$

If $T \subset L$, and $\Gamma \supset K$, then the set $\Gamma \cap L$ may be non-empty. If this is the case, the problem may be normal.

The non-uniqueness of the control u_j in (3.50) may be be deduced from the relation [4]

(3.51) $\quad \langle \underline{\pi}, \underline{\xi} \rangle = \text{constant} \geq 0$

By Lemma 2 of Appendix A, there exists a real, skew-symmetric D such that

(3.52) $\quad \underline{\pi} = [\beta I + D]\underline{\xi}$ where $\beta = \text{constant}/||\underline{\xi}||^2$ and D is not unique.

It appears that $\underline{\pi}$ can be chosen arbitrarily as long as (3.52) is satisfied. The non-uniqueness of (3.50) follows from the non-uniqueness of D.

In general, there exist no analytical techniques for synthesizing multiple-input time-optimal problems. All that is known is that there exists a non-unique bang-bang control which is time-optimal. The synthesis methods are, in most cases, by means of graphical procedures rather than by analytical techniques.

The techniques are illustrated in Examples 2 and 3 in the next section.

§4.4 Further Examples

EXAMPLE 2

Consider the following second order linear time-invariant system where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Since A is diagonalizable, then the conditions of Theorem 2 are satisfied and the problem is singular time-optimal. The control problem then consists of deriving a control law $\underline{u}^*$ that transfers any initial state $\underline{x}_o$ to the origin, in minimum time, subjected to the constraint $|u_j| \leq 1$, j=1 and 2.

The time-optimal problem is totally singular because G_1 and G_2 are singular matrices. Note that the system is proper.

The formal solution via the PMP is given by

$$u_1^* = -\text{sgn}\{ p_1 + p_2 \}$$
$$u_2^* = -\text{sgn}\{ p_2 \}$$

where
$$\underline{p} = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \underline{\pi}$$

For any initial co-state vector $\underline{\pi} \neq 0$, the solution is normal. Because $\underline{u}$ cannot change sign, there exists no normal solution which will steer the system from x_o to $\underline{0}$ unless $\underline{\xi}$ is on the γ_+ and γ_- curves. For $\underline{\pi}=0$, the problem is totally singular. However, the PMP [4] asserts that for optimality there must exist a non-zero solution of the co-state equation. Hence this case ($\underline{\pi}=0$) is impossible.

If $\pi_1 \neq 0$ and $\pi_2=0$, then u_1 is constant and cannot change sign, and u_2 is arbitrary. Since the system is proper, by Theorem

10 , the possible values of u_2 are restricted to +1 and -1. The non-unique bang-bang control is shown to be optimal in Fig. 4, for $\underline{\xi}$ in the region bounded by the curves γ_+, γ_- and the axis $x_2=0$.

On the other hand, if $\pi_1 = -\pi_2$, then u_1 is arbitrary and u_2 cannot change sign. The non-uniqueness of bang-bang control is shown to be optimal in Fig. 5, for $\underline{\xi}$ in the region bounded by the axis $x_2=0$ and the curves γ_- and γ_+ .

Using Thau's method, one would consider the reduced system equation

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -y_2 + u_2$$

where $\underline{y} = \underline{x} - \underline{x}_o$ and $\underline{x}_o$ corresponds to the trajectory having $u_2=0$. The problem is then reduced to determine u_2 so that $\underline{y}=0$ in minimum time or equivalently $\underline{x}=\underline{x}_o$ in minimum time. From 4.1 the states $\underline{y}$ which are controllable (i.e. all those with $y_1=0$) correspond to those states for which the minimum time condition is insufficient to determine u_2 uniquely. Clearly in this context $u_2 = -\mathrm{sgn}\,\{\pi_2 e^t\}$.

This method fails to account for a solution in the state plane exterior to the curves γ_+, γ_- and σ , since in these regions $\underline{u}$ cannot change sign. (see Fig. 6)

The curve γ_+ (γ_-) is defined as the locus of all points $\underline{x}$ which can be forced to $\underline{0}$ by the control $u_1=u_2=+1$ ($u_1=u_2=-1$), and the curve σ is similarly defined but for $u_1 = \pm 1$ and $u_2=0$.

[42]

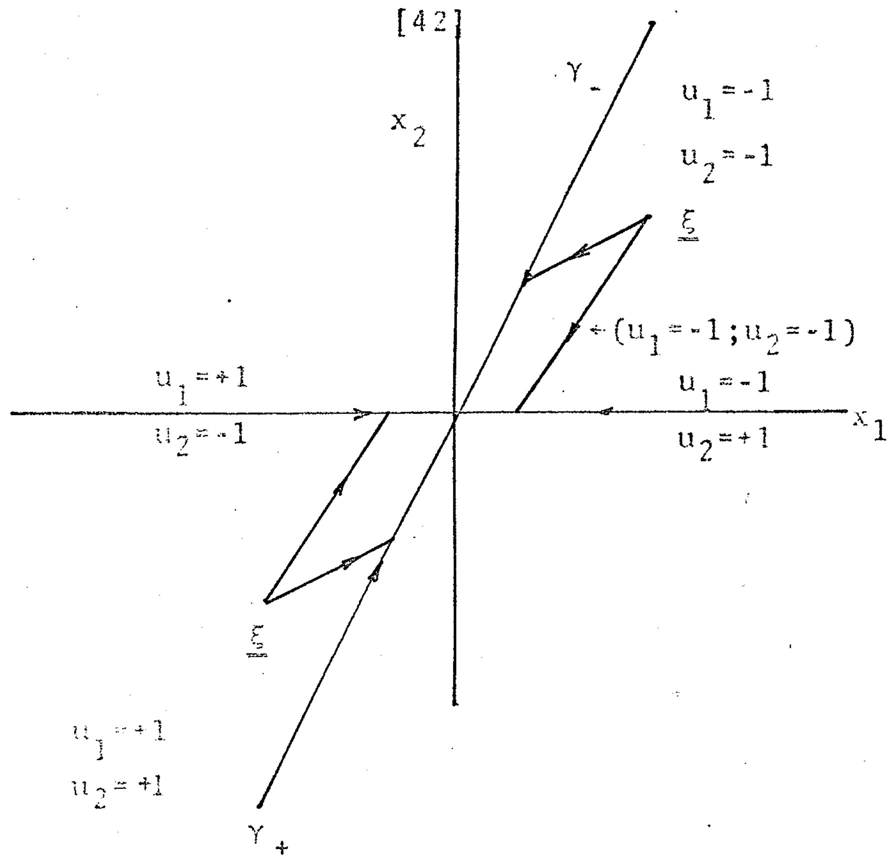


Fig.4 Example 2:-Singular optimal solutions for $\pi_1 \neq 0$ and $\pi_2 = 0$

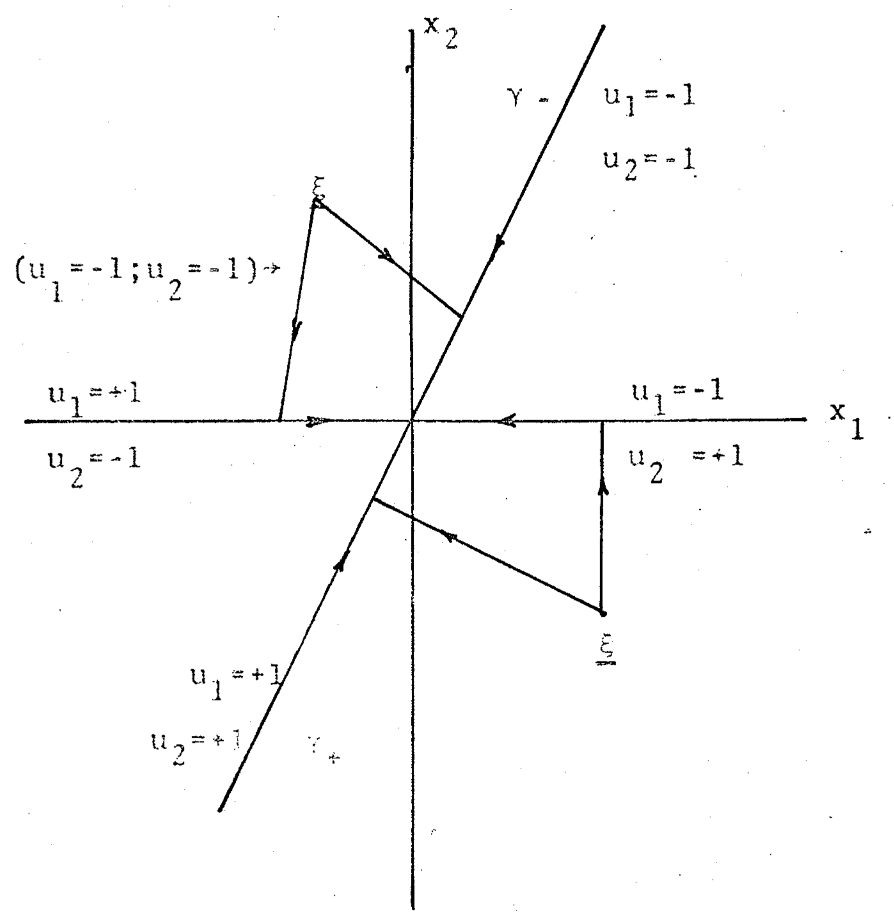


Fig.5 Singular optimal controls for $\pi_{\bar{1}} - \pi_2 \neq 0$

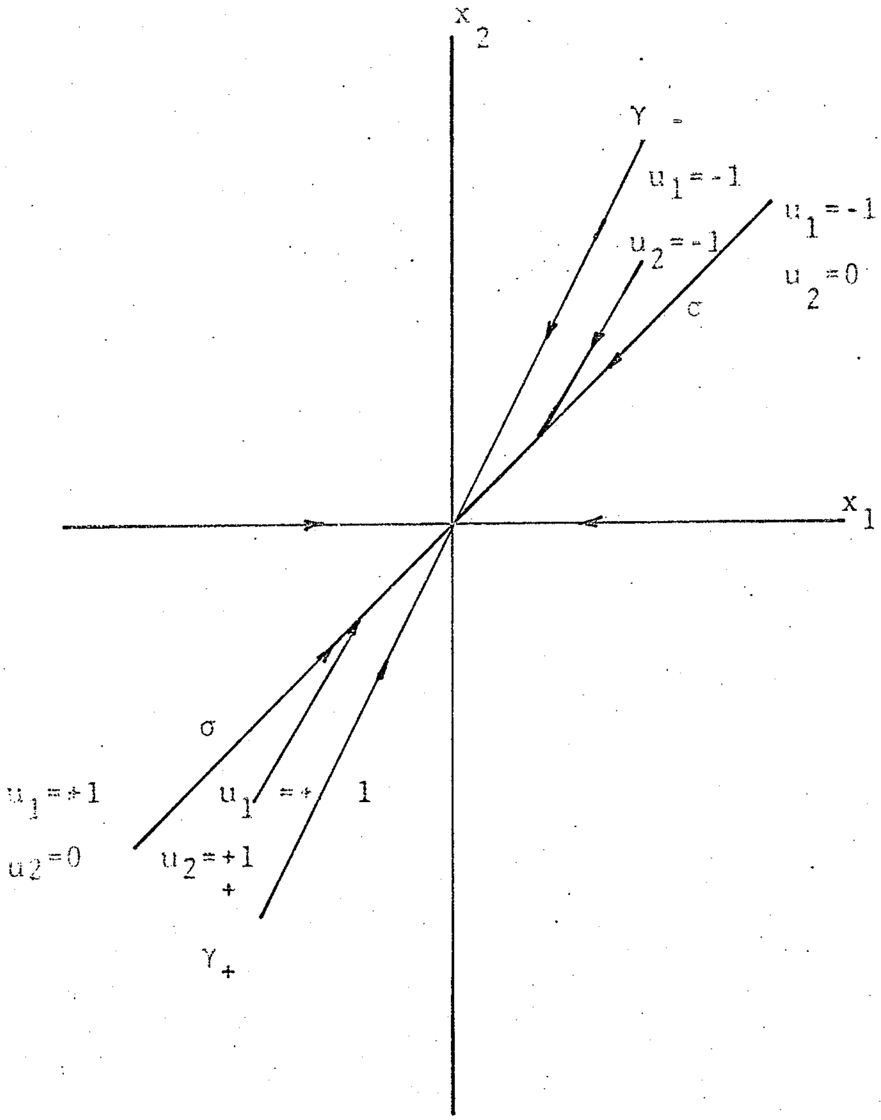


Fig. 6 Optimal solutions using Thau's-reserve force approach.

EXAMPLE 3

Consider the problem whose A and B matrices are

$$A=\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix} \qquad B=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

and $|u_j|\leq 1$, j=1 and 2.

The system is proper but non-normal because

$$G_2=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$ is a singular matrix.

The adjoint sytem is

$$\dot{p}_1=0$$

$$\dot{p}_2= -p_1 +p_2$$

and the solution is

$$\underline{p} = \begin{bmatrix}1 & 0\\ 1-e^t & e^t\end{bmatrix} \underline{\pi}$$

The controls u_1 and u_2 which minimize H are given by

$$u_1 = -\text{sgn } p_2$$

$$u_2 = -\text{sgn } p_1$$

If $\pi_1 \neq \pi_2 \neq 0$, u_1 can change sign at most once and u_2 cannot change sign. If $\pi_1 =0$ and $\pi_2 \neq 0$ then, the problem becomes singular : u_1 must not change sign and u_2 is arbitrary.

Since the matrix A is singular, and by Theorem 3, the condition that a time-optimal problem is singular is not sufficient. Therefore there exist initial states for which the problem is normal. Denote the sets X_- and X_+ to consist of all initial states for controls $u_1=-1$ ($|u_2|\leq 1$) and $u_1=+1$ ($|u_2|\leq 1$)

respectively such that $\underline{x}(T) = \underline{0}$. Then the union of sets X_- and X_+ consists of all initial states which are singular (see Fig.7). The set of singular states is open. It is clear that the complement of the singular region is the non-singular region which is closed.

The optimal control laws are illustrated in Fig.8. The solutions in $X_+ \cup X_-$ are singular solutions and they are non-unique, bang-bang steering functions. The solution exterior to $X_+ \cup X_-$ is unique.

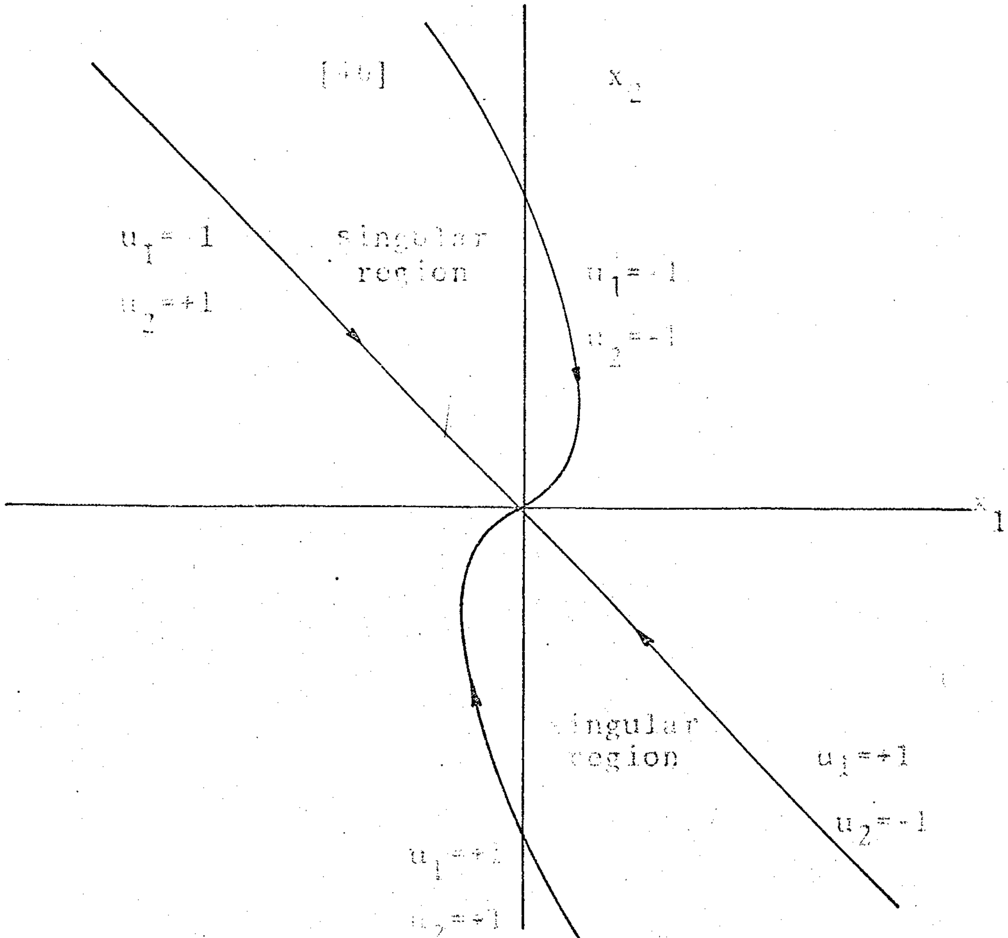


Fig.7 Singular regions for Example 3

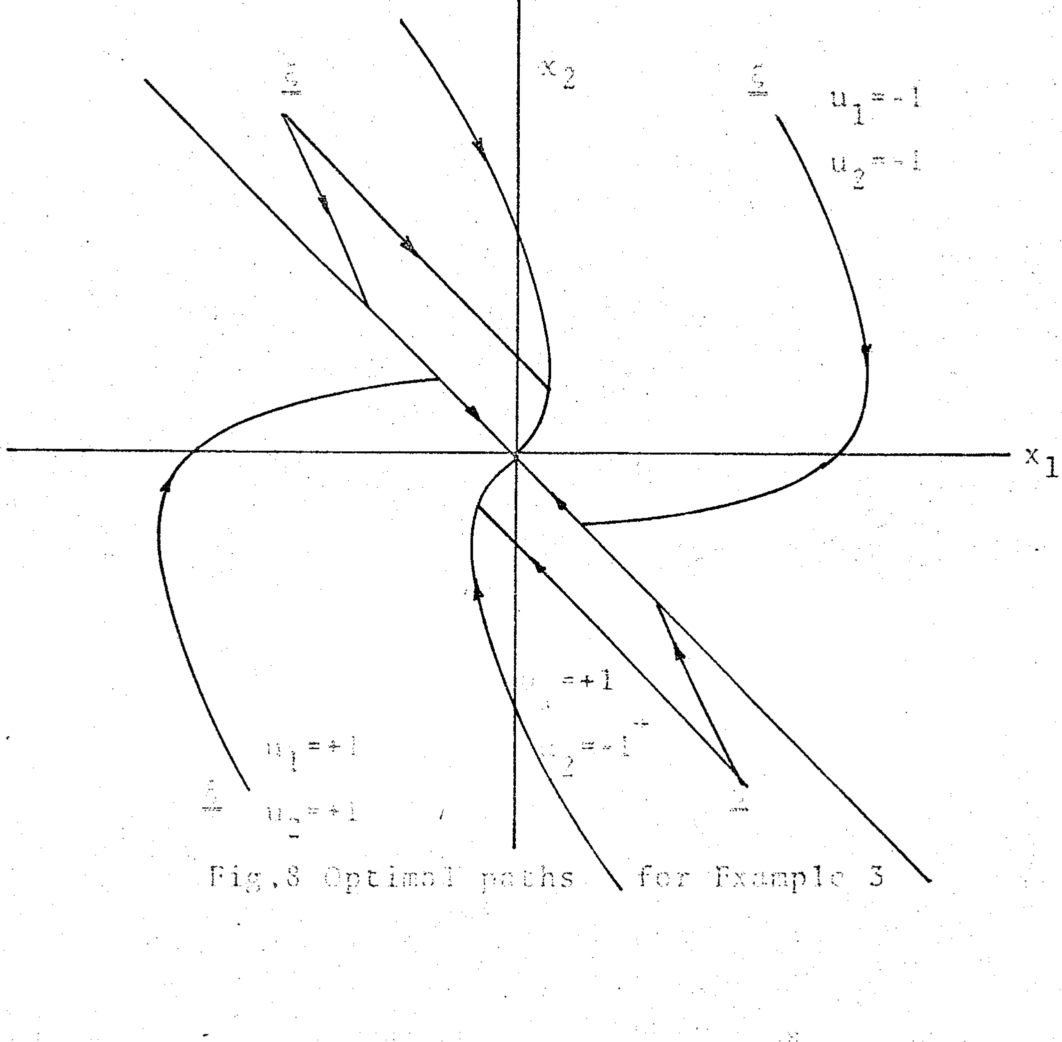


Fig.8 Optimal paths for Example 3

CHAPTER 4

SINGULAR FUEL-OPTIMAL PROBLEMS

§1. Introduction

Singular fuel-optimal problems arise quite frequently in optimal control processes. Investigations into the nature of singularity in the formulation of fuel-optimal problems have been carried out by several authors(Snow [32], Athans and Falb [4]). The direction this chapter will take is to examine the general solution if it exists, to a class of fuel-optimal problems. A necessary condition for singularity to occur is established in a manner like that of Athans and Falb [4]. It is shown that for linear time-invariant systems, singular controls cannot be optimal. However, for time-invariant non-linear systems, singular controls may be optimal.

§2. Choice of Index of Performance

There is at present a wide choice of performance criteria for fuel optimization. The particular chioce of performance criterion rests with the designers. In this paper the following cost functions are used.

(4.1) $$J(\underline{u}) = \int_0^T \sum_{j=1}^r |u_j| \, dt \qquad \text{T may be free or fixed}$$

(4.2) $$J(\underline{u}) = \int_0^T \left(k + \sum_{j=1}^r |u_j| \right) dt \qquad \text{T is free and } k>0$$

where $u_1, u_2, \ldots\ldots u_r$, are components of vector $\underline{u}$ and $\underline{u} \in \Omega$.

A formulation of the fuel-optimal problem involving (4.1) with no restrictions placed on the response time T often results in undesirable transition time. On the other hand, if T is fixed a priori, (4.1) allows the minimization of fuel only for some initial conditions less than some maximum. This unattractiveness of fuel optimization of (4.1) is overcome by defining the performance index as a weighted sum of time and fuel. Such a formulation allows fuel to be minimized for all initial states yet bounds the response time. Towards the latter half of this chapter, the cost function (4.2) is used entirely.

§ 3. The Singular Problem for a class of Linear Time-Invariant Systems

Consider the linear time-invariant system described by

(4.3) $$\dot{\underline{x}} = A\underline{x} + B\underline{u} \qquad |u_j| \leq 1 \;, j=1,2,\ldots\ldots r$$

It is desired to find a $\underline{u}$ which steers the system from an arbitrary state to the origin and in so doing minimizes either (4.1) or (4.2).

The Hamiltonian for fuel optimization is either

(4.4) $$H = p_o \sum_{j=1}^{r} |u_j| + \langle \underline{p},\ A\underline{x}+B\underline{u} \rangle \qquad \text{or}$$

(4.5) $$H = p_o \left(k + \sum_{j=1}^{r} |u_j| \right) + \langle \underline{p},\ A\underline{x}+B\underline{u} \rangle$$

for the cost function (4.1) or (4.2).

The canonical equations are identical with those of the minimum time problem. The formal solution via the PMP is given by

(4.6) $$\begin{aligned}\underline{u}^* &= -\mathrm{DEZ}\ \{B^T\underline{p}^*\} \\ &= -\mathrm{DEZ}\ \{\underline{q}^*\}\end{aligned}$$

or in component form

(4.7) $$\begin{aligned}u_j^* &= -\mathrm{dez}\ \{<\underline{b}_j, \underline{p}^*>\} \\ &= -\mathrm{dez}\ \{q_j\} \qquad\qquad j=1,2,\ldots\ldots.r.\end{aligned}$$

where the deadzone function is defined as follows:

$$\mathrm{dez}\ q = \begin{cases} 1 & \text{if } q > 1 \\ -1 & \text{if } q < -1 \\ 0 & \text{if } |q| < 1 \\ \text{undefined} & \text{if } |q| \equiv 1 \end{cases}$$

The optimal control $\underline{u}^*$ must satisfy the condition

(4.8) $$\sum_{j=1}^{r} |u_j^*| + <\underline{p}^*, A\underline{x}^* + B\underline{u}^*> \leq \sum_{j=1}^{r} |u_j| + <\underline{p}^*, A\underline{x}^* + B\underline{u}>$$

for all admissible $\underline{u}$ and for all $t\ \varepsilon[0,T]$ and * denotes the optimal solution, and $p_o = 1$ by PMP [2] .

The condition for singularity occurs when

(4.9) $|<\underline{b}_j, \underline{p}^*>| \equiv 1$ for some j and

for a finite interval of time , i.e. $\forall\ t\ \varepsilon[t_1, t_2]\ \varepsilon\ [0,T]$.

DEFINITION 9

Suppose that in the interval [0,T] or in the interval $[0,T_f]$ for fixed time case, there are one or more subintervals

$[t_1, t_2]_j$ such that for all j

(4.10) $|q_j| \equiv 1$ and $\forall\ t \in [t_1, t_2]_j$, $j=1,2,\ldots\ldots r$.

then a totally singular fuel-optimal problem is said to exist. If (4.10) holds for some $j, j=1,2,\ldots.r$, then the problem is said to be singular fuel-optimal.

The implication of this condition about the structure of the system may be understood by proceeding along the lines as in the time-optimal case. By repeated time differentiation of (3.9) and using the fact that $\dot{\underline{p}} = -A^T \underline{p}$, Athans [2] obtains

(4.11) $G^T A^T \underline{p} = 0 \quad \forall\ t \in [t_1, t_2]$ or in component form

$$G_j^T A^T \underline{p} = 0 \quad \forall\ t \in [t_1, t_2]$$

where G and G_j are the same matrices as the ones used in the time-optimal problem in Chapter 3.

The next theorem proves the necessary condition for fuel-optimal problem to be singular or totally singular.

<u>THEOREM 1 [4]</u>

A necessary condition for the fuel-optimal to be singular (totally singular) is that either G_j (G) and/or A is singular matrix.

Proof: $|q_j| \equiv 1\ \forall\ t \in [t_1, t_2]$ => $\underline{p} \neq 0$, then (4.11) implies that

det $G_j^T A^T (G^T A^T) = 0$ **implies that either**

G_j (G) and /or A must be a singular matrix

for all $t \in [t_1, t_2]$.

Thus, if the system is non-normal or completely uncontrollable, then the problem is necessarily singular. However, if the condition of normality is assumed, it is still possible to have a singular problem if A is singular.

It will be shown that if A is a non-singular matrix and if $G_j(G)$ is singular, then there are no singular (totally singular) controls for the fuel-optimal problem. So the problem is normal.

If det A $\neq 0$, then there exists a non-singular matrix P such that

$$\Lambda = P^{-1}AP$$

For simplicity, assume the eigenvalues are distinct . The solution to the adjoint system is

$$\underline{p} = e^{-\Lambda^T t}$$

The components of $\underline{p}$ are either monotone increasing or decreasing.† Therefore there are no initial co-state vectors $\underline{\pi}$ satisfying the singularity condition for any linear combination of p_i . Hence the problem is normal. This proof can be extended to any non-singular matrix A, but the steps involved will be lengthy and cumbersome.

THEOREM 2 [2]

If the problem is singular (totally singular) $\forall\ t \in [t_1,t_2]$, then it is singular (totally singular) $\forall\ t \in [0,T]$.

Proof: Let t_3 be any time such that $0 \leq t_3 \leq t_1$. Then for every $t \in [t_1,t_2]$ the solution of the adjoint system

† The eigenvalues of A are assumed to be real.

is

$$\underline{p} = e^{-A^T(t\ \ -t_3)}\ \underline{p}(t_3)$$

Substituting in (4.11) one obtains

$$G_j^T A^T (G^T A^T) e^{-A^T(t\ -t_3)} \underline{p}(t_3) = 0 \quad \forall\ t\ \varepsilon\ [t_1, t_2]$$

Since the matrix A^T and $\exp(-A^T(t - t_3))$ commute, and $\exp(-A^T(t - t_3))$ is non-singular, it follows that

$$G_j^T A^T (G^T A^T)\ \underline{p}(t_3) = 0$$

which impies that $\underline{p}(t_3)$ belongs to the null space of $G_j^T A^T (G^T A^T)$. This means that the problem is singular (totally singular) for all $t_3\ \varepsilon\ [0, t_1]$.

Similar reasoning can be used to show that $\forall\ t_4\ \varepsilon\ [t_2, T]$, $\underline{p}(t_4)$ also belongs to the null space of $G_j^T A^T (G^T A^T)$, hence the problem is still singular(totally singular).

Since the sufficiency condition implies the existence of a solution, the lack of it in the formulation of fuel-optimal presents additional computational and theoretical difficulties. It is conjectured that, if there exists a non-empty set $\hat{N}$ (of positive measure) such that $\underline{\xi} \varepsilon \hat{N}$, then the extremals are normal even though A is singular. To show this , V is denoted to be the set of all admissible extremal controls, i.e.

$$V = \{u_j(t): \quad -1 \leqq u_j(t) \leqq +1\ \forall\ t\ \varepsilon\ [0, T]\}$$

In view of Theorem 2, the extremal control is singular $\forall\ t\ \varepsilon\ [0, T]$. By denoting V_+ and V_- the sets of extremal controls that satisfy

$$0 \leq u_j(t) \leq +1 \qquad \text{when } q_j \equiv -1$$

$$-1 \leq u_j(t) \leq 0 \qquad \text{when } q_j \equiv +1$$

or more precisely

$$V_+ = \{ u(t): \quad 0 \leq u(\underline{t}) \leq +1,\ t \in [0,T] \}$$

$$V_- = \{ u(t): \quad -1 \leq u(t) \leq 0 ,\ t \in [0,T] \}$$

one may show that

(4.12) $\quad V_+ \cup V_- \in V \quad$ and

$V-(V_+ \cup V_-) \neq \phi \quad$ i.e $V_+ \cup V_-$ is a proper subset of V.

Note that $V_+ \cup V_-$ is the subset of all singular controls.

The general solution of (4.3) is

(4.13) $\quad \underline{x}(t) = e^{At} \left[\underline{\xi} + \int_0^t e^{-A\tau} B\underline{u}(\tau)\, d\tau \right]$

Now define the set of all initial states that can be forced to 0 by

$$X = \{ \underline{\xi}: \underline{\xi} = -\int_0^T e^{-At} B\underline{u}(t)\, dt,\ \underline{u}(t) \in V \}$$

$$X_+ = \{ \underline{\xi}: \underline{\xi} = -\int_0^T e^{-At} B\underline{u}(t)\, dt,\ \underline{u}(t) \in V_+ \}$$

$$X_- = \{ \underline{\xi}: \underline{\xi} = -\int_0^T e^{-At} B\underline{u}(t)\, dt,\ \underline{u}(t) \in V_- \}$$

It can be shown from (4.12) that

$$\hat{N} = X-(X_+ \cup X_-) \neq \phi$$

Thus the interior of $X_+ \cup X_-$ is the region of all singular states. The same reasoning holds also for totally singular problems.

A similar theorem to Theorem 5 of Chapter 3 is established for the fuel-optimal totally singular problem. This theorem may be used to relate the initial co-state vector $\underline{\pi}$ with the initial state vector $\underline{\xi}$. It should be noted the this theorem is only a necessary condition.

THEOREM 3

If the fuel-optimal problem is totally singular, then $\langle \underline{p}, A\underline{x} \rangle = 0$

Proof: The solution to (4.3) can be expressed as

(4.14)
$$\underline{x}(t) = \Phi(t,\tau)\underline{x}(\tau) + \int_{\tau}^{t} \Phi(t,\lambda)B\underline{u}(\lambda)\, d\lambda$$

where τ is arbitrary in the whole interval $[0,T]$. If for T and for an admissible $u(t') \neq 0$, $t \leq t' \leq T$, $\underline{x}(T)=0$, then (4.14) reduces to

(4.15)
$$\underline{x}(\tau) = -\int_{\tau}^{T} \Phi(\tau,\lambda)B\underline{u}(\lambda)\, d\lambda$$

Premultiplying (4.15) by A and taking the scalar product of the resulting equation with $\underline{p}(\tau)$ one would obtain

(4.16)
$$\langle \underline{p}(\tau), A\underline{x}(\tau)\rangle = -\int_{\tau}^{T} \langle \underline{p}(\tau), A\Phi(\tau,\lambda)B\underline{u}(\lambda)\rangle d\lambda$$
$$= -\int_{\tau}^{T} \langle \Phi^{T}(\tau,\lambda)A^{T}\underline{p}(\tau), B\underline{u}(\lambda)\rangle\, d\lambda$$

Since $\Phi^{T}(\tau,\lambda)$ and A^{T} commute, (4.16) can be written as

(4.17)
$$\langle \underline{p}(\tau), A\underline{x}(\tau)\rangle = -\int_{\tau}^{T} \langle A^{T}\Phi^{T}(\tau,\lambda)\underline{p}(\tau), B\underline{u}(\lambda)\rangle d\lambda$$

Using (3.9) and (3.10), (4.17) may be rewritten in a simpler form

(4.18)
$$\langle \underline{p}(\tau), A\underline{x}(\tau)\rangle = -\int_{\tau}^{T} \langle A^{T}\underline{p}(\lambda), B\underline{u}(\lambda)\rangle\, d\lambda$$
$$= -\int_{\tau}^{T} \langle B^{T}A^{T}\underline{p}(\lambda), \underline{u}(\lambda)\rangle\, d\lambda$$

Now $\dot{\underline{p}}(\lambda) = -A^{T}\underline{p}(\lambda)$

Therefore

$$\text{RHS} = + \int_{\tau}^{T} \langle B^T \dot{\underline{p}}(\lambda), \underline{u}(\lambda) \rangle \, d\lambda$$

The totally singular condition implies that

$$B^T \underline{p}(\lambda) = \begin{bmatrix} \pm 1 \\ \pm 1 \\ \cdot \\ \pm 1 \end{bmatrix} \quad \text{over the whole interval (by Theorem 2)}$$

Therefore $\frac{d}{d\lambda}(B^T \underline{p}(\lambda)) = \underline{0}$

Hence the right hand side of (4.18) is

$$\text{RHS} = + \int_{\tau}^{T} \langle \underline{0}, \underline{u}(\lambda) \rangle \, d\lambda = 0 \qquad \text{or}$$

(4.19) $\quad \langle \underline{p}(\tau), A\underline{x}(\tau) \rangle = 0$ over the whole interval $[0,T]$.

Thus the theorem is proved.

§4. Linear Single-Input Systems

For single-input systems, the assumption that the system is completely controllable is asserted for the material to be presented in this section. The condition that the problem is singular reduces to A being a singular matrix. A convenient characterization of the admissible control can be derived in terms of the initial states [4], by denoting

(4.20)
$$F(\underline{\xi}) = \int_0^T u(t) \, dt$$

$$|F(\underline{\xi})| = \left| \int_0^T u(t) \, dt \right|$$

$$|F(\underline{\xi})| \leqq \int_0^T |u(t)| \, dt = J(u)$$

with $u(t) \in \Omega$.

This means that the fuel required to force $\underline{\xi}$ to $\underline{0}$ cannot be smaller than $|F(\underline{\xi})|$ since $J(u) \geq |F(\underline{\xi})|$. Note $|F(\underline{\xi})|$ gives a lower bound for $J(u)$ for a given initial state. If a control u can be found which forces $\underline{\xi}$ to $\underline{0}$ with fuel $|F(\underline{\xi})|$, then this control is indeed optimal. Though $|F(\underline{\xi})|$ does not guarantee that it is the greatest lower bound for the fuel consumption, it is indeed so, as shall be demonstrated later.

THEOREM 4 [30]

An admissible control u in V is optimal if it does not change sign.

Proof: The equality in (4.20) holds if u does not change sign. Since $|F(\underline{\xi})|$ is the lower bound for $J(u)$, any such u must be optimal.

If $\underline{\xi}$ is in the interior of the subset $X_+ \cup X_-$ there are infinitely many extremal controls, each of which is admissible so long as they satisfy $|u| \leq 1$.

It has been shown [4] that the Hamiltonian along an extremal is zero for time not fixed and is constant ≥ 0 for time fixed a priori. This transversality condition may be used to rule out the possibility of singular arcs in the fuel-optimal problem.

THEOREM 5

For linear time-invariant systems, there exist no singular extremals for the cost functions (4.1) with T fixed a priori and for (4.2).

Proof: The Hamiltonians for these cost functions are

(4.21) $H = |u| + \langle \underline{p}, A\underline{x}+B\underline{u} \rangle$

(4.22) $H = k + |u| + \langle \underline{p}, A\underline{x}+B\underline{u} \rangle$

By using the results of Theorem 3 and the singularity condition (4.21) and (4.22) reduce to

(4.23) $H^* = 0$ and

(4.24) $H^* = k$ respectively.

For a singular control to be extremal, say the control given by

(4.25) $u_s = -\mathrm{sgn}\,\{B^T \underline{p}\}\ v(t)$

where $v(t) = \{v(t) : v(t) \in V_+$ and $v(t) \neq 0\}$, (4.23) must be a constant $\geqq 0$ and (4.24) must be zero . Therefore there are no singular extremals and the problem is normal.

This theorem indirectly proves that singular controls may be extremals if the cost function is (4.1) with T free. However for such extremals, Athans and Falb [4] indicate that singular controls cannot be optimal.The next theorem proves this fact.

THEOREM 6

For linear time-invariant systems and the cost function (4.1) with the response time unspecified, singular extremals cannot be optimal.

Proof: The control given by (4.25) satisfies equation (4.8) .. Assume it to be optimal, then by Theorem 2 and Theorem 4, this control is an admissible extremal control for all $t \in [0,T]$ and cannot change sign. It should be observed that (4.25) gives the polarity of the control but not its magnitude.

Since the system is linear and time-invariant, the trajectories are smooth and well-behaved towards the origin . The optimal-control sequence to the origin must be of the type $\{\ldots, +1\}$ or $\{\ldots, -1\}$ or $\{\ldots, u_s\}$. If a control sequence $\{0, u_s\}$ is applied and the switching occurs at the instant the path reaches the γ curve (where γ is the locus of all states that a control $u=+1$ or $u=-1$ will force the system to the origin), the origin will always be missed. The same argument holds if the switching is applied after the path crosses γ. However, if the control switches before the γ curve is reached, there are many admissible controls which get the system to the origin. If these controls are checked by substituting into (4.20) they are found to consume more fuel than the normal control sequence because F^* is a function of $\underline{\xi}$ only.

The control $u=0$ will force the system to the origin with $T = \infty$ (A is singular) and strictly speaking, such a u is not optimal. Thus the theorem is proved.

§5. Multiple-Input Systems

The necessary condition for singularity to occur in multiple-input systems is identical with that of the single-input systems.

Theorem 5 may be reworded to include multiple-input systems. A conjecture one will make is that singular controls cannot be optimal. A general proof of this statement is tedious if not impossible. It is asserted that there are no adjoint initial conditions that give an optimum by PMP.

Assume A to be singular and in the diagonal form, then one of the co-state variables is constant. Moreover, the components of $\underline{p}$ are strictly monotone increasing or decreasing unless the initial conditions are chosen such that one of p_i's is identically equal to plus or minus one, and the remaining p_i's = constant $\neq 0$.

For simplicity, consider a system of order 2. In order to satisfy the Hamiltonian which must be identically zero along an extremal, p_1 and p_2 can only have two values, namely $c \neq 0$ or ± 1 identically. These values are dictated by the structure of A, depending on how the state variables are defined.

Assume B is an identity matrix. If, for example the initial conditions are chosen such that $p_2 \equiv 1$ and $p_1 = c \neq 1$, then it is clear that u_1 is constant and cannot change sign. The problem may then be viewed as a single-input system which has been shown to be non-optimal. However, if $\underline{\pi}$ is such that the problem becomes totally singular, $\underline{u}$ is arbitrary though the polarity of the controls are known, and $|u_j| \leq 1$, j=1 and 2. If the initial state is in the interior of $X_+ \cup X_-$ the only optimal control is 0 or 1, which has at least two jumps; namely the sequences { 1, 0, 1 } or {1, 0, 1, 0}. To obtain these sequences, p_1 and p_2 would have to change the sign of their slopes at least once. This is impossible in view of Theorem 2 and Theorem 4 and $p_1=p_2=1$. Therefore the target $\underline{0}$ will never be reached. The same reasoning can be applied to the case when $p_1=p_2=-1$ or $p_1=-p_2=\pm 1$.

Another technique, using the Principle of Optimality and its geometric interpretation, may be easily used to verify that singular controls cannot be optimal.

§6. Examples on Singular Fuel-Optimal Problems

EXAMPLE 4

The system described by the following differential equation is considered for fuel optimization.

$$\ddot{x} + a\dot{x} = u \qquad |u| \leqq 1$$

The cost function is

$$J(u) = \int_0^T |u(t)|dt \quad \text{where } T \text{ may be free or fixed.}$$

The equivalent system equation is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -ax_2 + u \quad .$$

The Hamiltonian for this problem is

$$H = |u| + p_1 x_2 - p_2\, a x_2 + p_2 u$$

The co-state variables p_1 and p_2 are the solutions of the canonical equation

$$\dot{p}_1 = 0$$
$$\dot{p}_2 = -p_1 + ap_2$$

and

$$p = \begin{bmatrix} 1 & 0 \\ \frac{1}{a}(1-e^{at}) & e^{at} \end{bmatrix}$$

The control which absolutely minimizes the Hamiltonian is given by

$$\begin{array}{ll} u = 0 & \text{if } |p_2| < 1 \\ u = -\operatorname{sgn}\{p_2\} & \text{if } |p_2| > 1 \\ 0 \leqq u \leqq 1 & \text{if } p_2 = -1 \\ -1 \leqq u \leqq 0 & \text{if } p_2 = +1 \end{array}$$

For u=0, the curve α that goes through the origin is the equation $x_2 + ax_1 = 0$. The γ curve as defined in §4.4 is deduced to be a function of x_1 and x_2 and is given as follows:

$$\gamma = \{(x_1, x_2): x_1 = -\frac{x_2}{a} + \operatorname{sgn}\{x_2\} \frac{1}{a^2}\log(1+ a|x_2|)\}$$

These curves are shown in Fig. 9.

Case 1: Normal Solution--(T free)

A unique solution exists for initial states in the region bounded by the curves γ and α (shaded area). The optimal sequence is {0,1} for $x_2 < 0$ and {0,-1} for $x_2 > 0$. For initial states outside the shaded area a control sequence {-1,0} is applied, switching at the instant the curve α is reached. The response time to the origin is infinite. Strictly speaking, a fuel-optimal solution does not exist; however an ε- fuel-optimal solution exists.

Case 2 : Singular Solution --(T free)

Since A is a singular matrix, the problem is necessarily singular.

If $|\pi_1| = a$ and $\pi_2 = \frac{\pi_1}{a}$, then $|p_2| \equiv 1 \; \forall \; t \in [0,T]$. The control u_s defined by (4.25) is a candidate for fuel optimal control. The sequence $\{0, -\text{sgn}\{p_2\}v(t)\}$ for $\underline{x}$ belonging to the shaded area will take the system to the origin. In fact, there appears that many admissible controls exist. Investigations are carried out to check which control, among those that are admissible, uses the least fuel. The ideal minimum fuel $F^*(\underline{x})$ is $|x_2 + ax_1|$ (from the system equation). It is clear that none of these singular extremal controls consumes less fuel than the normal control. Hence singular solutions are non-optimal.

Case 3: T fixed a priori.

The possiblity of singular controls being extremals has been excluded by Theorem 5. The purpose of presenting this example is to verify the theorem.

The detailed solution to this problem is given in Snow's paper [32]. Fig 10 depicts the optimal trajectory for T = 1 second. It is shown that even though the trajectory enters the singular region (unshaded area) the optimal control is uniquely defined and the solution is therefore essentially normal.

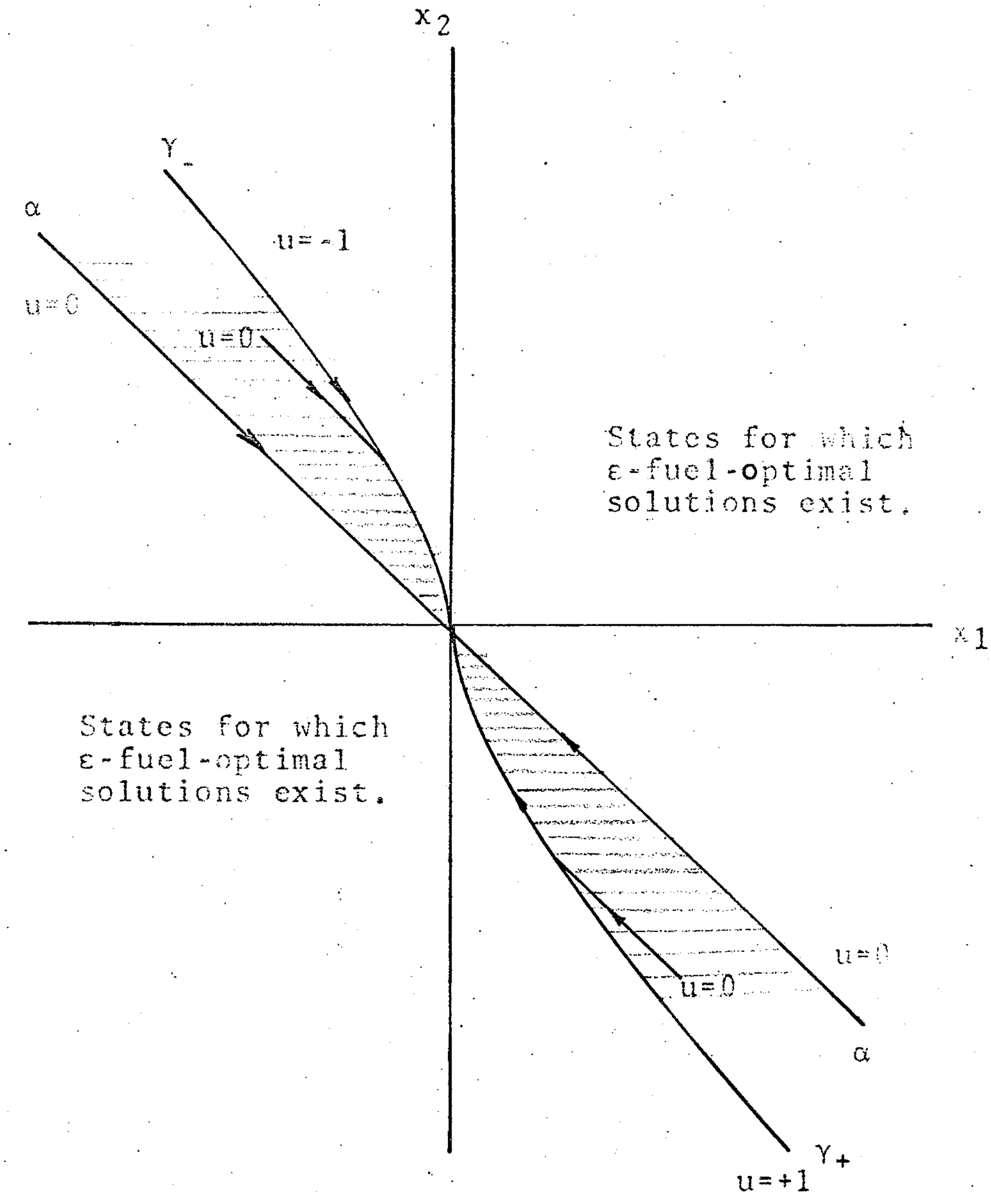


Fig. 9 Optimal trajectories for Example 4.

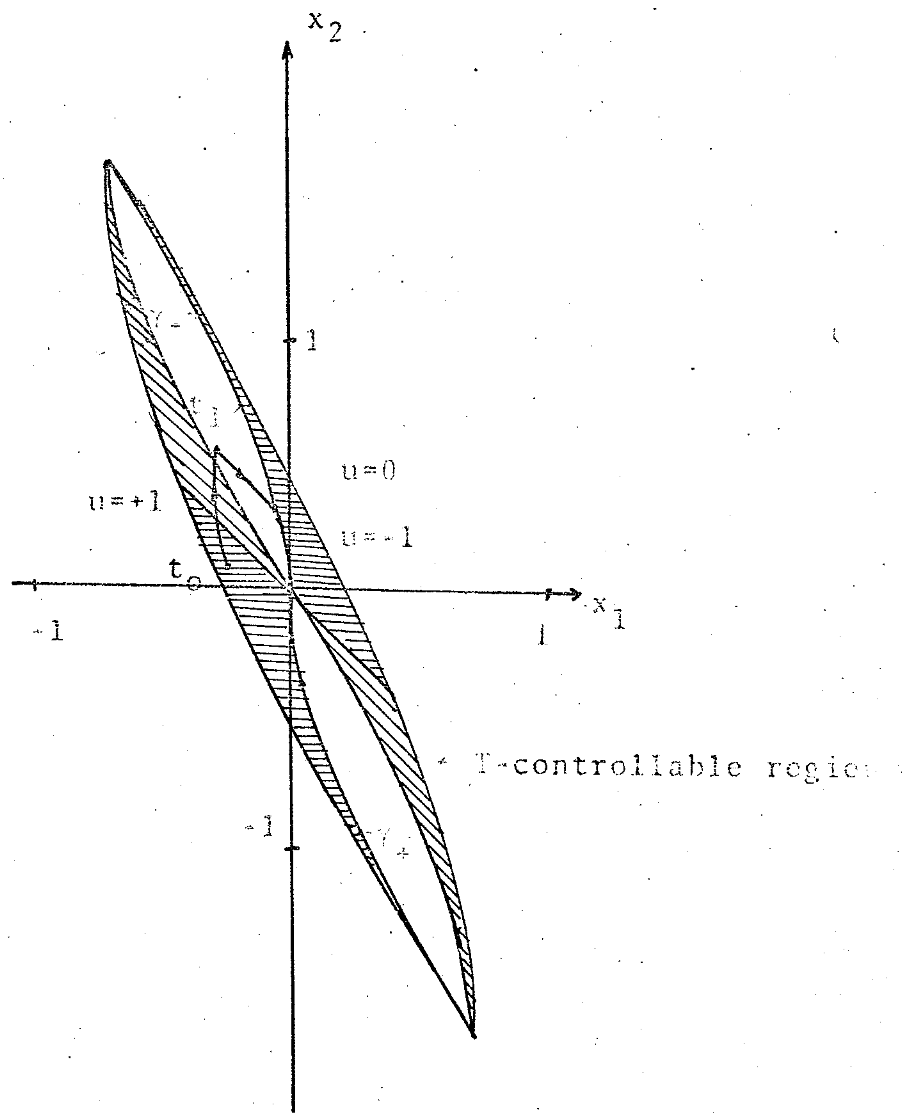


Fig.10 Optimal fuel control for T = 1 sec. and a =1. Unshaded regions are the singular states.

EXAMPLE 5

The system considered for time-optimal case in Example 3 will be solved for fuel-optimal solution. The cost function is $J(\underline{u}) = \int_0^T (|u_1| + |u_2|)\, dt$ with the response time unspecified.

The Hamiltonian for this problem is

$$H = |u_1| + |u_2| + p_1 x_2 - p_2 x_2 + p_1 u_2 + p_2 u_1$$

The controls which absolutely minimize the Hamiltonian are given by

$$u_1 = -\text{dez}\,\{p_2\}$$

$$u_2 = -\text{dez}\,\{p_1\}$$

where $p_1 = \pi_1$ and $p_2 = (1-e^t)\pi_1 + e^t \pi_2$.

If $|\pi_1| = 1$, then the problem is singular. The Hamiltonian must be identically zero along an extremal. The substitution of $|\pi_1| \equiv 1$ into H gives

$$p_2 = \frac{|u_1| + x_2 \operatorname{sgn}\{p_1\}}{x_2 - u_1}$$

For $u_1 = 0$, p_2 is identically sgn $\{p_1\}$ which implies that the problem is totally singular. Similarly, for $u_1 = -1$ and $p_1 = +1 \Rightarrow p_2 = 1$ and for $u_1 = +1$ and $p_1 = -1 \Rightarrow p_2 = -1$, total singularity is possible. However, because $p_2 = (1-e^t)\pi_1 + \pi_2 e^t$, it is either monotone increasing or decreasing depending on the signs of π_1 and π_2. Therefore these sub-cases are impossible unless $\pi_1 = \pi_2 = \pm 1$. For $u_1 = 1$ and $p_1 = +1$, PMP requires $p_2 < -1$ and consquently the above relation requires that, $0 < x_2 < +1$ if the necessary conditions of PMP are to be satisfied. For $u_1=-1$ and $|p_1|=-1$, it is required that, $-1 < x_2 < 0$. Since u_1 is specified and is constant in this restricted region, i.e. $|x_2| < 1$, the problem may be viewed as a single-input system which is shown to be singular non-optimal in Example 4.

Now, if $\pi_1 = \pi_2 = \pm 1$, then the problem becomes totally singular. It seems there exist many controls such that $|u_1| \leq 1$ and $|u_2| \leq 1$, that will drive the system to the origin. However these controls use more fuel than the normal control sequences as shown in Fig. 11. Thus they are non-optimal.

On the other hand, if $\pi_1, \pi_2 \neq \pm 1$, then the solution is non-singular. H=0 yields $|u_1| + |u_2| + \pi_2 u_1 + \pi_1 u_2 = 0$. Thus u_1 and u_2 are constant and cannot change sign. Therefore, unless the initial conditions are on the switching curve corresponds to $u_1 = u_2 = \mp 1$, there exist no controls that take the system to $\underline{0}$.

It is observed that u_2 can switch at most twice, for the normal case where $\pi_1 \neq \pi_2 \neq \pm 1$. The uniqueness of the normal solution follows the condition of normality [2].

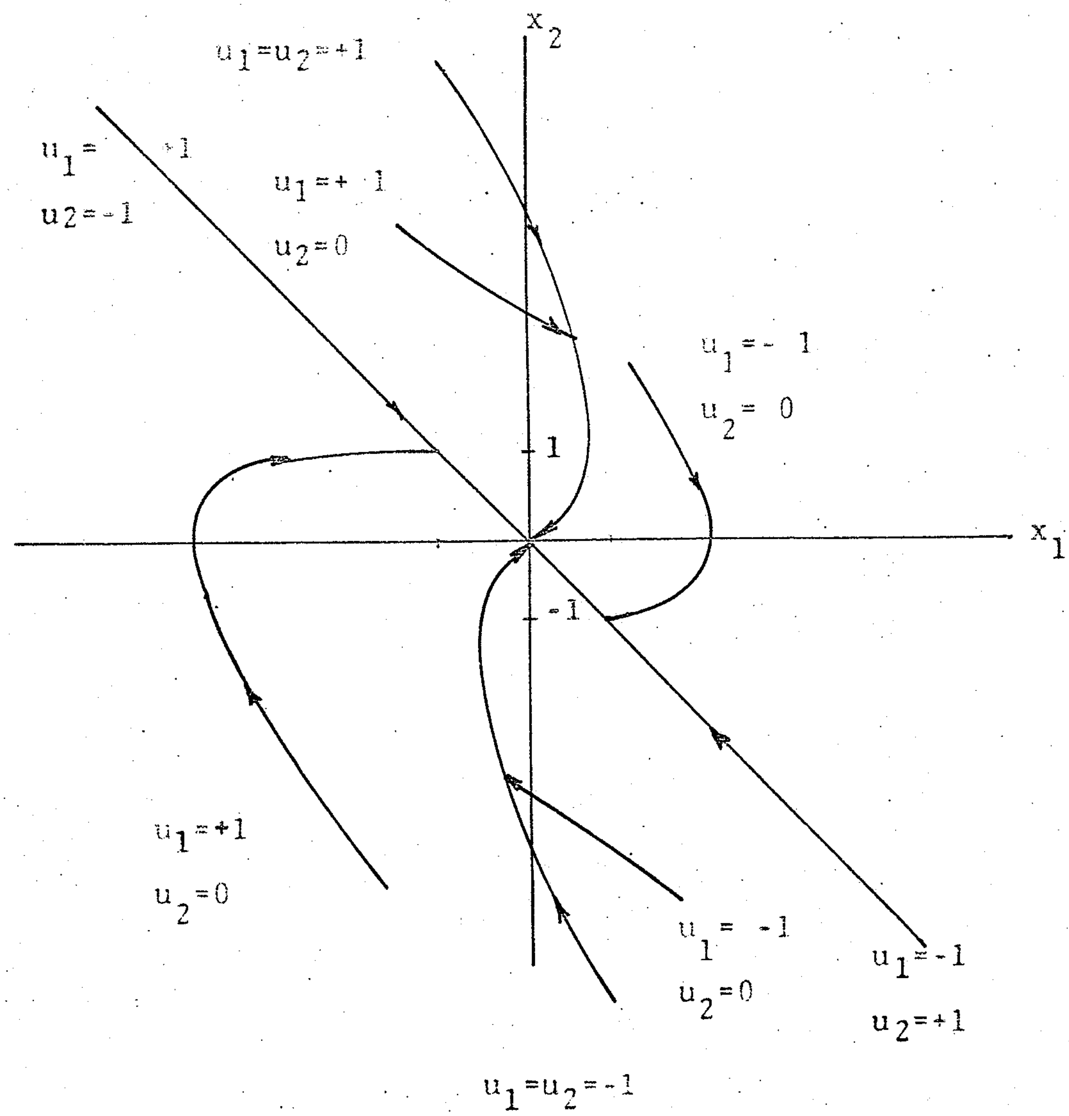


Fig.11 Fuel-optimal solutions for Example 5 for normal case.

§7. Singular Fuel-Optimal Problems for Non-Linear Systems

A characterization of a class of linear singular fuel optimal problems has been examined. It has been demonstrated by various examples, that for some problems, though there exist initial states for which the problems are essentially singular, singular solution cannot be optimal.

This section explores some nonlinear systems for which singular extremals may be optimal. The derivation of a necessary condition for singularity for nonlinear systems is, in general, tedious and laborious. It suffices to prove here the necessary condition for the singular fuel-optimal problem to a class of second order nonlinear systems,in particular, one described by the Lienard's equation:

(4.26) $\ddot{x} + f(x,\dot{x}) = u$ or equivalently in the state space form of

(4.27)
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -f(x_1,x_2) + u$$

Questions regarding existence of an optimal solution for second order nonlinear systems have been treated in detail by Lee and Markus [25] in their paper concerning nonlinear time-optimal problems. Some of their theorems are adapted and modified to extend to fuel-optimal problems.

It is assumed that $f(x,\dot{x})$ satisfies the following:

(a) $f(0,0) = 0$

(b) $\frac{\partial f}{\partial x}(x,\dot{x}) > 0$ and $\frac{\partial f}{\partial \dot{x}}(x,\dot{x}) > 0$ in R^2

If condition (a) is satisfied, then the domain of null controllability T is an open subset of R^2. If conditions (a) and (b) are satisfied, then the domain of null controllability T is the whole state plane R^2 and moreover, there exists an optimal $u \in \Omega$ which steers $\underline{\xi} \in R^2$ to $\underline{0}$. The proof of these assertions may be found in [23].

§ 8. Necessary Conditions for Singularity

The singular fuel-optimal problem for nonlinear systems was first reported by Athans and Canon [3]. Their technique is extended to the problem posed in this section.

The adjoint system induced by the PMP is

(4.28) $$\dot{p}_1 = f_{x_1} p_2$$
$$\dot{p}_2 = -p_1 + f_{x_2} p_2$$

where f_{x_1} and f_{x_2} denote the partial differentiation of $f(x_1,x_2)$ with respect to x_1 and x_2.

The optimal control which absolutely minimizes H is

(4.29) $$u^* = -\mathrm{dez}\{p_2\} \qquad |p_2| \neq 1$$

If $|p_2| \equiv 1 \ \forall \ t \in [t_1,t_2]$, u^* is not uniquely defined and the problem is said to be singular.

Assuming $p_2 \equiv 1 \ \forall \ t \in [t_1,t_2]$ implies

(4.30) $$-p_1 + p_2 f_{x_2} = 0 \qquad \forall \ t \in [t_1,t_2]$$

Differentiating (4.30), one obtains

(4.31) $$-\dot{p}_1 + \dot{p}_2 f_{x_2} + p_2 \dot{f}_{x_2} = 0 \qquad \forall \ t \in [t_1,t_2]$$

Substituting (4.27) and (4.28) in (4.31), one obtains

(4.32)
$$-p_1 f_{x_2} + p_2 \, [f^2_{x_2} - f_{x_1} + x_2 f_{x_2 x_1} - f f_{x_2 x_2}] + p_2 f_{x_2 x_2} u = 0$$

if $p_2 f_{x_2 x_2} \neq 0$, then

(4.33)
$$u = \frac{p_1 f_{x_2} - p_2 [f^2_{x_2} - f_{x_1} + x_2 f_{x_2 x_1} - f f_{x_2 x_2}]}{p_2 f_{x_2 x_2}}$$

If, on the other hand, $p_2 f_{x_2 x_2} = 0$, and since $u \neq 0$, then (4.32) reduces to

(4.34)
$$p_1 f_{x_2} - p_2 [f^2_{x_2} - f_{x_1} + x_2 f_{x_2 x_1}] = 0$$

Taking (4.30) and (4.34) and because these relations are linear in p_i, one may write in compact form, the following :

(4.35)
$$\begin{bmatrix} -1 & f_{x_2} \\ -f_{x_2} & f^2_{x_2} + x_2 f_{x_2 x_1} - f_{x_1} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = 0$$

or in vector form

(4.36) $\quad G(\underline{x})\underline{p} = 0$

To be optimal, PMP asserts that $\underline{p}$ must be a non-zero vector. For non-trivial solution of $\underline{p}$, (3.36) indicates that $\underline{p}$ must belong to the null space of $G(\underline{x})$,implying that $G(\underline{x})$ is a singular matrix.

Suppose that the time differentiations lead to an equation in which the coefficient of u is non-zero as assumed in (4.33).In such cases (4.33) provides a necessary condition on the extremal singular control; in essence, a relation between u, p_1, p_2 and x_1, x_2 has been found. Eq. (4.33) is admissible only if it satisfies the magnitude constraint $|\underline{u}| \leq 1$.

In any given problem, all the relations must be examined to see whether or not they are satisfied; if any of the relations are violated, then this represents a violation of the necessary conditions, and so singular controls cannot occur. Theorem 7 establishes the form of the nonlinear element necessary for singularity to occur.

THEOREM 7

A necessary condition for (4.27) to be singular fuel-optimal is that f is of the form :

(4.37) $\quad f = x_1 x_2 + g(x_2)$

where $g(x_2)$ is function of x_2.

Proof: $G(\underline{x})$ is singular implies

(4.38) $\quad f_{x_1} - x_2 f_{x_2 x_1} = 0$

A solution to this partial differential equation is

(4.39) $\quad f = x_1 x_2 + g(x_2)$

In general, it is not possible to prove that (4.37) is also a sufficient condition for the fuel-optimal problem to be singular. Since sufficiency conditions imply the existence of a solution, it is still required to show that singular extremals exist in this formulation of fuel optimization. Using again the min H function, a sequence of lemmas and theorems is presented to test for the possibility of singular extremals to occur for the cost functions (4.1) and (4.2).

LEMMA 1

Assume $f(x_1,x_2) = x_1x_2$, then there exist no singular extremals for the cost function $J(u) = \int_0^T (k + |u|)\,dt$, $k=c>0$.

Proof: The Hamiltonian for this problem is

$$H = k + |u| + p_1x_2 - p_2x_1x_2 + p_2u$$

Since T is unspecified $H^*=0$ along an extremal.

The adjoint system is

$$\dot{p}_1 = p_2x_2$$

$$\dot{p}_2 = -p_1 + p_2x_1$$

By hypothesis, $|p_2| \equiv 1 \;\forall\; t \in [t_1,t_2] \Rightarrow p_1 = p_2x_1 \;\forall\; t \in [t_1,t_2]$

Substituting this relation into H^*, one would have

$$H^* = k > 0, \quad \text{a contradiction}$$

therefore the problem is normal unless $k=0$

Lemma 1 shows that if $k=0$, then there may exist singular extremals. To prove that such singular extremals cannot be fuel-optimal, the following lemmas are verified.

LEMMA 2

If $|p_2| \equiv 1 \;\forall\; t \in [t_1,t_2]$, the control u_s defined by (4.25) is a candidate for fuel-optimal control.

Proof: $|p_2| \equiv 1 \Rightarrow p_1 = \text{sgn}\,\{p_2\}\, x_1 \;\forall\; t \in [t_1,t_2]$.

Therefore $H^* = |v(t)| + \text{sgn}\,\{p_2\}x_1x_2 - \text{sgn}\,\{p_2\}x_1x_2 - |p_2|v(t)$

$$= 0 \quad \forall\; t \in [t_1,t_2]$$

Thus the control (4.25) can be extremal.

LEMMA 3

Denote $F^*(\xi_1,\xi_2)$ the minimum fuel (if it exists) required to steer $\underline{\xi}$ to $\underline{0}$ in some (unspecified time) T, then the minimum $F^*(\xi_1,\xi_2)$ satisfies the relation

(4.39) $$F^*(\xi_1,\xi_2) \geqq \left|\frac{1}{2}\xi_1^2 + \xi_2\right|$$

It follows that if there exists a control $u^*(t)$ that forces $\underline{\xi}$ to $\underline{0}$ and requires fuel $\left|\frac{1}{2}\xi_1^2 + \xi_2\right|$, then $u^*(t)$ is optimal and $F^*(\xi_1,\xi_2) = \left|\frac{1}{2}\xi_1^2 + \xi_2\right|$.

Proof: From the system equation

(4.40) $$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -x_1x_2 + u$$

it is deduced that $u = \dot{x}_2 + x_1x_2$

Therefore

(4.41) $$\int_0^T u(t)dt = x_2 - \xi_2 + \frac{1}{2}(x_1^2 - \xi_1^2)$$

Since at $t = T, \underline{x}(T) = 0$, it follows that

(4.42) $$\left|\int_0^T u(t)\,dt\right| = \left|\frac{1}{2}\xi_1^2 + \xi_2\right|$$

$$F^*(\xi_1,\xi_2) = \int_0^T |u(t)|dt \geqq \left|\frac{1}{2}\xi_1^2 + \xi_2\right|$$

If this is so, it is necessary that

$$F^*(\xi_1,\xi_2) = \left|\frac{1}{2}\xi_1^2 + \xi_2\right| \text{ for optimality.}$$

LEMMA 4

$F^*(\xi_1,\xi_2)$ is the greatest lower bound on the fuel, hence it is the minimum fuel.

Proof: To establish this, it is necessary that the minimum fuel as a function of state is a solution to the Hamilton- Jacobi equation (see Athans and Falb [4] section 5-20).

In other words, the function $F^*(x_1,x_2) = |\frac{1}{2} x_1^2 + x_2|$ is a solution to

$$(4.43) \qquad \frac{\partial F^*}{\partial t} + |u^*(\underline{t})| + x_2^* \frac{\partial F^*}{\partial x_1^*} - x_1^* x_2^* \frac{\partial F^*}{\partial x_2^*} + u^*(t) \frac{\partial F^*}{\partial x_2^*} = 0$$

If $\underline{x}=0$, then $F^*(0,0) = 0$ and $u^*(t) = 0$, so that (4.43) is satisfied. If $\underline{x} \neq 0$, then

$$(4.44) \qquad \frac{\partial F^*}{\partial t} = 0 \qquad \frac{\partial F^*}{\partial x_1^*} = x_1^* \qquad \frac{\partial F^*}{\partial x_2^*} = \mathrm{sgn}\{x_2^*\}$$

$$u^*(t) = v(t) \qquad \text{and } u^*(t) = -\mathrm{sgn}\{x_2^*\}\, v(t)$$

Substituting (4.44) into (4.43), one would obtain for $x_2^* > 0$

$$|v(t)| + x_1^* x_2^* - \mathrm{sgn}\{x_2^*\}\, x_1^* x_2^* - \mathrm{sgn}\{x_2^*\}\, \mathrm{sgn}\{x_2^*\}\, v(t) = 0$$

Thus it is verified that $F^*(x_1,x_2) = |\frac{1}{2} x_1^2 + x_2|$ is the greatest lower bound on fuel and that $u^*(t)$ is a fuel-optimal solution.

Therefore, if the initial conditions are such that $F^* = 0$ then a control $u(t) \equiv 0$ is to be used to get the system to $\underline{0}$. If $u(t) = 0 \ \forall \ t \in [0,T]$, then the origin cannot be reached except on the curve β (where β is defined as the set of all states satisfying the equation $\frac{1}{2} x_1^2 + x_2 = 0$). However, as time progresses the system departs from the origin because there are no controls that will maintain it at the origin. This is obvious, by rewriting the equation as follows:

$$(4.45) \qquad \frac{dx_2}{dx_1} = \frac{-x_1 x_2 + u}{x_2}$$

For $u=0$ and $x_1 = x_2 = 0$, the slope is zero. Thus the trajectory will move away if perturbed. (see Fig. 12). Therefore, a minimum fuel does not exist for all initial states such that $\frac{1}{2}\xi_1^2 + \xi_2 = 0$. As can be seen from Fig. 15, there are additional states for which, strictly speaking, a fuel-optimal solution does not exist.

LEMMA 5

If $p_1 \neq \text{sgn}\{p_2\}x_1$, then $|p_2| = 1$ at most at two isolated times. The eight control sequences

(4.46) $\{-1\}, \{+1\}, \{-1,0\}, \{+1,0\}, \{0,-1\}, \{0,+1\}, \{-1,0,+1\}, \{+1,0,-1\}$

can be candidates for the fuel-optimal control.

Proof: The solutions of (4.27), (4.28) and (4.29) give the sequences of (4.46)

From these lemmas, it is now possible to prove the statement that for the cost function (4.2) with $k = 0$, there are no singular extremals which are optimal.

THEOREM 8

If $f(x_1, x_2) = x_1 x_2$ and the cost function is $\int_0^T |u(t)|dt$ with T unspecified, then singular controls cannot be optimal.

Proof: Denote the γ_+ curve to be the locus of all states which can be forced to the origin by the control $u=+1$ and also denote the γ_- curve to be the locus of all states which can be forced to the origin by the control $u=-1$, in positive time. (see Fig. 15)

Suppose $p_1 \neq \operatorname{sgn}\{p_2\}x_1$, so that Lemma 5 holds; then among the control sequences of (4.46), only the control sequence {+1} will force the state $(\xi_1,\xi_2) \in \gamma_+$ to the origin. If Lemma 2 holds, then the control u(t) must be given by (4.25). Further, suppose that $x_2'(t)$ and $x_1'(t)$ are the solutions of (4.40) with the initial state $(\xi_1,\xi_2) \in \gamma_+$ and with the control given by (4.25). Clearly, from (4.41), $x_2'(t)$ can be written as

$$(4.47) \qquad x_2'(t) = \xi_2 - \tfrac{1}{2}(x_1'(t)^2 - \xi_1^2) + \int_0^t [-\operatorname{sgn}\{p_2\}v(\tau)]\,d\tau$$

However, $\dot{x}_1' = x_2'$

and therefore x_1' may be expressed as

$$(4.48) \qquad x_1'(t) = \psi(\underline{\xi},t) + \int_0^t d\tau \int_0^\tau [-\operatorname{sgn}\{p_2\}v(\alpha)]\,d\alpha$$

where $\psi(\underline{\xi},t)$ is some function of $\underline{\xi}$ and t

Similarly the response due to control u=+1 can be written as

$$(4.49) \qquad x_1(t) = \psi(\underline{\xi},t) + \int_0^t d\tau \int_0^\tau 1\,d\alpha$$

From (4.48) and (4.49), it follows that

$$(4.50) \qquad x_1(t) - x_1'(t) = \int_0^t d\tau \int_0^\tau [1 + \operatorname{sgn}\{p_2\}v(\alpha)]\,d\alpha \geqq 0$$

which means that the trajectory generated by the control of (4.25) will always be to the left of γ_+, and so it will miss the origin. By process of elimination, it is concluded that u=+1 is the only control sequence that gets the sytsem to $\underline{0}$ for $(\xi_1,\xi_2) \in \gamma_+$. In view of Lemmas 4 and 5, u=+1 is the fuel-optimal solution.

If $\underline{\xi}$ is in the region bounded by γ_+ and β, there are many controls such that, $0 \leqq u(t) \leqq +1$ will get the system to the origin. However, by Lemma 4none of these many controls consumes less fuel than the sequence $\{0,+1\}$. The same is true for $(\xi_1,\xi_2) \varepsilon \gamma_-$ and for (ξ_1,ξ_2) in the region bounded by γ_- and β. For (ξ_1,ξ_2) exterior to the region bounded by the curves γ and β $(\gamma=\gamma_+ \cup \gamma_-)$, there exist no fuel-optimal solutions; however an ε-optimal solution exists.

The optimal solution is shown in Fig. 15. It should be remarked that the effect of this nonlinear element $f(x_1,x_2)= x_1x_2$ can be best visualized by comparing the system with the linear double integral plant solved in [4].

Thus, it has been shown that singular extremal controls for the cost function $\int_0^T |u(t)|\, dt$ with T free are non-optimal.

Lemma 1 has shown that for $f(x_1,x_2)= x_1x_2$ and the cost function (4.2), singular extremals do not exist. The next lemma proves that for the nonlinear element $f(x_1,x_2)= x_1x_2+\frac{1}{m}x_2^m$ the same conclusion can be reached.

LEMMA 6.

If $f(x_1,x_2) = x_1x_2 + \frac{1}{m} x_2^m$ and the cost function (4.2), then there exist no singular extremals; and so the problem is normal.

Proof: Mimic the proof of Lemma 1 to show that $H^*= k >0$, so the lemma is proved.

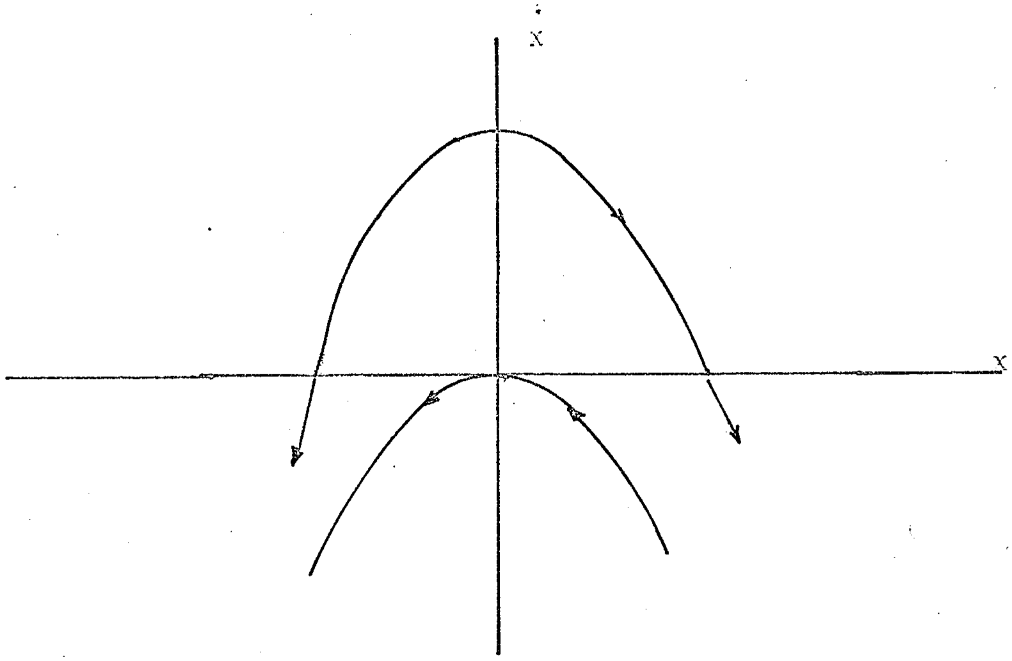


Fig. 12. Trajectories for the differential equation $\ddot{x} + x\dot{x} = 0$

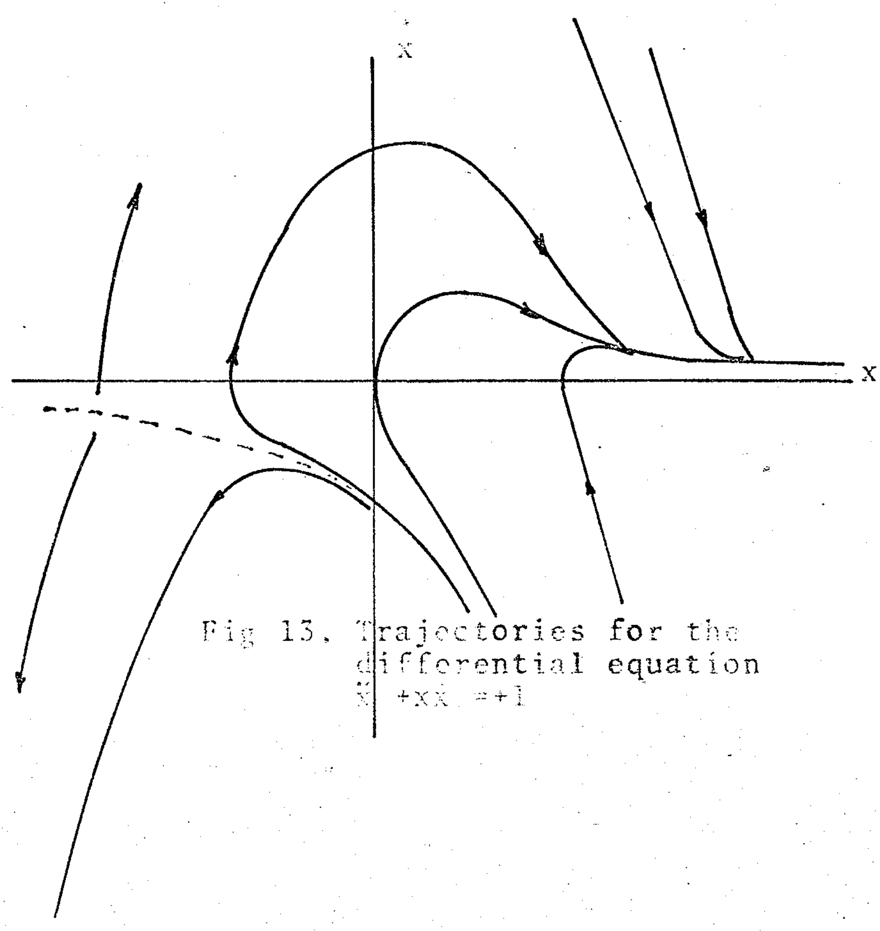


Fig 13. Trajectories for the differential equation $\ddot{x} + x\dot{x} = +1$

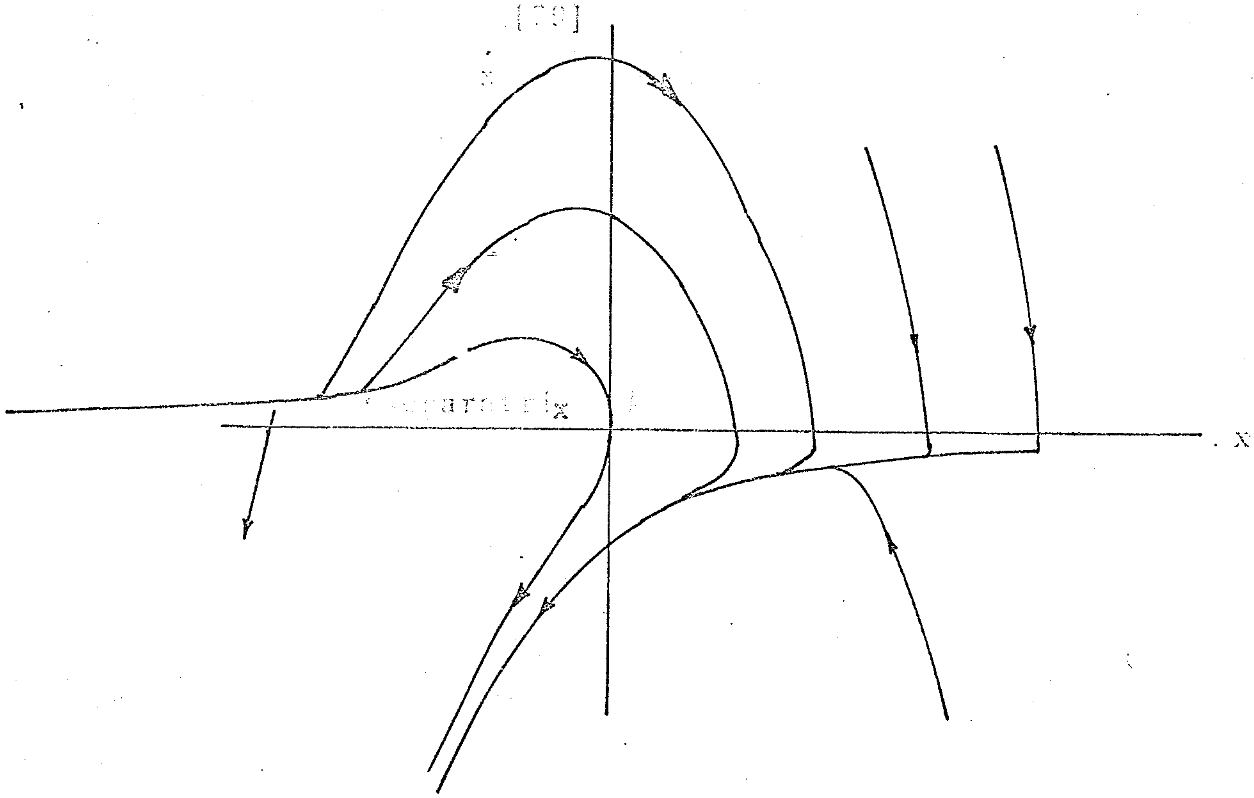


Fig. 14 Trajectories for the differential Equation $\ddot{x} + x\dot{x} = -1$

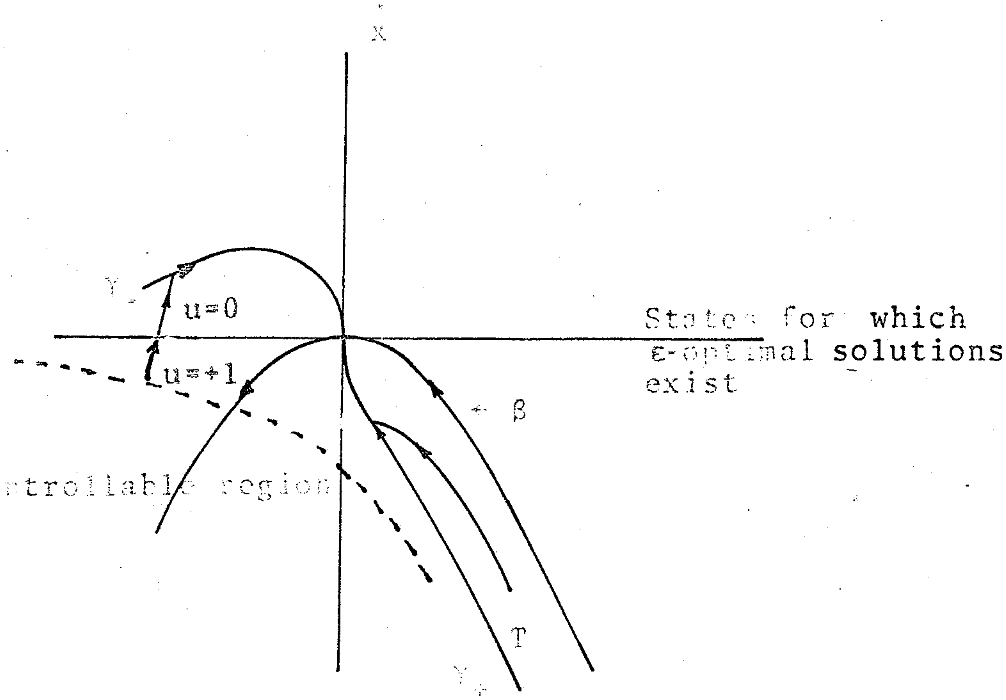


Fig.15 Optimal trajectories for the plant described by the differential equation $\ddot{x} - x^2 = u$. T is the domain of null controllability.

§ 9. Minimization of Time-Weighted Fuel for Plants Subjected to Nonlinear Friction Forces

The material presented so far indicated that singular controls cannot be optimal for plants subjected to linear friction forces or nonlinear drag of the form $f = x_1 x_2 + \frac{1}{m} x_2^m$. This section deals exclusively with plants subjected to nonlinear friction forces, which are proportional to velocity only. In general, the motion of the system is governed by the differential equation

$$(4.51) \qquad \ddot{x} + \frac{1}{m}\dot{x}^m = u \qquad |u| \leq 1$$

It is assumed that the drag is always in opposition to the direction of the motion. The singularity established in (4.35) is automatically satisfied. It is shown here that singular controls may be optimal.

THEOREM 9

Consider the control system described by (4.51) and the cost function $\int_0^T (k + |u(t)|)dt$, with T unspecified and $k > 0$. If $k \leq m-1$ and if $|p_2| \equiv 1 \ \forall \ t \in [t_1, t_2]$, the singular control u_s given by

$$(4.52) \qquad u_s = - \frac{k}{m-1} \operatorname{sgn}\{p_2\}$$

is a candidate for fuel-optimal control.

Proof: The equivalent system for (4.52) is

$$(4.53) \qquad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{1}{m} x_2^m + u \end{aligned}$$

The Hamiltonian function is

(4.54) $H = k + |u| + p_1 x_2 - p_2 \frac{1}{m} x_2^m + p_2 u$

The optimal control via the PMP is

(4.54) $u^* = -dez\ \{p_2\}$

and the co-state variables are the solutions of the canonical

(4.56)
$$\dot{p}_1 = 0$$
$$\dot{p}_2 = -p_1 + p_2 x_2^{m-1}$$

Singularity condition occurs if there is a subinterval $t \varepsilon [t_1, t_2]$ $\varepsilon [0,T]$ such that $|p_2| \equiv 1$.

Assume $|p_2| \equiv 1$, $\forall\ t\ \varepsilon\ [t_1, t_2]$, then $\dot{p}_2 = 0$. This implies that

(4.57) $p_1 = p_2\ x_2^{m-1} \qquad \forall\ t\ \varepsilon\ [t_1, t_2]$

Since p_1 is constant, (4.57) implies that x_2 is constant which in turn implies that $\dot{x}_2 = 0$ for all $t\ \varepsilon [t_1,\ t_2]$. If $\dot{x}_2 = 0$, then (4.53) implies that the control u is a constant given by

(4.58) $u_s = \frac{1}{m} x_2^m \qquad \forall\ t\ \varepsilon [t_1, t_2]$

Substituting (4.57) into (4.54), one obtains

(4.59)
$$x_2 = [\frac{-km}{m-1} \quad sgn\{p_2\}]^{\frac{1}{m}}$$

It follows then

(4.60)
$$u_s = -\frac{k}{m-1} \quad sgn\{\ p_2\ \}$$

If u_s is to be a candidate for fuel-optimal control, then

(4.61) $k \leqq m-1$.

If $k > m-1$, the singular control cannot be optimal because $|u| \leqq 1$ which contradicts the magnitude constraint on u.

The synthesis problem may be facilitated, if one defines

$$\omega = -\operatorname{sgn}\{\dot{p}_2\}\Big|_{|p_2|=1} \tag{4.62}$$

If for example, $\omega = 1$ at every point in the state space, then as t increases, the control can only switch from $u^* = -1$ to $-u_s$ or to 0. If the control u^* is initially 1, then no switching is allowed. Similar reasoning holds for $\omega = -1$.

Athans and Canon [3] have proved that p_2 is a continuous function $\forall\, t \in [0,T]$ and p_2 can be zero at most once. They also established that the optimal sequences to the origin must be of the type $\{\cdots\cdots, 0, 1\}$ or of the type $\{\cdots\cdots\cdots\cdots, 0, -1\}$

As a further demonstration that linear friction forces play a very insignificant role in the structure of singular fuel-optimal solutions, Example 6 is presented for comparison with the example solved by Athans and Canon [3].

In general, the existence of singular controls for fuel-optimal problems is an inherent property of nonlinear systems. It appears, therefore, that the exclusion of singular controls in the hypothesis of the theorems is unjustifiable. The possibility of singular controls must also be investigated for any given nonlinear problem for fuel minimization.

§10. An Example

EXAMPLE 6

Consider the nonlinear system with linear friction and quadratic drag. The dynamic equation describing the system is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -ax_2|x_2| - bx_2 + u \quad \text{where } a \text{ and } b > 0 \text{ and } |u| \leqq 1$$

The index of performance is $J(u) = \int_0^T (k + |u(t)|)dt$, $k > 0$ and the terminal time is free. The only difference between this problem and the one solved by Athans and Canon[3] is in the linear friction bx_2. Proceeding along the lines as indicated in Theorem 9, the singular condition implies

$$p_1 = (2\sqrt{ak} + b)\ \text{sgn}\{p_2\}$$
$$x_2 = \sqrt{\frac{k}{a}}\ \text{sgn}\{p_2\}$$
$$\text{sgn}\{p_2\} = \text{sgn}\{x_2\}$$

The control corresponds to singular extremals is given by

$$u_s = \left(k + b\sqrt{\frac{k}{a}}\right)\text{sgn}\{x_2\}$$

Thus it is clear the

$$\left(k + b\sqrt{\frac{k}{a}}\right) \leqq 1.$$

By means of (4. 62), the state space is subdivided into regions corresponding to

$$\omega = -\text{sgn}\{\dot{p}_2\}\Big|_{|p_2|=1} = \begin{cases} -\text{sgn}\{a|x\ | - \frac{k}{x_2}\} & \text{for } p_2 = +1 \\ -\text{sgn}\{-a|x\ | - \frac{k}{x_2}\} & \text{for } p_2 = -1 \end{cases}$$

In each of the regions the allowable switching direction is uniquely defined. It is clear that if $u^* = -1$ at any time while the representative points are in the upper plane $x_2 > 0$, then as t increases $u^* = -1$, as long as $x_2 > 0$. In fact, this control is maintained until $x_2 = -\frac{k}{a}$ as may be seen from the equation. A similar remark holds for the case $u^* = +1$ in the lower half plane $x_2 < 0$.

The optimality of the family of trajectories shown in Fig, 20 is identical with those established in [3].

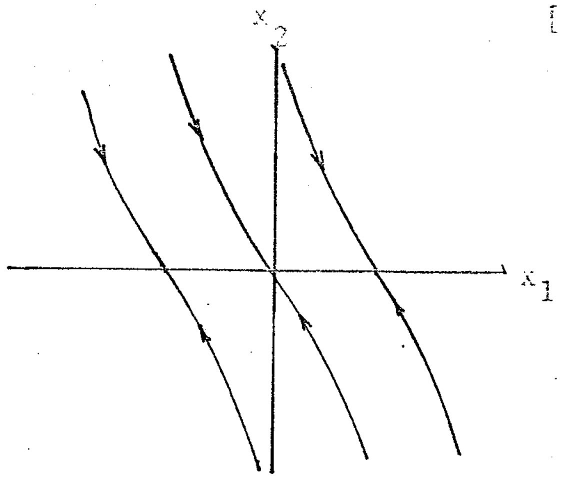


Fig. 16 Trajectories for u=0 in Example 6.

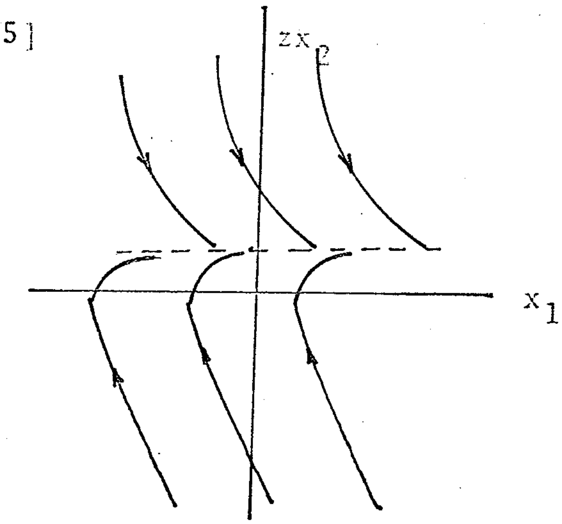


Fig. 17 Trajectories for u=+1 in Example 6

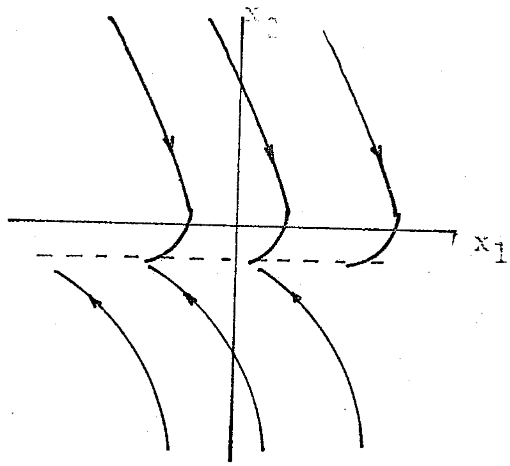


Fig. 18 Trajectories for u=-1 in Example 6.

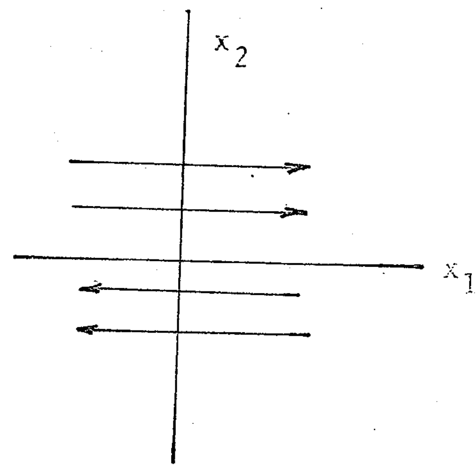


Fig. 19 Trajectories for $u = (k + b\sqrt{k/a})\ \mathrm{sgn}\{x_2\}$ in Example 6.

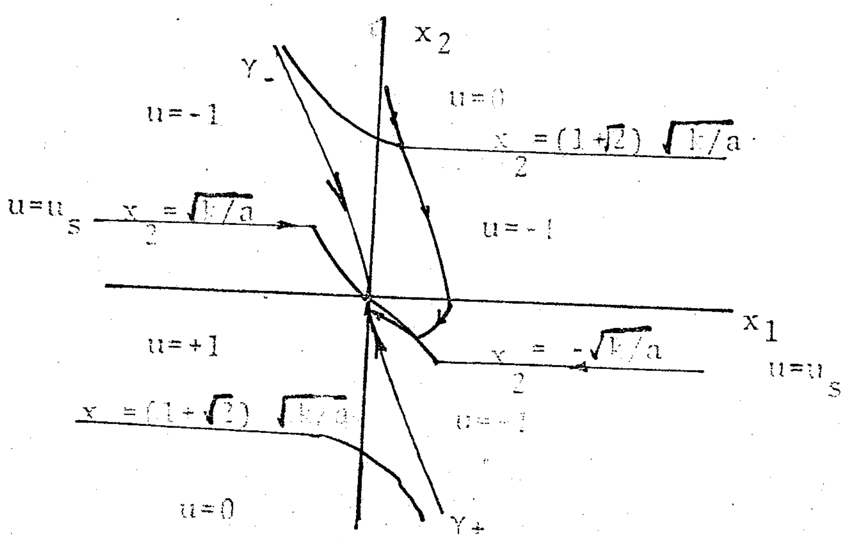


Fig.20 Family of singular and normal trajectories for Example 6. A typical optimal path is shown

CHAPTER 5

CONCLUSIONS AND REMARKS

It has been shown that for a class of optimization of control processes, singularity is an inherent property of the dynamical system and the function or functional to be minimized. In any given problem, the possibility of singular solution(s) should always be investigated especially in problems where the control variables appear linearly in the system equation or the performance criterion.

The optimum solutions for minimum time problems are obtained. For single-input systems, there exists no solution for some initial states (the uncontrollable states) and if all controllable states are constrained in the hyperplane $\langle p, x \rangle = 0$, then a unique bang-bang solution is found. The existence of several control laws is characteristic of multiple-input systems when the systems do not satisfy the normality conditions. For some plants, there exist initial states for which the problems are essentially normal, in which case, the solution is unique bang-bang control.

The considerations given in the fuel-optimal problem lead to the conclusion that for nonlinear systems, singular optimal controls may appear frequently in contradistinction to their non-occurrence in linear time invariant systems with real eigenvalues — even though there are initial states for which the problem is singular.

In conclusion, it is remarked that one cannot state a priori that singularity condition necessarily implies that the singular arc or subarc forms a part of the optimal solution, especially in fuel-optimal problems.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Athanassiades, M., and O. J. M. Smith, " Theory and Design of High Order Bang-Bang Control Systems," IRE Trans. Auto. Control, vol AC-6, pp. 125-134, 1961.

2. Athans, M., "On Uniqueness of Extremal Controls for a Class of Minimum Fuel Problems," IEEE Trans, Auto. Control, vol AC-11, pp. 660-668, 1966.

3. __________, and M.D. Canon, " On the Fuel-Optimal Singular Control of Non-Linear Second Order Systems," IEEE Trans. Auto. Control, vol AC-9, pp. 360-370, 1964.

4. __________, and P. L. Falb, "Optimal Control;", M cGraw-Hill Book Company, Inc., New York, 1966.

5. __________, and __________. "Time-Optimal Control for Plants with Numerator Dynamics," IRE Trans. Auto. Control, vol AC-7, pp. 47, 1962.

6. Bellman, R., I. Glicksberg, and O. Cross, "On Bang-Bang Control Problem," Quart. Appl. Math., vol.14, pp. 11-18, 1956.

7. Gibson, J. E., "Non-Linear Automatic Control", McGraw-Hill Book Company, Inc., New York, 1963.

8. Goh, B. D., "Necessary Conditions for Singular Extremals Involving Multiple Control variables," JSIAM Control, ser. A, vol 4, pp. 716-731, 1966.

9. __________, "Optimal Singular Control for Multi-Input Linear Systems," J. Math. Anal. Appl., vol, 20, pp. 534-539, 1967.

10. Halkin, H., "A Generalization of LaSalle's "Bang-Bang" Principle," J. SIAM Control, ser. A, vol, 2, pp. 199-202, 1965.

11. Haynes, G. W., " On the Optimality of a Totally Singular Vector Control: An Extension of Green's Theorem Approach to Higher Dimension," J. SIAM Control ser. A, vol, 4, pp. 662-677, 1966

12. Hermes, H., "Controllability and the Singular Problem," J. SIAM Control, ser. A, vol, 2, pp. 241-260, 1964.

13. Hermes, H., and G. Haynes, "On the Non-Linear Control Problem with the Control Appearing Linearly," IEEE SIAM Control, ser. A vol, 1, pp. 85-108, 1963.

14. Johnson, C. D. and J. E. Gibson, "Singular Solutions in Problems of Optimal Control," IEEE Trans. Auto., vol AC-8, pp. 4-14, 1963.

15. ____________, "Singular Solutions in the Problems of Optimal Control," in C. T. Leondes (ed), "Advances in Control Systems: Theory and Application," vol. II Academic Press Inc., New York, 1965

16. Kelly, H. J., "A Transformation Approach to Singular Subarcs in Optimal Trajectory and Control Problem," J. SIAM Control, ser. A, vol, 2, pp. 234-240, 1964.

17. ____________, "A Second Variation Test for Singular Extremals," J. SIAM Control, ser. A, vol, 2, pp. 1380-1382, 1964.

18. Kopp, R. E., and H. G. Moyer, "Necessary Conditions for Singular Extremals," J. AIAA , vol, 3, pp. 1439-1444, 1965.

19. Kreindler, E., " Contribution to the Theory of Time-Optimal Control," J. Franklin Inst., vol, 275, pp. 314-344, 1963.

20. LaSalle, J. P., "The Time-Optimal Control Problem," Contributions to Nonlinear Oscillation, Princeton University Press, vol, 6, pp. 1-24, 1959.

21. ____________, "The Bang-Bang Principle," Proc. IFAC Congress, Moscow, 1960, Butterworth & Co., Ltd., London, vol, 1, pp. 493-497.

22. Lee, E. B., "On the Time-Optimal Regulation of Plants with Numerator Dynamics," IRE Trans. Auto. Control, vol, 1, pp. 351-352, 1961.

23. ____________, and L. Markus, " On Necessary and Sufficient Conditions for Time-Optimal Control of Second Order Non-Linear Systems," Proc. IFAC Congress, Basle, 1963, Butterworth & Co., Ltd., London, vol, 2, pp. 300-305.

24. ____________, and ____________, " Foundations of Optimal Control Theory," J. Wiley & Sons Inc., New York, 1967

25. Neustadt, L. W., and B.H. Paiewonsky, "On Synthesis of Optimal Controls," Proc. IFAC Congress, Basle, 1963, Butterworth & Co., London, vol, 2, pp. 292-299.

26. Ogata, L., "State Space Analysis of Control Systems," Prentice Hall, New Jersey, 1966.

27. Oguztoreli, N. N., "Time -Lag Control Systems," Academic Press Inc.,New York, pp. 240-254, 1966

28. Pontryagin, L. S.,,V. G. Boltanskii, R. V. Gamkrelidze, and E. F. Mishchenko, "The Mathematical Theory of of Optimal Processes," InterScience Publishers, 1962.

29. Roher, R. A., ang M. Sobral, Jr., "Optimal Singular Solutions for Linear Multi-input Systems,". J. Basic Eng., vol, 88, pp. 323-328, 1966.

30. Snow, R. D., "Singular Optimal Controls for a Class of Minimum Effort Problems," J. SIAM Control, ser. A, vol, 2, pp. 203-219, 1964

31. Thau, F. E., "Optimal Time Control of Non-Normal Linear Systems," Inter. J. Control, vol, 1, pp. 363-373, 1965.

32. Wonham, W. M., and C. D. Johnson, "Optimal Bang-Bang Control with Quadratic Performance Index," Trans. ASME, ser. D, vol, 86, pp. 107-115, 1964.

33. Zadeh, L. A., and C. A. Desoer, "Linear System Theory: The State Space Approach," McGraw-Hill Book Company, Inc., New York, 1963.

APPENDIX A

The purpose of this appendix is to review the terminology and to present a number of results necessary for the development of the material in the thesis.

LINEAR TRANSFORMATION

DEFINITION: A linear transformation (L.T.) Ξ is a function whose domain is a linear vector space χ and whose range is in a linear vector space Y such that for any x_1 and x_2 in χ and any scalar α

$$y_1 = \Xi x_1 \Rightarrow \alpha y_1 = \Xi \alpha x_1$$

$$y_i = \Xi x_i, i=1,2 = \Xi(x_1 + x_2) \Rightarrow y_1 + y_2$$

For example consider the equation

$$Ax=y$$

Here $\chi = C^n$ is the usual n-dimensional vector space and Y is an m-dimensional vector space. If m=n, Y is usually identified with χ. In many applications where m<n , Y is considered to to a subspace of χ.

DEFINITION: The range or image of a L.T. R is the set R(R) defined by

$$R(R) = \{ y \varepsilon Y \mid y=Rx \text{ for some } x \varepsilon \chi \}$$

DEFINITION: The null space or kernel of a L.T. is the set N(R) defined by

$$N(R) = \{ x \varepsilon \chi \mid Rx=0 \}$$

DEFINITION: The rank of R is defined to be the dimension of its range or image and the nullity of R is defined to be the dimension of its kernel or nullity space.

The following fundamental relationship holds

Rank(R) + Nullity(R) = Dimensional (R)

Consider again the matrix equation

$Ax=y$

The solution of the equation may be viewed as the image of $y\epsilon Y$ under L.T. Furthermore the solution of the associated homogenous equation $Ax=0$ may be viewed as the kernel of linear mapping.

THEOREM A.1

The dimension (dim) of the solution space W of the homogenous system of linear equation $Ax=0$ is n-r where n is the number of unknowns and r is the rank of the matrix A.

Proof: Since dim(kerA)= dim(A) -Rank((A)

= n-r

PROOF OF THEOREM 3.3

Proof: The solution of (3.1) is

$$\underline{x}(t) = e^{At}(\underline{x}_o + \int_o^t e^{-A\tau}B\underline{u}(\tau)\, d\tau$$

At t = T*, $\underline{x}(T) = 0$. Therefore

$$\underline{x}_o = -\int_o^{T^*} e^{-A\tau}B\underline{u}(\tau)\, d\tau$$

By the Cayley-Hamilton Theorem [26], $e^{-A\tau}$ may be expressed as

$$e^{-A\tau} = \sum_{i=0}^{n-1} \alpha_i(\tau)\, A^i$$

Thus

$$\underline{x}_o = - \int_o^{T^*} \sum_{i=0}^{n-1} \alpha_i(\tau) A^i B \underline{u}(\tau) \; d\tau$$

or

$$\underline{x}_o = - \sum_{i=0}^{n-1} A^i B \int_o^{T^*} \alpha_i(\tau) \; \underline{u}(\tau) \; d\tau$$

$$= - \left[B \mid AB \mid \cdots\cdots A^{n-1}B \right] \int_o^{T^*} \begin{bmatrix} \alpha_1(\tau) & \underline{u}(\tau) \\ \alpha_2(\tau) & \underline{u}(\tau) \\ \cdot & \cdot \\ \alpha_{n-1}(\tau) & \underline{u}(\tau) \end{bmatrix} d\tau$$

If matrix A is of simple structure as assumed in previous derivation, then its characteristic polynomial is identical with its minimal polynomial. Since $\alpha_i(\tau)$ are linearly independent [33], the above relation is true and implies the matrix G defined by (3.7) has a rank =n. By definition of the range of a L.T., u has a solution if and only if it is in the range of G.

TWO LEMMAS ON INNER PRODUCTS

LEMMA 1

Let y and z be any two real r-vectors and assume $y \neq 0$. Then z satisfies the condition $\langle z, y \rangle = 0$ if and only if there exists a real skew-symmetric, rxr Λ, such that $z = \Lambda y$.

Proof; Sufficiency is trivial. To show necessity, let B be a rxr orthogonal matrix and y' be a column vector such that $y' = By = \text{col.}(||y||, 0, 0, \ldots\ldots, 0)$. Let $z' = Bz = \text{col.}(z'_1, \ldots\ldots\ldots, z'_r)$ where z'_i is the ith component of z'. Since B is orthogonal, $<z, y> = <z', y'> = 0$ which implies that $z'_1 = 0$.

Denote Q to be

$$Q = \begin{bmatrix} 0 & -z'_2 & -z'_3 & \ldots\ldots z'_r \\ z'_2 & 0 & 0 & \ldots\ldots 0 \\ \vdots & \vdots & \vdots & \ldots\ldots \vdots \\ z'_r & 0 & 0 & \ldots\ldots 0 \end{bmatrix}$$

and define $A = B^T QB/||y||$. Since $Q+Q^T=0$, then $A + A^T = 0 = B^T(Q+Q^T)\ ||y|| = 0$. Further $||y||Ay = B^T QBy = B^T Qy' = B^T ||y|| z' = ||y|| B^T Bz = ||y|| z$. Thus A is skew-symmetric rxr matrix such that $z = Ay$

LEMMA 2

Let y and z be any two real r-vector and assume $y \neq 0$. Then, if α is any scalar, $<z, y> = \alpha$ if and only if there exists a real skew-symmetric rxr matrix A such that $z = [\beta I + A]y$, where $\beta = \alpha/||y||^2$ and I is the identity matrix.

Proof: As in Lemma 1, the sufficiency condition follows by direct evaluation. To show necessity, let z be decomposed as $z = z' + z''$, where $<z', y> = 0$ and $z'' = \gamma y$, with γ being some scalar .

Since $\langle z, y\rangle = \alpha = \langle z'', y\rangle = \gamma||y||^2$. It follows that $\gamma = \alpha/||y||^2 = \beta$. Furthermore Lemma 1 implies that $z' = Ay$ where A is a skew-symmetric matrix, so $z = [\beta I + A]\, y$

The Lemmas give explicit (non-unique) solutions of the implicit algebraic equation $\langle z, y\rangle = \alpha$ for $y \neq 0$. When $y=0$ α must be zero for a solution to exist and z is arbitrary in this case. Of course, A is not unique.