BOUNDARY CONDITIONS IN THE MECHANICS OF RELATIVISTIC WAVE FIELDS

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CONTENTS

SUMMARY

In the usual discussion of relativistic wave fields with variational principles, only the field equations are obtained from the variational principle. It is shown, in this investigation, that the form of the boundary conditions as well may be obtained from the action-principle for these fields. The boundary conditions which are allowed depend on the form of the Lagrangian density which is used in the action-principle. The scalar field, the vector field and the Dirac field are discussed. The usual Lagrangian densities, for these fields, do not contain second derivatives. By generalizing the form of the Lagrangian density, for the scalar and vector fields, to contain second derivatives, it is shown that more boundary conditions are allowed than in the usual formulation. In particular, the use of the generalized Lagrangian density allows the possibility of linear homogeneous boundary conditions. This generalization also modifies the definition of the conjugate momentum fields, the stress-energy tensor and the charge-current vector. Thus the conjugate momentum fields for the vector field are defined so that the time components of these fields do not vanish identically. A symmetry in formulation between the scalar and the vector fields is obtained. The classical (unquantized) formulation for the scalar and the vector fields is completed with the modified field quantities.

INTRODUCTION

1.1 Field Theories and Variational Methods

The motions of wave fields are usually described by variational methods. These are analytical methods in which the field is represented by a function - called the Lagrange function - which is employed in a variational principle. If the Lagrange function and the principle are properly chosen they may describe the motion of the field completely.

In relativistic field theories we discuss physical systems which consist of field functions defined in given regions of space-time. The definitions of these field functions are stated in the form of partial differential equations which must be satisfied by the field functions at every point of their space-time regions. The solutions of these differential equations are restricted by conditions existing at the space-time boundaries of the fields. Since, in describing the motion of the system, we are usually interested in the progression, through time, of the physical system, it is customary to specify that the space-time region for the field be open with regard to the time co-ordinate. Then the field is given meaning in the following way. We seek those solutions of the differential equation of the field which can be separated into a product of two terms, one of which is time-dependent only and the other space-dependent only. Then we require that the space-dependent part of these solutions satisfy, for all values of time, certain conditions which we specify at the boundary. If the field region is finite the solutions of the field equation form a denumerably infinite set, the so-called normal modes of the system. The equations of motion for the field are the set of equations of motion for these normal modes. If a variational principle is to describe a relativistic wave field we ask that it give meaning to this field, that is, we ask that one principle be a unifying basis from which the entire denumerably infinite set of equations of motion for the normal modes follows directly.[1] This is equivalent to saying that the variational principle be required to yield the differential equation of the field and the conditions which are to hold at the space boundary of the field.

In the discussion of field theories by variational principles the boundary conditions are not usually recognized as a consequence of the principle. The variational method usually involves a fundamental quantity called 'action' and the principle associated with the method is then the statement that this action is stationary. The discussion above has shown that if this action-principle is to describe any wave-field completely we will require that it yield the partial differential equations of the field and also the space boundary conditions which may accompany these field equations. The manner in which the action-principle does this is readily seen if we examine the variational methods applied to a simple wave field, the real scalar field.

We assume that we have a field function ψ defined in a region τ, in space-time. τ is to be a cylinder which is open in the direction of the time-co-ordinate, x_4 ($=ict$). The intersection of any $x_4 =$ constant plane and τ forms a volume, v, in the three dimensional sub-space defined by the spatial co-ordinates, x_1, x_2, x_3. We require that this volume

[1] Variational principles are discussed very generally in " The Variational Principles of Mechanics" by Cornelius Lanczos (University of Toronto Press).

be fixed for all values of the time-co-ordinate x_4. Let the two-dimensional surface, in (x_1, x_2, x_3), which encloses V, be S. S is the space boundary of the field and is fixed for all values of time.

We represent the field by a Lagrange function, $\mathcal{L}$, which we call the Lagrangian density. $\mathcal{L}$ is a function of ψ and its space-derivatives and time-derivatives. It will be shown, in a later section, that relativistic considerations restrict the form of the dependence of $\mathcal{L}$, on ψ, to the following:

$$\mathcal{L} = \mathcal{L}(\psi, \partial_\alpha \psi, \partial_\alpha \partial_\alpha \psi) \qquad \alpha = 1,2,3,4$$

where $\partial_\alpha \equiv \frac{\partial}{\partial x_\alpha}$, x_1, x_2, x_3 are Cartesian space-co-ordinates

and $\partial_\alpha \partial_\alpha \psi \equiv \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2} + \frac{\partial^2 \psi}{\partial x_4^2}$

The notation, above, for the co-ordinates and for derivatives with respect to co-ordinates will be used throughout this investigation. The dummy index convention for the summation over repeated indices, as in $\partial_\alpha \partial_\alpha \psi$ above, will always be used unless otherwise stated. $\partial_\alpha \partial_\alpha \psi$ is frequently written $\Box \psi$ and called the Dalembertian of ψ.

In order to discuss the motion of the field with the aid of a variational principle, we define the action, I, for the field, in the following way:

$$I \equiv \int_{x_4^0}^{x_4^1} \int_V \mathcal{L}\, d\tau \qquad (1)$$

where x_4^0 and x_4^1 are two values of x_4

and $d\tau$ is the four-dimensional element of volume.

The action principle now states that, for arbitrary x_4^0 and x_4^1, the action, as defined in the above manner, is stationary in the following sense. Assume a variation, $\delta\psi$, in the functional dependence of ψ on its variables. Let $\delta\psi$ be completely arbitrary except at x_4^0 and x_4^1 where it vanishes, and on S where it is consistent with the boundary conditions for the field. Not only the variations $\delta\psi$ but also the variations $\delta\partial_\alpha\psi$ vanish at x_4^0 and x_4^1. However, only the vanishing of $\delta\psi$ and $\delta\partial_4\psi$, at x_4^0 and x_4^1, will be required in the variational procedure. If δI is the variation in the action, I, corresponding to the variation $\delta\psi$ in ψ, then the action principle may be stated, explicitly: for $\delta\psi$ which are variations about the correct functional dependence of ψ on its variables the corresponding δI vanishes. That is,

$$0 = \delta I = \delta \int_{x_4^0}^{x_4^1}\int_V \mathcal{L}\, d\tau = \int_{x_4^0}^{x_4^1}\int_V \delta\mathcal{L}\, d\tau \qquad (2)$$

If the variation, $\delta\psi$, is small we may expand $\delta\mathcal{L}$ in a Taylor's series, disregarding higher powers in $\delta\psi$ than the first.

Then $$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)}\partial_\alpha\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\gamma\partial_\gamma\psi)}\partial_\alpha\partial_\alpha\delta\psi$$

If this is substituted into the action principle the terms involving $\partial_\alpha\partial_\alpha(\delta\psi)$ and $\partial_\alpha(\delta\psi)$ may be integrated by parts to obtain:

$$0 = \delta I = \int_{x_4^0}^{x_4^1}\int_S \left\{ \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\gamma\partial_\gamma\psi)}\partial_\alpha\delta\psi - \partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\gamma\partial_\gamma\psi)}\delta\psi\right\} n_\alpha\, dS\, dx_4$$

$$+ \int_{x_4^0}^{x_4^1}\int_V \left\{\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)} + \partial_\alpha\partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\gamma\partial_\gamma\psi)}\right\}\delta\psi\, d\tau$$

where n_α is the unit-normal to S.

Since $\delta\psi$ is completely arbitrary for the whole region V - S for all values of X_4 between x_4^0 and x_4^1, we may write:

$$\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)} + \partial_\alpha\partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\gamma\partial_\gamma\psi)} = 0$$

which is the Euler-Lagrange equation for the field.

Then the action principle for the field may be written:

$$0 = \delta I = \int_{x_4^0}^{x_4^1}\int_S \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi)}\delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\gamma\partial_\gamma\psi)}\partial_\alpha\delta\psi - \partial_\alpha\frac{\partial \mathcal{L}}{\partial(\partial_\gamma\partial_\gamma\psi)}\delta\psi \right\} n_\alpha\, dS\, dx_4$$

and, because x_4^0 and x_4^1 are entirely arbitrary,

$$0 = \int_S \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi)}\delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\gamma\partial_\gamma\psi)}\partial_\alpha\delta\psi - \left(\partial_\alpha\frac{\partial \mathcal{L}}{\partial(\partial_\gamma\partial_\gamma\psi)}\right)\delta\psi \right\} n_\alpha\, dS \qquad (3b)$$

We require that $\mathcal{L}$ be of such a form that the Euler-Lagrange equation, above, is the differential equation for the field function ψ. We may then require, in addition, that the information about the boundary conditions which accompany this Euler-Lagrange equation be yielded by the surface integral above. In this investigation the form of this surface condition, as it arises in various wave fields, will be analysed.

The attitude, in the literature[2,3,4,5], is to require that the action principle yield the field equations while the boundary conditions are added to the formalism, in an hoc manner, whenever required. Thus, for example, the boundary integral in the preceding example is dismissed by requiring that $\delta\psi = 0$ on S. It is pointed out that various other aspects of the description of a field, e. g. the conservation theorems[3], arise as a consequence of the variational principle. However, it is not recognized that the variational principle for a wave field describes the motion of the wave field completely in the sense that it describes the motion of each of the normal modes for the field. If this is required of the action principle for wave fields and if this is recognized in the results of the action principle then the principle has given meaning to the wave field and has yielded all the information we can require of it. The results are

[2] G. Wentzel, "Quantum Theory of Fields" (Interscience Pub.), Chapter I.
[3] E. L. Hill, Reviews of Modern Physics, Vol. 23, pg.253-260, (1951).
[4] W. Pauli, Reviews of Modern Physics, Vol. 13, pg.203-232, (1941).
[5] Lanczos, (page 68, ref.1), does mention that the boundary conditions arise from the variational principle for a particular problem in classical mechanics.

the differential equation of the wave field and the form of the boundary conditions which accompany this. This information, obtained from the action principle, implies all the other information which we may obtain from the principle. Thus we are led to ascribe to a particular physical system a particular Lagrange function for which the conservation theorems are a natural consequence. The purpose of this investigation is to show, for several wave fields, that the boundary conditions which accompany the field equations, as well as the field equations themselves, are a consequence of the action principles for these fields. This discussion leads to a particularization of the Lagrange function which is to be associated with a given wave field.

In describing wave fields with action principles it is possible to show that a certain arbitrariness exists in the type of Lagrange function which will yield, at least, the differential equation of the field. Thus for some fields which are described by a field equation containing second-order time- and space-derivatives, the Lagrange function may be a function of: (a) the field function and its first order time- and space- derivatives only; (b) the field function and its second-order time- and space- derivatives only; (c) the field function and its first-order and second-order time- and space- derivatives. Each of these types of Lagrange functions may be associated with an action-principle and if one requires a meaningful description from each such action-principle then each type also yields a set of boundary conditions to accompany their common field equation. Even though the three types of Lagrange functions mentioned above have a common Euler-Lagrange equation they do not necessarily all lead to the same set of boundary conditions. Because of the usual attitude, in the literature, toward the variational principles for wave fields, i. e. the failure to recognize the boundary conditions as a consequence of the variational principles for the fields, the Lagrange functions of types (a), (b) and

(c) are usually regarded as equivalent in the discussion of wave fields. Then, because of the simplicity in the variational methods of type (a) and because of apparent difficulties[6] in the extended formalism of types (b) and (c), it is customary to employ only Lagrange functions of type (a). However, a discussion of the Lagrange functions of types (a), (b) and (c) - for the real and complex scalar fields and for the real and complex vector fields - and the recognition of the boundary conditions which occur for each of these types, suggests that in general only Lagrange functions of type (b) - rather than type (a) - should be used and that in some particular cases the use of types (a), (b) and (c) are equivalent in all respects.

Since the discussion of the boundary conditions arising from the action-principle - for the scalar and vector fields - suggested the use of Lagrange functions involving second order time- and space- derivatives, it is necessary to show that no difficulties occur in the field theory formalism for such Lagrange functions. Thus it is necessary to construct a Hamiltonian function in a natural manner and to obtain the equations of motion as the canonical equations for this Hamiltonian. One must also be able to set up the conservation theorems, that is, to define a suitable stress-energy tensor and a charge-current vector whose components enter into the required conservation theorems. The investigation discusses the manner in which this aspect of the formalism is constructed for the generalized Lagrange functions whose use is implied by the action principle. The result is a satisfactory formalism which possesses some advantages over that which is usually developed. A discussion of the differences in the formalism for different Lagrange functions leads to a closer identification of the form of the Lagrange function with the behaviour of the wave field at its boundary. The Lagrangian density must describe the boundary conditions

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6 G. Wentzel, (reference 2), page 16.

as well as the field if the action principle for the field is to yield the boundary conditions as well as the field equations.

1.2 The Relativistic Wave Fields

The relativistic wave fields which are considered in this discussion are the real and complex scalar fields, the real and complex vector fields, and the Dirac field. We shall be concerned with the motion of these fields in a vacuum, that is, without interaction. Since interaction will not be considered the pseudo-scalar and pseudo-vector fields will not be discussed. In a vacuum these fields satisfy exactly the same equations as the scalar and vector fields respectively.

For the complex scalar field the field function ψ and its complex conjugate ψ^* must satisfy the following differential equations in the space-time region of the field:

$$\begin{aligned} \partial_\alpha \partial_\alpha \psi &= K\psi \\ \partial_\alpha \partial_\alpha \psi^* &= K\psi^* \end{aligned} \qquad (1)$$

where K is a positive constant.

For the real scalar field we have only the field function ψ and the first of equations (1).

For the complex vector field ψ and ψ^* each have four components, that is,

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$$

$$\psi^* = (\psi_1^*, \psi_2^*, \psi_3^* \psi_4^*)$$

and the field equations are the set

$$\begin{aligned} \partial_\alpha \partial_\alpha \psi_\beta &= K\psi_\beta \\ \partial_\alpha \partial_\alpha \psi_\beta^* &= K\psi_\beta^* \end{aligned} \qquad \alpha, \beta = 1,2,3,4 \qquad (2)$$

with the auxiliary relations - the Lorentz conditions

$$\partial_\beta \psi_\beta = 0 \qquad \partial_\beta \psi_\beta^* = 0 \qquad (2a)$$

The field equations as well as the Lorentz conditions are contained in the following equations:

$$\partial_\alpha(\partial_\alpha\psi_\beta - \partial_\beta\psi_\alpha) = K\psi_\beta$$
$$\partial_\alpha(\partial_\alpha\psi_\beta^* - \partial_\beta\psi_\alpha^*) = K\psi_\beta^* \qquad (3)$$

for if we take the derivative, with respect to x_β of the first of equations (3), for example, then we have

$$\partial_\beta\partial_\alpha(\partial_\alpha\psi_\beta - \partial_\beta\psi_\alpha) = K\partial_\beta\psi_\beta$$

that is, $\partial_\beta\partial_\alpha\partial_\alpha\psi_\beta - \partial_\alpha\partial_\beta\partial_\beta\psi_\alpha = K\partial_\beta\psi_\beta \qquad (4)$

But we also have

$\partial_\alpha\partial_\beta\partial_\beta\psi_\alpha = \partial_\beta\partial_\alpha\partial_\alpha\psi_\beta$ by interchanging dummy indices.

Consequently, (4) may be written:

$$K\,\partial_\beta\psi_\beta = 0 \qquad K \neq 0$$

which is the Lorentz condition.

Therefore equations (3) are equivalent to the set (2) and (2a). If the Euler-Lagrange equations for the vector field are obtained in the form (3), then the variational principle for this field has provided not only the field equations but also the Lorentz condition.

The Dirac wave field is defined by a set of field equations which involve only first-order derivatives. The field equations are:

$$\frac{\hbar}{i}\dot{\psi}_\sigma^* + \frac{c\hbar}{i}\partial_k\psi_\rho^*\alpha_{\rho\sigma}^{(k)} - mc^2\psi_\rho^*\beta_{\rho\sigma} = 0 \qquad \hbar = \text{Planck's const} \times \frac{1}{2\pi}$$
$$\sigma,\rho = 1,2,3,4 \quad [7]$$
$$\frac{\hbar}{i}\dot{\psi}_\rho + \frac{c\hbar}{i}\alpha_{\rho\sigma}^{(k)}\partial_k\psi_\sigma + mc^2\beta_{\rho\sigma}\psi_\sigma = 0 \qquad k = 1,2,3 \quad (5)$$

where $\psi_\sigma^*, \psi_\rho, \alpha^{(k)}, \beta$ are matrices defined by

$$\psi_\sigma^* = \begin{pmatrix}\psi_1^*\\ \psi_2^*\\ \psi_3^*\\ \psi_4^*\end{pmatrix} \qquad \psi_\rho = (\psi_1, \psi_2, \psi_3, \psi_4)$$

$$\alpha^{(1)} = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix};\quad \alpha^{(2)} = \begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix};\quad \alpha^{(3)} = \begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix};\quad \beta = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

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7 $\dot{\psi}_\sigma^* \equiv \frac{d}{dt}\psi_\sigma^*$ we shall frequently use this notation for derivatives with respect to time. Also, unless otherwise stated, Greek

1.3 The Lagrange Functions For the Wave Fields

The scalar field equations and the vector field equations are linear, homogeneous, and of second order. Their form is also invariant under Lorentz transformations. The Euler-Lagrange equations, resulting from the action principle for these fields, is linear and homogeneous in $\mathcal{L}$, the Lagrangian density for the fields. The order, in $\mathcal{L}$, of the Euler-Lagrange equation is the same as the highest order derivative occurring in $\mathcal{L}$. This Euler-Lagrange equation is required to be the wave equation for the field described by $\mathcal{L}$. Since the vector and scalar field equations are of second order the Lagrange function for these fields cannot be of higher order than the second; since the field equations are homogeneous and linear, $\mathcal{L}$ must be homogeneous and quadratic; since the wave equations are invariant in form under Lorentz transformations, $\mathcal{L}$ must be invariant (the necessity for the invariance in form, of $\mathcal{L}$, is discussed and qualified in Appendix II). Consequently the only possible form of the Lagrange functions for these fields is given by:

$$\mathcal{L} = \frac{1}{2}\left(A\psi^2 + B(\partial_\alpha\psi)(\partial_\alpha\psi) + C\psi\partial_\alpha\partial_\alpha\psi\right) \tag{1}$$

for the real scalar field;

$$\mathcal{L} = A\psi\psi^* + B(\partial_\alpha\psi)(\partial_\alpha\psi^*) + \frac{C}{2}\left(\psi^*\partial_\alpha\partial_\alpha\psi + \psi\partial_\alpha\partial_\alpha\psi^*\right) \tag{2}$$

for the complex scalar field;

$$\mathcal{L} = A\psi_\beta\psi_\beta^* + B(\partial_\alpha\psi_\beta - \partial_\beta\psi_\alpha)(\partial_\alpha\psi_\beta^* - \partial_\beta\psi_\alpha^*) + \frac{C}{2}\left[\psi_\beta(\partial_\alpha\partial_\alpha\psi_\beta^* - \partial_\alpha\partial_\beta\psi_\alpha^*) + \psi_\beta^*(\partial_\alpha\partial_\alpha\psi_\beta - \partial_\alpha\partial_\beta\psi_\alpha)\right] \tag{3}$$

for the complex vector field;

where A, B, C are constants if we assume the homogeneity and isotropy of space - time.

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indices will be used to run through the values 1,2,3,4 while Latin indices will indicate only the 1, 2, 3 components.

The Dirac equation is of first-order. The Lagrange function for this field is therefore limited to first-order derivatives and therefore this field will be discussed more briefly than the scalar and vector fields. The Lagrangian density for the Dirac field will be given in the discussion of this field.

1.4 The Boundary Conditions for the Wave Fields

The motion of a wave field is discussed in terms of the normal modes for the field. These normal modes are a set of solutions of the field equations in terms of which an arbitrary function may be expanded. We have just discussed the differential equations for various relativistic wave fields. The normal modes will arise in a natural manner from the solutions of these field equations if certain conditions exist at the space boundary of the wave field. We shall examine the field equations to determine some of the boundary conditions that will yield normal modes. Having identified the boundary conditions and having recognized that these conditions are necessary in describing the motion of the wave field we shall expect that any variational principle which describes the motion of the field will yield, at least, some of these boundary conditions.

The normal modes, ψ_n, ψ_m^* are those solutions of the field equations which may be written in the form

$$\psi_n = e^{i\omega_n t} u_n(x_1, x_2, x_3)$$
$$\psi_m^* = e^{-i\omega_n t} u_m^*(x_1, x_2, x_3) \qquad (1)$$

where ω_n, ω_m are constants,

and in terms of which any arbitrary solutions, ψ, ψ^*, of the field equations may be expanded in the following way:

$$\psi = \sum_n c_n \psi_n$$
$$\psi^* = \sum_m c_m^* \psi_m^* \qquad (2)$$

where c_n and c_m are constants.

If this can be done for any arbitrary Ψ, Ψ^* then the u_n, u_m^* must form a complete orthogonal set of functions.

When the solutions (1) are substituted into the field equations these equations become partial differential equations for the u_n, u_m^* involving the ω_n, ω_m. That is, the equations 1.2(1), 1.2(2) may be written:

$$\partial_K \partial_K u_n + \left(\frac{\omega_n^2}{c^2} - k\right) u_n = 0$$

$$\partial_K \partial_K u_m^* + \left(\frac{\omega_m^2}{c^2} - k\right) u_m^* = 0 \qquad (3)$$

We require that u_n, u_m^* satisfy the conditions existing at the boundary s. Conversely, we require that the boundary conditions are such that ω_n may assume only a discrete set of values and that the u_n, u_m^* belonging to various such ω_n, ω_m, form a complete set of orthogonal functions. For $\omega_n \neq \omega_m$ we must then have,

$$\int_V u_n u_m^* \, dV = 0 \qquad (4)$$

which is the orthogonality condition. If various u's belong to the same ω_n then these can be made orthogonal by a Schmidt process. Using the field equations (3) the orthogonality condition (4) may be written:

$$\int_S \left\{ u_m^* \frac{\partial u_n}{\partial n} - u_n \frac{\partial u_m^*}{\partial n} \right\} dS = 0 \qquad (5)$$

where $\frac{\partial}{\partial n}$ is the normal derivative on S.

For the vector fields, $u_n = (u_{n_1}, u_{n_2}, u_{n_3}, u_{n_4})$ and $u_m^* = (u_{m_1}^*, u_{m_2}^*, u_{m_3}^*, u_{m_4}^*)$ so that the orthogonality condition may be written:

$$\int_S \left\{ u_{m\beta}^* \frac{\partial u_{n\beta}}{\partial n} - u_{n\beta} \frac{\partial u_{m\beta}^*}{\partial n} \right\} dS = 0 \qquad (6)$$

in which there is no summation on β.

Boundary conditions which satisfy (5) or (6), for the scalar and vector fields respectively, are:

$$\frac{\partial \psi^*}{\partial n}, \ \frac{\partial \psi}{\partial n} = 0 \quad \text{on } S.$$

$$\text{or} \quad \psi^*, \ \psi = 0 \quad \text{on } S. \qquad (7)$$

$$\text{or} \quad \psi, \psi^* \ \text{periodic} \quad \text{on } S.$$

For the vector field $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ and $\psi^* = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$. Each of the components has the boundary conditions above.

These are the boundary conditions which are usually employed in the discussion of the scalar and vector fields. They will be referred to as the 'usual boundary conditions' throughout this investigation.

In addition, linear homogeneous boundary conditions satisfy conditions (5) and (6). That is,

$$\frac{\partial \psi}{\partial n} = \gamma\psi \ ; \quad \frac{\partial \psi^*}{\partial n} = \gamma\psi^* \quad \text{on } S \qquad (8)$$

where γ is a constant

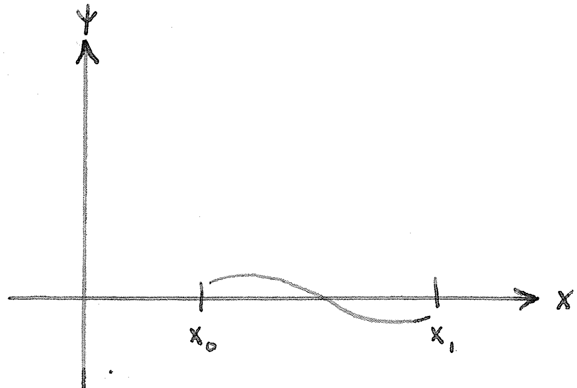
and $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$, $\psi^* = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$ for the vector field.

are boundary conditions which yield the normal modes. The scalar and vector fields will be discussed to show how both of the conditions (7) and (8) arise from the Lagrange functions which are employed in the variational principles for these fields. When this has been done we shall be able to say that the variational principles describe all those motions of the wave fields for which the boundary conditions are given by either (7) or (8).

The methods employed in the discussion of the vector and scalar fields - with Lagrange functions - may be introduced by a simple example. The vibrations of a stretched string constitute a 'field problem' which may be solved with an action principle. The method of analysis of this simple field, as developed in the next section, will indicate the manner in which the scalar and vector fields may be discussed and the way in which the results for these fields may interpreted.

1. 5 The Vibrations of a Uniform Stretched String

Assume that we have a uniform stretched string on the x-axis between



$x = x_0$ and $x = x_1$. If ρ is the uniform linear density of the string and if P is the tension in the string, the equation of motion can be shown to be:

$$\rho \frac{\partial^2 \psi}{\partial t^2} = P \frac{\partial^2 \psi}{\partial x^2} \qquad (1)$$

or

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}.$$

where

$$v = \sqrt{\frac{P}{\rho}}$$

We discuss the motion of this string in terms of the motion of its normal modes. We assume that

$$\psi_n = q_n(t) \; u_n(x) \qquad (2)$$

satisfies equation(1). Then equation (1) may be written:

$$\frac{1}{u_n} \frac{\partial^2 u_n}{\partial x^2} = \frac{1}{v^2 q_n} \frac{\partial^2 q_n}{\partial t^2} = \lambda_n, \text{ say} \qquad (3)$$

We seek all the u_n corresponding to λ_n allowed by the boundary conditions.

Then, for $\lambda_m \neq \lambda_n$ we have

$$\frac{\partial^2 u_n}{\partial x^2} = \lambda_n u_n$$

$$\frac{\partial^2 u_m}{\partial x^2} = \lambda_m u_m$$

If we multiply the first of these by u_m and the second of these by u_n, subtract the result for the second from the result for the first, and integrate the difference over the length of the string, we obtain:

$$\int_{x_0}^{x_1} \left\{ u_m \frac{\partial^2 u_n}{\partial x^2} - u_n \frac{\partial^2 u_m}{\partial x^2} \right\} dx = (\lambda_n - \lambda_m) \int_{x_0}^{x_1} u_n u_m \, dx$$

For the normal modes the orthogonality condition must hold, that is,

$$\int_{x_0}^{x_1} u_n u_m \, dx = 0$$

which may now be written

$$\int_{x_0}^{x_1} \left\{ u_m \frac{\partial^2 u_n}{\partial x^2} - u_n \frac{\partial^2 u_m}{\partial x^2} \right\} dx$$

or

$$\left(u_m \frac{\partial u_n}{\partial x} - u_n \frac{\partial u_m}{\partial x} \right) \Bigg|_{x_0}^{x_1} = 0 \qquad (4)$$

Condition (4) is the orthogonality condition which must be satisfied if we are to have normal modes for the problem of the stretched string. We would expect that a variational principle which describes the motion of the stretched string will also yield boundary conditions which satisfy (4).

We will now discuss two variational principles, both of which yield the desired equation of motion for the string. The variational principles differ in the Lagrangian density which they employ. We define

$$\mathcal{L}_1 \equiv \frac{P}{2} \psi \frac{\partial^2 \psi}{\partial x^2} - \frac{\rho}{2} \psi \frac{\partial^2 \psi}{\partial t^2}$$

$$\mathcal{L}_2 \equiv -\frac{P}{2} \left(\frac{\partial \psi}{\partial x} \right)^2 + \frac{\rho}{2} \left(\frac{\partial \psi}{\partial t} \right)^2 \qquad (5)$$

and then we define the action, I, between t_0 and t_1,

$$I_1 \equiv \int_{t_0}^{t_1} \int_{x_0}^{x_1} \mathcal{L}_1 \, dx \, dt \qquad I_2 \equiv \int_{t_0}^{t_1} \int_{x_0}^{x_1} \mathcal{L}_2 \, dx \, dt$$

The action-principles are then defined in terms of the variations in the functional dependence of ψ on its variables. That is, if δI is the variation in I corresponding to the variation $\delta\psi$ in the functional form of ψ, the action principles for the two Lagrange functions, $\mathcal{L}_1$ and $\mathcal{L}_2$, are respectively:

$$\begin{aligned} \delta I_1 &= 0 \\ \delta I_2 &= 0 \end{aligned} \tag{7}$$

where, for arbitrary t_0 and t_1, the $\delta\psi$ are variations about the correct form of ψ. The $\delta\psi$ vanish at $t = t_0$, $t = t_1$ and are consistent with the boundary conditions for the problem.

The action-principles may be written:

$$\begin{aligned} 0 &= \int_{t_0}^{t_1}\int_{x_0}^{x_1} \delta\mathcal{L}_1 \, dx\, dt \\ 0 &= \int_{t_0}^{t_1}\int_{x_0}^{x_1} \delta\mathcal{L}_2 \, dx\, dt \end{aligned} \tag{8}$$

Since
$$\mathcal{L}_1 = \mathcal{L}_1\left(\psi, \frac{\partial^2\psi}{\partial x^2}, \frac{\partial^2\psi}{\partial t^2}\right)$$

then, for small $\delta\psi$,

$$\delta\mathcal{L}_1 = \frac{\partial\mathcal{L}_1}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial^2\psi}{\partial x^2}\right)}\delta\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\mathcal{L}_1}{\partial\left(\frac{\partial^2\psi}{\partial t^2}\right)}\delta\frac{\partial^2\psi}{\partial t^2}$$

$$= \frac{P}{2}\left(\frac{\partial^2\psi}{\partial x^2}\right)\delta\psi - \frac{\rho}{2}\left(\frac{\partial^2\psi}{\partial t^2}\right)\delta\psi + \frac{P}{2}\psi\frac{\partial^2(\delta\psi)}{\partial x^2} - \frac{\rho}{2}\psi\frac{\partial^2(\delta\psi)}{\partial t^2}$$

The first of equations (8) may be written:

$$0 = \int_{t_0}^{t_1}\int_{x_0}^{x_1}\left\{\frac{1}{2}\left(P\frac{\partial^2\psi}{\partial x^2} - \rho\frac{\partial^2\psi}{\partial t^2}\right)\delta\psi + \frac{P}{2}\psi\frac{\partial^2(\delta\psi)}{\partial x^2} - \frac{\rho}{2}\psi\frac{\partial^2(\delta\psi)}{\partial t^2}\right\} dx\, dt \tag{9}$$

Integrating both of the last two terms by parts twice, we obtain:

$$0 = \int_{t_0}^{t_1}\left\{\frac{P}{2}\psi\frac{\partial(\delta\psi)}{\partial x} - \frac{P}{2}\frac{\partial\psi}{\partial x}\delta\psi\right\}\Big|_{x_0}^{x_1} dt + \int_{x_0}^{x_1}\left\{\frac{\rho}{2}\frac{\partial\psi}{\partial t}\delta\psi - \frac{\rho}{2}\psi\frac{\partial}{\partial t}\delta\psi\right\}\Big|_{t_0}^{t_1} dx$$
$$+ \int_{t_0}^{t_1}\int_{x_0}^{x_1}\left\{P\frac{\partial^2\psi}{\partial x^2} - \rho\frac{\partial^2\psi}{\partial t^2}\right\}\delta\psi\, dx\, dt$$

and from this, $P\frac{\partial^2\psi}{\partial x^2} - \rho\frac{\partial^2\psi}{\partial t^2} = 0$ as the Euler-Lagrange equation for the vibrating string. This is the required equation of motion for the string. The second integral in (10) vanishes because the action principle requires that the variations $\delta\psi$ vanish at $t = t_0$ and at $t = t_1$. Then equation (25) may be written:

$$0 = \frac{P}{2}\int_{t_0}^{t_1}\left\{\psi\frac{\partial(\delta\psi)}{\partial x} - \frac{\partial\psi}{\partial x}\delta\psi\right\}\Big|_{x_0}^{x_1} dt$$

or

$$0 = \frac{P}{2}\left\{\psi\frac{\partial(\delta\psi)}{\partial x} - \frac{\partial\psi}{\partial x}\delta\psi\right\}\Big|_{x_0}^{x_1} \qquad (11)$$

because of the arbitrariness of t_0 and t_1.

This condition accompanies the Euler-Lagrange equation of motion and will berecognized as the boundary condition which accompanies the wave equation.

Before we discuss this condition we shall examine the information yielded by the second of equations (8). Proceeding as for $\mathcal{L}_1$,

$$\delta\mathcal{L}_2 = \frac{\partial\mathcal{L}_2}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}_2}{\partial\left(\frac{\partial\psi}{\partial x}\right)}\delta\frac{\partial\psi}{\partial x} + \frac{\partial\mathcal{L}_2}{\partial\left(\frac{\partial\psi}{\partial t}\right)}\delta\frac{\partial\psi}{\partial t}$$

$$= -P\frac{\partial\psi}{\partial x}\frac{\partial(\delta\psi)}{\partial x} + \rho\frac{\partial\psi}{\partial t}\frac{\partial(\delta\psi)}{\partial t}$$

and the action-principle may be written:

$$0 = \int_{t_0}^{t_1}\int_{x_0}^{x_1}\left\{-P\frac{\partial\psi}{\partial x}\frac{\partial(\delta\psi)}{\partial x} + \rho\frac{\partial\psi}{\partial t}\frac{\partial}{\partial t}(\delta\psi)\right\}dx\,dt$$

Integrating both terms once, by parts, this becomes:

$$0 = -P\int_{t_0}^{t_1}\frac{\partial\psi}{\partial x}\delta\psi\Big|_{x_0}^{x_1}dt + \rho\int_{x_0}^{x_1}\frac{\partial\psi}{\partial t}\delta\psi\Big|_{t_0}^{t_1}dx \longrightarrow 0$$

$$+ \int_{t_0}^{t_1}\int_{x_0}^{x_1}\left\{P\frac{\partial^2\psi}{\partial x^2} - \rho\frac{\partial^2\psi}{\partial t^2}\right\}\delta\psi\,dx\,dt$$

so that for the Euler-Lagrange equation

$$P\frac{\partial^2\psi}{\partial x^2} - \rho\frac{\partial^2\psi}{\partial t^2} = 0$$

we obtain the boundary condition

$$0 = \frac{\partial\psi}{\partial x}\delta\psi\Big|_{x_0}^{x_1} \qquad (12)$$

The 'usual boundary conditions' for this problem are:

$$\psi = 0 \qquad \text{at } x = x_0,\ x = x_1$$

$$\text{or} \qquad \frac{\partial \psi}{\partial x} = 0 \qquad \text{at } x = x_0,\ x = x_1 \tag{13}$$

or ψ is periodic within $x_0 \leq x \leq x_1$, with period $x_1 - x_0$. All of these conditions satisfy both (11) and (12) and therefore the variational principle for either $\mathcal{L}_1$ or $\mathcal{L}_2$ allows the 'usual boundary conditions'.

But the linear, homogeneous boundary condition

$$\frac{\partial \psi}{\partial x} = \gamma \psi \qquad \text{at } x = x_0,\ x = x_1 \tag{14}$$

where γ is a constant,

satisfies (11) but not (12). Therefore a system involving a vibrating string with linear, homogeneous boundary conditions may be described only by a Lagrangian density of form $\mathcal{L}_1$, not by one of form $\mathcal{L}_2$. It can be shown (see Appendix I) that the linear homogeneous boundary condition is the only condition of the form

$$\psi \text{ is a function of } \frac{\partial \psi}{\partial x} \text{ at } x = x_0,\ x = x_1$$

that satisfies condition(11). Thus $\mathcal{L}_1$ generalizes the applications of the action principle to include those systems for which the boundary conditions are linear homogeneous ones.

In order to construct a Hamiltonian from the field functions it is necessary first to define a momentum field, π, which is conjugate to ψ. The conjugate momentum field is defined:

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \tag{15}$$

and then the Hamiltonian density, $\mathcal{H}$, is defined as

$$\mathcal{H} = \pi \dot{\psi} - \mathcal{L} \tag{16}$$

For $\mathcal{L}_2$, as defined above, we obtain:

$$\pi_2 = \rho \dot{\psi} \tag{17}$$

and consequently

$$\mathcal{H}_2 = \frac{\rho}{2}\dot{\psi}^2 + \frac{P}{2}\left(\frac{\partial\psi}{\partial x}\right)^2 \tag{18}$$

For Hamiltonian densities which contain first order or second order derivatives of the field function the canonical equations are defined in terms of functional derivatives of $\mathcal{H}$. Thus if we define the functional derivative, $\frac{\delta\mathcal{H}}{\delta\psi}$, to be

$$\frac{\delta\mathcal{H}}{\delta\psi} \equiv \frac{\partial\mathcal{H}}{\partial\psi} - \frac{\partial}{\partial x}\frac{\partial\mathcal{H}}{\partial\left(\frac{\partial\psi}{\partial x}\right)} + \frac{\partial^2}{\partial x^2}\frac{\partial\mathcal{H}}{\partial\left(\frac{\partial^2\psi}{\partial x^2}\right)} \tag{19}$$

the canonical equations of motion are:

$$\dot{\pi} = -\frac{\delta\mathcal{H}}{\delta\psi}$$

$$\dot{\psi} = \frac{\partial\mathcal{H}}{\partial\pi} \tag{20}$$

This form of the canonical equations will not be derived here. The derivation would proceed from the use of $\mathcal{H}$ in a variational principle. Similar canonical equations will be discussed for the scalar and vector fields.

$\mathcal{H}_2$ may be written

$$\mathcal{H}_2 = \frac{1}{2\rho}\pi_2^2 + \frac{P}{2}\left(\frac{\partial\psi}{\partial x}\right)^2 \tag{21}$$

Then the canonical equations, (20), yield:

$$\dot{\pi}_2 = P\frac{\partial^2\psi}{\partial x^2}$$, which is the equation of motion, (1).

and $$\dot{\psi} = \dot{\psi}$$, an identity.

Thus π_2 and ψ are conjugate with $\mathcal{H}_2$ as their Hamiltonian density.

The energy, E, of the vibrating string can be shown to be:

$$E = \frac{1}{2}\int_{x_0}^{x_1}\left\{\rho\left(\frac{\partial\psi}{\partial t}\right)^2 + P\left(\frac{\partial\psi}{\partial x}\right)^2\right\}dx \tag{22}$$

This is identical with the integrated Hamiltonian density $H_2\left(\equiv\int_{x_0}^{x_1}\mathcal{H}_2\,dx\right)$.

Thus the Hamiltonian density, $\mathcal{H}_2$, is identified with the energy density of the vibrating string.

We notice that $\mathcal{L}_1$ does not contain $\dot{\psi}$ and therefore π_1 may not be defined in the same way as π_2. However, we define:[8]

$$\pi_1 \equiv \rho \dot{\psi}$$

and

$$\begin{aligned} \mathcal{H}_1 &\equiv \frac{\rho}{2}\dot{\psi}^2 - \frac{P}{2}\psi\frac{\partial^2\psi}{\partial x^2} \\ &= \frac{1}{2\rho}\pi_1^2 - \frac{P}{2}\psi\frac{\partial^2\psi}{\partial x^2} \end{aligned} \qquad (23)$$

The canonical equations for $\mathcal{H}_1$ are identical with those for $\mathcal{H}_2$, that is, we obtain an identity and the equations of motion. Therefore π_1 and ψ are conjugate.

We shall discuss now the way in which $\mathcal{H}_1$ may be identified with the energy of the vibrating string.

Let $H_1 = \int_{x_0}^{x_1} \mathcal{H}_1 \, dx$

We observe that

$$\begin{aligned} H_1 - H_2 &= -\frac{P}{2}\int_{x_0}^{x_1}\left(\psi\frac{\partial^2\psi}{\partial x^2} + \left(\frac{\partial\psi}{\partial x}\right)^2\right)dx \\ &= -\frac{P}{2}\psi\frac{\partial\psi}{\partial x}\Big|_{x_0}^{x_1} \end{aligned} \qquad (24)$$

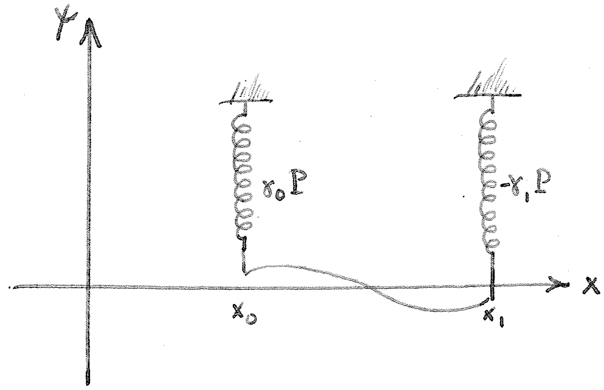
For the 'usual boundary conditions' $H_1 = H_2 = E$ and therefore for systems with these boundary conditions $\mathcal{H}_1$ may be identified with the energy density of the string. But if $\frac{\partial\psi}{\partial x} = \gamma\psi$ at $x = x_0$, $x = x_1$, where γ is a constant which may be different for $x = x_0$ and $x = x_1$, then

$$H_1 - H_2 = \frac{P}{2}\left(\gamma_0\psi^2(x_0) - \gamma_1\psi^2(x_1)\right) \qquad (25)$$

which is not necessarily zero. We shall identify this difference, $H_1 - H_2$, with the energy which we attribute to the linear, homogeneous boundary condition $\frac{\partial\psi}{\partial x} = \gamma\psi$.

[8] The validity of this definition, for the scalar and vector field, is discussed in Appendix II.

If we say that we have the linear homogeneous boundary conditions, (14), for the problem of the vibrating stretched string, this is equivalent to saying that we attach the following two massless springs to the string:



where the force constant of the spring at $x = x_0$ is $\gamma_0 P$, and the force constant of the spring at $x = x_1$ is $-\gamma_1 P$.

The vertical force exerted by the spring at $x = x_0$ is $-\gamma_0 P\psi$.

The vertical force exerted by the spring at $x = x_1$ is $\gamma_1 P\psi$.

The vertical force exerted by the string at $x = x_0$ is $P\frac{\partial\psi}{\partial x}$.

The vertical force exerted by the string at $x = x_1$ is $-P\frac{\partial\psi}{\partial x}$.

Then, for equilibrium,

$$\gamma_0 P\psi = P\frac{\partial\psi}{\partial x} \qquad \text{at } x = x_0$$

or

$$\frac{\partial\psi}{\partial x} = \gamma_0\psi \qquad \text{"}$$

and similarly

$$\frac{\partial\psi}{\partial x} = \gamma_1\psi \qquad \text{at } x = x_1$$

Therefore the system of the massless springs and the vibrating string is equivalent to a system of a vibrating string and linear homogeneous boundary conditions.

Since no mass is directly attached to the springs their energy is entirely potential energy. The potential energy of the spring at $x = x_0$ is

$$-\int_{\psi(x_0)=0}^{\psi(x_0)} (-\gamma_0 P\psi)\,d\psi = \frac{\gamma_0}{2} P\psi^2(x_0)$$

Similarly the the potential energy of the spring at x_1 is $-\frac{\gamma_1 P}{2}\psi^2_{(x_1)}$

The total energy, at the boundary, for the system is

$$\frac{P}{2}\left(\gamma_0\psi^2(x_0) - \gamma_1\psi^2(x_1)\right) = H_1 - H_2$$

Therefore we see that in all cases H_1 describes the energy of the physical system with which it is associated. $\mathcal{H}_1$ is the energy density of the system not only for the string between x_0 and x_1 but also at the boundary. We have seen that a physical system in which the boundary conditions themselves contain energy is described only by such Lagrangian densities as lead to Hamiltonian densities which, at the boundary, include the energy of the boundary conditions. In discussing the scalar and vector field equations we shall proceed in a manner which is quite similar to that followed in the simple example above. We shall require that $\mathcal{L}$ yield the boundary conditions as well as the field equations. Then we shall expect that the Hamiltonian function, associated with $\mathcal{L}$, describes the energy of these boundary conditions as well as the energy of the rest of the field.

CHAPTER II

THE SCALAR FIELD

2.1 The Action Principle for the Scalar Field

The variational principle for the real scalar field was discussed as an example (on page 2). The results obtained for this field are always a special case of the results for the complex scalar field. Consequently only the latter will be discussed in detail.

It was shown (page 10), for the complex scalar wave field, that the only form of the Lagrangian density which required consideration was given by:

$$\mathcal{L} = A\psi\psi^* + B(\partial_\alpha\psi)(\partial_\alpha\psi^*) + \frac{C}{2}(\psi\partial_\alpha\partial_\alpha\psi^* + \psi^*\partial_\alpha\partial_\alpha\psi) \qquad (1)$$

The action integral, I, for this field is defined by:

$$I = \int_{x_4^0}^{x_4^1}\int_V \mathcal{L}\, d\tau \qquad (2)$$

where $V, d\tau, x_4^0, x_4^1$ are as defined on pages 2 and 3.

Assume that $\delta\psi$, $\delta\psi^*$ are small variations in the functional dependence of ψ, ψ^* on their variables. If $\delta\psi, \delta\dot\psi, \delta\psi^*, \delta\dot\psi^*$ vanish at x_4^0, x_4^1 and are consistent with the boundary conditions on S, and if δI is the variation in I corresponding to such $\delta\psi$, $\delta\psi^*$, then the action principle may be stated in the following way: for arbitrary variations $\delta\psi$, $\delta\psi^*$ about the correct form of ψ, ψ^*, δI vanishes for arbitrary x_4^0 and x_4^1. That is,

$$\delta I = 0 \qquad (3)$$

This principle describes the motion of the complex scalar field completely.

Equation (3) may also be written:

$$O = \int_{x_4^0}^{x_4'} \int_V \delta\mathcal{L} \, d\tau \tag{4}$$

and $\mathcal{L} = \mathcal{L}(\psi, \psi^*, \partial_\alpha\psi, \partial_\alpha\psi^*, \partial_\alpha\partial_\alpha\psi, \partial_\alpha\partial_\alpha\psi^*)$

If the δ variations are small enough we may expand $\delta\mathcal{L}$ by a Taylor's series as

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\psi^*}\delta\psi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)}\partial_\alpha\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi^*)}\partial_\alpha(\delta\psi^*)$$

$$+ \frac{\partial\mathcal{L}}{\partial(\partial_\gamma\partial_\gamma\psi)}\partial_\alpha\partial_\alpha\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\gamma\partial_\gamma\psi^*)}\partial_\alpha\partial_\alpha\delta\psi^*$$

$$= (A\psi^* + \tfrac{C}{2}\partial_\alpha\partial_\alpha\psi^*)\delta\psi + (A\psi + \tfrac{C}{2}\partial_\alpha\partial_\alpha\psi)\delta\psi^* + B(\partial_\alpha\psi)(\partial_\alpha\delta\psi^*) \tag{5}$$

$$+ B(\partial_\alpha\psi^*)(\partial_\alpha\delta\psi) + \tfrac{C}{2}\psi^*\partial_\alpha\partial_\alpha\delta\psi + \tfrac{C}{2}\psi\partial_\alpha\partial_\alpha\delta\psi^*$$

Substitute (5) into (4). Then, integrating the terms containing $\partial_\alpha\delta\psi^*$, $\partial_\alpha\delta\psi$, once by parts, and the terms containing $\partial_\alpha\partial_\alpha\delta\psi$, $\partial_\alpha\partial_\alpha\delta\psi^*$ twice by parts and making use of the fact that $\delta\psi$, $\delta\psi^*$ vanish at x_4^0, x_4', (4) may be written:

$$O = \int_{x_4^0}^{x_4'}\int_S \left\{ B\left[(\partial_\alpha\psi^*)\delta\psi + (\partial_\alpha\psi)\delta\psi^*\right] + \tfrac{C}{2}\left[-(\partial_\alpha\psi)\delta\psi^* - (\partial_\alpha\psi^*)\delta\psi + \psi^*\partial_\alpha\delta\psi + \psi\partial_\alpha\delta\psi^*\right]\right\} n_\alpha \, dS \, dx_4$$

$$+ \int_{x_4^0}^{x_4'}\int_V \left\{\left[A\psi^* + (C-B)\partial_\alpha\partial_\alpha\psi^*\right]\delta\psi + \left[A\psi + (C-B)\partial_\alpha\partial_\alpha\psi\right]\delta\psi^*\right\} d\tau \tag{6}$$

where S is the space surface enclosing V

and n_α is the unit normal to the surface S.

We require that the variational principle, in the form (6), yield both the complex scalar field equations and the boundary conditions which may accompany these.

The arbitrariness of the variations $\delta\psi$, $\delta\psi^*$ in the entire region $V - S$ gives us the following Euler-Lagrange equations:

$$A\psi^* + (C-B)\partial_\alpha\partial_\alpha\psi^* = 0$$

$$A\psi + (C-B)\partial_\alpha\partial_\alpha\psi = 0 \qquad (7)$$

If we write $K = \frac{A}{B-C}$ then these are identically the complex scalar field equations 1.2 (1). With these equations, the principle in the form (6) then requires that the time-surface integral, $\int_{x_4^0}^{x_4'}\int_S$, be also set equal to zero. Since x_4^0 and x_4' are completely arbitrary we may set $\int_S\{\ \}dS = 0$. That is,

$$0 = \int_S \left\{ B\left[(\partial_\alpha\psi)\delta\psi^* + (\partial_\alpha\psi^*)\delta\psi\right] + \frac{C}{2}\left[-(\partial_\alpha\psi^*)\delta\psi - (\partial_\alpha\psi)\delta\psi^* + \psi^*\partial_\alpha\delta\psi + \psi\partial_\alpha\delta\psi^*\right]\right\} n_\alpha\, dS \qquad (8)$$

This equation should yield all the boundary conditions which may accompany the scalar field equations. Since $n_\alpha\partial_\alpha = \frac{\partial}{\partial n}$, the normal derivative on S, (8) may be written

$$0 = \int_S \left\{ (B - \tfrac{C}{2})\frac{\partial\psi^*}{\partial n}\delta\psi + (B - \tfrac{C}{2})\frac{\partial\psi}{\partial n}\delta\psi^* + \frac{C}{2}\psi\delta\frac{\partial\psi^*}{\partial n} + \frac{C}{2}\psi^*\delta\frac{\partial\psi}{\partial n} \right\} dS \qquad (9)$$

Inspection of (9) shows that all the 'usual boundary conditions' 1. 4 (7) satisfy (9) and consequently may accompany the scalar field equations. In addition, the linear, homogeneous boundary conditions 1.4 (8) satisfy (9) if and only if $B = 0$. The proof of this, as well as a discussion of the manner in which (9) favours the linear form, is given in Appendix I.

2.2 The Generalized Lagrangian Density for the Scalar Field.

The form of the Lagrangian density 2.1 (1) for the complex scalar field may now be further restricted. The discussion of those motions of the wave field for which the field has linear homogeneous boundary conditions requires $B = 0$ in $\mathcal{L}$. The stipulation, $B = 0$, in $\mathcal{L}$

does not, in any way, limit the usefulness of $\mathcal{L}$ in describing the motion of wave fields but rather it allows us to include, in the action principle, a larger number of cases of such motion. If B = 0 in τ then the 'usual boundary conditions' are allowed. These are among the conditions which give us the normal modes for the scalar field but these conditions are also allowed if we set B = o and in the latter case linear homogeneous boundary conditions are also allowed. Therefore, in discussing the scalar wave fields with an action principle of the form 2.1 (3), the Lagrangian densities which must be used to describe the largest number of cases of the motion of this field are of the form:

$$\mathcal{L} = A\psi\psi^* + \frac{C}{2}(\psi\partial_\alpha\partial_\alpha\psi^* + \psi^*\partial_\alpha\partial_\alpha\psi) \qquad (1)$$

for the complex scalar field, and similarly

$$\mathcal{L} = \frac{1}{2}A\psi^2 + \frac{1}{2}C\psi\partial_\alpha\partial_\alpha\psi \qquad (2)$$

for the real scalar field. These will be referred to as the 'generalized Lagrangian densities'.

Scalar wave fields are usually discussed with Lagrangian densities of the form $\mathcal{L}_2 = A\psi\psi^* - C(\partial_\alpha\psi)(\partial_\alpha\psi^*)$

for the complex scalar field

$$\mathcal{L}_2 = \frac{1}{2}A\psi^2 - \frac{1}{2}C(\partial_\alpha\psi)(\partial_\alpha\psi) \qquad (3)$$

for the real scalar field.

In future discussions we shall refer to these as the 'usual Lagrangian densities'. In the use of these Lagrangian densities the boundary condition integral 2.1 (9) is usually dismissed by requiring that $\delta\psi$, $\delta\psi^*$ vanish on S or that ψ, ψ^* are periodic on S. We saw that these were among the 'usual boundary conditions' for the scalar field. For all the usual boundary conditions the use of the 'generalized Lagrangian density' is equivalent to the use of the 'usual Lagrangian density'. The reason for this equivalence, in the action principle, is easily shown. The action integral, I, was defined:

$$I \equiv \int_{x_4^0}^{x_4^1}\int_V \mathcal{L}\, d\tau$$

Let $$I_1 \equiv \int_{x_4^0}^{x_4^1}\int_V \left\{ A\psi\psi^* + \frac{C}{2}\left(\psi\partial_\alpha\partial_\alpha\psi^* + \psi^*\partial_\alpha\partial_\alpha\psi\right)\right\} d\tau$$

in which the 'generalized Lagrangian density' is used,

and let $$I_2 \equiv \int_{x_4^0}^{x_4^1}\int_V \left\{ A\psi\psi^* - C(\partial_\alpha\psi)(\partial_\alpha\psi^*)\right\} d\tau$$

in which the 'usual Lagrangian density' is used. We may write I_1 in the form

$$I_1 = \int_{x_4^0}^{x_4^1}\int_V \left\{ A\psi\psi^* - C(\partial_\alpha\psi)(\partial_\alpha\psi^*) + \frac{C}{2}\partial_\alpha\left(\psi\partial_\alpha\psi^* + \psi^*\partial_\alpha\psi\right)\right\} d\tau$$

The integral of the divergence terms, in I_1, may be written as integrals over the space-time surface which encloses that part of τ lying between x_4^0 and x_4^1. That is,

$$I_1 = \int_{x_4^0}^{x_4^1}\int_V \left\{ A\psi\psi^* - C(\partial_\alpha\psi)(\partial_\alpha\psi^*)\right\} d\tau + \int_{x_4^0}^{x_4^1}\int_S \frac{C}{2}\left\{\psi\partial_k\psi^* + \psi^*\partial_k\psi\right\} n_k\, dS\, dx_4$$
$$+ \int_V \frac{C}{2}\left\{\psi\partial_4\psi^* + \psi^*\partial_4\psi\right\}\Big|_{x_4^0}^{x_4^1} dV$$

where n_k is the unit-normal to S.

Therefore

$$\delta I_1 = \delta\int_{x_4^0}^{x_4^1}\int_V \left\{ A\psi\psi^* - C(\partial_\alpha\psi)(\partial_\alpha\psi^*)\right\} d\tau + \delta\int_{x_4^0}^{x_4^1}\int_S \frac{C}{2}\left\{\psi\partial_k\psi^* + \psi^*\partial_k\psi\right\} n_k\, dS\, dx_4$$

$$+ \delta\int_V \frac{C}{2}\left\{\psi\partial_4\psi^* + \psi^*\partial_4\psi\right\}\Big|_{x_4^0}^{x_4^1} dV$$

The action-principle states that $\delta\psi$, $\delta\psi^*$, $\delta\partial_4\psi$, $\delta\partial_4\psi^*$, vanish at x_4^0 and x_4^1. Therefore

$$\delta\int_V \frac{C}{2}\left\{\psi\partial_4\psi^* + \psi^*\partial_4\psi\right\}\Big|_{x_4^0}^{x_4^1} dV = 0$$

Also, for all the 'usual boundary conditions'

$$\delta \int_{x_4^0}^{x_4'} \int_S \frac{c}{2} \{ \psi \partial_\alpha \psi^* + \psi^* \partial_\alpha \psi \} n_\alpha \, dS \, dx_4 = 0$$

Consequently, for the action principle under these boundary conditions

$$\delta I_1 = \delta \int_{x_4^0}^{x_4'} \int_V \{ A \psi \psi^* - c (\partial_\alpha \psi)(\partial_\alpha \psi^*) \} d\tau \equiv \delta I_2$$

This explains the equivalence of the 'generalized Lagrangian density' and the 'usual Lagrangian density' for a complex scalar field with the 'usual boundary conditions'.

The 'generalized Lagrangian density', however, allows more boundary conditions than the 'usual Lagrangian density'. This means that its use in the action principle constitutes a more general description of the motion of the scalar field. This suggests that the theory of scalar wave fields should be developed with the 'generalized Lagrangian density'. Such a developement is discussed in the next section.

2.3 The Generalized Field Quantities for the Scalar Field

A consideration of the boundary conditions arising from the action-principle for complex scalar fields yielded the result that the most general description of the motion of the field is obtained with a Lagrangian density of the form 2.2 (1). It was shown, in the preceding sections, that this Lagrange function led to a description of the motion of the wave field in terms of the motion of the normal modes of the field. In addition it is required of any useful Lagrangian density that it yield other field functions and other aspects of the motion. It is expected that a Lagrangian density will lead, in a natural manner, to the construction of a Hamiltonian function and the expressions for the momentum, angular momentum and charge density of the field. These functions should enter into the conservation theorems associated with the motion of the wave field. It can be shown[9] that these field quantities and their conservation theorems may be derived from the usual Lagrangian densities. The discussion in this section will develop this aspect of the classical (unquantized) theory for the generalized Lagrangian density.

We define the momentum fields π and π^* conjugate to ψ and ψ^*, and the Hamiltonian density, $\mathcal{H}$, for these fields by:

$$\pi \equiv \frac{C}{c^2}\dot{\psi}^* \qquad c = \text{vel. of light}$$

$$\pi^* = \frac{C}{c^2}\dot{\psi}$$

$$\begin{aligned} \mathcal{H} &\equiv (\pi\dot{\psi} + \pi^*\dot{\psi}^*) - \frac{C}{c^2}\dot{\psi}\dot{\psi}^* - A\psi\psi^* - \frac{C}{2}(\psi\partial_k\partial_k\psi^* + \psi^*\partial_k\partial_k\psi) \\ &= \frac{C}{c^2}\dot{\psi}\dot{\psi}^* - A\psi\psi^* - \frac{C}{2}(\psi\partial_k\partial_k\psi^* + \psi^*\partial_k\partial_k\psi) \\ &= \frac{c^2}{C}\pi\pi^* - A\psi\psi^* - \frac{C}{2}(\psi\partial_k\partial_k\psi^* + \psi^*\partial_k\partial_k\psi) \end{aligned} \tag{1}$$

It is shown, in Appendix II, that the definitions (1) are 'natural' for those fields which are described by the generalized Lagrangian density. It will be shown here that π and π^* are conjugate to ψ and ψ^* respectively and that $\mathcal{H}$ may be identified with the energy density for the complex scalar field.

The Hamiltonian density is generally constructed from the 'usual Lagrangian densities'. For these Lagrangian densities the conjugate momentum fields π_2 and π_2^*, and the Hamiltonian density, $\mathcal{H}$, are defined:

$$\pi_2 = \frac{C}{c^2}\dot{\psi}^*$$

$$\pi_2^* = \frac{C}{c^2}\dot{\psi}$$

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[9] G. Wentzel, (reference 2), Chapter I.

$$\mathcal{H}_2 \equiv \pi_2 \dot{\psi} + \pi_2 \dot{\psi}^* - \frac{C}{c^2} \dot{\psi}\dot{\psi}^* - A\psi\psi^* + C (\partial_k\psi)(\partial_k\psi^*)$$

On comparison with the similar functions defined for the 'generalized Lagrangian density' it is seen that the conjugate momenta are defined in exactly the same way for both types of Lagrangian densities, $\mathcal{L}$ and $\mathcal{L}_2$; while the two Hamiltonian densities, $\mathcal{H}$ and $\mathcal{H}_2$, differ only in their space-derivative terms. π_2 and π_2^* will be called the 'usual conjugate momenta' while $\mathcal{H}_2$ will be the 'usual Hamiltonian density'.

The canonical equations for the generalized Hamiltonian density are the equations of motion for the wave field. In order to obtain such canonical equations we define functional derivatives, $\frac{\delta \mathcal{H}}{\delta \psi}$, $\frac{\delta \mathcal{H}}{\delta \psi^*}$, $\frac{\delta \mathcal{H}}{\delta \pi}$ and $\frac{\delta \mathcal{H}}{\delta \pi^*}$, of $\mathcal{H}$ in the following way:

$$\frac{\delta \mathcal{H}}{\delta \psi} \equiv \frac{\partial \mathcal{H}}{\delta \psi} + \partial_j \partial_j \frac{\partial \mathcal{H}}{\partial(\partial_k \partial_k \psi)} \qquad (3)$$

with similar definitions for $\frac{\delta \mathcal{H}}{\delta \psi^*}$, $\frac{\delta \mathcal{H}}{\delta \pi}$, $\frac{\delta \mathcal{H}}{\delta \pi^*}$.

Then the canonical equations of motion are:

$$\dot{\psi} = \frac{\delta \mathcal{H}}{\delta \pi} \qquad \dot{\pi} = -\frac{\delta \mathcal{H}}{\delta \psi}$$

$$\dot{\psi}^* = \frac{\delta \mathcal{H}}{\delta \pi^*} \qquad \dot{\pi}^* = -\frac{\delta \mathcal{H}}{\delta \psi^*} \qquad (4)$$

Using the definition of $\mathcal{H}$, these equations are:

$$\dot{\psi} = \frac{\delta \mathcal{H}}{\delta \pi} = \frac{c^2}{C} \pi^* = \dot{\psi} \qquad \text{an identity}$$

$$\dot{\pi} = -\frac{\delta \mathcal{H}}{\delta \psi} = A\psi^* + \frac{C}{2} \partial_k \partial_k \psi^* + \frac{C}{2} \partial_k \partial_k \psi^*$$

which is the field equation for ψ^*.

Similarly, the other two canonical equations yield an identity and

$$A\psi - C\partial_\alpha\partial_\alpha\psi = 0$$

the other wave field equation.

In order to identify $\mathcal{H}$ with the energy density for the complex scalar field and in order to obtain someof the other field quantities for this wave field, a stress-energy tensor will be defined. The "generalized Hamiltonian density" for the complex scalar field was shown to be:

$$\mathcal{H} = -C(\partial_4\psi)(\partial_4\psi^*) - A\psi\psi^* - \frac{C}{2}(\psi\partial_k\partial_k\psi^* + \psi^*\partial_k\partial_k\psi)$$

Using the scalar wave field equations this may be written:

$$\mathcal{H} = -C(\partial_4\psi)(\partial_4\psi^*) + \frac{C}{2}(\psi\partial_4\partial_4\psi^* + \psi^*\partial_4\partial_4\psi) \qquad (5)$$

It is desired to construct a tensor, $T_{\mu\nu}$, of second rank whose 4-4 component, T_{44} is the negative of $\mathcal{H}$. That is

$$T_{44} = -\mathcal{H}. \qquad (6)$$

Such a tensor is, obviously,

$$T_{\mu\nu} \equiv \frac{C}{2}\left[(\partial_\mu\psi)(\partial_\nu\psi^*) + (\partial_\nu\psi)(\partial_\mu\psi^*) - \psi\partial_\mu\partial_\nu\psi^* - \psi^*\partial_\mu\partial_\nu\psi\right] \qquad (7)$$

If this is the correct form of $T_{\mu\nu}$ then we shall expect that some of the conservation theorems for the complex scalar field may be summarized by:

$$\partial_\mu T_{\mu\nu} = 0 \qquad (8)$$

The equation (8) follows from the definition of $T_{\mu\nu}$, (7) and the scalar field equations, for

$$\partial_\mu T_{\mu\nu} = \frac{C}{2}\Big\{(\partial_\mu\partial_\mu\psi)(\partial_\nu\psi^*) + (\partial_\mu\psi)(\partial_\mu\partial_\nu\psi^*) + (\partial_\mu\partial_\nu\psi)(\partial_\mu\psi^*)$$

$$+ (\partial_\nu\psi)(\partial_\mu\partial_\mu\psi^*) - (\partial_\mu\psi)(\partial_\mu\partial_\nu\psi^*) - \psi\partial_\mu\partial_\mu\partial_\nu\psi^*$$

$$- \psi^*\partial_\nu\partial_\mu\partial_\mu\psi - (\partial_\mu\psi^*)(\partial_\mu\partial_\nu\psi)\Big\}$$

$$= \frac{C}{2}\Big\{(\partial_\nu\psi^*)(\partial_\mu\partial_\mu\psi - \frac{A}{C}\psi) + (\partial_\nu\psi)(\partial_\mu\partial_\mu\psi^* - \frac{A}{C}\psi^*)\Big\}$$

$$= 0$$

In the formulation of the scalar field theory with the 'usual Lagrangian densities', the stress-energy tensor, $T_{\mu\nu}$, is defined:

$$T_{\mu\nu} = C\{(\partial_\mu\psi)(\partial_\nu\psi^*) + (\partial_\nu\psi)(\partial_\mu\psi^*)\} + \mathcal{L}\delta_{\mu\nu}$$

$$\delta_{\mu\nu} = 0 \quad \nu \neq \mu \qquad (9)$$
$$= 1 \quad \nu = \mu$$

This is consistent with the definition of the 'usual Hamiltonian density'. It will be observed that $\mathcal{L}$ does not occur explicitly in our definition of $T_{\mu\nu}$ as it does in the conventional formulation.

If we define $\quad S_k = \frac{c}{i} T_{k4} \quad k = 1,2,3$

then the time-component $(\nu = 4)$ of (8) is

$$\frac{\partial \mathcal{H}}{\partial t} + \partial_k S_k = 0$$

which is a continuity equation for the energy of the complex scalar field. $\mathcal{H}$ is then identified with the energy density of the scalar field and S_k with the energy flux density.

Except for linear homogeneous boundary conditions, the total energy H, of the field, is given by $H = \int_V \mathcal{H}\, dV$, and is identical with the total energy which is obtained for the usual Hamiltonian density, $\mathcal{H}_2$.

Let $H_2 = \int_V \mathcal{H}_2\, dV$

Then
$$H - H_2 = -c \int_V \left\{ \tfrac{1}{2}\left(\psi \partial_k \partial_k \psi^* + \psi^* \partial_k \partial_k \psi\right) + (\partial_k \psi)(\partial_k \psi^*) \right\} dV$$
$$= -c \int_S \left\{ \psi \partial_k \psi^* + \psi^* \partial_k \psi \right\} n_k \, dS \tag{11}$$

where S is the surface enclosing V

and n_k is the unit normal to S.

Therefore, for all the usual boundary conditions $H = H_2$. That is, the total energy for the field under these boundary conditions is equal to the usual definition of the total energy. For linear homogeneous boundary conditions the difference in energy, $H - H_2$, may be attributed to the energy of the boundary conditions themselves. This was done for the motion of the stretched string (see page 21).

Conventionally $\mathcal{H}_2$ is identified with the energy density of the field. For the usual boundary conditions the identity of H with H_2 allows us to identify $\mathcal{H}$ also with the energy density of the field. In order to do this for $\mathcal{H}$ under linear homogeneous boundary conditions, we define a surface energy density $\mathcal{E}_B$ so that

$$\mathcal{E}_B = -c\left(\psi \partial_k \psi^* + \psi^* \partial_k \psi\right) n_k \tag{12}$$

Then we may write equation (11) in the form

$$\int_V \mathcal{H}\, dV = \int_V \mathcal{H}_2\, dV + \int_S \mathcal{E}_B\, dS \tag{13}$$

Then $\mathcal{H}$ is clearly the energy density not only within V but also for the boundary conditions on S. $\mathcal{H}$ is the energy density for the whole system. The energy of the boundary conditions, (12) will not, in general, remain constant in time. That is, an exchange of energy will occur between the field and the boundary. However, the continuity equation for energy holds at every point of the field region. From it we should expect that the total energy of the system is conserved. The net flux of energy from the combined system of field plus boundary is given by:

$$\int_S S_k n_k \, dS = \int_S \frac{c}{i}\frac{c}{2}\left\{(\partial_k\psi)(\partial_4\psi^*) + (\partial_4\psi)(\partial_k\psi^*) - \psi\partial_4\partial_k\psi^* - \psi^*\partial_4\partial_k\psi\right\} n_k \, dS$$

$$= \frac{c}{2}\int_S \left\{-\dot{\psi}^*\frac{\partial\psi}{\partial n} - \dot{\psi}\frac{\partial\psi^*}{\partial n} + \psi\frac{\partial}{\partial t}\frac{\partial\psi^*}{\partial n} + \psi^*\frac{\partial}{\partial t}\frac{\partial\psi}{\partial n}\right\} dS \qquad (14)$$

$= 0$ for all the usual boundary conditions and for linear homogeneous boundary conditions.

Equation (10) allows us to write

$$-\int_S S_k n_k \, dS = \int_V \left(\frac{\partial}{\partial t}\mathcal{H}\right) dV = \int_V \frac{\partial \mathcal{H}_2}{\partial t} \, dV + \int_S \frac{\partial}{\partial t}\mathcal{E}_B \, dS$$

so that just as $\mathcal{H}$ is the energy density for the field and for the boundary conditions, S_k is the energy flux vector for the field energy and for the boundary energy. Therefore we have shown that the total energy of the system is conserved. However, energy may flow from the field into or out of the boundary. Essentially the generalization of the Lagrangian density has allowed us to use the action principle in the discussion of a wave field for which the energy is not conserved. It has done this by allowing a particular type of energy source at the boundary of the field.

The momentum of the wave field may also be defined from the stress-energy tensor. We define the momentum density of the scalar field by:

$$G_k = \frac{1}{ic} T_{4k} \qquad k = 1,2,3 \qquad (15)$$

where G_k is the k-component of momentum density.

Then the k-component of the general continuity equation, (8), is the conservation theorem for the k-component of the momentum of the field.

The conservation theorem for angular momentum follows from the definition of $T_{\mu\nu}$ if we define the angular momentum tensor[10], $M_{\mu\nu\lambda}$

$$M_{\mu\nu\lambda} = T_{\mu\nu} x_\lambda - T_{\mu\lambda} x_\nu \tag{16}$$

$$\text{then} \quad \partial_\mu M_{\mu\nu\lambda} = (\partial_\mu T_{\mu\nu}) x_\lambda + T_{\lambda\nu} - (\partial_\mu T_{\mu\lambda}) x_\nu - T_{\nu\lambda}$$

$$= T_{\lambda\nu} - T_{\nu\lambda} \qquad \text{if} \quad \partial_\mu T_{\mu\nu} = 0$$

Thus since $T_{\mu\nu}$ is symmetric, that is $T_{\mu\nu} = T_{\nu\mu}$, we have

$$\partial_\mu M_{\mu\nu\lambda} = 0 \tag{17}$$

The angular momentum density of the scalar field is defined as having components $\frac{1}{ic}M_{423}$, $\frac{1}{ic}M_{431}$, $\frac{1}{ic}M_{412}$. Equation (17) then expresses the conservation of angular momentum.

If the field has linear homogeneous boundary conditions an exchange of momentum or angular momentum may occur between the field and the boundary. The functions defined here include the momentum or angular momentum of both the field and the boundary. However, continuity equations hold for the whole system. The 'generalized' stress-energy tensor has led to all the conservation theorems expected from it. It remains to be shown that a charge conservation theorem can be set up for the 'generalized' formulation.

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10 G. Wentzel, (reference 2), Chapter I.

The complex scalar wave field can be interpreted as a charge carrying field in as much as it is possible to define an electric charge density, ρ, and an electric current density vector, s_k, which satisfy a continuity equation. If we let ϵ be a real constant with the dimensions of electric charge, and define

$$\rho \equiv -i\epsilon\,(\pi\psi - \pi^*\psi^*)$$

$$= -i\epsilon\frac{c}{c^2}\,(\psi\dot{\psi}^* - \psi^*\dot{\psi}) \qquad (18)$$

and $$s_k \equiv -i\epsilon\, c\,(\psi\partial_k\psi^* - \psi^*\partial_k\psi)$$

then, using the classical field equations 1.2 (1), it follows that

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \partial_k s_k = -i\epsilon c\,\{&\psi\partial_4\partial_4\psi^* - (\partial_4\psi)(\partial_4\psi^*) + (\partial_4\psi)(\partial_4\psi^*) \\ &- \psi^*\partial_4\partial_4\psi + \psi\partial_k\partial_k\psi^* + (\partial_k\psi)(\partial_k\psi^*) \\ &- (\partial_k\psi)(\partial_k\psi^*) - \psi^*\partial_k\partial_k\psi\}\end{aligned} \qquad (19)$$

$$= -i\epsilon c\left\{-\frac{A}{c}\psi\psi^* + \frac{A}{c}\psi\psi^*\right\}$$

$$= 0$$

which expresses the conservation of charge.

Since the generalized conjugate momentum fields are identical with the usual conjugate momentum fields, the charge density, ρ, and the current density, s_k, are identical with the charge density and current density which are derived from the usual Lagrange functions.[11] There is no exchange of charge between the field and the boundary.

For the complex scalar wave field, discussed in terms of the generalized Lagrangian density, all the classical field quantities have been defined.

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[11] G. Wentzel, (reference 2), Chapter I.

The definitions of the momentum fields, π , π^* , conjugate to the fields ψ , ψ^* , and the definitions of charge density ρ and current density, s_k , are those found in the usual discussions of this field. However, the expressions for the Hamiltonian density, $\mathcal{H}$, the energy flux density, S_k, the momentum and the angular momentum of the field have been modified. Quantization of the scalar wave field would proceed directly from the commutation rules for the quantities which have been defined. This quantization will not be carried out in this investigation.

CHAPTER III

THE VECTOR FIELD

3.1 The Action Principle for the Vector Field

The action principle for the vector field is quite analogous to the action principle for the scalar field. Only the complex vector field will be discussed in this section: the real vector field is always an obvious special case of the complex field.

It was shown, on page 10, that the only acceptable form of the Lagrangian density for the vector field is given by:

$$\mathcal{L} = A\psi_\beta \psi_\beta^* + B(\partial_\alpha\psi_\beta - \partial_\beta\psi_\alpha)(\partial_\alpha\psi_\beta^* - \partial_\beta\psi_\alpha^*)$$

$$+ \frac{C}{2}\left[\psi_\beta(\partial_\alpha\partial_\alpha\psi_\beta^* - \partial_\alpha\partial_\beta\psi_\alpha^*) + \psi_\beta^*(\partial_\alpha\partial_\alpha\psi_\beta - \partial_\alpha\partial_\beta\psi_\alpha)\right] \quad (1)$$

The action integral, I, is defined by:

$$I \equiv \int_{x_4^0}^{x_4^1}\int_V \mathcal{L}\, d\tau \quad (2)$$

and the action principle may be stated:

$$\delta I = 0 \quad (3)$$

where x_4^0 and x_4^1 and each $\delta\psi_\beta$, $\delta\psi_\beta^*$ are arbitrary in the manner which is given in the example on page 3. This principle describes the motion of the vector field completely. Equation (3) may also be written:

$$0 = \int_{x_4^0}^{x_4^1}\int_V \delta\mathcal{L}\, d\tau \quad (4)$$

Expanding $\delta\mathcal{L}$ in a Taylor's series we may write:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi_\beta}\delta\psi_\beta + \frac{\partial\mathcal{L}}{\partial\psi^*_\beta}\delta\psi^*_\beta + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi_\beta)}\partial_\alpha\delta\psi_\beta + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi^*_\beta)}\partial_\alpha(\delta\psi^*_\beta) + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\alpha\psi_\beta)}\partial_\gamma\partial_\gamma\delta\psi_\beta$$

$$+ \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\alpha\psi^*_\beta)}\partial_\gamma\partial_\gamma\delta\psi^*_\beta + \frac{\partial\mathcal{L}}{\partial(\partial_\beta\partial_\alpha\psi_\alpha)}\partial_\gamma\partial_\beta\delta\psi_\gamma + \frac{\partial\mathcal{L}}{\partial(\partial_\beta\partial_\alpha\psi^*_\alpha)}\partial_\gamma\partial_\beta\delta\psi^*_\gamma \qquad (5)$$

Substituting from (1) into (5) and then using (5) in the action principle (4), one obtains, after suitable partial integrations, the action principle in the form:

$$0 = \int_{x_4^0}^{x_4'}\int_V \left\{\left[A\psi^*_\beta + (C-2B)(\partial_\alpha\partial_\alpha\psi^*_\beta - \partial_\beta\partial_\alpha\psi^*_\alpha)\right]\delta\psi_\beta + \left[A\psi_\beta + (C-2B)(\partial_\alpha\partial_\alpha\psi_\beta - \partial_\beta\partial_\alpha\psi_\alpha)\right]\delta\psi^*_\beta\right\} d\tau \qquad (6)$$

$$+ \int_{x_4^0}^{x_4'}\int_S \Big\{(2B-\tfrac{C}{2})(\partial_\alpha\psi^*_\beta - \partial_\beta\psi^*_\alpha)\delta\psi_\beta + (2B-\tfrac{C}{2})(\partial_\alpha\psi_\beta - \partial_\beta\psi_\alpha)\delta\psi^*_\beta$$

$$+ \tfrac{C}{2}\psi^*_\beta(\partial_\alpha\delta\psi_\beta - \partial_\beta\delta\psi_\alpha) + \tfrac{C}{2}\psi_\beta(\partial_\alpha\delta\psi^*_\beta - \partial_\beta\delta\psi^*_\alpha)\Big\} n_\alpha\, dS$$

where S is the surface enclosing V

and n_α is the unit normal to S.

The $\int_{x_4^0}^{x_4'}\int_V$ integral in (6), for the completely arbitrary $\delta\psi_\beta$ and $\delta\psi^*_\beta$, yields the Euler-Lagrange equations of the motion of the wave field. These Euler-Lagrange equations are:

$$A\psi^*_\beta + (C-2B)(\partial_\alpha\partial_\alpha\psi^*_\beta - \partial_\alpha\partial_\beta\psi^*_\alpha) = 0$$

$$A\psi_\beta + (C-2B)(\partial_\alpha\partial_\alpha\psi_\beta - \partial_\alpha\partial_\beta\psi_\alpha) = 0 \qquad (7)$$

With $K = A / (2B - C)$ these are identically the field equations 1.2 (3) for the vector field. They are equivalent to the equations

$$A\psi^*_\beta + (C-2B)\partial_\alpha\partial_\alpha\psi^*_\beta = 0$$

$$A\psi_\beta + (C-2B)\partial_\alpha\partial_\alpha\psi_\beta = 0 \qquad (8)$$

and the Lorentz conditions
$$\partial_\beta \psi_\beta = 0$$
$$\partial_\beta \psi^*_\beta = 0$$

in terms of which the vector field is usually discussed.

With these Euler-Lagrange equations the action principle in the form (6) yields the boundary conditions for the vector field equations. Since x_4^0 and x_4^1 are entirely arbitrary, one obtains from (6)

$$0 = \int_S \Big\{(aB - \tfrac{c}{2})(\partial_k \psi^*_\beta - \partial_\beta \psi^*_k)\delta\psi_\beta + (aB - \tfrac{c}{2})(\partial_k \psi_\beta - \partial_\beta \psi_k)\delta\psi^*_\beta + \frac{c}{2}\psi^*_\beta(\partial_k \delta\psi_\beta - \partial_\beta \delta\psi_k) + \frac{c}{2}\psi_\beta(\partial_k \delta\psi^*_\beta - \partial_\beta \delta\psi^*_k)\Big\} n_k \, dS \tag{9}$$

All the boundary conditions to be allowed with the Euler-Lagrange equations (7) will satisfy the condition (9).

The Lorentz conditions may be used to remove some of the terms in (9). In order to do this we observe that the Lorentz conditions allow us to write:

$$0 = \int_{x_4^0}^{x_4^1}\int_V \Big\{ \delta\big[\psi^*_\beta \partial_\beta(\partial_\alpha \psi_\alpha) - \psi_\beta \partial_\beta(\partial_\alpha \psi^*_\alpha)\big]\Big\} d\tau \tag{10}$$

This may be rewritten as a surface integral over the space-time surface which encloses that part of τ which lies between x_4^0 and x_4^1. That is we obtain:

$$0 = \int_{x_4^0}^{x_4^1}\int_S \Big\{\delta(\psi^*_\beta \partial_\beta \psi_k - \psi_\beta \partial_\beta \psi^*_k)\Big\} n_k \, dS \, dx_4 + \int_V \Big\{\delta(\psi^*_\beta \partial_\beta \psi_4 - \psi_\beta \partial_\beta \psi^*_4)\Big\} n_4 \Big|_{x_4^0}^{x_4^1} dV \tag{11}$$

where n_k is the unit normal to S [illegible].

But the action principle states that the variations $\delta\psi_\beta$, $\delta\psi^*_\beta$, $\delta\partial_\beta\psi_4$, $\delta\partial_\beta\psi^*_4$, vanish at x_4^0 and x_4^1. Consequently

$$0 = \int_V \left\{ \delta\left(\psi_\beta^* \partial_\beta \psi_4 - \psi_\beta \partial_\beta \psi_4^*\right)\right\} \Big|_{x_4^0}^{x_4'} dV$$

whence

$$0 = \int_{x_4^0}^{x_4'} \int_S \left\{ \delta\left(\psi_\beta^* \partial_\beta \psi_k - \psi_\beta \partial_\beta \psi_k^*\right)\right\} n_k \, ds \, dx_4 \qquad (12)$$

which, for arbitrary x_4^0 and x_4^1, may be written:

$$\int_S \left\{ (\partial_\beta \psi_k)\delta\psi_\beta^* - \psi_\beta \partial_\beta \delta\psi_k^* \right\} n_k \, ds = \int_S \left\{ (\partial_\beta \psi_k^*)\delta\psi_\beta - \psi_\beta^* \partial_\beta \delta\psi_k \right\} n_k \, ds \qquad (13)$$

the left-hand side contains only the variations $\delta\psi_\beta^*$

the right-hand side contains only the variations $\delta\psi_\beta$ $\qquad \beta = 1,2,3,4.$

Since these variations are independent each side may be equated to zero separately, that is

$$\int_S \left\{ (\partial_\beta \psi_k)\delta\psi_\beta^* - \psi_\beta \partial_\beta \delta\psi_k^* \right\} n_k \, ds = 0$$

$$\int_S \left\{ (\partial_\beta \psi_k^*)\delta\psi_\beta - \psi_\beta \partial_\beta \delta\psi_k^* \right\} n_k \, ds = 0 \qquad (14)$$

Using (14), the boundary condition (9) may be written

$$0 = \int_S \Big\{ -\tfrac{c}{2}(\partial_k \psi_\beta^*)\delta\psi_\beta - \tfrac{c}{2}(\partial_k \psi_\beta)\delta\psi_\beta^* + \tfrac{c}{2}\psi_\beta^* \partial_k \delta\psi_\beta + \tfrac{c}{2}\psi_\beta \partial_k \delta\psi_\beta^* \qquad (15)$$

$$+ 2B(\partial_k \psi_\beta^* - \partial_\beta \psi_k^*)\delta\psi_\beta + 2B(\partial_k \psi_\beta - \partial_\beta \psi_k)\delta\psi_\beta^* \Big\} n_k \, ds$$

An examination of (15) reveals the following boundary conditions:

$$\psi_\beta = \psi_\beta^* = 0 \quad \text{on } S. \qquad (16)$$

$$\text{or} \quad \psi_\beta, \psi_\beta^* \quad \text{periodic on } S.$$

$$\beta = 1, 2, 3, 4$$

But, unless B = 0,

$$\text{neither} \quad \frac{\partial \psi_\beta^*}{\partial n} = 0 \qquad \frac{\partial \psi_\beta}{\partial n} = 0$$

$$\text{nor} \quad \frac{\partial \psi_\beta}{\partial n} = \gamma \psi_\beta \qquad \frac{\partial \psi_\beta^*}{\partial n} = \gamma \psi_\beta^* \qquad \text{on } S.$$

(where γ is a constant depending on β)

satisfy condition (15). Since these boundary conditions are expected of a useful Lagrangian density, we stipulate that B = 0. Then (15) may be written:

$$0 = -\frac{C}{2} \int_S \left\{ (\partial_\alpha \psi_\beta^*) \delta \psi_\beta + (\partial_\alpha \psi_\beta) \delta \psi_\beta^* - \psi_\beta^* \partial_\alpha \delta \psi_\beta - \psi_\beta \partial_\alpha \delta \psi_\beta^* \right\} n_\alpha \, dS \qquad (17)$$

which is satisfied by all the usual boundary conditions and also by linear homogeneous boundary conditions. The relation of the latter to the surface condition (17) is discussed in Appendix I.

3.2 The Generalized Lagrangian Density for Vector Fields

We shall call

$$\mathcal{L} = A \psi_\beta^* \psi_\beta + \frac{C}{2} \left\{ \psi_\beta (\partial_\alpha \partial_\alpha \psi_\beta^* - \partial_\beta \partial_\alpha \psi_\alpha^*) + \psi_\beta^* (\partial_\alpha \partial_\alpha \psi_\beta - \partial_\beta \partial_\alpha \psi_\alpha) \right\} \qquad (1)$$

the "generalized Lagrangian density" for vector fields. The Lagrangian density, $\mathcal{L}_2$, which is usually employed in the formulation of vector fields,

$$\mathcal{L}_2 = A \psi_\beta^* \psi_\beta - \frac{C}{2} (\partial_\alpha \psi_\beta - \partial_\beta \psi_\alpha)(\partial_\alpha \psi_\beta^* - \partial_\beta \psi_\alpha^*) \qquad (2)$$

will be called the 'usual Lagrangian density'. The Euler-Lagrange equations, 3.1 (7), show that both the generalized and the usual Lagrangian densities lead to the same field equations. However, the usual Lagrangian density allows only some of the usual boundary conditions while the generalized Lagrangian density allows all the usual boundary conditions and, in addition, linear homogeneous boundary conditions. Thus the use of the generalized Lagrangian density, (1), constitutes a generalization of the description of the motion of the vector field by an action principle.

In the next section the classical theory for vector fields will be developed for the generalized Lagrangian density (1).

3.3 The Generalized Field Quantities for the Vector Field

For the generalized Lagrangian density, 3.2 (1), the momentum fields, π_β and π_β^*, conjugate to ψ_β and ψ_β^* respectively, and the Hamiltonian density, $\mathcal{H}$, may be defined by:

$$\pi_\beta \equiv \frac{C}{c^2}\dot{\psi}_\beta^*$$

$$\pi_\beta^* \equiv \frac{C}{c^2}\dot{\psi}_\beta$$

$$\mathcal{H} = \frac{c^2}{C}\pi_\beta\pi_\beta^* - A\psi_\beta\psi_\beta^* - \frac{C}{2}\psi_\beta(\partial_k\partial_k\psi_\beta^* - \partial_\alpha\partial_\beta\psi_\alpha^*) \qquad (1)$$

$$- \frac{C}{2}\psi_\beta^*(\partial_k\partial_k\psi_\beta - \partial_\alpha\partial_\beta\psi_\alpha)$$

$$k = 1,2,3$$
$$\alpha,\beta = 1,2,3,4$$

The field quantities, (1), are derived from the generalized Lagrangian density, in a natural manner, in Appendix II. We shall refer to these quantities as the generalized field quantities for the vector field.

In the conventional formulation of the vector field the field quantities are constructed from the usual Lagrangian density, $\mathcal{L}_2$. Then the usual field quantities, $\pi_{\beta 2}$, $\pi_{\beta 2}^*$, $\mathcal{H}_2$ are defined:

$$\pi_{\beta_2} \equiv -\frac{c}{ic}\left(\partial_4 \psi^*_\beta - \partial_\beta \psi^*_4\right)$$

$$\pi^*_{\beta 2} \equiv -\frac{c}{ic}\left(\partial_4 \psi_\beta - \partial_\beta \psi_4\right)$$

$$\mathcal{H}_2 \equiv \pi_{\beta 2}\dot{\psi}_\beta + \pi^*_{\beta 2}\dot{\psi}^*_\beta - \mathcal{L}_2 - \partial_k F_k \qquad (2)$$

$$= C\left\{(\partial_j \psi^*_4 - \partial_4 \psi^*_j)\partial_4 \psi_j + (\partial_j \psi_4 - \partial_4 \psi_j)\partial_4 \psi^*_j\right\} - A\psi^*_\beta \psi_\beta$$

$$+ \frac{C}{2}(\partial_\alpha \psi^*_\beta - \partial_\beta \psi^*_\alpha)(\partial_\alpha \psi_\beta - \partial_\beta \psi_\alpha) - \partial_k F_k .$$

where F_k is some vector function of $\psi_1, \psi_2, \psi_3, \psi_4, \psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*$. It is usually assumed that divergences of the form $-\partial_k F_k$ can be added to $\mathcal{H}_2$ arbitrarily because the integrated Hamiltonian density, H_2 ($H_2 = \int_V \mathcal{H}_2 dV$), will not be affected by these terms if the field functions vanish at the boundary. $\mathcal{H}_2$ is used only for this special boundary condition.

It is seen, on inspection that the generalized π_β , π_β^* and $\mathcal{H}$ differ from the usual $\pi_{\beta_2}, \pi^*_{\beta 2}$ and $\mathcal{H}_2$. In particular, for the usual formulation both π_{4_2} and $\pi^*_{4_2}$ vanish identically while, for the generalized formulation, π_4 and π_4^* do not vanish in general. In the usual developement the vanishing of these two momentum fields introduces difficulties into the theory of vector fields. For example, if the canonical equations from a Hamiltonian density, for vector fields, are to yield the field equations in the form 3.1 (8) then these equations must arise for all the components of ψ, ψ^*, that is for $\psi_1, \psi_2, \psi_3, \psi_4$ $\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*$. However, with the usual conjugate momenta canonical equations would have to be set up involving the identically vanishing π_{4_2} and $\pi^*_{4_2}$. This construction would not have meaning. Even otherwise, it is usually necessary to assume a particular form of F_k if the remaining six canonical equations are to be the corresponding six field equations. Such difficulties are not encountered with the generalized conjugate momenta.

It would be possible, with the usual Lagrange function, to define the usual conjugate fields to be identical with the generalized conjugate fields (1). However, such definitions would not follow naturally from the usual Lagrangian density (see Appendix II). The merits of the generalized field quantities will receive further discussion in the following sections.

To obtain the canonical equations of motion from the generalized Hamiltonian density, $\mathcal{H}$, it is necessary to define functional derivatives, $\frac{\delta \mathcal{H}}{\delta \psi_\beta}$, $\frac{\delta \mathcal{H}}{\delta \psi^*_\beta}$, $\frac{\delta \mathcal{H}}{\delta \pi_\beta}$, $\frac{\delta \mathcal{H}}{\delta \pi^*_\beta}$, with respect to $\mathcal{H}$ in the following way:

$$\frac{\delta \mathcal{H}}{\delta \psi_\beta} \equiv \frac{\partial \mathcal{H}}{\partial \psi_\beta} + \partial_j \partial_j \frac{\partial \mathcal{H}}{\partial(\partial_k \partial_k \psi_\beta)} + \partial_\alpha \partial_\beta \frac{\partial \mathcal{H}}{\partial(\partial_\alpha \partial_\gamma \psi_\gamma)} \tag{3}$$

with similar definitions for $\frac{\delta \mathcal{H}}{\delta \psi^*_\beta}$, $\frac{\delta \mathcal{H}}{\delta \pi_\beta}$ and $\frac{\delta \mathcal{H}}{\delta \pi^*_\beta}$.

With these functional derivatives the canonical equations of motion are:

$$\dot{\psi}_\beta = \frac{\delta \mathcal{H}}{\delta \pi_\beta} \quad \text{(a)} \qquad \dot{\pi}_\beta = -\frac{\delta \mathcal{H}}{\delta \psi_\beta} \quad \text{(b)}$$

$$\dot{\psi}^*_\beta = \frac{\delta \mathcal{H}}{\delta \pi^*_\beta} \quad \text{(c)} \qquad \dot{\pi}^*_\beta = -\frac{\delta \mathcal{H}}{\delta \psi^*_\beta} \quad \text{(d)} \tag{4}$$

It is expected that these canonical equations will be the equations of motion of the vector wave field. This is readily verified.

(4a) yields $\dot{\psi}_\beta = \dot{\psi}_\beta$ an identity, and

(4b) yields
$$-C \partial_4 \partial_4 \psi^*_\beta = A \psi^*_\beta + \frac{C}{2}(\partial_k \partial_k \psi^*_\beta - \partial_\alpha \partial_\beta \psi^*_\alpha) + \frac{C}{2}(\partial_k \partial_k \psi^*_\beta - \partial_\alpha \partial_\beta \psi^*_\alpha)$$

which is the field equation for ψ^*_β in the form 3.1 (7).

similarly (4c) and (4d) yield, respectively:

$$\dot{\psi}^*_\beta = \dot{\psi}^*_\beta \qquad \text{an identity}$$

and $$0 = A\psi_\beta + C(\partial_\alpha\partial_\alpha\psi_\beta - \partial_\alpha\partial_\beta\psi_\alpha)$$

which is the equation of motion for the field ψ_β.

The canonical equations of motion, (4), have not been derived here from a variational principle involving the generalized Hamiltonian density $\mathcal{H}$. This could be done and such a discussion could also be used to obtain the boundary conditions to be allowed in the description of the motion. One would proceed to discuss $\mathcal{H}$ in a manner entirely analogous to the way in which $\mathcal{L}$ was discussed in the preceding sections.

If the generalized Hamiltonian density which we have constructed is to be entirely successful in the classical theory it remains to be shown that $\mathcal{H}$ may be interpreted as the energy density of the wave fields and that a continuity equation for energy may be set up. In order to do this, a stress-energy tensor, for the vector wave field, will be constructed from the definition of $\mathcal{H}$.

The vector field equations, 3.1 (8), may be used to write the generalized Hamiltonian function, (1), in the form

$$\mathcal{H} = -C(\partial_4\psi_\beta)(\partial_4\psi^*_\beta) + \frac{C}{2}(\psi_\beta\partial_4\partial_4\psi^*_\beta + \psi^*_\beta\partial_4\partial_4\psi_\beta) \tag{5}$$

A stress-energy tensor, $T_{\mu\nu}$, may now be defined

$$T_{\mu\nu} \equiv \frac{C}{2}\left\{(\partial_\mu\psi_\beta)(\partial_\nu\psi^*_\beta) + (\partial_\nu\psi_\beta)(\partial_\mu\psi^*_\beta) - \psi_\beta\partial_\mu\partial_\nu\psi^*_\beta - \psi^*_\beta\partial_\mu\partial_\nu\psi_\beta\right\} \tag{6}$$

It is seen, on inspection, that $T_{44} = -\mathcal{H}$ as required. We shall identify various field quantities with the components of $T_{\mu\nu}$. Some of the continuity equations for the vector field will be of the form:

$$\partial_\mu T_{\mu\nu} = 0 \tag{7}$$

This equation follows directly from the definition of $T_{\mu\nu}$ (6) and from the vector field equations 3.1 (7) (cf. proof of similar relation for the scalar field, page 32). The time-component of equation (7) will be recognized as a continuity equation for energy. The energy flux vector, S_k, for the vector field, is defined by:

$$S_k = \frac{c}{i} T_{k4} \qquad k = 1.2.3 \tag{8}$$

Then the time-component of equation (7) may be written:

$$\frac{\partial \mathcal{H}}{\partial t} + \partial_k S_k = 0 \tag{9}$$

The generalized Hamiltonian density, $\mathcal{H}$, will be identified with the energy density of the vector field through a comparison of $\mathcal{H}$ with the usual Hamiltonian density, $\mathcal{H}_2$. We may write

$$\mathcal{H} - \mathcal{H}_2 = -C \Big\{ (\partial_4 \psi_4^*)(\partial_4 \psi_4) + \tfrac{1}{2}(\psi_\alpha \partial_k \partial_k \psi_\alpha^*) + \tfrac{1}{2}(\psi_\alpha^* \partial_k \partial_k \psi_\alpha) + (\partial_j \psi_4^*)(\partial_4 \psi_j) + (\partial_j \psi_4)(\partial_4 \psi_j^*)$$
$$- (\partial_4 \psi_j^*)(\partial_4 \psi_j) + (\partial_4 \psi_\beta^* - \partial_\beta \psi_4^*)\partial_4 \psi_\beta + (\partial_j \psi_\beta^* - \partial_\beta \psi_j^*)\partial_j \psi_\beta \Big\} + \partial_k F_k \tag{10}$$

$$= -C(\partial_4 \psi_4)(\partial_4 \psi_4^*) + C(\partial_j \psi_k^*)(\partial_k \psi_j) + \partial_k F_k.$$

$$+ \left(-\frac{C}{2}\right)(\psi_\alpha \partial_k \partial_k \psi_\alpha^*) - \frac{C}{2} \psi_\alpha^* \partial_k \partial_k \psi_\alpha - C(\partial_k \psi_\alpha^*)(\partial_k \psi_\alpha)$$

To reduce this expression to a pure divergence we remember that the Lorentz conditions may be written:

$$\partial_k \psi_k = -\partial_4 \psi_4 \qquad \partial_k \psi_k^* = -\partial_4 \psi_4^*$$

Then we examine

$$\frac{c}{2}\partial_k\left\{\psi_j^*\partial_j\psi_k + (\partial_j\psi_k^*)\psi_j + \psi_k\partial_4\psi_4^* + \psi_k^*\partial_4\psi_4\right\}$$

$$= c(\partial_k\psi_j^*)(\partial_j\psi_k) + \frac{c}{2}\psi_j^*\partial_j(\partial_k\psi_k) + \frac{c}{2}\psi_j\partial_j(\partial_k\psi_k^*)$$

$$+\frac{c}{2}\left\{\psi_k\partial_k(\partial_4\psi_4^*) + 2(\partial_4\psi_4^*)(\partial_k\psi_k) + \psi_k^*\partial_k(\partial_4\psi_4)\right\} \qquad (12)$$

$$= c(\partial_k\psi_j^*)(\partial_j\psi_k) - c(\partial_4\psi_4^*)(\partial_4\psi_4)$$

Using (12) in (10), we have

$$\mathcal{H} - \mathcal{H}_2 = \frac{c}{2}\partial_k\left\{-\psi_\alpha\partial_k\psi_\alpha^* - \psi_\alpha^*\partial_k\psi_\alpha + \psi_j^*\partial_j\psi_k + \psi_j\partial_j\psi_k^* + \psi_k\partial_4\psi_4^* + \psi_k^*\partial_4\psi_4\right\} + \partial_k F_k \qquad (13)$$

which is a pure divergence.

The difference between the integrated Hamiltonians, $\int_V \mathcal{H}\,dV - \int_V \mathcal{H}_2\,dV$, is therefore an expression which may be evaluated at the boundary. If $\int_V \mathcal{H}_2\,dV$ is identified with the energy of the wave field, then, for the more general motions of the vector field $\int_V \mathcal{H}\,dV$ is identified with the energy of not only the wave field but also the boundary conditions.

The generalized Hamiltonian density will serve as the energy density of the vector field if we remember that at the boundary $\mathcal{H}$ contains terms which describe a surface energy density. However, the total energy flux through the boundary vanishes, that is,

$$\int_S S_k n_k\,dS = 0 \qquad (14)$$

This follows from the definition of S_k,

$$\int_S S_k n_k\,dS = \int_S \frac{e}{i}\frac{c}{2}\left\{(\partial_k\psi_\beta^*)(\partial_4\psi_\beta) + (\partial_k\psi_\beta)(\partial_4\psi_\beta^*) - \psi_\beta\partial_k\partial_4\psi_\beta^* - \psi_\beta^*\partial_k\partial_4\psi_\beta\right\} n_k\,dS$$

$$= \frac{c}{2}\int_S\left\{-\dot{\psi}_\beta^*\frac{\partial\psi_\beta}{\partial n} - \dot{\psi}_\beta\frac{\partial\psi_\beta^*}{\partial n} + \psi_\beta\frac{\partial}{\partial t}\left(\frac{\partial\psi_\beta^*}{\partial n}\right) + \psi_\beta^*\frac{\partial}{\partial t}\left(\frac{\partial\psi_\beta}{\partial n}\right)\right\} dS$$

$$= 0$$ for all the allowed boundary conditions.

Therefore the total energy of the system is conserved. However, as for the scalar field, an exchange of energy may occur between the field and the boundary. The generalization of the Lagrangian density has extended the use of the action principle to a special kind of non - conservative vector field.

The momentum density vector, G_k, for the vector field may be defined:

$$G_k = \frac{1}{ic} T_{4k} \qquad k = 1,2,3. \qquad (15)$$

from which it follows that the $\nu = k$ component of the general continuity equation (7) expresses the conservation of the k-component of the momentum of the wave field.

As for the scalar wave field, the conservation of angular momentum is an immediate consequence of the symmetry of $T_{\mu\nu}$.

By defining a charge density, ρ, and a current density, s_k, in terms of the vector field functions, and by constructing a continuity equation with ρ and s_k it is possible to interpret the vector field as a charge carrying field. As for the scalar field, we define:

$$\rho \equiv -i\epsilon(\pi_\beta \psi_\beta - \pi^*_\beta \psi^*_\beta) = -i\epsilon \frac{c}{c^2}(\psi_\beta \dot{\psi}^*_\beta - \psi^*_\beta \dot{\psi}_\beta)$$

$$s_k \equiv -i\epsilon c\left\{\psi_\beta \partial_k \psi^*_\beta - (\partial_k \psi_\beta) \psi^*_\beta\right\} \qquad (16)$$

where ϵ is a real constant with the dimensions of charge. Then, using the vector field equations, it follows immediately that

$$\frac{\partial \rho}{\partial t} + \partial_k s_k = 0 \qquad (17)$$

which expresses the conservation of charge.

Because the generalized conjugate momentum fields, π_β and π_β^*, are different from those usually employed in the literature, the definitions of charge density and current density, given above, also differ from the usual definitions of these quantities. Thus usually,

$$\rho' \equiv -\epsilon \frac{C}{c}\left(\psi_\beta \partial_4 \psi_\beta^* - \psi_\beta \partial_\beta \psi_4^* - \psi_\beta^* \partial_4 \psi_\beta + \psi_\beta^* \partial_\beta \psi_4\right)$$

$$s_k' \equiv -i\epsilon C\left(\psi_\beta \partial_k \psi_\beta^* - (\partial_\beta \psi_k^*)\psi_\beta - \psi_\beta^* \partial_k \psi_\beta + \psi_\beta^* \partial_\beta \psi_k\right) \tag{18}$$

Then we have

$$\begin{aligned} \rho - \rho' &= \epsilon \frac{C}{c}\left(\psi_\beta \partial_\beta \psi_4^* - \psi_\beta^* \partial_\beta \psi_4\right) \\ &= \epsilon \frac{C}{c} \partial_\beta \left(\psi_\beta \psi_4^* - \psi_\beta^* \psi_4\right) \end{aligned} \tag{19}$$

since $\partial_\beta \psi_\beta = 0$ and $\partial_\beta \psi_\beta^* = 0$

The total charges for ρ and ρ', that is, $\int_V \rho\, dV$ and $\int_V \rho'\, dV$ respectively, then differ only by a quantity which is evaluated at the boundary. This difference vanishes if the field functions vanish at the boundary.

The generalized Lagrange function for the vector field allowed a definition of the conjugate momentum fields for which the time-component of these fields did not vanish identically. It has been shown, in this section, that all the classical field quantities and conservation theorems may be derived along with such definitions of the conjugate momentum fields. The formulation was extended to include fields with linear homogeneous boundary conditions. More than that, the vector field quantities, as derived above, possess a remarkable similarity in form to their analogues in the complex scalar field. As a rule, if any generalized field quantity is defined for the scalar field its analogue in the vector field follows immediately by simply replacing the scalar field functions, ψ , ψ^* , by vector field functions, ψ_β , ψ^*_β , and by replacing products of scalar field functions by inner products of vector field functions. Thus

$$\begin{aligned} \psi &\longrightarrow \psi_\beta \\ \psi\psi^* &\longrightarrow \psi_\beta \psi^*_\beta \\ \frac{\partial \psi}{\partial n}\psi^* &\longrightarrow \frac{\partial \psi_\beta}{\partial n}\psi^*_\beta \end{aligned} \tag{20}$$

etc.

The only exceptions to this rule are the Lagrange function and the Hamiltonian function. These are logical exceptions because we require that the Lagrangian density and also the Hamiltonian density, for the vector field, lead to a Lorentz condition, a condition which has no analogue in the scalar field. The Hamiltonian functions and the Lagrange functions for the vector fields are at variance with the rule above only in those terms (that is, mixed-derivative terms), in the vector field quantities, which lead to this Lorentz condition.

The simple rule given above, for the transition in form from the scalar to the vector field does not apply for any of the field quantities in the usual developement of the classical theory for these fields.

CHAPTER IV

4.1 The Dirac Field

The Dirac field equations, 1.2 (5), involve no derivatives of higher order than the first and consequently the Lagrange function for this field may not be generalized to contain second-order derivatives. However, it may be shown, for this field also, that the form of the boundary conditions to be allowed in the description of the motion of the wave field are a direct consequence of the action principle for this field.

The Lagrangian density, $\mathcal{L}$, for the Dirac wave field may be chosen:

$$\mathcal{L} = -\psi_\rho^* \left\{ \frac{\hbar}{i}\dot{\psi}_\rho + \frac{c\hbar}{i}\alpha^{(k)}_{\rho\sigma}\partial_k\psi_\sigma + mc^2\beta_{\rho\sigma}\psi_\sigma \right\} \tag{1}$$

which, for reasons of simplicity, is unsymmetrical in ψ_ρ and ψ_ρ^*, but the discussion of the action-principle will show that this expression could be easily symmetrized. As for the scalar and vector fields, the action-principle here may be written:

$$0 = \int_{x_4^0}^{x_4'}\int_V \delta\mathcal{L}\, d\tau \tag{2}$$

where the variation $\delta\mathcal{L}$ means the variations in $\mathcal{L}$ of the functional dependence of all the ψ_ρ, ψ_ρ^* on their respective variables. These variations vanish at x_4^0 and x_4^1.

Using the form of $\mathcal{L}$ given in (1) and performing partial integrations, the usual variational procedure allows us to write (2) as:

$$0 = \int_{x_4^0}^{x_4'} \int_S \left\{ -\psi_\rho^* \frac{c\hbar}{i} \alpha_{\rho\sigma}^{(k)} \delta\psi_\sigma \right\} n_k \, dS \, dx_4$$

$$+ \int_{x_4^0}^{x_4'} \int_V \left\{ \left[\frac{\hbar}{i} \dot{\psi}_\rho + \frac{c\hbar}{i} \alpha_{\rho\sigma}^{(k)} \partial_k \psi_\sigma + mc^2 \beta_{\rho\sigma} \psi_\sigma \right] (-\delta\psi_\rho^*) \right. \tag{3}$$

$$\left. + \left[\frac{\hbar}{i} \dot{\psi}_\sigma^* + \frac{c\hbar}{i} \partial_k \psi_\rho^* \alpha_{\rho\sigma}^{(k)} - mc^2 \psi_\rho^* \beta_{\rho\sigma} \right] \delta\psi_\sigma \right\} d\tau$$

where $n_k = (n_1, n_2, n_3)$, the unit normal to S.

The Euler-Lagrange equations yielded by the second integral of (3) are identically the Dirac field equations in the form 1.2 (5). The first integral, which is the boundary condition integral, allows us to write:

either $\quad \psi_\sigma = 0 \quad$ on S $\qquad$ (4)

or $\quad n_k \psi_\rho^* \alpha_{\rho\sigma}^{(k)} = 0 \quad$ on s $\qquad$ (5)

or the periodicity of ψ_σ on S, as possible boundary conditions. By interchanging the places of ψ_ρ and ψ_ρ^* in the Lagrange function (1), one would have obtained

either $\quad \psi_\sigma^* = 0 \quad$ on S $\qquad$ (6)

or $\quad n_k \alpha_{\rho\sigma}^{(k)} \psi_\rho = 0 \quad$ on S $\qquad$ (7)

or the periodicity of ψ_σ^* on S.

It will now be shown that (4) is equivalent to (7) and that (5) is equivalent to (6). From the definitions of $\alpha_{\rho\sigma}^{(k)}$ and ψ_ρ^* as given by 1.2 (5), we may write (5) as

$$0 = (\psi_\rho^* \alpha_{\rho\sigma}^{(k)}) n_k = \begin{pmatrix} (n_1 \psi_4^* + i n_2 \psi_4^* + n_3 \psi_3^*) \\ (n_1 \psi_3^* - i n_2 \psi_3^* - n_3 \psi_4^*) \\ (n_1 \psi_2^* + i n_2 \psi_2^* + n_3 \psi_1^*) \\ (n_1 \psi_1^* - i n_2 \psi_1^* - n_3 \psi_2^*) \end{pmatrix} \tag{8}$$

and consequently

$$
\begin{aligned}
n_1 \psi_4^* + i n_2 \psi_4^* + n_3 \psi_3^* &= 0 \\
n_1 \psi_3^* - i n_2 \psi_3^* - n_3 \psi_4^* &= 0 \\
n_1 \psi_2^* + i n_2 \psi_2^* + n_3 \psi_1^* &= 0 \\
n_1 \psi_1^* - i n_2 \psi_1^* - n_3 \psi_2^* &= 0
\end{aligned}
\qquad (9)
$$

The first two equations of (9) are two linear homogeneous equations in ψ_4^* , ψ_3^*, whose determinant, ($n_1^2 + n_2^2 + n_3^2 = 1$), does not vanish. Consequently $\psi_3^* = \psi_4^* = 0$

Similarly, $\psi_1^* = \psi_2^* = 0$ from the third and fourth of (9)

Therefore the condition (5) is equivalent to (6). Similarly (4) is equivalent to (7). Of the usual boundary conditions the only ones allowed by the action-principle for the Dirac field are the vanishing of the field functions on S or their periodicity on S. The vanishing of the normal derivatives, on S, or linear homogeneous boundary conditions are excluded as possible boundary conditions for the Dirac Field.

4.2 Conclusion

The boundary conditions for the motion of the scalar field, the vector field and the Dirac field have been shown to be a consequence of the action-principle for these fields. It was shown, for the vector and scalar fields, that the use of Lagrangian densities containing second-order derivatives constituted a generalization in the sense that the description of the motion of these wave fields was extended to include those motions involving linear homogeneous boundary conditions. This generalization modified the form of various field quantities.

The use of linear homogeneous boundary conditions in the application of field theories to physical problems has not been investigated. It was pointed out that such boundary conditions allowed the exchange of certain field quantities between the field and the boundary. Any physical problem involving such boundary conditions would have to be discussed with the generalized field quantities which were derived in this investigation. All the usual field theory problems may be discussed with the generalized theory which has been developed. For all problems the formulation for vector fields became quite analogous in form to the formulation for scalar fields.

The modification of the field quantities, as suggested by the generalized Lagrangian densities, have been carried out for the classical (unquantized) theory of the scalar and vector fields. A natural extension of the work in this investigation would be the discussion of quantization with the modified field functions. Such an extension is contemplated.

Interactions were not considered in this investigation. The boundary conditions arising from the use of generalized Lagrangian densities in action-principles for the pseudoscalar and pseudovector fields could be discussed. In a vacuum the equations for these fields are identical with the scalar and vector fields respectively. It would also be of interest to discuss the electromagnetic field. In particular, it would be important to discuss the possibility of defining the momenta for this field in a manner analogous to the definitions of the generalized conjugate momenta for the vector field.

In the discussion of field theories with variational principles it should be remembered that the variational principle be required to yield all the information about the field which is necessary to describe the motion of the field.

APPENDIX I

LINEAR HOMOGENEOUS CONDITIONS FROM THE BOUNDARY INTEGRAL

The boundary condition integral for the complex scalar field was found to be (2.1 (9)):

$$0 = \int_S \left\{ (B-\tfrac{c}{2})\frac{\partial \psi^*}{\partial n}\delta\psi + (B-\tfrac{c}{2})\frac{\partial \psi}{\partial n}\delta\psi^* + \frac{c}{2}\psi\,\delta\frac{\partial \psi^*}{\partial n} + \frac{c}{2}\psi^*\delta\frac{\partial \psi}{\partial n}\right\} dS \qquad (1)$$

Among the boundary conditions satisfying this relation would be those conditions which express a relation between the field functions and their normal derivatives. That is,

$$\frac{\partial \psi}{\partial n} = G(\psi)$$
$$\frac{\partial \psi^*}{\partial n} = H(\psi^*) \qquad \text{on } S \qquad (2)$$

It will be shown that (1) requires that

$$G(\psi) = \gamma\psi$$
$$H(\psi^*) = \gamma\psi^* \qquad (3)$$

where γ is a constant.

Assume the relations (2). Then by Taylor's series expansions, for small $\delta\psi$, $\delta\psi^*$, we have

$$\delta\frac{\partial \psi}{\partial n} = \frac{dG}{d\psi}\delta\psi \quad ; \qquad \delta\frac{\partial \psi^*}{\partial n} = \frac{dH}{d\psi^*}\delta\psi^* \qquad (4)$$

This allows us to write (1) as:

$$0 = \int_S \left\{ \left[(B-\tfrac{c}{2})H + \frac{c}{2}\psi^*\frac{dG}{d\psi}\right]\delta\psi + \left[(B-c/2)G + \frac{c}{2}\psi\frac{dH}{d\psi^*}\right]\delta\psi^* \right\} dS \qquad (5)$$

and since the variations $\delta\psi$ and $\delta\psi^*$ are now completely arbitrary on S (this is consistent with (2)), this yields:

$$\left(B-\frac{C}{2}\right)H + \frac{C}{2}\psi^* \frac{dG}{d\psi} = 0$$

$$\left(B-\frac{C}{2}\right)G + \frac{C}{2}\psi \frac{dH}{d\psi^*} = 0 \qquad (6)$$

These are two differential equations which may be solved for H and G. The first equation of (6) may be written:

$$\frac{\left(B-\frac{C}{2}\right)H}{C/2\ \psi^*} = -\frac{dG}{d\psi} \qquad (7)$$

In (7) the left-hand side depends on ψ^* only while the right-hand side depends on ψ only. Therefore each side may be equated to a constant, $-\gamma$ say, Then

$$G = \gamma\psi$$

$$H = \frac{-C}{2(B-C/2)}\gamma\psi^* \qquad (8)$$

Similarly the second equation of (6) yields, for another arbitrary constant λ,

$$H = \lambda\psi^*$$

$$G = -\frac{C}{2(B-C/2)}\lambda\psi \qquad (9)$$

Equating the results of (8) and (9), we obtain

$$\gamma = -\frac{C}{2(B-C/2)}\lambda$$

$$\lambda = -\frac{C}{2(B-C/2)}\gamma \qquad (10)$$

that is

$$\left(B-\frac{C}{2}\right)^2 = \frac{C^2}{4} \qquad (11)$$

which holds for λ, γ not both zero. From this, either $B \equiv C$ or $B \equiv 0$. But from the Euler-Lagrange equations 2.1 (7), if $B \equiv C$ then $\psi = 0$ everywhere in V.

Therefore, for a non-trivial wave field, boundary conditions of the form (2) require that $B \equiv 0$. Also then, $\lambda = \gamma$ and the only possible form of the boundary conditions (2) are:

$$\frac{\partial \psi^*}{\partial n} = \gamma \psi^*$$
$$\frac{\partial \psi}{\partial n} = \gamma \psi \qquad (12)$$

For the vector field, a similar result is obtained for each component of $\psi (= \psi_1, \psi_2, \psi_3, \psi_4)$ and $\psi^* (= \psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$.

The conditions (2) and their restricted form (12) express a relation which holds separately for each point on the boundary. Thus γ is independent of time but may be a function of position on S. The linear homogeneous boundary conditions (12) are the only possible boundary conditions which are 'point-wise' relations in this manner.

APPENDIX II

THE CONSTRUCTION OF HAMILTONIANS FROM GENERALIZED LAGRANGIAN DENSITIES

A Hamiltonian function may be defined, for a wave field, if both the field functions and their conjugate momenta may be defined. We obtained, as the generalized Lagrangian densities:

$$\mathcal{L}_s = A\psi\psi^* + \frac{C}{2}\left(\psi\partial_\alpha\partial_\alpha\psi^* + \psi^*\partial_\alpha\partial_\alpha\psi\right)$$

$$\mathcal{L}_v = A\psi_\beta\psi^*_\beta + \frac{C}{2}\left[\psi_\beta(\partial_\alpha\partial_\alpha\psi^*_\beta - \partial_\beta\partial_\alpha\psi^*_\alpha) + \psi^*_\beta(\partial_\alpha\partial_\alpha\psi_\beta - \partial_\alpha\partial_\beta\psi_\alpha)\right] \quad (1)$$

where $\mathcal{L}_s$ is for the complex scalar field

and $\mathcal{L}_v$ for the complex vector field.

Usually the conjugate momentum fields are defined by

$$\pi_\beta \equiv \frac{\partial\mathcal{L}}{\partial\dot{\psi}_\beta} \qquad \pi^*_\beta = \frac{\partial\mathcal{L}}{\partial\dot{\psi}^*_\beta} \quad (2)$$

where $\beta = 1.2.3.4$ for the vector field

$= 1$ for the scalar field.

But neither of thegeneralized Lagrange functions above contains $\dot{\psi}_\beta$ or $\dot{\psi}^*_\beta$ explicitly.

Because of the special role played by x_4 in the action principles, any terms in $\mathcal{L}_s$ or $\mathcal{L}_v$ involving $\ddot{\psi}_\beta$ or $\ddot{\psi}^*_\beta$ ($\beta = 1.2.3.4$ for vector fields, $= 1$ for scalar fields) may be replaced by a suitable term involving $\dot{\psi}_\beta$ or $\dot{\psi}^*_\beta$, and the action principle remains unchanged under such an interchange. Thus we transform

$$\mathcal{L}_s \longrightarrow \mathcal{L}_s'$$
$$\mathcal{L}_v \longrightarrow \mathcal{L}_v' \quad (3)$$

where

$$\mathcal{L}_s' = A\psi\psi^* + \frac{c}{2}(\psi\partial_k\partial_k\psi^* + \psi^*\partial_k\partial_k\psi) - c(\partial_4\psi)(\partial_4\psi^*)$$

$$\mathcal{L}_v' = A\psi_\beta\psi_\beta^* + \frac{c}{2}\psi_\beta(\partial_k\partial_k\psi_\beta^* - \partial_\beta\partial_\alpha\psi_\alpha^*) + \frac{c}{2}\psi_\beta^*(\partial_k\partial_k\psi_\beta - \partial_\beta\partial_\alpha\psi_\alpha^*) - c(\partial_4\psi_\beta)(\partial_4\psi_\beta^*) \tag{4}$$

By employing $\mathcal{L}_s'$ and $\mathcal{L}_v'$ in the variational procedure it is seen at once that these Lagrange functions yield exactly the same Euler-Lagrange equations and exactly the same boundary condition integral as $\mathcal{L}_s$ and $\mathcal{L}_v$ respectively. Therefore $\mathcal{L}_s'$ and $\mathcal{L}_v'$ are equivalent to $\mathcal{L}_s$ and $\mathcal{L}_v$ respectively, in the discussion of scalar and vector wave fields with action-principles. In view of this equivalence π_β and π_β^* may be defined, unequivocally, as

$$\pi_\beta = \frac{\partial \mathcal{L}'}{\partial \dot{\psi}_\beta} \qquad \pi_\beta^* = \frac{\partial \mathcal{L}'}{\partial \dot{\psi}_\beta^*} \tag{5}$$

where $\beta = 1,2,3,4$ for vector fields

$= 1$ for scalar fields.

These constitute natural definitions of the conjugate momenta which were stated in the discussion of these wave fields.

It would be expected that the Hamiltonian function for the wave fields would follow from that form of the Lagrangian density which was employed in defining the conjugate momentum fields. Therefore we define the Hamiltonian density, $\mathcal{H}$, as

$$\mathcal{H} = \pi_\beta\dot{\psi}_\beta + \pi_\beta^*\dot{\psi}_\beta^* - \mathcal{L}' \tag{6}$$

$\beta = 1,2,3,4$ for vector fields

$= 1$ for scalar fields.

This is the natural definition of the Hamiltonian density for those wave fields which are described by a generalized Lagrange function. The definitions 2.3 (1), 3.3 (1) employed in the discussion of the scalar and vector fields follow immediately from (5) and (6).

The transformation $\mathcal{L}_s \to \mathcal{L}_s'$, above, has transformed all those terms in $\mathcal{L}_s$ which involve derivatives with respect to the time co-ordinate. This is not true of the transformation $\mathcal{L}_v \to \mathcal{L}_v'$. In the latter, time-derivatives occuring in terms of the form $\psi_\beta \partial_\beta \partial_\alpha \psi_\alpha^*$ or $\psi_\beta^* \partial_\beta \partial_\alpha \psi_\alpha$ have not been transformed and have not been employed in the definition of the conjugate momentum fields. In order to employ these terms in such definitions it would be necessary to define functional derivatives of $\mathcal{L}_v$ with respect to $\dot{\psi}_\beta$ or $\dot{\psi}_\beta^*$. However these terms occur in $\mathcal{L}_v$ only because they give rise to the Lorentz condition. Thus these terms have an auxiliary role in the variational procedure and their use in extending the definition of π_β and π_β^* need not be expected. It was shown that the definitions (5) and (6) were completely sufficient for the construction of a Hamiltonian whose canonical equations are the vector field equations in the form which includes the Lorentz condition. In view of this these definitions are not only natural but also sufficient.

The transformations $\mathcal{L}_s \longrightarrow \mathcal{L}_s'$, $\mathcal{L}_v \longrightarrow \mathcal{L}_v'$ correspond to transformations to Lagrange functions which are not relativistically invariant. This is a qualification of the relativistic invariance of $\mathcal{L}$. It arises because of the special role played by the time co-ordinate in the action principle. The space-derivatives, however, must enter into a generalized $\mathcal{L}$ uniformly.