A SURVEY OF ALTERNATIVE RINGS AND ALGEBRAS

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An Abstract of a Thesis

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The purpose of this paper is to survey some of the important discoveries made in the study of alternative rings and algebras. The study of rings and algebras can be divided into four cases: (1) rings with chain conditions, (2) finite dimensional algebras, (3) rings without chain conditions, (4) infinite dimensional algebras. In each case the radical, semi-simplicity, and simplicity of the set is considered. The associative structure theory is essentially complete. An equivalent alternative structure theory has been attempted, but many gaps still remain. We have shown in this survey, the relationships between the associative and alternative theories and pointed out the problems in alternative theory which are still unsolved. Since many difficulties arise when the associative law of multiplication is replaced by the alternative law, some authors have developed new and interesting techniques. We have attempted to

point out and discuss some of them.

For associative rings and algebras many important definitions of the radical have been obtained for the four cases. We have shown where some equivalent radicals have been developed for alternative rings and algebras. These radicals have many similar properties to those of the associative case. In associative theory, Jacobson was able to obtain the relation between the regular maximal right ideals of a ring (without chain conditions) and its radical. No parallel to this relation has as yet been shown for an alternative ring.

For case (1) the semi-simplicity and the simplicity have been studied by Zorn with the result that: A semi-simple alternative ring with chain conditions (such a ring is called a hypercomplex ring) on modules is a direct sum of simple hypercomplex rings which are in turn either associative, alternative fields or Cayley Algebras. The chain conditions placed on the modules are very complicated and we therefore wonder if it is not possible to obtain the same structure theory using a weaker hypothesis. The methods used for case (1) can be used also for case (2) yielding the same results.

The structure of associative division rings is still unclassified. However, for an alternative division ring Bruck and Kleinfield discovered that an alternative division ring is either (1) a field or skew field, (2) a Cayley Algebra over its centre. Since an alternative division algebra contains a unity its structure is immediately determined by the preceding results. Independent investigations have also been made by Zorn and Schafer with the same results.

Simple alternative rings with no chain conditions were studied by Albert. Although no author has as yet been able to obtain a structure for arbitrary simple rings, Albert was able to obtain one by assuming only the existence of an idempotent. Every simple alternative ring which contains an idempotent not its unity is either associative or a Cayley Algebra.

The structure of an arbitrary semi-simple alternative ring with no chain conditions is still unknown. Smiley was able to obtain a structure for semi-simple alternative rings which he calls special. A Special alternative ring has zero Jacobson radical if and only if it is a subdirect sum of special primitive alternative rings. For a much more generalized radical Brown and McCoy were able to obtain a structure for an arbitrary semi-simple ring. When this particular radical is zero the ring is a subdirect sum of rings each of which is right primitive. However the structure of primitive alternative rings is not known. Kaplansky was able to obtain a structure for semi-simple π regular rings. (A π regular ring is one in which, for every "a"

there exists an "x" and an integer "n" such that $a^n x a^n = a^n$)
A semi-simple π regular alternative ring is a subdirect sum of rings, each of which is (1) a division ring, (2) a Cayley matrix ring over a field or, (3) a primitive associative ring. Kaplansky suggests that when the theory of arbitrary rings is worked out these three will comprise precisely the primitive rings. This will parallel Jacobson's structure theory for arbitrary associative rings.

Since π regular rings contain all algebraic algebras, the structure of semi-simple algebraic algebras to compare with Jacobson's theory for the associative case is now complete.

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INTRODUCTION

The purpose of this paper is to survey some of the important discoveries made in the study of alternative rings and algebras. An alternative ring (or algebra) is the same as an associative ring except that the associative law of multiplication $(ab)c = a(bc)$ is replaced by the alternative law:

$a(bb) = (ab)b$; $a(ab) = (aa)b$ for every a, b of the ring. The structure theory of semi-simple and simple associative rings and algebras is essentially complete. One exception being that associative division rings are still not classified. An equivalent theory for alternative rings and algebras has been attempted, but many gaps still remain. We will show, in this survey, the relationships between the associative and alternative theories and point out the problems in alternative theory which are still unsolved. Many difficulties arise with the use of this weaker form of multiplication. Some authors have developed new and interesting techniques to overcome these difficulties and we will attempt to point out and discuss some of them.

The alternative law was first discovered by J. Kirmse in the form $(\bar{a}a)b = \bar{a}(ab)$ and $a(\bar{b}b) = (ab)\bar{b}$ where a, b & $\bar{a}$ are Cayley numbers and $\bar{a}$ is the conjugate of a. The name "alternate law" is due to Zorn [18][1] and is derived from the fact

1 A number in square brackets [ ] refers to the bibliography.

that the associator $(a,b,c) = (ab)c - a(bc)$ is an alternating function of its arguments. Today, the common form of the alternative law for a not necessarily associative ring C is $(ab)b = a(bb)$ and $a(ab) = (aa)b$ for all a, b of C. To overcome many of the difficulties of using this law, and to serve as a tool for replacing the associative law, Artin has discovered the very important and much used theorem:

"Every two elements of an alternative ring generate an associative subring."

The proof of this can be found in Zorn [18]. The method used to prove this is to show that the associator $(\langle D+B\rangle, \langle D+B\rangle, \langle D+B\rangle) = 0$ where $\langle D+B\rangle$ is the ring generated by two A-rings D and B of an alternative ring C. (B is an A-ring of C if $(B,B,C) = 0$)

Most of our knowledge of finite dimensional alternative algebras can be found in Schafer [12] and Dubisch and Perlis [7]. A discussion on the Dubisch and Perlis radical can be found in Chapter I. Max Zorn, [18], [19], [20] studied an alternative ring with chain conditions on modules. These modules were defined to replace the notion of a right ideal in associative theory. There is a close parallel between Zorn's structure theory for semi-simple and simple alternative rings with chain conditions on modules:

> "A semi-simple alternative ring is a direct sum of simple rings, where the simple rings are (1) associative, (2) alternative fields, (i.e. alternative zero divisor free rings with unit element in which $ax = b$ and $xa = b$ are solvable), (3) Cayley algebras over a field."

and the Wedderburn-Artin Structure theorems for semi-simple and simple rings with chain conditions on ideals:

> "A semi-simple associative ring with chain conditions on ideals is a direct sum of simple rings, which are in turn vector matrix rings over division rings."

A more detailed discussion of this point can be found in Chapter II. Zorn's structure theory for alternative rings with chain conditions on modules is true also for finite dimensional alternative algebras, since semi-simple and simple alternative algebras have a unity and therefore ideals in the algebra become ideals in the ring.

The structure of alternative fields or alternative division algebras has been studied by Zorn [19] and Schafer [12]. Their results imply that:

> "An alternative, not associative, central division algebra over a field F is an algebra of degree 2 and order 8 over F."

After studying a division algebra of degree 2 Schafer concluded that:

> "An alternative, not associative Algebra A, over an arbitrary field F is central simple, if and only if A is a Cayley Dickson Algebra over F."

The structure of finite dimensional alternative algebras and alternative rings with chain conditions is complete. Much progress has been made in the study of alternative rings without chain conditions to parallel Jacobson's theory [8] [9] for associative rings that:

> "A semi-simple associative ring is isomorphic to a subring of the complete direct sum of primitive rings where a primitive ring is isomorphic to a dense ring of linear transformations in a vector space over a division ring."

Smiley in [16] shows that Jacobson's definition of the radical as the join of all quasi-regular left ideals of an associative ring also applies to an alternative ring. A discussion of this radical can be found in Chapter I. Although Smiley did not succeed in finding a parallel for Jacobson's structure theory for arbitrary alternative rings, he was able to do so for rings which he calls special (i.e. an alternative ring where the set (1-a) A is a right ideal of A for every "a" in A). Smiley's results were [14]:

> "A special alternative ring has zero Jacobson radical (i.e. is semi-simple) if and only if it is a sub-direct sum of special primitive alternative rings."

He also showed that the Jacobson radical of a special alternative ring is the intersection of the regular maximal right ideals. This is a result which is true for all associative rings but as yet has not been proven true for all alternative rings. Special alternative rings include all direct sums whose summands are associative rings, alternative division rings, or alternative radical rings, but the Cayley Dickson algebra is not included.

Brown and McCoy have enlarged the Jacobson radical so that it will yield a certain type of structure theory for arbitrary non-associative rings [4] to parallel Jacobson's theory for associative rings. They define the radical N_r as the set of all elements "a" of a ring A for which every element "b" in (a) ((a) is the ideal generated by a) is such that the right ideal generated by the set (1-b)A is A. Brown has replaced Jacobson's quotient (I:A) of a right ideal I of A by the largest ideal I' contained in A. If I is a regular right ideal, $I' = (I:A)$, if (I:A) is an ideal of A. It can be shown that $R = N_r$ for special alternative rings. The duality of the Perlis-Jacobson radical of an alternative ring with respect to right and left multiplication is lost in this process of generalization. A consequence of this more general formulation is the fact proved by Brown and McCoy that every alternative ring has a greatest regular ideal.

Brown proves that $N_r = \cap M'$ as M ranges over the regular maximal right ideals of A. When $N_r = 0$, A is a subdirect sum of rings each of which is right primitive (i.e. contains a modular maximal right ideal M with $M' = 0$.) The main problem which still exists is that of finding a structure for a primitive alternative ring.

Kaplansky [10] made a study of semi-simple alternative rings without chain conditions but with a suitable hypotheses to provide a supply of idempotents. He defined a Zorn ring to be an alternative ring for which every element "a" is either nilpotent or has a right multiple "ab" which is a non zero idempotent. For a Zorn ring he proved that the radical (defined by Jacobson) is the intersection of the regular maximal right ideals and that it coincides with the set of properly nilpotent elements. Kaplansky was unable to obtain the structure of an arbitrary semi-simple alternative ring, but for a ring which he calls π regular (i.e. for every "a" there exists an "x" and an integer "n" such that $a^n x a^n = a^n$) he showed:

> "A semi-simple π regular alternative ring is a subdirect sum of rings each of which is a division ring, a Cayley Matrix ring over a field, or a primitive associative ring."

It seems possible to expect that when the theory of arbitrary rings is worked out, these three types will comprise precisely the primitive rings. π regular rings do not form a duly restrictive class since they include all finite rings and algebraic algebras. A discussion of Kaplansky's work can be found in Chapter V.

Although associative division rings are still not classified, R. H. Bruck and Erwin Kleinfield [5] [6] have shown that:

> "Every alternative division ring of characteristic not two is associative or a Cayley Matrix Algebra."

and Kleinfield has shown that:

> "Every alternative, not associative division ring of characteristic 2 is a Cayley Dickson division algebra of order 8 over its centre."

A discussion of their methods and results can be found in Chapter III. These results have answered some very important questions in the fields of geometry and topology as they apply to alternative theory.

A. A. Albert discussed simple alternative rings (see Chapter IV) with no chain conditions but which contain an idempotent element. He was able to show that:

> "Every simple alternative ring which contains an idempotent not its unity is either associative or a Cayley Algebra."

A structure theory for arbitrary alternative rings is not yet available.

Certain topological methods have been applied as a means of studying alternative rings. The main results are those of Albert, Jacobson, Taussky and Kaplansky. Kaplansky was able to prove that "A compact semi-simple alternative ring is a complete direct sum of finite simple rings." It has been suggested by Saunders MacLane that a cohomology theory exists for alternative rings and that this might be helpful in the study of primitive alternative rings. We will not discuss topological results in this paper but mention them only as a point of interest.

The main body of the thesis is divided into six chapters.

Chapter I - We discuss equivalent definitions of the radicals defined in associative theory, for alternative theory.

Chapter II - The structure of semi-simple and simple alternative rings with chain conditions on modules and how this compares with the structure of semi-simple and simple associative rings with chain conditions on ideals.

Chapter III - The structure of alternative division rings and the discussion of some very special techniques used by Bruck and Kleinfield in obtaining the structive theory.

Chapter IV - The structure of simple alternative rings without chain conditions but subject to the condition that the ring contains a non-zero idempotent. (as given by Albert [2].)

Chapter V - We discuss semi-simple alternative rings without chain conditions but with a suitable supply of idempotents, and show the parallel between the structure theory of semi-simple π regular alternative rings and semi-simple associative rings.

Chapter VI - Contains conclusions of the preceding chapters and an indication of the gaps which still remain in the theory.

NOTATION:

The chapters are numbered I II ..... etc.

The subsections are numbered (1), (2), (3), (4).....

A number $(\alpha.\beta)$ in front of a statement means subsection α number β.

The letter L indicates a lemma.

The letter D indicates a definition.

The letter T indicates a theorem.

These letters will be used for ease in reference.

CHAPTER I

THE RADICALS OF ALTERNATIVE RINGS AND ALGEBRAS

In associative theory, certain elements were found which complicated the study of the structure of rings and algebras. These elements form an ideal which is called the radical. When the radical was removed from the ring (or algebra), it was possible to develop a structure theory for the ring (algebra). We assume, therefore, that in alternative theory the first important concept to consider is the Radical.

SECTION I. IDENTITIES

We need now introduce certain important identities which will be used throughout the discussion of the radical.

D (1.1) The associator (a,b,c) is defined to be

$$(a,b,c) = (ab)c - a(bc)$$

The importance of the associator is noted in the fact that it is a measure of the lack of associativity. When it is an alternating function of its arguments for every element a,b,c in a ring, the ring is called alternative. Zorn [18] discovered the following identities about the associator for alternative rings:

(1.2) $(ab,c,d) - (a,bc,d) + (a,b,cd) = a(b,c,d) + (a,b,c)d$

(1.3) $(a,a,b) = (a,b,a) = (b,a,a) = 0;\ (a,b,c) = (b,c,a) = (c,a,b) = -(c,b,a)$

(1.4) $(a,ba,c) = -a(b,a,c)$

(1.5) $(ab,a,c) = (a,b,ac) = 0$

(1.6) $(ab,a,c) + (ba,a,c) - (b,aa,c) = 0$

The commutator (a,b) is defined to be $(a,b) = ab-ba$

D (1.7) The quasi sum is defined to be $a\circ b = a + b + ab$.

If $a + b + ab = 0$ then we say a is right quasi regular (abbreviated r.q.r.) and b is the right quasi inverse, r.q.i., of a.

Smiley [16] developed the following identities:

(1.8) $a\circ(b\circ c) - (a\circ b)\circ c = (a,b,c)$; $(a+b)\circ c = (a\circ c) + (b\circ c) - c$

$c(a\circ b) - (c\circ a)b = (c,a,b) + ca - ab$

(1.9) If $(a\circ b) = 0$ Then $(a,b,c) = 0$

(1.10) If $(a\circ b) = (c\circ a) = 0$ Then $ab = ca$ and $b = c$

(1.11) If $a + ab = 0$ and b is r.q.r. then $a = 0$

(1.12) If $e \in A$ is a non zero idempotent, then $-e$ is not r.q.r.

(1.13) If ab is r.q.r. then ba is r.q.r.

The importance of these identities is their use in developing a radical for the alternative case which parallels that of Jacobson [9] for the associative case. The technique used in proving these identities is that of using the relationship between the quasi sum and the associator. For example consider the proof of 1.13.

Proof: If ab is r.q.r. then there exists an element c such that $(ab)\circ c = 0$. If we let $(a,b,c) = \alpha$ then $b\alpha = b(a,b,c) = (ab,b,c) = 0$ by (1.9). Similarly $\alpha a = 0$, and therefore $(b,\alpha,a) = 0$. Now, since $ab + c + (ab)c = 0$ then $\alpha = (a,b,-(ab)c) = (a,b,\alpha - a(bc)) = -(a,b,bc)a = -\alpha ba$

Therefore $\alpha b+\alpha bab=0$ and $\alpha b=0$ by (1.11) and $\therefore \alpha=\alpha ba=0$.

Consider now $(ba)o(-ba-bca)=-b((ab)oc)a=0$.

Then ba is right quasi regular.

SECTION 2. EXISTENCE OF THE RADICAL

The study of the existence of the radical will be divided into four cases: (1) Rings with certain chain conditions, (2) finite dimensional algebras, (3) rings with no chain conditions, (4) infinite dimensional algebras.

CASE (1): The existence of a radical for rings with certain chain conditions was discussed in detail by Zorn [19]. The chain conditions of Zorn are not of the usual kind used in associative theory. The introduction of certain modules in an alternative ring, which become right ideals in an associative ring, provide a necessary equivalent for the right ideals in an associative ring. These modules are not right ideals in an alternative ring and therefore chain conditions on right ideals is not sufficient. When the following finite chain conditions are placed on the modules the ring is called hypercomplex.

The modules were defined as follows: A I-Module of R is a module aR for a in R; it consists of all elements of the form ax where x runs over all elements of R. A II-Module $A' = o/a$ consists of all elements which would be left annihilated by an element "a" of the ring; $aA' = a.\underset{a}{o}=0$.

(2.1) The chain conditions placed on these modules are:

(1) Every descending chain $aS \supset a^2S$ ... is finite

(2) Every ascending chain $0/a^i$ is finite (i is a power of a)

(3) Every descending chain $0/a_i$ is finite (i is a subscript)

To construct an ideal N which would be the radical of the hypercomplex ring R, Zorn first established a Peirce decomposition for R[1]. For a finite set of idempotents $e_1, \ldots, e_n$, where $e_i e_k = 0$ for $i \neq k$ we can find a direct sum $S = \sum S_{ik}$; $i,k = 0 \ldots n$, where the S_{ik} are modules consisting of elements x_{ik} and $e_j x_{ik} = \delta_{ji} x_{ij}$; $x_{ij} e_j = \delta_{kj} x_{ik}$

The component of x in S_{ik} is $e_i x e_k$ for $i \neq 0 \neq k$.

This Peirce decomposition looks exactly like the one for the associative case [3]. They differ only in certain multiplicative properties.

(2.3) $e = \sum_{i=1}^{m} e_i$

(2.4) $S_{ik} S_{kl} \subset S_{il}$; $S_{ik} S_{ik} \subset S_{ki}$ all other products $S_{ik} S_{lm}$ are zero. In the associative case $S_{ik} S_{kl} = 0$; $S_{ik} S_{ik} = S_{ki}$

(2.5) If N_{ik} is a module contained in S_{ik} and if $i \neq k$, then $N_{ik} S_{ik} = S_{ik} N_{ik}$.

(2.6) The product of three factors $(x_{ik} y_{kl}) z_{li}$ is associative except for a product of the type $(x_{ii} y_{ii}) z_{ii}$.

(2.7) $(x_{ik} y_{ik}) z_{ik}$ for $i \neq k$ retains its value when the factors are permuted cyclically.

1 Albert [2] discusses this Peirce decomposition. His techniques are more simple and will be found in detail in Chapter IV.

The power formulae (2.8) and (2.9) will be used not only to establish the existence of the radical, but will be of special use in semi-simple systems.

(2.8) $$(a_{ii}x)^m = (a_{ii}x_{ii})^{m-1}\sum_{j=0}^{n} a_{ii}x_{ij} \text{ for } i \neq k$$
$$(a_{ik}x)^m = (a_{ik}x_{ki})^{m-1}\sum_{j=0}^{n} a_{ik}x_{kj} + (a_{ik}x_{il})(a_{ik}x_{ki})^{n-1}$$
$$(a_{ik}x)^m e_i = (a_{ik}x_{ki})^m$$

For this Peirce decomposition the modules S_{oi} and S_{io} for i=0 ... n consist of all properly nilpotent elements. (A properly nilpotent element a is one whose multiples ax are all nilpotent and therefore all xa are nilpotent.)

A Zorn primitive ring is one which contains a unit which is the only idempotent of the ring.[2] Zorn showed that a primitive ring contains a radical which is a 2 ideal consisting of all the properly nilpotent elements. He also showed that the S_{ii} are hypercomplex (2.9) and primitive for (i = 1 ... n) their respective unit elements being the idempotents e_i. This corresponds to the associative case.

The ideal N which is the radical of the hypercomplex ring is defined to consist of all those elements whose components are singular (a_{ik} is singular if all elements $a_{ik}S_{ki}$ are nilpotent, otherwise a_{ik} is regular). If a_{ik} is

2 Zorn's primitive ring is not the same as the one discussed in the introduction. That primitive ring is one which contains a maximal right ideal I whose quotent ideal (I:A) = $\underline{0}$ (The correspondence $a \to \bar{a}$ is a homomorphism between A and $\bar{A}$ where a maps $x + I \to xa + I$. The kernel of this homomorphism is called (I:A)). Zorn's primitive ring is now obsolete.

singular then also all $S_{ki}a_{ik}$ are nilpotent. The ideal N is taken to be equal to $\sum N_{ik}$ (i,k = 0 ... n), where N_{ik} are the set of all singular elements in S_{ik}. The N_{ik} were shown to be modules [20], and by the use of certain multiplicative rules of the Peirce decomposition, N is shown to be a two-sided ideal. To be more explicit, Zorn developed the following:

(2.10) $N_{ik}S_{ki} \subset N_{ii}$ and $S_{kj}N_{jk} \subset N_{kk}$ by the definition of N_{ik}.

(2.11) $(N_{ij}S_{jk})S_{ki} \subset N_{ii}$ (by 2.6); $(N_{ki}S_{ki})S_{ki} \subset N_{ii}$ (by 2.7)

$(S_{ij}N_{jk})S_{ki} \subset N_{ii}$ (by 2.7); $(S_{ki}N_{ki})S_{ki} \subset N_{ii}$ (by 2.7)

Then $NS = \sum N_{ik} \sum S_{lm} = \sum N_{ij}S_{jk} \sum N_{ik}S_{ik}$ (i,j,k = 0 ... n, $i \neq k$) and by the above conditions this $\subseteq$ N. Similarly $SN \subset N$ and N is a two sided ideal. Zorn went on to show that N consists entirely of nilpotent elements, and therefore the radical exists for every hypercomplex ring.

CASE (2): The existence of the radical for a finite dimensional alternative algebra has been discussed in detail by R. Dubisch and S. Perlis [7]. They considered four radicals of an alternative algebra to correspond to specific radicals already defined for an associative algebra.

The first radical is the R radical:

D(2.12) It is defined for an alternative algebra A with unity, as the totality of elements r of A such that $g + r$ is regular whenever g is regular. (A regular element is one that has a two sided inverse.)

This radical was characterized in the associative case by S. Perlis. The "Theorem of Artin" discussed in the introduction is used in proving that the R radical is an ideal. In order to show the extensive use of Artin's Theorem we will state this proof as in [7].

<u>LEMMA 2.1</u> A product of regular elements of an alternative algebra is regular. Conversely if $g_1 g_2$ is regular so are g_1 and g_2.

Proof: The element $g_1^{-1} g_2^{-1} \epsilon A\langle g_1 g_2 1\rangle = A\langle g_1 g_2\rangle$. By Artin's theorem this is associative. Therefore $g_2^{-1} g_1^{-1}$ is the inverse for $g_1 g_2$. Conversely if $g_1 g_2$ has an inverse g, we know that $g \epsilon A\langle g_1 g_2\rangle$ and therefore $(g_1 g_2)g = g_1(g_2 g) = 1 = (gg_1)g_2$. Therefore g_1 and g_2 have inverses.

<u>LEMMA 2.2</u> If an alternative algebra A has a unity and the base field F has at least three elements, every element of A is a sum of regular elements. (Shoda's theorem)

To show that the R-radical is an ideal:

If g and g_i are regular in A, then $g + g_i r = g_i(g_i^{-1} g + r)$ is regular by (L 2.1). Therefore $g_i r \epsilon R$. But every $a \epsilon A$ is $= \sum g_i$ by (L 2.2). Therefore $ar \epsilon R$. Similarly $ra \epsilon R$. Therefore R is a two ideal of A.

The next radical defined is the Q radical:
This is the sum of all quasi regular right ideals of A (right ideals whose elements are all quasi regular). This radical is equivalent to the Jacobson radical [9] defined for arbitrary associative rings. In associative theory this radical is very important, since, when it is removed from the ring a complete structure theory can be found for the ring. By a proof again using the theorem of Artin, the Q radical of an alternative algebra with unity over an arbitrary field F, is found to be a quasi regular right ideal of A.

The third radical - the N radical - is defined as in Case I: It is the set of all properly nilpotent elements and the element 0.

The S radical of A is the intersection of all ideals B of A such that A-B is a direct sum of simple algebras. This radical was defined originally by Albert in his study of non-associative algebras. The existence of the N and S radicals is established by the fact that $Q = R = N = S$ for an algebra with unity.[3] [7]

CASE (3): The radical of an alternative ring with no chain conditions was defined by Smiley [16]. The right ideal r(a) generated by $a \epsilon A$ consists of the totality of elements of A

3 See section 3.

of the form $a_i + \sum aU_j$, where i is an integer and U_j the product of a finite number of right multiplications R_x; $aR_x = ax$.

D(2.13) The right radical $R_r(A)$ is defined as the totality of elements $a \in A$ for which each r(a) is a right quasi regular right ideal. The left radical $R_\ell(A)$ is defined similarly. Both radicals are defined in an equivalent manner to the Jacobson radical [9] for the associative case. Namely that the radical of a ring A is the join of all the quasi regular (left) right ideals of A. The technique used in proving that the right radical $R_r(A)$ is a right ideal, is a manipulation of the relations between the quasi sum (aob) and the associator (a,b,c) as follows:

LEMMA 2.3 If $a \in R_r(A)$ and b is right quasi regular, then $a + b$ is r.q.r.

Proof: Let $boc = 0$ for some $c \in A$, then $a + ac \in r(a)$ and therefore $(a + ac)od = 0$ for $d \in A$.

Now $(a + b)o(cod) = ((a + b)oc)od + (a + b, c, d)$ (by 1.8)

$= (a + ac)od + (a + b, c, d) = (a, c, d)$.

And also $(a + ac, c, d) = 0$. If $\alpha = (a, c, d)$ then

$\alpha + c\alpha = 0$ and $\alpha = 0$. (by 1.11)

Therefore $a + b$ is r.q.r.

LEMMA 2.4 The right radical $R_r(A)$ is a right ideal.

This is easily proven by L (2.3) considering $r(a + b)$ where $a, b \in R_r(A)$. To show that the right radical $R_r(A)$ is a left ideal a similar technique is used but the manipulations are slightly more complex. In this case $r(ba)$ is considered where $a \in R_{\lambda}(A)$ and $b \in A$. It can then be computed that $ba \in R_{\lambda}(A)$. Therefore it is concluded that $R_{\lambda}(A)$ is a left ideal.

The right radical can easily be shown to coincide with the left radical, and the radical is then defined as $R(A) = R_{\lambda}(A) = R_{\ell}(A)$ (a two ideal of A).

CASE (4): The existence of the radical of an algebra of possibly infinite order over a field F can now follow immediately. An ideal of the algebra A is an ideal of the ring A that is invariant under the scalar multiplication $x \rightarrow x\alpha$ for α in F. If A has an identity then $x(1\alpha) = (1\alpha)x = x\alpha$, and so any ideal of the ring A is an ideal of the algebra A. The preceding discussion for an alternative ring without chain conditions is then valid for an arbitrary alternative algebra A with unity. The radical of the algebra A is then defined as the totality of elements a for which each $r(a)$ is right quasi regular (similarly left quasi regular).

SECTION 3. RELATIONS BETWEEN THE RADICALS

<u>RELATION 3.1</u> The N radical of case 2 is equal to the R radical. The proof of this can be found in [7]. No special techniques are used there except perhaps another application of the important theorem of Artin.

Note: This establishes the existence of N, since N is now an ideal it is therefore a nil ideal.

RELATION 3.2 The Q radical of Case 2 = R radical = N radical for a right alternative algebra A with unity.

RELATION 3.3 The S radical of an alternative algebra A with unity = R = Q = N. The proof is given in [7] as follows:

Proof: We know that A-N is a direct sum of simple algebras (Zorn [18]). Therefore $S \leq N$. Since 1 is not in N, $A > N \geq S$. Now $A-S = B_1 \oplus B_2 \oplus \dots \oplus B_n$ where the B_i are simple with unity e_i. Each element $x \in N$ corresponds to a coset $[x] = [x_1] + \dots + [x_n]$ where $[x_i]$ in B_i are properly nilpotent in B_i and in A-S. As $[x]$ varies over N, $[x_i]$ varies over the ideal N_i of B_i. Therefore $N_i = 0$ or $N_i = B_i$. In the latter case $e_i \in N_i$, but e_i is not nilpotent. Therefore $N_i = 0$ and $N-S = 0$ or $N \leq S \leq N$ or $N = S$.

RELATION 3.4 The Q,N,S radicals also have meaning for an alternative algebra A_o without unity. If Q_o, N_o, S_o are the corresponding radicals for A_o it can be shown [7] that they equal Q,N,S respectively for the algebra $A = (A_o, 1)$ and that Q,N,S are identical.

RELATION 3.5 If we apply Zorn's chain conditions to the ring A, we find that the radical R(A) coincides with Zorn's radical. The proof of this is elementary and can be found in [16].

SECTION 4. PROPERTIES OF THE RADICAL

PROPERTY 3.1 The radical of A/R is zero for the radical R defined in case 2 sec. 2. The proof is simple using no unusual techniques.

The following four properties of the radical of an alternative algebra defined in case 4 sec. 2 are exactly the same as those for an associative algebra [9]. This is true since we know that if $a \circ b = 0$ then $(a,b,c) = 0$ for every $c \in A$.

PROPERTY 3.2 If A^* is an alternative algebra with unity, and containing A, such that $A^* = A + (1)$ (where (1) is a subring of A^* generated by 1). Then $R(A) = R(A^*) \cap A$. If also $A \cap (1) = 0$ and (1) is isomorphic to the ring of integers then $R(A) = R(A^*)$.

PROPERTY 3.3 $R(A - R(A)) = 0$

PROPERTY 3.4 If I is a right ideal of A and if each element of I is nilpotent then $I \subseteq R(A)$.

PROPERTY 3.5 If B is a subset of $R(A)$ closed under mult. and if $b \in B$, then for every positive integer n $b^{n-1}B \supset b^n B$ or $b^n = 0$.

There are some very important properties which can be proved for an associative ring but cannot be proved for an alternative ring. In the associative case Jacobson [9] proves certain important relations between the radical and the intersection of the maximal right ideals. To do this he makes use of the fact that $\{x + ax\}$ is a right ideal. We cannot prove this for the alternative case simply because $\{x + ax\}$ is not a right ideal. We can however, prove that a is right quasi regular if and only if every element of A has the form $ax + x$ for some $x \in A$.

CHAPTER II

THE STRUCTURE OF SEMI-SIMPLE AND SIMPLE HYPERCOMPLEX RINGS

SECTION 5. THE STRUCTURE OF SEMI-SIMPLE HYPERCOMPLEX RINGS

An associative semi-simple ring with descending chain conditions on ideals contains only a finite number of simple ideals within it, and is a direct sum of simple rings. In a like manner Zorn has shown that a semi-simple hypercomplex ring (i.e. an alternative ring with chain conditions on modules) is a direct sum of simple hypercomplex[1] rings.

D (5.1) We define a semi-simple ring as Zorn did [18] to be a ring which contains no properly nilpotent elements (i.e. no radical elements). This definition is the same as that for the associative ring.

The procedure used by Zorn in the decomposition of a semi-simple hypercomplex ring is very similar to that used for the associative semi-simple ring. The important exceptions to the procedure are the use of modules (D 2.1) instead of ideals, and the use of the theorem of Artin to replace the associative law.

Following Zorn we have:

1 This use of the word hypercomplex has no relation to a hypercomplex system.

<u>LEMMA 5.2</u> If M is a module and x an element such that $xM = M$ is a hypercomplex system, then there is no $m \neq 0$ such that $xm = 0$.

We will show this proof as an illustration of the extensive use of the theorem of Artin.

Proof: Since $xM = M$ then $x^n M = M$ by theorem of Artin. Now if $xm = 0$ for $m \neq 0$ and $m \in M$, then for any integer n there exists an element y_n of M such that $x^n y_n = m$. Then $x^{n+1} y_n = x(x^n y_n)$ (by Artin's theorem)

$$= xm = 0.$$

Then y_n is not in the II module $0/x^n$ but it is in $0/x^{n+1}$. This leads to a contradiction by the second chain theorem of hypercomplex rings and therefore $xm \neq 0$.

With the use of this theorem and again the use of the theorem of Artin we can prove:

<u>LEMMA 5.3</u> If $a \in C$ is not nilpotent then the I module aC contains an idempotent.

This is equivalent to proving that in an associative ring if a left ideal is not nilpotent it contains an idempotent. Therefore we see, that if a hypercomplex ring is not nilpotent (we know it then contains a non nilpotent element [20]) it contains an idempotent element.

D (5.4) Principal Idempotent -- an idempotent e is principal when the corresponding II module o/e contains no other module 0/e' as a proper subset (where $e' = (e')^2 \neq 0$). That is, e annihilates as little as possible.

<u>THEOREM 5.1</u> If the hypercomplex ring C is not nilpotent then it contains a principal idempotent e. In the Peirce decomposition with respect to e, $C = C_{11} + C_{10} + C_{01} + C_{00}$. All the members $\neq 0$ in the modules C_{10}, C_{01}, C_{00} are radical elements.

This proof closely parallels that for the associative case, the major difference being the use of modules rather than ideals. Zorn makes use, also of certain power formulae (2.8), and as a special illustration of this we will sketch the proof.

Proof: By L 5.2 we see that C contains an idempotent. By chain conditions for modules it must be a principal idempotent, call this e.

Refer to the Peirce decomposition[2]. $C = C_{11} + C_{10} + C_{01} + C_{00}$. The module 0/e is minimal and $= C_{01} + C_{00}$. The module C_{00} does not contain an idempotent e'. If it did, we could construct $e'' = e + e'$, where e'' is an idempotent and $0/e'' \subset 0/e$ is a proper subset. This

2 We refer here to Chapter IV where all the identities are proven for this particular Peirce decomposition. We will use them without further reference.

would contradict e as principal idempotent. Now $C_{01} + C_{00}$ does not contain an idempotent. (This can be shown easily by known relations of Peirce decomposition.)

We see that $C_{00} C \subset O/e$, therefore for every s_{00}, $s_{00} C \subset C_{01} + C_{00}$. If s_{00} is not nilpotent $s_{00}C$ contains an idempotent (L 5.3) and this is a contradiction. Therefore all elements of C_{00} must be nilpotent.

The power formulae (2.8) become:

$$(s_{01}x)^n = (s_{01}x_{01})(s_{01}x_{10})^{n-1} + (s_{01}x_{10})^{n-1}(s_{01}(x_{11} + x_{00}))$$
$$(s_{00}x)^n = (s_{00}x_{00})^{n-1}s_{00}x$$
$$(s_{10}x)^{n+1} = (s_{10}x_{10})(s_{10}x_{01})^n + (s_{10}x_{01})^n(s_{10}(x_{01} + x_{00}))$$

Since $s_{01}x_{10} \in C_{00}$ and $(s_{10}x_{01})^n = s_{10}(x_{01}s_{10})^{n-1}x_{01}$, we can see immediately that the modules C_{00}, C_{10}, C_{01} consist only of radical quantities and zero.

From this theorem it is immediately true that a semi-simple hypercomplex ring contains a principal unity.

<u>THEOREM 5.2</u> A semi-simple hypercomplex ring is a direct sum of simple rings.

Proof: This proof parallels the associative case as follows:

A two ideal of a hypercomplex ring is again a hypercomplex ring since the chain conditions hold easily. A radical quantity "a" of ideals A is also a radical quantity of C, since if $x \in C$ is arbitrary, $xax \in A$ and $a.xax = (ax)^2$ (by Artin's theorem)

Therefore ax is nilpotent and a is radical in C.

If the whole ring is semi-simple, then every two ideal is semi-simple. The ideal A then contains a principal unity e_1 for which $e_1 C \subset A$ and $Ce_1 \subset A$. But $e_1 C \supset e_1 A = A = Ae_1 \subset Ce_1$, therefore $e_1 C = A = Ce_1$. So, for arbitrary x of C

$$e_1 x = (e_1 x) e_1 = e_1 (x e_1) = x e_1 .$$

That is, e_1 is in the centre of C. Since 1 is in the centre $e_2 = 1 - e_1$, is in the centre.

The idempotents e_1 and e_2 are orthogonal idempotents with a corresponding Peirce decomposition:

$C = C_{11} + C_{21} + C_{12} + C_{22} + C_{01} + C_{10} + C_{02} + C_{20} + C_{00}$.

The only non zero modules are C_{11} and C_{22} for a semi-simple ring.

Therefore C is a direct sum of different ideals. There can only be a finite number of these because of the chain conditions. Therefore, a semi-simple hypercomplex ring is a direct sum of simple rings.

SECTION 6. THE STRUCTURE OF SIMPLE HYPERCOMPLEX RINGS

An associative simple ring is a total matrix ring over a division ring. A hypercomplex simple ring can be one of three things: (1) Associative, (2) Alternative field (this is a zero divisor free ring with a unity element, in which the equations $ax = b$ and $xa = b$ with $a \neq 0$ are solvable, (3) Vector matrix rings or in other words Cayley algebras.

The procedure in studying the structure for the alternative case will need to be different from the procedure for studying the associative case. We will give a short summary of Zorn's results [18], pointing out where the alternative case presents new difficulties.

<u>THEOREM 6.1</u> Every non-nilpotent ring C has a Peirce decomposition made up of n orthogonal idempotents $e_1, \ldots e_n$ whose sum is a principal idempotent, where none of the e_ν can be expressed as the direct sum of two orthogonal idempotents (these e_ν are called primitive as in the associative case).

Proof: This is similar to the associative case. The proof is established easily by the construction of a module chain which does not break off. This provides a contradiction to the fact that e has no decomposition into primitive idempotents.

Consider now a semi-simple system C, with a principal unity 1, and the following decomposition:

$$1 = e_1 + e_2 + \ldots e_n \text{ where } e_i \text{ are primitive.}$$

The C_{ii} of this decomposition are semi-simple and even simple. This is true since if C_{ii} contained a proper ideal it would contain an idempotent principal unity $\neq e_i$.

It is at this point that the greatest deviation is made from associative theory, for we must now consider three cases for the simple ring: (1) $n = 1$, i.e. where the simple ring contains only its principal unity as an idempotent element, (2) The case $n > 2$, (3) The case $n = 2$ i.e. $e = e_1 + e_2$. In associative theory these cases need not be considered.

CASE (1): C contains only its principal unity as an idempotent.

LEMMA 6.1 Every non-nilpotent element a of C has an inverse a^{-1} for which $aa^{-1} = a^{-1}a = 1$.

This proof is obvious by L 5.2.

LEMMA 6.2 In general $(a,a^{-1},C) = 0$ and a^{-1} is uniquely defined.

Proof: If $ay = 0$ for arbitrary y, then

$$(a,y,ax) = (aa,x,y) - a(a,x,y) = -(aa,y,x) + a(a,y,x) = 0.$$

Then $(a,y,a(a^{-1})^2) = (a,y,a^{-1}) = 0$.

Therefore $y = (a^{-1}a)y = a^{-1}(ay) = 0$. Therefore there are no divisors of zero.

Since $(a,ba,x) = a(a,b,x)$ (by 1.4), let $b = a^{-1}$ and get $a(a,a^{-1},x) = 0$. Since a is not a divisor of zero

$$(a,a^{-1},x) = 0.$$

LEMMA 6.3 C contains no nilpotent elements different from zero.

Proof: This is trivial using L 6.1.

From lemma's 6.1, 6.2 and 6.3 we can now state:

THEOREM 6.2 A semi-simple ring C with a primitive principal idempotent is an alternative field. (An alternative field is an alternative ring with a unity, in which every element has a unique inverse, the elements are not divisors of zero and every equation $ax = b$ or $xa = b$ for $a \neq 0$ is uniquely solvable.)

CASE (2): This is the case of a simple ring with the Peirce decomposition $C = \sum_{i,k=0}^{n} C_{ik}$ and a principal unity $1 = e_1 + \dots + e_n$, where $n > 2$.

LEMMA 6.4 The C_{ii} of the decomposition are division rings.

Proof: The C_{ii} are alternative fields (T 6.2). By using the multiplicative properties of the modules C_{iK} it can be shown that $(C_{ii}, C_{ii}, C_{ii}) = 0$, i.e. C_{ii} are associative.

LEMMA 6.5 If $C_{ik} \neq 0$, C_{ii} and C_{kk} are isomorphic. The isomorphism is represented by:

$$\Upsilon(x_{ii}) = e_{ki} x_{ii} e_{ik}\,; \quad \Upsilon^{-1}(y_{kk}) = e_{ik}\, y_{kk}\, e_{ki}.$$

THEOREM 6.3 If $n > 2$ in the Peirce decomposition then the simple system is associative.

Proof: This particular Peirce decomposition differs from the associative one in the following places: (see Chapter IV)

$C_{ik}^2 = 0$ is not necessarily true;

$(x_{ii}, y_{ii}, z_{ii}), (x_{ik}, y_{ik}, z_{ik}), (x_{ik}, y_{ik}, z_{ii}), (x_{il}, y_{ki}, z_{li})$.

Since we have L 6.4 we need only prove that $C_{ik}^2 = 0$ to make the ring associative.

The technique of proving $C_{ik}^2 = 0$ is rather original. Two new idempotents are created for which we get a Peirce decomposition of 4 modules which contain the original modules. To be more explicit the proof goes as follows:

Say there exists two elements e_{ik} and e_{ki} for which

$$e_{ik}\, e_{ki} = e_i \text{ and } e_{ki}\, e_{ik} = e_k.$$

Let $E_1 = e_i + e_k$ and $E_2 = \sum_{j \neq i,k} e_j$.

Let $1 = E_1 + E_2$ be a decomposition of 1 into orthogonal idempotents to which the Peirce decomposition $C = \widetilde{C}_{11} + \widetilde{C}_{12} + \widetilde{C}_{21} + \widetilde{C}_{22}$ belongs. $\widetilde{C}_{11}$ contains $C_{ii} + C_{ik} + C_{ki} + C_{kk}$. C_{21} contains in particular C_{li}.

It can be shown that $(\widetilde{a}_{\nu\nu}, \widetilde{b}_{\nu\nu})(\widetilde{C}_{\mu\nu}\, \widetilde{C}_{\mu\nu}) = 0 \quad \nu \neq \mu$

Therefore for $\nu = 1$ and $\mu = 2$, $\widetilde{a}_{11} = e_{ik}$ and $\widetilde{b}_{11} = e_{ki}$.

We know that $(e_{ik}, e_{ki}) = e_i - e_k$, then $(e_i - e_k)(\widetilde{C}_{21}\widetilde{C}_{21}) = 0$,

i.e. $e_i(\widetilde{C}_{21}\, \widetilde{C}_{21}) = e_k(\widetilde{C}_{21}\, \widetilde{C}_{21}) = 0$.

Therefore $(e_i + e_k)(\widetilde{C}_{21}\, \widetilde{C}_{21}) = \widetilde{C}_{21}\widetilde{C}_{21} = 0$, and therefore

$$C_{li}^2 = 0.$$

CASE (3): When $n = 2$, a simple alternative (non-associative) ring is a vector matrix ring over a field. (These are Cayley algebras.)

A vector matrix system is a set of elements $\begin{pmatrix}\alpha & A\\ B & \beta\end{pmatrix}$ where α and β are elements in a field K and A,B are vectors of a 3 dimensional vector system with co-ordinates in K.

The sum is defined as $\begin{pmatrix}\alpha & A\\ \beta & B\end{pmatrix}+\begin{pmatrix}\alpha' & A'\\ B' & \beta'\end{pmatrix}=\begin{pmatrix}\alpha+\alpha' & A+A'\\ B+B' & \beta+\beta'\end{pmatrix}$

The product is defined as $\begin{pmatrix}\alpha & A\\ B & \beta\end{pmatrix}\begin{pmatrix}\alpha' & A'\\ B' & \beta'\end{pmatrix}=\begin{pmatrix}\alpha\alpha'-AB' & \alpha A'+B\times B'+A\beta'\\ B\alpha'+\beta B'+A\times A' & \beta\beta'-BA'\end{pmatrix}$

The scalar product is AB and the vector product is $A\times B$. Every vector matrix can be built up linearly from 8 basal elements $e_1=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, $e_2=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$, $e_{12}^{\nu}=\begin{pmatrix}0 & i_\nu\\ 0 & 0\end{pmatrix}$, $e_{21}^{\nu}=\begin{pmatrix}0 & 0\\ i_\nu & 0\end{pmatrix}$, ν modulo 3, with coefficients in K. (The i_ν are orthogonal unit vectors.)

Where: $e_i e_j=\delta_{ij}e_i$, $e_i e_{ik}^{(\nu)}=e_{ik}^{\nu}e_k=e_{ik}^{\nu}$, $e_k e_{ik}^{\nu}=e_{ik}^{\nu}e_i=0$, $e_{ik}^{\nu}e_{ik}^{\nu}=0$, $e_{ik}^{\nu}e_{ik}^{\nu+1}=-e_{ik}^{\nu+1}e_{ik}^{\nu}=e_{ki}^{\nu+2}$, $e_{ik}^{\nu}e_{ki}^{\mu}=-\delta_{\nu\mu}e_i$, for $i,j,k=0,1$, $i\neq k$ and μ,ν modulo 3.

The simple ring with $n=2$ becomes decomposed into 4 modules C_{11}, C_{12}, C_{21}, C_{22}. It can be shown that C_{11} and C_{22} are commutative and therefore are commutative fields isomorphic to each other (L 6.5).

The field K consists of elements

$$a_{11}+\overset{\circ}{a}_{11}=\overset{*}{a}_{11}\ ,\quad a_{22}+\overset{\circ}{a}_{22}=\overset{*}{a}_{22}\ ,$$

where $\overset{\circ}{a}_{11}=e_{21}a_{11}e_{12}$ and $\overset{\circ}{a}_{22}=e_{12}a_{22}e_{21}$.

The field K is isomorphic to both C_{ii}, where $a_{ii}\longrightarrow\overset{*}{a}_{ii}$.

Then K lies in the centre of C. In particular we know that $(a_{ik}b_{ki})^{*}=(b_{ki}a_{ik})^{*}$ (by L 1.1, Chapter IV).

<u>LEMMA 6.6</u> Two elements a_{ik}, b_{ik} $i \neq k$ are linearly dependent w.r.t. K if and only if their product $a_{ik} b_{ik}$ vanishes.

To set up the vector matrix unities assume first two linearly independent elements e_{12}^{1}, e_{12}^{2} of C_{12} and an e_{12}^{3} such that $(e_{12}^{1} e_{12}^{2}) e_{12}^{3} = -e_2$. Therefore $e_{12}^{1}, e_{12}^{2}, e_{12}^{3}$ are linearly independent.

Then also $(e_{12}^{\nu} e_{12}^{\nu+1}) e_{12}^{\nu+2} = -e_2$ for ν modulo 3 and $e_{12}^{\nu}(e_{12}^{\nu+1} e_{12}^{\nu+2}) = -e_1$. From e_{12}^{ν} construct 3 linearly independent vectors e_{21}^{ν} of C_{21}. Set $e_{21}^{\nu} = e_{12}^{\nu+1} e_{21}^{\nu+2}$ and get $e_{ik}^{\nu} e_{ki}^{\mu} = -\delta_{\mu\nu} e_i$. Symmetrically we get $e_{12}^{\nu} = e_{21}^{\nu+1} e_{21}^{\nu+2}$.

The elements $e_{12}, e_{21}, e_{12}^{\nu}, e_{21}^{\nu}$ generate a hypercomplex system C^{*} over K, whose structure is completely determined by: $e_i e_j = \delta_{ij} e_i$, $e_i e_{ik}^{\nu} = e_{ik}^{\nu} e_k = e_{ik}^{\nu}$, $e_k e_{ik}^{\nu} = e_{ik}^{\nu} e_k = 0$ and $e_{ik}^{\nu} e_{ik}^{\nu} = 0$. $(i,j,k = 1,2; i \neq k; \nu$ modulo 3). Therefore these elements form a system of vector matrix unities.

The ring C can now be expressed as a linear combination of e_i and e_{ik}^{ν} with coefficients in K as follows:

$$x = x_{11}^{*} e_1 - \sum_{\mu \bmod 3} (x_{12} e_{21}^{\mu})^{*} e_{12}^{\mu} - \sum_{\mu \bmod 3} (x_{21} e_{12}^{\mu})^{*} e_{21}^{\mu} + x_{22}^{*} e_2 .$$

We can now say that for $n = 2$ a simple alternative (non assoc.) ring is a vector matrix ring over a field.

The theory used in obtaining the structure of a semi-simple and simple hypercomplex ring can be used in obtaining the same structure for a semi-simple and simple finite dimensional algebra, since these algebras contain a unity.

CHAPTER III

ALTERNATIVE DIVISION RINGS

An associative division ring is a ring which has a unity, and inverses for all non zero elements. An alternative ring is a division ring if the equations $ax = b$ and $ya = b$ have unique solutions x, y for $a \neq 0$.

The methods used by R. H. Bruck and Erwin Kleinfield [5] [6] in studying the structure of an alternative division ring are new as will be seen in the following discussion. Associative division rings are as yet not classified. The main results for alternative division rings found in Sections 9 and 10 are: An alternative division ring is either (I) a field or skew field or (II) a Cayley Dickson algebra over its centre.

SECTION 7. LINEAR FUNCTIONS AND IDENTITIES

The commutator $(x,y) = xy - yx$ has already been introduced. Two very important and easily calculated identities needed for future computations are:

(7.1) $\quad (xy,z) - x(y,z) - (x,z)y = (x,y,z) - (x,z,y) + (z,x,y)$

(7.2) $\quad (xy,z) = x(y,z) + (x,z)y + 3(x,y,z)$

A very curious linear function $f(w,x,y,z)$ is now introduced as follows:

(7.3) $\quad (wx,y,z) = (x,y,z)w + x(w,y,z) + f(w,x,y,z)$

If the ring were associative it is immediately obvious that $f(w,x,y,z) = 0$. We see that this function is non trivial for an alternative ring but does not exist for an associative ring.

We will follow Bruck and Kleinfield [5] in stating some important identities about this function.

<u>LEMMA 7.1</u> In an alternative ring R the function f is skew symmetric (i.e. if any two of w,x,y,z are equal then $f = 0$) and satisfies the identities:

(7.4) $3f(w,x,y,z) = (w,(x,y,z)) - (x,(y,z,w)) + (y,(z,w,x)) - (z,(w,x,y))$

(7.5) $f(w,x,y,z) = ((w,x),y,z) + ((y,z),w,x)$

The first part is easily proven using (1.2) substituting (7.3) into this, then permuting the elements cyclically. The second part of the proof consists of an easy computation using (7.3) and (7.4).

<u>LEMMA 7.2</u> For all x,y,z of an alternative ring we have:

(7.6) $(x^2,y,z) = x(x,y,z) + (x,y,z)x$

(7.7) $(x,xy,z) = (x,y,xz) = (x,y,z)x$

(7.8) $(x,yx,z) = (x,y,zx) = x(x,y,z)$

(7.9) $xy.zx = x(yz.x)$

(7.10) $x(y.xz) = (xy.x)z$; $(zx.y)x = z(x.yx)$

The equations (7.6), (7.7), (7.8) can be proven by easy substitution in (7.3). The equation (7.9) can be proven by using (7.8), and (7.10) by using (7.9).

We will show the proof of (7.6) to illustrate the usefulness of the function $f(w,x,y,z)$.

Proof: In (7.3) let $w = x$ and get

$$(xx,y,z) = (x,y,z)x + x(x,y,z) + f(x,x,y,z)$$

But $f(x,x,y,z) = 0$, therefore we have proven (7.6).

A new operator $g(u,v,w,x,y)$ is now defined for all u,v,w,x,y of R by:

$$f(uv,w,x,y) = uf(v,w,x,y) + f(u,w,x,y)v + (u,x,y)(v,w) + (u,w)(v,x,y) + g(u,v,w,x,y) \tag{7.10}$$

The linear function $g(u,v,w,x,y)$ is again zero for the associative case.

<u>LEMMA 7.3</u> The function $g(u,v,w,x,y)$ is skew symmetric in u,v,w and in x,y. Equivalently:

$$f(uv,u,x,y) = uf(v,u,x,y) + (u,x,y)(v,u) \tag{7.11}$$

$$f(uv,v,x,y) = f(u,v,x,y)v + (u,v)(v,x,y) \tag{7.12}$$

for all u,v,x,y of R.

The skew symmetry in x and y is obvious by definition. Equations (7.11) and (7.12) can be proven by easy computations beginning with equation (7.5).

<u>LEMMA 7.4</u> For all x,y,z of an alternative ring R,

$$((x,y,z),x,y) = (xy)(x,y,z) = -(x,y,z)(x,y) \tag{7.13}$$

$$((x,y,z)^2,x,y) = 0 \tag{7.14}$$

We will state this proof as an interesting illustration of the manipulation of the associator.

Proof: Let $u = (x,y,z)$ and $v = (x,y)$.

Let $p = (x^2,y,zy) = x(x,y,zy) + (x,y,zy)x$ (by L 7.2)

$= x.yu + yu.x$; and similarly $p = y.xu + y.ux$.

Therefore $0 = x.yu - y.xu + (y,u,x) = xy.u - (u,x,y) - yx.u$

$+ (y,x,u) + (y,u,x) = (x,y)u - (u,x,y)$

and $(u,x,y) = vu$. Similarly $(u,x,y) = -uv$

and $(u^2,x,y) = u(u,x,y) + (u,x,y)u = u.vu - uv.u = 0$.

LEMMA 7.5 Let x,y,z be any elements of an alternative ring R then (i) $((x,y,z)^4 x) = 0$, (ii) If (x,y,z) is not a divisor of zero $((x,y,z)^2,x) = 0$.

The proof of lemma 7.5 is very complicated but since it is a good illustration of the use of the function $f(w,x,y,z)$ we will display it as in [5].

Proof: Let $u = (x,y,z)$, $v = u^2$, $w = f(x,y,z,v)$. Then

$f(xv,x,y,z) = xf(v,x,y,z) + (x,y,z)(v,x)$ (by 7.11)

$= -xw + u(v,x)$

(7.15) Therefore $u(v,x) = xw + f(xv,x,y,z)$ and similarly

(7.16) $(v,x)u = wx + f(vx,x,y,z)$.

Now $f(xv,y,x,z) = xf(v,y,x,z) + (x,y)(v,x,z) + g(x,v,y,x,z)$

and $(v,x,z) = 0$ by (7.14), therefore

$f(xv,x,y,z) + xw = g(v,x,y,x,z)$. Similarly

$f(vx,x,y,z) + wx = -g(v,x,y,x,z)$, therefore

$u(v,x) = -(v,x)u$. (by (7.15) and (7.16))

Now $f(x,y,xv,z) = (xy,xv,z)-(y,xv,z)x-y(x,xv,z)$ by (7.3) and $(x,xv,z) = 0$. Therefore $f(xv,x,y,z) = (xv,y,z)x-(xv,xy,z)$. Also $(xv,y,z) = (v,y,z)x+v(y,z,x)+f(x,v,y,z) = vu+w$ and

$$(xv,xy,z) = (v,xy,z)x+v(x,xy,z)+f(x,v,xy,z)$$
$$= (xy,z,v)x+v.ux-f(xy,x,z,v) \text{ by (7.3 and (7.8).}$$

Therefore $f(xv,x,y,z) = (w-(xy,z,v))x+f(xy,x,z,v)$. Now $(xy,z,v) = y(x,z,v)+(y,z,v)x+f(x,y,z,v) = f(x,y,z,v) = w$ by (7.14) and $f(xy,x,z,v) = xf(y,x,z,v)+(xz,v)(y,x) = -xw$ by (7.3) and (7.14). Therefore $f(xv,x,y,z) = -xw$ and $u(v,x) = (v,x)u = 0$ by (7.15), (7.16). If u is not a divisor of zero then $(v,x) = 0$ and (ii) is true. Also $(u^4,x) = (v^2,x) = v(v,x)+(v,x)v = 0$ by (7.2) and (i) is true.

SECTION 8. NUCLEUS AND CENTRE OF AN ALTERNATIVE RING

D(8.1) The Nucleus of an alternative ring R is defined as the set of all n such that $(n,R,R) = 0$.

D(8.2) The centre C is the set of all c of R such that

$$(c,R) = (c,R,R) = 0$$

In an associative ring the nucleus of course is the whole ring and the centre is the set of all c in R such that $(c,R) = 0$. The significance of introducing the nucleus as well as the centre will become apparent in the discussion of an alternative ring without divisors of zero, where the nucleus will equal the centre or will be the whole ring.

We will discuss the various properties of the nucleus and centre as Bruck and Kleinfield did in [5].

<u>LEMMA 8.1</u> Let R be an alternative ring with centre C. Let S be a generating subset of R (i.e. no proper subring of R contains S). Then an element $p \in C$ if and only if $(p,S) = 0 = (p,S,S)$ and C is a subring of R.

Proof: If $p \in C$, then obviously $(p,S) = (p,S,S) = 0$. Conversely let $(p,S) = (p,S,S) = 0$. The technique used in proving the converse is to let X be the set of all x in R such that $(p,S,x) = 0$ and show that $X = R$. Similarly let y be the set of all y in R such that $(p,y,R) = 0$ and prove $Y = R$. Let z be the set of all z in R such that $(p,z) = 0$ and prove $Z = R$. The computations are simple manipulations of some of the identities in section 7.

<u>LEMMA 8.2</u> If R is an alternative ring with nucleus N and if a,b are elements of R such that $(a,b,R) = 0$, then (i) $(a,b) \in N$, (ii) If (a,b) is neither zero or a divisor of zero then $a,b \in N$, (iii) If $n \in N$ is neither zero nor a divisor of zero $nx \in N$ implies $x \in N$.

The proof of this is an easy computation again making use of the linear function $f(x,y,z,w)$.

<u>THEOREM 8.1</u> If R is an alternative ring with no divisors of zero, then if the nucleus is N and the centre C, then $N = R$ or $N = C$.

<u>Corallary</u> If $(R,R,R) \neq 0$, then $(a,b,R) = 0$ implies $(a,b) = 0$. The proof is straight forward using L 8.2.

The following lemma will play an important role in proving the main theorem about the structure of division rings of characteristic $\neq 2$.

<u>LEMMA 8.3</u> Let R be an alternative ring in which $(x,y) = 0$ if and only if $(x,y,R) = 0$. Let R contain elements a,b such that (a,b) is neither zero nor a divisor of zero. Then a necessary and sufficient condition that the element k of R belongs to the centre C of R is that $(k,a) = (k,b) = 0$.

The computations for this proof are heavy but the method is interesting. It is very similar to the method used in proving L 8.1. We will therefore sketch the proof as given in [5] leaving out most of the computations.

Proof: Let K be the set of all k in R such that $(k,a) = (k,b) = 0$. Let S be the set of all s in R such that $(K,s) = 0$. Since $C \subset K$ and $a,b \in S$, then $(K,S,R) = 0$ by hypotheses and $f(K,S,R,R) = 0$ by (7.3). Then $(K,(S,S,R)) = 0$ by (7.4) and therefore $(S,S,R) \subset S$ and $SS \subset S$ by (7.2).

By the above facts we can compute $((K,R,R),S,S) = 0$ and $(K,R,R)(S,S) = 0$. By hypotheses (a,b) is not a divisor of zero, therefore $(K,R,R) = 0$. Since $(x,y,R) = 0$ implies $(x,y) = 0$, then $(K,R) = 0$ and $K \subset C$.

Therefore $K = C$.

SECTION 9. ALTERNATIVE DIVISION RINGS OF CHARACTERISTIC NOT 2.

Every alternative division ring of characteristic not 2 is either (i) a skew field or (ii) a Cayley Dickson algebra over its centre. To prove this Bruck and Kleinfield [5] have proven something even stronger.

THEOREM 9.3 Let R be a non-associative alternative ring without divisors of zero and of characteristic not 2. Then the centre C of R contains a non zero element and R may be embedded in $R//C$ which is an alternative division algebra of order 8 over its centre $F = C//C$.

D (9.1) $R//C$ is defined as the set of all formal fractions x/c for $x \in R$ and $c \in C$, $c \neq 0$, where: (1) $x/c = y/d$ if and only if $dx = cy$, (2) $(x/c) + (y/d) = (dx + cy)/cd$, (3) $(x/c)(y/d) = (xy)/(cd)$.

The proof of theorem 9.3 raises no new difficulties and in some cases is even simpler than the proof for a division ring alone. This is true especially in the use of ordinary subrings rather than division subrings. We will proceed as in [5].

LEMMA 9.1 Let R be an alternative ring with no divisors of zero and of characteristic not 2 generated by three elements x,y,z such that $(x,y,z) \neq 0$. Let C be the centre of R, then (i) $px^2 - qx + r = 0$ for p,q,r in C and $p \neq 0$, (ii) $(x,y) \neq 0$.

The proof of lemma 9.1 is a good application of lemmas 7.4, 7.5 and 8.1. It also shows why the characteristic of the ring must be different from 2. To illustrate these points we will sketch the proof briefly.

Proof: Let $u = (x,y,z)$, $v = (x^2,y,z) = ux + xu$, $w = (x,xy,z) = ux$,

$$\text{then } u^2x^2 - (uv)x + w^2 = 0.$$

We must show that u^2, uv, and $w^2 \in C$.

If we let $S = \{xy, z\}$ then S generates R and L 8.1 can be applied. By L 7.4 and L 7.5 $u^2 \in C$.

We can compute v^2 to lie in the centres of the sub-rings generated by x^2,y,z; $x, xy+yx, z$; $x,y,xz+zx$, and therefore that $v^2 \in C$.

Similarly $(u+v)^2 \in C$ and therefore $2uv \in C$, i.e. $2(uv,R) = 2(uv,R,R) = 0$. But the characteristic is not 2 and therefore $uv \in C$.

By direct computation we can find $w^2 \in C$. This completes the proof of (i).

We may note here that this part can be proven for characteristic 2, but the computations are very involved.

Proof (ii): This part of the proof illustrates why the characteristic cannot be assumed to be 2.

Let y,z,x and $x+y,y,z$ generate R. Then we can get $px^2-qx+r=0$, $py^2-q'y+r'=0$ and $p(x+y)^2-q''y+r''=0$ for p,q,r,q',r',q'',r'' in C and $p\neq 0$.

Then $p(xy+yx)-sx-ty+c=0$, for s,t,c in C.

Assume $(x,y)=0$, then $sx-c=2pxy-ty$, and

$0=(sx-c,z,x)=(2pxy-ty,z,x)=2p(y,z,x)x-t(y,z,x)$

$=u(2px-t)$. Since $u\neq 0$ then $2px=t$.

But then $0=(t,y,z)=(2px,y,z)=2pu$.

This is a contradiction since $p\neq 0$ and the ring is of characteristic not 2. Therefore $(x,y)\neq 0$.

This proof obviously fails for characteristic 2. Also $(x,y)\neq 0$ is not true for characteristic 2, since every Cayley Dickson division algebra of characteristic 2 can be generated by three pairwise commutative elements.

THEOREM 9.1 Let R be a non associative alternative ring with no divisors of zero and of characteristic not 2. Then if C is the centre of R, every element x of R satisfies a quadratic equation $px^2-qx+r=0$ with p,q,r in C and $p\neq 0$.

The proof of this is an easy argument using T 8.1, L 9.1 and L 8.3.

The R//C of D (9.9) is an alternative ring with c/c as unity and centre F = C//C. The mapping $x\to(cx)/c$ is an isomorphism of R onto R//C and R may be imbedded into R//C by making the identification $x=(cx)/c$.

Corallary to Theorem 9.1: Under the hypotheses of T 9.1, R//C is a non-associative alternative ring with unit, without divisors of zero, of characteristic not 2 and with the field $F = C//C$ as centre. Also every element of R//C satisfies a quadratic equation $x^2 - \alpha x + \beta = 0$ with $\alpha, \beta \in F$.

We insert T 9.2 which was proven by A. A. Albert and can be found in [1].

THEOREM 9.2 Let R be an alternative ring with unit containing in its centre a field F of characteristic not 2 such that, if x is in R but not in F, x generates a quadratic field F(x) over F. Then R is an algebra of order 1,2,4,8 over F, order 8 corresponding to the case that R is not associative.

The set R//C satisfies the hypotheses of T 9.2 and therefore R//C is an algebra of order 8 over F. Also R//C has no divisors of zero and is therefore a division algebra. This proves T 9.4. An immediate result of theorem 9.4 is that if C is a field, R is a Cayley Dickson algebra over C. The main result follows at once.

> Every alternative division ring of characteristic not 2 is either (i) a field or skew field or (ii) a Cayley Dickson algebra over its centre.

Note: An interesting consequence of theorem 9.4 is that every ordered alternative ring is associative.

SECTION 10. ALTERNATIVE DIVISION RINGS OF CHARACTERISTIC 2.

Every alternative, not associative division ring R of characteristic 2 is a Cayley Dickson division algebra of order 8 over its centre. This was proven by Kleinfield [6]. His methods are the same as those already discussed in sections 7, 8, 9. For this reason the outline of Kleinfield's proof will be very brief.

In an alternative (not associative) division ring of characteristic 2 there are elements x,y,z such that $(x,y)=0$ and $(x,y,z)=v\neq 0$. If we let $x=a, y=b$ and $vz=c$, then $(a,b)=0, (a,c)=0, (b,c)=0$ and $u=(a,b,c)\neq 0$. Let $S=S[a,b,c]$ be the subring of R generated by a,b,c, and let K be the set of all k in R such that $(k,S)=(k,S,S)=0$.

<u>LEMMA 10.1</u> If $p\in R$ and $(p,a,b)=(p,a,c)=(p,b,c)=0$ then $p\in K$. This lemma which can be found in [6] is proven by an extensive use of the linear function $f(p,a,b,c)$ and the fact that the characteristic equals 2.

It is easily verified that $a^2, b^2, c^2, u\in K$. Furthermore, identities like $(a,b,c)=u$, $(ab,ac)=0$ and others obtained by permutations of a,b, and c permit us to set up a multiplication table for the elements $1,a,b,c,ab,ac,bc,a.bc$, which will show that these elements behave like basis elements of the Cayley Dickson algebra of characteristic 2.

<u>LEMMA 10.2</u> K is a field.

Proof: To prove that K is a field it must be shown that K is closed under multiplication, inverses exist, and K is associative and commutative. This proof is again a direct computation using $f(a,b,c,d)$. For example, to prove that K is closed under mult. Let $k,k' \in K$, then $(kk',a,b) = f(k,k',a,b)$ by (7.3) but $f(k,k',a,b) = ((k,a),k',b) + ((k',b),k,a)$ by (7.4)

$= 0$ by definition of K.

Therefore $(kk',a,b) = 0$, and by L 10.1 $kk' \in K$ and K is closed under multiplication.

The remaining computations are similar manipulations of the function $f(a,b,c,d)$ and its identities.

<u>LEMMA 10.3</u> If $(d,a) = 0 = (d,b)$ then $d = \alpha_0 + \alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 ab$, for some $\alpha_0 \ldots \alpha_4$ in K.

Proof: First it can be shown that $(d,c),(d,ab),(d,ac),(d,bc)$ are elements of K by manipulating the identities in section 7. Then form

$$d' = d-((d,c)u^{-1})ab-((d,ab)u^{-1})c-((d,ac)u^{-1})b-((d,bc)u^{-1})a.$$

It can be computed that:

$$(d',a) = (d',b) = (d',c) = (d',ab) = (d',ac) = (d',bc) = 0.$$

Therefore $(d',a,b) = (d',a,c) = (d',b,c) = 0$ by (7.2).

Therefore d' is an element of K by L 10.1. Therefore if $\alpha_0 = d'$, $\alpha_1 = (d,bc)u^{-1}$, $\alpha_2 = (d,ac)u^{-1}$, $\alpha_3 = (d,ab)u^{-1}$, $\alpha_4 = (d,c)u^{-1}$, then $d = \alpha_0 + \alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 ab$ for $\alpha_0 \ldots \alpha_4$ in K.

LEMMA 10.4 Any element d of R may be written as

$d = \beta_0 + \beta_1 a + \beta_2 b + \beta_3 c + \beta_4 ab + \beta_5 ac + \beta_6 bc + \beta_7 a.bc$ for $\beta_0 \ldots \beta_7$ in K.

Proof: By direct computation we find

$d(a,b,c) = (a,b,dc) + (a,b,d)c$, but (a,b,dc) and (a,b,d) satisfy the hypotheses of L 10.3, therefore we get

$$du = \alpha_0 + \alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 ab + \alpha_5 c + \alpha_6 ac + \alpha_7 bc + \alpha_8 c^2 + \alpha_9 ab.c.$$

Since $u, c^2 \in K$ if we multiply by u^{-1}, then

$$d = \beta_0 + \beta_1 a + \beta_2 b + \beta_3 c + \beta_4 ab + \beta_5 ac + \beta_6 bc + \beta_7 a.bc$$

$\beta_0 \ldots \beta_7$ in K.

We can show now that the centre of R coincides with K and that the elements 1,a,b,c,ab,ac,bc,a.bc are linearly independent over K and are therefore a set of basis elements for the Cayley Dickson division algebra R over K.

We conclude with:

THEOREM 10.1 Every alternative, not associative division ring R of characteristic 2 is a Cayley Dickson division algebra over its centre.

CHAPTER IV

SIMPLE ALTERNATIVE RINGS

The structure of an arbitrary simple alternative ring is as yet not known. However, Albert [2], with the aid of certain idempotents has been able to prove that every simple alternative ring which contains an idempotent not its unity is either associative or a Cayley Algebra.

The method used by Albert is very similar to that used by Zorn in his study of semi-simple and simple hypercomplex rings. Although their results are the same, Albert's methods are far more simple and more complete, and so we will discuss them in greater detail.

SECTION 11. PEIRCE DECOMPOSITION AND IDEALS

We will assume throughout the discussion that C contains an idempotent u not its unity.
Following Albert [2] we form the Peirce decomposition of C,

$$C = C_{11} + C_{10} + C_{01} + C_{00}.$$

This is a direct sum of modules where the C_{ij} consist of all x_{ij} such that $ux_{ij} = ix_{ij}$; $x_{ij}u = jx_{ij}$ where $i,j = 0,1$.
If $x = x_{11} + x_{10} + x_{01} + x_{00}$, then

$x_{11} = u(xu)$, $x_{10} = ux-u(xu)$, $x_{01} = xu-u(xu)$, $x_{00} = x-xu-ux-u(xu)$.

This decomposition looks exactly the same as in the associative case except for certain different multiplicative properties for the C_{ij}.

These properties are as follows.

Let $ux = \lambda x$, $xu = \mu x$, $uy = \alpha y$, $yu = \beta y$ and obtain:

$(xy)u = (\alpha+\beta-\mu)xy$, $\qquad u(xy) = (\lambda+\mu-\alpha)xy$,

$(yx)u = (\lambda+\mu-\beta)yx$, $\qquad u(yx) = (\alpha+\beta-\lambda)yx$,

$(\alpha-\mu)xy = (\beta-\lambda)yx$.

By giving $\lambda, \mu, \alpha, \beta$ appropriate values of 0 and 1 we can show that:

(11.1) C_{11} and C_{00} are orthogonal subrings of C.

(11.2) $C_{11}C_{10} \subseteq C_{10}$, $\qquad C_{10}C_{11} = 0$,

$C_{01}C_{11} \subseteq C_{01}$, $\qquad C_{11}C_{01} = 0$

(11.3) $C_{00}C_{10} = 0$, $\qquad C_{00}C_{01} \subseteq C_{01}$,

$C_{01}C_{00} = 0$, $\qquad C_{10}C_{00} \subseteq C_{10}$,

(11.4) $C_{10}C_{01} \subseteq C_{11}$, $\qquad C_{01}C_{10} \subseteq C_{00}$

The properties (11.1) (11.2) (11.3) (11.4) for the modules C_{ij} are properties satisfied by all associative rings.

However the following three properties will be those necessary to replace the identy $C_{10}^2 = C_{01}^2 = 0$ in the associative case:

(11.5) $x_{10}y_{10} = -y_{10}x_{10} = z_{01}$ for every x_{10}, y_{10} of C_{10} where z_{01} is in C_{01}.

(11.6) $x_{01}y_{01} = -y_{01}x_{01} = z_{10}$

(11.7) $x_{ij}^2 = 0$

<u>LEMMA 11.1</u> Let x,y,z be elements of the modules of C not all in the same subring C_{ii}. Then $(x,y,z)=0$ except when at least two of the elements are in the same C_{ij}.

This is a result proved by Zorn in [18]. When at least two of the elements are in the same C_{ij} or all three are in the same C_{ii} we can compute the identities:

(11.8) $z_{ji}(x_{ij}y_{ij}) = (x_{ij}z_{ji})y_{ij} = x_{ij}(y_{ij}z_{ji})$

(11.9) $(x_{ij}y_{ij})z_{ii} = (z_{ii}x_{ij})y_{ij} = x_{ij}(z_{ii}y_{ij})$

(11.10) $x_{ij}(y_{ji}z_{ji}) = z_{ji}(x_{ij}y_{ji}) - (z_{ji}x_{ij})y_{ji}$

(11.11) $x_{ij}(y_{ij}z_{ij}) = z_{ij}(x_{ij}y_{ij}) = y_{ij}(z_{ij}x_{ij})$

(11.12) $(x_{ij}y_{ij})z_{ij} = (z_{ij}x_{ij})y_{ij} = (y_{ij}z_{ij})x_{ij}$

These identities can be proven by lengthy calculations using the following identities which we have already shown to be true.

$z(xy+yx) = (zx)y + (zy)x, \quad (xy+yx)z = x(yz) + y(xz)$

$z(xy) + y(xz) = (zx)y + (yx)z$

The proofs of the remaining relations necessary to give L 11.1 are quite trivial.

<u>Corallary</u> The ring C is associative if and only if both C_{11} and C_{00} are associative and $C_{10}^2 = C_{01}^2 = 0$. This is obvious.

<u>LEMMA 11.2</u> The modules $C_{ij}C_{ij}^2$ and $C_{ji}^2 C_{ji}$ are ideals of C_{ii} contained in the centre of C_{ii} $(i,j=0,1, \ i\neq j)$.

This can be proved by an easy computation using L 11.1 and the relations (11.8) - (11.12).

<u>LEMMA 11.3</u> If B_i is an ideal of C_{ii}, then

$D_i = B_i + B_i C_{ij} + C_{ji} B_i + (C_{ji} B_i) C_{ij}$ $\quad (i, j = 0, 1, \quad i \neq j)$

is an ideal of C.

The proof of this is a lengthy computation with a straight-forward use of L 11.1 and its identities. Lemma 11.3 has a very interesting result for simple rings, since it tells us that if C_{ii} contains an ideal then C contains an ideal. This means that if C is simple C_{ii} is simple.

<u>LEMMA 11.4</u> If $C_{10}^2 C_{10} = C_{10} C_{10}^2 = C_{01}^2 C_{01} = C_{01} C_{01}^2 = 0$, then $G = C_{10}^2 + C_{01}^2$ is a proper ideal of C.

By an easy computation we see that G is an ideal of C and since $C_{11} \neq 0$ and C_{11} does not lie in G then G is a proper ideal of C.

SECTION 12. SIMPLE RINGS

<u>LEMMA 12.1</u> Let C be simple, then C_{11} is a simple associative ring and C_{00} is either zero or a simple associative ring.

Proof: We can easily compute that $C_{ij} C_{ji}$ is an associative ideal of C_{ii}. It follows immediately that

$B = C_{10} C_{01} + C_{10} + C_{01} + C_{01} C_{10}$ is an ideal of C.

Now since C is simple, $B = 0$ or $B = C$.

If $B = 0$, C_{00} is a proper ideal of C and therefore $C = C_{11}$ has u as a unity contrary to hypotheses.

Therefore $B = C$ and $C_{ij} C_{ji} = C_{ii}$, therefore the C_{ii} are associative. They are simple by L 11.3.

We proceed as Albert did, to show that the C_{ii} are fields. When C is simple then G of L 11.4 cannot be a proper ideal of C. Hence C is either associative or $C_{10}^2 + C_{01}^2 \neq 0$, i.e. one of the modules $C_{10}C_{10}^2$, $C_{01}^2 C_{01}$, $C_{10}^2 C_{10}$, $C_{01}C_{01}^2$ must not equal zero. Let us say $C_{ij}C_{ij}^2 \neq 0$. By L 11.2 $C_{ij}C_{ij}^2$ is a non-zero ideal of C_{ii} contained in the centre of C_{ii}. By L 12.1 $C_{ij}C_{ij}^2 = C_{ii}$. Therefore C_{ii} coincides with it's centre and is a field. It can be shown easily that $C_{ij}C_{ij}^2 \neq 0$ if and only if $C_{ij}^2 C_{ij} \neq 0$ and therefore that C_{11} and C_{00} are both fields.

<u>LEMMA 12.2</u> The rings C_{11} and C_{00} are isomorphic fields with unity quantities $u = e_{11}$ and e_{00} respectively, where $e = e_{11} + e_{00}$ is the unity of C, $e_{11} = e_{10}e_{01}$, $e_{00} = e_{01}e_{10}$ for quantities e_{ij} in C_{ij} such that $e_{01} = f_{10}g_{10}$ and f_{10}, g_{10} are in C_{10}.

Proof: The proof follows as in [2].

We choose f_{10} and g_{10} so that $x_{10}e_{01} = a_1 \neq 0$ is in the field C_{11}. Let b_1 in C_{11} be the inverse of a_1, then $b_1(x_{10}e_{01}) = e_{11} = (b_1x_{10})e_{01} = e_{10}e_{01}$. Therefore $e_{11}^2 = e_{11}(e_{10}e_{01}) = (e_{11}e_{10})e_{01} = (e_{10}(e_{01}e_{10}))e_{01} = e_{11}$ and so $e_{01}e_{10} = e_{00} \neq 0$. We can find $e_{00}^2 = e_{00}$ by using (11.9).

Therefore e_{00} is an idempotent of C_{00} and must be its unity.

Now we know that $B_i = C_{ii} \neq 0$ and applying L 11.3 we get $C_{11}C_{10} = C_{10}C_{00} = C_{10}$, $C_{01}C_{11} = C_{00}C_{01} = C_{01}$.

In particular $e_{00} x_{01} = x_{01}$ for every x_{01} of C_{01}, similarly $x_{10} e_{00} = x_{10}$ for every x_{10} of C_{10}.

We can easily show now that $e = e_{11} + e_{00}$ is the unity quantity of C.

Consider the mapping

$$x_{11} \to x_{11} T = e_{01}(x_{11} e_{10}) = (e_{01} x_{11}) e_{10}.$$

This is an isomorphism of C_{11} onto C_{00} such that:

(12.1) $y_{10}(x_{11} T) = x_{11} y_{10}$

(12.2) $(x_{11} T) z_{01} = z_{01} x_{11}$ for every x_{11} of C_{11}, y_{10} of C_{10}, z_{01} of C_{01}. The formulae (12.1) and (12.2) can be computed easily. Using (12.1) and (12.2) we can also compute $(x_{11} T)(y_{11} T) = (x_{11} y_{11}) T$.

Therefore, since C_{11} and C_{00} are fields, this proves that T is an isomorphism of C_{11} onto C_{00} where the inverse of T is given by

$$x_{11} = e_{10}(x_{00} e_{01}) = e_{10} y_{01}.$$

<u>LEMMA 12.3</u> If we let Z be the set of all elements $z = z_{11} + z_{11} T$ then Z is contained in the centre of C.

Proof: Commutativity of the elements can be shown easily. We know $z y_{ii} = y_{ii} z$. Now $z y_{10} = z_{11} y_{10} = y_{10}(z_{11} T) = y_{10} z$. Therefore $zy = yz$ for every y of C.

Associativity of the elements is shown as follows: We know, since $Z = C_{11} + C_{00}$, that the associators (z,x,y) are zero except possibly when x and y are in the same C_{ij}. (by L 11.1) But this can easily be shown by computing

$((z_{11} + z_{11} T)x_{10})y_{10} = (z_{11} + z_{11}T)(x_{10}y_{10})$.

Therefore Z is in the centre of C.

We have now constructed $C_{11} = e_{11}Z$ and $C_{00} = e_{00}Z$ to be one dimensional algebras over Z. By the use of our known relations it can be proven that the elements e_{ij}, f_{ij}, g_{ij} obey the following identities.

(12.3) $f_{10}\, g_{10} = e_{01}, g_{10}\, e_{10} = f_{01}, e_{10} f_{10} = g_{01}$

(12.4) $g_{01} f_{01} = e_{10}, e_{01}\, g_{01} = f_{01}, f_{01} e_{01} = g_{10}$

(12.5) $e_{ij} e_{ji} = f_{ij} f_{ji} = g_{ij}\, g_{ji} = e_{ii}$

(12.6) $e_{ij}\, f_{ji} = e_{ij}\, g_{ji} = f_{ij} e_{ji} = f_{ij}\, g_{ji} = g_{ij} e_{ij} = g_{ij} f_{ij} = 0$

<u>LEMMA 12.4</u> Let $h_{ij} h_{ji} = e_{ii}$ so that $h_{ji} h_{ij} = e_{jj}$, then $x_{ij} h_{ij} = 0$ if and only if $x_{ij} = \alpha h_{ij}$ for α in Z.

Proof: We know that $x_{ij}(e_{ii} + e_{jj}) = x_{ij} = (x_{ij}\, h_{ij})h_{ji} + (x_{ij}\, h_{ji})h_{ij}$.

If $x_{ij} h_{ij} = 0$, then $x_{ij} = \alpha h_{ij}$, where α is in Z.

Conversely, if $x_{ij} = \alpha h_{ij}$ then $x_{ij} h_{ij} = \alpha h_{ij}^2 = 0$.

<u>LEMMA 12.5</u> If $x_{ij} e_{ji} = x_{ij} f_{ji} = x_{ij}\, g_{ji} = 0$, then $x_{ij} = 0$.

Proof: If $x_{ij}\, h_{ji} = \alpha\, e_{ii}$ and $h_{ji}\, x_{ij} = \beta\, e_{jj}$, then

$h_{ji}(x_{ij} h_{ji}) = \alpha\, h_{ji} = (h_{ji} x_{ij})h_{ji} = \beta\, h_{ji}$.

If $h_{ji} \neq 0$ then $\alpha = \beta$.

Now $x_{ij} e_{ij} = \pm\, x_{ij}(f_{ji}\, g_{ji})$ by (12.3) and (12.4)

and $x_{ij} e_{ij} = \pm(g_{ji}(x_{ij}\, f_{ji}) - (g_{ji} x_{ij}) f_{ji})$ by (11.8).

Therefore $x_{ij}e_{ij} = 0$ and $x_{ij} = \alpha e_{ij}$ for α in Z by L 12.4. Similarly $x_{ij} = \beta f_{ij}$. If $\alpha \neq 0$, $\alpha e_{ij} f_{ij} = \alpha g_{ij} = \beta f_{ij}^2 = 0$ and $g_{ij} = 0$ which is a contradiction. Therefore $\alpha = 0$ i.e. $x_{ij} = 0$.

Lemma 12.5 implies that none of the g_{ij} are linear combinations of f_{ij} and e_{ij} and that $\alpha e_{ij} \neq f_{ij}$ for α in Z. If $g_{ij} = \alpha e_{ij} + \beta f_{ij}$, then $g_{ij}e_{ij} = \pm f_{ij} = \beta f_{ij} e_{ij} = \pm \beta g_{ij}$. This is a contradiction. We see therefore that e_{ij}, f_{ij}, g_{ij}, are linearly independent over Z.

THEOREM 12.1 A simple alternative (not associative) ring C is an eight dimensional Cayley algebra over Z.

Proof: We have shown that C contains an algebra D over Z with the multiplication table given by (12.3) - (12.6). We have shown by L 12.5 that e_{ij}, f_{ij}, g_{ij} are linearly independent over Z. It remains only to show that e_{ij}, f_{ij}, g_{ij} form a basis of C_{ij} over Z. If we let $x_{ij}e_{ji} = \alpha e_{ii}$, $x_{ij}f_{ji} = \beta e_{ii}$, $x_{ij}g_{ji} = \gamma e_{ii}$, for α, β, γ in Z, then for $y_{ij} = x_{ij} - (\alpha e_{ij} + \beta f_{ij} + \gamma g_{ij})$ we can compute:

$$y_{ij} e_{ji} = (\alpha - \alpha) e_{ii} = 0$$
$$y_{ij} f_{ji} = (\beta - \beta) e_{ii} = 0$$
$$y_{ij} g_{ji} = (\gamma - \gamma) e_{ii} = 0$$

Therefore by L 12.5 $y_{ij} = 0$, and $x_{ij} = \alpha e_{ij} + \beta f_{ij} + \gamma g_{ij}$, then e_{ij}, f_{ij}, g_{ij} form a basis of C_{ij} over Z.

Every Cayley algebra has a scaler extension which is isomorphic over its centre F to the algebra

$$C = e_{11}F + e_{00}F + C_{10} + C_{01}$$

where $C_{ij} = e_{ij}F + f_{ij}F + g_{ij}F$.

Therefore we have shown that a simple alternative (not associative) ring is an eight dimensional Caley algebra over Z.

CHAPTER V

SEMI-SIMPLE ALTERNATIVE RINGS

A structure theory for an arbitrary semi-simple alternative ring to parallel Jacobson's [9] has as yet not been discovered. Jacobson's results yield that an associative semi-simple ring is isomorphic to a subring of the complete direct sum of primitive rings (this is a subdirect sum), where a primitive ring is isomorphic to a dense ring of linear transformations on a right vector space over a division ring. The structure of an alternative primitive ring is completely unknown.

The closest parallel to Jacobson's structure theory for the alternative case has been made by Kaplansky [10]. He finds it necessary to apply a hypotheses which will give a supply of idempotents. For a π regular ring (i.e. a ring for which for every element a there exists an x and an integer n such that $a^n x a^n = a^n$). Kaplansky's results yield: A semi-simple π regular alternative ring is a subdirect sum of rings, each of which is a division ring, a Cayley matrix ring over a field or a primitive associative ring.

We will proceed as Kaplansky did in [10].

SECTION 13. PROPERTIES

The radical defined for the purpose of this study is the Smiley radical R(A)(see Chapter I). This is equivalent to the radical used by Jacobson for the associative study.

For this radical it is necessary to show also:

<u>THEOREM 13.1</u> If A is an alternative ring and I is a two-sided ideal in A, then $R(I) = R(A) \cap I$.

To prove this it is necessary to prove first,

<u>LEMMA 13.1</u> If A is an alternative ring and N is the set of all x with $axa = 0$ for all a in A, then N is a two ideal in A.

The proofs of these are straight forward using no new concepts and can be found in [10].

The Peirce decomposition discussed in Chapter IV will be needed along with the definitions,

(13.1) $D = C_{10}C_{10}^2 + C_{10}^2 C_{10} + C_{01}C_{01}^2 + C_{01}^2 C_{01}$

(13.2) $F = C_{10}C_{01} + C_{10} + C_{01} + C_{01}C_{10}$.

We now introduce the ideal H_i to be the subset of C_{ii} which both left annihilates C_{ij} and right annihilates C_{ji}. The main purpose of introducing H_i is that it can be found to contain all the non zero associators in C_{ii}.

(13.3) Let $H = H_1 + H_0$. Clearly H is an ideal.

Consider the mapping sending $x_{ii} \to$ the left multiple it effects on C_{ij}

i.e. $x_{ii} \to x_{ii}x_{ij}$.

This provides a homomorphism of the C_{ii} into the associative ring of endomorphisms of the additive group of C_{ij}. (Can be proven with the use of L 11.1.)

Therefore C_{ii} modulo the kernel is associative. Since H_i is the intersection of two such kernels C_{ii}/H_i is associative, i.e. H_i contains all the associators formed within C_{ii}.

It is now obvious using L 11.1 that:

<u>LEMMA 13.1</u> If $C_{10}^2 = C_{01}^2 = 0$, then A/H is associative. This is important since we can now say that if $C_{10}^2 = C_{01}^2 = 0$, then H contains all the non zero associators of A.

It can also be easily computed that

<u>LEMMA 13.2</u> The following statements are equivalent: (a) e is central, (b) $C_{10} = C_{01} = 0$, (c) e is in H.

<u>LEMMA 13.3</u> Suppose that A has no nil ideals, that e is non-central and that $D = 0$. Then there exists an ideal H' in A such that H' does not contain e and A/H' is a semi simple associative ring.

Proof: If e is non central, then by L 13.2 A/H contains a non zero idempotent. If we call H'/H the radical of A/H then A/H' is semi-simple and H' does not contain e.

If $D = 0$ (D defined by 13.1), then $C_{10}^2 + C_{01}^2$ is a nil ideal of A by L 11.4. Since A contains no nil ideals $C_{10}^2 = C_{01}^2 = 0$ and therefore by L 13.1 A/H is associative and therefore A/H' is a semi-simple associative ring.

By a lengthy computation using the results of Section 11 it can be computed that:

LEMMA 13.4 If J,M are ideals in C_{ii} and I,L the ideals they generate in A, then IL is the ideal generated by JM.

$$IL = JM + JMC_{ij} + C_{ji}JM + C_{ji}(JM)C_{ij}.$$

From this we see that if A has no nilpotent ideals then C_{ii} has no nilpotent ideals and in particular the centre of C_{ii} has no nilpotent ideals.

LEMMA 13.5 If $x \in C_{ij}$, $y \in C_{ji}$ such that (1) xy is an idempotent u in the centre of C_{ii}, (2) yx is in the centre of C_{jj}, (3) $ux = x$, $yu = y$, then $xy + yx$ is a central idempotent in A.

The proof is a technical manipulation of the multiplicative properties of this Peirce decomposition found in Chapter IV.

Let us now define a Cayley matrix ring. Take eight elements called Cayley matrix units e_{11}, e_{00}, e_{10}, e_{01}, f_{10}, g_{10}, g_{01}, f_{01}, satisfying the following rules:

$e_{11}^2 = e_{11}$, $e_{11}e_{10} = e_{10}$, $e_{11}e_{00} = e_{11}e_{01} = e_{10}^2 = 0$,

$f_{10}g_{10} = -g_{10}f_{10} = e_{01}$, $e_{10}e_{01} = e_{11}$, $e_{10}f_{01} = 0$, and all other rules derived by symmetry and cyclic permutation.

Let T be a commutative, associative ring. Form all linear combinations of the above Cayley matrix units with coefficients in T to get a Cayley matrix ring over T.

From the results of Chapter IV we can prove:

LEMMA 13.6 If A is an alternative ring with a unit 1, and if A contains a set of Cayley matrix units with $1 = e_{11} + e_{00}$; and if T is the set of all elements $x + e_{01}xe_{10}$ where x ranges over $e_{11}Ae_{11}$, then T is the centre of A and A is isomorphic to the Cayley matrix ring over T.

LEMMA 13.7 Let A be an alternative ring with unit, and suppose there are elements x,y,z in C with

$$x.yz = e, \quad yz.x = 1-e,$$

then $e_{11} = e$, $e_{00} = 1-e$, $e_{10} = x$, $f_{10} = y$, $g_{10} = z$, $e_{01} = yz$, $f_{01} = zx$, $g_{01} = xy$ are a set of Cayley matrix units.

This can be calculated easily. This lemma will be of important use in the construction of Cayley matrix units for Zorn and π regular rings.

LEMMA 13.8 Let T be a commutative associative ring with unit, and let A be a Cayley matrix ring over T. Then any right ideal in A is two sided and consists precisely of all elements with coefficients in a prescribed ideal of T.

Proof: Let J be a right ideal in A and I be the subset of T consisting of all elements appearing as coefficients in J.

We wish to show that if $c \in I$ then $c(e_{11} + e_{00}) \in J$. For if this is true, then for any Cayley matrix unit h_{ij},

$ch_{ij} = c(e_{11} + e_{00})h_{ij} \in J$, since J is a right ideal.
By a series of successive multiplications on any Cayley matrix unit with other Cayley units we can show that $c(e_{11} + e_{00})$ is in J.
J is a two ideal since: $\sum \beta_i k_{ij} \sum \alpha_i h_{ij} = \sum (\beta_i \alpha_i) k_{ij} h_{ij}$
where $\sum \beta_i k_{ij} \in A$ and $\sum \alpha_i h_{ij} \in J$
and $\beta_i \alpha_i \in I$ if $\alpha_i \in I$, since I is a two ideal, therefore

$$\sum \beta_i \alpha_i k_{ij} h_{ij} \in J.$$

We now define a Zorn ring which will supply us with necessary idempotents.

(13.4) An alternative ring A is called a Zorn ring, if for every non-nilpotent element a of A there exists an element $b \in A$ such that ab is a non zero idempotent. The Zorn ring gets its name from the study of the hypercomplex ring (by Zorn) which satisfies these conditions.

<u>LEMMA 13.9</u> Let A be a Zorn ring in which all idempotents are central. The radical of A is then exactly the set of all nilpotent elements.

Proof: Since in any Zorn ring the radical is a nil ideal, it is only necessary to prove that the set N of nilpotent elements is a right ideal. This proof is a simple application of the hypotheses.

It will now be useful to introduce the following definitions.

(13.5) An alternative ring is said to be regular if for every a there exists an x with $axa = a$.

(13.6) An alternative ring is said to be strongly regular if for every a there exists an x with $aax = a$. The important use of strong regularity is shown in L 13.10.

In order to obtain a closer parallel with the associative case it is necessary to strengthen the hypotheses on the ring to what we call π regularity. This will enable us to identify the regular maximal right ideals.

(13.7) An alternative ring is π regular if for every a there exists an x and an integer n such that $a^n x a^n = a^n$.

π regular rings are not restricted since they consist of all finite rings and algebraic algebras.

We see readily that regular implies π regular implies Zorn ring.

<u>LEMMA 13.10</u> The following statements are equivalent for an alternative ring A. (1) A is strongly regular, (2) A is regular with no nilpotent elements, (3) A is π regular with no nilpotent elements, (4) A is π regular, semi-simple and all idempotents are in the centre.

The proof is a simple application of the definitions and L 13.9.

This lemma shows the relation between strongly regular, regular, and π regular rings, it will be used extensively to prove the main theorem.

<u>LEMMA 13.11</u> If A is a Zorn ring and I is a two ideal in A, then I is a Zorn ring. If I is a nil ideal then A/I is a Zorn ring.

The techniques used in this proof are not unusual.

It can also be shown that:

<u>LEMMA 13.12</u> If A is a π regular alternative ring so is any ideal in A and any homomorphic image of A.

<u>LEMMA 13.13</u> If A is a strongly regular alternative ring, then any left or right ideal in A is two sided. If M is a maximal right ideal in A, then A/M is a division ring.

Proof: The argument goes as follows:

The right ideal K generated by a is also generated by the central idempotent ax where $aax = a$.

Therefore $yax = axy \in K$ and K is a two sided ideal.

The ring A/M is strongly regular and therefore by L 13.10 A is semi simple and all its idempotents are central. Therefore $A = C_{11}$ and the only idempotents are 0 and 1, so that if $a(ax) = a$, then $ax = 1$ and an inverse exists. Therefore A/M is a division ring.

Lemma 13.13 is very important since in considering the structure of semi-simple π regular rings the A/M of this lemma becomes a summand in the subdirect sum.

SECTION 14. THE STRUCTURE OF SEMI-SIMPLE ALTERNATIVE RINGS

The first theorem is very important since it shows how to build a direct summand which is a Cayley matrix ring. This is used as lemma 13.13 is to obtain a summand in the structure of semi-simple π regular rings.

<u>THEOREM 14.1</u> Let A be a Żorn ring with no nilpotent ideals and e an idempotent such that $D \neq 0$ (see 13.1). Then A contains a central idempotent v such that $ve \neq 0$ and vA is a Cayley matrix ring over a commutative semi-simple ring with unity.

Proof: Suppose $C_{ij} C_{ij}^2 \neq 0$, then $C_{ij} C_{ij}^2$ is an ideal of C_{ii} contained in its centre. By L 13.4 $C_{ij} C_{ij}^2$ has no nilpotent ideals. Choose a_1, b, c in C_{ij} so that $a_1.bc$ is a non zero element r. This element r is not nilpotent, therefore there exists an s in A such that rs is a non zero idempotent. Let s_{ii} be the C_{ii} component of s. Then rs_{ii} is a non zero idempotent u in the centre of C_{ii}.

Let $a = s_{ii}a_1$, then $a.bc = u$. Let $x = ua$, $y = ub$ and $z = uc$. It can now be computed that x and yz satisfy the hypotheses of L 13.5.

Therefore $v = x.yz + yz.x$ is a central idempotent in A. Clearly $ve \neq 0$.

If we now drop down to the direct summand vA and regard v as a unit element, then by lemmas 13.6 and 13.7 we see that vA is a Cayley matrix ring over a commutative associative ring T. Since T is also a Zorn ring it can have no nilpotent elements and is therefore semi-simple.

In order to prove that the radical of a Zorn ring is the intersection of the regular maximal right ideals, it is necessary first to construct a descending composition series as follows.

<u>THEOREM 14.2</u> Let A be a semi-simple Zorn ring. There exists a well ordered descending chain $\{I_\rho\}$ of ideals in A, starting at A and ending at I_λ with the properties:

(1) In I_λ all idempotents are central.

(2) Each $I_\rho / I_{\rho+1}$ is either associative or a Cayley matrix ring over a commutative semi-simple ring.

(3) If α is a limit ordinal, then I_α is the intersection of the preceding I's.

The argument here is interesting to follow. It is divided into three cases. (1) In I_λ all idempotents are central. (2) All idempotents are not central and $D \neq 0$. (3) All idempotents are not central and $D = 0$.

CASE (1): Suppose we have reached I_ρ, then by L 13.1 I_ρ is semi-simple. If all idempotents are central this ends the construction. If not, pick a non central idempotent e in I_ρ and form the Peirce decomposition of A relative to e.

CASE (2): $D \neq 0$. Take the central idempotent v of T 14.1 and let $K = (1-v)A$. If K contains I_ρ then $e \in K$ and $ev = 0$, which is a contradiction. Therefore K does not contain I_ρ.

Define $I_{\rho+1} = K \cap I_\rho$, then $I_\rho / I_{\rho+1} \cong I_{\rho+K}/K$.

The latter is an ideal in the Cayley matrix ring A/K.

Then $I_\rho / I_{\rho+1}$ is a Cayley matrix ring over a commutative semi-simple ring (by L 13.8).

CASE (3): $D = 0$. Consider the ideal H' of L 13.3. It does not contain I_ρ.

Define $I_{\rho+1} = H' \cap I_\rho$, then $I_\rho / I_{\rho+1} \cong (H' + I_\rho)/H$.

Therefore by L 13.3 $I_\rho / I_{\rho+1}$ is a semi-simple associative ring.

<u>THEOREM 14.3</u> For any element a of a Zorn ring the following three statements are equivalent. (1) The element a is in the radical. (2) The element a is properly nilpotent. (3) The element ab is right quasi regular for every b.

Proof: We already know that (1) implies (2) and (2) implies (3). To show that (3) implies (1) see [10] for a straightforward proof using the descending transfinite composition series of T 14.2.

D (14.1) A right ideal M is said to be regular if there exists an element g such that gx-x is in M.

THEOREM 14.4 In any alternative ring the radical is contained in every regular maximal right ideal.
This proof is the same as the corresponding proof for an associative ring.

The radical of an associative ring with unity is the intersection of the maximal right ideals. The radical of an arbitrary associative ring is the intersection of ideals whose difference rings are primitive [9].
In a similar manner:

THEOREM 14.5 The radical of a Zorn ring is the intersection of the regular maximal right ideals.

This is a very important result in alternative theory and so the proof will be worth noting.

Proof: The radical is contained in the intersection of the regular maximal right ideals (by T 14.4).
Conversely, if the intersection of the regular maximal right ideals is not in the radical, it cannot be a nil ideal and so must contain a non zero idempotent. We must therefore show that an idempotent e contained in every regular maximal right ideal is zero.

Assume by L 13.11 that A is semi-simple and form the Peirce decomposition relative to e.

CASE (1): $D \neq 0$. The set N is a maximal right ideal of vA therefore $N + (1-v)A$ is a regular maximal right ideal and e lies in every $N + (1-v)A$. Apply T 14.1. By L 13.8 we see that $\bigcap N's = 0$, hence $e \in (1-v)A$. Therefore $ev = 0$, and this is a contradiction to T 14.1. Therefore $e = 0$.

CASE (2): If $D = 0$, then e is non central. By associative theory e lies in the ideal H' of L 13.3. This is a contradiction. Therefore $e = 0$.

CASE (3): If e is central, take any maximal right ideal N in eA and observe that e is not in $N + (1-e)A$, for if it were, N would be all of A. Therefore e must equal 0. We conclude then that the regular maximal right ideals are in the radical.

In order to be able to identify the regular maximal right ideals we find it necessary at this point to strengthen our hypotheses from a Zorn ring to a π regular ring. It is necessary to construct an ascending composition series. This construction involves the taking of homomorphic images. Since π regularity is preserved under homomorphism and the Zorn property is not always preserved, we find it necessary to switch to π regularity.

<u>THEOREM 14.6</u> Let A be a π regular alternative ring. There exists a well ordered ascending chain of ideals $\{I_\rho\}$ in A, beginning at 0 and ending at I_λ with the properties:

(1) A/I_λ is strongly regular.

(2) Each $I_{\rho+1}/I_\rho$ is either nil, associative, or a Cayley matrix ring over a commutative ring with unity.

(3) For each limit ordinal α, I_α is the union of the preceding I's.

The method used in proving this is similar to other methods in this section already shown. It will therefore be omitted.

<u>THEOREM 14.7</u> Let A be a π regular alternative ring and M a regular maximal right ideal in A. Then either

(1) M is two sided and A/M is a division ring or a Cayley matrix ring over a field or

(2) If we write P for the set of all x with $Ax \subset M$, then $P \subset M$, P is an ideal in A, and A/P is a primitive associative ring.

Proof: Apply T 14.6. If M contains I_λ, then by L 13.13 M is a two sided ideal and A/M is a division ring.

Otherwise there exists an index ρ such that M contains I_ρ but not $I_{\rho+1}$. We will write I for $I_{\rho+1}$. We work mod. I_ρ. The ideal I can be (1) a nil ideal, (2) a Cayley matrix ring with unit, (3) associative.

CASE (1): I cannot be a nil ideal for if it were, then by T 14.4 I would be in M and this is a contradiction.

CASE (2): If I is a Cayley matrix ring with unit then I is a direct summand of A.

$A = I \oplus J$ and M is of the form $M = N \oplus J$

where N is a maximal right ideal in I.

Then by L 13.8 N is a two sided ideal and I/N is a Cayley matrix ring over a field. Hence M is two sided and $A/M \cong I/N$. Therefore A/M is a Cayley matrix ring over a field.

CASE (3): When I is associative, $M + I = A$.

All associators $(x,y,z) \in M$ for every $x,y,z \in A$.

For x in A let T_x be the right multiplication effected by x on the space of cosets modulo M.

Since all associators lie in M

$x \rightarrow T_x$ is a homomorphism of A into the associative ring of endomorphisms of the additive group A/M. The kernel is P defined in the hypotheses as the set of all x with $Ax \subset M$, and A/P is associative.

<u>CONCLUSION</u>

Theorems 14.6 and 14.7 imply that a semi-simple π regular alternative ring is a subdirect sum of rings each of which is a division ring, a Cayley matrix ring over a field or a primitive associative ring. It seems possible to surmise that when the theory of arbitrary alternative rings is worked out these last

three will comprise the primitive rings, so that a parallel to Jacobson's structure theory for associative rings would be available. Although even if this were so primitive alternative rings are still not classified.

It is of interest to note here that in the topological field, Kaplansky has proven that a compact semi-simple alternative ring is a complete direct sum of finite simple rings. This is equivalent to a semi-simple hypercomplex ring being a direct sum of simple rings. It is also equivalent to the associative structure theory for semi-simple rings using the Brown and McCoy radical.

CHAPTER VI

MAIN RESULTS AND GAPS IN THE THEORY

The study of rings and algebras can be divided into four cases. (1) Rings with chain conditions, (2) finite dimensional algebras, (3) rings with no chain conditions, (4) infinite dimensional algebras. In each case the radical, semi-simplicity and simplicity of the set is considered. The associative structure theory for these four cases is essentially complete. The equivalent alternative structure theory still contains many unsolved problems.

The results of Chapter I yield: The important radicals which exist for associative rings and algebras exist equivalently for alternative rings and algebras. Most of the properties of the radical are the same for both alternative and associative rings (algebras). The major gap in the theory of the radical exists for Case (3) above. This radical is defined by Smiley as the totality of elements $a \in A$ for which each right ideal $r(a)$ generated by a is right quasi regular. This is equivalent to the Jacobson radical for associative rings. However, Jacobson was able to study the relations between the regular maximal right ideals and the radical by making use of the fact that $\{x + ax\}$ is a right ideal. In alternative theory we know only that a is right quasi regular if and only if every element of A has the form ax+ x for some x in A. Therefore this important problem still remains unsolved:

QUESTION 1 What is the relation between the radical and the regular maximal right ideals of an alternative ring with no chain conditions?

Chapter II yields a structure for semi-simple and simple alternative rings with chain conditions on modules which is equivalent to the associative structure theory for rings with chain conditions on ideals.

For associative rings: A semi-simple ring with chain conditions on ideals is a direct sum of simple rings which are in turn total matrix rings over division rings.

For alternative rings: A semi-simple alternative ring with chain conditions on modules (this ring is called a hypercomplex ring) is a direct sum of simple hypercomplex rings which are in turn either associative, alternative fields or Cayley algebras. The chain conditions placed on the modules are very complicated and so we pose the problem:

QUESTION 2 Is it possible to obtain a structure theory for alternative rings with weaker chain conditions than those used by Zorn?

The structure of semi-simple and simple finite dimensional alternative algebras follows immediately from the above. Semi-simple and simple finite dimensional alternative algebras have unity elements. An ideal in an algebra with a unity is also an ideal in the corresponding ring. Therefore the theory of Chapter II can be applied to finite dimensional algebras.

Chapter III yields a structure for alternative fields or alternative division rings. An alternative division ring of characteristic not 2 is either a field or skew field or a Cayley algebra over its centre. An alternative division ring of characteristic 2 is a Cayley Dickson division algebra of order 8 over its centre. Since an alternative division algebra contains a unity, its structure is immediately determined by the preceding results. Independent investigations have also been made by Zorn and Schafer whose results imply that: An alternative, not associative, division algebra over a field F is an algebra of degree 2 and order 8 over F. This corresponds with the above results. It is of interest to note here that associative division rings are as yet not classified.

The results of Chapter IV yield a structure for simple alternative rings without chain conditions but subject to the existence of an idempotent. "Every simple alternative ring which contains an idempotent not its unity is either associative or a Cayley algebra." The following problem still exists.

<u>QUESTION 3</u> What is the structure of an arbitrary simple alternative ring with no chain conditions?

For associative rings without chain conditions Jacobson has developed a beautiful structure theory; An associative semi-simple ring is isomorphic to a subring of a complete direct sum of primitive rings (this is a subdirect sum), where a primitive ring is isomorphic to a dense ring of linear transformations on a right vector space over a division ring. Chapter V yields a close parallel to this theory for special rings called Zorn rings and π regular rings. For a Zorn ring, question 1 is answered, i.e. the radical of a Zorn ring is the intersection of the regular maximal right ideals. For a π regular ring a structure theory is developed to parallel Jacobson's. "A semi-simple π regular alternative ring is a subdirect sum of rings, each of which is a division ring, a Cayley matrix ring over a field, or a primitive associative ring."

Brown and McCoy worked out a structure theory for their own generalized radical. They found that a semi-simple

alternative ring (i.e. a ring with a zero Brown and McCoy radical) is a subdirect sum of right primitive rings. It seems plausible to expect that when the theory of arbitrary alternative rings is worked out for a Jacobson radical, the three types of rings in the subdirect sum of semi-simple π regular rings will comprise precisely the primitive rings.
This then suggests two very important unsolved problems.

<u>QUESTION 4</u> What is the structure of an arbitrary semi-simple alternative ring (w.r.t. Jacobson radical) without chain conditions? Is it a subdirect sum of primitive alternative rings?

Probably the most important problem is:

<u>QUESTION 5</u> What is the nature of primitive alternative rings?

Note: Since π regular rings contain all algebraic algebras, Kaplansky has developed a structure theory which also compares with Jacobson's for algebraic algebra.

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