Math Is Alive: The Metaphor of Living School Disciplines and Some of the Educational Implications

by

Stacy K. Stockton

A Thesis submitted to the Faculty of Graduate Studies of the University of Manitoba

in partial fulfillment of the requirements of the degree of

MASTER OF EDUCATION

Department of Curriculum, Teaching and Learning

Faculty of Education

University of Manitoba

Winnipeg

Abstract

This thesis is concerned with an unconventional understanding of the mathematics taught in schools and, thus, the role that students' engagement with mathematics plays for mathematics as a discipline. The discipline of mathematics has a rich philosophical history. In this thesis I explore mathematics through the lens of metaphor theory first developed by George Lakoff and Mark Johnson. This area of scholarship is used to develop the idea that all abstract concepts are conceived of metaphorically. From this perspective, the metaphors at the foundation of several major philosophies of mathematics are analyzed. This analysis concludes by asking if there might be another metaphor that allows for a different, more productive, understanding of school mathematics than commonly used mathematical metaphors. The metaphor "Math is Alive" is offered as an alternative through a detailed analysis of living systems in the tradition of Humberto Maturana and Francisco Varela's theory of autopoietic organization. The final chapter looks specifically at school mathematics as a part of mathematics as a discipline, and how "Math is Alive" might alter how educators view the role of children in mathematics.

Acknowledgements

This thesis would not have been possible without the guidance of my academic advisor, Thomas Falkenberg, and the support of my committee members, Barbara McMillan and Paul Betts.

Table of Contents Page

"It is inorganic, yet alive, and all the more alive for being inorganic"

(Deleuze & Guattari, 1987, p. 520).

I first discovered the science of complexity in a course on arts education. My exploration of its application to the arts in schools inspired my particular interest in autopoietic systems, and led me to a study of biological living systems. Maturana and Varela's *Tree of Knowledge* (1987), Capra's *Web of Life* (1996), and Deleuze and Guattari's *Thousand Plateaus* (1987) altered my understanding of being alive, and invited the possibility of living ideas.

As a teacher, I am constantly confronted with uncertainties and differences in paradigm when exploring conceptions of disciplines through discussions with others and through professional development and readings. Paradigmatic changes in disciplines over time and competing current conceptions became more apparent to me. I also began to notice curious resemblances between living systems and disciplines. This included similarities in their patterns of organization, similarities in the way they behaved, and similarities in their processes of evolution.

As I continued to read further, seeking more recent explorations of living concepts from a complexity perspective, I was unable to answer my questions about disciplines in particular, and what it would mean for education if they were conceived of as being alive. Just as Deleuze relies on his metaphor of rhizome to allow us to see our thinking about concepts differently, I have begun this thesis to ask if the metaphor of a living discipline might allow us to think differently about disciplines. Further, if we conceive of disciplines in this way, I wonder what this implies for the way that we currently engage with them in classrooms, and what this could become.

This inquiry will be philosophical, relying on philosophical inquiry as described by Burbules and Warnick (2006), Floden and Buchmann (1990), and to some extent, John Wilson (1963). It is not about finding answers, but about asking fundamental questions about what it means to be alive, and what we perceive our role to be in the production of disciplined knowledge. It is about the potential impact of these beliefs on disciplines themselves, which it would be impossible to empirically measure. This is the realm of philosophical inquiry. The methods I have selected represent several of Burbules and Warnick's (2006) ten methods of philosophical inquiry.

These will include the use of concept analysis (method 1, Burbules & Warnick, 2006, p. 491) through an extensive literature review on metaphor, the philosophy of mathematics, complex living systems, and school mathematics methods and metaphors. My thesis deals with the meanings that we assign to the terms we use in the contexts in which we use them, and how our conceptions guide our actions.

> Philosophical inquiry often relies on analyses of language or concepts. Inquiry will focus on terms that seem to play an important role in an argument or practice, trying to answer the question, what do these words or phrases mean or entail? (Floden & Buchmann, 1990, p. 3)

Chapter one focuses on metaphor and its importance of our understanding of abstract concepts. Chapters two and three explore the target and source domains in a living mathematics metaphor, current conceptions of mathematics and the biology of living systems.

Using information gathered regarding mathematics and living systems, a metaphor is then constructed in chapter four. This making of a metaphor will be shown to be a conceptual mapping, "an ambitious review of conceptual mapping may include a review of related concepts that share certain features with the primary object" (Burbules & Warnick, 2006, p. 492). The creation of a metaphor can identify correspondences in structure, maintenance, and reproduction in complex systems and mathematics, following method 10, “synthesizing disparate research from philosophy itself or other fields to find meaning and implications for educational theory and practice” (Burbules & Warnick, 2006, p. 491). Here I ask: what does living have to do with mathematics? Through this, I will be required to show that the vitality of a discipline is one of the “ends or purposes education should achieve,” reflecting method 6 (Burbules & Warnick, 2006, p. 491).

Finally, in chapter five I turn to method 7, “speculating about alternative systems or practices of education, whether utopian or programmatic, that contrast with and challenge conventional educational understandings and practices” (Burbules & Warnick, 2006, p. 491), in order to conceive of how this alternate view might impact the way mathematics is taught in schools. I will consider the implications of a living mathematics in educational practice, and how this might resolve difficulties with current perspectives, and create healthier relationships between children and mathematics.

CHAPTER ONE

Metaphor

"Language is a map, not a tracing"

(Deleuze & Guattari, 1987, p. 98).

Metaphor is a term most often used to describe a literary device. It is a type of figurative language used to compare objects with like properties. At least this is what I tell my grade six students. In 1980, Lakoff and Johnson challenged this assumption, by noticing the pervasiveness of everyday metaphors and their potential to shape our thinking (Kovecses, 2002, viii). These authors, as well as Gibbs (2008) and others continue to challenge our assumptions about metaphor comprehension, as knowledge about the inner workings of our brains increases. Interestingly, George Lakoff will reappear again later in this thesis when discussing philosophies of mathematics.

In cognitive linguistics, metaphors have come to be regarded as any conceptual domain defined in terms of another (Kovecses, 2002, p. 4). It is surprising to realize how often we use metaphor in our everyday discourse, and how difficult it is to describe many things without it. Some metaphors become embedded in a concept, and become a part of that concept to the point where we are barely able to understand them as metaphors at all, and the concepts themselves mean little without them. "Just living an everyday life gives you the experience and suitable brain activations to give rise to a huge system of the same primary metaphorical mappings that are learned around the world without any awareness" (Lakoff as referenced in Gibbs, 2008, p. 26).

In this chapter I will discuss some of Lakoff and Johnson's ongoing exploration of metaphor, categories of metaphors and their attributes, how metaphors develop, the potential for and of novel metaphors, and how this discourse is important to my thesis.

The Integrated Theory of Metaphor

Lakoff and Johnson (1999, p. 46) refer to their "integrated theory of metaphor" to explain how metaphor becomes a part of our fundamental understanding of things, and how our ability to use metaphor fluently develops. First, the integrated theory of metaphor includes Johnson's theory of conflation, wherein metaphors develop out of early experiences before infants are able to distinguish certain concepts from others. One example is love and warmth. Although love itself as an abstract concept has no temperature, the relationship between these two concepts is developed at an early age in a context where love and warmth are consistently correlated. Later, children are able to differentiate the two concepts, but "the cross-domain associations persist" (Lakoff & Johnson, 1999, p. 46).

Second, the integrated theory is also informed by Grady's theory of primary metaphor. This theory asserts that all metaphors come from primary metaphors. A primary metaphor "has a minimal structure and arises, naturally, automatically and unconsciously through everyday experience by means of conflation" (Lakoff & Johnson, 1999, p. 46).

Third, Narayanan's neural theory of metaphor contributes the argument that the neural connections developed during the period of conflation lead to the conceptual metaphorical structures we create as "simultaneous activations result in permanent neural connections being made across the neural networks that define conceptual domains" (Lakoff & Johnson, 1999, p. 46).

The fourth and final theory that informs the integrated theory of metaphor is Fauconnier and Turner's theory of conceptual blending. Here, connections continue to be made after the period of conflation when distinct conceptual domains are co-activated, allowing new inferences and connections to form. However, metaphor can exist without blending and blending without metaphor (Lakoff as referenced in Gibbs, 2008, p. 30). Yet, often when co-activations occur frequently, concepts become linked and one only has to think of one to think of the other. "Via the Hebbian Principle that neurons that fire together wire together, neural mapping circuits linking the two domains will be learned" (Lakoff as referenced in in Gibbs, 2008, p. 26).

Of importance to this thesis is the idea that a base, fundamental metaphor often gives rise to a complex array of related metaphors from the same source. It is also interesting to note that the age at which a metaphor is experienced and its persistence in ongoing experiences impacts how deeply embedded it becomes in the meaning that we apply to a concept.

Some Attributes of Metaphor

In a metaphor, such as love is warmth, how we choose which features of warmth apply to love is not entirely clear (Glucksberg, 2003, p. 92). The long held pragmatic model has us reject literal meanings before searching for possible similarities, while newer models deny this process of comparison, such as Lakoff and Johnson's (1999) reference to preexisting mappings of concepts, or Sam Glucksberg's categorical assertions.

In Lakoff and Johnson's (1999) view, all metaphors have source and target domains. In the above case, warmth is the source domain, and one's experience of it. The concept of love is the target domain. We are to understand love through our understanding of warmth. The conceptual, systematic relationship between source and target domains is referred to as a mapping (Kovecses, 2002, p. 6). This particular primary metaphor leads to metaphors of lovers as hot (or not), and relationship outcomes as temperatures, like when you are given the cold shoulder. Metaphor theorists such as Glucksberg prefer the terms "topic" and "vehicle" to describe the target/source relationship. Although this reflects some difference in views of how metaphors are realized and understood (Gibbs, 2003, p. 6; Glucksberg & McGlone, 1999, p. 1546), the significance of metaphor and its embedded nature remain the same. Here, mathematics is the target and living systems are the source domain. This thesis seeks to uncover what the potential is for this mapping.

It should also be noted that metaphorical mappings are asymmetrical, meaning that the source and target cannot be interchanged (Gibbs, 1996, p. 311). There is no consistency or meaningfulness between the idea that a car is a lemon and the idea that a lemon is a car. Glucksberg uses the example of surgeons and butchers, wherein both mappings can be understood metaphorically – the surgeon is a butcher and the butcher is a surgeon - but then have opposite meanings (Glucksberg, 2003, p. 95). I am not simply looking for similarities here, or to imply that living is like math, but instead to understand what it means to believe that math is alive (metaphorically speaking).

Categories of Metaphor

Metaphors can be distinguished in a variety of ways. One important first distinction is between the primary metaphors mentioned above and complex metaphors. Primary metaphors, as in Grady's (2008) theory, provide the basis for more complex metaphors in that they look at a broader conceptual domain in terms of another. For example, the metaphor 'love is warmth' is a primary metaphor that can be expressed in a variety of ways, including possibilities that are more specific or more complex. The general concept of warmth provides metaphorical possibilities, or entailments, for love.

Metaphors can also be distinguished according to their type. The type that we are most familiar with is structural metaphor. Structural metaphors define one item or concept in terms of another item. When I say, "Love is warmth." I am using a structural metaphor; one concept is structured in terms of another (Lakoff & Johnson, 1980, p.14). Another common type is the orientational metaphor. Orientational metaphors use direction and location as source domains, and align with basic human spatial concepts. For example, happiness is up and sadness is down, as in the case of "I'm feeling up today", or perhaps, "I am feeling down." Music uses the orientational metaphor of high and low to describe sound. We also talk about things in front of us, behind us, and backwards that could not possibly have this real capacity. When someone points out that a bad experience is behind him or her, do not go looking for it. Mathematics relies heavily on orientational metaphors to express concepts that really have nothing to do with spatial ideas, with numbers being higher or lower.

Ontological metaphors are ways of viewing experience as having the abilities of generalized entities and substances (Kovecses, 2002, p. 34). Examples of this include personification, where human abilities are the source for a non-human target. Other instances give the abilities of different general entities to concepts, such as Lakoff and Johnson's (1980, p. 28) example of a shattering experience. This includes metonymy, and container metaphors, such as when an idea gets boxed up.

Metaphors can also be categorized based on their conventionality (Kovecses, 2002, p. 29), or how commonly they occur. Kovecses (2002) lists examples such as "Argument is war; love is a journey; ideas are food; life is a journey" (p. 30). These metaphors are so common to our understanding that we use them in many ways without recognizing them as metaphors. In fact, most of the metaphors that we use fit into conventional categories, and unconventional metaphors can be quite shocking and disorientating, being often reserved for the figurative language we are accustomed to in poetry, such as when T.S. Eliot insists he "will show you fear in a handful of dust" (1922, pp. 473-485). While novel and unique metaphors can draw attention, conventional metaphors lie beyond notice with even greater powers to shape our thinking. Lakoff and Johnson (1999) warn against regarding conventional metaphors as 'dead', and reserve this language for instances where "the conceptual mapping has long since ceased to exist" (p. 124). They use the example of the word pedigree, which originally served as a metaphor for a grouse's foot, but no longer calls that to mind at all.

Metaphors for mathematics tend to be structural, and the metaphor "mathematics is a living system" ("mathematics is alive") is a structural metaphor. The metaphor "math is alive" is ontological, because it means that mathematics has the abilities of a living system; mathematics behaves as a living system does. The structural metaphor wherein mathematics is given the characteristics of a living system will be important as I explore what mathematics is, but the ontological metaphor will be of particular significance given a process oriented systems view of life. A living systems metaphor is both conventional and unconventional here, reflecting some metaphorical practices already in use, and challenging them with the distinct terminology of living systems theory.

Where Metaphors Come From

Embedded, everyday metaphors do more to shape our understanding of concepts than we realize, and their effect on how we experience our reality is deep and pervasive. “Like principles of phonology and grammar, conventional metaphors are relatively fixed, unconscious, automatic and so alive that they are used regularly without awareness or noticeable effect” (Lakoff & Johnson, 1999, p. 125). When it comes to abstract concepts, like love, evil or, say, mathematics, metaphors become even more important. Johnson (as referenced in Gibbs, 2008, p. 51) insists that “all our abstract concepts are metaphorically defined”. They allow us to reason about a “literal, nonmetaphorical skeleton” (Lakoff & Johnson, 1999, p. 125) with greater insight and flexibility, but they also shape and limit what potential reasoning can occur. “Most concepts are partially understood in terms of other concepts” (Lakoff & Johnson, 1980, p. 56). As in Fauconnier and Turner’s theory of conceptual blending (Lakoff as referenced in Gibbs, 2008, p. 30), the metaphors we use for abstract concepts affect our thinking and reasoning about that concept. In this context, the metaphors that we use to describe mathematics are deeply embedded in our understanding of what mathematics is.

This causes one to question why certain concepts may be favored over others as appropriate metaphors. Metaphors are grounded in correlations in experience (Lakoff & Johnson, 1999, p.

46). These correlations in experience are not to be seen as similarities (Kovecses, 2002, pp. 68-69). For example, the correlation between more and up is based on viewing rising levels in a container, but is not a similarity between more and up. The same is true for “love is warmth”. There is no actual similarity between love – a completely abstract feeling, and warmth – the presence of higher temperature, there are correlations in experience that create metaphorical associations. “Metaphors are rarely understood as comparisons. Instead they are usually understood exactly as they appear, as class-inclusive assertions” (Glucksberg as referenced in Gibbs, 2008, p. 68). This leads to the following insight that is central to this thesis: *Our experiences with mathematics will lead to the metaphors we favour, and the metaphors we favour will influence and define our experience of mathematics*. This insight provides the rationale for why the focus of this thesis is quite relevant to education more generally and mathematics education more specifically.

Some metaphors may develop very early when distinctions between these concepts have not yet developed, but they may also develop later as life experience accumulates. Lakoff and Johnson (1999) believe that many of our most basic primary metaphors and their associated mappings develop out of our physical experience of the world (pp. 54-57), and that these mappings form the basis for other more complex metaphors (Lakoff & Johnson, 1999, p. 49). Metaphors can be grounded in our own state of being in the world, as in “knowing is seeing” (Lakoff & Johnson, 1999, p. 48). They can also reflect perceived structural similarities, such as Kovecses (2001) example “ideas are food” (p. 72), wherein certain structural attributes of ideas are seen as comparable to food. Consider chewing on something that cannot be literally chewed. Another basis for metaphor is when the source is the root of the target. An excellent example is the metaphor of life being a play, when in fact a play is a representation of life (Kovecses, 2002, p. 74/75).

Because mathematics is an abstract concept, ideas about what it is have often been framed by metaphors, although it is not often that attention is drawn to this. In this thesis, I will explore the idea that current practice in school mathematics represents particular embedded metaphors that are long overdue for reconsideration. The metaphors we use matter, and we can choose to use new ones. Yet it is not as simple as saying ”look how mathematics is like a living system”. Mathematics is actually nothing like a living system. It has no biological matter, no cells or genes or organs or leaves or any of the things a living system can have. “Math is alive” is a metaphor based on what I perceive to be correlations in experience between mathematics and living systems that enable this class-inclusive assertion. Most of our ways of representing mathematics are metaphorical, so it fits right in. Given that metaphors often have their greatest impact when they are a consistent part of our early experience of the world, making the metaphors used with schoolchildren a part of dialogue about teaching is important. If metaphors gain the most traction not through language alone, but through our experiencing of the world as well, how might educators alter the language they use and the experiences they provide to encourage the development of the “math is alive” metaphor if they believe it is apt, useful and worthwhile? In this thesis I argue that is an apt, useful and worthwhile metaphor and that it has certain implications for the teaching of mathematics.

New Metaphors

Metaphors develop through our experience of the world. They are always, in some way, based on physical or cultural experience (Kovecses, 2005, p. 264). Many are so deeply important to our system of language that we cannot understand each other without them. Yet, some metaphors can change, and can be changed. Lakoff and Johnson (1980) assert that novel metaphors are usually structural (p. 152) and often based on a preexisting primary metaphor or fixed conventional mappings (Lakoff & Johnson, 1999, p. 149). Kovecses (2005, p. 261) discusses Lakoff and Turner's (1989) four ways in which creative metaphor is possible, including the extension and elaboration of current metaphors, as well as questioning and combining metaphors. Here again, we see metaphors growing from a pre-established metaphor based cognitive system. "Novel metaphors are often only novel as a linguistic expression but not as a deeper conceptual relation" (Katz as referenced in Katz, Cacciari, Gibbs, & Turner, 1998, p. 4). Glucksberg denies that there is always a need to reference preexisting conceptual structures for new metaphors. He admits that this is often the case, but adds to the discussion the idea that novel metaphors are often understood quickly and easily without reference to preexisting metaphors through our ability to quickly and readily identify apt categorical assertions (Glucksberg, 1999, pp. 1546, 1549 & 1554).

One example of a (relatively) new metaphor is "time is money". It is a structural metaphor, wherein one concept, time, is structured in terms of another, money. It is related to a larger mapping of metaphors, coming from "time is a substance", and it changes our thinking about time. Lakoff and Johnson (1980) said, "New metaphors have the power to create a new reality. This can begin to happen when we start to comprehend our experience in terms of a metaphor, and it becomes a deeper reality when we begin to act in terms of it" (p. 145). In regards to "time is money", Lakoff and Johnson (1980, p. 145) assert that the introduction of this metaphor to various cultures has contributed to the Westernization of those cultures. New metaphors are capable of highlighting, downplaying, and hiding aspects of the concepts they define (Lakoff & Johnson, 1980, p. 152). In *Metaphors We Teach By*, Elaine Brouwer, when discussing metaphor in education, states, "If a set of metaphors gains traction, it can lead to pervasive shifts in practice" (Brouwer as referenced in Badley & Van Brummelen, 2012, p 86).

The new metaphor "math is alive" is possible because it does, in fact, relate to primary metaphors that we use for mathematics. The implied notion that mathematics is a living system has structural similarities to philosophies, wherein math is a physical structure, and "math is alive" is also in some ways an expansion of preexisting primary ontological metaphors for knowledge wherein areas of knowledge have the characteristics of plant growth. Many other abstract concepts are already adopting metaphorical structures from living systems. In fact, metaphors of growth, plants, gardening, and nature are not uncommon in current educational language. The possibility that a living systems metaphor for mathematics may be able to grow out of these other metaphorical constructions is supported both by Lakoff and Johnson's theory of primary metaphor and Glucksberg's notion of apt categorical assertions. While I favour the former, the latter's willingness to recognize the potential power of new metaphors, however it is our brain comes to reconcile them, invites more potential for the implications of new metaphors.

> It should be clear that such a simple view [that all new metaphors are understood only through preexisting mappings] had to be wrong, for it strongly implies that good metaphors tell us nothing new. Yet this is precisely what good metaphors are for – to

say something new and informative in an interesting and striking way. (Camac & Glucksberg, 1984, p. 453)

The idea that mathematics is alive seems somewhat absurd at first, and this sense of novelty makes it noticeable and thought provoking, yet easily understood due to our preexisting metaphorical conceptual connections.

Conclusion

Beginning with our human ways of perceiving, we as a human society decide how to metaphorize the world. Through our child rearing, education, and living in the world we perpetuate metaphors that could just as easily have gone another way – that could still go another way. “Human conceptual systems are not monolithic. They allow alternative versions of concepts and multiple metaphorical perspectives of many important aspects of our lives” (Núñez as referenced in Gibbs, 2008, p. 360). Metaphors change, they change as paradigms shift and a need for new metaphors arises, and they shift paradigms when new inferences are built upon new metaphors. As was seen with ”time is money”, as well as many curricular metaphors in education, it is possible to purposefully introduce and reinforce new metaphors. “Even novel metaphors can be understood as rapidly as comparable literal expressions, provided that the novel metaphors are apt (Glucksberg as referenced in Gibbs, 2008, p. 69). Lakoff and Johnson, and the many others who have followed them, have shown that metaphors pervade our thinking, particularly when it comes to abstract concepts. “Such a system [of fundamental metaphors] will dominate your thought, your understanding of the world, and your actions" (Lakoff as referenced in Gibbs, 2008, p. 34). Gibbs asserts that it is possible to think of a concept metaphorically in more than one way. Concepts such as disciplines, become dependent on the metaphors we associate with them, but this also makes them malleable and changeable, "Multiple metaphors for concepts can be easily handled if we view concepts not as fixed, static structures, but as temporary representations that are dynamic and context-dependent" (Gibbs, 1996, p. 313). Mathematics is already defined and reasoned with using a variety of metaphorical concepts, and it can be expected that this variety will increase in the future.

CHAPTER TWO:

What Mathematics *Is* (metaphorically speaking)

Look at Mathematics, it's not a science, it's a monster slang, it's nomadic. Even in the realm of theory, especially in the realm of theory, any precarious and pragmatic framework is better than tracing concepts, with their breaks and progress changing nothing.

(Deleuze & Guattari, 1987, p.45)

In the metaphor "math is alive", mathematics is the target domain. As an abstract concept, mathematics has to be framed metaphorically in order to enable discourse about it. This thesis is concerned with the metaphors we use to frame our beliefs about what mathematics is and what that means for the ways we interact with it. Specifically, this thesis concerns school students and teachers, but before dealing with these participants, I will look more generally at the philosophy of mathematics and ways in which the idea of mathematics has been metaphorically framed historically and in the present. My assumption is that exploring philosophies of mathematics will provide an initial discourse which to make reference to when deconstructing research into the metaphors that teachers and students use to describe mathematics. I will start by framing the discourse of mathematics as a metaphorical enterprise. Following this I will differentiate what I see as three distinct yet overlapping groups of metaphors: those that are based on an objective view of reality and deny metaphor while relying on it heavily through physical descriptors, those that rely primarily on logical deduction using process-oriented language, game and time metaphors, but are not concerned with the nature of reality in general, and those that assume mathematics is a human endeavor, utterly metaphorical yet entirely representative of our fallible and biological selves.

Mathematics and Metaphor

Ralph Núñez refers to the ideas that constitute mathematics as “idealized mental abstractions” that rely on our use of conceptual metaphor to define them (Núñez as referenced in Cienki & Muller, 2008, p.96). Núñez, with George Lakoff (Lakoff & Núñez, 2000), use the terms grounding and linking metaphors to describe how we understand what mathematics is. According to Lakoff and Núñez (2000), grounding metaphors, like primary metaphors, represent a sort of fundamental cognition that directly connects to our human ways of interpreting the world. This focuses on basic arithmetic and includes arithmetic as object collection, as object construction, and as motion along a path (Lakoff & Núñez, 2000, p. 53). These metaphors are “grounded” in our human cognitive system. We define mathematics with our particular, human noticing of the world, and we use the contexts this noticing arises from to form our metaphorical conceptions. It is not that mathematics *is* a collection, creation, or movement of objects, but that we must rely on these tangible conceptions to grasp at the abstract connections we make. Our minds also allow us to focus on some metaphors over others, either consciously or unconsciously. We choose to see motion as a journey instead of growth, or a collection as a structure instead of a family, or an object as a form instead of a body. When discussing how humans conceptualize mathematics, Núñez says, “Mathematics is not purely literal; it is an imaginative, profoundly metaphorical enterprise, where the metaphorizing is, of course, generated, realized, and sustained by humans" (Núñez as referenced in Gibbs, 2008, p. 357).

Grounding metaphors form the basis for other so called linking metaphors, similar to the relation between primary and complex metaphors (Lakoff & Núñez, 2000, p. 53), which allow us to conceive of the much more intricate abstractions of mathematics in all of its complexity. As discussed in chapter one, humans use metaphors to understand a great many things, but in particular they rely heavily on metaphor when addressing abstractions. When considering the philosophy of mathematics, the need for metaphor becomes evident. Each philosophy will be seen to favour particular metaphors over others, while bumping up against longstanding beliefs generated from prior metaphors. Without metaphor, these philosophies could scarcely be understood at all. The philosophies we adopt affect our interactions with mathematics and our use of metaphors for concepts within mathematics, such as points on a line. Lakoff (as referenced in Lakoff & Núñez, 2000, p. xvi) discusses particular mathematical concepts such as infinity to show that metaphors are necessary for our cognitive ability to make sense of mathematics. In Norma Presmeg's research into the use of metaphor for mathematical concepts, she notes that "we cannot avoid the use of both metaphor and metonymy in mathematics" (Presmeg as referenced in Hoffman, Lenhard, & Seeger, 2005, p. 114). Lakoff and Núñez' (2000) understanding of metaphor has led them to also conclude that mathematics itself is embodied, meaning that it is a part of our biologically based experience of the world, as will be seen in the following discussion of philosophies of mathematics. I believe that acceptance of the metaphorical nature of how humans conceive of mathematics and believe in embodied mathematics are not co-dependent or exclusive of alternate metaphors. In fact, this philosophy of mathematics appears to invite and encourage alternate metaphors.

Philosophy of Mathematics

The philosophy of mathematics is brimming with variant ideas about the nature of mathematics, which I broadly see as belonging to three general groups: Objectivist philosophies, Deductive philosophies, and Humanist philosophies. I have borrowed the term “Humanism” in this context from Reuben Hersh’s (1999) *What is Mathematics, Really?*, as I see a collection of philosophies including Hersh’s and others that embody this perspective. Paul Ernest (1991, p. 18) uses the term “Fallibilist” to describe a similar collection of views as a move away from “absolutist” perspectives, including both the Objective and Deductive views I have differentiated.

From this standpoint, I have selected a few of the major philosophies from each of the three groupings I have created, looking for fundamental metaphors that may serve as reference points for the metaphorical ways in which educators and students refer to mathematics.

Objectivist Philosophies

Platonism. This philosophy is, not surprisingly, based on the Platonist worldview that implies abstract objects exist, and that ideas are somehow perfect forms belonging to a realm that is separate from human experience and physical reality. These forms are real, and have an objective, even if unperceivable, existence (Gowers as referenced in Hersh, 2006, p. 183; Jacquette, 2006, p. 238). In this sense, a circle has an existence outside of our application and understanding of circles. This philosophy is often associated with the idea that math just “is”. As humans, we strive to learn more and more math, but what we are learning is out there to be learned if we ever become able. “According to this conception mathematical assertions are true or false propositions, statements of fact about some definitive state of affairs, some objective reality, which exists independently of and prior to the mathematical act of investigating it” (Rotman as referenced in Hersh, 2006, p. 101). Current Platonist-leaning philosophers of mathematics such as Mark Balaguer argue against relativism on the basis that if mathematical understandings are not true in a literal sense, then nothing is, and the whole argument is a relatively moot point. “Now, maybe it bothers you to think that our mathematical and scientific theories are untrue. But it doesn’t bother me. The trick is to notice that . . . according to fictionalism, our mathematical and scientific theories are *virtually* true, or *for-all-practical-purposes* true, or some such thing (because they’re such that they *would* be true if there were abstract objects” (Balaguer, 2014, p. 8).

True Platonists would assert that they aren’t using metaphors at all so that mathematics is literally a form. I believe that this simply represents one particular metaphor that humans use to understand what mathematics is. We continue to use this metaphor because it is useful in the absence of alternate ways to describe the otherness of mathematics. With it come all the implications and complex metaphors associated with viewing mathematics this way including truth and falsehood and right and wrong.

Structuralism. A more recent version of an objectivist philosophy of mathematics is Structuralism. Michael Resnick (1975), a well-known Structuralist says,

> Since structures are abstract entities, Structuralism is a variant of Platonism. However, it is free from the objections I raised earlier against standard Platonism . . . The Structuralist does not claim that *all* structures are perceived directly without the aid of ordinary experience. He believes instead that the mathematical patterns are known in the same way, as, say, linguistic or musical patterns. (p. 34)

Structuralist ideas emerged in 1965 with the publication of Paul Benacerraf's (1965) paper "What Numbers Could Not Be". Benacerraf argued against some fundamental tenets of Platonism, rejecting the idea of abstract forms from another realm. However, he maintained that mathematics refers to mathematical structures and our description of them, and that these structures, in some way (there is debate about this among Structuralists), exist. Writing about numbers, Benacerraf states, "Only when we are considering a particular sequence as being, not the numbers, but *of the structure of the numbers* does the question of which element is, or rather *corresponds to*, 3 begin to make any sense" (1965, p.71). Stuart Shapiro, a self-professed "ante rem" Structuralist (Shapiro, 2009, p. 157) provides another clear and insightful brief commentary on the nature of Structuralism.

> A mathematical structure can, perhaps, be similarly construed as the form of a possible system of related objects, ignoring the features of the objects that are not relevant to the interrelations. The structure is thus completely described in terms of the interrelations. (Shapiro, 1983, p. 523)

As the Cambridge *Introduction to the Philosophy of Mathematics* points out, anything can play the role of the number two, so long as it maintains its place in the sequence of natural numbers; the number two has no value outside of its interrelations with other numbers (Colyvan, 2012, p. 40). In this way, Structuralism responds to some of the criticisms of Platonism and the ethereal existence subscribed to mathematical entities, while still asserting the possibility of mathematical truth. The preference of a specifically structural metaphor over the metaphor of a form is evidence of how the philosophy of mathematics uses the manipulation of metaphors to introduce new ideas about what mathematics is. Structure metaphors related to construction and building have enabled the growth of a variety of new ways of thinking about mathematics that are included later in this section, and the relational metaphors introduced here impact later process-based philosophies.

Deductive Philosophies

Logicism. Others have responded to the question of what mathematics is quite differently. Many have included some form of logic or deduction, sometimes in reference to proofs, in their understanding of mathematics. Logicists, like Gottlob Frege, believed that logical processes of inference could be used to prove mathematical concepts, and that this is the essence of what mathematics is, “arithmetic is a branch of logic and need not borrow any ground of proof whatever from experience or intuition” (Frege, 1917, p. 114). Frege was a proponent of analytic philosophy and part of the move towards logical positivism in philosophy, science, and mathematics in the early twentieth century. Here, it is possible to find an objective truth through the quality of one’s deduction, and this is directly applicable to mathematics. “According to

Logicism, mathematics is logic" (Colyvan, 2012, p. 5). Logicists following Frege believed that "The deduction of the simplest laws of numbers by logical means alone" was possible (Frege, 1917, p. 114), and that logic was infallible and led to truths. "[In the Logicist view] logical knowledge is thought to be more basic and less mysterious than mathematical knowledge" (Colyvan, 2012, p. 5). Frege was criticized for particular aspects of his proof of this notion, but even his criticizers, most notably Bertrand Russell, went on to continue his Logicist doctrine with some shifts in foundation (Colyvan, 2012, p. 6). Neo-Fregians such as Bob Hale and Crispin Wright claim to follow the basic tenets of Frege's approach, and argue that Frege was his own worst enemy in some of his more detailed explanations.

> The two main components in Frege's philosophy of Arithmetic . . . (i) his Platonist thesis that Arithmetic is a body of truths about independently existing objects . . . and (ii) his Logicist thesis that these truths are analytic . . . Neo-Fregians – such as Crispin Wright and myself – believe Frege to have been substantially right in both components of his view. (Hale, 2005, p. 21)

These writers turn to the philosophy of David Hume to show the underlying irrefutable arithmetical logic of their approach, "Hume's Principle: The number of Fs = the number of Gs if and only if there is a one-one correlation between the Fs and the Gs" (Hale, 2005, p. 21). Like Frege, Neo-Fregians maintain that a logical approach references objective truth, finding a different way to work around some of the pitfalls of Platonism while retaining its truthiness, "It is the assumption of this paper that there are necessary truths and that such truths can be known" (Hale,1994, p.300). In this logic-based approach, Platonic hints of truth and perfection remain.

"Note that one can be a Platonist and a Logicist at the same time" (Gowers as referenced in Hersh, 2006, p. 183). While Logicism could be grouped with the other objectivist philosophies mentioned, I have placed it in a separate category as I see it as a shift towards a Deductive approach that creates the stage for other similar philosophies where truth and reality cease to define mathematics. Logicism shuns the prefabricated metaphors of Platonism and Structuralism for metaphors of process, yet maintains its true/false metaphor. This sets the stage for the emergence of new metaphors that highlight mathematical processes and shift the focus away from issues of objectivity.

Formalism. Other philosophies of mathematics based on a Deductive approach allow more relativist possibilities where truth is not necessary or conceivable. In the case of Formalism, it simply does not matter whether there are any true or real implications. "In Hilbert's classic statement of the Formalist credo, mathematics consists of manipulating 'meaningless marks on paper'" (Rotman as referenced in Hersh, 2006, p. 100). While mathematics from a Formalist standpoint relies on self-reference, Formalism remains unclear about the objective existence of mathematics, concerning itself more with proof within the system than proof of the system itself (Rav as referenced in Hersh, 2006, p. 85). Formalists, following the path begun by David Hilbert, see mathematics as a sort of rule-based enterprise. "[Mathematics] must also be self-consistent; that is, starting from the rules themselves it must be impossible to deduce something declared to be false by those rules" (Courant & Hilbert, 1953, p. 2). Some branches of Formalism invite game structures to describe the relations of rules. Here, types of mathematics follow different sets of rules within semantic systems, and the mathematics within each type are sound when they absolutely follow the rules and the symbols are consistently applicable (Barton, 2008,

p. 69; Jacquette, 2006, p. 246; Peckhaus, 2003, p. 142). Alan Weir (2011), a current proponent of game theory Formalism in mathematics, also looks at the logic of language systems in a similar way, seeing a broader Formalism in semiotic systems. The mathematical philosopher Michael Gabbay (2010) rejects many notions of game theory in his Formalism, but maintains its linguistic notions and similarities: “knowledge of arithmetic, on my proposal, is obtained in much the same way as knowledge of the logical features of any language” (p. 234). Many recent Formalists prefer a strictly self-relational approach called “term Formalism”, wherein mathematical terms refer to themselves (Gabbay, 2010, p. 220). This type of Formalism rejects the game notion, including its relation to semiotic systems, seeing mathematics as fundamentally different from these systems. Haskell B. Curry (1953), an early proponent of term Formalism, states that, “mathematics, since it does not deal with signs and may be arrived at by abstraction from non-linguistic fields, is not a division of semiotics” (p. 176). Whatever variation of Formalism one prefers, many of the basic assumptions remain. For all Formalists, it is possible to derive mathematical consistencies in the form of proofs based on a chosen symbol set and clearly defined rules, although sometimes this may result in minor adjustments to the rules along the way. Again, process metaphors are preferred, but Formalism also invites different linguistic metaphors and game theory metaphors. Even term Formalism, while stoically attempting to reduce metaphor, still conceives of itself as reliable and consistent. This type of Formalism does, however, hint at the idea that perhaps mathematics determines itself, which is an idea that will reappear later when I discuss the living systems metaphor. In the philosophy of Formalism, mathematics is freed from the metaphorical realm of being as a form or structure and becomes a form of doing like a language or a game, yet it remains locked in closed system metaphors.

Intuitionism. Another significantly different way of looking at mathematics from a Deductive standpoint lies in the Intuitionist conception of mathematics. Although this philosophy is often seen as a refutation of the Formalist standpoint, its relationship to this view through opposition and discussion and its fundamental reliance on the figuring out of things leads me to include it with this group. Although the locus of the Deductive process is more situated in this context, the creation of a process of deduction remains. Intuitionism in mathematics arises from the works of L. E. J Brouwer (e.g., Brouwer, 1981), who separated himself from the Formalists by seeing mathematics as a constructive, human-based entity instead of a set of rules. Although current Intuitionist mathematicians such as Giovanni Sambin, Wim Veldman, Dirk van Dalen, and Mark van Atten sometimes conflict on particular aspects of this view, an underlying homage to Brouwer is always made. The use of the word "intuition" comes from the mental processes we use to assure ourselves of the consistency of things. It is through these mental processes that we construct our understanding of mathematics, so that mathematics and mathematical knowledge is a construction of the human mind. Intuitionists claim that the mathematics that exists is the mathematics that we are able to construct. "Mathematics in his [Brouwer's] view was a human enterprise; his aim was the revision of existing practice, by stressing that the fundamental objects, and indeed all objects, of mathematics are creations of the human mind" (van Dalen as quoted in van Atten, 2008, p. 4). While Intuitionism has some connection to Structuralism through its use of constructed elements, it differs in its conception of truth and the source of mathematics.

Brouwer believed that mathematics was not language based, but was its own mental process, deriving from our ways of perceiving and understanding time. One classic Intuitionist notion invokes the idea of a choice sequence, which is a sequence that is "created by the individual mathematician, by choosing one number at a time” (van Atten, 2003, p. 5). This sequence can never be known in advance of its creation and, like other mathematical objects, is under construction in time. “A choice sequence is generated freely, and at any time we have no more than a finite initial segment of it” (van Atten, 2003, p. 7). Intuitionism stood apart from Formalism and refuted the use of symbols, rules, and classical logic. Instead of seeing logic as a basis for mathematical thought, Brouwer saw mathematics as a basis for logical thought, but believed that what we consider logic was often polluted by language, which is emotive and cultural. Mathematical logic is a separate underlying constructed system that serves as a non-symbolic, language-less deduction of the mind (Franchella, 1995, pp. 306-307). This version of logic stands in opposition to the classic Aristotelian version upheld by Logicists and, to some extent, Formalists. A more recent Intuitionist, Charles McCarty, states that, “In overturning supposed logical laws on purely mathematical grounds, the Intuitionists have toppled a central pillar beneath conventional mathematics” (as quoted in van Atten, 2008, p. 49).

Intuitionism gave rise to a new metaphor by rejecting the eternal truths of Platonism, the rights and wrongs of Structuralism, the correct processes of Logicism, and the rules and languages of Formalism. At the same time, it grapples with retaining metaphors of purity and deduction. By situating mathematics as a type of human cognition and a process in time, Intuitionism introduced the new metaphor “math is time” and invited human and social metaphors into conversations about the nature of mathematics.

Humanist Philosophies

Social Constructivism. One Humanist perspective on mathematics sees mathematics as a social construct. Hersh (1995) refers to mathematics as a social concept, and a social entity (pp. 591 & 593), and Ernest (1991) describes his philosophy of mathematics as “Social Constructivism”. It is immediately apparent how metaphorical this is. Social Constructivists do not think it is possible to know whether or not mathematics has any sort of objective existence. Ernest (1992) clarifies by saying, “What is proposed is not the social construction of reality but the social construction of our knowledge of reality” (p. 96). Here math is seen as a shared social and historical construct, and a part of cultural initiation. Mathematics is consistent within cultures not because it refers to some underlying facts about the nature of reality, but because we have learned to perceive our reality in this way. “The uniformity of mathematical meaning amongst mathematicians, and a shared view of the structure of mathematics, results from an extended period of training in which students are indoctrinated with the ‘standard’ structure” (Ernest, 1992, p. 97). Mathematics makes sense because it is supported by the social reality in which we live, and not because it is somehow beyond that. Social Constructivists point to differences among cultures as evidence that there is no Platonist universal conception of mathematical forms. Cultural variance in the symbolism and language of mathematics affects the outcomes when Deductive reasoning is applied, whether it is a system of logic or intuition. In this view, mathematics is entirely a human construct and, furthermore, a construct of our minds and not a product of external reality.

However, Social Constructivists still see mathematics as existing beyond the level of the individual human and conceive of it as a shared endeavour distinctly influenced by three factors, or "fuzzy sets" as Ernest (1992, p. 93) refers to them. These include information-technology artefacts, persons, and linguistically based rules and activities (Ernest, 1992, p. 93). Information-technology artefacts are our cultural record of mathematics. Books, media, software, and other recorded communications or ideas enable us to preserve and repeat our understanding of mathematics, and they allow us to build and revise. The people of mathematics create these artifacts. When discussing the people of mathematics, Ernest often speaks of mathematicians of varying degree. He refers to research mathematicians as strong members of the set, and uses the example of a mathematics teacher as a weak member (Ernest, 1992, p. 94). Numerate citizens are classified as the weakest members of the set, which puts young students of mathematics even lower on the scale. The linguistically based rules and activities of mathematics both limit and enable the knowledge that can be derived from it. Ernest (1992) says,

> Mathematics rests on spoken (and thought and read) natural language, and mathematical symbolism is a refinement and extension on written natural language. Mathematical truths arise from the definitional truths of natural language, which is acquired by persons through social interaction. (Ernest, 1992, p. 94)

The language of mathematics makes sense to us (most of the time), because many of the rules and assumptions rest on a pre-established system of language, both historically and in the mind of the individual. Here there are hints of Formalism, however, in Social Constructivism, the language of mathematics is not entirely the same universally, creating different concepts of the logic and syntax of mathematics. For example, Bill Barton, a professor of mathematics and language at the University of Auckland, discusses number words in his book on the language of mathematics as an example of multiple understandings (Barton, 2008). He discusses research he was engaged in with Maori mathematicians when he noticed some significant differences in how the structure of their natural language influenced their mathematics. “In everyday talk numbers are usually used like adjectives . . . In Maori, prior to European contact, numbers in everyday talk were like actions” (Barton, 2008, p. 5). We take for granted the constructed nature of our most basic mathematical concepts, and the impact of this on our conception of mathematics as a whole. Social Constructivists also provide evidence of change in the history of mathematics as proof that there is no one universal mathematics or universally sound way of engaging with mathematics. Hersh (1995) says “Certainly it’s historical. The history of mathematics is a developed subject. Historians have studied mathematics back to the Babylonians” (p. 551). Mathematics has changed significantly over time, and these changes often reflect cultural or paradigmatic shifts in other areas. In discussing the history of number, Ernest (2006) says, “the history of number and counting is also the story of the development of semiotic systems of numeration and calculation” (p. 80). Just as we are able to document changes in language over time, we also recognize changes in the semiotics of mathematics. These changes can have profound implications and effects on society as a whole. It seems absurd to assume we could come to a time when mathematics just “is” and ceases to change, when the history of mathematics clearly demonstrates its constantly changing, fallible nature. Social Constructivists depart from any semblance of truth by having mathematics being built by humans. To do this, they repurpose mathematical metaphors, including the structural dialogue of Platonism and Formalism and the metaphors of time and humanity in Intuitionism. Here, forms are not static or objective and time is not just process, but a history. Yet with our metaphorical ability to build comes a dialogue of control and claiming of mathematics as product of our humanity.

Embodied mathematics. Another recent philosophy of mathematics that follows in the Humanist tradition and highlights particular aspects of Social Constructivism is the embodied philosophy of mathematics, pioneered by Rafael Núñez. While this "mind-based mathematics", as Lakoff and Núñez (2000) call it (p. 9), has some similarities to the Social Constructivism of Ernest and Hersh, it also seeks to differentiate itself from this philosophy. While Social Constructivists still see a mathematics "out there", albeit a human creation, mathematical philosophers that embrace embodiment see only the mathematics "in there", the constructions inside each mind with no existence beyond what each individual constructs for themselves. "Mind-based mathematics, as we describe it in this book, is not consistent with any of the existing philosophies of mathematics: Platonism, Intuitionism and Formalism. Nor is it consistent with recent post-modernist accounts of mathematics as a purely social construction" (Lakoff & Núñez, 2000, p. 9).

According to the philosophy of embodied mathematics, to understand what mathematics is, one must have some understanding of how the human mind works, and how it creates knowledge and understanding in a human. Proponents of the Embodied Mind theory believe that the human brain, like all living systems, has grown out of a need to survive and thrive. Over time the human brain has adapted to enable human beings to manipulate their world, and survive in a constantly changing, far from equilibrium environment. Our brain works through a series of neurons that, in a highly simplistic sense, operate via an input-output network. The input stimulates a series of metaphorical pathways that enable our thought processes (Devlin, 2008, pp. 369-372). Through this process, we come to understand and navigate our worlds. What mathematics is depends on the prior and ongoing evolution of this process. Here the brain is not some sort of perfect Turing machine, computing and calculating, but is instead a dynamic network of connections and responses that reflects its particular type of existence as a living system. This view of embodied cognition follows theories of mapping seen in representations of cognitive metaphor schemes. When we come across something novel, our brain is designed to try to perceive and understand it through prior knowledge. Although this is true of all knowledge, it is particularly important when dealing with human concepts and ideas that have no physical existence. Lakoff and Núñez (2000) remind us that, “For the most part, human beings conceptualize abstract concepts in concrete terms” (p. 5). We do this by relating the abstract concepts to concrete entities through metaphor. An often-cited example is the number line. Certainly, numbers do not physically sit on some infinite line increasing from left to right. However, when we call to mind this particular metaphor, it allows us to understand number in a way that is impossible without it. When we think of a number being higher or lower than another, we are calling up a metaphor that allows us to both think about and communicate our understanding of that number. However, as Núñez (2009) points out, the number line is not a universal truth but a recent Western cultural construct. (pp. 71, 72). Without these metaphors, mathematical thought could not exist. It is the evolution of our brain’s ability to use metaphors and connect concepts that enables and limits mathematics. It is our need to see mathematics as a concrete entity in order to make sense of it that causes it to seem real, infallible, and organized outside ourselves. “This is why mathematics is perceptually Platonistic . . . that is the only way the human brain can do it” (Devlin, 2008, p. 365). We have to work with what we have, but what we have is quite amazing. The human brain is a highly evolved system, but it is also only that and not a window to some greater understanding beyond itself. Mathematics is a product and a function of that brain. It is the result of both biology and indoctrination enabled by biology, and in the embodied perspective, it is nothing more than that. “Where does mathematics come from? It comes from us!” (Lakoff & Núñez, 2000, p. 9).

Embodied Mind theory enables the use of cognitive metaphors to describe the nature of mathematics. This theory asserts that we only think it has an existence of its own because our brain needs to think of it that way. We only think it makes sense because we made it to make sense. Embodied Mind theories of mathematics are the first to recognize that what we have to work with when describing mathematics is metaphors, yet new metaphors are not offered, and in its place is a sort of meta-cognitive analysis of the neuroscience of humans. From Embodied Mind theorists comes the understanding that our conception of mathematics is primarily metaphorical, which is an understanding so important to this thesis. Yet they have stopped short of identifying the potential of purposeful new metaphors. In Embodied Mind theory what is gained is recognition of the metaphorical nature of abstraction, but what is lost are metaphors for how to describe our relation to this mathematical entity that somehow dwells in us.

Conclusion

Mathematics is a metaphorical enterprise. Its fundamental nature as an abstract being in our minds demands that we cultivate comprehensible metaphors to express ourselves mathematically and when talking about mathematics. We are forced to name it as an entity in order to differentiate it from other conceptions, but we are at liberty to refine and evolve the metaphors we choose. Mathematics philosophy has shifted its grounding metaphors over time, creating new metaphors while holding on to others for either security or contrast. Mathematical reliability and its perseverance in human culture has been explained through Platonism as a result of its perfect form, through Structuralism as interrelations with a perfect fit, through Logicism as an inevitable process, through Formalism as the product of assumptions, through Intuitionism as an unfolding of time, through Social Constructivism as a grand human creation and through embodiment as a way to survive. The main concern has been a contrast between objectivist philosophies and those that embrace relativism, with objectivists being unable to deal with the obvious humanity of mathematics and relativists being unable to cope with its apparent otherness. It is my belief that new metaphors are still needed to both challenge our thinking about mathematics as a metaphorical conception, and offer the potential to reconcile some of the difficult byproducts of past and current philosophies. The next chapter will explore the literal meaning of being alive so that this metaphor may be applied appropriately, and to seek out the complex metaphors that arise from the biological dialogue of living.

CHAPTER 3:

Being Alive

The One is said with a single meaning of all the multiple. Being expresses in a single meaning all that differs. What we are talking about is not the unity of substance but the infinity of the modifications that are part of one another on this unique plane of life.

(Deleuze & Guattari, 1987, p. 276)

In the metaphor "math is alive", mathematics is the target and living systems are the source. Although living systems theory uses metaphor, and living can be seen as an abstract concept as well as a literal biological action, the following is an attempt to look at the latter literal meanings of the terms being used in this field of biology. A basic understanding of the literal scientific definitions of these terms precludes the use of living systems theory metaphorically. In order to understand the implications of conceptualizing mathematics as a living system, I have first analyzed what it is to be alive.

Living beings hold a special status for us. We recognize in them aspects of growth and change that fascinate us. When a seed becomes a flower, I am fascinated, and I am fascinated when babies are born. Scholars across fields of study have asked what it is to be alive for centuries and yet no clear scientific consensus exists (Benner, 2010, p. 1021). I theorize that being alive biologically is an autopoietic act, following the tradition of biological systems theory.

What is the difference between a live cat and a dead one? One scientific answer is 'systems biology'. A dead cat is a collection of its component parts. A live cat is the emergent behaviour of the system incorporating those parts. (Anonymous, 2005, as quoted in Robert as quoted in Hull & Ruse, 2007, p. 364)

Living systems theory sees living as a dynamic process, resistant to mechanistic reductionism, with any attempts to define it as observer-biased (Capra & Luisi, 2014, p. 43).

In the 1950s and 60s, Ludwig Von Bertalanffy formalized the biological view of systems theory through the study of what he called “general systems theory”. He believed that his theory had potential applications beyond biology, but maintained his focus on living, open systems. This gave biologists a language that had the potential to resolve unanswered questions about living beings. These animate objects refused to operate according to mechanistic models being literally applied by classical physics, and even appeared to occasionally reject what were thought to be universal laws in thermodynamics (Bertalanffy, 1950, p.26). They were, growing, changing, becoming, and decaying, in ways that often appeared random or unpredictable. A continual breaking down into parts yielded answers about structure, but neglected to explain the behaviour of those structures. In response, Bertalanffy proposed that the whole was more than the sum of its parts, and that, instead of deconstruction, reconstruction was needed – an analysis not just of the parts, but also of their interplay. These ideas have been taken up in various disciplines as Bertalanffy predicted, and continue to have a strong influence on the search for a scientific understanding of life.

The biologist James Grier Miller proposed in *Living Systems* that this view could be extended to a global perspective (Miller, 1978, p.5). Although he worked with similar systems concepts, his specific view of living systems is well known yet has not been taken up broadly in the study of living systems. Miller defines eight system levels with (exactly) twenty particular subsystems and includes a lengthy, hierarchical redefining of terms that can be seen to contradict current ideas of unity and interdependence (Bailey, 2005, p.45). However, Miller's concern for the roles of space and time, as well as the idea of nested systems continues to resonate.

While Miller was extrapolating on living systems ideas, others were working with small living systems – focusing on cells, bacteria, and the chemical reactions of life. The Belgian chemist Ilya Prigogine further developed the systems view of life through his work with dissipative structures in self-organizing systems (Capra & Luisi, 2014, p. 158). Although Miller briefly discusses ideas of sufficient and increasing complexity (Miller, 1978, pp. 18 & 293), Prigogine expanded the concept of complexity in science, adding to rigorous scientific scholarship on how complex systems behave, while steadfastly maintaining that a complex system demonstrates sufficient randomness to make certainty impossible (Prigogine, 1989, p. 399). Here, complexity refers to our inability to predict behaviour as a sum of the interaction of parts, because each interaction affects the other with multiple possible trajectories. Prigogine's work with complex systems enabled scholars such as Francisco Varela and Humberto Maturana to begin to develop a biological understanding of living beings as complex, autopoietic systems (Maturana & Varela, 1987 p. 43).

Characteristics of Living Systems

Sustainable living systems share three general characteristics: a particular pattern of organization in their structures, expressions of this pattern in their continual existence, and reliance on this pattern in how they replicate and evolve. This pattern of organization is specific to living systems (Capra & Luisi, 2014, p. 137). It is this pattern of organization – in structural characteristics, maintenance actions, and evolution – that shows intriguing similarities to the pattern of organization of a discipline such as mathematics. The goal of this chapter is to describe some fundamental aspects of living systems, so that I may provide evidence of the significance of these characteristics for a metaphorical understanding of mathematics in the next chapter.

Pattern of Organization

A living system is an autopoietic unit (Maturana & Varela, 1987, p. 43). Autopoietic systems are able to recreate themselves, they are “continually self-producing” (Maturana & Varela, 1987, p. 43). In living systems theory, the organization of structures that allows for autopoiesis is what differentiates living systems from the inanimate. The autopoietic goal is what governs our organization. Living systems sustain themselves autonomously in their environments through the circular logic of their organizational interactions, “It is a circular process: The structures can be formed only by its own operations because its own structures in turn determine the operations” (Luhmann as quoted in Clarke & Hansen, 2009, p.149).

Autopoiesis relies on the organization of structures and their actions rather than the specific structures themselves. “It is the organization of the system that is stressed: structure (i.e., the actual material components) is almost secondary, in the sense that an autopoietic unity can be realized with several different specific structures” (Luisi as referenced in Stein & Varela, 1993, p.20). There is no particular set of structures that makes up a living system, but there is commonality in their pattern of organization. Here I will explore characteristics of this pattern of organization that are common to all living systems when we regard autopoiesis as the defining attribute of life. This includes nesting, autonomy through boundary formation, being an open system with operational closure, extreme sensitivity to initial conditions, and instability, each of which I discuss in turn.

All living systems are nested systems. Although there are substantial critiques of Miller's subsystems hypothesis (Bailey, 2005, p. 45), our modern understanding of ecosystems comes from this view. Luhmann's (1984/1995) theory of social systems is also generated from this notion. From this living systems perspective (a) the interdependence of these systems resists hierarchical descriptions (Hansen, 2009, p. 128), although systems exist in a network of increasingly complex systems; and (b) systems within systems rely on each other for their own survival (Capra & Luisi, 2014, p. 359; Hansen, 2009, p. 128). Capra and Luisi (2014) describe these systems as "interlocked" (p. 347). Some current theories in biology even support the idea that some of this increasing complexity results from living systems engulfing others, such as in Lynn Margulis' symbiogenesis (Capra & Luisi, 2014, p. 201).

Maturana and Varela (1987) view life as cellular, with the cell, in all its variations, representing the simplest living system. The cell is a bounded unit, held together by a boundary of its own making. Within this boundary, the cell functions to reproduce its own components through maintenance functions. Living systems interact with their environment for survival, yet they are still distinct and autonomous with distinguishable boundaries. A living system is unique in that it produces and maintains its own boundary, so that there is always a relationship between "the inner" and "the outer" (the system's environment). "The most striking feature of an autopoietic system is that it pulls itself up by its own bootstraps and becomes distinct from the environment through its own dynamics, in such a way that both things are inseparable"

(Maturana & Varela, 1987, p. 46). It is this organization that defines their unity – that makes them distinct unto themselves. The boundary is not what separates the system from the outside world, but instead what enables its interaction with it, and it is the point at which the system operates by its own rules.

Living systems are characterized as open systems through this interdependence. An open system exchanges both matter and energy with its environment (Prigogine, 1980, p. 78). You can distinguish bacteria, or a human, or any other living system from its environment, even if the thing itself may be a part of a larger system, such as the vast array of bacteria that populate and enable the human digestive system. This is because open systems exhibit organizational closure. "This organizational closure implies that a living system is self-organizing in the sense that its order and behaviour are not imposed by the environment but are established by the system itself" (Capra, 1996, p. 167). A system exhibits autonomous actions as a unity, even though it is both made up of, and making up, other systems. This dichotomy of openness and closure is discussed in Clarke and Hansen's *Emergence and Embodiment*, where Hansen quotes Niklas Luhmann as follows: "Using boundaries, systems can open and close at the same time, separating initial interdependencies from system/environment interdependencies and relating both to each other" (Luhmann as quoted in Hansen, 2009, p. 114). Living systems remain "a distinguishable unity in the domain in which they exist" (Varela, 1997, p. 75). Although a tree may now look quite different from a tiny sprout that once popped out of the ground, we recognize it to be the same thing – a single living unity. The system produces its own boundary, even though that boundary is in a constant state of interaction with its own and other systems. It defines its identity and thus enables its identification as an autonomous unit. It also invites us to question the idea of a boundary more deeply – not just as a point of separation, but also as a point of contact –

autonomous, operationally closed, self-determining systems openly coupled to a volatile, unpredictable environment.

Although living systems exhibit organizational closure, they are fundamentally different from closed systems. A closed system does not change and reacts predictably. A closed system will follow a linear trajectory. In Miller's (1980) words, no "matter-energy or information transmissions" (p. 18) can be exchanged with a closed system. Prigogine (1980) later clarifies that a closed system can exchange energy, but not matter with the environment (p. 78). Given a set of initial conditions, you can predict the future of a closed system in a deterministic way. Newtonian physics is particularly good at providing answers about the behaviours of such systems.

> For example, if you kick a stone, it will react to the kick according to a linear chain of cause and effect, and its behaviour can be calculated by applying the basic laws of Newtonian mechanics. If you kick a dog, the dog will respond with structural changes according to its own nature and nonlinear pattern of organization – the resulting behaviour is generally unpredictable. (Capra & Luisi, 2014, p. 136)

Living systems have a set of initial conditions - the moment when they became an autonomous unity in a particular environment. While the initial conditions of a closed system are reliable predictors of its future states, this is not so for an open system (Kauffman, 1991, p. 81; Prigogine, 1989, p. 399). In a living system, initial conditions are not sufficient to predict the trajectory of the system over time (Bertalanffy, 1950, p. 25). Kenneth Showalter describes this as “extreme sensitivity to initial conditions” (as quoted in Wallaczek, 2000, p. 328), meaning that the smallest difference in initial conditions can set off exponential changes that result in great differences as the system grows. What first appears as virtually identical systems can become drastically differentiated over time. Despite vast similarities in genetic coding, apes, and even mice, are not like humans. Even twins with identical DNA grow less alike over time (Capra & Luisi, 2014, p. 196). It is not simply the structure of the code, but its actions, expressions, and interactions that make living systems what they are.

As the internal and external interactions of a non-linear system become increasingly chaotic over time; the system becomes increasingly disordered (Prigogine & Stengers, 1984, p. 128) – sometimes to the point of radical structural change through infinite numbers of potential pathways (Prigogine, 1989, p. 399). As a result of the complex interactions within a living system and its complex interactions with the environment, we can imagine potential futures, but can never know future states with absolute certainty.

Internally, a living system refuses reductionism and cannot be described as a particular collection of parts. Not only are different collections of particular structures sufficient to enable life, but these structures often change throughout the life of a system. The pattern of organization described here allows for life, but living is not accomplished by the presence of the parts, but by their interaction. This is an important distinction of the systems view of life; that the whole is greater than the sum of its parts. Like the dead cat, the pattern of organization of the structure of a system provides a potential landscape for life, but to understand life we must look not only at its properties, but also at its processes.

Maintenance

Prigogine (1980) named one of his books *From Being to Becoming*. Here, he stresses the importance of process over form in complex physical systems. These systems cannot be reduced or dissected, as we are so prone to do in science. What they are is in a constant state of flux, reacting to disequilibrium. In order to sustain themselves, living systems, as autopoietic unities, are constantly reproducing their own components. In order to do this, all living systems exhibit similarities in process. The fundamental process these systems are engaged in is self-organization. Characteristics of this process include self-sustenance in far from equilibrium conditions, energy and information exchange, and the presence of feedback loops operating with decentralized control. Autonomous self-organization is what allows a living system to stay alive by being able to continually recreate itself.

A living system exhibits particular processes in order to continually maintain its state of being alive through autopoiesis. The most significant process resulting from the pattern of organization of a living system is self-organization. This is "the capability of assuming an organized structure thanks to the inner rules of the system" (Capra & Luisi, 2014, p. 165). The continual, autonomous self-recreation of autopoiesis depends on the system's ability to organize itself for this purpose. The concept of self-organization first arose in cybernetics when scientists discovered that systems whose initial conditions were selected randomly would reach states of ordered patterning through what appeared to be spontaneous emergence (Capra, 1996, p. 84; Capra & Luisi, 2014, p. 96). Just as similar initial conditions can lead to vast difference, different initial conditions can result in striking similarity. This term was later applied to living systems when biologists noticed that despite wide fluctuations in environmental conditions, a living system is able to maintain its identity and regulate its inner state thought the self-organizing attributes of its pattern of organization. For example, the inner temperature of a human being remains relatively stable despite changes in the temperature in the outside environment. Living systems tend towards a state of homeostasis, defined in systems theory as a “state of dynamic balance” (Capra & Luisi, 2014, p. 91), wherein the system is never static and never able to reach infinite or absolute balance as it continually re-adjusts to conditions. Living systems are forced into non-equilibrium states due to disorder in the environment, sometimes acting far from equilibrium. “Not only does non-equilibrium lead to both order and disorder, but it also leads to events, because more possibilities appear than do in a state of equilibrium” (Prigogine, 1989, p. 399). This allows for the creation of novel structures and for creativity within the system. Capra (1996, p. 85) lists this creativity as the first of his three characteristics of self-organization.

His second characteristic of self-organization is that matter and energy are constantly being exchanged between a living system and its environment. The living system’s ability to regulate these transactions with its environment as they affect its internal conditions informs the process of self-organization. So, although a system identifies its autonomy by being organizationally closed and internally self-organizing, Clarke points out that these systems self-organize “in the presence of conditions provided for elsewhere, by environments that lend a necessary other to the self of self-organization” (Clarke, 2009, p. 42). Self-organization allows a living system to exist with and through its environment. Without this dynamic exchange, there is no life.

Capra’s third characteristic of self-organization is the existence of non-linearity and feedback loops. Living systems are non-linear in the sense that the actions of the system cannot be represented or described using a linear, casual chain of events or set of equations, hence, the impossibility of prediction. Living systems, always operating at some distance from equilibrium, depend on self-regulating mechanisms to attempt to regain homeostasis. These self-regulating mechanisms operate through multiple feedback loops. Prigogine (1980) asserts that “most biological reactions depend on feedback mechanisms” (p.95), and Beloussov restates Capra’s premise that living systems are “far from thermodynamic equilibrium and their elements are linked with strong feedback, described mathematically by nonlinear differential equations” (Beloussov as quoted in Stein & Varela, 1993, p. 153).

The concept of a feedback loop also appeared early in writing on cybernetics in the mid-twentieth century, pioneered by Norbert Wiener among others. Wiener connected his cybernetic theory to living systems and theorized two types of feedback loops, positive and negative (Capra, 1996, p. 52). Negative feedback loops are self-regulating. A common example is steering, wherein one steers in the opposite way when going off course. A modern example of Wiener’s “Steersman” concept would be if my car is going slightly to the left, I correct it by steering slightly to the right. The result is balanced and I avoid an accident. My hands feel that the wheel is going left, so my brain tells my hands to turn right, sometimes without even knowing it is happening. This is a negative feedback loop because the end result is balanced. Living systems rely on these conceptual loops to maintain homeostasis. Positive feedback loops also exist, wherein the car is going left, so I steer more left. The result is an amplified version of the original source. Capra (1996) reminds us of metaphors we have used for this process prior to scientific definitions; “vicious circles”, “self-fulfilling prophecies”, and “bandwagon effects” (p. 63). The dynamic feedback loops of open systems must self-regulate an ever-changing system in relation to an unpredictable environment. These systems rely on both negative and positive feedback to maintain their dynamic balance. Capra (2014) says “both types of feedback play important roles in the self-organization of dynamic systems. Self-balancing (negative) feedback loops maintain the system in a stable but continually fluctuating state, whereas self-amplifying (positive) feedback loops may lead to new emergent structures” (p. 159).

An important structural note about the role of feedback loops in living systems is that they exhibit decentralized control. This means that there is not one particular locus of control over the entire being, and that the component structures are locally self-regulating and dependently interconnected. “Living systems consist of compartments and microdomains each with its own steady state; complexes of a few molecules can act as efficient cyclic molecular machines with no immediate reference to the steady state of the surroundings” (Ho & Popp as quoted in Stein & Varela, 1993, p. 188). Speaking about neural networks in this context, Varela, Thompson, and Rosch (1991) state, “In this approach, each component operates only in its local environment, so that there is no external agent that, as it were, turns the system’s axle” (p. 88). The achievement of the feedback loop is that it allows for independent self-regulation of a system while requiring limited resources.

A living system must continually take in energy and use this energy to enable its self-organizing mechanisms. These feedback mechanisms work as part of a self-organizing system that enables new forms of order despite increasing disorder entering the system. Energy is always conserved in these transactions, following the first law of thermodynamics. The second law states that entropy always increases. The amount of useful energy in a system is always decreasing through the dissipation of energy during processes into heat, friction, etc. (Capra, 2014, p. 33). This increase in entropy has often been seen as a move towards disorder. However, biological systems seem to defy this rule in that they create order out of this increasing disequilibrium.

> Whereas the physical world runs down according to the second law of thermodynamics such that useful energy continually degrades into heat or random molecular motion (technically referred to as entropy), the biological world seems capable of doing just the opposite in organization by a flow of energy and matter, (Ho & Popp as quoted in Stein & Varela, 1993, p. 188)

Living systems take in energy through the consumption and metabolism of matter, and through this process living systems are able to both maintain and increase their organization.

> In an open system, and especially in a living organism, not only is there entropy production owing to irreversible processes, but the organism feeds, to use an expression of Schrödinger's, from negative entropy, importing complex organic molecules, using their energy, and rendering back the simpler end products to the environment. (Bertalanffy, 1950, p. 26)

Schrödinger (1967) theorized that a living system maintains its orderliness by "sucking orderliness from its environment" (p. 79). Prigogine refined these earlier concepts through the acknowledgment of the intimate relationship between order and disorder – that you cannot create one without the other.

The second law of thermodynamics, when applied to living systems, is problematic. Instead of running down, they grow and flourish. But how are these systems able to "suck order" from the environment, break it down into disordered parts, and use it to maintain orderliness within their own boundaries? They are able to do this because living systems have the capacity for knowledge and information. They know what to do, and they tell the system what to do in the form of information. “The biological sciences have adopted a theoretical perspective derived from information theory, together with developments in modern genetics and evolutionary science. This perspective holds that all biological processes involve the transfer of information” (Marcos as referenced in Arp & Terzis, 2011, p. 60). Feedback loops convey information about the outcome of a process or activity to the source (Capra & Luisi, 2014, p. 90). The system subsequently uses its available energy, and its ability to direct energy to enable work that needs to be done (Simms, 2003, p.402). “The perspective is developing that biological cells can be viewed as highly sophisticated information-processing devices that can discern complex patterns of extracellular stimuli” (Wallaczek, 2000, pp. 5-6).

An open system is still “information-tight” (Capra & Luisi, 2014, p. 93). As Capra and Luisi (2014) state, “all information needed for a fly to be a fly is contained inside the fly” (p. 133). The information about the fly extends only to the fly’s boundaries, and not beyond into its even more complex environment. The fly must exist in and exchange with this immense complexity – it must use information to sort through all the noise and create order. Remarkably, “self-organizing systems can also translate the external noise of their environments into systemic gain” (Clarke as quoted in Clarke & Hansen, 2009, p. 47). Living systems take advantage of their environments. We know how to use them for our personal gain down to the cellular level. It is not enough for us to take in energy; we need knowledge in the form of information in order to use it. Information is not matter or energy (Marcos as referenced in Arp & Terzis, 2011, p. 62), and the causal relationships it brings about in living systems are not entirely clear. How a gene “knows” to replicate or a cell knows to metabolize are questions about biological information. V.V. Egorov (2010), in proposing a new law of thermodynamics to account for the behaviour of living systems, defines information as “a thermodynamic function of the state of a living system, reflecting its ability for purposeful action” (p. 210). Egorov further notes that increases in information to offset increasing entropy imply increasing complexification and evolution. This is supported by Alvaro Moreno and Kepa Ruiz-Mirazo, who refer to living systems as “informational, genetically instructed organizations” who use their informational qualities to “reproduce their basic functional dynamics, bringing about an unlimited variety of equivalent systems, of ways of expressing those dynamics that are not subject to any predetermined upper bound of organizational complexity” (Moreno & Ruiz-Mirazo as quoted in Arp & Terzis, 2011, p. 168).

Living systems use their pattern of organization to regulate their state through self-organization – relying on information to respond to feedback loops, regulating these energy, matter, and information exchanges, and managing entropy despite increasing complexification. While this pattern of organization is designed to enable a living system to maintain its state of dynamic homeostasis, it is also designed to create and nurture change.

Growth, Adaptation and Evolution

How is it that living systems continually become? How have they kept going for so long through drastic change and even death? They do this through a pattern of organization that allows for adaptation and growth over a lifetime, reproduction, and the perseverance of favorable traits in evolutionary processes. A fundamental feature of our current definition of life is that it reproduces itself. This trait is what allows for bacteria to be included among living systems, but not viruses. The former can recreate itself through its own internal dynamics, but the latter cannot (Capra & Luisi, 2014, p. 138). Living systems are able to do this because they are autopoietic, dissipative structures, existing far from equilibrium, that use increasing entropy to produce organization and novelty through reproduction. Living systems are unities of multiplicities that depend on several conditions and processes for their growth and evolution that will be discussed here, including their autopoietic nature, time, irreversibility, distance from equilibrium, bifurcation points, and emergence.

A living system must first be an autopoietic unity before it can reproduce. However, Maturana and Varela (1987) note that while autopoiesis is necessary for the reproduction of living things, a living thing is not required to reproduce in order to be seen as living (p. 78). A mule cannot reproduce itself and is still regarded as alive. You might even know other humans who choose not to have children and yet appear very much alive. Regardless of evolutionary change, all living beings remain autopoietic. “For autopoietic theory, living systems conserve their organization, which means their functioning always restores homeostasis; evolution is merely structural change against this identity horizon” (Protevi as quoted in Clarke & Hansen, 2009, p. 97). While we recognize that reproduction and evolution are things that only living systems do – we also see that life can happen, if only for a moment, without them. Any type of living system can become extinct. However, reproduction and evolution cannot happen without life.

Autopoietic units remain coupled to their environments; open systems with operational closure, intimately linked to an environment that exists and changes in time. As one changes, so does the other. Adaptability and evolvability are significant features of living systems that produce a historical network of living systems (Protevi as referenced in Clarke & Hansen, 2009, p. 97), allowing them to exist through time, continually recreating their autopoietic pattern of organization in new and unexpected ways.

Capra notes that time plays two key roles in the processes of living beings – aging and evolution (Capra & Luisi, 2014, p. 137). In each case, from multiple potential paths, one is chosen, and that choice affects all potential options thereafter. Bailly and Longo (2011) use the example of reptilian jaw formations evolving into inner ear components in birds and mammals: "It would have been impossible to predict: it was not a necessity . . . but just one possibility to be formed, a generic path out of many compatible ones" (p. 122). This change affects all subsequent changes. When discussing biological time, Bailly and Longo (2011) refer to it as "the time of 'genesis of structure,' of constitutive process, because a bifurcation, or a catastrophe, can depend on the entire history of a system, and not only its state description at a given instant" (p. 116). Bailly and Longo (2011) also focus on the idea that these changes are co-constructed between an organism and its environment, each affecting the other, continually altering the course of history for all the systems and subsystems involved. "There is no such thing as the time of a single isolated dynamical system displaying linearity in its bifurcations. No such system exists. The genesis of structures proceeds in parallel, through interaction of a plurality of structures" (Bailly & Longo, 2011, p. 117). The far-from-equilibrium system takes in energy and uses it to construct itself, but it does this with a history, and in the context of its dependence. As our open system operates with its environment, it develops its own history, which changes its potential futures.

The history of all systems involved, as well as each system's search for its own autonomy, individuation, and survival all play out in time – but it is not as simple as the arrow of time in thermodynamics. The breaking down is accompanied by the building up, complexifying, and becoming that which something never was before (Longo & Montevil, 2014, p. 16). Here, time does not destroy, but instead is a function of building and becoming – Prigogine's (1989) "constructive role of time" (p. 398). Structurally, living systems exist in space, but it is their organization through time that enables their state of living. The initial conditions of a system do not determine the trajectory of the system, because internal changes over time can widely differentiate that trajectory. There are a variety of answers to the non-linear equations that occur far-from-equilibrium (Prigogine, 1989, p. 399), and these answers come in the form of irreversible changes that affect what the next equation will be.

Science struggles with irreversible processes. While water can melt, and freeze and melt and freeze again, some changes cannot be undone. Toast cannot be untoasted, popcorn cannot be un-popped, and living systems cannot undo changes in their state. Thermodynamics deals with the materials and products of these changes, but in classical thermodynamics, irreversible change always comes with waste and loss (Capra, 1996, p.184). Prigogine helped to alter this conception when he showed that points of maximum entropy, or disorder, can actually lead to increases in order – beneficial adaptation and not loss (Prigogine, 1980, p. 78; see also Capra, 1996, p. 184). Living systems are able to go beyond maintenance and empower change during the process of taking in the low-entropy energy of organic molecules, using it to build and grow in increasingly complex ways, and releasing the resulting high entropy energy as waste. Ludovisi, Pandolfi and Taticchi (2005) describe it by saying that biological systems

> use the energy (low entropy energy) embodied in the solar radiation and chemical compounds to build highly ordered structures. The irreversibility of both the anabolic and catabolic processes results in a production of entropy, which is released towards the environment as high-entropy energy (heat). (p. 35)

You cannot undo the processes of living – hence the dilemma of aging. Everything will, at some point, reach maximum entropy and die (Schrödinger, 1967, p. 76), instead of changing and becoming.

But prior to that, the system will do its best to maintain its steady state in a disordered, ever changing world. It will do things that cannot be undone as it develops, adapts, and prepares to reproduce. It will change its sameness and become different. In doing this, a living being engages in symmetry-breaking; creating difference from repetition and spontaneous unity formation, and making seemingly random choices from mirrors of statistical possibility (Prigogine & Stengers, 1984, p. 163). Prigogine (1980) implies that progress in our understanding of irreversibility relies on viewing it as a "symmetry breaking mechanism" (p. 196). Through these irreversible processes, an organism develops its own path – its own particular being in the world. The more complexified as a system the being is, the more possibilities for novelty. To create more order, more disorder is required (Capra, 1996, p.189). Inviting disorder can sometimes lead to higher levels of order within a complex, dissipative structure such as a living system. "Nonequilibrium is the source of order, nonequilibrium brings 'order out of chaos'" (Prigogine & Stengers, 1984, p. 287). From this chaos, the most striking irreversible processes occur.

A living system is an open system existing in an organized, autonomous state far from equilibrium in a delicate balance with its environment – operationally closed and distinct, while still situated, dependent, and open to suggestions from the outside world. It is this unique organization in time that allows for indeterminate growth and evolution. The distance from equilibrium allows for adaptation, in that living beings are constantly changing in order to adapt at any point in time: cooling, heating, eating, making waste, and the infinite number of other processes responding to environmental perturbances. Generally, these feedback loops aim for self-regulation – negative feedback steering the ship. However, positive feedback can also be beneficial to an organism because it is generational in nature – it generates new structures and responses. Prigogine (1989) notes that disequilibrium is what allows living systems to engage with their environment productively and 'make choices' that are advantageous to their survival;

> Out of equilibrium the system can see the totality of the system. One could almost say that matter in equilibrium is blind, and out of equilibrium it starts to see. Hence there are events, fluctuations which prepare for an event, amplification, sensitivity to the external world, historical perspectives due to other successive forms of organization, and the appearance of a series of new categories of phenomena, called attractors. (Prigogine, 1989, p. 399)

Systems in a state of equilibrium, like a frictionless pendulum, will have a predictable trajectory in phase space that can be described as an attractor. Systems far from equilibrium can have a variety of attractors (Capra, 1996, p. 1320). The frictionless pendulum is a periodic attractor, and there are also point attractors and strange attractors. The name 'attractor' serves as a metaphor to describe the system's return to its original state. It cycles through various stages with an apparent attraction towards an end point. Some strange attractors are less stable than others with a variety of points (Prigogine, 1989, p. 399). Complex dynamic systems, such as living systems, have multiple attractors including strange attractors, but because they are open systems, they are particularly sensitive to fluctuations in, and perturbations from, their environment. "In many non-linear systems, however, small changes of certain parameters may produce dramatic changes in the basic characteristics of the phase portrait. Attractors may disappear or change into one another, or new attractors may suddenly appear" (Capra, 1996, p. 136). It is difficult to predict the significance of small perturbations on a system, and positive feedback loops can amplify effects over time. "The inherent amplification properties of nonlinear systems thus represent one critical aspect that defines the system's sensitivity and the magnitude of its response to external perturbations" (Walleczek, 2000, p. 6).

Sometimes, when dissipative structures, such as living systems, reach extreme distances from equilibrium, a bifurcation point, previously known as a catastrophe (Capra & Luisi, 2014, p. 85), is met. At these points, the system appears to spontaneously move into a new highly ordered state. The system is a mixture of stability and instability designed to adapt by altering structures that determine its function. At certain points in their trajectories these far-from-equilibrium systems can reach points of instability where they exhibit mass organizational change. "When the flow of energy and matter through them increases, they may go through new instabilities and transform themselves into new emergent structures of increased complexity" (Capra & Luisi, 2014, p. 159). The point at which new states of order emerge is called a bifurcation point. These points are sometimes, but not always, predictable, due to the combined influence of internal dynamics and environmental change.

The potential unpredictability of the reorganization allows for emergent properties. Emergence – the spontaneous move to a new highly order state (Walleczek, 2000, p. 2) – is a part of the regular functioning of an individual living system, allowing it to respond to perturbations both in new ways and to new perturbations. Emergence is a part of a living system's autopoietic organization and processes, resulting in evolutionary change and phenomenological differences despite possible genealogical similarities. "Most biological problems have a vast number of different solutions, and many of these solutions harbor the seeds of innovation to solve other problems” (Wagner, 2005, p. 175). Emergence is also a higher order function related to the network connections of living structures, in the sense that the higher order processes of an ant colony emerge from the individual intricate interactions of individual ants (Trainor as referenced in Stein & Varela, 1993, p. 305; see also Capra & Luisi, 2014, p. 133). Miller’s living systems theory uses the word emergence strictly in this sense; the emergence of increasing hierarchy (Bailey, 2005, p. 34), and this view has been useful in social system theory and ecology (Gunaratne, 2007, p. 88). This definition of emergence, and the one used in association with spontaneous structural change, come from the same basic principle:

> Emergent properties are the novel properties that arise when a higher level of complexity is reached by putting together components of lower complexity. The properties are novel in the sense that they are not present in the parts: they emerge from the specific relationships and interactions among the parts in the organized ensemble. (Capra & Luisi, 2014, p. 154)

Such relationships or interaction might be an adaptive reaction or complexification in one living being, or an ensemble of living beings doing the same to create something greater than themselves. Here the focus is on emergence as spontaneous structural change – definition of emergence in living systems commonly associated with the work of Prigogine (Capra & Luisi, 2014, p. 116).

Emergence in this sense is also what allows for the formation of novel genealogical attributes when spontaneous emergence or reorganization occurs at the point of reproduction. “External selection works through competition; internal selection through coordination” (Jantsch, 1974, p.

5). Kauffman (1991, p. 82) notes that a new understanding of evolution is developing, one that still honors Darwin's ideas, but includes hypothesis about the role of self-organization – the internal coordination and problem solving of the autopoietic living system – in evolutionary processes. Self-organization is what allows the living system to accommodate its increasing entropy, using increasing disorder from the environment to form new, more highly complex forms of order internally (Capra & Luisi, 2014, p. 159; Davies, Rieper & Tuszynski, 2013, p. 1). Here, complexity is seen as a property of systems that allows for their behaviour to move beyond a simple analysis of parts. While Darwinian selection may account for the success or extinction of a trait, the possibilities that emerge are a product of self-organization. It is the organization of the system that allows for the capacity to reproduce and evolve, and the potential evolutions. Prigogine and Stengers (1984) remark that the emergence of life and its early evolution was not dependent on Darwinian selection principles, so much as it relied on the development of self-organizing processes; "It seems reasonable to assume that some of the first stages moving toward life were associated with the formation of mechanisms capable of absorbing and transforming chemical energy, so as to push the system into 'far-from-equilibrium' conditions" (Prigogine & Stengers, 1984, p. 191). These far-from equilibrium conditions are then favored through evolution due to their adaptive capacities. "Thus natural selection may favour and sustain living systems 'at the edge of chaos,' because these may be the best able to coordinate complex and flexible behaviour, best able to adapt and evolve" (Capra, 1996, p. 204).

At a bifurcation point, the emergent order can sometimes be unpredictable. Positive, amplifying feedback loops can lead to mass changes from what first seemed a small perturbation. The emergence of unpredictable change and new order leads to diversity among living systems of similar type or origin. This reach out to novelty and diversity accounts for the variety of species and types we see living on our planet. Our ecology is naturally creative. “Ecosystems contain many more species than would be ‘necessary’ if biological efficiency alone were the organizing principle. This ‘over-creativity’ of nature emerges naturally . . . in which ‘mutations’ and ‘innovations’ occur stochastically and are integrated into the system” (Prigogine, 1980, p. 128). Typically we have regarded evolution as a slow process that happens gradually. Research on bifurcation points implies that emergent forms of spontaneous order can happen quite suddenly, and that, in fact, most living systems go through periods of very little change followed by periods of large scale, cascading changes (Williams, 1985, p. 266). Maturana and Varela (1987) say “since every autopoietic system is a unity of many interdependencies, when one dimension in the system is changed, the whole organism undergoes correlative changes in many dimensions at the same time” (p. 116).

The autopoietic pattern of organization of living systems and their existence in time at a point far from equilibrium allows for the generative processes of irreversibility and bifurcation that lead to large scale cascading changes. This is how living systems are able to invent emergent patterns and structures that never existed before. Not only do these unities recreate themselves, they evolve towards increasingly complex unities in the tradition of complexity theory in systems; “Creativity – the generation of new forms – is a key property of all living systems. And since emergence is an integral part of the dynamics of open systems, open systems develop and evolve. Life constantly reaches out into novelty” (Capra & Luisi, 2014, p. 161).

Conclusion

The biology of living systems is amazing. The idea that molecules can self-organize into the highly complexified unities we see now invites new perspectives and possibilities into how we understand what it means to be alive. It also invites new languages and new comparisons. Living is no longer a deterministic clockwork, but a process of becoming. Autopoietic patterns of organization, tumultuous environmental dependencies, historical choices, and unpredictable futures characterize this new process-oriented notion of living.

It is no surprise that this collection of traits has been noticed and observed in other contexts. The biological metaphor of complex self-organization has been used in a variety of ways in business and educational networks through the inference that social networks are the human equivalent of ant colonies or bee hives. I will explore this metaphor somewhat differently. I intend to propose its relevance beyond systems existing in our physical world to abstract conceptual systems, and disciplines in particular. Instead of picking and choosing language that applies literally, I will consider the entirety of systems theory in my metaphor as though math is alive. I do not ask whether a discipline shares some literal similarities with a living system, but what the implications are of speaking about it as though it is alive. What is implied by this pattern of organization? How does it form an autopoietic structure from abstract parts and maintain itself in relation to its environment? What do adaptation, growth, reproduction and evolution look like over time? This chapter has presented a general outline of the key components of living according to current living systems theorists. The terms used have been intended to be taken literally. In the next chapter they become metaphors. If we look at the target domain of mathematics through the source of this living systems framework, does it make sense?

Does the language of living systems lead to metaphors that are obvious and logical, not obscure, but fundamental? Do these metaphors have consistencies with prior metaphors? Do they invite new understandings? I now provide my interpretation of this metaphor as one potential interpretation and an invitation for discussion about living mathematics.

CHAPTER FOUR:

Making a Metaphor

Furthermore, if we consider the plane of consistency we note that the most disparate of things and signs moves upon it: a semiotic fragment rubs shoulders with a chemical interaction, an electron crashes into a language, a black hole captures a genetic message, a crystallization produces a passion, the wasp and the orchid cross a letter . . . There is no ’like‘ here, we are not saying ’like an electron,’ ’like an interaction,’ etc. The plane of consistency is the abolition of all metaphor; all that consists is Real.

(Deleuze & Guattari, 1987, p. 90)

Mathematics has been assigned a variety of metaphors through humankind’s efforts to notice its attributes and devise ways of interacting with it. It is a form in Platonism, a structure in Structuralism, a construction in Constructivist philosophies like Intuitionism, a set of rules and a game in Formalism, and a human condition in Social Constructivism and Embodied Mind theories. In all of these, mathematics is not its own maker. It has no independent self, no acknowledgement of its will to exist, and no needs. Orientationally it only goes one way along a predictable trajectory, and ontologically it only responds and never provokes. These metaphors define our thinking about this abstract entity in the absence of a tangible, finite, concrete existence. They contribute to the evolution of mathematics, and making conscious our metaphorical understanding of mathematics enables us to choose our metaphors more wisely, and to be aware of the mathematical and social implications of our choices. It is within our human abilities to be thoughtful about the metaphors we use, to choose and explore new metaphors, and most importantly to be aware that they are indeed metaphors, even if we often use them without thinking about this at all. I propose that exploring the metaphor “math is alive” is possible because our understanding of mathematics sufficiently correlates in experience with our understanding of living systems. In this way describing mathematics as a living system can be seen as making a “class inclusive assertion” (Glucksberg as quoted in Gibbs, 2008, p. 68) regarding correspondences in characteristics and similar behavior in their fundamental features.

Potential Mappings

“Math is alive” works well as a new metaphor because the majority of new metaphors are structural (Lakoff & Johnson, 1980, p. 152), comparing one entity to another, and here mathematics is compared to a living system in a linguistically structural way. Although it gives rise to other orientational and ontological metaphors, the foundation is comfortably structural. Because it has a similar form to metaphors we already use for mathematics and follows the underlying structural metaphor “mathematics is a thing” that all philosophies of mathematics embrace, be it an outside structure or a human condition, it is also a conventional metaphor as most new metaphors are (Lakoff & Johnson, 1999, p. 149). Just as Plato compares mathematics to a type of form, “math is alive” compares mathematics to a type of system. It is a structural metaphor that aligns with our current use of metaphors for mathematics, and this makes it a good candidate to become a new metaphor, because we can use it to describe mathematics with the linguistic assumptions that we are familiar with and accustomed to. We already use a variety of growth-based metaphors in our language around disciplines, as we talk about fields of study, branches and roots. This metaphor extends and elaborates metaphors already in use, while also existing in comparison with these other metaphors. It has the potential to become a primary or grounding metaphor, because it speaks to the fundamental existence of mathematics. As a potential primary metaphor, living mathematics implies a variety of complex metaphors based on the specific characteristics of a living system.

I intend to demonstrate that the complex metaphors generated by the metaphor “math is alive” can be understood and make sense given the language of mathematics philosophy and the language of living systems. I will explore what mathematics is composed of and how it corresponds to the pattern of organization of a living system. I will also consider how mathematics maintains itself, at a state far from equilibrium in constant interplay with its environment. Finally, I will propose that mathematics adapts and evolves as a being in time corresponding to the experience of a living system. Just as the seed looks nothing like the oak tree it has become, sometimes new math seems unrecognizable at first. By the end of this section I hope to have shown that this metaphor not only works, but works well as a way in which to speak about mathematics. If you consider mathematics as a living system, you see its behaviours as something different than they appear through current metaphorical lenses. You begin to ask questions about how it lives and grows, how it changes unexpectedly, and what our role is and what it is not. When we live life with this metaphor we invite it to become a part of our everyday experience, and a part of the experience of very small children who see us treat mathematics as though it is alive. Through Johnson’s theory of conflation (Lakoff & Johnson, 1999, p. 46) children may truly be able to experience it as a primary metaphor. When discussing large mathematical change, Lakoff and Núñez (2000) said, “How do you accomplish such a complete paradigm shift? How do you utterly change from one mode of thought to another, from one system of concepts to another? The answer is, via conceptual metaphor" (p. 309).

Thus, what I try to do in this chapter is to use the defining properties of living systems as they were introduced in chapter two and try to demonstrate how well these properties can fit as properties of the target domain of the metaphor "math is alive", i.e. the discipline of mathematics.

Pattern of Organization

Mathematics has a pattern of organization that is consistent with the pattern of organization of living systems. Some current and past philosophies have attempted to define this pattern as a building as in Structuralism, and in Constructivism including Intuitionism, or even in the idea of a form presented in Platonism. Other philosophies have used a game with a set of rules as the primary metaphor on which to base our mathematical conceptions. Some newer philosophies such as Social Constructivism and theories of Embodied Mind situate their metaphors in human being and interaction as tool-like entities. In fact, the fundamental features of mathematics' pattern of organization correspond well with that of a living being, and this metaphor allows mathematics to be both a being unto itself and a feature of the human mind, resolving dichotomies created by other philosophies. In this next section I intend to reflect on class-inclusive assertions that reflect correspondences in our experiences and understanding of mathematics and living systems wherein mathematics exhibits nested autonomy, is an open system with operational closure, exhibits extreme sensitivity to initial conditions, and is inherently unstable. Given that we are able to make sense of these mappings and that they have meaning, this metaphor becomes applicable and potentially useful, which is what I want to demonstrate in this chapter.

Mathematics is an autonomous unity. If it were not, we would not be able to separate it from other like and unlike entities. For example, it is clear to us that mathematics is not a chair or a banana or a boardroom meeting. Likewise, we are able to separate mathematics from items in its own group. We know that mathematics is not science, language arts or music, although this is a trickier distinction as attributes of mathematics show up in these other concepts. Music, for example, can be analyzed mathematically, but application of mathematics to music is not necessarily inherently a part of music. Mathematics is also distinct from other living systems such as cells, trees, and humans. The distinctions in all areas are created by the existence of a boundary constructed by the entity itself: “An important characteristic of living systems is that their autopoietic organization includes the creation of a boundary that specifies the domain of the network’s operations and defines the system as a unit” (Capra, 1996, p. 98-99). Mathematics defines what it is to be mathematics. Leslie White’s discussion of mathematical growth and change from an anthropological perspective provides support for mathematics remaining an autonomous system:

> The process of mathematical growth is, as we have pointed out, one of interaction of mathematical elements upon each other. The process requires, of course, a basis in the brains of [humans] just as a telephone conversation requires wires, receivers, transmitters, etc. But we do not need to take the brains of [humans] into account in an explanation of mathematical growth and invention any more than we have to take the

telephone wires into consideration when we wish to explain the conversation it carries. (White as quoted in Hersh, 2006, p. 313)

The boundary is constructed so that it can interact with the environment and with the contents of the living system. But what is the environment of this type of living system? With other disciplines and systematic ideas, mathematics exists in the conceptual domain, or what White refers to above as “the brains of men [humans]”. According to Capra, Varela and Maturana (Capra, 2002, p. 34), thinking is the domain of living. Not in a conscious, Cartesian sense, but in the sense that cognition is the process by which living beings continue to live and reproduce. Mathematics as a living being not only exhibits cognition as it creates its own living, but manifests itself in the space created by mass human cognition; it is cognition within cognition. The mass mind of humans is intended quite literally, and in this context referes only to human cognition (taking for granted at this time the cognition of other living things). Humans exhibit cognition, and many humans together “think together” as a functional of the history and culture of communication. As will be noted there are geographical and social regions of this mass mind that are local, such as different occupations, locations, or intensities. For the purpose of this metaphor, human cognition within each human and as a part of communication between humans will be presented as a potential area or landscape environment.

There appears to be some similarity here between this abstract unity in the conceptual space I am describing and the heavenly realm of Platonic forms. A key difference between Platonic conceptions of a distinct form, untouchable in an abstract realm, and a living system is that this living form interacts with and defines itself in relation to its human cognitive environment. It designs its boundary to create the potential for selective exchanges and combined actions.

Mathematics is not separated from humanity as a distinct form or as a system unto itself as in Logicism or Formalism, nor is it solely a product or action of humans as in Social Constructive and Embodied theories. The metaphor of a living system allows mathematics to be distinct while still existing within an interactive space created by and modified by human cognition, consciously and unconsciously through cognitive and communicative acts. It seems clear to me that we do not have absolute control over what mathematics is, and that the system of mathematics itself determines its fundamental form and being in relation to its environment.

As a nested system, mathematics interacts with a variety of other living systems, assigning it a new, active ontological metaphor. In its environment, it exists within a system of subject matters, but also within a system of neurons and hardwiring. In this way, it differs from Luhmann's view of social systems that emphasizes a systems-within-systems approach (Luhmann, 2013; Mingers, 1992, p. 231). This is more consistent with Maturana and Varela's (1987) view of nesting and Capra and Luisi's (2014) use of the term "interlocked", which is non-hierarchical and allows for more metaphorical inclusion and possibility: "The interaction between the living organism and the environment is a dynamic one based on co-emergence, where the living organism and the environment become one through cognitive interactions" (Capra and Luisi, 2014, p. 143). Mathematics, in its environment of the mass human mind, interacts with its environment through its boundary mechanism, and through this is also able to interact with other living systems, and even physical objects in the environment. When we measure the couch, mathematics interacts with us and with it, and, importantly from this living system view, all those involved and alive will learn from this experience. The couch will learn nothing and eventually decay, but more on that later.

For mathematics to be an open system with operational closure, it needs to exchange matter and energy with its environment. It would be different from a closed system that follows a predictable trajectory and, in Prigogine's (1980) view, is only able to exchange energy with its environment, not matter or information (p.78). Platonic mathematics is a closed system. Everything is already in place, and we could know its inevitability if only we were wise enough to see it. It is separate from matter, and the stuff of the world does not interact with or change it (Gowers as referenced in Hersh, 2006, p. 183; Jacquette, 2006, p. 238). Structuralism, too, is limited to abstractions that can be built, and Formalism allows mathematics only to grow from itself, not from its interactions with others or the world. Living mathematics, on the other hand, metaphorically uses matter and energy in its human mass mind environment, as it would not exist without its applications to our perception of the physical world. Without a world for mathematics to interact with, it is unlikely it would have ever been noticed.

Next I ask what it means for mathematics to exhibit extreme sensitivity to initial conditions. Mathematics came to be in the environment of human minds. It is limited by the characteristics of this space, including human physiology and its relation to human perception and human need. Basic neural processes of subitizing and categorizing characterize early mathematics. Lakoff and Núñez (2000) note that "Very basic arithmetic uses at least the following capacities; subitizing, perception of simple arithmetic relationships, the ability to estimate numerosity with close approximation (for bigger arrays), and the ability to use symbols, calculate, and memorize short tables" (p. 26). All human cultures appear to have mathematics. In a living system view, all cultures appear to interact with mathematics. Yet different geographical areas of the mass mind begin to differentiate the part of mathematics in their particular environment, and mathematics does not develop the same as cultural factors emerge, highlighting some abilities and downplaying others, while ascribing certain metaphors that lead mathematics in one way or another. Unlike the inevitability of Intuitionism, living mathematics is not limited and defined by the logic of the mind, but is instead influenced and nurtured by it.

Barton (2009), who has explored linguistic cultural variations in the development of mathematics, says that,

> if mathematics had developed through a language where the path metaphor was dominant, then the mathematics that would have emerged would have been dominated by function, or some equivalent concept, rather than by sets. We have every reason to believe that this mathematics would have been just as powerful, and just as widely and effectively applicable: but it would have been different. (Barton, 2008 p. 94)

In Núñez's (2011) extensive research into number line he concludes there is "no innate number line in the human brain" – that this is a cultural feature of mathematics reflecting a particular orientational metaphor that is not seen in all cultures although it may have come from some similar initial noticing about number (p. 654). As a living system, even the smallest difference in initial conditions can lead to drastically different forms of mathematics, or parts of the mathematical living system. Barton (2008) has also asked "Would mathematics have developed differently if it had developed through languages in which numbers were verbal?" (p. 6). The anthropological study of mathematics says yes. Although the Intuitionists may have been correct that math itself is not a language, they were wrong to assume it is entirely separate from language and cultural expression. Parts of mathematics can develop in vastly different ways, even starting from very similar beginnings. This is also true of mathematics in its application. Richard Noss (2002), in his studies of epistemologies in mathematics, suggests that there are actually a variety of types of mathematics situated in various human actions that can operate as an identifiable part of the whole system while performing their own functions. “Abstractions constructed within concrete situations may compensate for their lack of universality by their gain in expressiveness” (Noss, 2002, p. 12). For example, engineers use and develop certain types of mathematics that are different from the types of mathematics used and developed by accountants. Although these parts of mathematics have similar initial conditions, the slightest deviation, such as with geographical location or occupation, can have a large-scale effect on how that part of the living being grows and becomes.

As well as being sensitive to initial conditions, mathematics as a living system is also inherently unstable, leading to large and unpredictable changes through time. The environment of human consciousness provides space for mathematics to interact with other concept systems and undergo change. Leslie White points out that “Ideas, like other cultural traits, interact with each other forming new syntheses and combinations. Two or three ideas coming together may form a new concept or synthesis” (White as quoted in Hersh, 2006, p. 312). White notes that human beings are not in control of this change; the change comes from within mathematics itself (White as referenced in Hersh, 2006, p. 307), again ascribing it a metaphorical ontology of power it has never had before. The system of mathematics reaches critically unstable points that bring forth this change, and this instability is caused by the metaphorical collision of concepts across the human mass mind as the environment of mathematics. It is mathematics’ pattern of organization that is designed to interact with the environment while defining itself that enables this change.

Mathematics has a pattern of organization consistent with a living system. It is autonomous because it is consistently identifiable as a certain type of thing across all areas of the mass mind and it maintains a boundary between it and the other. Mathematics is an open system with operational closure because all the information about what it is comes from mathematics itself, yet it relies on the environment of the human mass mind to survive. It exhibits extreme sensitivity to initial conditions, because same ideas can develop differently in different contexts, creating vastly different areas within mathematics. And finally, mathematics is unstable because it never reaches a state of homeostasis, but is always growing, changing, and complexifying as a result of its search for balance.

Maintenance

For a living system to be an apt metaphor for mathematics, mathematics must be engaged in the process of maintaining its boundaried, identifiable living pattern of organization despite environmental fluctuations, and in the face of entropy accumulation. The pattern of organization of mathematics allows for it to maintain itself as a living being, but the process by which this occurs is self-organization. To be a living being, mathematics must be autopoietic, which means that its sole purpose is to maintain itself through inner processes of self-regulation controlled by the system (Maturana & Varela, 1987, p. 43). Autopoietic living systems self-organize and are challenged to do this even in far from equilibrium states (Prigogine, 1989, p. 399). The fundamental characteristics of the process of self-organization can be seen as characteristics of mathematics and the way it behaves. I will now consider how the maintenance properties of living systems including matter and energy exchange, information use, the use of feedback loops with decentralized control, and the constructive use of entropy can be seen metaphorically in mathematics in significant and fundamental ways that enable the potential of a living systems metaphor.

Capra's (1996) first characteristic of self-organization is creativity (p. 85). In order to regulate and maintain themselves at far from equilibrium states and in unanticipated conditions, living systems must be able to produce novel responses and make adjustments to their very way of being. Again, we see new ontological metaphors entering the dialogue, as mathematics is metaphorically enabled to do things it has never done before in an ongoing process of self-determination. The process of self-organization allows living systems to find this state of dynamic balance. Mathematics changes, but it also is remarkably able to stay consistent even when faced with enormous pressure. What makes mathematics be mathematics never changes. The point of self-organization is to stay as much the same as possible in order to create the potential for beneficial growth and gain, even if the world around changes and challenges the survival of the system. Mathematics cannot create greater mathematics unless it is first able to maintain a consistent unity. Mathematics has mastered the ability to create the perception that it is a unity, and that it is the same thing it has always been, even when it is quite different. This accounts for the dominance of philosophical views of mathematics as a logically predetermined entity. This is due to its processes of self-organization, an open system able to constantly make adjustments in response to changes in the environment and inviting the possibility of difference through the dynamic nature of its quest for stasis. "Networks on the boundary between order and chaos may have the flexibility to adapt rapidly and successfully through the accumulation of useful variations. In such poised systems, most mutations have small consequences because of the systems' homeostatic nature" (Kauffmann as quoted in Capra, 1996, p. 205). The first trick to being alive is simply maintaining a state of ongoing unity.

To facilitate maintenance, a living system exchanges matter and energy with the environment. This is not to say that mathematics as a living system interacts with the matter of reality in the sensory way that humans do. Mathematics interacts with the reality that exists within mass human consciousness, as humans work together to define and relate perceptions and concepts, and the metaphorical matter and energy present in that environment. In this ontological metaphor, the matter is in mathematics' taking in and breaking down of physical reality. Our thinking provides energy. When we interact with mathematics both consciously and unconsciously, we power it by providing the ideas, questions, and preferences we have.

Yet the information for what it is to be mathematics and create mathematics is entirely contained within mathematics itself, like it is the case for Capra and Luisi's (2014) fly, wherein "all information needed for a fly to be a fly is contained inside the fly" (p. 133). Humans do not control what mathematics is, which is why Fallibilist theories break down. If we did, we could consciously make it anything we wanted it to be. Instead, all the information for what mathematics is, is contained by and determined by mathematics itself. It will be greatly affected by its dynamic interplay with the environment, but it will respond in its own way. "Mathematical realities thus have an existence independent of the individual mind, but are wholly dependent on the mind of the species" (White as quoted in Hersh, 2006, p. 307).

Mathematics is non-linear because its future cannot be predicted with any certainty. Previous orientational metaphors for mathematics have seen it proceeding only in one direction – forwards or upwards, but a living systems metaphor allows it to move and grow in multiple directions at once. A living system relies on information gathered by feedback mechanisms in its system to function within its environment, and these mechanisms operate in multiplicity with decentralized control. Feedback loops are mechanisms of information exchange within the system that tell it how to behave. Feedback loops in living systems operate on a small, local level, not requiring all the resources of the whole (Ho & Popp as referenced in Stein & Varela, 1993, p. 188; Varela, Thompson, & Rosch, 1991, p. 88). In mathematics, feedback loops also occur across multiple branches and applications including various cultures and occupations, and these parts of mathematics operate locally to maintain their relationship to the mass mind environment. In different areas of the mass mind, both in the mind itself and across its global landscape, mathematics proceeds in a particular way that does not always relate to any other part of mathematics, and is not controlled by an outside force. The role of feedback is to absorb cues from the environment and help to regulate exchanges in a way that preserves balance. Negative feedback is self-regulating (Capra, 1996, p. 59) and this occurs metaphorically in mathematics when concepts are reproduced the same over and over again with the goal of attaining the same result. Negative feedback loops generate the predictability and repetition of mathematical concepts and ideas. When you give mathematics seven and three and it gives back ten, negative feedback occurs keeping mathematics the same and defining its boundaries and self. Alternatively, positive feedback loops amplify perturbations to the system instead of balancing them, and can lead to sudden change that is not always predictable (Capra & Luisi, 2014, p. 159). The system is forced to create new information in order to maintain the information it already has. In mathematics, we see this as new branches of mathematics form and new applications arise in occupational or cultural usage to address changes in the environment of our mass mind. While negative feedback maintains mathematics by creating consistency and reliability, positive feedback allows it to respond to the environment by changing in order to remain consistent and reliable. As it changes, it complexifies in a systems view, becoming less predictable, and new ways of behaving become possible. Its trajectory is again changed, and it continues on with its particular non-linear way of being. Although feedback loops operate in local, decentralized ways, restructuring in any part of the system can set off a cascading effect on the whole and lead to even greater complexification, as mathematics struggles to remain a unity and keep on living.

Mathematics self-organizes because it has to take in matter and energy in order to regenerate and maintain itself in a changing and sometimes far-from-equilibrium mass mind environment. Collective human cognition can be seen as being in a far from equilibrium state when it is its most disordered or most conflicted, such as in the moments before a paradigm shift or an innovation. Despite changes in how or what humans think, mathematics has to try to stay recognizably the same. As mathematics engages in the processes of breaking down and organizing matter and energy it creates entropy. According to the second law of thermodynamics, the amount of useful energy in a system decreases as it dissipates through the systems processes and entropy increases, pushing the system to greater states of disorder. This is why closed systems slowly degrade and break down over time. As mentioned in chapter three, living systems are instead able to subvert the second law of thermodynamics by taking in complex organic molecules and low-entropy energy, extracting what is needed for the system, and excreting the unused components as simpler waste products and high-entropy energy in the form of heat. In this way, they gain available energy and complexify without the addition of disorder. Returning to Schrödinger's (1967) language, living systems "suck order from the environment" (p. 79) while avoiding the chaos that accompanies it. Is this not exactly what mathematics does? Mathematics is able to take in our reality through its interaction with the environment of our minds as a highly complex system, create order from this noise, and avoid being damaged in the process. It even leaves behind metaphorical waste matter: the mathematics textbook is not mathematics, nor is the spreadsheet or the blueprint or even thinking about an equation. This is where the simplified broken down products of mathematizing something are recorded. Like living beings, the environment can use the waste created. Our mass human mind and any other systems within it utilize the products of mathematics. Mathematics is an open system, allowing it to exchange with the environment to attain and construct the materials for its own regeneration. It should degrade from the entropy created in these processes, but instead it manages entropy production effectively, and even invites the creative potential of its turmoil.

Mathematics maintains itself, even in far-from-equilibrium states resulting from disorder and conflict in the human mass mind, because it behaves like a living system. When the maintenance language of living systems is applied to mathematics metaphorically, the resulting complex structural, orientational and ontological metaphors both make sense and invite new ways of imagining what mathematics is. Mathematics engages in the exchange of matter and energy with the environment as it takes in the energy of our thinking together, breaks down the matter of our mass human minds thinking and perceiving together, and uses the products to recreate its self. Mathematics is “information tight” (Capra & Luisi, 2014, p. 93) because it uses the information in its system to recreate itself and only the information in its system. Humans do not make mathematics. Mathematics makes itself. Mathematics has independent feedback loops operating with decentralized control as different areas of mathematics work in parallel to regulate mathematics and keep it consistent. Amplified positive feedback in one area can affect the whole, as increased disorder invites monumental change in order for mathematics to remain a unity. Mathematics defies entropy by “creating order out of chaos” (Prigogine & Stengers, 1984, p. 287). The most fundamental attribute of mathematics is that it seeks to break down and create order and consistency, but by doing so particularities and inconsistencies are often a by-product of the process. Mathematics, like a living system, is able to use this entropy to invite what was never possible or imaginable before. The complex metaphors contained in this dialogue on the maintenance of living systems correspond with our experience of mathematics deeply enough to assert the primacy of the metaphor “math is alive”.

Growth, Adaptation and Evolution

Living systems are designed to reproduce their own components, and this is the process that we recognize as living. However, the reproduction of self does not necessarily constitute reproducing another living entity of the same type. As Maturana and Varela (1987) have noted, living beings are alive even when they are not reproducing and may never reproduce (pp. 57-58). It is not because I have children that I am alive, although it is because I am alive that I have children. “It follows that the proper evaluation of the phenomenology of living systems, including reproduction and evolution, requires their proper evaluation as autopoietic unities” (Maturana & Varela, 1980, p. 96). It is the living being’s pattern of organization allowing for self-organization in an unpredictable environment that enables it to be alive, although it is a unique feature of living beings that they tend to reproduce new living beings of the same type, allowing for mass reorganization and evolutionary change. Metaphorically I ask how the living system’s processes of growth, adaptation, and evolution correspond to fundamental features of mathematics. This entails examining the characteristics of these processes shown in living systems, including their autopoietic nature, time, irreversibility, distance from equilibrium, bifurcation points, and emergence.

The only thing that mathematics makes is mathematics, and sometimes even more mathematics. Mathematics as a dynamic living system is complex, and as choices are made and feedback loops form to regulate and replicate the mathematics that exists, mathematics develops a history. This history affects all subsequent mathematics. The autopoietic nature of mathematics is what allows for it to change, even though autopoiesis is designed to preserve the living system. The system is in a state of constantly trying to reach an impossible homeostasis – hence the appearance of Platonism. It is important that there is a sense of what mathematics is and that this is a predictable and reliable being. The rose stays a rose and the fish stays a fish. Speaking about cells, Capra and Luisi (2014) say, “There are very many transformations taking place; however there is cellular self-maintenance – the fact that the cell maintains its individuality” (p. 130). But this process of trying to stay what one is is the very process that enables the being to live and grow and become different and unique. It is because mathematics must live in an environment that forces it to adapt and challenges its processes of self-regulation that it is forced to adjust and become, and begin to be drastically different than it was, but always in relation to what it has been. Through autopoiesis the seed becomes a tree, and we are amazed at how intricate it has become, but only sometimes surprised from day to day, when something emerges that seems to have never been there before, but is somehow able to burst into existence. Through mathematics’ interplay with its human mass mind environment it is forced to change in order to stay the same. Mathematics as a living being creates only itself and is therefore autopoietic, which is the fundamental factor in living beings enabling growth and adaptation.

> In fact, autopoiesis implies the subordination of all change in the autopoietic system to the maintenance of its autopoietic organization, and since this organization defines

> it as a unity, it implies total subordination of the phenomenology of the system to the maintenance of its unity. (Maturana & Varela, 1980, p. 97)

A living system becomes in every moment from what it is, yet it is never limited to a single trajectory. Capra (1996) says that “The behaviour of a dissipative structure far from equilibrium no longer follows any universal law but is unique to the system” (p. 182). This self-determined becoming is a powerful new ontological metaphor. Mathematics is not a closed system that degrades and breaks down over time, reaching maximum entropy and falling apart. Mathematics, like a living system, gains through time, taking in matter and energy and creating and recreating itself with increasing complexity and increasing potential for unexpected and monumental change. Mathematics is greater than it was when it began and has developed into an extremely complex, particular, and dynamic living being. As well, time cannot be undone. All adaptations of a system over time affect the future potential adaptations, so as mathematics complexifies and grows over time, the potential for future adaptations is altered. The response of the system is not the same, no matter what time it occurs at. Despite coming only from its own past, mathematics, like a living system, does not always respond predictably. The Intuitionists were astute to notice the unraveling of mathematics in time, but not astute enough to realize that the outcome was not an eventuality but a possibility. Living systems such as mathematics can change in different ways, responding differently to different perturbations over time, like Capra’s scenario where kicking a rock is compared to kicking a dog (Capra & Luisi, 2014, p. 136). The dog exhibits a certain reaction from an infinite variety of possible responses and this affects every response to follow, with a mixture of expected and unexpected results. Mathematics, like the dog, learns and is changed forever in ways we cannot predict with certainty, especially as time goes on. The rock does not learn or change, it only decays predictably until it is gone, just as the couch does in my previous example.

All systems eventually reach maximum entropy and die. The difference between closed and open systems is that closed systems have an absolutely predictable trajectory through time, while open systems regenerate and improve prior to decaying, and have the potential for multiple possible trajectories. Yet from all potentials, one trajectory is followed and is irreversible. Irreversibility is the symmetry breaking process by which a change cannot be undone (Prigogine, 1980, p. 196). This is how the history of a living system in time develops. “Living structure . . . is always a record of previous development” (Capra, 1996, p. 191). Mathematics has a history. It has a way of being that has developed through time and that cannot be undone.

> History of mathematics is an explanation, which gives us a picture that takes into account not only the final result of one mathematical theory, but its origin and development too. And that, I believe, lets us understand better, how mathematics was born and grew. (del Palacio as quoted in Hersh, 2006, p. 240).

Although mathematics itself is orientationally diverse in the “math is alive” metaphor, the arrow of time only goes one way. Changes, like those seen in the philosophy of mathematics as paradigms shift and collide, will not go away, and even in rejecting Platonism, we acknowledge that it has led to our conceptions today, and we metaphorically cling to it even now.

Living systems are able to facilitate these irreversible processes of gain (and not loss) by using entropy as a force of creativity instead of disarray. When the energy of our thinking becomes disordered or inconsistent, mathematics finds new ways to break it down and continue to release the high-entropy energy of our complexified thinking and the waste of our broken down mathematical by-products through our documentation, artifacts, applications, and ideas. Mathematics is enabled through the matter and energy it consumes to facilitate irreversible processes that enable complexity, creativity, and unpredictability.

As mathematics grows, complexifies and plods along its non-linear path, parts of the system may reach great distances from equilibrium as a result of irreversible processes pushing it in unforeseen directions. It is at these great distances that the greatest change is possible. Mathematics grows by responding with increasing complexity as it maintains itself dynamically and encounters novel situations. At times, mathematics is unable to process what the environment thrusts upon it. Positive feedback loops amplify, entropy spirals out of control and the system becomes chaotic. Constructive waste needs to be made. This happens in mathematics when it meets its greatest challenges; when it is forced to adapt or die due to changes in the environment. An engineer needs to build a building, a mathematician needs to create what seems to be an impossible proof, a politician needs to balance what seems to be an impossible budget, or a child needs to fit what seems to be an impossibly large number of toys in a bin. Because mathematics is a living system, it has an array of attractors, including strange attractors. An attractor is a tendency towards a certain state (Kauffman, 1991, p. 80). Mathematics must tend toward multiple states at any given time as its vast array of types and applications demand it in different forms. Metaphorically the attractors in mathematics are its particular tendencies in different contexts. When a system has multiple attractors like a living system and like mathematics it is more sensitive to changes in the environment as a minor fluctuation disturbs its entire phase portrait. Mathematics behaves like a living system in its ability to adapt when threatened with chaos, making significant changes in short periods of time. When mathematics struggles to take in its world, significant adaptation is possible.

As in living systems, the workings of the system (mathematics) over time cause it to become increasingly disordered as more ideas interact and create change and growth. Instability shows itself as contradiction and inconsistency in mathematics and the system seeks to undergo mass reconstruction. There have been many critical points in the history of mathematics, and there remain unsolvable problems and vague metaphorical assertions that keep mathematics from reaching stasis and make it dynamic and unstable like a living being. It is in these moments of extreme disequilibrium that a system is able to reorganize the most, allowing for growth and complexification. “The striking emergence of new structures and new forms of behaviour, which is the hallmark of self-organization, occurs only when the system is far from equilibrium” (Capra, 1996, p. 85). When everything is the most out of place, a vastly new order is possible. This is called a bifurcation point in a living system, and mathematics exhibits metaphorical bifurcations. These are times when part of the system undergoes a massive reorganization, and these changes can sometimes affect the entire system. Here, mathematics is challenged to adapt and complexify and the result is not always predictable and can never be undone. We do not know what will happen in the future as we continue to perturb mathematics, but the history of mathematics shows that it only grows and becomes more robust from this process.

Mathematics must be alive, because its limits are changeable and complexity increases, based on its relation to its environment, but beyond the environment’s control. Like a living system, it is enabled through its self-organizing, autopoietic processes to make sudden adaptations at bifurcation points in order to self-regulate and continue existing as a distinct entity, shown in the vast array of mathematical applications, cultural nuances, occupations, and sub-disciplines that exist. Hersh (1997) reminds us that mathematics encountered several crises in the nineteenth century leading to a questioning of geometry that led to the creation of new parts of math including infinite sets (p. 137). An example of mass change in the twentieth century occurred when Turing tried to solve a code, and no one at the time would have predicted the technological world that was to follow. Mathematics grew and changed and adapted both with its own limits and that of its mind environment, and over time the development of computing has drastically changed the trajectory of mathematics beyond the perturbation of one small part of the system for a single purpose (van Leeuwen & Wierdermann, 2000, p. 1139). There are many such examples spanning the history of mathematics. They are happening at this exact moment in some area of the discipline. Mathematics is alive, and because of this, a small theoretical or philosophical change can set off positive, amplifying feedback loops that can result in massive changes to the pattern of organization of the system while it preserves itself by continuing to assert its boundary and determine its own future.

When a significant change happens, and something new is created within a living system, this process is called emergence. “This spontaneous emergence of order at critical points of instability – often referred to simply as “emergence” – has been recognized as one of the hallmarks of life” (Capra & Luisi, 2014, p. 116). When a new type of mathematics springs into being, it emerges from the internal dynamics of the system. Mathematics makes more of itself differently, and often quite suddenly. Emergent change in living systems is often seen at the point of reproduction and is perceived as part of the process of evolution towards increasing complexity and creativity “The theory of autopoiesis shows that creativity – the generation of configurations that are constantly new – is a key property of all living systems. A special form of this creativity is the generation of diversity through reproduction” (Capra, 1996, p. 221). Living beings do not have to reproduce or evolve to be alive, but these processes are only done by living beings.

Mathematics is a living being that grows and adapts irreversibly in time, but is it the evolution of something else? Will it reproduce? What does evolving look like metaphorically for mathematics? It seems to be involved in this process with us. Unlike a biological living being, mathematics exists in an environment that is always on the move as human minds die and new minds are born. As such, mathematics constantly has to reproduce itself, but it does so bit by bit by bit. It takes a generation before mathematics has completed its reproduction, being entirely different than it was, and even then, it is still reproducing anew. As we use the products of mathematics and provide the necessities for it to grow, we keep it alive and aid in its reproduction by passing on this relationship to new minds as they become a part of our mass human mind and cultural cognition.

> The communication of ideas from person to person, the transmission of concept from one generation to another, placed in the minds of men (i.e. stimulated by their nervous systems) ideas which through interaction formed new synthesis which were passed on in turn to others" (White as quoted in Hersh, 2006, p. 338).

Mathematics stays alive as a single unity throughout its reproduction and evolution, and it is only on close historical and structural inspection that you realize you are looking at the offspring of a mathematics from long ago. A series of bifurcations, sometimes in the context of the current mass mind but often at the point of reproduction when new minds enter or gain access, allow mathematics to restructure, renew, complexify, and develop emergent properties never seen before yet clearly within the boundary of mathematics. "Ongoing structural change of living beings with conservation of their autopoeisis is occurring at every moment, continuously in many ways at the same time. It is the throbbing of all life" (Maturana & Varela, 1978, p. 100).

Mathematics grows, adapts, and evolves in ways that are remarkably similar to living systems. As an autopoietic entity, it maintains itself in a state of dynamic balance where its ability to adapt to environmental change invites creative responses. Like a living system, it exists in time and is subject to increased entropy but is somehow able to use its own far from equilibrium states to create new forms of increasingly complex order. The changes that have happened to mathematics over the years cannot be undone. Attempts to go back will never succeed because mathematics changes irreversibly as time goes on. The future of the system is both entirely dependent on its past and completely unpredictable at the same time. It embraces the arrow of time metaphor first seen in Intuitionism, yet invites new multiplicities of orientations through increasing complexity. Although every change limits future changes, it also enables possibilities that never existed before. Finally, through the ontological metaphor "math makes all/more/new math" mathematics reproduces and evolves in ways that are similar to a living system. Through its continual becoming in the minds of humans mathematics is able to make more significant changes as feedback loops are amplified when impossibilities are confronted, attractors change as the needs and abilities of humans shift, and bifurcations are reached when massive restructuring suddenly occurs.

Conclusion

Mathematics has significant parallels with living systems. It is possible to make statements about living systems that are also true for mathematics because it is possible to imagine mathematics this way and it makes sense. In fact, the correlations and relations are so evident and extensive that the idea that mathematics is alive could constitute a primary or grounding metaphor. From this fundamental relationship, a multitude of complex, contextual metaphors are possible, including structural, orientational and ontological metaphors. With the early introduction of this metaphor, a language can develop that changes the essence of mathematics in our mind from an early age when conflation based relationships may still be creating our reality. When the metaphor “math is alive” is used, it is implied that an experience of living systems corresponds with an experience of mathematics, and the characteristics of the process of living become potential metaphors. An exploration of these correspondences reveals that this metaphor does indeed provide a wealth of reasonable assertions about the nature of mathematics. First, mathematics has a pattern of organization consistent with the pattern of organization of a living system. Second, living systems must maintain themselves to avoid decay and flourish in their environment. Mathematics does this in the same way that living systems do. It takes in matter from the physical reality perceived and understood by human minds and breaks it down, putting it in mathematical order, and giving us back simple unifying consistencies of ideas jotted down in notebooks, labeled on designs or relayed in a conversation; exchanging what is waste to it with the environment that then often uses that waste for its own gain. Finally, mathematics can be seen as a living system when we consider its capacity to grow, adapt, and evolve. The pattern of organization of mathematics is designed to preserve its self through ongoing processes within the system and exchanges with the environment, but it is also designed to enable adaptation and change when it benefits the living system in its relationship with its environment.

> A living dissipative structure, such as an organism, needs a continual flow of air, water, and food from the environment through the system in order to stay alive and maintain its order. The vast network of metabolic processes keeps the system in a state far from equilibrium and, through its inherent feedback loops, given rise to bifurcations and thus to development and evolution. (Capra, 1996, p. 172)

Because mathematics makes itself, only it can make itself different. This often happens in living systems at the point of reproduction, and mathematics reproduces itself through generations of humans, gradually becoming completely different. Mathematics is also enabled to grow, adapt, and evolve because it has emergent properties. From the most challenging perturbations, the most creative and unique features of mathematics are able to emerge.

Mathematics currently relies on a variety of metaphors from games to forms to constructions, and the living systems metaphor is just as reasonable when an examination of the characteristics of these systems are compared with the characteristics of mathematics. Inviting the metaphor "math is alive" into the conversation is not only apt, but necessary as it highlights the limits of our current language and invites new perspectives on how we interact with it.

CHAPTER FIVE:

School Mathematics and Metaphor

"Words are not tools, but we give children language, pens, and notebooks as we give workers shovels and pickaxes."

(Deleuze & Guattari, 1987, p. 97)

The question of what mathematics is has been answered in a variety of ways in the philosophy of mathematics, with almost all of the conceptual views developed throughout recent history still playing a role. Formalist views, both objectivist and non-objectivist still abound, often standing in contrast to Humanist views. Structuralism is still often found, as one can occasionally find an Intuitionist position. However, in the public view, including the perspective taken in school mathematics, strict Platonism is still the most prominent philosophy of mathematics. I believe that current metaphors in mathematics education "highlight, downplay, and hide" (Lakoff & Johnson, 1980, p. 152) a variety of aspects of mathematics to and from students, and in fact, they do more to hide the potential of mathematics from students than to highlight or allow for new inferences to be built. "The system [of fundamental metaphors] will tend to make experiences and facts consistent with it noticeable and important, and experiences and facts inconsistent with it invisible" (Lakoff as quoted in Gibbs, 2008, p. 34).

I will first consider whether school mathematics is separate or not from what we consider to be academic mathematics (what mathematicians do) and how it is conceived of. Do philosophies of school mathematics reflect philosophies of academic mathematics, and how are they similar or different? Next, I will turn to metaphors used for mathematics in schools, returning to the idea that metaphors are fundamental to our understanding of abstract concepts, and using the understanding of what mathematics is or could be that was developed in chapter two to clarify the potential meaning of the metaphors being used and ground them in the tradition to which they refer. This includes a brief questioning of the way we understand school subjects in general, and an analysis of a selection of the research done into the metaphors used in school to describe mathematics by teachers and students. Educators state their philosophical beliefs about mathematics as a concept without even intending to through their regular use of metaphor in mathematical experiences. The language teachers choose perpetuates certain metaphors and the philosophies they draw on.

I can then ask in the second part of this chapter whether or not the metaphors we use are our best choices, or if a new metaphor might change the way we interact with mathematics in schools and enable students to realize their potential and find themselves in the mathematics that they live. I believe that I can show that a living systems metaphor would be more appropriate and that this metaphor can open up new and novel possibilities for thinking about mathematics at school.

School Mathematics as a Localized Area of Mathematics

Reuben Hersh, in his 1991 paper "Mathematics has a Front and a Back", discusses the difference between the public/school view of mathematics and the real work of mathematics using the metaphor of a front and back as there would be in a restaurant. "In general, the front is the region in which the public is admitted, where service is performed; the back is a region restricted to professionals, where preparations are made to provide a service" (Hersh, 1991, p. 127). The front does not have access to the back, where the decisions that affect them are being made, and the back only cares that the front is happy and able to make good decisions. Hersh (1991) identifies the members of the front and the back in the way they access and utilize mathematics. "The "front" of mathematics is mathematics in "finished" form, as it is presented to the public in classrooms, textbooks and journals. The "back" would be mathematics as it appears among working mathematicians, in informal settings, told to one another in an office behind closed doors" (Hersh, 1991, p. 128). Hersh (1991) goes on to identify the front of mathematics as myth-like. It is a belief in a certain way of doing math that is not representative of what actual mathematics is. "Mathematics, too, has its myths. One of the unwritten criteria separating the professional from the amateur, the insider from the outsider, is that the outsiders are taken in (deceived), and the insiders are not taken in" (1991, p.129).

Anne Watson describes the difference between what she calls "authentic" mathematics and school mathematics similarly to Hersh:

> Mathematics as a discipline, by contrast to school mathematics, is concerned with thought, structure, alternatives, abstract ideas, Deductive reasoning and an internal sense of validity and authority . . . The concerns of school mathematics pull learners in directions that differ from these. (Watson, 2008, p. 6)

Davis and Simmt (2006) argue that there indeed should be a "mathematics for teaching" that has its own individual traits distinct from mathematics as a discipline, although it would still be a part of this mathematics: "We argue further that mathematics-for-teaching might properly be regarded as a distinct branch of mathematics” (Davis & Simmt, 2006, p. 294). However, they also describe the act of teaching mathematics as a way of enabling students to be a part of the authentic mathematics that Watson refers to. Not a preconceived certainty, but a social discourse and conceptual exploration.

> For teachers, knowledge of established mathematics is inseparable to historical emergence of core concepts, interconnections among ideas, and from the knowledge of how mathematics is established. Of significance are insights into analogies and images that have come to be associated with different principles” (Davis and Simmt, 2006, p. 297).

Although Davis and Simmt argue for a distinct way of looking at the teaching of mathematics, the end goal appears to be the fostering of more authentic mathematical experience in the classroom, similar to Watson’s interest in reforming mathematical education: “that school students should be introduced to authentic mathematical activity such as is practiced by professional mathematicians” (Watson, 2008, p. 3).

The Pervasiveness of Platonism

Current mathematics comes with a variety of different philosophies, metaphors and conceptions. Certain views evolve from others, and some fade into the past. Pure Platonism, the view of an otherworldly perfect math, tends towards the latter in current academic mathematics, with Formalism’s game metaphor resolving its spiritual side. However, many scholars working with mathematics education claim that this view is still at the forefront of how we teach math in schools. Marcelo C. Borba and Ole Skovsmose (1997), working in the philosophy of mathematics education, have said,

> Phrases such as ‘it was mathematically proved’, ‘the numbers express the truth’, ‘the numbers speak for themselves’, ‘the equations show/assure that . . .’ are frequently used in the media and in schools. These phrases seem to express a view of mathematics as an ‘above-all’ referee, as a ‘judge’, one that is above humans, as a non-human device that can control human imperfection. (p. 17)

Peter Charles Taylor (1996), who advocates for conceptual change, refers to a system of myths when speaking about mathematics in schools:

> Modern school mathematics continues to be influenced strongly by the rationalist myth of cold reason. The pedagogical implications of this myth include a belief in the certainty of mathematical knowledge which leads to the perception that disembodied mathematical facts are knowable by means of an asocial cognitive activity of pure reason that transcends human lifeworlds. (p. 163)

When children ask mathematical questions, the answer too often involves a sense of “that is just how it is”. Even if that is how it is, expressing a willing ignorance of why gives mathematics a sort of power and authority over truth and reality. Mathematics is actually quite messy and complicated, but this can be avoided if one chooses the right question. The structure of schooling with large classes and assessment expectations requires that messiness be avoided. The best questions in schools are those that conform to Platonic expectations with straightforward right or wrong answers. This type of question functions well on evaluations of knowledge, and can be easily administered and explained to large groups.

> The teacher, the textbook, and the answer book make up a united authority which hides the background of the correction. It becomes unnecessary for the teacher to specify the authority that is behind different types of corrections. The students are not met with argumentation but with references to a seemingly uniform and consistent authority. (Borba & Skovsmose, 1997, p. 19)

In their research on teaching multiplication, Davis and Simmt (2006) found that teachers came to realize that there was no one right answer to the question of what multiplication is, and that the major stumbling blocks the researchers noticed in coming to this realization were that teachers themselves did not have enough experience with this way of thinking about mathematics and that most of the teachers did believe that “mathematics is a purely logical enterprise” (p. 303). Teachers, like students, are expected to know the right answer, and their job is to clear out any clutter that may imply that there is anything beyond this elegant simplicity. Stephen Lerman (1990), a philosopher of mathematics education, who puts himself in line with Lakatos’ Fallibilist philosophy describes the common understanding of school mathematics, “The teacher is the possessor of mathematical knowledge which the pupils must gain. That knowledge is certain, as are the methods used, and teaching becomes a conveying of that knowledge and those methods” (Lerman, 1990, p. 56). A heavy reliance on questions with single correct answers also reinforces truth-based absolute views of mathematics itself. “The usual mathematics curriculum adopted deals with one and only one correct solution, a fact that reinforces the idea that mathematics is free of human bias” (Borba & Skovsmose, 1997, p. 17). Reuben Hersh (1991), like Taylor, also speaks of the myths of public and school mathematics and also notes the problematic nature of the belief that “to every question there is an answer” (p. 128). The result is that the public view of mathematics is “formal, precise, ordered and abstract” (Hersh, 1991, p. 128), and its primary myths (or metaphors from my perspective) are about unity, objectivity, universality, and certainty (Hersh, 1991, p. 130). Anne Watson (2008) sees this need for certainty as another way in which school mathematics separates itself from the actual doing of mathematics, not as a part of mathematics, but a separate entity all together with very few ties to the reality of mathematical discourse, “There is a strong focus on answers and generalizations rather than structural insight or abstraction; there is avoidance by teachers, tests, and curricula, or the need for uncertain choices” (Watson, 2008, p. 6).

This allows for a pretty straightforward step-by-step approach to mastery, and only the question of what order to do it in remains for the developers of school mathematics programs and teachers of math. “The core activity in school mathematics is to learn to use mathematical tools and ways of working so that these can be used to learn more tools and ways of working later on” (Watson, 2008, p. 6). We learn school mathematics because we will have to do more school mathematics. Teachers work on the assumption that they must prepare students for what lies ahead; that the purpose of grade two math is to enable proficiency in grade three math, and the authenticity of that mathematical discourse is beside the point, and sometimes even detrimental to it. “Mathematical knowledge is seen as a library of accumulated experience, to be drawn upon and used by those who have access to it” (Lerman, 1990, p. 56). When students leave school, they often have no use for their accumulated library, finding that "most school students are taught and examined on mathematics of a kind that is done, both in the academy and in other workplaces, by machines" (Watson, 2008, p. 6). Children have no interest in authentic mathematics, having never been exposed to it and only having experienced a type of mathematics that is apart from them, predictable, unresponsive, and uninspiring.

Constructivism as Hidden Formalism

The term "Constructivism" stretches across a variety of fields of study, including mathematics and education. Constructivism in education has its roots in the works of Jean Piaget and the belief that humans build upon prior knowledge to construct new knowledge: "To put it simply, children construct their own knowledge" (Van de Walle & Lovin, 2006, p. 1). However, a Constructivist view in mathematics implies that a mathematical object must be able to be constructed in order to prove its existence. Intuitionism is often associated with Constructivist mathematics, although it is not the only form. While mathematics focuses on the construction of mathematics itself and the realization of its logical consequences, education uses this term in reference to the construction of an individual student's learning, and has no relation to the Intuitionist idea that mathematics itself is being constructed. The Constructivism of education is not about new mathematical ideas, it is about the way that the brain forms concepts, and where those concepts come from is secondary. Peter Charles Taylor (1996), a critic of Constructivism in its current form in mathematics education, states that "Conceptual change pedagogy shows a disturbing lack of concern for the socio-cultural and socio-emotional contexts within which lifeworld knowledge is constructed intersubjectively by learners in their actions with significant others" (p. 165). In most current educational practices, there is still a single right answer that the teacher already knows. The only difference for Constructivist education versus more traditional methods is that the idea cannot just be given or told because it is known, but must be constructed in the mind of the child through the teachers' choice of tasks that are intended to prove the point in the cleanest possible way with little room for alternate viewpoints to muck things up.

> Also missing from the Constructivist perspective of learning as conceptual change is a rational for empowering teachers and students as curriculum negotiators, a role that entails an advanced sense of agency and voice on the part of both the teacher and students. By default, the prevailing technical rationality of the traditional school culture predetermines the teacher's role as deliverer of curriculum and the students' roles as passive recipients. (Taylor, 1996, p. 165)

Students are not seen as constructors creating mathematics, they are not even seen as constructors discovering mathematics as Intuitionists do, but as constructors of their own storage system of a pre-established and unchangeable mathematics.

> There is a need to shift away from an overemphasis on students' subjective conceptions and thought processes, which is sometimes associated with Constructivist views of learning. According to these perspectives, learning consists of the elaboration of subjective knowledge structures in the learners mind, and the acquisition and elaboration of these is primary, whereas public mathematical

> activities such as working written mathematical tasks or assessment exercises is secondary. (Ernest, 1999, p. 79)

Here there is no need to involve students, or even teachers, in the constructing of the mathematics being done, but only in the constructing of their understanding of some pre-determined curriculum that continues to hearken back to a Platonic understanding of a fixed mathematics outside of humans and unchangeable. Taylor (1996) ultimately sees modern mathematical education as becoming Formalism (p. 164), with its game theory represented as the series of tests and worksheets one has to complete as the doing of mathematics.

Beyond Platonism

Some authors are currently working to alter this view of mathematics in education, although it appears very slow to change. These writers tend to align themselves with a Humanist or Fallibilist perspective of mathematics academically, and extend these beliefs to the educational sphere, instead of separating the act of learning from the act of creating mathematics.

> An important background development has been the emergence of Fallibilist perspectives in the philosophy of mathematics education. These views assert that the status of mathematical truth is determined, to some extent, relative to its contexts and is dependent, at least in part, on historical contingency. (Ernest, 1999, p. 67)

This philosophy of mathematics education denies the objective existence of mathematical truths and value-free mathematics. Here, mathematics can become without fixed or singular answers, and its becoming is a direct result of humans and in no way separable from the learning and processing of a culture of mathematics. “Notions of proof, truth, and rigour can be seen to be relative values, and there are conceivably alternate possibilities for the development of mathematics. There is no natural or logical necessity to the state of mathematical knowledge at present” (Lerman, 1990, p. 55). In this view, mathematics is not out there, but within us. It is not a series of inevitabilities, but a series of choices, and it is not something that we do, but something that we live.

> We propose that one important source of pedagogical problems in mathematics education are the philosophical foundations that have dominated our view of mathematics (objectivism, Platonism, Formalism). These philosophical commitments are necessarily (if unintentionally) transmitted in the teaching process, which can lead to the teaching of definitions and supposed eternal truths that capture mathematical essences, rather than mind based, embodied, human forms of sense making. (Ernest, 1999, p. 61)

Marilyn Frankenstein discusses the use of value-free or culture-free mathematics in education, and how viewing mathematics in this way, is not only limiting students ability to impact mathematics, but even limiting the legitimacy of the mathematics of some cultures and filtering through a lens of biased logic, “false assumptions about other peoples’ mathematical knowledge, coupled with lack of respect for their logic patterns, intersect with Eurocentric racism in Western considerations of what counts as 'mathematics'" (Frankenstein, 1990, p. 341). Frankenstein notes that mathematics is many things to many people, and not one (coincidentally Eurocentric) thing that some cultures are just late to arrive at.

Stephen Lerman supports the position that mathematical knowledge is coloured by the cultural narrative from which it was derived. If this is the case, then the history of the culture is embedded in mathematics itself and cannot be separated from mathematical knowledge in education in a Platonic or Formalist way.

> If one holds the view that mathematical knowledge is social in nature, then one cannot get away from involvement in values. The mathematical ideas that one is teaching would have originated in a certain time and place as a response to some social need, whether of the individual mathematician or of the wider community, scientific or otherwise. (Lerman, 1990, p. 60)

When considering mathematics as a semiotic system, Paul Ernest notes that it is imperative to go beyond mathematics in education as a Formalist game of terms, but instead to realize that semiotic system are social constructs. "At the heart of mathematics are its meanings, its purpose as a device for meaning making, and this is driven by its social and human aims and contexts" (Ernest as quoted in Hoffman, Lenhard & Seeger, 2005, p. 25). For Ernest, mathematics is a social construction in which learners play a part; a work in progress that reflects the context of the architect. Mathematicians play important roles in the architectural design of mathematics as a social construction. Ernest urges us not to see these agents as bias-free, objective observers, noticing and recording the mathematics around them, but as creators and artists bringing their own culturally situated humanity and educational experiences to the process of manipulating and changing mathematics. "The social context and professional communities of mathematicians play a crucial role in the creation and justification of mathematical knowledge" (Ernest, 1999, p. 68). As well, the validity of mathematical knowledge is judged beyond a single mathematician within a professional community with its own beliefs and expectations. "Both mathematical knowledge and mathematical knowers are judged within social institutions" (Ernest, 1999, p. 80).

Rafael Núñez brings his embodied views to education as he goes beyond mathematics being situated in social interactions to consider the situatedness of the mind itself in the realm of the human body. "Mathematics cannot be conceived as a pure and 'abstract' discipline. Our mathematical conceptual system, like the rest of our conceptual system, is grounded in our bodily functioning and experiences" (Núñez, 1999, p. 61). Núñez argues that we must teach mathematics as an embodied discipline; that this philosophy extends beyond scholars' conceptions to how we actually teach mathematics in schools.

> We should provide a learning environment in which mathematical ideas are taught and discussed with all their human, embodied, and social features. Students (and teachers) should know that mathematical theorems, proofs, and objects are about ideas, and that these ideas are situated and meaningful because they are grounded in our bodily experience as social animals. (Núñez, Edwards & Matos, 1999, p.62)

Brent Davis and Eileen Simmt (2006) have explored social dimensions of mathematics in the classroom through a complexity science based lens. Noticing the way mathematical understanding develops as a social and human act, they echo Núñez's noticing of the innate human nature of mathematics – that to do mathematics is to be human. A non-human mathematics is impossible and nonsensical for humans. "Mathematical knowing is rooted in our biological structure, framed by bodily experiences, elaborated within social interactions, enabled by cultural tools, and part of an ever-unfolding conversation of humans and the biosphere" (Davis & Simmt, 2006, p. 315). The Fallibilists offer an alternate perspective, both in the philosophy of mathematics and in the philosophy of mathematics education.

At this point in history, this view is relatively unfamiliar to most teachers of mathematics, particularly elementary school teachers, so that we continue to instill only Platonist and Formalist philosophies of mathematics in the minds of children, particularly when they are just beginning to form conceptions about what mathematics is and what role they may have to play in this knowledge. The majority of students leave the education system to become Hersh's (1991) "front" of mathematics; shaping the general public perception of what mathematics is, and perpetuating the objectivism they learned in school. The strength of the Platonist and Formalist leanings of mathematics education can be seen in the metaphors used by students and teachers to describe mathematics. These metaphors are very important in defining mathematics because, as an abstract concept, metaphorical descriptions are a fundamental part of how mathematics is understood.

Metaphors for Mathematics in Schools

Unfortunately, as children confront formal mathematics in schools and in public, they tend to be provided with metaphors that reinforce Platonic and Formalist philosophies, and these philosophies block the potential contributions of other Fallibilist views such as Social Constructivism and Embodied perspectives. Research into the metaphors used to discuss mathematics in schools supports the observation that mathematics is conceived in this public context as a combination of a Platonic form and a Formalist game. The structural metaphors of Constructivist views such as Structuralism and Intuitionism are also present, yet metaphors that support Fallibilism or an Embodied mathematics are largely absent, so there is no human connection to mathematics and nothing to bump up against these absolutist metaphors. These metaphors can be problematic at times because they tend to limit the potential for children and teachers to be active participants in what mathematics is; school children believe they cannot contribute to mathematics, so they do not attempt to contribute to mathematics, and therefore appear as non-contributors. This perpetuates the myth that their non-contributions are inherent in their role, and that they have nothing to do with what mathematics is.

Research into Metaphors for Mathematics

Amelie Schnick, Henry Neale, David Pugalee, and Victor Cifarelli (2008) researched metaphors used by high school students to describe mathematics. They specifically asked them to construct metaphors for this purpose, as opposed to listening for the metaphors they used in discourse. Schnick and her colleagues (2008) found that students used complex metaphors, such as being lost in a sewer system (p. 328), however several common themes emerged. Students saw mathematics generally as some sort of structure, journey or tool, and it often-included references to a need for perseverance or active involvement (p. 329), which the authors list as other major themes. The researchers found that encouraging the use of metaphor allowed students to express their beliefs about mathematics, which they normally did not have adequate language for (having been lead to believe there are no metaphors involved), and allowed them to discuss and explore their own views (p. 332). The authors concluded that “Metaphor theory holds promise as a tool for unpacking students’ deeply held beliefs about the nature of mathematics and their roles as learners” (Schnick et al, 2008, p. 333). In Amy Noelle Parks (2010) research on metaphor and hierarchy in mathematics education, she focuses on the prevalence of orientational journey metaphors to describe children’s learning of mathematics in textbooks and classrooms, and also noted that this metaphor is then learned as a function of the nature of mathematics, “These metaphors also, in effect, portray mathematics as a narrow path, which can be traveled in only one direction (up and ahead)” (Parks, 2010, p. 86). Vincenc Font and his colleagues (Font, Godino, Planas, & Acevedo, 2010) explored how metaphors were used in mathematical discourse in the classroom, looking more specifically at metaphors for mathematical concepts and not mathematics itself. In response to their research, they commented that

> If the teacher does not take care when using the verb “exist”, the students in this class may not remain within a position of external existence. Instead, they may change the language game (Wittgenstein, 1963) and assume the ‘external existence’ or reality of mathematical objects. (Font et al., 2010, p.16)

Teachers’ choices of mathematical metaphor have been explored by Andrew Noyes (2006), who identified four “root metaphors” (p. 902), choosing different language for primary or grounding metaphors, with three of them overlapping those noted by Schnick. Noyes (2006) adds the metaphor of a language to the group consisting of a journey, a tool (although it is a tool kit for Noyes), and a structure (p. 902). Noyes (2006), like Schnick, has also grouped a collection of more specific and complex metaphors as coming from the roots, as the specific metaphors given by the teachers were varied and sometimes included reference to more than one root. For example, a network or a building (Noyes, 2006, p. 905) were identified as structure metaphors. Noyes (2006) believes that there is a connection between teachers' metaphorical choices and the way in which they teach, noting that the toolkit metaphor is prevalent among new teachers, as is an instrumentalist view of practice (p.907).

In her research on metaphors elementary mathematics teachers use to complete the phrase "math is . . .", Gladys Sterenberg (2007) found that the teachers who participate in her study initially used the metaphors of a battle, a mountain, and a bridge, that Schnick and Noyes may have classified as journey and structure metaphors, but later focused more on the language metaphor (Sterenberg, 2007, p. 93).

Alba Gonzalez Thompson (1984) researched the link between teachers' conceptions of mathematics and their practice. Although metaphor is not directly explored, Thompson did find that beliefs about what mathematics is influenced teaching practices: "Teachers' beliefs, views, and preferences about mathematics and its teaching, regardless of whether or not they are consciously or unconsciously held, play a significant, albeit subtle, role in shaping the teachers' characteristic pattern of instructional behavior" (Thompson, 1984, p. 125).

Math is Alive

Mathematics has a conceptual history. Its paradigm has not changed so much as it has multiplied with old, objectivist philosophies like Platonism thriving side by side with Formalist and Fallibilist perspectives. These philosophies are seen in mathematics education, with an over-emphasis on Platonism and Formalism in the guise of child-friendly Constructivism with virtually no reference to Social Constructivist or Embodied mathematics. I established the significance of metaphor in mathematics discourse, and expected to find that Platonist, Formalist, and Intuitionist metaphors would be used more frequently than Fallibilist, relative metaphors. My inquiry yielded several overlapping themes in theses metaphors. Structures, tools, journeys, and language were often cited as root or theme metaphors mirroring primary and grounding metaphors. Structures and tools are building metaphors, and to see mathematics as a structure to be built implies an Intuitionist or structural perspective; mathematics is what has been or can be constructed. The journey metaphor seems more romantic, but both a path and a destination are implied. Amy Noelle Park's metaphor of mathematics as a journey straight and upwards comes to mind. Here mathematics retains its sense of being constructed while creating a sense of a predetermined world to visit and explore. The romance is in the implication that personal experience counts for something, and there is a slight sense of relativism in the journey metaphor that structures and tools do not offer. I believe that the final common metaphor, language, is primarily a Formalist metaphor. It is the grammar and rules of language that allow communication, and it is the same for the language of mathematics. The only difference with math is that we tend to believe it is a sort of universal language instead of the product of a particular culture, difficult to understand for those outside of the culture and evolving with different idioms and structures that invite varied meanings and perspectives. The language of mathematics is seen as the same everywhere. The Pythagorean Theorem is true for all the right triangles you find no matter where they are or who finds them, and this sort of universal rulebook is seen as the basis for the language of mathematics. And, like language, you have to go out into the world and learn it. When you know the language, you can communicate, and what you communicate about is the universal truths of the world. But anyone who has tried desperately to read Beowulf knows that even language changes drastically over time. Rules are made to be broken, but this is not part of the language metaphor. Neither is the fact that children change the world. All these metaphors, structures, tools, journeys and languages are conceptually problematic. David Pimm, in his research on metaphor in mathematics, notes the importance of how metaphors can evolve and yet retain old meanings and how the history of a concept can be perpetuated through metaphor as the concept itself changes. “It (the known meaning of an old metaphor in mathematics) certainly has the effect of altering the balance between the various connotations of the concept” (Pimm, 1988, p. 33).

Our metaphorical conceptions of what mathematics is have a significant effect on how we teach and learn in classrooms, yet we continue to use outdated objectivist metaphors without even attempting to balance that with newer Fallibilist claims that mathematics may not be as “out there” as we were led to believe. We often do not even notice that we are using metaphors at all. As Pimm states,

> If the metaphoric quality of certain conceptual extensions in mathematics is not made clear to children, then specific meanings and observations (whether intuitive or

consciously formulated) about the original setting will be carried over to the new setting where they are often inappropriate. (Pimm, 1981, p. 40)

We should choose our metaphors consciously, and not simply reiterate old metaphors or make reference to them without intention. We should consider making new metaphors.

What primary/grounding/root/theme metaphor could capture mathematics as a dynamic, complex feedback system that could evolve and change under the right conditions? Conditions that include using all the parts of the system in the most effective possible way; conditions that create a “healthy” system of mathematics? Is it possible that a living systems metaphor could provide a robust metaphor for growing mathematics in classrooms and beyond? Does this metaphor have a range of effective and provocative applications? Does it resist contradiction? Mathematics is being taught to our children as though it is a task to be done; a beautiful journey at best and some sort of pre-fabricated warehouse to build at worst. We tell children, our most creative citizens, that their creativity is not needed here: this is mathematics, and mathematics just “is”. I propose that instead of asking them to reconstruct it, travel along it, or fluently speak it, we ask children to grow it. They need to notice it has been growing all along and its life is becoming their responsibility. By changing the way we speak about mathematics, we may be able to develop a more fertile landscape for a flourishing, lively mathematics. I believe that the conceptual metaphor of a living system offers new possibilities for philosophical perceptions of mathematics in schools, and that this will lead to a greater, more inclusive mathematics for all.

In this final section I will consider the potential implications of the metaphor “math is alive” in the education of children. To do this I will revisit the mappings created in chapter four that present one possible interpretation of how the source domain “alive” can be understood as a class inclusive assertion that correlates with humanity's experience of the target domain "math". This foundational metaphor uses the particular language of living systems theory to map complex metaphors. Here I discuss how the metaphorical pattern of organization of mathematics, its maintenance functions, and its ability to grow, adapt and reproduce might impact what we do in classrooms. In this philosophical inquiry, guided by Burbules and Warnick's (2006) ten methods of philosophical inquiry, I have analyzed the concepts of metaphor, mathematics and living systems. I have engaged in conceptual mapping in proposing these metaphors, and in this final discourse I turn to method seven; "speculating about alternative systems or practices of education, whether utopian or programmatic, that contrast with and challenge conventional educational understandings and practices" (Burbules & Warnick, 2006, p. 491). At this time, I can only speculate what the implications for practice could be if this metaphor were to be included in the way that we think about and interact with mathematics in classrooms. I can only imagine how it might affect children to know that they matter and that mathematics depends on them for its very survival. We know what happens to children when they are confronted with Platonist metaphors that disenfranchise them and disempower mathematics. We know mathematics is not a building, a tool, an ethereal form, a game or a language, yet we have no issues applying this metaphorical language to the point that we lose the sense that these are metaphors at all. I believe that the mappings created in chapter four contrast and challenge these metaphorical conventions in mathematics education by confronting authenticity and questioning our relationship with mathematics, by asking us to both give up control and take responsibility, and by asking us to nurture creativity and complexification in a discipline that maintains itself through order and simplification.

Pattern of Organization

Mathematics as a living being has a distinct pattern of organization that is particular to living systems. The idea of a pattern of organization breaks with the notion of a simple structural model because it recognizes that a variety of components can make up a living system and many living systems have drastically different components. This metaphor stretches this idea from biology to conceptions. Here I will explore what it means that children's minds are part of the environment of mathematics and how classrooms access that environment. When considering the pattern of organization of mathematics as a living system, teachers must admit that they recognize mathematics is dynamic and unstable, and that they have been actively avoiding these parts of mathematics when planning for teaching. Recognizing chaos in mathematics by exploring competing metaphors and terms allows students to interact with mathematics as a living system more meaningfully and with mutual benefit.

Mathematics is autonomous with a boundary of its own making. Although we may have constructed a variety of ways to facilitate conscious interaction with mathematics through the use of physical tools, technology and language and symbols, these things are not mathematics itself. Mathematics is its own being, and in this sense the Platonists were right to see it as a beautiful form, but they neglected to realize that its spiritual becoming is unnecessary, because math becomes itself and determines its own reach and limits. Mathematics as autonomous and open challenges Platonist, Deductive and Fallibilist perspectives, because recognizing the boundary it creates to determine itself defies our control, yet our role as the mindscape environment of mathematics makes it our responsibility to be conscious of how we interact with this open system.

This potential for interaction is not present in current educational practice. In schools, we assume mathematics is the generation of universal truths discovered by professional mathematicians, but having always been true. If teachers instead assume that their students can and do interact with mathematics, even though it is its own unity, we invite conversations about what sort of being it is, how it behaves, and how we influence it. In these conversations, we find out more about children's authentic mathematical interactions and conflicts.

Fallibilism offers a way for us to consider human impact in mathematics, but Fallibilist perspectives that mathematics is within us and not a separate bounded unity are rarely considered in how mathematics is represented to children because of the difficulty presented by the relativism they invite. A living system metaphor resolves some of this difficulty by showing that mathematics is both beyond our control as an autonomous being, and dependent on us for its survival through the boundary it constructs to interact with us. We no longer have to fear telling children that mathematics is made up like Santa Claus, just because we question its status as a universal truth. With a living systems metaphor, children are invited to acknowledge what mathematics is while recognizing the relationship that they have with it. Schnick and her colleagues (2008) have shown that students do believe that they interact with mathematics, but they lack a language with which to explore this belief (p. 329). Acknowledging this relationship enables classrooms to consider the environment that they provide.

Additionally, when we frame mathematics as an abstract form instead of as a living system we ignore the responsibility we have to grow and support mathematics. We assume mathematics is to be discovered by mathematicians as it is and always has been. Children should be aware that they are part of the environment of this open system with academics and others working with mathematics, and that mathematics depends on them for its survival. When considering the mindscape of children as part of the mass human mind environment of mathematics, we must first wonder if we are really even considering how children actually interact with mathematics. This is in line with Anne Watson's (2008) noticing that children need to be engaged with mathematics authentically through real challenges in their minds if they are to be involved with actual mathematics (p. 6). We must make classroom mathematics a relevant part of how children actually interact with mathematics by honouring children's culture more in our planning for classroom learning. Children authentically impact mathematics through their use of technology, through their play choices, through their social interactions, and through their everyday needs and interests. As teachers, we must try to access the area of our mass mind where children and mathematics mingle together before we can contribute to the vitality of that environment. We must spend less time telling children what mathematics is, and more time finding out where it dwells in them, so that we can actually be a part of what their mathematics is becoming.

As the Fallibilists have realized, in order to understand the relationship between humans and mathematics, you must take into account the biology of thinking in humans. What they are wrong about is assuming that mathematics is inherently based on human physiology and a human act. "Mathematics as we know it is limited and structured by the human brain and human mental capacities. The only mathematics we know or can know is a brain-and-mind-based mathematics" (Lakoff & Núñez, 2000, p. 1). Mathematics is embodied in the sense that it resides biologically in our minds, but I argue that we are capable of understanding that it is beyond us at the same time, nested and networked and dependent on itself to define what it is. In a living systems metaphor, mathematics can be its own self while living in its environment, the neurochemistry of humans. Students and teachers alike rarely discuss their brains and the impact of logical thinking, pattern seeking, or memory in their interactions with mathematics.

Mathematics in schools resists any form of Fallibilism and therefore does not have access to a discussion of how humans impact mathematics by being human. A living metaphor focuses on the relationship between human minds and mathematics, and invites discussions of biology, embodiment and humanity; such discussion is evident in philosophy of mathematics but still missing in education and other disciplines.

Marcelo C. Borba and Ole Skovsmose (1997, p. 17) discussed the notion that Platonist teaching makes the mistake of assuming that mathematics is the same for all people at all times if only we can see it. Instead of recognizing truth and objectivity as metaphorical products of mathematical maintenance and the way that mathematics behaves, we mistakenly assume they are characteristics of mathematics itself. As in Hersh's (1991) front and back metaphor, children are part of the public who are only permitted in the front and are shielded from what goes on in the back. As discussed, Font, Godino, Planas, and Acevedo (2010, p.16) have shown that teachers continue to present mathematics as having an objective immutable existence. In fact, the history of mathematics shows that it changes over time and creates internal inconsistencies, and that it grows differently in different geographical regions of the mass mind including a variety of academic contexts, professions and applications, and varied social and cultural elements existing in the environment. In a living system, even a small difference in initial conditions can lead to vastly different development both across and within systems. From a tiny group of cells, living allows the differentiation needed to produce all the parts of the system. Drastic differences can result in the mathematics we see being used in networks of cultures, countries, occupations and applications including in the realm of childhood education (Ernest, 1991, p. 263). Yet we aim to teach mathematics as though it is universally applicable and without an environmental context or history. Using standardized curricula or a "common core" and standardized testing is an example of our inability to recognize that we are dealing with differentiated mathematics. What a child in school A knows or needs to know is not the same as a child in school B, and when we refuse to honour these differences in the way that we teach, we fail to recognize our role as the environment for mathematics.

Even though mathematics presents the façade of reliability in its unity as all living creatures strive to do, it is always to some degree unstable. This instability and unpredictability is healthy and necessary for any living system. In education, we fear situations that are unstable or unpredictable in mathematics. We know the answers to all the questions that we ask and we cling to uniformity, tradition and technicality in language through an unseen Formalist tradition. Anything without an answer key is avoided and spontaneity is discouraged and constrained by a standardized curriculum that must be unraveled in a specific amount of time. Instead of seeking mathematical interaction, teachers avoid real living mathematics by trying to keep students in areas of little growth and change, maintaining ideas from hundreds if not thousands of years ago. If math is alive, then the job of a classroom teacher is reframed to notice it, nurture it, and tend to inconsistencies, pushing them to points of resolution. We need to seek out children's interactions with mathematics instead of preparing them for what we pretend to believe is adult mathematics, or even just what happens next in school, as Watson (2008, p. 6) and Lerman (1990, p. 56) have shown is often the case.

In doing this we must also honour the many mathematical languages that children bring with them into classrooms. From their families, their peers, and technology they pick up various words for mathematical expressions and actions, and we take it upon ourselves as teachers to weed out unnecessary language and provide a sort of "newspeak" that will solve all their mathematical problems. This is not to say that new metaphors for concepts should not be introduced, but that a wealth of metaphors is necessary to provide a depth of understanding about mathematics itself and the concepts and operations associated with it. When we limit our metaphors, we imply that we are not using metaphors at all, instead of seeking to uncover the hidden associations in all our mathematical terminology. Recognizing and encouraging the use of multiple metaphors embraces the previously discussed research of Amelie Schick, Henry Neale, David Pugalee, and Victor Cifarelli (2008) showing that students desperately wanted languages to express their relationship with mathematics.

One of the more elegant aspects of the metaphor “math is alive” is that it is so blatantly metaphorical that it forces us to realize how foolish we have been not to notice it has all been metaphors all along, and that complicating, not clarifying leads to a deeper understanding of mathematics and a more mutually interactive and beneficial relationship between mathematics and the minds of children. Accepting living mathematics as dynamic and unstable allows teachers to let go of the implication of control seen in the Platonist and Formalist philosophical leanings of education and embrace the chaos of the unknown and the unsolvable and the multiple languages and metaphors needed to fully and consciously interact and influence this process.

When math is alive, it has a pattern of organization consistent with a living system, regardless of what the parts that make it up are composed of. When we acknowledge this being living in the minds of children, we can see that perfection, reliability, and standardization are impossible and repressive.

Maintenance

The pattern of organization of a living mathematics generates the self-organization metaphor. Here I will discuss how classrooms might be impacted by recognizing that children provide what is necessary for it to continually create and recreate itself. They provide the matter and energy it consumes to fuel its self-production, and we must consider what sort of matter and energy school mathematics contributes, if any. As a living being, mathematics is enabled to build up and create instead of break down and destroy from its approach to entropy in the environment and within itself. Although education takes the Platonist stance that we can have complete faith in the absolute truth of math, we teach as though we have very little faith at all that it will not fall apart in an instant. The self-organization metaphor allows us to appreciate mathematical consistency and nurture this in classrooms.

Living mathematics is an open system, which is how it is able to regulate itself in the far from equilibrium state caused by conflict, inconsistency, and changing collective human cognition, and how it continues to be itself through dynamic internal consistency. Because of its dynamism, it does not need its environment to always be consistent or predictable. In fact, only complexity and indeterminacy can push mathematics to become and grow. Teachers attempt to present children with a mathematics that is already organized and abstract, but real mathematics is messy and full of indecisive terms; consider Brouwer's Intuitionist struggle with the excluded middle (Franchella, 1995, p. 308). In putting together a loot bag for a party the factors are infinite. What sort of toys? What sort of budget? How many of each? How many people? What relation of people? What does the child like? What sort of family does the child come from? We try to edit out the actual interactions with mathematics that are occurring. As Stephen Lerman (1990, p. 56) noted with concern, we do not want to clutter up mathematics. However, if mathematics is a living system, it can deal with clutter just fine, and thrives on a little bit of healthy clutter now and then. Mathematics will self-organize even when the minds of children interacting with it, are in far-from-equilibrium states, confused and befuddled, yet actually working to interact with mathematics that can organize and produce for them.

Mathematics requires a healthy variety of consumables to continue to live and grow and avoid decay. This is how the system is able to engage in autopoiesis and maintain itself. Mathematics is powered and empowered by the fuel it takes from our mass mind environment. Through our role as the environment of mathematics we provide the quality, novelty and quantity of the matter and energy that goes into mathematics. Whether or not we apply it to new experiences, and the type and amount of mathematical thinking that we do matters. Children are a large segment of the mass mind environment of mathematics. Mathematics may grow and maintain its state of living even if their mathematical interactions do not benefit it, just as other living systems rely on their ability to self-organize and stay alive even when a part of them is neglected by diverting resources. However, it is unlikely to thrive in these circumstances. When an entire area of mathematics such as the part that dwells in children is barely surviving, the entirety of mathematics is affected. Living mathematics becomes more robust when we consider the importance of novelty, quality, variety, and complexity in our mind environment as we interact with mathematics. Are we providing mathematics in classrooms with a rich variety of healthy matter and energy?

The matter that is taken in by mathematics consists of the physical and mental constructions we apply it to. If we place mathematics primarily in the domain of textbooks and worksheets, we cannot provide novelty or depth and mathematics in schools becomes limp and grey, surviving but not thriving as a part of mathematics nobody really likes. When we feed it the same matter over and over again we neglect any change that might impact the system. What does mathematics gain from the worksheet you are about to feed it? From the word problem you demand a specific solution for? From the measuring activity you are going out to do (you know, the one you did ahead of time to make sure it would work)?

A living mathematics metaphor asks us to be critical not only of how children's mathematical learning is affected by our educational choices, but what the effect on mathematics itself is. Many teachers are already on to the idea that drill style devices like "mad minutes" may not be healthy for children, but here we ask, too, if they are healthy for mathematics. Mathematics is more likely to be impacted and to undergo complexification and growth when students are enabled to interact with it authentically through their actual thoughts, interests and concerns because this exposes mathematics to a greater variety of matter, including matter it may have never been in contact with before. When children's ideas interact with mathematics, mathematics is enabled to organize and combine, which is what it does best.

Children provide mathematics with energy through their thinking. In classrooms, there is a tendency to encourage all students to demonstrate the same type of thinking; there is no healthy variety, no chaos, nothing too stimulating, only the essential minimum required for mathematics to maintain the boundary and distinguish itself in its environment. Healthy, robust mathematics takes its energy from a variety of sources in a variety of ways, so it benefits mathematics when children bring their unique childhood thinking, new ideas and different ways of approaching topics that are unique to them and cannot come from the teacher or from mathematics itself.

Mathematics knows how to keep being mathematics despite unpredictability and novelty. We do not have to spend so much time worrying that it will fall apart in the minds of children when we recognize that we do not control mathematics' ability to be itself. We have come to believe that difference in mathematics is wrong. We only concern ourselves with the most effective way of arriving in the same place. We recognize that mathematics is information-tight, but our metaphors do not allow us to include recognition of growth, change and environmental adaptability. This Platonist/Formalist view can be reframed through the structural and ontological self-organization metaphors generated by a living mathematics as developing an understanding of how mathematics behaves. This allows us to respect the consistency and reliability of mathematics, while recognizing the potential for innovation. When mathematics is alive, it still controls what it is, while being open to creating a more complex self through its taking in of environmental matter and energy to maintain itself autopoietically.

In a living system like mathematics, we see that the feedback that regulates the system is localized. Different parts of the system, occupational, academic or otherwise, work with autonomy and independence while playing a role in the health and function of the system as a whole. Mathematics senses what is available in each environment and responds (or doesn't respond) appropriately. Feedback in a system is primarily negative and regulatory, and there is actually something to be said for dependable consistency in an environment. Living systems rely on the environment to stay the same more than it changes. We do want mathematics to reveal its consistencies to children through their experience. This cannot be done by seeking out the consistencies in advance for children, but by allowing them to experience this in their authentic interactions with a self-organizing mathematics that seeks to stay itself. On occasion, this may provoke positive feedback where mathematics is challenged to be itself. Here it is possible to impact the entire being of mathematics through the potential amplification of change. We must let go of our fear of Fallibilism and accept that we cannot absolutely control how mathematics reacts, but embrace the idea that we can create a healthy environment that is likely to produce beneficial adaptation.

Children’s minds are part of the environment of mathematics, and because their experiences are different from those before them, they will inevitably introduce novelty, creativity and complexity that the feedback mechanisms of mathematics will attempt to regulate. It is up to educators to decide whether or not they recognize the importance of these interactions and approach mathematics in schools in such a way as to contribute to the health and vitality of the relationship between mathematics and children’s mind environments.

Entropy is required to complexify the behavior of mathematics as it struggles to find itself in disorder, yet it can also be destructive. In education, we should prevent chaos from building up, undigested and unstable, until it causes more damage than good to the living system. When entropy is left to build up instead of used to construct, frustration emerges in students as they desperately cling to a notion of one right answer. It is not about finding ways to relate the curriculum to children’s lives, but the honesty of dealing with the mess of thinking that comes along with children’s lives and often lies outside of or challenges the curriculum. This entropy created by children’s knowing there is a more authentic mathematics beyond their reach cannot be left behind to build up and cause the decay of relationships with mathematics. When educators believe in mathematics’ ability to maintain itself and prevent contradiction and dissolution, children are allowed to confront mathematics really with all its parts, redundancies and tendencies. They see that mathematics is regulatory and more reliable than not, and that there is nothing to fear.

When the living system of mathematics runs smoothly, it takes in matter and energy from its environment and lets off high entropy energy and simple unused, deconstructed waste products.

Mathematics does not need to keep the paper you wrote on or the program you used. It emits the waste of its solutions through ideas relayed in language and symbols. Ideally, this waste is then useful to the environment, but in the case of school mathematics it usually is not. As a parent and teacher, I have thrown out thousands of worksheets. I have seen whole binders put in the trash as the textbooks head back to the shelves, inspected for any sign of use, because no one is allowed to write in those. All these ideas generated by children become useless, and any ideas that might have sprung forth from them are quashed by their irrelevance. Mathematics wants to be generative. It can only benefit mathematics if the environment it resides in is reciprocally healthy and vital, but education contributes little to this process because what mathematics in education produces is generally useless.

When the “math is alive” metaphor is applied to mathematics education, the metaphors that can be created from considering the way it maintains itself as a self-organizing autopoietic entity highlight inadequacies currently inherent in our current Platonist and Formalist approaches. Mathematics is able to maintain itself in far-from-equilibrium states, so we do not need to fear chaos and unpredictability in the classroom. Mathematics will know what to do, because it alone can be itself. Mathematics is able to use localized feedback to regulate these interactions, and through children’s role as a part of the environment of mathematics they can influence this feedback through the types of interactions educators access. When we provide mathematics with what it needs to live well, we also enable it to change and grow in unexpected, unique and amazing ways.

Growth, Adaptation and Evolution

Mathematics evolves through the consequences of small changes, as trajectories are irreversibly altered and old methods and beliefs die away with their environments. We cannot stop children from changing mathematics, we can only decide if school mathematics is a part of that interaction. Students, instead of academics, should create change in school mathematics, and from that they may influence mathematics as a whole. The trajectory can be changed irreversibly, and with each reproduction of mathematics from one generation to the next, mass reorganization is possible. The conflation of metaphors in the minds of children creates the possibilities that they are able to access, and defines their interactions with mathematics. When we invite metaphors of growth, change, reproduction and evolution into our conversations we not only change what is possible for children's interaction with mathematics, but also what is possible for mathematics through their conscious, purposeful interactions with this living system.

Children's minds become a part of the environment of a mathematics that has existed for a long time. Although it has reproduced itself entirely several times over, it will not be wholly reproduced in their lifetimes. This is why it is important for them to understand that mathematics has a context and a history so that they can see purpose in starting from where we are in this relationship. This is not just how it is, but how it has become. We have failed to teach children anything about the history of mathematics, just as we have failed to recognize that mathematics history is a part of what it is generally. When we approach mathematics Platonically as though it is a universal truth, or from a Formalist perspective where learning the rules is the primary goal of instruction, we neglect to acknowledge our role in the development of this living system, and the opportunity we have to impact it at the current time. If mathematics is presented as a living system, we are forced to analyze it according to our expectations of living systems and acknowledge the life that it has lived so far. Students begin to see that because of the way that mathematics reproduces and evolves through new minds, their mind matters. They are a part of the environment of mathematics and they are a part of the ongoing history of mathematics, because it lives and grows through its interactions with all humans. As students come to know mathematics, they appreciate the complexity that has developed throughout history and see the change and growth that has occurred, forcing them beyond Platonism. Instead of exploring or discovering mathematics, a living metaphor enables children to metaphorically interact and grow with mathematics and, most importantly, to see that it reacts to their intention to affect it. Changes to mathematics cannot be undone, but these changes are evidence that more change can happen.

When a living system is forced to react, it can reach a point of instability from the amplification of positive feedback that causes the system to undergo mass reorganization. This is a bifurcation point, and they are sometimes, but not always predictable. What we know is that mathematics undergoes significant change at the point of paradoxical shifts in the human mass mind cognitively and communicatively, and when it is forced to confront matter and energy, it does not yet have the ability to break down, either in the more abstract forms of academic math or the more concrete forms of occupational math. We also see that big changes come from small, localized beginnings. Mathematics must be exposed to change and be challenged in order for it to grow new parts and reconfigure old ones, adding complexity while maintaining function. For school mathematics to be a relevant part of the environment of mathematics, students must bring with them their new ways of thinking and the problems that they authentically have. When mathematics is confronted with these inconsistencies, it realizes that it must adapt. Through bifurcation, a tiny adaptation can lead to cascading changes throughout the entire system, so that any part of mathematics, including classroom mathematics, is capable of altering the entire system of mathematics. To provoke this we must dwell at the point of impact in the real worlds of children at this point in history, because they are the keepers of thoughts that have never been thought before.

Mathematics has the greatest potential for change at the point of reproduction as it moves to new minds. Children's experiences with mathematics are uniquely important because of this. The metaphors that we use and the interactions we present provide the very foundation of their relationship with mathematics. When educators show that there is always a right answer, they are teaching them to be Platonists. When educators give students a problem based on a process they have taught, they are teaching them to be Formalists. Neither of these philosophies is sufficient in its ability to explain how mathematics that tries so hard to stay the same can change. There has been significant change in school mathematics, and we have seen bifurcations that have led to adaptations. These adaptations tend to happen in quick succession, followed by long periods of relative stasis. New mathematics is one such adaptation, as is the back to basics movement that led to a common core approach. There has also been a significant impact on math in schools through the presence of a Constructivist mindset towards brain behavior and social behavior that has unintentionally brought with it Formalist and Intuitionist implications. We are at a point where we know that it is not a pure language or a true form, but it is easier not to notice. We are well aware that children's ways of interacting with the world will change it, but we instead see it as our duty to simplify and reduce. This denial of novelty and chaos attempts to define the boundary of mathematics instead of letting it define its own boundary, and limits change at the point of reproduction. In childhood, when the next generation of metaphors are still shaped by conflation and deeply embedded, when children are first coming to know mathematics and create a relationship with it, this is the place where mathematics can change the most, grow the most, and complexify the most. Educators must seize this opportunity to introduce the idea of living mathematics to children. When we use this metaphor, we empower children and educators alike to interact with mathematics more authentically, and we allow mathematics to grow and flourish as an open, living system.

Conclusion

This thesis started as an idea. Disciplines change and grow, so they must be alive. I believed that I could show that this was not only possible but apparent. I began to see limits in my teaching driven by old metaphors. In mathematics, in particular, an ethereal, untouchable discipline was unavoidable given the language and experiences provided in schools. Mathematics becomes a source of frustration for children in schools as a pre-determined, limited, inaccessible form void of personal meaning, ambivalence, history, artistry or vitality.

I recognized early on that a discipline is not a biological system, but this hardly seemed problematic given the current language used to talk about what mathematics is. The history of mathematics philosophy is rife with metaphors or forms, constructions and behaviours that cannot be and are not what mathematics actually is as an abstract concept unto itself. An exploration of how we understand abstract concepts through metaphor brought me to Lakoff and Johnson and other researchers in the field where a dialogue about the mappings that metaphors generate and the way that metaphors enter our conceptions altered my understanding of how new metaphors emerge and how current metaphors can suppress. This exploration supported my belief that abstract concepts such as mathematics rely heavily on our metaphorical understandings to define themselves, which are often deeply embedded in our ideas about ideas to the point where we cease to name these associations as metaphors.

When I came back to the philosophy of mathematics, I came with a new perspective on how its metaphors formed and multiplied, growing out of and back into each other, and leaving infinite tracings behind of old objectivist mappings. I recognized mathematics in the descriptions of Platonism. I remembered encountering its Formalist rules with no meaning. I even recognized Intuitionism and Social Constructivism from my teacher education courses when disseminating how to adequately deconstruct a concept for reconstruction (although it was never so that it could be reconstructed differently). Aspects of Humanist philosophies were lacking or absent. I discovered that in the philosophy of mathematics there has been a move in academia towards social and cognitive views of mathematics, despite little evidence in the public consciousness. Reuben Hersh's idea of the front and back of mathematics gave a voice to my noticing of the gaps in public awareness and the areas of mathematics to which only a select few were admitted. Despite this small and exclusive locus of control, mathematics still appeared to have the tendency to respond and change unexpectedly, often in response to cultural and geographical changes beyond academia in the history of our collective, networked experience. My exploration of the philosophy of mathematics supported my beliefs that our conceptions of mathematics are largely metaphorical and that metaphors can change and grow, even if the process is slow. It also confronted me with our perception of the fundamental consistency and reliability of mathematics, and our tendency to revert to Platonism to cope with our belief that mathematics is a separate entity from us. I began to see a living systems metaphor as a way to reconcile these dichotomies, so that mathematics could be fundamentally trying to stay the same while enabling its ability to grow, and that it could be its own self and dependent on us at the same time.

Biological living systems are complicated. There is no clear consensus as to what it means to be alive. I chose living systems theory as the source domain for my metaphor because it provided a consistent and applicable language for my attempt to conceive of mathematics as alive. I started with Maturana and Varela and was led from there to Capra, Kaufmann, Luhmann, Prigogine and others. I realized that because a living system can only be defined by the characteristics of its system and not by its parts, which may differ completely, it was possible that something completely different, like mathematics, could be a living system as well. Living systems emerged for me as networked yet autonomous, and bounded yet dependent. The maintenance functions that allow living systems to continually recreate themselves as they are and avoid decay reminded me of our human need to turn to math for predictability, reliability, truth and objectivity. This sameness is good for a system. Without it, the system's unity and identity are destroyed. Because a system works within time to sustain itself, it is enabled to grow and change when it needs to in a volatile environment. Breaking points emerge where sameness no longer works and the build-up of entropy created by its failure causes new ideas to burst into being. As I read about living systems, I could not help but see mathematics as one of these varied creatures, and mappings began to emerge.

When mathematics is seen as a living system, the pattern of organization of mathematics allows it to be both a separate entity from humans and intimately related to humans. Mathematics determines what mathematics is, and it enables the interactions that allow it to grow and become through a boundary of its own making. We notice the boundary of mathematics, but we do not choose it; the boundary comes from mathematics itself. However, we do have opportunities to interact with mathematics through each of our individual roles in its environment of global human cognition, and these interactions and transactions can be consciously influenced by us to some extent, even if mathematics does with it what it will. Mathematics is impacted by a variety of human mathematical endeavours and has a complexly related array of parts that do not always interact directly with each other. Academic, occupational, and educational mathematics often segregate themselves by developing their own languages and contexts, creating localized self-regulating feedback. However, all parts of mathematics have the ability, through the amplification of positive feedback loops, to impact the whole of mathematics in profound ways. The extreme complexity of the concept of a global human mass mind environment proposed in this discussion lends itself to change and unpredictability, and mathematics must continually react, reaching for homeostasis but never getting there, and kept alive by this dynamic interplay. As this goes on, mathematics develops a history. It has changed and developed through time, adding new branches and restructuring those that arose early in human civilizations. “The history of mathematics shows that mathematics is a dynamic process, which often develops through torturous and tormented paths not determined a priori, and proceeds through false starts and standstills, routine periods and sudden turnings” (Celucci, 2006, p. 22). If we only look back or outside of our own culture, it becomes obvious that objectivist philosophies of mathematics are neglecting obvious differences, changes and restructurings that have occurred and continue to occur. Mathematics becomes different in order to stay the same in a changing environment. It seems the same to us, but this is only because we are different too.

I have become different in my teaching. As an example, I can reflect upon my recent encounters with transformations on a coordinate grid, specifically rotation and rotational symmetry. I chose this topic for teaching because it is a part of the provincial curriculum that guides me. Specifically, I am referring to outcome 6.SS.6: “Perform a combination of transformations (translations, rotations, or reflections) on a single 2-D shape, and draw and describe the image” and outcome 6.SS.7: “Perform a combination of successive transformations of 2-D shapes to create a design, and identify and describe the transformations” (Manitoba Education, 2013). I know a trick for making wonderful yarn bookmarks with a design resulting from rotational symmetry, so I include this idea even though symmetry isn’t specified.

Now that I see mathematics as having a pattern of organization that resists stability and depends on the strength of its unity and not the specificity of its parts, I am more comfortable seeing my curriculum as a guide and as a place to start and not as a prescription. I am confident to get out my yarn, and I realized that when the students ask what weaving might have to do with mathematics, they only do this because they have been taught to look for a closed boundary, not one of open interconnections. You simply have to ask them the same question back and a response is almost immediate. Mathematics is not lost, but clearly defined as they notice the consistency in the movement and pattern.

At the same time, mathematics needs to maintain itself and wants to stay the same. Change can be gradual or immediate, but it cannot be constantly ongoing. The self-organizing properties of mathematics are designed to try to move towards balance. Worksheets were done, textbooks were inspected and many grids were drawn and used on paper, respecting the current state of school mathematics as a local area of the whole. An array of traditional language and symbols were used. Feedback was often negative and predictable. Yet there were also moments when I came across something messy or complicated and I didn’t walk away or attempt to simplify. Given my new understanding that mathematics will maintain itself, even when the environment enters a chaotic state, I started to embrace inconsistencies. I proceeded with the belief that mathematics will assert its unity and identity in tricky situations. Preparing for rotational symmetry, I came across inconsistencies in my sources, with some using degrees and others fractions. My first instinct was to clarify what was in my curriculum and choose one, but my new understanding forced me to balance knowledge of local expectations and assumptions with the reality that there is more than one way to talk mathematically about moving around an axis. As it turns out, my students were quite flexible about the unit of rotation. They were also open to a variety of experiences on and off paper. They sometimes struggled to reconcile things they had been told, but rarely struggled to reconcile what they believed they knew. When I asked them to describe the rotational symmetry in their woven bookmarks, they were frustrated that I refused to tell them the exact wording I expected as an answer, yet all of them were able to provide a description with thoughtful mathematical detail. It is not my responsibility to show mathematics to them, it is our responsibility together to nurture their relationship with it.

I have also been changed by my analysis of the way mathematics as a living system might metaphorically grow, adapt, and evolve. In my rotation example, students easily reconciled different ways of measuring rotation partially because they could understand and connect fractions and degrees. In my ongoing resource search, I found a piece that called counterclockwise “anti-clockwise”, a term even I was unfamiliar with. I could have put these examples aside to avoid confusion, but instead I investigated and confirmed that this is a British term. My students found this fascinating, and their immediate “whys” situated them in the history and culture of mathematics. I have also ceased to believe that academic thought is more valuable and that students are incapable of creating real perturbations because I recognize school mathematics as a part of mathematics that is capable of change and of changing the whole. In this spirit, I have tried to stop forcing them into the front of mathematics, to reference Hersh, and to invite them into how it works in the back. During our experiences with rotation, I stopped worrying so much when they worked in groups that they might copy each other, and instead focused on ways to nurture and support their group learning so that they saw it as a time to bounce ideas off each other and to deepen their understanding by hearing others discuss the concept; to work with mathematics as mathematicians do. Sometimes they worked independently but often with others, struggling together with their relationships to mathematics. Finally, I have tried to notice and celebrate mathematics' tiny evolutions as I see each new class living in a different moment than the one before. I have shared artistic uses of rotation before, but this is the first time many students went home and looked up different methods and designs on the internet. Students get their information from multitudes of sources where inconsistencies abound. What they need from mathematics and what mathematics needs from them is different than ever before. Instead I have been trying to encourage my students to build enough momentum to perturb the system, cause bifurcations and begin a cascade of significant change.

In this last chapter, I have looked at what this means to me as a teacher in a school with children. I started by researching current conceptions of mathematics in classrooms, including those that acknowledged the importance of metaphor in this process, although they were few and far between. As I had anticipated, mathematics in schools was generally conceived of as Platonist or Formalist and more of a mathematical initiation than mathematics itself. Children were not provoked to impact mathematics. School mathematics has no history, no culture, no point of interaction, and it seems to ignore children completely. Yet, when educators and high school students were asked to describe mathematics metaphorically, they found this acknowledgement of metaphor liberating, because they could manipulate their own language around this abstract entity. We can, and we want to, find new ways to talk about our relationship with mathematics, and we can begin this with children.

Through my perception of mathematics as alive, I hypothesized some general ways in which educators could view their teaching of mathematics differently that might impact how they plan for (or do not plan for) classroom experiences. The pattern of organization of mathematics inspired my view that a deeper appreciation of authenticity is required through honouring children's culture in the classroom, so that mathematics can be challenged by its real, messy environment, as well as the belief that children's relationship with mathematics should be framed as interactive instead of absorptive. Children play, and they should be encouraged to play with math. Mathematics education needs to invite the clamour of children. In a living system metaphor, chaos does not cause decay, but instead vitality and growth. The maintenance functions of mathematics led me to the idea that classrooms must accept chaos and indeterminacy through the knowledge that mathematics is designed to adapt and stay consistent. Teachers must let go of some control for mathematics to show it can truly control itself. As we let go of control, we must accept responsibility as well. Education is a part of children's mathematical lives, and children's mathematical lives impact mathematics. Even when we try not to let children affect mathematics, they do through the quantity and quality of matter and energy they provide to its autopoietic processes. This includes the social values reflected in our mathematical teaching choices. When considering the growth, adaptation and evolution of mathematics, in education we must allow for complexification and change in mathematics. This means ceasing to narrow our language and instead nurturing its expansion, inviting conflicting ideas and solutions, and engaging consciously with multiple metaphors. It means acknowledging and teaching about paradigms and conflicts in the history of mathematics, and knowing that the trajectory of mathematics comes from its past, but still has many potential futures that change with every moment and every person born who at first is a child.

Finally, perceiving that mathematics grows, adapts, and evolves asks educators to capitalize on children's natural creativity – their ability to see the world without the preconceptions and assumptions of the older generation and use new metaphors. Children have the potential to create new foundations that in turn enable new possibilities for mathematics never thought of before. When children are invited to view mathematics as living and recognize their role as its environment and their ability to influence mathematics through their contributions to its autopoietic process, they can become conscious of their change-making and attempt to perturb the system into brilliant new forms. When the parts change, the whole is changed as well, and a whole new mathematics is born from a single classroom moment cascading out into the world.

References

Bailey, K. D. (2005). Emergence, drop-back and reductionism in living systems theory. *Axiomathes, 15*(1), 29-45. doi: 10.1007/s10516-004-1910-x.

Bailly, F., & Longo, G. (2011). *Mathematics and the natural sciences: The physical singularity of life*. London, England: Imperial College Press.

Balaguer, M. (2014). A guide for the perplexed: What mathematicians need to know to understand philosophers of mathematics. *The Mathematical Intelligencer, 36*(2). 3-8. doi: 10.1007/s00283-013-9406-4

Barton, B. (2008). *The language of mathematics telling mathematical tales* (Mathematics Education Library, v. 44). Dordrecht: Springer.

Benacerraf, P. (1965). What numbers could not be. *The Philosophical Review*, *74*(1), 47-73.

Benner, S. A. (2010). Defining life. *Astrobiology, 10*(10), 1021-1030. doi:10.1089/ast.2010.0524.

Borba, M., & Skovsmose, O. (1997). The ideology of certainty in mathematics education. *For the Learning of Mathematics*, *17*(3), 17-23.

Brouwer, E. (2012) Curriculum as a journey towards wisdom. In K. Badly & H. Van Brummelen (Eds.), *Metaphors we teach by: How metaphors shape what we do in classrooms* (pp. 68-88). Eugene, OR: Wipf and Stock Publishers.

Brouwer, L. E. J. (1981). *Brouwer's Cambridge lectures on Intuitionism* (D. Van Dalen, Ed.). New York, NY: Cambridge University Press.

Burbules, N. & Warnik, B. R. (2006). Philosophical inquiry. In J. Green, G. Camilli, P. B. Elmore, & American Educational Research Association. *Handbook of complementary methods in education research*. Washington, D.C.: Lawrence Erlbaum Associates.

Camac, M. K., & Glucksberg, S. (1984). Metaphors do not use associations between concepts, they are used to create them. *Journal of Psycholinguistic Research, 13*(6). 443-455. doi:10.1007/BF01068178

Capra, F. (1996). *The web of life*. New York, NY: Anchor Books.

Capra, F. (2002). *The hidden connections: Integrating the biological, cognitive, and social dimensions of life into a science of sustainability* (1st ed.). New York, NY: Doubleday.

Capra, F., & Luisi. P. L. (2014). *The systems view of life: A unifying vision.* New York, NY: Cambridge University Press.

Cellucci, C. (2006). “Introduction” to Filosofia e matematica. In R. Hersh, (Ed.), *18 Unconventional essays on the nature of mathematics* (pp. 17-36). Dordrecht: Springer-Verlag New York.

Clarke, B. (2009). Heinz von Foerster’s demons: The emergence of second order systems theory. In B. Clarke & M. B. N. Hansen (Eds.), *Emergence and embodiment: New essays on second order systems theory* (pp. 34-61). London, England: Duke University Press.

Colyvan, M. (2012). *An Introduction to the philosophy of mathematics* (Cambridge Introductions to Philosophy). Cambridge: Cambridge University Press.

Courant, R., & Hilbert, D. (1953). *Methods of mathematical physics*. New York, NY: Interscience.

Curry, H. (1953). Mathematics, syntactics and logic. *Mind: A Quarterly Review of Psychology and Philosophy, 62*, 172.

Davies, P. C.W., Rieper, E., & Tuszynski, J. A. 2012. Self-organization and entropy reduction in a living cell. *BioSystems, 111*(1), 1-10. doi: 10.1016/j.biosystems.2012. 10.005

Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics, 61*(3), 293-319. doi: 10.1007/s10649-006-2372-4

Deleuze, G., & Guattari, F. (1987). *A thousand plateaus: Capitalism and schizophrenia*. Minneapolis, MN: University of Minnesota Press.

del Palacio, A.C.A. (2006). What is philosophy of mathematics looking for. In R. Hersh, (Ed.), *18 Unconventional essays on the nature of mathematics* (pp. 236-249). Dordrecht: Springer-Verlag New York.

Devlin, K. (2008). A mathematician reflects on the useful and reliable illusion of reality in mathematics. *Erkenntnis, 68*(3), 359-379. doi: 10.1007/s10670-008-9105-2

Eliot, T.S. (1922). The wasteland. *The Dial*, 73(5), (473-485). Dial Publishing Company: New York, NY.

Egorov, V.V. (2010). Information as a thermodynamic function of a living system. *Russian Agricultural Sciences, 36*(3), 208-210. doi: 10.3103/S106836741003016X

Ernest, P. (2006). A semiotic perspective of mathematical activity: The case of number. *Educational Studies in Mathematics*, *61*, 67-2), p.67-101. doi: 10.1007/s10649-006-6423-7

Ernest, P. (2005). Agency and creativity in the semiotics of learning mathematics. In M. Hoffmann, J. Lenhard, & F. Seeger (Eds.), *Activity and Sign Grounding Mathematics Education* (pp. 23-34). Dordrecht: Springer-Verlag New York.

Ernest, P. (1999). Forms of knowledge in mathematics and mathematics education: Philosophical and rhetorical perspectives. *Educational Studies in Mathematics, 38*(1), 67-83. doi: 10.1023/A:1003577024357

Ernest, P. (1992). The nature of mathematics: Towards a social constructivist account..*Science & Education, 1*(1), 89-100. doi: 10.1007/BF00430212

Ernest, P. (1991). *Philosophy of mathematics education*. Independence, MO: Taylor and Francis.

Fauconnier, G., & Turner, M. (2008). Rethinking metaphor. In R.W. Gibbs (Ed.), *The Cambridge handbook of metaphor and thought* (pp. 53-66). New York, NY: Cambridge University Press.

Font, V., Godino, J., Planas, N., & Acevedo, J. (2010). The object metaphor and synecdoche in mathematics classroom discourse. *For the Learning of Mathematics, 30*(1), 15-19.

Franchella, M. (1995). L.E.J. Brouwer: Toward intuitionistic logic. *Historia Mathematica, 22*(3), 304-322. doi: 10.1006/hmat.1995.1026

Frankenstein, M. (1990). Incorporating race, gender, and class issues into a critical mathematical literacy curriculum. *Journal of Negro Education, 59*(3), 336-47.

Frege, G. (1917). Class, function, concept, relation. *The Monist, 27*(1), 114.

Gabbay, M. (2010). A Formalist philosophy of mathematics part I: Arithmetic. *Studia Logica*, *96*(2), 219-238. doi: 10.1007/s11225-010-9283-1

Gibbs, R.W. (2003). Embodied experience and linguistic meaning. *Brain and Language, 84*(1). 1-15. doi:10.1016/S0093-934X(02)00517-5

Gibbs, R. W. (1996). Why many concepts are metaphorical. *Cognition, 61*(3), 309-319. doi:10.1016/S0010-0277(96)00723-8

Glucksberg, S. (2008). How metaphors create categories – quickly. In R. W. Gibbs (Ed.), *The Cambridge handbook of metaphor and thought* (pp. 67-83). New York, NY: Cambridge University Press.

Glucksberg, S. (2003). The psycholinguistics of metaphor. *Trends in Cognitive Science, 7*(2), 92-96. doi: 10.1016/S1364-6613(02)00040-2

Glucksberg, S., & McGlone. M. S. (1999). When love is not a journey: What metaphors mean. *Journal of Pragmatics, 31*(12), 1541-1558. doi:10.1016/S0387-2166(99)00003-X

Gonzalez Thompson, A. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics*, *15*(2), 105-127.

Gowers, W.T. (2006). Does mathematics need a philosophy? In Hersh, R. (Ed.), *18 Unconventional essays on the nature of mathematics* (pp. 182-200). Dordrecht: Springer-Verlag New York.

Gunaratne, S. (2007). Systems approaches and communication research: The age of entropy. *Communications, 32*(1), 79-96. doi: 10.1515/COMMUN.2007.004

Hale, B. (2005). Real numbers and set theory – Extending the neo-Fregean programme beyond arithmetic. *Synthese, 147*(1), 21-41. doi: 10.1007/s11229-004-6205-7

Hale, B. (1994). Is Platonism epistemologically bankrupt? *The Philosophical Review, 103*(2). 299-325. doi: 10.2307/2185739

Hansen, M.B.N. (2009). System-environment hybrids. In B. Clarke, B. & M. B. N. Hansen (Eds.), *Emergence and embodiment: New essays on second order systems theory* pp. (113-142). London, England: Duke University Press.

Hersh, R. (1997). *What is mathematics, really?* New York, NY: Oxford University Press.

Hersh, R. (1995). Fresh breezes in the philosophy of mathematics. *The American Mathematical Monthly, 102*(7), 589.

Hersh, R. (1991). Mathematics has a front and a back. *Synthese, 88*(2), 127-133. doi: 10.1007/BF00567741

Ho, M-W., & Popp, F-P. (1993). Biological organization, coherence, and light emission from living organisms. SFI studies in the sciences of complexity, Lect. Note Vol. III. In W. Stein & F. J. Varela (Eds.), *Thinking about biology* (pp. 183-214). Reading, MA: Addison Wesley Publishing Company.

Jacquette, D. (2006). Applied mathematics in the sciences. *Croatian Journal of Philosophy,* (17), 237-267.

Jatsch, E. (1974). Organising the human world: An evolutionary outlook. *Futures, 6*(1), 4-15. doi: 10.1016/0016-3287(74)90003-2

Johnson, M. (2008). Philosophy's debt to metaphor. In Gibbs, R. W. (Ed.), *The Cambridge handbook of metaphor and thought* (pp. 39-52). New York, NY: Cambridge University Press.

Katz. A. N. (1998). Figurative language and figurative thought. In A. N. Katz, C. Cacciara, R. W. Gibbs, & M. Turner (Eds.), *Figurative language and thought* (pp. 3-43). New York, NY: Oxford University Press.

Kauffman, S. A. (1991). Antichaos and adaptation. *Scientific American, 265*(2), 78-84. doi: 10.1038/scientificamerican0891-782

Kovecses, Z. (2005). *Metaphor in culture: Universality and variation*. New York, NY: Cambridge University Press.

Kovecses, Z. (2002). *Metaphor: A practical introduction*. New York, NY: Oxford University Press.

Lakoff, G. (2008). The neural theory of metaphor. In R. W. Gibbs (Ed.), *The Cambridge Handbook of metaphor and thought* (pp. 17-38). New York, NY: Cambridge University Press.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Lakoff, G., & Johnson, M. (1999). *Philosophy in the flesh*. New York, NY: Basic Books.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. Chicago, IL: University of Chicago Press.

Lakoff, G., & Turner, M. (1989). *More than cool reason: A field guide to poetic metaphor.* Chicago: Chicago University Press.

Lerman, S. (1990). Alternative perspectives of the nature of mathematics and their influence on the teaching of mathematics. *British Educational Research Journal,16*(1), 53-61. doi: 10.1080/0141192900160105

Longo, G., & Montevil, M. (2014). *Perspectives on organisms: Biological time, symmetries and singularities*. New York, NY: Springer.

Ludovisi, A., Pandolfi, P. P., & Illuminata Taticchi, M. (2005). The strategy of ecosystem development: specific dissipation as an indicator of ecosystem maturity. *Journal of Theoretical Biology 235*(1). 33–43. doi: 10.1016/j.jtbi.2004. 12.017

Luhmann, N. (2013). *Introduction to systems theory* (paperback ed.). (Gilgen, P. Trans). Cambridge, UK: Polity.

Luhmann, N. (2009). Self-organization and autopoesis. In B. Clarke & M. B. N. Hansen (Eds.), *Emergence and embodiment: New essays on second order systems theory* (pp. 143-156). London, England: Duke University Press.

Luhmann, N. (1995). *Social systems* (J. Bednarz, Jr., Trans.). Stanford, CA: Stanford University Press. (Original published 1984)

Luisi, P. L. (1993). Defining the transition to life: Self-replicating bounded structures and chemical autopoesis. SFI studies in the sciences of complexity, Lect. Note Vol. III. In W. Stein & F.J. Varela (Eds.), *Thinking about Biology* (pp. 17-40). Reading, MA: Addison Wesley Publishing Company.

Manitoba Education. (2013). *Kindergarten to Grade 8 mathematics: Manitoba curriculum framework of outcomes*. Winnipeg, MB: Government of Manitoba.

Marcos, A. (2011). Bioinformation as a triadic relation. In G. Terzis & R. Arp (Eds.), *Information and Living Systems: Philosophical and Scientific Perspectives* (pp. 55-90). Cambridge, MT: The MIT Press.

Maturana, H. (1999). The organization of the living: A theory of living organization. *International Journal of Human-Computer Studies, 51*(2), 149-168. doi: 10.1006/ijhc.1974.0304

Maturana, H. R., & Varela, F. J. (1987). *The tree of knowledge: The Biological Roots of Human Understanding*. Boston, MA: New Science Library.

Maturana, H., & Varela, F. J. (1980). *Autopoiesis and cognition: The realization of the living* (Boston studies in the philosophy of science; v. 42). Dordrecht: D. Reidel Pub.

McCarty, C. (2008). The new Intuitionism. In M. van Atten, P. Boldini, M. Bourdeau, and G. Heinzmann (Eds.), *One hundred years of Intuitionism (1907-2007)* (pp. 37-49). Boston, MA: Birkhauser Verlag AG.

Miller, J. G. (1978). *Living systems*. New York: NY: McGraw-Hill Book Company.

Mingers, J. (2002). Can social systems be autopoietic? Assessing Luhmann's social theory. *Sociological Review, 50*(2), 278-299. doi: 10.1111/1467-954X.00367

Moreno, A., & Ruiz-Mirazo, K. (2011) The informational nature of biological causality. In Terzis, G. & Arp, R. (Eds.), *Information and living systems: Philosophical and scientific perspectives* (pp. 157-176). Cambridge, MT: The MIT Press.

Noss, R. (2002). Mathematical epistemologies at work. *For the Learning of Mathematics, 22*(2), 2-13.

Noyes, A. (2006). Using metaphor in mathematics teacher preparation. *Teaching & Teacher Education: An International Journal of Research and Studies,22*(7), 898-909.

Núñez, R. (2011). No innate number line in the human brain. *Journal of Cross - Cultural Psychology, 42*(4), 651. doi: 10.1177/0022022111406097

Núñez, R. (2009), Numbers and arithmetic: Neither hardwired nor out there. *Biological Theory 4*(1). 68-83.

Núñez R. (2008). A fresh look at the foundations of mathematics: Gesture and the psychological reality of conceptual metaphor. In A. Cienki & C. Müller (Eds.), *Gesture and metaphor (pp. 93—114).* Amsterdam: John Benjamins.

Núñez, R. (2008). Conceptual metaphor, human cognition and the nature of mathematics. In Gibbs, R. (Ed.), *The Cambridge handbook of metaphor and thought* (Xiii-Xiv) (pp. 339-362). Cambridge: Cambridge University Press.

Núñez, R., Edwards, E., & Matos, L .F. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics, 39*(1), 45-65. doi: 10.1023/A:1003759711966

Parks, A. N. (2010), Metaphors of hierarchy in mathematics education discourse: The narrow path. *Journal of Curriculum Studies, 42*(1). 79-97. doi: 10.1080/00220270903167743

Peckhaus, V. (2003). The pragmatism of Hilbert's program. *Synthese, 137*(1-2) 141-156. doi: 10.1023/A:1026235118657

Pimm, D. (1988). Mathematical metaphor. *For the Learning of Mathematics 8(*1). 30-34.

Pimm, D. (1981). Metaphor and analogy in mathematics. *For the Learning of Mathematics, 1*(3). 47-50.

Presmeg, N. (2005). Metaphor and metonymy in processes of semiosis in mathematics education. In M. Hoffman, J. Lenhard & F. Seeger (Eds), *Activity and sign grounding mathematics education* (pp. 105-115). Dordrecht: Springer-Verlag New York.

Presmeg, N. (1998). Metaphoric and metonymic signification in mathematics. *Journal of Mathematical Behavior, 17*(1), 25-32.

Prigogine, I. (1980). *From being to becoming: Time and complexity in the physical sciences.* San Francisco, CA: W. H. Freeman and Company.

Prigogine, I., & Stengers, I. (1984). *Order out of chaos*. New York, NY: Bantam Books.

Prigogine, I. (1989). The philosophy of instability. *Futures, 21*(4), 396-400. doi: 10.1016/S0016-3287(89)80009-6

Prigogine, I. (1989). What is entropy? *Naturwissenschaften, 76*(1), 1-8. doi: 10.1007/BF00368303

Protevi, J. (2009). Beyond autopoesis. In B. Clarke, B. & M. B. N. Hansen (Eds.), *Emergence and embodiment: New essays on second order systems theory* (pp. 94-112). London, England: Duke University Press.

Rav, Y. (2006). Philosophical problems of mathematics in the light of evolutionary epistomology. In R. Hersh, (Ed.), *18 unconventional essays on the nature of mathematics* (pp. 71-96). Dordrecht: Springer-Verlag New York.

Resnik, M. (1975). Mathematical knowledge and pattern cognition. *Canadian Journal of Philosophy, 5*(1), 25-39. doi: 10.1080/00455091.1975.10716095

Robert, J. S. (2007). Molecular and systems biology and bioethics. In D. L. Hull & M. Ruse (Eds.), *The Cambridge companion to the philosophy of biology* (pp. 361-371). New York, NY: Cambridge University Press.

Rotman, B. (2006). Towards a semiotics of mathematics. In R. Hersh (Eds.), *18 unconventional essays on the nature of mathematics* (pp. 97-127). Dordrecht: Springer-Verlag New York.

Schick, A., Neale, H., Pugalee, D., & Cifarelli, V. (2008). Using metaphors to unpack student beliefs about mathematics. *School Science and Mathematics. 108* (7). 326-333.

Schrödinger, E. (1967). *What is life? The physical aspect of the living cell & mind and matter*. London, England: Cambridge University Press.

Shapiro. (2009). Life on the ship of Neurath: Mathematics in the philosophy of mathematics. *Croatian Journal of Philosophy*, (26), 149-166.

Shapiro, S. (1983). Mathematics and reality. *Philosophy of Science, 50*(4), 523-548. doi: 10.1086/289138

Showalter, K. (2000). Controlling chaos in dynamical systems. In J. Walleczek (Ed.), *Self-organized biological dynamics and non-linear control: Towards understanding complexity,* *chaos and emergent function in living systems* (pp. 328-340). Cambridge, United Kingdom: Cambridge University Press.

Simms, J. R. (2003). Living systems science methodology for managing complexity and change. *Systems Research and Behavioral Science*, *20*(1), 401-408. doi: 10.1002/sres.563

Sterenberg, G. (2008). Investigating teachers' images of mathematics. *Journal of Mathematics Teacher Education*, *11*(2), 89-105. doi: 10.1007/s10857-007-9062-8

Taylor, P. (1996). Mythmaking and mythbreaking in the mathematics classroom. *Educational Studies in Mathematics, 31*(1), 151-173.

Trainor, L.E.H. (2011). Modeling the behaviour of ant colonies as an emergent property of a system of ant-ant interactions. In G. Terzis & R. Arp (Eds.), *Information and living systems: Philosophical and scientific perspectives* (pp. 303-313). Cambridge, MT: The MIT Press.

Van Atten, M. (2003). Brouwer, as never read by Husserl. *Synthese, 137*(1), 3-19. doi: 10.1023/A:1026270631861

Van Dalen, D. (2008). Another look at Brouwer's dissertation. In M. van Atten, P. Boldini, M. Bourdeau & G. Heinzmann (Eds.), *One hundred years of intuitionism (1907-2007)* (pp. 3-20). Boston, MA: Birkhauser Verlag AG.

Van de Walle, J., & Lovin, L. H. (2006). *Teaching student-centered mathematics. Grades 5-8* (Van de Walle, John A. Van de Walle professional mathematics series; v. 3). Boston: Pearson Allyn & Bacon.

Van Leeuwen, J., & Wieredermann, J. (2000). The Turing machine paradigm in contemporary computing. *Mathematics Unlimited 2001 and Beyond*. LNCS. 1139-1155. Springer-Verlag.

Varela, F. J. (1997). Patterns of life: Intertwining identity and cognition. *Brain and Cognition, 34*(1), 72-87. doi: 10.1006/brcg.1997.0907.

Varela, F., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. Cambridge, MA.: MIT Press.

Von Bertalanffy, L. (1950). The theory of open systems in physics and biology. *Science (New Series), 111*(2872), 23-29.

Wagner, A. (2005). *Robustness and evolvability in living systems.* Princeton, NJ: Princeton University Press.

Watson, A. (2008). School mathematics as a special kind of mathematics. *For the Learning of Mathematics, 28*(3), 3-7.

Weber, B. (2010). What is life: Defining life in the context of emergent complexity. *Origins of Life and Evolution of Biospheres, 40(*2), 221-229. doi: 10.1007/s11084-010-9203-4

Weir, A. (2011). *Truth through proof: A Formalist foundation for mathematics*. New York, NY: Oxford University Press.

White, L. (2006). The locus of mathematical reality: An anthropological footnote. In R. Hersh, (Ed.), *18 unconventional essays on the nature of mathematics* (pp. 304-319). Dordrecht: Springer-Verlag New York.

Williams, T. (1985). A science of change and complexity. *Futures*, 17(3), 263-268. doi: 10.1016/0016-3287(85)90037-0

Wittgenstein, L. (1963). *Philosophical investigations* (English text reprinted ed.). Oxford: Blackwell.