FOUR PROBLEMS IN COMBINATORIAL DESIGNS.

by

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Table of Contents.

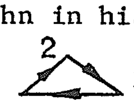
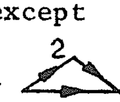
CHAPTER I

Introduction

This thesis considers four different problems in combinatorial design: Steiner Systems, Directed Triple Systems, Handcuffed Designs and Howell Designs.

In Chapter II we determine the number of isomorphism classes in the Steiner System $S(3, 4, 14)$. Since $S(3, 4, 14)$ is the first quadruple system which could have more than one isomorphism class, a series of computer programs have been developed to compute Steiner Systems. By considering the possible ways in which the two non-isomorphic systems of type $S(2, 3, 13)$ may be embedded in $S(3, 4, 14)$ it has been determined that apart from isomorphism there are exactly four non-isomorphic systems of type $S(3, 4, 14)$. Their automorphism groups have been determined. Furthermore, none of these can be embedded in an $S(4, 5, 15)$ so that $S(4, 5, 15)$ does not exist.

The main purpose of the remaining chapters is to consider the construction of some combinatorial designs. Basically, we use direct construction methods, although some recursive constructions are given.

In Chapter III and Chapter IV we consider the block designs on a graph. First to be studied are directed triple systems which are examples of block designs on directed graphs. These can be seen as a natural generalization of Steiner Triple Systems. N.S. Mendelsohn in his paper [5] has shown that the cyclic triple systems on the graph 1  3 exist for all positive integral v such that $v \not\equiv 2 \bmod 3$ except $v = 1$ and $v = 6$. For triple systems another possible graph is 1  3 . We call triple systems on this graph directed triple systems. In Chapter III we prove that for this kind of triple system the necessary conditions for existence are also sufficient. i.e. For all positive integral v, the conditions are $v \not\equiv 2 \mod 3$ and $v \neq 1$.

Another kind of block design on graphs is the handcuffed design. A handcuffed design with parameters v, k, λ consists of a system of ordered k-subsets of a v-set, called handcuffed blocks. In a block $\{A_1, A_2, \cdots, A_k\}$ each element is assumed to be handcuffed to its neighbours and the block contains $k-1$ handcuffed pairs $(A_1, A_2), (A_2, A_3), \cdots, (A_{k-1}, A_k)$. These pairs are considered unordered. The collection of handcuffed blocks constitute a handcuffed design if the following are satisfied: (1) each element of the v-set appears amongst the blocks the same number of times (and at most once in a block) and (2) each pair of distinct elements of the v-set are handcuffed in exactly λ of the blocks.

We denote by $H(v, k, \lambda)$ the class of all handcuffed designs with parameters v, k, λ and say $H(v, k, \lambda)$ exists if there is a design with parameters v, k, λ.

In Chapter IV we prove that the necessary conditions for $H(v, k, \lambda)$ to exist are also sufficient in the following cases: (a) $\lambda = 1$ or 2; (b) $k = 3$; (c) k is even $k = 2h$, and $(\lambda, 2h-1) = 1$; (d) k is odd, $k = 2h+1$, and $(\lambda, 4h) = 1$ or $(\lambda, 4h) = 2$.

In Chapter V we consider the Howell Designs. These can be seen as a generalization of Room Designs. Howell Designs have been used to construct schedules for bridge movements for a long time, but to consider them in the most general way is a development of the last two years. In this chapter we generalized the starter-adder method, the multiplication theorem, the recursive construction theorem, which have been useful tools in the study of Room Designs. Especially, we can use the generalized starter-adder method to construct almost every type of Howell Designs except some of lower order sides, and such types we have proved cannot be constructed by the starter-adder method. The method for constructing the starter-adder for the most general case is given. For $H(n+k, 2n)$ only finitely many lower values of n have to use special methods. Complete information is given for $k = 0, \cdots, 10$, the only unknown cases being $H(8, 10)$ and $H(2n+1, 2n)$, where n is even and $n \geq 10$.

CHAPTER II

On the Steiner Systems $S(3, 4, 14)$ and $S(4, 5, 15)$

1. Introduction. Let $T = \langle Q, B, t, k, n\rangle$ be a system where Q is an n-set, B is a collection of k-subsets of the elements of Q called blocks, such that any t-subset of Q belongs to exactly one block. We call T a Steiner system and denote by $S(t, k, n)$ the set of all T with parameters t, k, n. Instead of writing $T \in S(t, k, n)$ we will often say T is an $S(t, k, n)$. Problems of importance are the number of distinct isomorphism classes belonging to $S(t, k, n)$ and the automorphism groups of a T, where $T \in S(t, k, n)$. $T = \langle Q, B, t, k, n\rangle$ and $T^* = \langle Q^*, B^*, t, k, n\rangle$ are isomorphic if there is a bijection from Q to Q^* which maps the blocks of B onto the blocks of B^*. An automorphism of T is a permutation of Q which also permutes the blocks of B.

If $T \in S(2, 3, n)$ we refer to T as a triple system. A triple system exists if and only if $n \equiv 1 \bmod 6$ or $n \equiv 3 \bmod 6$, and the number of blocks is

$$b = \frac{n(n-1)}{6}.$$

Each element appears r times amongst the blocks where

$$r = \frac{n-1}{2}.$$

For $n = 3, 7, 9$ there is only one system, apart from isomorphism. For $n = 13$ there are two isomorphism classes and for $n = 15$ the number of isomorphism classes is 80. The number of classes increases very rapidly with n . In private correspondence with Jean Doyen the author has been informed that R. M. Wilson has proved that for $n = 19$ at least 2,450 classes exist, for $n = 21$ the number is greater than 2,160,980 and for $n = 25$, the number is greater than 163,929,929,318,400.

If $T \in S(3, 4, n)$, we refer to T as a quadruple system. In [3], H. Hanani shows that a quadruple system exists if and only if $n \equiv 2 \bmod 6$ or $n \equiv 4 \bmod 6$. In a quadruple system on n elements if there are b blocks, and if each element appears in r blocks and each pair of elements appears in λ blocks then

$$b = \frac{n(n-1)(n-2)}{24}, \quad r = \frac{(n-1)(n-2)}{6}, \quad \lambda = \frac{n-2}{2}.$$

For $n = 4, 8, 10$ there is only one system apart from isomorphism [1]. For $n = 14$, in [1], Bays and De Weck had conjectured the existence of at least two isomorphism classes. This work goes back to 1935. The computational problem is formidable.

The main purpose of this chapter is to determine the number of isomorphism classes in $S(3, 4, 14)$ and the automorphism groups of each class. The result is that the number of classes is 4. These, together with their automorphism groups, are given in the main body of the paper and the appendices. The results are used to show that the class of $S(4, 5, 15)$ is empty. The results have been obtained using a number of computer algorithms described in what follows.

2. <u>Initiating the Computation.</u> The computation consists of two main stages - the construction of the systems and then the reduction by means of isomorphisms.

First, it is clear that the blocks of an S(3, 4, 14) which contain a fixed element form an S(2, 3, 13) when that element is removed from each of the blocks. If the fixed element removed is e, the corresponding S(2, 3, 13) is called the system associated with e. An S(3, 4, 14) contains 14 associated S(2, 3, 13) . To start a computation we take an S(2, 3, 13) on the elements 2, 3, 4, ···, 14 and to each of the triples we append the element 1. Since there are only two non-isomorphic S(2, 3, 13) there are essentially only two possible starts for an S(3, 4, 14) . We can obtain all non-isomorphic S(3, 4, 14) by finding the systems obtained from these two starts and reducing the systems thus found by isomorphism.

The two non-isomorphic S(2, 3, 13) on elements (2, 3, ···, 14) are denoted by A and B and are given by:

A	A	B	B
2 3 4	5 8 10	2 3 4	5 8 10
2 5 6	5 11 14	2 5 6	5 11 14
2 7 8	5 12 13	2 7 8	5 12 13
2 9 10	6 7 11^*	2 9 10	6 7 14^*
2 11 12	6 9 12	2 11 12	6 9 12
2 13 14	6 10 14^*	2 13 14	6 10 11^*
3 5 7	7 9 13	3 5 7	7 9 13
3 6 8	7 10 12	3 6 8	7 10 12
3 9 11	8 9 14	3 9 11	8 9 14
3 10 13	8 11 13	3 10 13	8 11 13
3 12 14		3 12 14	
4 5 9		4 5 9	
4 6 13		4 6 13	
4 7 14^*		4 7 11^*	
4 8 12		4 8 12	
4 10 11^*		4 10 14^*	

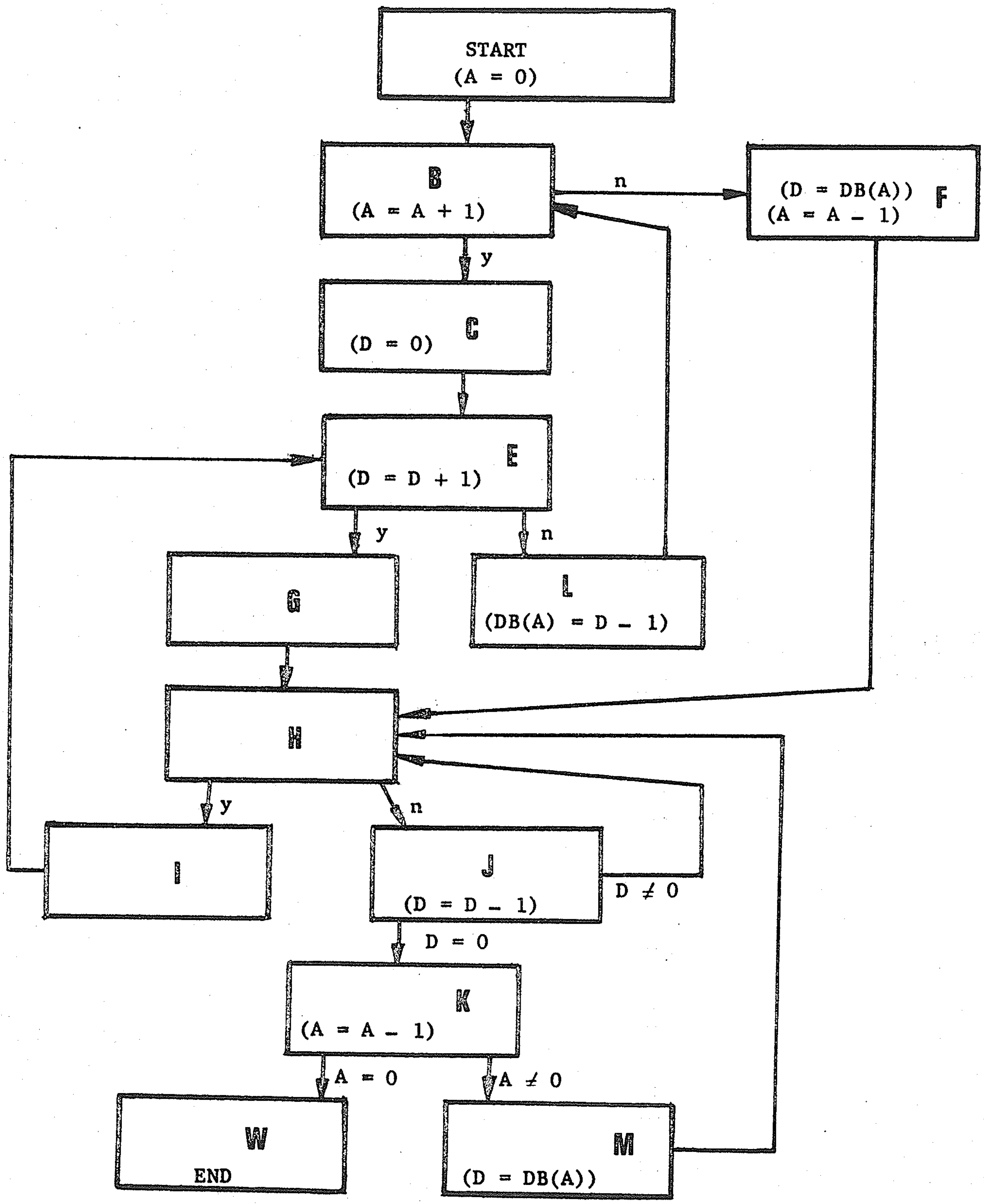
These systems differ only in four triples which are followed by an asterisk and result from two interchanges of 11 and 14. System A has an automorphism group of order 6 while system B has one of order 39. To distinguish system A from system B in isomorphic copies the notion of interlacing due to F. N. Cole [2] is used. We say x and y are interlaced if a, b, c, d exist such that $x\,a\,b$, $x\,c\,d$, $y\,a\,c$, $y\,b\,d$ are triples of the system. Cole discovered that for system B every element interlaces with six other elements while with system A every element interlaces with 3 or 4 other elements. Hence, to discriminate A from B we need only compute the interlacings of a single element. When an $S(3, 4, 14)$ is completed we will assign a letter A or B to each symbol according as the associated $S(2, 3, 13)$ are of type A or type B. We will also call the quadruples obtained by adjoining 1 to each of the blocks A or B as a starting subquadruple system.

3. <u>The Main Program.</u> The program "comb 4" is written to find all possible quadruple systems from a starting subquadruple system. It can be used for $n = 4, 8, 10, 14$ and 16. (On the basis of experience an improved version is being written.)

There are two important control parameters in the program "comb 4", one for the level of quadruples A, and one for the level of triples D. (Unfortunately the letter A has been used in two senses, the control parameter and the type of $S(2, 3, 13)$. The context guarantees that there will be no confusion.) The exact role of parameters A and D will be clarified on examination of the program "comb 4". The basic code may be outlined as follows.

Routines	Operations performed	Successor Criterion	If yes go to	If no go to
Start	$A = 0$; let starting quadruple system be $Q(0)$.	No choice	B	B
B	$A = A + 1$. Find least element which appears less than r times in $Q(A-1)$ and denote it by $E(A)$.	Is there any $E(A)$ found?	C	F
C	$D = 0$; collect all blocks which contain $E(A)$ in $Q(A-1)$ and remove $E(A)$ from these blocks. Then there is a subtriple system as a starting system for the $S(2, 3, 13)$ associated with $E(A)$. Let the subsystem be $T(A, 0)$.	No choice	E	E
E	$D = D + 1$; find the least element which is not $E(A)$ and appears in $T(A, D-1)$ less than λ times and let it be $ED(A, D)$.	Is there any $ED(A, D)$ found?	G	L
F	$A = A - 1$; $D = DB(A)$. Print out the completed quadruple system $Q(A)$.	No choice	H	H
G	Determine elements available for use with present choice of $ED(A, D)$. Let the set of all such elements be $AB(A, D)$.	No choice	H	H

Routines	Operations performed	Successor Criterion	If yes go to	If no go to
H	Permute AB(A, D) until a case is possibly reached such that every pair can be put in a block with the present ED(A, D), but no pair would appear twice in T(A, D) and no triple would appear twice in Q(A) .	Is any new subsystem formed?	I	J
I	Store the new **blocks with** the other completed ones, then call the new subsystem T(A, D) .	No choice	E	E
J	D = D − 1	Is D = 0?	K	H
K	A = A − 1	Is A = 0?	W	M
L	DB(A) = D − 1, and the S(2,3,13) is completed. Add E(A) to each block of T(A, DB(A)) then store these new quadruples with other completed ones. Call the new subquadruple system Q(A).	No choice	B	B
M	D = DB(A)	No choice	H	H
W	END. All possible quadruple systems starting with this subsystem have been found.			


START
(A = 0)
B
(A = A + 1)
n
(D = DB(A))
(A = A – 1)
F
y
C
(D = 0)
E
(D = D + 1)
y
n
G
L
(DB(A) = D – 1)
H
y
n
I
J
(D = D – 1)
D ≠ 0
D = 0
K
(A = A – 1)
A = 0
A ≠ 0
W
END
M
(D = DB(A))

Step H involving, as it does, a systematic series of permutations is relatively complicated and will not be described here. As the whole program "comb 4" appears as Appendix I, the reader can cull out the details there.

The program "comb 4" is written in Fortran IV. When using it with IBM 360/65 and Fortran H compiler, the time taken from each start to the final result is 40 hours CPU time. The program requires 200 K core storage. All completed quadruple systems are punched out as data cards for future use.

The program is now being revised to avoid certain redundancies and it is expected that the modification will be several orders of magnitude faster. The revision will be more practical for a study of S(3, 4, 16).

4. Results of the Computation. Using triple system A as a starting point 26 quadruple systems denoted by $A_1, A_2, \cdots, A_{26}$ were obtained. With triple system B the number obtained was 65 which are denoted by $B_1, B_2, \cdots, B_{65}$. Appendix II gives a listing of the systems $A_1, A_2, \cdots, A_{26}$ and B_1, B_2.

For each system it was next determined which of the 14 associated triple systems was of type A or of type B. The computation is very rapid since the interlacing property is easily checked.

It turned out that there are three distinct classes as follows:

I. All fourteen of the associated triple systems are of type A.

II. Twelve are of type A and two of type B.

III. Six are of type A and eight are of type B.

As there are no quadruple systems in which all fourteen of the associated triple systems are of type B, it follows that any quadruple system which resulted from a start of type B must be isomorphic to one which resulted from a start of type A. It is thus unnecessary to analyse further the 65 quadruple systems $B_1, B_2, \cdots, B_{65}$.

In terms of classes I, II, III the twenty-six systems $A_1, A_2, \cdots, A_{26}$ break up as follows:

I: $A_1, A_2, A_4, A_6, A_8, A_9, A_{23}, A_{24}$.

II: $A_3, A_7, A_{10}, A_{11}, A_{12}, A_{13}, A_{15}, A_{17}, A_{19}, A_{20}, A_{25}, A_{26}$.

III: $A_5, A_{14}, A_{16}, A_{18}, A_{21}, A_{22}$.

5. Reduction to Isomorphism Classes. It is obvious that two systems coming from different classes I, II, III are non-isomorphic. The question is do classes I, II, III divide further. By means of computations described in what follows, it turns out that II and III each contain a single isomorphism class but that class I splits into two isomorphism classes:

I_a: $A_1, A_2, A_4, A_6, A_8, A_9$.

I_b: A_{23}, A_{24}.

The non-isomorphism of I_a and I_b is further confirmed by the fact that systems from these classes have different automorphism groups. In the confirmation of the isomorphism of the different systems in types I_a, I_b, II, III explicit permutations are obtained which map one system of a given class onto another.

The computation used is based on the method used by H. S. White [4] for the discrimination of triple systems. White's method produces automorphism groups and the permutations which map one system onto another.

In present day terminology the extension of White's methods to general Steiner systems may be described as follows.

A mapping of a finite set into itself can always be represented as a set of disjoint figures, each figure being a cycle with attached trees. We refer to these attached trees as appendices. Also if a cycle is divided into parts, each part with the attached appendices is called a section.

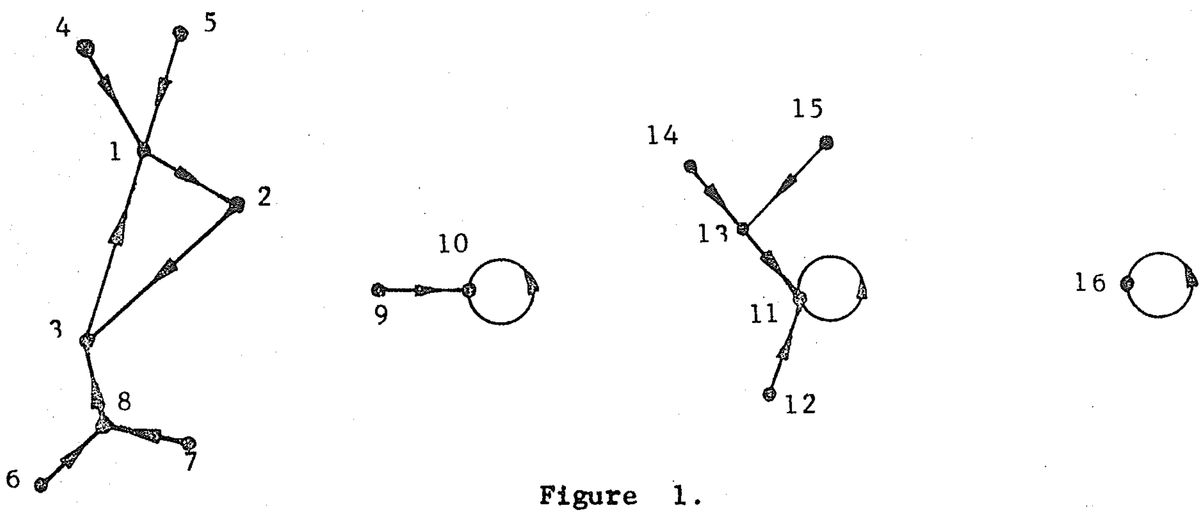


Figure 1.

For example, Figure 1 represents the mapping $1 \to 2$, $2 \to 3$, $3 \to 1$, $4 \to 1$, $5 \to 1$, $6 \to 8$, $7 \to 8$, $8 \to 3$, $9 \to 10$, $10 \to 10$, $11 \to 11$, $12 \to 11$, $13 \to 11$, $14 \to 13$, $15 \to 13$, $16 \to 16$. We then can look upon a quadruple system as an algebra belonging to an equational system with a single ternary operator. The construction is as follows. Suppose we have a system on n elements and $b = \frac{n(n-1)}{6}$ blocks. If $\{a, b, c, d\}$ are the members of a block define a ternary operator f by $f(a, b, c) = d$ and the following universal identities

$$f(x, y, z) = f(y, x, z) = f(x, z, y)$$

$$f(x, x, x) = x$$

$$f(x, x, y) = y$$

$$f(x, y, f(x, y, z)) = z \quad .$$

The basic idea of White's is the following. We can use the operator f to define a mapping φ of all the $\binom{n}{4}$ unordered quadruples on n elements into themselves by

$$\varphi\{a, b, c, d\} = \{f(b, c, d), f(a, c, d), f(a, b, d), f(a, b, c)\} \; .$$

In this mapping the only fixed elements are the quadruples of the system and these are the one-cycles in the associated figures. It is clear that if two quadruple systems are isomorphic the collection of figures associated with the two mappings of all $\binom{n}{4}$ quadruples into themselves are isomorphic and hence, in particular, have the same geometric structures.

In the actual calculations we proceeded as follows. From each of the classes I_a, I_b, II, III we choose a representative. These are respectively, A_1, A_{24}, B_1, B_2 . The reason for using B_1 and B_2 to represent classes II and III is that the computer takes less time to reproduce these. For each representative system the mapping φ was determined. These were used both for automorphisms and for finding a permutation which maps a system on one isomorphic to it.

The permutations mapping the representatives onto the other systems in each class are given as follows:

$A_1 \rightarrow A_2$: (8, 13)(1, 4, 11, 10)(2, 7, 5, 6)(3, 12, 9, 14)

A_4 : (10)(7, 12, 14, 8)(1, 9, 5, 6, 13, 2, 11, 4, 3)

A_6 : (8, 12)(1, 2, 11, 14, 6, 13, 4, 5, 9, 3, 10, 7)

A_8 : (11)(5, 14, 8, 6)(1, 2, 10, 4, 9, 12, 7, 13, 3)

A_9 : (5, 8)(1, 2, 7, 4, 9, 13, 10, 12, 14, 11, 3, 6)

$A_{24} \rightarrow A_{23}$: (1, 5)(4, 13, 14)(8, 10)(9, 12, 11)(2)(3)(6)(7)

$B_1 \rightarrow A_3$: (12)(2, 9, 8)(1, 3, 10, 4, 6, 11, 7, 13, 14, 5)

A_7 : (4)(5)(6)(1, 11, 10, 9, 2, 8, 13, 12, 3, 14, 7)

A_{10} : (1, 6)(4, 5, 9, 8, 11)(2, 12, 3, 7, 14, 10, 13)

A_{11} : (9)(1, 14, 11, 6, 3, 4)(2, 7, 10, 5, 8, 13, 12)

A_{12} : (2)(5, 8)(1, 3, 4)(6, 14, 10, 13, 12, 7, 11, 9)

A_{13} : (5)(7)(10)(2, 14, 4)(1, 6, 3, 9)(8, 13, 11, 12)

A_{15} : (11)(1, 9, 6, 13)(2, 10, 5, 14)(3, 4, 7, 12, 8)

A_{17} : (3)(8)(10, 14)(1, 4, 2)(5, 6, 13, 11, 9, 7, 12)

A_{19} : (9)(1, 12, 13, 11, 6, 3)(2, 7, 8, 14, 5, 4, 10)

A_{20} : (5)(8)(1, 12, 6, 11, 13, 14)(2, 3, 9, 4, 10, 7)

A_{25} : (4)(10, 11)(3, 5, 8, 14)(1, 7, 6, 9, 13, 2, 12)

A_{26} : (10)(8, 9)(1, 2, 3, 11, 14, 13, 6, 4, 7, 12, 5)

$B_2 \rightarrow A_5$: (14)(2, 8, 3, 5)(1, 6, 10, 12, 9, 11, 4, 13, 7)

A_{14} : (5)(1, 2, 11, 8, 6, 7, 13, 9, 12, 14, 4, 3, 10)

A_{16} : (7, 13)(1, 4, 10)(2, 11, 8, 3, 4, 6, 9, 5, 12)

A_{18} : (1, 7)(2, 8)(3, 13, 14, 6, 11, 10, 5, 9, 4, 12)

A_{21} : (1, 3, 12, 7)(2, 8, 6, 4, 13, 9, 11, 10, 5, 14)

A_{22} : (1, 8)(2, 13, 3, 9, 4, 5, 7, 11, 6, 14, 12, 10)

6. <u>Determination of the Automorphism Groups.</u> In the generalized H. S. White figures it is only necessary to look at the longest cycles and their appendices.

For class I_a the system A_1 has two cycles of length 35, each of which is divided into seven sections. A complete picture of these two 35-cycles and their sections are given in Appendix 3. We denote the sections by $1\cdot1, 1\cdot2, \cdots, 1\cdot7$ and $2\cdot1, 2\cdot2, 2\cdot3, \cdots 2\cdot7$. It is easy to check that:

(1) Only the identity maps a section onto itself

(2) For any two sections there is exactly one permutation which maps the first onto the second.

Hence there are at most 14 permutations which can map a 35-cycle onto itself or onto the other 35-cycle. The automorphism group G_{A_1} of A_1 has order at most 14. After determining the fourteen mappings of the 35-cycles it is easy to check that these permutations are automorphisms. Actually the 14 automorphisms are generated by two generators x and y which are given by

$$x = (1, 5, 12, 14, 13, 2, 11)(3, 10, 4, 9, 8, 6, 7)$$

$$y = (1, 8)(2, 7)(3, 13)(4, 12)(5, 9)(6, 11)(10, 14) .$$

The way in which x and y map the 14 sections onto each other is given by the following diagram (Figure 2).

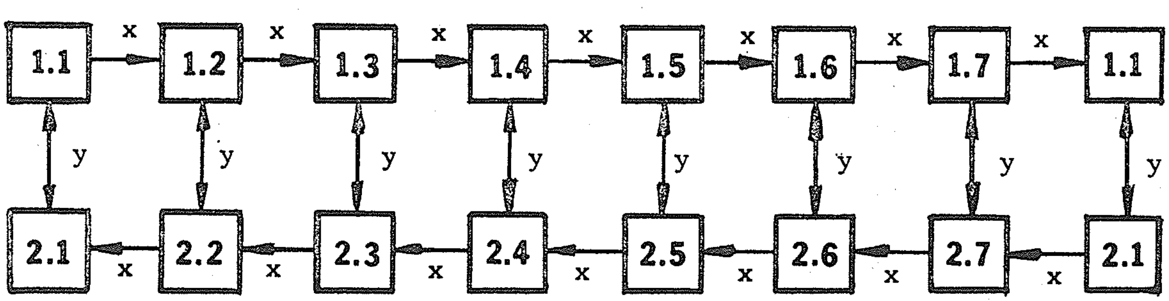


Figure 2.

x maps the first 35-cycle from left to right and the second from right to left and y interchanges corresponding sections. It is easy to see that $x^y = x^{-1}$.

Hence G_{A_1} has presentation

$$G_{A_1} = \{x, y ; x^7, y^2, x^y = x^{-1}\}$$

Hence G_{A_1} is the dihedral group of order 14. It is transitive on the elements.

The other classes I_b, II and III are treated similarly. Complete details of cycles and sections appear in Appendices 4, 5 and 6.

We summarize the results as follows:

Class I_b. Group $G_{A_{24}}$. This group is generated by

$x = (1, 10, 11, 2, 4, 3, 9)(5, 13, 7, 12, 6, 14, 8)$ and

$y = (1, 8)(2, 5, 3, 12, 9, 13)(4, 14, 11, 7, 10, 6)$.

This group is transitive on the elements, is of order 42 and has presentation

$$G_{A_{24}} = \{x, y; x^7, y^6, x^y = x^{-2}\} .$$

Class II. Group G_{B_1}. This group is generated by

$x = (1)(10)(2, 7, 8)(3, 14, 11)(4, 6, 13)(9, 12, 5)$

$y = (1, 10)(2, 12)(3, 6)(4, 14)(5, 8)(7, 9)(11, 13)$.

The group is intransitive and has presentation

$$G_{B_1} = \{x, y; x^3, y^2, x^y = x^{-1}\} \cong S_3 .$$

The point orbits are $\{1, 10\}, \{2, 5, 7, 8, 9, 12\}, \{3, 4, 6, 11, 13, 14\}$.

<u>Class III.</u> Group G_{B_2} . This group is generated by

$$x = (4)(5)(1, 3, 2)(6, 7, 8)(9, 12, 11)(10, 14, 13)$$

$$y = (1, 9)(2, 12)(3, 11)(4, 5)(6, 10)(7, 13)(8, 14) .$$

This group is intransitive and has presentation

$$G_{B_2} = \{x, y; x^3, y^2, x^y = x^{-1}\} \cong S_3 .$$

In spite of the fact that G_{B_1} and G_{B_2} are permutation isomorphic the systems B_1 and B_2 are non-isomorphic as proved previously. The orbits of G_{B_2} are $\{4, 5\}, \{1, 2, 3, 9, 11, 12\}\{6, 7, 8, 10, 13, 14\}$.

7. <u>The non-existence of S(4, 5, 15) .</u> By adding an element to the blocks of an S(3, 4, 14) we get a starting subsystem for an S(4, 5, 15) . As there are exactly four non-isomorphic S(3, 4, 14) then if an S(4, 5, 15) exists it must arise from one of these starts.

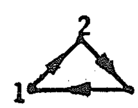
A program "comb 5" similar to "comb 4" has been written to construct an S(4, 5, 15) from a given start. Its accuracy was verified by using an S(3, 4, 10) to construct an S(4, 5, 11) and the results checked by hand. On using the program with each of the S(3, 4, 14) starts it was found that none of them led to an S(4, 5, 15) . Hence, S(4, 5, 15) does not exist. The program "comb 5" using 350K core can be used to study S(4, 5, 17). However these computations will be undertaken when the program is improved.

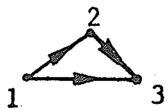
CHAPTER III

Directed Triple Systems

1. Introduction.

Directed triple systems are an example of block designs on directed graphs. A block design on a directed graph can be defined as follows. Let G be a directed graph of k vertices which contains no loops. Let S be a set of v elements. A collection of k-subsets of S with an assignment of the elements of each k-subset to the vertices of G is called a block design on G of order v if the following is satisfied. Any ordered pair of elements of S is assigned λ times to an edge of G. For example, if $S = \{a,b,c,d,e\}$ and $G = 1 \rightarrow 2 \leftarrow 3$ and b a e; c a d; a b c; d b e; a c d; b c e; a d b; c d e; a e d; b e c; is a collection of 3-subsets so written that in each subset the first element is assigned to the vertex 1, the second to 2 and the third to 3, then the collection is a block design on G with $\lambda = 1$.

In [5], it has been shown that, for the graph 

the design exists for all v such that $v \not\equiv 2 \bmod 3$ with the exceptions $v = 1$ and $v = 6$ (for technical reasons it is better to consider $v = 1$ as a non-exception). Such block designs are called cyclic triple systems (or generalized triple systems). Algebraically such cyclic triple systems correspond to a variety of quasigroups with identities $x^2 = x$ and $x(yx) = y$. In this chapter we investigate designs on the graph G given by 

. Such designs we designate as directed triple systems if $\lambda = 1$.

2. Directed Triple Systems.

Suppose that there is a directed triple system of order v. If we consider the triples of the system without reference to the graph G we obtain an associated B.I.B.D. with parameters, v, $k = 3$, $\lambda = 2$, $b = \frac{v(v-1)}{3}$ and $r = v - 1$. If this associated B.I.B.D. has no repeated blocks we say that the directed triple system is pure. When we write a triple with square brackets as $[a,b,c]$ we mean that in the graph G, a is assigned to the vertex 1, b to the vertex 2 and c to the vertex 3 and hence the triple $[a,b,c]$ will be said to contain the ordered pairs ab, ac, bc. In a cyclic triple system we will use the notation $\{a,b,c\}$ and say the triple $\{a,b,c\}$ contains the ordered pairs ab, bc and ca. It is clear then that $\{a,b,c\} = \{b,c,a\} = \{c,a,b\}$ but a corresponding result does not hold for directed triples. Suppose that $\{a,b,c\}$ and $\{b,a,d\}$ are two cyclic triples. Then it is obvious that the directed triples $[b,c,a]$ and $[a,d,b]$ contain the same ordered pairs as do the cyclic triples. Hence, we have proved the following lemma.

Lemma 1. Let S be a cyclic triple system of order v. Suppose that the blocks of S can be paired in such a way that in each pair of blocks one block contains an ordered pair ab and the second an ordered pair ba. Then the same blocks can be used to form a directed triple system of order v. In this case we say that the directed triple system is derived from the cyclic triple system.

There are various methods by which directed triple systems may be derived from cyclic triple systems. Besides pair replacement we can replace a set of three cyclic triples by three directed triples. In fact, the set of cyclic triples $\{a,b,c\}$, $\{a,e,b\}$ and $\{a,c,d\}$ contain the same ordered pairs as the set of directed triples $[b,a,c]$, $[a,e,b]$ and $[c,d,a]$. In what follows, our directed triple systems will be derived by a combination of pair replacements and replacements of sets of three triples.

Lemma 2. If $v \equiv 1$ or 3 mod 6 there is a directed triple system of order v.

Proof. Let S be a Steiner triple system of order v. To each block $a\ b\ c$ of S take the two blocks $[a\ b\ c]$, $[c\ b\ a]$. This collection of blocks forms a directed triple system of order v.

It is known [1], that there is no cyclic triple system of order 6. However, the directed triple system of order 6 exists as the following example shows. We use the triples $[1\ 2\ 3]$, $[2\ 1\ 4]$, $[3\ 2\ 5]$, $[4\ 2\ 6]$, $[4\ 1\ 5]$, $[5\ 3\ 4]$, $[3\ 1\ 6]$, $[6\ 4\ 3]$, $[5\ 6\ 1]$, $[6\ 5\ 2]$. This example also has the property that it is pure.

In what follows, we will be interested in directed triple systems which are derivable from cyclic triple systems. However, in all cases the results are more generally applicable to all directed triple systems. The reason is based on the following observation. Let S be a set and T a subset. Let S^* be the collection of directed triples of a directed triple system on S and suppose that T^* is a sub collection of S^* consisting entirely of directed triples of elements from T and such that T^* is a directed triple system. Let T^{**} be any other directed triple system on the elements of T. Then $S^{**} = (S^* - T^*) \cup T^{**}$ is a directed triple system on S. We call this the replacement of T^* by T^{**} in S^*.

3. Embedding Theorems.

Theorem 1. Let S be a cyclic triple system on v elements from which a directed triple system can be derived. Then there exists a cyclic triple system on $2v + 1$ elements from which a directed triple system can be derived.

Proof. Let $\{1,2,3,\cdots,v\}$ be the set elements on which S is a cyclic triple system and let elements $\{v+1, v+2,\cdots,2v+1\}$ be a set of $v+1$ new elements. We consider $\{v+1, v+2,\cdots, 2v+1\}$ as a complete set of residues mod $v+1$. To the cyclic triple system S adjoin the cyclic triples

$\{1,\ v+i,\ v+i+1\}$ $\qquad i = 1,2,\cdots,v+1$

$\{2,\ v+i,\ v+i+2\}$ $\qquad i = 1,2,\cdots,v+1$

$\{k,\ v+i,\ v+i+k\}$ $\qquad i = 1,2,\cdots,v+1$

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$\{v,\ v+i,\ 2v+i\}$ $\qquad i = 1,2,\cdots,v+1.$

Case 1. v even. Consider the triples of types $\{k,\ v+i,\ v+i+k\}$ and $\{v+1-k,\ v+j,\ 2v+1+j-k\}$. By choosing $j = k+i$ we obtain cyclic triples $\{k,\ v+i,\ v+i+k\}$ and $\{v+1-k,\ v+i+k,\ v+i\}$. These may be replaced by directed triples $[v+i+k,\ k,\ v+i]$ and $[v+i,\ v+1-k,\ v+i+k]$. As k ranges from 1 to $\frac{v}{2}$ and i from 1 to $v+1$ we obtain a complete replacement of cyclic triples by directed triples.

Case 2. v odd. The proof is the same as for v even but we call attention to the set of cyclic triples $\left\{\frac{v+1}{2},\ v+i,\ \frac{3v+2i+1}{2}\right\}$ $i = 1,2,\cdots,v+1$. Pairing the triples corresponding to i and $i+\frac{v+1}{2}$ we obtain cyclic triples $\left\{\frac{v+1}{2},\ v+i,\ \frac{3v+2i+1}{2}\right\}$ and $\left\{\frac{v+1}{2},\ \frac{3v+2i+1}{2},\ v+i\right\}$ and the corresponding directed triples $\left[v+i,\ \frac{v+1}{2},\ \frac{3v+2i+1}{2}\right]$ and $\left[\frac{3v+2i+1}{2},\ \frac{v+1}{2},\ v+i\right]$. This indicates that when v is odd the extended triple system is not pure.

Corollary. Any directed triple system on v elements can be embedded in a directed triple system on $2v+1$ elements. If v is even and the original system is pure then the embedding system so constructed is pure. (Note that this includes the embedding of a system on 6 elements in one on 13 elements even though the cyclic system on 6 elements does not exist).

Theorem 2. Let S be a cyclic triple system on v elements from which a directed triple system can be derived. Then S can be embedded in a cyclic system on $2v+4$ elements from which a directed triple system can be derived.

Proof. Let the system be on elements $1,2,\cdots,v$ and let $v+1$, $v+2$, $\cdots$, $2v+4$, be new elements. Add the cyclic triples, $\{k, v+i, v+k+2+i\}$; $i = 1,2,\cdots,v+4$, $k = 1,2,\cdots, v-2$ (here the second and third components are taken $\mathrm{mod}(v+4)$ amongst the residues $v+1$, $v+2$, $\cdots$, $2v+4$), $\{v-1, v+i, 2v+i+2\}$; $i = 1,2,\cdots,v+4$ (second and third components taken $\mathrm{mod}(v+4)$ amongst residues $v+1$, $v+2$, $\cdots$, $2v+4$), $\{v, v+i, v+i-1\}$; $i = 1, 2, \cdots, v+4$ (second and third components taken $\mathrm{mod}(v+4)$ amongst residues $v+1$, $v+2$, $\cdots$, $2v+4$), $\{v+i, v+i+1, v+i+3\}$; $i = 1, 2, \cdots, v+4$ and all components taken mod $v+4$ amongst residues $v+1$, $v+2$, $\cdots$, $2v+4$. For $k = 2,3,4, \cdots, \left[\frac{v}{2}\right]$ we can pair the triples which start with k with those which start with $v-k$ as follows. The triples starting with k are of form $\{k, v+i, v+i+k+2\}$. The triples starting with $v-k$, are of form $\{v-k, v+i+k+2, v+i\}$. The corresponding pairs of directed triples are $[v+i+k+2, k, v+i]$ and $[v+i, v-k, v+i+k+2]$. We are now left with four sets of triples, namely:

$S_1 \;=\; \{1,\ v+i,\ v+i+3\}$; $i = 1,2,\cdots,\ v+4$ (second and third components in range $v+1$, to $2v+4$ mod $v+4$).

$S_{v-1} = \{v-1,\ v+i,\ v+i-2\}$; $i = 1,2,\cdots,v+4$ (second and third components in range $v+1$ to $2v+4$ mod $v+4$).

$S_v \;=\; \{v,\ v+i,\ v+i-1\}$; $i = 1,2,\cdots,v+4$ (second and third components in range $v+1$ to $2v+4$ mod $v+4$) and $\hat{S} = \{v+i,\ v+i+1,\ v+i+3\}$; $i = 1,2,\cdots,v+4$ (all components in the range $v+1$ to $2v+4$ mod $v+4$).

We now distinguish 4 cases.

Case 1. $v \equiv 3$ mod 4.

Pair the triples of S_1 with $\hat{S}$ by pairing $\{1,\ v+i,\ v+i+3\}$ with $\{v+i,\ v+i+1,\ v+i+3\}$ for all i except that $\{1,\ 2v+4,\ v+3\}$ is not paired with $\{2v+4,\ v+1,\ v+3\}$ and $\{1,\ 2v+1,\ 2v+4\}$ is not paired with $\{2v+1,\ 2v+2,\ 2v+4\}$.

Pair the triples of S_{v-1} amongst themselves as follows. The triple $\{v-1,\ 2v+4,\ 2v+2\}$ is not paired; the triple $\{v-1,\ v+1,\ 2v+3\}$ is paired with $\{v-1,\ 2v+3,\ 2v+1\}$. There remain triples associated with values of $i = 2,3,\cdots,v+2$. Since $v \equiv 3$ mod 4 the number of these values is $v+1 \equiv 0$ mod 4.

Divide the values of i into consecutive sets of four, e.g. 2,3,4,5; 6,7,8,9; etc. In each set of four values of i pair the triples associated with the first and third value and those associated with the second and fourth values.

Pair the triples of S_v as follows. Leave the triple $\{v, v+1, 2v+4\}$ unpaired. For even i, $i = 2,4,6,\cdots, 2v+3$ pair the triples $\{v, v+i, v+i-1\}$ and $\{v, v+i+1, v+i\}$.

Now all triples have been paired except the following (we have written these cyclically with $2v+4$ first): $\{2v+4, v+3, 1\}$, $\{2v+4, v, v+1\}$, $\{2v+4, v+1, v+3\}$ and $\{2v+4, 2v+1, 2v+2\}$, $\{2v+4, 1, 2v+1\}$, $\{2v+4, 2v+2, v-1\}$. Triple replacement is now used on the first three and the second three triples.

Case 2. $v \equiv 1 \bmod 4$.

This is the same as $v \equiv 3 \bmod 4$ except for the pairing of elements of S_{v-1}. In this case we leave $\{v-1, 2v+4, 2v+2\}$ unpaired and the values of $i = 1,2,3,\cdots,2v+3$ are divided in sets of four 1,2,3,4; 5,6,7,8; etc. and in each set the triples corresponding to the first and third and the second and fourth values of i are paired.

Case 3. $v \equiv 0 \bmod 4$.

The pairing is exactly the same as the case $v \equiv 3 \bmod 4$ except that no triples are left unpaired.

Case 4. $v \equiv 2 \bmod 4$.

The pairing of triples is the same as in previous cases except that in S_1 and $\hat{S}$ the pairs $\{1, 2v, 2v+3\}$, $\{2v, 2v+1, 2v+3\}$ and $\{1, 2v+1, 2v+4\}$, $\{2v+1, 2v+2, 2v+4\}$ are not matched and in S_{v-1} the triples $\{v-1, 2v+3, 2v+1\}$ and $\{v-1, 2v+4, 2v+2\}$ are omitted from pair matchings. This leaves unpaired $\{1, 2v, 2v+3\}$, $\{v-1, 2v+3, 2v+1\}$, $\{2v, 2v+1, 2v+3\}$ and $\{1, 2v+1, 2v+4\}$, $\{v-1, 2v+4, 2v+2\}$ $\{2v+1, 2v+2, 2v+4\}$. We then apply the replacements to each of these two sets of three triples.

Corollary: A directed triple system on v elements can be embedded in a directed triple system on $2v+4$ elements.

Theorem 3. Let S be a cyclic triple system on v elements where $v \equiv 1 \bmod 3$, from which a directed triple system can be derived. Then S can be embedded in a cyclic triple system on $2v+2$ elements from which a directed triple system can be derived.

Proof. Let S be on $1,2,\cdots,v$ and add elements $v+1, v+2, \cdots, 2v+2$. The number $v+2$ of added elements is divisible by 3. Adjoin the following sets of triples.

$S_1 = \{1, v+i, v+i+1\}$ $i = 1,2,\cdots,v+2$; second and third components in range $v+1$, to $2v+2 \bmod v+2$. For $k = 2,3,\cdots,v-1$,

$S_k = \{k, v+i, v+i+k+1\}$, $i = 1,2,\cdots,v+2$; second and third components in range $v+1$ to $2v+2 \bmod v+2$.

S_v $= \{v, v+i, v+i-1\}$; $i = 1,4,7,\cdots,v$, second and third components in the range $v+1$ to $2v+2$ mod $v+2$:

$= \{v, v+i, v+i+2\}$; i in range 2 to $v+2$ and $i \not\equiv 1$ mod 3; and second and third components in range $v+1$ to $2v+2$ mod $v+2$).

$\hat{S}$ $= \{v+i, v+i+2, v+i+1\}$, $i = 1,4,7,\cdots,v$ all components in range $v+1$ to $2v+2$mod $v+2$.

The sets S_k, $k = 2,3,\cdots,v-2$ are matched by matching S_k with S_{v-k} using the pairing $\{k, v+i, v+i+k+1\}$ with $\{v-k, v+i+k+1, v+i\}$.

The sets S_1, $\hat{S}_{v-1}$, S_v and $\hat{S}$ are matched into pairs and sets of three as follows:

(a) Sets of three using a triple from $\hat{S}$ together with two triples of S_1; $\{1, v+i, v+i+1\}$, $\{1, v+i+1, v+i+2\}$, $\{v+i, v+i+2, v+i+1\}$ for $i = 1,4,7,10,\cdots,v$.

(b) Pairs consisting of a single triple from S_1 and a single triple from S_v; $\{1, v+i, v+i+1\}$ and $\{v, v+i+1, v+i\}$ for $i = 3,6,9,\cdots, v+2$.

(c) Pairs consisting of two triples from S_v; $\{v, v+i, v+i+2\}$, $\{v, v+i+2, v+i+4\}$ for $i = 3,6,9,\cdots v$.

(d) Sets of three using triples from S_{v-1}; $\{v-1, v+i, 2v+i\}$, $\{v-1, 2v+i, 2v+i-2\}$, $\{v-1, v+i+2, v+i\}$ for $i = 3,6,9,\cdots, 2v+2$.

This results in a complete matching.

<u>Corollary</u>. If $v \equiv 1$ mod 3 a directed triple system on v elements can be embedded in one on $2v+2$ elements.

Theorem 4. A directed triple system on v elements exists for all positive integral v except $v = 1$ or $v \equiv 2 \bmod 3$.

Proof. No system on v elements can exist if $v \equiv 2 \bmod 3$ since the corresponding B.I.B.D. parameters would not all be integral. By lemma 2, the directed triple system exists for $v \equiv 1$ or $3 \bmod 6$. If $v = 4$, the triples $[1,4,2]$, $[2,3,1]$, $[4,1,3]$, $[3,2,4]$ form a directed triple system. If $v = 6$ the example in section 2 shows that the system exists. There remain the general cases $v \equiv 0 \bmod 12$, $v \equiv 4 \bmod 12$, $v \equiv 6 \bmod 12$, $v \equiv 10 \bmod 12$. We use induction and theorems 2 and 3.

Case 1. $v \equiv 0 \bmod 12$. Here $v = 12u = 2\{6(u-1)+4\} + 4$ and the result follows from theorem 2.

Case 2. $v \equiv 4 \bmod 12$. Then $v = 4 + 12u = 2(6u+1) + 2$ and the result follows from theorem 3.

Case 3. $v \equiv 6 \bmod 12$. Then $v = 6 + 12u = 2(6u+1) + 4$ and the result follows from theorem 2.

Case 4. $v \equiv 10 \bmod 12$. Then $v = 10 + 12u = 2(6u+4) + 2$ and the result follows from theorem 3.

Corollary. For all v, $v \neq 1$, $v \neq 6$, $v \not\equiv 2 \bmod 3$ a directed triple system on v elements exists which is derivable from a cyclic triple system.

Proof. The only case to which the above proof does not apply is $v = 16 = 2(6) + 4$ since no cyclic triple system on six elements exists. For $v = 16$ the following example completes the proof. In this example the paired triples are separated by semicolons:

{1,2,3}, {2,1,4} ; {3,4,1}, {4,3,2}; {5,6,7}, {6,5,8}; {7,8,5}, {8,7,6};
{9,10,11}, {10,9,12}; {11,12,9}, {12,11,10}; {13,14,15}, {14,13,16};
{15,16,13}, {16,15,14};

{1,5,9}, {5,1,13}; {9,13,1}, {13,9,5}; {2,6,10}, {6,2,14}; {10,14,2},
{14,10,6};

{3,7,11}, {7,3,15}; {11,15,3}, {15,11,7}; {4,8,12}, {8,4,16};
{12,16,4}, {16,12,8};

{1,6,11}, {6,1,16}; {11,16,1}, {16,11,6}; {1,7,12}, {7,1,14};
{12,14,1}, {14,12,7};

{1,8,10}, {8,1,15}; {10,15,1}, {15,10,8}; {2,5,12}, {5,2,15};
{12,15,2}, {15,12,5};

{2,7,9}, {7,2,16}; {9,16,2}, {16,9,7}; {2,8,11}, {8,2,13};
{11,13,2}, {13,11,8};

{3,5,10}, {5,3,16}; {10,16,3}, {16,10,5}; {3,6,12}, {6,3,13};
{12,13,3}, {13,12,6};

{3,8,9}, {8,3,14}; {9,14,3}, {14,9,8}; {4,5,11}, {5,4,14};
{11,14,4}, {14,11,5};

{4,6,9}, {6,4,15}; {9,15,4}, {15,9,6}; {4,7,10}, {7,4,13};
{10,13,4}, {13,10,7}.

4. Remarks on Pure Systems.

The embedding theorems of the previous section do not necessarily produce pure systems from pure systems. Only embeddings given by our constructions are considered. We summarize the results here. In embedding a pure system on v elements into one of $2v+1$ elements the resultant system is pure, if and only if v is even. Our embedding of a system on v elements into one of $2v+2$ elements is pure if and only if v is odd. Our embedding of a system on v elements into one on $2v+4$ elements is pure if and only if v is odd.

There is a construction which always yields a pure system in cases $v = p$, p prime and $p \equiv 1 \bmod 3$ and $v = p^2$, p prime and $p \equiv 2 \bmod 3$. The proof is based on theorems 6 , 7 and 9 of [5]. Details are omitted.

CHAPTER IV

Handcuffed Designs

<u>Introduction.</u> There has recently appeared in the literature the notion of a block design on a graph. As the concept is quite new a description of this notion is given here. Let G be a graph on k vertices without loops or multiple edges. Denote the vertices by the integers $1, 2, 3, \cdots, k$. Let S be a v-set and let T be a collection of k-subsets of S. In the notation $\{v_1, v_2, \cdots, v_k\}$ it is assumed that the subscripts are an indexing of the elements by the vertices of G. If $\{v_1, v_2, \cdots, v_k\} \in T$ we call $\{v_1, v_2, \cdots, v_k\}$ a block and say the pair (v_i, v_j) is incident on the block if i and j are joined by an edge in G. We call (S, T, G) a block design on G if every element of S is in the same number r of the blocks, and if every pair of distinct elements of S are incident on the same number λ of the blocks. In the case where G is the graph given by

we call the system a handcuffed design. In a handcuffed design the only independent parameters are v, k, λ. We denote by $H(v, k, \lambda)$ the set of all handcuffed designs with parameters v, k, λ and $Q \in H(v, k, \lambda)$ means that Q is a handcuffed design with parameters v, k, λ. We say $H(v, k, \lambda)$ exists if there is a $Q \in H(v, k, \lambda)$. In a handcuffed design we will write a block as an ordered k-tuple $\{A_1, A_2, \cdots, A_k\}$ where each pair of consecutive elements is incident on the block. We call a pair A_i, A_{i+1} which is incident on a block a handcuffed pair. In each block $\{A_1, A_2, \cdots, A_k\}$ the elements $A_2, A_3, A_4, \cdots, A_{k-1}$ interior elements. We usually refer to the blocks as handcuffed blocks.

2. Necessary Conditions for the Existence of $H(v, k, \lambda)$.

Let $Q \in H(v, k, \lambda)$ and suppose $Q = (S, T. G)$. Let b be the number of blocks and let r be the number of times each element appears amongst the blocks. Let α be an element in S , and let u_α be the number of times α appears as an interior element of a block. By definition of λ α appears handcuffed λ times to each of the remaining $v-1$ elements of S . Also if α is an interior member of a block it is handcuffed to two other elements of the block; otherwise it is handcuffed to one other element in the block. Hence, $2u_\alpha + (r-u_\alpha) = \lambda(v-1)$ or $u_\alpha = \lambda(v-1) - r$. Thus u_α is independent of α . We now replace u_α by u and write our first necessary condition as

$$r + u = \lambda(v-1) \quad . \tag{1}$$

By counting appearances of elements in blocks in two ways we obtain

$$vr = kb \quad . \tag{2}$$

By counting handcuffed pairs in two ways we obtain

$$\frac{\lambda v(v-1)}{2} = (k-1)b \ . \tag{3}$$

Equations (1), (2), (3) are necessary conditions for the existence of a handcuffed design with parameters v, k, λ . Solving for b, r and u we obtain

$$b = \frac{\lambda v(v-1)}{2(k-1)}, \quad r = \frac{k\lambda(v-1)}{2(k-1)}, \quad u = \frac{\lambda(v-1)(k-2)}{2(k-1)} .$$

If $\lambda = 1$, we call the design a Steiner Handcuffed Design.

3. Steiner Handcuffed Designs.

In this section we show that when $\lambda = 1$ the necessary condition for the existence of a design is also sufficient. With $\lambda = 1$ our equations become $b = \frac{v(v-1)}{2(k-1)}$, $r = \frac{k(v-1)}{2(k-1)}$ and $u = \frac{(v-1)(k-2)}{2(k-1)}$. The case $k = 1$ is vacuous in as much as there are no handcuffed pairs. If $k = 2$, $b = \frac{v(v-1)}{2}$, $r = v-1$ and $u = 0$. Hence S is a v-set and T is the set of all 2-subsets of S.

The first non-trivial case is $k = 3$. Then $b = \frac{v(v-1)}{4}$, $r = \frac{3(v-1)}{4}$, $u = \frac{v-1}{4}$. Hence $v = 4u + 1$ or $v \equiv 1 \mod 4$.

<u>Theorem 1.</u> For $k = 3$, a Steiner Handcuffed Design exists if and only if $v \equiv 1 \mod 4$.

<u>Proof.</u> We have already proved the "only if" part. Suppose now $v = 4u + 1$. We use induction on u. If $u = 1$, $v = 5 = b$ and $r = 3$. The following exhibits the design: $\{1, 5, 2\}$, $\{2, 1, 3\}$, $\{3, 2, 4\}$, $\{4, 3, 5\}$, $\{5, 4, 1\}$. (Denote this design by Q_1 and let $S_1 = \{1, 2, 3, 4, 5\}$.) Suppose the design exists for $u = M$ and denote such a design by Q_M on a set $S_M = \{1, 2, 3, \cdots, 4M+1\}$. We now construct a design Q_{M+1} on $S_{M+1} = S_M \cup \{4M+2, 4M+3, 4M+4, 4M+5\}$. The blocks of Q_{M+1} are the following.

(1) All the blocks of Q_M .

(2) The blocks of a design isomorphic to Q_1 on the set $\{4M+1, 4M+2, 4M+3, 4M+4, 4M+5\}$.

(3) For $h = 1, 3, 5, \cdots, 2M-1$ the blocks

$\{4M+2, 2h-1, 4M+3\}$

$\{4M+2, 2h, 4M+3\}$

$\{2h-1, 4M+4, 2h\}$

$\{2h-1, 4M+5, 2h\}$

totaling $4M$ blocks.

(4) For $h = 2, 4, 6, \cdots, 2M$ the blocks

$\{4M+4, 2h-1, 4M+5\}$

$\{4M+4, 2h, 4M+5\}$

$\{2h-1, 4M+2, 2h\}$

$\{2h-1, 4M+3, 2h\}$

totalling $4M$ blocks.

It is a straightforward, though tedious verification, that the blocks of Q_{M+1} are a Steiner Handcuffed Design.

We now proceed to the general value of k . We distinguish two cases, namely, k odd and k even. Suppose k is odd. Put $k = 2h+1$, $h = 1, 2, 3, \cdots$. The case $h = 1$ has been disposed of in Theorem 1. The equations for b, r, u in terms of v, h, are $b = \frac{v(v-1)}{4h}$, $r = \frac{(2h+1)(v-1)}{4h}$, $u = \frac{(v-1)(2h-1)}{4h}$. Then $v = \frac{4h}{2h-1}u + 1$. Since v is an integer $(2h-1)|u$. Put $u = m(2h-1)$. Then $v = 4hm + 1$ so that

$v \equiv 1 \mod 4h$. Conversely if $v \equiv 1 \mod 4h$ we can solve for b, r, u and obtain integral values. This suggests the following theorem.

Theorem 2. The design $H(v, k, 1)$ exists where $k = 2h+1$ if and only if $v \equiv 1 \mod 4h$.

Proof. The "only if" part having been proved in the previous paragraph we proceed to prove the if part. Put $v = 4hm + 1$. Then $u = m(2h-1)$, $r = m(2h+1)$, $b = m(4hm+1)$. We use induction on m . If $m = 1$, $v = 4h+1$, $u = 2h-1$, $r = 2h+1 = u+2$, $b = 4h+1$. Let $S_1 = \{1, 2, 3, \cdots, 4h+1\}$ and denote the design to be constructed by Q_1 . We take the first block of Q_1 to be $\{1, 4h+1, 2, 4h, 3, \cdots, 3h+2, h+1\}$. Note that the absolute value of the differences of consecutive pairs are all distinct. We now translate this block mod $(4h+1)$ obtaining the set of $4h+1$ blocks

$$\begin{array}{l}
\{1,\ 4h+1,\ 2,\ 4h,\ \cdots,\ 3h+2,\ h+1\} \\
\{2,\ 1,\ 3,\ 4h+1,\ \cdots,\ 3h+3,\ h+2\} \\
\{3,\ 2,\ 4,\ 1,\ \cdots,\ 3h+4,\ h+3\} \\
\vdots \\
\{4h+1,\ 4h,\ 1,\ 4h-1,\ \cdots,\ 3h+1,\ h\} .
\end{array}$$

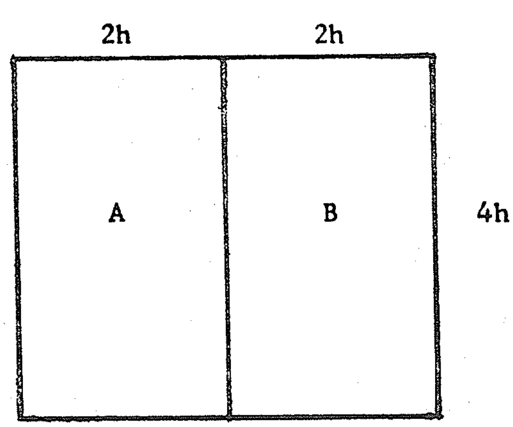
The total number of handcuffed pairs is $2h(4h+1)$ and no pair is repeated as this would imply that in the first block two sets of consecutive pairs would have the same absolute value of their differences. Hence the set of blocks constitute a Steiner Handcuffed Design. Now assume that Q_M exists on the set $S_M = \{1, 2, 3, \cdots, 4hM+1\}$. Let $S_{M+1} = S_M \cup \{4hM+2, 4hM+3, \cdots, 4hM+4h+1\}$.

We construct Q_{M+1} as follows. First, take all the blocks of Q_M. Secondly, add the blocks of a design isomorphic to Q_1 based on the set $\{4hM+1, 4hM+2, \cdots, 4hM+4h+1\}$. Before, writing down the remaining blocks, we first list the pairs which are not accounted for in the blocks already designated. There are the pairs $(i, 4hM+j)$ where $i = 1, 2, \cdots, 4hM$ and $j = 2, 3, \cdots, 4h+1$. We divide these pairs into M sets $T_0, T_1, T_2, \cdots, T_{M-1}$. The set T_m will contain the pairs $(i, 4hM+j)$ where $i = 4hm+1, 4hm+2, \cdots, 4hm+4h$; $j = 2, 3, \cdots, 4h+1$. We write the set T_0 as an array N_0 as follows:

$$\begin{array}{l}
(1, 4hM+2), (2, 4hM+2), \cdots, (4h, 4hM+2) \\
(1, 4hM+3), (2, 4hM+3), \cdots, (4h, 4hM+3) \\
\vdots \\
(1, 4hM+4h+1), (2, 4hM+4h+1), \cdots, (4h, 4hM+4h+1) .
\end{array}$$

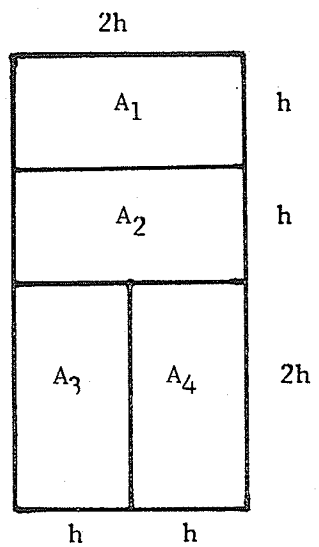
Ignoring brackets we look upon N_0 as a matrix with $4h$ rows and $4h$ columns. We now divide N_0 into two parts A and B where A consists of the first $2h$ columns and B the remaining $2h$ columns.

$N_0 =$



Now A is broken up into four parts A_1, A_2, A_3, A_4 given by the following diagram:

$A =$



We consider the cases h odd and h even separately. If h is odd we write down h blocks of the design Q_{M+1} as follows:

$$\{1,\ 4hM+2,\ 2,\ 4hM+3,\ \cdots,\ 4hM+h+1,\ h+1\}$$
$$\{3,\ 4hM+2,\ 4,\ 4hM+3,\ \cdots,\ 4hM+h+1,\ h+3\}$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$\{2h-1,\ 4hM+2,\ 2h,\ 4hM+3,\cdots,\ 4hM+h+1,\ h-1\}\ .$$

It is readily verified that the handcuffed pairs in these blocks are precisely the pairs in the matrix A_1 . Call this set of blocks C . If in the blocks of C we replace each $4hM+j$ by $4hM+h+j$ we get a new set of blocks which we add to our design and the handcuffed pairs in these blocks are the pairs in the matrix A_2 . Call the new blocks D .

If h is even we still use the same blocks C to account for the pairs of A_1 but to get the pairs of A_2 we replace $4hM+j$ by $4hM+h+j$ and $2i$ by $2i-1$ and $2i-1$ by $2i$ in the blocks of C to get the blocks of D.

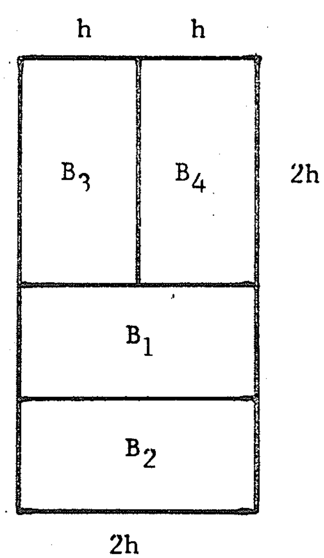
We now obtain sets of blocks E and F whose handcuffed pairs are the pairs of A_3 and A_4 respectively. E is obtained from C by replacing

$$\begin{array}{lll} i & \to 4hM+2h+1+i, & i=1,2,\cdots,2h \\ 4hM+j & \to j-1, & j=2,3,\cdots,h+1 . \end{array}$$

Also F is obtained from D by the replacements

$$\begin{array}{lll} i & \to 4hM+2h+1+i, & i=1,2,\cdots,2h \\ 4hM+h+j & \to h+j-1, & j=2,3,\cdots,h+1 . \end{array}$$

We now consider the matrix B. We divide B into four submatrices B_1, B_2, B_3, B_4 according to the diagram

$B =$ 

From the sets of blocks C, D, E, F we obtain sets C^*, D^*, E^*, F^* by the substitutions

$$\begin{aligned}
i &\longrightarrow 2h+i, && i = 1, 2, \cdots, 2h \\
2h+i &\longrightarrow i, && i = 1, 2, \cdots, 2h \\
4hM+j &\longrightarrow 4hM+2h+j, && j = 2, 3, \cdots, 2h+1 \\
4hM+2h+j &\longrightarrow 4hM+j, && j = 2, 3, \cdots, 2h+1 .
\end{aligned}$$

A direct verification shows that the handcuffed pairs in C^*, D^*, E^*, F^* contain all pairs in B_1, B_2, B_3, B_4 respectively exactly once.

By repeating with the sets $T_1, T_2, \cdots, T_{M-1}$ the same operation as was carried out for T_0 we obtain new sets of blocks. The totality of blocks obtained completes the construction of Q_{M+1} .

<u>Theorem 3.</u> A necessary and sufficient condition for $H(v, k, 1)$ to exist when $k = 2h$, $h = 2, 3, 4, \cdots$ is $v \equiv 1 \mod (2h-1)$.

<u>Proof.</u> The equations for b, r, u become

$$b = \frac{v(v-1)}{2(2h-1)}, \qquad r = \frac{2h(v-1)}{2(2h-1)}, \qquad u = \frac{(v-1)(h-1)}{2h-1}$$

Hence $v = \dfrac{(2h-1)}{(h-1)} u + 1$. Since u is an integer $(h-1)|u$ so that $u = m(h-1)$, $v = (2h-1)m + 1$, $r = hm$, $b = \dfrac{\{(2h-1)m+1\}m}{2}$. Also $((2h-1)m+1)m$ is always an even integer so b is an integer . From $v = (2h-1)m+1$ it follows that $v \equiv 1 \mod (2h-1)$. Conversely, if $v \equiv 1 \mod (2h-1)$ by putting $v = (2h-1)m+1$, the values of u, r, b are all integers. Hence, the condition is necessary.

For sufficiency, let $v = (2h-1)m+1$ and use induction on m. For $m = 1$, $v = k = 2h$, $b = h$, $u = h-1$. The set of blocks

$$\{1, 2h, 2, 2h-1, \quad \cdots, h, h+1\}$$
$$\{2, 1, 3, 2h, \quad \cdots, h+1, h+2\}$$
$$\vdots \qquad\qquad \vdots$$
$$\{h, h-1, h+1, h-2, \cdots, 2h-1, 2h\}$$

form a design Q_1 on the set $S_1 = \{1, 2, \cdots, 2h\}$. The proof is identical with that used in Theorem 2. Suppose now that Q_M exists on a set $S_M = \{1, 2, \cdots, (2h-1)M+1\}$. Put $S_{M+1} = S_M \cup \{(2h-1)M+2, (2h-1)M+3, \cdots, (2h-1)M+2h\}$. The design Q_{M+1} are taken as follows. First, include all blocks of Q_M. Secondly, take the blocks of a design isomorphic to Q_1 based on the set $\{(2h-1)M+1, (2h-1)M+2, \cdots, (2h-1)M+2h\}$. There remain pairs $(i, (2h-1)M+j)$, $i = 1, 2, \cdots, (2h-1)M$, $j = 2, 3, 4, \cdots, 2h$ yet to be accounted for. Divide these pairs into M sets $T_1, T_2, \cdots, T_M$ where

$$T_1 = \{(i, (2h-1)M+j) \mid i = 1, 2, \cdots, 2h-1; \quad j = 2, 3, \cdots, 2h\}$$
$$T_2 = \{(i, (2h-1)M+j) \mid i = 2h, 2h+1, \cdots, 2(2h-1); \quad j = 2, 3, \cdots, 2h\}$$
$$\vdots$$
$$T_r = \{(i, (2h-1)M+j) \mid i = (2r-2)h+2-r, \cdots, r(2h-1); j = 2, 3, \cdots, 2h\}$$
$$\vdots$$
$$T_M = \{(i, (2h-1)M+j) \mid i = (2M-2)h+2-M, \cdots, M(2h-1); j = 2, 3, \cdots, 2h\}.$$

We first dispose of the pairs in T_1 . Take the $(2h-1)$ blocks

$$\{1,\ (2h-1)M+2,\ 2h-1,\ (2h-1)M+3,\ \cdots,\ h+1,\ (2h-1)M+h+1\}$$
$$\{2,\ (2h-1)M+3,\ 1,\ (2h-1)M+4,\ \cdots,\ h+2,\ (2h-1)M+h+2\}$$
$$\vdots$$
$$\{2h-1,\ (2h-1)M+2h, 2h-2, (2h-1)M+2, \cdots,\ h,\ (2h-1)M+h\}\ .$$

A direct verification shows that the handcuffed pairs in these blocks are precisely the pairs in T_1 . In the same way, by translating i, we obtain blocks corresponding to each of $T_2, T_3, \cdots, T_M$ and these complete the design.

4. Handcuffed Designs for $\lambda > 1$.

For $\lambda > 1$ the information on the existence of handcuffed designs is incomplete. However, the theorems below give added information.

A trivial lemma which is useful is the following. It depends on the fact that we are allowed to repeat blocks.

Lemma. If $H(v, k, \lambda_1)$ exists and $H(v, k, \lambda_2)$ exists then $H(v, k, \lambda_1 + \lambda_2)$ exists.

Proof. Take $Q_1 \in H(v, k, \lambda_1)$ and $Q_2 \in H(v, k, \lambda_2)$ where Q_1 and Q_2 are on the same set S . The union of the blocks in Q_1 and Q_2 are the blocks of a $H(v, k, \lambda_1 + \lambda_2)$.

Corollary. If $H(v, k, 1)$ exists then so does $H(v, k, \lambda)$. In particular if $k = 2h+1$ and $(\lambda, 4h) = 1$, then since $v = \dfrac{4hu}{(2h-1)\lambda} + 1$, $(2h-1)\lambda \mid u$ so that $u = m\lambda(2h-1)$ or $v = 4hm + 1$ and $v \equiv 1 \mod 4h$. Now by Theorem 2, $H(v, k, 1)$ exists so that $H(v, k, \lambda)$ exists.

Theorem 4. If k is even, $k = 2h$ and $(\lambda, 2h-1) = 1$. A necessary and sufficient condition for $H(v, k, \lambda)$ to exist is $v \equiv 1 \mod (2h-1)$.

Proof. To show sufficiency note that if $v \equiv 1 \mod 2h-1$ then $H(v, k, 1)$ exists, hence $H(v, k, \lambda)$ exists. Now suppose $(\lambda, 2h-1) = 1$. The equation $u = \frac{\lambda(v-1)(k-2)}{2(k-1)} = \frac{\lambda(v-1)(h-1)}{2h-1}$ implies $v = \frac{(2h-1)u}{\lambda(h-1)} + 1$. Since $(\lambda, 2h-1) = 1$ and $(h-1, 2h-1) = 1$ it follows that $\lambda(h-1) | u$. Put $u = m\lambda(h-1)$. Hence $v = 1 + (2h-1)m$ or $v \equiv 1 \mod (2h-1)$.

Corollary. If $k = 2h$ then $H(v, k, 2)$ exists if and only if $v \equiv 1 \mod (2h-1)$.

Theorem 5. A necessary and sufficient condition for $H(v, k, 2)$ to exist if $k = 2h+1$ is $v \equiv 1 \mod 2h$.

Proof. If $k = 2h+1$ then $u = \frac{(v-1)(2h-1)}{2h}$, so that $v = 1 + \frac{2hu}{2h-1}$. Hence, $(2h-1)|u$. Put $u = m(2h-1)$ $v = 1 + 2hm$ or $v \equiv 1 \mod 2h$. Hence the condition is necessary.

To show sufficiency, we put $v = 1 + 2hm$ and use induction on m. (Note that even values of m are covered by Theorem 2, but we will not use this.) If $m = 1$ we construct a design Q_1 on $S_1 = \{1, 2, \cdots, 2h+1\}$. In fact the blocks

$$\{1, 2h+1, 2, 2h, \cdots, h, h+2, h+1\}$$
$$\{2, 1, 3, 2h+1, \cdots, h+1, h+3, h+2\}$$
$$\vdots$$
$$\{2h+1, 2h, 1, 2h-1, \cdots, h-1, h+1, h\}$$

constitute the required design. The proof is the same as that used in Theorems 2 and 3. Now assume that Q_M exists on a set $S_M = \{1, 2, 3, \cdots, 1+2hM\}$ and let $S_{M+1} = S_M \cup \{2hM+2, 2hM+3, \cdots, 2hM+2h+1\}$ The blocks of Q_{M+1} are obtained as follows. First, include the blocks of Q_M. Then add the blocks of a design isomorphic to Q_1 on the set $S = \{2hM+1, 2hM+2, \cdots, 2hM+2h+1\}$. The pairs which have not been handcuffed in the blocks used so far are $\{(\ell+i, 2hM+j) \mid \ell = 0, 1, \cdots, M-1, i = 1, 2, \cdots, 2h, j = 2, 3, \cdots, 2h+1\}$. Divide these into M subsets $T_0, T_1, \cdots, T_{M-1}$ where $T_\ell = \{(\ell+i, 2hM+j) \mid i = 1, 2, \cdots, 2h, j = 2, 3, \cdots, 2h+1\}$. We write the pairs of T_ℓ as an array N of $2h$ rows as follows

$$\begin{array}{l}
(\ell+1,\ 2hM+2),\ (\ell+2,\ 2hM+2),\ \cdots,\ (\ell+2h,\ 2hM+2) \\
(\ell+1,\ 2hM+3),\ (\ell+2,\ 2hM+3),\ \cdots,\ (\ell+2h,\ 2hM+3) \\
\vdots \\
(\ell+1,\ 2hM+2h+1),\ (\ell+2,\ 2hM+2h+1),\cdots,\ (\ell+2h,\ 2hM+2h+1)\ .
\end{array}$$

Denote the first h rows of N by A and the last h rows by B.

We now distinguish two cases h odd and h even. If h is odd the handcuffed blocks of

$$\begin{array}{l}
\{\ell+1,\ 2hM+2,\ \ell+2,\ 2hM+3,\ \cdots,\ 2hM+h+1,\ \ell+h+1\} \\
\{\ell+3,\ 2hM+2,\ \ell+4,\ 2hM+3,\ \cdots,\ 2hM+h+1,\ \ell+h+3\} \\
\vdots \\
\{\ell+2h-1,\ 2hM+2,\ \ell+2h,\ 2hM+3,\ \cdots,\ 2hM+h+1,\ \ell+h-1\}
\end{array}$$

contains the pairs of A once so we repeat these blocks in our design in order to handcuff the pairs of A twice.

Again the handcuffed blocks

$$\{\ell+1,\ 2hM+h+2,\ \ell+2,\ 2hM+h+3,\ \cdots,\ 2hM+2h+1,\ \ell+h+1\}$$
$$\{\ell+3,\ 2hM+h+2,\ \ell+4,\ 2hM+h+3,\ \cdots,\ 2hM+2h+1,\ \ell+h+3\}$$
$$\vdots$$
$$\{\ell+2h-1,\ 2hM+h+2,\ \ell+2h,\ 2hM+h+3,\ \cdots,\ 2hM+2h+1,\ \ell+h-1\}$$

when repeated once handcuff the pairs of B twice. Repeating the operation with all the T_i's completes the design.

If h is even, a different procedure is used to handcuff the pairs of A and B twice. We use the two sets of h handcuffed blocks viz.

$$\{\ell+1,\ 2hM+2,\ \ell+2,\ 2hM+3,\ \cdots,\ 2hM+h+1,\ \ell+h+1\}$$
$$\{\ell+3,\ 2hM+2,\ \ell+4,\ 2hM+3,\ \cdots,\ 2hM+h+1,\ \ell+h+3\}$$
$$\vdots$$
$$\{\ell+2h-1,\ 2hM+2,\ \ell+2h,\ 2hM+3,\cdots,\ 2hM+h+1,\ \ell+h-1\}$$

and

$$\{\ell+2,\ 2hM+2,\ \ell+3,\ 2hM+3,\ \cdots,\ 2hM+h+1,\ \ell+h+2\}$$
$$\{\ell+4,\ 2hM+2,\ \ell+5,\ 2hM+3,\ \cdots,\ 2hM+h+1,\ \ell+h+2\}$$
$$\vdots$$
$$\{\ell+2h,\ 2hM+2,\ \ell+1,\ 2hM+3,\ \cdots,\ 2hM+h+1,\ \ell+h\}$$

to handcuff the pairs of A twice and a similar set of $2h$ handcuffed blocks to handcuff the pairs of B twice.

Again, carrying out these operations for all the T_i completes the design.

<u>Corollary.</u> Using our lemma it follows that if k is odd and $(\lambda, 4h) = 2$ then λ is even and hence $H(v, k, \lambda)$ exists.

For designs with $k = 3$, complete information exists as given by the following theorem.

Theorem 6. If $k = 3$, the following are necessary and sufficient conditions for $H(v, 3, \lambda)$ to exist.

(1) If $(\lambda, 4) = 1$, $v \equiv 1 \mod 4$.

(2) If $(\lambda, 4) = 2$, $v \equiv 1 \mod 2$.

(3) If $(\lambda, 4) = 4$, $v \geq 3$.

Proof. If $(\lambda, 4) = 1$, we have $v = \frac{4u}{\lambda} + 1$ so that $\lambda \mid u$. Put $u = m\lambda$ which implies $v = 4m + 1$ or $v \equiv 1 \mod 4$. Hence by Theorem 1 $H(v, 3, 1)$ exists and by our lemma $H(v, 3, \lambda)$ exists.

If $(\lambda, 4) = 2$ then $\lambda = 2(2h+1)$ and $r = \frac{2u}{2h+1} + 1$ or $2h+1$ divides u. Put $u = (2h+1)m$ $v = 2m+1$ or $v \equiv 1 \mod 2$. By Theorem 5, $H(v, 3, 2)$ exists. Hence by our lemma $H(v, 3, 2(2h+1)) = H(v, 3, \lambda)$ exists.

Suppose now $(\lambda, 4) = 4$. Then $\lambda = 4t$ $v = \frac{4u}{4t} + 1 = \frac{u}{t} + 1$. Put $u = mt$ and $v = m+1$. Hence there is no restriction on v except $v \geq 3$. If v is odd then by Theorem 5, $H(v, 3, 2)$ exists so $H(v, 3, 4t) = H(v, 3, \lambda)$ exists. There remains the case $v = 2n$, $n = 2, 3, 4, 5, \cdots$. In this case we show $H(v, 3, 4)$ exists from which it will follow that $H(v, 3, \lambda)$ exists.

Generally, with $v = 2n$, $u = 2n-1$, $r = 3(2n-1)$, $b = 2n(2n-1)$ and a $Q \in H(2n, 3, 4)$ is constructed as follows. We now write down $(2n-1)$ triples $(a_1\ b_1\ c_1)$ $(a_2\ b_2\ c_2)$ $(a_3\ b_3\ c_3)$ $\cdots$ $(a_{2n-1}\ b_{2n-1}\ c_{2n-1})$ such that the successive differences $b_1 - a_1$, $b_1 - c_1$, $b_2 - a_2$, $b_2 - c_2$, $\cdots$, $b_{2n-1} - a_{2n-1}$, $b_{2n-1} - c_{2n-1}$ are the sequence of numbers

$2n-1, 2n-2, 2n-3, \cdots, 2, 1, n, 2n-1, 2n-2, \cdots, n+1, n-1, n-2, \cdots, 2, 1$

and $a_1 = 1$, $a_2 = c_1$, $a_3 = c_2$, $\cdots$, $a_{2n-1} = c_{2n-2}$. These conditions uniquely determine the triples. If each of the triples is translated 2n times mod 2n we get the set of all blocks of the design. The following examples illustrate the construction

$n = 2$;

1 4 2	2 3 1	1 4 3
2 1 3	3 4 2	2 1 4
3 2 4	4 1 3	3 2 1
4 3 1	1 2 4	4 3 2

$n = 3$;

1 6 2	2 5 3	3 4 1	1 6 2	2 4 3
2 1 3	3 6 4	4 5 2	2 1 3	3 5 4
3 2 4	4 1 5	5 6 3	3 2 4	4 6 5
4 3 5	5 2 6	6 1 4	4 3 5	5 1 6
5 4 6	6 3 1	1 2 5	5 4 6	6 2 1
6 5 1	1 4 2	2 3 6	6 5 1	1 3 2

CHAPTER V

On Howell Designs

1. Introduction.

Let n and s be integers where $n \leq s \leq 2n - 1$. A square of side s such that each cell is empty or contains an unordered pair of integers from amongst $1, 2, \cdots, 2n$ is called a Howell design of type $H(s, 2n)$, provided; 1) each integer from 1 to $2n$ appears exactly once in each row and each column and, 2) every unordered pair of integers appears at most once in a cell of the square. For a design of type $H(s, 2n)$ we call it a design of side s and of order $2n$. The range of possible values of s is $n \leq s \leq 2n - 1$.

In fact such designs are being used in the scheduling of bridge movements. Their history goes back many years and such designs of lower order can be found in any "Bridge Director's Manual".

One of the extreme cases of Howell designs occurs when $s = 2n - 1$. In that case every unordered pair appears exactly once. This type of Howell design is called a Room Design. Around 1900 Professor E.C. Howell of M.I.T. found the Room Squares of orders from 8 to 30 and they are published in Bridge World Inc. 1933. Hence, we use his name in the designation of the general design $H(s, 2n)$. It must be remembered, however, that he only was interested in the case $s = 2n - 1$. Stanton, Mullin and their students have been working in Room squares for some time and solved almost all problems concerning existence for that case. The problem of existence of Room designs for all values of n would be completely solved if a square of type $H(257, 258)$ could be constructed. This would still leave open the problem of isomorphism classes.

2. <u>The Extreme Case $s = n$.</u>

Another extreme case is when $s = n$. In this case every cell of the square would contain one unordered pair of integers. An obvious case is $n = 1 = s$ $2n = 2$. The square is simply $(1, 1')$.

It is easy to see that if there are two orthogonal latin squares of order n then there would be a Howell design of type $H(n, 2n)$. In fact let the first latin square contain $1, \cdots, n$ and the second one contain $1', \cdots, n'$. Then take the elements in corresponding cells of both squares as the pair in the corresponding cell of a new $n \times n$ square. It satisfies all the conditions for a Howell Design.

Orthogonal Latin Squares exist for all positive integers except 2 and 6 so $H(n, 2n)$ exists for all positive integers n except possibly $n = 2$ and $n = 6$. It is not difficult to show that there exist no $H(2, 4)$; but for $n = 6$ there exists a $H(6, 12)$ as follows:

1, 7	2, 8	3, 9	4, 10	5, 11	6, 12
2, 3	12, 7	4, 5	8, 9	6, 1	10, 11
11, 12	3, 4	7, 8	5, 6	9, 10	1, 2
5, 10	6, 11	1, 12	2, 7	3, 8	4, 9
6, 9	1, 10	2, 11	3, 12	4, 7	5, 8
4, 8	5, 9	6, 10	1, 11	2, 12	3, 7

Hence we have:

<u>Theorem 1</u>: $H(n, 2n)$ exists for all positive integers n except $n = 2$.

In what follows we shall give constructions for $H(n, 2n)$ independent of orthogonal latin squares.

3. Construction of Some Designs for $n < s < 2n - 1$.

From these two extreme cases we know that Howell Designs widely exist. Now we are going to see what will happen when $n < s < 2n - 1$. We first consider small values of n, $n = 2$; there is no s for which $n < s < 2n - 1$. In the extreme cases neither $H(2, 4)$ nor $H(3, 4)$ exist. $n = 3$; the only case $n < s < 2n - 1$ is $H(4, 6)$ which exists as follows:

1, 6		4, 5	2, 3
3, 4	2, 6		1, 5
2, 5	4, 1	3, 6	
	3, 5	1, 2	4, 6

$n = 4$; there are two possible such cases $s = 5$, $s = 6$. After a complete investigation by computer it is known that $H(5, 8)$ does not exist but $H(6, 8)$ exists as follows:

		1, 2	3, 4	5, 6	7, 8
		5, 7	6, 8	1, 3	2, 4
4, 5	1, 8			2, 7	3, 6
3, 7	2, 6			4, 8	1, 5
2, 8	3, 5	4, 6	1, 7		
1, 6	4, 7	3, 8	2, 5		

Before we go to any higher order we want to describe a modification of the starter-adder method which is originally described by Mullin and Stanton [6]. This starter-adder method was used by Stanton, Mullin and Nemeth to construct many Room squares. For the square $H(2n-1, 2n)$ the starter-adder method is the following. We consider the integers mod $(2n-1)$ and append a symbol ∞ with the property that $\infty + 1 = \infty$. The set of pairs $(\infty, 0), (a_1, b_1), (a_2, b_2), \cdots, (a_{n-1}, b_{n-1})$ is called a starter if $a_1, a_2, \cdots, a_{n-1}, b_1, b_2, \cdots, b_{n-1}$ are all the $2n-2$ non-zero residues mod $(2n-1)$ and if $\pm(a_1 - b_1), \pm(a_2 - b_2), \cdots, \pm(a_{n-1} - b_{n-1})$ are all distinct. The set of elements $c_0, c_1, c_2, \cdots, c_{n-1}$ are said to be an adder if $c_0 + 0, c_1 + a_1, c_1 + b_1, c_2 + a_2, c_2 + b_2, \cdots, c_{n-1} + a_{n-1}, c_{n-1} + b_{n-1}$ are all the residues mod $(2n-1)$ and if $c_0, c_1, \cdots, c_{n-1}$ are all distinct. With a starter-adder one can construct an $H(2n-1, 2n)$ as follows. First, we place (a_i, b_i) in the cell occupying the first row and $(2n - c_i)$th column, and $(\infty, 0)$ in the first row $(2n - c_0)$th column. Then if the pair (x, y) occupies the cell in the ith row and jth column, then the pair $(x + 1, y + 1)$ occupies the cell in the $(i + 1)$th row and $(j + 1)$th column (row and column indices are taken mod $(2n - 1)$. If the cell in the ith row, jth column is empty then so is the cell in the $(i + 1)$th row and $(j + 1)$th column. In case of Room square there is only one infinity, but now in our modification for any $H(s, 2n)$ there are $(2n - s)$ infinities. Let the $2n$ elements be $1, \cdots, s, \infty_{s-n+1}, \cdots, \infty_n$, where the non-infinity elements are taken mod s and consider the pairs:

$$(x_i, y_i), \quad i = 1, \cdots, s - n \quad \text{where } x_i, y_i \in (1, \cdots, s)$$

such that $\pm(x_i - y_i)$ are all distinct.

(∞_j, z_j), $z_j \in (1, \cdots, s)$ $z_j \neq x_i$, $z_j \neq y_i$

$j = s - n + 1, \cdots, n$ $i = 1, \cdots, s - n$

where $\{x_i\} \cup \{y_i\} \cup \{z_j\} = \{1, 2, \cdots, s\}$.

We call these following pairs a starter:

$$(x_1, y_1), \cdots, (x_{s-n}, y_{s-n}), (\infty_{s-n+1}, z_{s-n+1}), \cdots, (\infty_n, z_n) .$$

Let $a_i, a_j \in (1, \cdots, s)$, $i = 1, \cdots, s - n$, $j = s - n + 1, \cdots, n$ such that $x_i + a_i$, $y_i + a_i$, $z_j + a_j$ are all distinct. Then we call set $a_1, a_2, \cdots, a_n$ an adder. The construction of $H(s, n)$ now proceeds in the same way as in the original starter-adder method. The square constructed by this method would contain no repeated pairs, and no repeated elements in the same row or the same column.

If we can find such a set of starter-adder for positive integers s and $2n$, then there exists a design $H(s, 2n)$, but this is not a necessary condition, we will see later. For convenience of notation we will omit the ∞'s in writing down a starter i.e. instead of $(x_1, y_1), (x_2, y_2), \cdots, (x_{s-n}, y_{s-n}), (\infty_{s-n+1}, z_{s-n+1}), \cdots, (\infty_n, z_n)$ we will write $(x_1, y_1), (x_2, y_2), \cdots, (x_{s-n}, y_{s-n}), z_{s-n+1}, \cdots, z_n$. Also, we will not necessarily list the z's at the end of the table. For instance for $n = 5$, $s = 6$ the following is a starter: 6, 5, 4, (2, 3), 1 . The notation is ambiguous, but the ambiguity is inconsequential since it is immaterial which subscripts are appended to the ∞-symbols provided they are all different i.e. 6, 5, 4, (2, 3), 1 could represent $(\infty_4, 6)$ $(\infty_2, 5)$, $(\infty_3, 4)$, $(2, 3)$, $(\infty_5, 1)$.

4. Construction of Some Infinite Classes of Designs.

Using this modified starter-added method we can easily construct many infinite classes of Howell designs. Henceforth, we denote the design which can be constructed by this method by: $H_S(s, 2n)$.

Theorem 2: For all positive odd integers n $H_S(n, 2n)$ exists.

Proof: (In this paper, for convenience we write an integer $k \equiv 0 \mod m$ as "m" rather than "0" in all the constructions of starter-adder method.) We give the starter-adder as follows:

$$\text{adder:} \quad n,\ n-1,\ \ldots,\ \frac{n+3}{2},\ \frac{n+1}{2},\ \frac{n-1}{2},\ \ldots,\ 4,\ 3,\ 2,\ 1$$

$$\text{starter:} \quad n,\ n-1,\ \ldots,\ \frac{n+3}{2},\ \frac{n+1}{2},\ \frac{n-1}{2},\ \ldots,\ 4,\ 3,\ 2,\ 1$$

$$\text{sum:} \quad \underbrace{n,\ \ldots\ldots\ldots\ 3,\quad 1,}_{\text{odd}}\ \underbrace{n-1,\ \ldots,\ 8,\ 6,\ 4,\ 2}_{\text{even}}$$

From Theorem 2 we know that $H(n, 2n)$ when n is odd, can be constructed without using orthogonal latin squares.

Theorem 3: For positive odd integers $n \geq 3$ $H_S(n+1, 2n)$ exists.

Proof: The starter-adder and sum are given explicitly as follows:

$$\begin{array}{llllllll}
\text{adder:} & n+1,\ n,\ \cdots,\ \frac{n+1}{2}+3, & \frac{n+1}{2}+2, & \frac{n+1}{2}+1, & \frac{n+1}{2},\ \cdots, & 3, & 2, & 1 \\
\text{starter:} & n+1,\ n,\ \cdots,\ \frac{n+1}{2}+3, & X & \frac{n+1}{2}+2, & \frac{n+1}{2}+1, \cdots, & 4, & (2,\ 3), & 1 \\
\text{sum:} & n+1,\ n-1, \cdots,\ 6, & & 3, & 1,\ \cdots, & 7, & 4,\ 5, & 2
\end{array}$$

For convenience all possible adders are listed. An X under an adder means that it has not been used and in fact corresponds to an empty cell in the first row.

<u>Theorem 4</u>: For any positive even integer n $H_S(n, 2n)$ and $H_S(n + 1, 2n)$ do not exist.

<u>Proof</u>: For $H_S(n, 2n)$, if there exist a starter-adder

adder:	$n,\ n-1,\ \cdots,\ 3,\ 2,\ 1$
starter:	$a_n,\ a_{n-1},\ \cdots,\ a_3,\ a_2,\ a_1$
sum:	$s_n,\ s_{n-1},\ \cdots,\ s_3,\ s_2,\ s_1$

Since $a_i \in (1, \cdots, n)$ $i = 1, \cdots, n$ and all distinct

$s_i \in (1, \cdots, n)$ $i = 1, \cdots, n$ and all distinct

so we have

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} i \quad \text{and} \quad \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} i$$

but $\quad s_i = a_i + i \quad$ so $\quad \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} i = \sum_{i=1}^{n} s_i$

$$\Rightarrow \quad 2\sum_{i=1}^{n} i = \sum_{i=1}^{n} i \quad \Rightarrow \quad \sum_{i=1}^{n} i \equiv 0 \mod n\ .$$

Now if n is even

$$\sum_{i=1}^{n} i = n + (n-1) + \cdots + 1 = \frac{n}{2}(n+1) \equiv \frac{n}{2} \not\equiv 0 \mod n\ .$$

Hence there does not exist a starter-adder for $H(n, 2n)$.

In the case of $H(n + 1, 2n)$ without loss of generality, we can assume that the empty cell in the first row is the last one i.e. $(n + 1)$-th.

If there exist a starter-adder:

let $(x_1, y_1), (x_2\ \infty_2), \cdots, (x_n, \infty_n)$ be the starter

$$x_i, y_1 \in (1, \cdots, n + 1) \qquad i = 1, \cdots, n$$

and $a_1, \cdots, a_n \in (1, \cdots, n)$ be adders.

x_i, y_1 are all distinct, $a_1, \cdots, a_n$ are all distinct.

$$(y_1 + a_1) + (x_1 + a_1) + (x_2 + a_2) + \cdots + (x_n + a_n)$$

$$= \quad y_1 + a_1 + \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} x_i$$

but $y_1 + a_1 + \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} x_i = \sum_{i=1}^{n+1} i \equiv 0 \mod (n + 1)$

$\because$ n is even $\quad \therefore \sum_{i=1}^{n} x_i + y_1 = \sum_{i=1}^{n+1} i \equiv 0 \mod (n + 1)$

$$\therefore \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} i \equiv 0 \mod (n + 1)$$

so $a_1 \equiv 0 \mod (n + 1)$ which contradicts the fact that $a_1 \in \{1, 2, \cdots, n\}$.

We now proceed to describe a method of constructing the designs of type $H(n, 2n)$ with n = even without using Orthogonal Latin Squares. Let $s_1 = \{1, \cdots, n\}$, $s_2 = \{n + 1, \cdots, 2n\}$ be the $2n$ elements. We distinguish two cases.

Case 1. $n = 2t$ and t is even. We arrange the $2n$ elements in a table as follows:

Here the first column gives the value of D_i, $i = 1, \cdots, t, t+1, \cdots, n$ where D_i represents the difference (mod n) of the elements in a pair in the ith row.

$$\left.\begin{array}{lllll}
n & (1, n+1) & (2, n+2) & (3, n+3) & \ldots (n, 2n) \\
n-2 & (2, 2n) & (3, n+1) & (4, n+2) & \ldots (1, 2n-1) \\
n-4 & (3, 2n-1) & (4, 2n) & (5, n+1) & \ldots (2, 2n-2) \\
n-6 & (4, 2n-2) & (5, 2n-1) & (6, 2n) & \ldots (3, 2n-3) \\
\vdots & \vdots \quad \vdots & \vdots \quad \vdots & \vdots \quad \vdots & \vdots \quad \vdots \\
2 & (t, 2n-t+2) & (t+1, 2n-t+3) & (t+2, 3t+4) & \ldots (t-1, 3t+1)
\end{array}\right\} t \text{ rows}$$

$$\left.\begin{array}{lllll}
n-1 & (t+2, 3t+1) & (t+3, 3t+2) & (t+4, 3t+3) & \cdots (t+1, 3t) \\
n-3 & (t+3, 3t) & (t+4, 3t+1) & (t+5, 3t+2) & \cdots (t+2, 3t-1) \\
\vdots & \vdots \quad \vdots & \vdots \quad \vdots & \vdots \quad \vdots & \vdots \quad \vdots \\
i = \frac{3}{2}t,\ t+1 & (\frac{3}{2}t+1, \frac{5}{2}t+2) & (\frac{3}{2}t+2, \frac{5}{2}t+3) & (\frac{3}{2}t+3, \frac{5}{2}t+4) & \cdots (\frac{3}{2}t, \frac{5}{2}t+1) \\
\vdots & \vdots \quad \vdots & \vdots \quad \vdots & \vdots \quad \vdots & \vdots \quad \vdots \\
3 & (n, n+3) & (1, n+4) & (2, n+5) & \ldots (n-1, n+2) \\
i = n,\ t+1 & (t+1, n+2) & (t+2, n+3) & (t+3, n+4) & \cdots (t, n+1)
\end{array}\right\} t \text{ rows}$$

In this arrangement, ignoring the first column of D_i's obviously in each row and in each column, each element appears only once. We only have to consider whether there are any repeated pairs. The repeated pairs can only occur in such rows where they have same D_i. In this case the first t rows have values of

D_i which are even and distinct. The second t rows have values of D_i which are odd and all but two are distinct. The two rows that have the same $D_i = t + 1$, are the n-th row and $\frac{3}{2}t$-th row. We rearrange the table to avoid repeats using the $(\frac{3}{2}t-1)$th and $(\frac{3}{2}t)$th rows as follows:

the old rows,

$$i=\tfrac{3}{2}t-1 \qquad (\tfrac{3}{2}t,\tfrac{5}{2}t+3)\ (\tfrac{3}{2}t+1,\tfrac{5}{2}t+4)\ (\tfrac{3}{2}t+2,\tfrac{5}{2}t+5)\cdots(\tfrac{3}{2}t-1,\tfrac{5}{2}t+2)$$

$$i=\tfrac{3}{2}t \qquad (\tfrac{3}{2}t+1,\tfrac{5}{2}t+2)(\tfrac{3}{2}t+2,\tfrac{5}{2}t+3)\ (\tfrac{3}{2}t+3,\tfrac{5}{2}t+4)\cdots(\tfrac{3}{2}t,\tfrac{5}{2}t+1)$$

are replaced by the new rows

$$i=\tfrac{3}{2}t-1 \qquad (\tfrac{3}{2}t,\tfrac{3}{2}t+1)\ (\tfrac{5}{2}t+3,\tfrac{5}{2}t+4)\ (\tfrac{3}{2}t+2,\tfrac{3}{2}t+3)\cdots(\tfrac{5}{2}t+1,\tfrac{5}{2}t+2)$$

$$i=\tfrac{3}{2}t \qquad (\tfrac{5}{2}t+2,\tfrac{5}{2}t+3)(\tfrac{3}{2}t+1,\tfrac{3}{2}t+2)\ (\tfrac{5}{2}t+4,\tfrac{5}{2}t+5)\cdots(\tfrac{3}{2}t-1,\tfrac{3}{2}t)$$

Obviously in each new row every element occurs exactly once and since each element is still in the same column, each column contains every element exactly once. Now there are no repeated pairs since all pairs in these new rows are pairs in which both elements come from s_1 or from s_2, while in the original rows each pair consists of one element from s_1 and one from s_2. For example in $H(4, 8)$ just rearrange the last two rows as follows:

1, 5	2, 6	3, 7	4, 8		1, 5	2, 6	3, 7	4, 8
2, 8	3, 5	4, 6	1, 7		2, 8	3, 5	4, 6	1, 7
4, 7	1, 8	2, 5	3, 6	$\Rightarrow$	3, 4	7, 8	1, 2	5, 6
3, 6	4, 7	1, 8	2, 5		6, 7	4, 1	8, 5	2, 3

Case 2. $n = 2t$, $t = \text{odd}$, $t \neq 1$.

Similar to Case 1 but the repeated $D_i = t+1$ would be even and the rows which have the repeated D_i would be n-th row and $\frac{t+1}{2}$-th row. We rearrange the $\frac{t-1}{2}$-th row and $\frac{t+1}{2}$-row to avoid the repeated pairs, doing as follows:

The old rows

$$i = \frac{t-1}{2} \quad (\tfrac{t-1}{2}, n+\tfrac{3t+5}{2}), (\tfrac{t+1}{2}, n+\tfrac{3t+7}{2}), (\tfrac{t+3}{2}, n+\tfrac{3t+9}{2}), \ldots (\tfrac{t-3}{2}, n+\tfrac{3t+3}{2})$$

$$i = \frac{t+1}{2} \quad (\tfrac{t+1}{2}, n+\tfrac{3t+3}{2}), (\tfrac{t+3}{2}, n+\tfrac{3t+5}{2}), (\tfrac{t+5}{2}, n+\tfrac{3t+7}{2}), \ldots (\tfrac{t-1}{2}, n+\tfrac{3t+1}{2})$$

become the new rows

$$i = \frac{t-1}{2} \quad (\tfrac{t-1}{2}, \tfrac{t+1}{2}), (n+\tfrac{3t+5}{2}, n+\tfrac{3t+7}{2}), (\tfrac{t+3}{2}, \tfrac{t+5}{2}), \ldots, (n+\tfrac{3t+1}{2}, n+\tfrac{3t+5}{2})$$

$$i = \frac{t+1}{2} \quad (n+\tfrac{3t+3}{2}, n+\tfrac{3t+5}{2}), (\tfrac{t+1}{2}, \tfrac{t+3}{2}), (n+\tfrac{3t+7}{2}, n+\tfrac{3t+9}{2}), \ldots, (\tfrac{t-3}{2}, \tfrac{t-1}{2}).$$

An example for this case is H(6, 12).

6	1, 7	2, 8	3, 9	4,10	5,11	6,12		1, 2	7, 8	3, 4	9,10	5, 6	11,12
4	2,12	3, 7	4, 8	5, 9	6,10	1,11		12, 7	2, 3	8, 9	4, 5	10,11	6, 1
2	3,11	4,12	5, 7	6, 8	1, 9	2,10		3,11	4,12	5, 7	6, 8	1, 9	2,10
5	5,10	6,11	1,12	2, 7	3, 8	4, 9	⇒	5,10	6,11	1,12	2, 7	3, 8	4, 9
3	6, 9	1,10	2,11	3,12	4, 7	5, 8		6, 9	1,10	2,11	3,12	4, 7	5, 8
4	4, 8	5, 9	6,10	1,11	2,12	3, 7		4, 8	5, 9	6,10	1,11	2,12	3, 7

so the existence of designs H(n, 2n) n = even, indeed does not have to depend on the Orthogonal Latin Squares.

We know already from Theorem 1 that H(n, 2n), $n \neq 2$ exists so the starter-adder method is not a necessary conditon. In case of H(n+1, 2n) n even, the following designs have been constructed by computer:

H(7, 12) i.e. n = 6

	1, 7	2, 8	3, 9	4,10	5,11	6,12
11,12		9,10	7, 8	5, 6	3, 4	1, 2
8, 9	10,12		4,11	2, 7	1, 6	3, 5
5,10	6,11	1,12		3, 8	2, 9	4, 7
1, 4	2, 3	5, 7	6,10		8,12	9,11
3, 7	5, 9	4, 6	2,12	1,11		8,10
2, 6	4, 8	3,11	1, 5	9,12	7,10	

H(9, 16) i.e. n = 8

	8,13	2, 6	7,12	3, 4	10,11	14,15	5, 9	1,16
1, 9		11,12	4,14	2, 5	3, 6	7,10	13,16	8,15
4, 8	11,16		6, 9	7,15	5,14	2,12	1, 3	10,13
5,10	2,15	1, 8		14,16	9,13	3,11	4, 7	6,12
12,16	1, 6	7, 9	11,13		4,15	5, 8	2,10	3,14
6,14	3, 7	5,13	2, 8	1,10		9,16	12,15	4,11
11,15	4, 5	10,14	3,16	12,13	1, 7		6, 8	2, 9
2, 3	9,14	4,16	10,15	6,11	8,12	1,13		5, 7
7,13	10,12	3,15	1, 5	8, 9	2,16	4, 6	11,14	

The general cases $H(n+k, 2n)$ $k \geq 2$ will be discussed in what follows.

5. The Multiplication Theorems.

We explain next the multiplication method for the construction of higher order Howell designs by means of the use of lower order designs. The method given here can be seen as a generalization of Horton's method which was originally used in the construction of Room Squares.

<u>Definition</u>: We say a Howell design of type $H(s, 2n)$ satisfies a $*$-condition on a set of $(2n - s)$ integers if there are $(2n - s)$ integers amongst $1, 2, \cdots, 2n$ such that no pair of them occupy a cell.

Many Howell designs satisfy the $*$-conditon, for example:

(a) All designs $H_S(s, 2n)$ constructed by the starter-adder method must satisfy the $*$-conditon since the infinites constitute a set of $2n - s$ elements no two of which occupy a cell.

(b) All Room Squares i.e. $H(2n-1, 2n)$ satisfy the $*$-condition. Since $2n - s = 1$.

(c) All designs of type $H(2n-2, 2n)$ satisfy the $*$-condition. Consider an element "x" in such a design. It is paired with $2n-2$ other elements so there remains an element y with which x is not paired in a cell.

(d) All $H(n, 2n)$ which come from Orthogonal Latin Squares satisfy the $*$-condition (obvious), but the design $H(n, 2n)$ n = even constructed by the method given in the previous section without using Orthogonal Latin Squares need not satisfy the $*$-condition.

For $H(n, 2n)$, all $H(3, 6)$, $H(4, 8)$, constructed by any method whatever, satisfy the $*$-condition. But for $H(5, 10)$ we know that there are at least two non-isomorphic Howell designs of this type. One of these comes from two

Orthogonal Latin Squares (or by the starter-adder method) and must satisfy the $*$-condition. A second one constructed by computer is as follows:

1, 6	2, 3	9, 10	4, 5	7, 8
8, 9	1, 7	2, 4	3, 10	5, 6
5, 3	4, 10	1, 8	6, 7	2, 9
2, 10	6, 8	5, 7	1, 9	3, 4
4, 7	5, 9	3, 6	2, 8	1, 10

It does not satisfy the $*$-condition, as is easily checked.

For $n = 6$, $H(6, 12)$ exists as shown previously but no design of type $H(6, 12)$ can satisfy the $*$-condition. If there are any of them satisfying this condition, then it would imply the existence of a pair of Orthogonal Latin Squares of order 6. Henceforth, we denote a Howell design of type $H(s, 2n)$ which satisfies the $*$-condition by $H^*(s, 2n)$. (An $H_S(s, 2n)$ must be an $H^*(s, 2n)$; whether we use "S" or "*" depends on what condition we wish to emphasize.)

<u>Theorem 6</u>: (Multiplication Theorem) If $H^*(s_1, 2n_1)$, and $H(s_2, 2n_2)$ exist and a pair of Orthogonal Latin Squares of order s_1 exist, then $H(s_1 s_2, (2n_2 - 1)s_1 + (2n_1 - s_1))$ exists.

<u>Proof</u>: Let H_1 be the design of type $H^*(s_1, 2n_1)$ and having elements $(1, 2, \cdots, 2n_1)$ H_2 the design of type $H(s_2, 2n_2)$ having elements $(1, 2, \cdots, 2n_2 - 1, \infty)$.

Without loss of generality we can assume that H_2 is such a Howell design whose main diagonal consists of those cells containing (∞, i) $i = 1, \cdots, s_2$.

Let L, R be two Orthogonal Latin Squares of order s_1 on $(1, \cdots, s_1)$. We enlarge each cell of H_2 to a $s_1 \times s_1$ square by the following rules:

(1) Each empty cell in H_2 is replaced by a $s_1 \times s_1$ empty square.

(2) The cell containing (∞, i) in the main diagonal of H_2 is replaced by a A_i, $i = 1, \cdots, s_2$ where A_i is an H_1 all of whose elements except $(s_1 + 1, \cdots, 2n_1)$ are subscripted by i and leave $s_1 + 1, \cdots, 2n_1$ non-subscripted.

(3) Each cell not empty and not in the main diagonal of H_2 must be a cell containing (j, k), $j \neq k$ is replaced by the matrix $Q_{j,k}$ where $Q_{j,k}$ is the $s_1 \times s_1$ square with entry (ℓ_j, r_k) where ℓ, r occurs in the corresponding cell of L and R respectively and every ℓ subscripted by j every r subscripted by k .

Now we have a square of side $s_1 s_2$ which we call H and:

(1) The design H is of order $(2n_2 - 1)s_1 + (2n_1 - s_1)$. There are s_1 elements of H_1 subscripted by $1, \cdots, 2n_2 - 1$ and there are $(2n_1 - s_1)$ elements non-subscripted so the total is $(2n_2 - 1)s_1 + (2n_1 - s_1)$ elements.

(2) Each column in H intersects only one A_i, all of whose elements are subscripted by i and the non-subscripted elements which appear in the cells of A_i, and each element subscripted by j, k $j \neq i$, $k \neq i$ appears

in some cell of one of $Q_{j,k}$. Since H_1, H_2 are Howell designs and $Q_{j,k}$ comes from a pair L, R of Orthogonal Latin Squares, then every element must appear exactly once in each column. The same holds for each row.

(3) There are no repeated pairs.

(a) Obviously there are not any repeated pairs amongst $Q_{j,k}$ and A_i. This is also true for any pair of different $Q_{j,k}$.

(b) If any repeated pair appears in same A_i, this implies that there is a repeated pair in H_1, a contradiction. Similarly for $Q_{j,k}$.

(c) If any repeated pair appears in A_i, A_j $i \neq j$ then the only possible repeated pairs are those pairs containing both elements which are non-subscripted but H_1 satisfies $*$-condition on these $2n-s$ elements so no repeated pairs appear.

This completes the proof that $H(s_1 s_2, (2n_2 - 1)s_1 + (2n_1 - s_1))$ exists.

<u>Theorem 7</u>: If H_1 and H_2 in the previous theorem both satisfy the $*$-condition so does $H(s_1 s_2, (2n_2 - 1)s_1 + (2n_1 - s_1))$.

<u>Proof:</u> We can choose all non-subscripted elements and all elements subscripted by $s_2 + 1, \cdots, 2n_2 - 1$ to be the set of elements on which we apply the $*$-condition. Then:

(a) the total number of these elements is

$$(2n_2 - 1)s_1 + (2n_1 - s_1) - s_1 s_2 = (2n_2 - s_2 - 1)s_1 + 2n_1 - s_1 .$$

(b) Since H_2 is of side s_2 then no non-subscripted element appears together with an element subscripted by $s_2 + 1, \cdots, 2n_2 - 1$. Since all non-subscripted elements only appear in the diagonal of H_2, they can be paired only with subscripted elements in the diagonal where the subscripts are amongst $i = 1, \cdots, s_2$.

(c) Since H_2 satisfies the $*$-condition no cell of H_2 contains (j, k) where $j, k \in (s_2 + 1, \cdots, 2n_2 - 1)$. So no pair of elements subscripted by $(s_2 + 1, \cdots, 2n_2 - 1)$ can appear in a cell.

(d) Since H_1 satisfies the $*$-condition, no pair of non-subscripted elements can appear in a cell.

<u>Corollary 8</u>: If $H(s, 2n)$ exists and Orthogonal Latin Squares of order k exist then $H(sk, 2kn)$ exists and if $H(s, 2n)$ satisfies the $*$-condition, so does $H(sk, 2kn)$.

<u>Proof</u>: Since Orthogonal Latin Squares of order k exist, it follows that $H^*(k, 2k)$ exists. Hence by Theorem 6 $H(sk, (2n-1)k+k)$ exists but $(2n-1)k+k = 2kn$ so $H(sk, 2kn)$ exists. If $H(s, 2n)$ also satisfies the $*$-condition then by Theorem 7 so does $H(sk, 2kn)$.

<u>Corollary 9</u>: If $H^*(s, 2n)$ exists, and k is a positive integer $k \neq 2, 6$ then $H^*(sk, 2kn)$ exists.

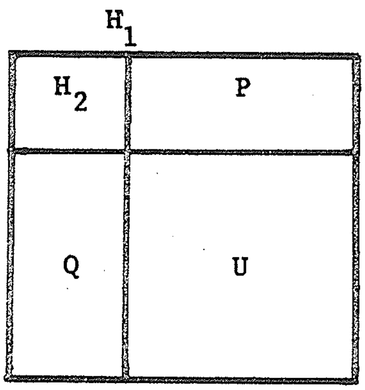
<u>Proof</u>: Orthogonal Latin Squares exist for all positive integers k except $k = 2$ and $k = 6$.

<u>Theorem 10</u>: If $H_0(2n-1,\ 2n)$, $H_1(s_1,\ 2n_1)$ exist and

1. $H_2(s_2,\ 2n_2)$ is a subsquare of H_1 such that $s_1 - s_2 = 2n_1 - 2n_2$.

2. Orthogonal Latin Squares L, R of order $s_1 - s_2$ exist.

3. Let $H_1(s_1,\ 2n_1)$ be as in the figure:



and suppose no two of the elements of H_2 appear together in any cell of P, Q, U (we say that P, Q and U have the $*$-property on elements of H_2) .

Then: $H((2n-1)(s_1 - s_2) + s_2,\ (2n_1 - 2n_2)(2n-1) + 2n_2)$ exists.

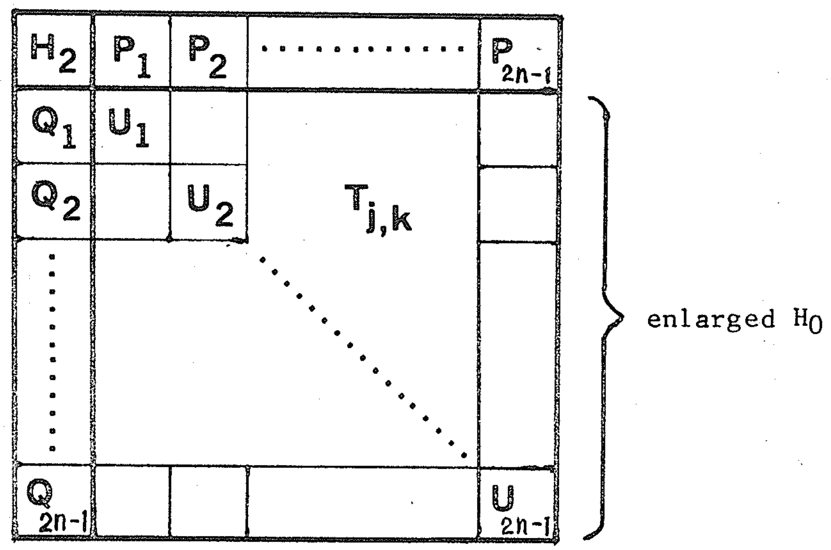
<u>Proof</u>: Let H_0 be a Room square such that every cell in the main diagonal contains a pair (∞, i) , $i = 1, 2, \cdots, 2n-1$.

(1) Replace every empty cell of H_0 by an empty matrix of size $(s_1 - s_2) \times (s_1 - s_2)$.

(2) Replace every cell in the diagonal of H_0 by U_i where U_i is obtained from U by adding the subscript i to elements which do not come from H_2 .

(3) Replace every cell in H_0 containing j, k $j \neq k$ by $T_{j,k}$ $j, k \in (1, 2, \cdots, 2n-1)$ where $T_{j,k}$ has the entry (ℓ_j, r_k) where ℓ, r are the entries of Orthogonal Latin Squares L and R in the corresponding cells.

Then add H_2, P_i, Q_i, $i = 1, \cdots, 2n-1$ to the enlarged H_0 as in the figure and call the new square H .



where P_i and Q_i are the P and the Q in H_1 with every element, except those elements of H_2, subscripted by i ,

Now we check the following:

(1) H now has side $(2n-1)(s_1 - s_2) + s_2$.

(2) H has $(s_1 - s_2)$ elements subscripted by $(2n-1)$ distinct numbers and $2n_2$ elements are non-subscripted so the total is $(2n_1 - 2n_2)(2n-1) + 2n_2$ elements since $s_1 - s_2 = 2n_1 - 2n_2$.

(3) Each column of H contains every element once. Since H_0 is a Room square and all Q_i come from Q in H_1 , then the first s_2 columns contains each non-subscripted element once in some cell of H_2 and each subscripted element once in some cell of Q_i, and each other column must intersect only one U_i and one P_i, so that all non-subscripted elements and the elements subscripted by i would appear in P_i or U_i and all other elements in some $T_{j,k}$.

Similarly for rows.

(4) There are no repeated pairs.

(a) No repeated pairs can appear in any of H_2, Q_i, P_i, U_i and $T_{j,k}$. This follows since all elements in H_2 are non-subscripted and since all non-subscripted elements in Q_i, P_i, U_i are elements of H_2, no two of them appear together because of the $*$-property, and in $T_{j,k}$ there are no non-subscripted elements, so no repeated pairs can appear between any two of H_2, Q_i, P_i, U_i, $T_{j,k}$.

(b) For different Q_i, Q_j, P_i, P_j or U_i, U_j where $i \neq j$ there are no repeated pairs. Since the $*$-property holds there are no non-subscripted pairs and all subscripted elements are distinct in Q_i and Q_j (or P_i and P_j, U_i and U_j) if $i \neq j$, so no repeated pairs can appear amongst them.

(c) There are no repeated pairs in different $T_{j,k}$. Since H_0 is a Room square no pair (j, k) would appear more than once.

__Corollary.__ The $H((2n-1)(s_1-s_2)+s_2,\ (2n_1-2n_2)(2n-1)+2n_2)$ constructed in the theorem satisfies all the conditions mentioned for $H_1(s_1, 2n_1)$ in the Theorem 1. Furthermore, if H_2 satisfies the $*$-condition so does the $H((2n-1)(s_1-s_2)+s_2,\ (2n_1-2n_2)(2n-1)+2n_2$.

__Proof:__ (1) H_2 is a subsquare of H and $(2n-1)(s_1-s_2)+s_2-s_2$

$$= (2n_1-2n_2)(2n-1)+2n_2-2n_2 .$$

(2) U = enlarged H_0 .

$$P = P_1 \cup P_2 \cup \cdots \cup P_{2n-1} .$$

$$Q = Q_1 \cup Q_2 \cup \cdots \cup Q_{2n-1} .$$

All of them have $*$-property on elements of H_2 .

The Corollary, asserts that the new design constructed in the theorem can be reused again to construct further ones.

To construct an $H_1(s_1, 2n_1)$ satisfying the conditions demanded of it in Theorem 10 is in general very difficult. However the multiplication theorem (Theorem 6) gives us one such method. We start with a pair of designs $H^*(s_2, 2n_2)$ and $H^*(2n-1, 2n)$ and use Theorem 6 to construct an $H_1(s_2(2n-1), (2n-1)s_2 + 2n_2 - s_2)$. Put $s_2(2n-1) = s_1$ and $(2n-1)s_2 + 2n_2 - s_2 = 2n_1$. We now have a square $H(s_1, 2n_1)$ containing a subsquare $H(s_2, 2n_2)$. Using the same methods as used in the proof of Theorem 7 it follows readily that $H_1(s_1, 2n_1)$ satisfies the conditions of Theorem 10.

Example.

Take $H^*(s_2, 2n_2) = H^*(3, 6)$ and $H^*(2n-1, 2n) = H^*(7, 8)$ and Orthogonal Latin Squares as in the diagrams on the following page:

∞, 1			3, 6		2, 7	4, 5
5, 6	∞, 2			4, 7		3, 1
4, 2	6, 7	∞, 3			5, 1	
	5, 3	7, 1	∞, 4			6, 2
7, 3		6, 4	1, 2	∞, 5		
	1, 4		7, 5	2, 3	∞, 6	
		2, 5		1, 6	3, 4	∞, 7

$H^*(7, 8) = H^*(2n-1, 2n)$

6, 3	5, 2	4, 1
4, 2	6, 1	5, 3
5, 1	4, 3	6, 2

$H^*(3, 6) = H^*(s_2, 2n_2)$

3, 3	2, 2	1, 1
1, 2	3, 1	2, 3
2, 1	1, 3	3, 2

Orthogonal
Latin Squares.

The construction yields an $H(21, 24)$ with a subsquare $H(3, 6)$ satisfying the appropriate conditions.

6. General Constructions Using Starter-Adder Method.

In this section we describe a recursive construction using the starter-adder method and show that $H_S(n+k, 2n)$ exists for all $k \geq 2$ and all $n \geq N(k)$. Our construction gives an estimate of $N(k)$ which, in general, is too large.

Theorem 12: $H_S(n+2, 2n)$ exists for every positive integer $n \geq 5$.

Proof: For n odd and $n \geq 5$, the following table exhibits a starter-adder.

Adder:	$n+2$,	$n+1$,	n,	$\cdots$,	$\frac{n+1}{2}+2$,	$\frac{n+1}{2}+1$,	$\cdots$,	5,	4	3, 2, 1
Starter:	1,	$(n+2, n)$,	$n-1$,	$\cdots$,	$\frac{n+1}{2}+1$,	$\frac{n+1}{2}$,	$\cdots$,	4,	$(n+1, 3)$,	2, x, x
Sum:	1,	$(n+1, n-1)$,	$n-3$,	$\cdots$,	2,	$n+2$,	$\cdots$,	9,	$(3, 7)$,	5,
$\pm(x_i - y_i)$:		± 2,							± 4	

Note that the sum and $\pm(x_i - y_i)$ are taken mod $(n+2)$.

For n even, and $n \geq 8$, the following exhibits a starter-adder:

Adder:	$n+2,$	$n+1,$	$\cdots,$	$\frac{n}{2}+3,$	$\frac{n}{2}+2,$	$\frac{n}{2}+1,$	$\cdots,$	$4,$	$3,$	$2,$	1
Starter:	$1,$	$(n+2, n),$	$\cdots,$	$\frac{n}{2}+2,$	$x,$	$\frac{n}{2}+1,$	$\cdots,$	$4,$	$(n+1, 3),$	$2,$	x
Sum:	$1,$	$(n+1, n-1),$	$\cdots,$	$3,$	$\cdots,$	$n+2,$	$\cdots,$	$8,$	$(2, 6),$	4	
$\pm(x_i - y_i)$:		± 2							± 4		

For $n = 6$, $H_S(8, 12)$ has the following starter-adder:

Adder:	$8,$	$7,$	$6,$	$5,$	$4,$	$3,$	$2,$	1
Starter:	$1,$	$x,$	$(2, 5),$	$x,$	$8,$	$7,$	$(3, 4),$	6
Sum:	$1,$	$,$	$(8, 3),$		$4,$	$2,$	$(5, 6),$	7
$\pm(x_i - y_i)$:			± 3				± 1	

Let $N(k)$ be the least positive integer, such that for any $n \geq N(k)$, $H_S(n+k, 2n)$ exists. That $N(k)$ exists is proved in Theorem 13. Thus, for $k = 2$, we have $N(2) = 5$. We will use $N'(k)$ for an estimate of $N(k)$ such that $N'(k) \geq N(k)$.

<u>Theorem 13</u>: For any $k \geq 2$, there is an integer $N(k)$ such that $H_S(n+k, 2n)$ exists for all positive integers $n \geq N(k)$.

<u>Proof</u>: For convenience we put $s = n+k$, so that $2n = 2s - 2k$. Then $H_S(n+k, 2n) = H_S(s, 2s-2k)$. Our theorem can be stated equivalently as follows: for $k \geq 2$ there is an $S(k)$ such that for all $s \geq S(k)$, $H_S(s, 2s-2k)$ exists. It is clear that $S(k) = N(k)+k$. In our proof

we will consider s odd and s even separately. We will use $S_o'(k)$ for an estimate of $S(k)$ with s restricted to odd values of k and $S_e'(k)$ for an estimate of $S(k)$ with s restricted to even values of k and take $S'(k) = \max(S_e'(k), S_o'(k))$ and $N'(k) = S'(k) + k$. (If $|S_e'(k) - S_o'(k)| = 1$, then a better value of $S'(k)$ is $\min(S_e'(k), S_o'(k))$. The proof is by induction on k. We start with $k = 2$ and for fixed s write down a starter-adder. We then show that a starter-adder can be constructed for the value $k+1$ by modifying the starter-adder for k and that the process does not stop until $s < S_o'(k)$ if s is odd or $s < S_e'(k)$ if s is even.

<u>Case 1.</u> s odd. We start with $k = 2$ and putting $A = \frac{s+1}{2}$, we write down the starter-adder:

Adder:	s,	$s-1$,	$\cdots$,	$A+3$,	$A+2$,	$A+1$,	A,	$\cdots$,	5,	4,	3,	2,	1
Starter:	1,	$(s, s-2)$,	$\cdots$,	$A+2$,	$A+1$,	A,	$A-1$,	$\cdots$,	4,	$(s-1, 3)$,	2,	x,	x
Sum:	1,	$(s-1, s-3)$,	$\cdots$,	6,	4	2,	s,	$\cdots$,	9,	$(3, 7)$,	5		
$\pm(x_i - y_i)$:		± 2								± 4			

Call this the initial starter-adder. We refer to a starter pair like $(s, s-2)$ as a doubleton and a starter like 2 as a singletom. We modify this to a starter-adder for $k = 3$ by means of the following two steps:

<u>Step 1</u>: Remove the singleton starter 2 from the adder 3 and place it under adder 2. The sum 5 is replaced by the sum 4. (In notation to be described later $RS(3) = 2$, $FA(3) = 3$, $SA(3) = 2$, $FS(3) = 5$, $SA(3) = 2$, $SS(3) = 4$).

<u>Step 2</u>: Remove the singleton starter $A+1$ from the adder $A+2$ (where the sum is $4 = SS(3)$) and combine it with the singleton starter $A+2$ to form a doubleton starter $(A+2, A+1)$ and forming a new $\pm(x_i - y_i) = \pm 1$. (Here $LS(3) = A+1$, $FLA(3) = A+2$, $SLA(3) = A+2$ in the notation to be described.)

We arrive at a starter-adder for a value $k = 3$ which we call the final starter-adder. This is given by:

Adder:	s,	$s-1$,	$\cdots$,	$A+3$,	$A+2$,	$A+1$,	$\cdots$,	5,	4,	$3, 2, 1$
Starter:	1,	$(s, s-2)$,	$\cdots$,	$(A+2, A+1)$,	x	, A,	$\cdots$,	4,	$(s-1, 3)$,	$x, 2, x$
Sum:	1,	$(s-1, s-3)$,	$\cdots$,	$(6, 5)$,		2,	$\cdots$,	9,	$(3, 7)$,	$, 4,$
$\pm(x_i - y_i)$:		± 2,		± 1,					± 4	

We will now show how steps 1 and 2 can be iterated to go from the starter-adder with a value $k-1$, to the starter-adder with a value k. Note that in going from $k = 2$ to $k = 3$ we created a new doubleton pair with $\pm(x_i - y_i) = \pm 1$. We will carry out the steps in such a way that in going from $k = 3$ to $k = 4$ the values of $\pm(x_i - y_i)$ for the new doubleton pair will be ± 3 and generally in going from $k-1$ to k the new doubleton pair will have $\pm(x_i - y_i) = \pm(2k-5)$. Calling the starter-adder for $k-1$ the initial starter-adder and the starter-adder for k the final starter we introduce the following notation:

$RS(k)$ is the starter in the initial starter-adder which is moved in Step 1.

$FA(k)$ is the adder from which $RS(k)$ has been removed in Step 1.

$SA(k)$ is the adder to which $RS(k)$ has been moved in Step 1.

$FS(k) = FA(k) + RS(k)$ = initial sum associated with the starter $RS(k)$.

$SS(k) = SA(k) + RS(k)$ = final sum associated with the starter $RS(k)$.

$LS(k)$ is the starter for which the sum in the initial starter-adder is equal to $SS(k)$. $LS(k)$ is the starter to be removed in Step 2.

$SLA(k)$ is the adder to which $LS(k)$ is moved. The starter under $SLA(k)$ together with $LS(k)$ form a doubleton pair in the final starter-adder.

$FLA(k)$ is the adder from which $LS(k)$ has been removed.

$FLS(k) = FLA(k) + LS(k) = SS(k)$.

$SLS(k) = SLA(k) + LS(k)$.

Note in the above equalities it is understood that they are taken mod s .

We subject these functions to the following condition (not all independent):

(1) $SA(k) = FA(k-1)$.

(2) $SS(k) \equiv FLS(k) \mod s$.

(3) $SLA(k) = FLA(k) + 2k-5$.

(4) $SLS(k) = FLS(k) + 2k-5$.

(5) $RS(k) = FA(k) - 1$ (holds only for s odd).

(6) $LS(k) = FLA(k) - 1$.

We summarize Steps 1 and 2 as follows:

Step 1. Remove $RS(k)$ from $FA(k)$ to $SA(k)$. Note that these numbers are determined as follows:

$SA(k) = FA(k-1)$ and take $FA(k) = SA(k) + (2k-5)$ and $RS(k)$ appears under $FA(k)$ in the initial starter-adder.

As a consequence $FA(k)$ has no starter associated with it in the final starter-adder and $SS(k) = FS(k) - (2k-5)$.

<u>Step 2.</u> Determine $LS(k)$ as follows. Take $FLS(k) \equiv SS(k) \bmod s$ and $LS(k)$ is the starter associated with the sum $FLS(k)$ in the initial starter-adder. Remove the starter $LS(k)$ from the adder $FLA(k)$ to the adder $FLA(k) + (2k-5) = SLA(k)$. It will pair up with the adder initially under $SLA(k)$ to form a doubleton with $\pm (x_i - y_i) = 2k-5$.

When Steps 1 and 2 are completed it will be verified that $FLA(k) = SLA(k-1) + 1$ and $SLS(k) \equiv FS(k) \bmod s$. This will show that the final configuration is actually a starter-adder and that we can proceed from k to $k+1$ (if $s \geq S_o(k+1)$.

Now $FA(3) = 3$ and by induction $FA(k) = 2 + \sum_{i=3}^{k} (2i-5) = 2 + (k-2)^2$.

Also
$$FS(k) = FA(k) + RS(k) = 2FA(k) - 1 \quad \text{by} \quad (5)$$
$$= 3 + 2(k-2)^2$$
$$SS(k) = FS(k) - (2k-5) = 8 - 2k + 2(k-2)^2$$

Again
$$SS(k) \equiv FLS(k) \bmod s \quad \text{by} \quad (2)$$
$$\equiv FLA(k) + LS(k) \bmod s$$
$$\equiv 2FLA(k) - 1 \bmod s \quad \text{by} \quad (6) .$$

Since $SS(k)$ is even and $2FLA(k) - 1$ is odd $SS(k) = 2FLA(k) - 1 - s$

or
$$2FLA(k) = 8 - 2k + 2(k-2)^2 + 1 + s$$
$$FLA(k) = 4 - k + (k-2)^2 + A \quad (\text{since } A = \tfrac{1+s}{2}) .$$

Hence $SLA(k) = k - 1 + (k-2)^2 + A$.

Replace k by $k-1$

getting $SLA(k-1) = k - 2 + (k-3)^2 + A$

$$= k - 2 + (k-2) - 1)^2 + A$$

$$= k - 2 + (k-2)^2 - 2(k-2) + 1 + A$$

$$= 3 - k + (k-2)^2 + A \; .$$

Hence, $FLA(k) = SLA(k-1) + 1$.

Again, $SLS(k) = FLS(k) + (2k-5)$ by (4)

$$= SS(k) + (2k-5) + s$$

$$= FS(k) + s$$

Hence $SLS(k) \equiv FS(k) \bmod s$.

To determine $S_o'(k)$ note that we can only go from k to $k+1$ if

$$s \geq SLA(k) + 2$$

$$\geq k + 1 + (k-2)^2 + A$$

$$\geq k + 1 + (k-2)^2 + \frac{s+1}{2} \; .$$

Hence $\frac{s}{2} \geq k + 1 + (k-2)^2 + \frac{1}{2}$,

or $s \geq 2k + 3 + 2(k-2)^2 = 2k^2 - 6k + 11$.

Hence we may take $S_o'(k) = 2k^2 - 6k + 11$. In particular $S_o'(3) = 11$.

Case 2. s even. For $s \geq 10$ we have the starter-adder with $k = 2$, $A = \frac{s}{2}$ given by

Adder:	s,	$s-1$,	$\cdots$,	$A+3$,	$A+2$,	$A+1$,	A,	$\cdots$,	4,	3,	2,	1
Starter:	1,	$(s, s-2)$,	$\cdots$,	$A+2$,	$A+1$,	x ,	A,	$\cdots$,	4,	$(s-1, 3)$,	2,	x
Sum:	1,	$(s-1, s-3)$,	$\cdots$,	5 ,	3 ,	,	s,	$\cdots$,	8,	(2, 6),	4	

The calculation is exactly the same as in the odd case but condition (5) must be replaced by (5′) $RS(k) = FA(k)$ s, even.

Again we must show that $FLA(k) = SLA(k-1) + 1$ and $SLS(k) \equiv FS(k) \bmod s$, in order that we can proceed from k to $k+1$ (if $s \geq S_e(k+1)$) and to prove our final configuration is a starter-adder.

Now $FA(3) = 2$ and by induction $FA(k) = 1 + \sum_{i=3}^{k} (2k-5) = 1 + (k-2)^2$.

Also $SA(k) = FA(k-1)$ for $k > 3$

$$= 1 + (k-3)^2 \quad \text{(also true for } k = 3).$$

Again,
$$FS(k) = FA(k) + RS(k)$$
$$= 2FA(k) \quad \text{by } 5'$$
$$= 2 + 2(k-2)^2$$
$$SS(k) = RS(k) + SA(k)$$
$$= FA(k) + SA(k)$$
$$= 1 + (k-2)^2 + 1 + (k-3)^2$$
$$= 2 + (k-2)^2 + (k-3)^2 .$$

$SS(k)$ is odd since the sum of two consecutive squares is odd.

Now
$$SS(k) = FLS(k) - s \quad \text{by } (2)$$
$$= FLA(k) + LS(k) - s$$
$$= 2FLA(k) - 1 - s \quad \text{by } (6)$$

Hence $2 + (k-2)^2 + (k-3)^2 = 2FLA(k) - 1 - s$

or
$$FLA(k) = \frac{3 + (k-2)^2 + (k-3)^2}{2} + \frac{s}{2} .$$

This reduces to

$$FLA(k) = k - 1 + A + (k-3)^2 .$$

Also
$$SLA(k) = FLA(k) + (2k-5) \quad \text{by } (3)$$
$$= A + k - 1 + (k-2)^2$$

or
$$SLA(k-1) = A + k - 2 + (k-3)^2$$

Hence $FLA(k) = SLA(k-1) + 1$

$$SLS(k) = FLS(k) + 2k - 5 \quad \text{by} \quad (4)$$

$$= 2FLA(k) - 1 + 2k - 5$$

$$= s + 2k - 3 + 2(k-3)^2 + 2k - 5$$

$$= s + 2 + 2(k-2)^2$$

Hence $SLS(k) \equiv FS(k) \bmod s$.

To determine $S_e'(k)$ note that

$$s \geq 2 + SLA(k)$$

$$= 2 + A + k - 1 + (k-2)^2 \ .$$

Since $A = \frac{s}{2}$ this becomes

$$s \geq 2 + 2k + 2(k-2)^2 = 2k^2 - 6k + 10 \ .$$

Hence $S_e'(k)$ may be taken as $2k^2 - 6k + 10$.

Since $S_e'(k) - S_o'(k) = 1$ we may take

$$S'(k) = S_e'(k) = 2k^2 - 6k + 10 \ .$$

Hence $N'(k) = 2k^2 - 7k + 10$.

7. Some best possible results and some information obtained from a computer.

In section six the value $N'(k)$ is usually not best possible. In this section we sill improve some of the values of $N'(k)$ and summarize in a table the results obtained. The following table gives the estimates of $S'(k)$ and $N'(k)$ obtained in Section 6 and better estimates $S''(k)$ and $N''(k)$ obtained by special constructions which are similar to those in Section 6, and by computer constructions.

k	S′(k)	N′(k)	S″(k)	N″(k)
0	-	-	3	3
1	-	-	-	-
2	-	-	6	4
3	10	7	9	6
4	18	14	9	5
5	30	25	11	6
6	46	40	13	7
7	66	59	15	8
8	90	82	17	9
9	118	109	19	10
10	150	140	21	11

In the next table we list the known results for $n = 1$ to 15 and $k = 0$ to 10.

2n	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	2n > 30
n	1	x	3	{4}	5	{6}	7	{8}	9	{10}	11	{12}	13	{14}	15	all exist n = odd by S–A n = even by O.L.S.
n + 1		x	4	x	6	{7}	8	{9}	10	{?}	12	{?}	14	{?}	16	n = odd all exist by S–A n = even cannot be obtained by S–A.
n + 2			x	{6}	7	8	9	10	11	12	13	14	15	16	17	all exist S(2) = 6.
n + 3				7	{?}	9	10	11	12	13	14	15	16	17	18	all exist S″(3) = 9 .
n + 4					9	10	11	12	13	14	15	16	17	18	19	all exist S(4) = 9.
n + 5						11	12	13	14	15	16	17	18	19	20	all exist S(5) = 11.
n + 6							13	14	15	16	17	18	19	20	21	all exist S(6) = 13.
n + 7								15	16	17	18	19	20	21	21	all exist S(7) = 15.
n + 8									17	18	19	20	21	22	23	all exist S(8) = 17.
n + 9										19	20	21	22	23	24	all exist S(9) = 19.
n + 10											21	22	23	24	25	all exist S(10) = 21.

All sides listed here can be constructed by the starter-adder method unless the following entries appear:

x : has been proved that it does not exist.

{i}: cannot be constructed by the starter-adder method, but does exist.

{?}: not yet known whether the design exists, but has been proved that it cannot be constructed by starter-adder method.

<u>Validation of the table.</u>

$k+0$ and $k=1$, have been treated in a previous section.

$k = 2$. $H(5, 6)$ does not exist. $H(6, 8)$ is given in Section 3.

For all $n \geq 5$ i.e. $s \geq 7$, the general method is given in Theorem 12. Hence $S(2) = 6$.

$k = 3$. By Theorem 13, $S'(3) = 10$. $H(8, 10)$ cannot be constructed by starter-adder and its existence has not yet been established. For $H(7, 8)$ the following is a starter-adder.

Starter:	(2, 4),	(5, 6),	(7, 3),	1
Adder:	3 ,	5 ,	6 ,	7

For $H(9, 12)$ the following is a starter-adder.

Starter:	(2, 4),	(5, 6),	(7, 3),	8,	9,	1
Adder:	1 ,	3 ,	4 ,	5,	6,	9

Hence $S''(3) = 9$ and $S(3) = 9$ or $S(3) = 8$.

$k = 4$. For s even $S_e'(4) = 18$ by Theorem 13. For s odd $S_o''(4) = 13$ using the following starter-adder.

Adder:	s, ···,	14,	13 ,	12,	11,	10 ,	9,	8,	7,	6,	5 ,	4,	3,	2 ,	1
Starter:	s, ···,	14,	(5, 13),	12,	11,	(6, 10),	x,	x,	x,	x,	(7, 9),	4,	3,	(2, 8),	1

for $s = 9, 10, \cdots, 16$ a computer starter-adder is given in the next table. Hence $S(4) = 9$.

k = 5. $S_o(5) \le 23$ from the following starter-adder with s odd $s \ge 23$ and $A = \frac{s+1}{2}$.

Adder:	s,	$(s-1)$,	$\cdots$,	$A+9$,	$A+8$,	$A+7$,	$A+6$,	$A+5$,	$A+3$,
Starter:	1,	$(s, s-2)$,	$\cdots$,	$(A+8, A+5)$,	$A+7$,	x,	x,	$A+4$,	$(A+2, A+1)$,

(Continued)

Adder:	$A+2$,	$A+1$,	A,	$\cdots$,	11,	10,	9,	8,	7,	6,	5,	4,	3,	2,	1
Starter:	$A+6$,	A,	$A-1$,	$\cdots$,	10,	(9, 4),	8,	x,	6,	5,	7,	$(s-1, 3)$,	x,	2,	x

$S_e(5) \le 22$ from the following starter-adder with s even, $s \ge 22$, $A = \frac{s}{2}$.

Adder:	s,	$s-1$,	$\cdots$,	$A+10$,	$A+9$,	$A+8$,	$A+7$,	$A+6$,	$\cdots$,	$A+2$,	$A+1$,
Starter:	1,	$(s, s-2)$,	$\cdots$,	$A+9$,	$(A+8, A+3)$,	x,	x,	$A+5$,	$\cdots$,	$A+1$,	$A+7$,

(Continued)

Adder:	A,	$\cdots$,	12,	11,	10,	9,	8,	7,	6,	5,	4,	3,	2,	1
Starter:	A,	$\cdots$,	12,	(11, 4),	10,	9,	(8, 5),	7,	x,	x,	x,	$(s-1, 3)$,	2,	6

For $11 \le s \le 21$, the computer values of the starter-adder are given in the next table. Hence $S(5) = 11$.

k = 6. We obtain $S_e''(6) = 22$ from the following starter-adder with s even and $A = \frac{s}{2}$, $s \ge 22$.

Adder:	s,	$s-1$,	$\cdots$,	$A+10$,	$A+9$,	$A+8$,	$A+7$,	$A+6$,	$A+5$,	$A+4$,
Starter:	1,	$(s, s-2)$,	$\cdots$,	$A+9$,	$(A+8, 13)$,	x,	x,	$A+5$,	$A+4$,	x,

(Continued)

Adder:	$A+3$,	$A+2$,	$A+1$,	A,	$\cdots$,	12,	11,	10,	9,	8,	7,	6,	5,
Starter:	$A+2$,	$A+1$,	$(A+6, A+7)$,	A,	$\cdots$,	12,	(11, 4),	10,	9,	(8, 5),	7,	x,	x,

(Continued)

Adder:	4,	3,	2,	1
Starter:	6,	$(s-1, 3)$,	2,	x

Also, $S''_o(6) = 25$ is obtained from the following starter-adder with s odd, $s \geq 25$, $A = \frac{s+1}{2}$.

Adder:	s,	s − 1,	⋯,	A + 11,	A + 10,	A + 9,	A + 8,	A + 7,	A + 6,	A + 5,	A + 4,
Starter:	1,	(s, s − 2),	⋯,	A + 10,	(A + 9, 4),	x,	x,	A + 6,	A + 5,	A + 7,	A + 3,

(Continued)

Adder:	A + 3,	A + 2,	A + 1,	A,	⋯,	14,	13,	12,	11,	10,	9,	8,
Starter:	(a + 2, A + 1),	A + 8,	A,	A − 1,	⋯,	13,	(12, 5),	11,	10,	(9, 6),	8,	x,

(Continued)

Adder:	7,	6,	5,	4,	3,	2,	1
Starter:	x,	x,	4,	(s − 1, 3),	7,	2,	x

For $13 \leq s, \leq 23$, the computer starter-addersare given in the next table. Hence $S(6) = 13$.

k = 7. We obtain $S''_o(7) = 29$, from the following starter-adder with s odd, $s \geq 29$, $A = \frac{s+1}{2}$.

Adder:	s,	s − 1,	⋯,	A + 13,	A + 12,	A + 11,	A + 10,	A + 9,	A + 8,
Starter:	1,	(s, s − 2),	⋯,	A + 12,	(A + 11, A + 8),	(A + 10, A + 3),	A + 9,	x,	A + 7,

(Continued)

Adder:	A + 7,	A + 6,	A + 5,	A + 4,	A + 3,	A + 2,	A + 1,	A,	⋯,	13,	12,
Starter:	x,	A + 5,	A + 4,	x,	(A + 2, A + 1),	A + 6,	A,	A − 1,	⋯,	12,	(11, 5),

(Continued)

Adder:	11,	10,	9,	8,	7,	6,	5,	4,	3,	2,	1
Starter:	x,	(9, 4),	x,	10,	6,	5,	x,	(s − 1, 3),	8,	2,	7

$S''_e(7) = 28$, from the following starter-adder with s even, $s \geq 28$, $A = \frac{s}{2}$.

Adder:	s,	$s-1$,	$\cdots$,	$A+13$,	$A+12$,	$A+11$,	$A+10$,	$A+9$,
Starter:	1,	$(s, s-2)$,	$\cdots$,	$A+12$,	$(A+11, A+4)$,	$(A+10, A+9)$,	x,	$A+8$,

(Continued)

Adder:	$A+8$,	$A+7$,	$A+6$,	$A+5$,	$A+4$,	$A+3$,	$A+2$,	$A+1$,	A,	$\cdots$,	13,	12,	11,
Starter:	$A+7$,	$A+6$,	$A+5$,	x,	x,	$A+2$,	$A+1$,	$A+3$,	A,	$\cdots$,	13,	(12, 7),	11,

(Continued)

Adder:	10,	9,	8,	7,	6,	5,	4,	3,	2,	1
Starter:	x,	9,	x,	x,	6,	(5, 2),	(10, 4),	$(s-1, 3)$,	x,	8

For $15 \le s \le 28$, the computer starter-adders are given in the next table. Hence $S(7) = 15$.

$k = 8$. We obtain $S''_o(8) = 31$, from the following starter-adder with s odd, $s \ge 31$, $A = \frac{s+1}{2}$.

Adder:	s,	$s-1$,	$\cdots$,	$A+13$,	$A+12$,	$A+11$,	$A+10$,	$A+9$,
Starter:	1,	$(s, s-2)$,	$\cdots$,	$A+12$,	$A+11$,	$(A+10, A+3)$,	$A+9$,	$(A+8, A+7)$,

(Continued)

Adder:	$A+8$,	$A+7$,	$A+6$,	$A+5$,	$A+4$,	$A+3$,	$A+2$,	$A+1$,	A,	$\cdots$,	16,
Starter:	x,	$A+6$,	$(A+5, A+2)$,	$A+4$,	x,	x,	$A+1$,	A,	$A-1$,	$\cdots$,	15,

(Continued)

Adder:	15,	14,	13,	12,	11,	10,	9,	8,	7,	6,	5,	4,	3,	2,	1
Starter:	(14, 8),	(13, 5),	12,	x,	10,	x,	x,	x,	6,	x,	11,	$(s-1, 3)$,	2,	(4, 9),	7

$S''_e(8) = 30$, from the following starter-adder with, s even, $s \ge 30$, $A = \frac{s}{2}$.

Adder:	s	s − 1	$\cdots$	A + 9	A + 8	A + 7	A + 6	A + 5	A + 4	A + 3
Starter:	1	(s, s − 2)	$\cdots$	A + 8	x	(A + 6, A + 3)	A + 5	A + 4	x	(A + 2, A + 1

(Continued)

Adder:	A + 2	A + 1	A	$\cdots$	16	15	14	13	12	11	10	9	8	7
Starter:	x	A + 7	A	$\cdots$	16	(15, 9)	(14, 6)	13	x	(11, 4)	x	x	8	7

(Continued)

Adder:	6	5	4	3	2	1
Starter:	12	x	x	(s − 1, 3)	(5, 10)	2

For $17 \le s \le 29$, the computer starter-adders are given in the next table. Hence $S(8) = 17$.

k = 9. Here $S''_e(9) = 34$, from the following starter-adder with s even, $s \ge 34$, $A = \frac{s}{2}$.

Adder:	s	s − 1	$\cdots$	A + 16	A + 15	A + 14	A + 13	A + 12	A + 11
Starter:	1	(s, s − 2)	$\cdots$	A + 15	(A + 14, A + 5)	A + 13	A + 12	A + 11	A + 10

(Continued)

Adder:	A + 10	A + 9	A + 8	A + 7	A + 6	A + 5	A + 4	A + 3
Starter:	(A + 9, A + 4)	x	A + 7	x	x	A + 8	(A + 6, A + 3)	(A + 2, A + 1)

(Continued)

Adder:	A + 2	A + 1	A	$\cdots$	17	16	15	14	13	12	11	10	9	8	7	6
Starter:	x	x	A	$\cdots$	17	16	15	(14, 8)	13	(12, 5)	x	x	9	x	x	6

(Continued)

Adder:	5	4	3	2	1
Starter:	11	4	(s − 1, 3)	7	(2, 10)

$S''_o(9) = 35$, from the following starter-adder with s odd, $s \geq 35$, $A = \frac{s+1}{2}$.

Adder:	s,	$s-1$,	$\cdots$,	$A+16$,	$A+15$,	$A+14$,	$A+13$,	$A+12$, $A+11$,
Starter:	1,	$(s, s-2)$,	$\cdots$,	$A+15$,	$(A+14, A+5)$,	$A+13$,	x,	$A+11$, $A+10$,

(Continued)

Adder:	$A+10$,	$A+9$,	$A+8$,	$A+7$,	$A+6$,	$A+5$,	$A+4$,	$A+3$,	$A+2$,	$A+1$,
Starter:	$A+9$,	$A+8$,	$A+6$,	x,	$A+12$,	$A+4$,	x,	$(A+2, A+7)$,	$A+1$,	$(A+3, A)$,

(Continued)

Adder:	A,	$\cdots$,	19,	18,	17,	16,	15,	14,	13,	12,	11, 10,	9,
Starter:	$A-1$,	$\cdots$,	18,	(17, 11),	(16, 9),	15,	x,	13,	(12, 4),	x,	5, x,	14,

(Continued)

Adder:	8,	7,	6,	5,	4,	3,	2,	1
Starter:	x,	(7, 6),	2,	x,	$(s-1, 3)$,	x,	10,	8

For $19 \leq s \leq 33$, the next table gives the computer starter-adder. Hence $S(9) = 19$.

$k = 10$. Here $S''_e(10) = 34$ from the following starter-adder with s even, $s \geq 34$, $a = \frac{s}{2}$.

Adder:	s,	$s-1$,	$\cdots$,	$A+15$,	$A+14$,	$A+13$,	$A+12$,	$A+11$, $A+10$,
Starter:	1,	$(s, s-2)$,	$\cdots$,	$A+14$,	$A+13$,	$(A+12, A+4)$,	$A+7$,	$A+10$, x,

(Continued)

Adder:	$A+9$,	$A+8$,	$A+7$,	$A+6$,	$A+5$,	$A+4$,	$A+3$,	$A+2$,
Starter:	$(A+5, A+11)$,	x,	x,	x,	x,	$(A+6, A+3)$,	$(A+2, A+1)$,	$A+9$,

(Continued)

Adder:	$A+1$,	A,	$\cdots$,	17,	16,	15,	14,	13,	12,	11, 10,	9,	8,
Starter:	$A+8$,	A,	$\cdots$,	17,	(16, 6),	(15, 8),	14,	13,	12,	x, x,	7,	(10, 8),

(Continued)

Adder:	7,	6,	5,	4,	3	, 2,	1
Starter:	x,	9,	x,	4,	$(s-1, 3)$,	x,	(2, 11)

$S''_o(10) = 35$ from the following starter-adder, s odd, $s \geq 35$, $A = \frac{s+1}{2}$.

Adder:	s,	$s-1$	, $\cdots$,	$A+15$,	$A+14$	, $A+13$,	$A+12$,	$A+11$,	$A+10$,	
Starter:	1,	$(s, s-2)$,	$\cdots$,	$A+14$,	$(A+13, A+5)$,	$A+12$,	x	, $A+10$,	x	,

(Continued)

Adder:	$A+9$,	$A+8$	, $A+7$,	$A+6$,	$A+5$,	$A+4$,	$A+3$,	$A+2$,
Starter:	$A+8$,	$(A+7, A+4)$,	$A+6$,	x	, $A+11$,	$A+3$,	x	, $(A+2, A+1)$,

(Continued)

Adder:	$A+1$	, A	, $\cdots$,	18,	17	, 16	, 15,	14,	13,	12,	11, 10,
Starter:	$(A+9, A)$,	$A-1$,	$\cdots$,	17,	(16, 6),	(15, 8),	14,	13,	12,	x,	10, 9,

(Continued)

Adder:	9,	8	, 7,	6,	5,	4	, 3,	2,	1
Starter:	x,	(2, 7),	x,	x,	4,	$(s-1, 3)$,	x,	x,	(5, 11)

For $21 \leq s \leq 33$, the computer starter-adders are given in the next table. Hence $S(10) = 21$.

<u>Conjecture.</u> The following conjecture is almost certainly true, namely: $S(k) = 2k+1$ for all $k \geq 4$.

Table of computer calculated values of H(s, 2n).

k = 4, S = 9,...,16;	k = 8, S = 17,...,29;
k = 5, S = 11,...,21;	k = 9, S = 19,...,33;
k = 6, S = 13,...,23;	k = 10, S = 21,...,33.
k = 7, S = 15,...,27;	

```
**********     K =    4     **********

               H( 9,10)
S=   1  2   3  5   6  9   4  8
A=     1      3      7      6
S=  7
A=  2

               H(10,12)
S=   1  2   3  5   6  9  10  4
A=     1      3      8      5
S=  7  8
A=  4  2

               H(11,14)
S=   1  2   3  5   6  9  10  4
A=     1      2      3      7
S=  7  8 11
A=  8 11 10

               H(12,16)
S=   1  2   3  5   6 10  11  4
A=     1      2      3      7
S=  7  8  9 12
A=  5  8 11 10

               H(13,18)
S=   1  2   3  5   6  9  10  4
A=     1      2      3      4
S=  7  8 11 12 13
A=  6  9  8 12 10

               H(14,20)
S=   1  2   3  5   6  9  10 14
A=     1      2      3      4
S=  4  7  8 11 12 13
A=  6  8  5  9 10 12

               H(15,22)
S=   1  2   3  5   6  9  10 14
A=     1      2      3      5
S=  4  7  8 11 12 13 15
A=  4  6  8 15  9 12 14
```

```
                H(16,24)
S=    1  2   3  5   6  9  10 14
A=     1      2      3      6
S=   4  7  8 11 12 13 15 16
A=   4  7  9 11 14 16 12 15


**********    K =    5    **********

                H(11,12)
S=    1  2   3  5   6  9  10  4   7 11
A=     1      4      2     11     5
S=   8
A=   9

                H(12,14)
S=    1  2   3  5   6 10  11  8   9  4
A=     1      2      6     10     4
S=   7 12
A=   3 11

                H(13,16)
S=    1  2   3  5   6  9  10  4   7 11
A=     1      2      3      4     6
S=   8 12 13
A=  11 12 10

                H(14,18)
S=    1  2   3  5   6  9  10 14   4 12
A=     1      2      4      8    11
S=   7  8 11 13
A=   5  6  9 12

                H(15,20)
S=    1  2   3  5   6  9  10 14  15  7
A=     1      2      3      5     6
S=   4  8 11 12 13
A=  10  8 14 11 13

                H(16,22)
S=    1  2   3  5   6  9  10 14  15  4
A=     1      2      3      6     7
S=   7  8 11 12 13 16
A=   8  5 15 12  4 14

                H(17,24)
S=    1  2   3  5   6  9  10 14  15  4
A=     1      2      3      4     6
S=   7  8 11 12 13 16 17
A=   8  9  5 13 10 14 11
```

```
                  H(18,26)
S=     1   2    3   5    6   9    10  14    15   4
A=       1        2        3        4        7
S=   7   8  11  12  13  16  17  18
A=   6   9   8  12  13  17  11  16

                  H(19,28)
S=     1   2    3   5    6   9    10  14    15   4
A=       1        2        3        4       12
S=   7   8  11  12  13  16  17  18  19
A=   6   7   9   5  10  14   8  11  19

                  H(20,30)
S=     1   2    3   5    6   9    10  14    15  20
A=       1        2        3        4        6
S=   4   7   8  11  12  13  16  17  18  19
A=   7   8  11   9  12  15  14  16  19  17

                  H(21,32)
S=     1   2    3   5    6   9    10  14    15  20
A=       1        2        3        4        5
S=   4   7   8  11  12  13  16  17  18  19  21
A=   6   8   9  11   7  14  13  15  16  18  21


**********      K  =    6      **********

                  H(13,14)
S=     1   2    3   5    6  10    11   8     9   4     7  13
A=       1        2        4         13        8        6
S=  12
A=  10

                  H(14,16)
S=     1   2    3   5    6   9    10  14     4  12    13   8
A=       1        2        4         8        11       12
S=   7  11
A=   5   3

                  H(15,18)
S=     1   2    3   5    6   9    10  14    15   7     8  13
A=       1        2        3        11        8        6
S=   4  11  12
A=   9   5  14

                  H(16,20)
S=     1   2    3   5    6   9    10  14    15   4     7  13
A=       1        2        3         6        9        4
S=   8  11  12  16
A=   7  15  10  14
```

```
                H(17,22)
S=    1  2   3  5   6  9   10 14  15  4    7 12
A=     1      2      3      4      6      16
S=   8 11 13 16 17
A=   7  5 12 14 17

                H(18,24)
S=    1  2   3  5   6  9   10 14  15  4    7 12
A=     1      2      3      4      7      12
S=   8 11 13 16 17 18
A=   5  6 13 17 11 16

                H(19,26)
S=    1  2   3  5   6  9   10 14  15  4    7 12
A=     1      2      3      4     12       8
S=   8 11 13 16 17 18 19
A=   5  6 10  9 13 11 19

                H(20,28)
S=    1  2   3  5   6  9   10 14  15 20   4 11
A=     1      2      3      4      6       9
S=   7  8 12 13 16 17 18 19
A=   8 11  5 15 14  7 13 17

                H(21,30)
S=    1  2   3  5   6  9   10 14  15 20   21  7
A=     1      2      3      4      5       6
S=   4  8 11 12 13 16 17 18 19
A=   7  8 10 17  9 15 19 20 21

                H(22,32)
S=    1  2   3  5   6  9   10 14  15 20   21  7
A=     1      2      3      4      6       9
S=   4  8 11 12 13 16 17 18 19 22
A=   7  5 11  8 10 12 15 19 22 17

                H(23,34)
S=    1  2   3  5   6  9   10 14  15 20   21  4
A=     1      2      3      4      7      12
S=   7  8 11 12 13 16 17 18 19 22 23
A=   6  9  8 11 16 18 21 13  5 22 20


**********     K =    7     **********

                H(15,16)
S=    1  2   3  5   6  9   10 15    4 11   12  8   13  7
A=     1      2      4      9      12       3      14
S= 14
A= 15
```

```
                     H(16,18)
S=    1  2    3  5    6  9    10 16    4 15     7 14     8 12
A=      1       2       3        4       12        8        5
S= 11 13
A= 15 11

                     H(17,20)
S=    1  2    3  5    6  9    10 14   15  4     7 16    17 12
A=      1       2       3        4        6       16       13
S=  8 11 13
A=  9  5 15

                     H(18,22)
S=    1  2    3  5    6  9    10 14   15  4     7 13    16  8
A=      1       2       3        4        7        6        8
S= 11 12 17 18
A= 15  5 16 10

                     H(19,24)
S=    1  2    3  5    6  9    10 14   15  4     7 12    13 19
A=      1       2       3        4       12        8        6
S=  8 11 16 17 18
A=  9 18  7 13 14

                     H(20,26)
S=    1  2    3  5    6  9    10 14   15 20     4 11    12 18
A=      1       2       3        4        6        9       12
S=  7  8 13 16 17 19
A=  8 11 15 20 14 18

                     H(21,28)
S=    1  2    3  5    6  9    10 14   15 20    21  7     8 16
A=      1       2       3        4        5        6       13
S=  4 11 12 13 17 18 19
A=  7  8 10 18 21 19 17

                     H(22,30)
S=    1  2    3  5    6  9    10 14   15 20    21  7     8 17
A=      1       2       3        4        6        9        5
S=  4 11 12 13 16 18 19 22
A=  7  8 11 15 16 21 18 20

                     H(23,32)
S=    1  2    3  5    6  9    10 14   15 20    21  4     7 16
A=      1       2       3        4        7       12        8
S=  8 11 12 13 17 18 19 22 23
A=  5  6  9 10 14 11 15 21 19

                     H(24,34)
S=    1  2    3  5    6  9    10 14   15 20    21  4     7 13
A=      1       2       3        4        5        7        6
S=  8 11 12 16 17 18 19 22 23 24
A=  8 10 11 14 15 16 20 19 23 24
```

```
            H(25,36)
S=    1  2    3  5    6  9   10 14   15 20   21  4    7 13
A=      1       2       3       4       5      12      6
S=  8 11 12 16 17 18 19 22 23 24 25
A=  7 10 11 13  9 17 23 14  8 25 22

            H(26,38)
S=    1  2    3  5    6  9   10 14   15 20   21  4    7 13
A=      1       2       3       4       5       6       8
S=  8 11 12 16 17 18 19 22 23 24 25 26
A=  9 11  7 10 13 14 15 17 19 25 12 24

            H(27,40)
S=    1  2    3  5    6  9   10 14   15 20   21 27    4 11
A=      1       2       3       4       5       6      11
S=  7  8 12 13 16 17 18 19 22 23 24 25 26
A=  9 13  7 10 12 14 17 18 22 15 16 26 27


**********    K =    8     **********

            H(17,18)
S=    1  2    3  5    6  9   10 15   16  7    8 14   17 13    4 11
A=      1       2       3       8      14      17      15       6
S= 12
A=  4

            H(18,20)
S=    1  2    3  5    6  9   10 14   15  4    7 17   18 12   13  8
A=      1       2       3       4      11      12      16       9
S= 11 16
A= 13 15

            H(19,22)
S=    1  2    3  5    6  9   10 14   15  4    7 12   13 19    8 17
A=      1       2       3       4      12       8       6      15
S= 11 16 18
A= 19 13 18

            H(20,24)
S=    1  2    3  5    6  9   10 14   15 20    4 11   12 18   19  7
A=      1       2       3       4       6       9      18      12
S=  8 13 16 17
A=  7 15  8 20

            H(21,26)
S=    1  2    3  5    6  9   10 14   15 20   21  7    8 16   17 11
A=      1       2       3       4       5       6      13      20
S=  4 12 13 18 19
A= 11  7  9 14 19
```

```
            H(22,28)
S=    1  2    3  5    6  9   10 14   15 20   21  7    8 17   18 11
A=      1       2       3       4       6       9      5      21
S=  4 12 13 16 19 22
A=  7  8 10 12 18 19

            H(23,30)
S=    1  2    3  5    6  9   10 14   15 20   21  4    7 17   18 11
A=      1       2       3       4       7      12      6      13
S=  8 12 13 16 19 22 23
A=  9  8 16  5 19 20 11

            H(24,32)
S=    1  2    3  5    6  9   10 14   15 20   21  4    7 13   16 24
A=      1       2       3       4       5       7      6       8
S=  8 11 12 17 18 19 22 23
A=  9 10 11 13 16 20 18 23

            H(25,34)
S=    1  2    3  5    6  9   10 14   15 20   21  4    7 13   16 23
A=      1       2       3       4       5      12      6       8
S=  8 11 12 17 18 19 22 24 25
A=  7 10 14 18 11 17 20 24 22

            H(26,36)
S=    1  2    3  5    6  9   10 14   15 20   21  4    7 13   16 23
A=      1       2       3       4       5       6      8       7
S=  8 11 12 17 18 19 22 24 25 26
A=  9 11 12 15 16 18 20 21 14 26

            H(27,38)
S=    1  2    3  5    6  9   10 14   15 20   21 27    4 11   12 22
A=      1       2       3       4       5       6     11       9
S=  7  8 13 16 17 18 19 23 24 25 26
A= 10  8 13 12  7 17 18 15 16 21 24

            H(28,40)
S=    1  2    3  5    6  9   10 14   15 20   21 27   28  7    8 16
A=      1       2       3       4       5       7      8      11
S=  4 11 12 13 17 18 19 22 23 24 25 26
A=  6 10 12 16  9 14 20 19 22 27 25 18

            H(29,42)
S=    1  2    3  5    6  9   10 14   15 20   21 27   28  7    8 17
A=      1       2       3       4       5       6      9       7
S=  4 11 12 13 16 18 19 22 23 24 25 26 29
A= 13  8 10 15 14 11 20 18 12 26 17 29 23
```

```
**********    K =   9    **********

              H(19,20)
S=   1  2   3  5   6  9   10 14   15  4    7 12   13 19    8 17   18 11
A=    1      2      3      4      12       8       6      15      18
S= 16
A= 14

              H(20,22)
S=   1  2   3  5   6  9   10 14   15 20    4 11   12 18   19  7    8 17
A=    1      2      3      4       6       9      18      12      20
S= 13 16
A= 11 19

              H(21,24)
S=   1  2   3  5   6  9   10 14   15 20   21  7    8 18   19 13   16  4
A=    1      2      3      4       5       6      14      18      13
S= 11 12 17
A=  8  9 19

              H(22,26)
S=   1  2   3  5   6  9   10 14   15 20   21  7    8 17   18 11   12 22
A=    1      2      3      4       6       9       5      21      11
S=  4 13 16 19
A= 15  7 12 18

              H(23,28)
S=   1  2   3  5   6  9   10 14   15 20   21  4    7 17   18 11   12 23
A=    1      2      3      4       7      12       6      13      17
S=  8 13 16 19 22
A= 11  8 18 19 21

              H(24,30)
S=   1  2   3  5   6  9   10 14   15 20   21  4    7 13   16 24    8 17
A=    1      2      3      4       5       7       6       8      13
S= 11 12 18 19 22 23
A= 11 22 23 20 18 24

              H(25,32)
S=   1  2   3  5   6  9   10 14   15 20   21  4    7 13   16 23   24  8
A=    1      2      3      4       5      12       6       8      18
S= 11 12 17 18 19 22 25
A= 10 11 19 17 21  7 22

              H(26,34)
S=   1  2   3  5   6  9   10 14   15 20   21  4    7 13   16 23   24  8
A=    1      2      3      4       5       6       8       7      18
S= 11 12 17 18 19 22 25 26
A= 11 12 15 19 20 23  9 17
```

```
                 H(27,36)
S=    1  2    3  5    6  9   10 14   15 20   21 27   4 11   12 22   23  7
A=     1       2       3       4       5       6      11       9      12
S=  8 13 16 17 18 19 24 25 26
A=  8 13  7 20 10 19 16 26 18

                 H(28,38)
S=    1  2    3  5    6  9   10 14   15 20   21 27   28  7    8 16   17 26
A=     1       2       3       4       5       7       8      11       6
S=  4 11 12 13 18 19 22 23 24 25
A=  9 10 14 16 20 25 17 22 26 27

                 H(29,40)
S=    1  2    3  5    6  9   10 14   15 20   21 27   28  7    8 17   18 25
A=     1       2       3       4       5       6       9       7      10
S=  4 11 12 13 16 19 22 23 24 26 29
A= 13  8 11 16 14 20 18 19 26 29 22

                 H(30,42)
S=    1  2    3  5    6  9   10 14   15 20   21 27   28  7    8 16   17 24
A=     1       2       3       4       5       7       8      11       6
S=  4 11 12 13 18 19 22 23 25 26 29 30
A=  9 10 12 16 13 19 18 24 21 15 23 26

                 H(31,44)
S=    1  2    3  5    6  9   10 14   15 20   21 27   28  4    7 16   17 25
A=     1       2       3       4       5       8       9      10       6
S=  8 11 12 13 18 19 22 23 24 26 29 30 31
A=  7 11 12 14 21 13 19 24 26 16 23 29 30

                 H(32,46)
S=    1  2    3  5    6  9   10 14   15 20   21 27   28  4    7 16   17 24
A=     1       2       3       4       5       6      12       8      14
S=  8 11 12 13 18 19 22 23 25 26 29 30 31 32
A=  9 10  7 13 11 17 20 22 18 28 26 30 31 32

                 H(33,48)
S=    1  2    3  5    6  9   10 14   15 20   21 27   28  4    7 17   18 25
A=     1       2       3       4       5       6       9      12      14
S=  8 11 12 13 16 19 22 23 24 26 29 30 31 32 33
A=  7 10 11 13  8 15 19 20 25 18 21 31 24 32 30


**********    K =   10     **********

                 H(21,22)
S=    1  2    3  5    6  9   10 14   15 20   21  8   11 17   18  7   12 19    4 13
A=     1       2       3       4       5      13      21       9      10       6
S= 16
A= 20
```

```
              H(22,24)
S=    1  2    3  5    6  9    10 14    15 20    21  7    8 18    19  4    11 17    22 13
A=      1       2       3       4        6        9        5       13       11       20
S= 12 16
A=  7 21

              H(23,26)
S=    1  2    3  5    6  9    10 14    15 20    21  4    7 17    18 11    12 23     8 16
A=      1       2       3       4        7       15       14       22       11        8
S= 13 19 22
A= 16 19 21

              H(24,28)
S=    1  2    3  5    6  9    10 14    15 20    21  4    7 13    16 24     8 19    22 12
A=      1       2       3       4        5        7        6        8       15       18
S= 11 17 18 23
A= 10 24 21 23

              H(25,30)
S=    1  2    3  5    6  9    10 14    15 20    21  4    7 13    16 23    24  8    11 22
A=      1       2       3       4        5       12        6        8       18       24
S= 12 17 18 19 25
A= 10 19 11 21 23

              H(26,32)
S=    1  2    3  5    6  9    10 14    15 20    21  4    7 13    16 23    24  8    11 19
A=      1       2       3       4        5        6        8        7       18       13
S= 12 17 18 22 25 26
A= 10 17 25 15 14 19

              H(27,34)
S=    1  2    3  5    6  9    10 14    15 20    21 27    4 11    12 22    23  7     8 16
A=      1       2       3       4        5        6       11        9       12        8
S= 13 17 18 19 24 25 26
A= 13 20 10 19 16 25 18

              H(28,36)
S=    1  2    3  5    6  9    10 14    15 20    21 27    28  7    8 16    17 26     4 19
A=      1       2       3       4        5        7        8       11        6       20
S= 11 12 13 18 22 23 24 25
A= 10 14 16 23 22 15 21 25

              H(29,38)
S=    1  2    3  5    6  9    10 14    15 20    21 27    28  7    8 17    18 25    26 11
A=      1       2       3       4        5        6        9        7       10       29
S=  4 12 13 16 19 22 23 24 29
A= 13 11 16 14 20 26 19 27 21

              H(30,40)
S=    1  2    3  5    6  9    10 14    15 20    21 27    28  7    8 16    17 24    25 11
A=      1       2       3       4        5        7        8       11        6       13
S=  4 12 13 18 19 22 23 26 29 30
A=  9 10 16 22 12 19 23 25 18 26
```

H(31,42)

S= 1 2 3 5 6 9 10 14 15 20 21 27 28 4 7 16 17 25 26 8
A= 1 2 3 4 5 8 9 10 6 13
S= 11 12 13 18 19 22 23 24 29 30 31
A= 11 7 15 14 22 20 23 31 18 28 30

H(32,44)

S= 1 2 3 5 6 9 10 14 15 20 21 27 28 4 7 16 17 24 25 8
A= 1 2 3 4 5 6 12 8 14 11
S= 11 12 13 18 19 22 23 26 29 30 31 32
A= 10 16 9 24 13 7 20 19 26 28 18 30

H(33,46)

S= 1 2 3 5 6 9 10 14 15 20 21 27 28 4 7 17 18 25 26 8
A= 1 2 3 4 5 6 9 12 14 8
S= 11 12 13 16 19 22 23 24 29 30 31 32 33
A= 10 11 13 15 22 21 7 20 19 25 26 18 28

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APPENDIX 1

Program (COMB 4)

```
      IMPLICIT INTEGER (A-Y)
      COMMON /B1/ DB,PPN,JJ,OLT,COUN,C,HHN,CHEC,CAP,HP,HK,B,IK,A,NN,U,V
      DIMENSION L(16),B(140,4),C(15,35,4),BOCUL(16),OCUL(16),
     1 PPN(15), HHN(15,14), CHEC(15,14,70,2),COUN(15,14),
     2 TLL(16),OL(16),OLT(15,14,16),HP(15,14), IK(15,14),OCH(16),
     3 HK(15,14),CAP(15,14),JJ(15,14),DB(15)
      ZTT=0.
      CALL $TRTM(ZTIME)
      READ (5,209) ZZ
  209 FORMAT (F8.0)
      READ (5,55) YY
   55 FORMAT (I1)
      READ (5,50) T, (L(I),I=1,T)
      WRITE (6,51) T, (L(I), I=1,T)
   50 FORMAT (17I4)
   51 FORMAT (' T = ',I4,/16I4)
      S=(T-1)/2
      U=(T-1)*(T-2)/6
      V=T*(T-1)*(T-2)/24
      IF(YY .EQ. 0) GO TO 204
      CALL READ1
      A=A+1
      GO TO 205
  204 CONTINUE
      NN=0
      READ (5,59) PP
   59 FORMAT (I4)
      READ (5,58,END=300) ((B(P,G),G=1,4),P=1,PP)
   58 FORMAT (24I3)
      WRITE (6,53) ((B(P,G),G=1,4),P=1,PP)
   53 FORMAT (9(I4,I3,I3,I3))
      A=1
      PPN(A)=PP
  205 CONTINUE
      WRITE (6,25) PPN(A)
   25 FORMAT ('OPPN = ',I4)
  122 CONTINUE
      DO 1 I=1,T
      BOCUL(I)=0
    1 CONTINUE
      I=0
  101 I=I+1
      IF (I .GT. T) GO TO 102
      PA=PPN(A)
      DO 2 P=1,PA
      DO 3 G=1,4
      IF (L(I) .NE. B(P,G)) GO TO 3
      BOCUL(I)=BOCUL(I)+1
    3 CONTINUE
    2 CONTINUE
      GO TO 101
  102 CONTINUE
      I=0
  103 I=I+1
      IF (I .GT. T) GO TO 1301
      IF (BOCUL(I) .LT. U) GO TO 104
```

```
      GO TO 103
  104 LL=U-BOCUL(I)
      E=L(I)
      H=0
      P1=PPN(A)
      DO 4 P=1,P1
      IF (B(P,1) .EQ. E .OR. B(P,2) .EQ. E .OR. B(P,3) .EQ. E
     1 .OR. B(P,4) .EQ. E) GO TO 105
      GO TO 4
  105 H=H+1
      DO 5 G=1,4
      C(A,H,G)=B(P,G)
    5 CONTINUE
    4 CONTINUE
      D=1
      HHN(A,D)=H
      CALL REAR (C,E,HHN,A,D)
      H3=HHN(A,D)
      HP(A,D)=HHN(A,D)+1
      H1=HP(A,D)
      DO 6 H=H1,U
      C(A,H,1)=E
    6 CONTINUE
  119 DO 14 I=1,T
      OCUL(I)=0
   14 CONTINUE
      I=0
  106 I=I+1
      IF (I .GT. T) GO TO 107
      CALL CHECKA (C,L,OCUL,I,HHN,A,D)
      GO TO 106
  107 CONTINUE
      I=0
  190 I=I+1
      IF (I .GT. T) GO TO 120
      IF (OCUL(I) .GE. S) GO TO 190
      KK=S-OCUL(I)
      EA=L(I)
      IQ=I
      HK(A,D)=HHN(A,D)+KK
      HP(A,D)=HHN(A,D)+1
      H1=HP(A,D)
      H2=HK(A,D)
      DO 7 H=H1,H2
      C(A,H,2)=EA
      OCUL(IQ)=OCUL(IQ)+1
    7 CONTINUE
      K=0
      HA=HHN(A,D)
      DO 8 H=1,HA
      K=K+1
      CHEC(A,D,K,1)=C(A,H,3)
      CHEC(A,D,K,2)=C(A,H,4)
    8 CONTINUE
      P=0
  109 P=P+1
```

```
      IF (P .GT. PPN(A)) GO TO 111
      IF (B(P,2) .EQ. EA) GO TO 110
      GO TO 109
  110 K=K+1
      CHEC(A,D,K,1)=B(P,3)
      CHEC(A,D,K,2)=B(P,4)
      GO TO 109
  111 IK(A,D)=K
      Q=0
      H=0
   81 H=H+1
      IF (H .GT. HHN(A,D)) GO TO 154
      IF (EA .EQ. C(A,H,2)) GO TO 151
      IF (EA .EQ. C(A,H,3)) GO TO 152
      IF (EA .EQ. C(A,H,4)) GO TO 153
      GO TO 81
  151 Q=Q+1
      OCH(2*Q-1)=C(A,H,3)
      OCH(2*Q)=C(A,H,4)
      GO TO 81
  152 Q=Q+1
      OCH(2*Q-1)=C(A,H,2)
      OCH(2*Q)=C(A,H,4)
      GO TO 81
  153 Q=Q+1
      OCH(2*Q-1)=C(A,H,2)
      OCH(2*Q)=C(A,H,3)
      GO TO 81
  154 QQ=Q
      I=0
      J=0
  155 I=I+1
      IF (I .GT. T) GO TO 116
      IF (OCUL(I) .GE. S) GO TO 155
      Q=0
  157 Q=Q+1
      IF (Q .GT. QQ) GO TO 156
      IF (L(I) .EQ. OCH(2*Q-1) .OR. L(I) .EQ. OCH(2*Q)) GO TO 155
      GO TO 157
  156 J=J+1
      OL(J)=L(I)
      GO TO 155
  116 JJ(A,D)=J
      JA=JJ(A,D)
      DO 9 J=1,JA
      OLT(A,D,J)=OL(J)
    9 CONTINUE
      CAP(A,D)=1
      JA=JJ(A,D)
      DO 15 F=1,JA,2
      CAP(A,D)=CAP(A,D)*F
   15 CONTINUE
      COUN(A,D)=0
      GO TO 128
  117 COUN(A,D)=COUN(A,D)+1
      IF (COUN(A,D) .EQ. CAP(A,D)) GO TO 140
```

```
  129 CALL COMB (JJ,OLT,CCUN,A,D)
  128 JA=JJ(A,D)
      DO 10 J=1,JA
      TLL(J)=OLT(A,D,J)
   10 CONTINUE
      CALL PERM (JJ,TLL,A,D)
      PASS=0
      CALL CHECKC (JJ,TLL,PASS,CHEC,IK,A,D)
      IF (PASS .EQ. 0) GO TO 118
      IF (COUN(A,D) .LT. CAP(A,D)) GO TO 117
  140 D=D-1
      IF (D .GT. 0) GO TO 117
      A=A-1
      IF (A .GT. 0) GO TO 108
      GO TO 150
  108 D=DB(A)
      GO TO 117
  118 Y=0
      H1=HP(A,D)
      H2=HK(A,D)
      DO 11 H=H1,H2
      Y=Y+1
      C(A,H,3)=TLL(2*Y-1)
      C(A,H,4)=TLL(2*Y)
   11 CONTINUE
      D=D+1
      HHN(A,D)=HK(A,D-1)
      GO TO 119
  120 DB(A)=D-1
      P=PPN(A)
      H=HHN(A,1)
  125 H=H+1
      IF (H .GT. U) GO TO 124
      P=P+1
      DO 13 G=1,4
      B(P,G)=C(A,H,G)
   13 CONTINUE
      GO TO 125
  124 CONTINUE
      ZT=ZTIME
      CALL $TPTM(ZT)
      ZTT=ZTT+ZT
      IF (ZTT .GE. ZZ) GO TO 301
      CALL $TRTM(ZTIME)
      IF (P .GE. V) GO TO 130
      A=A+1
      PPN(A)=P
      GO TO 122
  150 WRITE (6,56)
   56 FORMAT ('0 THAT IS ALL')
      GO TO 204
 1301 A=A-1
  130 NN=NN+1
      WRITE (6,603) NN
  603 FORMAT ('ONN = ',I8)
      DO 17 P=1,V
```

```
      WRITE (6,57) P, (B(P,G), G=1,4)
   17 CONTINUE
   57 FORMAT (' P = ',I4,3X,4(I4))
      WRITE (7,54) ((B(P,G),G=1,4),P=1,V)
   54 FORMAT (24I3)
      D=DB(A)
      GO TO 117
  301 PPN(A+1)=P
      CALL WRITE1
  300 CALL EXIT
      END


      SUBROUTINE REAR (C,E,HHN,A,D)
      IMPLICIT INTEGER (A-Z)
      DIMENSION C(15,35,4),L(16),HHN(15,14)
      K=HHN(A,D)
      DO 1 H=1,K
      DO 2 G=2,4
      IF (E .EQ. C(A,H,G)) GO TO 3
      GO TO 2
    3 GG=G+1
    4 GG=GG-1
      IF (GG .EQ. 1) GO TO 5
      C(A,H,GG)=C(A,H,GG-1)
      GO TO 4
    5 C(A,H,1)=E
    2 CONTINUE
    1 CONTINUE
      RETURN
      END


      SUBROUTINE CHECKA (C,L,OCUL,I,HHN,A,D)
      IMPLICIT INTEGER (A-Z)
      DIMENSION C(15,35,4),L(16),OCUL(16),HHN(15,14)
      K=HHN(A,D)
      DO 1 H=1,K
      DO 2 G=1,4
      IF (L(I) .NE. C(A,H,G)) GO TO 2
      OCUL(I)=OCUL(I)+1
    2 CONTINUE
    1 CONTINUE
      RETURN
      END


      SUBROUTINE COMB (JJ,OLT,KOUN,R,S)
      IMPLICIT INTEGER (A-Z)
      DIMENSION L(14),JJ(15,14),OLT(15,14,14),KOUN(15,14)
      K=JJ(R,S)
      COUN=KOUN(R,S)
      DO 13 I=1,K
      L(I)=OLT(R,S,I)
   13 CONTINUE
      N=K-2
```

```
      STOR=L(N)
      J=K-1
      DO 10 I=N,J
      L(I)=L(I+1)
   10 CONTINUE
      L(K)=STOR
      A=COUN-COUN/3*3
      B=COUN-COUN/15*15
      C=COUN-COUN/105*105
      D=COUN-COUN/945*945
      E=COUN-COUN/10395*10395
      F=COUN-COUN/135135*135135
      IF (F .EQ. 0) GO TO 6
      IF (E .EQ. 0) GO TO 5
      IF (D .EQ. 0) GO TO 4
      IF (C .EQ. 0) GO TO 3
      IF (B .EQ. 0) GO TO 2
      IF (A .EQ. 0) GO TO 1
      GO TO 11
    1 G=2
      GO TO 7
    2 G=4
      GO TO 7
    3 G=6
      GO TO 7
    4 G=8
      GO TO 7
    5 G=10
      GO TO 7
    6 G=12
    7 N=N-2
      STOR=L(N)
      DO 20 I=N,J
      L(I)=L(I+1)
   20 CONTINUE
      L(K)=STOR
      M=K-2-G
      IF (N .EQ. M) GO TO 11
      GO TO 7
   11 DO 12 I=1,K
      OLT(R,S,I)=L(I)
   12 CONTINUE
      RETURN
      END


      SUBROUTINE PERM (JJ,TLL,A,D)
      IMPLICIT INTEGER (A-Z)
      DIMENSION TLL(14),JJ(15,14)
      P=JJ(A,D)/2
      H=0
   15 H=H+1
      IF (H .GT. P) RETURN
      IF (TLL(2*H-1) .LE. TLL(2*H)) GO TO 15
      CPU=TLL(2*H-1)
      TLL(2*H-1)=TLL(2*H)
```

```
      TLL(2*H-1)=TLL(2*H)
      TLL(2*H)=CPU
      GO TO 15
      END


      SUBROUTINE CHECKC (JJ,TLL,PASS,CHEC,IK,A,D)
      IMPLICIT INTEGER (A-Z)
      DIMENSION TLL(14),CHEC(15,14,70,2),JJ(15,14),IK(15,14)
      H=0
      P=JJ(A,D)/2
   11 H=H+1
      IF (H .GT. P) RETURN
      K=0
   13 K=K+1
      IF (K .GT. IK(A,D)) GO TO 11
      IF (TLL(2*H-1) .EQ. CHEC(A,D,K,1) .AND.
     1 TLL(2*H) .EQ. CHEC(A,D,K,2)) GO TO 12
      GO TO 13
   12 PASS=1
      RETURN
      END


      SUBROUTINE READ1
      IMPLICIT INTEGER (A-Z)
      COMMON /B1/ DB,PPN,JJ,OLT,COUN,C,HHN,CHEC,CAP,HP,HK,B,IK,A,NN,U,V
      DIMENSION DB(15),PPN(15),JJ(15,14),OLT(15,14,16),COUN(15,14),
     1          C(15,35,4),HHN(15,14),CHEC(15,14,70,2),CAP(15,14),
     2          HP(15,14),HK(15,14),B(140,4),IK(15,14)
      REWIND 8
      READ (8) NN
      READ (8) A,(DB(AA),AA=1,A)
      A2=A+1
      READ (8) (PPN(AA),AA=1,A2)
      DO 1 AA=1,A
      DF=DB(AA)
      DO 2 DD=1,DF
      READ (8)  JJ(AA,DD),COUN(AA,DD),HHN(AA,DD),CAP(AA,DD),HP(AA,DD),
     1          HK(AA,DD),IK(AA,DD)
      J1=JJ(AA,DD)
      DO 3 J=1,J1
      READ (8)  OLT(AA,DD,J)
    3 CONTINUE
      K1=IK(AA,DD)
      DO 4 K=1,K1
      DO 5 R=1,2
      READ (8)  CHEC(AA,DD,K,R)
    5 CONTINUE
    4 CONTINUE
    2 CONTINUE
      DO 6 H=1,U
      DO 7 G=1,4
      READ (8)  C(AA,H,G)
    7 CONTINUE
    6 CONTINUE
```

```
  1 CONTINUE
    P1=PPN(A+1)
    DO 8 P=1,P1
    DO 9 G=1,4
    READ (8)  B(P,G)
  9 CONTINUE
 50 FORMAT (' P = ',I4,3X,4(I4))
    WRITE (6,50) P,(B(P,G),G=1,4)
  8 CONTINUE
    WRITE (6,51)
 51 FORMAT ('0 **** NEW PRG STAR HERE *****')
    RETURN
    END


    SUBROUTINE WRITE1
    IMPLICIT INTEGER (A-Z)
    COMMON /B1/ DB,PPN,JJ,OLT,COUN,C,HHN,CHEC,CAP,HP,HK,B,IK,A,NN,U,V
    DIMENSION DB(15),PPN(15),JJ(15,14),OLT(15,14,16),COUN(15,14),
   1          C(15,35,4),HHN(15,14),CHEC(15,14,70,2),CAP(15,14),
   2          HP(15,14),HK(15,14),B(140,4),IK(15,14)
    WRITE (6,55)
 55 FORMAT (' TIMER INTERRUPT ROUTINE')
    WRITE (6,201) A,DB(A)
201 FORMAT ('0A = ',I4,5X,'D = ',I4)
    REWIND 8
    WRITE (8) NN
    WRITE (8) A,(DB(AA),AA=1,A)
    A2=A+1
    WRITE (8) (PPN(AA),AA=1,A2)
    DO 1 AA=1,A
    DF=DB(AA)
    DO 2 DD=1,DF
    WRITE (8) JJ(AA,DD),COUN(AA,DD),HHN(AA,DD),CAP(AA,DD),HP(AA,DD),
   1          HK(AA,DD),IK(AA,DD)
    J1=JJ(AA,DD)
    DO 3 J=1,J1
    WRITE (8) OLT(AA,DD,J)
  3 CONTINUE
    K1=IK(AA,DD)
    DO 4 K=1,K1
    DO 5 R=1,2
    WRITE (8) CHEC(AA,DD,K,R)
  5 CONTINUE
  4 CONTINUE
  2 CONTINUE
    DO 6 H=1,U
    DO 7 G=1,4
    WRITE (8) C(AA,H,G)
  7 CONTINUE
  6 CONTINUE
  1 CONTINUE
    P1=PPN(A+1)
    DO 8 P=1,P1
    DO 9 G=1,4
    WRITE (8) B(P,G)
```

```
 9 CONTINUE
   WRITE (6,50) P,(B(P,G),G=1,4)
50 FORMAT (' P = ',I4,3X,4(I4))
 8 CONTINUE
   END FILE 8
   RETURN
   END
```

APPENDIX 2

The List of Quadruple Systems

A1,···, A26 & B1, B2

THE SYSTEM ***** A 1 *****

TRIPLE SYSTEM														
ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	A	A	A	A	A	A	A	A	A	A	A	A

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	8
P =	28	2	3	9	12
P =	29	2	3	7	13
P =	30	2	3	10	11
P =	31	2	3	6	14
P =	32	2	4	5	10
P =	33	2	4	7	11
P =	34	2	4	8	14
P =	35	2	4	6	9
P =	36	2	4	12	13
P =	37	2	5	7	9
P =	38	2	5	11	13
P =	39	2	5	12	14
P =	40	2	6	7	12
P =	41	2	6	10	13
P =	42	2	6	8	11
P =	43	2	7	10	14
P =	44	2	8	9	13
P =	45	2	8	10	12
P =	46	2	9	11	14
P =	47	3	4	5	14
P =	48	3	4	6	10
P =	49	3	4	11	13
P =	50	3	4	7	12
P =	51	3	4	8	9
P =	52	3	5	6	11
P =	53	3	5	10	12
P =	54	3	5	9	13
P =	55	3	6	7	9
P =	56	3	6	12	13
P =	57	3	7	8	10
P =	58	3	7	11	14
P =	59	3	8	11	12
P =	60	3	8	13	14
P =	61	3	9	10	14
P =	62	4	5	6	12
P =	63	4	5	7	13
P =	64	4	5	8	11
P =	65	4	6	7	8
P =	66	4	6	11	14
P =	67	4	7	9	10
P =	68	4	8	10	13
P =	69	4	9	11	12
P =	70	4	9	13	14
P =	71	4	10	12	14
P =	72	5	6	7	10
P =	73	5	6	8	13
P =	74	5	6	9	14
P =	75	5	7	8	14
P =	76	5	7	11	12
P =	77	5	8	9	12
P =	78	5	9	10	11
P =	79	5	10	13	14
P =	80	6	7	13	14
P =	81	6	8	9	10
P =	82	6	8	12	14
P =	83	6	9	11	13
P =	84	6	10	11	12
P =	85	7	8	9	11
P =	86	7	8	12	13
P =	87	7	9	12	14
P =	88	7	10	11	13
P =	89	8	10	11	14
P =	90	9	10	12	13
P =	91	11	12	13	14

THE SYSTEM ***** A 2 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	A	A	A	A	A	A	A	A	A	A	A	A

P = 1	1	2	3	4
P = 2	1	2	5	6
P = 3	1	2	7	8
P = 4	1	2	9	10
P = 5	1	2	11	12
P = 6	1	2	13	14
P = 7	1	3	5	7
P = 8	1	3	6	8
P = 9	1	3	9	11
P = 10	1	3	10	13
P = 11	1	3	12	14
P = 12	1	4	5	9
P = 13	1	4	6	13
P = 14	1	4	7	14
P = 15	1	4	8	12
P = 16	1	4	10	11
P = 17	1	5	8	10
P = 18	1	5	11	14
P = 19	1	5	12	13
P = 20	1	6	7	11
P = 21	1	6	9	12
P = 22	1	6	10	14
P = 23	1	7	9	13
P = 24	1	7	10	12
P = 25	1	8	9	14
P = 26	1	8	11	13
P = 27	2	3	5	8
P = 28	2	3	9	13
P = 29	2	3	6	14
P = 30	2	3	7	12
P = 31	2	3	10	11
P = 32	2	4	5	10
P = 33	2	4	8	11
P = 34	2	4	9	14
P = 35	2	4	6	7
P = 36	2	4	12	13
P = 37	2	5	7	9
P = 38	2	5	11	13
P = 39	2	5	12	14
P = 40	2	6	8	13
P = 41	2	6	9	11
P = 42	2	6	10	12
P = 43	2	7	10	13
P = 44	2	7	11	14
P = 45	2	8	9	12
P = 46	2	8	10	14
P = 47	3	4	5	11
P = 48	3	4	9	12
P = 49	3	4	6	10
P = 50	3	4	7	8
P = 51	3	4	13	14
P = 52	3	5	6	13
P = 53	3	5	9	14
P = 54	3	5	10	12
P = 55	3	6	7	9
P = 56	3	6	11	12
P = 57	3	7	10	14
P = 58	3	7	11	13
P = 59	3	8	9	10
P = 60	3	8	11	14
P = 61	3	8	12	13
P = 62	4	5	6	12
P = 63	4	5	7	13
P = 64	4	5	8	14
P = 65	4	6	8	9
P = 66	4	6	11	14
P = 67	4	7	9	10
P = 68	4	7	11	12
P = 69	4	8	10	13
P = 70	4	9	11	13
P = 71	4	10	12	14
P = 72	5	6	7	14
P = 73	5	6	8	11
P = 74	5	6	9	10
P = 75	5	7	8	12
P = 76	5	7	10	11
P = 77	5	8	9	13
P = 78	5	9	11	12
P = 79	5	10	13	14
P = 80	6	7	8	10
P = 81	6	7	12	13
P = 82	6	8	12	14
P = 83	6	9	13	14
P = 84	6	10	11	13
P = 85	7	8	9	11
P = 86	7	8	13	14
P = 87	7	9	12	14
P = 88	8	10	11	12
P = 89	9	10	11	14
P = 90	9	10	12	13
P = 91	11	12	13	14

THE SYSTEM ***** A 3 *****

TRIPLE SYSTEM														
ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	B	B	A	A	A	A	A	A	A	A	A	A

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	8
P =	28	2	3	9	13
P =	29	2	3	7	14
P =	30	2	3	10	11
P =	31	2	3	6	12
P =	32	2	4	5	13
P =	33	2	4	6	14
P =	34	2	4	7	9
P =	35	2	4	10	12
P =	36	2	4	8	11
P =	37	2	5	7	11
P =	38	2	5	9	12
P =	39	2	5	10	14
P =	40	2	6	7	10
P =	41	2	6	11	13
P =	42	2	6	8	9
P =	43	2	7	12	13
P =	44	2	8	10	13
P =	45	2	8	12	14
P =	46	2	9	11	14
P =	47	3	4	5	6
P =	48	3	4	7	11
P =	49	3	4	12	13
P =	50	3	4	10	14
P =	51	3	4	8	9
P =	52	3	5	9	14
P =	53	3	5	10	12
P =	54	3	5	11	13
P =	55	3	6	7	13
P =	56	3	6	11	14
P =	57	3	6	9	10
P =	58	3	7	8	10
P =	59	3	7	9	12
P =	60	3	8	11	12
P =	61	3	8	13	14
P =	62	4	5	7	10
P =	63	4	5	11	12
P =	64	4	5	8	14
P =	65	4	6	7	12
P =	66	4	6	8	10
P =	67	4	6	9	11
P =	68	4	7	8	13
P =	69	4	9	10	13
P =	70	4	9	12	14
P =	71	4	11	13	14
P =	72	5	6	7	9
P =	73	5	6	10	13
P =	74	5	6	12	14
P =	75	5	6	8	11
P =	76	5	7	8	12
P =	77	5	7	13	14
P =	78	5	8	9	13
P =	79	5	9	10	11
P =	80	6	7	8	14
P =	81	6	8	12	13
P =	82	6	9	13	14
P =	83	6	10	11	12
P =	84	7	8	9	11
P =	85	7	9	10	14
P =	86	7	10	11	13
P =	87	7	11	12	14
P =	88	8	9	10	12
P =	89	8	10	11	14
P =	90	9	11	12	13
P =	91	10	12	13	14

THE SYSTEM ***** A 4 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	A	A	A	A	A	A	A	A	A	A	A	A

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	8
P =	28	2	3	9	13
P =	29	2	3	11	14
P =	30	2	3	6	12
P =	31	2	3	7	10
P =	32	2	4	5	13
P =	33	2	4	8	14
P =	34	2	4	7	9
P =	35	2	4	10	12
P =	36	2	4	6	11
P =	37	2	5	7	11
P =	38	2	5	9	12
P =	39	2	5	10	14
P =	40	2	6	7	13
P =	41	2	6	9	14
P =	42	2	6	8	10
P =	43	2	7	12	14
P =	44	2	8	9	11
P =	45	2	8	12	13
P =	46	2	10	11	13
P =	47	3	4	5	14
P =	48	3	4	6	7
P =	49	3	4	8	13
P =	50	3	4	9	10
P =	51	3	4	11	12
P =	52	3	5	6	9
P =	53	3	5	10	12
P =	54	3	5	11	13
P =	55	3	6	10	11
P =	56	3	6	13	14
P =	57	3	7	8	11
P =	58	3	7	12	13
P =	59	3	7	9	14
P =	60	3	8	9	12
P =	61	3	8	10	14
P =	62	4	5	6	10
P =	63	4	5	8	11
P =	64	4	5	7	12
P =	65	4	6	8	9
P =	66	4	6	12	14
P =	67	4	7	8	10
P =	68	4	7	11	13
P =	69	4	9	11	14
P =	70	4	9	12	13
P =	71	4	10	13	14
P =	72	5	6	7	14
P =	73	5	6	8	13
P =	74	5	6	11	12
P =	75	5	7	8	9
P =	76	5	7	10	13
P =	77	5	8	12	14
P =	78	5	9	10	11
P =	79	5	9	13	14
P =	80	6	7	8	12
P =	81	6	7	9	10
P =	82	6	8	11	14
P =	83	6	9	11	13
P =	84	6	10	12	13
P =	85	7	8	13	14
P =	86	7	9	11	12
P =	87	7	10	11	14
P =	88	8	9	10	13
P =	89	8	10	11	12
P =	90	9	10	12	14
P =	91	11	12	13	14

THE SYSTEM ***** A 5 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	B	A	B	B	B	A	B	B	A	B	A	B	A

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	8
P =	28	2	3	9	14
P =	29	2	3	6	12
P =	30	2	3	7	13
P =	31	2	3	10	11
P =	32	2	4	5	12
P =	33	2	4	8	13
P =	34	2	4	9	11
P =	35	2	4	6	14
P =	36	2	4	7	10
P =	37	2	5	7	11
P =	38	2	5	9	13
P =	39	2	5	10	14
P =	40	2	6	7	9
P =	41	2	6	8	10
P =	42	2	6	11	13
P =	43	2	7	12	14
P =	44	2	8	9	12
P =	45	2	8	11	14
P =	46	2	10	12	13
P =	47	3	4	5	14
P =	48	3	4	6	7
P =	49	3	4	8	11
P =	50	3	4	12	13
P =	51	3	4	9	10
P =	52	3	5	6	10
P =	53	3	5	11	13
P =	54	3	5	9	12
P =	55	3	6	9	13
P =	56	3	6	11	14
P =	57	3	7	8	9
P =	58	3	7	10	14
P =	59	3	7	11	12
P =	60	3	8	10	12
P =	61	3	8	13	14
P =	62	4	5	6	11
P =	63	4	5	10	13
P =	64	4	5	7	8
P =	65	4	6	8	9
P =	66	4	6	10	12
P =	67	4	7	9	12
P =	68	4	7	11	13
P =	69	4	8	10	14
P =	70	4	9	13	14
P =	71	4	11	12	14
P =	72	5	6	7	12
P =	73	5	6	8	13
P =	74	5	6	9	14
P =	75	5	7	9	10
P =	76	5	7	13	14
P =	77	5	8	9	11
P =	78	5	8	12	14
P =	79	5	10	11	12
P =	80	6	7	8	14
P =	81	6	7	10	13
P =	82	6	8	11	12
P =	83	6	9	10	11
P =	84	6	12	13	14
P =	85	7	8	10	11
P =	86	7	8	12	13
P =	87	7	9	11	14
P =	88	8	9	10	13
P =	89	9	10	12	14
P =	90	9	11	12	13
P =	91	10	11	13	14

THE SYSTEM ***** A 6 *****

TRIPLE SYSTEM														
ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	A	A	A	A	A	A	A	A	A	A	A	A

P	= 1	1	2	3	4
P	= 2	1	2	5	6
P	= 3	1	2	7	8
P	= 4	1	2	9	10
P	= 5	1	2	11	12
P	= 6	1	2	13	14
P	= 7	1	3	5	7
P	= 8	1	3	6	8
P	= 9	1	3	9	11
P	= 10	1	3	10	13
P	= 11	1	3	12	14
P	= 12	1	4	5	9
P	= 13	1	4	6	13
P	= 14	1	4	7	14
P	= 15	1	4	8	12
P	= 16	1	4	10	11
P	= 17	1	5	8	10
P	= 18	1	5	11	14
P	= 19	1	5	12	13
P	= 20	1	6	7	11
P	= 21	1	6	9	12
P	= 22	1	6	10	14
P	= 23	1	7	9	13
P	= 24	1	7	10	12
P	= 25	1	8	9	14
P	= 26	1	8	11	13
P	= 27	2	3	5	9
P	= 28	2	3	10	14
P	= 29	2	3	6	12
P	= 30	2	3	8	13
P	= 31	2	3	7	11
P	= 32	2	4	5	13
P	= 33	2	4	12	14
P	= 34	2	4	6	11
P	= 35	2	4	7	10
P	= 36	2	4	8	9
P	= 37	2	5	7	14
P	= 38	2	5	8	12
P	= 39	2	5	10	11
P	= 40	2	6	7	13
P	= 41	2	6	9	14
P	= 42	2	6	8	10
P	= 43	2	7	9	12
P	= 44	2	8	11	14
P	= 45	2	9	11	13
P	= 46	2	10	12	13
P	= 47	3	4	5	6
P	= 48	3	4	7	8
P	= 49	3	4	9	10
P	= 50	3	4	11	12
P	= 51	3	4	13	14
P	= 52	3	5	8	14
P	= 53	3	5	10	12
P	= 54	3	5	11	13
P	= 55	3	6	7	10
P	= 56	3	6	11	14
P	= 57	3	6	9	13
P	= 58	3	7	9	14
P	= 59	3	7	12	13
P	= 60	3	8	9	12
P	= 61	3	8	10	11
P	= 62	4	5	7	12
P	= 63	4	5	10	14
P	= 64	4	5	8	11
P	= 65	4	6	7	9
P	= 66	4	6	10	12
P	= 67	4	6	8	14
P	= 68	4	7	11	13
P	= 69	4	8	10	13
P	= 70	4	9	11	14
P	= 71	4	9	12	13
P	= 72	5	6	7	8
P	= 73	5	6	9	10
P	= 74	5	6	11	12
P	= 75	5	6	13	14
P	= 76	5	7	9	11
P	= 77	5	7	10	13
P	= 78	5	8	9	13
P	= 79	5	9	12	14
P	= 80	6	7	12	14
P	= 81	6	8	9	11
P	= 82	6	8	12	13
P	= 83	6	10	11	13
P	= 84	7	8	9	10
P	= 85	7	8	11	12
P	= 86	7	8	13	14
P	= 87	7	10	11	14
P	= 88	8	10	12	14
P	= 89	9	10	11	12
P	= 90	9	10	13	14
P	= 91	11	12	13	14

THE SYSTEM ***** A 7 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	A	A	A	A	A	A	B	A	B	A	A	A

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	9
P =	28	2	3	7	10
P =	29	2	3	8	14
P =	30	2	3	6	11
P =	31	2	3	12	13
P =	32	2	4	5	11
P =	33	2	4	10	12
P =	34	2	4	8	13
P =	35	2	4	9	14
P =	36	2	4	6	7
P =	37	2	5	7	14
P =	38	2	5	8	12
P =	39	2	5	10	13
P =	40	2	6	8	10
P =	41	2	6	12	14
P =	42	2	6	9	13
P =	43	2	7	9	12
P =	44	2	7	11	13
P =	45	2	8	9	11
P =	46	2	10	11	14
P =	47	3	4	5	8
P =	48	3	4	9	10
P =	49	3	4	11	13
P =	50	3	4	6	14
P =	51	3	4	7	12
P =	52	3	5	6	13
P =	53	3	5	10	14
P =	54	3	5	11	12
P =	55	3	6	7	9
P =	56	3	6	10	12
P =	57	3	7	8	13
P =	58	3	7	11	14
P =	59	3	8	9	12
P =	60	3	8	10	11
P =	61	3	9	13	14
P =	62	4	5	6	10
P =	63	4	5	12	14
P =	64	4	5	7	13
P =	65	4	6	8	9
P =	66	4	6	11	12
P =	67	4	7	8	10
P =	68	4	7	9	11
P =	69	4	8	11	14
P =	70	4	9	12	13
P =	71	4	10	13	14
P =	72	5	6	7	12
P =	73	5	6	9	14
P =	74	5	6	8	11
P =	75	5	7	8	9
P =	76	5	7	10	11
P =	77	5	8	13	14
P =	78	5	9	10	12
P =	79	5	9	11	13
P =	80	6	7	8	14
P =	81	6	7	10	13
P =	82	6	8	12	13
P =	83	6	9	10	11
P =	84	6	11	13	14
P =	85	7	8	11	12
P =	86	7	9	10	14
P =	87	7	12	13	14
P =	88	8	9	10	13
P =	89	8	10	12	14
P =	90	9	11	12	14
P =	91	10	11	12	13

THE SYSTEM ***** A 8 *****

TRIPLE SYSTEM														
ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	A	A	A	A	A	A	A	A	A	A	A	A

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	9
P =	28	2	3	8	10
P =	29	2	3	6	11
P =	30	2	3	7	14
P =	31	2	3	12	13
P =	32	2	4	5	8
P =	33	2	4	9	11
P =	34	2	4	10	12
P =	35	2	4	7	13
P =	36	2	4	6	14
P =	37	2	5	7	12
P =	38	2	5	11	13
P =	39	2	5	10	14
P =	40	2	6	7	9
P =	41	2	6	10	13
P =	42	2	6	8	12
P =	43	2	7	10	11
P =	44	2	8	9	13
P =	45	2	8	11	14
P =	46	2	9	12	14
P =	47	3	4	5	10
P =	48	3	4	11	13
P =	49	3	4	8	14
P =	50	3	4	6	9
P =	51	3	4	7	12
P =	52	3	5	6	14
P =	53	3	5	8	13
P =	54	3	5	11	12
P =	55	3	6	7	13
P =	56	3	6	10	12
P =	57	3	7	8	11
P =	58	3	7	9	10
P =	59	3	8	9	12
P =	60	3	9	13	14
P =	61	3	10	11	14
P =	62	4	5	6	12
P =	63	4	5	13	14
P =	64	4	5	7	11
P =	65	4	6	7	10
P =	66	4	6	8	11
P =	67	4	7	8	9
P =	68	4	8	10	13
P =	69	4	9	10	14
P =	70	4	9	12	13
P =	71	4	11	12	14
P =	72	5	6	7	8
P =	73	5	6	9	13
P =	74	5	6	10	11
P =	75	5	7	9	14
P =	76	5	7	10	13
P =	77	5	8	9	11
P =	78	5	8	12	14
P =	79	5	9	10	12
P =	80	6	7	12	14
P =	81	6	8	9	10
P =	82	6	8	13	14
P =	83	6	9	11	14
P =	84	6	11	12	13
P =	85	7	8	10	14
P =	86	7	8	12	13
P =	87	7	9	11	12
P =	88	7	11	13	14
P =	89	8	10	11	12
P =	90	9	10	11	13
P =	91	10	12	13	14

THE SYSTEM ***** A 9 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	A	A	A	A	A	A	A	A	A	A	A	A

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	10
P =	28	2	3	8	11
P =	29	2	3	9	12
P =	30	2	3	6	13
P =	31	2	3	7	14
P =	32	2	4	5	7
P =	33	2	4	6	8
P =	34	2	4	9	11
P =	35	2	4	12	14
P =	36	2	4	10	13
P =	37	2	5	8	12
P =	38	2	5	11	13
P =	39	2	5	9	14
P =	40	2	6	7	9
P =	41	2	6	10	12
P =	42	2	6	11	14
P =	43	2	7	10	11
P =	44	2	7	12	13
P =	45	2	8	9	13
P =	46	2	8	10	14
P =	47	3	4	5	13
P =	48	3	4	8	14
P =	49	3	4	7	9
P =	50	3	4	10	12
P =	51	3	4	6	11
P =	52	3	5	6	14
P =	53	3	5	8	9
P =	54	3	5	11	12
P =	55	3	6	7	12
P =	56	3	6	9	10
P =	57	3	7	8	10
P =	58	3	7	11	13
P =	59	3	8	12	13
P =	60	3	9	13	14
P =	61	3	10	11	14
P =	62	4	5	6	12
P =	63	4	5	10	14
P =	64	4	5	8	11
P =	65	4	6	7	10
P =	66	4	6	9	14
P =	67	4	7	8	13
P =	68	4	7	11	12
P =	69	4	8	9	10
P =	70	4	9	12	13
P =	71	4	11	13	14
P =	72	5	6	7	8
P =	73	5	6	9	13
P =	74	5	6	10	11
P =	75	5	7	9	11
P =	76	5	7	12	14
P =	77	5	7	10	13
P =	78	5	8	13	14
P =	79	5	9	10	12
P =	80	6	7	13	14
P =	81	6	8	9	11
P =	82	6	8	12	14
P =	83	6	8	10	13
P =	84	6	11	12	13
P =	85	7	8	9	12
P =	86	7	8	11	14
P =	87	7	9	10	14
P =	88	8	10	11	12
P =	89	9	10	11	13
P =	90	9	11	12	14
P =	91	10	12	13	14

THE SYSTEM ***** A10 *****

TRIPLE SYSTEM														
ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	A	A	A	B	A	A	A	A	A	A	B	A

P	=	1	1	2	3	4
P	=	2	1	2	5	6
P	=	3	1	2	7	8
P	=	4	1	2	9	10
P	=	5	1	2	11	12
P	=	6	1	2	13	14
P	=	7	1	3	5	7
P	=	8	1	3	6	8
P	=	9	1	3	9	11
P	=	10	1	3	10	13
P	=	11	1	3	12	14
P	=	12	1	4	5	9
P	=	13	1	4	6	13
P	=	14	1	4	7	14
P	=	15	1	4	8	12
P	=	16	1	4	10	11
P	=	17	1	5	8	10
P	=	18	1	5	11	14
P	=	19	1	5	12	13
P	=	20	1	6	7	11
P	=	21	1	6	9	12
P	=	22	1	6	10	14
P	=	23	1	7	9	13
P	=	24	1	7	10	12
P	=	25	1	8	9	14
P	=	26	1	8	11	13
P	=	27	2	3	5	10
P	=	28	2	3	8	11
P	=	29	2	3	6	9
P	=	30	2	3	7	14
P	=	31	2	3	12	13
P	=	32	2	4	5	8
P	=	33	2	4	9	13
P	=	34	2	4	10	14
P	=	35	2	4	6	11
P	=	36	2	4	7	12
P	=	37	2	5	7	9
P	=	38	2	5	11	13
P	=	39	2	5	12	14
P	=	40	2	6	7	13
P	=	41	2	6	8	14
P	=	42	2	6	10	12
P	=	43	2	7	10	11
P	=	44	2	8	9	12
P	=	45	2	8	10	13
P	=	46	2	9	11	14
P	=	47	3	4	5	13
P	=	48	3	4	11	14
P	=	49	3	4	6	12
P	=	50	3	4	7	9
P	=	51	3	4	8	10
P	=	52	3	5	6	11
P	=	53	3	5	9	12
P	=	54	3	5	8	14
P	=	55	3	6	7	10
P	=	56	3	6	13	14
P	=	57	3	7	8	12
P	=	58	3	7	11	13
P	=	59	3	8	9	13
P	=	60	3	9	10	14
P	=	61	3	10	11	12
P	=	62	4	5	6	14
P	=	63	4	5	7	11
P	=	64	4	5	10	12
P	=	65	4	6	7	8
P	=	66	4	6	9	10
P	=	67	4	7	10	13
P	=	68	4	8	9	11
P	=	69	4	8	13	14
P	=	70	4	9	12	14
P	=	71	4	11	12	13
P	=	72	5	6	7	12
P	=	73	5	6	10	13
P	=	74	5	6	8	9
P	=	75	5	7	8	13
P	=	76	5	7	10	14
P	=	77	5	8	11	12
P	=	78	5	9	10	11
P	=	79	5	9	13	14
P	=	80	6	7	9	14
P	=	81	6	8	10	11
P	=	82	6	8	12	13
P	=	83	6	9	11	13
P	=	84	6	11	12	14
P	=	85	7	8	9	10
P	=	86	7	8	11	14
P	=	87	7	9	11	12
P	=	88	7	12	13	14
P	=	89	8	10	12	14
P	=	90	9	10	12	13
P	=	91	10	11	13	14

THE SYSTEM ***** A11 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	A	A	B	A	A	A	A	A	A	A	A	B

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	11
P =	28	2	3	6	12
P =	29	2	3	7	10
P =	30	2	3	8	13
P =	31	2	3	9	14
P =	32	2	4	5	13
P =	33	2	4	11	14
P =	34	2	4	10	12
P =	35	2	4	6	8
P =	36	2	4	7	9
P =	37	2	5	7	12
P =	38	2	5	10	14
P =	39	2	5	8	9
P =	40	2	6	7	14
P =	41	2	6	9	11
P =	42	2	6	10	13
P =	43	2	7	11	13
P =	44	2	8	10	11
P =	45	2	8	12	14
P =	46	2	9	12	13
P =	47	3	4	5	8
P =	48	3	4	9	12
P =	49	3	4	13	14
P =	50	3	4	6	10
P =	51	3	4	7	11
P =	52	3	5	6	14
P =	53	3	5	9	13
P =	54	3	5	10	12
P =	55	3	6	7	9
P =	56	3	6	11	13
P =	57	3	7	8	14
P =	58	3	7	12	13
P =	59	3	8	9	10
P =	60	3	8	11	12
P =	61	3	10	11	14
P =	62	4	5	6	11
P =	63	4	5	12	14
P =	64	4	5	7	10
P =	65	4	6	7	12
P =	66	4	6	9	14
P =	67	4	7	8	13
P =	68	4	8	9	11
P =	69	4	8	10	14
P =	70	4	9	10	13
P =	71	4	11	12	13
P =	72	5	6	7	13
P =	73	5	6	8	12
P =	74	5	6	9	10
P =	75	5	7	8	11
P =	76	5	7	9	14
P =	77	5	8	13	14
P =	78	5	9	11	12
P =	79	5	10	11	13
P =	80	6	7	8	10
P =	81	6	8	9	13
P =	82	6	8	11	14
P =	83	6	10	11	12
P =	84	6	12	13	14
P =	85	7	8	9	12
P =	86	7	9	10	11
P =	87	7	10	13	14
P =	88	7	11	12	14
P =	89	8	10	12	13
P =	90	9	10	12	14
P =	91	9	11	13	14

THE SYSTEM ***** A12 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	B	A	A	A	A	A	A	A	A	A	B	A

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	11
P =	28	2	3	10	12
P =	29	2	3	6	13
P =	30	2	3	7	9
P =	31	2	3	8	14
P =	32	2	4	5	8
P =	33	2	4	9	12
P =	34	2	4	11	13
P =	35	2	4	10	14
P =	36	2	4	6	7
P =	37	2	5	7	10
P =	38	2	5	12	14
P =	39	2	5	9	13
P =	40	2	6	8	12
P =	41	2	6	9	14
P =	42	2	6	10	11
P =	43	2	7	11	14
P =	44	2	7	12	13
P =	45	2	8	9	11
P =	46	2	8	10	13
P =	47	3	4	5	14
P =	48	3	4	6	9
P =	49	3	4	7	10
P =	50	3	4	8	11
P =	51	3	4	12	13
P =	52	3	5	6	10
P =	53	3	5	9	12
P =	54	3	5	8	13
P =	55	3	6	7	14
P =	56	3	6	11	12
P =	57	3	7	8	12
P =	58	3	7	11	13
P =	59	3	8	9	10
P =	60	3	9	13	14
P =	61	3	10	11	14
P =	62	4	5	6	11
P =	63	4	5	10	12
P =	64	4	5	7	13
P =	65	4	6	8	10
P =	66	4	6	12	14
P =	67	4	7	8	9
P =	68	4	7	11	12
P =	69	4	8	13	14
P =	70	4	9	10	13
P =	71	4	9	11	14
P =	72	5	6	7	12
P =	73	5	6	13	14
P =	74	5	6	8	9
P =	75	5	7	8	14
P =	76	5	7	9	11
P =	77	5	8	11	12
P =	78	5	9	10	14
P =	79	5	10	11	13
P =	80	6	7	8	13
P =	81	6	7	9	10
P =	82	6	8	11	14
P =	83	6	9	11	13
P =	84	6	10	12	13
P =	85	7	8	10	11
P =	86	7	9	12	14
P =	87	7	10	13	14
P =	88	8	9	12	13
P =	89	8	10	12	14
P =	90	9	10	11	12
P =	91	11	12	13	14

THE SYSTEM ***** A13 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	A	A	A	B	A	A	A	B	A	A	A	A

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	12
P =	28	2	3	7	13
P =	29	2	3	8	9
P =	30	2	3	10	14
P =	31	2	3	6	11
P =	32	2	4	5	13
P =	33	2	4	12	14
P =	34	2	4	6	10
P =	35	2	4	8	11
P =	36	2	4	7	9
P =	37	2	5	7	10
P =	38	2	5	9	11
P =	39	2	5	8	14
P =	40	2	6	7	12
P =	41	2	6	8	13
P =	42	2	6	9	14
P =	43	2	7	11	14
P =	44	2	8	10	12
P =	45	2	9	12	13
P =	46	2	10	11	13
P =	47	3	4	5	11
P =	48	3	4	12	13
P =	49	3	4	9	14
P =	50	3	4	8	10
P =	51	3	4	6	7
P =	52	3	5	6	14
P =	53	3	5	8	13
P =	54	3	5	9	10
P =	55	3	6	9	13
P =	56	3	6	10	12
P =	57	3	7	8	14
P =	58	3	7	9	12
P =	59	3	7	10	11
P =	60	3	8	11	12
P =	61	3	11	13	14
P =	62	4	5	6	12
P =	63	4	5	10	14
P =	64	4	5	7	8
P =	65	4	6	8	9
P =	66	4	6	11	14
P =	67	4	7	10	13
P =	68	4	7	11	12
P =	69	4	8	13	14
P =	70	4	9	10	12
P =	71	4	9	11	13
P =	72	5	6	7	9
P =	73	5	6	10	13
P =	74	5	6	8	11
P =	75	5	7	11	13
P =	76	5	7	12	14
P =	77	5	8	9	12
P =	78	5	9	13	14
P =	79	5	10	11	12
P =	80	6	7	8	10
P =	81	6	7	13	14
P =	82	6	8	12	14
P =	83	6	9	10	11
P =	84	6	11	12	13
P =	85	7	8	9	11
P =	86	7	8	12	13
P =	87	7	9	10	14
P =	88	8	9	10	13
P =	89	8	10	11	14
P =	90	9	11	12	14
P =	91	10	12	13	14

THE SYSTEM ***** A14 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	B	B	A	B	A	A	B	A	B	B	B	A	B

P = 1	1	2	3	4
P = 2	1	2	5	6
P = 3	1	2	7	8
P = 4	1	2	9	10
P = 5	1	2	11	12
P = 6	1	2	13	14
P = 7	1	3	5	7
P = 8	1	3	6	8
P = 9	1	3	9	11
P = 10	1	3	10	13
P = 11	1	3	12	14
P = 12	1	4	5	9
P = 13	1	4	6	13
P = 14	1	4	7	14
P = 15	1	4	8	12
P = 16	1	4	10	11
P = 17	1	5	8	10
P = 18	1	5	11	14
P = 19	1	5	12	13
P = 20	1	6	7	11
P = 21	1	6	9	12
P = 22	1	6	10	14
P = 23	1	7	9	13
P = 24	1	7	10	12
P = 25	1	8	9	14
P = 26	1	8	11	13
P = 27	2	3	5	12
P = 28	2	3	8	13
P = 29	2	3	7	9
P = 30	2	3	10	11
P = 31	2	3	6	14
P = 32	2	4	5	8
P = 33	2	4	9	11
P = 34	2	4	6	12
P = 35	2	4	7	13
P = 36	2	4	10	14
P = 37	2	5	7	11
P = 38	2	5	10	13
P = 39	2	5	9	14
P = 40	2	6	7	10
P = 41	2	6	11	13
P = 42	2	6	8	9
P = 43	2	7	12	14
P = 44	2	8	10	12
P = 45	2	8	11	14
P = 46	2	9	12	13
P = 47	3	4	5	13
P = 48	3	4	8	14
P = 49	3	4	9	10
P = 50	3	4	6	11
P = 51	3	4	7	12
P = 52	3	5	6	9
P = 53	3	5	10	14
P = 54	3	5	8	11
P = 55	3	6	7	13
P = 56	3	6	10	12
P = 57	3	7	8	10
P = 58	3	7	11	14
P = 59	3	8	9	12
P = 60	3	9	13	14
P = 61	3	11	12	13
P = 62	4	5	6	14
P = 63	4	5	7	10
P = 64	4	5	11	12
P = 65	4	6	7	9
P = 66	4	6	8	10
P = 67	4	7	8	11
P = 68	4	8	9	13
P = 69	4	9	12	14
P = 70	4	10	12	13
P = 71	4	11	13	14
P = 72	5	6	7	12
P = 73	5	6	8	13
P = 74	5	6	10	11
P = 75	5	7	8	9
P = 76	5	7	13	14
P = 77	5	8	12	14
P = 78	5	9	10	12
P = 79	5	9	11	13
P = 80	6	7	8	14
P = 81	6	8	11	12
P = 82	6	9	10	13
P = 83	6	9	11	14
P = 84	6	12	13	14
P = 85	7	8	12	13
P = 86	7	9	10	14
P = 87	7	9	11	12
P = 88	7	10	11	13
P = 89	8	9	10	11
P = 90	8	10	13	14
P = 91	10	11	12	14

THE SYSTEM ***** A15 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	A	A	B	A	A	A	B	A	A	A	A	A

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	12
P =	28	2	3	11	13
P =	29	2	3	7	14
P =	30	2	3	8	10
P =	31	2	3	6	9
P =	32	2	4	5	11
P =	33	2	4	7	12
P =	34	2	4	8	9
P =	35	2	4	10	13
P =	36	2	4	6	14
P =	37	2	5	7	9
P =	38	2	5	10	14
P =	39	2	5	8	13
P =	40	2	6	7	13
P =	41	2	6	8	11
P =	42	2	6	10	12
P =	43	2	7	10	11
P =	44	2	8	12	14
P =	45	2	9	11	14
P =	46	2	9	12	13
P =	47	3	4	5	8
P =	48	3	4	9	13
P =	49	3	4	10	14
P =	50	3	4	7	11
P =	51	3	4	6	12
P =	52	3	5	6	13
P =	53	3	5	9	14
P =	54	3	5	10	11
P =	55	3	6	7	10
P =	56	3	6	11	14
P =	57	3	7	8	9
P =	58	3	7	12	13
P =	59	3	8	11	12
P =	60	3	8	13	14
P =	61	3	9	10	12
P =	62	4	5	6	7
P =	63	4	5	10	12
P =	64	4	5	13	14
P =	65	4	6	8	10
P =	66	4	6	9	11
P =	67	4	7	8	13
P =	68	4	7	9	10
P =	69	4	8	11	14
P =	70	4	9	12	14
P =	71	4	11	12	13
P =	72	5	6	8	14
P =	73	5	6	9	10
P =	74	5	6	11	12
P =	75	5	7	8	11
P =	76	5	7	12	14
P =	77	5	7	10	13
P =	78	5	8	9	12
P =	79	5	9	11	13
P =	80	6	7	8	12
P =	81	6	7	9	14
P =	82	6	8	9	13
P =	83	6	10	11	13
P =	84	6	12	13	14
P =	85	7	8	10	14
P =	86	7	9	11	12
P =	87	7	11	13	14
P =	88	8	9	10	11
P =	89	8	10	12	13
P =	90	9	10	13	14
P =	91	10	11	12	14

THE SYSTEM ***** A16 *****

TRIPLE SYSTEM														
ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	B	A	B	B	B	A	B	A	A	B	B	A	B

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	13
P =	28	2	3	6	14
P =	29	2	3	7	11
P =	30	2	3	10	12
P =	31	2	3	8	9
P =	32	2	4	5	11
P =	33	2	4	6	12
P =	34	2	4	7	9
P =	35	2	4	10	14
P =	36	2	4	8	13
P =	37	2	5	7	10
P =	38	2	5	8	12
P =	39	2	5	9	14
P =	40	2	6	7	13
P =	41	2	6	8	10
P =	42	2	6	9	11
P =	43	2	7	12	14
P =	44	2	8	11	14
P =	45	2	9	12	13
P =	46	2	10	11	13
P =	47	3	4	5	6
P =	48	3	4	7	13
P =	49	3	4	9	14
P =	50	3	4	8	10
P =	51	3	4	11	12
P =	52	3	5	8	11
P =	53	3	5	9	12
P =	54	3	5	10	14
P =	55	3	6	7	12
P =	56	3	6	9	13
P =	57	3	6	10	11
P =	58	3	7	8	14
P =	59	3	7	9	10
P =	60	3	8	12	13
P =	61	3	11	13	14
P =	62	4	5	7	12
P =	63	4	5	10	13
P =	64	4	5	8	14
P =	65	4	6	7	10
P =	66	4	6	11	14
P =	67	4	6	8	9
P =	68	4	7	8	11
P =	69	4	9	10	12
P =	70	4	9	11	13
P =	71	4	12	13	14
P =	72	5	6	7	8
P =	73	5	6	9	10
P =	74	5	6	11	13
P =	75	5	6	12	14
P =	76	5	7	9	11
P =	77	5	7	13	14
P =	78	5	8	9	13
P =	79	5	10	11	12
P =	80	6	7	9	14
P =	81	6	8	11	12
P =	82	6	8	13	14
P =	83	6	10	12	13
P =	84	7	8	9	12
P =	85	7	8	10	13
P =	86	7	10	11	14
P =	87	7	11	12	13
P =	88	8	9	10	11
P =	89	8	10	12	14
P =	90	9	10	13	14
P =	91	9	11	12	14

THE SYSTEM ***** A17 *****

TRIPLE SYSTEM														
ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	A	B	A	A	A	A	A	A	A	A	A	B

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	13
P =	28	2	3	7	14
P =	29	2	3	8	9
P =	30	2	3	10	12
P =	31	2	3	6	11
P =	32	2	4	5	8
P =	33	2	4	9	12
P =	34	2	4	11	13
P =	35	2	4	10	14
P =	36	2	4	6	7
P =	37	2	5	7	12
P =	38	2	5	9	14
P =	39	2	5	10	11
P =	40	2	6	8	10
P =	41	2	6	12	14
P =	42	2	6	9	13
P =	43	2	7	9	11
P =	44	2	7	10	13
P =	45	2	8	11	14
P =	46	2	8	12	13
P =	47	3	4	5	10
P =	48	3	4	11	14
P =	49	3	4	6	12
P =	50	3	4	8	13
P =	51	3	4	7	9
P =	52	3	5	6	9
P =	53	3	5	11	12
P =	54	3	5	8	14
P =	55	3	6	7	10
P =	56	3	6	13	14
P =	57	3	7	8	12
P =	58	3	7	11	13
P =	59	3	8	10	11
P =	60	3	9	10	14
P =	61	3	9	12	13
P =	62	4	5	6	11
P =	63	4	5	12	14
P =	64	4	5	7	13
P =	65	4	6	8	14
P =	66	4	6	9	10
P =	67	4	7	8	10
P =	68	4	7	11	12
P =	69	4	8	9	11
P =	70	4	9	13	14
P =	71	4	10	12	13
P =	72	5	6	7	14
P =	73	5	6	8	13
P =	74	5	6	10	12
P =	75	5	7	8	11
P =	76	5	7	9	10
P =	77	5	8	9	12
P =	78	5	9	11	13
P =	79	5	10	13	14
P =	80	6	7	8	9
P =	81	6	7	12	13
P =	82	6	8	11	12
P =	83	6	9	11	14
P =	84	6	10	11	13
P =	85	7	8	13	14
P =	86	7	9	12	14
P =	87	7	10	11	14
P =	88	8	9	10	13
P =	89	8	10	12	14
P =	90	9	10	11	12
P =	91	11	12	13	14

THE SYSTEM ***** A18 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	B	B	A	A	B	B	B	B	A	B	B	A

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	13
P =	28	2	3	8	14
P =	29	2	3	6	9
P =	30	2	3	7	12
P =	31	2	3	10	11
P =	32	2	4	5	14
P =	33	2	4	6	7
P =	34	2	4	8	10
P =	35	2	4	9	11
P =	36	2	4	12	13
P =	37	2	5	7	9
P =	38	2	5	10	12
P =	39	2	5	8	11
P =	40	2	6	8	12
P =	41	2	6	10	13
P =	42	2	6	11	14
P =	43	2	7	10	14
P =	44	2	7	11	13
P =	45	2	8	9	13
P =	46	2	9	12	14
P =	47	3	4	5	12
P =	48	3	4	8	13
P =	49	3	4	9	10
P =	50	3	4	7	11
P =	51	3	4	6	14
P =	52	3	5	6	11
P =	53	3	5	10	14
P =	54	3	5	8	9
P =	55	3	6	7	13
P =	56	3	6	10	12
P =	57	3	7	8	10
P =	58	3	7	9	14
P =	59	3	8	11	12
P =	60	3	9	12	13
P =	61	3	11	13	14
P =	62	4	5	6	10
P =	63	4	5	11	13
P =	64	4	5	7	8
P =	65	4	6	8	9
P =	66	4	6	11	12
P =	67	4	7	9	12
P =	68	4	7	10	13
P =	69	4	8	11	14
P =	70	4	9	13	14
P =	71	4	10	12	14
P =	72	5	6	7	12
P =	73	5	6	8	13
P =	74	5	6	9	14
P =	75	5	7	10	11
P =	76	5	7	13	14
P =	77	5	8	12	14
P =	78	5	9	10	13
P =	79	5	9	11	12
P =	80	6	7	8	14
P =	81	6	7	9	10
P =	82	6	8	10	11
P =	83	6	9	11	13
P =	84	6	12	13	14
P =	85	7	8	9	11
P =	86	7	8	12	13
P =	87	7	11	12	14
P =	88	8	9	10	12
P =	89	8	10	13	14
P =	90	9	10	11	14
P =	91	10	11	12	13

THE SYSTEM ***** A19 *****

TRIPLE SYSTEM														
ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	B	A	A	A	A	A	A	A	A	A	B	A	A

P = 1	1	2	3	4
P = 2	1	2	5	6
P = 3	1	2	7	8
P = 4	1	2	9	10
P = 5	1	2	11	12
P = 6	1	2	13	14
P = 7	1	3	5	7
P = 8	1	3	6	8
P = 9	1	3	9	11
P = 10	1	3	10	13
P = 11	1	3	12	14
P = 12	1	4	5	9
P = 13	1	4	6	13
P = 14	1	4	7	14
P = 15	1	4	8	12
P = 16	1	4	10	11
P = 17	1	5	8	10
P = 18	1	5	11	14
P = 19	1	5	12	13
P = 20	1	6	7	11
P = 21	1	6	9	12
P = 22	1	6	10	14
P = 23	1	7	9	13
P = 24	1	7	10	12
P = 25	1	8	9	14
P = 26	1	8	11	13
P = 27	2	3	5	13
P = 28	2	3	9	14
P = 29	2	3	7	10
P = 30	2	3	8	11
P = 31	2	3	6	12
P = 32	2	4	5	7
P = 33	2	4	8	10
P = 34	2	4	6	11
P = 35	2	4	9	13
P = 36	2	4	12	14
P = 37	2	5	8	14
P = 38	2	5	9	11
P = 39	2	5	10	12
P = 40	2	6	7	14
P = 41	2	6	8	9
P = 42	2	6	10	13
P = 43	2	7	9	12
P = 44	2	7	11	13
P = 45	2	8	12	13
P = 46	2	10	11	14
P = 47	3	4	5	11
P = 48	3	4	7	12
P = 49	3	4	8	9
P = 50	3	4	6	10
P = 51	3	4	13	14
P = 52	3	5	6	14
P = 53	3	5	8	12
P = 54	3	5	9	10
P = 55	3	6	7	9
P = 56	3	6	11	13
P = 57	3	7	8	13
P = 58	3	7	11	14
P = 59	3	8	10	14
P = 60	3	9	12	13
P = 61	3	10	11	12
P = 62	4	5	6	12
P = 63	4	5	8	13
P = 64	4	5	10	14
P = 65	4	6	7	8
P = 66	4	6	9	14
P = 67	4	7	9	11
P = 68	4	7	10	13
P = 69	4	8	11	14
P = 70	4	9	10	12
P = 71	4	11	12	13
P = 72	5	6	7	10
P = 73	5	6	8	11
P = 74	5	6	9	13
P = 75	5	7	8	9
P = 76	5	7	11	12
P = 77	5	7	13	14
P = 78	5	9	12	14
P = 79	5	10	11	13
P = 80	6	7	12	13
P = 81	6	8	10	12
P = 82	6	8	13	14
P = 83	6	9	10	11
P = 84	6	11	12	14
P = 85	7	8	10	11
P = 86	7	8	12	14
P = 87	7	9	10	14
P = 88	8	9	10	13
P = 89	8	9	11	12
P = 90	9	11	13	14
P = 91	10	12	13	14

THE SYSTEM ***** A20 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	A	A	A	A	B	A	A	A	A	B	A	A

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	13
P =	28	2	3	10	14
P =	29	2	3	6	11
P =	30	2	3	7	9
P =	31	2	3	8	12
P =	32	2	4	5	11
P =	33	2	4	12	14
P =	34	2	4	6	10
P =	35	2	4	7	13
P =	36	2	4	8	9
P =	37	2	5	7	10
P =	38	2	5	9	12
P =	39	2	5	8	14
P =	40	2	6	7	12
P =	41	2	6	8	13
P =	42	2	6	9	14
P =	43	2	7	11	14
P =	44	2	8	10	11
P =	45	2	9	11	13
P =	46	2	10	12	13
P =	47	3	4	5	14
P =	48	3	4	6	9
P =	49	3	4	8	10
P =	50	3	4	11	13
P =	51	3	4	7	12
P =	52	3	5	6	10
P =	53	3	5	11	12
P =	54	3	5	8	9
P =	55	3	6	7	14
P =	56	3	6	12	13
P =	57	3	7	8	13
P =	58	3	7	10	11
P =	59	3	8	11	14
P =	60	3	9	10	12
P =	61	3	9	13	14
P =	62	4	5	6	7
P =	63	4	5	8	13
P =	64	4	5	10	12
P =	65	4	6	8	14
P =	66	4	6	11	12
P =	67	4	7	8	11
P =	68	4	7	9	10
P =	69	4	9	11	14
P =	70	4	9	12	13
P =	71	4	10	13	14
P =	72	5	6	8	11
P =	73	5	6	12	14
P =	74	5	6	9	13
P =	75	5	7	8	12
P =	76	5	7	13	14
P =	77	5	7	9	11
P =	78	5	9	10	14
P =	79	5	10	11	13
P =	80	6	7	8	9
P =	81	6	7	10	13
P =	82	6	8	10	12
P =	83	6	9	10	11
P =	84	6	11	13	14
P =	85	7	8	10	14
P =	86	7	9	12	14
P =	87	7	11	12	13
P =	88	8	9	10	13
P =	89	8	9	11	12
P =	90	8	12	13	14
P =	91	10	11	12	14

THE SYSTEM ***** A21 *****

TRIPLE SYSTEM														
ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	B	A	A	A	B	B	A	B	B	B	B	B

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	13
P =	28	2	3	10	14
P =	29	2	3	6	11
P =	30	2	3	7	12
P =	31	2	3	8	9
P =	32	2	4	5	7
P =	33	2	4	8	10
P =	34	2	4	11	13
P =	35	2	4	12	14
P =	36	2	4	6	9
P =	37	2	5	8	12
P =	38	2	5	9	14
P =	39	2	5	10	11
P =	40	2	6	7	14
P =	41	2	6	8	13
P =	42	2	6	10	12
P =	43	2	7	9	11
P =	44	2	7	10	13
P =	45	2	8	11	14
P =	46	2	9	12	13
P =	47	3	4	5	10
P =	48	3	4	7	11
P =	49	3	4	8	14
P =	50	3	4	6	12
P =	51	3	4	9	13
P =	52	3	5	6	14
P =	53	3	5	8	11
P =	54	3	5	9	12
P =	55	3	6	7	13
P =	56	3	6	9	10
P =	57	3	7	8	10
P =	58	3	7	9	14
P =	59	3	8	12	13
P =	60	3	10	11	12
P =	61	3	11	13	14
P =	62	4	5	6	8
P =	63	4	5	11	12
P =	64	4	5	13	14
P =	65	4	6	7	10
P =	66	4	6	11	14
P =	67	4	7	8	13
P =	68	4	7	9	12
P =	69	4	8	9	11
P =	70	4	9	10	14
P =	71	4	10	12	13
P =	72	5	6	7	12
P =	73	5	6	10	13
P =	74	5	6	9	11
P =	75	5	7	8	14
P =	76	5	7	9	10
P =	77	5	7	11	13
P =	78	5	8	9	13
P =	79	5	10	12	14
P =	80	6	7	8	9
P =	81	6	8	10	11
P =	82	6	8	12	14
P =	83	6	9	13	14
P =	84	6	11	12	13
P =	85	7	8	11	12
P =	86	7	10	11	14
P =	87	7	12	13	14
P =	88	8	9	10	12
P =	89	8	10	13	14
P =	90	9	10	11	13
P =	91	9	11	12	14

THE SYSTEM ***** A22 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	A	B	B	B	B	B	B	B	A	A	B	A

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	13
P =	28	2	3	11	14
P =	29	2	3	7	12
P =	30	2	3	8	9
P =	31	2	3	6	10
P =	32	2	4	5	10
P =	33	2	4	7	11
P =	34	2	4	8	13
P =	35	2	4	9	14
P =	36	2	4	6	12
P =	37	2	5	7	14
P =	38	2	5	8	12
P =	39	2	5	9	11
P =	40	2	6	7	9
P =	41	2	6	11	13
P =	42	2	6	8	14
P =	43	2	7	10	13
P =	44	2	8	10	11
P =	45	2	9	12	13
P =	46	2	10	12	14
P =	47	3	4	5	6
P =	48	3	4	7	9
P =	49	3	4	10	12
P =	50	3	4	13	14
P =	51	3	4	8	11
P =	52	3	5	8	14
P =	53	3	5	9	12
P =	54	3	5	10	11
P =	55	3	6	7	14
P =	56	3	6	9	13
P =	57	3	6	11	12
P =	58	3	7	8	10
P =	59	3	7	11	13
P =	60	3	8	12	13
P =	61	3	9	10	14
P =	62	4	5	7	8
P =	63	4	5	11	13
P =	64	4	5	12	14
P =	65	4	6	7	10
P =	66	4	6	11	14
P =	67	4	6	8	9
P =	68	4	7	12	13
P =	69	4	8	10	14
P =	70	4	9	10	13
P =	71	4	9	11	12
P =	72	5	6	7	13
P =	73	5	6	9	14
P =	74	5	6	10	12
P =	75	5	6	8	11
P =	76	5	7	9	10
P =	77	5	7	11	12
P =	78	5	8	9	13
P =	79	5	10	13	14
P =	80	6	7	8	12
P =	81	6	8	10	13
P =	82	6	9	10	11
P =	83	6	12	13	14
P =	84	7	8	9	11
P =	85	7	8	13	14
P =	86	7	9	12	14
P =	87	7	10	11	14
P =	88	8	9	10	12
P =	89	8	11	12	14
P =	90	9	11	13	14
P =	91	10	11	12	13

THE SYSTEM ***** A23 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	A	A	A	A	A	A	A	A	A	A	A	A

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	13
P =	28	2	3	11	14
P =	29	2	3	10	12
P =	30	2	3	6	7
P =	31	2	3	8	9
P =	32	2	4	5	14
P =	33	2	4	6	9
P =	34	2	4	10	13
P =	35	2	4	7	12
P =	36	2	4	8	11
P =	37	2	5	7	10
P =	38	2	5	9	11
P =	39	2	5	8	12
P =	40	2	6	8	13
P =	41	2	6	12	14
P =	42	2	6	10	11
P =	43	2	7	9	14
P =	44	2	7	11	13
P =	45	2	8	10	14
P =	46	2	9	12	13
P =	47	3	4	5	11
P =	48	3	4	6	12
P =	49	3	4	7	8
P =	50	3	4	9	13
P =	51	3	4	10	14
P =	52	3	5	6	10
P =	53	3	5	9	12
P =	54	3	5	8	14
P =	55	3	6	9	14
P =	56	3	6	11	13
P =	57	3	7	9	10
P =	58	3	7	11	12
P =	59	3	7	13	14
P =	60	3	8	10	11
P =	61	3	8	12	13
P =	62	4	5	6	8
P =	63	4	5	10	12
P =	64	4	5	7	13
P =	65	4	6	7	10
P =	66	4	6	11	14
P =	67	4	7	9	11
P =	68	4	8	9	10
P =	69	4	8	13	14
P =	70	4	9	12	14
P =	71	4	11	12	13
P =	72	5	6	7	9
P =	73	5	6	11	12
P =	74	5	6	13	14
P =	75	5	7	8	11
P =	76	5	7	12	14
P =	77	5	8	9	13
P =	78	5	9	10	14
P =	79	5	10	11	13
P =	80	6	7	8	14
P =	81	6	7	12	13
P =	82	6	8	9	11
P =	83	6	8	10	12
P =	84	6	9	10	13
P =	85	7	8	9	12
P =	86	7	8	10	13
P =	87	7	10	11	14
P =	88	8	11	12	14
P =	89	9	10	11	12
P =	90	9	11	13	14
P =	91	10	12	13	14

THE SYSTEM ***** A24 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	A	A	A	A	A	A	A	A	A	A	A	A

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	14
P =	28	2	3	6	7
P =	29	2	3	8	9
P =	30	2	3	10	11
P =	31	2	3	12	13
P =	32	2	4	5	13
P =	33	2	4	8	14
P =	34	2	4	9	11
P =	35	2	4	7	12
P =	36	2	4	6	10
P =	37	2	5	7	10
P =	38	2	5	8	11
P =	39	2	5	9	12
P =	40	2	6	8	12
P =	41	2	6	9	13
P =	42	2	6	11	14
P =	43	2	7	9	14
P =	44	2	7	11	13
P =	45	2	8	10	13
P =	46	2	10	12	14
P =	47	3	4	5	8
P =	48	3	4	9	10
P =	49	3	4	11	14
P =	50	3	4	6	12
P =	51	3	4	7	13
P =	52	3	5	6	10
P =	53	3	5	11	12
P =	54	3	5	9	13
P =	55	3	6	9	14
P =	56	3	6	11	13
P =	57	3	7	8	11
P =	58	3	7	9	12
P =	59	3	7	10	14
P =	60	3	8	10	12
P =	61	3	8	13	14
P =	62	4	5	6	14
P =	63	4	5	7	11
P =	64	4	5	10	12
P =	65	4	6	7	9
P =	66	4	6	8	11
P =	67	4	7	8	10
P =	68	4	8	9	13
P =	69	4	9	12	14
P =	70	4	10	13	14
P =	71	4	11	12	13
P =	72	5	6	7	12
P =	73	5	6	8	13
P =	74	5	6	9	11
P =	75	5	7	8	9
P =	76	5	7	13	14
P =	77	5	8	12	14
P =	78	5	9	10	14
P =	79	5	10	11	13
P =	80	6	7	8	14
P =	81	6	7	10	13
P =	82	6	8	9	10
P =	83	6	10	11	12
P =	84	6	12	13	14
P =	85	7	8	12	13
P =	86	7	9	10	11
P =	87	7	11	12	14
P =	88	8	9	11	12
P =	89	8	10	11	14
P =	90	9	10	12	13
P =	91	9	11	13	14

THE SYSTEM ***** A25 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	A	A	A	A	A	B	A	A	A	B	A	A	A

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	14
P =	28	2	3	6	10
P =	29	2	3	11	13
P =	30	2	3	7	12
P =	31	2	3	8	9
P =	32	2	4	5	8
P =	33	2	4	7	9
P =	34	2	4	10	13
P =	35	2	4	11	14
P =	36	2	4	6	12
P =	37	2	5	7	11
P =	38	2	5	10	12
P =	39	2	5	9	13
P =	40	2	6	7	13
P =	41	2	6	8	14
P =	42	2	6	9	11
P =	43	2	7	10	14
P =	44	2	8	10	11
P =	45	2	8	12	13
P =	46	2	9	12	14
P =	47	3	4	5	6
P =	48	3	4	7	11
P =	49	3	4	10	12
P =	50	3	4	9	13
P =	51	3	4	8	14
P =	52	3	5	8	13
P =	53	3	5	9	12
P =	54	3	5	10	11
P =	55	3	6	7	9
P =	56	3	6	11	14
P =	57	3	6	12	13
P =	58	3	7	8	10
P =	59	3	7	13	14
P =	60	3	8	11	12
P =	61	3	9	10	14
P =	62	4	5	7	12
P =	63	4	5	11	13
P =	64	4	5	10	14
P =	65	4	6	7	10
P =	66	4	6	8	11
P =	67	4	6	9	14
P =	68	4	7	8	13
P =	69	4	8	9	10
P =	70	4	9	11	12
P =	71	4	12	13	14
P =	72	5	6	7	8
P =	73	5	6	9	10
P =	74	5	6	11	12
P =	75	5	6	13	14
P =	76	5	7	9	14
P =	77	5	7	10	13
P =	78	5	8	9	11
P =	79	5	8	12	14
P =	80	6	7	12	14
P =	81	6	8	9	13
P =	82	6	8	10	12
P =	83	6	10	11	13
P =	84	7	8	9	12
P =	85	7	8	11	14
P =	86	7	9	10	11
P =	87	7	11	12	13
P =	88	8	10	13	14
P =	89	9	10	12	13
P =	90	9	11	13	14
P =	91	10	11	12	14

THE SYSTEM ***** A26 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	A	B	A	A	A	A	A	A	A	B	A	A	A	A

P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	14
P =	15	1	4	8	12
P =	16	1	4	10	11
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	11
P =	21	1	6	9	12
P =	22	1	6	10	14
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	14
P =	28	2	3	6	13
P =	29	2	3	7	11
P =	30	2	3	9	12
P =	31	2	3	8	10
P =	32	2	4	5	8
P =	33	2	4	9	11
P =	34	2	4	12	13
P =	35	2	4	10	14
P =	36	2	4	6	7
P =	37	2	5	7	9
P =	38	2	5	10	12
P =	39	2	5	11	13
P =	40	2	6	8	12
P =	41	2	6	9	14
P =	42	2	6	10	11
P =	43	2	7	10	13
P =	44	2	7	12	14
P =	45	2	8	9	13
P =	46	2	8	11	14
P =	47	3	4	5	12
P =	48	3	4	11	13
P =	49	3	4	8	14
P =	50	3	4	6	9
P =	51	3	4	7	10
P =	52	3	5	6	10
P =	53	3	5	8	11
P =	54	3	5	9	13
P =	55	3	6	7	12
P =	56	3	6	11	14
P =	57	3	7	8	9
P =	58	3	7	13	14
P =	59	3	8	12	13
P =	60	3	9	10	14
P =	61	3	10	11	12
P =	62	4	5	6	14
P =	63	4	5	7	11
P =	64	4	5	10	13
P =	65	4	6	8	11
P =	66	4	6	10	12
P =	67	4	7	8	13
P =	68	4	7	9	12
P =	69	4	8	9	10
P =	70	4	9	13	14
P =	71	4	11	12	14
P =	72	5	6	7	13
P =	73	5	6	8	9
P =	74	5	6	11	12
P =	75	5	7	8	12
P =	76	5	7	10	14
P =	77	5	8	13	14
P =	78	5	9	10	11
P =	79	5	9	12	14
P =	80	6	7	8	14
P =	81	6	7	9	10
P =	82	6	8	10	13
P =	83	6	9	11	13
P =	84	6	12	13	14
P =	85	7	8	10	11
P =	86	7	9	11	14
P =	87	7	11	12	13
P =	88	8	9	11	12
P =	89	8	10	12	14
P =	90	9	10	12	13
P =	91	10	11	13	14

THE SYSTEM ***** B 1 *****

TRIPLE SYSTEM ASSOCI. WITH	1	2	3	4	5	6	7	8	9	10	11	12	13	14
IS OF TYPE	B	A	A	A	A	A	A	A	A	B	A	A	A	A

P = 1	1	2	3	4
P = 2	1	2	5	6
P = 3	1	2	7	8
P = 4	1	2	9	10
P = 5	1	2	11	12
P = 6	1	2	13	14
P = 7	1	3	5	7
P = 8	1	3	6	8
P = 9	1	3	9	11
P = 10	1	3	10	13
P = 11	1	3	12	14
P = 12	1	4	5	9
P = 13	1	4	6	13
P = 14	1	4	7	11
P = 15	1	4	8	12
P = 16	1	4	10	14
P = 17	1	5	8	10
P = 18	1	5	11	14
P = 19	1	5	12	13
P = 20	1	6	7	14
P = 21	1	6	9	12
P = 22	1	6	10	11
P = 23	1	7	9	13
P = 24	1	7	10	12
P = 25	1	8	9	14
P = 26	1	8	11	13
P = 27	2	3	5	8
P = 28	2	3	9	12
P = 29	2	3	11	13
P = 30	2	3	6	14
P = 31	2	3	7	10
P = 32	2	4	5	12
P = 33	2	4	7	13
P = 34	2	4	8	9
P = 35	2	4	6	10
P = 36	2	4	11	14
P = 37	2	5	7	11
P = 38	2	5	9	13
P = 39	2	5	10	14
P = 40	2	6	7	12
P = 41	2	6	8	13
P = 42	2	6	9	11
P = 43	2	7	9	14
P = 44	2	8	10	11
P = 45	2	8	12	14
P = 46	2	10	12	13
P = 47	3	4	5	13
P = 48	3	4	7	14
P = 49	3	4	8	11
P = 50	3	4	6	12
P = 51	3	4	9	10
P = 52	3	5	6	10
P = 53	3	5	11	12
P = 54	3	5	9	14
P = 55	3	6	7	11
P = 56	3	6	9	13
P = 57	3	7	8	9
P = 58	3	7	12	13
P = 59	3	8	10	12
P = 60	3	8	13	14
P = 61	3	10	11	14
P = 62	4	5	6	11
P = 63	4	5	8	14
P = 64	4	5	7	10
P = 65	4	6	7	8
P = 66	4	6	9	14
P = 67	4	7	9	12
P = 68	4	8	10	13
P = 69	4	9	11	13
P = 70	4	10	11	12
P = 71	4	12	13	14
P = 72	5	6	7	9
P = 73	5	6	8	12
P = 74	5	6	13	14
P = 75	5	7	8	13
P = 76	5	7	12	14
P = 77	5	8	9	11
P = 78	5	9	10	12
P = 79	5	10	11	13
P = 80	6	7	10	13
P = 81	6	8	9	10
P = 82	6	8	11	14
P = 83	6	10	12	14
P = 84	6	11	12	13
P = 85	7	8	10	14
P = 86	7	8	11	12
P = 87	7	9	10	11
P = 88	7	11	13	14
P = 89	8	9	12	13
P = 90	9	10	13	14
P = 91	9	11	12	14

THE SYSTEM ***** B.2 *****

TRIPLE SYSTEM ASSOCI. WITH IS OF TYPE	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	B	B	B	B	B	A	A	A	B	A	B	B	A	A

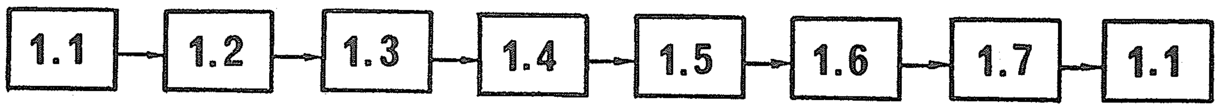
P =	1	1	2	3	4
P =	2	1	2	5	6
P =	3	1	2	7	8
P =	4	1	2	9	10
P =	5	1	2	11	12
P =	6	1	2	13	14
P =	7	1	3	5	7
P =	8	1	3	6	8
P =	9	1	3	9	11
P =	10	1	3	10	13
P =	11	1	3	12	14
P =	12	1	4	5	9
P =	13	1	4	6	13
P =	14	1	4	7	11
P =	15	1	4	8	12
P =	16	1	4	10	14
P =	17	1	5	8	10
P =	18	1	5	11	14
P =	19	1	5	12	13
P =	20	1	6	7	14
P =	21	1	6	9	12
P =	22	1	6	10	11
P =	23	1	7	9	13
P =	24	1	7	10	12
P =	25	1	8	9	14
P =	26	1	8	11	13
P =	27	2	3	5	8
P =	28	2	3	9	12
P =	29	2	3	11	13
P =	30	2	3	10	14
P =	31	2	3	6	7
P =	32	2	4	5	11
P =	33	2	4	6	12
P =	34	2	4	7	9
P =	35	2	4	10	13
P =	36	2	4	8	14
P =	37	2	5	7	13
P =	38	2	5	9	14
P =	39	2	5	10	12
P =	40	2	6	8	10
P =	41	2	6	11	14
P =	42	2	6	9	13
P =	43	2	7	10	11
P =	44	2	7	12	14
P =	45	2	8	9	11
P =	46	2	8	12	13
P =	47	3	4	5	12
P =	48	3	4	13	14
P =	49	3	4	6	11
P =	50	3	4	7	10
P =	51	3	4	8	9
P =	52	3	5	6	14
P =	53	3	5	9	13
P =	54	3	5	10	11
P =	55	3	6	9	10
P =	56	3	6	12	13
P =	57	3	7	8	13
P =	58	3	7	9	14
P =	59	3	7	11	12
P =	60	3	8	10	12
P =	61	3	8	11	14
P =	62	4	5	6	10
P =	63	4	5	8	13
P =	64	4	5	7	14
P =	65	4	6	7	8
P =	66	4	6	9	14
P =	67	4	7	12	13
P =	68	4	8	10	11
P =	69	4	9	10	12
P =	70	4	9	11	13
P =	71	4	11	12	14
P =	72	5	6	7	12
P =	73	5	6	11	13
P =	74	5	6	8	9
P =	75	5	7	8	11
P =	76	5	7	9	10
P =	77	5	8	12	14
P =	78	5	9	11	12
P =	79	5	10	13	14
P =	80	6	7	9	11
P =	81	6	7	10	13
P =	82	6	8	11	12
P =	83	6	8	13	14
P =	84	6	10	12	14
P =	85	7	8	9	12
P =	86	7	8	10	14
P =	87	7	11	13	14
P =	88	8	9	10	13
P =	89	9	10	11	14
P =	90	9	12	13	14
P =	91	10	11	12	13

APPENDIX 3

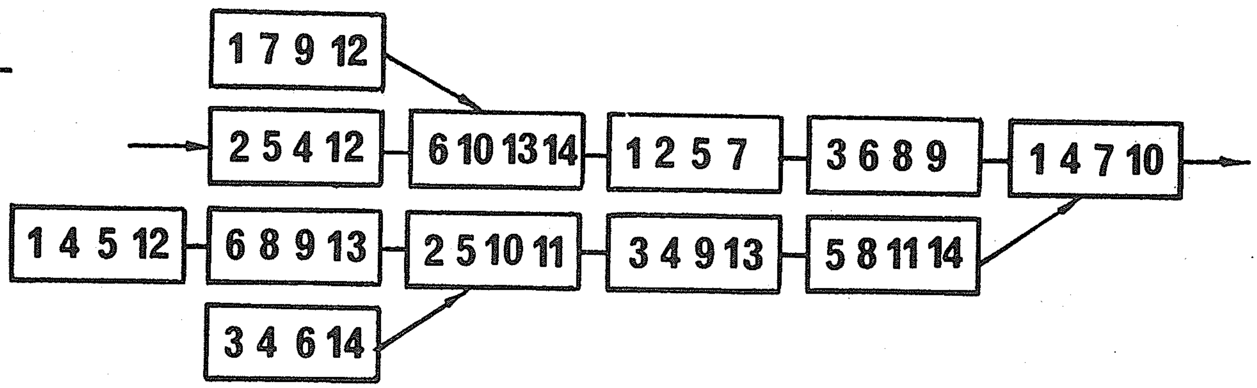
Two 35-cycles & All Their Appendices

of Quadruple System A1

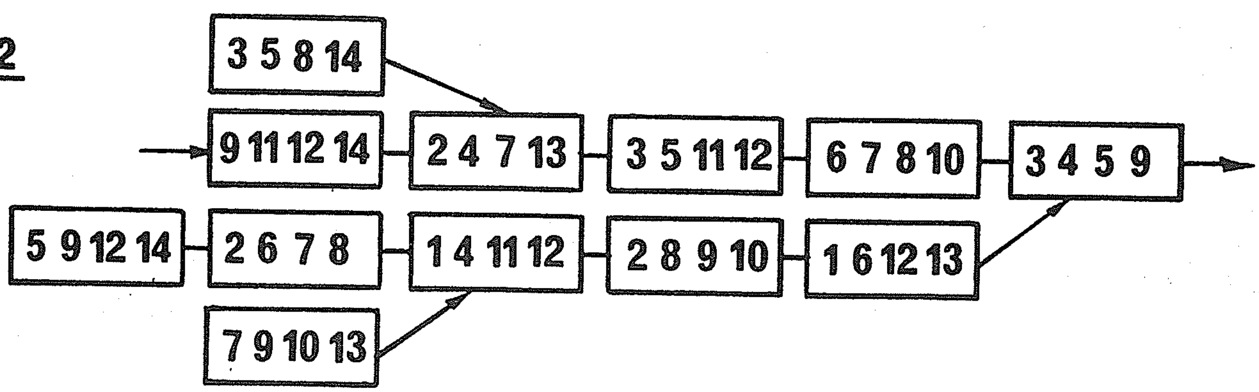
First 35-cycle



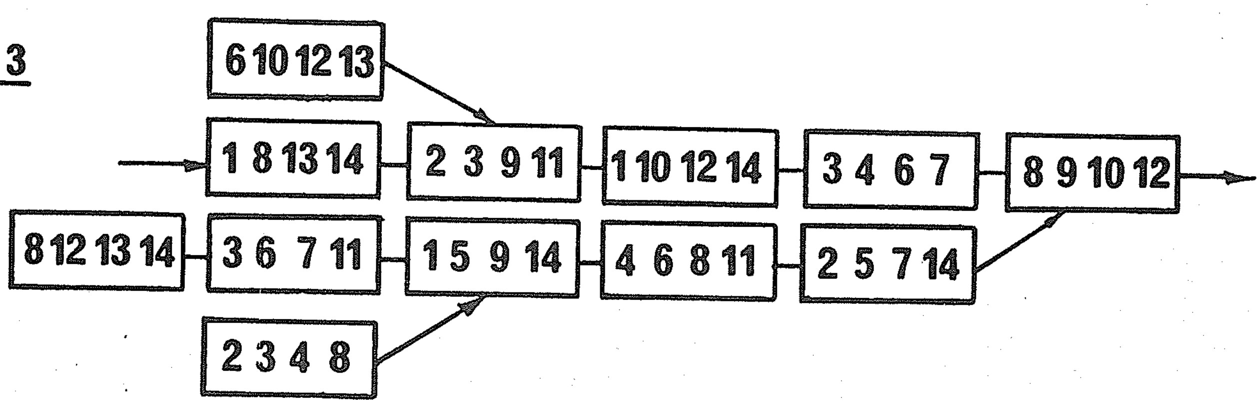
1.1



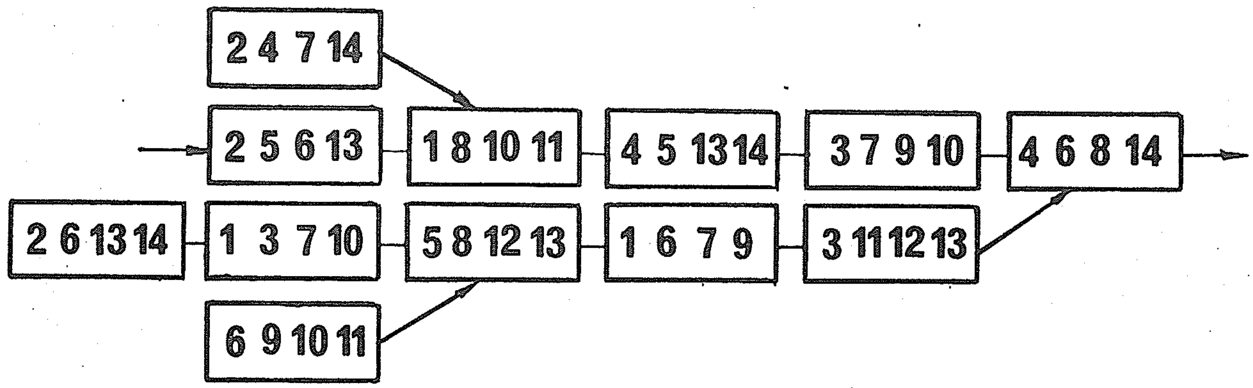
1.2



1.3



1.4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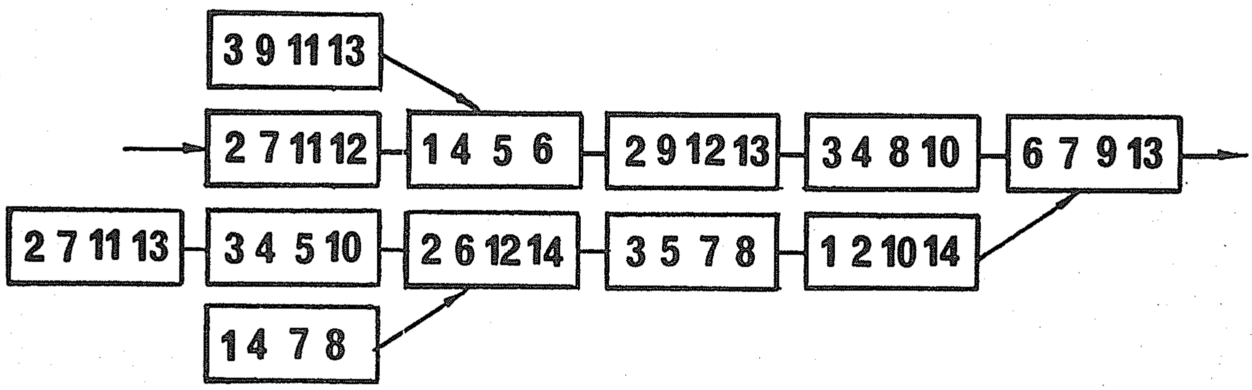
2 5 6 13
1 8 10 11
4 5 13 14
3 7 9 10
4 6 8 14
2 6 13 14
1 3 7 10
5 8 12 13
1 6 7 9
3 11 12 13
6 9 10 11

1.5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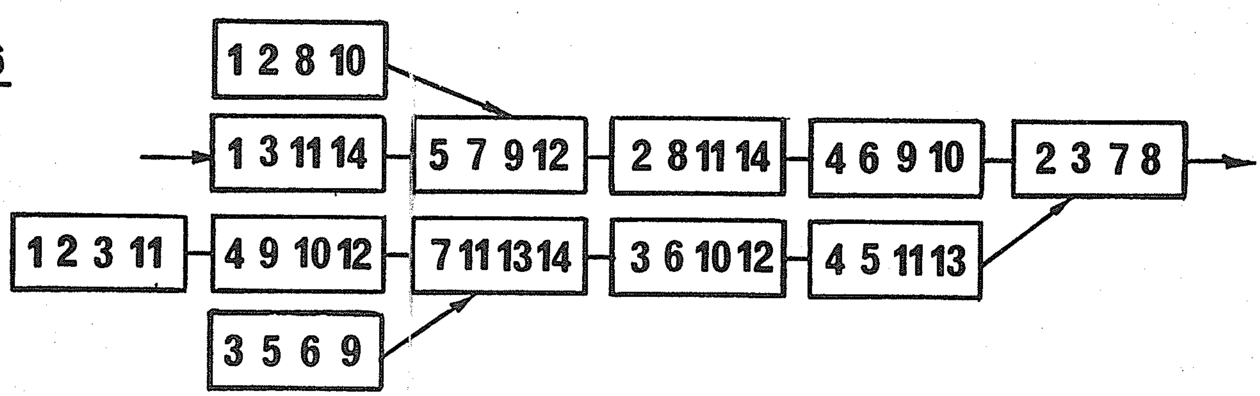
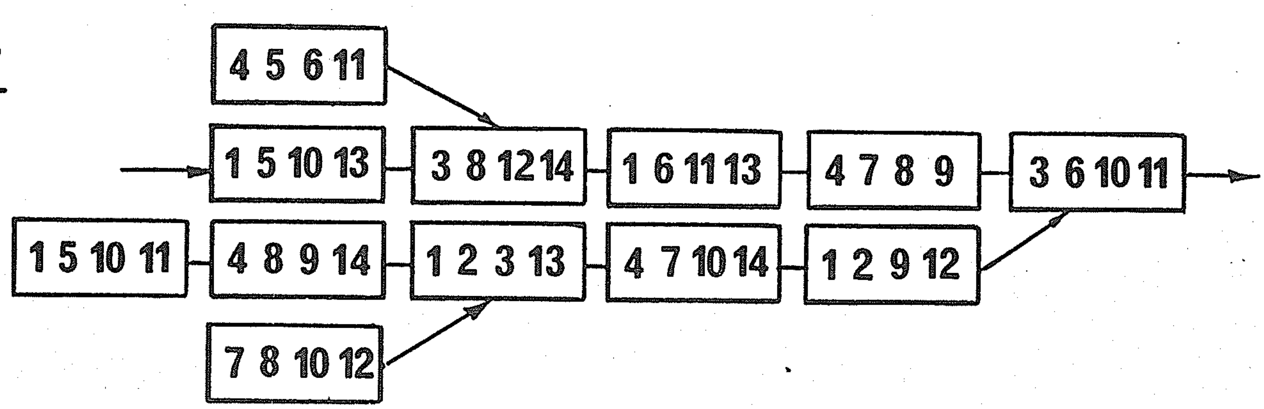
2 7 11 12
1 4 5 6
2 9 12 13
3 4 8 10
6 7 9 13
2 7 11 13
3 4 5 10
2 6 12 14
3 5 7 8
1 2 10 14
1 4 7 8

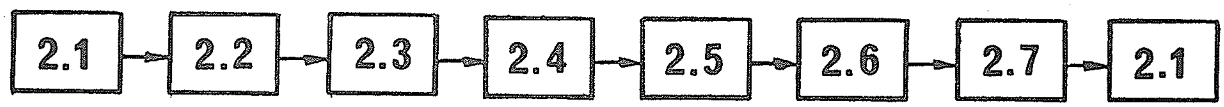
1.6


1 2 8 10
1 3 11 14
5 7 9 12
2 8 11 14
4 6 9 10
2 3 7 8
1 2 3 11
4 9 10 12
7 11 13 14
3 6 10 12
4 5 11 13
3 5 6 9

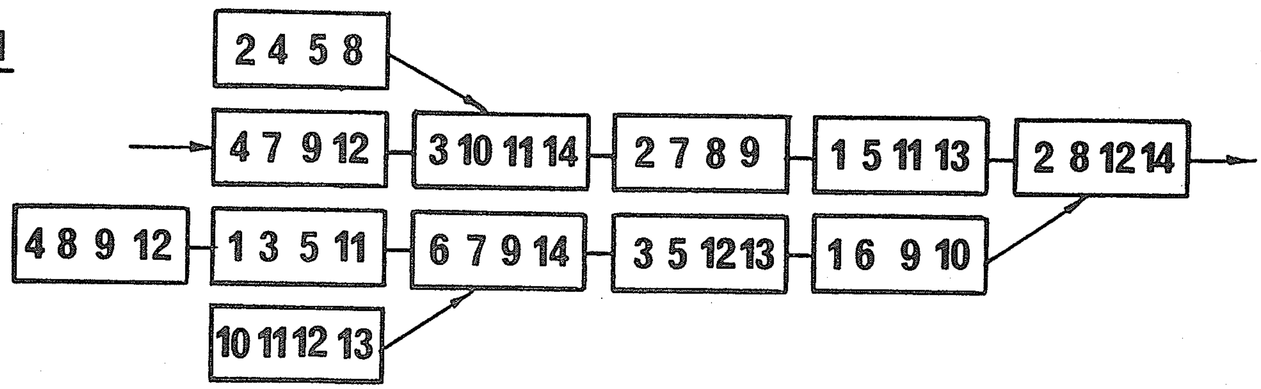
1.7


4 5 6 11
1 5 10 13
3 8 12 14
1 6 11 13
4 7 8 9
3 6 10 11
1 5 10 11
4 8 9 14
1 2 3 13
4 7 10 14
1 2 9 12
7 8 10 12

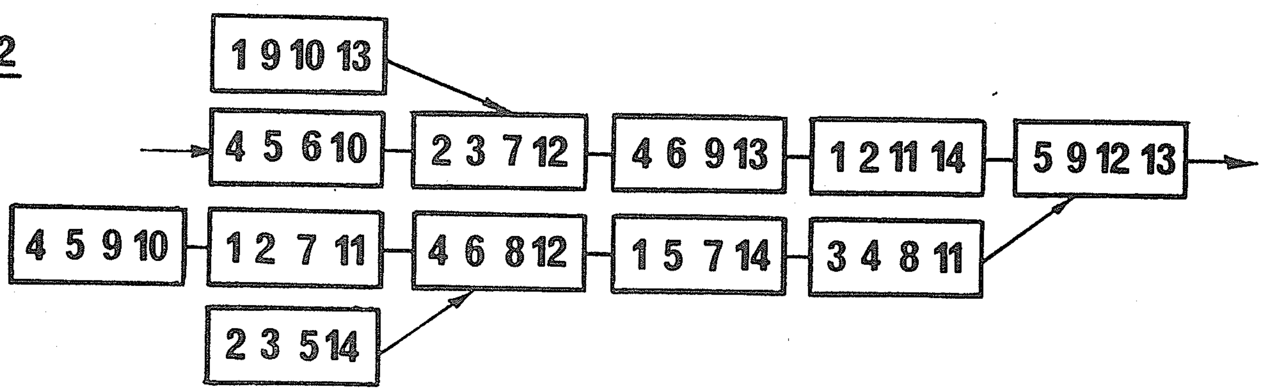
Second 35-cycle



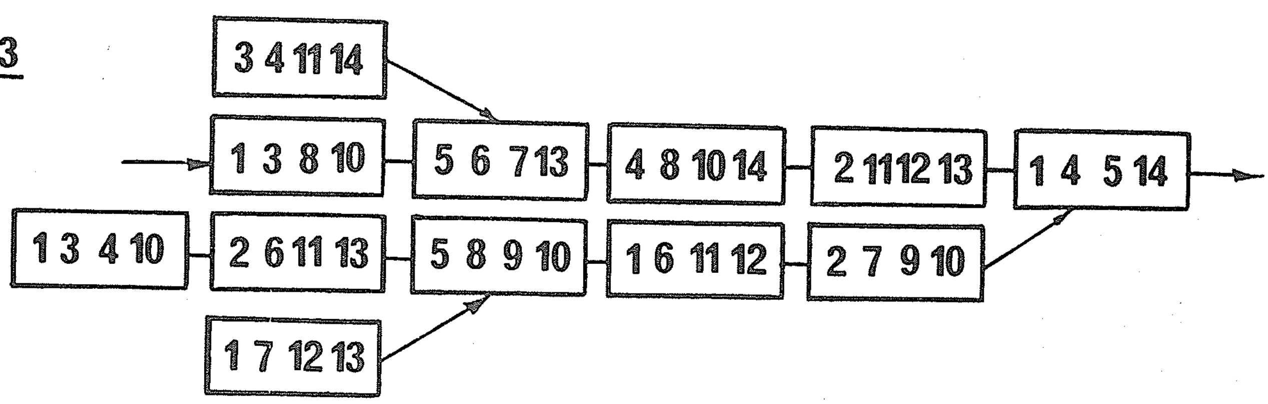
2.1



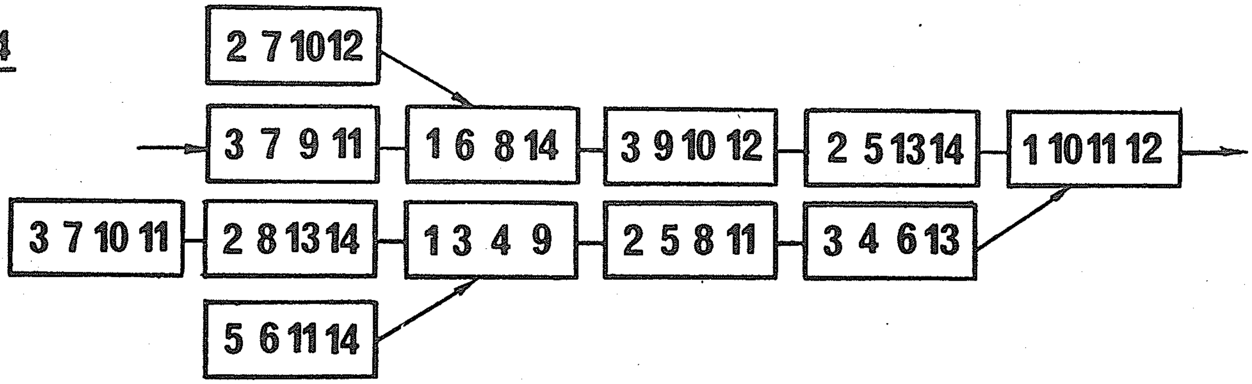
2.2



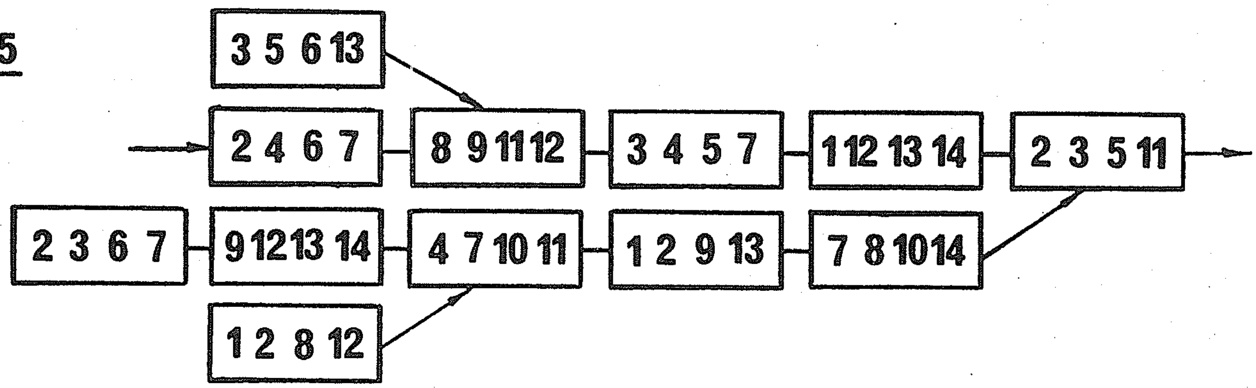
2.3



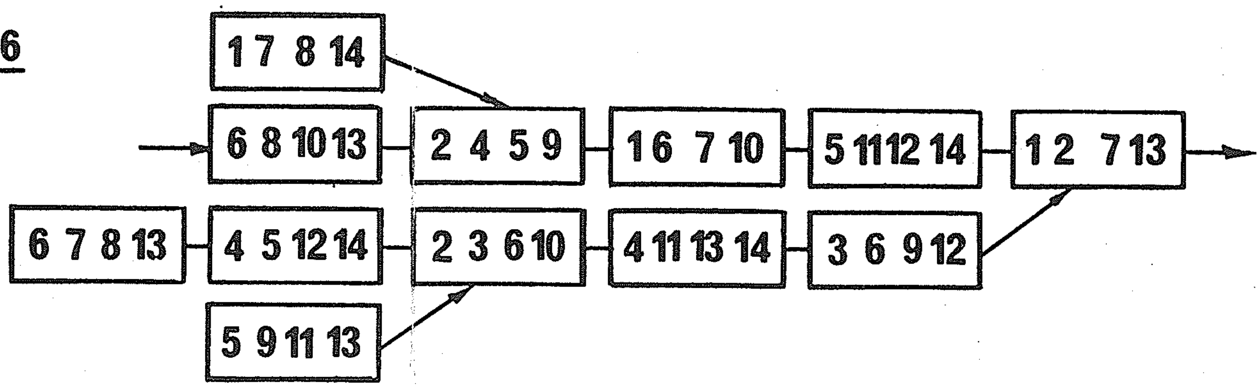
2.4


2 7 1012
3 7 9 11
1 6 8 14
3 9 10 12
2 5 13 14
1 10 11 12
3 7 10 11
2 8 13 14
1 3 4 9
2 5 8 11
3 4 6 13
5 6 11 14

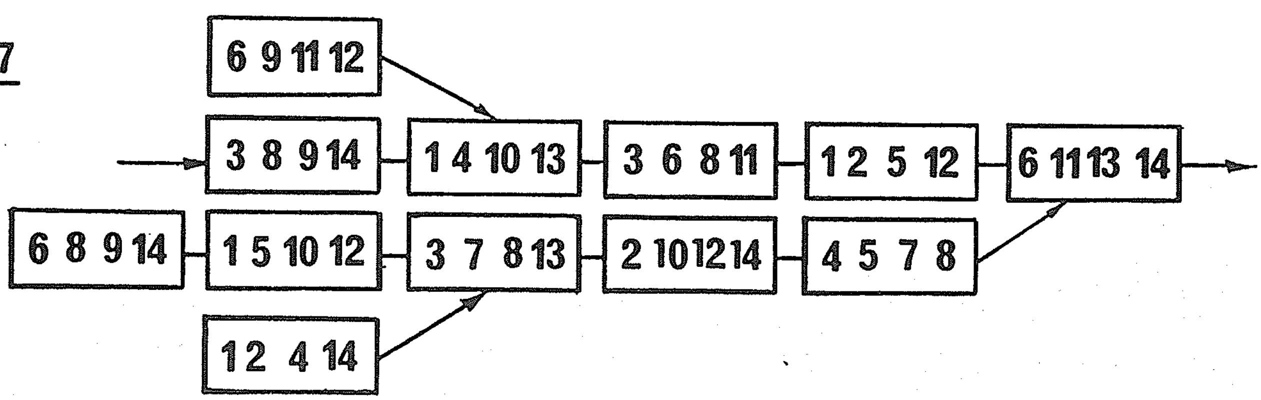
2.5


3 5 6 13
2 4 6 7
8 9 11 12
3 4 5 7
1 12 13 14
2 3 5 11
2 3 6 7
9 12 13 14
4 7 10 11
1 2 9 13
7 8 10 14
1 2 8 12

2.6


1 7 8 14
6 8 10 13
2 4 5 9
1 6 7 10
5 11 12 14
1 2 7 13
6 7 8 13
4 5 12 14
2 3 6 10
4 11 13 14
3 6 9 12
5 9 11 13

2.7

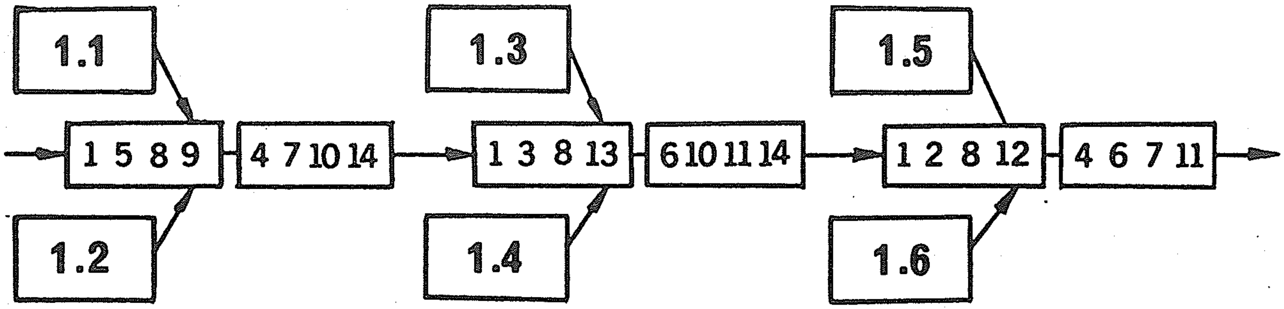

6 9 11 12
3 8 9 14
1 4 10 13
3 6 8 11
1 2 5 12
6 11 13 14
6 8 9 14
1 5 10 12
3 7 8 13
2 10 12 14
4 5 7 8
1 2 4 14

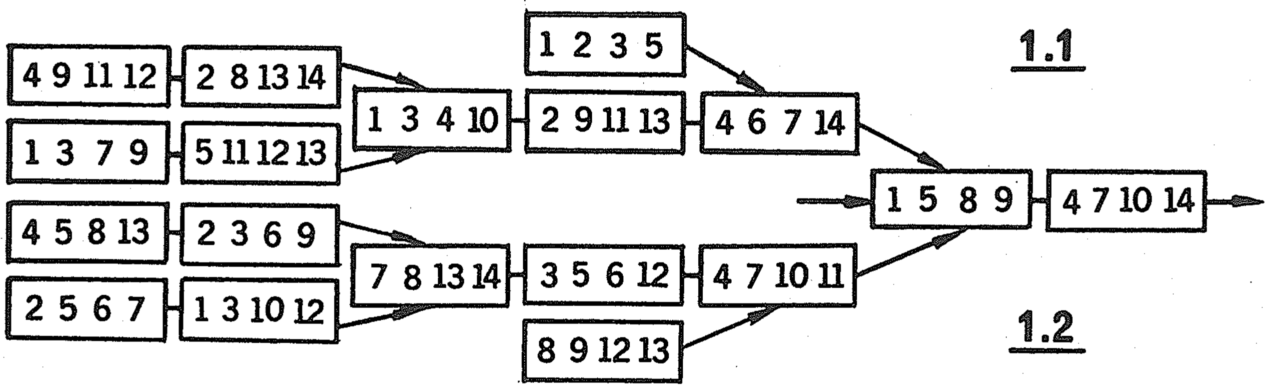
APPENDIX 4

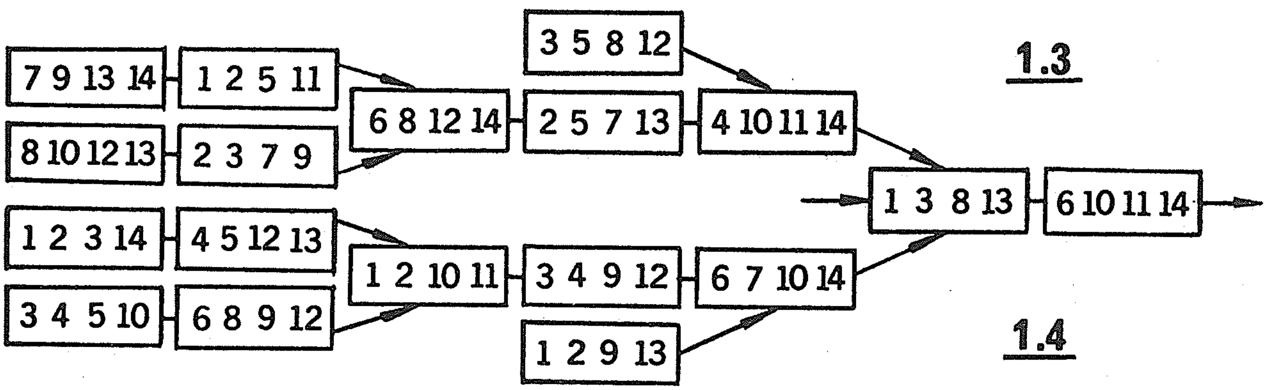
Seven 6-cycles & All Their Appendices

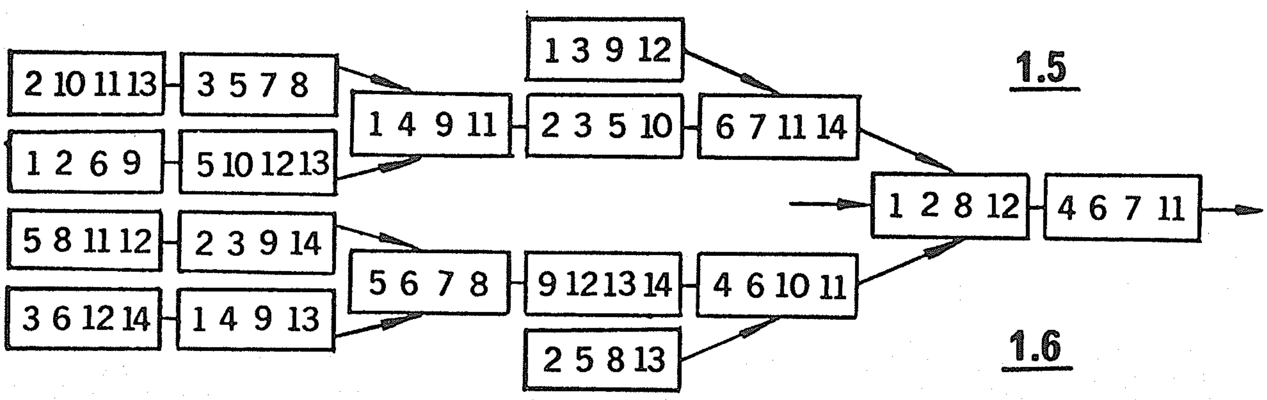
of Quadruple System A24

FIRST 6-CYCLE

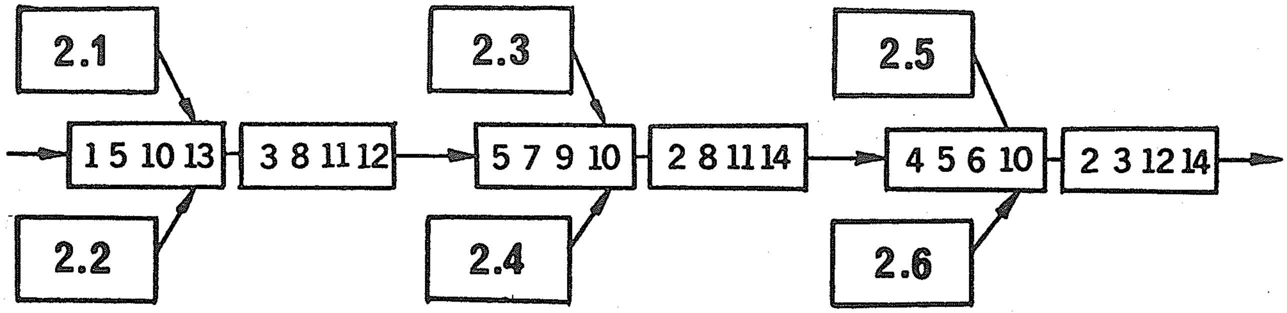


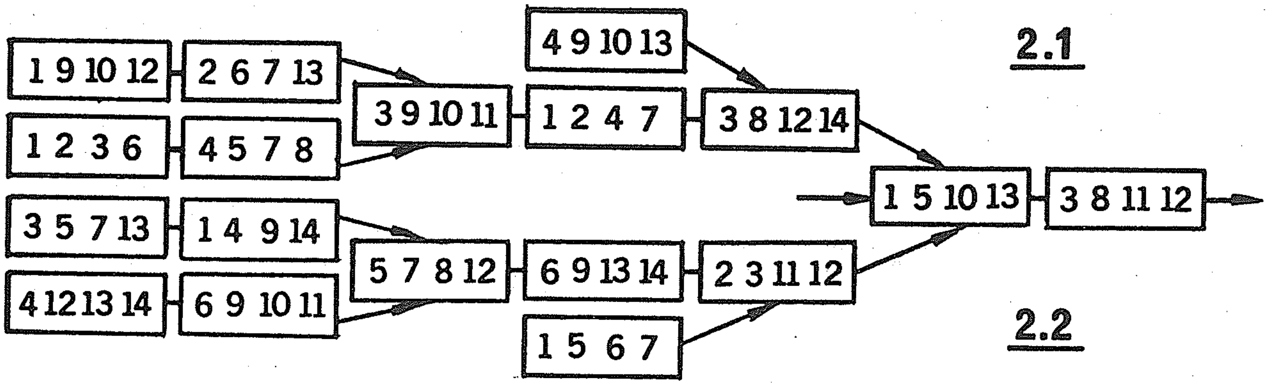


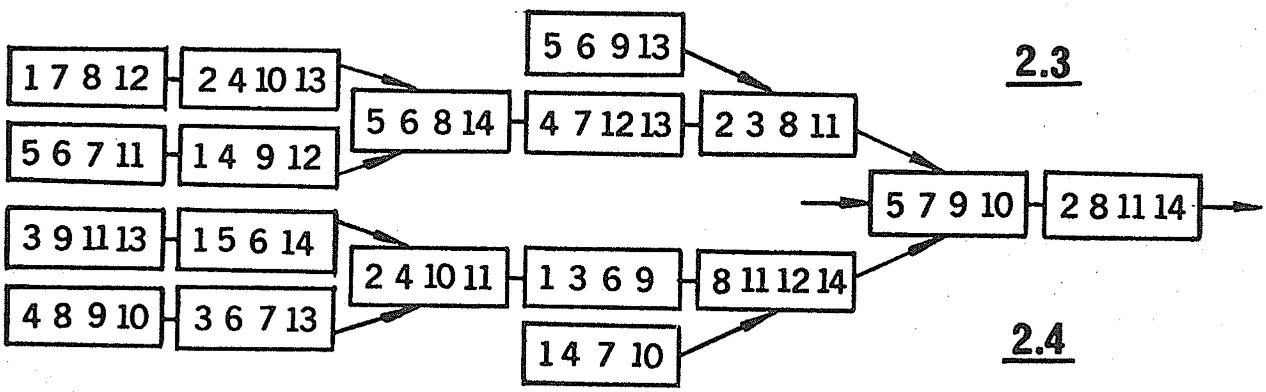


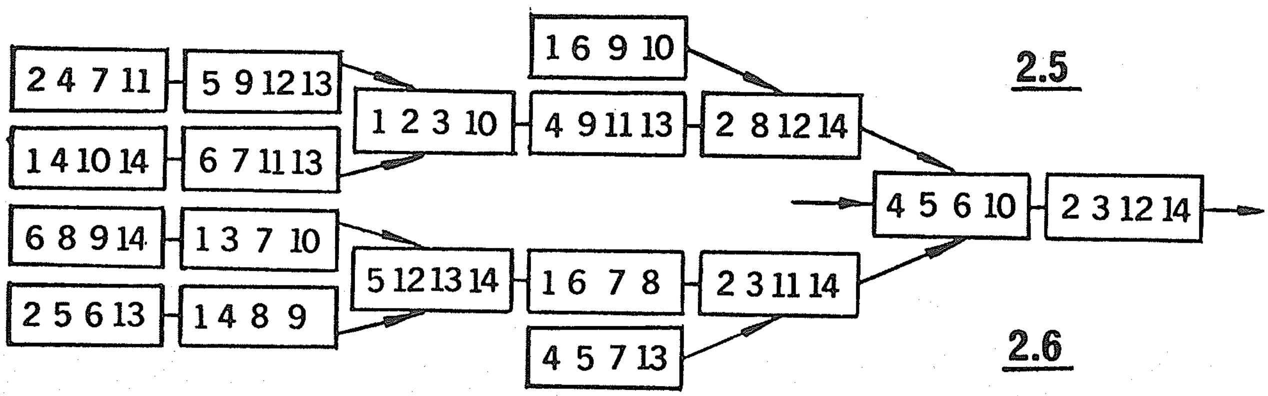


SECOND 6-CYCLE

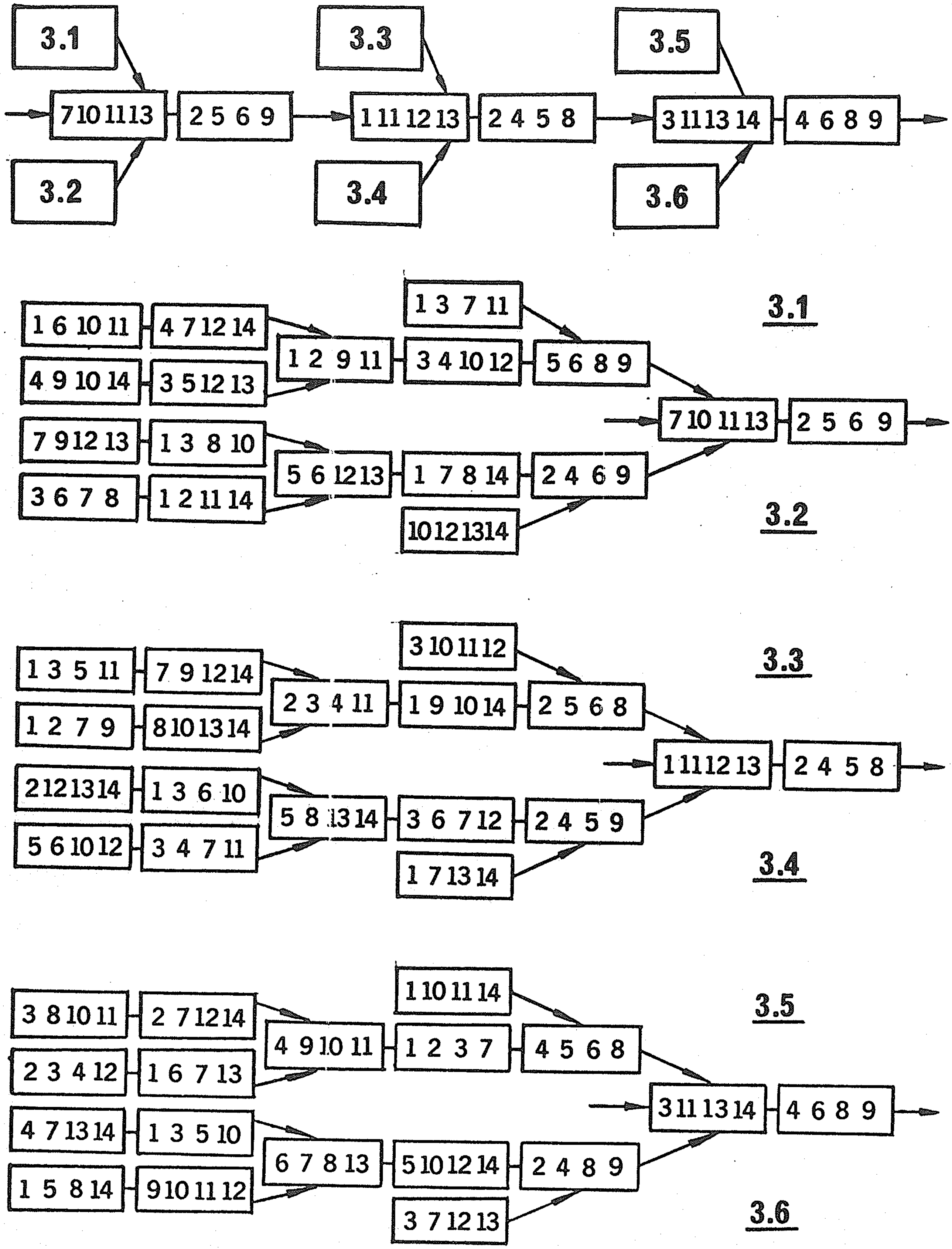




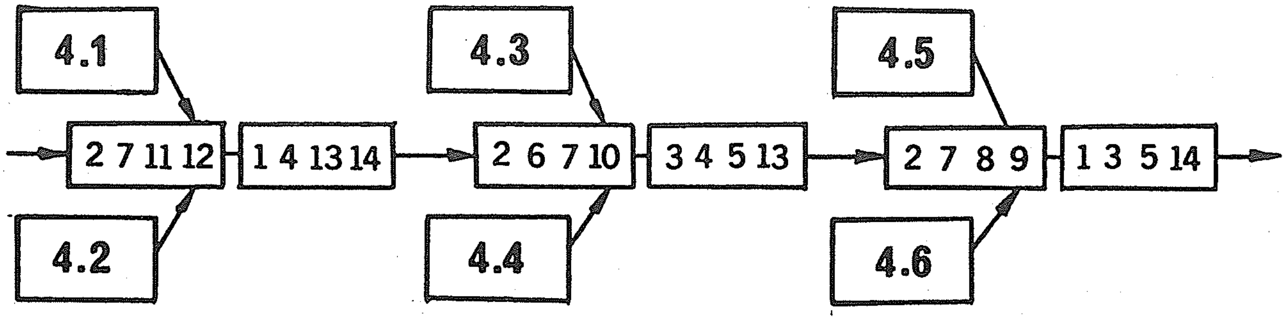


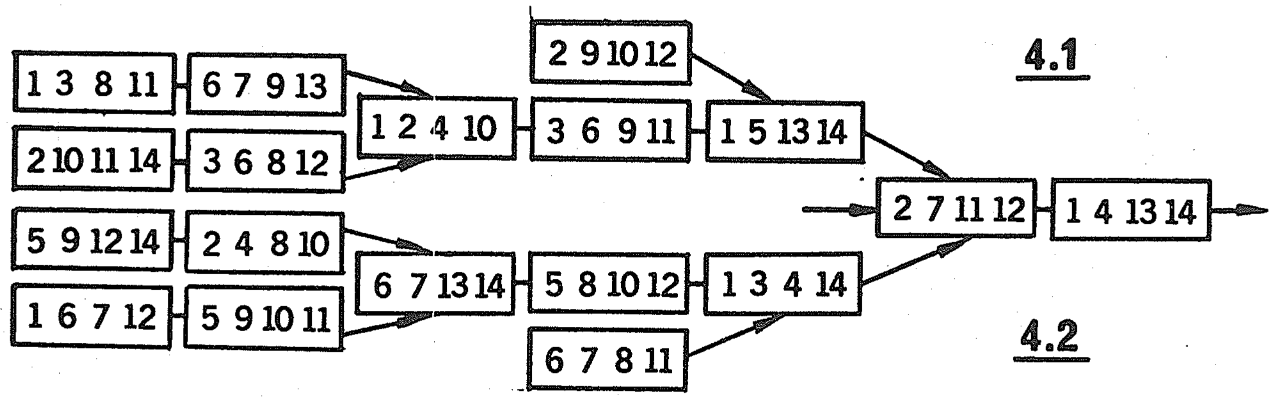


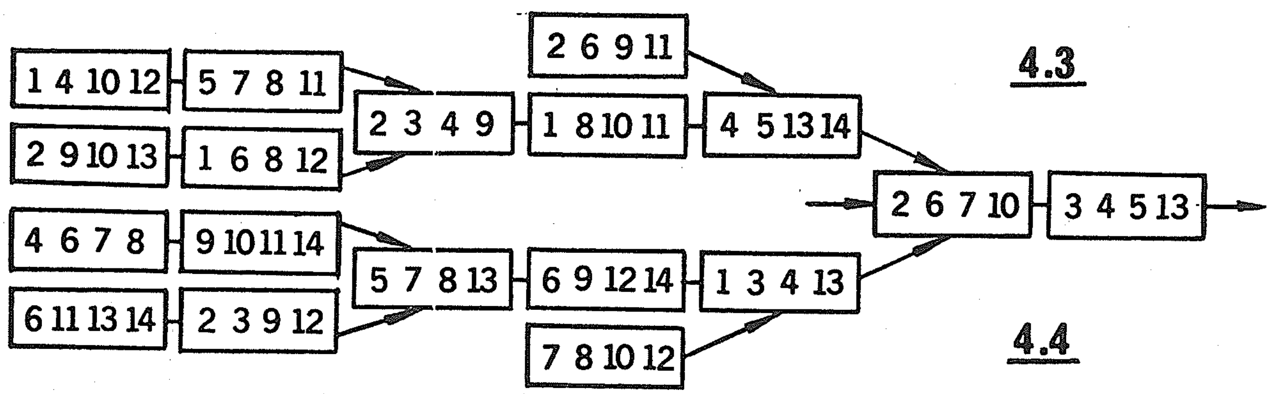
THIRD 6-CYCLE

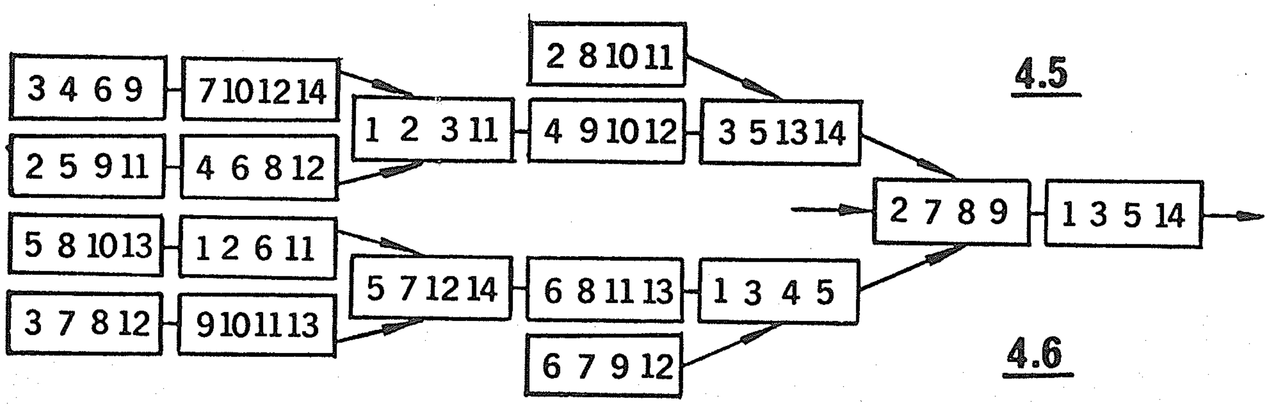


FOURTH 6-CYCLE

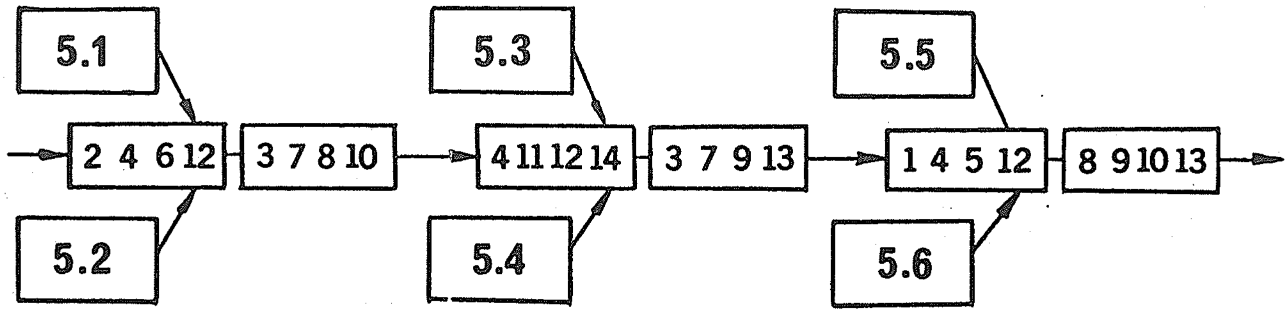


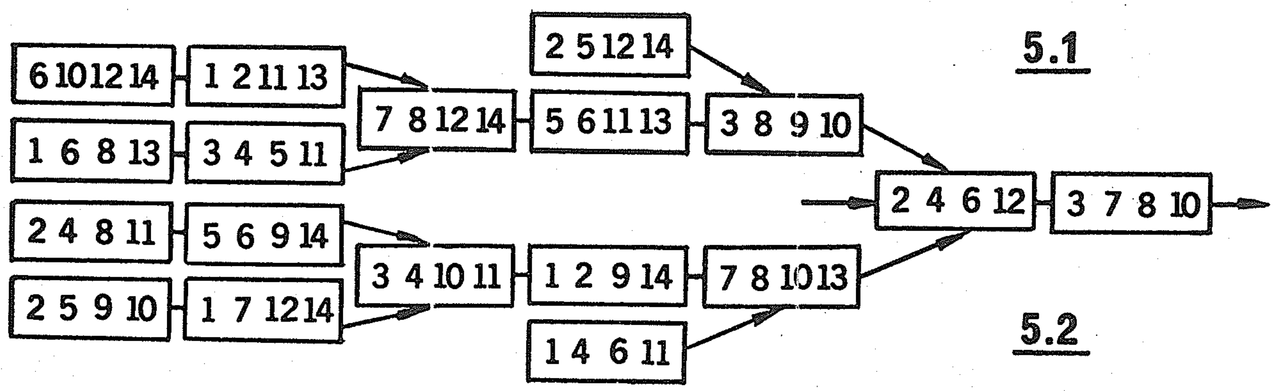


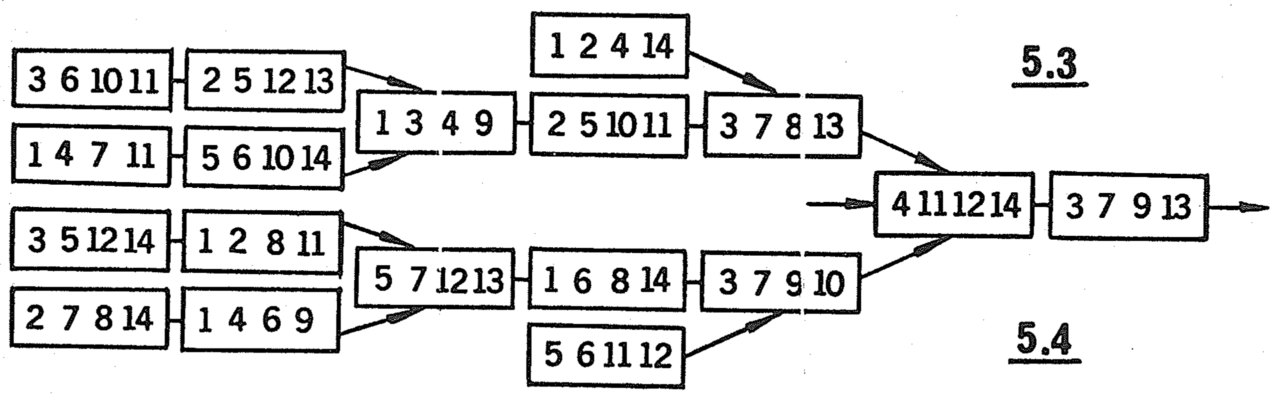


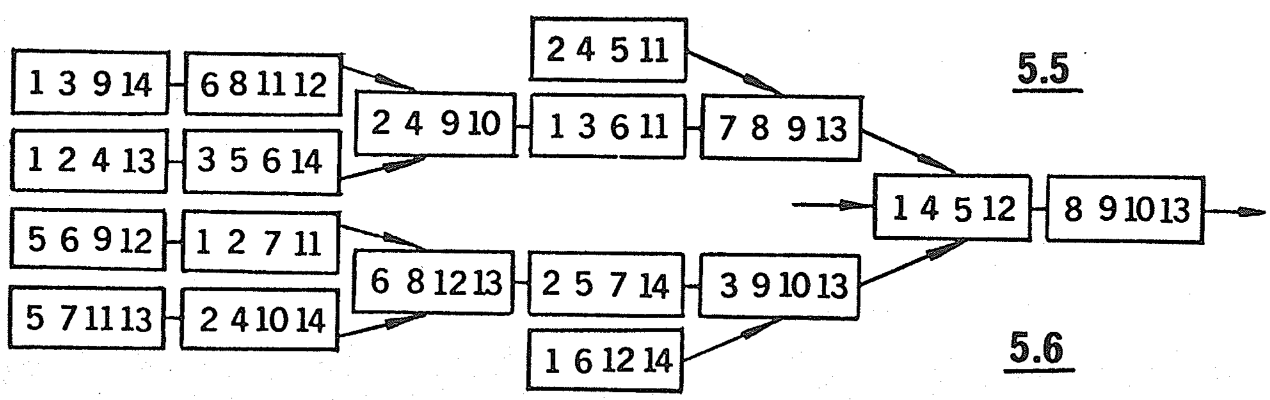


FIFTH 6-CYCLE

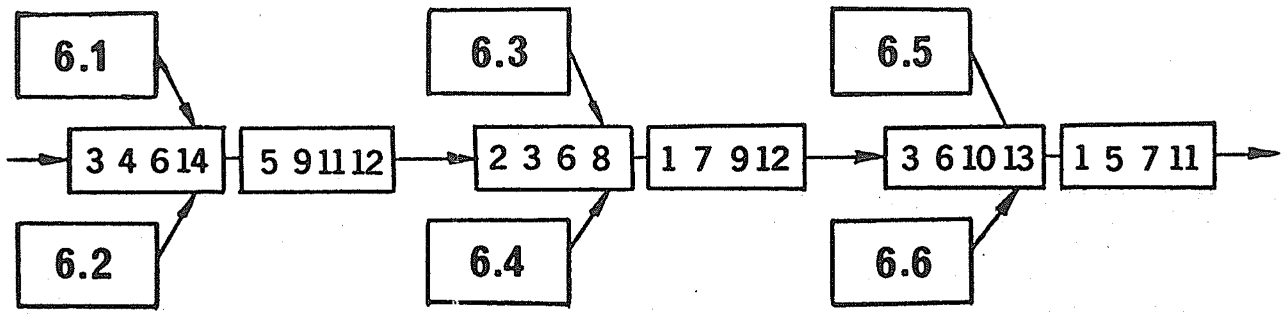


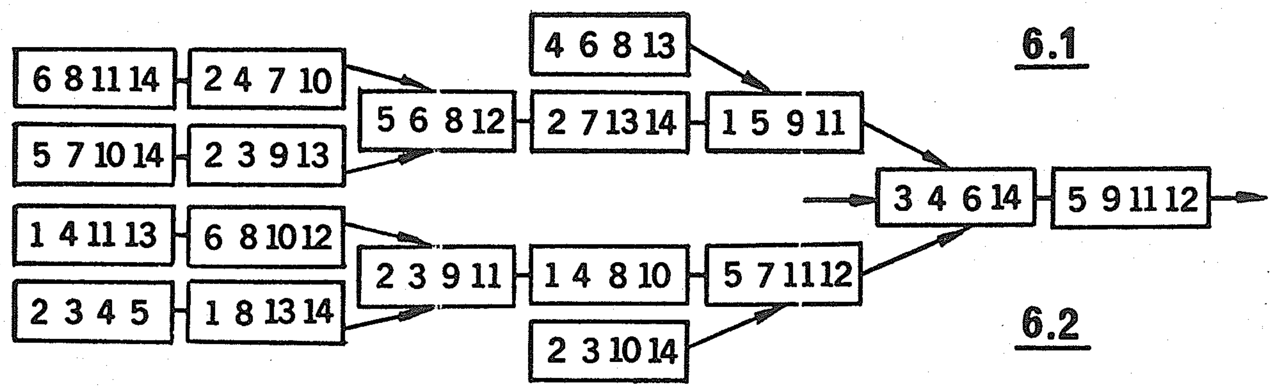


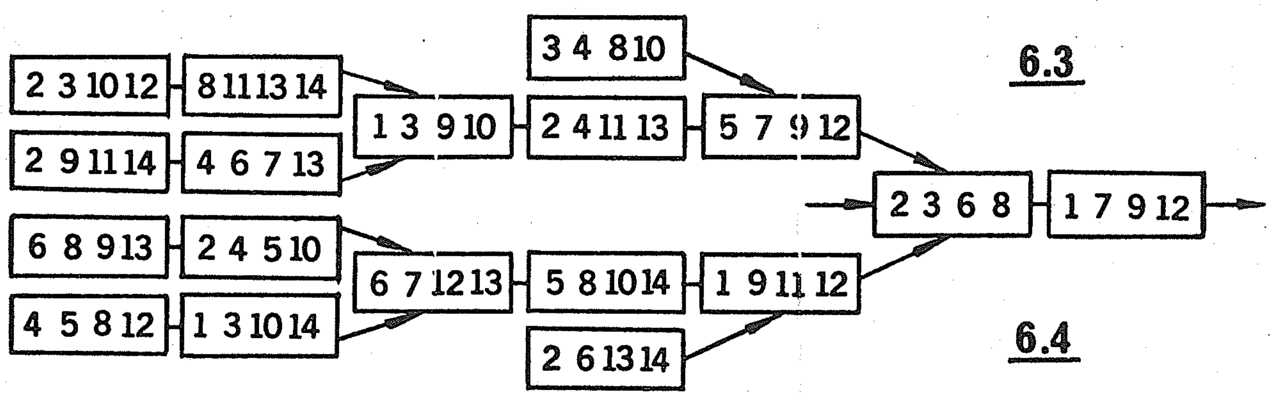


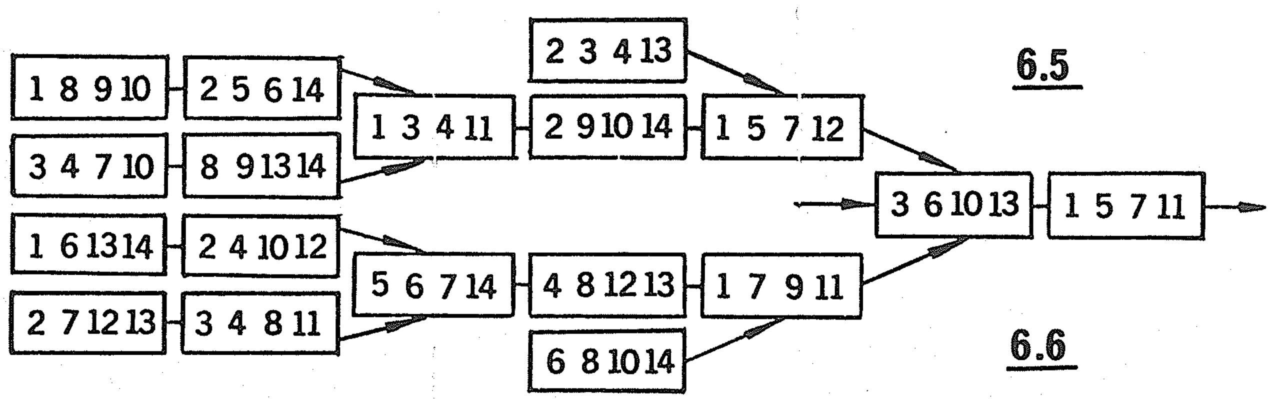


SIXTH 6-CYCLE

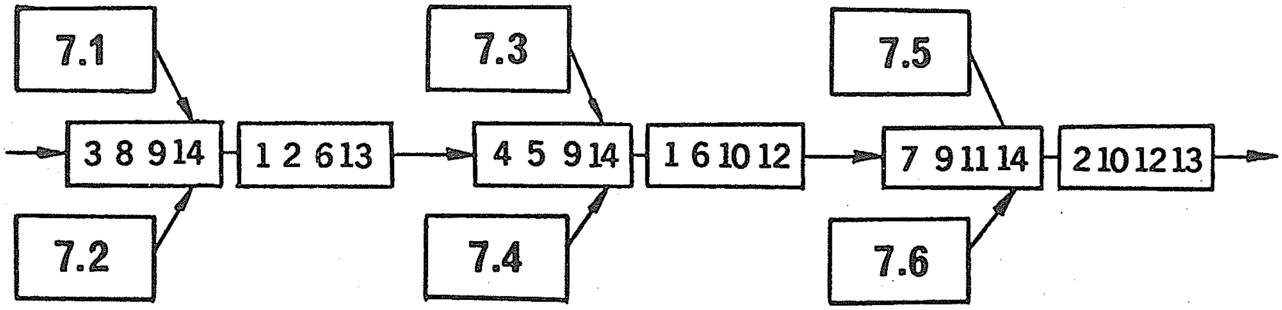


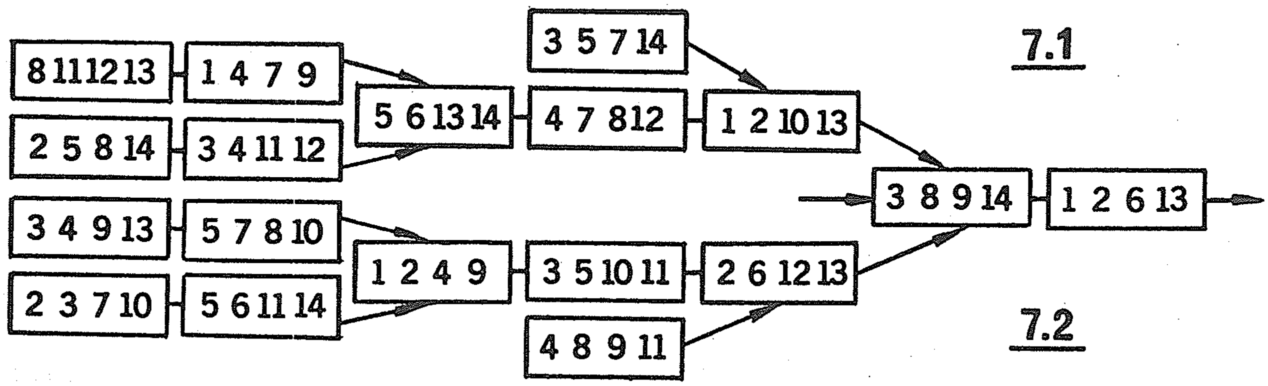






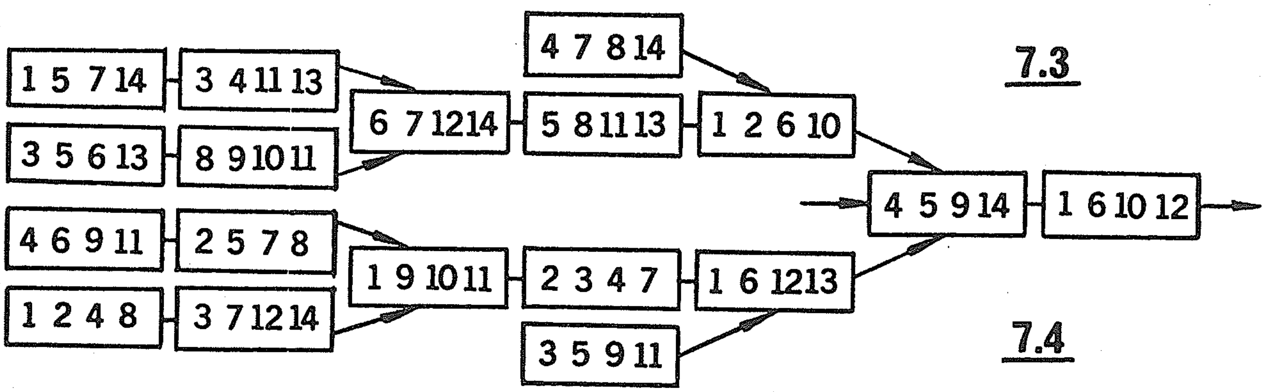
SEVENTH 6-CYCLE





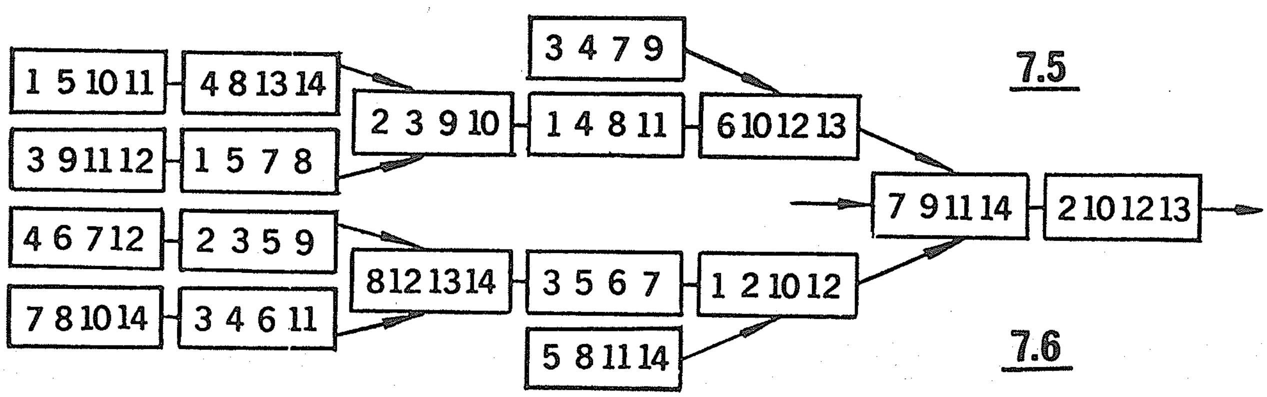
7.1

7.2



7.3

7.4



7.5

7.6

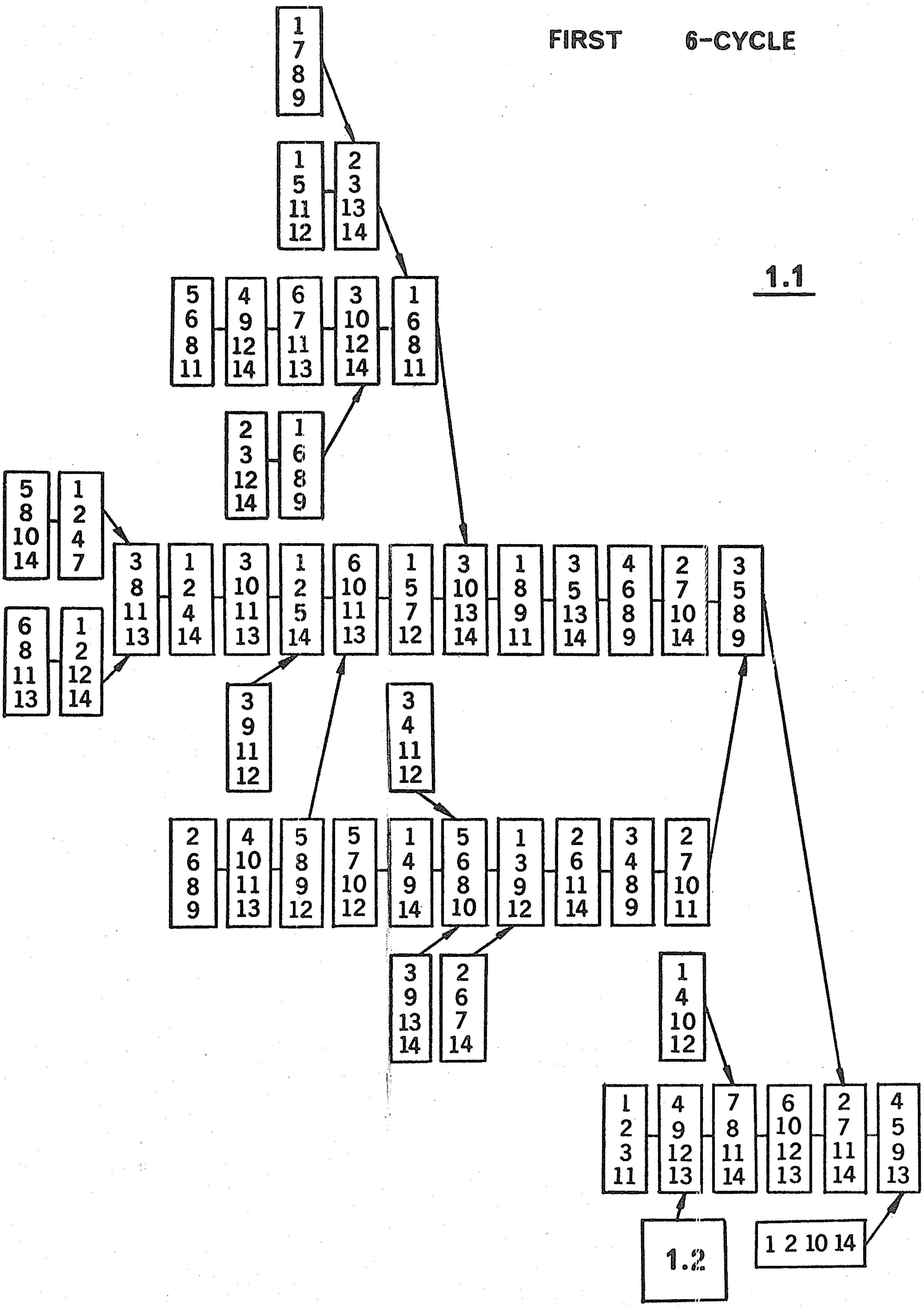
APPENDIX 5

Three 6-cycles & All Their Appendices

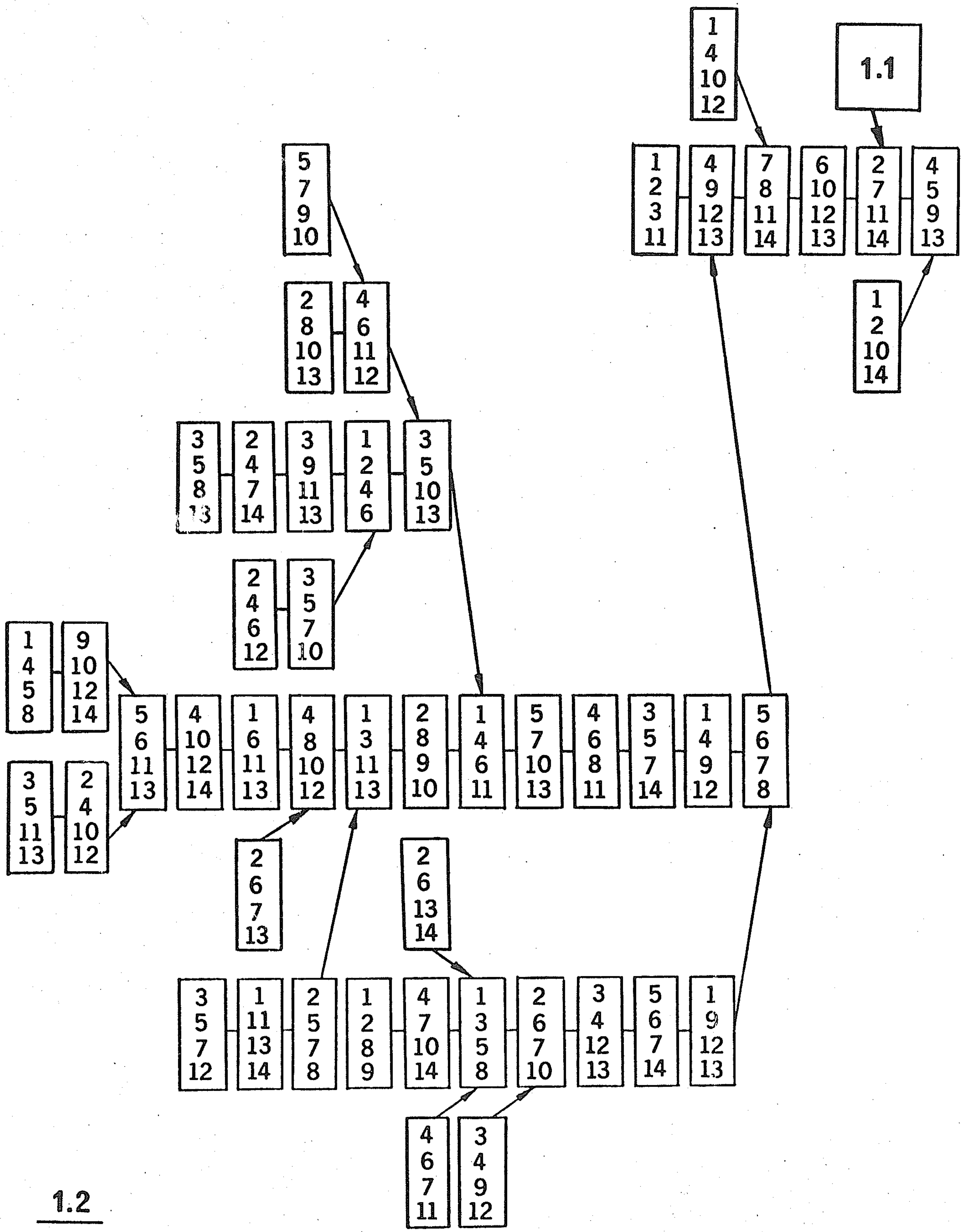
of Quadruple System B1

FIRST 6-CYCLE

1.1

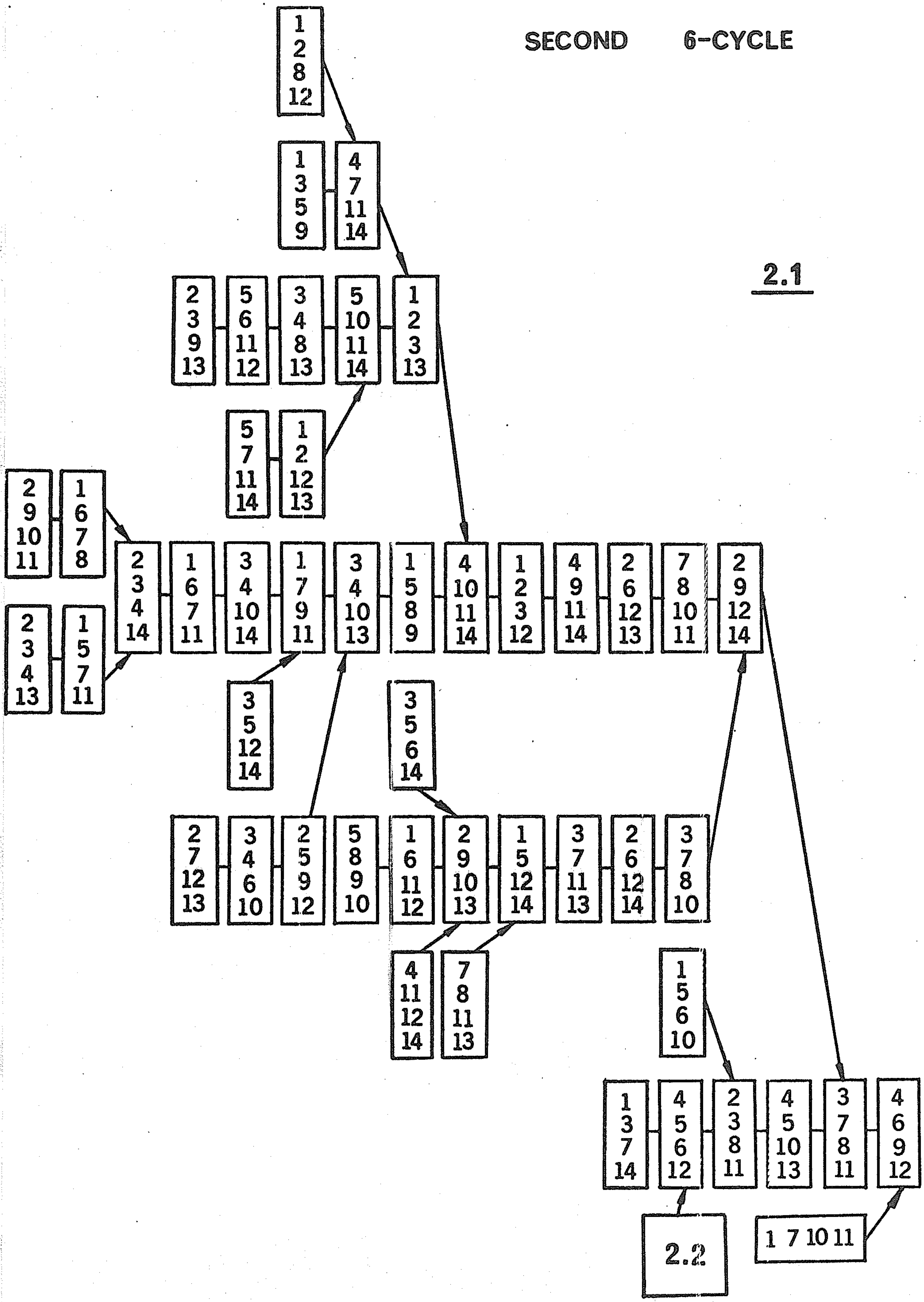


FIRST 6-CYCLE

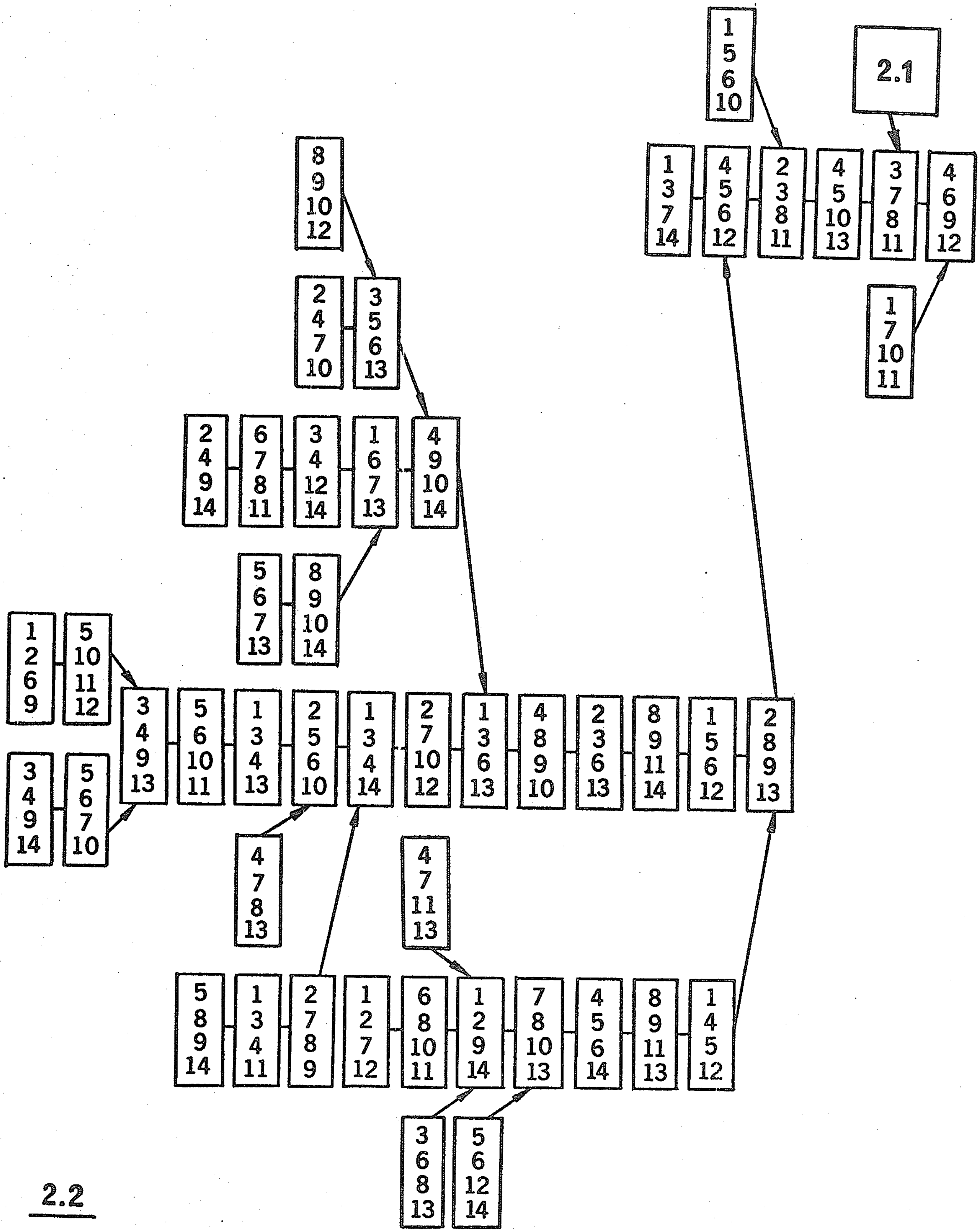


1.2

SECOND 6-CYCLE

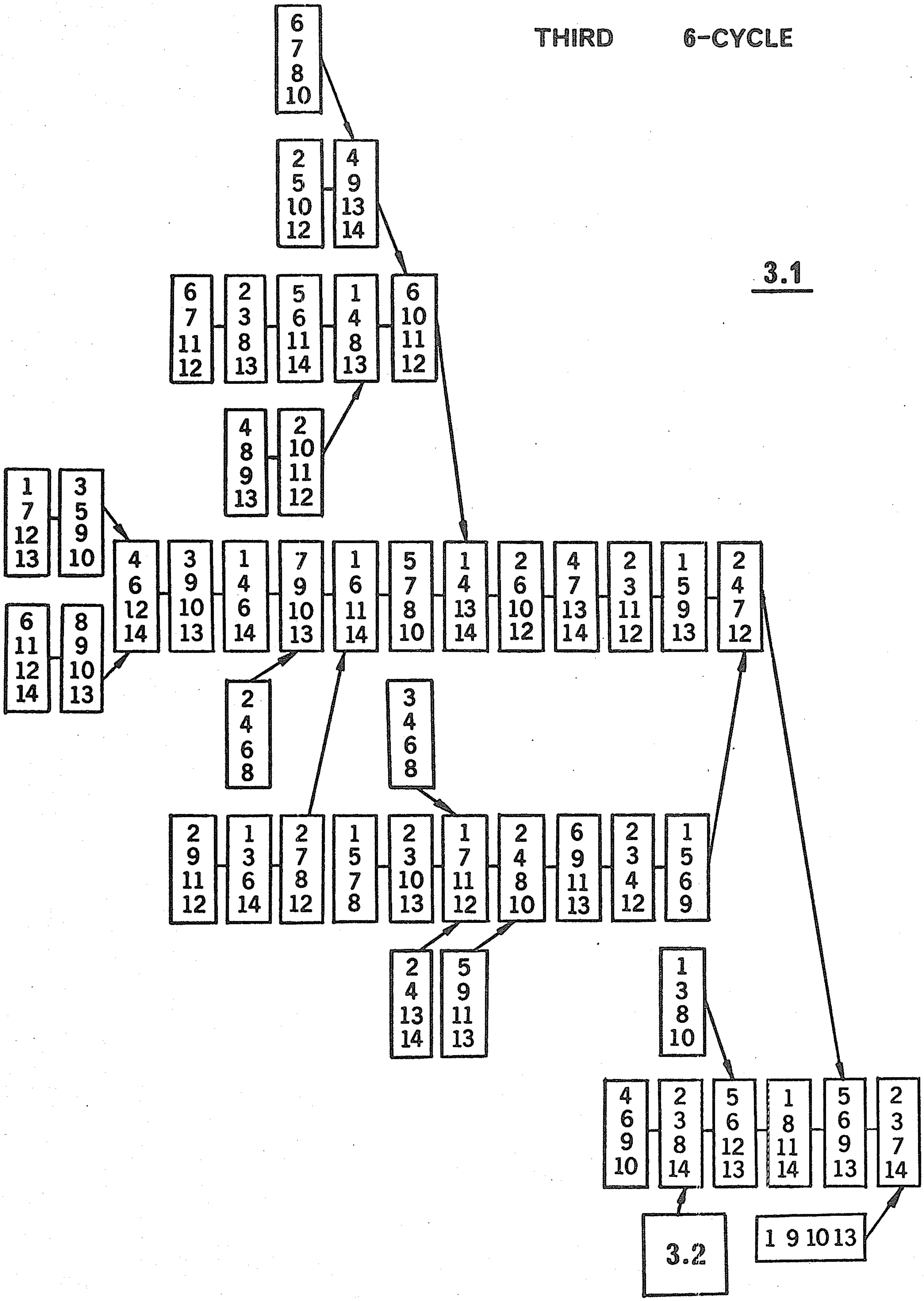


SECOND 6-CYCLE

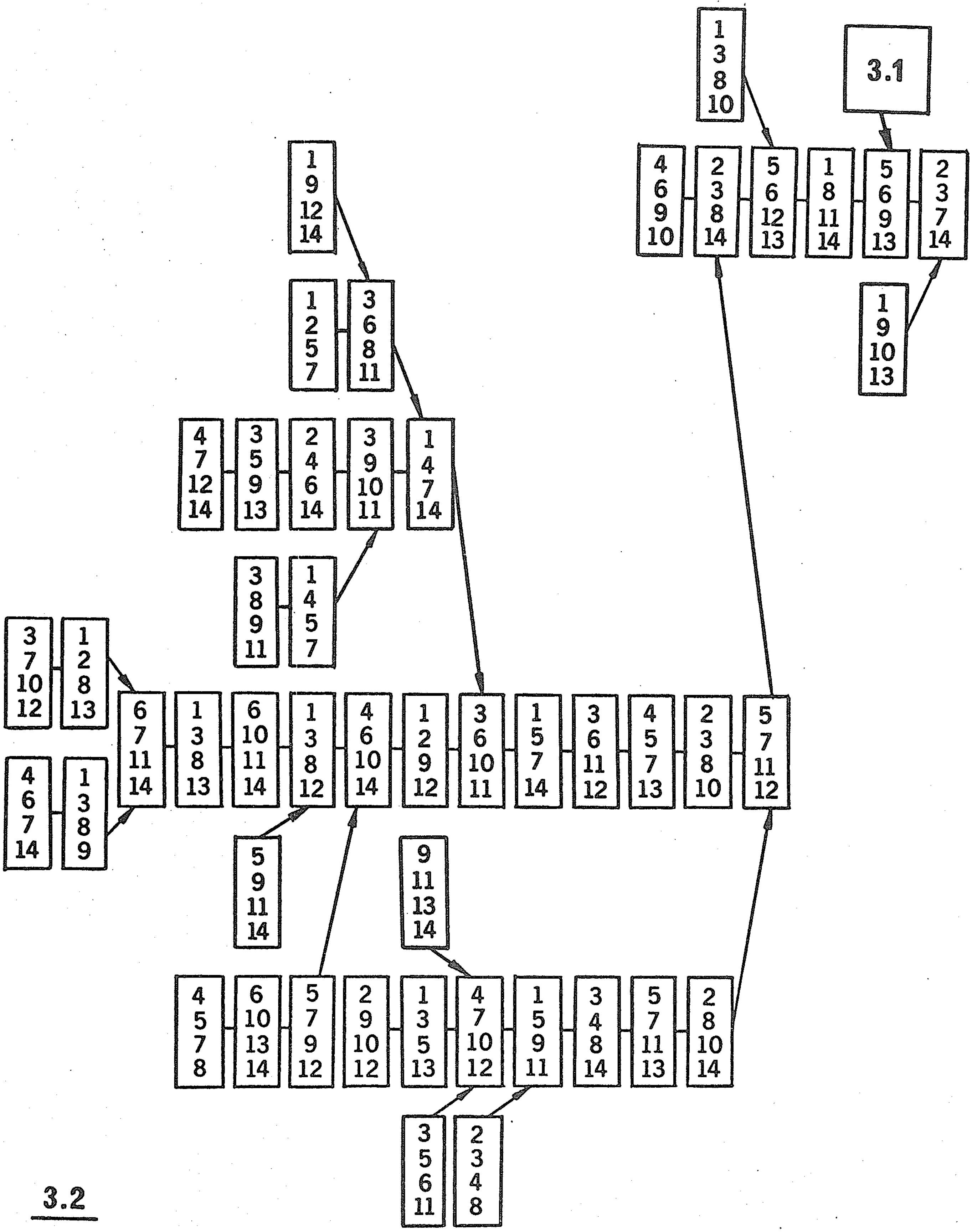


2.2

THIRD 6-CYCLE



THIRD 6-CYCLE

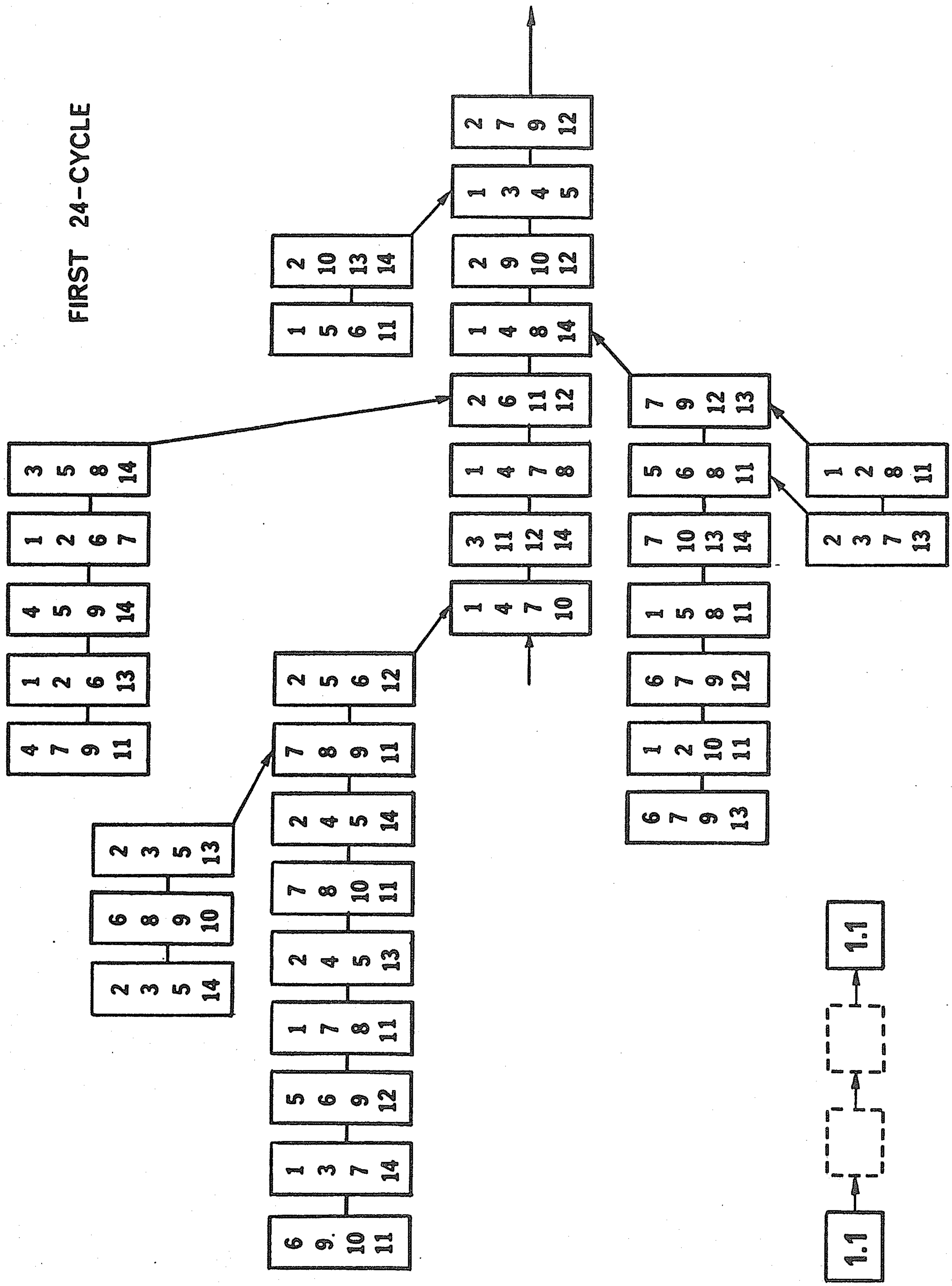


3.2

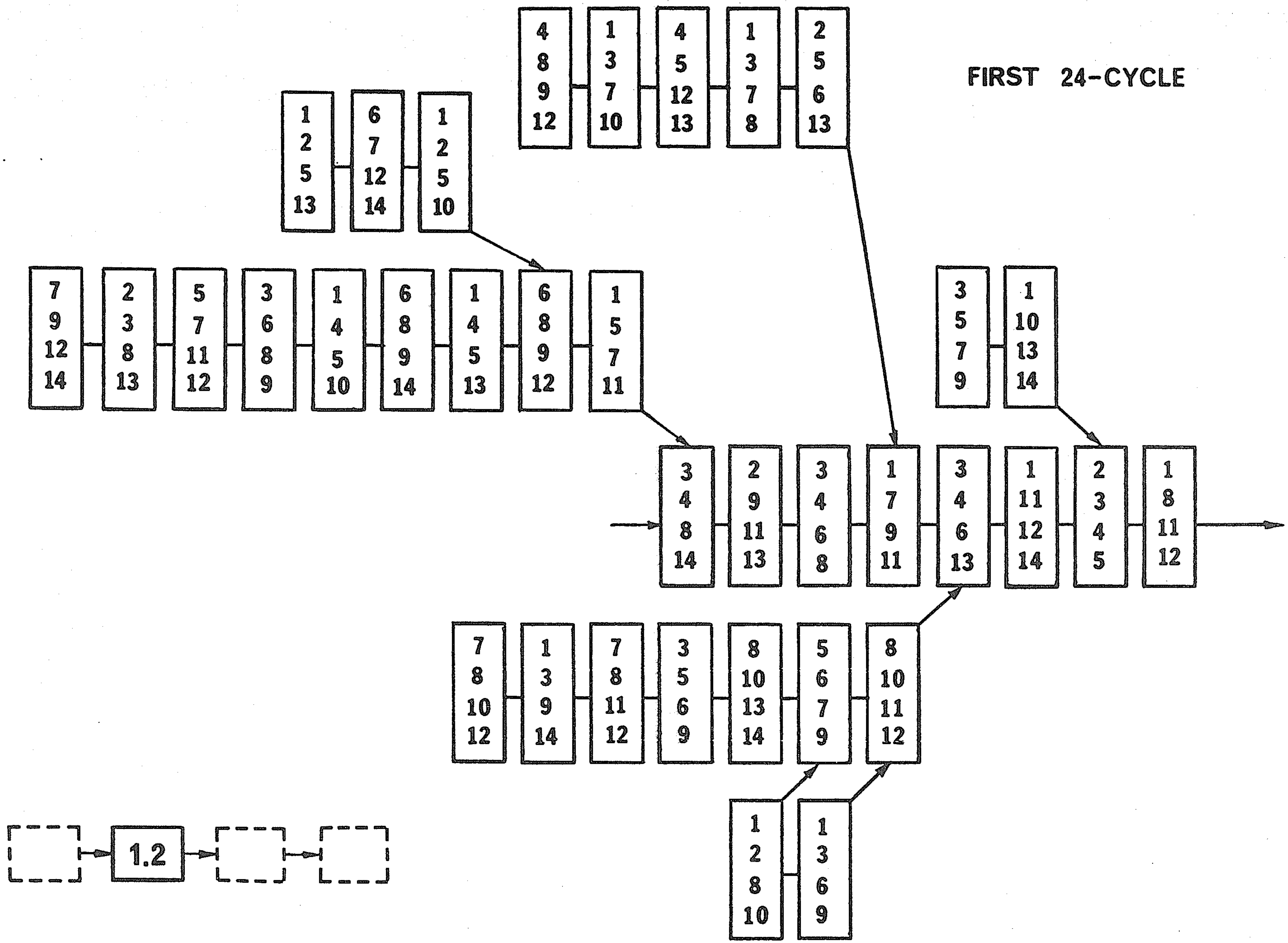
APPENDIX 6

Two 24-cycles & All Their Appendices

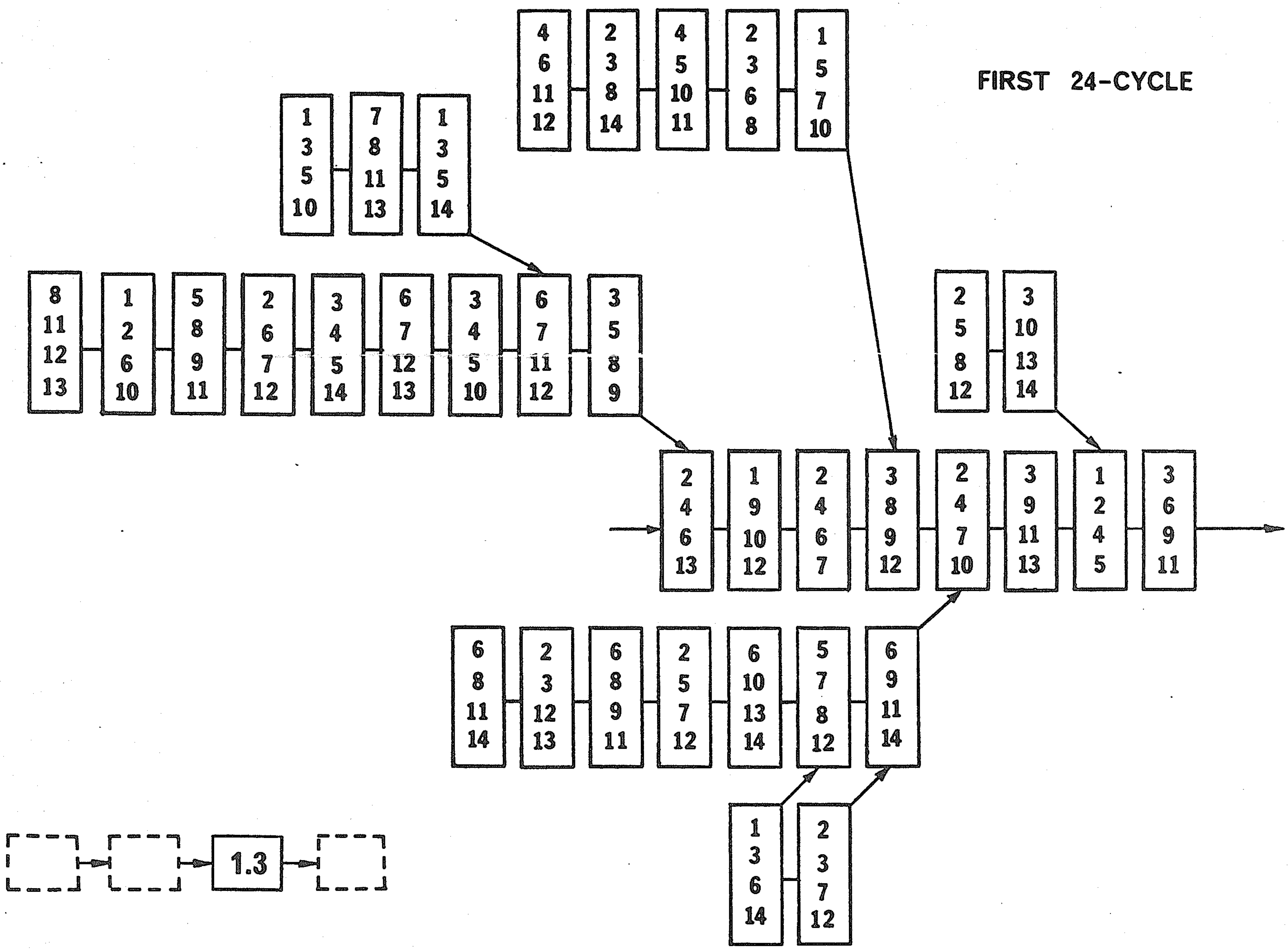
of Quadruple System B 2


FIRST 24-CYCLE
1.1
1.1

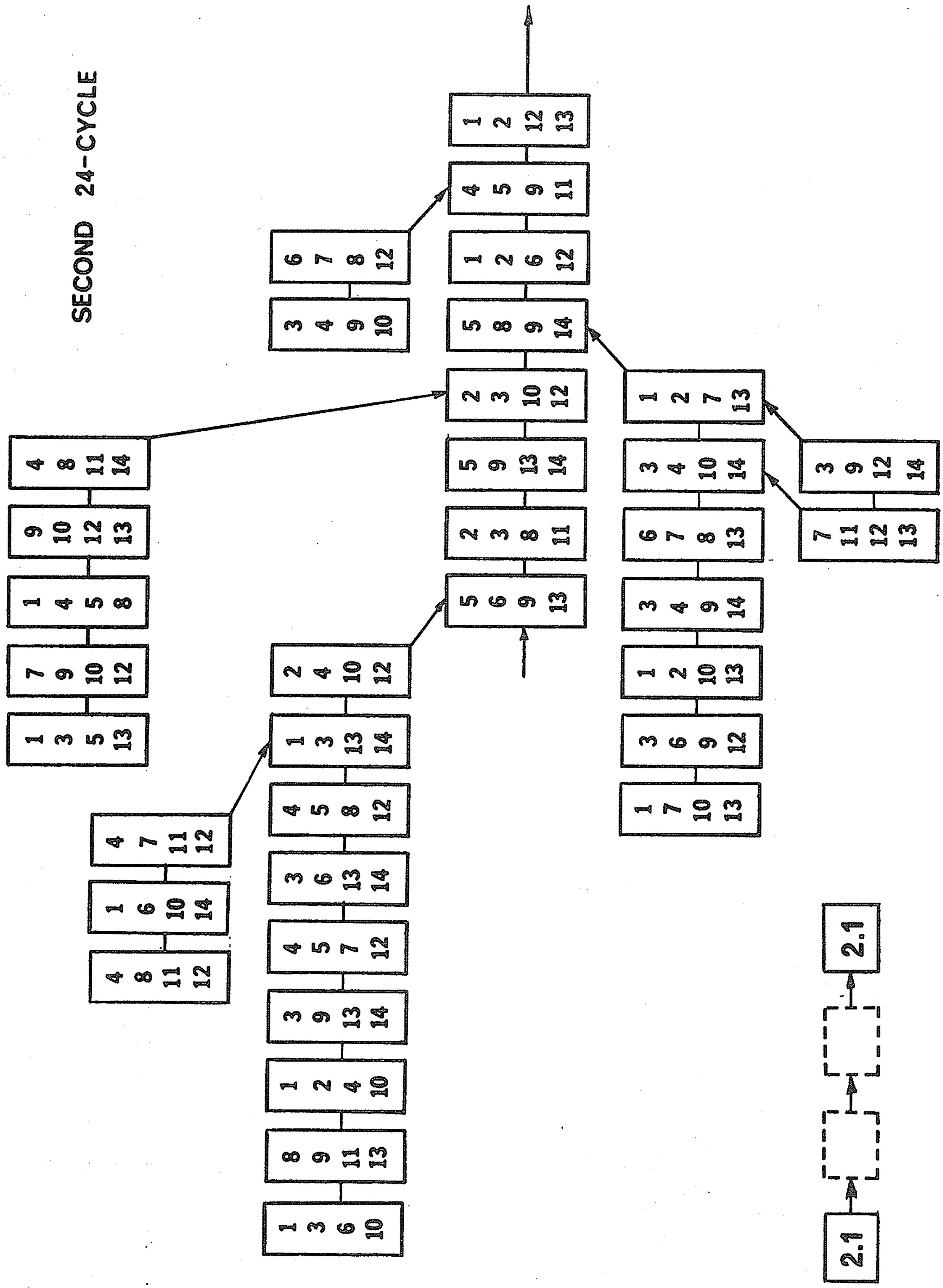
FIRST 24-CYCLE



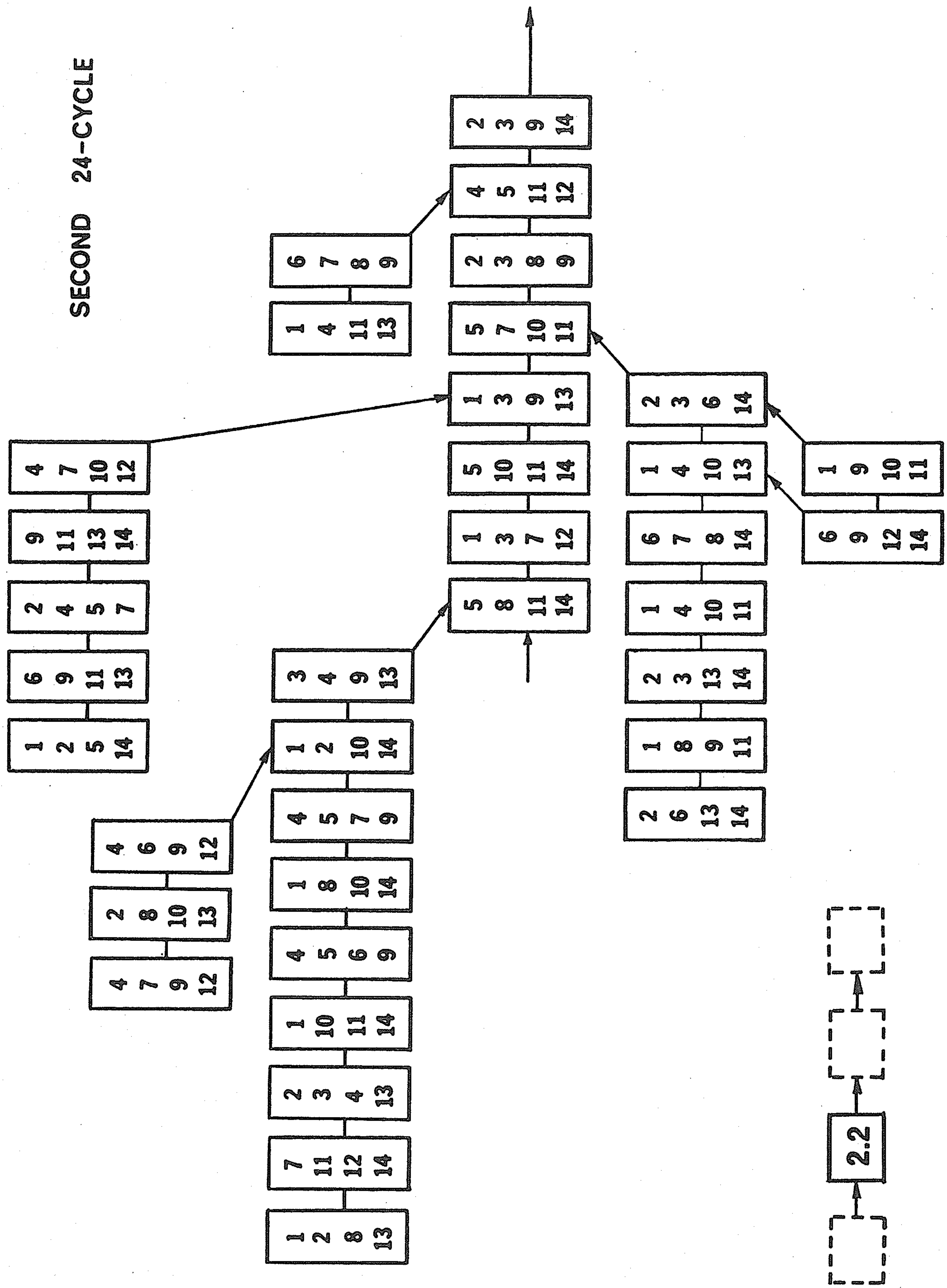
FIRST 24-CYCLE

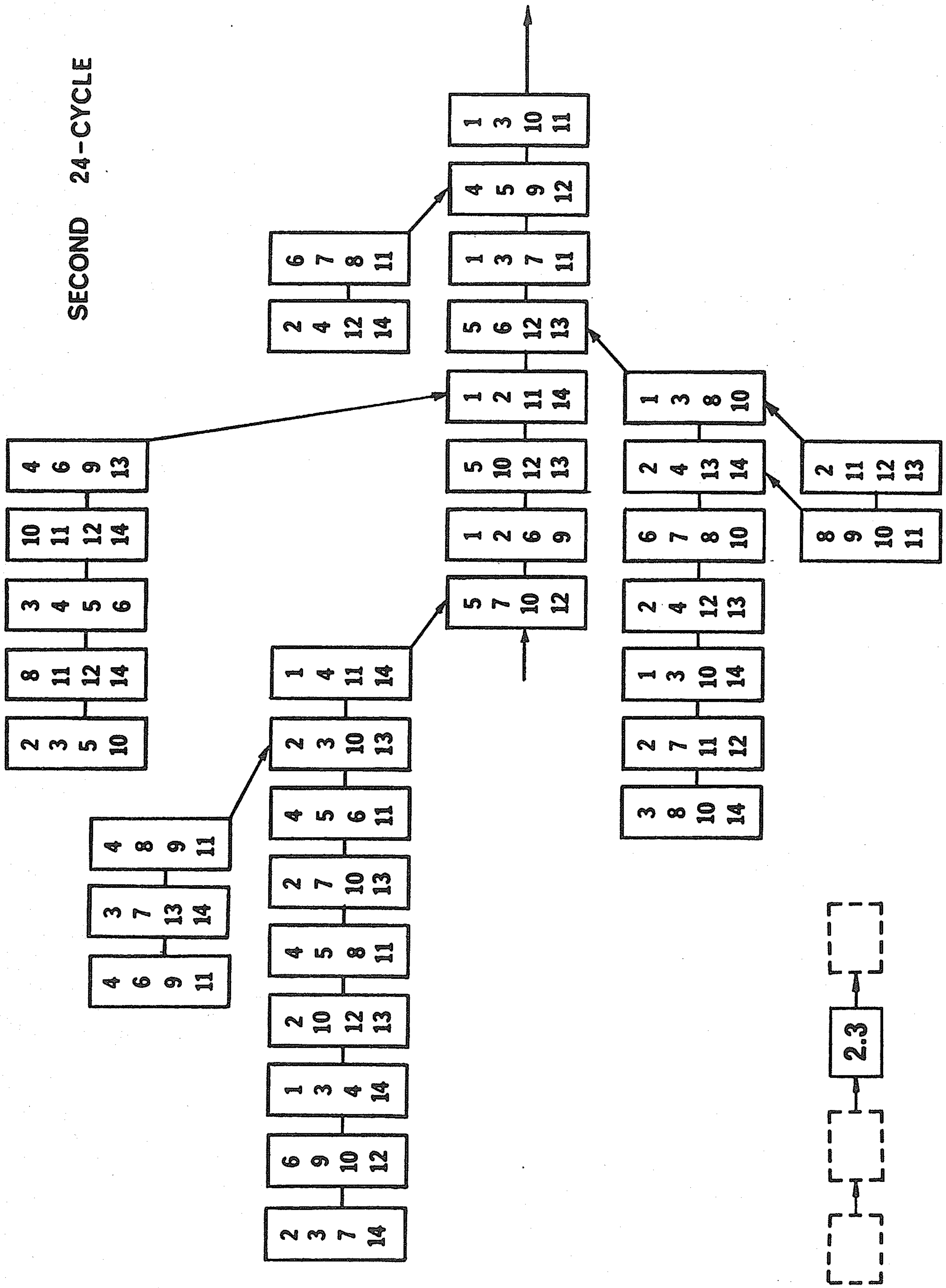


SECOND 24-CYCLE



SECOND 24-CYCLE





SECOND 24-CYCLE

APPENDIX 7

Program (COMB 5)

```
      IMPLICIT INTEGER (A-Y)
      COMMON / / A,D,C,EA(10),ED(10,10),EC(10,10,10),PASS,B(273,5),
     1 L(15),A1(15),D1(15),C1(15),APN(11),AHN(10,11),AKN(10,10,10),
     2 BC(10,10,26,5),OCH(12),OLT(10,10,10,12),JJ(10,10,10),
     3 CAP(10,10,10),COUN(10,10,10),TLL(12),KA(10,10,10),DB(10),
     4 CB(10,10),BD(10,91,5)
      COMMON /BL1/ CH(100,2),IS
      ZTT=0
      READ (5,79) ZZ
   79 FORMAT (F8.0)
      CALL $TRTM(ZTIME)
      DO 1 I=1,15
      L(I)=I
    1 CONTINUE
      READ (5,64) DIREC
   64 FORMAT (I1)
      IF (DIREC .EQ. 0) GO TO 3
      CALL READ2
      GO TO 90
    3 READ (5,59) ((B(P,G),G=1,5),P=1,91)
   59 FORMAT (35I2)
      WRITE (6,61) ((B(P,G),G=1,5),P=1,91)
   61 FORMAT (8(I4,4I3))
      A=1
    5 APN(A)=91
  122 CALL FINDA
      WRITE (6,62) (A1(I),I=1,15)
   62 FORMAT ('0A1 = ',15I4)
      DO 6 I=1,15
      IF (A1(I) .LT. 91) GO TO 7
    6 CONTINUE
      GO TO 130
    7 EA(A)=L(I)
      H=0
      CALL TOMA(H)
      D=1
      AHN(A,D)=H
      CALL REARRA
      HP=AHN(A,D)+1
      DO 8 H=HP,91
      BD(A,H,1)=EA(A)
    8 CONTINUE
    9 CALL FINDD
      WRITE (6,63) (D1(I),I=1,15)
   63 FORMAT (' D1 = ',15I4)
      DO 11 I=1,15
      IF (D1(I) .LT. 26) GO TO 12
   11 CONTINUE
      GO TO 120
   12 ED(A,D)=L(I)
   90 CONTINUE
      ZT=ZTIME
      CALL $TPTM(ZT)
      ZTT=ZTT+ZT
      IF (ZTT .GE. ZZ) GO TO 161
      CALL $TRTM(ZTIME)
```

```
      K=0
      CALL TOMD(K)
      C=1
      AKN(A,D,C)=K
      CALL REARRD
      KP=AKN(A,D,C)+1
      DO 13 K=KP,26
      BC(A,D,K,1)=EA(A)
      BC(A,D,K,2)=ED(A,D)
   13 CONTINUE
   15 CALL FINDC
      DO 16 I=1,15
      IF (C1(I) .LT. 6) GO TO 17
   16 CONTINUE
      GO TO 110
   17 KC=6-C1(I)
      EC(A,D,C)=L(I)
      KA(A,D,C)=AKN(A,D,C)+KC
      K2=KA(A,D,C)
      KP=AKN(A,D,C)+1
      DO 18 K=KP,K2
      BC(A,D,K,3)=EC(A,D,C)
   18 CONTINUE
      Q=0
      K1=AKN(A,D,C)
      DO 19 K=1,K1
      IF (EC(A,D,C) .EQ. BC(A,D,K,3)) GO TO 20
      IF (EC(A,D,C) .EQ. BC(A,D,K,4)) GO TO 21
      IF (EC(A,D,C) .EQ. BC(A,D,K,5)) GO TO 22
      GO TO 19
   20 Q=Q+1
      OCH(2*Q-1)=BC(A,D,K,4)
      OCH(2*Q)=BC(A,D,K,5)
      GO TO 19
   21 Q=Q+1
      OCH(2*Q-1)=BC(A,D,K,3)
      OCH(2*Q)=BC(A,D,K,5)
      GO TO 19
   22 Q=Q+1
      OCH(2*Q-1)=BC(A,D,K,3)
      OCH(2*Q)=BC(A,D,K,4)
   19 CONTINUE
      QQ=Q
      J=0
      DO 23 I=1,15
      IF (L(I) .EQ. EC(A,D,C)) GO TO 23
      IF (C1(I) .GE. 6) GO TO 23
      DO 24 Q=1,QQ
      IF (L(I) .EQ. OCH(2*Q-1) .OR. L(I) .EQ. OCH(2*Q)) GO TO 23
   24 CONTINUE
      J=J+1
      OLT(A,D,C,J)=L(I)
   23 CONTINUE
      JJ(A,D,C)=J
      J1=JJ(A,D,C)
      CAP(A,D,C)=1
```

```
      DO 26 F=1,J1,2
      CAP(A,D,C)=CAP(A,D,C)*F
   26 CONTINUE
      COUN(A,D,C)=0
      CALL CHECKO
      GO TO 27
   28 COUN(A,D,C)=COUN(A,D,C)+1
      IF (COUN(A,D,C) .EQ. CAP(A,D,C)) GO TO 29
      CALL COMB
      J1=JJ(A,D,C)
      J2=J1/2*2
      IF (J1 .NE. J2) GO TO 162
   27 DO 30 J=1,J1
      TLL(J)=OLT(A,D,C,J)
   30 CONTINUE
      CALL PERM
      PASS=0
      CALL CHECKC
      IF (PASS .EQ. 0) GO TO 31
      IF (COUN(A,D,C) .LT. CAP(A,D,C)) GO TO 28
   29 C=C-1
      IF (C .GT. 0) GO TO 128
      D=D-1
      IF (D .GT. 0) GO TO 32
      A=A-1
      IF (A .GT. 0) GO TO 33
      GO TO 150
  128 CALL CHECKO
      GO TO 28
   33 D=DB(A)
   32 C=CB(A,D)
      CALL CHECKO
      GO TO 28
   31 Y=0
      K1=AKN(A,D,C)+1
      K2=KA(A,D,C)
      DO 34 K=K1,K2
      Y=Y+1
      BC(A,D,K,4)=TLL(2*Y-1)
      BC(A,D,K,5)=TLL(2*Y)
   34 CONTINUE
      C=C+1
      AKN(A,D,C)=KA(A,D,C-1)
      GO TO 15
  110 CB(A,D)=C-1
      H=AHN(A,D)
      K1=AKN(A,D,1)+1
      DO 35 K=K1,26
      H=H+1
      DO 36 G=1,5
      BD(A,H,G)=BC(A,D,K,G)
   36 CONTINUE
   35 CONTINUE
      D=D+1
      AHN(A,D)=H
      GO TO 9
```

```
120 CONTINUE
    WRITE (6,51) A,D
 51 FORMAT ('0A = ',I4,5X,' D = ',I4)
 50 FORMAT (' H = ',I4,5X,5I4)
    WRITE (6,50) (H,(BD(A,H,G),G=1,5),H=1,91)
    DB(A)=D-1
    P=APN(A)
    H1=AHN(A,1)+1
    DO 37 H=H1,91
    P=P+1
    DO 38 G=1,5
    B(P,G)=BD(A,H,G)
 38 CONTINUE
 37 CONTINUE
    A=A+1
    APN(A)=P
    GO TO 122
130 IF (P .GE. 273) GO TO 140
150 WRITE (6,56)
    GO TO 160
 56 FORMAT ('0 THERE IS NO SUCH A SYSTEM ')
140 DO 39 P=1,273
    WRITE (6,57) P,(B(P,G),G=1,5)
 39 CONTINUE
 57 FORMAT (' P = ',I4,5X,5I4)
    WRITE (7,54) ((B(P,G),G=1,5),P=1,273)
 54 FORMAT (35I2)
    GO TO 160
162 WRITE (6,77) A,D,C
 77 FORMAT  ('0ERROR ON OLT, THE NUMBER IS NOT EVEN',3I4)
    GO TO 160
161 CALL WRITE2
160 CALL EXIT
    END

    SUBROUTINE FINDA
    IMPLICIT INTEGER (A-Y)
    COMMON / / A,D,C,EA(10),ED(10,10),EC(10,10,10),PASS,B(273,5),
   1 L(15),A1(15),D1(15),C1(15),APN(11),AHN(10,11),AKN(10,10,10),
   2 BC(10,10,26,5),OCH(12),OLT(10,10,10,12),JJ(10,10,10),
   3 CAP(10,10,10),COUN(10,10,10),TLL(12),KA(10,10,10),DB(10),
   4 CB(10,10),BD(10,91,5)
    DO 1 I=1,15
    A1(I)=0
  1 CONTINUE
  2 P1=APN(A)
    DO 3 I=1,15
    DO 4 P=1,P1
    DO 5 G=1,5
    IF (L(I) .NE. B(P,G)) GO TO 5
    A1(I)=A1(I)+1
  5 CONTINUE
  4 CONTINUE
  3 CONTINUE
    RETURN
    END
```

```
      SUBROUTINE FINDD
      IMPLICIT INTEGER (A-Y)
      COMMON / / A,D,C,EA(10),ED(10,10),EC(10,10,10),PASS,B(273,5),
     1 L(15),A1(15),D1(15),C1(15),APN(11),AHN(10,11),AKN(10,10,10),
     2 BC(10,10,26,5),OCH(12),OLT(10,10,10,12),JJ(10,10,10),
     3 CAP(10,10,10),COUN(10,10,10),TLL(12),KA(10,10,10),DB(10),
     4 CB(10,10),BD(10,91,5)
      DO 1 I=1,15
      D1(I)=0
    1 CONTINUE
    2 H1=AHN(A,D)
      DO 3 I=1,15
      DO 4 H=1,H1
      DO 5 G=1,5
      IF (L(I) .NE. BD(A,H,G)) GO TO 5
      D1(I)=D1(I)+1
    5 CONTINUE
    4 CONTINUE
    3 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE FINDC
      IMPLICIT INTEGER (A-Y)
      COMMON / / A,D,C,EA(10),ED(10,10),EC(10,10,10),PASS,B(273,5),
     1 L(15),A1(15),D1(15),C1(15),APN(11),AHN(10,11),AKN(10,10,10),
     2 BC(10,10,26,5),OCH(12),OLT(10,10,10,12),JJ(10,10,10),
     3 CAP(10,10,10),COUN(10,10,10),TLL(12),KA(10,10,10),DB(10),
     4 CB(10,10),BD(10,91,5)
      DO 1 I=1,15
      C1(I)=0
    1 CONTINUE
    2 K1=AKV(A,D,C)
      DO 3 I=1,15
      DO 4 K=1,K1
      DO 5 G=1,5
      IF (L(I) .NE. BC(A,D,K,G)) GO TO 5
      C1(I)=C1(I)+1
    5 CONTINUE
    4 CONTINUE
    3 CONTINUE
      RETJRN
      END
```

```
      SUBROUTINE TOMA(H)
      IMPLICIT INTEGER (A-Y)
      COMMON / / A,D,C,EA(10),ED(10,10),EC(10,10,10),PASS,B(273,5),
     1 L(15),A1(15),D1(15),C1(15),APN(11),AHN(10,11),AKN(10,10,10),
     2 BC(10,10,26,5),OCH(12),OLT(10,10,10,12),JJ(10,10,10),
     3 CAP(10,10,10),COUN(10,10,10),TLL(12),KA(10,10,10),DB(10),
     4 CB(10,10),BD(10,91,5)
      P1=APN(A)
```

```
      DO 1 P=1,P1
      DO 2 G=1,5
      IF (B(P,G) .EQ. EA(A)) GO TO 3
    2 CONTINUE
      GO TO 1
    3 H=H+1
      DO 4 G=1,5
      BD(A,H,G)=B(P,G)
    4 CONTINUE
    1 CONTINUE
      RETURN
      END


      SUBROUTINE TOMD(K)
      IMPLICIT INTEGER (A-Y)
      COMMON / / A,D,C,EA(10),ED(10,10),EC(10,10,10),PASS,B(273,5),
     1 L(15),A1(15),D1(15),C1(15),APN(11),AHN(10,11),AKN(10,10,10),
     2 BC(10,10,26,5),OCH(12),OLT(10,10,10,12),JJ(10,10,10),
     3 CAP(10,10,10),COUN(10,10,10),TLL(12),KA(10,10,10),DB(10),
     4 CB(10,10),BD(10,91,5)
      H1=AHN(A,D)
      DO 1  H=1,H1
      DO 2 G=1,5
      IF (BD(A,H,G) .EQ. ED(A,D)) GO TO 3
    2 CONTINUE
      GO TO 1
    3 K=K+1
      DO 4 G=1,5
      BC(A,D,K,G)=BD(A,H,G)
    4 CONTINUE
    1 CONTINUE
      RETURN
      END


      SUBROUTINE PERM
      IMPLICIT INTEGER (A-Y)
      COMMON / / A,D,C,EA(10),ED(10,10),EC(10,10,10),PASS,B(273,5),
     1 L(15),A1(15),D1(15),C1(15),APN(11),AHN(10,11),AKN(10,10,10),
     2 BC(10,10,26,5),OCH(12),OLT(10,10,10,12),JJ(10,10,10),
     3 CAP(10,10,10),COUN(10,10,10),TLL(12),KA(10,10,10),DB(10),
     4 CB(10,10),BD(10,91,5)
      P=JJ(A,D,C)/2
      DO 1 H=1,P
      IF (TLL(2*H-1) .LE. TLL(2*H)) GO TO 1
      CPU=TLL(2*H-1)
      TLL(2*H-1)=TLL(2*H)
      TLL(2*H)=CPU
    1 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE REARRA
      IMPLICIT INTEGER (A-Y)
      COMMON / / A,D,C,EA(10),ED(10,10),EC(10,10,10),PASS,B(273,5),
     1 L(15),A1(15),D1(15),C1(15),APN(11),AHN(10,11),AKN(10,10,10),
     2 BC(10,10,26,5),OCH(12),OLT(10,10,10,12),JJ(10,10,10),
     3 CAP(10,10,10),COUN(10,10,10),TLL(12),KA(10,10,10),DB(10),
     4 CB(10,10),BD(10,91,5)
      HH=AHN(A,D)
      DO 1 H=1,HH
      DO 2 G=2,5
      IF (EA(A) .EQ. BD(A,H,G)) GO TO 3
      GO TO 2
    3 GG=G+1
    4 GG=GG-1
      IF (GG .EQ. 1) GO TO 5
      BD(A,H,GG)=BD(A,H,GG-1)
      GO TO 4
    5 BD(A,H,1)=EA(A)
    2 CONTINUE
    1 CONTINUE
      RETURN
      END

      SUBROUTINE REARRD
      IMPLICIT INTEGER (A-Y)
      COMMON / / A,D,C,EA(10),ED(10,10),EC(10,10,10),PASS,B(273,5),
     1 L(15),A1(15),D1(15),C1(15),APN(11),AHN(10,11),AKN(10,10,10),
     2 BC(10,10,26,5),OCH(12),OLT(10,10,10,12),JJ(10,10,10),
     3 CAP(10,10,10),COUN(10,10,10),TLL(12),KA(10,10,10),DB(10),
     4 CB(10,10),BD(10,91,5)
      KK=AKN(A,D,C)
      DO 1 K=1,KK
      DO 2 G=2,5
      IF (ED(A,D) .EQ. BC(A,D,K,G)) GO TO 3
      GO TO 2
    3 GG=G+1
    4 GG=GG-1
      IF (GG .EQ. 2) GO TO 5
      BC(A,D,K,GG)=BC(A,D,K,GG-1)
      GO TO 4
    5 BC(A,D,K,2)=ED(A,D)
    2 CONTINUE
    1 CONTINUE
      RETURN
      END

      SUBROUTINE CHECKO
      IMPLICIT INTEGER (A-Y)
      COMMON / / A,D,C,EA(10),ED(10,10),EC(10,10,10),PASS,B(273,5),
     1 L(15),A1(15),D1(15),C1(15),APN(11),AHN(10,11),AKN(10,10,10),
     2 BC(10,10,26,5),OCH(12),OLT(10,10,10,12),JJ(10,10,10),
     3 CAP(10,10,10),COUN(10,10,10),TLL(12),KA(10,10,10),DB(10),
     4 CB(10,10),BD(10,91,5)
      COMMON /BL1/ CH(100,2),IS
      K1=AKN(A,D,C)
```

```
      H1=AHN(A,D)
      P1=APN(A)
      S=0
      DO 1 K=1,K1
      S=S+1
      CH(S,1)=BC(A,D,K,4)
      CH(S,2)=BC(A,D,K,5)
    1 CONTINUE
      DO 2 H=1,H1
      IF (BD(A,H,3) .NE. EC(A,D,C)) GO TO 2
      S=S+1
      CH(S,1)=BD(A,H,4)
      CH(S,2)=BD(A,H,5)
    2 CONTINUE
      DO 3 P=1,P1
      IF (B(P,2) .NE. ED(A,D) .OR. B(P,3) .NE. EC(A,D,C)) GO TO 3
      S=S+1
      CH(S,1)=B(P,4)
      CH(S,2)=B(P,5)
    3 CONTINUE
      IS=S
      RETURN
      END


      SUBROUTINE CHECKC
      IMPLICIT INTEGER (A-Y)
      COMMON / / A,D,C,EA(10),ED(10,10),EC(10,10,10),PASS,B(273,5),
     1 L(15),A1(15),D1(15),C1(15),APN(11),AHN(10,11),AKN(10,10,10),
     2 BC(10,10,26,5),OCH(12),OLT(10,10,10,12),JJ(10,10,10),
     3 CAP(10,10,10),COUN(10,10,10),TLL(12),KA(10,10,10),DB(10),
     4 CB(10,10),BD(10,91,5)
      COMMON /BL1/ CH(100,2),IS
      P=JJ(A,D,C)/2
      DO 1 H=1,P
      DO 2 S=1,IS
      IF (TLL(2*H-1) .EQ. CH(S,1) .AND. TLL(2*H) .EQ. CH(S,2)) GO TO 3
    2 CONTINUE
    1 CONTINUE
      GO TO 4
    3 PASS=1
    4 RETURN
      END


      SUBROUTINE COMB
      IMPLICIT INTEGER (A-Y)
      COMMON / / A,D,C,EA(10),ED(10,10),EC(10,10,10),PASS,B(273,5),
     1 L(15),A1(15),D1(15),C1(15),APN(11),AHN(10,11),AKN(10,10,10),
     2 BC(10,10,26,5),OCH(12),OLT(10,10,10,12),JJ(10,10,10),
     3 CAP(10,10,10),COUN(10,10,10),TLL(12),KA(10,10,10),DB(10),
     4 CB(10,10),BD(10,91,5)
      DIMENSION LL(12)
      KK=JJ(A,D,C)
      KOUN=COUN(A,D,C)
      DO 13 I=1,KK
```

```
      LL(I)=OLT(A,D,C,I)
   13 CONTINUE
      N=KK-2
      STOR=LL(N)
      J=KK-1
      DO 10 I=N,J
      LL(I)=LL(I+1)
   10 CONTINUE
      LL(KK)=STOR
      AA=KOJN-KOUN/3*3
      BB=KOUN-KOUN/15*15
      CC=KOUN-KOUN/105*105
      DD=KOUN-KOUN/945*945
      EE=KOUN-KOUN/10395*10395
      IF (EE .EQ. 0) GO TO 5
      IF (DD .EQ. 0) GO TO 4
      IF (CC .EQ. 0) GO TO 3
      IF (BB .EQ. 0) GO TO 2
      IF (AA .EQ. 0) GO TO 1
      GO TO 11
    1 G=2
      GO TO 7
    2 G=4
      GO TO 7
    3 G=6
      GO TO 7
    4 G=8
      GO TO 7
    5 G=10
    7 N=N-2
      STOR=LL(N)
      DO 20 I=N,J
      LL(I)=LL(I+1)
   20 CONTINUE
      LL(K<)=STOR
      M=KK-2-G
      IF (N .EQ.M) GO TO 11
      GO TO 7
   11 DO 12 I=1,KK
      OLT(A,D,C,I)=LL(I)
   12 CONTINUE
      RETURN
      END


      SUBROUTINE WRITE2
      IMPLICIT INTEGER (A-Y)
      COMMON / / A,D,C,EA(10),ED(10,10),EC(10,10,10),PASS,B(273,5),
     1 L(15),A1(15),D1(15),C1(15),APN(11),AHN(10,11),AKN(10,10,10),
     2 BC(10,10,26,5),OCH(12),OLT(10,10,10,12),JJ(10,10,10),
     3 CAP(10,10,10),COUN(10,10,10),TLL(12),KA(10,10,10),DB(10),
     4 CB(10,10),BD(10,91,5)
      WRITE (6,55)
   55 FORMAT ('OTIMER INTERRUPT ROUTINE')
      WRITE  (6,56) A,D
   56 FORMAT ('0***** A = ',I4,' D = ',I4)
```

```
      REWIND 8
      WRITE (8) A,D
      IF (A .EQ. 1) GO TO 9
      A2=A-1
      DO 1 AA=1,A2
      WRITE (8) DB(AA),EA(AA),APN(AA),((BD(AA,H,G),G=1,5),H=1,91)
      DF=DB(AA)
      DO 2 DD=1,DF
      WRITE (8) CB(AA,DD),AHN(AA,DD),ED(AA,DD),
     1 ((BC(AA,DD,K,G),G=1,5),K=1,26)
      CF=CB(AA,DD)
      DO 3 CC=1,CF
      WRITE (8) AKN(AA,DD,CC),CAP(AA,DD,CC),COUN(AA,DD,CC),JJ(AA,DD,CC),
     1 KA(AA,DD,CC),EC(AA,DD,CC)
      J1=JJ(AA,DD,CC)
      DO 4 J=1,J1
      WRITE (8) OLT(AA,DD,CC,J)
    4 CONTINUE
    3 CONTINUE
    2 CONTINUE
    1 CONTINUE
    9 WRITE  (8) APN(A),EA(A)
      IF (D .EQ. 1) GO TO 10
      D2=D-1
      DO 5 DD=1,D2
      WRITE (8) CB(A,DD),ED(A,DD),AHN(A,DD),
     1 ((BC(A,DD,K,G),G=1,5),K=1,26)
      CF=CB(A,DD)
      DO 6 CC=1,CF
      WRITE (8) AKN(A,DD,CC),CAP(A,DD,CC),COUN(A,DD,CC),JJ(A,DD,CC),
     1 KA(A,DD,CC),EC(A,DD,CC)
      J1=JJ(A,DD,CC)
      DO 7 J=1,J1
      WRITE (8) OLT(A,DD,CC,J)
    7 CONTINUE
    6 CONTINUE
    5 CONTINUE
   10 WRITE (8) AHN(A,D),ED(A,D)
      H1=AHN(A,D)
      WRITE (8) ((BD(A,H,G),G=1,5),H=1,H1)
      P1=APN(A)
      DO 8 P=1,P1
      WRITE (8) (B(P,G),G=1,5)
      WRITE (6,50) P,(B(P,G),G=1,5)
   50 FORMAT (' P = ',I4,3X,5I4)
    8 CONTINUE
      WRITE (6,52) (H,(BD(A,H,G),G=1,5),H=1,H1)
   52 FORMAT (' H = ',I4,3X,5I4)
      END FILE 8
      RETURN
      END


      SUBROUTINE READ2
      IMPLICIT INTEGER (A-Y)
      COMMON / / A,D,C,EA(10),ED(10,10),EC(10,10,10),PASS,B(273,5),
```

```
   1 L(15),A1(15),D1(15),C1(15),APN(11),AHN(10,11),AKN(10,10,10),
   2 BC(10,10,26,5),OCH(12),OLT(10,10,10,12),JJ(10,10,10),
   3 CAP(10,10,10),COUN(10,10,10),TLL(12),KA(10,10,10),DB(10),
   4 CB(10,10),BD(10,91,5)
     REWIND 8
     READ (8) A,D
     IF (A .EQ. 1) GO TO 9
     A2=A-1
     DO 1 AA=1,A2
     READ (8)  DB(AA),EA(AA),APN(AA),((BD(AA,H,G),G=1,5),H=1,91)
     DF=DB(AA)
     DO 2 DD=1,DF
     READ (8)  CB(AA,DD),AHN(AA,DD),ED(AA,DD),
   1 ((BC(AA,DD,K,G),G=1,5),K=1,26)
     CF=CB(AA,DD)
     DO 3 CC=1,CF
     READ (8)  AKN(AA,DD,CC),CAP(AA,DD,CC),COUN(AA,DD,CC),JJ(AA,DD,CC),
   1 KA(AA,DD,CC),EC(AA,DD,CC)
     J1=JJ(AA,DD,CC)
     DO 4 J=1,J1
     READ (8)  OLT(AA,DD,CC,J)
   4 CONTINUE
   3 CONTINUE
   2 CONTINUE
   1 CONTINUE
   9 READ (8)   APN(A),EA(A)
     IF (D .EQ. 1) GO TO 10
     D2=D-1
     DO 5 DD=1,D2
     READ (8)  CB(A,DD),ED(A,DD),AHN(A,DD),
   1 ((BC(A,DD,K,G),G=1,5),K=1,26)
     CF=CB(A,DD)
     DO 6 CC=1,CF
     READ (8)  AKN(A,DD,CC),CAP(A,DD,CC),COUN(A,DD,CC),JJ(A,DD,CC),
   1 KA(A,DD,CC),EC(A,DD,CC)
     J1=JJ(A,DD,CC)
     DO 7 J=1,J1
     READ (8) OLT(A,DD,CC,J)
   7 CONTINUE
   6 CONTINUE
   5 CONTINUE
  10 READ (8)  AHN(A,D),ED(A,D)
     H1=AHN(A,D)
     READ (8)  ((BD(A,H,G),G=1,5),H=1,H1)
     P1=APN(A)
     DO 8 P=1,P1
     READ (8)  (B(P,G),G=1,5)
     WRITE (6,50) P,(B(P,G),G=1,5)
  50 FORMAT (' P = ',I4,3X,5I4)
   8 CONTINUE
     WRITE (6,52) (H,(BD(A,H,G),G=1,5),H=1,H1)
  52 FORMAT (' H = ',I4,3X,5I4)
     WRITE (6,51)
  51 FORMAT ('0***** NEW PGM STAR HERE')
     RETURN
     END
```