PRICING FINANCIAL DERIVATIVES WITH FUZZY ALGEBRAIC MODELS: A THEORETICAL AND COMPUTATIONAL APPROACH.

BY

Srimantoorao Semischetty Appadoo

A Thesis
Submitted to the Faculty of Graduate Studies
In Partial Fulfillment of the Requirements for the Degree of

DOCTOR OF PHILOSOPHY

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Dedication

TO MY LATE MUM AND DAD

Abstract

The thesis is comprised of seven chapters. The first chapter is introductory in nature and pertains to a brief review of the related work to the proposed study. It also contains the summary of the research work presented in this thesis.

Chapter 2 provides a literature survey of the work done by various researchers on triangular fuzzy numbers, trapezoidal fuzzy numbers and option pricing under fuzzy environment. Some of the work done by Carlsson and Fuller [30] on possibilitic mean and variance of fuzzy numbers is highlighted. The work done by Fuller and Majlender [53] on weighted possibilistic mean and variance of fuzzy numbers is also discussed to some extents. At the end of Chapter 2, a summary of the thesis is provided.

In Chapter 3, we introduce the O(m, n)-Trapezoidal Type Fuzzy Numbers, and establish some of their properties along with some examples.

In Chapter 4 for O(m, n)-Trapezoidal Type Fuzzy Numbers we derive expressions for possibilistic mean and possibilistic variance, weighted possibilistic mean and weighted possibilistic variance, expressions for possibilistic covariance and weighted possibilistic covariance. Some applications are provided in the form of examples using weighted

functions.

In Chapter 5, we make use of O(m, n)-Trapezoidal Type Fuzzy Numbers to discuss the fuzzy binomial option pricing model and derive expression for the fuzzy risk neutral probabilities, along with fuzzy expression for the fuzzy call prices. As a consequence, we obtain weighted intervals for the risk neutral probabilities and for the expected fuzzy call price. Numerical examples are provided to illustrate the results.

In Chapter 6, we present the fuzzy binomial option pricing model using LR-Fuzzy numbers and obtain expressions for the fuzzy risk neutral probabilities and for the fuzzy call prices in terms of LR-fuzzy numbers.

In the last chapter of the thesis, we present the contributions made in the thesis and conclusion along with some recommendations for future directions on the problems considered in the thesis.

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Notations

The following notation and terminology are used throughout this thesis.

A or $\tilde{A}$	Notation for a fuzzy number.
$M_*(A)$	Lower possibilistic mean value of $\tilde{A}$.
$M^*(A)$	Upper possibilistic mean value of $\tilde{A}$.
$M(A)$	Interval value possibilistic mean value of $\tilde{A}$.
$\overline{M}(A)$	Crisp possibilistic mean value of $\tilde{A}$.
$E_*(A)$	Lower probability mean value of $\tilde{A}$.
$E^*(A)$	Upper probability mean value of $\tilde{A}$.
$E(A)$	Interval value probability mean value of $\tilde{A}$.
$\overline{E}(A)$	Crisp probability mean value of $\tilde{A}$.
$M_f^-(A)$	Lower f-weighted possibilistic mean value of $\tilde{A}$.
$M_f^+(A)$	Upper f-weighted possibilistic mean value of $\tilde{A}$.
$M_f(A)$	f-weighted interval-valued possibilistic mean value of $\tilde{A}$.
$\overline{M}_f(A)$	f-weighted possibilistic mean value of $\tilde{A}$.
S	Stock price at time $t = 0$.
K	Exercise Price of the call option.
T	Time to expiration.
u	Upward Movement in the Stock Price.

d	Downward Movement in the Stock Price.
p_u	Probability of an Upward Movement in the Stock Price.
p_d	Probability of a Downward Movement in the Stock Price.
C_d	Value of the call option in the downward state.
C_u	Value of the call option in the upward state.
C_0	Current price of the call option..
$\tilde{C}_u$	Fuzzy price of the derivative in the up state.
$\tilde{C}_d$	Fuzzy price of the derivative in the down state.
$\tilde{d}$	Fuzzy down movement in the stock price.
$\tilde{u}$	Fuzzy up movement in the stock price.
$\tilde{C}$	Fuzzy current price of the call option.

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Chapter 1

Introduction

Historically, probability theory has been used to form theoretical foundations for reasoning and decision making in situations involving uncertainty. However, one is often faced with situations in which decisions are required to be made on the basis of ill-defined variables, and imprecise (vague) data. Fuzzy algebra is a simple and useful way to propagate impreciseness through a cascade of calculations. It has been used to model systems that are hard to define precisely. In option pricing the volatility has been modeled by a fuzzy number by Carlsson and Fuller [31].

As a methodology, it incorporates imprecision, subjective risk assessment, vague data information, and sensitivity analysis into the model formulation and solution process. In finance and management science, uncertainty is usually handled through the probability theory, which sometimes encounters difficulties. In general, probability calculus is not well adapted to an imprecise corpus of knowledge, where as fuzzy calculus appears to be a more supple technique that provides pragmatic answers to problems under fuzzy environment. The use of fuzzy set theory, introduced by Zadeh [136] as a methodology for modeling and analyzing certain financial problems, is of particular interest to a number of researchers in option pricing ([3], [5], [9], [17], [98]) due to fuzzy set theory's ability to both quantify and quantify those problems that involve vagueness and imprecision. Option pricing theory can be traced back to Bachelier [7]. Binomial option pricing ([33], [38]) is a simple but powerful technique that can be used to solve many complex option pricing problems. In this thesis we consider such problem of option pricing and on the lines of Muzzioli and Torricelli [98] we discuss the option pricing when payoffs are described by O(m, n)-Trapezoidal Type Fuzzy Numbers and LR-Fuzzy Numbers.

1.1 Fuzzy Sets Theory

In this chapter, some of the fundamentals of basic fuzzy set theory that we use in this thesis are reviewed. Starting with basic definitions about fuzzy sets, we provide a method for extending some non fuzzy mathematical concepts to the fuzzy framework. Using the extension principle we carry operations on fuzzy numbers.

We now introduce certain terminology, notation, definitions and prerequisites that will be used in the sequel.

Fuzzy Set

Let X be a classical set of objects, called the universe, whose generic elements are denoted by X. The membership in a crisp subset A of X is viewed as characteristic function $\mu_A(x)$ from X to $\{0, 1\}$ such that

$$\mu_A(x) = \begin{cases} 0 & \text{for } x \notin A \\ 1 & \text{for } x \in A \end{cases}$$

where $\{0,\ 1\}$ is called a valuation set ([13],[44],[136],[137]).

If the valuation set is allowed to be the closed real interval $[0,\ 1]$, then A is called a fuzzy set as proposed by Zadeh [141].

$\mu_A(x)$ is the degree of membership of X in A. The closer the value of $\mu_A(x)$ is to 1, the more x belongs to A. Therefore, a fuzzy set A is completely characterized by the set of ordered pairs:

$A = \{(x, \mu_A(x) \quad x \in X\}$ where $\mu_A(x)$ maps X to the membership space [0,1]. Elements with zero degree of membership are usually not listed.

Definition 1.1.1 *Let A be a fuzzy set in X. The height $h(A)$ of A is defined as*

$$h(A) = \sup_{x \in X} \mu_A(x)$$

Normal Fuzzy Set.

If $h(A) = 1$, then the fuzzy set A is called a normal fuzzy set.

$\alpha-$ Cut.

An α-cut denoted by $A(\alpha)$ is the crisp set of elements $X \in \Re$ whose degree of belonging to the fuzzy set A is at least $\alpha \in (0, 1]$.

This means $A(\alpha) = \{x \in X \mid \mu(x) \geq \alpha, \alpha \in (0, 1]\}$. The $\alpha-$ cut is the crisp set $A(\alpha)$ that contains all elements of the universal set $X \in \Re$ whose membership grades in A are greater than or equal to the specified value of α, $\alpha \in (0, 1]$. Sometimes, in the literature $\alpha-$ cut is also referred to as $\gamma-$level set, in which case we use γ instead of α.

Support of a Fuzzy Set

Let A be a fuzzy set in X. Then the support of A, denoted by S(A), is the crisp set given by

$$S(A) \quad = \quad \{x \in X : \mu_A(x) > 0\}$$

Intersection of Fuzzy Sets

Intersection of two fuzzy sets A and B is a fuzzy set C denoted by C = A ∩ B, whose membership function is related to those of A and B by

$$\mu_C(\mathrm{x}) = \mathrm{Min}\ [\ \mu_A(\mathrm{x}),\ \ \mu_B(\mathrm{x})\ ] \quad \forall\ \ x \in \mathrm{X}.$$

1.2 Algebraic Operations on Fuzzy Sets

In addition to the set theoretical operations, we can also define a number of combinations of fuzzy sets and relate them to one another. Here we present some more important operations among them.

1.3 Convexity of Fuzzy Sets

The notion of convexity can be extended to fuzzy sets in such a way as to preserve many of the properties that it has in case of crisp sets. In what follows, we assume that X is the n-dimensional space R^n. We have the following two definitions of convexity of a fuzzy sets.

Convex Fuzzy Set. A fuzzy set A is convex if and only if the sets $A_\alpha = \{x \in X \mid \mu_A\ (x) \geq \alpha\}$ for all $\alpha \in (\ 0,1]$ is a convex set. The second definition of convexity of a fuzzy set is as follows: A fuzzy set A is said to be a convex set if $\mu(\lambda x_1+(1-\lambda)x_2) \geq \min(\mu(x_1),\ \mu(x_2)), x_1, x_2 \in X, \quad \lambda \in (0,1]$, i.e. if $\mu(x)$ is a quasi-concave function on X. The definition of a convex fuzzy set leads us to the following definitions of a fuzzy number.

1.4 Fuzzy Number

Definition 1.4.1 *A Fuzzy Number A is a fuzzy set on the real line $\Re$, that possesses the following properties.*

(1) A is a normal, convex fuzzy set on $\Re$,

(2) The $\alpha-$cut A_α is a closed interval for every $\alpha \in (0,1]$,

(3) The support of A, $S(A) = \{x \mid \mu_A(x) > 0\}$, is bounded.

Definition 1.4.2 *A fuzzy set A in $\Re$ is called a fuzzy number if it satisfies the following conditions.*

(i) A is normal,

(ii) A_α is a closed interval for every $\alpha \in (0,1]$,

(iii) μ_A is upper semicontinuous, and

(iv) the support of A is bounded.

Fuzzy arithmetic is based on two properties of fuzzy numbers:

- Each fuzzy set and thus, each fuzzy number can be fully and uniquely represented by its $\alpha-$level sets.

- $\alpha-$ cut of each fuzzy number are closed intervals of real numbers for all $\alpha \in (0, 1]$.

A fuzzy number can be characterized by an interval of confidence at level α, ([13], [77], [142]) as follows.

$A(\alpha) = [a_1^{(\alpha)},\ a_2^{(\alpha)}]$ which has the property

$\alpha \leq \alpha' \qquad \Longrightarrow A(\alpha') \subset A(\alpha)$.

These properties enable us to define an arithmetic operation on fuzzy numbers in terms of arithmetic operations on their $\alpha-$level sets. (i.e. arithmetic operations on closed intervals). In what follows we shall use the notation A or $\tilde{A}$ for a fuzzy number without making any distinction between them. The interpretation will be clear from the context.

1.5 Fuzzy Arithmetic Based on Operations on Closed Intervals.

Let $A = [a, b] \in \Re$ and $B = [c, d] \in \Re$ be two fuzzy intervals then we define the arithmetic operations on them as follows.

Addition $A + B = [a + c, b + d]$

Subtraction $A - B = [a - d, b - c]$

Multiplication $AB = [\text{Min}(ac, ad, bc, bd),\ \text{Max}(ac, ad, bc, bd)]$

Inverse of A $\quad A^{-1} = [\text{Min}(\frac{1}{a}, \frac{1}{b}),\ \text{Max}(\frac{1}{a}, \frac{1}{b})]$

Division $\frac{A}{B} = [\text{Min}(\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}),\ \text{Max}(\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d})]$

Let A and B be two fuzzy numbers such that $A_\alpha = [\mathrm{a}_1^{(\alpha)},\ \mathrm{a}_2^{(\alpha)}]$ is the $\alpha-$ cut of A and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ is the $\alpha-$ cut of B. Let $*$ denote any of the arithmetic operations $+$, $-$, $.$, $/$, $\wedge$ and $\vee$ on fuzzy numbers. Then, we define a fuzzy set A $*$ B in R, by defining its $\alpha-$ level sets $(\mathrm{A} * \mathrm{B})_\alpha$ as $(\mathrm{A} * \mathrm{B})_\alpha = \mathrm{A}_\alpha * \mathrm{B}_\alpha$ for any $\alpha \in [0,1]$. Since $(\mathrm{A} * \mathrm{B})_\alpha$ is a closed interval for each $\alpha \in [0,1]$ and A and B are fuzzy numbers, A $*$ B is also a fuzzy number.

The multiplication of fuzzy number $\mathrm{A} \subset \mathrm{R}$ by an ordinary number $\mathrm{k} \in \mathrm{R}^+$ can also be defined as $(\mathrm{k}^*\ \mathrm{A}_\alpha) = \mathrm{k}\ (\cdot\ )\ \mathrm{A}_\alpha = [\ \mathrm{ka}_1^{(\alpha)},\ \ \mathrm{ka}_2^{(\alpha)}\ ]$ or equivalently, $\mu_{\mathrm{k.A}}(\mathrm{x}) = \mu_{\mathrm{A}}(\frac{x}{k})$ $\forall x \in \Re$.

Definition 1.5.1 *Triangular Fuzzy Number (T. F. N.) A T.F.N. can be represented completely by a triplet $\tilde{A} = (a_1, a_2, a_3)$, whose membership function is defined as follows,*

$$\mu(x) = \begin{cases} 0 & x \leq a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & a_1 \leq x \leq a_2 \\ \dfrac{x - a_3}{a_2 - a_3} & a_2 \leq x \leq a_3 \\ 0 & x \geq a_3\ . \end{cases} \tag{1.5.1}$$

Alternatively [[77] , p. 26, 27], defining the interval of confidence at level- α as,

$$A_\alpha = [a_1^\alpha, a_2^\alpha],$$

we characterize the T.F.N. (a_1, a_2, a_3) as

$$A_\alpha = [a_1 + \alpha(a_2 - a_1), a_3 + \alpha(a_2 - a_3)] \quad \forall\, \alpha \in \ (0, 1]. \tag{1.5.2}$$

Definition 1.5.2 *Trapezoidal Fuzzy Number (Tr.F.N.) A Tr.F.N. can be represented completely by a quadruplet $\tilde{A} = (a_1, a_2, a_3, a_4)$, whose membership function is defined as follows*

$$\mu(x) = \begin{cases} 0 & x \leq a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & a_1 \leq x \leq a_2 \\ 1 & a_2 \leq x \leq a_3 \\ \dfrac{x - a_4}{a_3 - a_4} & a_3 \leq x \leq a_4 \\ 0 & x \geq a_4 \ . \end{cases} \tag{1.5.3}$$

Alternatively [[77] , p. 26, 27], defining the interval of confidence at level- α as,

$$A_\alpha \ = \ [a_1^\alpha, a_2^\alpha],$$

we characterize the Tr.F.N.(a_1, a_2, a_3, a_4) as

$$A_\alpha \ = \ [a_1 + \alpha(a_2 - a_1), a_4 + \alpha(a_3 - a_4)] \quad \forall\, \alpha \in \ [0, 1]. \tag{1.5.4}$$

Definition 1.5.3 *LR-Fuzzy Number. A fuzzy number $\tilde{M}$ is of the LR-*type *if there exist shape functions L and R and four parameters $(\underline{m}, \overline{m}) \in \bigcup\{-\infty, +\infty\}$, α, β and the membership function of $\tilde{M}$ is*

$$\mu_{\tilde{M}}(x) = \begin{cases} L\left(\dfrac{\underline{m} - x}{\alpha}\right) & \forall\, x \leq \underline{m},\ \alpha > 0 \\ 1 & \forall\, \underline{m} \leq x \leq \overline{m},\ \alpha > 0 \\ R\left(\dfrac{x - \overline{m}}{\beta}\right) & \forall\, x \geq \overline{m},\ \beta > 0\ . \end{cases}$$

The LR-fuzzy number is then denoted by $\tilde{M} = (\underline{m}, \overline{m}, \alpha, \beta)_{LR}$, where α is the left spread and β is the right spread respectively.

This definition is very general and allows quantification of quite different types of information.

If $\tilde{M}$ *is supposed be be a real crisp number for* $m \in R$, *then*

$\tilde{M} = (m, m, 0, 0)_{LR}$, $\forall L$ *and* $\forall R$.

If $\tilde{M}$ *is a crisp interval,then*

$\tilde{M} = (a, b, 0, 0)_{LR}$, $\forall L$ *and* $\forall R$ *and* $a \neq b$.

If $\tilde{M}$ *is a trapezoidal fuzzy number, then*

$$L(x) = R(x) = Max\ (0, 1 - x) \qquad (1.5.5)$$

and

$$L^{-1}(x) = R^{-1}(x) = Max\ (0, 1 - x) \qquad (1.5.6)$$

are implied Zimmermann[142].

If L and R are strictly decreasing functions then we can easily compute the γ*-level sets of* $\tilde{M}$. $[\underline{m}, \overline{m}]$ *is the peak of M and* $\underline{m}$ *and* $\overline{m}$ *are the lower and upper modal values. Also, L, R:* $[0,1] \to [0,1]$ *with L(0) = R(0) = 1 and L(1) = R(1) = 0 are non-increasing, continuous mappings [44].*

Assumption underlying LR-fuzzy numbers.

(a) We assume that $L^{-1}(.)$ and $R^{-1}(.)$ exist and are finite.

In view of assumption (a) we now state the following definition of a γ-level set.

The γ-level set for which assumption (a) holds is given by the following expression [30].

$$[M]^{\gamma} = [M_1(\gamma), M_2(\gamma)] = [\underline{m} - \alpha L^{-1}(\gamma), \overline{m} + \beta R^{-1}(\gamma)],\ \gamma \in (0, 1]. \qquad (1.5.7)$$

Let $\widetilde{M} = [a,\ b,\ \alpha,\ \beta]_{LR}$ and $\widetilde{N} = [c,\ d,\ \gamma,\ \delta]_{LR}$ be two Tr.F.N's of L-R type, where $\widetilde{M} > 0$ and $\widetilde{N} > 0$ then, we have the following (Zimmermann [142]).

$$\widetilde{M} + \widetilde{N} = [a+c,\ b+d,\ \alpha+\gamma,\ \beta+\delta] \tag{1.5.8}$$

$$\widetilde{M} * \widetilde{N} \approx [ac,\ bd,\ a\gamma + c\alpha,\ b\delta + d\beta] \tag{1.5.9}$$

$$\frac{\widetilde{M}}{\widetilde{N}} \approx \left[\frac{a}{d},\ \frac{b}{c},\ \frac{a\delta + d\alpha}{d(d+\delta)},\ \frac{b\gamma + c\beta}{c(c-\gamma)}\right]. \tag{1.5.10}$$

We will refer to the operations given by (1.5.8)-(1.5.10) in deriving the fuzzy risk neutral probabilities and fuzzy call price in Chapter 5.

1.5.1 Zadeh's Extension Principle.

Zadeh's extension principle is often referred to in the fuzzy literature as the sup min extension principle. This principle allows us to extend any point operations to operations involving fuzzy sets and is stated as follows (Bector and Chandra [10]).

Definition 1.5.4 *(Zadeh's extension principle). In terms of the notation introduced above, the extension principle of Zadeh states that*

(i) $\mu_{f(A)}(y) = \sup_{x \in X,\ f(x)=y} (\mu_A(x))$, *for all* $A \in F(X)$, *and*

(ii) $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$, *for all* $B \in F(Y)$.

Sometimes the function f maps n-tuple in X to a point in Y i.e. $X = X_1 \times X_2 \times \ldots \times X_n$ and $f : X \to Y$ given by $y = f(x_1, x_2, \ldots x_n)$. Let $A_1, A_2, \ldots A_n$ be n fuzzy sets in $X_1, X_2, \ldots X_n$ respectively. The extension principle of Zadeh allows to extend the crisp function $y = f(x_1, x_2, \ldots x_n)$ to act on n fuzzy subsets of X, namely $A_1, A_2, \ldots A_n$ such that $B = f(A_1, A_2, \ldots A_n)$.

Here the fuzzy set B is defined by

$$B = \{(y, \mu_B(y)) : y = f(x_1, \ldots, x_n), (x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n\}$$

and

$$\mu_B(y) = \sup_{x \in X,\, y=f(x)} \min(\mu_{A_1}(x_1), \ldots \mu_{A_n}(x_n)).$$

1.6 Possibility Theory.

In this section, we review the preliminary concepts of possibility and necessity measures.

Dubois and Prade [45] studied the ranking of fuzzy numbers in the setting of possibility theory. To discuss this, suppose we have two fuzzy number A and B. Then in accordance with the extension principle of Zadeh, the crisp inequality $x \leq y$ can be extended to obtain the Truth value of the assertion that A is less than or equal to B, as follows (Bector and Chandra [10]).

$$T(A(\leq)B) = \sup_{x \leq y} (\min(\mu_A(x), \mu_B(x))).$$

This truth value $T(A(\leq)B)$ is also called the *grade of possibility of dominance* of B on A and is denoted by Poss$(A(\leq)B)$.

Similarly, the grade (or degree) of possibility that the assertion "A is greater than or equal to B" is true, is given by

$$\text{Poss}(A(\geq)B) = \sup_{x \geq y} (\min(\mu_A(x), \mu_B(x))).$$

Also, the degree of possibility that the assertion "A is equal to B" is denoted by Poss $(A(=)B)$, and is defined as

$$\text{Poss}(A(=)B) = \sup_{x} (\min(\mu_A(x), \mu_B(x))).$$

The above discussion motivates us to define $A(\leq)B$ if and only if Poss$(A(\leq)B) \geq$ Poss$(B(\leq)A)$. Here it may be noted that for the case when $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ are T.F.N's then $a_2 \leq b_2$ gives Poss$(A(\leq)B) = 1$ and Poss$(B(\leq)A) =$ height $(A \cap B) \leq 1$.

Therefore for the case of T.F.N's it can be defined that $A(\leq)B$ with respect to Poss$(A(\leq)B)$ if $a_2 \leq b_2$.

Related with the number "Poss$(A(\leq)B)$" there is another number "Necc$(A(\leq)B)$" which measures the grade (or degree) of necessity of dominance of B on A, given by

$$\text{Necc}(A(\leq)B) = 1 - \text{Poss}(A(\geq)B)$$

The number "Necc$(A(\leq)B)$" can also be used for ranking of fuzzy numbers. For this, we can define $A(\leq)B$ if and only if Necc$(A(\leq)B) \geq Necc(B(\leq)A)$.

In case $A = (a_l, a_2, a_3)$ and $B = (b_l, b_2, b_3)$ are T.F.N's then by actual computation of Necc$(A(\geq)B)$ it can be defined that $A(\leq)B$ with respect to Necc$(A(\leq)B)$ approach if $a_1 + a_2 \leq b_1 + b_2$.

1.7 Mean Value and Variance of Fuzzy Numbers.

Viewing the fuzzy numbers as random sets Dubois and Prade [43] defined their interval valued expectation and introduced their mean value as a closed interval bounded by the expectations calculated from its upper and lower distribution functions. Recently, Carlsson and Fuller [30] introduced the notion of possibilistic mean value, interval-valued possibilistic mean, crisp possibilistic mean value and crisp(possibilistic) variance. On the line of Carlsson and Fuller [30], Fuller and Majlender [53] introduced the concepts of weighted possibilistic mean and variance of fuzzy numbers. In this section we discuss possibilistic mean value, possibilistic variance, weighted mean weighted variance and weighted covariance of a special type of fuzzy numbers.

1.7.1 Possibilistic Mean Value and Variance of Fuzzy Numbers.

In the section we review the concepts of possibilitic mean and possibilistic variance of a fuzzy number.

Consider two fuzzy numbers $\tilde{A}$ and $\tilde{B} \in \mathbf{F}$ such that their α-cuts are written as $A(\alpha) = [a_1(\alpha),\, a_2(\alpha)]$ and $B(\alpha) = [b_1(\alpha),\, b_2(\alpha)]$, $\alpha \in (0, 1]$. Goetschel and Voxman [56] introduce a method for ranking fuzzy numbers as

$$\tilde{A} \;\leq\; \tilde{B} \qquad \Longleftrightarrow \int_0^1 \alpha(a_1(\alpha) + a_2(\alpha))d\alpha \;\leq\; \int_0^1 \alpha(b_1(\alpha) + b_2(\alpha))d\alpha \quad (1.7.1)$$

In Goetschel and Voxman [56], (1.7.1) is motivated in part by the desire to give less importance to the lower levels of fuzzy numbers. Taking the weight of the arithmetic mean of $a_1(\alpha)$ and $a_2(\alpha)$ as α, Carlsson and Fuller [30], define the level-weighted average of the arithmetic means of all α-cuts of the fuzzy number $\tilde{A}$ by expression(1.7.2).

$$\overline{M}(A) \;=\; \int_0^1 \alpha(a_1(\alpha) + a_2(\alpha))d\alpha = \frac{\displaystyle\int_0^1 \frac{\alpha(a_1(\alpha) + a_2(\alpha))}{2} d\alpha}{\displaystyle\int_0^1 \alpha\, d\alpha} \qquad (1.7.2)$$

$$= \frac{1}{2}\left(\frac{\displaystyle\int_0^1 \alpha\, a_1(\alpha)\, d\alpha}{\displaystyle\int_0^1 \alpha\, d\alpha} + \frac{\displaystyle\int_0^1 \alpha\, a_2(\alpha)\, d\alpha}{\displaystyle\int_0^1 \alpha\, d\alpha} \right)$$

$$= \frac{M_*(A) + M^*(A)}{2} \qquad (1.7.3)$$

The first quantity, denoted by $M_*(A)$ can be reformulated as

$$M_*(A) = \frac{\int_0^1 \alpha\, a_1(\alpha)\, d\alpha}{\int_0^1 \alpha\, d\alpha} = \frac{\int_0^1 \text{Poss}[A \le a_1(\alpha)]\, a_1(\alpha)\, d\alpha}{\int_0^1 \text{Poss}[A \le a_1(\alpha)]\, d\alpha}$$

$$= \frac{\int_0^1 \text{Poss}[A \le a_1(\alpha)]\, \min[A(\alpha)]\, d\alpha}{\int_0^1 \text{Poss}[A \le a_1(\alpha)]\, d\alpha} \qquad (1.7.4)$$

where Poss denotes possibility. As in Carlsson and Fuller [30],

$\text{Poss}[A \le a_1(\alpha)] = \Pi(-\infty, a_1(\alpha)) = \sup_{u \le a_1(\alpha)} A(u) = \alpha.$

Thus, $M_*(A)$ is nothing else but the lower possibility-weighted average of the α-cut.

In a similar manner, $M^*(A)$, the upper possibilistic mean value of $\tilde{A}$, is given by

$$M^*(A) = \frac{\int_0^1 \alpha\, a_2(\alpha)\, d\alpha}{\int_0^1 \alpha\, d\alpha} = \frac{\int_0^1 \text{Poss}[A \ge a_2(\alpha)]\, a_2(\alpha)\, d\alpha}{\int_0^1 \text{Poss}[A \ge a_2(\alpha)]\, d\alpha}$$

$$= \frac{\int_0^1 \text{Poss}[A \ge a_2(\alpha)]\, \max[A(\alpha)]\, d\alpha}{\int_0^1 \text{Poss}[A \ge a_2(\alpha)]\, d\alpha} \qquad (1.7.5)$$

where, as in Carlsson and Fuller [31],

$\text{Poss}[A \ge a_2(\alpha)] = \Pi(a_2(\alpha), \infty) = \sup_{u \ge a_2(\alpha)} A(u) = \alpha.$

In view of expression (1.7.4) and (1.7.5), the lower and upper possibilistic mean values of fuzzy number A are defined as follows.

$$M(A) = [M_*(A), M^*(A)] \qquad (1.7.6)$$

Since, $M(A)$ is represented as an interval in (1.7.6), therefore, in the sequel we shall call $M(A)$ as the interval-valued possibilistic mean of fuzzy number $\tilde{A}$. Carlsson and Fuller [30] define the crisp possibilistic mean value of fuzzy number A as the arithmetic mean of its lower possibilistic and upper possibilistic mean values, i.e.

$$\overline{M}(A) = \frac{M_*(A) + M^*(A)}{2} \tag{1.7.7}$$

In view of (1.7.2)-(1.7.7), Carlsson and Fuller [30] proved the following two important theorems.

Theorem 1.7.1 (Carlsson and Fuller[30]) *Let A and B be two fuzzy numbers and let $\lambda \in \Re$ be a real number. Then*

$$M(A+B) = M(A) + M(B) \qquad M(\lambda A) = \lambda M(A)$$

i.e

$$M_*(A+B) = M_*(A) + M_*(B) \qquad M^*(A+B) = M^*(A) + M^*(B) \text{ and}$$

$$[M_*(\lambda A), M^*(\lambda A)] = \begin{cases} [\lambda M_*(A), \lambda M^*(A)] & \text{if } \lambda \geq 0 \\ [\lambda M^*(A), \lambda M_*(A)] & \text{if } \lambda < 0 \end{cases}$$

where the addition and multiplication by a scalar of fuzzy numbers is defined by the sup-min extension principle [136].

Theorem 1.7.2 (Carlsson and Fuller[30]) *Let $A(\alpha) = [a_1(\alpha), a_2(\alpha)]$ and $B(\alpha) = [b_1(\alpha), b_2(\alpha)]$ be two fuzzy numbers and let $\lambda \in \Re$ be a real number. Then*

$$\overline{M}(A+B) = \overline{M}(A) + \overline{M}(B)$$

and

$$\overline{M}(\lambda A) = \lambda \overline{M}(A)$$

1.7.2 Relation Between Interval-Valued Expectation and Interval Value Possibilistic Mean

We now discuss an important relationship between the interval-valued expectation $E(A) = [E_*(A), E^*(A)]$ introduced by Dubois and Prade [44] and the interval-valued possibilistic mean introduced by Carlsson and Fuller [30]. For any fuzzy number $\tilde{A}$ the α-cut can be written as,

$$A(\alpha) = [a_1(\alpha),\ a_2(\alpha)]$$

The lower probability mean value of $\tilde{A}$ is [44],

$$E_*(A) = \int_0^1 a_1(\alpha)d\alpha \qquad (1.7.8)$$

Similarly, the upper probability mean value of $\tilde{A}$ is [44],

$$E^*(A) = \int_0^1 a_2(\alpha)d\alpha. \qquad (1.7.9)$$

On the lines of Dubois and Prade [44], the interval value probability mean value of $\tilde{A}$ is defined as follows,

$$E(A) = [E_*(A), E^*(A)], \qquad (1.7.10)$$

and the crisp probability mean value of $\tilde{A}$ is

$$\overline{E}(A) = \left[\frac{E_*(A) + E^*(A)}{2}\right]. \qquad (1.7.11)$$

There is a strong relationship between probability mean of a fuzzy number and the possibilistic mean of a fuzzy number. Since the support of A is bounded, therefore, the lower and upper possibilistic mean values are obtained by Carlsson and Fuller

[30] as

$$M_*(A) = \int_0^1 \alpha a_1(\alpha)d\alpha, \tag{1.7.12}$$

$$M^*(A) = \int_0^1 \alpha a_2(\alpha)d\alpha, \tag{1.7.13}$$

The crisp possibilistic mean value of $\tilde{A}$ is

$$M(A) = [M_*(A), M^*(A)]. \tag{1.7.14}$$

The interval value possibilistic mean value of $\tilde{A}$ is given by Carlsson and Fuller [30], is

$$\overline{M}(A) = \left[\frac{M_*(A) + M^*(A)}{2}\right]. \tag{1.7.15}$$

We can now state the following lemma, given by Carlsson and Fuller [30].

Lemma 1.7.1 *If $A \in F$ is a fuzzy number with strictly increasing and strictly decreasing (and continuous) functions then its interval-valued possibilistic mean is a proper subset of its interval-valued probabilistic mean, i.e* $M(A) \subset E(A)$.

According to Carlsson and Fuller [30], Lemma 1.8.1 reflects on the fact that points with small membership degrees are considered to be less important in the definition of lower and upper possibilistic mean values than in the definition of probabilistic ones. It is important to point out that in the limiting case when $\tilde{A}$ is a fuzzy number having equal spreads $(a_2 - a_1 = a_4 - a_3)$, the possibilistic and probabilistic mean values are equal. That is,

$$E(A) = M(A) \tag{1.7.16}$$

1.7.3 Possibilistic Variance of Fuzzy Number

The possibilistic variance of a fuzzy number $\tilde{A}$, where $\tilde{A} \in F$, is defined by Carlsson and Fuller [30] as

$$Var(A) = \int_0^1 \frac{1}{2}\alpha \left(a_2(\alpha) - a_1(\alpha)\right)^2 d\alpha. \tag{1.7.17}$$

and the standard deviation of $\tilde{A}$ is

$$\sigma_A = \sqrt{Var(A)}.$$

1.8 Weighted Possibilistic Mean and Variance of Fuzzy Number.

Interval-valued expectation of fuzzy numbers is defined by Dubois and Prade [44] viewing them as random sets. Viewing the fuzzy numbers as random sets, Dubois and Prade [44] define their interval valued expectation, whereas, Carlsson and Fuller [30] define a possibilistic interval-valued mean value of fuzzy numbers, viewing them as possibility distributions. Furthermore, weighted possibilistic mean, variance and weighted interval-valued possibilistic mean value of fuzzy numbers are all introduced in Fuller and Majlender [53].

Definition 1.8.1 (Fuller and Majlender [53]) *Let $A \in F$ be a fuzzy number with $A(\alpha) = [a_1(\alpha), a_2(\alpha)], \alpha \in [0,1]$. A function $f : [0,1] \to \Re$ is said to be a weighted function if f is non-negative, monotone increasing and satisfies the following normalization condition.*

$$\int_0^1 f(\alpha) d\alpha = 1 \tag{1.8.1}$$

Definition 1.8.2 (Fuller and Majlender [53]) *The f-weighted possibilistic mean value of fuzzy number A is defined as*

$$\overline{M}_f(A) = \frac{\int_0^1 \alpha(a_1(\alpha) + a_2(\alpha))d\alpha}{2} f(\alpha)d\alpha \tag{1.8.2}$$

For example, if $f(\alpha) = 2\alpha, \alpha \in [0,1]$ then

$$\overline{M}_f(A) = \int_0^1 \frac{a_1(\alpha) + a_2(\alpha)}{2} 2\alpha d\alpha = \int_0^1 [a_1(\alpha) + a_2(\alpha)]\, \alpha d\alpha = M(A) \tag{1.8.3}$$

This yields that f-weighted possibilistic mean value defined by (1.8.2) can be considered as a generalization of possibilistic mean value introduced by Carlsson and Fuller [30].

Definition 1.8.3 (Fuller and Majlender [53]) *Let f be a weighting function and let A be a fuzzy number. Then we define the f-weighted interval-valued possibilistic mean of $\tilde{A}$ is defined as*

$$M_f(A) = [M_f^-(A), M_f^+(A)] \tag{1.8.4}$$

where

$$M_f^-(A) = \int_0^1 a_1(\alpha) f(\alpha) d\alpha, \tag{1.8.5}$$

$$M_f^+(A) = \int_0^1 a_2(\alpha) f(\alpha) d\alpha, \tag{1.8.6}$$

$$Pos[A \leq a_1(\alpha)] = \sup_{u \leq a_1(\alpha)} A(u) = \alpha, \tag{1.8.7}$$

$$Pos[A \geq a_2(\alpha)] = \sup_{u \geq a_2(\alpha)} A(u) = \alpha, \tag{1.8.8}$$

$M_f^-(A)$ is the f-weighted average of the minimum of the α-cuts and $M_f^+(A)$ is the f-weighted average of the maximum of the α-cut.

In view of (1.8.4) -(1.8.8) the following two theorems are proved in Fuller and Majlender [53].

Theorem 1.8.1 (Fuller and Majlender [53]) . *Let $\tilde{A}, \tilde{B} \in F$, f be a weighting function, and λ be a real number. Then*

$$\begin{aligned} M_f(A+B) &= M_f(A) + M_f(B) \\ M_f(\lambda B) &= \lambda M_f(A) \end{aligned}$$

From this and (1.8.2) we observe that the f-weighted possibilistic mean of $\tilde{A}$ is the arithmetic mean of its f-weighted lower and upper possibilistic mean values, i.e.

$$\overline{M}_f(A) = \frac{M_f^-(A) + M_f^+(A)}{2} \tag{1.8.9}$$

Theorem 1.8.2 (Fuller and Majlender [53]) *If $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers and $\lambda \in \Re$, then the following relationship holds.*

$$\overline{M}_f(A+B) = \overline{M}_f(A) + \overline{M}_f(B) \tag{1.8.10}$$

$$\overline{M}_f(\lambda A) = \lambda \overline{M}_f(A) \tag{1.8.11}$$

1.8.1 Weighted Possibilistic Variance and Covariance.

Definition 1.8.4 (Fuller and Majlender [53]) *Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers and let f be a weighted function. We define the f-weighted possibilistic variance of $\tilde{A}$ by*

$$Var_f(A) = \int_0^1 \left(\frac{a_2(\alpha) - a_1(\alpha)}{2}\right)^2 f(\alpha) d\alpha \tag{1.8.12}$$

and the f-weighted covariance of $\tilde{A}$ and $\tilde{B}$ is defined as

$$Cov_f(A) = \int_0^1 \left(\frac{a_2(\alpha) - a_1(\alpha)}{2}\right)\left(\frac{b_2(\alpha) - b_1(\alpha)}{2}\right) f(\alpha) d\alpha \qquad (1.8.13)$$

From (1.8.12) *and* (1.8.13) *we observe that the f-weighted possibilistic variance and covariance can be considered as a generalization of possibilistic variance and covariance.*

1.9 Option Pricing Model

An option is a contract between a buyer and a seller where the buyer of the contract obtains the right to trade an underlying asset, for a specified price, called the exercise price, on or before the maturity date.

An option that provides the holder the right to buy the underlying asset is known as a call option.

Let r be the risk free rate, u be the up jump factors, d be the down jump factor and S be the stock price at time $t = 0$. All of r, u, d and S are assumed to be crisp. Then, the stock price at time $t = 1$ are obtained by multiplying S with the jump factors u and d, and are Su and Sd respectively.

Binomial Option Pricing Model

In the binomial option pricing model introduced by Cox, Ross and Rubenstein [38], we model stock prices with respect to discrete time, assuming that at each particular step, the stock price will change to one of two possible values, namely, d and u, where $0 < d < u$ such that over a single period of time, the stock price can only move from its current price to any of the two possible values. Assuming that S is the current price of the stock, then in the next period the price will be either

$$C_u = \text{Max}(Su - K, 0) \qquad \text{or} \qquad C_d = \text{Max}(Sd - K, 0) \tag{1.9.1}$$

Thus, the value of a one-period call option on a stock governed by a binomial lattice is

$$C_0 = \left[\frac{p_u \text{ Max } (Su - K, 0) + p_d \text{ Max } (Sd - K, 0)}{1 + r}\right] = \left[\frac{p_u\, C_u + p_d\, C_d}{1 + r}\right] \tag{1.9.2}$$

where, C_u and C_d are defined in (1.9.1). The above CRR model is a well known and widely used model for valuing standard option.

A Two Period Binomial Option Pricing Model

In the above CRR model adding another period to the binomial tree yields a two periods binomial option pricing model. Such an action increase the number of possible outcomes at expiration (see [33] for more details).

In such a model we assume that at the end of the first period the stock price has risen to Su. During the second period the price of the stock could go either up or down, in which case the stock price would end up as either Su^2 or Sud. If, on the other hand, the stock price has gone down in the first period to Sd then during the second period it either goes down again or goes back up, in which case the price of the stock ends up at either Sd^2 or Sdu. Therefore, in this case, the prices for a two periods binomial option pricing at expiration are,

$$C_{u^2} = \text{Max}[(Su^2 - K,\ 0)] \tag{1.9.3}$$

$$C_{ud} = \text{Max}[(Sud - K,\ 0)] \tag{1.9.4}$$

$$C_{d^2} = \text{Max}[(Sd^2 - K,\ 0)] \tag{1.9.5}$$

Thus, the price of a two period call option is given by,(see [33] for more details).

$$C_0 = \left[\frac{p_u^2 C_{u^2} + 2p_u p_d C_{ud} + p_d^2 C_{d^2}}{(1+r)^2}\right]$$

Similarly, for the n periods binomial option pricing model, the call price is given by [33].

$$C_0 = \left[\frac{\sum_{j=o}^{n} \frac{n!}{j!(n-j)!}\, p_u^j\, p_d^{n-j} \text{Max}\left[(\ S\ u^j d^{n-j} - K,\ 0\ )\right]}{(1+r)^n}\right] \tag{1.9.6}$$

Binomial Option Pricing Model Assumptions:

In order to derive the price of a call option in a vague environment we make the following assumptions which are similar to those made by Muzzioli and Torricelli [98].

1. All investors have homogeneous beliefs.
2. Markets are frictionless i.e. markets have no transaction costs, no taxes, no restrictions on short sales and asset are infinitely divisible.
3. Every investor acts as a price taker.

4. Interest rates are positive. The interest rate is equal to r percent per time period.

5. No arbitrage opportunities are allowed. This condition is expressed by the following formula, $d_3 < (1+r) < u_1$.

6. The market is complete.

1.10 Organization of the Thesis

Chapter 1 provides an introduction to the concepts of decision making and presents the motivations and the need for a comprehensive methodology for the binomial option pricing model. Chapter 2 deals with the literature review of the related work done by other researchers relevant to this research. In Chapter 3, we introduce O(m,n)-Tr.T.F.N's as a generalization of Tr.T.F.N's and discuss their varying algebraic properties. In Chapter 4, on the lines of Carlsson and Fuller [30] and Fuller and Majlender [53], we discuss their possibilistic mean, variance and covariance as well as the weighted possibilistic mean, variance and covariance for O(m,n)-Tr.T.F.N's. Chapter 5 deals with the binomial option pricing model using O(m,n)-Tr.T.F.N's as input parameters. Various cases are discussed for different values of m and n. Chapter 6 deals with the fuzzy binomial option pricing model using LR-fuzzy numbers. A number of results for O(m,n)-Tr.T.F.N's and for trapezoidal fuzzy numbers can be deduced as special cases of the results proved in this chapter. Finally, the conclusion and the discussion on the contributions made by the thesis, along with some recommendations for further research, are given in Chapter 7.

Chapter 2

Literature Survey

The main objective of this chapter is to provide a survey of the literature dealing with fuzzy numbers, Binomial Option pricing Model and Fuzzy Binomial Option pricing Model.

2.1 Fuzzy Numbers

In most fuzzy financial applications, linear membership functions are used to model impreciseness, vagueness, fuzziness and arbitrariness in the various parameters of the model. Linear and piecewise linear membership functions are easily manipulated by fuzzy operators. The other major reason for using linear membership function is to avoid complex non-linear computations in the analysis of various results. Linear membership functions are not always appropriate in various financial applications as, many times, they do not represent the linguistic terms being modeled. Membership functions are the building blocks of fuzzy set theory. However, there are many difficulties associated in selecting the solution of a problem written with linear membership function. Therefore, various types of membership functions have been proposed in the literature (for example, a tangent type of a membership function [83], an interval linear membership function [61], an exponential membership function [28], inverse tangent membership function [115], logistic type of membership function [125], concave piecewise linear membership function [72] and piecewise linear membership function [65]). Unfortunately, most of those membership functions are not flexible enough to capture impreciseness, vagueness, fuzziness, randomness and arbitrariness. The need for an efficient membership function has long been felt. Medasani et al. [95] have highlighted the importance of having membership functions that can be easily tuned and adjusted. Other authors (for example, Medaglia et al. [94], Medasani et al. [95], Appadoo et al. [2] and others) have expressed the need for membership functions that are easy to use and be manipulated. Moreover, the parameters associated with the membership functions should be easily tweaked until the performance is acceptable. In this thesis, we propose a family of fuzzy numbers to overcome some of the shortcomings associated with linear membership functions and some of the fuzzy numbers mentioned above. Furthermore, we discuss some important properties of the proposed fuzzy numbers.

2.2 Binomial Option Pricing Model

Various extensions to the original binomial model have been proposed in the literature, such as Boyle [19], Nelson and Ramaswamy [102], Hull and White [67], Tian [120], and Leisen and Reimer [85], just to mention a few. The sole motivation for these subsequent models is either improving the rate of convergence or for pricing more complex derivatives. Another extension of the binomial option pricing model is the trinomial option pricing model of Kamrad and Ritchken [75] which is a generalization of the binomial model described by Cox, Ross and Rubinstein [38]. In the available literature binomial and trinomial extensions are commonly referred to as tree-based models. However, binomial option pricing model under crisp assumptions lack flexibility in the sense that the jump size of the binomial tree is fixed for a given set of option parameters and time increments. Johnson, Pawlukiewicz, and Mehta [73] develop a binomial option pricing model that is dependent on skewness.

Other researchers have developed alternative models for option pricing when the distribution of the stock return is not normally distributed or where the stock price follows a jump-diffusion processes (for example, Cox and Ross [39] and Merton [93]).

2.3 Fuzzy Binomial Option Pricing Model

Recently there has been growing interest in using fuzzy models in Finance, Economics and Actuarial Science (for example see Muzzioli and Torricelli [98] Ostaszewski [104], Wu [126], Appadoo et al. [2]). Fuzzy option pricing model have been studied in Muzzioli et al. [98] and in Wu [126] using fuzzy sets theory of Zadeh [137]. A nonlinear shape fuzzy number for the fuzzy binomial option pricing model has been proposed in Appadoo et al. [2]. In a variety of financial models a number of parameters in general are vague, arbitrary and subjective in nature. Such models have been the subject of extensive study ever since Buckley published a paper in 1987 [25].

The difficulty in such models arise from uncertainty which cannot be represented by probability theory alone. A mathematical model for pricing American put option and European options with uncertainty has been proposed by Yoshida [132]. In his model [132], randomness and fuzziness in the parameters are evaluated by both probabilistic expectation and fuzzy expectation defined by a possibility measure from the viewpoint of fuzzy expectation, taking into account the decision-makers subjective judgement. European options with uncertainty are also discussed under appropriate assumptions. In 2001, Zmeskal [138] proposed a fuzzy algebraic approach to price a call option. Zmeskal [138] used fuzzy numbers for the input data. More recently, an application of fuzzy sets theory to the Black Scholes option pricing model has been proposed by Wu [126]. Fuzzy interest rate, fuzzy volatility, and fuzzy stock price have been used in the model. Under these assumptions, the European option price at time t turns out to be a fuzzy number, thus, allowing us to choose the European option price at his (her) acceptable degree of belief. The pricing models of European option using the real interval limited Choquet integral for a nonnegative measurable function over a real fuzzy measure space has also been investigated by Kaino and Hirota [74]. A fuzzy approach to real options has been the subject of study of Carlsson and Fuller [31]. Carlsson and Fuller [31] introduce a (heuristic) real option rule in a fuzzy framework, where the discounted cash flow of the expected cash flow and expected costs are estimated by trapezoidal fuzzy numbers. They determine an optimal exercise time using possibilistic mean value and variance of fuzzy numbers. Furthermore, they believe that uncertainty cannot be dealt as a stochastic phenomenon when working with decisions on giga-investments. Thus, possibility theory becomes an alternative way to handle future uncertainty.

Recently there has been growing interest in using fuzzy supported finance modelling. Appadoo et. al [5] propose a crisp risk free rate assisted by Capital Asset Pricing Model(CAPM) return in the fuzzy binomial option pricing model. Their model is geared towards a more natural and intuitive way to deal with fuzziness, uncertainty and arbitrariness. The classical binomial option pricing model becomes a special case of the proposed model. The generality and validity of the proposed fuzzy supported option pricing model is highlighted in their paper.

2.4 Summary of the Thesis

Chapters 1 and 2 contain, respectively, the introduction and the literature survey relevant to the thesis.

Chapter 3 Algebra of O(m,n)-Tr.T.F.N

Recently there has been rapid growth in the application of fuzzy set theory to financial problems. Implementation issues have led to the development of addition, subtraction, multiplication, division, and the inverse of various types of fuzzy numbers. The shape of a membership function always presents the knowledge about the grade of the elements in the fuzzy set. In this chapter, we introduce O(m,n)-Tr.T.F.N's. The algebra underlying the O(m,n)-Tr.T.T.N is discussed in details.

Chapter 4 Moment Properties of O(m,n)-Tr.T.F.N

On the line of Carlsson and Fuller [30] and Fuller and Majlender [53], we derive expressions for the possibilistic mean, possibilistic variance and possibilistic covariance as well as weighted possibilistic mean, weighted possibilistic variance and weighted possibilistic covariance using O(m,n)-Tr.T.F.N's. The advantage of using O(m,n)-Tr.T.F.N relative to other types of fuzzy numbers is also discussed.

Chapter 5 Binomial Option pricing Model with O(m,n)-Tr.T.F.N's.

In this chapter, on the lines of Muzzioli et al. [98], we use O(m,n)-Tr.T.F.N's to the fuzzy binomial option pricing model. Numerical examples are provided to validate the results. Furthermore, we consider different cases when parameters m and n assume different values.

Chapter 6 Binomial Option With LR-Fuzzy Numbers.

In this chapter we discuss the binomial option pricing model under fuzzy environment, using LR-Fuzzy numbers.

Chapter 7 Conclusion, Contribution and Recommendations.

In this chapter, we present the contributions made in the thesis. Conclusion along with some recommendations for further research are also given.

Chapter 3

Algebra of O(m,n)-Trapezoidal Type Fuzzy Numbers

In this chapter we introduce O(m,n)-Trapezoidal Type Fuzzy Numbers (Tr.T.F.N's) as a generalization of Trapezoidal Fuzzy Numbers (Tr.F.N's) and discuss their various algebraic properties. Some numerical examples are given to reinforce the results.

3.1 O(m,n)-Trapezoidal Type Fuzzy Numbers

In this section we introduce O(.,.)- Trapezoidal Type Fuzzy Numbers (O(.,.)-Tr.T.F.N's).

Definition 3.1.1 *A fuzzy number $A = [a_1, a_2, a_3, a_4]_{O(m,n)}$, $a_1 < a_2 < a_3 < a_4$ is said to be O(m,n)-Trapezoidal Type Fuzzy Number (O(m,n)-Tr.T.F.N.) if its membership function is given as*

$$\mu(x) = \begin{cases} 0 & x \leq a_1 \\ 1 - \left(\dfrac{a_2 - x}{a_2 - a_1}\right)^m & a_1 \leq x \leq a_2 \\ 1 & a_2 \leq x \leq a_3 \\ 1 - \left(\dfrac{a_3 - x}{a_3 - a_4}\right)^n & a_3 \leq x \leq a_4 \\ 0 & x \geq a_4 \end{cases} \tag{3.1.1}$$

Alternatively, following [[77], p. 26, 27], defining the α-cut (interval of confidence at level- α) as, $A_\alpha = [a_1^\alpha, a_2^\alpha]$, we characterize the O(m,n)-Tr.T.F.N. $[a_1, a_2, a_3, a_4]_{O(m,n)}$ *as*

$$A_\alpha = [a_2 - (a_2 - a_1)(1-\alpha)^{\frac{1}{m}},\ a_3 - (a_3 - a_4)(1-\alpha)^{\frac{1}{n}}], \forall,\ \alpha, \in\ (0,1] \tag{3.1.2}$$

by setting $1 - \left(\dfrac{a_2 - x}{a_2 - a_1}\right)^m = \alpha$ *and* $1 - \left(\dfrac{a_3 - x}{a_3 - a_4}\right)^n = \alpha$ *respectively.*

An O(m,n)-Tr.T.F.N is said to be symmetric if it satisfied the following two conditions.

(a) $a_2 - a_1 = a_4 - a_3$

(b) $m = n$

This is different from triangular fuzzy numbers (Definition 1.5.1) and trapezoidal fuzzy numbers (Definition 1.5.2) which requires condition (b), with $m = n = 1$.

Definition 3.1.2 *A fuzzy number* $A = [a_1, a_2, a_3, a_4]_{O(2,2)}$, $a_1 < a_2 < a_3 < a_4$ *is said to be O(2,2)-Trapezoidal Type Fuzzy Number (O(2,2)-Tr.T.F.N.) if its membership function is given as*

$$\mu(x) = \begin{cases} 0 & x \leq a_1 \\ 1 - \left(\dfrac{a_2 - x}{a_2 - a_1}\right)^2 & a_1 \leq x \leq a_2 \\ 1 & a_2 \leq x \leq a_3 \\ 1 - \left(\dfrac{a_3 - x}{a_3 - a_4}\right)^2 & a_3 \leq x \leq a_4 \\ 0 & x \geq a_4 \end{cases} \tag{3.1.3}$$

Alternatively, following [[77], p. 26, 27], defining the α-cut (interval of confidence at level- α) as, $A_\alpha = [a_1^\alpha, a_2^\alpha]$, we characterize the O(2,2)-Tr.T.F.N. $[a_1, a_2, a_3, a_4]_{O(2,2)}$ *as*

$$A_\alpha \;=\; [a_2 - (a_2 - a_1)(1-\alpha)^{\frac{1}{2}},\; a_3 - (a_3 - a_4)(1-\alpha)^{\frac{1}{2}}]\;\forall\;\alpha \in\;(0,1]. \quad (3.1.4)$$

by setting $1 - \left(\dfrac{a_2 - x}{a_2 - a_1}\right)^2 = \alpha$ *and* $1 - \left(\dfrac{a_3 - x}{a_3 - a_4}\right)^2 = \alpha$ *respectively.*

An O(2,2)-Tr.T.F.N is said to be symmetric if it satisfied the following two conditions.

(a) $a_2 - a_1 = a_4 - a_3$

(b) $m = n$

Definition 3.1.3 *A fuzzy number* $A = [a_1, a_2, a_3, a_4]_{O(0.5,0.5)}$, $a_1 < a_2 < a_3 < a_4$ *is said to be O(0.5,0.5)-Trapezoidal Type Fuzzy Number (O(0.5,0.5)-Tr.T.F.N.) if its membership function is given as*

$$\mu(x) = \begin{cases} 0 & x \le a_1 \\ 1 - \left(\dfrac{a_2 - x}{a_2 - a_1}\right)^{0.5} & a_1 \le x \le a_2 \\ 1 & a_2 \le x \le a_3 \\ 1 - \left(\dfrac{a_3 - x}{a_3 - a_4}\right)^{0.5} & a_3 \le x \le a_4 \\ 0 & x \ge a_4 \end{cases} \quad (3.1.5)$$

Alternatively, following [[77], p. 26, 27], defining the α-cut (interval of confidence at level- α) as, $A_\alpha = [a_1^\alpha, a_2^\alpha]$, we characterize the O(0.5,0.5)-Tr.T.F.N. $[a_1, a_2, a_3, a_4]_{O(0.5,0.5)}$ *as*

$$A_\alpha \;=\; [a_2 - (a_2 - a_1)(1-\alpha)^2,\; a_3 - (a_3 - a_4)(1-\alpha)^2]\;\forall\;\alpha \in\;(0,1]. \quad (3.1.6)$$

by setting $1-\left(\frac{a_2-x}{a_2-a_1}\right)^{0.5}=\alpha$ *and* $1-\left(\frac{a_3-x}{a_3-a_4}\right)^{0.5}=\alpha$ *respectively.*

An O(0.5,0.5)-Tr.T.F.N is said to be symmetric if it satisfied the following two conditions.

(a) $a_2-a_1=a_4-a_3$

(b) $m=n$

3.2 Algebraic Operations on O(m,n)-Trapezoidal Type Fuzzy Number

In this section we discuss various operations and state and prove certain properties of O(m, n)-Tr.T.F.N's.

Let $\tilde{A}$ and $\tilde{B}$ be two O(m, n)-Tr.T..F.N's, such that

$$\begin{aligned}\tilde{A} &= [a_1,a_2,a_3,a_4]_{(m,n)}\\ \tilde{B} &= [b_1,b_2,b_3,b_4]_{(m,n)}\end{aligned}$$

Then the $\alpha-$cut for each of $\tilde{A}$ and $\tilde{B}$ is defined as follows.

$$A(\alpha) = [a_1(\alpha),a_2(\alpha)]=[a_2-(a_2-a_1)(1-\alpha)^{\frac{1}{m}},a_3-(a_3-a_4)(1-\alpha)^{\frac{1}{n}}] \tag{3.2.1}$$

$$B(\alpha) = [b_1(\alpha),b_2(\alpha)]=[b_2-(b_2-b_1)(1-\alpha)^{\frac{1}{m}},b_3-(b_3-b_4)(1-\alpha)^{\frac{1}{n}}] \tag{3.2.2}$$

3.2.1 Addition $\tilde{A}(+)\tilde{B}$ of two O(m,n)-Tr.T.F.N.'s

Writing $\tilde{A}$ and $\tilde{B}$ in terms of their α-cuts, we have.

$$\begin{aligned}
A(\alpha)+B(\alpha) &= [a_1(\alpha),a_2(\alpha)]+[b_1(\alpha),b_2(\alpha)]\\
&= [a_2-(a_2-a_1)(1-\alpha)^{\frac{1}{m}},\ \ a_3-(a_3-a_4)(1-\alpha)^{\frac{1}{n}}]+\\
&\quad [b_2-(b_2-b_1)(1-\alpha)^{\frac{1}{m}},\ \ b_3-(b_3-b_4)(1-\alpha)^{\frac{1}{n}}]\\
&= [(a_2+b_2-((a_2+b_2)-(a_1+b_1))\,(1-\alpha)^{\frac{1}{m}},\\
&\quad a_3+b_3-((a_3+b_3)-(a_4+b_4))(1-\alpha)^{\frac{1}{n}}] \qquad (3.2.3)\\
&= [a_1(\alpha)+b_1(\alpha),a_2(\alpha)+b_2(\alpha)]\\
&= [c_2-(c_2-c_1)(1-\alpha)^{\frac{1}{m}},c_3-(c_3-c_4)(1-\alpha)^{\frac{1}{n}}]
\end{aligned}$$

where $c_1=a_1+b_1$, $c_2=a_2+b_2$, $c_3=a_3+b_3$, $c_4=a_4+b_4$.

The following observations are made on the addition of two O(m.n)-Tr.T.F.N's.

(a) Sum of two O(m.n)-Tr.T.F.N's is commutative, and the addition of two O(m.n)-Tr.T.F.N's results in an O(m.n)-Tr.T.F.N.

(b) If we set $m=1$ and $n=1$ in expression (3.2.3), we obtained the α-cut of a Tr.F.N. Thus, Tr.F.N is a special case of O(m.n)-Tr.T.F.N.

(c) Another important observation can be made is that if we let m and n get very large in (3.2.3),then $lim_{m\to\infty}(a_1(\alpha)+b_1(\alpha))=a_1+b_1$ and $\lim_{n\to\infty}(a_2(\alpha)+b_2(\alpha))=a_4+b_4$. This implies that when m and n get very large, the limits of the α-cut converge to the sum of their end points and do not depend on α.

(d) Similarly, it is obvious that when m and n get very small, the limits of the α-cut given by (3.2.3) converge to their interior points (a_2+b_2) and (a_3+b_3) respectively, and do not depend on α.

3.2.2 Membership Function of $\tilde{A} + \tilde{B}$

Consider two O(m,n)-Tr.T.F.N's $\tilde{A}$ and $\tilde{B}$ respectively. The sum of two O(m,n)-trapezoidal type fuzzy numbers is given by expression (3.2.3) and is as follows.

$$\begin{aligned} A(\alpha) + B(\alpha) &= [(a_2 + b_2 - ((a_2 + b_2) - (a_1 + b_1))(1-\alpha)^{\frac{1}{m}}, \\ &\qquad a_3 + b_3 - ((a_3 + b_3) - (a_4 + b_4))(1-\alpha)^{\frac{1}{n}}] \end{aligned}$$

In order to find the membership function for the fuzzy number $\tilde{AB}$, we need to find the interior points as well as the end points. To find the interior points, we set $\alpha = 1$ in expression (3.2.3), and to find the end points, we set $\alpha = 0$ in expression (3.2.3). Thus we have,

$$A(1) + B(1) = [a_2 + b_2,\ a_3 + b_3] \tag{3.2.4}$$

and

$$A(0) + B(0) = [a_1 + b_1,\ a_4 + b_4] \tag{3.2.5}$$

In view of (3.2.4) and (3.2.5), the fuzzy number $\tilde{A} + \tilde{B}$ can be written as

$\tilde{A} + \tilde{B} = [a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3,\ a_4 + b_4]_{O(m,n)}$

The α-cut of the left hand part of fuzzy number $A_1(\alpha) + B_1(\alpha)$ can be rewritten as follows:

$$A_1(\alpha) + B_1(\alpha) = a_2 + b_2 - ((a_2 + b_2) - (a_1 + b_1))(1-\alpha)^{\frac{1}{m}} \tag{3.2.6}$$

The α-cut of the right hand part of fuzzy number $\tilde{A} + \tilde{B}$ can be rewritten as follows,

$$A_2(\alpha) + B_2(\alpha) = a_3 + b_3 - ((a_3 + b_3) - (a_4 + b_4))(1-\alpha)^{\frac{1}{n}} \tag{3.2.7}$$

To find the membership function of $\tilde{A}+\tilde{B}$, we set

$A_1(\alpha)+B_1(\alpha)=x$ and $A_2(\alpha)+B_2(\alpha)=x$ and solve for α as follows.

$$a_2+b_2-((a_2+b_2)-(a_1+b_1))(1-\alpha)^{\frac{1}{m}} \quad = \quad x \tag{3.2.8}$$

Solving for α, we obtain the following expression

$$\alpha \quad = \quad 1-\left[\frac{(a_2+b_2)-x}{((a_2+b_2)-(a_1+b_1))}\right]^m \tag{3.2.9}$$

Similarly, if we set $A_2(\alpha)+B_2(\alpha)=x$, then we have

$$\alpha \quad = \quad 1-\left[\frac{(a_3+b_3)-x}{((a_3+b_3)-(a_4+b_4))}\right]^n \tag{3.2.10}$$

Using (3.2.9) and (3.2.10) we have the membership function for $\tilde{A}+\tilde{B}$

$$\mu_{A+B}(x)=\begin{cases} 0 & x\leq a_1+b_1 \\ 1-\left[\frac{(a_2+b_2)-x}{((a_2+b_2)-(a_1+b_1))}\right]^m & a_1+b_1\leq x\leq\ a_2+b_2 \\ 1 & a_2+b_2\leq x\leq\ a_3+b_3 \\ 1-\left[\frac{(a_3+b_3)-x}{((a_3+b_3)-(a_4+b_4))}\right]^n & a_3+b_3\leq x\leq a_4+b_4 \\ 0 & x\geq a_4+b_4 \end{cases}$$

Example 1 *In this example, we consider the addition of two O(m,n)-Tr.T.F.N's and discuss the effect on the membership function when m and n vary. Let $\tilde{A}$ and $\tilde{B}$ be two O(m,n)-Tr.T.F.N's, such that m and n ¿ 0 and*

$$\widetilde{A} \quad = \quad [a_1,a_2,a_3,a_4]_{(m,n)}=[1,5,6,9]_{(m,n)}$$

$$\widetilde{B} \quad = \quad [b_1,b_2,b_3,b_4]_{(m,n)}=[2,3,5,8]_{(m,n)}$$

Using expression (3.2.1) and (3.2.2), the α-cut for $\tilde{A}$ and $\tilde{B}$ are

$$A(\alpha) = [5 - 4(1-\alpha)^{\frac{1}{m}}, 6 + 3(1-\alpha)^{\frac{1}{n}}] \tag{3.2.11}$$

$$B(\alpha) = [3 - (1-\alpha)^{\frac{1}{m}}, 5 + 3(1-\alpha)^{\frac{1}{n}}] \tag{3.2.12}$$

Using expression (3.2.3), we write the sum of $\tilde{A}$ and $\tilde{B}$ as

$$A(\alpha) + B(\alpha) = [(8 - 5(1-\alpha)^{\frac{1}{m}}, 11 + 6(1-\alpha)^{\frac{1}{n}}]. \tag{3.2.13}$$

If we substitute different values of m and n in expression (3.2.13) we obtain the following tables and graphs.

Table 3.1: Addition of two O(m, n)-Tr.T.F.N's, where $m = 2, n = 2$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$A_1(\alpha) + B_1(\alpha)$	3.00	3.26	3.53	3.82	4.13	4.46	4.84	5.26	5.76	6.42	8.00
$A_2(\alpha) + B_2(\alpha)$	17.00	16.69	16.37	16.02	15.65	15.24	14.79	14.29	13.68	12.90	11.00

Table 3.2: Addition of two O(m, n)-Tr.T.F.N's, where $m = 0.5, n = 0.5$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$A_1(\alpha) + B_1(\alpha)$	3.00	3.95	4.80	5.55	6.20	6.75	7.20	7.55	7.80	7.95	8.00
$A_2(\alpha) + B_2(\alpha)$	17	15.86	14.84	13.94	13.16	12.5	11.96	11.54	11.24	11.06	11.00

Table 3.3: Addition of two O(m, n)-Tr.T.F.N's, where $m = 2, n = 0.5$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$A_1(\alpha) + B_1(\alpha)$	3.00	3.26	3.53	3.82	4.13	4.46	4.84	5.26	5.76	6.42	8.00
$A_2(\alpha) + B_2(\alpha)$	17.00	15.86	14.84	13.94	13.16	12.50	11.96	11.54	11.24	11.06	11.00

Table 3.4: Addition of two O(m, n)-Tr.T.F.N's, where $m = 0.5, n = 2$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$A_1(\alpha) + B_1(\alpha)$	3	3.95	4.80	5.55	6.20	6.75	7.20	7.55	7.80	7.95	8.00
$A_2(\alpha) + B_2(\alpha)$	17.00	16.69	16.37	16.02	15.65	15.24	14.79	14.29	13.68	12.90	11.00

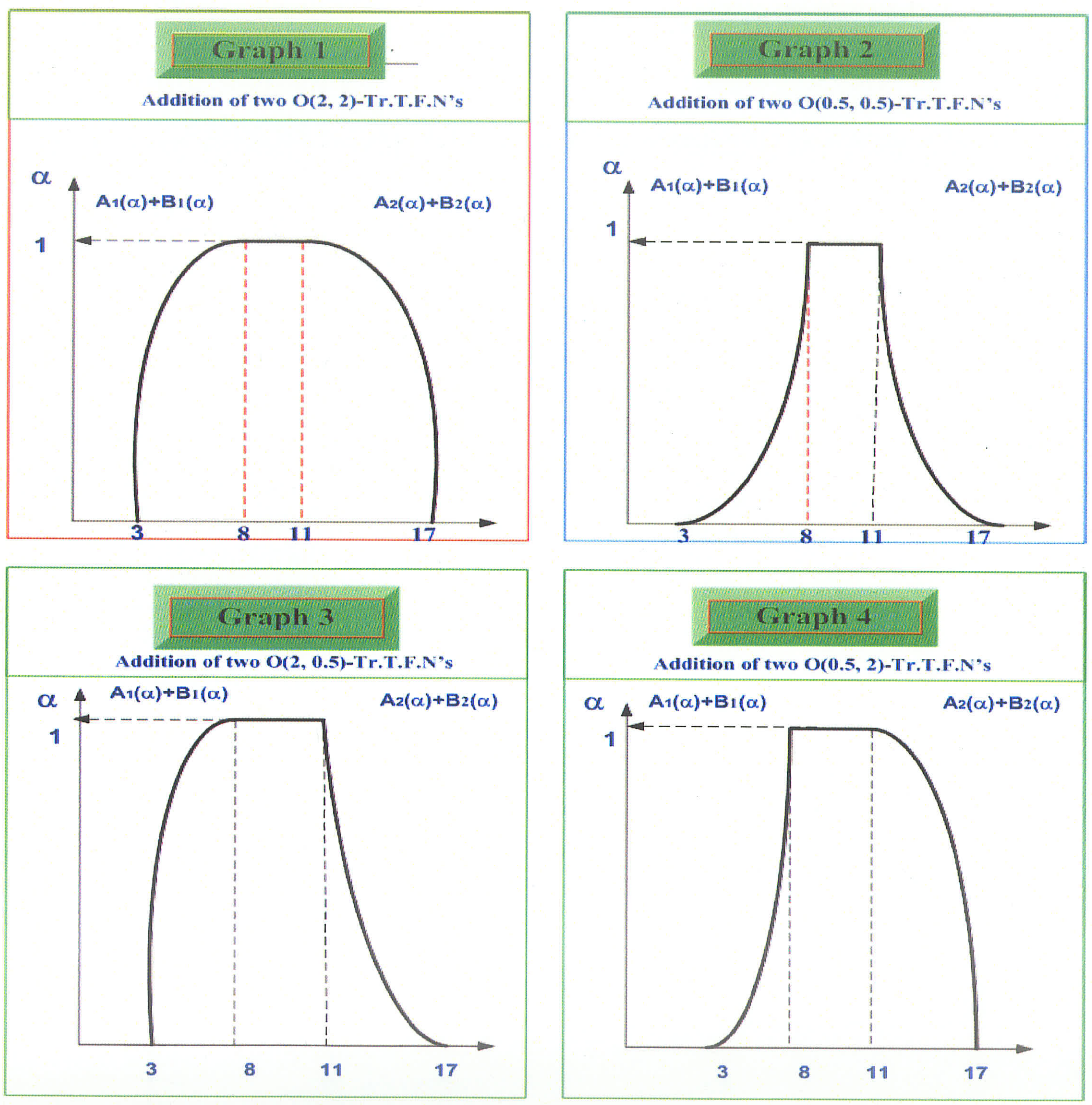

Figure 3.1: Addition of two O(,)-Tr.T.F.N's.

We now make the following observations.

(a) When $m = 2$ and $n = 2$, Graph 1 bulges outward and is often interpreted in linguistic term as "more or less" or "dilatation" (or concave). This means that we do dot have much confidence in the middle portion of the membership function.

(b) When $m = 0.5$ and $n = 0.5$, Graph 2 bulges inward and is often interpreted in linguistic term as "very" or "concentration" (or convex). This mean that we have more confidence in the middle portion of the membership function than we do at the end points.

(c) When $m = 2$ and $n = 0.5$, the shape Graph 3 of the membership function is different from those describe in (a) and (b). It bulges outward on the left hand side and bulges inward on the right hand side. In linguistic terms this can be interpreted as a mixture of "concentration" (or convex) and "dilatation" (or concave). This mean that we are "more or less" satisfied on the left hand side of the membership function and "very" satisfied on the right hand side.

(d) When $m = 0.5$ and $n = 2$, the shape of Graph 4 of the membership function is different from (a), (b) or (c). In this case it bulges inward on the left hand side of the membership function and bulges outward on the right hand side. In linguistic terms this is interpreted as a mixture of "concentration" (or convex) and "dilatation" (or concave). This means that we are very satisfied on the left hand side of the membership function and "more or less" satisfied on the right hand side.

Thus, we observe that due to the flexibility that is associated with O(m,n)-Tr.T.F.N's, the left hand side of the membership function can be estimated independently from the right hand side. In actual practice, users choose the shape of the membership functions from a collection of commonly used membership functions including triangular and trapezoidal membership functions. The added benefit of O(m,n)-Tr.T.F.N's is that the parameters m and n can be easily manipulated to tune and adjust the shape, without having to change the supports or the cores of the membership function. Thus, O(m,n)-Tr.T.F.N's can be used to produce membership function for a number of imprecise concepts.

3.2.3 Difference $\tilde{A}(-)\tilde{B}$ of two O(m,n)-Tr.T.F.N's

Writing $\tilde{A}$ and $\tilde{B}$ in terms of their α-cuts, we have,

$$\begin{aligned} A(\alpha) - B(\alpha) &= [a_1(\alpha), a_2(\alpha)] + [-b_2(\alpha), -b_1(\alpha)] \\ &= [a_1(\alpha) - b_2(\alpha),\ a_2(\alpha) - b_1(\alpha)] \\ &= [a_2 - (a_2 - a_1)(1-\alpha)^{\frac{1}{m}}, a_3 - (a_3 - a_4)(1-\alpha)^{\frac{1}{n}}] + \\ &\quad [-b_3 + (b_3 - b_4)(1-\alpha)^{\frac{1}{n}}, -b_2 + (b_2 - b_1)(1-\alpha)^{\frac{1}{m}}] \\ &= [(a_2 - b_3 - ((a_2 - a_1)(1-\alpha)^{\frac{1}{m}}) - (b_3 - b_4)(1-\alpha)^{\frac{1}{n}})), \\ &\quad (a_3 - b_2 - ((a_3 - a_4)(1-\alpha)^{\frac{1}{n}} - (b_2 - b_1)(1-\alpha)^{\frac{1}{m}}))] \end{aligned} \quad (3.2.14)$$

If we set $m = n$ in expression (3.2.14), we obtain the following results.

$$\begin{aligned} A(\alpha) - B(\alpha) &= [(a_2 - b_3 - ((a_2 - a_1) - (b_3 - b_4))(1-\alpha)^{\frac{1}{m}}, \\ &\quad (a_3 - b_2 - ((a_3 - a_4) - (b_2 - b_1))(1-\alpha)^{\frac{1}{m}}] \qquad (3.2.15) \\ &= [d_2 - (d_2 - d_1)(1-\alpha)^{\frac{1}{m}}, d_3 - (d_3 - d_4)(1-\alpha)^{\frac{1}{m}}] \end{aligned}$$

where, $d_1 = (a_1 - b_4), \;\; d_2 = (a_2 - b_3), \;\; d_3 = (a_3 - b_2), \;\; d_4 = (a_4 - b_1)$.

In this case we make the following observations.

(a) Note that α-cut given by (3.2.14) is not the α-cut of a O(m,n)-Tr.T.F.N's. Thus, we conclude that the difference of two O(m,n)-Tr.T.F.N's does not yield an O(m,n)-Tr.T.F.N.

(b) If we set $m = 1$ and $n = 1$ in expression (3.2.14), we obtain the α-cut of a Tr.F.N.

(c) If we set $m = n$ in (3.2.14), then $\tilde{A} - \tilde{B}$ yields an O(m,m)-Tr.T.F.N.

(d) If we let m and n become large in (3.2.14), then we have

$\lim_{m,n\to\infty}(a_2 - b_3 - ((a_2 - a_1)(1-\alpha)^{\frac{1}{m}}) - (b_3 - b_4)(1-\alpha)^{\frac{1}{n}})) = a_1 - b_4$

$\lim_{m,n\to\infty}(a_3 - b_2 - ((a_3 - a_4)(1-\alpha)^{\frac{1}{n}}) - (b_2 - b_1)(1-\alpha)^{\frac{1}{m}})) = a_4 - b_1$

When m and n get very large, the limits of the α-cut converge to the difference of their end points and do not depend on α.

(e) Similarly, it is obvious that when m and n get very small, the limits of the α-cut given by (3.2.14) converge to $a_3 - b_2$ and $a_2 - b_3$ and do not depend on α.

In order to illustrate the difference of two O(m,n)-Tr.T.F.N's, we consider the following example.

Example 2 *Consider the same two O(m,n)-Tr.T.F.N's of Example 1 Using expression (3.2.1) and (3.2.2), the α-cut for fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are given by expression (3.2.11) and (3.2.12) respectively.*

Using expression(3.2.14), we write the substraction of those two fuzzy numbers $\tilde{A}$ and

$\tilde{B}$ *as follows.*

$$
\begin{aligned}
A(\alpha) - B(\alpha) &= [(a_2 - b_3 - ((a_2 - a_1)(1-\alpha)^{\frac{1}{m}}) - (b_3 - b_4)(1-\alpha)^{\frac{1}{n}})), \\
&\quad (a_3 - b_2 - ((a_3 - a_4)(1-\alpha)^{\frac{1}{n}} - (b_2 - b_1)(1-\alpha)^{\frac{1}{m}}))] \\
&= [(5 - 5 - ((5-1)(1-\alpha)^{\frac{1}{m}}) - (5-8)(1-\alpha)^{\frac{1}{n}})), \\
&\quad (6 - 3 - ((6-9)(1-\alpha)^{\frac{1}{n}} - (3-2)(1-\alpha)^{\frac{1}{m}}))] \\
&= [-4(1-\alpha)^{\frac{1}{m}} - 3(1-\alpha)^{\frac{1}{n}}, 3 + 3(1-\alpha)^{\frac{1}{n}} + (1-\alpha)^{\frac{1}{m}}] \quad (3.2.16)
\end{aligned}
$$

If we substitute different values of m and n in expression (3.2.16) we obtain the following tables and graphs.

Table 3.5: Subtraction of two O(m,n)-Tr.T.F.N's, where $m = 2$, $n = 2$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$A_1(\alpha) - B_1(\alpha)$	-7.00	-6.64	-6.26	-5.86	-5.42	-4.95	-4.43	-3.83	-3.13	-2.21	0.00
$A_2(\alpha) - B_2(\alpha)$	9.00	8.69	8.37	8.02	7.65	7.24	6.79	6.29	5.68	4.90	3.00

Table 3.6: Subtraction of two O(m,n)-Tr.T.F.N's, where $m = 0.5$, $n = 0.5$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$A_1(\alpha) - B_1(\alpha)$	-7.00	-5.67	-4.48	-3.43	-2.52	-1.75	-1.12	-0.63	-0.28	-0.07	0.00
$A_2(\alpha) - B_2(\alpha)$	9.00	7.86	6.84	5.94	5.16	4.5	3.96	3.54	3.24	3.06	3.00

Table 3.7: Subtraction of two O(m,n)-Tr.T.F.N's, where $m = 2$, $n = 0.5$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$A_1(\alpha) - B_1(\alpha)$	-7.00	-6.22	-5.50	-4.82	-4.18	-3.58	-3.01	-2.46	-1.91	-1.29	0.00
$A_2(\alpha) - B_2(\alpha)$	9.00	8.28	7.60	6.98	6.40	5.87	5.38	4.91	4.46	3.98	3.00

Table 3.8: Subtraction of two O(m,n)-Tr.T.F.N's, where $m = 0.5$, $n = 2$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$A_1(\alpha) - B_1(\alpha)$	-7.00	-6.09	-5.24	-4.47	-3.76	-3.12	-2.54	-2.00	-1.50	-0.99	0.00
$A_2(\alpha) - B_2(\alpha)$	9.00	8.28	7.60	6.98	6.40	5.87	5.38	4.91	4.46	3.98	3.00

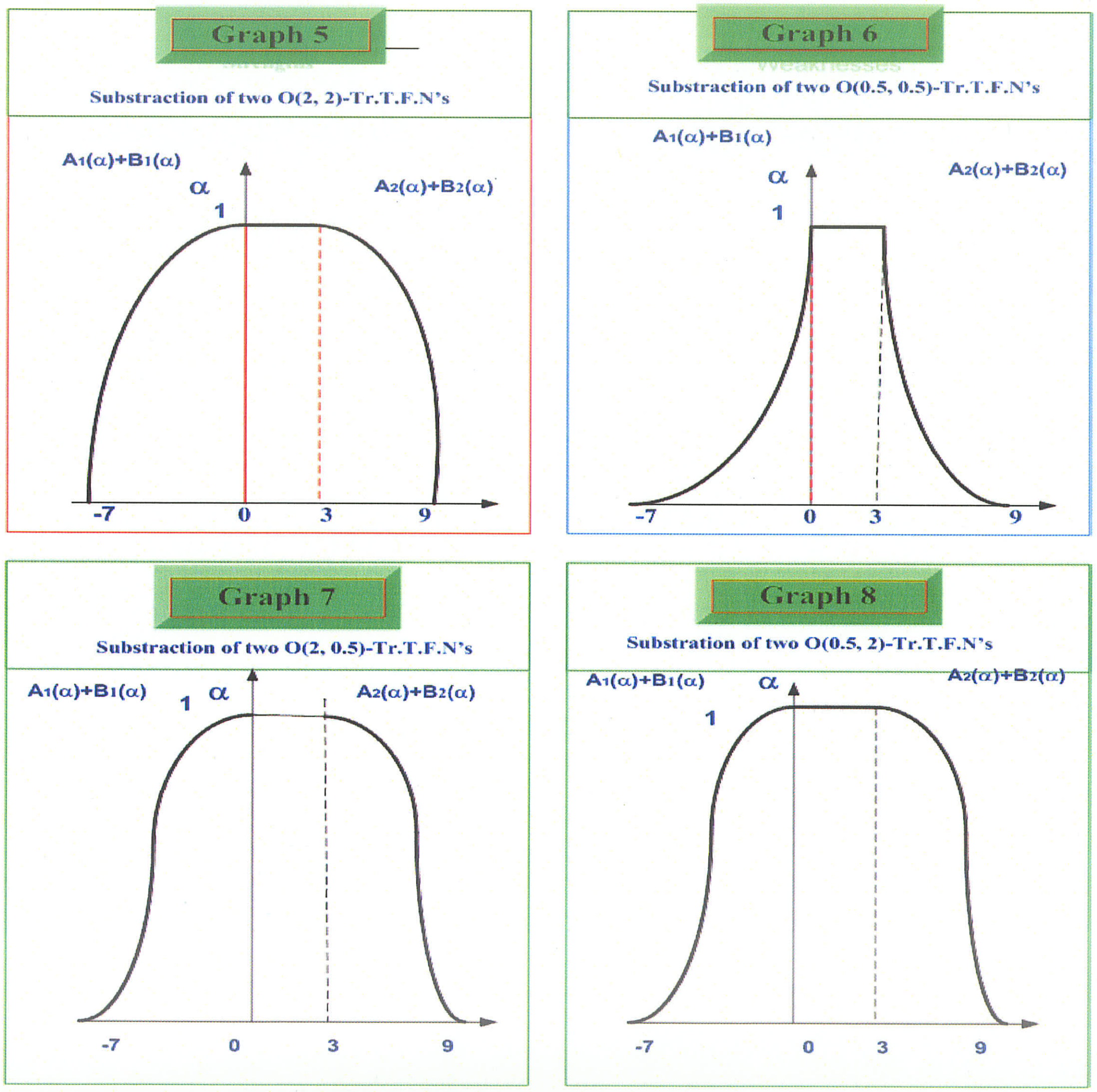

Figure 3.2: Substraction of two O(,)-Tr.T.F.N's.

We now make the following observations.

(a) When $m = 2$ and $n = 2$, the graph bulges outward and is often interpreted in linguistic term as "more or less" or "dilatation" (or concave). The form of the membership function is very similar to the membership function of the addition of two O(m,n)-Tr.T.F.N's.

(b) When $m = 0.5$ and $n = 0.5$, the graph bulges inward and is often interpreted in linguistic term as "very" or "concentration" (or convex). This mean we have more confidence in the middle portion of the membership function than at its end points.

(c) When $m = 2$ and $n = 0.5$, the shape of the graph is different to those describe in (a) and (b). From (3.2.16), if we consider the left hand side of the membership function, we have,

$$\frac{d^2}{d\alpha^2}(-4\sqrt{1-\alpha} - 3(1-\alpha)^2) = -\frac{1 - 6\sqrt{(1-\alpha)} + 6\alpha\sqrt{(1-\alpha)}}{(-1+\alpha)\sqrt{(1-\alpha)}}$$

$\frac{d^2}{d\alpha^2}(-4\sqrt{1-\alpha}-3(1-\alpha)^2) = 0$, yields $\alpha = 0.69$. When $\alpha > 0.69$, $\frac{d^2}{d\alpha^2}(-4\sqrt{1-\alpha}-3(1-\alpha)^2) > 0$, else $\frac{d^2}{d\alpha^2}(-4\sqrt{1-\alpha}-3(1-\alpha)^2) < 0$. Thus, in this case a change in the direction of concavity results in an inflection point.

Similarly, from (3.2.16), if we consider the right hand side of the membership function, we have,

$$\frac{d^2}{d\alpha^2}(3 + 3(1-\alpha)^2 + \sqrt{1-\alpha}) = \frac{1}{4}\frac{-24\sqrt{(1-\alpha)} + 24\alpha\sqrt{(1-\alpha)} + 1}{(-1+\alpha)\sqrt{(1-\alpha)}}$$

$\frac{d^2}{d\alpha^2}(3 + 3(1-\alpha)^2 + \sqrt{1-\alpha}) = 0$, yields $\alpha = 0.87$. When $\alpha > 0.87$, $\frac{d^2}{d\alpha^2}(3 + 3(1-\alpha)^2 + \sqrt{1-\alpha}) < 0$, else $\frac{d^2}{d\alpha^2}(3 + 3(1-\alpha)^2 + \sqrt{1-\alpha}) > 0$. Since, there is a change in the direction of concavity, we conclude that there is an inflection point at $\alpha = 0.87$.

(d) When $m = 2$ and $n = 0.5$, the shape of the graph is different from those described in (c). From (3.2.16), if we consider the left hand side of the membership function, we have,

$$\frac{d^2}{d\alpha^2}(-4\,(1-\alpha)^2 - 3\sqrt{1-\alpha}) \quad = \quad -\frac{1}{4}\frac{-32\sqrt{(1-\alpha)} + 32\alpha\sqrt{(1-\alpha)} + 3}{(-1+\alpha)\sqrt{(1-\alpha)}}$$

$\frac{d^2}{d\alpha^2}(-4\,(1-\alpha)^2 - 3\sqrt{1-\alpha}) = 0$, yields $\alpha = 0.79$.
$\frac{d^2}{d\alpha^2}(-4\,(1-\alpha)^2 - 3\sqrt{1-\alpha}) > 0$, when $\alpha > 0.79$ and $\frac{d^2}{d\alpha^2}(-4\,(1-\alpha)^2 - 3\sqrt{1-\alpha}) < 0$, for $\alpha < 0.79$. Since there is a change in the direction of concavity, we conclude that there is a point of inflection at $\alpha = 0.79$. Similarly, if we consider the right hand side of the membership function of (3.2.16), it can easily be shown that there is a point of inflection at $\alpha = 0.48$.

Thus, O(m,n)-Tr.T.F.N's allow us to introduce fuzziness at various places in the membership function. After a particular value of m and n have been selected, the user can still go back and fine tune the membership. This added interaction and flexibility associated in the use of O(m,n)-Tr.T.F.N's are desirable features when designing imprecise concepts.

3.2.4 Multiplication $\tilde{A}(.)\tilde{B}$ of two O(m,n)-Tr.T.F.N's

Writing $\tilde{A}$ and $\tilde{B}$ in terms of their α-cuts, we have,

$$\begin{aligned}
A(\alpha)(.)B(\alpha) &= [a_1(\alpha), a_2(\alpha)](.)[b_1(\alpha), b_2(\alpha)] \\
&= [a_1(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)] \\
&= [a_2 - (a_2 - a_1)(1-\alpha)^{\frac{1}{m}}, a_3 - (a_3 - a_4)(1-\alpha)^{\frac{1}{n}}](.) \\
&\quad [b_2 - (b_2 - b_1)(1-\alpha)^{\frac{1}{m}}, b_3 - (b_3 - b_4)(1-\alpha)^{\frac{1}{n}}] \\
&= [(a_2 - (a_2 - a_1)(1-\alpha)^{\frac{1}{m}})(b_2 - (b_2 - b_1)(1-\alpha)^{\frac{1}{m}}), \\
&\quad (a_3 - (a_3 - a_4)(1-\alpha)^{\frac{1}{n}})(b_3 - (b_3 - b_4)(1-\alpha)^{\frac{1}{n}})]
\end{aligned}$$

Thus,

$$\begin{aligned}
A(\alpha)(.)B(\alpha) &= [a_2b_2 + (1-\alpha)^{\frac{1}{m}}(a_2b_1 - 2a_2b_2 + a_1b_2) + (1-\alpha)^{\frac{2}{m}}(a_2 - a_1)(b_2 - b_1), \\
&\quad a_3b_3 + (1-\alpha)^{\frac{1}{n}}(a_4b_3 - 2a_3b_3 + a_3b_4) + (1-\alpha)^{\frac{2}{n}}(a_3 - a_4)(b_3 - b_4)]
\end{aligned} \tag{3.2.17}$$

Note that the product of two O(m,n)-Tr.T.F.N's is not an O(m,n)-Tr.T.F.N.

If we set values of $m = n = 1$ in (3.2.17), we obtain the results for the product of two trapezoidal fuzzy numbers as follows,

$$\begin{aligned}
A(\alpha)(.)B(\alpha) &= [a_1b_1 + \alpha(a_2b_1 + a_1b_2 - 2a_1b_1) + \alpha^2(a_2b_2 - a_2b_1 - a_1b_2 + a_1b_1), \\
&\quad a_4b_4 + \alpha(a_4b_3 + a_3b_4 - 2a_4b_4) + \alpha^2(a_3b_3 - a_3b_4 - a_4b_3 + a_4b_4)].
\end{aligned} \tag{3.2.18}$$

We now make the following observations on the product of two O(m.n)-Tr.T.F.N's.

(a) Note that the α-cut given by (3.2.17) is not the α-cut of an O(m,n)-Tr.T.F.N's. Thus, the product of two O(m,n)-Tr.T.F.N's is not an O(m,n)-Tr.T.F.N.

(b) If we set $m = 1$ and $n = 1$ in expression (3.2.14), we obtain the α-cut of a Tr.F.N.

(c) On the other hand, if we set $m = n$ in (3.2.17), then $\tilde{A}(.)\tilde{B}$ yields an O(m,m)-Tr.T.F.N.

(d) From (3.2.14)we have

$\lim_{m\to\infty} a_2b_2 + (1-\alpha)^{\frac{1}{m}}(a_2b_1 - 2a_2b_2 + a_1b_2) + (1-\alpha)^{\frac{2}{m}}(a_2 - a_1)(b_2 - b_1) = a_1b_1$

$\lim_{n\to\infty} a_3b_3 + (1-\alpha)^{\frac{1}{n}}(a_4b_3 - 2a_3b_3 + a_3b_4) + (1-\alpha)^{\frac{2}{n}}(a_3 - a_4)(b_3 - b_4) = a_4b_4$

When m and n get very large, the limits of the α-cut converge to the product of their end points and do not depend on α.

(e) Similarly, it is obvious that when m and n get very small, the limits of the α-cut given by (3.2.17) converge to their interior points a_2b_2 and a_3b_3 and do not depend on α.

Note that the α-cut given by (3.2.17) is not linear. However, under appropriate assumptions the α-cut given by (3.2.17) can be approximated by linear membership function.

3.2.5 Membership Function of $\tilde{A}(.)\tilde{B}$.

Let $\tilde{A}$ and $\tilde{B}$ be two O(m,n)-Tr.T.F.N's. Using $\tilde{A}$ and $\tilde{B}$ respectively. The product of two O(m,n)-trapezoidal type fuzzy numbers is given by expression (3.2.17) and is as follows.

$$\begin{aligned} A(\alpha)(.)B(\alpha) &= [a_2b_2 + (1-\alpha)^{\frac{1}{m}}(a_2b_1 - 2a_2b_2 + a_1b_2) + (1-\alpha)^{\frac{2}{m}}(a_2 - a_1)(b_2 - b_1), \\ &\quad a_3b_3 + (1-\alpha)^{\frac{1}{n}}(a_4b_3 - 2a_3b_3 + a_3b_4) + (1-\alpha)^{\frac{2}{n}}(a_3 - a_4)(b_3 - b_4)] \end{aligned}$$

In order to find the membership function for the fuzzy number $\widetilde{AB}$, we need to find the interior points as well as the end points. To find the interior points, we set $\alpha = 1$ in expression (3.2.17), and to find the end points, we set $\alpha = 0$ in expression (3.2.17). Thus we have,

$$A(1)(.)B(1) = [a_2b_2,\ a_3b_3] \tag{3.2.19}$$

and

$$\begin{aligned} A(0)(.)B(0) &= [a_2b_2 + (a_2b_1 - 2a_2b_2 + a_1b_2) + (a_2 - a_1)(b_2 - b_1), \\ &\quad a_3b_3 + (a_4b_3 - 2a_3b_3 + a_3b_4) + (a_3 - a_4)(b_3 - b_4)] \\ &= [a_1b_1, a_4b_4] \end{aligned} \tag{3.2.20}$$

In view of (3.2.20) and (3.2.19), the fuzzy number $\tilde{A}(.)\tilde{B}$ can be written as

$\tilde{A}(.)\tilde{B} = [a_1b_1, a_2b_2,\ a_3b_3, a_4b_4]$

The α-cut of the left hand part of fuzzy number $\tilde{A}(.)\tilde{B}$ can be rewritten as follows:

$$\begin{aligned} A_1(\alpha)B_1(\alpha) &= (a_2 - a_1)(b_2 - b_1)\left((1-\alpha)^{\frac{1}{m}}\right)^2 + \\ &\quad (a_2b_1 - 2a_2b_2 + a_1b_2)\left((1-\alpha)^{\frac{1}{m}}\right) + a_2b_2 \end{aligned} \tag{3.2.21}$$

The α-cut in (3.2.17) of the right hand part of fuzzy number $\widetilde{AB}$ can be rewritten as follows,

$$\begin{aligned} A_2(\alpha)B_2(\alpha) &= (a_3 - a_4)(b_3 - b_4)\left((1-\alpha)^{\frac{1}{n}}\right)^2 + \\ &\quad (a_4b_3 - 2a_3b_3 + a_3b_4)\left((1-\alpha)^{\frac{1}{n}}\right) + a_3b_3 \end{aligned} \tag{3.2.22}$$

Note that (3.2.21) and (3.2.22) have a quadratic structure in $(1-\alpha)^{\frac{1}{m}}$ and $(1-\alpha)^{\frac{1}{n}}$. To find the membership function, we set

$A_1(\alpha)B_1(\alpha) = x$ and $A_2(\alpha)B_2(\alpha) = x$ and solve for α as follows.

$$\begin{aligned}
(a_2 - a_1)(b_2 - b_1)\left((1-\alpha)^{\frac{1}{m}}\right)^2 + (a_2b_1 - 2a_2b_2 + a_1b_2)\left((1-\alpha)^{\frac{1}{m}}\right) + a_2b_2 &= x \\
(a_2 - a_1)(b_2 - b_1)\left((1-\alpha)^{\frac{1}{m}}\right)^2 + (a_2b_1 - 2a_2b_2 + a_1b_2)\left((1-\alpha)^{\frac{1}{m}}\right) + a_2b_2 - x &= 0
\end{aligned} \tag{3.2.23}$$

Solving for $(1-\alpha)^{\frac{1}{m}}$

$$(1-\alpha)^{\frac{1}{m}} = \frac{-(a_2b_1 - 2a_2b_2 + a_1b_2) + \sqrt{(a_2b_1 - a_1b_2)^2 + 4(a_2 - a_1)(b_2 - b_1)x)}}{2(a_2 - a_1)(b_2 - b_1)}$$

$$(1-\alpha) = \left[\frac{-(a_2b_1 - 2a_2b_2 + a_1b_2) + \sqrt{(a_2b_1 - a_1b_2)^2 + 4(a_2 - a_1)(b_2 - b_1)x)}}{2(a_2 - a_1)(b_2 - b_1)}\right]^m$$

$$\alpha = 1 - \left[\frac{-(a_2b_1 - 2a_2b_2 + a_1b_2) + \sqrt{(a_2b_1 - a_1b_2)^2 + 4(a_2 - a_1)(b_2 - b_1)x)}}{2(a_2 - a_1)(b_2 - b_1)}\right]^m \tag{3.2.24}$$

Similarly, if we set $A_2(\alpha)B_2(\alpha) = x$, then we have

$$\alpha = 1 - \left[\frac{-(a_4b_3 - 2a_3b_3 + a_3b_4) + \sqrt{(a_4b_3 - a_3b_4)^2 + 4\,(a_3 - a_4)(b_3 - b_4)x}}{2\,(a_3 - a_4)(b_3 - b_4)}\right] \tag{3.2.25}$$

Using (3.2.24) and (3.2.25) we have the membership function for $\tilde{A}(.)\tilde{B}$.

$$\mu_{AB}(x) = \begin{cases}
0 & x \le a_1b_1 \\
1 - \left[\frac{-(a_2b_1 - 2a_2b_2 + a_1b_2) + \sqrt{(a_2b_1 - a_1b_2)^2 + 4(a_2 - a_1)(b_2 - b_1)x)}}{2(a_2 - a_1)(b_2 - b_1)}\right]^m & a_1b_1 \le x \le a_2b_2 \\
1 & a_2b_2 \le x \le a_3b_3 \\
1 - \left[\frac{-(a_4b_3 - 2a_3b_3 + a_3b_4) + \sqrt{(a_4b_3 - a_3b_4)^2 + 4\,(a_3 - a_4)(b_3 - b_4)x}}{2\,(a_3 - a_4)(b_3 - b_4)}\right]^n & a_3b_3 \le x \le a_4b_4 \\
0 & x \ge a_4b_4
\end{cases}$$

Example 3 *Let $\tilde{A}$ and $\tilde{B}$ be two O(m,n)-Tr.T.F.N's. Using expression (3.2.1), (3.2.2) and (3.2.17), we have*

$$A(\alpha)(.)B(\alpha) = [15 - 17\,(1-\alpha)^{\frac{1}{m}} + 4\,(1-\alpha)^{\frac{2}{m}}\,,\; 30 + 33\,(1-\alpha)^{\frac{1}{n}} + 9\,(1-\alpha)^{\frac{2}{n}}] \tag{3.2.26}$$

For different values of m and n in expression (3.2.26) we obtain the following tables and graphs.

Table 3.9: Multiplication of two O(m,n)-Tr.T.F.N's, where $m = 2$, $n = 2$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$A_1(\alpha)(*)B_1(\alpha)$	2.00	2.47	2.99	3.58	4.23	4.98	5.85	6.89	8.20	10.02	15.00
$A_2(\alpha)(*)B_2(\alpha)$	72.00	69.41	66.72	63.91	60.96	57.83	54.47	50.77	6.56	41.34	30.00

Table 3.10: Multiplication of two O(m,n)-Tr.T.F.N's, where $m = 0.5$, $n = 0.5$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$A_1(\alpha)(*)B_1(\alpha)$	2.00	3.85	5.76	7.63	9.40	11.00	12.38	13.50	14.33	14.83	15.00
$A_2(\alpha)(*)B_2(\alpha)$	72.00	62.63	54.81	48.33	43.05	38.81	35.51	33.04	31.33	30.33	30.00

Table 3.11: Multiplication of two O(m,n)-Tr.T.F.N's, where $m = 2$, $n = 0.5$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$A_1(\alpha)(*)B_1(\alpha)$	2.00	2.47	2.99	3.58	4.23	4.98	5.85	6.89	8.20	10.02	15.00
$A_2(\alpha)(*)B_2(\alpha)$	72.00	62.63	54.81	48.33	43.05	38.81	35.51	33.04	31.33	30.33	30.00

Table 3.12: Multiplication of two O(m,n)-Tr.T.F.N's, where $m = 0.5$, $n = 2$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$A_1(\alpha)(*)B_1(\alpha)$	2.00	3.85	5.76	7.63	9.40	11.00	12.38	13.50	14.33	14.83	15.00
$A_2(\alpha)(*)B_2(\alpha)$	72.00	69.41	66.72	63.91	60.96	57.83	54.47	50.77	46.56	41.34	30.00

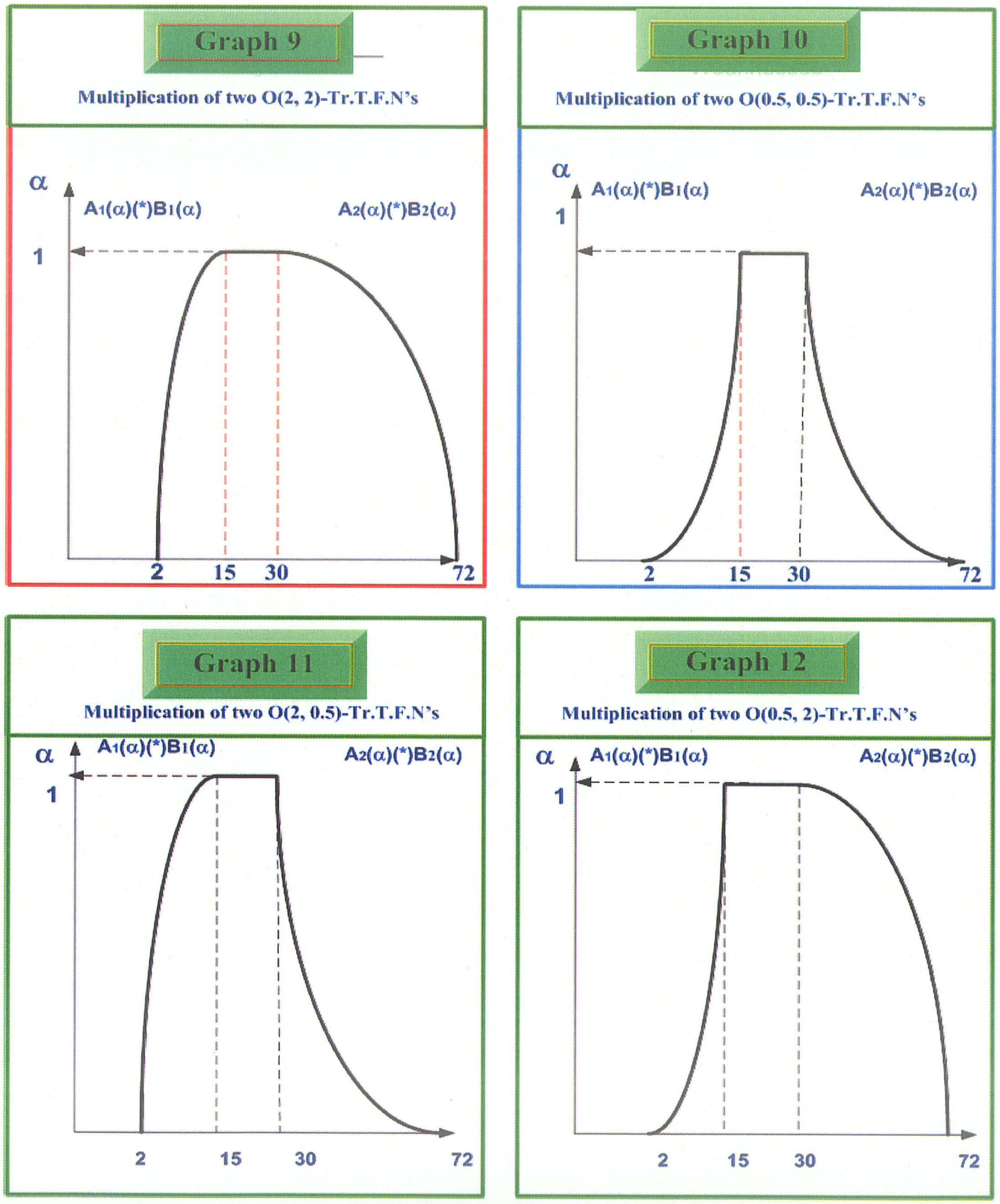

Figure 3.3: Multilication of two O(,)-Tr.T.F.N's.

We make the following observations on the product of two O(m,n)-Tr.T.F.N's.

(a) When $m = 2$ and $n = 2$, the graph bulges outward and is often interpreted in linguistic term as "more or less" or "dilatation" (or concave).

(b) When $m = 0.5$ and $n = 0.5$, the graph bulges inward and is often interpreted in linguistic term as "very" or "concentration" (or convex). We have more confidence in the middle portion of the membership function than we do at the end points.

(c) When $m = 2$ and $n = 0.5$, the shape of the graph is different from those describe in (b) and (c). The graph bulges outward on the left hand side of the membership function and bulges inward on the right hand side of the membership function. In linguistic terms this is interpreted as a mixture of "dilatation" (or concave) and "concentration" (or convex).

(d) When $m = 0.5$ and $n = 2$, the shape of the graph is different from those described in (b) or (c). The graph shrinks inward on the left hand side of the membership function and bulges outward on the right hand side of the membership function. In linguistic terms this is interpreted as a mixture of "dilatation" (or concave) and "concentration" (or convex).

O(m,n)-Tr.T.F.N's are well suited for a broad spectrum of fuzzy financial modeling. The shape of the membership functions can be modified by varying the values of m and n. Furthermore, O(m,n)-Tr.T.F.N's are more flexible to describe the vagueness in the fuzzy parameters for the fuzzy binomial option pricing model considered in Chapter 5.

3.2.6 Division $\tilde{A}(:)\tilde{B}$ of two O(m,n)-Tr.T.F.N 's

Writing $\tilde{A}$ and $\tilde{B}$ in terms of their α-cuts, we have,

$$(\tilde{A}(\alpha)(:)\tilde{B}(\alpha)) = [a_1(\alpha), a_2(\alpha)](.)\left[\frac{1}{b_2(\alpha)}, \frac{1}{b_1(\alpha)}\right] = \left[\frac{a_1(\alpha)}{b_2(\alpha)}, \frac{a_2(\alpha)}{b_1(\alpha)}\right]$$

$$= \left[\frac{a_2-(a_2-a_1)(1-\alpha)^{\frac{1}{m}}}{b_3-(b_3-b_4)(1-\alpha)^{\frac{1}{n}}}, \frac{a_3-(a_3-a_4)(1-\alpha)^{\frac{1}{n}}}{b_2-(b_2-b_1)(1-\alpha)^{\frac{1}{m}}}\right] \quad (3.2.27)$$

If we set $m = n = 1$ in expression (3.2.27), we obtain the following expression

$$(\tilde{A}(\alpha)(:)\tilde{B}(\alpha)) = \left[\frac{a_1+(a_2-a_1)\alpha}{b_4+(b_3-b_4)\alpha}, \frac{a_4+(a_3-a_4)\alpha}{b_1+(b_2-b_1)\alpha}\right] \quad (3.2.28)$$

We now make the following observations on the division of two O(m.n)-Tr.T.F.N's:

(a) The expression given by (3.2.27) is not an O(m,n)-Tr.T.F.N. However, (3.2.27) can be approximated by a linear shape fuzzy numbers under certain assumptions.

(b) If we set $m = 1$ and $n = 1$ in expression (3.2.27), we obtain the α-cut of the division of two Tr.T.F.N's.

(c) Another important observation that can be made is that if we let m and n get very large in (3.2.27), then

$$\lim_{m,n\to\infty} \frac{a_2-(a_2-a_1)(1-\alpha)^{\frac{1}{m}}}{b_3-(b_3-b_4)(1-\alpha)^{\frac{1}{n}}} = \frac{a_1}{b_4}$$

and

$$\lim_{m,n\to\infty} \frac{a_3-(a_3-a_4)(1-\alpha)^{\frac{1}{n}}}{b_2-(b_2-b_1)(1-\alpha)^{\frac{1}{m}}} = \frac{a_4}{b_1}$$

When m and n get very large, the limits of the α-cut converge to the division of their end points and do not depend on α.

(d) Similarly, it is obvious that when m and n get very small, the limits of the α-cut given by (3.2.27) converge to $\frac{a_2}{b_3}$ and $\frac{a_3}{b_2}$ respectively, and do not depend on α.

Example 4 *Let $\tilde{A}$ and $\tilde{B}$ be two O(m,n)-Tr.T.F.N's, then*

$$(\tilde{A}(\alpha)(:)\tilde{B}(\alpha)) = \left[\frac{5-4(1-\alpha)^{\frac{1}{m}}}{5+3(1-\alpha)^{\frac{1}{n}}}, \frac{6+3(1-\alpha)^{\frac{1}{n}}}{3-(1-\alpha)^{\frac{1}{m}}}\right] \tag{3.2.29}$$

For different values of m and n in expression (3.2.29) we obtain the following Tables and graphs.

Table 3.13: Division of two O(m,n)-Tr.T.F.N's, where $m = 2$, $n = 2$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$A_1(\alpha)(*)B_1(\alpha)$	0.13	0.15	0.19	0.22	0.26	0.30	0.36	0.42	0.51	0.63	1.00
$A_2(\alpha)(*)B_2(\alpha)$	4.50	4.31	4.12	3.93	3.74	3.54	3.34	3.12	2.88	2.59	2.00

Table 3.14: Division of two O(m,n)-Tr.T.F.N's, where $m = 0.5$, $n = 0.5$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$A_1(\alpha)(*)B_1(\alpha)$	0.13	0.24	0.35	0.47	0.59	0.70	0.80	0.88	0.95	0.99	1.00
$A_2(\alpha)(*)B_2(\alpha)$	4.50	3.85	3.36	2.98	2.68	2.45	2.28	2.15	2.07	2.02	2.00

Table 3.15: Division of two O(m,n)-Tr.T.F.N's, where $m = 2$, $n = 0.5$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$A_1(\alpha)(*)B_1(\alpha)$	0.13	0.16	0.21	0.26	0.31	0.38	0.45	0.53	0.63	0.74	1.00
$A_2(\alpha)(*)B_2(\alpha)$	4.50	4.11	3.76	3.45	3.18	2.94	2.74	2.56	2.40	2.25	2.00

Table 3.16: Division of two O(m,n)-Tr.T.F.N's, where $m = 0.5$, $n = 2$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$A_1(\alpha)(*)B_1(\alpha)$	0.13	0.22	0.32	0.40	0.49	0.56	0.63	0.70	0.76	0.83	1.00
$A_2(\alpha)(*)B_2(\alpha)$	4.50	4.04	3.68	3.39	3.15	2.95	2.78	2.63	2.48	2.32	2.00

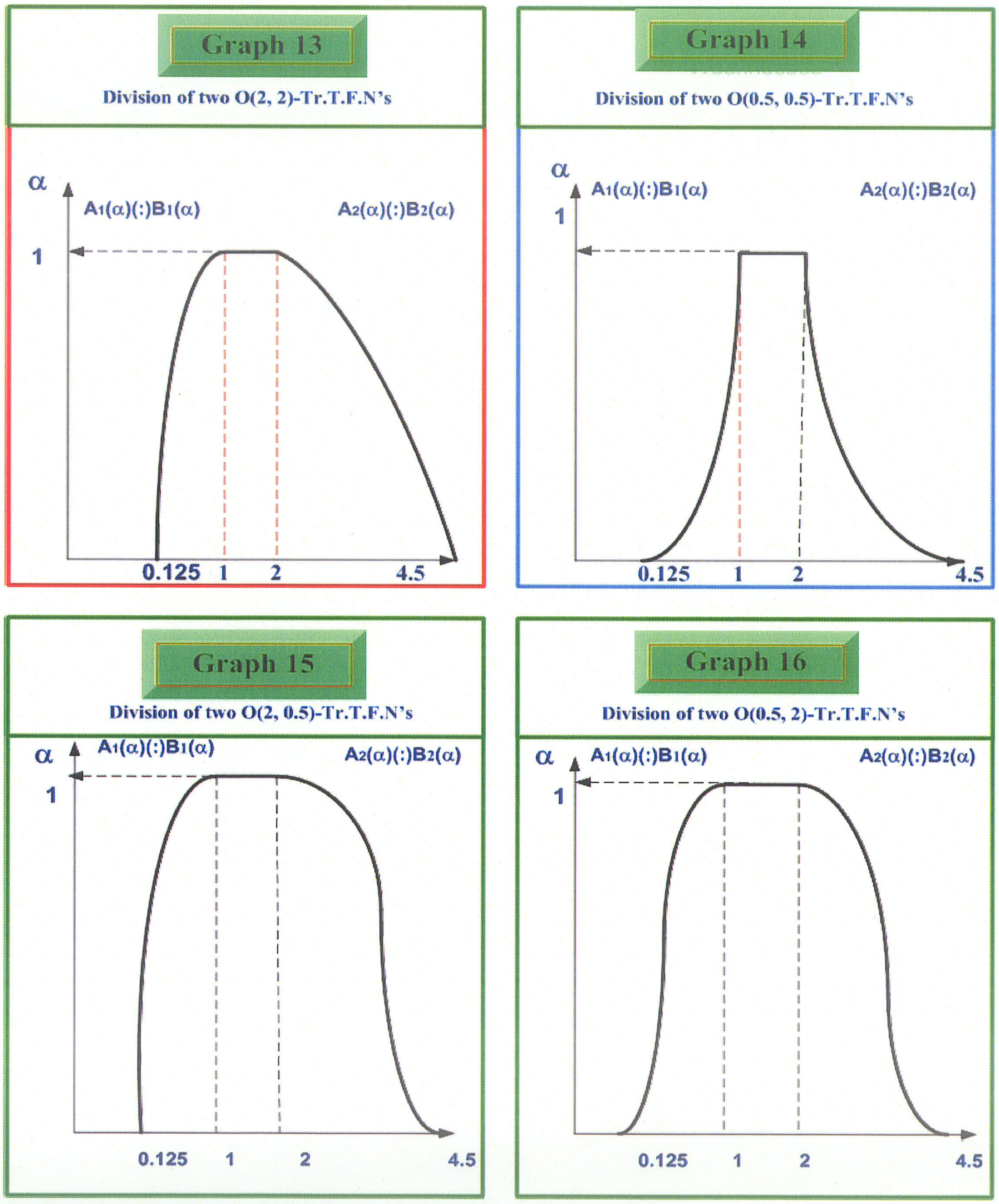

Figure 3.4: Division of two O(,)-Tr.T.F.N's.

We, now make the following observations.

(a) When $m = 2$ and $n = 2$, the graph bulges outward and is often interpreted in linguistic term as "more or less" or "dilatation" (or concave).

(b) When $m = 0.5$ and $n = 0.5$, the graph bulges inward and is often interpreted in linguistic term as "very" or "concentration" (or convex). We have more confidence in the middle portion of the membership function than at its end points.

(c) When $m = 2$ and $n = 0.5$, the shape of the graph is different than those describe in (b) and (c). There is no point of inflection on either sides of the membership function.

(d) When $m = 0.5$ and $n = 2$, the graph bulges outward on both sides of the membership function. There is no point of inflection on either sides of the membership function.

3.3 Conclusion

In this chapter a class of O(m,n)-Tr.T.F.N's is introduced as a generalization of the Tr.T.F.N's, and some of their properties are discussed using $+, -, \times$ and $\div$ operations. Some numerical examples are also provided to reinforce the results.

Chapter 4

Moment Properties of O(m,n)-Trapezoidal Type Fuzzy Numbers

Carlsson and Fuller [30] and Fuller and Majlender [53] discuss the possibilistic mean, interval valued possibilistic mean, possibilistic variance and possibilistic covariance for Tr.F.N's. Carlsson and Fuller [31] show the usefulness of possibilistic mean value and variance of fuzzy numbers by applying it to a fuzzy real option pricing model. In the present chapter we derive, on the lines of Carlsson and Fuller [30] and Fuller and Majlender [53] similar results for O(m, n)-Tr.T.F.N's.

4.1 Possibilistic Mean of an O(m,n)-Tr.T.F.N

Let $\tilde{A} = [a_1, a_2, a_3, a_4]_{O(m,n)}$ be an O(m,n)-Tr.T.F.N such that the α-cut of A is given by

$$A(\alpha) \;=\; [a_2 - (a_2 - a_1)(1-\alpha)^{\frac{1}{m}}, a_3 - (a_3 - a_4)(1-\alpha)^{\frac{1}{n}}]$$

On the lines of Carlsson and Fuller [30] and Carlsson and Majlender [53], we have the lower possibilistic mean $M_*(A)$ as

$$M_*(A) \;=\; 2\int_0^1 \alpha a_1(\alpha) d\alpha = 2\int_0^1 \alpha\left(a_2 - (a_2 - a_1)(1-\alpha)^{\frac{1}{m}}\right) d\alpha$$

$$
\begin{aligned}
&= 2a_2\int_0^1 \alpha d\alpha - 2(a_2-a_1)\int_0^1 \alpha(1-\alpha)^{\frac{1}{m}}d\alpha \\
&= a_2 + 2(a_2-a_1)\left[\frac{m\alpha}{1+m}(1-\alpha)^{\frac{1+m}{m}} + \frac{m^2}{(1+m)(1+2m)}(1-\alpha)^{\frac{1+2m}{m}}\right]_0^1 \\
&= a_2 - \left[\frac{2m^2(a_2-a_1)}{(1+m)(1+2m)}\right] \qquad (4.1.1)
\end{aligned}
$$

Similarly, the upper posibilistic mean

$$
\begin{aligned}
M^*(A) &= 2\int_0^1 \alpha a_2(\alpha)d\alpha \\
&= a_3 - \left[\frac{2n^2(a_3-a_4)}{(n+1)(2n+1)}\right] \qquad (4.1.2)
\end{aligned}
$$

The average of $\overline{M}(A)$ of $M_*(A)$ and $M^*(A)$ is

$$
\begin{aligned}
\overline{M}(A) &= \frac{M_*(A)+M^*(A)}{2} = \frac{a_2 - \left[\frac{2m^2(a_2-a_1)}{(1+m)(1+2m)}\right] + a_3 - \left[\frac{2n^2(a_3-a_4)}{(n+1)(2n+1)}\right]}{2} \\
&= \frac{a_2+a_3}{2} - \left[\frac{m^2(a_2-a_1)}{(1+m)(1+2m)}\right] - \left[\frac{n^2(a_3-a_4)}{(1+n)(1+2n)}\right]. \qquad (4.1.3)
\end{aligned}
$$

When $\tilde{A}$ is a symmetric fuzzy number, such that $(a_2-a_1)=(a_4-a_3)$ and $m=n$, then from (4.1.3) we have,

$$
\overline{M}(A) = \frac{a_2+a_3}{2} \qquad (4.1.4)
$$

Note that in this case $\overline{M}(A)$ is independent of m and n. We make the following observations on the possibilistic mean of an O(m,n)-Tr.T.F.N.

(a) When $m=1$, the lower possibilistic mean $M_*(A)$ is $\frac{a_2}{3}+\frac{a_1}{6}$.

(b) When $n=1$, the upper possibilistic mean $M^*(A)$ is $\frac{a_3}{3}+\frac{a_4}{6}$.

(c) When $m\to\infty$, i.e, $\lim_{m\to\infty} a_2 - \left[\frac{m^2(a_2-a_1)}{(1+m)(1+2m)}\right] = a_1$. This mean that the lower possibilistic mean $M_*(A)$ converges to a_1.

(d) When $n \to \infty$, i.e, $\lim_{n\to\infty} a_3 - \left[\frac{2n^2(a_3-a_4)}{(n+1)(2n+1)}\right] = a_4$. This mean that the upper possibilistic mean $M_*(A)$ converges to a_4.

(e) $\lim_{m,n\to\infty} \frac{a_2+a_3}{2} - \left[\frac{m^2(a_2-a_1)}{(1+m)(1+2m)}\right] - \left[\frac{n^2(a_3-a_4)}{(1+n)(1+2n)}\right] = \frac{a_1+a_4}{2}$.

(f) When $m = n = 0$, the possibilistic mean of A is $\frac{a_2+a_3}{2}$.

Also, note that $\lim_{m,n\to\infty} \overline{M}(A)$ is independent of m and n.

4.1.1 Interval Valued Probabilistic Mean of an O(m,n)-Tr.T.F.N.

Let $\tilde{A} = (a_1, a_2, a_3, a_4)_{O(m,n)}$ be an O(m,n)-Tr.T.F.N such that $m, n > 0$, and $m \neq n$. We now show that $M(A) \subset E(A)$. As in Carlsson and Fuller [30] the lower probability mean of A is given by,

$$\begin{aligned} E_*(A) &= \int_0^1 a_1(\alpha)d\alpha = \int_0^1 [a_2 - (a_2-a_1)(1-\alpha)^{\frac{1}{m}}]d\alpha \\ &= a_2 - (a_2-a_1)\int_0^1 (1-\alpha)^{\frac{1}{m}}d\alpha \\ &= a_2 + (a_2-a_1)\left[\frac{m}{1+m}(1-\alpha)^{\frac{1+m}{m}}\right]_0^1 \\ &= a_2 - (a_2-a_1)\left[\frac{m}{1+m}\right] \qquad (4.1.5) \end{aligned}$$

Similarly, the upper probability mean of $\tilde{A}$ is given below,

$$\begin{aligned} E^*(A) &= \int_0^1 a_2(\alpha)d\alpha = \int_0^1 [a_3 - (a_3-a_4)(1-\alpha)^{\frac{1}{n}}]d\alpha \\ &= a_3 - (a_3-a_4)\left[\frac{n}{1+n}\right] \qquad (4.1.6) \end{aligned}$$

From (4.1.1) and (4.1.2), the interval value possibility mean of $\tilde{A}$ is as follows,

$$\begin{aligned} M(A) &= [M_*(A),\ M^*(A)] \\ &= \left[a_2 - (a_2-a_1)\frac{2m^2}{(1+m)(1+2m)},\ a_3 - (a_3-a_4)\frac{2n^2}{(n+1)(2n+1)}\right] \end{aligned}$$

From (4.1.5) and (4.1.6), the interval value probability mean of $\tilde{A}$ is as follows,

$$E(A) = [E_*(A),\ E^*(A)] = \left[a_2 - (a_2 - a_1)\frac{m}{1+m},\ a_3 - (a_3 - a_4)\frac{n}{1+n}\right].$$

We now state the following Lemma:

Lemma 4.1.1 *If $A \in F$ is a fuzzy number with strictly increasing and strictly decreasing (and continuous) functions then its interval-valued possibilistic mean is a proper subset of its interval-valued probabilistic mean, i.e $M(A) \subset E(A)$.*

Lemma 4.1.1 is similar to Lemma 1.7.1 and its proof follows on the same lines of Carlsson and Fuller [30]. In view of Lemma 4.1.1, we have $M(A) \subset E(A)$.

We now make the following observations on the interval valued probabilistic mean and possibilistic mean of an O(m,n)-Tr.T.F.N.

(a) When $m = 1$, the lower probability mean $E_*(A)$ is $\dfrac{a_1 + a_2}{2}$.

(b) When $m \to \infty$, then $\lim_{m\to\infty} a_2 - \dfrac{m(a_2 - a_1)}{1+m} = a_1$. The lower probability mean $E_*(A)$ converges to a_1.

(c) At $n = 0$, the upper probability mean $E^*(A)$ is equal to a_3.

(d) When $n = 1$, the upper probability mean $E^*(A)$ is equal to $\dfrac{a_3 + a_4}{2}$.

(e) When $n \to \infty$, then $\lim_{n\to\infty} a_3 - \dfrac{n(a_3 - a_4)}{1+n} = a_4$. The upper probability mean $E^*(A)$ converges to a_4.

(f) When $\tilde{A}$ is a symmetric fuzzy number, i.e., $m = n$ and $a_2 - a_1 = a_4 - a_3$ then, the crisp probability mean of $\tilde{A}$ is,

$$\begin{aligned} \overline{E}(A) &= \frac{E_*(A) + E^*(A)}{2} = \frac{a_2 + a_3}{2} - (a_2 - a_1)\left(\left[\frac{m}{1+m}\right] - \left[\frac{n}{1+n}\right]\right) \\ &= \frac{a_2 + a_3}{2} \end{aligned} \tag{4.1.7}$$

Note also that the probability mean of $\tilde{A}$ is independent of m and n. $\overline{E}(A)$ given by (4.1.7) is equal to $\overline{M}(A)$ which is given by (4.1.4). This property is observed by Carlsson and Fuller [30] also for triangular fuzzy numbers.

(g) It is easy to see that,

$$\lim_{m,\, n\to\infty} \frac{a_2+a_3}{2} - (a_2-a_1)\left(\left[\frac{m}{1+m}\right] - \left[\frac{n}{1+n}\right]\right) = \frac{a_2+a_3}{2}$$

Thus we see that $\lim_{m,\, n\to\infty} \overline{E}(A)$ is is independent of m and n and converges to $\dfrac{a_2+a_3}{2}$.

4.1.2 Possibilistic Variance of O(m,n)-Tr.T.F.N.

If A is an O(m,n)-Tr.T.F.N, then using (1.7.17) the possibilistic variance of A can be computed as follows.

$$\begin{aligned}
Var(A) &= \int_0^1 \frac{1}{2}\alpha\,(a_2(\alpha)-a_1(\alpha))^2\,d\alpha \\
&= \frac{1}{2}\int_0^1 \alpha\left[\left(a_3-(a_3-a_4)(1-\alpha)^{\frac{1}{n}}\right)-\left(a_2-(a_2-a_1)(1-\alpha)^{\frac{1}{m}}\right)\right]^2 d\alpha \\
&= \frac{1}{2}\int_0^1 \alpha\left[(a_3-a_2)-(a_3-a_4)(1-\alpha)^{\frac{1}{n}}+(a_2-a_1)(1-\alpha)^{\frac{1}{m}}\right]^2 d\alpha
\end{aligned}$$

$$\begin{aligned}
= {} & \frac{1}{2}(a_3-a_2)^2\int_0^1 \alpha d\alpha+\frac{1}{2}(a_3-a_4)^2\int_0^1 \alpha(1-\alpha)^{\frac{2}{n}}d\alpha+\frac{1}{2}(a_2-a_1)^2\int_0^1 \alpha(1-\alpha)^{\frac{2}{m}}d\alpha \\
& -(a_3-a_2)(a_3-a_4)\int_0^1 \alpha(1-\alpha)^{\frac{1}{n}}d\alpha+(a_3-a_2)(a_2-a_1)\int_0^1 \alpha(1-\alpha)^{\frac{1}{m}}\,d\alpha \\
& -(a_3-a_4)(a_2-a_1)\int_0^1 \alpha(1-\alpha)^{\frac{1}{n}+\frac{1}{m}}\,d\alpha
\end{aligned} \tag{4.1.8}$$

Thus,

$$
\begin{aligned}
Var(A) &= \frac{1}{4}(a_3-a_2)^2 + \left[\frac{n^2(a_3-a_4)^2}{4(2+n)(1+n)}\right] + \left[\frac{m^2(a_2-a_1)^2}{4(2+m)(1+m)}\right] \\
&- \left[\frac{n^2(a_3-a_2)(a_3-a_4)}{(1+n)(1+2n)}\right] + \left[\frac{m^2(a_3-a_2)(a_2-a_1)}{(1+m)(1+2m)}\right] \\
&- \left[\frac{n^2m^2(a_3-a_4)(a_2-a_1)}{(m+n+2mn)(m+n+mn)}\right] \qquad (4.1.9)
\end{aligned}
$$

The following result that follows as special cases of (4.1.9) can be found in [30].

(a) If $m=n$ in expression (4.1.9) we obtain the variance of an O(m,m)-Tr.T.F.N as

$$
\begin{aligned}
Var(A) &= \frac{(a_3-a_2)^2}{4} + \frac{m^2(a_3-a_2)\left((a_2-a_1)+(a_4-a_3)\right)}{(1+m)(1+2m)} + \\
&\frac{m^2((a_4-a_3)+(a_2-a_1))^2}{4(2+m)(1+m)} \qquad (4.1.10)
\end{aligned}
$$

(b) If we set $(a_2-a_1)=(a_4-a_3)=\alpha$ in expression (4.1.10), we obtain the following result for the variance of a symmetric O(m,m)-Tr.T.F.N:

$$
Var(A) = \frac{(a_3-a_2)^2}{4} + \frac{2m^2\alpha(a_3-a_2)}{(1+m)(1+2m)} + \frac{m^2\alpha^2}{(2+m)(1+m)} \qquad (4.1.11)
$$

(c) If we let $m=1$ in expression (4.1.10), we obtain the following result for the variance of a Tr.F.N.

$$
Var(A) = \frac{(a_3-a_2)^2}{4} + \frac{(a_3-a_2)\left((a_2-a_1)+(a_4-a_3)\right)}{6} + \frac{((a_4-a_3)+(a_2-a_1))^2}{24} \qquad (4.1.12)
$$

(d) If we set $m=1$ and $(a_2-a_1)=(a_4-a_3)=\alpha$ in expression (4.1.10), we obtain the following result for the variance of a symmetric Tr.F.N.

$$
Var(A) = \frac{(a_3-a_2)^2}{4} + \frac{\alpha(a_3-a_2)}{3} + \frac{\alpha^2}{6} \qquad (4.1.13)
$$

(e) If we set $m = 1$, $a_4 = a_3$ and $a_3 = a_2$ in expression (4.1.10), we obtain the following result for the variance of a T.F.N

$$Var(A) = \frac{((a_3 - a_2) + (a_2 - a_1))^2}{24} \tag{4.1.14}$$

(f) Now, if $a_3 - a_2 = a_2 - a_1 = \alpha$ in expression (4.1.14), we obtain the variance of a symmetric T.F.N as follows.

$$Var(A) = \frac{\alpha^2}{6} \tag{4.1.15}$$

Let $A \in F$, where F is the family of all O(m,n)-Tr.T.F.N's and let θ be a real number. If the fuzzy number A is shifted by θ, then in the following theorem we show that the possibilistic mean of $\tilde{A}$ shifts by θ and the variance of an O(m,n)-Tr.T.F.N is invariant to shifting.

Theorem 4.1.1 *Let $A \in F$, where F is the family of all O(m,n)-Tr.T.F.N's and let θ be a real number. If fuzzy number A is shifted by θ, where, $B(\alpha) = [a_1(\alpha) + \theta,\ a_2(\alpha) + \theta]$ such that,*

(a) $\overline{M}(B) = \overline{M}(A) + \theta$,

(b) $Var(B) = Var(A)$.

Proof. Let $A(\alpha) = [a_1(\alpha), a_2(\alpha)]$, $\alpha \in [0, 1]$. Then, we have

$B(\alpha) = A(\alpha) + \theta = [a_1(\alpha),\ a_2(\alpha)] + \theta = [a_1(\alpha) + \theta,\ a_2(\alpha) + \theta]$,

Thus, using expression (1.7.2) for the crisp possibilistic mean of a fuzzy number, we obtain that,

$$
\begin{aligned}
\overline{M}(B) &= \int_0^1 \alpha\ [a_1(\alpha) + \theta + \ a_2(\alpha) + \theta]\, d\alpha \\
&= \int_0^1 \alpha\ [a_1(\alpha) + \ a_2(\alpha)]\, d\alpha + 2\theta \int_0^1 \alpha d\alpha \\
&= \int_0^1 \alpha\ [a_1(\alpha) + \ a_2(\alpha)]\, d\alpha + \theta = \overline{M}(A) + \theta \qquad (4.1.16)
\end{aligned}
$$

Thus, (4.1.16) yields that if fuzzy number A is shifted by θ then, the possibilistic mean of A shifts by θ as well. Similarly, for part (b) of the theorem we have

$$
\begin{aligned}
Var(B) &= \frac{1}{2}\int_0^1 \alpha\,[(a_2(\alpha) + \theta\ ) - (a_1(\alpha) + \theta)]^2\, d\alpha = \frac{1}{2}\int_0^1 \alpha\,[(a_2(\alpha)\ - a_1(\alpha))]^2\, d\alpha \\
&= Var(A) \qquad (4.1.17)
\end{aligned}
$$

From(4.1.17), it follows that variance of an O(m,n)-Tr.T.F.N is invariant to shifting. We observe here that Theorem 4.1.1 is a generalization of the results given by Carlsson and Fuller [30] and is independent of the type of fuzzy numbers but is dependent on the α-cut only.

4.1.3 Possibilistic Covariance of Two O(m,n)-Trapezoidal Type Fuzzy Numbers.

On the lines of Carlsson and Fuller [30] and Fuller and Majlender [53], the possibilistic covariance between two fuzzy numbers, $\tilde{A}$ and $\tilde{B}$ is defined as follows,

$$
\begin{aligned}
Cov(A,B) &= \frac{1}{2}\int_0^1 \alpha\,[(a_2(\alpha) - a_1(\alpha))(b_2(\alpha) - b_1(\alpha))]\, d\alpha \\
&= \frac{1}{2}\int_0^1 \alpha\quad [a_2(\alpha)b_2(\alpha) - a_2(\alpha)b_1(\alpha) - a_1(\alpha)b_2(\alpha) + a_1(\alpha)b_1(\alpha)]\, d\alpha \\
&= \frac{1}{2}\int_0^1 \alpha\ a_2(\alpha)b_2(\alpha)d\alpha - \frac{1}{2}\int_0^1 \alpha\ \ a_2(\alpha)b_1(\alpha)d\alpha - \frac{1}{2}\int_0^1 \alpha a_1(\alpha)b_2(\alpha)d\alpha \\
&\quad + \frac{1}{2}\int_0^1 \alpha\ \ a_1(\alpha)b_1(\alpha)d\alpha \qquad (4.1.18)
\end{aligned}
$$

If $\tilde{A}$ and $\tilde{B}$ are two O(m,n)-Tr.T.F.N's of the type given by (3.2.1) and (3.2.2), then the α-cut for $\tilde{A}$ and $\tilde{B}$ are as follows.

$$\begin{aligned} A(\alpha) &= [a_1(\alpha), a_2(\alpha)] = [a_2 - (a_2 - a_1)(1-\alpha)^{\frac{1}{m}}, a_3 - (a_3 - a_4)(1-\alpha)^{\frac{1}{n}}] \\ B(\alpha) &= [b_1(\alpha), b_2(\alpha)] = [b_2 - (b_2 - b_1)(1-\alpha)^{\frac{1}{m}}, b_3 - (b_3 - b_4)(1-\alpha)^{\frac{1}{n}}] \end{aligned}$$

Below, we enumerate various possibilities of possibilistic covariance between two O(m,n)-Tr.T.F.N's.

(a) Using (4.1.18), we obtain the possibilistic covariance between two O(m,n)-Tr.T.F.N's as follows,

$$\begin{aligned} Cov(A,B) &= \left[\frac{(a_3-a_2)(b_3-b_2)}{4}\right] + \\ &\left[\frac{n^2[(b_4-b_3)(a_3-a_2)+(a_4-a_3)(b_3-b_2)]}{2(1+n)(1+2n)}\right] + \\ &\left[\frac{n^2(b_4-b_3)(a_4-a_3)+m^2(b_2-b_1)(a_2-a_1)}{4(2+n)(1+n)}\right] + \\ &\left[\frac{m^2[(b_2-b_1)(a_3-a_2)+(a_2-a_1)(b_3-b_2)]}{2(1+m)(1+2m)}\right] + \\ &\left[\frac{n^2m^2[(b_2-b_1)(a_4-a_3)+(a_2-a_1)(b_4-b_3)]}{2(m+n+2mn)(m+n+mn)}\right]. \end{aligned} \tag{4.1.19}$$

(b) If we set $m = n = 1$ in the expression for the possibilistic covariance given by (4.1.19), then we obtain the following result for the possibilistic covariance between two Tr.F.N's, as obtained by Carlsson and Fuller [30] and Fuller and

Majlender [53].

$$
\begin{aligned}
Cov(A,B) &= \left[\frac{(a_3-a_2)(b_3-b_2)}{4}\right] + \\
&\left[\frac{(a_3-a_2)((b_4-b_3)+(b_2-b_1))+(b_3-b_2)((a_4-a_3)+(a_2-a_1))}{12}\right] + \\
&\left[\frac{(b_4-b_3)((a_4-a_3)+(a_2-a_1))+(b_2-b_1)((a_2-a_1)+(a_4-a_3))}{24}\right]
\end{aligned}
\tag{4.1.20}
$$

(c) If we set $a_4 = a_3,\ a_3 = a_2,\ b_4 = b_3$ and $b_3 = b_2$ in expression (4.1.20), we obtain the possibilistic covariance between two Tr.F.N's

$$
\begin{aligned}
Cov(A,B) &= \left[\frac{(b_3-b_2)((a_3-a_2)+(a_2-a_1))+(b_2-b_1)((a_2-a_1)+(a_3-a_2))}{24}\right] \\
&= \left[\frac{[(a_3-a_2)+(a_2-a_1)][(b_3-b_2)+(b_2-b_1)]}{24}\right]
\end{aligned}
\tag{4.1.21}
$$

(d) If we assume that $\tilde{A}$ and $\tilde{B}$ are symmetric fuzzy numbers, i.e., $(a_2-a_1) = (a_3-a_2) = \alpha$ and $(b_2-b_1) = (b_3-b_2) = \beta$. Then, we obtain the following results for the covariance between two symmetric fuzzy numbers.

$$
Cov(A,B) = \left[\frac{(\alpha+\alpha)(\beta+\beta)}{24}\right] = \left[\frac{(2\alpha)(2\beta)}{24}\right] = \frac{\alpha\beta}{6} \tag{4.1.22}
$$

The results given by expression (4.1.22) is identical to the possibilistic covariance between two symmetric T.F.N's reported by Carlsson and Fuller [30].

The following is a generalization of Theorem 4.1 proved by Carlsson and Fuller [30].

Theorem 4.1.2 *Let λ and $\mu \in \Re$ and let A an B be two $O(m, n)$-Tr.T.F.N's. Then,*

$$
Var(\lambda A + \mu B) = \lambda^2 Var(A) + \mu^2 Var(B) + 2|\lambda\mu| Cov(A,B) \tag{4.1.23}
$$

where the addition and multiplication by a scalar of fuzzy numbers is defined by the extension principle given in Chapter 1.

Proof: Let, $A(\alpha) = [a_1(\alpha), a_2(\alpha)]$ and $B(\alpha) = [b_1(\alpha), b_2(\alpha)]$, $\forall\ \alpha\ \in\ (0,1]$.
Suppose that $\lambda\ > 0$ and $\mu\ > 0$, then the α-cut for $\lambda A + \mu B$ is given by,

$$\begin{aligned}\lambda A(\alpha) + \mu B(\alpha) &= [\lambda a_1(\alpha), \lambda a_2(\alpha)] + [\mu b_1(\alpha), \mu b_2(\alpha)] \\ &= [\lambda a_1(\alpha) + \mu b_1(\alpha), \lambda a_2(\alpha) + \mu b_2(\alpha)] \qquad (4.1.24)\end{aligned}$$

Thus,

$$\begin{aligned}Var(\lambda A + \mu B) &= \frac{1}{2}\int_0^1 (\lambda a_2(\alpha) + \mu b_2(\alpha)) - (\lambda a_1(\alpha) + \mu b_1(\alpha))^2 d\alpha \\ &= \frac{1}{2}\int_0^1 [\lambda(a_2(\alpha) - a_1(\alpha)) + \mu(b_2(\alpha) - b_1(\alpha))]^2\, d\alpha \\ &= \lambda^2 \int_0^1 \frac{1}{2}(a_2(\alpha) - a_1(\alpha))^2 d\alpha + \mu^2 \int_0^1 \frac{1}{2}(b_2(\alpha) - b_1(\alpha))^2 d\alpha + \\ &\quad 2\lambda\mu \int_0^1 \frac{1}{2}(a_2(\alpha) - a_1(\alpha))(b_2(\alpha) - b_1(\alpha)) d\alpha \\ &= \lambda^2 Var(A) + \mu^2 Var(B) + 2\lambda\mu Cov(A+B) \qquad (4.1.25)\end{aligned}$$

Now, assume that $\lambda\ < 0$ and $\mu\ < 0$, then the $Var(\lambda A + \mu B)$ is as follows.

$$\begin{aligned}\lambda A(\alpha) + \mu B(\alpha) &= [\lambda a_2(\alpha), \lambda a_1(\alpha)] + [\mu b_2(\alpha), \mu b_1(\alpha)] \\ &= [\lambda a_2(\alpha) + \mu b_2(\alpha), \lambda a_1(\alpha) + \mu b_1(\alpha)]\end{aligned}$$

Thus,

$$\begin{aligned}Var(\lambda A + \mu B) &= \frac{1}{2}\int_0^1 (\lambda a_1(\alpha) + \mu b_1(\alpha)) - (\lambda a_2(\alpha) + \mu b_2(\alpha))^2 d\alpha \\ &= \frac{1}{2}\int_0^1 [\lambda(a_1(\alpha) - a_2(\alpha)) + \mu(b_1(\alpha) - b_2(\alpha))]^2\, d\alpha \\ &= \lambda^2 \int_0^1 \frac{1}{2}(a_1(\alpha) - a_2(\alpha))^2 d\alpha + \mu^2 \int_0^1 \frac{1}{2}(b_1(\alpha) - b_2(\alpha))^2 d\alpha + \\ &\quad 2\lambda\mu \int_0^1 \frac{1}{2}(a_1(\alpha) - a_2(\alpha))(b_1(\alpha) - b_2(\alpha)) d\alpha \\ &= \lambda^2 Var(A) + \mu^2 Var(B) + 2\lambda\mu Cov(A+B) \qquad (4.1.26)\end{aligned}$$

Suppose now that $\lambda > 0$ and $\mu < 0$. Then we get

$$\begin{aligned}\lambda A(\alpha) + \mu B(\alpha) &= [\lambda a_1(\alpha), \lambda a_2(\alpha)] + [\mu b_2(\alpha), \mu b_1(\alpha)] \\ &= [\lambda a_1(\alpha) + \mu b_2(\alpha), \lambda a_2(\alpha) + \mu b_1(\alpha)]\end{aligned}$$

Thus,

$$\begin{aligned}Var(\lambda A + \mu B) &= \frac{1}{2}\int_0^1 (\lambda a_2(\alpha) + \mu b_1(\alpha)) - (\lambda a_1(\alpha) + \mu b_2(\alpha))^2 d\alpha \\ &= \frac{1}{2}\int_0^1 \left[\lambda(a_2(\alpha) - a_1(\alpha)) + \mu(b_1(\alpha) - b_2(\alpha))\right]^2 d\alpha \\ &= \lambda^2 \int_0^1 \frac{1}{2}(a_2(\alpha) - a_1(\alpha))^2 d\alpha + \mu^2 \int_0^1 \frac{1}{2}(b_1(\alpha) - b_2(\alpha))^2 d\alpha - \\ &\quad 2\lambda\mu \int_0^1 \frac{1}{2}(a_2(\alpha) - a_1(\alpha))(b_2(\alpha) - b_1(\alpha)) d\alpha \\ &= \lambda^2 Var(A) + \mu^2 Var(B) - 2\lambda\mu Cov(A + B) \qquad (4.1.27)\end{aligned}$$

Similar reasoning holds for the case when $\lambda < 0$, and $\mu > 0$. It is important to point out that the above theorem is proved for O(m,n)-Tr.T.F.N's, yet the proof is independent of O(m,n)-Tr.T.F.N's and is dependent upon the α-cuts only. Therefore, it appears that the theorem may be proved without the assumption of O(m,n)-Tr.T.F.N's.

In this section on the line of Fuller and Majlender [53] we discuss moment properties of various weighted functions. We derive expressions for weighted possibilistic mean, weighted possibilistic variance and weighted possibilistic covariance related to O(m,n)-Tr.T.F.N's.

Using the theory of weighted functions discussed in this chapter, an alternative fuzzy binomial option pricing model (Chapter 5) can be developed. In the fuzzy binomial option pricing model developed in the next chapter (Chapter 5), instead of taking probability expectation for the call price, we could as well opt for possibilistic or weighted possibilistic expectation. These techniques using possibility theory concepts are generalizations of already existing methods and will be the subject of our future research.

In this section we provide some application of the results proved using some weighted functions.

4.1.4 Application 1

Let $f(\alpha) = (\omega+1)\alpha^{\omega}$ be a weighted function and $\tilde{A} = (a_1, a_2, a_3, a_4)_{O(m,n)}$ be an O(m,n)-Tr.T.F.N with interior points, a_2 and a_3 and end points a_1 and a_4. Then the α-cut of $\tilde{A}$ is given by

$$A(\alpha) = [a_1(\alpha), a_2(\alpha)] = [a_2 - (a_2 - a_1)(1-\alpha)^{\frac{1}{m}}, a_3 - (a_3 - a_4)(1-\alpha)^{\frac{1}{n}}].$$

The lower weighted possibilistic mean values of $\tilde{A}$ is obtained as follows.

Using (1.8.5), we have

$$\begin{aligned}
M_{\bar{f}}(A) &= \int_0^1 a_1(\alpha) f(\alpha) d\alpha = \int_0^1 [a_2 - (a_2 - a_1)(1-\alpha)^{\frac{1}{m}}](\omega+1)\alpha^{\omega} d\alpha \\
&= a_2(\omega+1)\int_0^1 \alpha^{\omega} d\alpha - (a_2 - a_1)(\omega+1)\int_0^1 (1-\alpha)^{\frac{1}{m}}\alpha^{\omega} d\alpha \\
&= a_2(\omega+1)\int_0^1 \alpha^{\omega} d\alpha - (a_2 - a_1)(\omega+1)\int_0^1 (1-\alpha)^{(\frac{1}{m}+1)-1}\alpha^{(\omega+1)-1} d\alpha \\
&= a_2(\omega+1)\left[\frac{\alpha^{\omega+1}}{\omega+1}\right]_0^1 - (\omega+1)(a_2 - a_1)\left[\frac{\Gamma(\frac{1}{m}+1)\Gamma(\omega+1)}{\Gamma(\frac{1}{m}+\omega+2)}\right] \\
&= a_2 - (\omega+1)(a_2 - a_1)\left[\frac{\Gamma(\frac{1}{m}+1)\Gamma(\omega+1)}{\Gamma(\frac{1}{m}+\omega+2)}\right] \qquad (4.1.28)
\end{aligned}$$

(i)

$$\lim_{\omega\to 0} M_f^-(A) = \lim_{\omega\to 0} a_2 - (\omega+1)(a_2-a_1)\left[\frac{\Gamma(\frac{1}{m}+1)\Gamma(\omega+1)}{\Gamma(\frac{1}{m}+\omega+2)}\right]$$
$$= a_2 - \frac{m(a_2-a_1)}{1+m}$$

(ii)

$$\lim_{m\to\infty} M_f^-(A) = \lim_{m\to\infty} a_2 - (\omega+1)(a_2-a_1)\left[\frac{\Gamma(\frac{1}{m}+1)\Gamma(\omega+1)}{\Gamma(\frac{1}{m}+\omega+2)}\right]$$
$$= a_1$$

Similarly, the upper weighted possibilistic mean values of $\tilde{A}$ are as follows.

Using (1.8.6) we have

$$M_f^+(A) = a_3 - (\omega+1)(a_3-a_4)\left[\frac{\Gamma(\frac{1}{n}+1)\Gamma(\omega+1)}{\Gamma(\frac{1}{n}+\omega+2)}\right] \tag{4.1.29}$$

and

$$\lim_{\omega\to 0} M_f^+(A) = \lim_{\omega\to 0} a_3 - (\omega+1)(a_3-a_4)\left[\frac{\Gamma(\frac{1}{n}+1)\Gamma(\omega+1)}{\Gamma(\frac{1}{n}+\omega+2)}\right]$$
$$= a_3 - \frac{n(a_3-a_4)}{1+n}$$
$$\lim_{n\to\infty} M_f^+(A) = \lim_{n\to\infty} a_3 - (\omega+1)(a_3-a_4)\left[\frac{\Gamma(\frac{1}{n}+1)\Gamma(\omega+1)}{\Gamma(\frac{1}{n}+\omega+2)}\right]$$
$$= a_4$$

Using (1.8.4) the f-weighted interval-valued possibilistic mean of $\tilde{A}$ is as follows.

$$[M_{\overline{f}}(A), M_{\underset{f}{+}}(A)] = \left[\begin{array}{c} a_2 - (\omega+1)(a_2-a_1)\left(\dfrac{\Gamma(\frac{1}{m}+1)\Gamma(\omega+1)}{\Gamma(\frac{1}{m}+\omega+2)}\right) \\ , \\ a_3 - (\omega+1)(a_3-a_4)\left(\dfrac{\Gamma(\frac{1}{n}+1)\Gamma(\omega+1)}{\Gamma(\frac{1}{n}+\omega+2)}\right) \end{array}\right] \quad (4.1.30)$$

The f-weighted possibilistic mean of the fuzzy number $\tilde{A}$, defined by (1.8.2), is the arithmetic mean of its f-weighted lower and upper possibilistic mean values, i.e.,

$$\begin{aligned} \overline{M}_f(A) &= \frac{M_f^-(A)+M_f^+(A)}{2} \\ &= \left[\begin{array}{c} \dfrac{a_2+a_3}{2} - \left(\dfrac{(\omega+1)(a_2-a_1)\Gamma(\frac{1}{m}+1)\Gamma(\omega+1)}{2\Gamma(\frac{1}{m}+\omega+2)}\right) \\ , \\ -\left(\dfrac{(\omega+1)(a_3-a_4)\Gamma(\frac{1}{n}+1)\Gamma(\omega+1)}{2\Gamma(\frac{1}{n}+\omega+2)}\right) \end{array}\right] \end{aligned} \quad (4.1.31)$$

and

$$\begin{aligned} \lim_{\omega\to 0}\overline{M}_f(A) &= \lim_{\omega\to 0}\frac{a_2+a_3}{2} - \left[\frac{(\omega+1)(a_2-a_1)\Gamma(\frac{1}{m}+1)\Gamma(\omega+1)}{2\Gamma(\frac{1}{m}+\omega+2)}\right] \\ &\quad - \left[\frac{(\omega+1)(a_3-a_4)\Gamma(\frac{1}{n}+1)\Gamma(\omega+1)}{2\Gamma(\frac{1}{n}+\omega+2)}\right] \\ &= \frac{a_2+a_3}{2} - \left(\frac{m(a_2-a_1)}{2(1+m)}\right) - \left(\frac{n(a_3-a_4)}{2(1+n)}\right) \end{aligned}$$

If we assume that $\tilde{A}$ is a Tr.F.N, then, on the lines of Carlsson and Fuller [30] and Fuller and Majlender [53] we have the following results:

(a) If we set $m = 1$ in expression (4.1.28), the f-weighted lower possibilistic mean value of A is converges to

$$\begin{aligned} M_{\overline{f}}(A) &= a_2 - (\omega+1)(a_2 - a_1)\left[\frac{\Gamma(2)\Gamma(\omega+1)}{\Gamma(\omega+3)}\right] \\ &= a_2 - \frac{a_2 - a_1}{\omega+2} \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{\omega\to 0} M_f^-(A) &= \lim_{\omega\to 0} a_2 - \frac{a_2 - a_1}{\omega+2} = \frac{a_1 + a_2}{2} \\ \lim_{\omega\to\infty} M_f^-(A) &= \lim_{\omega\to\infty} a_2 - \frac{a_2 - a_1}{\omega+2} = a_2 \end{aligned}$$

(b) If $n = 1$ in expression (4.1.29), the f-weighted upper possibilistic mean value of $\tilde{A}$ is

$$\begin{aligned} M_f^+(A) &= a_3 - (\omega+1)(a_3 - a_4)\left[\frac{\Gamma(2)\Gamma(\omega+1)}{\Gamma(\omega+3)}\right] \\ &= a_3 - \frac{a_3 - a_4}{\omega+2} \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{\omega\to 0} M_f^+(A) &= \lim_{\omega\to 0} a_3 - \frac{a_3 - a_4}{\omega+2} = \frac{a_3 + a_4}{2} \\ \lim_{\omega\to\infty} M_f^+(A) &= \lim_{\omega\to\infty} a_3 - \frac{a_3 - a_4}{\omega+2} = a_3 \end{aligned}$$

(c) Given the weighted function $(\omega+1)\alpha^{\omega}$ and the fuzzy number $\tilde{A}$ then the f-weighted interval-valued possibilistic mean $\tilde{A}$ is given below as

$$M_f(A) = [M_{\overline{f}}(A), M_{\underset{f}{+}}(A)] = \left[a_2 - \frac{a_2 - a_1}{\omega+2}, \;\; a_3 + \frac{a_4 - a_3}{\omega+2}\right]$$

(d) The f-weighted possibilistic mean of $\tilde{A}$, is the arithmetic mean of its f-weighted lower and upper possibilistic mean values, i.e.

$$\overline{M}_f(A) = \frac{1}{2}\left[a_2 + a_3 - \frac{a_2 - a_1}{\omega + 2} + \frac{a_4 - a_3}{\omega + 2}\right] = \frac{1}{2}\left[a_2 + a_3 + \left(\frac{a_4 - a_3 - a_2 + a_1}{\omega + 2}\right)\right]$$

(e) When $\omega \longrightarrow \infty$, the f-weighted possibilistic mean of $\tilde{A}$, is as follows,

$$\lim_{\omega \to \infty} \overline{M}_f(A) = \lim_{\omega \to \infty} \frac{1}{2}\left[a_2 + a_3 + \left(\frac{a_4 - a_3 - a_2 + a_1}{\omega + 2}\right)\right] = \frac{a_2 + a_3}{2}$$

and

$$\lim_{\omega \to 0} \overline{M}_f(A) = \lim_{\omega \to 0} \frac{1}{2}\left[a_2 + a_3 + \left(\frac{a_4 - a_3 - a_2 + a_1}{\omega + 2}\right)\right]$$

$$= \frac{a_2 + a_3 + a_4 + a_1}{4}$$

4.1.5 Application 2

Let A be an O(m,n)-Tr.T.F.N and let $f(\alpha) = (\omega - 1)\left(\frac{1}{\sqrt[\omega]{1-\alpha}} - 1\right)$ be a weighted function, then on the lines of Carlsson and Fuller [30] and Fuller and Majlender [53], the f-weighted lower and upper possibilistic mean values of fuzzy number $\tilde{A}$ are obtained as follows.

$$\begin{aligned} M_{\overline{f}}(A) &= \int_0^1 [a_2 - (a_2 - a_1)(1-\alpha)^{\frac{1}{m}}](\omega - 1)\left(\frac{1}{\sqrt[\omega]{1-\alpha}} - 1\right) d\alpha \\ &= a_2 - \left[\frac{m^2(a_2 - a_1)(\omega - 1)}{(\omega - m + m\omega)(1+m)}\right] \end{aligned} \quad (4.1.32)$$

Thus, we have

$$\begin{aligned} \lim_{\omega \to 0} M_f^-(A) &= \lim_{\omega \to 0} a_2 - \left[\frac{m^2(a_2 - a_1)(\omega - 1)}{(\omega - m + m\omega)(1+m)}\right] \\ &= a_2 - \left[\frac{m(a_2 - a_1)}{(1+m)}\right], \end{aligned} \quad (4.1.33)$$

$$\begin{aligned}\lim_{\omega\to\infty} M_f^-(A) &= \lim_{\omega\to\infty} a_2 - \left[\frac{m^2(a_2-a_1)(\omega-1)}{(\omega-m+m\omega)(1+m)}\right]\\ &= a_2 - \left[\frac{m^2(a_2-a_1)}{(1+m)^2}\right], \qquad (4.1.34)\end{aligned}$$

$$\lim_{m\to\infty} M_f^-(A) = \lim_{m\to\infty} a_2 - \left[\frac{m^2(a_2-a_1)(\omega-1)}{(\omega-m+m\omega)(1+m)}\right] = a_1, \qquad (4.1.35)$$

$$\lim_{m\to 0} M_f^-(A) = \lim_{m\to 0} a_2 - \left[\frac{m^2(a_2-a_1)(\omega-1)}{(\omega-m+m\omega)(1+m)}\right] = a_2. \qquad (4.1.36)$$

The upper weighted possibilistic mean is

$$\begin{aligned}M_f^+(A) &= \int_0^1 [a_3-(a_3-a_4)(1-\alpha)^{\frac{1}{n}}](\omega-1)\left(\frac{1}{\sqrt[\omega]{1-\alpha}}-1\right)d\alpha\\ &= a_3 + \left[\frac{n^2(a_4-a_3)(\omega-1)}{(\omega-n+n\omega)(1+n)}\right]. \qquad (4.1.37)\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{\omega\to 0} M_f^+(A) &= \lim_{\omega\to 0} a_3 + \left[\frac{n^2(a_4-a_3)(\omega-1)}{(\omega-n+n\omega)(1+n)}\right]\\ &= a_3 + \left[\frac{n(a_4-a_3)}{(1+n)}\right] \qquad (4.1.38)\end{aligned}$$

$$\begin{aligned}\lim_{\omega\to\infty} M_f^+(A) &= \lim_{\omega\to\infty} a_3 + \left[\frac{n^2(a_4-a_3)(\omega-1)}{(\omega-n+n\omega)(1+n)}\right]\\ &= a_3 + \left[\frac{n^2(a_4-a_3)}{(1+n)^2}\right] \qquad (4.1.39)\end{aligned}$$

$$\lim_{n\to\infty} M_f^+(A) = \lim_{n\to\infty} a_3 + \left[\frac{n^2(a_4-a_3)(\omega-1)}{(\omega-n+n\omega)(1+n)}\right] = a_4 \qquad (4.1.40)$$

$$\lim_{n\to 0} M_f^+(A) = \lim_{n\to 0} a_3 + \left[\frac{n^2(a_4-a_3)(\omega-1)}{(\omega-n+n\omega)(1+n)}\right] = a_3 \qquad (4.1.41)$$

From (4.1.32) and (4.1.37) we obtain

$$\begin{aligned}M_f(A) &= [M_f^-(A), M_f^+(A)]\\ &= \left[a_2 - \left(\frac{m^2(a_2-a_1)(\omega-1)}{(\omega-m+m\omega)(1+m)}\right),\ a_3 + \left(\frac{n^2(a_4-a_3)(\omega-1)}{(\omega-n+n\omega)(1+n)}\right)\right]\end{aligned}$$
$$(4.1.42)$$

That is,

$$\overline{M}_f(A) = \frac{1}{2}\left[a_2 + a_3 + \frac{n^2(\omega-1)(a_4-a_3)}{(\omega-n+n\omega)(1+n)} - \frac{m^2(a_2-a_1)(\omega-1)}{(\omega-m+m\omega)(1+m)}\right] \tag{4.1.43}$$

Now, when $m = n$, $a_4 - a_3 = a_2 - a_1 = \alpha$, then the expression (4.1.43) becomes

$$\overline{M}_f(A) = \frac{1}{2}[a_2 + a_3] \tag{4.1.44}$$

and

$$\begin{aligned}\lim_{\omega\to\infty} \overline{M}_f(A) &= \lim_{\omega\to\infty} \frac{1}{2}\left[a_2 + a_3 + \frac{n^2(\omega-1)(a_4-a_3)}{(\omega-n+n\omega)(n+1)} - \frac{m^2(a_2-a_1)(\omega-1)}{(\omega-m+m\omega)(1+m)}\right] \\ &= \frac{1}{2}\left[a_2 + a_3 - \frac{m^2\,(a_2-a_1)}{(1+2m+m^2)} + \frac{n^2\,(a_4-a_3)}{(1+2n+n^2)}\right],\end{aligned} \tag{4.1.45}$$

$$\begin{aligned}\lim_{m,n\to\infty} \overline{M}_f(A) &= \lim_{m,n\to\infty} \frac{1}{2}\left[a_2 + a_3 + \frac{n^2(a_4-a_3)(\omega-1)}{(\omega-n+n\omega)(1+n)} - \frac{m^2(a_2-a_1)(\omega-1)}{(\omega-m+m\omega)(1+m)}\right], \\ &= \frac{a_4+a_1}{2}\end{aligned} \tag{4.1.46}$$

$$\begin{aligned}\lim_{\omega\to 0} \overline{M}_f(A) &= \lim_{\omega\to 0} \frac{1}{2}\left[a_2 + a_3 + \frac{n^2(a_4-a_3)(\omega-1)}{(\omega-n+n\omega)(1+n)} - \frac{m^2(a_2-a_1)(\omega-1)}{(\omega-m+m\omega)(1+m)}\right] \\ &= \frac{1}{2}\left[a_2 + a_3 + \frac{n(a_4-a_3)}{(1+n)} - \frac{m(a_2-a_1)}{(1+m)}\right].\end{aligned} \tag{4.1.47}$$

(a) If we set $m = 1$ in expression (4.1.32), the f-weighted lower possibilistic mean value of A becomes

$$M_f^-(A) = a_2 - \left[\frac{(a_2-a_1)(\omega-1)}{2(2\omega-1)}\right], \tag{4.1.48}$$

and

$$\lim_{\omega\to 0} M_f^-(A) = \lim_{\omega\to 0} a_2 - \left[\frac{(a_2-a_1)(\omega-1)}{2(2\omega-1)}\right] = \left[\frac{a_2+a_1}{2}\right], \tag{4.1.49}$$

$$\lim_{\omega\to\infty} M_f^-(A) = \lim_{\omega\to\infty} a_2 - \left[\frac{(a_2-a_1)(\omega-1)}{2(2\omega-1)}\right] = \frac{3a_2+a_1}{4}. \tag{4.1.50}$$

(b) If we set $n = 1$ in expression (4.1.37), the f-weighted upper possibilistic mean value of A is becomes

$$M_f^-(A) = a_2 - \left[\frac{(a_2 - a_1)(\omega - 1)}{2(2\omega - 1)}\right], \tag{4.1.51}$$

and

$$\lim_{\omega \to \infty} M_f^-(A) = \lim_{\omega \to \infty} a_2 - \left[\frac{(a_2 - a_1)(\omega - 1)}{2(2\omega - 1)}\right] = \frac{3a_2 + a_1}{4}, \tag{4.1.52}$$

$$\lim_{\omega \to 0} M_f^-(A) = \lim_{\omega \to 0} a_2 - \left[\frac{(a_2 - a_1)(\omega - 1)}{2(2\omega - 1)}\right] = \frac{a_2 + a_1}{2}. \tag{4.1.53}$$

(c) Given the weighted function $f(\alpha) = (\omega - 1)\left(\frac{1}{\sqrt[\omega]{1 - \alpha}} - 1\right)$ and the fuzzy number $\tilde{A}$, then the f-weighted interval-valued possibilistic mean A is

$$[M_f^-(A),\ M_f^+(A)] = \left[a_2 - \frac{(a_2 - a_1)(\omega - 1)}{2(2\omega - 1)},\ a_3 + \frac{(a_4 - a_3)(\omega - 1)}{2(2\omega - 1)}\right]. \tag{4.1.54}$$

(d) The f-weighted possibilistic mean of $\tilde{A}$,

$$\overline{M}_f(A) = \frac{1}{2}\left[a_2 + a_3 + \frac{(\omega - 1)(a_1 + a_4 - (a_3 + a_2))}{2(2\omega - 1)}\right]. \tag{4.1.55}$$

(e) When $\omega \longrightarrow \infty$, the f-weighted possibilistic mean of $\tilde{A}$ is

$$\begin{aligned} \lim_{\omega \to \infty} \overline{M}_f(A) &= \lim_{\omega \to \infty} \frac{1}{2}\left[a_2 + a_3 + \frac{(\omega - 1)(a_1 + a_4 - (a_3 + a_2))}{2(2\omega - 1)}\right] \\ &= \left[\frac{a_1 + a_2 + a_3 + a_4}{4}\right], \end{aligned} \tag{4.1.56}$$

and

$$\begin{aligned} \lim_{\omega \to 0} \overline{M}_f(A) &= \lim_{\omega \to 0} \frac{1}{2}\left[a_2 + a_3 + \frac{(\omega - 1)(a_1 + a_4 - (a_3 + a_2))}{2(2\omega - 1)}\right] \\ &= \left[\frac{a_1 + a_2 + a_3 + a_4}{4}\right]. \end{aligned} \tag{4.1.57}$$

4.1.6 Application 3

Let $\tilde{A}$ be an O(m,n)-Tr.T.F.N and let $(\omega+1)\alpha^{\omega}$ be a weighted function, then the f-weighting possibilistic variance is computed as follows.

$$\begin{aligned}
Var_f(A) &= \int_0^1 \left(\frac{a_2(\alpha)-a_1(\alpha)}{2}\right)^2 f(\alpha)d\alpha \\
&= \frac{(\omega+1)}{4}\int_0^1 \left[(a_3-a_2)+(a_2-a_1)(1-\alpha)^{\frac{1}{m}}-(a_3-a_4)(1-\alpha)^{\frac{1}{n}})\right]^2 \alpha^{\omega}d\alpha
\end{aligned}$$

$$\begin{aligned}
&= \frac{(a_3-a_2)^2}{4}+\left(\frac{(\omega+1)(a_2-a_1)^2}{4}\right)\left[\frac{\Gamma(\omega+1)\Gamma(\frac{2}{m}+1)}{\Gamma(\omega+\frac{2}{m}+2)}\right]+ \\
&\left(\frac{(\omega+1)(a_3-a_4)^2}{4}\right)\left[\frac{\Gamma(\omega+1)\Gamma(\frac{2}{n}+1)}{\Gamma(\omega+\frac{2}{n}+2)}\right]+ \\
&\left(\frac{(\omega+1)(a_3-a_2)(a_2-a_1)}{2}\right)\left[\frac{\Gamma(\omega+1)\Gamma(\frac{1}{m}+1)}{\Gamma(\omega+\frac{1}{m}+2)}\right]- \\
&\left(\frac{(\omega+1)(a_3-a_2)(a_3-a_4)}{2}\right)\left[\frac{\Gamma(\omega+1)\Gamma(\frac{1}{n}+1)}{\Gamma(\omega+\frac{1}{m}+2)}\right]- \\
&\left(\frac{(\omega+1)(a_2-a_1)(a_3-a_4)}{2}\right)\left[\frac{\Gamma(\omega+1)\Gamma(\frac{1}{m}+\frac{1}{n}+1)}{\Gamma(\omega+\frac{1}{m}+\frac{1}{n}+2)}\right] \qquad (4.1.58)
\end{aligned}$$

From (4.1.58) we obtain

$$\begin{aligned}
\lim_{\omega\to 0} Var_f(A) &= \frac{(a_3-a_2)^2}{4}+\frac{m(a_2-a_1)^2}{4(2+m)} \\
&+\frac{n(a_3-a_4)^2}{4(2+n)}+\frac{m(a_3-a_2)(a_2-a_1)}{2(1+m)}- \\
&\frac{m(a_3-a_2)(a_3-a_4)\frac{1}{n}!}{2(1+m)\frac{1}{m}!}-\frac{(a_2-a_1)(a_3-a_4)}{2}\left[\frac{mn}{m+n+mn}\right] \\
&\qquad (4.1.59)
\end{aligned}$$

(a) If we set $m = n = 1$ in (4.1.58) and use some of the properties of Gamma function, then,

$$\begin{aligned} Var_f(A) &= \left[\frac{(a_3-a_2)^2}{4}\right] + \left[\frac{(a_3-a_2)\left((a_2-a_1)+(a_4-a_3)\right)}{2\,(\omega+2)}\right] + \\ &\quad \left[\frac{((a_2-a_1)+(a_4-a_3))^2}{2\,(\omega+2)\,(\omega+3)}\right] \end{aligned} \tag{4.1.60}$$

(b) If we set $\alpha = (a_2 - a_1)$ and $\beta = (a_4 - a_3)$ in (4.1.60), then $Var_f(A)$ can be re-written as

$$\begin{aligned} Var_f(A) &= \left[\left(\frac{a_3-a_2}{2}\right)^2 + \frac{(a_3-a_2)(\alpha+\beta)}{2\,(\omega+2)} + \left(\frac{\alpha+\beta}{2\,(\omega+2)}\right)^2\right] + \\ &\quad \left[\frac{(\alpha+\beta)^2}{2\,(\omega+2)\,(\omega+3)} - \frac{(\alpha+\beta)^2}{4\,(\omega+2)^2}\right] \\ &= \left[\frac{a_3-a_2}{2} + \frac{\alpha+\beta}{2\,(\omega+2)}\right]^2 + \left[\frac{(\alpha+\beta)^2\,(\omega+1)}{4\,(\omega+2)^2\,(\omega+3)}\right] \end{aligned} \tag{4.1.61}$$

(c) If $\alpha = \beta$ in (4.1.61), we obtain the variance of a symmetric Tr.F.N.

$$Var_f(A) = \left[\frac{a_3-a_2}{2} + \frac{\alpha}{(\omega+2)}\right]^2 + \left[\frac{\alpha^2\,(\omega+1)}{(\omega+2)^2\,(\omega+3)}\right] \tag{4.1.62}$$

(d) When $\omega \longrightarrow \infty$, the limit of the variance given by (4.1.61) becomes

$$\begin{aligned} \lim_{\omega\to\infty} Var_f(A) &= \lim_{\omega\to\infty} \left[\frac{a_3-a_2}{2} + \frac{\alpha+\beta}{2\,(\omega+2)}\right]^2 + \left[\frac{(\alpha+\beta)^2\,(\omega+1)}{4\,(\omega+2)^2\,(\omega+3)}\right] \\ &= \frac{(a_3-a_2)^2}{4} \end{aligned} \tag{4.1.63}$$

(e) When $\omega \longrightarrow 0$, the limit of the variance given by (4.1.61) becomes

$$\begin{aligned} \lim_{\omega\to 0} Var_f(A) &= \lim_{\omega\to 0} \left[\frac{a_3-a_2}{2} + \frac{\alpha+\beta}{2(\omega+2)}\right]^2 + \left[\frac{(\alpha+\beta)^2\,(\omega+1)}{4(\omega+2)^2(\omega+3)}\right] \\ &= \left[\frac{(a_3-a_2)^2}{4} + \frac{(\alpha+\beta)^2}{12} + \frac{(a_3-a_2)(\alpha+\beta)}{4}\right] \end{aligned} \tag{4.1.64}$$

(f) Thus, if $\tilde{A}$ is a symmetric fuzzy number, then the expression (4.1.64) becomes

$$\lim_{\omega\to 0} Var_f(A) = \left[\frac{(a_3-a_2)^2}{4} + \frac{\alpha^2}{3} + \frac{(a_3-a_2)\alpha}{2}\right] \tag{4.1.65}$$

4.1.7 Application 4

Let $\tilde{A}$ and $\tilde{B}$ be two O(m,n)-Tr.T.F.N's. Let $f(\alpha) = (\omega+1)\alpha^{\omega}$ be a weighted function then the power-weighted covariance between $\tilde{A}$ and $\tilde{B}$ is computed as follows.

$$Cov_f(A,B) = \int_0^1 \left(\frac{a_2(\alpha)-a_1(\alpha)}{2}\right)\left(\frac{b_2(\alpha)-b_1(\alpha)}{2}\right) f(\alpha)d\alpha$$

$$\begin{aligned}
Cov_f(A,B) = {} & \left[\frac{a_3b_3}{4}\right] - \left[\frac{(\omega+1)\left[a_3(b_3-b_4)+b_3(a_3-a_4)\right]}{4}\right]\left[\frac{\Gamma(\omega+1)\Gamma(\frac{1}{n}+1)}{\Gamma(\omega+\frac{1}{n}+2)}\right] + \\
& \left[\frac{(\omega+1)(a_3-a_4)(b_3-b_4)}{4}\right]\left[\frac{\Gamma(\omega+1)\Gamma(\frac{2}{n}+1)}{\Gamma(\omega+\frac{2}{n}+2)}\right] - \left[\frac{a_3b_2}{4}\right] + \\
& \left[\frac{(\omega+1)a_3(b_2-b_1)}{4}\right]\left[\frac{\Gamma(\omega+1)\Gamma(\frac{1}{m}+1)}{\Gamma(\omega+\frac{1}{m}+2)}\right] + \\
& \left[\frac{(\omega+1)b_2(a_3-a_4)}{4}\right]\left[\frac{\Gamma(\omega+1)\Gamma(\frac{1}{n}+1)}{\Gamma(\omega+\frac{1}{n}+2)}\right] - \\
& \left[\frac{(\omega+1)(a_3-a_4)(b_2-b_1)}{4}\right]\left[\frac{\Gamma(\omega+1)\Gamma(\frac{1}{m}+\frac{1}{n}+1)}{\Gamma(\omega+\frac{1}{m}+\frac{1}{n}+2)}\right] - \\
& \left[\frac{a_2b_3}{4}\right] + \left[\frac{(\omega+1)a_2(b_3-b_4)}{4}\right]\left[\frac{\Gamma(\omega+1)\Gamma(\frac{1}{n}+1)}{\Gamma(\omega+\frac{1}{n}+2)}\right] + \\
& \left[\frac{(\omega+1)b_3(a_2-a_1)}{4}\right]\left[\frac{\Gamma(\omega+1)\Gamma(\frac{1}{m}+1)}{\Gamma(\omega+\frac{1}{m}+2)}\right] -
\end{aligned}$$

$$\left[\frac{(\omega+1)(a_2-a_1)(b_3-b_4)}{4}\right]\left[\frac{\Gamma(\omega+1)\Gamma(\frac{1}{m}+\frac{1}{n}+1)}{\Gamma(\omega+\frac{1}{m}+\frac{1}{n}+2)}\right]+$$

$$\left[\frac{a_2b_2}{4}\right]-\left[\frac{(\omega+1)\,(a_2(b_2-b_1)+b_2(a_2-a_1))}{4}\right]\left[\frac{\Gamma(\omega+1)\Gamma(\frac{1}{m}+1)}{\Gamma(\omega+\frac{1}{m}+2)}\right]+$$

$$\left[\frac{(\omega+1)(a_2-a_1)(b_2-b_1)}{4}\right]\left[\frac{\Gamma(\omega+1)\Gamma(\frac{2}{m}+1)}{\Gamma(\omega+\frac{2}{m}+2)}\right] \quad (4.1.66)$$

Thus,

$$\begin{aligned}\lim_{\omega\to 0} Cov_f(A) \quad = \quad & \left[\frac{(b_2-b_3)\,(a_2-a_3)}{4}\right]+\\ & n\left[\frac{(b_3-b_4)(a_2-a_3)+(a_3-a_4)(b_2-b_3)}{4(1+n)}\right]\\ & +m\left[\frac{(a_3-a_2)(b_2-b_1)+(b_3-b_2)(a_2-a_1)}{4\,(1+m)}\right]\\ & -mn\left[\frac{(a_3-a_4)(b_2-b_1)+(a_2-a_1)(b_3-b_4)}{4(m+n+mn)}\right]\\ & +m\left[\frac{(a_2-a_1)(b_2-b_1)}{4(2+m)}\right]+n\left[\frac{(a_3-a_4)(b_3-b_4)}{4\,(2+n)}\right] \end{aligned} \quad (4.1.67)$$

If $m, n \longrightarrow \infty$ in (4.1.66) then,

$$\lim_{m,n\to\infty} Cov_f(A) \quad = \quad \frac{(b_4-b_1)\,(a_4-a_1)}{4} \quad (4.1.68)$$

(a) If we set $m = n = 1$ and $(a_2 - a_1) = \alpha$, $(b_2 - b_1) = \alpha_1$, $(a_4 - a_3) = \beta$, $(b_4 - b_3) = \beta_1$ in (4.1.66) and use some of the properties of Gamma function, then,

$$\begin{aligned} Cov_f(A,B) \quad = \quad & \left[\frac{(a_3-a_2)}{2}+\frac{(\alpha+\beta)}{2\,(\omega+2)}\right]\left[\frac{(b_3-b_2)}{2}+\frac{(\alpha_1+\beta_1)}{2\,(\omega+2)}\right]+\\ & \left[\frac{(\omega+1)\,(\alpha_1+\beta_1)\,(\alpha+\beta)}{4\,(\omega+2)^2\,(\omega+3)}\right] \end{aligned} \quad (4.1.69)$$

(b)

$$\lim_{\omega\to\infty} Cov_f(A,B) \quad = \quad \left[\frac{(a_3-a_2)}{2}\frac{(b_3-b_2)}{2}\right] \quad (4.1.70)$$

(c) If $\tilde{A}$ is a triangular number then the weighted covariance given by (4.1.69) reduce to

$$\begin{aligned} Cov_f(A,B) &= \left[\frac{(\alpha+\beta)}{2(\omega+2)}\right]\left[\frac{(\alpha_1+\beta_1)}{2(\omega+2)}\right] + \left[\frac{(\omega+1)(\alpha_1+\beta_1)(\alpha+\beta)}{4(\omega+2)^2(\omega+3)}\right] \\ &= \left[\frac{(\alpha+\beta)(\alpha_1+\beta_1)}{2(\omega+2)(\omega+3)}\right] \end{aligned} \tag{4.1.71}$$

It may be pointed out here that results proved in (4.1.69)- (4.1.71) are similar to the results proved by Fuller and Carlsson [30] and Fuller and Majlender [53] for T.F.N's. Also,

$$\lim_{\omega\to\infty}\left[\frac{(\alpha+\beta)(\alpha_1+\beta_1)}{2(\omega+2)(\omega+3)}\right] = 0$$

$$\lim_{\omega\to 0}\left[\frac{(\alpha+\beta)(\alpha_1+\beta_1)}{2(\omega+2)(\omega+3)}\right] = \frac{(\alpha+\beta)(\alpha_1+\beta_1)}{12}$$

(d) If $\tilde{A}$ and $\tilde{B}$ are symmetric triangular fuzzy number then the weighted covariance from (4.1.69) reduce to

$$Cov_f(A,B) = \left[\frac{(\alpha+\beta)^2}{(\omega+2)(\omega+3)}\right] \tag{4.1.72}$$

and

$$\lim_{\omega\to\infty}\left[\frac{(\alpha+\beta)^2}{2(\omega+2)(\omega+3)}\right] = 0$$

$$\lim_{\omega\to 0}\left[\frac{(\alpha+\beta)^2}{2(\omega+2)(\omega+3)}\right] = \frac{(\alpha+\beta)^2}{12}$$

4.2 Conclusion

In this chapter using possibility approach some of the moment properties of O(m,n)-Tr.T.F.N's are discussed. Expressions for their possibilistic mean, possibilistic variance, possibilistic variance and possibilistic covariance are also derived. Furthermore, interval valued probability and possibilistic mean of O(m,n)-Tr.T.F.N's are also discussed and some numerical examples are provided to reinforce the results. Some applications are provided in the form of examples using weighted functions.

Chapter 5

Binomial Option Pricing Model with O(m, n)-Tr.T.F.N's.

Binomial option pricing [6] is a simple but powerful technique that can be used to solve many complex lattice option-pricing problems. Due to the vague fluctuation of financial markets, it is natural to consider fuzzy parameters in the binomial option pricing model where the stock price at each state may take imprecise values. Using O(m, n)-Tr.T.F.N's application of fuzzy sets theory to the binomial option pricing model is proposed in this chapter. The imprecision in the stock price movements yields both the risk-neutral probabilities and the stock price as weighted intervals instead of one crisp value. The proposed model is flexible and allows for additional insight into the binomial option pricing model.

We assume that the price of a stock at time $t = 0$ is S, whereas at $t = 1$ we obtain its price by multiplying S with the jump factors. Let the α-cuts for a fuzzy increase $\tilde{u} = [u_1, u_2, u_3, u_4]_{O(m,n)}$ and a fuzzy decrease $\tilde{d} = [d_1, d_2, d_3, d_4]_{O(m,n)}$, respectively, in the stock price be given by α-cuts

$u(\alpha) = [u_1(\alpha), u_2(\alpha)]$, and $d(\alpha) = [d_1(\alpha), d_2(\alpha)]$ such that we have

$$u_1(\alpha) = [u_2 - (u_2 - u_1)(1-\alpha)^{\frac{1}{m}}], \qquad 0 \leq \alpha \leq 1 \tag{5.0.1}$$

$$u_2(\alpha) = u_3 - (u_3 - u_4)(1-\alpha)^{\frac{1}{n}})], \qquad 0 \leq \alpha \leq 1 \tag{5.0.2}$$

$$d_1(\alpha) = [d_2 - (d_2 - d_1)(1-\alpha)^{\frac{1}{m}}], \qquad 0 \leq \alpha \leq 1 \tag{5.0.3}$$

$$d_2(\alpha) = d_3 - (d_3 - d_4)(1-\alpha)^{\frac{1}{n}})], \qquad 0 \leq \alpha \leq 1. \tag{5.0.4}$$

Below we now state and prove a theorem that provides the results for the fuzzy risk-neutral probabilities needed to price a fuzzy call option under O(m, n)-Tr.T.F.N's.

5.1 Main Results

Theorem 5.1.1 *Let $\tilde{d} = [d_1, d_2, d_3, d_4]_{O(m,n)}$ and $\tilde{u} = [u_1, u_2, u_3, u_4]_{O(m,n)}$, respectively, represent fuzzy decrease and fuzzy increase in the stock price. Let,*

(a) $d(\alpha) = [d_1(\alpha),\ d_2(\alpha)]$, $\forall$, $0 \leq \alpha \leq 1$, be the $\alpha-$ cut for $\tilde{d}$,

(b) $u(\alpha) = [u_1(\alpha),\ u_2(\alpha)]$, $\forall$, $0 \leq \alpha \leq 1$, be the $\alpha-$ cut for $\tilde{u}$,

(c) $p_d(\alpha) = [p_{d1}(\alpha),\ p_{d2}(\alpha)]$, $\forall$, $0 \leq \alpha \leq 1$, be the fuzzy risk neutral probability associated with $\tilde{d}$,

(d) $p_u(\alpha) = [p_{u1}(\alpha),\ p_{u2}(\alpha)]$, $\forall$, $0 \leq \alpha \leq 1$, be the fuzzy risk neutral probability associated with $\tilde{u}$, and

(e) r be the risk free rate, assumed to be a constant (crisp).

Then, the

(i) fuzzy risk neutral probability associated with a fuzzy downward movement in the stock price is given by

$$p_d(\alpha) = [p_{d1}(\alpha), p_{d2}(\alpha)] = \left[\frac{u_1(\alpha)-(1+r)}{u_1(\alpha)-d_1(\alpha)}, \frac{u_2(\alpha)-(1+r)}{u_2(\alpha)-d_2(\alpha)}\right] \quad (5.1.1)$$

where,

$$p_{d1}(\alpha) = \left[\frac{(u_2-1-r)-(u_2-u_1)(1-\alpha)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-\alpha)^{\frac{1}{m}}}\right] \quad (5.1.2)$$

$$p_{d2}(\alpha) = \left[\frac{(u_3-1-r)-(u_3-u_4)(1-\alpha)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)(1-\alpha)^{\frac{1}{n}}}\right] \quad (5.1.3)$$

(ii) fuzzy risk neutral probability associated with a fuzzy upward movement in the stock price is given by

$$p_u(\alpha) = [p_{u1}(\alpha), p_{u2}(\alpha)] = \left[\frac{(1+r)-d_2(\alpha)}{u_2(\alpha)-d_2(\alpha)}, \frac{(1+r)-d_1(\alpha)}{u_1(\alpha)-d_1(\alpha)}\right] \quad (5.1.4)$$

where

$$p_{u1}(\alpha) = \left[\frac{(1+r-d_3)+(d_3-d_4)(1-\alpha)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)(1-\alpha)^{\frac{1}{n}})}\right] \quad (5.1.5)$$

$$p_{u2}(\alpha) = \left[\frac{(1+r-d_2)-(d_2-d_1)(1-\alpha)^{\frac{1}{m}})}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-\alpha)^{\frac{1}{m}}}\right] \quad (5.1.6)$$

***Proof*:**

As given above, the α-cut for a fuzzy increase and a fuzzy decrease in the stock price, respectively are

$$u(\alpha) = [u_1(\alpha), u_2(\alpha)] = [u_2-(u_2-u_1)(1-\alpha)^{\frac{1}{m}}, u_3-(u_3-u_4)(1-\alpha)^{\frac{1}{n}})]$$

$$d(\alpha) = [d_1(\alpha), d_2(\alpha)] = [d_2-(d_2-d_1)(1-\alpha)^{\frac{1}{m}}, d_3-(d_3-d_4)(1-\alpha)^{\frac{1}{n}})]$$

$$(5.1.7)$$

Also, we have

$$p_d(\alpha) + p_u(\alpha) = 1 \tag{5.1.8}$$

$$\frac{[d_1(\alpha), d_2(\alpha)]}{1+r} p_d(\alpha) + \frac{[u_1(\alpha), u_2(\alpha)]}{1+r} p_u(\alpha) = 1 \tag{5.1.9}$$

Following [4] and [98], this leads to the following two systems of equations.

$$p_{d1}(\alpha) + p_{u2}(\alpha) = 1 \tag{5.1.10}$$

$$\frac{d_1(\alpha)}{1+r} p_{d1}(\alpha) + \frac{u_1(\alpha)}{1+r} p_{u2}(\alpha) = 1 \tag{5.1.11}$$

and

$$p_{d2}(\alpha) + p_{u1}(\alpha) = 1 \tag{5.1.12}$$

$$\frac{d_2(\alpha)}{1+r} p_{d2}(\alpha) + \frac{u_2(\alpha)}{1+r} p_{u1}(\alpha) = 1 \tag{5.1.13}$$

Equations (5.1.10)-(5.1.13) can be re-arranged so as to yield the following two sets of equations.

$$p_{d1}(\alpha) + p_{u2}(\alpha) = 1 \tag{5.1.14}$$

$$d_1(\alpha) p_{d1}(\alpha) + u_1(\alpha) p_{u2}(\alpha) = 1 + r \tag{5.1.15}$$

and

$$p_{d2}(\alpha) + p_{u1}(\alpha) = 1 \tag{5.1.16}$$

$$d_2(\alpha) p_{d2}(\alpha) + u_2(\alpha) p_{u1}(\alpha) = 1 + r \tag{5.1.17}$$

Solving equations (5.1.14)-(5.1.17), we obtain the following α-cuts for the risk neutral probability $p_d(\alpha)$ for a down movement in the stock price as well as the risk neutral probability $p_u(\alpha)$ for an up movement in the stock price.

$$p_d(\alpha) = [p_{d1}(\alpha), p_{d2}(\alpha)] = \left[\frac{u_1(\alpha)-(1+r)}{u_1(\alpha)-d_1(\alpha)}, \frac{u_2(\alpha)-(1+r)}{u_2(\alpha)-d_2(\alpha)}\right] \quad (5.1.18)$$

$$p_u(\alpha) = [p_{u1}(\alpha), p_{u2}(\alpha)] = \left[\frac{(1+r)-d_2(\alpha)}{u_2(\alpha)-d_2(\alpha)}, \frac{(1+r)-d_1(\alpha)}{u_1(\alpha)-d_1(\alpha)}\right] \quad (5.1.19)$$

Remark 5.1.1 *It is important to observe here that the above proof to find $p_d(\alpha)$ and $p_u(\alpha)$ is not dependent on the type of fuzzy number. Also, it may be noted that expressions (5.1.18) and (5.1.19) for $p_d(\alpha)$ and $p_u(\alpha)$ are also independent of the type of fuzzy numbers.*

1. From (5.1.7), (5.1.7) and (5.1.18), we obtain the following expression for the risk neutral probability $p_d(\alpha)$ for a fuzzy decrease in the stock price.

$$\begin{aligned} p_{d1}(\alpha) &= \left[\frac{u_1(\alpha)-(1+r)}{u_1(\alpha)-d_1(\alpha)}\right] \\ &= \left[\frac{\left(u_2-(u_2-u_1)(1-\alpha)^{\frac{1}{m}}\right)-(1+r)}{\left(u_2-(u_2-u_1)(1-\alpha)^{\frac{1}{m}}\right)-\left(d_2-(d_2-d_1)(1-\alpha)^{\frac{1}{m}}\right)}\right] \\ &= \left[\frac{(u_2-1-r)-(u_2-u_1)(1-\alpha)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-\alpha)^{\frac{1}{m}}}\right] \end{aligned} \quad (5.1.20)$$

$$p_{d2}(\alpha) = \left[\frac{u_2(\alpha)-(1+r)}{u_2(\alpha)-d_2(\alpha)}\right]$$

$$= \left[\frac{(u_3-(u_3-u_4)(1-\alpha)^{\frac{1}{n}})-(1+r)}{(u_3-(u_3-u_4)(1-\alpha)^{\frac{1}{n}})-(d_3-(d_3-d_4)(1-\alpha)^{\frac{1}{n}})}\right]$$

$$= \left[\frac{(u_3-1-r)-(u_3-u_4)(1-\alpha)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)(1-\alpha)^{\frac{1}{n}}}\right] \tag{5.1.21}$$

2. From (5.1.7), (5.1.7) and (5.1.19) we obtain the following expression for the risk neutral probability $p_u(\alpha)$ for a fuzzy increase in the stock price

$$p_{u1}(\alpha) = \left[\frac{(1+r)-d_2(\alpha)}{u_2(\alpha)-d_2(\alpha)}\right]$$

$$= \left[\frac{(1+r)-(d_3-(d_3-d_4)(1-\alpha)^{\frac{1}{n}})}{(u_3-(u_3-u_4)(1-\alpha)^{\frac{1}{n}})-(d_3-(d_3-d_4)(1-\alpha)^{\frac{1}{n}})}\right]$$

$$= \left[\frac{(1+r-d_3)+(d_3-d_4)(1-\alpha)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)(1-\alpha)^{\frac{1}{n}}}\right] \tag{5.1.22}$$

$$p_{u2}(\alpha) = \left[\frac{(1+r)-d_1(\alpha)}{u_1(\alpha)-d_1(\alpha)}\right]$$

$$= \left[\frac{(1+r)-(d_2-(d_2-d_1)(1-\alpha)^{\frac{1}{m}}}{(u_2-(u_2-u_1)(1-\alpha)^{\frac{1}{m}})-(d_2-(d_2-d_1)(1-\alpha)^{\frac{1}{m}})}\right]$$

$$= \left[\frac{(1+r-d_2)+(d_2-d_1)(1-\alpha)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-\alpha)^{\frac{1}{m}}}\right] \tag{5.1.23}$$

This proves the Theorem.

Corollary 5.1.1 . *Let $p_d(\alpha) = [p_{d1}(\alpha), p_{d2}(\alpha)]$ be the fuzzy risk neutral probability of an increase in the stock price and let $p_d(\alpha) = [p_{d1}(\alpha), p_{d2}(\alpha)]$ be the fuzzy risk neutral probability of a decrease in the stock price, then from (5.1.20) - (5.1.23) the following complimentary conditions hold true for $0 \le \alpha \le 1$. Therefore,*

$$\left[\frac{(u_2-1-r)-(u_2-u_1)(1-\alpha)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-\alpha)^{\frac{1}{m}}}\right]+ \left[\frac{(1+r-d_2)+(d_2-d_1)(1-\alpha)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-\alpha)^{\frac{1}{m}}}\right]=1$$

$$\left[\frac{(u_3-1-r)-(u_3-u_4)(1-\alpha)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)(1-\alpha)^{\frac{1}{n}}}\right]+ \left[\frac{(1+r-d_3)+(d_3-d_4)(1-\alpha)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)(1-\alpha)^{\frac{1}{n}}}\right]=1$$

Muzzioli and Torricelli [98], discussed the complimentary relationship for the fuzzy risk neutral probabilities when the parameters in the option pricing model are triangular fuzzy numbers. However, from (5.1.18) and (5.1.19) we observe that the complimentary hold true irrespective of the fuzzy number.

5.1.1 Properties of the Fuzzy Risk Neutral Probabilities

In this section, we discuss some of the properties of the fuzzy risk neutral probabilities.

$$\begin{aligned}\lim_{m\to\infty} p_{d1}(\alpha) &= \lim_{m\to\infty}\left[\frac{(u_2-1-r)-(u_2-u_1)(1-\alpha)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-\alpha)^{\frac{1}{m}}}\right]\\ &= \left[\frac{u_1-1-r}{u_1-d_1}\right] \qquad (5.1.24)\end{aligned}$$

$$\begin{aligned}\lim_{m\to 0} p_{d1}(\alpha) &= \lim_{m\to 0}\left[\frac{(u_2-1-r)-(u_2-u_1)(1-\alpha)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-\alpha)^{\frac{1}{m}}}\right]\\ &= \left[\frac{u_2-1-r}{u_2-d_2}\right] \qquad (5.1.25)\end{aligned}$$

For $m = 1$

$$p_{d1}(\alpha) = \left[\frac{u_1 + (u_2 - u_1)\alpha - (1+r)}{u_1 - d_1 + (u_2 - u_1 - d_2 + d_1)\alpha}\right] \tag{5.1.26}$$

The result obtained in (5.1.26) is analogous to the result obtained by Appadoo et. al [4] for $p_{d1}(\alpha)$, and coincides with $p_{d1}(\alpha)$ for a Tr.F.N.

$$\begin{aligned} \lim_{n\to\infty} p_{d2}(\alpha) &= \lim_{n\to\infty} \left[\frac{(u_3 - 1 - r) - (u_3 - u_4)(1-\alpha)^{\frac{1}{n}}}{u_3 - d_3 - (u_3 - u_4 - d_3 + d_4)(1-\alpha)^{\frac{1}{n}}}\right] \\ &= \left[\frac{u_4 - 1 - r}{u_4 - d_4}\right] \end{aligned} \tag{5.1.27}$$

$$\begin{aligned} \lim_{n\to 0} p_{d2}(\alpha) &= \lim_{n\to\infty} \left[\frac{(u_3 - 1 - r) - (u_3 - u_4)(1-\alpha)^{\frac{1}{n}}}{u_3 - d_3 - (u_3 - u_4 - d_3 + d_4)(1-\alpha)^{\frac{1}{n}}}\right] \\ &= \left[\frac{u_3 - 1 - r}{u_3 - d_3}\right] \end{aligned} \tag{5.1.28}$$

For $n = 1$.

$$p_{d2}(\alpha) = \left[\frac{u_4 - 1 - r + (u_3 - u_4)\alpha}{u_4 - d_4 + (u_3 - u_4 + d_4 - d_3)\alpha}\right] \tag{5.1.29}$$

The result obtained in (5.1.29) is analogous to the result obtained by Appadoo et. al [4] for $p_{d2}(\alpha)$ and coincides with $p_{d2}(\alpha)$ for a Tr.F.N. Also, if we set $u_4 = u_3$, $d_4 = d_3$, $u_3 = u_2$ and $d_3 = d_2$ in (5.1.29), we obtained the result for $p_{d2}(\alpha)$ for T.F.N's as follows.

$$p_{d2}(\alpha) = \left[\frac{u_3 + (u_2 - u_3)\alpha - (1+r)}{u_4 - d_3 + (u_2 - u_3 + d_3 - d_2)\alpha}\right] \tag{5.1.30}$$

The result obtained in (5.1.30) is analogous to the result obtain by Muzzioli and Torricelli [98] for $p_{d2}(\alpha)$. Similarly,

$$\begin{aligned} \lim_{n\to\infty} p_{u1}(\alpha) &= \lim_{n\to\infty} \left[\frac{(1 + r - d_3) + (d_3 - d_4)(1-\alpha)^{\frac{1}{n}}}{u_3 - d_3 - (u_3 - u_4 - d_3 + d_4)(1-\alpha)^{\frac{1}{n}}}\right] \\ &= \left[\frac{1 + r - d_4}{u_4 - d_4}\right] \end{aligned} \tag{5.1.31}$$

$$\begin{aligned}
\lim_{n\to 0}\ p_{u1}(\alpha) &= \lim_{n\to 0}\left[\frac{(1+r-d_3)+(d_3-d_4)\,(1-\alpha)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)\,(1-\alpha)^{\frac{1}{n}}}\right] \\
&= \left[\frac{1+r-d_3}{u_3-d_3}\right] \qquad (5.1.32)
\end{aligned}$$

$$\begin{aligned}
p_{u1}(\alpha) &= \left[\frac{(1+r-d_3)+(d_3-d_4)\,(1-\alpha)}{u_3-d_3-(u_3-u_4-d_3+d_4)\,(1-\alpha)}\right] \\
&= \left[\frac{1+r-d_4+(d_4-d_3)\alpha}{u_4-d_4-(u_4-u_3-d_4+d_3)\alpha}\right] \qquad (5.1.33)
\end{aligned}$$

The result obtained in (5.1.33) is analogous to the result obtained by Appadoo et. al [4] for $p_{u1}(\alpha)$. Also, if we set $u_4 = u_3,\ d_4 = d_3,\ u_3 = u_2$ and $d_3 = d_2$ in (5.1.33), for $n = 1$ we obtained the result for $p_{u1}(\alpha)$ for T.F.N's.

$$p_{u1}(\alpha) = \left[\frac{1+r-d_3+(d_3-d_2)\alpha}{u_3-d_3-(u_3-u_2-d_3+d_2)\alpha}\right] \qquad (5.1.34)$$

The result obtained in (5.1.34) is analogous to the result obtained by Muzzioli and Torricelli [98] for $p_{u1}(\alpha)$. Again,

$$\begin{aligned}
\lim_{m\to\infty} p_{u2}(\alpha) &= \lim_{m\to\infty}\left[\frac{(1+r-d_2)+(d_2-d_1)\,(1-\alpha)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)\,(1-\alpha)^{\frac{1}{m}}}\right] \\
&= \left[\frac{1+r-d_1}{u_1-d_1}\right] \qquad (5.1.35)
\end{aligned}$$

$$\begin{aligned}
\lim_{m\to 0} p_{u2}(\alpha) &= \lim_{m\to 0}\left[\frac{(1+r-d_2)+(d_2-d_1)\,(1-\alpha)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)\,(1-\alpha)^{\frac{1}{m}}}\right] \\
&= \left[\frac{1+r-d_2}{u_2-d_2}\right] \qquad (5.1.36)
\end{aligned}$$

For $m = 1$

$$\begin{aligned}
p_{u2}(\alpha) &= \left[\frac{(1+r-d_2)+(d_2-d_1)\,(1-\alpha)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)\,(1-\alpha)^{\frac{1}{m}}}\right] \\
&= \left[\frac{1+r-d_1-(d_2-d_1)\alpha}{u_1-d_1+(u_2-u_1-d_2+d_1)\alpha}\right] \qquad (5.1.37)
\end{aligned}$$

The result obtained in (5.1.37) is analogous to the result obtained by Appadoo et. al [4] for $p_{u2}(\alpha)$.

5.1.2 Membership Function for $p_d(\alpha)$ and $p_u(\alpha)$

It is important to point out here that $\tilde{p}_d$ and $\tilde{p}_u$ given in (5.1.20)- (5.1.23) represent fuzzy numbers. In this section we obtain their respective membership functions. Muzzioli and Torricelli [98], obtained results similar to (5.1.20)- (5.1.23) using triangular fuzzy numbers whereas in the curent section we consider O(m,n)-Tr.T.F.N.'s. In order to find the two ends points and the two interior points which describe the risk neutral probability of a fuzzy decrease in the stock price corresponding to the O(m, n)- Tr.T.F.N., we set $\alpha = 0$ and $\alpha = 1$ in (5.1.20) and (5.1.21). The results are

$$p_{d1}(0) = \left[\frac{(u_2-1-r)-(u_2-u_1)(1-0)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-1)^{\frac{1}{m}}}\right] = \left[\frac{u_1-1-r}{u_1-d_1}\right] \quad (5.1.38)$$

$$p_{d2}(0) = \left[\frac{(u_3-1-r)-(u_3-u_4)(1-0)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)(1-1)^{\frac{1}{n}}}\right] = \left[\frac{u_4-1-r}{u_4-d_4}\right] \quad (5.1.39)$$

$$p_{d1}(1) = \left[\frac{(u_2-1-r)-(u_2-u_1)(1-1)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-1)^{\frac{1}{m}}}\right] = \left[\frac{u_2-1-r}{u_2-d_2}\right] \quad (5.1.40)$$

$$p_{d2}(1) = \left[\frac{(u_3-1-r)-(u_3-u_4)(1-1)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)(1-1)^{\frac{1}{n}}}\right] = \left[\frac{u_3-1-r}{u_3-d_3}\right] \quad (5.1.41)$$

Similarly, to find the two ends points and the two interior points, we set $\alpha = 0$ and $\alpha = 1$ in (5.1.22) and (5.1.23). This yields

$$p_{u1}(0) = \left[\frac{(1+r-d_3)+(d_3-d_4)(1-0)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)(1-0)^{\frac{1}{n}}}\right] = \left[\frac{1+r-d_4}{u_4-d_4}\right] \quad (5.1.42)$$

$$p_{u2}(0) = \left[\frac{(1+r-d_2)+(d_2-d_1)(1-0)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-0)^{\frac{1}{m}}}\right] = \left[\frac{1+r-d_1}{u_1-d_1}\right] \quad (5.1.43)$$

$$p_{u1}(1) = \left[\frac{(1+r-d_3)+(d_3-d_4)(1-1)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)(1-\alpha)^{\frac{1}{n}}}\right] = \left[\frac{1+r-d_3}{u_3-d_3}\right] \quad (5.1.44)$$

$$p_{u2}(1) = \left[\frac{(1+r-d_2)+(d_2-d_1)(1-1)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-\alpha)^{\frac{1}{m}}}\right] = \left[\frac{1+r-d_2}{u_2-d_2}\right] \quad (5.1.45)$$

In view of (5.1.38)-(5.1.41) and (5.1.42)-(5.1.45), the ends points and the interior points of the risk neutral probabilities are

$$p_d = \left[\frac{u_1-1-r}{u_1-d_1}, \frac{u_2-1-r}{u_2-d_2}, \frac{u_3-1-r}{u_3-d_3}, \frac{u_4-1-r}{u_4-d_4}\right] \quad (5.1.46)$$

and

$$p_u = \left[\frac{1+r-d_4}{u_4-d_4}, \frac{1+r-d_3}{u_3-d_3}, \frac{1+r-d_2}{u_2-d_2}, \frac{1+r-d_1}{u_1-d_1}\right]. \quad (5.1.47)$$

Below, we determine the membership functions for the fuzzy risk neutral probabilities $\tilde{p}_d$ and $\tilde{p}_u$ in the stock price. Since,

$$p_{d1}(\alpha) = \left[\frac{(u_2-1-r)-(u_2-u_1)(1-\alpha)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-\alpha)^{\frac{1}{m}}}\right], \text{ therefore for}$$

$\dfrac{u_1-1-r}{u_1-d_1} \leq \tilde{p}_d \leq \dfrac{u_2-1-r}{u_2-d_2}$, we set $p_{d1}(\alpha) = p_d$ and we solve for α, therefore,

$$\left[\frac{(u_2-1-r)-(u_2-u_1)(1-\alpha)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-\alpha)^{\frac{1}{m}}}\right] = p_d.$$

Solving for α, we get

$$\alpha \;=\; 1-\left(\frac{p_d(u_2-d_2)-(u_2-1-r)}{(p_d(u_2-u_1-d_2+d_1)-(u_2-u_1))}\right)^m \tag{5.1.48}$$

Similarly, for $\dfrac{u_3-1-r}{u_3-d_3} \le p_d \le \dfrac{u_4-1-r}{u_4-d_4}$

we set,

$$\left[\frac{(u_3-1-r)-(u_3-u_4)\,(1-\alpha)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)\,(1-\alpha)^{\frac{1}{n}}}\right] = p_d,$$

Solving for α, we get

$$\alpha \;=\; 1-\left(\frac{p_d(u_3-d_3)-(u_3-1-r)}{(p_d(u_3-u_4-d_3+d_4)-(u_3-u_4))}\right)^n \tag{5.1.49}$$

Therefore, the membership function for the fuzzy risk neutral probability $\tilde{p}_d$ for a fuzzy downward movement in the stock price is

$$\mu(p_d) = \begin{cases} 0 & p_d \le \dfrac{u_1-1-r}{u_1-d_1} \\ 1-\left(\dfrac{p_d(u_2-d_2)-(u_2-1-r)}{(p_d(u_2-u_1-d_2+d_1)-(u_2-u_1))}\right)^m & \dfrac{u_1-1-r}{u_1-d_1} \le p_d \le \dfrac{u_2-1-r}{u_2-d_2} \\ 1 & \dfrac{u_2-1-r}{u_2-d_2} \le p_d \le \dfrac{u_3-1-r}{u_3-d_3} \\ 1-\left(\dfrac{p_d(u_3-d_3)-(u_3-1-r)}{(p_d(u_3-u_4-d_3+d_4)-(u_3-u_4))}\right)^n & \dfrac{u_3-1-r}{u_3-d_3} \le p_d \le \dfrac{u_4-1-r}{u_4-d_4} \\ 0 & p_d \ge \dfrac{u_4-1-r}{u_4-d_4} \end{cases} \tag{5.1.50}$$

Similarly, the membership function for the fuzzy risk neutral probability $\tilde{p}_u$ for a fuzzy upward movement is given by

$$\mu(p_u) = \begin{cases} 0 & \tilde{p}_u \leq \dfrac{1+r-d_4}{u_4-d_4} \\ 1-\left(\dfrac{p_u\,(u_3-d_3)-(1+r-d_3)}{(p_u(u_3-u_4-d_3+d_4)+(d_3-d_4))}\right)^n & \dfrac{1+r-d_4}{u_4-d_4} \leq p_u \leq \dfrac{1+r-d_3}{u_3-d_3} \\ 1 & \dfrac{1+r-d_3}{u_3-d_3} \leq p_u \leq \dfrac{1+r-d_2}{u_2-d_2} \\ 1-\left(\dfrac{p_u(u_2-d_2)-(1+r-d_2)}{(p_u(u_2-u_1-d_2+d_1)+(d_2-d_1))}\right)^m & \dfrac{1+r-d_2}{u_2-d_2} \leq p_u \leq \dfrac{1+r-d_1}{u_1-d_1} \\ 0 & p_u \geq \dfrac{1+r-d_1}{u_1-d_1} \end{cases} \tag{5.1.51}$$

5.1.3 Characteristics of the Fuzzy Risk Neutral Probabilities of a Fuzzy Downward and a Fuzzy Upward Movement in a Stock Price.

We now analyze the behavior of the risk neutral fuzzy probabilities of a fuzzy upward movement and a fuzzy downward movement in a stock price, under the following assumptions, A_1.

Assumptions A_1. We assume that the following inequalities hold.

$u_1 < u_2 < u_3 < u_4, \qquad d_1 < d_2 < d_3 < d_4,$

$d_1 < d_2 < d_3 < d_4 < 1+r < u_1 < u_2 < u_3 < u_4,$

$0 < p_{d1} < 1, \qquad 0 < p_{d2} < 1, \qquad 0 < p_{d3} < 1, \qquad 0 < p_{d4} < 1$

$0 < p_{u1} < 1, \qquad 0 < p_{u2} < 1, \qquad 0 < p_{u3} < 1, \qquad 0 < p_{u4} < 1.$

Now,

$$p_{d1}(\alpha) = \left[\frac{(u_2-1-r)-(u_2-u_1)\,(1-\alpha)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)\,(1-\alpha)^{\frac{1}{m}}}\right] = \frac{N_1}{D_1} \tag{5.1.52}$$

where,

$$N_1 = (u_2 - 1 - r) - (u_2 - u_1)(1-\alpha)^{\frac{1}{m}}$$

$$D_1 = u_2 - d_2 - (u_2 - u_1 - d_2 + d_1)(1-\alpha)^{\frac{1}{m}}$$

This yields,

$$\frac{dD_1}{d\alpha} = \frac{d}{d\alpha}\left(u_2 - d_2 - (u_2 - u_1 - d_2 + d_1)(1-\alpha)^{\frac{1}{m}}\right) = (u_2 - u_1 - d_2 + d_1)\frac{(1-\alpha)^{\frac{1}{m}-1}}{m}$$

$$\frac{dN_1}{d\alpha} = \frac{d}{d\alpha}\left((u_2 - 1 - r) - (u_2 - u_1)(1-\alpha)^{\frac{1}{m}}\right) = (u_2 - u_1)\frac{(1-\alpha)^{\frac{1}{m}-1}}{m}$$

Using the assumptions A_1, we get

$$\frac{dp_{d1}(\alpha)}{d\alpha} = \left[\frac{D_1(u_2 - u_1) - N_1(u_2 - u_1 - d_2 + d_1)}{mD_1^2(1-\alpha)^{1-\frac{1}{m}}}\right] > 0 \tag{5.1.53}$$

Again,

$$\frac{d^2p_{d1}(\alpha)}{d\alpha^2} = \left[\frac{(mD_1^2(1-\alpha)^{1-\frac{1}{m}})\left[\frac{d}{d\alpha}((D_1 - N_1)(u_2 - u_1) + N_1(d_2 - d_1))\right]}{(mD_1^2(1-\alpha)^{1-\frac{1}{m}})^2}\right] - \left[\frac{((D_1 - N_1)(u_2 - u_1) + N_1(d_2 - d_1))\left[\frac{d}{d\alpha}mD_1^2(1-\alpha)^{1-\frac{1}{m}}\right]}{(mD_1^2(1-\alpha)^{1-\frac{1}{m}})^2}\right] \tag{5.1.54}$$

It can be easily shown that

$$\left[\frac{d}{d\alpha}((D_1 - N_1)(u_2 - u_1) + N_1(d_2 - d_1))\right] = 0 \tag{5.1.55}$$

in (5.1.54).

Thus, it implies that the expression (5.1.54) can be rewritten as

$$\frac{d^2p_{d1}(\alpha)}{d\alpha^2} = -\left[\frac{((D_1 - N_1)(u_2 - u_1) + N_1(d_2 - d_1))\left[\frac{d}{d\alpha}mD_1^2(1-\alpha)^{1-\frac{1}{m}}\right]}{(mD_1^2(1-\alpha)^{1-\frac{1}{m}})^2}\right] \tag{5.1.56}$$

$$
\begin{aligned}
\frac{d^2p_{d1}(\alpha)}{d\alpha^2} &= -\left[\frac{\frac{dp_{d1}(\alpha)}{d\alpha}}{D_1^2\,(1-\alpha)^{1-\frac{1}{m}}}\right]\left[\frac{d}{d\alpha}D_1^2\,(1-\alpha)^{1-\frac{1}{m}}\right] \\
&= -\left[\frac{\frac{dp_{d1}(\alpha)}{d\alpha}}{D_1^2\,(1-\alpha)^{1-\frac{1}{m}}}\right]\left[(1-\alpha)^{1-\frac{1}{m}}\frac{dD_1^2}{d\alpha}+D_1^2\frac{d\,(1-\alpha)^{1-\frac{1}{m}}}{d\alpha}\right] \\
&= -\left[\frac{\frac{dp_{d1}(\alpha)}{d\alpha}}{D_1\,(1-\alpha)^{1-\frac{1}{m}}}\right]\left[\frac{2}{m}\,(u_2-u_1-d_2+d_1)+D_1\frac{\frac{1}{m}-1}{\sqrt[m]{1-\alpha}}\right]
\end{aligned}
\tag{5.1.57}
$$

Similarly,

$$
\frac{dp_{d2}(\alpha)}{d\alpha} = \left[\frac{D_2\,(u_3-u_4)-N_2\,(u_3-u_4-d_3+d_4)}{nD_2{}^2\,(1-\alpha)^{1-\frac{1}{n}}}\right]<0 \tag{5.1.58}
$$

where,

$$
\begin{aligned}
N_2 &= (u_3-1-r)-(u_3-u_4)\,(1-\alpha)^{\frac{1}{n}} \\
D_2 &= u_3-d_3-(u_3-u_4-d_3+d_4)\,(1-\alpha)^{\frac{1}{n}}
\end{aligned}
$$

and

$$
\frac{d^2p_{d2}(\alpha)}{d\alpha^2} = -\left[\frac{\frac{dp_{d2}(\alpha)}{d\alpha}}{D_2\,(1-\alpha)^{1-\frac{1}{n}}}\right]\left[\frac{2}{n}\,(u_3-u_4-d_3+d_4)+D_2\left(\frac{\frac{1}{n}-1}{\sqrt[n]{1-\alpha}}\right)\right]. \tag{5.1.59}
$$

$$
\frac{dp_{u1}(\alpha)}{d\alpha} = \left[\frac{(D_3-N_3)\,(d_4-d_3)+N_3(u_4-u_3)}{nD_3{}^2\,(1-\alpha)^{1-\frac{1}{n}}}\right]>0, \tag{5.1.60}
$$

where,

$$
\begin{aligned}
N_3 &= (1+r-d_3)+(d_3-d_4)\,(1-\alpha)^{\frac{1}{n}} \\
D_3 &= u_3-d_3-(u_3-u_4-d_3+d_4)\,(1-\alpha)^{\frac{1}{n}},
\end{aligned}
$$

and

$$\frac{d^2p_{u1}(\alpha)}{d\alpha^2} = -\left[\frac{\frac{dp_{u1}(\alpha)}{d\alpha}}{D_3\,(1-\alpha)^{1-\frac{1}{n}}}\right]\left[\frac{2}{n}\,(u_3-u_4-d_3+d_4)+D_3\left(\frac{\frac{1}{n}-1}{\sqrt[n]{1-\alpha}}\right)\right]. \tag{5.1.61}$$

Also,

$$\frac{dp_{u2}(\alpha)}{d\alpha} = \left[\frac{D_4\,(d_1-d_2)-N_4(u_2-u_1-d_2+d_1)}{mD_4{}^2\,(1-\alpha)^{1-\frac{1}{m}}}\right] < 0 \tag{5.1.62}$$

where,

$$\begin{aligned} N_4 &= (1+r-d_2)+(d_2-d_1)\,(1-\alpha)^{\frac{1}{m}}\,, \\ D_4 &= u_2-d_2-(u_2-u_1-d_2+d_1)\,(1-\alpha)^{\frac{1}{m}}\,, \end{aligned}$$

and

$$\frac{d^2p_{u2}(\alpha)}{d\alpha^2} = -\left[\frac{\frac{dp_{u2}(\alpha)}{d\alpha}}{D_4\,(1-\alpha)^{1-\frac{1}{m}}}\right]\left[\frac{2}{m}\,(u_2-u_1-d_2+d_1)\,+D_4\left(\frac{\frac{1}{m}-1}{\sqrt[m]{1-\alpha}}\right)\right]. \tag{5.1.63}$$

It may be pointed out that the second derivative of each of the fuzzy risk neutral probability of an upward movement and a fuzzy risk neutral probability of a downward movement in the stock price could be positives or negatives. If the second derivative of a fuzzy risk neutral probability is positive (negative, respectively), it implies that the corresponding probability function is convex (concave, respectively) in nature.

5.2 Call Option Value With O(m,n)-Tr.T.F.N's.

Suppose that there is a call option on a stock with exercise price K and expiration at the end of Period 1. As before, we take $u(\alpha)$ as the α-cut of a fuzzy increase $\tilde{u}$ in the stock price, $d(\alpha)$ as the α-cut of a fuzzy decrease $\tilde{d}$ in the stock price and we assume that arbitrage opportunity is not allowed. Then, the α-cut for $\tilde{d}$ and $\tilde{u}$ respectively, are

$$\begin{aligned} d(\alpha) &= [d_1(\alpha),\ d_2(\alpha)] \\ &= [d_2 - (d_2 - d_2)\,(1-\alpha)^{\frac{1}{m}}\,, d_3 - (d_3 - d_4)\,(1-\alpha)^{\frac{1}{n}}], \qquad (5.2.1) \\ u(\alpha) &= [u_1(\alpha),\ u_2(\alpha)] \\ &= [u_2 - (u_2 - u_2)\,(1-\alpha)^{\frac{1}{m}}\,, u_3 - (u_3 - u_4)\,(1-\alpha)^{\frac{1}{n}}]. \qquad (5.2.2) \end{aligned}$$

Assume that $\tilde{C}$ is a current fuzzy price of a call option on a stock whose crisp exercise price is K. Also assume that when the call option expires, it is worth either $\tilde{C}_u$ or $\tilde{C}_d$, where $\tilde{C}_u$ and $\tilde{C}_d$ are fuzzy quantities whose values are subjected to arbitrariness and subjectivity. In this case, the α-cut for $\tilde{C}_u$ is found to be

$$\begin{aligned} C_u(\alpha) &= [C_{u1}(\alpha), C_{u2}(\alpha)] \\ &= [\mathrm{Max}(Su_1(\alpha) - K, 0),\ \ \mathrm{Max}(Su_2(\alpha) - K, 0)]\,. \qquad (5.2.3) \end{aligned}$$

From (5.2.2) and (5.2.3), we get

$$C_{u1}(\alpha) = \mathrm{Max}(Su_1(\alpha) - K,\ \ 0) \qquad (5.2.4)$$

$$C_{u2}(\alpha) = \mathrm{Max}(Su_2(\alpha) - K,\ \ 0). \qquad (5.2.5)$$

Similarly, the α-cuts for $\tilde{C}_d$ is given by

$$\begin{aligned} C_d(\alpha) &= [C_{d1}(\alpha),\ C_{d2}(\alpha)] \\ &= [\mathrm{Max}(Sd_1(\alpha) - K,\ \ 0),\ \ \mathrm{Max}(Sd_2(\alpha) - K, 0)] \qquad (5.2.6) \end{aligned}$$

From (5.2.1) and (5.2.6), we get

$$C_{d1}(\alpha) \;=\; \text{Max}(Sd_1(\alpha) - K, \;\; 0) \tag{5.2.7}$$

and

$$C_{d2}(\alpha) \;=\; \text{Max}(Sd_2(\alpha) - K, \;\; 0) \tag{5.2.8}$$

If $Sd_1(\alpha)$ and $Sd_2(\alpha)$ are less than K, that is, the fuzzy stock price goes down, then the call option expires out-of-the-money. If $Su_1(\alpha)$ and $Su_2(\alpha)$ are greater than K, that is the fuzzy stock goes up, then the call option expires in-the-money. Note that if both fuzzy stock prices result in the option expiring in-the-money, then the call option will not be very speculative. However, in this case, the fuzzy model will still be able to price it.

Substituting expressions (5.2.2) for $u(\alpha)$ into expression (5.2.3), the expression for the price of the call in the up state, yields the following expressions for the price of the derivative in the up state, under the assumption that $\tilde{u}$ is an O(m,n)-Tr.T.F.N.

$$\begin{aligned}
C_u(\alpha) &= [\text{Max}(Su_1(\alpha) - K, \;\; 0), \text{Max}(Su_2(\alpha) - K, \;\; 0)] \\
&= \begin{bmatrix} \text{Max}(S(u_2 - (u_2 - u_1)(1-\alpha)^{\frac{1}{m}}) - K, \;\; 0), \\ \text{Max}(S(u_3 - (u_3 - u_4)(1-\alpha)^{\frac{1}{n}}) - K, \;\; 0) \end{bmatrix} \\
&= \left[S\left(u_2 - (u_2 - u_1)(1-\alpha)^{\frac{1}{m}}\right) - K, \;\; S\left(u_3 - (u_3 - u_4)(1-\alpha)^{\frac{1}{n}}\right) - K\right]
\end{aligned} \tag{5.2.9}$$

such that,

$$C_{u1}(\alpha) \;=\; S\left(u_2 - (u_2 - u_1)(1-\alpha)^{\frac{1}{m}}\right) - K \tag{5.2.10}$$

and

$$C_{u2}(\alpha) = S\left(u_3 - (u_3 - u_4)(1-\alpha)^{\frac{1}{n}}\right) - K \tag{5.2.11}$$

Under the assumption that $\tilde{d}$ is an O(m,n)-Tr.T.F.N., (5.2.1) and (5.2.6), yield the following expressions for the price of the derivative in the down state.

$$C_d(\alpha) = [\text{Max}(Sd_1(\alpha) - K, \ \ 0), \ \ \text{Max}(Sd_2(\alpha) - K, \ \ 0)]$$

$$= \begin{bmatrix} \text{Max}(S(d_2 - (d_2 - d_1)(1-\alpha)^{\frac{1}{m}}) - K, \ \ 0), \\ \text{Max}(S(d_3 - (d_3 - d_4)(1-\alpha)^{\frac{1}{n}}) - K, \ \ 0) \end{bmatrix} = [0, 0]$$

It may be pointed out here that when $Sd_1(\alpha)$ and $Sd_2(\alpha)$ are less than K, then the fuzzy stock price goes down and the call option expires out-of-the money. Hence, in this case the maximum value of $C_{d1}(\alpha) = 0$ and the maximum value of $C_{d2}(\alpha) = 0$. This yields,

$$C_{d1}(\alpha) = 0 \text{ and } C_{d2}(\alpha) = 0 \tag{5.2.12}$$

5.3 Expected Fuzzy Call Option Value With O(m,n)-Tr.T.F.N's.

Let $\tilde{C}$ be a fuzzy number that characterizes the fuzzy current price of the call option and let $C(\alpha)$ be the α-cut for the fuzzy current price of the option. Then, the expected fuzzy call price is given by [98],

$$C(\alpha) = \frac{1}{1+r}\tilde{E}(\tilde{C}) \tag{5.3.1}$$

where $\tilde{E}$ stands for the expectation under the fuzzy risk neutral probabilities.

Therefore,

$$\begin{aligned}[C_1(\alpha), C_2(\alpha)] &= \frac{1}{1+r}[C_{u1}(\alpha), C_{u2}(\alpha)][p_{u1}(\alpha), p_{u2}(\alpha)] \\ &+ \frac{1}{1+r}[C_{d1}(\alpha), C_{d2}(\alpha)][p_{d1}(\alpha), p_{d2}(\alpha)] \end{aligned} \tag{5.3.2}$$

Since, the fuzzy call option has zero payoff in the down state, the expected fuzzy option pricing formula given by equation (5.3.2) simplifies to

$$[C_1(\alpha), C_2(\alpha)] = \frac{1}{1+r}[C_{u1}(\alpha), C_{u2}(\alpha)][p_{u1}(\alpha), p_{u2}(\alpha)] \tag{5.3.3}$$

where $C_{u1}(\alpha)$ and $C_{u2}(\alpha)$ are as in (5.2.4) and (5.2.5) respectively. Also, $p_{u1}(\alpha)$ and $p_{u2}(\alpha)$ are given by (5.1.22) and (5.1.23) respectively.

Thus,

$$\tilde{C} = \frac{1}{1+r}\tilde{E}(\tilde{C}) = \frac{1}{1+r}[C_{u1}(\alpha)p_{u1}(\alpha), C_{u2}(\alpha)p_{u2}(\alpha)]$$

yields

$$C(\alpha) = \left[\begin{array}{c} \dfrac{S\left(u_2-(u_2-u_1)(1-\alpha)^{\frac{1}{m}}\right)-K}{1+r}\dfrac{(1+r-d_3)+(d_3-d_4)(1-\alpha)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)(1-\alpha)^{\frac{1}{n}}}, \\ \dfrac{S\left(u_3-(u_3-u_4)(1-\alpha)^{\frac{1}{n}}\right)-K}{1+r}\dfrac{(1+r-d_2)+(d_2-d_1)(1-\alpha)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-\alpha)^{\frac{1}{m}}} \end{array}\right].$$

This leads to

$$C_1(\alpha) = \frac{C_{u1}(\alpha)}{1+r}p_{u1}(\alpha) \quad \text{and} \quad C_2(\alpha) = \frac{C_{u2}}{1+r}p_{u2}(\alpha)$$

Plugging the values of $C_{u1}(\alpha)$ from (5.2.10) and $p_{u1}(\alpha)$ from (5.1.22) we obtain the following expression for the left hand part of the α-cut for the fuzzy current price of the call option.

$$C_1(\alpha) = \left[\frac{S\left(u_2 - (u_2 - u_1)(1-\alpha)^{\frac{1}{m}}\right) - K}{1+r} \frac{(1+r-d_3) + (d_3 - d_4)(1-\alpha)^{\frac{1}{n}}}{u_3 - d_3 - (u_3 - u_4 - d_3 + d_4)(1-\alpha)^{\frac{1}{n}}}\right] \tag{5.3.4}$$

Similarly, from (5.2.11) and $p_{u1}(\alpha)$ from (5.1.22) we obtain the right hand part of the α-cut

$$C_2(\alpha) = \left[\frac{S\left(u_3 - (u_3 - u_4)(1-\alpha)^{\frac{1}{n}}\right) - K}{1+r} \frac{(1+r-d_2) + (d_2 - d_1)(1-\alpha)^{\frac{1}{m}}}{u_2 - d_2 - (u_2 - u_1 - d_2 + d_1)(1-\alpha)^{\frac{1}{m}}}\right] \tag{5.3.5}$$

It is easy to show that as α increases the call option interval of price shrinks, and at $\alpha = 1$, the interval is the smallest. Similarly, at $\alpha = 0$, the call price interval is the largest. Therefore, $[C_1(\alpha), C_2(\alpha)]$ gives us a weighted expected value interval for the call price $\tilde{C}$. This is an important property for financial applications as it allows us to determine the most useful outcome of the call price $\tilde{C}$.

5.3.1 Membership Function for the Fuzzy Call Option Value.

In order to find the two exterior points and the two interior points which describe the fuzzy call price corresponding to the O(m,n)-Tr.T.F.N we set $\alpha = 0$ and $\alpha = 1$ in (5.3.4) and (5.3.5) respectively.

$$C_1(0) = \left[\frac{Su_1 - k}{1+r} \quad \frac{1+r-d_4}{u_4 - d_4}\right], \quad C_2(0) = \left[\frac{Su_4 - k}{1+r} \quad \frac{1+r-d_1}{u_1 - d_1}\right]$$

$$C_1(1) = \left[\frac{Su_2 - k}{1+r} \quad \frac{1+r-d_3}{u_3 - d_3}\right], \quad C_2(1) = \left[\frac{Su_3 - k}{1+r} \quad \frac{1+r-d_2}{u_2 - d_2}\right] \tag{5.3.6}$$

From (5.3.6) the fuzzy call price is given by

$$\tilde{C} = \begin{bmatrix} \frac{Su_1 - K}{1+r}\frac{1+r-d_4}{u_4-d_4}, & \frac{Su_2 - K}{1+r}\frac{1+r-d_3}{u_3-d_3}, \\ \frac{Su_3 - K}{1+r}\frac{1+r-d_2}{u_2-d_2}, & \frac{Su_4 - K}{1+r}\frac{1+r-d_1}{u_1-d_1} \end{bmatrix} \quad (5.3.7)$$

In view of (5.3.4)-(5.3.7), we determine the membership functions for $\tilde{C}$ as follows. From (5.3.4), setting $C_1(\alpha) = \tilde{C}$, we get

$$\left[\frac{S\left(u_2 - (u_2 - u_1)(1-\alpha)^{\frac{1}{m}}\right) - K}{1+r} \quad \frac{(1+r-d_3) + (d_3 - d_4)(1-\alpha)^{\frac{1}{n}}}{u_3 - d_3 - (u_3 - u_4 - d_3 + d_4)(1-\alpha)^{\frac{1}{n}}}\right] = \tilde{C} \quad (5.3.8)$$

This yields the following equation

$$\begin{aligned} &\left((Su_2 - K)(1+r-d_3) - \tilde{C}(u_3 - d_3)(1+r)\right) + \\ &\left((Su_2 - K)(d_3 - d_4) + \tilde{C}(u_3 - u_4 - d_3 + d_4)(1+r)\right)(1-\alpha)^{\frac{1}{n}} - \\ &S(u_2 - u_1)(1+r-d_3)(1-\alpha)^{\frac{1}{m}} - S(u_2 - u_1)(d_3 - d_4)(1-\alpha)^{\frac{1}{m}+\frac{1}{n}} = 0 \end{aligned} \quad (5.3.9)$$

Similarly, setting $C_2(\alpha) = \tilde{C}$ yields the following equation

$$\begin{aligned} &(Su_3 - K)(1+r-d_2) - \tilde{C}(1+r)(u_2 - d_2) + \\ &\left((Su_3 - K)(d_2 - d_1) + \tilde{C}(1+r)(u_2 - u_1 - d_2 + d_1)\right)(1-\alpha)^{\frac{1}{m}} - \\ &(1+r-d_2)S(u_3 - u_4)(1-\alpha)^{\frac{1}{n}} - S(u_3 - u_4)(d_2 - d_1)(1-\alpha)^{\frac{1}{m}+\frac{1}{n}} = 0 \end{aligned} \quad (5.3.10)$$

Special Cases:

(a) Setting $m = n$ in (5.3.9) yields

$$A_1 (1-\alpha)^{\frac{2}{m}} - (C_1 - B_1)(1-\alpha)^{\frac{1}{m}} - D_1 \;=\; 0 \tag{5.3.11}$$

where

$$\begin{aligned}
A_1 &= S(u_2 - u_1)(d_3 - d_4) \\
B_1 &= (Su_2 - K)(d_3 - d_4) + \widetilde{C}(u_3 - u_4 - d_3 + d_4)(1+r) \\
C_1 &= S(u_2 - u_1)(1 + r - d_3) \\
D_1 &= (Su_2 - K)(1 + r - d_3) - \widetilde{C}(u_3 - d_3)(1+r)
\end{aligned}$$

Thus, solving expression (5.3.11) for α, yields

$$\begin{aligned}
(1-\alpha)^{\frac{1}{m}} &= \frac{(C_1 - B_1) + \sqrt{(C_1 - B_1)^2 + 4A_1 D_1}}{2A_1} \\
\alpha &= 1 - \left(\frac{(C_1 - B_1) + \sqrt{(C_1 - B_1)^2 + 4A_1 D_1}}{2A_1}\right)^m \qquad (5.3.12)
\end{aligned}$$

Similarly if we set $m = n$ in expression (5.3.10), we obtain

$$A_2 (1-\alpha)^{\frac{2}{m}} - (B_2 - C_2)(1-\alpha)^{\frac{1}{m}} - D_2 \;=\; 0 \tag{5.3.13}$$

where

$$\begin{aligned}
A_2 &= S(u_3 - u_4)(d_2 - d_1) \\
B_2 &= (1 + r - d_2)\, S(u_3 - u_4) \\
C_2 &= ((Su_3 - K)(d_2 - d_1) - \widetilde{C}(1+r)(u_2 - u_1 - d_2 + d_1)) \\
D_2 &= (Su_3 - K)(1 + r - d_2) + \widetilde{C}(1+r)(u_2 - d_2)
\end{aligned}$$

Thus, solving expression (5.3.13) for α, yields

$$\alpha = 1 - \left(\frac{(B_2 - C_2) + \sqrt{(B_2 - C_2)^2 + 4A_2D_2}}{2A_2} \right)^m \tag{5.3.14}$$

This gives the membership function of $\tilde{C}$ as.

$$\mu(\tilde{C}) = \begin{cases} 0 & \tilde{C} \leq \frac{Su_1 - K}{1+r}\frac{1+r-d_4}{u_4-d_4} \\ 1 - \left(\frac{(C_1 - B_1) + \sqrt{(C_1 - B_1)^2 + 4A_1D_1}}{2A_1} \right)^m & \frac{Su_1 - K}{1+r}\frac{1+r-d_4}{u_4-d_4} \leq \tilde{C} \leq \frac{1+r-d_3}{u_3-d_3} \\ 1 & \frac{Su_2 - K}{1+r}\frac{1+r-d_3}{u_3-d_3} \leq \tilde{C} \leq \frac{Su_3 - K}{1+r}\frac{1+r-d_2}{u_2-d_2} \\ 1 - \left(\frac{(B_2 - C_2) + \sqrt{(B_2 - C_2)^2 + 4A_2D_2}}{2A_2} \right)^m & \frac{Su_3 - K}{1+r}\frac{1+r-d_2}{u_2-d_2} \leq \tilde{C} \leq \frac{Su_4 - K}{1+r}\frac{1+r-d_2}{u_1-d_1} \\ 0 & \tilde{C} \geq \frac{Su_4 - K}{1+r}\frac{1+r-d_2}{u_1-d_1} \end{cases} \tag{5.3.15}$$

(b) If $\tilde{u}$ and $\tilde{d}$ are two symmetric O(m,n)-Tr.T.F.N's, with $u_2 - u_1 = \alpha_1$, $u_4 - u_3 = \beta_1$, $d_2 - d_1 = \alpha_2$ and $d_4 - d_3 = \beta_2$ then the membership function $\mu(\tilde{C})$ given by (5.3.15) will have the following values for $A_1, B_1, C_1, D_1, A_2, B_2, C_2$, and D_2.

$$\begin{aligned} A_1 &= -S\alpha_1\beta_2 \\ B_1 &= -(Su_2 - K)\beta_2 + \widetilde{C}(\beta_2 - \beta_1)(1+r) \\ C_1 &= S\alpha_1(1+r-d_3) \\ D_1 &= (Su_2 - K)(1+r-d_3) - \widetilde{C}(u_3 - d_3)(1+r) \end{aligned}$$

$$\begin{aligned} A_2 &= -S\beta_1\alpha_2 \\ B_2 &= -(1+r-d_2)S\beta_1 \\ C_2 &= (Su_3 - K)\alpha_2 - \widetilde{C}(1+r)(\alpha_1 - \alpha_2) \\ D_2 &= (Su_3 - K)(1+r-d_2) + \widetilde{C}(1+r)(u_2 - d_2) \end{aligned}$$

(c) Now, if $\tilde{u}$ and $\tilde{d}$ are two symmetric O(m,n)-Tr.T.F.N's having equal spreads, i.e., $u_2 - u_1 = u_4 - u_3 = d_2 - d_1 = d_4 - d_3 = \beta$, i.e., $(\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \beta)$, then the membership function $\mu(\tilde{C})$ given by (5.3.15) will have the following values for $A_1, B_1, C_1, D_1, A_2, B_2, C_2$ and D_2.

$$
\begin{aligned}
A_1 &= -S\beta^2 \\
B_1 &= -(Su_2 - K)\beta \\
C_1 &= S\beta(1 + r - d_3) \\
D_1 &= (Su_2 - K)(1 + r - d_3) - \widetilde{C}(u_3 - d_3)(1 + r)
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= -S\beta^2 \\
B_2 &= -(1 + r - d_2)S\beta \\
C_2 &= (Su_3 - K)\beta \\
D_2 &= (Su_3 - K)(1 + r - d_2) + \widetilde{C}(1 + r)(u_2 - d_2)
\end{aligned}
$$

5.3.2 Characteristics of the Call Price.

We now analyze the behavior of the fuzzy call price. To do so we calculate their derivatives and discuss their behavior under the following assumptions.

Assumptions A_2.

$d_1 < d_2 < d_3 < d_4 \quad < 1 + r < u_1 < u_2 < u_3 < u_4$

$0 < p_{di} < 1, \quad 0 < p_{ui} < 1 \quad i = 1, 2, 3, 4$

$Su_i > K, \quad Sd_i < K \quad$ and $\quad i = 1, 2, 3, 4.$

Now, (5.3.4) can be rewritten as follows,

$$
C_1(\alpha) = \frac{C_{u1}(\alpha)}{1 + r} p_{u1}(\alpha) = \frac{1}{1 + r}\left[C_{u1}(\alpha) \;\; p_{u1}(\alpha)\right]
$$

$$\begin{aligned}
\frac{dC_1(\alpha)}{d\alpha} &= \frac{1}{1+r}\left[\frac{dC_{u1}(\alpha)}{d\alpha}p_{u1}(\alpha) + \frac{dp_{u1}(\alpha)}{d\alpha}C_{u1}(\alpha)\right] \\
&= \frac{1}{1+r}\left(\frac{S\,(u_2-u_1)}{m\,(1-\alpha)^{1-\frac{1}{m}}}\right)p_{u1}(\alpha) + \\
&\quad \frac{1}{1+r}\frac{dp_{u1}(\alpha)}{d\alpha}\left(S(u_2-(u_2-u_1)\,(1-\alpha)^{\frac{1}{m}}) - K\right) > 0 \qquad (5.3.16)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^2C_1(\alpha)}{d\alpha^2} &= \frac{1}{1+r}\left[p_{u1}(\alpha)\frac{d^2C_{u1}(\alpha)}{d\alpha^2} + \frac{dC_{u1}(\alpha)}{d\alpha}\frac{dp_{u1}(\alpha)}{d\alpha}\right] + \\
&\quad \frac{1}{1+r}\left[C_{u1}(\alpha)\frac{d^2p_{u1}(\alpha)}{d\alpha^2} + \frac{dC_{u1}(\alpha)}{d\alpha}\frac{dp_{u1}(\alpha)}{d\alpha}\right] \\
&= \frac{1}{1+r}\left[p_{u1}(\alpha)\frac{d^2C_{u1}(\alpha)}{d\alpha^2} + 2\frac{dC_{u1}(\alpha)}{d\alpha}\frac{dp_{u1}(\alpha)}{d\alpha} + C_{u1}(\alpha)\frac{d^2p_{u1}(\alpha)}{d\alpha^2}\right]
\end{aligned}$$

(5.3.17)

Similarly, (5.3.5) can be rewritten as

$$C_2(\alpha) = \frac{C_{u2}(\alpha)}{1+r}p_{u2}(\alpha) = \frac{1}{1+r}\left[C_{u2}(\alpha)\ \ p_{u2}(\alpha)\right]$$

Thus,

$$\begin{aligned}
\frac{dC_2(\alpha)}{d\alpha} &= \frac{1}{1+r}\left[\frac{dC_{u2}(\alpha)}{d\alpha}p_{u2}(\alpha) + \frac{dp_{u2}(\alpha)}{d\alpha}C_{u2}(\alpha)\right] \\
&= \frac{1}{1+r}\left(\frac{S\,(u_3-u_4)}{n\,(1-\alpha)^{1-\frac{1}{n}}}\right)p_{u2}(\alpha) + \\
&\quad \frac{1}{1+r}\left(\frac{dp_{u2}(\alpha)}{d\alpha}(S(u_3-(u_3-u_4)\,(1-\alpha)^{\frac{1}{n}}) - K)\right) < 0 \qquad (5.3.18)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^2C_2(\alpha)}{d\alpha^2} &= \frac{1}{1+r}\left[p_{u2}(\alpha)\frac{d^2C_{u2}(\alpha)}{d\alpha^2}+\frac{dC_{u2}(\alpha)}{d\alpha}\frac{dp_{u2}(\alpha)}{d\alpha}\right]+ \\
&\quad \frac{1}{1+r}\left[C_{u2}(\alpha)\frac{d^2p_{u2}(\alpha)}{d\alpha^2}+\frac{dC_{u2}(\alpha)}{d\alpha}\frac{dp_{u2}(\alpha)}{d\alpha}\right] \\
&= \frac{1}{1+r}\left[p_{u2}(\alpha)\frac{d^2C_{u2}(\alpha)}{d\alpha^2}+2\frac{dC_{u2}(\alpha)}{d\alpha}\frac{dp_{u2}(\alpha)}{d\alpha}+C_{u2}(\alpha)\frac{d^2p_{u2}(\alpha)}{d\alpha^2}\right]
\end{aligned} \tag{5.3.19}$$

From (5.1.60) and (5.1.61), we observe that $\frac{dp_{u_1}(\alpha)}{d\alpha} > 0$, and $\frac{d^2p_{u_1}(\alpha)}{d\alpha^2}$ may be > 0, or < 0. In view of assumptions A1 and A2, we observe that $\frac{dC_1(\alpha)}{d\alpha} > 0$, whereas $\frac{d^2C_1(\alpha)}{d\alpha^2}$ can be positive or negative. If, $\frac{d^2C_1(\alpha)}{d\alpha^2}$ is positive (negative, respectively), it implies that the left hand side or right hand side of the fuzzy numbeı that characterize the fuzzy call price is convex (concave, respectively). Similarly, using (5.3.18) and (5.3.19), we observe that $\frac{dC_2(\alpha)}{d\alpha} > 0$, whereas $\frac{d^2C_2(\alpha)}{d\alpha^2}$ can be positive or negative. If, $\frac{d^2C_2(\alpha)}{d\alpha^2}$ is positive (negative, respectively), it implies that the left hand side or right hand side of the fuzzy number that characterize the fuzzy call price is convex (concave, respectively).

We now find limits of $C_1(\alpha)$ and $C_2(\alpha)$ when both m, n approach zero and when both m, n approach infinity.

$$\begin{aligned}
\lim_{m,n\to\infty} C_1(\alpha) &= \lim_{m,n\to\infty}\left[\begin{array}{c}\dfrac{S\left(u_2-(u_2-u_1)(1-\alpha)^{\frac{1}{m}}\right)-K}{1+r} \\ \\ \dfrac{(1+r-d_3)+(d_3-d_4)(1-\alpha)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)(1-\alpha)^{\frac{1}{n}}}\end{array}\right] \\
&= \left[\frac{Su_1-K}{1+r}\ \frac{1+r-d_4}{u_4-d_4}\right]
\end{aligned} \tag{5.3.20}$$

When both m and $n \to \infty$, $C_1(\alpha)$ converges to the left end point of $\tilde{C}$. This mean that the amount of fuzziness in the model increases.

$$\lim_{m,n\to 0} C_1(\alpha) = \lim_{m,n\to 0} \left[\begin{array}{c} \dfrac{S\left(u_2 - (u_2 - u_1)(1-\alpha)^{\frac{1}{m}}\right) - K}{1+r} \\ \\ \dfrac{(1+r-d_3) + (d_3 - d_4)(1-\alpha)^{\frac{1}{n}}}{u_3 - d_3 - (u_3 - u_4 - d_3 + d_4)(1-\alpha)^{\frac{1}{n}}} \end{array} \right]$$

$$= \left[\frac{Su_2 - K}{1+r} \quad \frac{1+r-d_3}{u_3 - d_3} \right] \tag{5.3.21}$$

When each of m and n go to zero, $C_1(\alpha)$ converges to the left interior point of $\tilde{C}$, thus as m and n get smaller and smaller the amount of fuzziness in the model decreases. Similarly,

$$\lim_{m,n\to \infty} C_2(\alpha) = \lim_{m,n\to \infty} \left[\begin{array}{c} \dfrac{S\left(u_3 - (u_3 - u_4)(1-\alpha)^{\frac{1}{n}}\right) - K}{1+r} \\ \\ \dfrac{(1+r-d_2) + (d_2 - d_1)(1-\alpha)^{\frac{1}{m}}}{u_2 - d_2 - (u_2 - u_1 - d_2 + d_1)(1-\alpha)^{\frac{1}{m}}} \end{array} \right]$$

$$= \left[\frac{Su_4 - K}{1+r} \quad \frac{1+r-d_1}{u_1 - d_1} \right] \tag{5.3.22}$$

When m and n become large, $C_2(\alpha)$ converges to the right end point of $\tilde{C}$, this means that the amount of fuzziness in the model increases.

$$\lim_{m,n\to 0} C_2(\alpha) = \lim_{m,n\to \infty} \left[\begin{array}{c} \dfrac{S\left(u_3 - (u_3 - u_4)(1-\alpha)^{\frac{1}{n}}\right) - K}{1+r} \\ \\ \dfrac{(1+r-d_2) + (d_2 - d_1)(1-\alpha)^{\frac{1}{m}}}{u_2 - d_2 - (u_2 - u_1 - d_2 + d_1)(1-\alpha)^{\frac{1}{m}}} \end{array} \right]$$

$$= \left[\frac{Su_3 - K}{1+r} \quad \frac{1+r-d_2}{u_2 - d_2} \right] \tag{5.3.23}$$

Thus we see that as each of m and n goes to zero, $C_2(\alpha)$ converges to the right interior point of $\tilde{C}$. This means that as m and n get small the amount of fuzziness in the model decreases.

5.4 Numerical Examples.

In this section we illustrates our results with the help of various numerical examples and highlight some of the salient point in the fuzzy option pricing model developed in this chapter. We also assume that the initial price of the stock is S and the exercise price of the call option, K, are crisp and known. For, $0 \leq \alpha \leq 1$, let

$$\tilde{u} = [u_1, u_2, u_3, u_4]_{O(m,n)} = [1.12, 1.13, 1.15, 1.17]_{O(m,n)} \tag{5.4.1}$$

$$\tilde{d} = [d_1, d_2, d_3, d_4]_{O(m,n)} = [0.65, 0.75, 0.85, 0.95]_{O(m,n)} \tag{5.4.2}$$

$$r = 0.0633 \tag{5.4.3}$$

$$S = 65 \tag{5.4.4}$$

$$K = 59 \tag{5.4.5}$$

We observe that for the above data the following hold.

$$d_1 < d_2 < d_3 < d_4 < 1 + r < u_1 < u_2 < u_3 < u_4, \tag{5.4.6}$$

$$0 < p_{di} < 1, \quad 0 < p_{ui} < 1 \quad i = 1, 2, 3, 4 \tag{5.4.7}$$

$$Su_i > K, \quad Sd_i < K \text{ and } \quad i = 1, 2, 3, 4 \tag{5.4.8}$$

For different values of m and n we discuss the following cases.

(a) **Case 1**, when $m = n = 2$,

(b) **Case 2**, when $m = 2$ and $n = 0.5$,

(c) **Case 3**, when $m = 0.5$ and $n = 2$,

(d) **Case 4**, when $m = n = 0.5$.

5.4.1 Case 1.

In view of (5.4.1)-(5.4.8) and for $m = 2$ and $n = 2$, we now compute the fuzzy price of the call option. Using expressions (5.1.2),(5.1.3),(5.1.5) and (5.1.6), we obtain the following risk neutral probabilities that we shall use to price the fuzzy call option value.

$$p_{d1}(\alpha) = \left[\frac{(u_2 - 1 - r) - (u_2 - u_1)\sqrt{1-\alpha}}{u_2 - d_2 - (u_2 - u_1 - d_2 + d_1)\sqrt{1-\alpha}}\right] = \left[\frac{0.0667 - 0.01\sqrt{1-\alpha}}{0.38 + 0.09\sqrt{1-\alpha}}\right]$$

$$p_{d2}(\alpha) = \left[\frac{(u_3 - 1 - r) - (u_3 - u_4)\sqrt{1-\alpha}}{u_3 - d_3 - (u_3 - u_4 - d_3 + d_4)\sqrt{1-\alpha}}\right] = \left[\frac{0.0867 + 0.02\sqrt{1-\alpha}}{0.3 - 0.08\sqrt{1-\alpha}}\right]$$

$$p_{u1}(\alpha) = \left[\frac{(1 + r - d_3) + (d_3 - d_4)\sqrt{1-\alpha}}{u_3 - d_3 - (u_3 - u_4 - d_3 + d_4)\sqrt{1-\alpha)}}\right] = \left[\frac{0.2133 - 0.1\sqrt{1-\alpha}}{0.3 - 0.08\sqrt{1-\alpha}}\right]$$

$$p_{u2}(\alpha) = \left[\frac{(1 + r - d_2) + (d_2 - d_1)\sqrt{1-\alpha)}}{u_2 - d_2 - (u_2 - u_1 - d_2 + d_1)\sqrt{1-\alpha}}\right] = \left[\frac{0.313 + 0.1\sqrt{1-\alpha}}{0.38 + 0.09\sqrt{1-\alpha}}\right]$$

Characteristics of the Risk Neutral Probabilities when $m = n = 2$. For, $0 \le \alpha \le 1$ we have

$$\frac{d}{d\alpha}[p_{d1}(\alpha)] = \frac{1}{\sqrt{1-\alpha}}\left[\frac{49.015}{(38 + 9\sqrt{1-\alpha})^2}\right] > 0$$

$$\frac{d^2}{d\alpha^2}[p_{d1}(\alpha)] = \frac{1}{(1-\alpha)^{\frac{3}{2}}}\left[\frac{24.508(38 + 27\sqrt{1-\alpha})}{(38 + 9\sqrt{1-\alpha})^3}\right] > 0$$

$$\frac{d}{d\alpha}[p_{d2}(\alpha)] = -\frac{1}{\sqrt{1-\alpha}}\left[\frac{16.17}{(-15 + 4\sqrt{1-\alpha})^2}\right] < 0$$

$$\frac{d^2}{d\alpha^2}[p_{d2}(\alpha)] = \frac{1}{(1-\alpha)^{\frac{3}{2}}}\left[\frac{-24.255(-5 + 4\sqrt{1-\alpha})}{(-15 + 4\sqrt{1-\alpha})^3}\right] < 0$$

$$\frac{d}{d\alpha}[p_{u1}(\alpha)] = \frac{1}{\sqrt{1-\alpha}}\left[\frac{16.17}{\left(-15+4\sqrt{1-\alpha}\right)^2}\right] > 0$$

$$\frac{d^2}{d\alpha^2}[p_{u1}(\alpha)] = \frac{1}{(1-\alpha)^{\frac{3}{2}}}\left[\frac{24.255(-5+4\sqrt{1-\alpha})}{\left(-15+4\sqrt{1-\alpha}\right)^3}\right] > 0$$

$$\frac{d}{d\alpha}[p_{u2}(\alpha)] = -\frac{1}{\sqrt{1-\alpha}}\left[\frac{49.15}{\left(38+9\sqrt{1-\alpha}\right)^2}\right] < 0$$

$$\frac{d^2}{d\alpha^2}[p_{u2}(\alpha)] = -\frac{1}{(1-\alpha)^{\frac{3}{2}}}\left[\frac{24.575(38+27\sqrt{1-\alpha})}{\left(38+9\sqrt{1-\alpha}\right)^3}\right] < 0$$

Note that the first derivatives of the left hand side of the various fuzzy risk neutral probabilities are greater than zero and the derivatives of the right hand side of the fuzzy risk neutral probabilities are less than zero. This indicates that as the left hand part of the α-cut characterizes an increasing function and the right hand part of the α-cut characterize a decreasing function. However, the second derivative for $p_{d1}(\alpha)$ is greater than zero, which indicates convex character of $p_{d1}(\alpha)$. The second derivative of $p_{d2}(\alpha)$ is less than zero which indicates concave character.

Table ?? summarize the fuzzy risk neutral probabilities of a down movement in the stock price, the fuzzy risk neutral probabilities of an up movement in the stock price for different values of α, where $0 \leq \alpha \leq 1$.

Table 5.1: Kurtosis of Sign RCA model and RCA Model.

α	$p_{u1}(\alpha)$	$p_{u2}(\alpha)$	$p_{d1}(\alpha)$	$p_{d2}(\alpha)$
0	0.515	0.879	0.121	0.485
0.1	0.528	0.876	0.123	0.472
0.2	0.542	0.874	0.125	0.458
0.3	0.556	0.871	0.128	0.444
0.4	0.571	0.868	0.131	0.429
0.5	0.586	0.865	0.134	0.414
0.6	0.602	0.861	0.138	0.398
0.7	0.619	0.857	0.143	0.381
0.8	0.638	0.851	0.148	0.362
0.9	0.661	0.844	0.156	0.339
1	0.711	0.824	0.176	0.289

It is interesting to point out that the fuzzy algebraic approach to option pricing preserves the complimentary condition imposed by probability theory and at the same time does not disturb the stochastic structure of the risk neutral probabilities. In other words, the complimentary conditions

(a) $p_{u1}(\alpha) + p_{d2}(\alpha) = 1$

(b) $p_{u2}(\alpha) + p_{d1}(\alpha) = 1$

are satisfied.

From Table(??) we obtain Figure (5.1) and Figure (5.2) respectively.

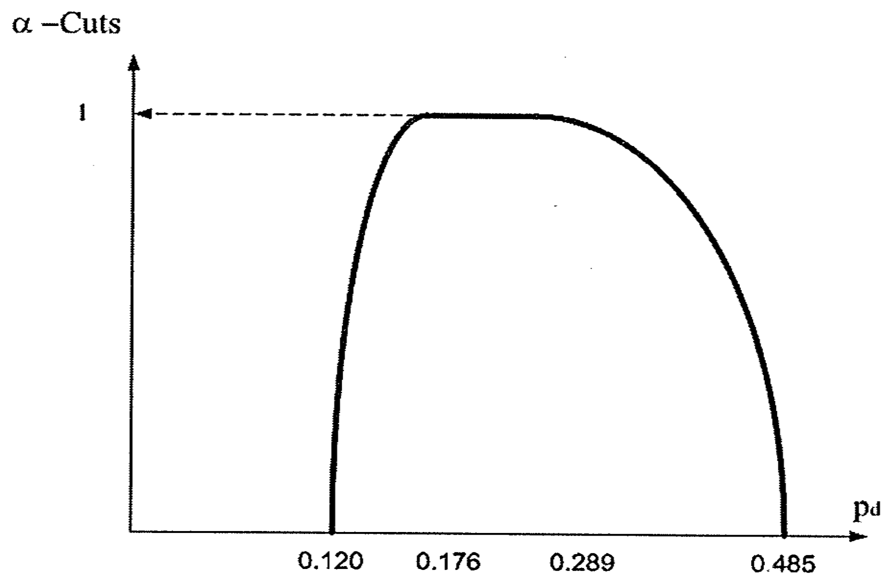

Figure 5.1: p_d, Fuzzy Risk Neutral Probabilities of a fuzzy downward movement in the stock price when $m = n = 2$.

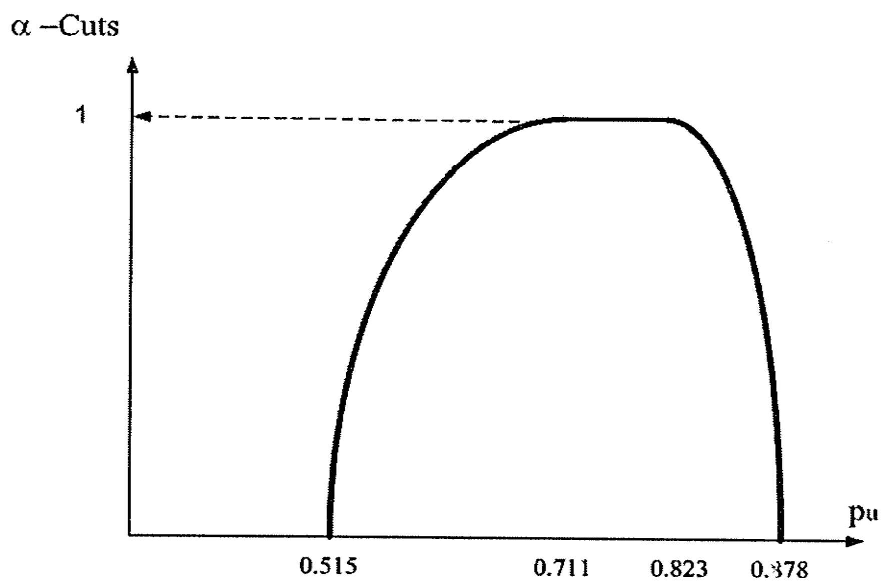

Figure 5.2: p_u, Fuzzy Risk Neutral Probabilities of a fuzzy upward movement in the stock price when $m = n = 2$.

When $m = n = 2$, the graphs of $p_d(\alpha)$ and $p_u(\alpha)$ are spread out. This mean that the amount of fuzziness in the risk neutral probabilities are more than it would have been if Tr.F.N's were chosen for $\tilde{u}$ and $\tilde{d}$. Increasing the values of m and n in the model increases the amount of fuzziness in the model. Estimation of membership functions is an important step in many applications of fuzzy sets theory. Thus eliciting membership functions is one of the fundamental issues associated with the application of fuzzy set theory.

Fuzzy Call Option Value Calculation with O(2,2)-Tr.T.F.N's.

Substituting values of $r, S, K, \tilde{u}$ and $\tilde{d}$ in formulas (5.3.4) and (5.3.5), we obtain the fuzzy current price of the call option as follows.

$$
\begin{aligned}
C_1(\alpha) &= \left[\frac{S(u_2-(u_2-u_1)\sqrt{1-\alpha})-K}{1+r} \quad \frac{(1+r-d_3)+(d_3-d_4)\sqrt{1-\alpha}}{(u_3-d_3)-(u_3-u_4-d_3+d_4)\sqrt{1-\alpha}}\right] \\
&= \left[\frac{14.45-0.65\sqrt{1-\alpha}}{1.0633} \frac{0.2133-0.1\sqrt{1-\alpha}}{0.3-0.08\sqrt{1-\alpha}}\right]
\end{aligned}
\tag{5.4.9}
$$

$$
\begin{aligned}
C_2(\alpha) &= \left[\frac{S(u_3-(u_3-u_4)\sqrt{1-\alpha})-K}{1+r} \quad \frac{(1+r-d_2)-(d_2-d_1)\sqrt{1-\alpha}}{u_2-d_2-(u_2-u_1-d_2+d_1)\sqrt{1-\alpha}}\right] \\
&= \left[\frac{15.75+1.3\sqrt{1-\alpha}}{1.0633} \frac{0.3133+0.1\sqrt{1-\alpha}}{0.38+0.09\sqrt{1-\alpha}}\right]
\end{aligned}
\tag{5.4.10}
$$

In Table (5.2), the fuzzy call option values are tabulated for different values of α, $0 \leq \alpha \leq 1$ and $m = n = 2$. Table (5.2) gives the $\alpha-$cut closed intervals $C_1(\alpha), C_1(\alpha)$ of the fuzzy call price of a call option along with different degrees of truth.

Table 5.2: Fuzzy Call Option Values.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$C_1(\alpha)$	6.68	6.88	7.07	7.27	7.49	7.71	7.94	8.20	8.50	8.86	9.66
$C_2(\alpha)$	14.10	14.01	13.91	13.81	13.69	13.57	13.43	13.27	13.08	12.83	12.21

From Table 5.2 we obtain Figure 5.3 that depicts the fuzzy call values for the option pricing problem.

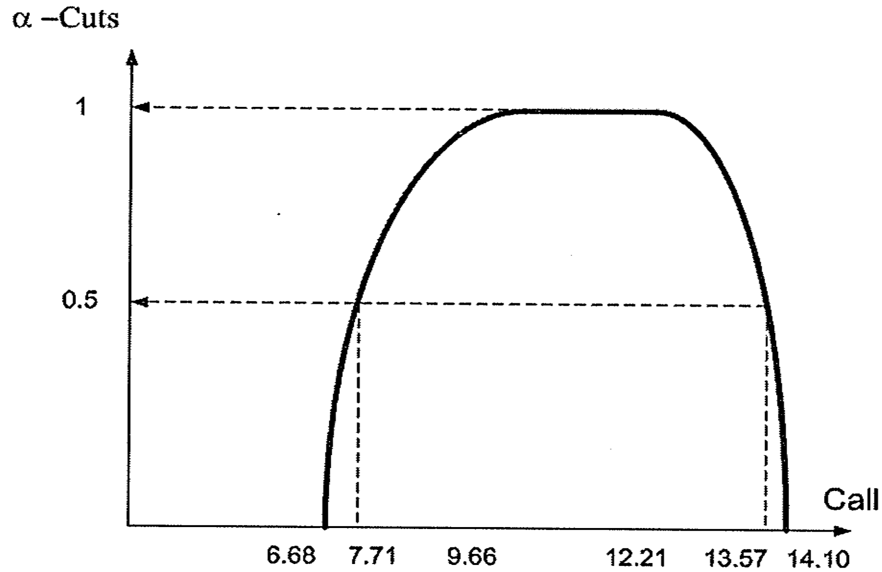

Figure 5.3: Case 1: Fuzzy Call Option Values.

In Case 1 we consider a call option on a stock under conditions listed in (5.4.1)-(5.4.8). Table 5.2 gives the degrees of truth for different fuzzy call option values $C_1(\alpha)$ and $C_2(\alpha)$. For example, if any call option price is taken in the range of [7.71, 13.57], then the degree of truth associated with the value of this call option price is at least 0.5. Therefore, if an analyst is comfortable with the degree of truth of at least 0.5, then he(she) can take this option price for his(her) later use. A similar interpretation holds for the other interval in Table 5.2.

Characteristics of the Fuzzy Call Option Value Considered in Case 1 , where $m = 2$ and $n = 2$.

We now analyze the behavior of the fuzzy call price, considered in Case 1. To do so we calculate their derivatives.

For $0 \leq \alpha \leq 1$, (5.4.9) yields.

$$\frac{dC_1(\alpha)}{d\alpha} = \frac{1}{1+r}\left[\frac{dC_{u1}(\alpha)}{d\alpha}p_{u1}(\alpha) + \frac{dp_{u1}(\alpha)}{d\alpha}C_{u1}(\alpha)\right] > 0 \qquad (5.4.11)$$

Using (5.3.17) and (5.4.11), we have

$$\begin{aligned}\frac{d^2C_1(\alpha)}{d\alpha^2} &= \left(\frac{0.1625\left(0.2133 - 0.1\sqrt{1-\alpha}\right)(1-\alpha)^{-\frac{3}{2}}}{1.0633\left(0.3 - 0.08\sqrt{1-\alpha}\right)}\right) + \\ &\quad \left(\frac{10.511}{1.0633\,(1-\alpha)\left(-15+4\sqrt{1-\alpha}\right)^2}\right) + \\ &\quad \left(\frac{24.255\left(14.45 - 0.65\sqrt{1-\alpha}\right)\left(-5+4\sqrt{1-\alpha}\right)(1-\alpha)^{-\frac{3}{2}}}{1.0633\left(-15+4\sqrt{1-\alpha}\right)^3}\right) > 0\end{aligned}$$

Since, $\frac{d^2C_1(\alpha)}{d\alpha^2} > 0$, therefore the curvature for the left hand side of $C_{u1}(\alpha)$ is convex. Similarly, from (5.3.18), the expression for $\frac{dC_2(\alpha)}{d\alpha}$ is always negative, irrespective of the value of α, $0 \leq \alpha \leq 1$. That is

$$\frac{dC_2(\alpha)}{d\alpha} = \frac{1}{1+r}\left[\frac{dC_{u2}(\alpha)}{d\alpha}p_{u2}(\alpha) + \frac{dp_{u2}(\alpha)}{d\alpha}C_{u2}(\alpha)\right] < 0$$

On the other hand, the $\frac{d^2C_2(\alpha)}{d\alpha^2}$ is as follows,

$$\begin{aligned}\frac{d^2C_2(\alpha)}{d\alpha^2} &= \left(\frac{\left(0.3133 + 0.1\sqrt{1-\alpha}\right)\left(-1.325\,(1-\alpha)^{-\frac{3}{2}}\right)}{1.0633\left(0.38 + 0.09\sqrt{1-\alpha}\right)}\right) - \\ &\quad \left(\frac{49.015}{\left(38.0 + 9.0\sqrt{1-\alpha}\right)^2\sqrt{1-\alpha}}\right) + \\ &\quad \left(\frac{2\,(-0.65)\,(-49.015)\,(1-\alpha)^{-\frac{1}{2}}}{1.0633\sqrt{1-\alpha}\left(38.0+9.0\sqrt{1-\alpha}\right)^2}\right) + \\ &\quad \left(\frac{24.508\left(15.75 + 1.3\sqrt{1-\alpha}\right)\left(38.0 + 27.0\sqrt{1-\alpha}\right)(1-\alpha)^{-\frac{3}{2}}}{1.0633\left(38.0 + 9.0\sqrt{(1-\alpha)}\right)^3}\right) < 0\end{aligned}$$

Since, $\frac{d^2\ C_2(\alpha)}{d\alpha^2} < 0$, it implies that the right hand side of $C(\alpha)$ considered in Case 1 is concave.

Membership Function of the Fuzzy Call Option Considered in Case 1.

We now compute the membership function of the fuzzy call price considered in Case 1. In order to find the two ends points and the two interior points which describe the fuzzy call price corresponding to the O(2,2)-Tr.T.F.N in Case 1, we set $\alpha = 0$ and $\alpha = 1$ in (5.4.9) and (5.4.10), respectively.

This yields

$C_1(0) = 6.68,\ C_1(1) = 9.66,\ \ C_2(0) = 14.10,\ \ C_2(1) = 12.21.$

Thus, the fuzzy number that describe the fuzzy call price in (5.4.9) is given by $\tilde{C} = [6.68, 9.66, 12.21, 14.10]$.

In view of the above discussion, we determine the membership function as follows.

From (5.4.9), setting $C_1(\alpha) = \tilde{C}$, we get

$$\left[\frac{2.8987 - 1.4894\sqrt{1-\alpha} + 0.06113\left(\sqrt{1-\alpha}\right)^2}{0.3 - 0.08\sqrt{1-\alpha}}\right] = \tilde{C}$$

Cross multiplication yields, the following equation.

$$0.06113\left(\sqrt{1-\alpha}\right)^2 - \left(1.4894 - 0.08\tilde{C}\right)\sqrt{1-\alpha} + \left(2.8987 - 0.3\tilde{C}\right) = 0$$

Thus,

$$\alpha = 1 - \left(\frac{\left(1.4894 - 0.08\tilde{C}\right) + \sqrt{1.5095 - 0.16495\tilde{C} + 0.0064\tilde{C}^2}}{0.12226}\right)^2$$

Similarly, from (5.4.10), setting $C_2(\alpha) = \tilde{C}$, we get

$$0.12226\left(\sqrt{1-\alpha}\right)^2 - (0.09\tilde{C} - 1.8643)\sqrt{1-\alpha} + (4.6407 - 0.38\tilde{C}) = 0$$

and

$$\alpha = 1 - \left(\frac{(0.09C - 1.8643) + \sqrt{0.0081C^2 - 0.14974C + 1.2061}}{0.12226}\right)^2$$

This gives the membership function $\tilde{C}$.

$$\mu(\tilde{C}) = \begin{cases} 0 & \tilde{C} \leq 6.68 \\ 1 - \left(\dfrac{\left(1.4894 - 0.08\tilde{C}\right) + \sqrt{1.5095 - 0.16495\tilde{C} + 0.0064\tilde{C}^2}}{0.12226} \right)^2 & 6.68 \leq \tilde{C} \leq 9.66 \\ 1 & 9.66 \leq \tilde{C} \leq 12.21 \\ 1 - \left(\dfrac{(0.09\tilde{C} - 1.8643) + \sqrt{0.0081\tilde{C}^2 - 0.14974\tilde{C} + 1.2061}}{0.12226} \right)^2 & 12.21 \leq \tilde{C} \leq 14.10 \\ 0 & \tilde{C} \geq 14.10 \end{cases}$$

5.4.2 Case 2.

In the problem considered in (5.4.1)-(5.4.5) and $m = 2$ and $n = 0.5$, we compute the fuzzy call price. Using expressions (5.1.2),(5.1.3),(5.1.5) and (5.1.6), we compute the risk neutral probabilities as follows.

$$p_{d1}(\alpha) = \left[\frac{0.0667 - 0.01\sqrt{1-\alpha}}{0.38 + 0.09\sqrt{1-\alpha}} \right] \tag{5.4.12}$$

$$p_{d2}(\alpha) = \left[\frac{0.0867 + 0.02\,(1-\alpha)^2}{0.3 - 0.08\,(1-\alpha)^2} \right] \tag{5.4.13}$$

$$p_{u1}(\alpha) = \left[\frac{0.2133 - 0.1\,(1-\alpha)^2}{0.3 - 0.08\,(1-\alpha)^2} \right] \tag{5.4.14}$$

$$p_{u2}(\alpha) = \left[\frac{0.3133 + 0.1\sqrt{1-\alpha}}{0.38 + 0.09\sqrt{1-\alpha}} \right] \tag{5.4.15}$$

We use the fuzzy risk neutral probabilities given by expressions (5.4.12)-(5.4.15) in expressions (5.3.4) and (5.3.5) to compute the fuzzy call price for different values of α, $0 \leq \alpha \leq 1$.

Characteristics of the Risk Neutral Probability for Case 2.

We find the derivatives of each of those risk neutral probabilities and checks their behavior as α varies, where, $0 \leq \alpha \leq 1$.

$$
\begin{aligned}
\frac{dp_{d1}(\alpha)}{d\alpha} &= \frac{1}{\sqrt{1-\alpha}}\left[\frac{49.015}{\left(38.0+9.0\sqrt{1-\alpha}\right)^2}\right] > 0 \\
\frac{d^2p_{d1}(\alpha)}{d\alpha^2} &= \left[\frac{24.508\left(38.0+27.0\sqrt{1-\alpha}\right)}{(1-\alpha)^{\frac{3}{2}}\left(38.0+9.0\sqrt{1-\alpha}\right)^3}\right] > 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\, p_{d2}(\alpha)}{d\alpha} &= \left[\frac{-64.68(1-\alpha)}{(-11.0-8.0\alpha+4.0\alpha^2)^2}\right] < 0 \\
\frac{d\,^2p_{d2}(\alpha)}{d\alpha^2} &= \left[\frac{-194.04\,(9.0-8.0\alpha+4.0\alpha^2)}{(-11.0-8.0\alpha+4.0\alpha^2)^3}\right] > 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\, p_{u1}(\alpha)}{d\alpha} &= \left[\frac{64.68(1-\alpha)}{(-11.0-8.0\alpha+4.0\alpha^2)^2}\right] > 0 \\
\frac{d^2\, p_{u1}(\alpha)}{d\alpha^2} &= \left[\frac{194.04\,(9.0-8.0\alpha+4.0\alpha^2)}{(-11.0-8.0\alpha+4.0\alpha^2)^3}\right] < 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{dp_{u2}(\alpha)}{d\alpha} &= \frac{1}{\sqrt{1-\alpha}}\left[\frac{-49.015}{\left(38.0+9.0\sqrt{1-\alpha}\right)^2}\right] < 0 \\
\frac{d^2p_{u2}(\alpha)}{d\alpha^2} &= \left[\frac{-24.508\left(38.0+27.0\sqrt{1-\alpha}\right)}{(1-\alpha)^{\frac{3}{2}}\left(38.0+9.0\sqrt{1-\alpha}\right)^3}\right] < 0
\end{aligned}
$$

Note that the first derivatives of the left hand side of the fuzzy risk neutral probabilities are greater than zero and the derivatives of the right hand side of the fuzzy risk neutral probabilities are less than zero. However, the second derivative for $p_{d1}(\alpha)$ and $p_{d2}(\alpha)$ are greater than zero, which indicates convex character . The second derivative of $p_{u1}(\alpha)$ and $p_{u2}(\alpha)$ are less than zero which indicates concave character.

Table 5.3 summarizes the fuzzy risk neutral probabilities in the stock price for different values of α, where $0 \leq \alpha \leq 1$.

Table 5.3: Fuzzy Risk Neutral Probabilities

α	$p_{u1}(\alpha)$	$p_{u2}(\alpha)$	$p_{d1}(\alpha)$	$p_{d2}(\alpha)$
0	0.515	0.879	0.121	0.485
0.1	0.563	0.876	0.123	0.438
0.2	0.600	0.874	0.125	0.400
0.3	0.630	0.871	0.128	0.370
0.4	0.654	0.868	0.131	0.346
0.5	0.673	0.865	0.134	0.328
0.6	0.687	0.861	0.138	0.313
0.7	0.698	0.857	0.143	0.302
0.8	0.705	0.851	0.148	0.295
0.9	0.710	0.844	0.156	0.290
1	0.711	0.824	0.176	0.289

From Table(5.3) we obtain Figure (5.4) and Figure (5.4) respectively.

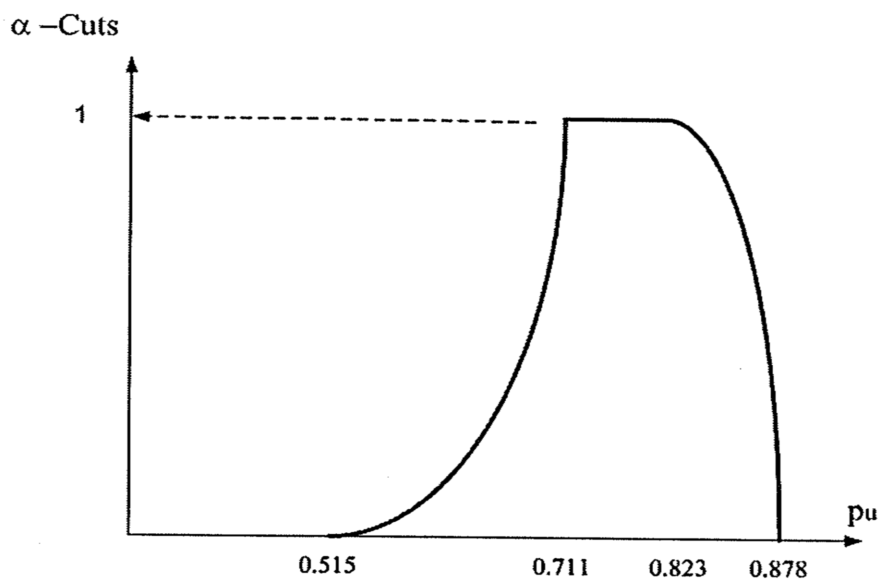

Figure 5.4: p_u, Fuzzy Risk Neutral Probabilities of an up movement in the stock price when $m = 2$ and $n = 0.5$.

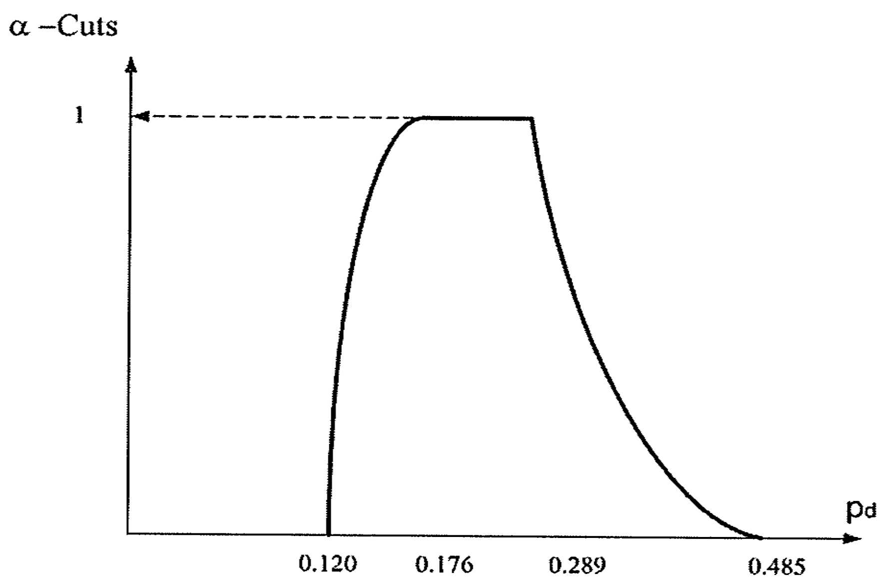

Figure 5.5: p_d, Fuzzy Risk Neutral Probabilities of a down movement in the stock price when $m = 2$ and $n = 0.5$.

When $m = 2$ and $n = 0.5$, the graph of $p_u(\alpha)$ bulges inward on the left hand side and bulge outward on the right hand side. Thus, there is less fuzziness on the left hand of the membership function. On the other hand, the graph of $p_d(\alpha)$ bulges outward on the left hand side and bulge inward on the right hand side. Thus, there is more fuzziness on the left hand of the membership function than at the right hand side.

Fuzzy Call Option Value Calculation with O(2,0.5)-Tr.T.F.N's.

Substituting values of $r, S, K, \tilde{u}$ and $\tilde{d}$ in formulas (5.3.4) and (5.3.5), we obtain the fuzzy current price of the call option.

$$C_1(\alpha) = \left[\frac{14.45 - 0.65\sqrt{1-\alpha}}{1.0633} \quad \frac{0.2133 - 0.1\,(1-\alpha)^2}{0.3 - 0.08\,(1-\alpha)^2}\right] \tag{5.4.16}$$

$$C_2(\alpha) = \left[\frac{15.75 + 1.3\,(1-\alpha)^2}{1.0633} \quad \frac{0.3133 + 0.1\sqrt{1-\alpha}}{0.38 + 0.09\sqrt{1-\alpha}}\right] \tag{5.4.17}$$

If we substitute different values of α, $0 \leq \alpha \leq 1$ we obtain the following table.

In Table (5.4), the fuzzy Call are tabulated.

Table 5.4: Case 2: Fuzzy Call Option Values.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$C_1(\alpha)$	6.68	7.32	7.83	8.24	8.57	8.85	9.07	9.25	9.39	9.51	9.66
$C_2(\alpha)$	14.10	13.86	13.64	13.44	13.25	13.09	12.93	12.79	12.66	12.52	12.21

From Table 5.4 we obtain Figure 5.6 that depicts the call values obtained when $m = 2$ and $n = 2$ for the option pricing problem.

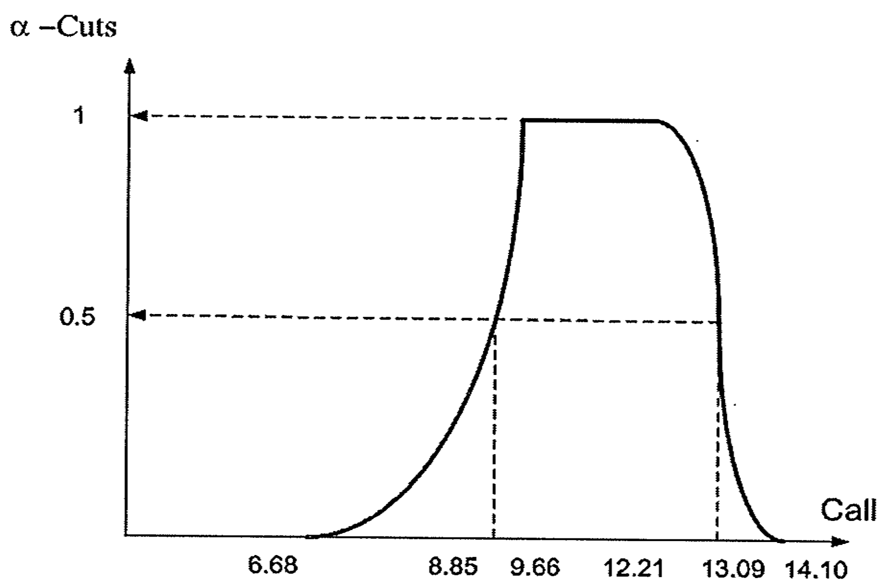

Figure 5.6: Fuzzy Call Option Values with O(2, 0.5)-Tr.T.F.N.

In Case 2 we consider a call option on a stock under conditions listed in the problem considered in (5.4.1)-(5.4.8). Table 5.4 gives the degrees of truth in different fuzzy call option values $C_1(\alpha)$ and $C_1(\alpha)$. For example, if any call option price is taken in the range of [8.85, 13.09], then the degree of truth associated with the value of this call option price is at least 0.5. Therefore, if an analyst is comfortable with the degree of truth of at least 0.5, then he(she) can take this option price for his(her) later use.

Characteristics of the Fuzzy Call Option Value Considered in Case 2, where $m = 2$ and $n = 0.5$..

We now analyze the behavior of the fuzzy call price, considered in Case 2. To do so we calculate their derivatives.

For $0 \leq \alpha \leq 1$.

$$\frac{dC_1(\alpha)}{d\alpha} = \frac{1}{1+r}\left[\frac{dC_{u1}(\alpha)}{d\alpha}p_{u1}(\alpha) + \frac{dp_{u1}(\alpha)}{d\alpha}C_{u1}(\alpha)\right] > 0 \tag{5.4.18}$$

On the other hand for $0 \leq \alpha \leq 1$.

$$\begin{aligned}\frac{d^2C_1(\alpha)}{d\alpha^2} &= \frac{1}{1+r}\left[p_{u1}(\alpha)\frac{d^2C_{u1}(\alpha)}{d\alpha^2} + 2\frac{dC_{u1}(\alpha)}{d\alpha}\frac{dp_{u1}(\alpha)}{d\alpha} + C_{u1}(\alpha)\frac{d^2p_{u1}(\alpha)}{d\alpha^2}\right] \\ &= \left(\frac{1}{1.0633}\right)\left(\frac{0.2133 - 0.1\,(1-\alpha)^2}{0.3 - 0.08\,(1-\alpha)^2}\right)\left(0.1625\,(1-\alpha)^{-\frac{3}{2}}\right) + \\ &\quad \left(\frac{2}{1.0633}\right)\left(0.325\,(1-\alpha)^{-\frac{1}{2}}\right)\left(\frac{64.68(1-\alpha)}{(-11.0 - 8.0\alpha + 4.0\alpha^2)^2}\right) + \\ &\quad \left(\frac{(14.45 - 0.65\sqrt{1-\alpha})}{1.0633}\right)\left(\frac{194.04\,(9.0 - 8.0\alpha + 4.0\alpha^2)}{(-11.0 - 8.0\alpha + 4.0\alpha^2)^3}\right) < 0\end{aligned} \tag{5.4.19}$$

Similarly, for $0 \leq \alpha \leq 1$.

$$\frac{dC_2(\alpha)}{d\alpha} = \frac{1}{1+r}\left[\frac{dC_{u2}(\alpha)}{d\alpha}p_{u2}(\alpha) + \frac{dp_{u2}(\alpha)}{d\alpha}C_{u2}(\alpha)\right] < 0 \tag{5.4.20}$$

and

$$\begin{aligned}
\frac{d^2C_2(\alpha)}{d\alpha^2} &= \frac{1}{1+r}\left[p_{u2}(\alpha)\frac{d^2C_{u2}(\alpha)}{d\alpha^2}+2\frac{dC_{u2}(\alpha)}{d\alpha}\frac{dp_{u2}(\alpha)}{d\alpha}+C_{u2}(\alpha)\frac{d^2p_{u2}(\alpha)}{d\alpha^2}\right]\\
&= \frac{1}{1.0633}\left(\frac{0.3133+0.1\sqrt{1-\alpha}}{0.38+0.09\sqrt{1-\alpha}}\right)2.6+\\
&\quad \left(\frac{2\left(-2.6+2.6\alpha\right)(1-\alpha)^{-\frac{1}{2}}}{1.0633}\right)\left(\frac{-49.015}{\left(38.0+9.0\sqrt{1-\alpha}\right)^2}\right)+\\
&\quad \frac{\left(15.75+1.3(1-\alpha)^2\right)}{1.0633}\left(\frac{-24.508\left(38.0+27.0\sqrt{1-\alpha}\right)(1-\alpha)^{-\frac{3}{2}}}{\left(38.0+9.0\sqrt{1-\alpha}\right)^3}\right)
\end{aligned} \tag{5.4.21}$$

From (5.4.21), we see that $C_2(\alpha)$ has a point of inflection at $\alpha = 0.76$, therefore, it is convex for $\alpha < 0.76$ and concave at $\alpha > 0.76$ (because $\frac{d^2C_2(\alpha)}{d\alpha^2} > 0$ for $\alpha < 0.76$ and $\frac{d^2C_2(\alpha)}{d\alpha^2} < 0$ for $\alpha > 0.76$).

5.4.3 Case 3.

In this case we take $m = 0.5$ and $n = 2$ in the problem considered in (5.4.1)-(5.4.5) and compute the fuzzy price of the call option. Using expressions (5.1.2), (5.1.3), (5.1.5) and (5.1.6), we compute the risk neutral probabilities as follows.

$$\begin{aligned}
p_{d1}(\alpha) &= \left[\frac{(u_2-1-r)-(u_2-u_1)(1-\alpha)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-\alpha)^{\frac{1}{m}}}\right]\\
&= \left[\frac{0.0667-0.01(1-\alpha)^2}{0.38+0.09(1-\alpha)^2}\right]
\end{aligned}$$

$$\begin{aligned}
p_{d2}(\alpha) &= \left[\frac{(u_3-1-r)-(u_3-u_4)(1-\alpha)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)(1-\alpha)^{\frac{1}{n}}}\right]\\
&= \left[\frac{0.0867+0.02\sqrt{1-\alpha}}{0.3-0.08\sqrt{1-\alpha}}\right]
\end{aligned}$$

$$\begin{aligned}
p_{u1}(\alpha) &= \left[\frac{(1+r-d_3)+(d_3-d_4)(1-\alpha)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)(1-\alpha)^{\frac{1}{n}}}\right]\\
&= \left[\frac{0.2133-0.1\sqrt{1-\alpha}}{0.3-0.08\sqrt{1-\alpha}}\right]
\end{aligned}$$

$$p_{u2}(\alpha) = \left[\frac{0.3133 + 0.1\,(1-\alpha)^2}{0.38 + 0.09\,(1-\alpha)^2}\right]$$

Characteristics of the Risk Neutral Probabilities.

To study the behavior of the fuzzy call price, we find the derivatives of each of those risk neutral probabilities and checks their behavior as α varies, where, $0 \le \alpha \le 1$.

$$\frac{dp_{d1}(\alpha)}{d\alpha} = \frac{d}{d\alpha}\left[\frac{0.0667 - 0.01\,(1-\alpha)^2}{0.38 + 0.09\,(1-\alpha)^2}\right] = \left[\frac{196.06\,(1-\alpha)}{(47.0 - 18.0\alpha + 9.0\alpha^2)^2}\right] > 0$$

$$\frac{d^2p_{d1}(\alpha)}{d\alpha^2} = \frac{d^2}{d\alpha^2}\left[\frac{0.0667 - 0.01\,(1-\alpha)^2}{0.38 + 0.09\,(1-\alpha)^2}\right] = \left[\frac{196.06\,(-11.0 - 54.0\alpha + 27.0\alpha^2)}{(47.0 - 18.0\alpha + 9.0\alpha^2)^3}\right] < 0$$

$$\frac{dp_{d2}(\alpha)}{d\alpha} = \frac{d}{d\alpha}\left[\frac{0.0867 + 0.02\,(1-\alpha)^{\frac{1}{2}}}{0.3 - 0.08\,(1-\alpha)^{\frac{1}{2}}}\right] = \left[\frac{-16.17}{\left(-15.0 + 4.0\sqrt{1-\alpha}\right)^2\sqrt{1-\alpha}}\right] < 0$$

$$\frac{d^{\,2}p_{d2}(\alpha)}{d\alpha^2} = \frac{d^2}{d\alpha^2}\left[\frac{0.0867 + 0.02\,(1-\alpha)^{\frac{1}{2}}}{0.3 - 0.08\,(1-\alpha)^{\frac{1}{2}}}\right] = \left[\frac{-24.255\left(-5.0 + 4.0\sqrt{1-\alpha}\right)}{(1-\alpha)^{\frac{3}{2}}\left(-15.0 + 4.0\sqrt{1-\alpha}\right)^3}\right] < 0$$

$$\frac{d\,p_{u1}(\alpha)}{d\alpha} = \frac{d}{d\alpha}\left[\frac{0.2133 - 0.1\sqrt{1-\alpha}}{0.3 - 0.08\sqrt{1-\alpha}}\right] = \left[\frac{16.17}{\left(-15.0 + 4.0\sqrt{1-\alpha}\right)^2\sqrt{1-\alpha}}\right] > 0$$

$$\frac{d^2\,p_{u1}(\alpha)}{d\alpha^2} = \frac{d^2}{d\alpha^2}\left[\frac{0.2133 - 0.1\sqrt{1-\alpha}}{0.3 - 0.08\sqrt{1-\alpha}}\right] = \left[\frac{24.255\left(-5.0 + 4.0\sqrt{1-\alpha}\right)}{(1-\alpha)^{\frac{3}{2}}\left(-15.0 + 4.0\sqrt{1-\alpha}\right)^3}\right] > 0$$

$$\frac{dp_{u2}(\alpha)}{d\alpha} = \frac{d}{d\alpha}\left[\frac{0.3133 + 0.1\,(1-\alpha)^2}{0.38 + 0.09\,(1-\alpha)^2}\right] = \left[\frac{-196.06(1-\alpha)}{(47.0 - 18.0\alpha + 9.0\alpha^2)^2}\right] < 0$$

$$\frac{d^2p_{u2}(\alpha)}{d\alpha^2} = \frac{d^2}{d\alpha^2}\left[\frac{0.3133 + 0.1\,(1-\alpha)^2}{0.38 + 0.09\,(1-\alpha)^2}\right] = \left[\frac{-196.06\,(-11.0 - 54.0\alpha + 27.0\alpha^2)}{(47.0 - 18.0\alpha + 9.0\alpha^2)^3}\right] > 0$$

Note that the first derivatives of the left hand side of the fuzzy risk neutral probabilities are greater than zero and the derivatives of the right hand side of the fuzzy risk neutral probabilities are less than zero. The second derivative for $p_{d1}(\alpha)$ and $p_{d2}(\alpha)$ are less than zero, which indicates their concave character. On similar lines we conclude that $p_{u1}(\alpha)$ and $p_{u2}(\alpha)$ are convex in character.

Table 5.5 summarizes the fuzzy risk neutral probabilities in the stock price for different values of α, where $0 \leq \alpha \leq 1$.

Table 5.5: Fuzzy Risk Neutral Probabilities.

α	$p_{u1}(\alpha)$	$p_{u2}(\alpha)$	$p_{d1}(\alpha)$	$p_{d2}(\alpha)$
0	0.52	0.88	0.12	0.49
0.1	0.53	0.87	0.13	0.47
0.2	0.54	0.86	0.14	0.46
0.3	0.56	0.85	0.15	0.44
0.4	0.57	0.85	0.15	0.43
0.5	0.59	0.84	0.16	0.41
0.6	0.60	0.83	0.17	0.40
0.7	0.62	0.83	0.17	0.38
0.8	0.64	0.83	0.17	0.36
0.9	0.66	0.83	0.17	0.34
1	0.71	0.82	0.18	0.29

From Table 5.5 we obtain Figures 5.7 and 5.8 which depict the fuzzy risk neutral probabilities for different values of α, $0 \leq \alpha \leq 1$.

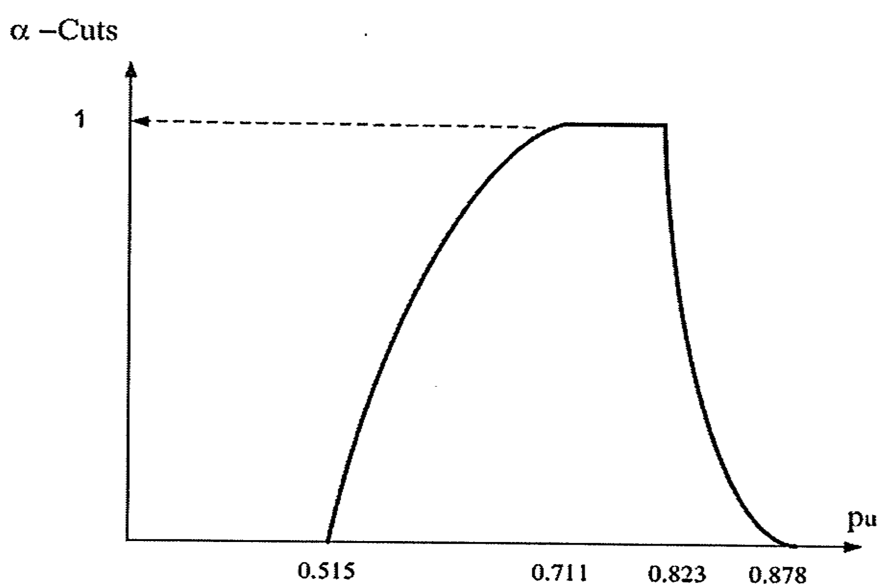

Figure 5.7: p_u, Probability of an up movement in the stock price.

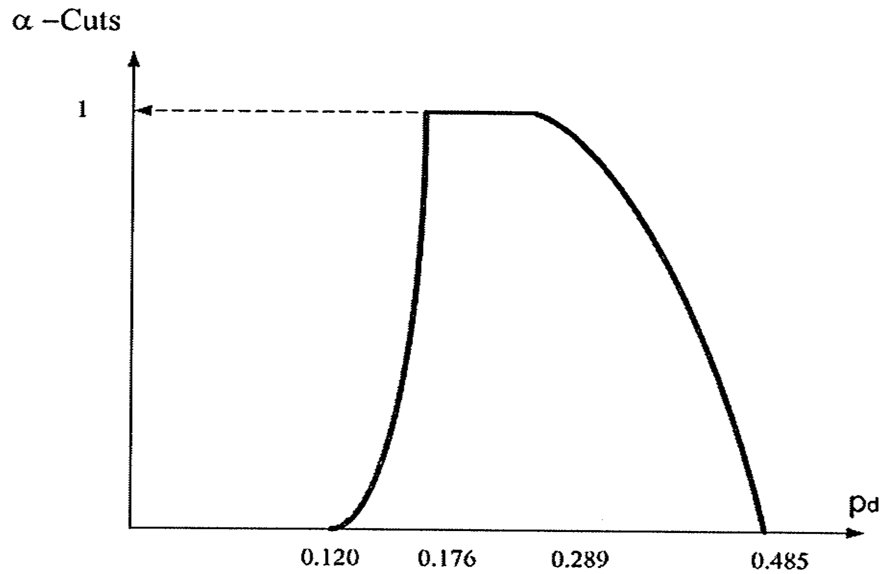

Figure 5.8: p_d, Probability of a down movement in the stock price.

Fuzzy Call Option Value Calculation with O(0.5,2)-Tr.T.F.N's.

Substituting values of $r, S, K, \tilde{u}$ and $\tilde{d}$ in formulas (5.3.4) and (5.3.5), we obtain the fuzzy current price of the call option.

$$\begin{aligned} C_1(\alpha) &= \left[\frac{S\left(u_2-(u_2-u_1)(1-\alpha)^{\frac{1}{m}}\right)-K}{1+r}\,\frac{(1+r-d_3)+(d_3-d_4)(1-\alpha)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)(1-\alpha)^{\frac{1}{n}}}\right] \\ &= \left[\frac{14.45-0.65\,(1-\alpha)^2}{1.0633}\,\frac{0.2133-0.1\sqrt{1-\alpha}}{0.3-0.08\sqrt{1-\alpha}}\right] \end{aligned}$$

$$\begin{aligned} C_2(\alpha) &= \left[\frac{S\left(u_3-(u_3-u_4)(1-\alpha)^{\frac{1}{n}}\right)-K}{1+r}\,\frac{(1+r-d_2)+(d_2-d_1)(1-\alpha)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-\alpha)^{\frac{1}{m}}}\right] \\ &= \left[\frac{15.75+1.3\sqrt{1-\alpha}}{1.0633}\,\frac{0.3133+0.1\,(1-\alpha)^2}{0.38+0.09\,(1-\alpha)^2}\right] \end{aligned}$$

In Table (5.6), the fuzzy call price are tabulated. By substituting different values of α in the above expressions for $C_1(\alpha)$ and $C_2(\alpha)$ we obtain the following tables for the fuzzy call option values.

Table 5.6: Fuzzy Call Option Values.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$C_1(\alpha)$	6.68	6.92	7.16	7.39	7.63	7.87	8.12	8.38	8.65	8.98	9.66
$C_2(\alpha)$	14.10	13.91	13.71	13.53	13.35	13.18	13.01	12.86	12.70	12.54	12.21

Using Table (5.6), we obtain Figure (5.9) which depicts the fuzzy call values for the option pricing problem.

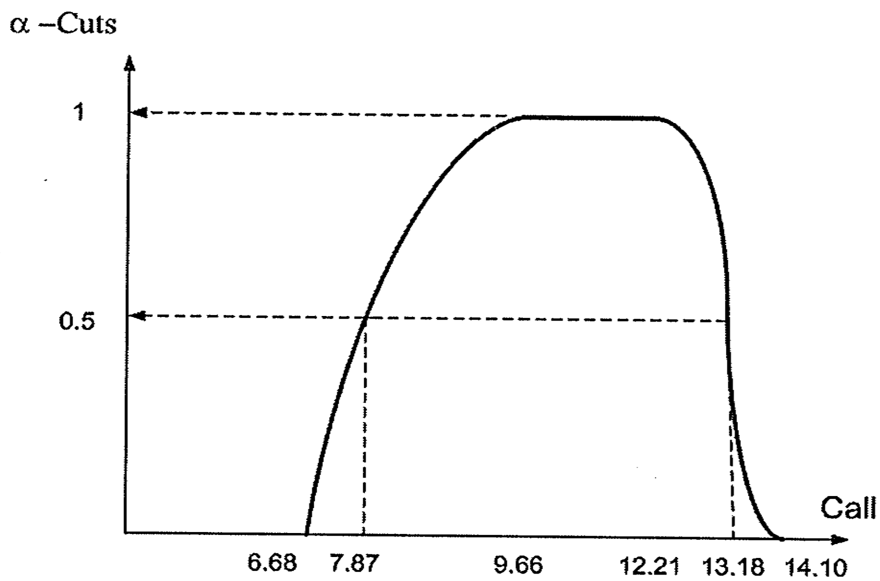

Figure 5.9: Fuzzy Call Option Values with O(0.5, 2)

In Case 3 we consider a call option on a stock under conditions listed in (5.4.1)-(5.4.8). Table 5.6 gives the degree of truth for different fuzzy call option values $C_1(\alpha)$ and $C_1(\alpha)$. For example, if any call option price is taken in the range of [7.87, 13.18], then the degree of truth associated with the value of this call option price is at least 0.5. Therefore, if an analyst is comfortable with the degree of truth of at least 0.5, then he(she) can take this option price for his(her) later use.

Characteristics of the Fuzzy Call Option Value Considered in Case 3, where $m = 0.5$ and $n = 2$.

We now analyze the behavior of the fuzzy call price, considered in Case 3.

Consider expression (5.4.16), the fuzzy call price considered in Case 2. Expression (5.4.16), is always greater than zero, irrespective of the value of α. for $0 \leq \alpha \leq 1$.

$$\frac{dC_1(\alpha)}{d\alpha} = \frac{1}{1+r}\left[\frac{dC_{u1}(\alpha)}{d\alpha}p_{u1}(\alpha) + \frac{dp_{u1}(\alpha)}{d\alpha}C_{u1}(\alpha)\right] > 0$$

$$\begin{aligned}
\frac{d^2C_1(\alpha)}{d\alpha^2} &= \frac{1}{1+r}\left[p_{u1}(\alpha)\frac{d^2C_{u1}(\alpha)}{d\alpha^2} + 2\frac{dC_{u1}(\alpha)}{d\alpha}\frac{dp_{u1}(\alpha)}{d\alpha} + C_{u1}(\alpha)\frac{d^2p_{u1}(\alpha)}{d\alpha^2}\right] \\
&= \frac{-1.3}{1.0633}\left(\frac{0.2133 - 0.1\sqrt{1-\alpha}}{0.3 - 0.08\sqrt{1-\alpha}}\right) + \\
&\quad \frac{1}{1.0633}\left(\frac{42.042 - 42.042\alpha\,(1-\alpha)^{-\frac{1}{2}}}{\left(-15.0 + 4.0\sqrt{1-\alpha}\right)^2}\right) + \\
&\quad \frac{\left(14.45 - 0.65\,(1-\alpha)^2\right)}{1.0633}\left(\frac{\left(-121.28 + 97.02\sqrt{(1-\alpha)}\right)(1-\alpha)^{-\frac{3}{2}}}{\left(-15.0 + 4.0\sqrt{1-\alpha}\right)^3}\right) < 0
\end{aligned} \tag{5.4.22}$$

The expression given by (5.4.22) is negative irrespective of the value of α, for $0 \leq \alpha \leq 1$. Therefore, $C_1(\alpha)$ is concave in nature for $0 \leq \alpha \leq 1$.

Similarly, for $0 \leq \alpha \leq 1$..

$$\frac{dC_2(\alpha)}{d\alpha} = \frac{1}{1+r}\left[\frac{dC_{u2}(\alpha)}{d\alpha}p_{u2}(\alpha) + \frac{dp_{u2}(\alpha)}{d\alpha}C_{u2}(\alpha)\right] < 0$$

From expression (5.3.19), we have

$$\begin{aligned}
\frac{d^2C_2(\alpha)}{d\alpha^2} &= \frac{1}{1+r}\left[p_{u2}(\alpha)\frac{d^2C_{u2}(\alpha)}{d\alpha^2}+2\frac{dC_{u2}(\alpha)}{d\alpha}\frac{dp_{u2}(\alpha)}{d\alpha}+C_{u2}(\alpha)\frac{d^2p_{u2}(\alpha)}{d\alpha^2}\right]\\
&= \left(\frac{-0.325\,(1-\alpha)^{-\frac{3}{2}}}{1.0633}\right)\left(\frac{0.3133+0.1\,(1-\alpha)^2}{0.38+0.09\,(1-\alpha)^2}\right)+\\
&\quad \left(\frac{-1.3\,(1-\alpha)^{-\frac{1}{2}}}{1.0633}\right)\left(\frac{-196.06(1-\alpha)}{(47.0-18.0\alpha+9.0\alpha^2)^2}\right)+\\
&\quad \left(\frac{(15.75+1.3\sqrt{1-\alpha})}{1.0633}\right)\left(\frac{-196.06\,(-11.0-54.0\alpha+27.0\alpha^2)}{(47.0-18.0\alpha+9.0\alpha^2)^3}\right)
\end{aligned} \tag{5.4.23}$$

Therefore, from (5.4.23), we have a point of inflection at $\alpha = 0.74$, it is convex at $\alpha < 0.74$ and concave for $\alpha > 0.74$. (because $\dfrac{d^2C_2(\alpha)}{d\alpha^2} > 0$ for $\alpha < 0.74$ and $\dfrac{d^2C_2(\alpha)}{d\alpha^2} < 0$ for $\alpha > 0.74$).

5.4.4 Case 4.

In the problem considered in (5.4.1)-(5.4.5) and $m = 0.5$ and $n = 0.5$, we compute the fuzzy call price. Using expressions (5.1.2),(5.1.3),(5.1.5) and (5.1.6), we compute the risk neutral probabilities as follows.

$$\begin{aligned}
p_{d1}(\alpha) &= \left[\frac{(u_2-1-r)-(u_2-u_1)\,(1-\alpha)^2}{u_2-d_2-(u_2-u_1-d_2+d_1)\,(1-\alpha)^2}\right]\\
&= \left[\frac{0.0667-0.01\,(1-\alpha)^2}{0.38+0.09\,(1-\alpha)^2}\right]
\end{aligned}$$

$$\begin{aligned}
p_{d2}(\alpha) &= \left[\frac{(u_3-1-r)-(u_3-u_4)\,(1-\alpha)^2}{u_3-d_3-(u_3-u_4-d_3+d_4)\,(1-\alpha)^2}\right]\\
&= \left[\frac{0.0867+0.02\,(1-\alpha)^2}{0.3-0.08\,(1-\alpha)^2}\right]
\end{aligned}$$

$$p_{u1}(\alpha) = \left[\frac{(1+r-d_3)+(d_3-d_4)(1-\alpha)^2}{u_3-d_3-(u_3-u_4-d_3+d_4)(1-\alpha)^2}\right]$$
$$= \left[\frac{0.2133-0.1(1-\alpha)^2}{0.3-0.08(1-\alpha)^2}\right]$$

$$p_{u2}(\alpha) = \left[\frac{(1+r-d_2)+(d_2-d_1)(1-\alpha)^2}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-\alpha)^2}\right]$$
$$= \left[\frac{0.3133+0.1(1-\alpha)^2}{0.38+0.09(1-\alpha)^2}\right]$$

Characteristics of the Risk Neutral Probabilities.

To study the behavior of the fuzzy call price, we find the derivatives of each of those risk neutral probabilities and checks their behavior as α varies, where, $0 \leq \alpha \leq 1$.

$$\frac{dp_{d1}(\alpha)}{d\alpha} = \frac{d}{d\alpha}\left[\frac{0.0667-0.01(1-\alpha)^2}{0.38+0.09(1-\alpha)^2}\right] = \left[\frac{196.06(1-\alpha)}{(47.0-18.0\alpha+9.0\alpha^2)^2}\right] > 0$$
$$\frac{d^2p_{d1}(\alpha)}{d\alpha^2} = \frac{d^2}{d\alpha^2}\left[\frac{0.0667-0.01(1-\alpha)^2}{0.38+0.09(1-\alpha)^2}\right] = \left[\frac{196.06(-11.0-54.0\alpha+27.0\alpha^2)}{(47.0-18.0\alpha+9.0\alpha^2)^3}\right] < 0$$

$$\frac{d\,p_{d2}(\alpha)}{d\alpha} = \frac{d}{d\alpha}\left[\frac{0.0867+0.02(1-\alpha)^2}{0.3-0.08(1-\alpha)^2}\right] = \left[\frac{-64.68(1-\alpha)}{(-11.0-8.0\alpha+4.0\alpha^2)^2}\right] < 0$$
$$\frac{d\,^2p_{d2}(\alpha)}{d\alpha^2} = \frac{d^2}{d\alpha^2}\left[\frac{0.0867+0.02(1-\alpha)^2}{0.3-0.08(1-\alpha)^2}\right] = \left[\frac{-194.04(9.0-8.0\alpha+4.0\alpha^2)}{(-11.0-8.0\alpha+4.0\alpha^2)^3}\right] > 0$$

$$\frac{d\,p_{u1}(\alpha)}{d\alpha} = \frac{d}{d\alpha}\left[\frac{0.2133-0.1(1-\alpha)^2}{0.3-0.08(1-\alpha)^2}\right] = \left[\frac{64.68(1-\alpha)}{(-11.0-8.0\alpha+4.0\alpha^2)^2}\right] > 0$$
$$\frac{d^2\,p_{u1}(\alpha)}{d\alpha^2} = \frac{d^2}{d\alpha^2}\left[\frac{0.2133-0.1(1-\alpha)^2}{0.3-0.08(1-\alpha)^2}\right] = \left[\frac{194.04(9.0-8.0\alpha+4.0\alpha^2)}{(-11.0-8.0\alpha+4.0\alpha^2)^3}\right] < 0$$

$$\frac{dp_{u2}(\alpha)}{d\alpha} = \frac{d}{d\alpha}\left[\frac{0.3133+0.1(1-\alpha)^2}{0.38+0.09(1-\alpha)^2}\right] = \left[\frac{-196.06(1-\alpha)}{(47.0-18.0\alpha+9.0\alpha^2)^2}\right] < 0$$
$$\frac{d^2p_{u2}(\alpha)}{d\alpha^2} = \frac{d^2}{d\alpha^2}\left[\frac{0.3133+0.1(1-\alpha)^2}{0.38+0.09(1-\alpha)^2}\right] = \left[\frac{-196.06(-11.0-54.0\alpha+27.0\alpha^2)}{(47.0-18.0\alpha+9.0\alpha^2)^3}\right] > 0$$

Note that the first derivatives of the left hand side of the fuzzy risk neutral probabilities are greater than zero and the second derivatives of the right hand side of the fuzzy risk neutral probabilities are less than zero which indicates its concave character . The second derivative of $p_{u2}(\alpha)$ is greater than zero which would indicate convex character.

Table 5.7 summarizes the fuzzy risk neutral probabilities in the stock price for different values of α, where $0 \leq \alpha \leq 1$.

Table 5.7: Fuzzy Risk Neutral Probabilities.

α	$p_{u1}(\alpha)$	$p_{u2}(\alpha)$	$p_{d1}(\alpha)$	$p_{d2}(\alpha)$
0	0.515	0.879	0.121	0.485
0.1	0.563	0.871	0.129	0.438
0.2	0.600	0.862	0.138	0.400
0.3	0.630	0.854	0.146	0.370
0.4	0.654	0.847	0.153	0.346
0.5	0.673	0.840	0.160	0.328
0.6	0.687	0.835	0.165	0.313
0.7	0.698	0.830	0.170	0.302
0.8	0.705	0.827	0.173	0.295
0.9	0.710	0.825	0.175	0.290
1	0.711	0.824	0.176	0.289

Using Table (5.7), we obtained Figure 5.10 and 5.11.

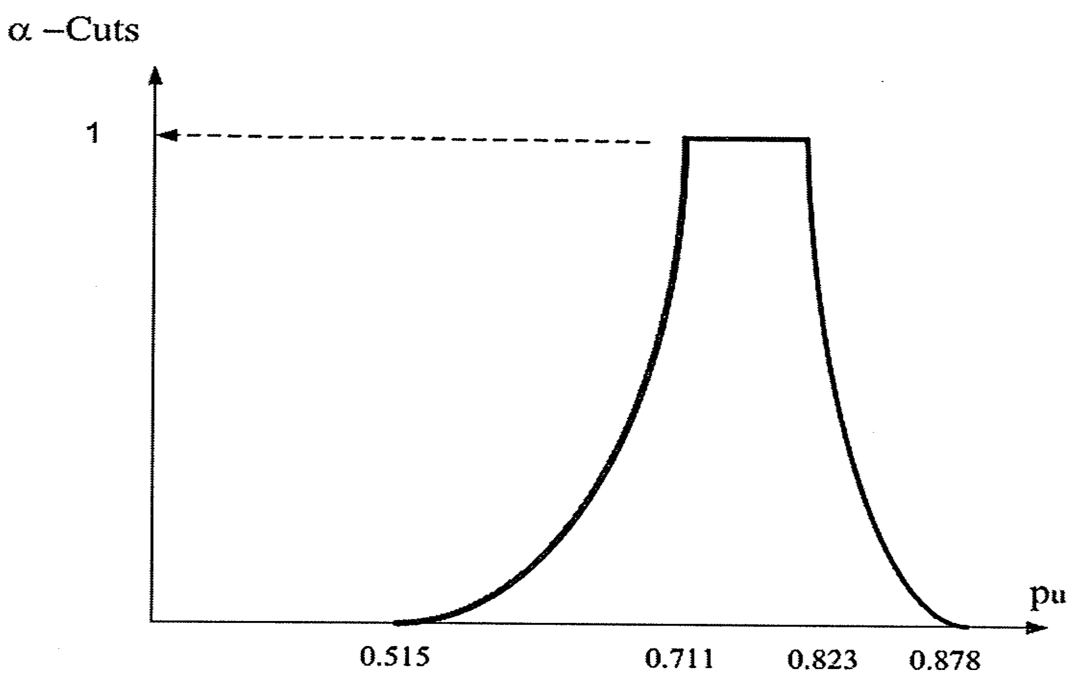

Figure 5.10: p_u, Probability of an up movement in the stock price.

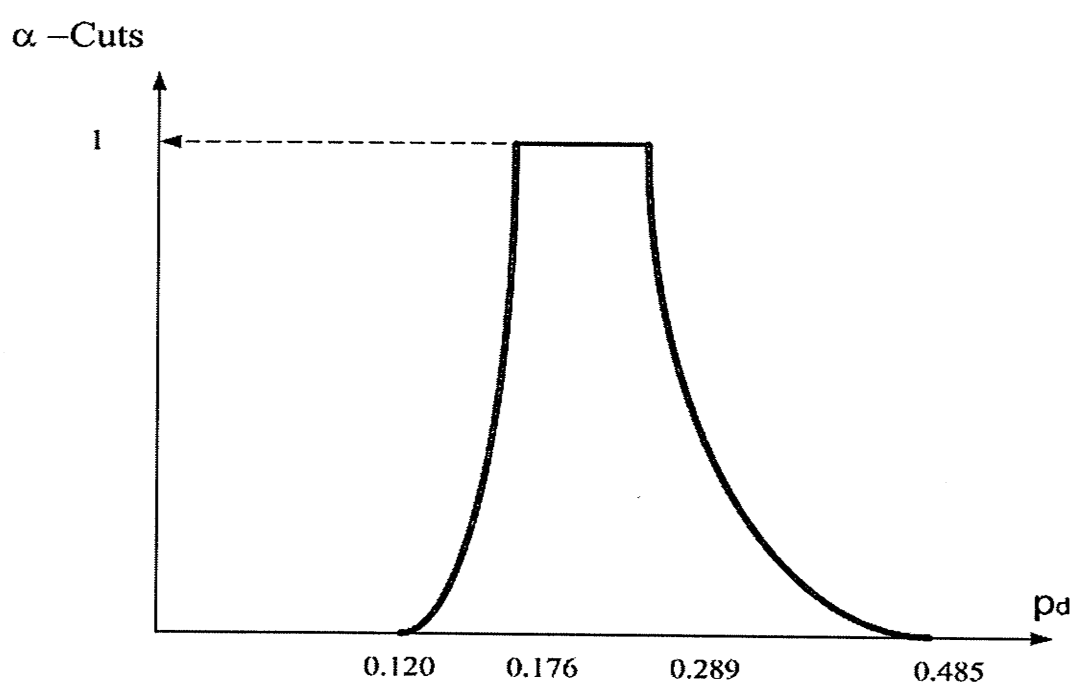

Figure 5.11: p_d, Probability of a down movement in the stock price.

Fuzzy Call Option Value Calculation with O(0.5,0.5)-Tr.T.F.N's.

Substituting values of $r, S, K, \tilde{u}$ and $\tilde{d}$ in formulas (5.3.4) and (5.3.5), we obtain the fuzzy current price of the call option.

$$C_1(\alpha) = \left[\frac{S\left(u_2-(u_2-u_1)(1-\alpha)^{\frac{1}{m}}\right)-K}{1+r} \quad \frac{(1+r-d_3)+(d_3-d_4)(1-\alpha)^{\frac{1}{n}}}{u_3-d_3-(u_3-u_4-d_3+d_4)(1-\alpha)^{\frac{1}{n}}}\right]$$

$$= \left[\frac{14.45-0.65(1-\alpha)^2}{1.0633} \quad \frac{0.2133-0.1(1-\alpha)^2}{0.3-0.08(1-\alpha)^2}\right] \tag{5.4.24}$$

$$C_2(\alpha) = \left[\frac{S\left(u_3-(u_3-u_4)(1-\alpha)^{\frac{1}{n}}\right)-K}{1+r} \quad \frac{(1+r-d_2)+(d_2-d_1)(1-\alpha)^{\frac{1}{m}}}{u_2-d_2-(u_2-u_1-d_2+d_1)(1-\alpha)^{\frac{1}{m}}}\right]$$

$$= \left[\frac{15.75+1.3(1-\alpha)^2}{1.0633} \quad \frac{0.3133+0.1(1-\alpha)^2}{0.38+0.09(1-\alpha)^2}\right] \tag{5.4.25}$$

In Table (5.8), the fuzzy call price are tabulated. By substituting different values of α in the above expressions for $C_1(\alpha)$ and $C_2(\alpha)$ we obtain the following tables for the fuzzy call option values.

Table 5.8: Fuzzy Call Option Values.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$C_1(\alpha)$	6.68	7.37	7.92	8.37	8.74	9.04	9.27	9.44	9.57	9.64	9.66
$C_2(\alpha)$	14.10	13.76	13.45	13.17	12.92	12.71	12.53	12.39	12.29	12.23	12.21

Using Table (5.8), we obtain Figure 5.12 which depicts the fuzzy call values for the option pricing problem.

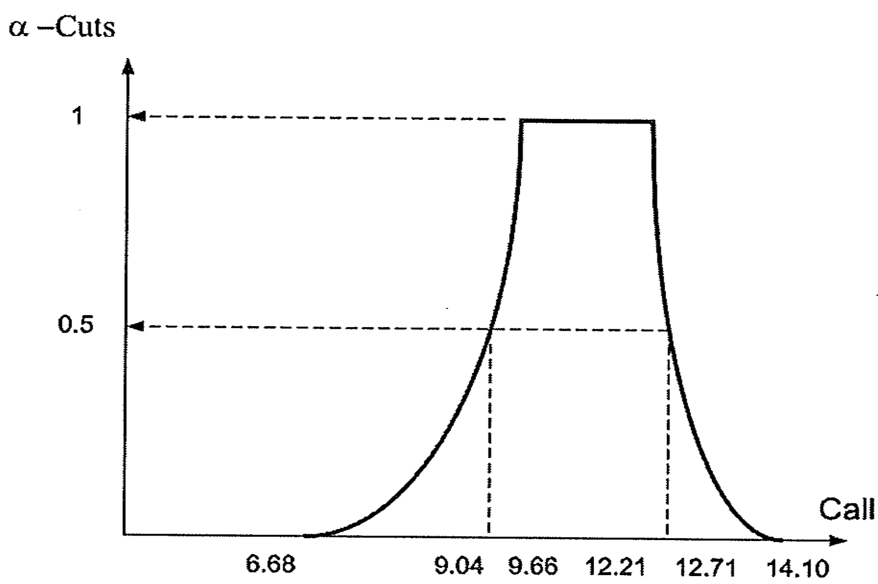

Figure 5.12: Fuzzy Call Option Values with O(0.5, 0.5)

In Case 4 we consider a call option on a stock under conditions listed in (5.4.1)-(5.4.8). Table 5.8 gives the degree of truth for different fuzzy call option values $C_1(\alpha)$ and $C_1(\alpha)$. For example, if any call option price is taken in the range of [9.04, 12.71], then the degree of truth associated with the value of this call option price is at least 0.5. Therefore, if an analyst is comfortable with the degree of truth of at least 0.5, then he(she) can take this option price for his(her) later use.

Characteristics of the Fuzzy Call Option Value Considered in Case 4, where $m = 0.5$ and $n = 0.5$.

We now analyze the behavior of the fuzzy call price, considered in Case 4. From (5.4.16) we have for α, for $0 \leq \alpha \leq 1$.

$$\frac{dC_1(\alpha)}{d\alpha} = \frac{1}{1+r}\left[\frac{dC_{u1}(\alpha)}{d\alpha}p_{u1}(\alpha) + \frac{dp_{u1}(\alpha)}{d\alpha}C_{u1}(\alpha)\right] > 0 \qquad (5.4.26)$$

Also, for α, for $0 \leq \alpha \leq 1$.

$$\begin{aligned}
\frac{d^2C_1(\alpha)}{d\alpha^2} &= \frac{1}{1+r}\left[p_{u1}(\alpha)\frac{d^2C_{u1}(\alpha)}{d\alpha^2} + 2\frac{dC_{u1}(\alpha)}{d\alpha}\frac{dp_{u1}(\alpha)}{d\alpha} + C_{u1}(\alpha)\frac{d^2p_{u1}(\alpha)}{d\alpha^2}\right] \\
&= \frac{-1.3}{1.0633}\left(\frac{0.2133 - 0.1\,(1-\alpha)^2}{0.3 - 0.08\,(1-\alpha)^2}\right) + \\
&\quad \frac{2\,(1.3 - 1.3\alpha)}{1.0633}\left(\frac{64.68(1-\alpha)}{(-11.0 - 8.0\alpha + 4.0\alpha^2)^2}\right) + \\
&\quad \frac{(14.45 - 0.65\,(1-\alpha)^2)}{1.0633}\left(\frac{194.04(9.0 - 8.0\alpha + 4.0\alpha^2)}{(-11.0 - 8.0\alpha + 4.0\alpha^2)^3}\right) < 0
\end{aligned} \tag{5.4.27}$$

Hence, $C_1(\alpha)$ is concave in character for $0 \leq \alpha \leq 1$.

Similarly, for, $0 \leq \alpha \leq 1$.

$$\frac{dC_2(\alpha)}{d\alpha} = \frac{1}{1+r}\left[\frac{dC_{u2}(\alpha)}{d\alpha}p_{u2}(\alpha) + \frac{dp_{u2}(\alpha)}{d\alpha}C_{u2}(\alpha)\right] < 0$$

and

$$\begin{aligned}
\frac{d^2C_2(\alpha)}{d\alpha^2} &= \frac{1}{1+r}\left[p_{u2}(\alpha)\frac{d^2C_{u2}(\alpha)}{d\alpha^2} + 2\frac{dC_{u2}(\alpha)}{d\alpha}\frac{dp_{u2}(\alpha)}{d\alpha} + C_{u2}(\alpha)\frac{d^2p_{u2}(\alpha)}{d\alpha^2}\right] \\
&= \frac{2.6}{1.0633}\left(\frac{0.3133 + 0.1\,(1-\alpha)^2}{0.38 + 0.09\,(1-\alpha)^2}\right) + \\
&\quad \frac{2\,(-2.6 + 2.6\alpha)}{1.0633}\left(\frac{-196.06(1-\alpha)}{(47.0 - 18.0\alpha + 9.0\alpha^2)^2}\right) + \\
&\quad \frac{(15.75 + 1.3\,(1-\alpha)^2)}{1.0633}\left(\frac{-196.06\,(-11.0 - 54.0\alpha + 27.0\alpha^2)}{(47.0 - 18.0\alpha + 9.0\alpha^2)^3}\right) > 0
\end{aligned} \tag{5.4.28}$$

This means that, $C_2(\alpha)$ is convex in character for $0 \leq \alpha \leq 1$.

Membership Function of the Fuzzy Call Option Considered in Case 4.

We now compute the membership function of the fuzzy call price considered in Case 4. In order to find the two exterior points and the two interior points which describe the fuzzy call price corresponding to the O(0.5,0.5)-Tr.T.F.N in Case 4, we set $\alpha = 0$ and $\alpha = 1$ in (5.4.24) and (5.4.25), respectively.

Thus, the fuzzy number that describe the fuzzy call price in (5.4.24) is as given by $\tilde{C} = [6.68, 9.66, 12.21, 14.10]$. In view of the above discussion, we determine the membership function as follows. From (5.4.24), setting $C_1(\alpha) = \tilde{C}$, we get

$$\left[\frac{14.45 - 0.65\,(1-\alpha)^2}{1.0633} \quad \frac{0.2133 - 0.1\,(1-\alpha)^2}{0.3 - 0.08\,(1-\alpha)^2}\right] = \tilde{C}$$

Cross multiplication yields, the following equation for α.

$$\alpha = 1 - \sqrt{\left(\frac{\left(1.4894 - 0.08\tilde{C}\right) + \sqrt{1.5095 - 0.16495\tilde{C} + 0.0064\tilde{C}^2}}{0.12226}\right)}$$

Similarly, from (5.4.25), setting $C_2(\alpha) = \tilde{C}$, we get

$$\alpha = 1 - \sqrt{\left(\frac{(0.09\tilde{C} - 1.8643) + \sqrt{0.0081\tilde{C}^2 - 0.14974\tilde{C} + 1.2061}}{0.12226}\right)}$$

This gives the membership function $\tilde{C}$ of (see Figure(5.12)) as

$$\mu(\tilde{\tilde{C}}) = \begin{cases} 0 & \tilde{C} \leq 6.68 \\ 1 - \sqrt{\left(\frac{\left(1.4894 - 0.08\tilde{C}\right) + \sqrt{1.5095 - 0.16495\tilde{C} + 0.0064\tilde{C}^2}}{0.12226}\right)} & 6.68 \leq \tilde{C} \leq 9.66 \\ 1 & 9.66 \leq \tilde{C} \leq 12.21 \\ 1 - \sqrt{\left(\frac{(0.09\tilde{C} - 1.8643) + \sqrt{0.0081\tilde{C}^2 - 0.14974\tilde{C} + 1.2061}}{0.12226}\right)} & 12.21 \leq \tilde{C} \leq 14.10 \\ 0 & \tilde{C} \geq 14.10 \end{cases}$$

Conclusion

In this chapter, on the lines of Muzzioli and Torricelli [98] and Appadoo et al. [2], we discuss the option pricing when payoffs are described by O(m, n)-Tr.T.F.N. numbers. We believe that this approached can be extended to price a wide variety of options with different types of fuzzy pay-off patterns.

Chapter 6

Fuzzy Option Pricing Model Using LR-Fuzzy Numbers

In this chapter we derive and discuss the results for fuzzy binomial risk neutral option pricing model using LR-fuzzy number. The model provides a reasonable range of option price to choose from. Certain results derived in Chapter 5 of this thesis and in [2], [4] and [98] may be viewed as special cases of the results derived in this chapter.

Suppose the price of a stock at time $t = 0$ is S, whereas at $t = 1$ we obtain its price by multiplying S with the jump factors. Let the γ-level sets for a fuzzy increase $\tilde{u}$ and a fuzzy decrease $\tilde{d}$, respectively, in the stock price be given by γ-level sets $u(\gamma) = [u_1(\gamma), u_2(\gamma)]$, and $d(\gamma) = [d_1(\gamma), d_2(\gamma)]$ such that using (1.5.7) we have

$$d_1(\gamma) = \underline{d} - \alpha_1 L^{-1}(\gamma) \tag{6.0.1}$$

$$d_2(\gamma) = \overline{d} + \beta_1 R^{-1}(\gamma) \tag{6.0.2}$$

$$u_1(\gamma) = \underline{u} - \alpha_2 L^{-1}(\gamma) \tag{6.0.3}$$

$$u_2(\gamma) = \overline{u} + \beta_2 R^{-1}(\gamma) \tag{6.0.4}$$

where, $\alpha_1 > 0$ is the left spread of a fuzzy decrease in the stock price, $\beta_1 > 0$ is the right spread of a fuzzy decrease in the stock price, $\alpha_2 > 0$ is the left spread of a fuzzy increase in the stock price and $\beta_2 > 0$ is the right spread of a fuzzy increase in the stock price. Furthermore, L and R are left and right functions: $[0,1] \to [0,1]$, with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$ and are non-increasing, continuous mappings.

Below we now state a theorem that provides the results for the fuzzy risk-neutral probabilities needed to price a fuzzy call option under LR-fuzzy numbers. Since its proof follows the lines of the proof of Theorem 4.1.1, therefore we do not prove it here.

6.1. Main Results

Theorem 6.1.1 *Let $\tilde{u} = (\underline{u}, \overline{u}, \alpha_2, \beta_2)_{LR}$ and $\tilde{d} = (\underline{d}, \overline{d}, \alpha_1, \beta_1)_{LR}$ represent, respectively, a fuzzy increase and a fuzzy decrease in S. At $t = 1$ Let,*

(a) $u(\gamma) = [u_1(\gamma),\ u_2(\gamma)]$, $\forall$, $0 \le \gamma \le 1$, be the γ-level set for a fuzzy increase in S,

(b) $d(\gamma) = [d_1(\gamma),\ d_2(\gamma)]$, $\forall$, $0 \le \gamma \le 1$, be the γ −level set for a fuzzy decrease in S.

(c) $p_d(\gamma) = [p_{d1}(\gamma),\ p_{d2}(\gamma)]$, $\forall$, $0 \le \gamma \le 1$, be the associated fuzzy risk neutral probability for a fuzzy decrease in S,

(d) $p_u(\gamma) = [p_{u1}(\gamma),\ p_{u2}(\gamma)]$, $\forall$, $0 \le \gamma \le 1$, be the associated fuzzy risk neutral probability for a fuzzy increase in S, and

(e) r be the risk free rate, assumed to be constant.

Then

(i) the fuzzy risk neutral probability associated with a fuzzy downward movement in the stock price is given by,

$$p_d(\gamma) = \left[\frac{\underline{u} - (1+r) - \alpha_1 L^{-1}(\gamma)}{\underline{u} - \underline{d} + L^{-1}(\gamma)(\alpha_2 - \alpha_1)}, \frac{\overline{u} - (1+r) + \beta_1 R^{-1}(\gamma)}{\overline{u} - \overline{d} + R^{-1}(\gamma)(\beta_1 - \beta_2)}\right] \quad (6.1.1)$$

such that,

$$p_{d1}(\gamma) = \left[\frac{\underline{u} - (1+r) - \alpha_1 L^{-1}(\gamma)}{\underline{u} - \underline{d} + L^{-1}(\gamma)(\alpha_2 - \alpha_1)}\right] \quad (6.1.2)$$

$$p_{d2}(\gamma) = \left[\frac{\overline{u} - (1+r) + \beta_1 R^{-1}(\gamma)}{\overline{u} - \overline{d} + R^{-1}(\gamma)(\beta_1 - \beta_2)}\right] \quad (6.1.3)$$

and

(ii) the fuzzy risk neutral probability associated with a fuzzy upward movement in the stock price is given by,

$$p_u(\gamma) = \left[\frac{(1+r-\overline{d}) - \beta_2 R^{-1}(\gamma)}{\overline{u} - \overline{d} + R^{-1}(\gamma)(\beta_1 - \beta_2)}, \frac{(1+r-\underline{d}) + \alpha_2 L^{-1}(\gamma)}{\underline{u} - \underline{d} + L^{-1}(\gamma)(\alpha_2 - \alpha_1)}\right] \quad (6.1.4)$$

such that,

$$p_{u1}(\gamma) = \left[\frac{(1+r-\overline{d}) - \beta_2 R^{-1}(\gamma)}{\overline{u} - \overline{d} + R^{-1}(\gamma)(\beta_1 - \beta_2)}\right] \quad (6.1.5)$$

$$p_{u2}(\gamma) = \left[\frac{(1+r-\underline{d}) + \alpha_2 L^{-1}(\gamma)}{\underline{u} - \underline{d} + L^{-1}(\gamma)(\alpha_2 - \alpha_1)}\right]. \quad (6.1.6)$$

Corollary 6.1.1 *Let $p_{u1}(\gamma)$ and $p_{u2}(\gamma)$ be the fuzzy risk neutral probabilities of a fuzzy increase in the stock price and let $p_{d1}(\gamma)$ and $p_{d2}(\gamma)$ be the risk neutral probabilities of* *a fuzzy decrease in the stock price, then the following complimentary conditions hold.*

$$\left[\frac{\underline{u}-(1+r)-\alpha_1 L^{-1}(\gamma)}{\underline{u}-\underline{d}+L^{-1}(\gamma)(\alpha_2-\alpha_1)}\right]+\left[\frac{(1+r-\underline{d})+\alpha_2 L^{-1}(\gamma)}{\underline{u}-\underline{d}+L^{-1}(\gamma)(\alpha_2-\alpha_1)}\right] = 1$$

$$\left[\frac{(1+r-\overline{d})-\beta_2 R^{-1}(\gamma)}{\overline{u}-\overline{d}+R^{-1}(\gamma)(\beta_1-\beta_2)}\right]+\left[\frac{\overline{u}-(1+r)+\beta_1 R^{-1}(\gamma)}{\overline{u}-\overline{d}+R^{-1}(\gamma)(\beta_1-\beta_2)}\right] = 1$$

Muzzioli and Torricelli [98], discussed the complimentary relationship when the parameters in the option pricing model are triangular fuzzy numbers. Our approach is more general and it can be adapted to handle different kind of fuzzy numbers, including fuzzy numbers with certain kind of nonlinear membership functions, that includes O(m,n)-Tr.T.F.N's also.

6.1.1 Membership Function for $p_d(\gamma)$ and $p_u(\gamma)$

In order to find the two exterior points and the two interior points which describe the risk neutral probability of a fuzzy decrease in the stock price corresponding to the LR-fuzzy numbers, we set $\gamma = 0$ and $\gamma = 1$ in (6.1.2)) and (6.1.3)). This yields,

$$p_{d1}(0) = \left[\frac{\underline{u}-(1+r)-\alpha_1 L^{-1}(0)}{\underline{u}-\underline{d}+L^{-1}(0)(\alpha_2-\alpha_1)}\right] \tag{6.1.7}$$

$$p_{d2}(0) = \left[\frac{\overline{u}-(1+r)+\beta_1 R^{-1}(0)}{\overline{u}-\overline{d}+R^{-1}(0)(\beta_1-\beta_2)}\right] \tag{6.1.8}$$

$$p_{d1}(1) = \left[\frac{\underline{u}-(1+r)-\alpha_1 L^{-1}(1)}{\underline{u}-\underline{d}+L^{-1}(1)(\alpha_2-\alpha_1)}\right] \tag{6.1.9}$$

$$p_{d2}(1) = \left[\frac{\overline{u}-(1+r)+\beta_1 R^{-1}(1)}{\overline{u}-\overline{d}+R^{-1}(1)(\beta_1-\beta_2)}\right] \tag{6.1.10}$$

Similarly, to find the two exterior points and the two interior points which describe the risk neutral probability of a fuzzy increase in the stock price corresponding to the LR-fuzzy numbers we set $\gamma = 0$ and $\gamma = 1$ in (6.1.5)) and (6.1.6)). This yields

$$p_{u1}(0) = \left[\frac{(1+r-\overline{d}) - \beta_2 R^{-1}(0)}{\overline{u} - \overline{d} + R^{-1}(0)(\beta_1 - \beta_2)}\right] \tag{6.1.11}$$

$$p_{u2}(0) = \left[\frac{(1+r-\underline{d}) + \alpha_2 L^{-1}(0)}{\underline{u} - \underline{d} + L^{-1}(0)(\alpha_2 - \alpha_1)}\right] \tag{6.1.12}$$

$$p_{u1}(1) = \left[\frac{(1+r-\overline{d}) - \beta_2 R^{-1}(1)}{\overline{u} - \overline{d} + R^{-1}(1)(\beta_1 - \beta_2)}\right] \tag{6.1.13}$$

$$p_{u2}(1) = \left[\frac{(1+r-\underline{d}) + \alpha_2 L^{-1}(1)}{\underline{u} - \underline{d} + L^{-1}(1)(\alpha_2 - \alpha_1)}\right] \tag{6.1.14}$$

In view of (6.1.7))- (6.1.14)), the fuzzy probabilities of a down movement and of an up movement are given respectively, by

$$\tilde{p}_d = \begin{bmatrix} \dfrac{\underline{u} - (1+r) - \alpha_1 L^{-1}(0)}{\underline{u} - \underline{d} + L^{-1}(0)(\alpha_2 - \alpha_1)}, & \dfrac{\underline{u} - (1+r) - \alpha_1 L^{-1}(1)}{\underline{u} - \underline{d} + L^{-1}(1)(\alpha_2 - \alpha_1)}, \\ \dfrac{\overline{u} - (1+r) + \beta_1 R^{-1}(1)}{\overline{u} - \overline{d} + R^{-1}(1)(\beta_1 - \beta_2)}, & \dfrac{\overline{u} - (1+r) + \beta_1 R^{-1}(0)}{\overline{u} - \overline{d} + R^{-1}(0)(\beta_1 - \beta_2)} \end{bmatrix} \tag{6.1.15}$$

and

$$\tilde{p}_u = \begin{bmatrix} \dfrac{(1+r-\overline{d}) - \beta_2 R^{-1}(0)}{\overline{u} - \overline{d} + R^{-1}(0)(\beta_1 - \beta_2)}, & \dfrac{(1+r-\overline{d}) - \beta_2 R^{-1}(1)}{\overline{u} - \overline{d} + R^{-1}(1)(\beta_1 - \beta_2)}, \\ \dfrac{(1+r-\underline{d}) + \alpha_2 L^{-1}(1)}{\underline{u} - \underline{d} + L^{-1}(1)(\alpha_2 - \alpha_1)}, & \dfrac{(1+r-\underline{d}) + \alpha_2 L^{-1}(0)}{\underline{u} - \underline{d} + L^{-1}(0)(\alpha_2 - \alpha_1)} \end{bmatrix} \tag{6.1.16}$$

We now determine the membership functions for the fuzzy risk neutral probabilities of downward movement and the fuzzy risk neutral probabilities of an upward movement in a stock price.

Since

$$p_{d1}(\gamma) = \left[\frac{\underline{u} - \alpha_1 L^{-1}(\gamma) - (1+r)}{(\underline{u} - \alpha_1 L^{-1}(\gamma)) - (\underline{d} - \alpha_2 L^{-1}(\gamma))}\right]$$

therefore, for

$$\frac{\underline{u} - (1+r) - \alpha_1 L^{-1}(0)}{\underline{u} - \underline{d} + L^{-1}(0)(\alpha_2 - \alpha_1)} \leq \tilde{p}_d \leq \frac{\underline{u} - (1+r) - \alpha_1 L^{-1}(1)}{\underline{u} - \underline{d} + L^{-1}(1)(\alpha_2 - \alpha_1)},$$

we set

$$\left[\frac{\underline{u} - \alpha_1 L^{-1}(\gamma) - (1+r)}{(\underline{u} - \alpha_1 L^{-1}(\gamma)) - (\underline{d} - \alpha_2 L^{-1}(\gamma))}\right] = \tilde{p}_d$$

$$\begin{aligned}
\underline{u} - \alpha_1 L^{-1}(\gamma) - (1+r) &= \tilde{p}_d \left[(\underline{u} - \alpha_1 L^{-1}(\gamma)) - (\underline{d} - \alpha_2 L^{-1}(\gamma))\right] \\
\underline{u} - \alpha_1 L^{-1}(\gamma) - (1+r &= \tilde{p}_d \underline{u} - \tilde{p}_d \alpha_1 L^{-1}(\gamma) - \tilde{p}_d \underline{d} + \tilde{p}_d \alpha_2 L^{-1}(\gamma) \\
\underline{u} - (1+r) - \underline{u}\tilde{p}_d + \underline{d}\tilde{p}_d &= -\tilde{p}_d \alpha_1 L^{-1}(\gamma) + \tilde{p}_d \alpha_2 L^{-1}(\gamma) + \alpha_1 L^{-1}(\gamma) \\
\underline{u} - (1+r) + \underline{d}\tilde{p}_d - \underline{u}\tilde{p}_d &= L^{-1}(\gamma)\left(\alpha_1 + \tilde{p}_d(\alpha_2 - \alpha_1)\right) \\
\gamma &= L\left[\frac{\underline{u} - (1+r) + \tilde{p}_d(\underline{d} - \underline{u})}{\tilde{p}_d(\alpha_2 - \alpha_1) + \alpha_1}\right] \qquad (6.1.17)
\end{aligned}$$

Similarly, for $\dfrac{\overline{u} - (1+r) + \beta_1 R^{-1}(1)}{\overline{u} - \overline{d} + R^{-1}(1)(\beta_1 - \beta_2)} \leq \tilde{p}_d \leq \dfrac{\overline{u} - (1+r) + \beta_1 R^{-1}(0)}{\overline{u} - \overline{d} + R^{-1}(0)(\beta_1 - \beta_2)}$

setting,

$$\left[\frac{\overline{u} + \beta_1 R^{-1}(\gamma) - (1+r)}{(\overline{u} + \beta_1 R^{-1}(\gamma)) - (\overline{d} + \beta_2 R^{-1}(\gamma))}\right] = \tilde{p}_d$$

yields

$$\gamma = R\left(\frac{\tilde{p}_d(\overline{u} - \overline{d}) - \overline{u} + (1+r)}{(\beta_1 + \tilde{p}_d(\beta_2 - \beta_1))}\right) \qquad (6.1.18)$$

Therefore, the membership function for the down movement probability $\tilde{p}_d$ is

$$\mu(\tilde{p}_d) = \begin{cases} 0 & \tilde{p}_d \leq \dfrac{\underline{u} - (1+r) - \alpha_1 L^{-1}(0)}{\underline{u} - \underline{d} + L^{-1}(0)(\alpha_2 - \alpha_1)} \\ L\left(\dfrac{\underline{u} - (1+r) + \tilde{p}_d(\underline{d} - \underline{u})}{\tilde{p}_d(\alpha_2 - \alpha_1) + \alpha_1}\right) & \dfrac{\underline{u} - (1+r) - \alpha_1 L^{-1}(0)}{\underline{u} - \underline{d} + L^{-1}(0)(\alpha_2 - \alpha_1)} \leq \tilde{p}_d \leq \dfrac{\underline{u} - (1+r) - \alpha_1 L^{-1}(1)}{\underline{u} - \underline{d} + L^{-1}(1)(\alpha_2 - \alpha_1)} \\ 1 & \dfrac{\underline{u} - (1+r) - \alpha_1 L^{-1}(1)}{\underline{u} - \underline{d} + L^{-1}(1)(\alpha_2 - \alpha_1)} \leq \tilde{p}_d \leq \dfrac{\overline{u} - (1+r) + \beta_1 R^{-1}(1)}{\overline{u} - \overline{d} + R^{-1}(1)(\beta_1 - \beta_2)} \\ R\left(\dfrac{\tilde{p}_d(\overline{u} - \overline{d}) - \overline{u} + (1+r)}{(\beta_1 + \tilde{p}_d(\beta_2 - \beta_1))}\right) & \dfrac{\overline{u} - (1+r) + \beta_1 R^{-1}(1)}{\overline{u} - \overline{d} + R^{-1}(1)(\beta_1 - \beta_2)} \leq \tilde{p}_d \leq \dfrac{\overline{u} - (1+r) + \beta_1 R^{-1}(0)}{\overline{u} - \overline{d} + R^{-1}(0)(\beta_1 - \beta_2)} \\ 0 & \tilde{p}_d \geq \dfrac{\overline{u} - (1+r) + \beta_1 R^{-1}(0)}{\overline{u} - \overline{d} + R^{-1}(0)(\beta_1 - \beta_2)} \end{cases} \tag{6.1.19}$$

The computation for the membership function for the fuzzy risk neutral probabilities of an upward movement in the stock price is as follows. Set

$$\begin{aligned} \left[\frac{(1+r-\overline{d}) - \beta_2 R^{-1}(\gamma)}{\overline{u} - \overline{d} + R^{-1}(\gamma)(\beta_1 - \beta_2)}\right] &= \tilde{p}_u \\ (1+r-\overline{d}) - \beta_2 R^{-1}(\gamma) &= \tilde{p}_u\left(\overline{u} - \overline{d} + R^{-1}(\gamma)(\beta_1 - \beta_2)\right) \\ (1+r-\overline{d}) - \tilde{p}_u\overline{u} + \tilde{p}_u\overline{d} &= \tilde{p}_u R^{-1}(\gamma)(\beta_1 - \beta_2) + \beta_2 R^{-1}(\gamma) \\ (1+r-\overline{d}) + \tilde{p}_u(\overline{d} - \overline{u}) &= R^{-1}(\gamma)(\tilde{p}_u(\beta_1 - \beta_2) + \beta_2) \\ R^{-1}(\gamma) &= \left[\frac{(1+r-\overline{d}) + \tilde{p}_u(\overline{d} - \overline{u})}{(\tilde{p}_u(\beta_1 - \beta_2) + \beta_2)}\right] \\ \gamma &= \left[R\left(\frac{(1+r-\overline{d}) + \tilde{p}_u(\overline{d} - \overline{u})}{(\tilde{p}_u(\beta_1 - \beta_2) + \beta_2)}\right)\right] \end{aligned}$$

similarly, setting

$$\left[\frac{(1+r-\underline{d}) + \alpha_2 L^{-1}(\gamma)}{\underline{u} - \underline{d} + L^{-1}(\gamma)(\alpha_2 - \alpha_1)}\right] = \tilde{p}_u$$

yields

$$\gamma = L\left(\frac{(1+r-\underline{d}) + \tilde{p}_u(\underline{d} - \underline{u})}{(\tilde{p}_u(\alpha_2 - \alpha_1) - \alpha_2)}\right)$$

Therefore, the membership function for the fuzzy risk neutral probability of an upward movement in the stock price is

$$\mu(\tilde{p}_u) = \begin{cases} 0 & \tilde{p}_u \leq \dfrac{(1+r-\overline{d})-\beta_2 R^{-1}(0)}{\overline{u}-\overline{d}+R^{-1}(0)(\beta_1-\beta_2)} \\ R\left(\dfrac{(1+r-\overline{d})+\tilde{p}_u(\,\overline{d}-\overline{u})}{(\tilde{p}_u(\beta_1-\beta_2)+\beta_2)}\right) & \dfrac{(1+r-\overline{d})-\beta_2 R^{-1}(0)}{\overline{u}-\overline{d}+R^{-1}(0)(\beta_1-\beta_2)} \leq \tilde{p}_u \leq \dfrac{(1+r-\overline{d})-\beta_2 R^{-1}(1)}{\overline{u}-\overline{d}+R^{-1}(1)(\beta_1-\beta_2)} \\ 1 & \dfrac{(1+r-\overline{d})-\beta_2 R^{-1}(1)}{\overline{u}-\overline{d}+R^{-1}(1)(\beta_1-\beta_2)} \leq \tilde{p}_u \leq \dfrac{(1+r-\underline{d})+\alpha_2 L^{-1}(1)}{\underline{u}-\underline{d}+L^{-1}(1)(\alpha_2-\alpha_1)} \\ L\left(\dfrac{(1+r-\underline{d})+\tilde{p}_u\,(\underline{d}-\underline{u})}{(\tilde{p}_u(\alpha_2-\alpha_1)-\alpha_2)}\right) & \dfrac{(1+r-\underline{d})+\alpha_2 L^{-1}(1)}{\underline{u}-\underline{d}+L^{-1}(1)(\alpha_2-\alpha_1)} \leq \tilde{p}_u \leq \dfrac{(1+r-\underline{d})+\alpha_2 L^{-1}(0)}{\underline{u}-\underline{d}+L^{-1}(0)(\alpha_2-\alpha_1)} \\ 0 & \tilde{p}_u \geq \dfrac{(1+r-\underline{d})+\alpha_2 L^{-1}(0)}{\underline{u}-\underline{d}+L^{-1}(0)(\alpha_2-\alpha_1)} \end{cases} \tag{6.1.20}$$

6.2 Characteristics of the LR-Fuzzy Risk Neutral Probability of a Down or up movement in the Stock Price.

We now discuss the behavior of the risk neutral fuzzy probabilities of the stock price. In order to determine the shape of the two probabilities, we compute their values at $\gamma = 0$ and $\gamma = 1$ and then we analyze their behavior as γ varies. We now make additional assumptions that the functions L^{-1} and R^{-1} are finite and twice differentiable with respect to γ. Furthermore, we make the following assumptions.

$$[\alpha_1\,(\underline{d}-(1+r))+\alpha_2\,((1+r)-\underline{u})]\,\frac{dL^{-1}(\gamma)}{d\gamma} \;>\; 0 \tag{6.2.1}$$

$$[\beta_1(1+r)-\overline{d})+\beta_2(\overline{u}-(1+r))]\,\frac{dR^{-1}(\gamma)}{d\gamma} \;<\; 0 \tag{6.2.2}$$

$$[\beta_2(1+r)-\overline{u})+\beta_1(\overline{d}-(1+r))]\,\frac{dR^{-1}(\gamma)}{d\gamma} \;>\; 0 \tag{6.2.3}$$

$$[\alpha_2(\underline{u}-(1+r))+\alpha_1((1+r)-\underline{d}]\,\frac{dL^{-1}(\gamma)}{d\gamma} \;<\; 0. \tag{6.2.4}$$

From (6.1.2) we have

$$p_{d1}(\gamma) = \left[\frac{\underline{u}-(1+r)-\alpha_1 L^{-1}(\gamma)}{\underline{u}-\underline{d}+L^{-1}(\gamma)(\alpha_2-\alpha_1)}\right] = \frac{N_5}{D_5}$$

where,

$$N_5 = \underline{u}-(1+r)-\alpha_1 L^{-1}(\gamma)$$

$$D_5 = \underline{u}-\underline{d}+L^{-1}(\gamma)(\alpha_2-\alpha_1)$$

such that

$$\frac{dN_5}{d\gamma} = -\alpha_1\frac{dL^{-1}(\gamma)}{d\gamma}$$
$$\frac{dD_5}{d\gamma} = (\alpha_2-\alpha_1)\frac{dL^{-1}(\gamma)}{d\gamma}$$

therefore,

$$\begin{aligned}
\frac{dp_{d1}(\gamma)}{d\gamma} &= \left[\frac{-D_5\alpha_1\dfrac{dL^{-1}(\gamma)}{d\gamma}-N_5(\alpha_2-\alpha_1)\dfrac{dL^{-1}(\gamma)}{d\gamma}}{D_5^2}\right] \\
&= \left[\frac{(\alpha_1(N_5-D_5)-N_5\alpha_2)\,\dfrac{dL^{-1}(\gamma)}{d\gamma}}{D_5^2}\right] \\
&= \left[\frac{(\alpha_1(-1-r+\underline{d}-L^{-1}(\gamma)\alpha_2)-(\underline{u}-(1+r)-\alpha_1 L^{-1}(\gamma))\,\alpha_2)\,\dfrac{dL^{-1}(\gamma)}{d\gamma}}{D_5^2}\right] \\
&= \left[\frac{(\alpha_1\,(\underline{d}-1-r)+\alpha_2\,(r-\underline{u}+1))\dfrac{dL^{-1}(\gamma)}{d\gamma}}{(\underline{u}-\underline{d}+L^{-1}(\gamma)(\alpha_2-\alpha_1))^2}\right] \qquad (6.2.5)
\end{aligned}$$

which is positive under assumption (6.2.1).

Similarly, under assumptions (6.2.3), (6.2.2) and (6.2.4) respectively, we have

$$\frac{dp_{d2}(\gamma)}{d\gamma} = \left[\frac{(\beta_1(1+r)-\overline{d})+\beta_2(\overline{u}-(1+r)))\,\dfrac{dR^{-1}(\gamma)}{d\gamma}}{(\overline{u}-\overline{d}+R^{-1}(\gamma)(\beta_1-\beta_2))^2}\right] < 0 \qquad (6.2.6)$$

$$\frac{dp_{u1}(\gamma)}{d\gamma} = \left[\frac{(\beta_2(1+r)-\overline{u})+\beta_1(\overline{d}-(1+r)))\dfrac{dR^{-1}(\gamma)}{d\gamma}}{\left(\overline{u}-\overline{d}+R^{-1}(\gamma)(\beta_1-\beta_2)\right)^2}\right] > 0 \qquad (6.2.7)$$

and

$$\frac{dp_{u2}(\gamma)}{d\gamma} = \left[\frac{(\alpha_2(\underline{u}-(1+r))+\alpha_1((1+r)-\underline{d}))\dfrac{dL^{-1}(\gamma)}{d\gamma}}{\underline{u}-\underline{d}+L^{-1}(\gamma)(\alpha_2-\alpha_1)^2}\right] < 0 \qquad (6.2.8)$$

To find the convexity or concavity character of the fuzzy risk neutral probabilities $\tilde{p}_d$ and $\tilde{p}_u$ we need to find the second derivatives of each of p_{d1}, p_{d2}, p_{u1} and p_{u2}. If the second derivative is positive (negative, respectively), it implies that the corresponding probability is convex (concave, respectively) in nature.

6.2.1 Call Option Value With LR-Fuzzy Numbers.

Suppose that there is a call option on a stock with exercise price K and expiration at the end of Period 1. In view of the above notation we take $u(\gamma)$ as the γ-level set of a fuzzy increase $\tilde{u}$ in the stock price, $d(\gamma)$ as the γ-cut of a fuzzy decrease $\tilde{d}$ in the stock price.

We assume that arbitrage opportunity is not allowed. Then, the γ-level set for $\tilde{u}$ and $\tilde{d}$ are respectively given by

$$u(\gamma) = [u_1(\gamma), u_2(\gamma)] = [\underline{u}-\alpha_1 L^{-1}(\gamma), \overline{u}+\beta_1 R^{-1}(\gamma)] \qquad (6.2.9)$$

$$d(\gamma) = [d_1(\gamma), d_2(\gamma)] = [\underline{d}-\alpha_2 L^{-1}(\gamma), \overline{d}+\beta_2 R^{-1}(\gamma)] \qquad (6.2.10)$$

Assume that $\tilde{C}$ is a fuzzy current price of a call option on a stock whose crisp exercise price is K. Also, assume that when the call option expires, it is worth either $\tilde{C}_u$ or $\tilde{C}_d$, where $\tilde{C}_u$ and $\tilde{C}_d$ are fuzzy quantities whose values are subjected to arbitrariness, subjectivity and fuzziness. In this case the γ-level set for

$$\begin{aligned} C_u(\gamma) &= [C_{u1}(\gamma), C_{u2}(\gamma)] \\ &= [Max(Su_1(\gamma) - K, 0), \quad Max(Su_2(\gamma) - K, 0)] \end{aligned} \tag{6.2.11}$$

(6.2.9) and (6.2.11) yield the following two equations.

$$C_{u1}(\gamma) = Max(Su_1(\gamma) - K, 0) \tag{6.2.12}$$

$$C_{u2}(\gamma) = Max(Su_2(\gamma) - K, 0) \tag{6.2.13}$$

Similarly, the γ-level set for $\tilde{C}_d$ is given by

$$C_d(\gamma) = [C_{d1}(\gamma), C_{d2}(\gamma)] \tag{6.2.14}$$

$$= [Max(Sd_1(\gamma) - K, 0), \quad Max(Sd_2(\gamma) - K, 0)] \tag{6.2.15}$$

From (6.2.10) and (6.2.15), we get

$$C_{d1}(\gamma) = Max(Sd_1(\gamma) - K, 0) \tag{6.2.16}$$

and

$$C_{d2}(\gamma) = Max(Sd_2(\gamma) - K, 0) \tag{6.2.17}$$

If $Sd_1(\gamma)$ and $Sd_2(\gamma)$ are less than K, that is, the fuzzy stock price goes down, then the call option will expire out- of-the-money. If $Su_1(\gamma)$ and $Su_2(\gamma)$ are greater than K, that is the fuzzy stock goes up, then the call option expires in the-money. Substituting expressions (6.2.9) for $u(\gamma)$ into expression (6.2.11), the expression for the price of the call in the up state, under the assumption that $\tilde{u}$ is a LR-fuzzy number yields

$$\begin{aligned} C_u(\gamma) &= [Max(Su_1(\gamma) - K, 0), \ Max(Su_2(\gamma) - K, 0)] \\ &= [Max(S(\underline{u} - \alpha_1 L^{-1}(\gamma)) - K, 0), \ Max(S(\overline{u} + \beta_1 R^{-1}(\gamma)) - K, 0)] \\ &= [S(\underline{u} - \alpha_1 L^{-1}(\gamma)) - K, \ \ S(\ \overline{u} + \beta_1 R^{-1}(\gamma)) - K] \end{aligned}$$

such that

$$C_{u1}(\gamma) = \left[S(\underline{u} - \alpha_1 L^{-1}(\gamma)) - K\right] \tag{6.2.18}$$

yields

$$\frac{dC_{u1}(\gamma)}{d\gamma} = -S\alpha_1 \frac{L^{-1}(\gamma)}{d\gamma} \tag{6.2.19}$$

Note that expression (6.2.19) is needed to analyze the behavior of the call price. From expression (6.2.19), it can be observed that $\frac{dC_{u1}(\gamma)}{d\gamma} > 0$, if $\frac{L^{-1}(\gamma)}{d\gamma} < 0$.

Again,

$$C_{u2}(\gamma) = \left[S(\ \overline{u} + \beta_1 R^{-1}(\gamma)) - K\right] \tag{6.2.20}$$

gives

$$\frac{dC_{u2}(\gamma)}{d\gamma} = S\beta_1 \frac{R^{-1}(\gamma)}{d\gamma} \tag{6.2.21}$$

Expression (6.2.21) is also needed to analyze the behavior of the call price. From expression (6.2.21), it can be observed that $\frac{dC_{u2}(\gamma)}{d\gamma} < 0$, if $\frac{R^{-1}(\gamma)}{d\gamma} < 0$.

Assuming that $\tilde{d}$ is a LR-fuzzy number, (6.2.10) and (6.2.15), yield the following expressions for the price of the derivative in the down state.

$$\begin{aligned} C_d(\gamma) &= [Max(Sd_1(\gamma) - K, 0), \quad Max(Sd_2(\gamma) - K, 0)] \\ &= [Max(S(\underline{d} - \alpha_2 L^{-1}(\gamma)) - K, 0), \quad Max((\overline{d} + \beta_2 R^{-1}(\gamma)) - K, 0)] \end{aligned}$$

It may be pointed out here that when $Sd_1(\gamma)$ and $Sd_2(\gamma)$ are less than K, that is when the fuzzy stock price goes down, then the call option expires out-of-the money.

In this case the maximum value of $C_{d1}(\gamma) = 0$ and the maximum value of $C_{d2}(\gamma) = 0$. This yields

$$C_{d1}(\gamma) = 0 \qquad C_{d2}(\gamma) = 0 \tag{6.2.22}$$

6.2.2 Expected Fuzzy Call Option Value With LR-Fuzzy Numbers.

Let $\tilde{C}$ be a fuzzy number that characterizes the fuzzy current price of the call option and $\tilde{C}_1$ be the fuzzy payoff of the call option at time $t = 1$. Let $C(\gamma)$ be the α-cut for the fuzzy current price of the option. Then, the expected fuzzy call price is given by

$$\tilde{C} = \frac{1}{1+r}\tilde{E}(\tilde{C})$$

that is,

$$[C_1(\gamma), C_2(\gamma)] = \frac{1}{1+r}[C_{u1}(\gamma), C_{u2}(\gamma)][p_{u1}(\gamma), p_{u2}(\gamma)] + \frac{1}{1+r}[C_{d1}(\gamma), C_{d2}(\gamma)][p_{d1}(\gamma), p_{d2}(\gamma)] \tag{6.2.23}$$

where $\tilde{E}$ stands for the expectation under the risk-neutral probabilities and $\tilde{C}_1$ is the fuzzy payoff of the call option at time $t = 1$.

Since, the fuzzy call option has zero payoff in the down state, the fuzzy option pricing formula given by equation (6.2.23), simplifies to

$$[C_1(\gamma), C_2(\gamma)] = \frac{1}{1+r}[C_{u1}(\gamma), C_{u2}(\gamma)][p_{u1}(\gamma), p_{u2}(\gamma)] \tag{6.2.24}$$

$C_{u1}(\gamma)$ and $C_{u2}(\gamma)$ are as in (6.2.18) and (6.2.20) respectively. Also, $p_{u1}(\gamma)$ and $p_{u2}(\gamma)$ that describe the fuzzy probabilities of an up movement in the stock price are given by (6.1.5) and ((6.1.6) respectively.

Thus,

$$\tilde{C} = \frac{1}{1+r}\tilde{E}(\tilde{C}) = \frac{1}{1+r}[C_{u1}(\gamma)p_{u1}(\gamma), C_{u2}(\gamma)p_{u2}(\gamma)]$$

yields

$$\tilde{C} = \begin{bmatrix} \dfrac{S(\underline{u} - \alpha_1 L^{-1}(\gamma)) - K}{1+r} & \dfrac{(1+r-\overline{d}) - \beta_2 R^{-1}(\gamma)}{\overline{u} - \overline{d} + R^{-1}(\gamma)(\beta_1 - \beta_2)}, \\ \\ \dfrac{S(\overline{u} + \beta_1 R^{-1}(\gamma)) - K}{1+r} & \dfrac{(1+r-\underline{d}) + \alpha_2 L^{-1}(\gamma)}{\underline{u} - \underline{d} + L^{-1}(\gamma)(\alpha_2 - \alpha_1)} \end{bmatrix}. \qquad (6.2.25)$$

This leads to

$$C_1(\gamma) = \frac{C_{u1}(\gamma)}{1+r}\, p_{u1}(\gamma) \qquad C_2(\gamma) = \frac{C_{u2}(\gamma)}{1+r} p_{u2}(\gamma)$$

Plugging the values of $C_{u1}(\gamma)$ from (6.2.18) and $p_{u1}(\gamma)$ from (6.1.5) we obtain the following expression for the left hand part of the γ-level set for the fuzzy current price of the call option.

$$C_1(\gamma) = \left[\frac{S(\underline{u} - \alpha_1 L^{-1}(\gamma)) - K}{1+r} \; \frac{(1+r-\overline{d}) - \beta_2 R^{-1}(\gamma)}{\overline{u} - \overline{d} + R^{-1}(\gamma)(\beta_1 - \beta_2)}\right] \qquad (6.2.26)$$

Similarly, from (6.2.20) and $p_{u2}(\gamma)$ from (6.1.6) we obtain the right hand part of the γ-level set

$$C_2(\gamma) = \left[\frac{S(\overline{u} + \beta_1 R^{-1}(\gamma)) - K}{1+r} \; \frac{(1+r-\underline{d}) + \alpha_2 L^{-1}(\gamma)}{\underline{u} - \underline{d} + L^{-1}(\gamma)(\alpha_2 - \alpha_1)}\right] \qquad (6.2.27)$$

Now, $[C_1(\gamma), C_2(\gamma)]$ gives us a weighted expected value interval for the call price. If we assume that $C_1(\gamma)$ and $C_2(\gamma)$ are decreasing functions of γ then (6.2.26) and (6.2.27) yield that as γ increases the call option interval $[C_1(\gamma), C_2(\gamma)]$ of price shrinks, and at $\gamma = 1$, the interval is the smallest. Similarly, at $\gamma = 0$, the call price interval is the largest. This is an important property for financial applications as it allows us to determine the most useful outcomes of the call price.

In order to determine the membership function we have to find the two exterior points and the two interior points which describe the fuzzy call price corresponding to the LR- fuzzy number, we set $\gamma = 0$ and $\gamma = 1$ in (6.2.26) and (6.2.27) respectively. Thus,

$$C_1(0) = \left[\frac{S(\underline{u} - \alpha_1 L^{-1}(0)) - K}{1+r} \quad \frac{(1+r-\overline{d}) - \beta_2 R^{-1}(0)}{\overline{u} - \overline{d} + R^{-1}(0)(\beta_1 - \beta_2)}\right] \tag{6.2.28}$$

$$C_2(0) = \left[\frac{S(\ \overline{u} + \beta_1 R^{-1}(0)) - K}{1+r} \quad \frac{(1+r-\underline{d}) + \alpha_2 L^{-1}(0)}{\underline{u} - \underline{d} + L^{-1}(0)(\alpha_2 - \alpha_1)}\right] \tag{6.2.29}$$

$$C_1(1) = \left[\frac{S(\underline{u} - \alpha_1 L^{-1}(1)) - K}{1+r} \quad \frac{(1+r-\overline{d}) - \beta_2 R^{-1}(1)}{\overline{u} - \overline{d} + R^{-1}(1)(\beta_1 - \beta_2)}\right] \tag{6.2.30}$$

$$C_2(1) = \left[\frac{S(\ \overline{u} + \beta_1 R^{-1}(1)) - K}{1+r} \quad \frac{(1+r-\underline{d}) + \alpha_2 L^{-1}(1)}{\underline{u} - \underline{d} + L^{-1}(1)(\alpha_2 - \alpha_1)}\right] \tag{6.2.31}$$

From (6.2.28)-(6.2.31) we have that the fuzzy call price is given by

$$\tilde{C} = [C_1(0), C_1(1), C_2(1), C_2(0)]$$

$$\tilde{C} = \begin{bmatrix} \dfrac{S(\underline{u} - \alpha_1 L^{-1}(0)) - K}{1+r} \quad \dfrac{(1+r-\overline{d}) - \beta_2 R^{-1}(0)}{\overline{u} - \overline{d} + R^{-1}(0)(\beta_1 - \beta_2)}, \\[2em] \dfrac{S(\underline{u} - \alpha_1 L^{-1}(1)) - K}{1+r} \quad \dfrac{(1+r-\overline{d}) - \beta_2 R^{-1}(1)}{\overline{u} - \overline{d} + R^{-1}(1)(\beta_1 - \beta_2)}, \\[2em] \dfrac{S(\ \overline{u} + \beta_1 R^{-1}(1)) - K}{1+r} \quad \dfrac{(1+r-\underline{d}) + \alpha_2 L^{-1}(1)}{\underline{u} - \underline{d} + L^{-1}(1)(\alpha_2 - \alpha_1)}, \\[2em] \dfrac{S(\ \overline{u} + \beta_1 R^{-1}(0)) - K}{1+r} \quad \dfrac{(1+r-\underline{d}) + \alpha_2 L^{-1}(0)}{\underline{u} - \underline{d} + L^{-1}(0)(\alpha_2 - \alpha_1)} \end{bmatrix} \tag{6.2.32}$$

For the membership functions for $\tilde{C}$ we set $C_1(\gamma) = \tilde{C}$, and $C_2(\gamma) = \tilde{C}$. Therefore,

$$\left[\frac{S(\underline{u} - \alpha_1 L^{-1}(\gamma)) - K}{1+r} \quad \frac{(1+r-\overline{d}) - \beta_2 R^{-1}(\gamma)}{\overline{u} - \overline{d} + R^{-1}(\gamma)(\beta_1 - \beta_2)}\right] = \tilde{C} \tag{6.2.33}$$

and

$$\left[\frac{S(\ \overline{u} + \beta_1 R^{-1}(\gamma)) - K}{1+r} \quad \frac{(1+r-\underline{d}) + \alpha_2 L^{-1}(\gamma)}{\underline{u} - \underline{d} + L^{-1}(\gamma)(\alpha_2 - \alpha_1)}\right] = \tilde{C}. \tag{6.2.34}$$

Solving (6.2.33) and (6.2.34) for γ yields the membership function for $\tilde{C}$.

6.2.3 Characteristics of the Fuzzy Call Price with LR-Fuzzy Numbers

Making appropriate assumptions on $C_1(\gamma)$, $C_2(\gamma)$, $\dfrac{dC_1(\gamma)}{d\gamma}$, $\dfrac{dC_2(\gamma)}{d\gamma}$, $\dfrac{d^2C_1(\gamma)}{d\gamma^2}$ and $\dfrac{d^2C_2(\gamma)}{d\gamma^2}$ we can, as in Chapter 5 and as in Section 6.2 discuss the behavior of the fuzzy call price with LR-Fuzzy Numbers.

6.3 Conclusion

In this chapter we demonstrated how call options can be valued under fuzzy environment using LR-fuzzy numbers. The approach can be easily extended to price a wide variety of options with different types of pay-off patterns. The fuzzy binomial option pricing approach considered in this chapter is quite general and the methodology developed by Muzzioli [98] and certain results of Chapter 5 and 4 of this thesis may be viewed as a special case of the results developed in this chapter. This methodology provides an intuitive and easy way to look at the vagueness in the stock price movement and our result include the results of the standard binomial option pricing model as special case.

Chapter 7

Conclusion, Contribution and Recommendations.

In the present chapter, we state the contributions and conclusions made in this thesis. Finally, we give some recommendations for further research.

7.1 Contribution and Conclusion.

In the present thesis, we introduce O(m, n).Tr.T.F.N's and discuss their various algebraic properties in Chapter 3. Some numerical examples are provided to highlight those properties. We, also point out some of their advantages. Furthermore, we discuss moment properties of O(m, n).Tr.T.F.N's in Chapter 4 and derive expression for possibilistic mean, possibilistic variance, possibilistic covariance, weighted possibilistic mean, weighted possibilistic variance and weighted possibilistic covariance. Some examples are provided to reinforce the results. In Chapter 5 we consider an important problem in the field of fuzzy binomial option pricing model and model it using O(m, n).Tr.T.F.N's in order to capture the impreciseness present in the model. The main contribution of this chapter are the derivations of the fuzzy risk neutral probabilities and the weighted interval associated with the call price. The behavior of those probabilities and the call price is also discussed and is supported by numerical examples. In Chapter 6, the problem considered in Chapter 5 is extended using LR-fuzzy numbers. Some results are derived for the fuzzy risk neutral probabilities and the fuzzy call option values using LR-fuzzy numbers. Based on the results presented in this thesis, we conclude that the fuzzy sets theory approach offers the added advantage of flexibility when dealing with uncertainty involved in the binomial option pricing model. Certain fuzzy techniques have been applied to some existing deterministic models and that have resulted in more flexible solutions than normally obtained with their counterparts under crisp environment.

7.2 Recommendations for Future Research.

The results developed in this thesis for O(m, n).Tr.T.F.N's can be utilized in a portfolio selection problem, where one can use a possibilistic variance-covariance matrix. Also, one can generate the possibilistic variance-covariance matrix using a weighted function. Using the theory of weighted functions discussed in Chapter 4, an alternative fuzzy binomial option pricing model can be developed. In the fuzzy binomial option pricing model developed in Chapter 5, instead of taking probability expectation for the call price, we could as well opt for possibilistic or weighted possibilistic expectation. We believe that the work presented in this thesis will initiate further research in a number of areas (for example, portfolio selection). Using the concept of lower possibilistic mean and upper possibilistic mean we can construct two possibilistic variance-covariance matrices. This will generate two possibilistic quadratic programming models. Last but not least, one can introduce other kind of fuzzy numbers and derive results similar to the one discussed in Chapter 3-6.

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