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Research Article

The Refined Positive Definite and Unimodal Regions for the Gram-Charlier and Edgeworth Series Expansion

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Gram-Charlier and Edgeworth Series Expansions are used in the field of statistics to approximate probability density functions. The expansions have proven useful but have experienced limitations due to the values of the moments that admit a proper probability density function. An alternative approach in developing the boundary conditions for the boundary of the positive region for both series expansions is investigated using Sturm's theorem. The result provides a more accurate representation of the positive region developed by others.

1. Introduction

The Gram-Charlier and Edgeworth Series Expansions are frequently used in statistics to approximate probability density functions. Shenton [1] investigated the efficiency of the Gram-Charlier Type A distribution, while Draper and Tierney [2] provided exact formulae for the Edgeworth Expansion up to the 10th moment (cumulants). Hald [3] provided a thorough historical review of the cumulants and the Gram-Charlier Series. Hall [4] used the Edgeworth Expansion to investigate and develop properties of the bootstrap method. Chen and Sitter [5] derived the Edgeworth Expansion for stratified sampling without replacement from a finite population. Recently, the Edgeworth Expansion was used to investigate the robustness of various process capability indices [6, 7]. Berberan-Santos [8, 9] and Cohen [10] derive and illustrate the relationship between probability density functions and their respective cumulants using the Gram-Charlier and subsequently the Edgeworth expansions.

Although the expansions are useful, there are limitations on the values of the moments (or cumulants) that admit a proper probability density function (pdf). For example, if the first four moments are used, in both the Gram-Charlier and the Edgeworth Expansions there are regions where the resulting pdf is negative and not unimodal. The negative values of the pdf violate a basic condition associated with proper pdfs. Barton and Dennis [11] and Draper and Tierney [12] provided numerical solutions to the conditions where the Edgeworth Expansion and Gram-Charlier series are nonnegative and unimodal. Balitskaya and Zolotuhina [13] determined the positive region analytically for the Edgeworth Expansion up to the 4th moment. However the plot provided in Barton and Dennis [11] is not precise enough for practical use. We will use an alternative approach to develop the boundary of the positive and unimodal regions for both series expansions. Mathematica [14] code is provided that allows verification of the moment ratio combinations that result in a proper pdf. This method will accurately determine the positive and unimodal regions of the expansions. Tabulated values are provided and plots included illustrate the proposed method. The results can then be compared with the regions developed by Barton and Dennis [11].

2. The Hermite Polynomial

Defining $\phi(x) = (1/\sqrt{2\pi})e^{-(x^2/2)}$ for $-\infty < x < \infty$ and $D = d/dx$, to be the pdf for the standard normal distribution and the differentiation operator, respectively, Hermite polynomials $H_r(x)$ are defined to be

$$(-D)^r \phi(x) = H_r(x)\phi(x) \quad \text{for } r \geq 0. \tag{2.1}$$

The first seven polynomials are

$$\begin{gathered} H_0(x) = 1, \qquad H_1(x) = x, \qquad H_2(x) = x^2 - 1, \\ H_3(x) = x^3 - 3x, \qquad H_4(x) = x^4 - 6x^2 + 3, \\ H_5(x) = x^5 - 10x^3 + 15x, \\ H_6(x) = x^6 - 15x^4 + 45x^2 - 15, \\ H_7(x) = x^7 - 21x^5 + 105x^3 - 105x. \end{gathered} \tag{2.2}$$

Two properties of the Hermite polynomials that will be used in the manuscript include (see Stuart and Ord [15])

$$\begin{gathered} DH_r(x) = rH_{r-1}(x), \quad \text{for } r \geq 1, \\ H_r(x) - xH_{r-1}(x) + (r-1)H_{r-2}(x) = 0, \quad \text{for } r \geq 2. \end{gathered} \tag{2.3}$$

3. Gram-Charlier Series: Type A

Kotz and Johnson [16] defined the Gram-Charlier Type A Series as follows: if $f(x)$ is a pdf with cumulants $\kappa_1, \kappa_2, \ldots$, the function is

$$\begin{aligned} g(x) &= \exp\left[\sum_{j=1}^{\infty} \varepsilon_j \left\{\frac{(-D)^j}{j!}\right\}\right] f(x) \\ &= f(x) - \varepsilon_1 D f(x) + \frac{1}{2}\left(\varepsilon_1^2 + \varepsilon_2\right) D^2 f(x) - \frac{1}{6}\left(\varepsilon_1^3 + 3\varepsilon_1\varepsilon_2 + \varepsilon_3\right) D^3 f(x) \\ &\quad + \frac{1}{24}\left(\varepsilon_1^4 + 6\varepsilon_1^2\varepsilon_2 + 4\varepsilon_1\varepsilon_3 + \varepsilon_4\right) D^4 f(x) + \cdots \\ &= \sum_{r=0}^{\infty} c_r (-1)^{-r} D^r \phi(x) \end{aligned} \tag{3.1}$$

with cumulants $\kappa_1 + \varepsilon_1, \kappa_2 + \varepsilon_2, \ldots$. In the case where $f(x)$ is the normal pdf (i.e., $\phi(x)$), we have

$$D^r \phi(x) = (-1)^r H_r(x) \phi(x),$$

$$g(x) = \sum_{r=0}^{\infty} c_r H_r(x) \phi(x) = \left(c_0 H_0(x) + c_1 H_1(x) + c_2 H_2(x) + c_3 H_3(x) + c_4 H_4(x) \cdots\right) \phi(x), \tag{3.2}$$

where the c_r's are defined to be

$$\begin{gathered} c_0 = 1, \qquad c_1 = 0, \qquad c_2 = \frac{1}{2}(\lambda_2 - 1), \qquad c_3 = \frac{1}{6}\lambda_3, \\ c_4 = \frac{1}{24}(\lambda_4 - 6\lambda_2 + 3), \qquad c_5 = \frac{1}{120}(\lambda_5 - 10\lambda_3), \\ c_6 = \frac{1}{720}(\lambda_6 - 15\lambda_4 + 45\lambda_2 - 15) \end{gathered} \tag{3.3}$$

and λ_r represents moment ratios defined as follows:

$$\lambda_r(x) = \mu_r \{\mu_2\}^{-r/2} \quad \text{where } \mu_r = E\left((X - E(X))^r\right). \tag{3.4}$$

Shenton [1] used the terms up to the 4th moment to represent the Gram-Charlier Type A Series. For $\lambda_1 = 0$ and $\lambda_2 = 1$,

$$\begin{aligned} g(x) &= \left(c_0 H_0(x) + c_1 H_1(x) + c_2 H_2(x) + c_3 H_3(x) + c_4 H_4(x)\right) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \\ &= \left(1 + \frac{\lambda_3}{6} H_3(x) + \left(\frac{\lambda_4 - 3}{24}\right) H_4(x)\right) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \\ &= \left(1 + \frac{\lambda_3}{6}\left(x^3 - 3x\right) + \left(\frac{\lambda_4 - 3}{24}\right)\left(x^4 - 6x^2 + 3\right)\right) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \end{aligned} \tag{3.5}$$

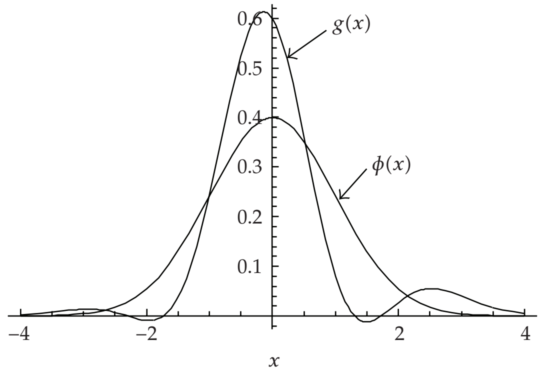

Figure 1: $\phi(x)$ and $g(x)$ for $\lambda_3 = 1$ and $\lambda_4 = 7$.

represents the Gram-Charlier approximation of a standardized pdf, with mean 0 and variance of 1. In the case of the normal pdf (i.e., $\lambda_3 = 0$, $\lambda_4 = 3$), the approximation is exactly the standard normal distribution.

It can be shown that in the case of the normal distribution, (3.5) is such that

$$\int_{-\infty}^{\infty} g(x)dx = 1. \tag{3.6}$$

Using $g(x)$ as defined in (3.5) and by changing the values of λ_3 and λ_4 one can examine their impact on the general shape of the resulting pdf. Figure 1 illustrates the impact on $g(x)$ associated with changes in λ_3 and λ_4.

In general as λ_3 increases, $g(x)$ becomes more skewed, while as λ_4 increases, $g(x)$ becomes more peaked and multimodal. This is why λ_3 is often used as a measure of skewness and λ_4 as a measure of peakedness (kurtosis). For several combinations of λ_3 and λ_4, $g(x)$ will pass through the x-axis to produce an improper probability density function. For example, when $\lambda_3 = 1$, $\lambda_4 = 7$, $g(x)$ crosses the x-axis at

$$x = -\sqrt{3},\ \sqrt{3},\ \frac{1}{2}\left(-1 - \sqrt{13}\right),\ \frac{1}{2}\left(-1 + \sqrt{13}\right), \tag{3.7}$$

resulting in a portion of the pdf not being positive definite (see Figure 1).

There are many combinations of λ_i's that result in $g(x) < 0$. In order for the approximation to be valid, the polynomial

$$1 + \frac{\lambda_3}{6}\left(x^3 - 3x\right) + \left(\frac{\lambda_4 - 3}{24}\right)\left(x^4 - 6x^2 + 3\right) \tag{3.8}$$

must be nonnegative for all x. It is sufficient to say that the above is true when (3.8) has no real root (i.e., it does not touch the x-axis) and the coefficient of x^4 is positive. Shenton [1] obtained the solution analytically using the theory of equations. He stated that for

$$B(x) = 24 + 4a_4 H_3(x) + a_4 H_4(x) > 0, \tag{3.9}$$

the condition of a positive polynomial for $-\infty < x < \infty$ is $4a_4^3 - a_4^4 + 4a_3 a_4^2 - 3a_3^2 a_4^2 + 4a_3^4 > 0$.

There are also values of λ_3 and λ_4 where the Gram-Charlier and Edgeworth Expansion produce a multimodal pdf. In order to determine the regions, we will use the approach developed by Barton and Dennis [11] where by letting

$$\begin{aligned}
\frac{dg(x)}{dx} &= \frac{d}{dx}\sum_{r=0}^{\infty} c_r H_r(x)\phi(x) \\
&= \phi(x)\left\{\frac{d}{dx}\left[1 + \sum_{r=1}^{n} c_r H_r(x)\right] - \left[1 + \sum_{r=1}^{n} c_r H_r(x)\right]x\right\} \\
&\Longrightarrow \phi(x)\left\{H_1 + \sum_{r=1}^{n} c_r H_{r+1}(x)\right\},
\end{aligned} \tag{3.10}$$

then in order for $g(x)$ to be unimodal, $g'(x)$ can only have one real root.

We will provide an alternative approach in obtaining the boundary values using Sturm's theorem and compare the results with those values obtained by Barton and Dennis [11] and Draper and Tierney [2].

4. Sturm's Theorem

Let $p(x)$ represent a polynomial in x and define $p_0(x), p_1(x), \ldots, p_r(x)$ to be

$$\begin{aligned}
&p_0(x) = p(x), \qquad p_1(x) = p'(x), \\
&\quad\vdots \\
&p_r(x) = -\text{remainder}\ \left(p_{r-2}(x), p_{r-1}(x)\right) \quad \text{for } r \geq 2.
\end{aligned} \tag{4.1}$$

The resulting $p_0(x), p_1(x), \ldots, pr(x)$ is said to represent Sturm's sequence. Sturm's theorem states that if $p(x) = 0$ is an algebraic equation with real coefficients and without multiple roots and if a and b are real numbers, $a < b$, and neither are a root of the given equation, then the number of real roots of $p(x) = 0$ between a and b is equal to $\nu_a - \nu_b$ where ν_a and ν_b are the variations of sign in Sturm's sequence at a and b.

Using Sturm's theorem, one can then determine the region where a polynomial is positive for a specific range of x. If we set $a = -\infty$, $b = \infty$, we can determine how many real roots there are for $g(x)$. If the polynomial is always positive, it implies that $\nu_a - \nu_b = 0$. Using Mathematica [14] we can tabulate the boundaries of the positive regions and plot them.

Table 1: Positive region boundary (λ_4, λ_3) for Gram-Charlier Series.

3.005, 0.018	3.58, 0.541	4.155, 0.8105	4.73, 0.9735	5.305, 1.046	5.88, 1.017	6.455, 0.836
3.01, 0.03	3.585, 0.544	4.16, 0.812	4.735, 0.9745	5.31, 1.046	5.885, 1.016	6.46, 0.8335
3.015, 0.0405	3.59, 0.547	4.165, 0.814	4.74, 0.9755	5.315, 1.046	5.89, 1.0155	6.465, 0.8305
3.02, 0.05	3.595, 0.5495	4.17, 0.816	4.745, 0.9765	5.32, 1.0465	5.895, 1.0145	6.47, 0.828
3.025, 0.059	3.6, 0.5525	4.175, 0.8175	4.75, 0.9775	5.325, 1.0465	5.9, 1.0135	6.475, 0.825
3.03, 0.0675	3.605, 0.5555	4.18, 0.8195	4.755, 0.9785	5.33, 1.047	5.905, 1.013	6.48, 0.8225
3.035, 0.0755	3.61, 0.5585	4.185, 0.8215	4.76, 0.9795	5.335, 1.047	5.91, 1.012	6.485, 0.8195
3.04, 0.0835	3.615, 0.5615	4.19, 0.823	4.765, 0.9805	5.34, 1.047	5.915, 1.011	6.49, 0.817
3.045, 0.091	3.62, 0.564	4.195, 0.825	4.77, 0.9815	5.345, 1.0475	5.92, 1.0105	6.495, 0.814
3.05, 0.098	3.625, 0.567	4.2, 0.8265	4.775, 0.9825	5.35, 1.0475	5.925, 1.0095	6.5, 0.811
3.055, 0.105	3.63, 0.57	4.205, 0.8285	4.78, 0.9835	5.355, 1.0475	5.93, 1.0085	6.505, 0.8085
3.06, 0.112	3.635, 0.5725	4.21, 0.83	4.785, 0.9845	5.36, 1.048	5.935, 1.0075	6.51, 0.8055
3.065, 0.119	3.64, 0.5755	4.215, 0.832	4.79, 0.9855	5.365, 1.048	5.94, 1.0065	6.515, 0.8025
3.07, 0.1255	3.645, 0.578	4.22, 0.8335	4.795, 0.986	5.37, 1.048	5.945, 1.006	6.52, 0.7995
3.075, 0.132	3.65, 0.581	4.225, 0.8355	4.8, 0.987	5.375, 1.048	5.95, 1.005	6.525, 0.7965
3.08, 0.138	3.655, 0.584	4.23, 0.837	4.805, 0.988	5.38, 1.0485	5.955, 1.004	6.53, 0.7935
3.085, 0.144	3.66, 0.5865	4.235, 0.8385	4.81, 0.989	5.385, 1.0485	5.96, 1.003	6.535, 0.7905
3.09, 0.1505	3.665, 0.5895	4.24, 0.8405	4.815, 0.99	5.39, 1.0485	5.965, 1.002	6.54, 0.787
3.095, 0.156	3.67, 0.592	4.245, 0.842	4.82, 0.991	5.395, 1.0485	5.97, 1.001	6.545, 0.784
3.1, 0.162	3.675, 0.595	4.25, 0.844	4.825, 0.9915	5.4, 1.0485	5.975, 1	6.55, 0.781
3.105, 0.168	3.68, 0.5975	4.255, 0.8455	4.83, 0.9925	5.405, 1.0485	5.98, 0.999	6.555, 0.7775
3.11, 0.1735	3.685, 0.6	4.26, 0.847	4.835, 0.9935	5.41, 1.049	5.985, 0.998	6.56, 0.7745
3.115, 0.179	3.69, 0.603	4.265, 0.849	4.84, 0.9945	5.415, 1.049	5.99, 0.997	6.565, 0.771
3.12, 0.1845	3.695, 0.6055	4.27, 0.8505	4.845, 0.995	5.42, 1.049	5.995, 0.996	6.57, 0.768
3.125, 0.19	3.7, 0.608	4.275, 0.852	4.85, 0.996	5.425, 1.049	6, 0.995	6.575, 0.7645
3.13, 0.1955	3.705, 0.611	4.28, 0.854	4.855, 0.997	5.43, 1.049	6.005, 0.994	6.58, 0.761
3.135, 0.201	3.71, 0.6135	4.285, 0.8555	4.86, 0.998	5.435, 1.049	6.01, 0.9925	6.585, 0.758
3.14, 0.206	3.715, 0.616	4.29, 0.857	4.865, 0.9985	5.44, 1.049	6.015, 0.9915	6.59, 0.7545
3.145, 0.211	3.72, 0.619	4.295, 0.859	4.87, 0.9995	5.445, 1.049	6.02, 0.9905	6.595, 0.751
3.15, 0.2165	3.725, 0.6215	4.3, 0.8605	4.875, 1.0005	5.45, 1.049	6.025, 0.9895	6.6, 0.7475
3.155, 0.2215	3.73, 0.624	4.305, 0.862	4.88, 1.001	5.455, 1.049	6.03, 0.9885	6.605, 0.744
3.16, 0.2265	3.735, 0.6265	4.31, 0.8635	4.885, 1.002	5.46, 1.049	6.035, 0.987	6.61, 0.74
3.165, 0.2315	3.74, 0.629	4.315, 0.865	4.89, 1.0025	5.465, 1.049	6.04, 0.986	6.615, 0.7365
3.17, 0.2365	3.745, 0.6315	4.32, 0.867	4.895, 1.0035	5.47, 1.049	6.045, 0.985	6.62, 0.733
3.175, 0.241	3.75, 0.6345	4.325, 0.8685	4.9, 1.0045	5.475, 1.049	6.05, 0.9835	6.625, 0.729
3.18, 0.246	3.755, 0.637	4.33, 0.87	4.905, 1.005	5.48, 1.049	6.055, 0.9825	6.63, 0.7255
3.185, 0.2505	3.76, 0.6395	4.335, 0.8715	4.91, 1.006	5.485, 1.049	6.06, 0.9815	6.635, 0.7215
3.19, 0.2555	3.765, 0.642	4.34, 0.873	4.915, 1.0065	5.49, 1.049	6.065, 0.98	6.64, 0.7175
3.195, 0.26	3.77, 0.6445	4.345, 0.8745	4.92, 1.0075	5.495, 1.0485	6.07, 0.979	6.645, 0.714
3.2, 0.265	3.775, 0.647	4.35, 0.876	4.925, 1.008	5.5, 1.0485	6.075, 0.9775	6.65, 0.71
3.205, 0.2695	3.78, 0.6495	4.355, 0.8775	4.93, 1.009	5.505, 1.0485	6.08, 0.9765	6.655, 0.706
3.21, 0.274	3.785, 0.652	4.36, 0.8795	4.935, 1.0095	5.51, 1.0485	6.085, 0.975	6.66, 0.702
3.215, 0.2785	3.79, 0.6545	4.365, 0.881	4.94, 1.0105	5.515, 1.0485	6.09, 0.974	6.665, 0.6975
3.22, 0.283	3.795, 0.657	4.37, 0.8825	4.945, 1.011	5.52, 1.048	6.095, 0.9725	6.67, 0.6935
3.225, 0.2875	3.8, 0.6595	4.375, 0.884	4.95, 1.012	5.525, 1.048	6.1, 0.971	6.675, 0.6895
3.23, 0.292	3.805, 0.6615	4.38, 0.8855	4.955, 1.0125	5.53, 1.048	6.105, 0.97	6.68, 0.685
3.235, 0.296	3.81, 0.664	4.385, 0.887	4.96, 1.0135	5.535, 1.048	6.11, 0.9685	6.685, 0.681
3.24, 0.3005	3.815, 0.6665	4.39, 0.8885	4.965, 1.014	5.54, 1.0475	6.115, 0.967	6.69, 0.6765
3.245, 0.305	3.82, 0.669	4.395, 0.89	4.97, 1.015	5.545, 1.0475	6.12, 0.966	6.695, 0.672
3.25, 0.309	3.825, 0.6715	4.4, 0.8915	4.975, 1.0155	5.55, 1.0475	6.125, 0.9645	6.7, 0.6675

Table 1: Continued.

3.255, 0.3135	3.83, 0.674	4.405, 0.8925	4.98, 1.016	5.555, 1.047	6.13, 0.963	6.705, 0.663
3.26, 0.3175	3.835, 0.676	4.41, 0.894	4.985, 1.017	5.56, 1.047	6.135, 0.9615	6.71, 0.6585
3.265, 0.3215	3.84, 0.6785	4.415, 0.8955	4.99, 1.0175	5.565, 1.047	6.14, 0.96	6.715, 0.654
3.27, 0.326	3.845, 0.681	4.42, 0.897	4.995, 1.018	5.57, 1.0465	6.145, 0.9585	6.72, 0.649
3.275, 0.33	3.85, 0.6835	4.425, 0.8985	5, 1.019	5.575, 1.0465	6.15, 0.9575	6.725, 0.6445
3.28, 0.334	3.855, 0.6855	4.43, 0.9	5.005, 1.0195	5.58, 1.0465	6.155, 0.956	6.73, 0.6395
3.285, 0.338	3.86, 0.688	4.435, 0.9015	5.01, 1.02	5.585, 1.046	6.16, 0.9545	6.735, 0.6345
3.29, 0.342	3.865, 0.6905	4.44, 0.903	5.015, 1.021	5.59, 1.046	6.165, 0.953	6.74, 0.6295
3.295, 0.346	3.87, 0.6925	4.445, 0.904	5.02, 1.0215	5.595, 1.0455	6.17, 0.9515	6.745, 0.6245
3.3, 0.35	3.875, 0.695	4.45, 0.9055	5.025, 1.022	5.6, 1.0455	6.175, 0.95	6.75, 0.6195
3.305, 0.354	3.88, 0.697	4.455, 0.907	5.03, 1.0225	5.605, 1.045	6.18, 0.9485	6.755, 0.614
3.31, 0.358	3.885, 0.6995	4.46, 0.9085	5.035, 1.0235	5.61, 1.045	6.185, 0.9465	6.76, 0.609
3.315, 0.362	3.89, 0.702	4.465, 0.91	5.04, 1.024	5.615, 1.0445	6.19, 0.945	6.765, 0.6035
3.32, 0.366	3.895, 0.704	4.47, 0.911	5.045, 1.0245	5.62, 1.0445	6.195, 0.9435	6.77, 0.598
3.325, 0.3695	3.9, 0.7065	4.475, 0.9125	5.05, 1.025	5.625, 1.044	6.2, 0.942	6.775, 0.5925
3.33, 0.3735	3.905, 0.7085	4.48, 0.914	5.055, 1.0255	5.63, 1.0435	6.205, 0.9405	6.78, 0.587
3.335, 0.3775	3.91, 0.711	4.485, 0.9155	5.06, 1.026	5.635, 1.0435	6.21, 0.9385	6.785, 0.5815
3.34, 0.381	3.915, 0.713	4.49, 0.9165	5.065, 1.027	5.64, 1.043	6.215, 0.937	6.79, 0.5755
3.345, 0.385	3.92, 0.7155	4.495, 0.918	5.07, 1.0275	5.645, 1.043	6.22, 0.9355	6.795, 0.5695
3.35, 0.3885	3.925, 0.7175	4.5, 0.9195	5.075, 1.028	5.65, 1.0425	6.225, 0.9335	6.8, 0.5635
3.355, 0.3925	3.93, 0.7195	4.505, 0.9205	5.08, 1.0285	5.655, 1.042	6.23, 0.932	6.805, 0.5575
3.36, 0.396	3.935, 0.722	4.51, 0.922	5.085, 1.029	5.66, 1.0415	6.235, 0.9305	6.81, 0.5515
3.365, 0.3995	3.94, 0.724	4.515, 0.9235	5.09, 1.0295	5.665, 1.0415	6.24, 0.9285	6.815, 0.545
3.37, 0.4035	3.945, 0.7265	4.52, 0.9245	5.095, 1.03	5.67, 1.041	6.245, 0.927	6.82, 0.5385
3.375, 0.407	3.95, 0.7285	4.525, 0.926	5.1, 1.0305	5.675, 1.0405	6.25, 0.925	6.825, 0.532
3.38, 0.4105	3.955, 0.7305	4.53, 0.927	5.105, 1.031	5.68, 1.04	6.255, 0.923	6.83, 0.5255
3.385, 0.414	3.96, 0.733	4.535, 0.9285	5.11, 1.0315	5.685, 1.04	6.26, 0.9215	6.835, 0.5185
3.39, 0.418	3.965, 0.735	4.54, 0.93	5.115, 1.032	5.69, 1.0395	6.265, 0.9195	6.84, 0.5115
3.395, 0.4215	3.97, 0.737	4.545, 0.931	5.12, 1.0325	5.695, 1.039	6.27, 0.918	6.845, 0.5045
3.4, 0.425	3.975, 0.739	4.55, 0.9325	5.125, 1.033	5.7, 1.0385	6.275, 0.916	6.85, 0.497
3.405, 0.4285	3.98, 0.7415	4.555, 0.9335	5.13, 1.0335	5.705, 1.038	6.28, 0.914	6.855, 0.4895
3.41, 0.432	3.985, 0.7435	4.56, 0.935	5.135, 1.034	5.71, 1.0375	6.285, 0.912	6.86, 0.482
3.415, 0.4355	3.99, 0.7455	4.565, 0.936	5.14, 1.0345	5.715, 1.0375	6.29, 0.9105	6.865, 0.4745
3.42, 0.439	3.995, 0.7475	4.57, 0.9375	5.145, 1.035	5.72, 1.037	6.295, 0.9085	6.87, 0.4665
3.425, 0.4425	4, 0.7495	4.575, 0.9385	5.15, 1.0355	5.725, 1.0365	6.3, 0.9065	6.875, 0.4585
3.43, 0.4455	4.005, 0.752	4.58, 0.94	5.155, 1.036	5.73, 1.036	6.305, 0.9045	6.88, 0.45
3.435, 0.449	4.01, 0.754	4.585, 0.941	5.16, 1.0365	5.735, 1.0355	6.31, 0.9025	6.885, 0.4415
3.44, 0.4525	4.015, 0.756	4.59, 0.942	5.165, 1.0365	5.74, 1.035	6.315, 0.9005	6.89, 0.4325
3.445, 0.456	4.02, 0.758	4.595, 0.9435	5.17, 1.037	5.745, 1.0345	6.32, 0.8985	6.895, 0.4235
3.45, 0.459	4.025, 0.76	4.6, 0.9445	5.175, 1.0375	5.75, 1.034	6.325, 0.8965	6.9, 0.4145
3.455, 0.4625	4.03, 0.762	4.605, 0.946	5.18, 1.038	5.755, 1.0335	6.33, 0.8945	6.905, 0.4045
3.46, 0.466	4.035, 0.764	4.61, 0.947	5.185, 1.0385	5.76, 1.033	6.335, 0.8925	6.91, 0.395
3.465, 0.469	4.04, 0.766	4.615, 0.948	5.19, 1.0385	5.765, 1.0325	6.34, 0.89	6.915, 0.3845
3.47, 0.4725	4.045, 0.768	4.62, 0.9495	5.195, 1.039	5.77, 1.0315	6.345, 0.888	6.92, 0.374
3.475, 0.476	4.05, 0.77	4.625, 0.9505	5.2, 1.0395	5.775, 1.031	6.35, 0.886	6.925, 0.363
3.48, 0.479	4.055, 0.772	4.63, 0.9515	5.205, 1.04	5.78, 1.0305	6.355, 0.884	6.93, 0.3515
3.485, 0.482	4.06, 0.774	4.635, 0.953	5.21, 1.0405	5.785, 1.03	6.36, 0.8815	6.935, 0.3395
3.49, 0.4855	4.065, 0.776	4.64, 0.954	5.215, 1.0405	5.79, 1.0295	6.365, 0.8795	6.94, 0.327
3.495, 0.4885	4.07, 0.778	4.645, 0.955	5.22, 1.041	5.795, 1.029	6.37, 0.877	6.945, 0.314
3.5, 0.492	4.075, 0.78	4.65, 0.9565	5.225, 1.0415	5.8, 1.028	6.375, 0.875	6.95, 0.3
3.505, 0.495	4.08, 0.782	4.655, 0.9575	5.23, 1.0415	5.805, 1.0275	6.38, 0.8725	6.955, 0.2855

Table 1: Continued.

3.51, 0.498	4.085, 0.784	4.66, 0.9585	5.235, 1.042	5.81, 1.027	6.385, 0.8705	6.96, 0.27
3.515, 0.5015	4.09, 0.786	4.665, 0.9595	5.24, 1.0425	5.815, 1.0265	6.39, 0.868	6.965, 0.2535
3.52, 0.5045	4.095, 0.788	4.67, 0.9605	5.245, 1.0425	5.82, 1.0255	6.395, 0.8655	6.97, 0.2355
3.525, 0.5075	4.1, 0.79	4.675, 0.962	5.25, 1.043	5.825, 1.025	6.4, 0.8635	6.975, 0.2155
3.53, 0.5105	4.105, 0.7915	4.68, 0.963	5.255, 1.043	5.83, 1.0245	6.405, 0.861	6.98, 0.1935
3.535, 0.514	4.11, 0.7935	4.685, 0.964	5.26, 1.0435	5.835, 1.0235	6.41, 0.8585	6.985, 0.1685
3.54, 0.517	4.115, 0.7955	4.69, 0.965	5.265, 1.044	5.84, 1.023	6.415, 0.856	6.99, 0.138
3.545, 0.52	4.12, 0.7975	4.695, 0.966	5.27, 1.044	5.845, 1.022	6.42, 0.8535	6.995, 0.0985
3.55, 0.523	4.125, 0.7995	4.7, 0.967	5.275, 1.0445	5.85, 1.0215	6.425, 0.8515	
3.555, 0.526	4.13, 0.801	4.705, 0.9685	5.28, 1.0445	5.855, 1.0205	6.43, 0.849	
3.56, 0.529	4.135, 0.803	4.71, 0.9695	5.285, 1.045	5.86, 1.02	6.435, 0.8465	
3.565, 0.532	4.14, 0.805	4.715, 0.9705	5.29, 1.045	5.865, 1.019	6.44, 0.8435	
3.57, 0.535	4.145, 0.8065	4.72, 0.9715	5.295, 1.0455	5.87, 1.0185	6.445, 0.841	
3.575, 0.538	4.15, 0.8085	4.725, 0.9725	5.3, 1.0455	5.875, 1.0175	6.45, 0.8385	

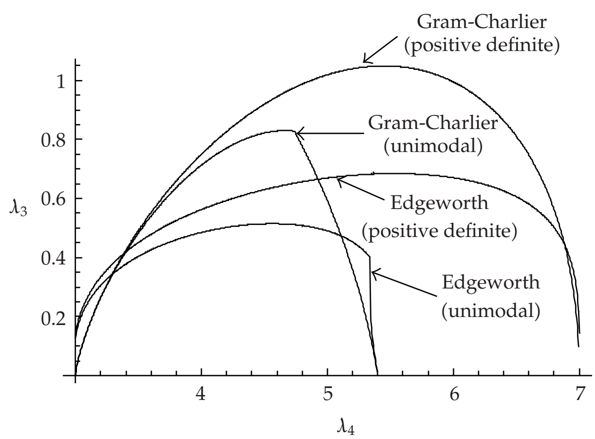

Figure 2: Positive definite and unimodal regions for $g(x)$.

The tabulated values, along with the plot, provide all the necessary information needed to describe the combination of moment ratios that result in the Gram-Charlier and Edgeworth Expansion being positive definite. The plots alone may not have sufficient definition to describe those regions near the boundary conditions. Extensive tables with precise detail can be developed for those regions where accuracy is important using this technique. The source code for determining the tabulated values has been included in the Appendices.

Figure 2 illustrates the boundaries for the positive and unimodal regions for values combination of λ_3 and λ_4 for both the Gram-Charlier Series and Edgeworth Expansions. Table 1 includes the boundary values of λ_3, λ_4 where the Gram-Charlier series changes from positive definite to non-positive definite.

Table 2: Unimodal region boundary (λ_4, λ_3) for Gram-Charlier Series.

3.01, 0.03	3.355, 0.388	3.7, 0.591	4.045, 0.7275	4.39, 0.808	4.735, 0.8285	5.08, 0.507
3.015, 0.0405	3.36, 0.3915	3.705, 0.5935	4.05, 0.729	4.395, 0.809	4.74, 0.827	5.085, 0.5015
3.02, 0.05	3.365, 0.395	3.71, 0.596	4.055, 0.7305	4.4, 0.8095	4.745, 0.823	5.09, 0.496
3.025, 0.059	3.37, 0.3985	3.715, 0.5985	4.06, 0.7325	4.405, 0.8105	4.75, 0.819	5.095, 0.49
3.03, 0.0675	3.375, 0.402	3.72, 0.6005	4.065, 0.734	4.41, 0.811	4.755, 0.815	5.1, 0.4845
3.035, 0.0755	3.38, 0.4055	3.725, 0.603	4.07, 0.7355	4.415, 0.812	4.76, 0.811	5.105, 0.4785
3.04, 0.0835	3.385, 0.409	3.73, 0.6055	4.075, 0.737	4.42, 0.8125	4.765, 0.807	5.11, 0.4725
3.045, 0.091	3.39, 0.4125	3.735, 0.6075	4.08, 0.7385	4.425, 0.813	4.77, 0.8025	5.115, 0.467
3.05, 0.098	3.395, 0.416	3.74, 0.61	4.085, 0.74	4.43, 0.814	4.775, 0.7985	5.12, 0.461
3.055, 0.105	3.4, 0.4195	3.745, 0.6125	4.09, 0.7415	4.435, 0.8145	4.78, 0.7945	5.125, 0.455
3.06, 0.112	3.405, 0.423	3.75, 0.6145	4.095, 0.7425	4.44, 0.815	4.785, 0.79	5.13, 0.449
3.065, 0.1185	3.41, 0.426	3.755, 0.617	4.1, 0.744	4.445, 0.816	4.79, 0.786	5.135, 0.443
3.07, 0.125	3.415, 0.4295	3.76, 0.619	4.105, 0.7455	4.45, 0.8165	4.795, 0.782	5.14, 0.437
3.075, 0.1315	3.42, 0.433	3.765, 0.6215	4.11, 0.747	4.455, 0.817	4.8, 0.7775	5.145, 0.431
3.08, 0.138	3.425, 0.436	3.77, 0.6235	4.115, 0.7485	4.46, 0.8175	4.805, 0.7735	5.15, 0.4245
3.085, 0.144	3.43, 0.4395	3.775, 0.626	4.12, 0.75	4.465, 0.818	4.81, 0.769	5.155, 0.4185
3.09, 0.15	3.435, 0.4425	3.78, 0.628	4.125, 0.751	4.47, 0.819	4.815, 0.765	5.16, 0.412
3.095, 0.156	3.44, 0.446	3.785, 0.63	4.13, 0.7525	4.475, 0.8195	4.82, 0.7605	5.165, 0.406
3.1, 0.1615	3.445, 0.449	3.79, 0.6325	4.135, 0.754	4.48, 0.82	4.825, 0.7565	5.17, 0.3995
3.105, 0.1675	3.45, 0.4525	3.795, 0.6345	4.14, 0.7555	4.485, 0.8205	4.83, 0.752	5.175, 0.393
3.11, 0.173	3.455, 0.4555	3.8, 0.6365	4.145, 0.7565	4.49, 0.821	4.835, 0.7475	5.18, 0.3865
3.115, 0.1785	3.46, 0.4585	3.805, 0.639	4.15, 0.758	4.495, 0.8215	4.84, 0.7435	5.185, 0.38
3.12, 0.184	3.465, 0.462	3.81, 0.641	4.155, 0.7595	4.5, 0.822	4.845, 0.739	5.19, 0.3735
3.125, 0.1895	3.47, 0.465	3.815, 0.643	4.16, 0.7605	4.505, 0.8225	4.85, 0.7345	5.195, 0.367
3.13, 0.195	3.475, 0.468	3.82, 0.645	4.165, 0.762	4.51, 0.823	4.855, 0.73	5.2, 0.36
3.135, 0.2	3.48, 0.471	3.825, 0.6475	4.17, 0.763	4.515, 0.823	4.86, 0.7255	5.205, 0.3535
3.14, 0.2055	3.485, 0.474	3.83, 0.6495	4.175, 0.7645	4.52, 0.8235	4.865, 0.721	5.21, 0.3465
3.145, 0.2105	3.49, 0.477	3.835, 0.6515	4.18, 0.766	4.525, 0.824	4.87, 0.7165	5.215, 0.3395
3.15, 0.2155	3.495, 0.4805	3.84, 0.6535	4.185, 0.767	4.53, 0.8245	4.875, 0.712	5.22, 0.3325
3.155, 0.2205	3.5, 0.4835	3.845, 0.6555	4.19, 0.7685	4.535, 0.825	4.88, 0.7075	5.225, 0.3255
3.16, 0.2255	3.505, 0.4865	3.85, 0.6575	4.195, 0.7695	4.54, 0.825	4.885, 0.703	5.23, 0.3185
3.165, 0.2305	3.51, 0.4895	3.855, 0.6595	4.2, 0.7705	4.545, 0.8255	4.89, 0.6985	5.235, 0.3115
3.17, 0.235	3.515, 0.492	3.86, 0.6615	4.205, 0.772	4.55, 0.826	4.895, 0.694	5.24, 0.304
3.175, 0.24	3.52, 0.495	3.865, 0.6635	4.21, 0.773	4.555, 0.8265	4.9, 0.6895	5.245, 0.297
3.18, 0.245	3.525, 0.498	3.87, 0.6655	4.215, 0.7745	4.56, 0.8265	4.905, 0.685	5.25, 0.2895
3.185, 0.2495	3.53, 0.501	3.875, 0.6675	4.22, 0.7755	4.565, 0.827	4.91, 0.68	5.255, 0.282
3.19, 0.254	3.535, 0.504	3.88, 0.6695	4.225, 0.7765	4.57, 0.827	4.915, 0.6755	5.26, 0.2745
3.195, 0.2585	3.54, 0.507	3.885, 0.6715	4.23, 0.778	4.575, 0.8275	4.92, 0.671	5.265, 0.2665
3.2, 0.2635	3.545, 0.5095	3.89, 0.6735	4.235, 0.779	4.58, 0.8275	4.925, 0.666	5.27, 0.259
3.205, 0.268	3.55, 0.5125	3.895, 0.6755	4.24, 0.78	4.585, 0.828	4.93, 0.6615	5.275, 0.251
3.21, 0.2725	3.555, 0.5155	3.9, 0.6775	4.245, 0.781	4.59, 0.828	4.935, 0.6565	5.28, 0.243
3.215, 0.277	3.56, 0.518	3.905, 0.679	4.25, 0.7825	4.595, 0.8285	4.94, 0.652	5.285, 0.235
3.22, 0.281	3.565, 0.521	3.91, 0.681	4.255, 0.7835	4.6, 0.8285	4.945, 0.647	5.29, 0.227
3.225, 0.2855	3.57, 0.524	3.915, 0.683	4.26, 0.7845	4.605, 0.829	4.95, 0.642	5.295, 0.2185
3.23, 0.29	3.575, 0.5265	3.92, 0.685	4.265, 0.7855	4.61, 0.829	4.955, 0.6375	5.3, 0.21
3.235, 0.294	3.58, 0.5295	3.925, 0.6865	4.27, 0.7865	4.615, 0.829	4.96, 0.6325	5.305, 0.2015
3.24, 0.2985	3.585, 0.532	3.93, 0.6885	4.275, 0.7875	4.62, 0.8295	4.965, 0.6275	5.31, 0.193
3.245, 0.3025	3.59, 0.535	3.935, 0.6905	4.28, 0.7885	4.625, 0.8295	4.97, 0.6225	5.315, 0.184
3.25, 0.307	3.595, 0.5375	3.94, 0.692	4.285, 0.7895	4.63, 0.8295	4.975, 0.6175	5.32, 0.175

Table 2: Continued.

3.255, 0.311	3.6, 0.54	3.945, 0.694	4.29, 0.7905	4.635, 0.8295	4.98, 0.6125	5.325, 0.166
3.26, 0.315	3.605, 0.543	3.95, 0.6955	4.295, 0.7915	4.64, 0.8295	4.985, 0.6075	5.33, 0.1565
3.265, 0.319	3.61, 0.5455	3.955, 0.6975	4.3, 0.7925	4.645, 0.83	4.99, 0.6025	5.335, 0.147
3.27, 0.3235	3.615, 0.548	3.96, 0.6995	4.305, 0.7935	4.65, 0.83	4.995, 0.5975	5.34, 0.1375
3.275, 0.3275	3.62, 0.551	3.965, 0.701	4.31, 0.7945	4.655, 0.83	5, 0.5925	5.345, 0.1275
3.28, 0.3315	3.625, 0.5535	3.97, 0.703	4.315, 0.7955	4.66, 0.83	5.005, 0.5875	5.35, 0.1175
3.285, 0.3355	3.63, 0.556	3.975, 0.7045	4.32, 0.7965	4.665, 0.83	5.01, 0.5825	5.355, 0.107
3.29, 0.339	3.635, 0.5585	3.98, 0.7065	4.325, 0.7975	4.67, 0.83	5.015, 0.577	5.36, 0.0965
3.295, 0.343	3.64, 0.5615	3.985, 0.708	4.33, 0.7985	4.675, 0.83	5.02, 0.572	5.365, 0.0855
3.3, 0.347	3.645, 0.564	3.99, 0.7095	4.335, 0.799	4.68, 0.83	5.025, 0.5665	5.37, 0.0745
3.305, 0.351	3.65, 0.5665	3.995, 0.7115	4.34, 0.8	4.685, 0.83	5.03, 0.5615	5.375, 0.063
3.31, 0.3545	3.655, 0.569	4, 0.713	4.345, 0.801	4.69, 0.8295	5.035, 0.556	5.38, 0.0515
3.315, 0.3585	3.66, 0.5715	4.005, 0.7145	4.35, 0.802	4.695, 0.8295	5.04, 0.551	5.385, 0.039
3.32, 0.3625	3.665, 0.574	4.01, 0.7165	4.355, 0.8025	4.7, 0.8295	5.045, 0.5455	5.39, 0.0265
3.325, 0.366	3.67, 0.5765	4.015, 0.718	4.36, 0.8035	4.705, 0.8295	5.05, 0.54	5.395, 0.0135
3.33, 0.3695	3.675, 0.579	4.02, 0.7195	4.365, 0.8045	4.71, 0.829	5.055, 0.5345	
3.335, 0.3735	3.68, 0.5815	4.025, 0.7215	4.37, 0.805	4.715, 0.829	5.06, 0.529	
3.34, 0.377	3.685, 0.584	4.03, 0.723	4.375, 0.806	4.72, 0.829	5.065, 0.5235	
3.345, 0.381	3.69, 0.5865	4.035, 0.7245	4.38, 0.8065	4.725, 0.8285	5.07, 0.518	
3.35, 0.3845	3.695, 0.589	4.04, 0.726	4.385, 0.8075	4.73, 0.8285	5.075, 0.5125	

If we use additional terms for the Gram-Charlier Series, then

$$\begin{aligned} g(x) &= \left(c_0H_0(x) + c_1H_1(x) + c_2H_2(x) + c_3H_3(x) + c_4H_4(x) + c_5H_5(x)\right)\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \\ &= \left(1 + \frac{\lambda_3}{6}H_3(x) + \left(\frac{\lambda_4 - 3}{24}\right)H_4(x) + \left(\frac{\lambda_5 - 10\lambda_3}{120}\right)H_5(x)\right)\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \\ &= \left(1 + \frac{\lambda_3}{6}\left(x^3 - 3x\right) + \left(\frac{\lambda_4 - 3}{24}\right)\left(x^4 - 6x^2 + 3\right) + \left(\frac{\lambda_5 - 10\lambda_3}{120}\right)\left(x^5 - 10x^3 + 15x\right)\right) \\ &\quad \times \frac{1}{\sqrt{2\pi}}e^{-x^2/2}. \end{aligned} \tag{4.2}$$

If the coefficient of x^5 is nonzero, the polynomial will always cross the x-axis, rendering the associated pdf $g(x)$ invalid. When

$$\frac{\lambda_5 - 10\lambda_3}{120} = 0, \tag{4.3}$$

the polynomial is as defined in (3.5). Therefore, if we include λ_5 in the series, we are restricted to the condition $\lambda_5 = 10\lambda_3$ in order for $g(x)$ to result in a proper pdf.

If we introduce one more term in the Gram-Charlier Series, we get

$$
\begin{aligned}
g(x) &= (c_0H_0(x) + c_1H_1(x) + c_2H_2(x) + c_3H_3(x) + c_4H_4(x) + c_5H_5(x) + c_6H_6(x))\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \\
&= \left(1 + \frac{\lambda_3}{6}H_3(x) + \left(\frac{\lambda_4 - 3}{24}\right)H_4(x) + \left(\frac{\lambda_5 - 10\lambda_3}{120}\right)H_5(x)\right. \\
&\quad \left. + \left(\frac{\lambda_6 - 15\lambda_4 + 45\lambda_2 - 15}{720}\right)H_6(x)\right)\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \\
&= \left(1 + \frac{\lambda_3}{6}\left(x^3 - 3x\right) + \left(\frac{\lambda_4 - 3}{24}\right)\left(x^4 - 6x^2 + 3\right) + \left(\frac{\lambda_5 - 10\lambda_3}{120}\right)\left(x^5 - 10x^3 + 15x\right)\right. \\
&\quad \left. + \left(\frac{\lambda_6 - 15\lambda_4 + 45\lambda_2 - 15}{720}\right)\left(x^6 - 15x^4 + 45x^2 - 15\right)\right)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}.
\end{aligned}
\tag{4.4}
$$

Since it is a 6-degree polynomial, it is possible to find the ranges of λ_3, λ_4, λ_5, and λ_6 that produce a positive definite region. Applying Sturm's theorem, tables of values can be tabulated. If we can assume that the moments of the approximation match up to and including the 4th moment for the standard normal distribution (i.e., $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 0$, $\lambda_4 = 3$), we can find the values of λ_5, λ_6 that result in a positive definite pdf.

For the unimodal problem, since

$$
\begin{aligned}
\frac{dg(x)}{dx} &= \frac{d}{dx}\sum_{r=0}^{\infty} c_rH_r(x)\phi(x) = \phi(x)\left\{H_1 + \sum_{r=1}^{n} c_rH_{r+1}(x)\right\} \\
&= x + \frac{\lambda_3}{6}\left(x^4 - 6x^2 + 3\right) + \frac{\lambda_4 - 3}{24}\left(x^5 - 10x^3 + 15x\right)
\end{aligned}
\tag{4.5}
$$

and we only want $g'(x)$ to have one real root, setting $\nu_a - \nu_b = 1$ will calculate the boundary values of λ_3, λ_4 for unimodality. The results have been tabulated in Table 2 and illustrated in Figure 2.

5. Edgeworth Expansion

The Edgeworth Expansion can be defined as $g(x) = \sum_{j=0}^{\infty} c_jH_j(x)\phi(x)$ where

$$
\begin{aligned}
c_0 = 1, \qquad c_1 = c_2 = 0, \qquad c_3 = \frac{1}{6}\lambda_3, \\
c_4 = \frac{1}{24}(\lambda_4 - 3), \qquad c_5 = 0, \qquad c_6 = \frac{1}{72}{\lambda_3}^2.
\end{aligned}
\tag{5.1}
$$

Barton and Dennis [11] examined the positive definite regions using the terms up to and including the 4th moment, defining

$$\begin{aligned} g(x) &= \left(c_0H_0(x) + c_1H_1(x) + c_2H_2(x) + c_3H_3(x) + c_4H_4(x) + c_5H_5(x) + c_6H_6(x)\right)\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \\ &= \left(1 + \frac{\lambda_3}{6}H_3(x) + \left(\frac{\lambda_4 - 3}{24}\right)H_4(x) + \left(\frac{\lambda_3^2}{72}\right)H_6(x)\right)\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \\ &= \left(1 + \frac{\lambda_3}{6}\left(x^3 - 3x\right) + \left(\frac{\lambda_4 - 3}{24}\right)\left(x^4 - 6x^2 + 3\right) + \frac{\lambda_3^2}{72}\left(x^6 - 15x^4 + 45x^2 - 15\right)\right) \\ &\quad \times \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \end{aligned} \tag{5.2}$$

and provided the parametric equation for the combinations that satisfied the positive definite criteria. In order to ensure that $g(x)$ is positive definite, the following must be true:

$$1 + \frac{\lambda_3}{6}\left(x^3 - 3x\right) + \left(\frac{\lambda_4 - 3}{24}\right)\left(x^4 - 6x^2 + 3\right) + \frac{{\lambda_3}^2}{72}\left(x^6 - 15x^4 + 45x^2 - 15\right) > 0. \tag{5.3}$$

Using Sturm's theorem, we can obtain the boundaries of the positive region by fixing $\nu_a - \nu_b = 0$. The boundary values of λ_3, λ_4 have been tabulated (Table 3) and drawn in Figure 2.

For the unimodal problem, since

$$\begin{aligned} \frac{dg(x)}{dx} &= \frac{d}{dx}\sum_{r=0}^{\infty} c_r H_r(x)\phi(x) = \phi(x)\left\{H_1 + \sum_{r=1}^{n} c_r H_{r+1}(x)\right\} = x + \frac{\lambda_3}{6}\left(x^4 - 6x^2 + 3\right) \\ &\quad + \frac{\lambda_4 - 3}{24}\left(x^5 - 10x^3 + 15x\right) + \frac{\lambda_3^2}{72}\left(x^7 - 21x^5 + 105x^3 - 105x\right) \end{aligned} \tag{5.4}$$

and since we want $g'(x)$ to have exactly one real root, we set $\nu_a - \nu_b = 1$ and can tabulate the boundary combinations to ensure that $g(x)$ is unimodal (see Table 4). The results are depicted in Figure 2. Draper and Tierney [2] pointed out that the original unimodal graph from Barton and Dennis [11], that used a numerical optimization nonlinear least square method to find the condition for the positive definite region, was incorrect. This is readily confirmed from Figure 2.

With the inclusion of the 5th moment, the Edgeworth Expansion becomes a polynomial of degree nine:

$$\begin{aligned} g(x) = \Bigg(1 &+ \frac{\lambda_3}{6}H_3(x) + \left(\frac{\lambda_4 - 3}{24}\right)H_4(x) + \left(\frac{{\lambda_3}^2}{72}\right)H_6(x) + \frac{\lambda_5}{120}H_5(x) \\ &+ \frac{\lambda_3(\lambda_4 - 3)}{144}H_7(x) + \left(\frac{\lambda_3^3}{1296}\right)H_9(x)\Bigg)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}. \end{aligned} \tag{5.5}$$

Table 3: Positive region boundary (λ_4, λ_3) for Edgeworth Expansion.

3, 0.1377	3.575, 0.4723	4.15, 0.5865	4.725, 0.6509	5.3, 0.6815	5.875, 0.6774	6.45, 0.6167
3.005, 0.1517	3.58, 0.4737	4.155, 0.5872	4.73, 0.6513	5.305, 0.6817	5.88, 0.6771	6.455, 0.6157
3.01, 0.1632	3.585, 0.475	4.16, 0.588	4.735, 0.6517	5.31, 0.6818	5.885, 0.6769	6.46, 0.6148
3.015, 0.1732	3.59, 0.4763	4.165, 0.5887	4.74, 0.6521	5.315, 0.6819	5.89, 0.6767	6.465, 0.6138
3.02, 0.182	3.595, 0.4776	4.17, 0.5894	4.745, 0.6525	5.32, 0.682	5.895, 0.6764	6.47, 0.6128
3.025, 0.1901	3.6, 0.4789	4.175, 0.5901	4.75, 0.6529	5.325, 0.6821	5.9, 0.6762	6.475, 0.6117
3.03, 0.1975	3.605, 0.4802	4.18, 0.5908	4.755, 0.6533	5.33, 0.6828	5.905, 0.6759	6.48, 0.6107
3.035, 0.2044	3.61, 0.4815	4.185, 0.5916	4.76, 0.6537	5.335, 0.6828	5.91, 0.6757	6.485, 0.6097
3.04, 0.2109	3.615, 0.4828	4.19, 0.5923	4.765, 0.6541	5.34, 0.6828	5.915, 0.6754	6.49, 0.6086
3.045, 0.217	3.62, 0.4841	4.195, 0.593	4.77, 0.6545	5.345, 0.6828	5.92, 0.6752	6.495, 0.6075
3.05, 0.2228	3.625, 0.4854	4.2, 0.5937	4.775, 0.6548	5.35, 0.6828	5.925, 0.6749	6.5, 0.6065
3.055, 0.2283	3.63, 0.4866	4.205, 0.5944	4.78, 0.6552	5.355, 0.6828	5.93, 0.6747	6.505, 0.6054
3.06, 0.2336	3.635, 0.4879	4.21, 0.5951	4.785, 0.6556	5.36, 0.6829	5.935, 0.6744	6.51, 0.6042
3.065, 0.2387	3.64, 0.4891	4.215, 0.5957	4.79, 0.656	5.365, 0.6829	5.94, 0.6741	6.515, 0.6031
3.07, 0.2435	3.645, 0.4903	4.22, 0.5964	4.795, 0.6563	5.37, 0.6830	5.945, 0.6738	6.52, 0.602
3.075, 0.2482	3.65, 0.4916	4.225, 0.5971	4.8, 0.6567	5.375, 0.6831	5.95, 0.6736	6.525, 0.6008
3.08, 0.2528	3.655, 0.4928	4.23, 0.5978	4.805, 0.6571	5.38, 0.6832	5.955, 0.6733	6.53, 0.5997
3.085, 0.2572	3.66, 0.494	4.235, 0.5985	4.81, 0.6574	5.385, 0.6833	5.96, 0.673	6.535, 0.5985
3.09, 0.2614	3.665, 0.4952	4.24, 0.5992	4.815, 0.6578	5.39, 0.6833	5.965, 0.6727	6.54, 0.5973
3.095, 0.2656	3.67, 0.4964	4.245, 0.5998	4.82, 0.6582	5.395, 0.6835	5.97, 0.6724	6.545, 0.5961
3.1, 0.2696	3.675, 0.4976	4.25, 0.6005	4.825, 0.6585	5.4, 0.6836	5.975, 0.6721	6.55, 0.5948
3.105, 0.2735	3.68, 0.4988	4.255, 0.6012	4.83, 0.6589	5.405, 0.6836	5.98, 0.6718	6.555, 0.5936
3.11, 0.2773	3.685, 0.5	4.26, 0.6018	4.835, 0.6592	5.41, 0.6837	5.985, 0.6715	6.56, 0.5923
3.115, 0.2811	3.69, 0.5011	4.265, 0.6025	4.84, 0.6596	5.415, 0.6837	5.99, 0.6711	6.565, 0.5911
3.12, 0.2847	3.695, 0.5023	4.27, 0.6031	4.845, 0.6599	5.42, 0.6838	5.995, 0.6708	6.57, 0.5898
3.125, 0.2882	3.7, 0.5035	4.275, 0.6038	4.85, 0.6603	5.425, 0.6839	6, 0.6705	6.575, 0.5884
3.13, 0.2917	3.705, 0.5046	4.28, 0.6044	4.855, 0.6606	5.43, 0.6839	6.005, 0.6702	6.58, 0.5871
3.135, 0.2951	3.71, 0.5058	4.285, 0.6051	4.86, 0.661	5.435, 0.684	6.01, 0.6698	6.585, 0.5858
3.14, 0.2984	3.715, 0.5069	4.29, 0.6057	4.865, 0.6613	5.44, 0.684	6.015, 0.6695	6.59, 0.5844
3.145, 0.3017	3.72, 0.508	4.295, 0.6064	4.87, 0.6616	5.445, 0.6841	6.02, 0.6692	6.595, 0.583
3.15, 0.3049	3.725, 0.5092	4.3, 0.607	4.875, 0.662	5.45, 0.6841	6.025, 0.6688	6.6, 0.5816
3.155, 0.308	3.73, 0.5103	4.305, 0.6076	4.88, 0.6623	5.455, 0.6842	6.03, 0.6685	6.605, 0.5802
3.16, 0.3111	3.735, 0.5114	4.31, 0.6083	4.885, 0.6626	5.46, 0.6842	6.035, 0.6681	6.61, 0.5787
3.165, 0.3141	3.74, 0.5125	4.315, 0.6089	4.89, 0.6629	5.465, 0.6843	6.04, 0.6677	6.615, 0.5773
3.17, 0.3171	3.745, 0.5136	4.32, 0.6095	4.895, 0.6633	5.47, 0.6843	6.045, 0.6674	6.62, 0.5758
3.175, 0.32	3.75, 0.5147	4.325, 0.6101	4.9, 0.6636	5.475, 0.6843	6.05, 0.667	6.625, 0.5743
3.18, 0.3229	3.755, 0.5158	4.33, 0.6108	4.905, 0.6639	5.48, 0.6844	6.055, 0.6666	6.63, 0.5728
3.185, 0.3257	3.76, 0.5169	4.335, 0.6114	4.91, 0.6642	5.485, 0.6844	6.06, 0.6662	6.635, 0.5712
3.19, 0.3285	3.765, 0.518	4.34, 0.612	4.915, 0.6645	5.49, 0.6844	6.065, 0.6658	6.64, 0.5696
3.195, 0.3313	3.77, 0.519	4.345, 0.6126	4.92, 0.6649	5.495, 0.6845	6.07, 0.6655	6.645, 0.568
3.2, 0.334	3.775, 0.5201	4.35, 0.6132	4.925, 0.6652	5.5, 0.6845	6.075, 0.6651	6.65, 0.5664
3.205, 0.3366	3.78, 0.5212	4.355, 0.6138	4.93, 0.6655	5.505, 0.6845	6.08, 0.6647	6.655, 0.5648
3.21, 0.3393	3.785, 0.5222	4.36, 0.6144	4.935, 0.6658	5.51, 0.6845	6.085, 0.6642	6.66, 0.5631
3.215, 0.3418	3.79, 0.5233	4.365, 0.615	4.94, 0.6661	5.515, 0.6845	6.09, 0.6638	6.665, 0.5614
3.22, 0.3444	3.795, 0.5243	4.37, 0.6156	4.945, 0.6664	5.52, 0.6845	6.095, 0.6634	6.67, 0.5597
3.225, 0.3469	3.8, 0.5254	4.375, 0.6162	4.95, 0.6667	5.525, 0.6845	6.1, 0.663	6.675, 0.5579
3.23, 0.3494	3.805, 0.5264	4.38, 0.6168	4.955, 0.667	5.53, 0.6845	6.105, 0.6626	6.68, 0.5561

Table 3: Continued.

3.235, 0.3518	3.81, 0.5274	4.385, 0.6174	4.96, 0.6673	5.535, 0.6845	6.11, 0.6621	6.685, 0.5543
3.24, 0.3542	3.815, 0.5284	4.39, 0.618	4.965, 0.6676	5.54, 0.6845	6.115, 0.6617	6.69, 0.5525
3.245, 0.3566	3.82, 0.5295	4.395, 0.6185	4.97, 0.6678	5.545, 0.6845	6.12, 0.6612	6.695, 0.5506
3.25, 0.359	3.825, 0.5305	4.4, 0.6191	4.975, 0.6681	5.55, 0.6845	6.125, 0.6608	6.7, 0.5487
3.255, 0.3613	3.83, 0.5315	4.405, 0.6197	4.98, 0.6684	5.555, 0.6845	6.13, 0.6603	6.705, 0.5468
3.26, 0.3636	3.835, 0.5325	4.41, 0.6203	4.985, 0.6687	5.56, 0.6845	6.135, 0.6599	6.71, 0.5449
3.265, 0.3659	3.84, 0.5335	4.415, 0.6208	4.99, 0.669	5.565, 0.6845	6.14, 0.6594	6.715, 0.5429
3.27, 0.3681	3.845, 0.5345	4.42, 0.6214	4.995, 0.6693	5.57, 0.6845	6.145, 0.6589	6.72, 0.5408
3.275, 0.3704	3.85, 0.5354	4.425, 0.622	5, 0.6695	5.575, 0.6845	6.15, 0.6585	6.725, 0.5388
3.28, 0.3725	3.855, 0.5364	4.43, 0.6225	5.005, 0.6698	5.58, 0.6844	6.155, 0.658	6.73, 0.5367
3.285, 0.3747	3.86, 0.5374	4.435, 0.6231	5.01, 0.6701	5.585, 0.6844	6.16, 0.6575	6.735, 0.5345
3.29, 0.3769	3.865, 0.5384	4.44, 0.6237	5.015, 0.6703	5.59, 0.6844	6.165, 0.657	6.74, 0.5323
3.295, 0.379	3.87, 0.5393	4.445, 0.6242	5.02, 0.6706	5.595, 0.6843	6.17, 0.6565	6.745, 0.5301
3.3, 0.3811	3.875, 0.5403	4.45, 0.6247	5.025, 0.6709	5.6, 0.6843	6.175, 0.656	6.75, 0.5279
3.305, 0.3832	3.88, 0.5413	4.455, 0.6253	5.03, 0.6711	5.605, 0.6843	6.18, 0.6555	6.755, 0.5256
3.31, 0.3852	3.885, 0.5422	4.46, 0.6258	5.035, 0.6714	5.61, 0.6842	6.185, 0.6549	6.76, 0.5232
3.315, 0.3873	3.89, 0.5432	4.465, 0.6264	5.04, 0.6716	5.615, 0.6842	6.19, 0.6544	6.765, 0.5208
3.32, 0.3893	3.895, 0.5441	4.47, 0.6269	5.045, 0.6719	5.62, 0.6841	6.195, 0.6539	6.77, 0.5184
3.325, 0.3913	3.9, 0.545	4.475, 0.6275	5.05, 0.6721	5.625, 0.6841	6.2, 0.6533	6.775, 0.5159
3.33, 0.3933	3.905, 0.546	4.48, 0.628	5.055, 0.6724	5.63, 0.684	6.205, 0.6528	6.78, 0.5133
3.335, 0.3952	3.91, 0.5469	4.485, 0.6285	5.06, 0.6726	5.635, 0.684	6.21, 0.6523	6.785, 0.5107
3.34, 0.3971	3.915, 0.5478	4.49, 0.629	5.065, 0.6729	5.64, 0.6839	6.215, 0.6517	6.79, 0.5081
3.345, 0.3991	3.92, 0.5487	4.495, 0.6296	5.07, 0.6731	5.645, 0.6838	6.22, 0.6511	6.795, 0.5054
3.35, 0.401	3.925, 0.5497	4.5, 0.6301	5.075, 0.6733	5.65, 0.6838	6.225, 0.6506	6.8, 0.5026
3.355, 0.4029	3.93, 0.5506	4.505, 0.6306	5.08, 0.6736	5.655, 0.6837	6.23, 0.65	6.805, 0.4997
3.36, 0.4047	3.935, 0.5515	4.51, 0.6311	5.085, 0.6738	5.66, 0.6836	6.235, 0.6494	6.81, 0.4968
3.365, 0.4066	3.94, 0.5524	4.515, 0.6316	5.09, 0.674	5.665, 0.6836	6.24, 0.6488	6.815, 0.4939
3.37, 0.4084	3.945, 0.5533	4.52, 0.6322	5.095, 0.6743	5.67, 0.6835	6.245, 0.6482	6.82, 0.4908
3.375, 0.4102	3.95, 0.5542	4.525, 0.6327	5.1, 0.6745	5.675, 0.6834	6.25, 0.6476	6.825, 0.4877
3.38, 0.412	3.955, 0.555	4.53, 0.6332	5.105, 0.6747	5.68, 0.6833	6.255, 0.647	6.83, 0.4845
3.385, 0.4138	3.96, 0.5559	4.535, 0.6337	5.11, 0.6749	5.685, 0.6832	6.26, 0.6464	6.835, 0.4812
3.39, 0.4156	3.965, 0.5568	4.54, 0.6342	5.115, 0.6752	5.69, 0.6831	6.265, 0.6457	6.84, 0.4779
3.395, 0.4173	3.97, 0.5577	4.545, 0.6347	5.12, 0.6754	5.695, 0.683	6.27, 0.6451	6.845, 0.4744
3.4, 0.4191	3.975, 0.5585	4.55, 0.6352	5.125, 0.6756	5.7, 0.6829	6.275, 0.6445	6.85, 0.4708
3.405, 0.4208	3.98, 0.5594	4.555, 0.6357	5.13, 0.6758	5.705, 0.6828	6.28, 0.6438	6.855, 0.4672
3.41, 0.4225	3.985, 0.5603	4.56, 0.6362	5.135, 0.676	5.71, 0.6827	6.285, 0.6432	6.86, 0.4634
3.415, 0.4242	3.99, 0.5611	4.565, 0.6366	5.14, 0.6762	5.715, 0.6826	6.29, 0.6425	6.865, 0.4595
3.42, 0.4259	3.995, 0.562	4.57, 0.6371	5.145, 0.6764	5.72, 0.6825	6.295, 0.6418	6.87, 0.4555
3.425, 0.4276	4, 0.5628	4.575, 0.6376	5.15, 0.6766	5.725, 0.6824	6.3, 0.6412	6.875, 0.4514
3.43, 0.4292	4.005, 0.5637	4.58, 0.6381	5.155, 0.6768	5.73, 0.6823	6.305, 0.6405	6.88, 0.4471
3.435, 0.4309	4.01, 0.5645	4.585, 0.6386	5.16, 0.677	5.735, 0.6822	6.31, 0.6398	6.885, 0.4426
3.44, 0.4325	4.015, 0.5654	4.59, 0.639	5.165, 0.6772	5.74, 0.682	6.315, 0.6391	6.89, 0.438
3.445, 0.4341	4.02, 0.5662	4.595, 0.6395	5.17, 0.6774	5.745, 0.6819	6.32, 0.6384	6.895, 0.4333
3.45, 0.4357	4.025, 0.567	4.6, 0.64	5.175, 0.6776	5.75, 0.6818	6.325, 0.6376	6.9, 0.4283
3.455, 0.4373	4.03, 0.5678	4.605, 0.6405	5.18, 0.6778	5.755, 0.6816	6.33, 0.6369	6.905, 0.4232
3.46, 0.4389	4.035, 0.5687	4.61, 0.6409	5.185, 0.678	5.76, 0.6815	6.335, 0.6362	6.91, 0.4178
3.465, 0.4405	4.04, 0.5695	4.615, 0.6414	5.19, 0.6781	5.765, 0.6814	6.34, 0.6354	6.915, 0.4121
3.47, 0.442	4.045, 0.5703	4.62, 0.6418	5.195, 0.6783	5.77, 0.6812	6.345, 0.6347	6.92, 0.4063

Table 3: Continued.

3.475, 0.4436	4.05, 0.5711	4.625, 0.6423	5.2, 0.6785	5.775, 0.6811	6.35, 0.6339	6.925, 0.4001
3.48, 0.4451	4.055, 0.5719	4.63, 0.6427	5.205, 0.6787	5.78, 0.6809	6.355, 0.6332	6.93, 0.3935
3.485, 0.4466	4.06, 0.5727	4.635, 0.6432	5.21, 0.6788	5.785, 0.6808	6.36, 0.6324	6.935, 0.3866
3.49, 0.4481	4.065, 0.5735	4.64, 0.6437	5.215, 0.679	5.79, 0.6806	6.365, 0.6316	6.94, 0.3793
3.495, 0.4496	4.07, 0.5743	4.645, 0.6441	5.22, 0.6792	5.795, 0.6804	6.37, 0.6308	6.945, 0.3715
3.5, 0.4511	4.075, 0.5751	4.65, 0.6445	5.225, 0.6794	5.8, 0.6803	6.375, 0.63	6.95, 0.3631
3.505, 0.4526	4.08, 0.5759	4.655, 0.645	5.23, 0.6795	5.805, 0.6801	6.38, 0.6292	6.955, 0.3541
3.51, 0.4541	4.085, 0.5767	4.66, 0.6454	5.235, 0.6797	5.81, 0.6799	6.385, 0.6284	6.96, 0.3443
3.515, 0.4556	4.09, 0.5774	4.665, 0.6459	5.24, 0.6798	5.815, 0.6798	6.39, 0.6275	6.965, 0.3336
3.52, 0.457	4.095, 0.5782	4.67, 0.6463	5.245, 0.68	5.82, 0.6796	6.395, 0.6267	6.97, 0.3216
3.525, 0.4584	4.1, 0.579	4.675, 0.6467	5.25, 0.6801	5.825, 0.6794	6.4, 0.6258	6.975, 0.3082
3.53, 0.4599	4.105, 0.5798	4.68, 0.6472	5.255, 0.6803	5.83, 0.6792	6.405, 0.625	6.98, 0.2926
3.535, 0.4613	4.11, 0.5805	4.685, 0.6476	5.26, 0.6804	5.835, 0.679	6.41, 0.6241	6.985, 0.2741
3.54, 0.4627	4.115, 0.5813	4.69, 0.648	5.265, 0.6806	5.84, 0.6788	6.415, 0.6232	6.99, 0.2506
3.545, 0.4641	4.12, 0.582	4.695, 0.6484	5.27, 0.6807	5.845, 0.6786	6.42, 0.6223	6.995, 0.2175
3.55, 0.4655	4.125, 0.5828	4.7, 0.6488	5.275, 0.6809	5.85, 0.6784	6.425, 0.6214	7, 0.144
3.555, 0.4669	4.13, 0.5835	4.705, 0.6493	5.28, 0.681	5.855, 0.6782	6.43, 0.6205	
3.56, 0.4683	4.135, 0.5843	4.71, 0.6497	5.285, 0.6811	5.86, 0.678	6.435, 0.6196	
3.565, 0.4696	4.14, 0.585	4.715, 0.6501	5.29, 0.6813	5.865, 0.6778	6.44, 0.6186	
3.57, 0.471	4.145, 0.5858	4.72, 0.6505	5.295, 0.6814	5.87, 0.6776	6.445, 0.6177	

A 9-degree polynomial will always have a real root; hence the pdf approximation will always have some combination of λ_3 and λ_4 that results in negative coordinates. Draper and Tierney [2] developed an algorithm to produce Edgeworth Expansions up to the 10th moment.

Using the same approach developed previously and assuming that all moment up to the 4th moment match, (i.e., $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 0$, $\lambda_4 = 3$),

$$g(x) = \left(1 + \frac{\lambda_5}{120}H_5(x) + \left(\frac{\lambda_6 - 15}{720}\right)H_6(x)\right)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}. \tag{5.6}$$

We see that $g(x)$ is the same as the Gram-Charlier Series, and hence similar conditions for λ_3 and λ_4 apply.

6. Comments

We have developed a general procedure to find the positive definite and unimodal regions for the series expansions due to Gram-Charlier and Edgeworth. The method can be used to provide accurate results and boundaries for the region of λ_3 and λ_4. With the technology available, we can easily produce the results accurately. Also the Mathematica functions are developed for evaluating whether the moment ratio combinations produce a proper pdf. It can be shown that expansions can also be obtained in terms of derivatives of the Gamma (via Laguerre polynomials) and Beta distributions (via Jacobi polynomials). The paper also illustrates the fact that the probability density function is valid only for certain ranges of the λ_i's. If we are outside these ranges, we cannot safely apply the Edgeworth Expansion or the Gram-Charlier Series approximations. For example, if we use this method to investigate

Table 4: Unimodal region boundary (λ_4, λ_3) for Edgeworth Expansion.

3, 0.1265	3.345, 0.3615	3.69, 0.4425	4.035, 0.4885	4.38, 0.5115	4.725, 0.5115	5.07, 0.48
3.005, 0.14	3.35, 0.363	3.695, 0.4435	4.04, 0.489	4.385, 0.5115	4.73, 0.5115	5.075, 0.4795
3.01, 0.151	3.355, 0.3645	3.7, 0.4445	4.045, 0.4895	4.39, 0.5115	4.735, 0.511	5.08, 0.4785
3.015, 0.16	3.36, 0.366	3.705, 0.445	4.05, 0.49	4.395, 0.512	4.74, 0.511	5.085, 0.4775
3.02, 0.1685	3.365, 0.3675	3.71, 0.446	4.055, 0.4905	4.4, 0.512	4.745, 0.511	5.09, 0.477
3.025, 0.176	3.37, 0.369	3.715, 0.447	4.06, 0.491	4.405, 0.512	4.75, 0.5105	5.095, 0.476
3.03, 0.183	3.375, 0.3705	3.72, 0.448	4.065, 0.4915	4.41, 0.512	4.755, 0.5105	5.1, 0.475
3.035, 0.1895	3.38, 0.372	3.725, 0.4485	4.07, 0.492	4.415, 0.5125	4.76, 0.51	5.105, 0.474
3.04, 0.195	3.385, 0.3735	3.73, 0.4495	4.075, 0.4925	4.42, 0.5125	4.765, 0.51	5.11, 0.473
3.045, 0.201	3.39, 0.375	3.735, 0.45	4.08, 0.4925	4.425, 0.5125	4.77, 0.5095	5.115, 0.472
3.05, 0.206	3.395, 0.3765	3.74, 0.451	4.085, 0.493	4.43, 0.513	4.775, 0.5095	5.12, 0.471
3.055, 0.211	3.4, 0.378	3.745, 0.452	4.09, 0.4935	4.435, 0.513	4.78, 0.509	5.125, 0.47
3.06, 0.216	3.405, 0.3795	3.75, 0.4525	4.095, 0.494	4.44, 0.513	4.785, 0.509	5.13, 0.469
3.065, 0.2205	3.41, 0.381	3.755, 0.4535	4.1, 0.4945	4.445, 0.513	4.79, 0.5085	5.135, 0.468
3.07, 0.225	3.415, 0.382	3.76, 0.454	4.105, 0.495	4.45, 0.513	4.795, 0.5085	5.14, 0.467
3.075, 0.2295	3.42, 0.3835	3.765, 0.455	4.11, 0.4955	4.455, 0.5135	4.8, 0.508	5.145, 0.466
3.08, 0.2335	3.425, 0.385	3.77, 0.4555	4.115, 0.4955	4.46, 0.5135	4.805, 0.5075	5.15, 0.465
3.085, 0.2375	3.43, 0.3865	3.775, 0.4565	4.12, 0.496	4.465, 0.5135	4.81, 0.5085	5.155, 0.464
3.09, 0.2415	3.435, 0.3875	3.78, 0.457	4.125, 0.4965	4.47, 0.5135	4.815, 0.5075	5.16, 0.4625
3.095, 0.245	3.44, 0.389	3.785, 0.458	4.13, 0.497	4.475, 0.5135	4.82, 0.507	5.165, 0.4615
3.1, 0.2485	3.445, 0.39	3.79, 0.4585	4.135, 0.4975	4.48, 0.514	4.825, 0.5065	5.17, 0.4605
3.105, 0.252	3.45, 0.3915	3.795, 0.4595	4.14, 0.4975	4.485, 0.514	4.83, 0.5065	5.175, 0.459
3.11, 0.2555	3.455, 0.393	3.8, 0.46	4.145, 0.498	4.49, 0.514	4.835, 0.506	5.18, 0.458
3.115, 0.259	3.46, 0.394	3.805, 0.461	4.15, 0.4985	4.495, 0.514	4.84, 0.5055	5.185, 0.4565
3.12, 0.262	3.465, 0.3955	3.81, 0.4615	4.155, 0.499	4.5, 0.514	4.845, 0.5055	5.19, 0.4555
3.125, 0.2655	3.47, 0.3965	3.815, 0.4625	4.16, 0.4995	4.505, 0.514	4.85, 0.505	5.195, 0.454
3.13, 0.2685	3.475, 0.398	3.82, 0.4635	4.165, 0.4995	4.51, 0.514	4.855, 0.5045	5.2, 0.453
3.135, 0.2715	3.48, 0.399	3.825, 0.464	4.17, 0.5	4.515, 0.5145	4.86, 0.504	5.205, 0.4515
3.14, 0.2745	3.485, 0.4005	3.83, 0.4645	4.175, 0.5005	4.52, 0.5145	4.865, 0.504	5.21, 0.45
3.145, 0.277	3.49, 0.4015	3.835, 0.465	4.18, 0.5005	4.525, 0.5145	4.87, 0.5035	5.215, 0.4485
3.15, 0.28	3.495, 0.403	3.84, 0.466	4.185, 0.501	4.53, 0.5145	4.875, 0.503	5.22, 0.447
3.155, 0.283	3.5, 0.404	3.845, 0.4665	4.19, 0.5015	4.535, 0.5145	4.88, 0.5025	5.225, 0.4455
3.16, 0.2855	3.505, 0.405	3.85, 0.467	4.195, 0.5015	4.54, 0.5145	4.885, 0.5025	5.23, 0.444
3.165, 0.288	3.51, 0.4065	3.855, 0.468	4.2, 0.502	4.545, 0.5145	4.89, 0.502	5.235, 0.4425
3.17, 0.291	3.515, 0.4075	3.86, 0.4685	4.205, 0.5025	4.55, 0.5145	4.895, 0.5015	5.24, 0.441
3.175, 0.2935	3.52, 0.4085	3.865, 0.469	4.21, 0.5025	4.555, 0.5145	4.9, 0.501	5.245, 0.4395
3.18, 0.296	3.525, 0.41	3.87, 0.47	4.215, 0.503	4.56, 0.5145	4.905, 0.5005	5.25, 0.438
3.185, 0.2985	3.53, 0.411	3.875, 0.4705	4.22, 0.5035	4.565, 0.5145	4.91, 0.5	5.255, 0.436
3.19, 0.301	3.535, 0.412	3.88, 0.471	4.225, 0.5035	4.57, 0.5145	4.915, 0.4995	5.26, 0.4345
3.195, 0.303	3.54, 0.413	3.885, 0.472	4.23, 0.504	4.575, 0.5145	4.92, 0.499	5.265, 0.4325
3.2, 0.3055	3.545, 0.4145	3.89, 0.4725	4.235, 0.5045	4.58, 0.5145	4.925, 0.4985	5.27, 0.431
3.205, 0.308	3.55, 0.4155	3.895, 0.473	4.24, 0.5045	4.585, 0.5145	4.93, 0.498	5.275, 0.429
3.21, 0.31	3.555, 0.4165	3.9, 0.4735	4.245, 0.505	4.59, 0.5145	4.935, 0.4975	5.28, 0.427
3.215, 0.3125	3.56, 0.4175	3.905, 0.4745	4.25, 0.505	4.595, 0.5145	4.94, 0.497	5.285, 0.425
3.22, 0.3145	3.565, 0.4185	3.91, 0.475	4.255, 0.5055	4.6, 0.5145	4.945, 0.4965	5.29, 0.423
3.225, 0.317	3.57, 0.4195	3.915, 0.4755	4.26, 0.506	4.605, 0.5145	4.95, 0.496	5.295, 0.421
3.23, 0.319	3.575, 0.4205	3.92, 0.476	4.265, 0.506	4.61, 0.5145	4.955, 0.4955	5.3, 0.419
3.235, 0.321	3.58, 0.4215	3.925, 0.4765	4.27, 0.5065	4.615, 0.514	4.96, 0.495	5.305, 0.417

Table 4: Continued.

3.24, 0.323	3.585, 0.423	3.93, 0.4775	4.275, 0.5065	4.62, 0.514	4.965, 0.4945	5.31, 0.4145
3.245, 0.325	3.59, 0.424	3.935, 0.478	4.28, 0.507	4.625, 0.514	4.97, 0.494	5.315, 0.4125
3.25, 0.3275	3.595, 0.425	3.94, 0.4785	4.285, 0.507	4.63, 0.514	4.975, 0.4935	5.32, 0.41
3.255, 0.3295	3.6, 0.426	3.945, 0.479	4.29, 0.5075	4.635, 0.514	4.98, 0.493	5.325, 0.4075
3.26, 0.331	3.605, 0.427	3.95, 0.4795	4.295, 0.5075	4.64, 0.514	4.985, 0.492	5.33, 0.405
3.265, 0.333	3.61, 0.428	3.955, 0.48	4.3, 0.508	4.645, 0.514	4.99, 0.4915	5.335, 0.4025
3.27, 0.335	3.615, 0.429	3.96, 0.4805	4.305, 0.508	4.65, 0.5135	4.995, 0.491	5.34, 0.4
3.275, 0.337	3.62, 0.43	3.965, 0.4815	4.31, 0.5085	4.655, 0.5135	5, 0.4905	5.345, 0.1875
3.28, 0.339	3.625, 0.431	3.97, 0.482	4.315, 0.5085	4.66, 0.5135	5.005, 0.4895	5.35, 0.163
3.285, 0.3405	3.63, 0.4315	3.975, 0.4825	4.32, 0.509	4.665, 0.5135	5.01, 0.489	5.355, 0.142
3.29, 0.3425	3.635, 0.4325	3.98, 0.483	4.325, 0.509	4.67, 0.5135	5.015, 0.4885	5.36, 0.1225
3.295, 0.3445	3.64, 0.4335	3.985, 0.4835	4.33, 0.5095	4.675, 0.513	5.02, 0.4875	5.365, 0.105
3.3, 0.346	3.645, 0.4345	3.99, 0.484	4.335, 0.5095	4.68, 0.513	5.025, 0.487	5.37, 0.0885
3.305, 0.348	3.65, 0.4355	3.995, 0.4845	4.34, 0.5095	4.685, 0.513	5.03, 0.4865	5.375, 0.0725
3.31, 0.3495	3.655, 0.4365	4, 0.485	4.345, 0.51	4.69, 0.5125	5.035, 0.4855	5.38, 0.057
3.315, 0.3515	3.66, 0.4375	4.005, 0.4855	4.35, 0.51	4.695, 0.5125	5.04, 0.485	5.385, 0.0425
3.32, 0.353	3.665, 0.438	4.01, 0.486	4.355, 0.5105	4.7, 0.5125	5.045, 0.484	5.39, 0.028
3.325, 0.3545	3.67, 0.439	4.015, 0.4865	4.36, 0.5105	4.705, 0.5125	5.05, 0.4835	5.395, 0.01
3.33, 0.3565	3.675, 0.44	4.02, 0.487	4.365, 0.5105	4.71, 0.512	5.055, 0.4825	
3.335, 0.358	3.68, 0.441	4.025, 0.4875	4.37, 0.511	4.715, 0.512	5.06, 0.482	
3.34, 0.3595	3.685, 0.442	4.03, 0.488	4.375, 0.511	4.72, 0.512	5.065, 0.481	

```
f:= proc (b1, b2, x)
1+sqrt(b1)/6 * (x^3-3x) + (b2-3)/24 *(x^4-6x^2+3)
end;
readlib(sturm):
for i from 3 by 0.001 to 7 do
        test:=0
        for j from 0.0001 by 0.0001 while test=0 do
        test:=sturm(sturmseq(f(j,i,x),x),x,-100,100):
            if test <> 0
            then lprint(evalf(i,4),evalf(j-0.0001,9));
fi; od; od; quit;
f:= proc (b1, b2, x)
  x+sqrt(b1)/6 *(x^4-6*x^2+3) + (b2-3)/24 * (x^5 - 10*x^3 + 15 * x)
end;
readlib(sturm):
for i from 0.01 by 0.001 to 0.7 do
  for j from 3 by 0.005 to 5.6 do
   test:=sturm(sturmseq(f(i, j, x),x), x, -100, 100);
   if test = 1
   then lprint(evalf(i,4), evalf(j,4))
   fi; od; od;
```

Algorithm 1: Mathematica source code.

the robustness of departures from normality, we must restrict our conclusions to specific values of λ_3 and λ_4. Similar limitations occur when examining the relationships between probability density functions and their cumulants.

Appendix

See Algorithm 1.

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