Spectra associated with

commutative rings.

by

ICHIRO WATANABE

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CONTENTS

Introduction.

The study of the spectrum Spec(A) of a commutative ring A with unity, that is, the set of all prime ideals of A endowed with the so called Zariski topology is very important in Algebraic Geomety. The study of Spec(A) as a topological space is in itself a very interesting subject. One of the key results in this area is due to Hochster (cf. [Hoc,1]) (**Note**: All items indicated like this within square brackets refer to books or papers listed in the bibliography) where he showed that there exists a category $\mathscr{S}$ of topological spaces called spectral spaces with spectral maps as morphisms, such that Spec is a full functor from the category $\mathscr{C}$ of commutative rings with unity and ring homomorphisms as morphisms to the category $\mathscr{S}$ of spectral spaces. He further showed that Spec can be inverted on some subcategories of $\mathscr{S}$ but not on all the subcategories of $\mathscr{S}$. The object of this thesis is to present in detail the study of spectral spaces, and minspectral spaces and maxspectral spaces (the latter classes of spaces correspond respectively to the minimal prime spectra and maximal ideal spectra of commutative rings). For reasons of restricting the length of this thesis the study of other classes of related spaces such as Jacobson spectral spaces, Jacobson spaces and Goldman prime spectra are presented in summary form only.

In chapter 1, section 1, the topological properties of Spec(A) are established. Note (1.2.2) shows that Spec(A) is a spectral space in the sense of the definition in section 2. In section 2, the study of spectral spaces begins. To the spectral topology on a spectral spaces there is a related compact (we use this terminology to indicate Hausdorff quasi-compact) topology called the patch topology, which is very useful. Proposition (1.2.13) shows how spectral spaces and compact spaces are related. This relationship is so significant that the class of compactifiable spaces (which are characterized as subspaces of compact spaces) has an analogue in the class of spectral spaces. These are the so called spectralifiable spaces studied in section 3 (cf. (1.3.3)). Theorem (1.3.4) shows that a topological space is spectralifiable iff it is "spectrally embeddable" in a spectral space.

In chapter 2 our principal object is to show that every spectral space is indeed the spectrum of some commutative ring with unity (cf. (2.1.26.1)). In fact we show that Spec is a full functor from $\mathscr{C}$ to $\mathscr{S}$ (cf. (2.2.1)). The proof of this fact is very technical and involves the construction of some categories and functors (cf. from (2.1.1) to (2.1.4) for an overview and also (2.1.28)) and is presented in section 1. In section 2 we show some applictions in (2.2.1), (2.2.2) and (2.2.3). A very interesting application is (2.2.10) where we present the proof that a topological space is the underlying space of a prescheme iff it is a locally spectral space.

In chapter 3, section 1 we study the topological properties of minimal prime specturm of a commutative ring and (3.2.6) in section 2 provides topological characterization of such spaces.

Chapter 4 describes the topological properties of the maximal ideal spectrum of a commutative ring with unity and (4.2.2) establishes its topological characterization.

In chapter 5 we present in summary form some further results in the area relating to the Jacobson ideal spectrum and Goldman ideal spectrum in section 1 and the answers to the question of compactness of the minimal prime spectrum is given in section 2.

Chapter 1. Prime spectra and spectral spaces.

Section 1. Prime Spectra.

The word "ring" stands for a commutative ring with unity 1.

Definition (1.1.0). — The *prime spectrum* of a ring A, denoted by Spec(A), is the set of all proper prime ideals of A endowed with the topology whose closed sets are the subsets $V(E) = \{[P] \in \mathrm{Spec}(A) : E \subseteq P\}$, where E is a subset of A. It is well known that $\{V(E) : E \subseteq A\}$ is indeed the set of all closed subsets of a topology on Spec(A). For a prime ideal P of A, we use the notation [P] when it is thought as a point of the topological space X = Spec(A), and for a point x in X, denote x as j_x when x is considered as a prime ideal of A. If E is a singleton set {f}, we use the notation V(f) rather than V({f}).

The topology thus defined is called the *spectral topology* or the *Zariski toplogy* on Spec(A).

Let A, A' be two rings, h: A $\rightarrow$ A' a ring homomorphism, and X = Spec(A), X' = Spec(A'), then the map ${}^{a}h$: X' $\rightarrow$ X from X' into X defined by ${}^{a}h([P']) = [h^{-1}(P')]$ is called the *associated map* of the homomorphism h. ([Bou,2], II, sec.4, n° 3)

Proposition (1.1.0.1). — The associated map ${}^a h$: $X' \to X$ is continuous. ([Bou,2], II, sec.4, n^0 3, Prop.13)

[Pf] Let E be any subset of A; then the subset $({}^a h)^{-1}(V(E))$ of X' is the set of prime ideals P' of A' such that $E \subseteq h^{-1}(P')$. This inclusion relation is equivalent to $h(E) \subseteq P'$, therefore $({}^a h)^{-1}(V(E)) = \{[P'] \in X': h(E) \subseteq P'\} = V(h(E))$, that is, the inverse image of any closed subset of X by ${}^a h$ is closed in X'. ■

Definition (1.1.1). — Let *rad(E)* be the *radical of the ideal of A generated by E*, that is, the set of elements f of A such that f^n, for some $n \in \mathbf{N}$, belongs to the ideal generated by E. An ideal J of A equal to its own radical, i.e., J = rad(J), is called a *radical ideal* of A.

Note (1.1.2). — (i) $\mathrm{rad}(E) = \cap \{P: [P] \in V(E)\}$.

(ii) $V(0) = X$, $V(1) = \varnothing$.

(iii) For subsets E, E' of A, $E' \subseteq E$ implies $V(E) \subseteq V(E')$.

(iv) For any family $\{E_i: i \in I\}$ of subsets of A, $V(\cup \{E_i: i \in I\}) = V(\Sigma\{E_i: i \in I\}) = \cap\{V(E_i): i \in I\}$, where $\Sigma\{E_i: i \in I\}$ represents the ideal generated by the union of E_i.

(v) $V(E) = V(\mathrm{rad}(E))$, for any subset E of A.

(vi) $V(E) \cup V(E') = V(EE')$, for any subsets E, E' of A.

Let $D(E) = X - V(E)$, for any subset E of A. Then

(vii) $D(f) = \{[P] \in \mathrm{Spec}(A): f \notin P\}$, for any element f of A.

(viii) $D(fg) = D(f) \cap D(g)$, for any elements f, g of A.

Proposition (1.1.3). — The sets $D(f) = \{[P] \in Spec(A): f \notin P\}$, for all $f \in A$, form an open basis of the spectral topology on $X = Spec(A)$. ([Bou,2], II, sec.4, n° 3, p.126)

[Pf] Clearly, $D(f)$ is open for any $f \in A$. For any open subset U of X, there is a subset E of A such that $U = X - V(E)$. By (1.1.2), (iv), $U = X - V(E) = X - \cap\{V(f): f \in E\} = \cup\{D(f): f \in E\}$. ■

Proposition (1.1.4). — For any subset Y of $X = Spec(A)$, define $I(Y)$ as $\cap\{P: [P] \in Y\}$. Then

(i) $I(\varnothing) = A$.

(ii) For $Y, Y' \subseteq X$, $Y \subseteq Y'$ implies $I(Y) \supseteq I(Y')$.

(iii) For any family $\{Y_\lambda: \lambda \in L\}$ of subsets of X, $I(\cup\{Y_\lambda: \lambda \in L\}) = \cap\{I(Y_\lambda): \lambda \in L\}$. ([Bou,2], II, sec.4, n° 3)

Proposition (1.1.5). — Let A be a ring, J an ideal of A and Y a subset of $X = Spec(A)$.

(i) The set $V(J)$ is closed in X and $I(Y)$ is a radical ideal.

(ii) $I(V(J))$ is the radical of J, and $V(I(Y)) = Cl_X Y$, the closure of Y in X.

(iii) The correspondences I and V define order-reversing bijections between the set of all closed subsets of X and the set of all radical ideals of A and are inverses to each other. ([Bou,2], II, sec.4, n° 3, Prop.11).

[Pf] (i): By (1.1.0), $V(J)$ is closed in X; since $I(Y) = \cap\{P \subseteq A: [P] \in Y\}$ and since the radical of $I(Y)$ is the intersection of prime ideals containing $I(Y)$, $I(Y) = \mathrm{rad}(I(Y))$. ■

(ii): $I(V(J)) = \cap\{P: [P] \in V(J)\} = \cap\{P: J \subseteq P$, P is a prime ideal of A$\}$ = rad(J). Next if a closed set V(E) contains Y, then for any $[P] \in Y$, $E \subseteq P$. Thus $E \subseteq I(Y)$. Then by (1.1.2), (iii), $V(I(Y)) \subseteq V(E)$. This shows that $V(I(Y))$ is the smallest closed subset of X containing Y, that is, the closure of Y. ■

(iii): The correspondences V and I are decreasing by (1.1.2), (iii) and (1.1.4), (ii) respectively. If J is a radical ideal, then $I(V(J)) = \mathrm{rad}(J) = J$; if Y is a closed subset of X, $Y = \mathrm{Cl}Y = V(I(Y))$. ■

Proposition (1.1.6). — Let $\{f_\lambda : \lambda \in L\}$ be a family of elements of A, and let $g \in A$. Then, $D(g) \subseteq \cup \{D(f_\lambda): \lambda \in L\}$ iff there exists an integer $n > 0$ such that g^n belongs to the ideal generated by f_λ, that is, g belongs to $\mathrm{rad}(\Sigma\{f_\lambda: \lambda \in L\})$. ([Bou,2],II, sec.4, n° 3, Cor to Prop.11).

[Pf] The relation $D(g) \subseteq \cup\{D(f_\lambda): \lambda \in L\}$ is equivalent to $V(g) \supseteq \cap\{V(f_\lambda): \lambda \in L\} = V(\Sigma\{f_\lambda: \lambda \in L\})$. By applying the correspondence I we have $(g) \subseteq \mathrm{rad}(g) = I(V(g)) \subseteq I(V(\Sigma\{f_\lambda: \lambda \in L\})) = \mathrm{rad}(\Sigma\{f_\lambda: \lambda \in L\})$. ■

Proposition (1.1.7). — For any ideal J of A, Spec(A/J) is identified canonically to the closed subset V(J) of Spec(A). ([Gro-Die,2], (I, 1.2.4))

[Pf] There is a canonical bijective correspondence preserving the order structure of inclusion between the set of ideals (resp. prime ideals) of A/J and the set of ideals (resp. prime ideals) of A containing J. This correspondence provides what is required. ■

Proposition (1.1.8). — Let N be the *nilradical* (or *prime radical* in [Lam]) of a ring A (i.e., the set of nilpotent elements of A). Then the topological spaces Spec(A) and Spec(A/N) are homeomorphic. ([Gro-Die,1], (I, 1.1.12))

[Pf] Since the nilradical of a ring A is the radical of the zero-ideal, $\mathrm{Spec}(A/N) \cong V(N) = V(\mathrm{rad}(0)) = V(0) = \mathrm{Spec}(A)$ by (1.1.7). ■

Proposition (1.1.9). — Let $[P] \in \mathrm{Spec}(A)$. Then $\mathrm{Cl}\{[P]\} = V(P)$. The set $\{[P]\}$ is closed in Spec(A) iff P is a maximal ideal (in this case [P] is called a *closed point*). ([Bou,2], II, sec.4, n° 3, Cor.6 to Prop.11)

[Pf] Since P is prime, $I(\{[P]\}) = P$. So that by (1.1.5), (ii), $\mathrm{Cl}\{[P]\} = V(I\{[P]\})$ $= V(P)$. It is clear from definition of V(P) that the singleton set $\{[P]\}$ is closed in Spec(A) iff P is a maximal ideal of A. ■

Proposition (1.1.10). — The prime spectrum X = Spec(A) of a ring A is T_0. ([Gro-Die,1], (I, 1.1.8))

[Pf] Let $[P_1] \neq [P_2]$. Then $P_1 \neq P_2$. There are two cases possible: in the case that $P_1 \not\subset P_2$, we have $P_2 \notin V(P_1) = \mathrm{Cl}\{[P_1]\}$; in the case that $P_1 \subset P_2$, we have $P_2 \not\subset P_1$, then $P_1 \notin V(P_2) = \mathrm{Cl}\{[P_2]\}$. This proves that Spec(A) is T_0. ■

Definition (1.1.11). — A ring A is called *Noetherian* if the ideals in A satisfy the ascending chain condition, that is, for any ascending chain of ideals $J_0 \subseteq J_1 \subseteq ... \subseteq J_n \subseteq ...$ there exists an integer $m \geq 0$ such that if $n \geq m$ then $J_m = J_n$.

A topological space X is called *Noetherian* if the closed subsets of X satisfy the descending chain condition (or equivalently, the open subsets of X satisfy the ascending chain condition).

Proposition (1.1.11.1). — A ring A is Noetherian iff every ideal of A is finitely generated. ■

Proposition (1.1.11.2). — If every prime ideal of a ring A is finitely generated, then A is Noetherian. ([Kap], Th.8) ■

Note (1.1.12.1). — The converse of (1.1.11.2) need not be true: for example, a non discrete valuation domain of rank 1 has an infinite ascending chain of ideals and so is not a Noetherian ring but its prime spectrum is a doubleton space, and so obviously is a Noetherian space.

Proposition (1.1.11.3). — (i) Every subspace of a Noetherian space is Noetherian.

(ii) Let $\{E_i: i \in I\}$ be a finite cover of a topological space X. If spaces E_i of X are Noetherian, then X is Noetherian. ([Bou,2, II, sec.4, n° 2, Prop.8)

[Pf] (i): Let X be a Noetherian space, E a subspace of X, and $\{F_n: n = 0,1,2,\ldots\}$ be a descending chain of closed subsets of E. Then for any n, $F_n = Cl_E F_n = Cl_X F_n \cap E$. $\{Cl_X F_n: n = 0,1,2,\ldots\}$ is a descending chain of closed subsets of X, since $F_{n+1} \subseteq F_n$ implies $Cl_X F_{n+1} \subseteq Cl_X F_n$ for any n. Since X is Noetherian, there exists an integer $m \in \mathbf{N}$ such that $n \geq m$ implies $ClF_n = ClF_m$, so with this m, $n \geq m$ implies $F_n = F_m$, thus E is Noetherian. ∎

(ii): Let $\{G_n: n = 0,1,2,\ldots\}$ be a descending chain of closed subsets of X. By hypothesis, for any $i \in I$, there exists an $n_i \in \mathbf{N}$ such that if $n \geq n_i$ then $G_{n_i} \cap E_i = G_n \cap E_i$, since $\{G_n \cap E_i: n = 0,1,2,\ldots\}$ is a descending chain of closed subsets of E_i. Furthermore since I is finite, there exists $m \in \mathbf{N}$ such that if $n \geq m$, then $G_n \cap E_i = G_m \cap E_i$ for any $i \in I$ (for example $m = \max\{n_i: i \in I\}$). Now $\{E_i: i \in I\}$ is a cover of X, so that $G_n = \cup \{G_n \cap E_i: i \in I\}$ for any n = 0, 1, 2,..., that is, for $n \geq m$, $G_n = G_m$. Thus X is Noetherian. ∎

Proposition (1.1.11.4). — A topological space X is Noetherian iff every open subset of X is *quasi-compact* (that is, for any open cover $\{U_i: i \in I\}$ of X there exists a finite subset F of I such that $\{U_i: i \in F\}$ covers X). ([Bou,2], II, sec.4, n° 2, Prop.9)

[Pf] Suppose X is Noetherian and Y is open in X. Then Y, as a subspace, is Noetherian. So we need only to show that X is quasi-compact.

Let $\{U_i: i \in I\}$ be an open cover of X, then the set E of finite unions of U_i, $i \in I$, is nonempty since each U_i belongs to this set. Since X is Noetherian every ascending chain (with respect to set inclusion) of members of E terminates at some finite stage. Thus every ascending chain has an upper bound. So that, by Zorn's lemma, there exists a maximal member $V = \cup\{U_j: j \in H\}$ of E, where H is a finite subset of I. Thus for any $i \in I$, $V \cap U_i = U_i$ by the maximality of V. This implies that $X = \cup\{U_i: i \in I\} = \cup\{V \cap U_i: i \in I\} = V \cap (\cup\{U_i: i \in I\}) = V \cap X = V$. Therefore X is quasi-compact.

Conversely, suppose that every open subset of X is quasi-compact. Let $\{U_n: n \in N\}$ be an increasing sequence of open subsets of X. The union V of the U_n is open and hence quasi-compact; as $\{U_n\}$ is an open covering of V, there is a finite subfamily of $\{U_n\}$ which is a covering of V and hence $V = U_n$ for some index n, which proves that the sequence $\{U_n\}$ terminates at some finite stage. ■

Proposition (1.1.12). — If a ring A is Noetherian, then Spec(A) is a Noetherian topological space. ([Bou,2], II, sec.4, Cor.7 to Prop.11)

Note (1.1.12.1). — The converse of (1.1.12) need not be true: for example, a non discrete valuation domain of rank 1 has an infinite ascending chain of ideals and so is not a Noetherian ring but its prime spectrum is a doubleton space, and so obviously is a Noetherian space.

Proposition (1.1.13). — For any $f \in A$, the subset D(f) of Spec(A) is quasi-compact; in particular, the space Spec(A) = D(1) is quasi-compact. ([Bou,2], II, sec.4, n° 3, Prop.12)

[Pf] Since $\{D(g): g \in A\}$ forms an open basis of Spec(A), it suffices to consider the case that an open cover of D(f) is $\{D(g_i): i \in I, g \in A\}$ and show that there exists a finite subcover. By (1.1.6) $D(f) \subseteq \cup\{D(g_i): i \in I\}$ iff f belongs to the radical ideal $rad(\Sigma\{g_i: i \in I\})$ generated by $\{g_i: i \in I\}$, that is, there exist an integer $n > 0$ and a finite subset H of I such that $f^n \in rad(\Sigma\{ g_i: i \in I\})$. Then $D(f) \subseteq \cup\{D(g_i): i \in H\}$. This shows that D(f) is quasi-compact. ■

Proposition (1.1.14). — Let A, A' be two rings, h: $A \to A'$ be a ring homomorphism, X = Spec(A) and X' = Spec(A') . Then the inverse image of a quasi-compact open subset Y of X by ${}^a h$, the associated map of h, is quasi-compact (open) in X'.

[Pf] Since Y is quasi-compact, Y can be taken as a finite union of $D(f_i)$, i = 1,...,n. So in order to show that $({}^a h)^{-1}(Y)$ is quasi-compact, it suffices to show that for any $f \in A$, $({}^a h)^{-1}(D(f))$ is quasi-compact. Now in the proof of (1.1.0.1), we showed $({}^a h)^{-1}(V(E)) = V(h(E))$ for any subset E of A, therefore $({}^a h)^{-1}(D(f)) = D(h(f))$. This is quasi-compact in X' by (1.1.13). ■

Proposition (1.1.15). — Let h be as in (1.1.14). If for each $f' \in A'$, there are a unit u' of A' and an element f in A such that $f' = u'(h(f))$, then ${}^a h$ is a homeomorhphism of X' onto ${}^a h(X')$. ([Gro-Die,1], (I, 1.2.4))

[Pf] Let E' be a subset of A'. By hypothesis, for each $f' \in E'$ there exist $f \in A$ and a unit $u' \in A'$ such that $f' = u'(h(f))$. Since u' is a unit, $u'(h(f)) = f' \in P'$, where P' is a (prime) ideal, is equivalent to $h(f) \in P'$. Thus V(E') =

$V(h(E))$, where $E = \{f \in A$: there exist a unit $u' \in A'$ such that $u'h(f) \in E'\}$. Now we are ready to prove that ${}^a h$ is injective. Consider two prime ideals P_1', P_2' of A'. There exist two subsets E_1, E_2 corresponding to P_1', P_2' respectively in the manner as above, that is, $E_1 = \{f \in A$: there exists a unit $u' \in A'$ such that $u'(h(f)) = f' \in P_1'\}$ and $E_2 = \{f \in A$: there exists a unit $u \in A'$ such that $u'(h(f) = f' \in P_2'\}$.

Since P_1' is a prime ideal and u' is invertible $u'h((f)) \in P_1'$ is equivalent to $h(f) \in P_1'$, so that in fact $E_1 = h^{-1}(P_1')$. Similarly, $E_2 = h^{-1}(P_2')$. Suppose ${}^a h([P_1']) = {}^a h([P_2'])$ then $h^{-1}(P_1') = h^{-1}(P_2')$, i.e., $E_1 = E_2$, and so $V(P_1') = V(h(E_1)) = V(h(E_2)) = V(P_2')$. This implies $P_1' = P_2'$ since X' is T_0. Thus ${}^a h$ is injective.

Finally any closed subset of X' is of the form $V(E')$, $E' \subseteq A'$. Let $E \subseteq A$ be defined with respect to E' as above. By a routine verification, ${}^a h(V(E')) = V(h^{-1}(E')) \cap {}^a h(X')$. So that ${}^a h$ is a closed map. Consequently, it is a homeomorphism of X' onto ${}^a h(X')$. ■

Remark (1.1.15.1). — If h in (1.1.15) above is surjective, then the condition in (1.1.15) is fufilled when we set the unit u' of A' as the unity of A', and $E = h^{-1}(E')$ for any subset $E'$of A.

This observation provides another proof of (1.1.8) since the canonical map $h: A \to A/N$ is surjective.

Definition (1.1.16). — A ring A is called a *local ring* if A has exactly one maximal ideal M. Then M is the set of non-invertible elements of A. M is also the *Jacobson radical* of A ($=_{\text{def}}$ the intersection of all maximal ideals of A).

The quotient field A/M of A by its Jacobson radical (that is, the only maximal ideal of the local ring A) is called the *residue field* of A.

Let A be a ring, and S a subset of A closed under multiplication and containing the unity of A. Such an S is called a *multiplicative subset* of A.

Let us define an equivalence relation ~ on $A \times S$ as : (a, s) ~ (a', s') iff there exists a $t \in S$ such that $t(a's - s'a) = 0$. Then $(A \times S)/\sim$ is a ring called the *ring of fractions* of A with respect to S and is denoted by $S^{-1}A$; let us denote by a/s the equivalence class of (a, s).

The map $i_{A,S}: A \to S^{-1}A$ sending a to a/1 is called the *canonical homomorphism* and has the following properties:

(i) for any $s \in S$, $i_{A,S}(s)$ is invertible in $S^{-1}A$;

(ii) if $i_{A,S}(a) = 0$ then $as = 0$ for some $s \in S$;

(iii) every element of $S^{-1}A$ is of the form $i_{A,S}(a)i_{A,S}(s)^{-1}$ for some $a \in A$ and $s \in S$.

Note (1.1.16.1). — Let $g: A \to B$ be a ring homomorphism such that g(s) is invertible for all $s \in S$. Then there is a unique ring homomorphism $f: S^{-1}A \to B$ such that $g = f \circ i_{A,S}$. (cf. [Ati-Mac], Prop.3.1)

Note (1.1.16.2). — Let P be a prime ideal, and S = A - P, then S is a multiplicative subset of A. Thus we have the ring of fractions $S^{-1}A$, and we denote it by A_P. Let s be a nonzero element of A, then $S = \{s^n : n \in \mathbb{N}\}$, where $\mathbb{N}$ contains 0, is a multiplicative subset of A, which is denoted by A_s instead of $S^{-1}A$. If A is an integral domain then the canonical homomorphism is injective and if S = A - {0}, then $S^{-1}A$ is the field of fractions of A.

Proposition (1.1.16.3). — Let S be a multiplicative subset of a ring A, and $i_{A,S}: A \to S^{-1}A$ be the canonical homomorphism. Then the associated map ${}^a i_{A,S}$ is a homeomorphism of $\mathrm{Spec}(S^{-1}A)$ onto the subspace Y of Spec(A) consisting of prime ideals disjoint from S. ([Bou,2], II, sec.4, n° 3, Cor. to Prop.13)

[Pf] Since any element of $S^{-1}A$ is of the form $i_{A,S}(a)i_{A,S}(s)^{-1}$, where $a \in A$ and $s \in S$, our setting satisfies the hypothesis of (1.1.15). Thus ${}^a i_{A,S}$ is a homeomorphism of $\mathrm{Spec}(S^{-1}A)$ onto $Y = {}^a i_{A,S}(\mathrm{Spec}(S^{-1}A))$. So it remains to show that any element of Y is a prime ideal which is disjoint from S.

Let P be a proper prime ideal of $S^{-1}A$. By definition ${}^a i_{A,S}([P]) = [i_{A,S}^{-1}(P)]$. Since $i_{A,S}$ is a ring homomorphism, for any elements $a, b \in i_{A,S}^{-1}(P)$ and $c \in A$, $a - b \in i_{A,S}^{-1}(P)$ and $ca \in i_{A,S}^{-1}(P)$. If $ab \in i_{A,S}^{-1}(P)$ then $i_{A,S}(a)i_{A,S}(b) = i_{A,S}(ab) \in P$, so $i_{A,S}(a) \in P$ or $i_{A,S}(b) \in P$. Thus $a \in i_{A,S}^{-1}(P)$ or $b \in i_{A,S}^{-1}(P)$. So $i_{A,S}^{-1}(P)$ is a prime ideal of A. If $i_{A,S}^{-1}(P) \cap S \neq \varnothing$, then $s \in i_{A,S}^{-1}(P) \cap S$

implies $i_{A,S}(s)$ is invertible in $S^{-1}A$; but then $1 = i_{A,S}(s)i_{A,S}(s)^{-1} \in P$, which is impossible as P is a proper prime ideal. ∎

Proposition (1.1.16.4). — Let A be a ring, S be a multiplicative set and J be an ideal of A. If we identify $S^{-1}J$ with the ideal generated by J in $S^{-1}A$, then $S^{-1}(A/J) \cong S^{-1}A/S^{-1}J$. ([Bou,2], II, sec.2, n° 4, p.89) ∎

Proposition (1.1.16.5). — Let A be a ring, P a prime ideal of A. Then the ring A_P is local; its maximal ideal is PA_P, the ideal generated by the canonical image of P in A_P; its residue field is canonically isomorphic with the field of fractions of A/P. ([Bou,2], II, sec.3, n° 1, Prop.2)

[Pf] By (1.1.16.3), every prime ideal Q' of $A_P = S^{-1}A$ corresponds bijectively to a prime ideal P' of A which is disjoint from S = A - P. Then P' is contained in P. Thus PA_P is the maximal ideal of A_P. Therefore A_P is a local ring with the maximal ideal PA_P.

Let $f: A \to A/P$ be the canonical surjection. Then f(S) is the set of nonzero elements of the integral domain A/P. Thus $f(S)^{-1}(A/P)$ is the field of fractions of A/P, and is identified with $S^{-1}(A/P)$ by the map $a/f(s) \mapsto a/s$, where $a \in A/P$ and $s \in S$. Now $S^{-1}(A/P)$ is identified with $A_P/(PA_P)$. On the other hand $A_P/(PA_P)$ is the residue field by definition. So we have a canonical identification between the residue field of A at P and the field of fractions of A/P. ∎

Corollary (1.1.16.6). — Let A be a ring, s an element of A and f: $A \to A_s$ the canonical map f(a) = a/1, for any $a \in A$. Then ${}^a f$: Spec(A_s) $\to$ Spec(A) is a homeomorphism from Spec(A_s) onto D(s).

[Pf] By (1.1.16.3), ${}^a f$ is the homeomorphism from Spec(A_s) onto the subset of Spec(A) consisting of prime ideals of A disjoint from the multiplicative subset generated by s, but this subset of Spec(A) is D(s). ∎

Definition (1.1.17). — A topological space X is said to be *irreducible* if very finite intersection of nonempty open subsets of X is nonempty. A subset E of X is called *irreducible* if as a subspace of X it is irreducible. Every maximal irreducible subset of X is called an *irreducible component* of X.

Proposition (1.1.17.1). — Let X be a nonempty topological space. Then the following conditions are equivalent:

(i) X is irreducible;

(ii) every nonempty open subset of X is dense in X;

(iii) every open subset of X is connected. ([Bou,2], II, sec.4, n° 1, Prop.1)

[Pf] (i) $\Leftrightarrow$ (ii): A dense subset of X is a subset with which every open subset has nonempty intersection, thus (i) and (ii) are equivalent. ∎

(iii) $\Rightarrow$ (i): Suppose X is not irreducible. Then, there are two nonempty disjoint open subsets U_1 and U_2 of X, and then $U_1 \cup U_2$ is a disconnected open subset of X. ∎

(i) $\Rightarrow$ (iii): Let U be a disconnected open subset of X. Then U is a union of two disjoint open subsets U'and U" of U. But since U is open in X, U'and U" are disjoint open subsets of X, so X is not irreducible. ∎

Note (1.1.17.2). — Let E be a subset of a topological space X. Then the following are equivalent:

(i) E is irreducible;

(ii) If two open subsets U, V of X meet E (i.e., they have nonempty intersection with E), then $U \cap V$ also meets E;

(iii) for any two closed subsets F, G of X if $E \subseteq F \cup G$ then $E \subseteq F$ or $E \subseteq G$, or by finite induction, for a finite family $F_1, F_2, \ldots, F_n$ of closed subsets of X, if $E \subseteq \cup F_i$ then $E \subseteq F_i$ for some $i = 1,2,\ldots,n$.

Proposition (1.1.17.3). — Let E be a subset of a topological space X. Then E is irreducible iff ClE is irreducible. ([Bou,2], II, sec.4, n° 1, Prop.2)

[Pf] An open subset U of X meets E iff U meets ClE, for if U meets ClE then U is a neighbourhood of each point that belongs to $U \cap \mathrm{Cl}E$ in X, thus U meets E; the other implication is obvious.

Now let U, V be open subsets of X. It is clear that $U \cap V \cap E \neq \varnothing$ is equivalent to $U \cap V \cap \mathrm{Cl}(E) \neq \varnothing$. So by (1.1.17.2), the irreducibility of E is equivalent to the irreduciblity of ClE. ∎

Proposition (1.1.17.4). — Let X and Y be topological spaces and f a continuous map of X into Y. If a subset E of X is irreducible, f(E) is irreducible. ([Bou,2], II, sec.4, n^{0} 1, Prop.4)

[Pf] Let U, V be two open subsets of Y which have nonempty intersection with f(E). Then $f^{-1}(U)$ and $f^{-1}(V)$ are open and have nonempty intersection with E. Since E is irreducible, $f^{-1}(U) \cap f^{-1}(V) \cap E \neq \emptyset$. Thus an element x in this intersection has its image f(x) in $U \cap V \cap f(E)$, that is, $U \cap V$ meets f(E); so that by (1.1.17.2), f(E) is irreducible. ■

Proposition (1.1.17.5). — Let $U \neq \emptyset$ be an open subset of a topological space X. The map α defined by $\alpha(V) = Cl_X V$ is a bijection of the set of irreducible subsets of U closed in U onto the set of closed irreducible subsets of X intersecting U, and its inverse map β is defined by $\beta(Z) = Z \cap U$. ([Bou,2], II, sec.4, n^{0} 1, Prop.7)

[Pf] Let V be a closed irreducible subset of U. Since U is open in X, V is also an irreducible subset of X. Thus its closure $Cl_X V$ in X is irreducible subset of X by (1.1.17.3). Conversely, let Z be an irreducible closed subset of X which meets U. Then $Z \cap U$ is a nonempty closed irreducible subset of U. Thus we have two correspondences in both directions.

Let Z be a closed irreducible subset of X, then $Z \cap U$ if nonempty, is dense in Z by (1.1.17.1), that is, $Cl_Z(Z \cap U) = Z$. But then $Z = Cl_Z(Z \cap U) = Cl_X(Z \cap U) \cap Z = Cl_X(Z \cap U)$, since Z is closed in X and $Z \cap U \subseteq Z$. Therefore the composition $\alpha \circ \beta$ is the identity map on the set of irreducible closed subsets of X. Let V be a closed irreducible subset of U. Then clearly $V = Cl_X V \cap U$. So $\beta \circ \alpha$ is the identity map. ■

Proposition (1.1.18). — Let A be a ring. Then a subset Y of Spec(A) is irreducible iff $I(Y) = \cap\{j_y \subseteq A: y \in Y\}$ is a prime ideal. ([Bou,2], II, sec.4, n^o 3, Prop.14)

[Pf] At first note that for an element $f \in A$, $f \in I(Y)$ is equivalent to $Y \subseteq V(f)$.

Suppose that Y is irreducible. Let $f, g \in A$ be two elements such that $fg \in I(Y)$. Then $Y \subseteq V(fg) = V(f) \cup V(g)$. Since Y is irreducible and V(f), V(g) are closed, $Y \subseteq V(f)$ or $Y \subseteq V(g)$, that is $f \in I(Y)$ or $g \in I(Y)$. Thus I(Y) is a prime ideal.

Conversely suppose I(Y) is prime. Then $ClY = V(I(Y)) = Cl\{[I(Y)]\}$, since I(Y) being prime implies $I(\{[I(Y)]\}) = I(Y)$. Since the closure of a one-point set $\{[I(Y)]\}$ is irreducible, so is ClY, and Y is also irreducible. ■

Definition (1.1.18.1). — When $Cl\{[P]\}=Y$, the point [P] is called a *generic point* of Y.

Proposition (1.1.19). The prime spectrum $X = Spec(A)$ of a ring A is irreducible iff the quotient A/N of A by its nilradical N is an integral domain. ([Bou,2], II, sec.4, n^o 3, Cor.1 to Prop.14)

[Pf] Since $I(X)$ is the radical of the zero ideal (0) which is the nilradical N of A, X is irreducible iff N is prime by (1.1.18), thus iff A/N is an integral domain. ■

Proposition (1.1.20). — The map α defined by $\alpha([P]) = V(P)$ is a bijection of $X = \mathrm{Spec}(A)$ onto the set of the closed irreducible subspaces of X; in particular, any irreducible component of a closed subspace Y of X is of the form $V(P)$, where P is minimal among prime ideals of A which contain $I(Y)$. ([Bou,2], II, sec.4, n^{o} 3, Cor.2 to Prop.14)

[Pf] For any prime ideal P of A, $I(V(P)) = P$, that is, $V(P)$ is irreducible by (1.1.18); we know that $V(P)$ is closed, and V is a bijection when restricted to the set of all radical ideals of A by (1.1.5), (iii). In particular, for two distinct prime ideals P and Q, we have $V(P) \neq V(Q)$, that is, α is injective. Now for any closed irreducible subset Y of X, $Y = \mathrm{Cl}(Y) = V(I(Y))$, since Y is closed. Furthermore $I(Y)$ is prime since Y is irreducible, that is, Y is an image of a prime ideal by the map α, so the map is surjective; therefore α: $[P] \mapsto V(P)$ is a bijective map of X onto closed irreducible susbspaces of X.

On the other hand, $Y \supseteq V(P) \supseteq V(Q)$ iff $Q \supseteq P = I(V(P)) \supseteq I(Y)$; thus $V(P)$ is an irreducible component of Y, i.e., a maximal irreducible subset of Y, iff P is minimal among the prime ideals that contain $I(Y)$. ■

Proposition (1.1.21). — The prime spectrum $X = \mathrm{Spec}(A)$ of a ring A is connected iff 0 and 1 are the only idempotent elements of A. ([Bou,2], II, sec.4, n° 3, cor.2 to Prop.15)

[Pf] Suppose that X is not connected and is a disjoint union of nonempty open subsets Y_1 and Y_2. Since Y_1 and Y_2 are also closed, there exist two radical ideals I_1 and I_2 such that $Y_1 = V(I_1)$, $Y_2 = V(I_2)$. By hypothesis we have at first $V(I_1 + I_2) = V(I_1) \cap V(I_2) = Y_1 \cap Y_2 = \emptyset = V(1)$, thus there exist $e \in I_1$, $f \in I_2$ such that $e + f = 1$ (cf. (1.1.6)); secondly $V(I_1 I_2) = V(I_1) \cup V(I_2) = Y_1 \cup Y_2 = X = V(0)$, thus we have some integer $n > 0$ such that $(ef)^n = e^n f^n = 0$. Since $V(e^n) = V(e)$ and $V(f^n) = V(f)$, $V(e^n + f^n) = V(e^n) \cap V(f^n) = V(e) \cap V(f) = \emptyset = V(1)$. So that there exist $u, v \in A$ such that $ue^n + vf^n = 1$ and then $0 = uv(e^n f^n) = ue^n vf^n = ue^n(1 - ue^n) = ue^n - (ue^n)^2$, that is, ue^n is an idempotent. Similarly, vf^n is idempotent. If $ue^n = 0$, then $vf^n = 1$, so $\emptyset = V(1) = V(vf^n) = V(v) \cup V(f^n) = V(v) \cup V(f) \supseteq V(v) \cup Y_2$, that is, $Y_2 = \emptyset$, a contradiction. If $ue^n = 1$, then by a similar argument, we have the contradiction $Y_1 = \emptyset$. Thus ue^n is an idempotent different from 0 and 1.

Conversely if there exists an idempotent e distinct from 0 and 1, we get $V(e) \neq \emptyset$, $V(1 - e) \neq \emptyset$, and $V(e) \cup V(1 - e) = V(e(1 - e)) = V(e - e^2) = V(0) = X$, $V(e) \cap V(1 - e) = V(e + 1 - e) = V(1) = \emptyset$, that is , $V(e)$ and $V(1 - e)$ are disjoint nonempty clopen subsets of X whose union is X, thus X is not connected. ■

Section 2. Spectral spaces and the patch topology.

Definition (1.2.1). — A topological space X is called *spectral* if it satisfies the following four conditions: (i) X is a T_0-space; (ii) X is quasi-compact; (iii) X has an open basis consisting of quasi-compact open subsets of X; and (iv) every closed irreducible subset of X has a generic point.

A continuous map between two spectral spaces is called a *spectral map* if its inverse image of any quasi-compact open subset is quasi-compact.

A subspace Y of a spectral space X is called a *spectral subspace* of X if the inclusion map is spectral. ([Hoc, 1])

Note (1.2.2). — The prime spectrum of a ring is a spectral space (1.1.7) (for (i)), (1.1.13) (for (ii)) and (1.1.3) (for (iii)), and (1.1.18) (for (iv)); the associated map of a ring homomorphism is a spectral map by (1.1.14).

Definition (1.2.3). — By the *patch topology* we mean the topology defined on a spectral space X with a subbasis for closed sets consisting of the closed sets and quasi-compact open subsets in the original space X (or equivalently, which has the quasi-compact open sets and their complements as an open subbasis), and the space considered to be a topological space with the patch topology is denoted by *con(X)*. A subset of X closed in con(X) is called a *patch* of X. ([Hoc,1])

Proposition (1.2.4). — The patch topology is compact (that is, Hausdorff and quasi-compact). ([Hoc,1], Th.1)

[Pf] Let X be a spectral space. (i) con(X) is Hausdorff: for any distinct points x, y in X, there is an open neighbourhood U of one of the two points, say x, not containing the other, y. Since quasi-compact open sets form an open basis of X, we can take U as quasi-compact open in X, and thus U is a clopen neighbourhood of x in con(X) that doesn't contain y. So that we have an open neighbourhood X - U of y in con(X) that is disjoint from the open neighbourhood U of x. ∎

(ii) con(X) is quasi-compact: Let $\mathscr{U}$ be a maximal family of subbasic closed subsets of con(X) with (FIP). So $\mathscr{U}$ consists of quasi-compact open sets and closed subsets of X. We plan to show that $\cap\mathscr{U} \neq \varnothing$ (then by the Alexander subbasis theorem, con(X) is quasi-compact). Let $\mathscr{V}$ be the subfamily of $\mathscr{U}$ consisting of closed sets. Clearly $\mathscr{V}$ has (FIP) and since X is quasi-compact, $F = \cap\mathscr{V} \neq \varnothing$. Clearly F is closed in X. If $F \notin \mathscr{U}$, then $\mathscr{U}' = \mathscr{U} \cup \{F\} \supsetneq \mathscr{U}$ consists of subbasic closed sets of con(X) with (FIP), contradicting the maximality of $\mathscr{U}$. Thus $F \in \mathscr{U}$. Next we show F is irreducible. For if $F = F' \cup F''$, F', F'' closed in X, $F' \neq \varnothing$, $F'' \neq \varnothing$, consider $\mathscr{U} \cup \{F', F''\}$. This is a family of subbasic closed subsets of con(X), and strictly contains $\mathscr{U}$. If we verify that $\mathscr{U} \cup \{F', F''\}$ has (FIP), that will contradict the maximality of $\mathscr{U}$ and so it follows that one of F', F'' is empty, and thus F is irreducible.

Let $\{U_1,\dots,U_n\} \subseteq \mathscr{U} - \mathscr{V}$. Now $U_1 \cap \dots \cap U_n \cap F \neq \varnothing$. Suppose $U_1 \cap \dots \cap U_n \cap F'' = \varnothing$ (hence $U_1 \cap \dots \cap U_n \cap F' \neq \varnothing$). Then we claim that for every

$\{V_1,\ldots, V_m\} \subseteq \mathscr{U} - \mathscr{V}$, $V_1 \cap \ldots \cap V_m \cap F' \neq \varnothing$. Now $V_1 \cap \ldots \cap V_m \cap F \neq \varnothing$ by (FIP) of $\mathscr{U}$. $V_1 \cap \ldots \cap V_m \cap F \supseteq V_1 \cap \ldots \cap V_m \cap U_1 \cap \ldots \cap U_n \cap F$ and this is equal to $(V_1 \cap \ldots \cap V_m \cap U_1 \cap \ldots \cap U_n \cap F') \cup (V_1 \cap \ldots \cap V_m \cap U_1 \cap \ldots \cap U_n \cap F'') = V_1 \cap \ldots \cap V_m \cap \ldots \cap U_1 \cap \ldots \cap U_n \cap F'$ as $U_1 \cap \ldots \cap U_n \cap F'' = \varnothing$. Thus $V_1 \cap \ldots \cap V_m \cap F' \supseteq V_1 \cap \ldots \cap V_m \cap U_1 \cap \ldots \cap U_n \cap F' \neq \varnothing$. This proves our claim. So F is irreducible. Finally since F is a closed irreducible subset of the spectral space, F has a (unique) generic point x, i.e., F = $Cl_X\{x\}$. We show that for each $U \in \mathscr{U}$, $x \in U$. Since $x \in F$, clearly x is contained in every member of $\mathscr{V}$. Let $U \in \mathscr{U} - \mathscr{V}$. Then U is open in X, so $U \cap F \neq \varnothing$. This implies $x \in U$ since $F = Cl\{x\}$. ■

Proposition (1.2.5). — Let X and X' be spectral spaces.

(i) For any spectral subspace Y of X, con(Y) is the subset Y of con(X) with its subspace topology.

(ii) A map f: $X \to X'$ is spectral iff it is continuous in both original and patch topologies.

(iii) For a spectral map f: $X \to X'$, f(X) is a patch in X'. ([Hoc,1], p.45)

[Pf] (i) Let Y be a spectral subspace of X. Since the inclusion map is spectral, for any quasi-compact open subset V of X, $V \cap Y$ is quasi-compact (open) in Y. Therefore a basis of open sets for Y is $\{V \cap Y: V \in \mathscr{U}\}$, where $\mathscr{U}$ is a basis of open sets for X. So each member of the subbasis of open sets for con(Y) (which consists of the family of an quasi-compact open subset of Y and their complements in Y) can be obtained as $S \cap Y$, where S is quasi-compact open in X or the complement of a quasi-compact open subset in X. So con(Y) is the subset Y of con(X) with the induced topology. ∎

(ii): Let f: $X \to X'$. Let both f: $X \to X'$, and f: con(X) $\to$ con(X') be continuous. We show that f: $X \to X'$ is spectral. Let U' quasi-compact open in X'. Since f: $X \to X'$ is continuous, $f^{-1}(U')$ is open in X. Since U' and its complement are basic open sets in con(X'), they are both clopen in con(X). Therefore $f^{-1}(U')$ is clopen in con(X) as f: con(X) $\to$ con(X') is continuous. So that $f^{-1}(U')$ is quasi-compact in con(X) and so quasi-compact in X, because X has a coarser topology than con(X). So f: $X \to X'$ is spectral. Conversely let f: $X \to X'$ be spectral. Then f: $X \to X'$ is continuous . We check that f: con(X) $\to$ con(X') is continuous. Let F' be a subbasic closed subset of con(X'). Then F' is either closed in X', so $f^{-1}(F')$ is closed in X or F' is quasi-compact open in X', so $f^{-1}(F')$ is quasi-compact as f is spectral, and thus in either case $f^{-1}(F')$ is closed in con(X). This shows that f: con(X) $\to$ con(X') is continuous. ∎

(iii): Let f: $X \to X'$ be spectral. Then f: con(X) $\to$ con(X') is continuous. Now con(X) is compact, so that f(X) is compact in con(X'), thus closed in con(X'). By definition, f(X) is a patch of X'. ∎

Proposition (1.2.6). — A subspace Y of a spectral space X is a spectral subspace of X iff it is a patch in X. ([Hoc,1], p.45)

[Pf] If Y is a spectral subspace of X, then the inclusion map i: $Y \to X$ is spectral and so by (1.2.5), (iii), it follows that Y is a patch in X.

Conversely, let Y be a patch in X. Since every quasi-compact subset F of X is closed in the Hausdorff space con(X), $F \cap Y$ is closed in con(Y) by (1.2.5), (i). So $F \cap Y$ is quasi-compact in con(Y) and also in the coarser topology of Y, that is, $i^{-1}(F) = F \cap Y$ is quasi-compact in Y. Therefore for the inclusion map i: $Y \to X$, preimages of quasi-compact open subsets are quasi-compact, that is, i is spectral. Hence Y is a spectral subspace of X. ■

Definition. (1.2.6.1). — Let x, y be two points of a topological space X. Then y (resp. x) is said to be a *specialization* of x (resp. *generalization* of y) if $y \in \mathrm{Cl}\{x\}$. Let Y be a subset of X, then let us denote by *sp(Y)* (resp. *gen(Y)*) the set of specializations (resp. generalizations) of all elements of Y, that is, $\mathrm{sp}(Y) = \{z \in X: z \in \mathrm{Cl}\{y\}$ for some $y \in Y\}$, and $\mathrm{gen}(Y) = \{z \in X: y \in \mathrm{Cl}\{z\}$ for some $y \in Y\}$. We note that $Y = \mathrm{sp}(Y)$ for any closed Y.

Proposition (1.2.7). — A spectral subspace Y of a spectral space X is closed iff it is stable under specialization, that is, $Y = \mathrm{sp}(Y)$. ([Hoc,1], Cor. to Th.1)

[Pf] Since obviously $\mathrm{Cl}Y \supseteq \mathrm{sp}(Y)$, it suffices to show $\mathrm{Cl}Y \subseteq \mathrm{sp}(Y)$. Let $y \in \mathrm{Cl}Y$ and $\mathscr{U}$ be the family of all quasi-compact open neighbourhoods of y. Let $\mathscr{V} = \mathscr{U} \cup \{Y\}$. Then $\mathscr{V}$ is a family of closed sets of con(X) which has (FIP), since any neighbourhood of y intersects Y. By the compactness of X, $\cap\mathscr{V} = Y$

$\cap(\cap\mathscr{U}) \neq \varnothing$. So there exists an $x \in Y$, such that $x \in \cap\mathscr{U}$, that is, $y \in Cl\{x\}$, so $y \in sp(Y)$. ■

Definition (1.2.8). — A topological space is said to be *quasi-Hausdorff* if any two distinct points either have disjoint neighbouhoods or are in the closure of a third point.

Proposition (1.2.8.1). — Every spectral space is quasi-Hausdorff. ([Hoc,1], Cor. to Th.1, p.45)

[Pf] Let x, y be two points of X, $\mathscr{U}_x$, $\mathscr{U}_y$ be the set of all quasi-compact open neighbourhoods of x, y respectively. If no two neighbourhoods U in $\mathscr{U}_x$ and V in $\mathscr{U}_y$ are disjoint, then $\mathscr{U}_x \cup \mathscr{U}_y$ is a family of closed sets in con(X) with (FIP). So there exists a point $z \in \cap(\mathscr{U}_x \cup \mathscr{U}_y)$. Since $\mathscr{U}_x$, $\mathscr{U}_y$ are bases of neighbourhoods of x, y respectively, $x, y \in Cl\{z\}$. ■

Definition (1.2.9). — Clearly the class of spectral spaces with spectral maps as morphisms forms a category, which we denote by $\mathscr{S}$.

Proposition (1.2.9.1). — In $\mathscr{S}$, arbitrary products, images, inverse limits and finite coproducts exist and all the underlying spaces are respectively counterparts in the category $\mathscr{T}$ of topological spaces with continuous maps as morphisms. ([Hoc,1], Th.7)

[Pf] The categorical products: Let $\{X_i: i \in I\}$ be a family of spectral spaces. Then it is easy to verify that $X = \prod\{X_i: i \in I\}$ with the product topology is a spectral space, each projection pr_i: $X \to X'$ is spectral and X is the categorical product of the X_i's.

The statement about arbitrary inverse limits immediately follows.

Because images with spectral maps of spectral spaces are patches (cf. (1.2.5), (iii)) and a subspace is spectral iff it is a patch (cf. (1.2.6)), spectral images in $\mathscr{S}$ are spectral spaces.

Let $X_1,\ldots, X_n$, be spectral spaces. Let X be the disjoint union of $X_1,\ldots, X_n$. Then it can be easily shown that X is a spectral space and the inclusion map i_j: $X_j \to X$ is spectral for any $j = 1,\ldots, n$. It is easy to prove that X is the coproduct of $X_1,\ldots, X_n$. ■

Remark (1.2.9.2). — If $\{X_i: i \in I\}$ is not a finite family of spectral spaces, the disjoint union X of $\{X_i\}$ need not be spectral. However we shall prove arbitrary coproducts, quotients, and direct limits exist in $\mathscr{S}$ in (1.3.5), (1.3.7) and (1.3.8).

Proposition (1.2.10). — A topological space X is spectral iff X is isomorphic to a patch of a product of copies of the topological space $W = \{0, 1\}$ whose open sets are $\varnothing$, $\{0\}$, and W. ([Hoc,1], Prop.9)

[Pf] At first note that W is spectral with $\{\{0\}, W\}$ being the open basis and $\{\{1\}, W\}$ being the set of all closed irreducible subsets.

$\Rightarrow$: Let V be the set of all spectral maps from X to W, W^V be the product space of copies of W indexed by V, and e: $X \to W^V$ defined by $e(x)(f) = f(x)$ be the evaluation map. Then W^V is spectral (because it is a product of spectral spaces) and e is spectral (because e is continuous with respect to both the original and the patch topologies since for any $x \in X$ and $f \in V$, $(pr_f \circ e)(x) = f(x)$, where both pr_f (the f-th projection map) and f are continuous with respect to the original and patch topologies).

Next, for any two distinct points x, y of X there exists a quasi-compact open subset U of X which contains one of x, y and not the other, since X is T_0. Say $x \in U$, $y \notin U$. Define a map $f: X \to W$ by $f(U) = \{0\}$, $f(X - U) = \{1\}$. Then since the only nonempty open sets in W are $\{0\}$ and W, and $f^{-1}(\{0\}) = U$, $f^{-1}(W) = X$, f is spectral. And $e(x)(f) = f(x) = 0 \neq 1 = f(y) = e(y)(f)$, thus $e(x) \neq e(y)$, that is, e is injective.

Suppose E is a closed subset of X and $y \in X - E$. Define a map $f: X \to W$ by $f(E) = \{1\}$, $f(X - E) = \{0\}$, then a similar argument as above shows that $f \in V$ and $f(y) = 0 \notin \{1\} = Cl(f(E))$, that is, V separates points from closed sets. Let $x \in X$ and U be an open neighbourhood of x in X. Choose $f \in V$ such that $f(x) = pr_f(e(x)) \notin Cl(f(X - U))$, and let $B = \{y \in W^V: pr_f(y) \notin Cl(f(X - U))\}$, then $e(x) \in B$ and $B = pr_f^{-1}(W - Cl(f(X - U)))$ is open in W^V. Since $f(x) \notin Cl(f(X - U))$, we get $Cl(f(X - U)) \neq W$, that is, $Cl(f(X - U)) = \{1\} = f(X - U)$, $f(U) = \{0\}$. Thus from the injectivity of e, it follows obviously that $e^{-1}(B \cap e(X)) = e^{-1}(B) \cap X = e^{-1}(pr_f(W - Cl(f(X - U))) = e^{-1}(pr_f^{-1}(\{0\})) = f^{-1}(\{0\}) = U$. So by considering e as a bijection of X onto e(X), we have $B \cap e(X) = e(e^{-1}(B \cap e(X))) = e(U)$, that is,

$e(U)$ is open in $e(X)$. This shows that e is an open map, and consequently e is a homeomorphism of X onto $e(X)$. As a result, e is a spectral embedding of X into W^V. Thus X is isomorphic to the patch $e(X)$ in W^V. ∎

$\Leftarrow$: The converse is immediate from (1.2.6) taking into account that W^V is a spectral space by (1.2.9.1). ∎

Proposition (1.2.11). — A topological space is spectral iff it is an inverse limit of finite T_0 spaces. ([Hoc,1], Prop.10)

[Pf] Obviously any finite T_0 space is quasi-compact and has an open basis consisting of quasi-compact open subsets. Let $\{x_1, \ldots, x_n\} = E$ be a closed irreducible subset of a finite T_0 space X, and U be the intersection of all the open subsets of E. Since E is irreducible, U is not empty and is finite. If U contains more than one point, U cannot satisfy T_0 separation axiom, thus U is a singleton and $Cl_E U = E$, that is, E has its generic point. It follows that any finite T_0 space is spectral. So an inverse limit of finite T_0 spaces is spectral by (1.2.9.1).

Conversely, let X be a spectral space, then by (1.2.10), X is isomorphic to a patch of W^V. Let $e: X \to W^V$ be the embedding. We are going to construct an inverse system $(X_S, f_{S,T}; I)$ relative to the set I, the set of finite subsets of V ordered by inclusion, such that $X = \text{inv.lim.}X_S$. For any $S \in I$ let $\alpha_S: W^V \to W^S$ be the map such that for any element $f \in W^V$, $\alpha_S(f) \in W^S$, where $f(s) = \alpha_S(f)(s)$, $s \in S$, i.e., α_S is the coordinate-wise projection. Define $f_S = \alpha_S \circ e$, X_S

$= f_S(X)$ and for $S, T \in I$ with $S \subseteq T$ define $f_{S,T}$ to be the restriction of $h_{S,T}$ to X_T, where $h_{S,T} : W^T \to W^S$ is a map such that $h_{S,T}((w_t)_{t \in T}) = (w_s)_{s \in S}$ for $(w_t)_{t \in T} \in W^T$. Then for any $x \in X$, $f_S(e(x)) = f_{S,T}(f_T(e(x))$ for $S, T \in I$ with $S \subseteq T$; and for $S, T, U \in I$ with $S \subseteq T \subseteq U$, $f_{S,U} = f_{S,T} \circ f_{T,U}$. Hence $X = \text{inv.lim.}(X_S, f_{S,T}; I)$ and so X is realized as an inverse limit of finite T_0 spaces. ■

From the following up to (1.2.14) we generalize the definition of con(X) to arbitrary T_0 spaces instead of spectral spaces as we did earlier in (1.2.3) above.

Definition (1.2.12). — Let X be a T_0 space. (i) By *con(X)* we mean the topological space whose underlying set is X with topology for which the closed subsets and the quasi-compact open sets of X form a subbasis for closed sets. A closed set in con(X) is called a *patch* in X. (ii) We can define a partial order on X thus: If $x, y \in X$, $x \leq y$ means $y \in Cl\{x\}$. (This is indeed a partial order on X.)

Proposition (1.2.12.1). — If a quasi-compact T_0 space X has an open basis consisting of quasi-compact open subsets of X stable under finite intersection, then the following are equivalent:

(i) X is spectral.

(ii) Every nonempty closed irreducible subset of X has a generic point.

(iii) If a family of quasi-compact open subsets of a closed subset of X has (FIP), then it has nonempty intersection.

(iv) con(X) is compact with a basis consisting of clopen subsets.

(v) con(X) is quasi-compact.

(vi) If a family of patches in X has (FIP), then it has nonempty quasi-compact intersection in X.

(vii) Every closed subspace is quasi-Hausdorff and every lower directed set has a greatest lower bound in its closure. ([Hoc,1], Prop.6) (Note: The partially ordered set X is said to be *lower directed* if for any x, y, there is $z \in X$ such that $z \leq x, y$.)

[Pf] (i) $\Leftrightarrow$ (ii): Obvious by the definition of spectral space (1.2.1). ∎

(i) $\Rightarrow$ (v): By (1.2.4), we have this implication. ∎

(v) $\Rightarrow$ (vi): A patch is a closed subset of con(X), so a family of patches with (FIP) has a nonempty intersection, which is closed in the quasi-compact space con(X), so is quasi-compact in con(X) and then also in coarser topology of X. ∎

(vi) $\Rightarrow$ (iv): By the Alexander Subbasis Theorem, the assertion (vi) implies that con(X) is quasi-compact. By hypothesis, X has an open basis $\mathscr{B}$ consisting of quasi-compact open subsets of X closed under finite intersection. Let con($\mathscr{B}$) be the collection of subsets of X generated by the members of $\mathscr{B}$ and their complements in X under finite intersection. We claim con($\mathscr{B}$) is a basis for open sets of con(X) and the proof of this claim would establish the above implication. Let U be open in con(X) and $x \in U$. Then $F = X - U$ is closed in con(X) and $x \notin F$, i.e., $F_1 \cap \ldots \cap F_n \supseteq F$, $x \notin F_i$ for any i, where each F_i is closed in X or a quasi-compact open set in X. So x belongs to an open subset of U or x belongs to the complement of a quasi-compact open subset of X contained in U, that is, x belongs to some quasi-compact open subset of X contained in U (as $\mathscr{B}$ is an open basis of X) or x belongs to the complement of quasi-compact open set of X contained in U. Thus in any case x belongs to some member of con($\mathscr{B}$) contained in U. So con($\mathscr{B}$) is a basis for open sets of con(X). This completes the proof of our claim. It is clear that con(X) is Hausdorff. So con(X) is compact. ∎

(iv) $\Rightarrow$ (iii): Let Y be a closed subset of X. Then Y is a patch of X and any quasi-compact open subset of Y is closed in con(Y). Thus a family of quasi-compact open subsets of Y is a family of closed subsets of con(Y). If this family has (FIP) then it has nonempty intersection. ∎

(iii) $\Rightarrow$ (ii): Let Y be a nonempty closed irreducible subset of X. Take the family $\mathscr{U}$ of all the quasi-compact open subsets of Y, then $\mathscr{U}$ has (FIP), since Y is irreducible. Therefore $\cap\mathscr{U} \neq \varnothing$. If $\cap\mathscr{U}$ contains two distinct points, then there exists a basic open subset U of X that contains only one of them. But then $U \cap Y$ is a basic open subset of Y which is quasi-compact open, and U

$\cap Y \notin \mathscr{U}$, a contradiction since $U \cap Y \in \mathscr{U}$ by the definition of $\mathscr{U}$. Thus $\cap\mathscr{U}$ reduces to a set of one point which is obviously a generic point of Y. ∎

(i) (equivalently, (ii)) $\Rightarrow$ (vii): Every spectral space X is quasi-Hausdorff by (1.2.8.1). If Y is a closed subset of X, then Y is a patch in X and so Y is a spectral subspace of X. Hence Y is quasi-Hausdorff. Let E be a lower directed set. Let U, V be two relatively open subsets of E. Let $x \in U$, $y \in V$. Then there exists $z \in E$ such that $z \leq x, y$, that is $x, y \in Cl\{z\}$. Thus any neighbourhood of x and y contanins z, so that $z \in U \cap V$. Therefore E is irreducible, and so is ClE. Let z_0 be a generic point of ClE, then obviously $E \subseteq ClE = Cl\{z_0\}$, so that z_0 is a lower bound of E. If there exists another lower bound z' of E, then E is contained in Cl{z'}. Then we have $Cl\{z_0\} \subseteq Cl\{z'\}$, and so $z' \leq z_0$, that is, z_0 is the greatest lower bound of E. ∎

(vii) $\Rightarrow$ (ii): Let Y be a nonempty closed irreducible subset of X. Since Y is closed in X so it is quasi-Hausdorff. Let x, y be any two distinct points of Y. Since Y is irreducible, x and y have no disjoint open neighbourhoods relative to Y and hence in X. Since X is quasi-Hausdorff, they are in the closure of a third point z, that is, $x, y \in Cl\{z\}$, so that $x, y \geq z$. Thus Y is lower directed, and so by hypothesis, there exists a greatest lower bound z_0 of Y in Y = ClY. Since for any $y \in Y$ $z_0 \leq y$, $Y \subseteq Cl\{z_0\}$. Therefore z_0 is a generic point of Y. ∎

Proposition (1.2.13). — Let X be a compact space, and let $\mathscr{U} = \{U_i : i \in I\}$ be a family of clopen subsets of X. Let X' be the topological space whose underlying set is X and with the topology whose open subbasis is $\mathscr{U}$. Then X' is spectral iff it is T_0, in which case X is the patch space of X', that is, X = con(X'). Conversely, every spectral space arises from its patch space in this way. ([Hoc,1], Prop.7)

[Pf] Let $\mathscr{V}$ be the family of all finite intersections of the members of $\mathscr{U}$. Then any member of $\mathscr{V}$ is a clopen subset of X, namely, compact open subset of X. Obviously $\mathscr{V}$ is an open basis of X' and the original topology on X is finer than the topology on X'. So any member of $\mathscr{V}$ is quasi-compact open in X' and X' is quasi-compact. Now if X' is T_0, then it satisfies the hypothesis of (1.2.12). So in that case if we can show that the patch space of X' is X, then by (1.2.12), (iv), we know that X' is spectral. But since the patch space of X' is compact (so Hausdorff) and coarser than the compact (Hausdorff) topology on X, the patch topology on X' coincides with the topology on X. By hypothesis, X' is T_0 and so it follows that X' is spectral and con(X') = X. ■

Proposition (1.2.14). — A topological space X' with a basis of quasi-compact open sets is spectral iff its patch space is compact. ([Hoc,1], Cor. to Prop.7)

[Pf] If X' is spectral then its patch space is compact by (1.2.4). Conversely, suppose the patch space con(X') is compact. If we show X' is T_0, then by taking the family of quasi-compact open subsets of X' as $\{U_i: i \in I\}$ in (1.2.13), we will have shown that X' is spectral.

Take two distinct points x, y of X'. By hypothesis, con(X') is Hausdorff. So there exists a basic open subset U of con(X') such that $x \in U$, $y \notin U$. U is a finite intersection of quasi-compact open subsets $A_0,\dots, A_n$ of X' and the complements of quasi-compact open subsets $B_0,\dots, B_m$ of X'. So that there exist A_i or B_j such that $y \notin A_i$ or $y \notin X' - B_j$. In the first case, $x \in A_i$, $y \notin A_i$, and in the second case, $x \notin B_j$, $y \in B_j$; thus X is T_0. ■

Definition (1.2.15). — We recall the following definitions:

Let $\mathscr{C}$ be a category, A, B, C $\in$ Obj($\mathscr{C}$). (1°) Let $f: A \to C$, $g: B \to C$ morphisms. If there exist $D \in \mathrm{Obj}(\mathscr{C})$, $\pi_1:D \to A$, $\pi_2: D \to B$ such that $f \circ \pi_1 = g \circ \pi_2$ and if there exist $U \in \mathrm{Obj}(\mathscr{C})$, $p_1: U \to A$, $p_2: U \to B$ with $f \circ p_1 = g \circ p_2$, then there is a unique morphism $t: U \to D$ such that $p_1 = \pi_1 \circ t$ and $p_2 = \pi_2 \circ t$. Then D is called the *fibered product* (or *pull back*) $A \times_C B$ of A and B over C. (It is not difficult to see that D is unique up to "canonical" isomorphism.)

(2°) Let $f: C \to A$, $g: C \to B$ be morphisms. If there exist $D \in \mathrm{Obj}(\mathscr{C})$, $i_1 : A \to D$, $i_2: B \to D$ such that $i_1 \circ f = i_2 \circ g$ and whenever there exist $U \in \mathrm{Obj}(\mathscr{C})$, $j_1 : A \to U$, $j_2: B \to U$ with $j_1 \circ f = j_2 \circ g$, then there exists a unique morphism $t: D \to U$ such that $j_1 = t \circ i_1$, $j_2 = t \circ i_2$. Then D is called a *fibered sum* (or *push out*) $A +_C B$ of A and B over C. Again it is well known that $A +_C B$ is unique up to "canonical" isomorphism.

Note (1.2.15.1). — (i) If A, B and C are rings and $u: A \to C$, $v: B \to C$ are ring homomorphisms, then the ring $\{(a, b) \in A \times B: u(a) = v(b)\} = A \times_C B$ as a subring of $A \times B$ along with π_1, π_2 the restrictions of projections of $A \times B$ to A and B, respectively.

(ii) If A, B and C are rings and $u: C \to A$, $v: C \to B$ are ring homomorphisms , then $A +_C B = A \otimes_C B$.

(iii) If X and Y are topological spaces and Z is a closed subset of Y, $\alpha: Z \to X$ is a continuous map, then $X +_Z Y = X \cup_\alpha Y$, where $X \cup_\alpha Y$ is the quotient topological space of disjoint sum $X + Y$ of X and Y modulo the equivalence relation generated by $z \sim \alpha(z)$ for all $z \in Z$ (often called *attaching space of Y to X over the closed set Z by the continuous map* α, or *adjunction space* of X and Y). Let $q: X + Y \to X \cup_\alpha Y$ be the quotient map. It is known that: (1°) $q|_X$ is a homeomorphism and q(X) is closed in $X \cup_\alpha Y$; (2°) $q|_{(Y - Z)}$ is a homeomorphism and $q(Y - Z)$ is open in $X \cup_\alpha Y$.

(iv) Let $\mathscr{C}$ be the category of all the preschemes. Let X = Spec(A), Y = Spec(B), Z = Spec(C), where A, B, C are objects in the category of commutative rings $\mathscr{C}$, and $f: C \to A$, $g: C \to B$ are ring homomorphisms, then $\mathrm{Spec}(A \otimes_C B) = X \times_Z Y$. ([Har], chap.III, Th.3.3)

(v) As a special case, we have $\mathrm{Spec}(A \times B) = \mathrm{Spec}(A) \cup \mathrm{Spec}(B)$ (disjoint).

Proposition (1.2.16). — Let u: A $\to$ C, v: B $\to$ C be ring homomorphisms and v surjective. Then $\mathrm{Spec}(A) +_{\mathrm{Spec}(C)} \mathrm{Spec}(B) = \mathrm{Spec}(A) \cup_\alpha \mathrm{Spec}(B)$, where $\alpha = {}^a u$. ([Fon], Th.1.4)

[Pf] Let U be a spectral space, p_1: Spec(B) $\to$ U and p_2: Spec(A) $\to$ U be continuous maps such that $p_1 \circ {}^a v = p_2 \circ {}^a u$. Then there exist a ring D with U = Spec(D) and ring homomorphisms f: D $\to$ A, g: D $\to$ B with ${}^a f = p_1$, ${}^a g = p_2$ (as we prove later in Chap.2 the existence of such a ring and homomorphisms independently of this theorem). Then we have $f \circ u = g \circ v$. So by the universality property of $A \times_C B$ there exists a unique ring homomorphism t: D $\to A \times_C B$ and ${}^a t$: $\mathrm{Spec}(A \times_C B) \to \mathrm{Spec}(D)$, that is, $\mathrm{Spec}(A \times_C B)$ satisfies the universality property. Thus $\mathrm{Spec}(A \times_C B) = \mathrm{Spec}(A) +_{\mathrm{Spec}(C)} \mathrm{Spec}(B)$. So it sufices to prove that $\mathrm{Spec}(A \times_C B) \cong \mathrm{Spec}(A) \cup_\alpha \mathrm{Spec}(B)$.

Claim (1.2.16.1). — Let A, B, C, u and v as above, and $D = A \times_C B$. Let u': D $\to$ B, v': D $\to$ A be the restrictions to D of the projection maps. Set X = Spec(A), Y = Spec(B), Z = Spec(C), W = Spec(D), $\alpha = {}^a u$: Z $\to$ X, $\beta = {}^a v$: Z $\to$ Y, $\alpha' = {}^a u'$: Y $\to$ W, and $\beta' = {}^a v'$: X $\to$ W.

$$\begin{tikzcd}
A \arrow[dr, "u"] & \\
D \arrow[u, "v'"] \arrow[r] \arrow[d, "u'"'] & C, \\
B \arrow[ur, "v"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \arrow[dl, "\beta'"'] \arrow[d] & \\
W & X \cup_\alpha Y \arrow[l] & Z. \arrow[l] \arrow[ul, "\alpha"'] \arrow[dl, "\beta"] \\
& Y \arrow[ul, "\alpha'"] \arrow[u] &
\end{tikzcd}$$

Then v' is a surjective homomorphism, and so $\beta' = {}^a v'$ is a closed embedding (cf. (1.1.15.1) or (1.1.8)).

[Pf] For any $a \in A$, $u(a) \in C$. Since v is surjective by hypothesis, there is $b \in B$ such that $v(b) = u(a)$. Hence $(a, b) \in A \times_C B$ and $v'(a, b) = pr_1 |_D(a, b) = a$. Thus v' is surjective. ▮

Claim (1.2.16.2). — Let $\mathbf{b} = \mathrm{Ker}(v)$ and $\mathbf{d} = \mathrm{Ker}(v')$, then (1°) $u'|_{\mathbf{d}}: \mathbf{d} \to \mathbf{b}$ is an isomorphism of modules (subordinate to $u': D \to B$), (2°) the *conductor* (is by definition the ideal $\mathrm{Ann}_A(B/\mathrm{Im}(f))$ for a ring homomorphism $f: A \to B$, in our case, $\mathrm{Ann}_D(B/u'(D))$, contains $\mathbf{d}$, and (3°) for every $h \in \mathbf{d}$, the canonical homomorphism $D_h \to B_{u'(h)}$ is an isomorphism, where D_h is the ring of fractions of D with the multiplicative set $\{h^n : n \in \mathbf{N}\}$.

[Pf] (1°): $v(u'(\mathbf{d})) = u(v'(\mathbf{d})) = u(0) = 0$. Thus $u'(\mathbf{d}) \subseteq \mathbf{b}$. Let $b \in \mathbf{b}$, then $v(b) = 0$, so that for any $a \in \ker(u)$, $(a, b) \in \mathbf{d}$, and $u'(a, b) = pr_2 |_D(a, b) = b$. Thus $\mathbf{b} \subseteq u'(\mathbf{d})$, so we have $u'(\mathbf{d}) = \mathbf{b}$. Next let $d, d' \in \mathbf{d}$, $d = (a, b)$, $d' = (a', b')$. Suppose $u'(d) = u'(d')$, that is, $b = pr_2 |_D(a, b) = u'(a, b) = u'(d) = u'(d') = b'$. Since $d, d' \in \mathbf{d} = \ker(v')$, $a = pr_1 |_D(a, b) = v'(d) = 0 = v'(d') = a'$. Hence $d = d'$, and so $u' |_{\mathbf{d}}$ is a bijection of $\mathbf{d}$ onto $\mathbf{b}$. Obviously it is an isomorphism of modules. ▮

(2°): Since $B/\mathbf{b} \cong B/u'(\mathbf{d})$, and $\mathbf{b} = \mathrm{Ann}(B/\mathbf{b}) \supseteq \mathrm{Ann}(B/u'(D))$, $u'(\mathbf{d}) \subseteq \mathrm{Ann}(B/u'(D))$. Therefore $\mathbf{d} \subseteq \mathrm{Ann}_D(B/u'(D))$. ▮

(3°): Let $\varphi_h: D_h \to B_{u'(h)}$ be the map defined by $\varphi_h(d/h^n) = u'(d)/u'(h)^n$. Now $\mathbf{d} \subseteq \mathrm{Ann}_D(B/u'(D))$ means that for any $b \in B$ and $h \in \mathbf{d}$, $bu'(h) \in u'(D)$, that is, there exists $d \in D$ such that $bu'(h) = u'(d)$. Let $b/u'(h)^m$ be an arbitrary element of $B_{u'(h)}$, where $h \in \mathbf{d}$. Then $u'(h)b/u'(h)^{m+1} = u'(d)/u(h)^{m+1} \in \varphi_h(D_h)$. Thus φ_h is surjective. Let d/h^n, $d'/h^m \in D_h$. Suppose $\varphi_h(d/h^n) = \varphi_h(d'/h^m)$. Then by definition, there exists $k \in \mathbf{N}$ such that $0 = u'(h)^k(u'(d)u'(h)^m - u'(h)^n u'(d')) = u'(dh^{m+k} - h^{n+k}d')$. Obviously $dh^{m+k} - h^{n+k} \in \mathbf{d}$ since $h \in \mathbf{d}$. By (1.2.16.2), u' is bijective when restricted to $\mathbf{d}$, so $dh^{m+k} - h^{n+k}d' = 0$, that is, $d/h^m = d/h^n$. This shows that φ_h is injective, and so it is a bijection. It is obviously a ring homomorphism. ∎

Claim (1.2.16.3). — Let P be a prime ideal of $D = A \times_C B$ with $P \not\supseteq \mathbf{d}$. Then there exists a unique prime ideal Q of B such that $u'^{-1}(Q) = P$ with $Q \not\supseteq \mathbf{b}$ and $B_Q \cong D_P$.

[Pf] Let $S = D - P$, $S' = u'(S)$, then S and S' are multiplicative sets in D and B respectively. Since $P \not\supseteq \mathbf{d}$, there exists $h \in \mathbf{d} - P \subseteq D - P = S$. Note that $u'(h) \in \mathbf{b} \cap S'$. By (1.2.16.2), u' extends to the isomorphism $u'_h: D_h \to B_{u'(h)}$. By ([Bou,2], II, sec.2, n° 1, Prop.2 and Cor.4 to Prop.2), u'_h can extend to the isomorphism $u'_S: D \to S'^{-1}B$ by the change of multiplicative sets, where D_h is regarded as a subring of $S^{-1}D$. Since P is a prime ideal of D with $P \cap S = \varnothing$, there exists a unique prime ideal P' of $S^{-1}D$ such that $u'_S{}^{-1}(Q') = P'$. Let Q be the prime ideal of B such that $Q = Q' \cap B$. Then $Q \not\supseteq \mathbf{b}$. Since u'_S is an extension of the map u', $u'^{-1}(Q) = P$. Thus there exists a bijective correspondence between the prime ideals P of D such that $P \not\supseteq \mathbf{d}$ and the prime ideals Q of B such that $Q \not\supseteq \mathbf{b}$. ∎

Remark (1.2.16.4). — We know from (1.2.16.1), ${}^a v'$ is a closed embedding of Spec(A) to Spec(D) and v is onto by hypothesis so ${}^a v$ is also a closed embedding. Thus Spec(C) can be identified with the set of all the prime ideals Q of B such that $Q \supseteq \mathbf{b} = \ker(v)$, that is, a closed subset $V(\mathbf{b})$ of Spec(B). Moreover $\alpha = {}^a u$: Spec(C) $\to$ Spec(A) is continuous, so it is meaningful to talk about Spec(A) $\cup_\alpha$ Spec(B).

Now we want to show Spec($A \times_C B$) $\cong$ Spec(A) $\cup_\alpha$ Spec(B), that is, we want to construct a continuous closed bijective map σ: Spec(A) $\cup_\alpha$ Spec(B) $\to$ Spec($A \times_C B$).

As Spec(A) $\cup_{{}^a u}$ Spec(B) = Spec(A) $+_{\mathrm{Spec(C)}}$ Spec(B) and ${}^a u' \circ {}^a v = {}^a v' \circ {}^a u$, that is, $\alpha' \circ \beta = b' \circ \alpha$, by the universality of Spec(A) $+_{\mathrm{Spec(C)}}$ Spec(B), we have a unique morphism σ: Spec(A) $\cup_\alpha$ Spec(B) $\to$ Spec($A \times_C B$). Note that σ is continuous as it is a morphism.

Proposition (1.2.16.5). — σ: Spec(A) $\cup_\alpha$ Spec(B) $\to$ Spec(D) is a bijection.

[Pf] Let P be a prime ideal of $D = A \times_C B$. If $P \supseteq \mathbf{d}$, then P corresponds to a unique prime ideal of Spec(A) with respect to the surjective homomorphism v': $D \to A$ (cf. (1.1.7) or (1.1.15.1)), namely $V(\mathbf{d}) \cong \mathrm{Spec}(D/\mathbf{d}) \cong \mathrm{Spec}(A)$. Now Spec(C) is identified with a closed subset of Spec(B) as the subspace of all prime ideals of B which contain $\mathbf{b}$, that is, the kernel of the surjective homomorphism v: $B \to C$. So the set of all prime ideals Q of B such that $Q \not\supseteq \mathbf{b}$ is the open subset Spec(B) - Spec(C). Thus σ is the bijection from Spec(B) - Spec(C) onto the set of all prime ideals P of D such that $P \not\supseteq \mathbf{d}$. ▮

Claim (1.2.16.6). — The map σ is a closed map.

[Pf] Let F' be a closed subet in Spec(A) $\cup_\alpha$ Spec(B). Let $\sigma(F') = F$. We show that F is closed in Spec(D). Let X = Spec(A), Y = Spec(B), Z = Spec(C), W = Spec(D). Let $i_X: X \to X + Y$ and $i_Y: Y \to X + Y$ be the canonical injection, p: $X + Y \to X \cup_\alpha Y$ be the quotient map. Let $\alpha = {}^a u: Z \to X$, $\alpha' = {}^a v': Y \to W$, $\beta' = {}^a u': X \to W$. Then the following diagram commutes:

$$\begin{tikzcd}
& X \arrow[dl, "i_X"'] \arrow[d, "f_X"] \arrow[dr, "\beta'"] & \\
X + Y \arrow[r, "p"] & X \cup_\alpha Y \arrow[r, "\sigma"] & W. \\
& Y \arrow[ul, "i_Y"] \arrow[u, "f_Y"'] \arrow[ur, "\alpha'"'] &
\end{tikzcd}$$

Since $\alpha' = \sigma \circ p \circ i_Y$, we get $\alpha'(F) = i_Y^{-1}(p^{-1}(\sigma^{-1}(F))) = i_Y^{-1}(p^{-1}(F')) = p^{-1}(F') \cap Y$, that is, $\alpha'^{-1}(F)$ is closed in Y, since X + Y is disjoint. Similarly, $\beta'(F)$ is closed in X. We need show that if $F \subseteq W$ is such that $\alpha'^{-1}(F)$ is closed in Y and $\beta'^{-1}(F)$ is closed in X then F is closed in W. First , since $\alpha'^{-1}(F)$ is closed in Y = Spec(B), $\alpha'^{-1}(F) = V(I)$ for some ideal I of B. Similarly, $\beta'^{-1}(F) = V(J)$ for some ideal J of A. Then we have a ring homomorphism h: $A \times_C B \to (A/J) \times (B/I)$ composed the canonical injection $A \times_C B \to A \times B$ and the canonical surjection $A \times B \to (A/J) \times (B/I)$. Then ${}^a h(\mathrm{Spec}((A/J) \times (B/I)) = {}^a h(\mathrm{Spec}(A/J) \cup \mathrm{Spec}(B/I))$ (disjoint) $\subseteq W = \mathrm{Spec}(A \times_C B)$. Next we show $F = {}^a h(\mathrm{Spec}(A/J) \cup \mathrm{Spec}(B/I))$. Let $x \in F$. Since $W = X \cup \alpha'(Y)$, either $x \in F \cap X$ or $x \in F \cap \alpha'(Y)$. In the first case, $\beta'^{-1}(x) \in \beta'^{-1}(F) = V(J)$, that is, $j_{\beta'^{-1}(x)} \supseteq J$. So $h^{-1}(j_x) \supseteq J \times I$, and $x \in {}^a h(\mathrm{Spec}(A/J))$. The steps above can be reversed. In the second case, we have $\alpha'^{-1}(x) \in \alpha'^{-1}(F) = V(I)$, that is, $j_{\alpha'^{-1}(x)} \supseteq I$, and so $x \in {}^a h(\mathrm{Spec}(B/I)$. Again the steps above can be reversed. Therefore $F = {}^a h(\mathrm{Spec}(A/J) \cup {}^a h(\mathrm{Spec}(B/I))$. Let $x \in \mathrm{Cl}\{y\}$, for some y in F. Then $j_x \supseteq j_y$, $y \in F$. If $y \in {}^a h(\mathrm{Spec}(A/J))$, then $j_y \supseteq J$, and so $j_x \supseteq j_y \supseteq J$. Thus $x \in F$. Similarly if $y \in {}^a h(\mathrm{Spec}(B/I))$, we obtain $x \in F$. Therefore if $x \in \mathrm{Cl}\{y\}$ for some y in F then $x \in F$. Consequently F is closed under specialization and so by (1.2.7), F is closed in W. ∎

Section 3. IQO sets and spectralification.

Definition (1.3.1). — Let X be a topological space. By an *IQO set* we mean an open subset U of X such that for any quasi-compact Q of X, $Q \cap U$ is quasi-compact.

Remark (1.3.1.1). — Finite unions and finite intersections of IQO sets are IQO. If X is quasi-compact then every IQO set of X is quasi-compact open.

Definition (1.3.2). — Let X, Y be topological spaces and $f: X \to Y$ continuous. If for any IQO set E of Y, $f^{-1}(E)$ is IQO in X, then f is called *spectral.*

Remark (1.3.2.1). — If X, Y are spectral spaces, the above definition of spectral maps coincides with the concept of spectral maps defined in (1.2.1)

[Pf] Since Y is spectral, Y is quasi-compact. So every IQO set E of Y is quasicompact open. If $f: X \to Y$ is spectral in the sense of (1.2.1), $f^{-1}(E)$ is quasi-compact open. In order to show that $f^{-1}(E)$ is IQO, we must show that its intersection with any quasi-compact open subset of X is quasi-compact. In fact we prove that the intersection of any two quasi-compact open subset of X is quasi-compact open. For if U, V are quasi-compact open in X, then U, V are patches in the compact patch topology, that is, U, V are closed in the patch topology. Hence $U \cap V$ is closed and so compact in con(X). Therefore $U \cap V$ is quasi-compact in the (coarser) given topology on X. Thus f is spectral in the sense of above definition.

Conversely, let X, Y be spectral, f: $X \to Y$ spectral in the sense of the definition above. Let E be quasi-compact open in Y. By the foregoing argument which shows that in a spectral space the intersection of any two quasi-compact open sets is quasi-compact open, E is IQO in Y. Since f is spectral as in the definition above, $f^{-1}(E)$ is IQO. Since X is spectral, X is quasi-compact. So by (1.3.1.1), $f^{-1}(E)$ is quasi-compact open in X. Thus f is spectral in the sense of (1.2.1). ■

Remark (1.3.2.2). — Let f: $X \to Y$ be continuous, and $f(X) \subseteq \cup\{E_i: i = 1,\dots,n\}$, where E_i is quasi-compact open in Y for any i. If for any quasi-compact open V inY, $f^{-1}(V)$ is IQO in X, then f is spectral.

[Pf] Let E be IQO in Y. Then $E = \cup\{E_i \cap E: i = 1,\dots,n\}$, and each $E_i \cap E$ is quasi-compact open as E is IQO. Thus $f^{-1}(E) = \cup\{f^{-1}(E_i \cap E): i = 1,\dots,n\}$ is IQO as a finite union of IQO sets (cf. (1.3.1.1)).■

The following is an immediate corollary.

Corollary (1.3.2.3). — Let X, Y be topological spaces, Y spectral and f: $X \to Y$ a continuous map. Then f is spectral if for any quasi-compact open V inY, $f^{-1}(V)$ is IQO in X.

Definition (1.3.3). — Let X, X' be topological spaces and g: $X \to X'$ a spectral embedding. Then the pair (X', g) is called a *spectralification* of X if X' is spectral and if for any pair (Y, f) of a spectral space Y and a spectral map

$f: X \to Y$, there is a unique spectral map $f': X' \to Y$ such that $f = f' \circ g$. We say X is *spectralifiable* if X has a spectralification (X', g).

Theorem (1.3.4). — The following conditions on a topological space X are equivalent:

(i) The space X is spectralifiable;

(ii) the space X is T_0 and the IQO sets form an open basis;

(iii) the space X is T_0 and for any closed subset A of X and $x \in X - A$, there exists a spectral map $f: X \to W$ such that $f(x) = \{0\}$ and $f(A) = \{1\}$, where $W = \{0, 1\}$ with $\{0\}$ and W being nonempty open sets;

(iv) the space X can be spectrally embedded in some spectral space.

[Pf] (i) $\Rightarrow$ (iv): Obvious by definition.

(iv) $\Rightarrow$ (ii): Let X be a spectral embedding in a spectral space X'. Since X' is T_0, X is T_0. Let U be open in X, and $x \in U$. Then the supposition that X' is spectral implies that there exists a quasi-compact open subset E of X' such that $x \in E$ and $E \cap X \subseteq U$. By (1.3.1.1), E is IQO in X'. Since X is spectrally embedded in X' the preimage with respect to the embedding of E in X which is $E \cap X$ is IQO in X. So the IQO sets of X form an open basis.

(ii) $\Rightarrow$ (iii): Let A be closed in X and $x \notin A$. So there exists an IQO set U of X such that $x \in U \subseteq X - A$. Note that $U \cap A = \varnothing$. Define $f: X \to W$ thus: $f(y) = 0$ if $y \in U$ and 1 if $y \notin U$. The quasi-compact open (hence IQO) sets in W are

$\varnothing$, $\{0\}$, W. Since $f^{-1}(\{0\}) = U$ and U is IQO, it follows that f is spectral, and $f(x) = 0$, $f(A) = \{1\}$.

(iii) $\Rightarrow$ (ii): It is enough to show that if U is open in X, and $x \in U$ then there exists an IQO set E in X such that $x \in E \subseteq U$. If $U = X$, then there is nothing to prove as X is IQO and so we need only set $E = X$. If $U \subsetneq X$ and x does not belong to X - U, a closed set of X, there exists a spectral map f: $X \to W$ such that $f(x) = 0$ and $f(X - U) = \{1\}$. Let $E = f^{-1}(\{0\})$. As f is spectral, E is IQO in X, and clearly $x \in E \subseteq U$.

(ii) (or equivalently (iii)) $\Rightarrow$ (iv): Let V be the set of all the spectral maps f: $X \to W$. Let W^V be the product of copies of W for each $f \in V$. Since W is spectral, with the product topology, W^V is a spectral space. Let e be the evaluation map $X \to W^V$, that is, $e(x)(f) = pr_f(e(x)) = f(x)$, then e is clearly continuous. If $x \neq y$, $x, y \in X$, we can assume without loss of generality $x \notin Cl\{y\}$. Then by (iii) there exists a spectral map f: $X \to W$ such that $f(x) = 0$ and $f(Cl\{y\}) = 1$. So $f(x) \neq f(y)$. This shows that $e(x) \neq e(y)$, and e is injective.

Let B be open in X, and $x \in B$. Then by (iii), there exists a spectral map f: $X \to W$ such that $f(x) = 0$ and $f(X - B) = \{1\}$. Let $U = \{\xi \in W^V: pr_f(\xi) = 0\}$. So U is open in the product topology on W^V. Clearly $e(x) \in U \cap e(X) \subseteq e(B)$. So e is an open map from X onto e(X). Finally we claim that e is spectral. For since W^V is spectral by (1.3.1.1) it suffices to show that for any quasi-compact open set Q of W^V, $e^{-1}(Q)$ is IQO in X. Since Q is quasi-compact open subset in the product topology, Q is a finite union of basic open subsets $E_i = \Pi\{U_v: v \in V\}$, $i = 1,\dots,n$, where for each i, at most finitely many U_v are the proper open (quasi-compact open) subset $\{0\}$ of the v-th factor space W and all others are whole factor spaces, copies of W. So $e^{-1}(E_i) = \cap\{v^{-1}(U_v): v \in V'\}$, where $U_v = \{0\}$ for $v \in V' \subseteq V$ and V' is a finite subset of V, is a finite intersection of IQO sets because each v is spectral and {0} is IQO in W. Thus $e^{-1}(E_i)$ is IQO by (1.3.1.1). Consequently, by (1.3.1.1), $e^{-1}(Q) = \cup\{e^{-1}(E_i): i = 1,\dots,n\}$ is IQO. ∎

(ii) ⇒ (i): We have the spectral embedding $e: X \to W^V$ in the paragraph above. Let $X' = Cl_{con(X)}e(X)$. Let us restrict the range of e to X' and call this spectral map also as $e: X \to X'$. We claim (X', e) is the spectralification of X. Let Y be a spectral space, $f: X \to Y$ a spectral map. We construct a spectral map $f': X' \to Y$ such that $f = f' \circ e$. Let V' be the set of all spectral maps from Y to W. Since Y is spectral, it satisfies (ii) (for when Y is spectral IQO sets are quasi-compact open, cf. (1.3.2)), and we have a spectral embedding $e': Y \to W^{V'}$. Let $f^*: V' \to V$ be defined by $f^*(v') = v' \circ f$ for every $v' \in V'$, and $f^{**}: W^V \to W^{V'}$ defined by $f^{**}(q) = q \circ f^*$ for every $q \in W^V$ (cf. [Kel], pp.152-153, Lem.3.3). Let $v' \in V'$. Then $(pr_x \circ f^*)(v') = pr_x(v' \circ f) = v'(f(x)) = pr_{f(x)}(v')$ for every $x \in X$. So $pr_x \circ f^* = pr_{f(x)}$. Next we show that f^{**} is spectral. Since $f^{**}(q) = q \circ f^*$, for any $v' \in V'$, we get $(pr_{v'} \circ f^{**})(q) = pr_{v'}(q \circ f^*) = q(f^*(v')) = pr_{f(v')}(q)$. Thus $pr_{v'} \circ f^{**} = pr_{f(v')}$ for every $v' \in V'$, and this shows that f^{**}

is spectral since $pr_{f(v')}$ is spectral (the proof is similar to [Kel], Th.3.3). Now we have the commutativity $e' \circ f = f^{**} \circ e$. Because for any $x \in X$ and $v' \in V'$, $((f^{**} \circ e)(x))(v') = f^{**}(e(x))(v') = (e(x) \circ f^*)(v') = e(x)(f^*(v')) = e(x)(v' \circ f) = (v' \circ f)(x) = v'(f(x)) = pr_{v'}(f(x))$. Thus $f^{**} \circ e = e' \circ f$ as we claimed. We want to show that $e'^{-1} \circ f^{**} |_{X'}$ is the map f' we are looking for. In order to do it, we need show that $e'(Y) \supseteq f^{**}(X')$. Since f^{**} is spectral, it is a continuous closed map with respect to the patch topology. So $f^{**}(X')$ is closed in the patch topology and hence a patch in the spectral topology on $W^{V'}$. And since e' is a spectral embedding, $e'(Y)$ is a patch in $W^{V'}$. So that $f^{**}(X') = f^{**}(Cl_{con(W^V)}e(X)) = Cl_{con(W^{V'})}(f^{**} \circ e)(X) = Cl_{con(W^{V'})}(e' \circ f)(X) = Cl_{con(W^{V'})}(e'(f(X)) \subseteq Cl_{con(W^{V'})}e'(Y) = e'(Y)$ (the second equality holds as f^{**} is continuous and closed). It remains to show that $f' = e'^{-1} \circ f^{**} |_{X'}$ is an extension of f to X'. Since $f^{**}(X')$ is a patch in $W^{V'}$ and $f^{**}(X')$ is contained in the patch $e'(Y)$, $f^{**}(X')$ is a patch in $e'(Y)$. Thus $e^{-1} |_{f^{**}(X')}$ is spectral. Since $f^{**}: X' \to f^{**}(X')$ is spectral, f' is spectral. ■

Proposition (1.3.5). — Arbitrary coproducts of spectral spaces is spectralifiable. ([Hoc,1], Th.7)

[Pf] Let $\{U_i : i \in I\}$ be a family of spectral spaces. Since quasi-compact open sets in U_i are IQO and form an open basis of U_i, the collection of all IQO sets in U_i, $i \in I$, form an open basis of the coproduct space of the family

$X = \Sigma\{U_i: i \in I\}$. Thus by the previous theorem, the coproduct of the family $\{U_i: i \in I\}$ is spectralifiable. ■

Definition (1.3.6). — Let X be a spectral space, E an equivalence relation on X. A spectral space structure on X/E is called the *quotient spectral space structure with respect to E* if (1) the quotient map p: $X \to X/E$ is spectral and (2) whenever g: $X \to X'$ is a surjective spectral map which is constant on equivalence classes in X with respect to E, there exists a unique spectral map h: $X/E \to X'$ such that $g = h \circ p$.

Proposition (1.3.7). — Quotients exist in the category $\mathcal{S}$ ([Hoc,1], Th.7)

[Pf] Let X be a spectral space, E an equivalence relation on X. Let D be the set of all spectral maps from X to spectral spaces whose underlying sets are subsets of X and which are constant on the equivalence classes with respect to E. Clearly D is not empty. For any $f \in D$, let Y_f denote the range space of X with respect to f. Consider $\Pi\{Y_f: f \in D\}$ which is a spectral space by (1.2.9). Let e: $X \to \Pi\{Y_f: f \in D\}$ be the evaluation map. Note that e is spectral. Because $pr_f \circ e = f$ and both f (because it is spectral) and pr_f are continuous in the spectral and patch topologies it follows that e is continuous in the spectral and patch topologies. So by (1.2.5), (ii), e is spectral. Let e_0: $X \to e(X) = Y$. Then Y is a patch in $\Pi\{Y_f: f \in D\}$ and so has a spectral space structure. Then (Y, e_0) is the quotient spectral space with respect to the equivalence relation E. For let g: $X \to X'$ be a given surjective spectral map constant on every equivalence class with respect to E. For each point $x' \in X'$ choose a point from every class that is contained in $g^{-1}(x')$. Let us denote by Z_g the subset of X consisting of points thus chosen. Then we have the surjection $h_g: Z_g \to X'$, with the property $h_g^{-1}(U') = Z_g \cap g^{-1}(U')$ for any $U' \subseteq X'$. Since g is spectral, it is continuous with respect to both spectral and the patch topologies. So h_g is spectral with respect to the subspace topology of X on Z_g. Thus for every $g: X \to X'$ we have its representative $f: X \to Y_f = Z_g$ in D. Now $f = pr_f|_Y \circ e_0$, so if we define $h: Y \to X'$ as $h = h_g \circ pr_f|_Y$, where Z_g defined as above and $pr_f|_Y$ is the restriction of the projection map $\Pi\{T_f: f \in D\} \to Y_f$ to Y, then we have $g = h \circ e_0$. This shows that (Y, e_0) is the quotient object of X with respect to E. ■

Corollary (1.3.8). — Direct limits exist in the category $\mathscr{S}$ ([Hoc,1], Th.7)

[Pf] Since an arbitrary coproduct of spectral spaces is realized as a space that can be embedded into a spectral space by (1.3.5) and quotients exist in $\mathscr{S}$, by (1.3.7), this is an immediate corollary. ■

Chapter 2. Topological characterization.

Let $\mathscr{C}$ denote the category of commutative rings with unity as objects and unitary homomorphism as morphisms. Let $\mathscr{T}$ denote the category of topological spaces as objects and continuous functions as morphisms. Let us denote by $\mathscr{S}$ the subcategory of $\mathscr{T}$ consisting of spectral spaces as objects and spectral maps as morphisms. Then Spec: $\mathscr{C} \to \mathscr{T}$ is a contravariant functor and the image of Spec is clearly contained in $\mathscr{S}$. The object of this section is to show that every object and morphism in $\mathscr{S}$ is in the image under Spec of an object and morphism in $\mathscr{C}$, that is, Spec is a full functor from $\mathscr{C}$ to $\mathscr{S}$.

We noticed that if A is a ring (commutative and with unity), then spec(A) is a T_0 topological space such that (i) it is quasi-compact, (ii) its quasi-compact open subsets form an open basis, and (iii) every closed irreducible subset has a unique generic point (cf. (1.2.2)).

We show that given T_0 topological space X with the properties (i), (ii) and (iii), there is a ring A such that X = Spec(A).

We also noticed that if f: A $\to$ B is a ring homomorphism from A to B, then the adjoint map ^{a}f: Spec(B) $\to$ Spec(A) is a spectral map.

In what follows we show that if X, X' are spectral spaces and f: X $\to$ X' a spectral map then there exist rings A, A' and a homomorphism φ: A' $\to$ A such that X = Spec(A), X' = Spec(A') and f = ${}^a\varphi$ = Spec(φ), that is, every spectral space and spectral map is in the image of Spec.

It should be noticed that this does not mean there exists a contravariant functor from $\mathscr{S}$ to $\mathscr{C}$ which is the inverse of the functor Spec. What is implied is the following:

Let $\mathscr{R} = \{X, X'$ objects of $\mathscr{S}$, id_X, $id_{X'}$ and $f: X \to X'$ morphisms of $\mathscr{S}\}$. Then $\mathscr{R}$ is a subcategory of $\mathscr{S}$. What is asserted is that there is a functor from $\mathscr{R}$ to $\mathscr{C}$ which is the inverse of Spec.

In what follws we also examine some examples of the following question: Given a subcategory $\mathscr{R}$ of $\mathscr{S}$, can we invert Spec on $\mathscr{R}$?

We show that there are some subcategories of $\mathscr{S}$ on which Spec can be inverted and there are subcategories of $\mathscr{S}$ on which Spec cannot be inverted. The process is quite technical and for a quick summary refer to (2.1.28) below.

Section 1. Topological characterization of spectral spaces.

Definition (2.1.1). — We say that the Spec: $\mathscr{C} \to \mathscr{T}$ is *invertible* on a subcategory $\mathscr{R}$ of $\mathscr{S}$ if there is a (contravariant) functor F: $\mathscr{R} \to \mathscr{C}$ whose composite with Spec is isomorphic with the inclusion functor incl from $\mathscr{R}$ into $\mathscr{S}$, that is, for any object X of $\mathscr{R}$ there are isomorphisms $f(X)$: $(\mathrm{Spec} \circ F)(X) \to \mathrm{incl}(X)$ and $f(X')$: $(\mathrm{Spec} \circ F) \to \mathrm{incl}(X')$ in $\mathscr{S}$ such that for any morphism u: $X \to X'$ in $\mathscr{S}$, $f(X') \circ (\mathrm{Spec} \circ F)(u) = \mathrm{incl}(u) \circ f(X)$. Then we shall also say that F is a *space preserving* functor from $\mathscr{T}$ to $\mathscr{C}$.

Definition (2.1.2). — Let X be a spectral space and let *Q(X)* denote the set of all quasi-compact open subsets of X. Define a contravariant functor Q: $\mathscr{S} \to \mathscr{E}$, where $\mathscr{E}$ denotes the category of sets and functions, as follows: if f: $X \to X'$ is a spectral map, $Q(f)$: $Q(X') \to Q(X)$ is defined by $Q(f)(U') = f^{-1}(U')$ (since f is spectral, for any quasi-compact open subset U' of X', $f^{-1}(U')$ is quasi-compact in X.)

Definition (2.1.3). — Let X be a spectral space, E a set and g: $E \to Q(X)$ a function such that g(E) is a basis for open sets of X. Then the triple G(X) = (X, E, g) is called the *space with indeterminates associated with X*.

Let X, X' $\in$ Obj($\mathscr{S}$). Then a pair (f, r) is said to be a *morphism of the spaces with indeterminates* : (X, E, g) $\to$ (X', E', g') if (1^o) f: $X \to X'$ is spectral, (2^o) r: $E' \to E$ is an injection, and (3^o) $Q(f) \circ g' = g \circ r$. The class consisting of spaces with indeterminates and morphisms obviously forms a category, which we shall denote by $\mathscr{J}$. Let G(X) = (X, E, g), G(X') = (X', E', g') be the spaces with indeterminates associated with X, X' respectively, and G(f) = (f, r), where r: E' $\to$ E an injection. Then G is a functor from $\mathscr{S}$ into $\mathscr{J}$, which we also call a *space preserving* functor.

Remark (2.1.3.1). — In what follows we show that given any field κ, there exists a contravariant functor L: $\mathscr{J} \to \mathscr{C}$ such that for any object (X, E, g) of $\mathscr{J}$, L(X, E, g) = R is a κ-algebra with X = Spec(R). Furthermore, R can be viewed as the set of functions on X whose values are polynomials over κ.

Our technique of inverting Spec on a subcategory $\mathscr{R}$ of $\mathscr{S}$ is as follows:

Construct a space preserving functor G: $\mathscr{R} \to \mathscr{J}$ and use the functor L: $\mathscr{J} \to \mathscr{C}$ to obtain the contravariant functor LG: $\mathscr{R} \to \mathscr{C}$ which inverts Spec on $\mathscr{R}$. (cf. section 2 below for such applications).

Definition (2.1.4). — A triple **A** = (X, {A(x)}, A) consisting of a spectral space X, a family of integral domains {A(x): x $\in$ X} and a ring A (without radical) isomorphic to a subring of $\prod$ {A(x): x $\in$ X} (so that every element of A can be regarded as a function X $\to \cup$ {A(x): x $\in$ X}), is called a *spring* if (i) for any x $\in$ X, A(x) = {a(x): a $\in$ A}, (ii) for any a $\in$ A, *d(a)* = {x $\in$ X: a(x) $\neq$ 0} is quasi-compact open in X, and (iii) {d(a): a $\in$ A} is a basis for open sets of X.

Let **A** = (X, {A(x)}, A), **A** = (X', {A'(x')}, A') be springs, f: X $\to$ X' a spectral map and h: A' $\to$ A a ring homomorphism. Then for any x $\in$ X, there is a ring homomorphism h_x: $A'(f(x)) \to A(x)$ defined by $h_x(a'(f(x)) = h(a')(x)$. Let us denote by v_x: $A \to A(x)$ defined by $v_x(a) = a(x)$. Define $v'_{x'}$ similarly.

A triple $(f, \{h_x\}, h)$, where f: $X \to X'$ is a spectral map, h: $A' \to A$ is a ring homomorphism and $\{h_x\}_{x \in X}$ is defined as above is said to be a morphism of springs from $\mathbb{A} = (X, \{A(x)\}, A)$ to $\mathbb{A}' = (X', \{A'(x')\}, A')$ if for any $x \in X$, $v_x \circ h = h_x \circ v'_{f(x)}$ and for any $a' \in A'$, $f^{-1}(d(a')) = d(h(a'))$.

This definition of morphisms makes the class of springs into a category denoted by $\mathscr{A}$.

Example (2.1.5). — Let $(X, E, g) \in \mathrm{Obj}(\mathscr{S})$ and κ an arbitrary field. Let $\{t(e)$: $e \in E\}$ be algebraically independent indeterminates over κ. Let $K = \kappa[t(e)$: $e \in E]$. For any $e \in E$, $T(e)$: $X \to K$ is defined thus: $T(e)(x) = t(e)$ if $x \in g(e)$ and 0 if $x \notin g(e)$. Let K^X denote the commutative ring of all functions from X to K. Let $A = \kappa[T(e)$: $e \in E]$ be the subring of K^X generated by the functions $\{T(e)$: $e \in E\}$.

Let $S \in A$ be a nonzero monomial in $T(e)$'s, say, $S = \lambda T(e_1)...T(e_k)$, $\lambda \in \kappa - \{0\}$. Then for $x \in X$, $S(x) = \lambda t(e_1)...t(e_k)$ (nonzero monomial in the independent indeterminates $\{t(e)\}$ over κ) or 0 according as $x \in g(e_1) \cap ... \cap g(e_k)$ or not. Thus $\{x \in X$: $S(x) \neq 0\} = g(e_1) \cap ... \cap g(e_k)$, which is quasi-compact open in X because by the hypothesis $g(E)$ is an open basis consisting of quasi-compact open subsets of X.

Let a be an arbitrary element of A, $a = \Sigma\{\lambda_{(j,1)\ldots(j,i_j)}T(e_{j,1})\ldots T(e_{j,i_j}): j = 1,\ldots,k\}$. For any $x \in X$ the value a(x) is a polynomial in t(e)'s, that is, a finite sum of monomials in t(e)'s. So it follows from above remark that $d(a) = \{x \in X: a(x) \neq 0\}$ is a finite union of quasi-compact open sets in X. Hence it is a quasi-compact open subset of X. For later use define *z(a)* = X - d(a). Since $\{g(e): e \in E\}$ is an open basis of X and each d(a) is a finite union of finite intersections of g(e)'s, $\{d(a): a \in A\}$ is an open basis of X.

Let A(x) be the subring of K generated by $\{t(e): e \in E, x \in g(e)\}$ for any $x \in X$. Each A(x) is clearly an integral domain. For each $x \in X$, we have a ring homomorphism $v_x: A \to A(x)$, defined by $v_x(T(e)) = T(e)(x)$. It is easily seen that $v_x(T(e_1)T(e_2)) = v_x(T(e_1)v_x(T(e_2))$. Thus v_x can be extended to a ring homomorphism $A \to \Pi\{A(x): x \in X\}$. Since A(x) is an integral domain $\ker(v_x) = P_x$ (say) is a prime ideal of A.

Next, we verify that A is isomorphic to a subring of $\Pi\{A(x): x \in X\}$. We check for any $a \in A - \{0\}$ there exists $x \in X$ such that $v_x(a) = a(x) \neq 0$. Now a is of the form $\Sigma\{\lambda_{(j,1)\ldots(j,i_j)}T(e_{j,1})\ldots T(e_{j,i_j}): j = 1,\ldots,k\}$ and since $a \neq 0$, there exists $x \in X$ such that $\Sigma\{\lambda_{(j,1)\ldots(j,i_j)}T(e_{j,1})(x) \ldots T(e_{j,i_j})(x): j = 1,\ldots,k\} \neq 0\ldots(\blacklozenge)$. By definition $v_x(a) = \Sigma\{\lambda_{(j.1)\ldots(j,i_j)}v_x(T(e_{j,1})) \ldots v_x(T(e_{j,i_j})): j = 1,\ldots,k\}(x) = \Sigma\{\lambda_{(j,1)\ldots(j,i_j)}T(e_{j,1})(x) \ldots T(e_{j,i_j})(x): j = 1,\ldots,k\} \neq 0$ by $(\blacklozenge)$. So if we define the evaluation map $v: A \to \Pi\{A(x): x \in X\}$ by $v(a) = \Pi\{v_x(a): x \in X\}$, v is clearly an injective ring homomorphism.

Thus starting with a spectral space X and a field κ we obtain a triple (X, {A(x)}, A) where {A(x)} is a family of integral domains indexed by X and A is a ring which is isomorphic to a subring of $\prod\{A(x): x \in X\}$. Clearly A is a ring without radical. We call this spring the associated spring of (X, E, g), and this association is indeed a functor from $\mathscr{I}$ to $\mathscr{A}$. But for our problem of inverting Spec on a subcategory of $\mathscr{S}$, we need construct a different functor H (cf. (2.1.17) below) based on this association.

Proposition (2.1.6). — Let **A** = (X, {A(x)}, A) be a spring. Then X is homeomorphic to a dense patch of Spec(A).

[Pf] Define φ: X $\rightarrow$ Spec(A) as follows: For $x \in X$, let $\varphi(x)$ be the subset of A consisting of all functions in A which vanish at x, that is, $\varphi(x) = \{a \in A: a(x) = 0\}$. Then $\varphi(x)$ is obviously a prime ideal of A. In Spec(A), a basic open set could be taken, without loss of generality, to be $D(a) = \{[P] \in \mathrm{Spec}(A): a \in P\}$, $a \in A$ (cf. (1.1.3)). We have following equivalent assertions: $\varphi(x) \in D(a) \Leftrightarrow a \notin \varphi(x) \Leftrightarrow a(x) \neq 0 \Leftrightarrow x \in d(a)$. So $\varphi^{-1}(D(a)) = d(a)$, which is quasi-compact open, and $\varphi(d(a)) = D(a) \cap \varphi(X)$. This shows that φ is a spectral and open map of X onto $\varphi(X)$. Thus $\varphi(X)$ is a patch in Spec(A).

Next let $x, y \in X$, $x \neq y$. Since X is T_0, one of them is not in the closure of the other, say $x \notin \mathrm{Cl}\{y\}$. Then there exists a basic quasi-compact open set U(x) such that $y \notin U(x) = g(e)$, $x \in U(x)$ for some $e \in E$. We have $t(e) = T(e)(x)$

$\neq T(e)(y) = 0$, thus $T(e) \notin \varphi(x)$ while $T(e) \in \varphi(y)$, that is, $\varphi(x) \neq \varphi(y)$. So φ is injective. Consequently φ is a spectral embedding.

Let D(a) be a basic open set in Spec(A) for some nonzero element a of A. Then there exists $x \in X$ such that $a(x) \neq 0$, that is, $\varphi(x) \in D(a)$ by the previous equivalent assertions. So $\varphi(X)$ is dense in Spec(A). ■

Remark (2.1.6.1). — Note $\varphi^{-1}(V(a)) = z(a)$.

Proposition (2.1.7). — Let (X, A) be a pair of a spectral space X and a ring A without radical such that X is homeomorphic to a dense patch of Spec(A). Let for each $x \in X$, $A(x) = A/j_x$, where j_x is x regarded as an ideal of A (cf. (1.1.1)). Note that A(x) is an integral domain. Then (X, {A(x)}, A) is a spring.

[Pf] From the definition of $A(x) = A/j_x$, it is clear that the condition (i) for springs is satisfied. Next, for any $a \in A$, D(a) is quasi-compact open in Spec(A). Let φ be the homeomorphism of X to a dense patch of Spec(A), then $D(a) \cap \varphi(X)$ is quasi-compact open in $\varphi(X)$. So $d(a) = \varphi^{-1}(D(a))$ is quasi-compact open in X, this is the condition (ii). Since $\{D(a): a \in A\}$ is an open basis for Spec(A), $\{d(a): a \in A\}$ is an open basis of X. Hence condition (iii) of definition of the springs is satisfied. ■

Note (2.1.7.1). — We notice that the pair (f, h) consisting of a spectral map $f: X \to X'$ and a ring homomorphism $h: A' \to A$ determines a spring morphism $(f, \{h_x\}, h)$ from (X, {A(x)}, A) to (X', {A'(x'), A'), where for each x

$\in X$, h_x is the canonical homomorphism from $A'(x') = A'/h^{-1}(j_x)$ to $A(x) = A/j_x$. Thus if we define pairs like (f, h) as morphisms $(X, A) \to (X', A')$ then we get a category equivalent to $\mathscr{A}$.

Remark (2.1.7.2). — Thus we may refer to a spring either as a triple (X, {A(x)}, A) as in (2.1.4), or equivalently as a pair (X, A) as in (2.1.7) above.

Definition (2.1.8). — Let **A** = (X, {A(x)}, A) be a spring. **A** is said to be *affine* if X = Spec(A).

Proposition (2.1.9). — A spring **A** is affine if the condition (*) below is satisfied:

(*) For each $\{a, b_1,..., b_n\} \subseteq A$, $\cup\{d(b_i): i = 1,...,n\} \supseteq d(a) \Rightarrow a \in \mathrm{rad}(b_1,..., b_n)$

[Pf] Let **A** satisfy (*). Suppose **A** is not affine. Then $\varphi(X) \subsetneq \mathrm{Spec}(A)$, where φ is defined as in the proof of (2.1.6). Since $\varphi(X)$ is a patch in Spec(A) (cf. (1.2.5),(iii)), $\varphi(X)$ is a proper closed set in the patch topology of Spec(A). Recall that quasi-compact open sets and closed sets of Spec(A) form a subbasis for closed sets of the patch topology of Spec(A), thus $\varphi(X)$ being a proper closed set implies that there exist $\{a, b_1,..., b_n\} \subseteq A$ such that $\varphi(X) \subseteq V(a) \cup (\cup\{D(b_i): i = 1,...,n\}) \subsetneq \mathrm{Spec}(A)$... (I).

Since φ is injective, $X = \varphi^{-1}(\varphi(X)) = \varphi^{-1}(V(a)) \cup (\cup\{\varphi^{-1}(D(b_i)): i = 1,...,n\}) = z(a) \cup (\cup\{d(b_i): i = 1,...,n\})$ (cf. the proof of (2.1.7)), so that $\cup\{d(b_i): i = 1,...,n\} \supseteq$

$d(a)$. Then by (*), $a \in \mathrm{rad}(b_1, \ldots, b_n)$, so $V(a) \supseteq V(b_i): i = 1,\ldots,n\} = \mathrm{Spec}(A) - \cup\{D(b_i): i = 1,\ldots,n\}$, a contradiction to (I). ■

Definition (2.1.10). — Let $\mathbb{A} = (X, \{A(x)\}, A)$ be a spring. If $A(x)$ are all subrings of some common ring K (in which case each element of A can be interpreted as a function on X) and having only a finite numbers of distinct values, then we say that the spring $\mathbb{A}$ is *simple*.

Note (2.1.10.1). — Let X be a spectral space, $\mathbb{A} = (X, \{A(x)\}, A)$ its associated spring. We note that all $A(x)$ are subrings of K and A is a ring of functions on X into K and each element $a \in A$ as a function on X has only a finite number of distinct values in K. So the associated spring of a spectral space is simple.

Proposition (2.1.11). — If a spring $\mathbb{A} = (X, \{A(x)\}, A)$ is simple and satisfies the following condition (**), then it is affine:

(**) $d(a) \subseteq d(b) \Rightarrow a \in \mathrm{rad}(b)$. ([Hoc,1], Prop.1)

[Pf] It suffices to show that (**) implies the condition (*) in (2.1.9). Let $a, b_1,\ldots, b_n \in A$ such that $d(a) \subseteq d(b_1) \cup \ldots \cup d(b_n)$. In order to show that $a \in \mathrm{rad}(b_1,\ldots, b_n)$ it is enough to show that there exist $b \in (b_1,\ldots, b_n)$ such that $d(b) = d(b_1) \cup\ldots\cup d(b_n)$. By finite induction it suffices to show it in the case $n = 2$. Now each b_j, $j = 1, 2$, can be interpreted as a function on X with only a finite number of distinct values in the ring K since $\mathbb{A}$ is simple. Let $W_j(1),\ldots, W_j(k_j)$ denote the mutually disjoint subsets of $d(b_j)$ on which b_j has mutually different constant nonzero values. Choose from $W_j(i_j)$, $i_j = 1,\dots,k_j$, a point $y_{W_j(i_j)}$. Also choose from every nonempty intersection of $W_1(i_1)$ and $W_2(i_2)$ a point $y_{W_1(i_1),W_2(i_2)}$. Let Y be the set of all of these y's so chosen. Then Y is a finite subset of X and b_1, b_2 do not vanish at any point in Y. Let $a \in [b_1, b_2]$ (which designates the unitary subring generated by b_1, b_2 in A), and $a = \alpha b_1 + \beta b_2 + \gamma b_1 b_2$, where α, β, γ are integers. Then a as a function on X vanishes outside $d(b_1) \cup d(b_2)$. Let $x \in d(_1) - d(b_2)$. Thus we observe that depending on $x \in d(b_1) - d(b_2)$, $x \in d(b_2) - d(b_1)$ or $x \in d(b_1) \cap d(b_2)$ the value of a at x is the same as its value at some point in Y of the form $y_{W_1(i_1)}$, $y_{W_2(i_2)}$ or $y_{W_1(i_1),W_2(i_2)}$ respectively. Let $J = (b_1, b_2) \cap [b_1, b_2]$. We note that J is a subset stable under addition and multiplication. Now $b_1 b_2 \in J$ and $b_1 b_2(y) = b_1(y) b_2(y) \neq 0$ for any $y \in Y$. Therefore $b_1 b_2 \notin \varphi(y)$. Thus for any $y \in Y$ the prime ideal $\varphi(y) \not\supseteq J$. So by ([Bou,2], II, sec.1, n° 1, Prop.2), $\cup\{\varphi(y): y \in Y\} \not\supseteq J$. Let $c \in J - \cup\{\varphi(y): y \in Y\}$. We verify $d(c) = d(b_1) \cup d(b_2)$. Let $c = \alpha b_1 + \beta b_2 + \gamma b_1 b_2$ and $x \in d(b_1) \cup d(b_2)$. Now $c(x) = c(y)$ for some $y \in Y$ by the earlier observation. Since $c \in J - \cup\{\varphi(y): y \in Y\}$, $c(y) \neq 0$. So $c(x) \neq 0$. Therefore $x \in d(c)$. If $x \notin d(b_1) \cup d(b_2)$ then $b_1(x)$, $b_2(x)$ are both 0 and so $c(x) = 0$, that is , $x \notin d(c)$. Thus $d(c) = d(b_1) \cup d(b_2)$. ■

Definition (2.1.12). — Let (X, {A(x)}, A) be a spring, a, b ∈ A with $d(a) \subseteq d(b)$ and $A^*(x)$ the field of fractions of A(x). Then *a#b* is the element in $\Pi\{A^*(x): x \in X\}$ such that $a\#b(x) = 0$ if $x \notin d(b)$ and $a(x)/b(x)$ if $x \in d(b)$.

Proposition (2.1.13). — Let $\mathbb{A}$ = (X, {A(x)}, A) be a spring. Then $\mathbb{A}$ satisfies (**) if $\mathbb{A}$ satisfies the following condition (***):

(***) if a, b $\in$ A, d(a) $\subseteq$ d(b) implies that there exists a nonnegative integer n such that $a^n\#b \in$ A.

[Pf] Since A is isomorphic to a subring of $\prod\{A(x): x \in X\}$, we may identify every element of A with its image in $\prod\{A(x): x \in X\}$. Now $b(a^n\#b)(x) = 0$ if x $\notin$ d(b), $a^n(x)$ if x $\in$ d(b). Since by hypothesis a#b $\in$ A, this implies a $\in$ rad(b). ■

Remark (2.1.14). — In summary, starting with a spectral space X we associated with it a spring (X, {A(x)}, A). In general X is only homeomorphic to a dense patch of Spec(A). So we are interested in constructing a spring (X, {A'(x'), A'), where the underlying space is the space X which we started with and is actually homeomorphic to Spec(A'), that is, an affine spring whose underlying space is X. So we developed sufficient conditions for a spring to be affine. First we introduced (*) in (2.1.9). Then we showed that simplicity along with (**) implies affineness (cf. (2.1.11)). Finally we show that if $\mathbb{A}$ is a simple and satisfies (***), then we can "extend" the ring A to M(A) so that M(A) satisfies (***) without disturbing X and in this manner obtain an affine spring.

Definition (2.1.15). — Let $\mathbb{A}$ = (X, {A(x)}, A), $\mathbb{A}'$ = (X', {A'(x')}, A') be springs. If X = X' and for any x $\in$ X , A(x) $\subseteq$ A'(x) $\subseteq$ A*(x) then $\mathbb{A}'$ is said to be an *extension* of $\mathbf{A}$. Let B be a subset of $\prod\{A^*(x): x \in X\}$ and $C = A[B]$ be the subring of $\prod\{A^*(x): x \in X\}$ generated by A and B, A identified as subring of $\prod\{A^*(x): x \in X\}$. If for any $c \in C$, $d(c)$ is quasi-compact open in X, then we say that B *induces an extension* of the spring $\mathbf{A}$. Let for every $x \in X$, $C(x) = \{c(x): c \in C\}$. Note that $(X, \{C(x)\}, C)$ need not be a spring, because $d(c)$ may not be quasi-compact open in X.

Remark (2.1.15.1). — Let $\mathbf{A}$ be a spring as above, $a, b \in A$, $d(a) \subseteq d(b)$, so that a#b is defined. Does there exist an integer n such that a#b induces an extension of $\mathbf{A}$? In what follows we will show for a given spring $\mathbf{A}$ there is an extension $M(\mathbf{A})$ such that whatever be $a, b \in M(A)$ with $d(a) \subseteq d(b)$, there exists an integer n such that $a^n\#b \in M(A)$. This would imply in the case of a spectral space X for the associated spring $\mathbf{A} = (X, \{A(x)\}, A)$ such an extension exists and so satisfies (***). Furthermore it is simple. So by (2.1.11) and (2.1.13) we obtain an affine spring $M(\mathbf{A}) = (X, M(A))$, that is, $X = \mathrm{Spec}(M(A))$ (cf. (2.1.26.1)). In what follows we describe the construction of $M(\mathbf{A})$.

Definition. (2.1.16). — Let $\mathbf{A} = (X, \{A(x)\}, A)$ be a spring. Let $\sigma(X) = \{(y, x) \in X \times X: x \in \mathrm{Cl}\{y\}\}$. By an *index* v on $\mathbf{A}$ we mean a family of additive valuations $v_p: A^*(y) \to \mathbf{Z}$, where $\mathbf{Z}$ is the ordered group of integers with respect to addition and for any $p = (y, x) \in \sigma(X)$ subject to the following conditions (I_1), (I_2):

(I_1) For all $a \in A$ and $p = (y, x) \in \sigma(X)$ such that $y \in d(a)$, $0 \le v_p(a(y))$ with equality iff $a(x) \ne 0$.

(I_2) For all $a \in A$ there is an integer $N > 0$ such that if $p = (y, x) \in \sigma(X)$ and $y \in d(a)$, then $v_p(a(y)) \le N$.

A spring $\mathbf{A}$ together with an index v for it is called an *indexed spring*, and is denoted by $(\mathbf{A}, v)$.

A morphism of indexed springs $(\mathbf{A}, v)$ to $(\mathbf{A}', v')$ is a morphism of springs (f, h) with the additional condition $v_p(h(a')(y)) = v'_{f(p)}(a'(f(x)))$, where $a' \in A'$, $p = (y, x) \in \sigma(X)$ with $y \in d(h(a'))$, and $f(p) = (f(y), f(x))$ (as is easily verified that $(f \times f)(\sigma(X)) \subseteq \sigma(X'))$.

Indexed springs and morphisms of indexed springs form a category, which we denote by $\mathscr{B}$.

Remark (2.1.17). — The spring associated with a spectral space (described in (2.1.5)) can be made into an indexed spring thus:

Recall that the spring associated to a spectral space X is the triple (X, $\{A(x)\}$, A) defined as follows: Let κ be a fixed field and $K = \kappa[t(e): e \in E]$, where $\{t(e): e \in E\}$ is a family of indeterminates not belonging to κ, indexed by E. Let $A = \kappa[T(e): e \in E]$, where $T(e): X \to K$ is the function defined by $T(e)(x) = t(e)$ if $x \in g(e)$ and 0 if $x \notin g(e)$. Let $A(x) = \kappa[t(e): e \in E, x \in g(e)] = \{a(x): a \in A\}$. Note that if $p = (y, x) \in \sigma(X)$, then $x \in g(e)$ implies $y \in g(e)$. So $A(x) \subseteq A(y)$, and for any $a \in A$, a(y) can be regarded as a polynomial over

$A(x)$ with indeterminates $\{t(e): e \in E_p\}$, where $E_p = \{e \in E: y \in g(e), x \notin g(e)\}$. In this case $a(x)$ occurs as the constant term of the polynomial $a(y)$ in $A(x)[t(e): e \in E_p] = A(y)$. Now define v as follows: Let $p = (y, x) \in \sigma(X)$, $v = (v_p)$ be the family of functions $v_p: A^*(y) \to \mathbf{Z}$, $p \in \sigma(X)$ which assigns to each $a(y) \in A(y)$ the order of $a(y)$ regarding it as a polynomial over $A(x)$ with indeterminates $\{t(e): e \in E_p\}$ (i.e., the minimal degree of nonzero monomials in $a(y)$).

First we show that for any $p = (y, x) \in \sigma(X)$, $v_p: A(y) \to \mathbf{Z}$ is the additive valuation with the properties:

(i) $v_p(T(e)(x)) = v_p(t(e)) = 0$ if $x \in g(e)$, and 1 if $x \notin g(e)$.

(ii) For distinct monomials $s_1,\ldots, s_n$ in $A(y)$ and for elements $\lambda_1,\ldots,\lambda_n$ of $\kappa - \{0\}$, $v_p(\Sigma\{\lambda_i s_i: i = 1,\ldots,n\}) = \min\{v_p(s_i): i = 1,\ldots,n\}$.

Furthermore v_p extends to $v_p: A^*(y) \to \mathbf{Z}$ by $v_p(a(y)/b(y)) = v_p(a(y)) - v_p(b(y))$ for $a(y)/b(y) \in A^*(y) - \{0\}$.

[Pf] By definition, for any two elements $a(y)$, $b(y)$ of $A(y)$, $v_p(a(y)b(y)) = v_p(a(y)) + v_p(b(y))$, $v_p((a + b)(y)) = \min\{v_p(a), v_p(b)\}$, $v_p(a(y)) \geq 0$, and $v_p(1) = 0$. So that v_p is an additive valuation. It is also verified that v_p extends to $A^*(y) \to \mathbf{Z}$ with the definition: for an element $a(y)/b(y)$ of $A^*(y)$, $v_p(a(y)/b(y)) = v_p(a) - v_p(b)$.

Second, in order to show that the spring $\mathbf{A} = (X, \{A(x)\}, A)$ associated with a spectral space X is an indexed spring with the index v defined as above, it is sufficient to show that for any $p = (y, x) \in \sigma(X)$, v_p defined above is indeed an index on the spring **A** which satisfies the conditions (I_1), (I_2) in the defintion (2.1.5). Note that any $a \in A$ is of the form $\Sigma\{\lambda_{(j,1)\ldots(j,i_j)}T(e_{j,1})\ldots T(e_{j,i_j}): j = 1,\ldots, k\}$, thus if $y \in d(a)$, $a(y)$ can be written in the form $a(y) = a(x) + \mu_1 s_1 + \ldots + \mu_m s_m$, where $\mu_\lambda \in A(x)$ and s_λ is a monomial in $T(e_{1,1})(y),\ldots, T(e_{1,i_1})(y),\ldots, T(e_{j,1})(y),\ldots, T(e_{j,i_j})(y),\ldots, T(e_{n,1})(y),\ldots,$ and $T(e_{n,i_n})(y)$, $\lambda = 1,\ldots, m$, where m is taken to be the greatest degree of nonzero monomials s_λ.

(I_1): Clearly for any $a \in A$ and $p = (y, x) \in \sigma(X)$, we obtain $v_p(a(y)) \geq 0$. Since the order of a polynomial $a(y)$ is 0 iff its constant term $a(x)$ is nonzero, that is, iff $x \in d(a)$. (I_2): Take N in (I_2) as m, that is, the greatest degree of nonzero monomials in $a(y)$. ■

Now we shall define a covariant functor $H: \mathscr{I} \to \mathscr{B}$ which preserves the space.

Remark and Definition (2.1.18). — Let $G(X) = (X, E, g) \in \mathrm{Obj}(\mathscr{I})$. Then by (2.1.17), we can assign to $G(X)$ an indexed spring $(\mathbf{A}, v) = ((X, \{A(x)\}, A), v)$. Let H be this correspondence $H: \mathscr{I} \to \mathscr{B}$ and denote by $H(G(X))$ this corresponding indexed spring. Let $G(X) = (X, E, g)$, $G(X') = (X', E', g') \in \mathrm{Obj}(\mathscr{I})$ and (f, r) a morphism in the category $\mathscr{I}$. Let $h_0: K' \to K$ be a κ-homomorphism defined by $h_0(t(e')) = t(r(e'))$, where $K = \kappa[t(e): e \in E]$ and $K' = \kappa[t(e'): e' \in E']$. Note that $r: E' \to E$ is an injection. Let $\eta: K \to A$ (resp. η':

$K' \to A$) be defined by $\eta(t(e)) = T(e)$ (resp. $\eta'(t(e')) = T(e')$). If we define a function $h: A' \to A$ by $h \circ \eta' = \eta \circ h_0$, then h is a κ-homomorphism which arises from h_0. Furthermore since the family $\{t(e'): e' \in E'\}$ generates K', obviously h is unique. Conversely by defining a function h_0 with the identities above, we have a unique κ-homomorphism h_0. Next if we define h by $h \circ \eta' = \eta \circ h_0$. Then h is a unique κ-homomorphism from A' to A (uniqueness is shown by the fact that $\{T(e'): e' \in E'\}$ generates A').

We show that $(f, \{h_x\}, h)$ is a spring morphism. It is not difficult to show that $\nu_x: A \to A(x)$, $\nu'_{f(x)}: A' \to A'(f(x))$ and h_x are ring homomorphisms, so the proofs are omitted. So we show that $\nu_x \circ h = h_x \circ \nu'_{f(x)}$ and $f^{-1}(d(a')) = d(h(a'))$. Since $h(T(e')) = h(\eta'(t(e'))) = \eta h_0(t(e')) = \eta(t(r(e')) = T(r(e'))$, we obtain $(\nu_x \circ h)(T(e')) = T(r(e'))(x) = 1$ if $x \in g(r(e')) = f^{-1}(g'(e')) = (Q(f) \circ g')(e')$ and 0 if $x \notin g(r(e')) = f^{-1}(g'(e'))$. And $(h_x \circ \nu'_{f(x)}(T(e')) = h_x(T(e')(f(x))) = h_x(1) = 1$ if $f(x) \in g'(e')$ and $h_x(0) = 0$ if $f(x) \notin g'(e')$. So that $\nu_x \circ h = h_x \circ \nu'_{f(x)}$, by the equivalent relations $x \in g(r(e')) \Leftrightarrow x \in f^{-1}(g'(e'))$ ... (#) induced by $Q(f) \circ g' = g \circ r$. Next by observing $d(T(e)) = g(e)$, $d(T(e_1)...T(e_n)) = g(e_1) \cap ... \cap g(e_n)$ and for distinct nonzero monomials $s_1,..., s_n$ in K and $\lambda_1,...,\lambda_n \in \kappa - \{0\}$, we have $d(\eta(\Sigma\{\lambda_i s_i: i = 1,...,n\})) = \cup\{d(\eta(s_i)): i = 1,...,n\}$. So $f^{-1}(d(T(e'))) = f^{-1}(g'(e')) = Q(f)(g'(e')) = g(r(e')) = d(T(r(e'))) = d(\eta(t(r(e')))) = d(\eta(h_0(t(e'))) = d(h(\eta'(t(e'))))$ $= d(h(T(e')))$, and thus for any $a' \in A'$, $f^{-1}(d(a')) = d(h(a'))$. Therefore the triple $(f, \{h_x\}, h)$ is a morphism of springs. Finally we show $(f, \{h_x\}, h)$ is a morphism of indexed spring. We need to show $v_p(h(a')(y) = v'_{f(x)}(a'(f(y))$ for every $a' \in A'$ and $p = (y, x) \in \sigma(x)$. Take $T(e') \in A'$. Since $h(T(e')) = h(\eta'(t(e'))) = \eta(h_0(t(e'))) = \eta(t(r(e'))) = T(r(e'))$, $v_p(h(T(e'))(y)) = 0$ if $x \in g(r(e'))$ and 1 if $x \notin g(r(e'))$. But again by (#), $v'_{f(p)}(t(e')(f(y)))$ takes the same value as $v_p(h(T(e'))(y))$, so that $v_p(T(e'))(y)) = v'_{f(p)}(T(e')(f(y)))$. Since $T(e')$'s generate A', this suffices to prove that $(f,\{h_x\}, h)$ is a morphism of indexed springs. As a consequence, we can associate to a morphism (f, r) of the category $\mathscr{I}$ a morphism $(f, \{h_x\}, h)$ of the category $\mathscr{B}$, thus we have established a functor $H: \mathscr{I} \to \mathscr{B}$ by defining $H(f, r) = (f, \{h_x\}, h)$.

Remark (2.1.18.1). — Since every element $a \in A$ is a polynomial with respect to $\{T(e): e \in E\}$ over κ, say $a = \Sigma\{\lambda_j T(e_{j,1})...T(e_{j,i_j}): j = 1,...,n\}$, its value at any $x \in X$ cannot exceed $n.\max\{i_j: j = 1,...,n\}$. This shows that for any $X \in Obj(\mathscr{S})$, $H(G(X))$ is a simple indexed spring.

Definition (2.1.19). — Let $(\mathbf{A}, v)$ and $(\mathbf{A}', v')$ be indexed springs with $\mathbf{A}'$ an extension of $\mathbf{A}$. So for any $y \in Y$, $A(y) \subseteq A'(y) \subseteq A^*(y)$ and for each $p = (y, x) \in \sigma(X)$, v_p, v'_p are additive valuations on $A^*(y)$. If $v_p = v'_p$ for any $p \in \sigma(X)$ then we say the indexed spring $(\mathbf{A}', v')$ is a *v-extension* of the indexed spring $(\mathbf{A}, v)$.

Proposition (2.1.20). — Let $(\mathbf{A}, v)$ be an indexed spring, and let $a, b \in A$ be such that $d(a) \subseteq d(b)$ (so a#b is defined) then the following conditions on the pair (a, b) are equivalent.

(i) for each $p = (y,x) \in \sigma(X)$ with $y \in d(a)$, we have $v_p(a) \geq v_p(b)$, and equality holds only if $x \in d(b)$.

(ii) a#b induces a v-extension of (A, v). ([Hoc,1], Th.3)

[Pf] (ii) $\Rightarrow$ (i): Let v be an index for A[a#b]. Let a, b $\in$ A such that $d(a) \subseteq d(b)$, $p = (y, x) \in \sigma(X)$ and $y \in d(a)$. Now $v_p((a\#b)(y)) \geq 0$ as v is an index on A[a#b]. Since $y \in d(a) \subseteq d(b)$, $(a\#b)(y) = a(y)/b(y)$. So $0 \leq v_p((a\#b)(y)) = v_p(a(y)) - v_p(b(y))$, that is, $v_p(a(y)) \geq v_p(b(y))$. If $v_p(a(y)) = v_p(b(y))$ then $v_p((a\#b)(y)) = 0$, so by the condition (I_1) for the index, $(a\#b)(x) \neq 0$; this implies $x \in d(a)$ and so $x \in d(b)$. ∎

(i) $\Rightarrow$ (ii): First we prove that A[a#b] is a spring extension of A. We need show that for any $r \in$ A[a#b], d(r) is quasi-compact open. Let $r = \Sigma\{a_i(a\#b)^i: i = 0,1,\dots,n\}$, where $a_i \in$ A, and $c = b^n r = \Sigma\{a_i a^i b^{n-i}: i = 0,1,\dots,n\}$. Then $c \in$ A. For any $x \in X$, either $a(x) = 0$ or $\neq 0$. If $a(x) = 0$, then $r(x) = a_0(x)$, so $r(x) \neq 0$ is equivalent to $a_0(x) \neq 0$. And if $a(x) \neq 0$, then $d(a) \subseteq d(b)$ yields $b(x) \neq 0$, and so $r(x) \neq 0$ is equivalent to $c(x) \neq 0$, that is, $d(r) = (d(c) \cap d(a)) \cup (d(a_0) \cap z(a))$; by a similar argument we have also $d(r) = (d(c) \cap d(b)) \cup (d(a_0) \cap z(b))$. It results that d(r) is a patch as it is a finite union of finite intersections of quasi-compact open sets. It follows immediately that d(r) is quasi-compact.

In order to show that d(r) is open, it suffices to show that z(r) is closed in X, for which we show that z(r) is a patch and $Cl_X z(r) = sp(z(r))$, then apply (1.2.7) to get the result required.

Now by an argument similar to the one for $d(r)$ above, we obtain two identities for $z(r)$, $z(r) = (z(c) \cap d(b)) \cup (z(a_0) \cap z(b))$ ... (1), and $z(r) = (z(c) \cap d(a)) \cup (z(a_0) \cap z(a))$ ... (2). It follows that $z(r)$ is a patch. Let $y \in z(r)$ and $x \in Cl\{y\}$. If we show that $x \in z(r)$ then (1.2.7) can be applied to conclude that $z(r)$ is closed in X. Since $z(a_0) \cap z(a)$ is closed in X, if $y \in z(a_0) \cap z(a)$ then $x \in Cl\{y\} \subseteq z(a_0) \cap z(a) \subseteq z(r)$ by (2). So assume the other case that $y \in z(c) \cap d(a) \subseteq z(c) \cap d(b)$. We check the two cases when $x \in d(b)$ and $x \in z(b)$. Since $z(c) \cap d(b)$ is closed in $d(b)$, so if $x \in d(b)$, $x \in Cl\{y\} \cap d(b) \subseteq z(c) \cap d(b) \subseteq z(r)$. Next if $x \in z(b)$, by the condition (i), we have $v_p(a) > v_p(b)$, where $p = (y, x) \in \sigma(X)$. Then $r(y) = 0$ implies $0 = \Sigma\{a_i(y)(a(y)^i/b(y)^i)$ since $y \in d(a) \subseteq d(b)$. Therefore by multiplying $b(y)^n$on both hand sides, we have $0 = \Sigma\{a_i(y)a(y)^i b(y)^{n-i}$. And thus $- b(y)^n a_0(y) = \Sigma\{a_i(y)a(y)^i b(y)^{n-i}$. So operating v_p on both hand sides, we have $v_p(- b(y)^n a_0(y)) = v_p(\Sigma\{a_i(y)a(y)^i b(y)^{n-i})$. Now the left hand side is $nv_p(b) + v_p(a_0)$, and the right hand side is $\min\{v_p(a_i) + iv_p(a) + (n - i)v_p(b): i = 1,\ldots,n\} > \min\{v_p(a_i) + nv_p(b): i = 1,\ldots,n\} \geq nv_p(b)$, since $v_p(a_i) \geq 0$ for any i, and $v_p(a) > v_p(b)$. So $nv_p(b) + v_p(a_0) > nv_p(b)$, and $v_p(a_0) > 0$. Thus by (I_1), $a_0(x) = 0$ and so $x \in z(a_0)$. Therefore $x \in z(a_0) \cap z(b) \subseteq z(r)$. Consequently, every possible case we have $x \in z(r)$. This completes the proof that a#b induces a spring extension of **A**.

It remains to show that v is an index for A[a#b]. We need show for any $r \in A[a\#b]$ and $y \in d(r)$, $v_p(r) \geq 0$, with equality iff $r(x) \neq 0$ (for (I_1)), and there is an integer $N > 0$ such that $v_p(r) \leq N$ (for (I_2)). At first recall that $d(r) = d(c) \cap d(b)) \cup (d(a_0) \cap z(b))$ and $c = b^n r \in A$. Since $y \in d(r)$ by assumption, $y \in d(c) \cap d(b))$ or $y \in d(a_0) \cap z(b)$. Suppose $y \in d(c) \cap d(b)$. By the form of c, the assertion that for any $u \in X$, $c(u) \neq 0$ implies $u \in d(b^n) = d(b)$, that is, $d(c) \subseteq d(b^n)$. So $c\#b^n$ has meaning and equals to r. Now $v_p(r(y)) = v_p(c(y)) - v_p(b^n(y)) \geq 0$ and equality only if $x \in d(b^n) = d(b)$ by (i). But $x \in d(b)$ is equivalent to $v_p(b(y)) = 0$, so that $v_p(r(y)) = 0$ also implies $v_p(c(y)) = 0$, that is, $x \in d(c)$. Therefore if $v_p(r(y)) = 0$, then $x \in d(b) \cap d(c) \subseteq d(r)$. Let $v_p(c(y)) \leq N$. Then $v_p(r(y)) = v_p(c(y)) - v_p(b^n(y)) = v_p(c(y)) - nv_p(b(y)) \leq N$ because $v_p(b(y)) \geq 0$ by $b \in A$. Next suppose $y \in d(a_0) \cap z(b)$. Then from the form of r, $r(y) = a_0(y)$. Clearly $v_p(r(y)) = v_p(a_0(y)) \geq 0$ with equality only if $x \in d(a_0)$. Since $y \in d(a_0) \cap z(b) \subseteq z(b)$, $x \in Cl\{y\} \subseteq z(b)$. Thus if $v_p(r(y)) = 0$, then $x \in d(a_0) \cap z(b) \subseteq d(r)$. Therefore in every possible case, v satisfies the condition of being an index on A[a#b]. ■

Definition (2.1.21). — Let $(\mathbf{A}, v) = ((X, \{A(x)\}, A), v)$ be an indexed spring. Define $G(\mathbf{A}, v) = \{(a, b) \in A \times A : a\#b$ induces a v-extension of $(\mathbf{A}, v)\}$. In other words, by (2.1.20), $G(\mathbf{A}, v) = \{(a, b) \in A \times A$: $d(a) \subseteq d(b)$ and for any $p = (y, x) \in \sigma(X)$, $v_p(a(y)) \geq v_p(b(y))$ with equality only if $x \in d(b)\}$. Let us denote by $(\mathbf{A}^1, v)$ the indexed spring whose underlying ring is $A^1 = A[a\#b: (a, b) \in G(\mathbf{A}, v)]$.

Note (2.1.21.1). — We can show by a similar argument as in (2.1.20) and with the finite induction that the indexed spring (A^1, v) is well defined and it is a v-extension of (A, v). For, every element $r \in A^1$ belongs to some $A[a_1\#b_1, ..., a_n\#b_n]$, a subring of A^1 generated by a finite number of $a_i\#b_i$, where $(a_i, b_i) \in G(A, v)$. Note also that $G(A^1, v) \cap (A \times A) = G(A, v)$.

Definition (2.1.22). — When a v-extension (A^n, v) of (A, v), has been defined, let (A^{n+1}, v) be the v-extension of (A, v) induced by $G(A^n, v)$. And let M(A, v) be the v-extension induced by the union of the underlying rings A^n of all the A^n. Denote by M(A) the underlying ring of M(A, v).

Note (2.1.22.1). — It is easy to show for any n, $G(A^{n+1}, v) \cap (A^n \times A^n) = G(A^n, v)$, this implies that $\{(A^n, v): n = 0,1,...\}$ is an ascending chain of v-extensions of (A, v) with respect to inclusion. Therefore for any $a, b \in M(A)$, there exists an integer $n \geq 0$ such that $a, b \in A^n$, and it is possible to think of the v-extension M(A, v) of (A, v).

Proposition (2.1.23). — If an indexed spring (A, v) is simple, so is M(A, v). ([Hoc,1], Th.4)

[Pf] If $a, b \in A$ have finite image, so does a#b, because every element $r \in M(A)$ is a polylnomial of a finte number of $a_i\#b_i$ over A, where $(a_i, b_i) \in G(A^n, v)$ for some n for every i. So each $r \in M(A)$ has only finitely many values on X. ■

Proposition (2.1.24). — M(A, v) satisfies the condition (***) of (2.1.13).

[Pf] Let a, b $\in$ M(A) and d(a) $\subseteq$ d(b), then a, b $\in$ A^n for some n, and there exists N $\in$ **N**, such that for any p = (y,x) $\in$ σ(X), $v_p(b(y)) \leq N$. We want to show $(a^{N+1}, b) \in G(A^n, v)$. Because, then $a^{N+1}\#b \in A^{n+1} \subseteq M(A)$. First $d(a^{N+1}) = d(a) \subseteq d(b)$, so $a^{N+1}\#b$ is defined. It remains to show that $v_p(a^{N+1}(y)) \geq v_p(b(y))$ with equality only if x $\in$ d(b). If $v_p(a(y)) = 0$ then x $\in$ d(a) $\subseteq$ d(b) because v is an index on (**A**, v). So $v_p(b(y)) = 0$, and x $\in$ d(b). Therefore $v_p(a^{N+1}(y)) = v_p(b(y))$ and x $\in$ d(b). If $v_p(a(y)) \neq 0$, then $v_p(a(y)) \geq 1$. So $v_p(b(y)) \leq N < N + 1 \leq (N + 1)v_p(a(y)) = v_p(a^{N+1}(y))$. This shows $(a^{N+1}, b) \in G(\mathbf{A}^n, v)$ by (1.3.18). Consequently $a^{N+1}\#b \in$ M(A) and so there exists c $\in$ M(A) such that $a^{N+1} = bc$, that is, a $\in$ rad(b), which is to be shown. ■

Proposition (2.1.25). — Let (f, h) be a morphism of indexed springs from (**A**, v) to (**A**', v'). Let (a', b') $\in$ G(**A**',v'). Then (h(a'), h(b')) $\in$ G(**A**, v) and there is a unique extension h_1 of h to a homomorphism from A'[a'#b'] to A[h(a')#h(b')] such that h_1(a'#b') = h(a')#h(b') and for this h_1, (f, h_1) is a morphism of indexed springs from (**A**[h(a')#h(b')], v) to (**A**'[a'#b'], v'). Moreover if ($\mathbf{A}_1$, v), ($\mathbf{A}'_1$,v') extend (**A**, v), (**A**'[a'#b'], v') respectively and h_2: $A'_1 \rightarrow A_1$ is a homomorphism agreeing with h on A' such that (f, h_2) is a morphism of indexed springs from ($\mathbf{A}_1$, v) to ($\mathbf{A}'_1$, v), then h(a')#h(b') $\in$ A_1 and h_2(a'#b') = h(a')#h(b'). ([Hoc,1], Prop.2)

[Pf] By (a', b') $\in$ G(**A**', v'), we get $v'_{f(p)}(a'(y)) \geq v'_{f(p)}(b'(y))$ with equality only if f(x) $\in$ d(b'), where f(p) = (f(x), f(y)) $\in$ σ(X) and f(y) $\in$ d(a'). Since (f, h) is a morphism of indexed springs, $f(y) \in d(a')$ implies $y \in f^{-1}(d(a')) = d(h(a'))$ and also $f(x) \in d(b')$ implies $x \in f^{-1}(d(b') = d(h(b'))$. So $v_p(h(a')) = v'_{f(p)}(a') \geq v'_{f(p)}(b'y)) = v_p(h(b'))$ with equality $x \in d(h(b'))$. Therefore $(h(a'), h(b')) \in G(\mathbf{A}, v)$. Next by defining h_1: $A'[a'\#b'] \to A[h(a')\#h(b')]$ by $h_1(a') = a'$ and $h_1(a'\#b') = h(a')\#h(b')$, we have an extension h_1 of h to $A'[a'\#b']$. Observing that a'#b' generates A'[a'#b'], the uniqueness follows. ■

Definition (2.1.26). — Let ($\mathbf{A}$, v), ($\mathbf{A}'$, v') be indexed springs. Let (f, h) a morphism from ($\mathbf{A}$, v) to ($\mathbf{A}'$, v'). Then by the previous proposition, we have a homomorphism h_2: M(A') → M(A) which is an extension of h. Let M(f, h) = (f, h_2). Then M is a space preserving functor $\mathscr{B} \to \mathscr{B}$ which assigns to a simple indexed spring an affine spring.

Remark (2.1.26.1). — Let X be a spectral space. We defined H(G(X)) in (2.1.17) and we observed in (2.1.18) that H(G(X)) is an indexed simple spring. So by (2.1.23), M(H(G(X))) is also an indexed simple spring. On the other hand, by (2.1.24), M(H(G(X))) satisfies the condition (***)) of (2.1.13), so that it satisfies the condition (**) in (2.1.11) and it results from (2.1.11) that M(H(G(X))) = ((X, {M(A)(x)}, M(A), v) is affine, that is, X = Spec(M(A)).

Conclusion (2.1.27). — Let F: $\mathscr{B} \to \mathscr{C}$ be the forgetful functor defined by f($\mathbf{A}$, v) = A, where ($\mathbf{A}$, v) = ((X, {A(x)}, A), v) constructed in (2.1.18). Then the functor L referred in (2.1.3.1) can be taken as FMH: $\mathscr{I} \to \mathscr{C}$. ([Hoc,1], Th.5)

(2.1.28). — Description of the categories and functors involved.

(1^0) $\mathscr{E}$: The category of sets and functions.

(2^0) $\mathscr{C}$: The category of commutative rings and ring homomorphisms.

(3^0) $\mathscr{T}$: The category of topological spaces and continuous maps.

(4^0) $\mathscr{S}$: The category of spectral spaces and spectral maps.

(5^0) $Q : \mathscr{S} \to \mathscr{E}$ the contravariant functor such that

$Q(X) = \{U \subseteq X$: U is quasi-compact and open$\}$ for $X \in \mathrm{Obj}(\mathscr{S})$,

$Q(f)$: $Q(X') \to Q(X)$ defined by $Q(f)(U') = f^{-1}(U')$, for $f \in \mathrm{Hom}_{\mathscr{S}}(X, X')$.

(6^0) σ: $\mathscr{S} \to \mathscr{E}$ the covariant functor such that

$\sigma(X) = \{(y, x) \in X \times X: x \in \mathrm{Cl}\{y\}\}$ for $X \in \mathrm{Obj}(\mathscr{S})$,

$\sigma(f)$: $\sigma(X) \to \sigma(X')$ defined by $\sigma(f) = (f(y), f(x))$ for $f \in \mathrm{Hom}_{\mathscr{S}}(X, X')$.

(7^0) $\mathscr{I}$: The category of spaces with indeterminates.

Objects: $G(X) = (X, E, g)$, where X is a spectra space, E is a set and g: $E \to Q(X)$ the function whose image $g(E)$ is an open basis of X.

Morphisms: (f, r): $(X, E, g) \to (X', E', g')$, where f: $X \to X'$ is a spectral map, r: $E' \to E$ is an injective function such that $g \circ r = Q(f) \circ g'$.

(8^0) $G : \mathscr{S} \to \mathscr{I}$ the functor such that:

$G(X) = (X, E, g)$, for $X \in \mathrm{Obj}(\mathscr{S})$,

$G(f) = (f, r)$: $(X, E, g) \to (X', E', g')$.

(9^0) $\mathscr{A}$: The category of springs and spring morphisms.

Objects: springs $\mathbf{A} = (X, \{A(x)\}, A)$.

Morphisms: (f, $\{h_x\}$, h): $\mathbf{A} \to \mathbf{A}'$, where f: $X \to X'$ is a spectral map, h: $A' \to A$ is a ring homomorphism such that $f^{-1}(d(a')) = d(h(a'))$, and h_x: $A'(f(x)) \to A(x)$ such that $h_x(a'(f(x))) = h(a')(x)$ (or $v_x \circ h = h_x \circ v'_{f(x)}$) for any $a' \in A'$.

(10°) $\mathscr{B}$: The category of indexed springs and those morphisms.

Objects: indexed spring $(\mathbf{A}, v) = ((X, \{A(x)\}, A), v)$.

Morphisms: (f, h): $(\mathbf{A}, v) \to (\mathbf{A}', v')$, where (f, $\{h_x\}$, h) is a spring morphism with the property $v_p(h(a')(y)) = v_{f(p)}(a(f(y)))$, for $a' \in A'$, $p = (y, x) \in \sigma(X)$, $y \in d(h(a'))$ and $f(p) = (f(y), f(x)) \in \sigma(X')$.

(11°) H : $\mathscr{G} \to \mathscr{B}$ the covariant functor whose image objects are simple such that $H(G(X)) = ((X, \{A_x\}, A), v)$, defined in (2.1.18) and with the property (*) in (2.1.9), for $G(X) = (X, E, g)$.

$H(f, r) = (f, \{h_x\}, h)$: $G(X) \to G(X')$, where h: $A' \to A$ is defined as follows: Let $K = \kappa[t(e): e \in E]$, $K' = \kappa[t'(e'): e' \in E']$, and η: $K \to A = \kappa[T(e): e \in E]$ (resp. η': $K' \to A' = \kappa[T'(e'): e' \in E']$) be a κ-homomorphism $\eta(t(e)) = T(e)$ (resp. $\eta'(t'(e')) = T'(e')$) and for each $x \in X$, h_x is defined as above. We have a κ-homomorphism h_0: $K' \to K$ such that $h_0(t'(e')) = t(r(e'))$. Then we obtain a unique κ-homomorphism h: $A' \to A$ such that $h \circ \eta' = \eta \circ h_0$, and (f, $\{h_x\}$, h) = H(f, r) is a morphism of indexed spring.

(12°) M : $\mathscr{B} \to \mathscr{B}$ the covariant functor such that M(A,v) is defined as in (2.1.22) and with the property (**) in (2.1.11).

$M(f, h) = (f, h^*): M(\mathbb{A},v) \to M(\mathbb{A}',v')$, where h^* is the extension of the κ-homomorphism h to $M(A') \to M(A)$.

(13^0) $F : \mathscr{B} \to \mathscr{C}$ the forgetful functor such that

$F(\mathbb{A},v) = F((X, \{A(x)\}, A),v) = A$ for $(\mathbb{A},v) \in \mathrm{Obj}(\mathscr{B})$,

$F(f, h) = h: A' \to A$ for $(f, h) \in \mathrm{Hom}((\mathbb{A},v), (\mathbb{A}',v))$.

(14^0) Relation of categories (on the next sheet).

Categories Objects

$\mathscr{S}$ X

$\downarrow$ G

$\mathscr{I}$ $G(X) = (X, E, g)$ space with indeterminates

$\downarrow$ H

$\mathscr{B}$ $H(G(X)) = (\mathbf{A}, v) = ((X, \{A(x)\}, A), v)$ simple indexed spring

$\downarrow$ M

$\mathscr{B}$ $M(H(G(X))) = (M(\mathbf{A}), v) = ((X, \{M(A)(x)\}, M(A)), v)$ indexed

$\downarrow$ spring simple and affine

$\mathscr{C}$ $LG(X) = F(M(H(G(X)))) = M(A)$ the ring whose spectrum is X

$\downarrow$ Spec

$\mathscr{S}$ $\mathrm{Spec}(LG(X)) = \mathrm{Spec}(M(A)) = X$ spectral space

(15^{0}) Construction (on the next sheet).

$\sigma(X) \xrightarrow{(\sigma(f))} \sigma(X')$ $\quad \sigma(f)(y, x) = (f(y), f(x)) \in \sigma(X')$, where $(y, x) \in \sigma(X)$

$\sigma\uparrow \qquad\qquad \uparrow\sigma$

$X \xrightarrow{(f)} X'$ $\quad$ given spectral spaces X, X' and spectral map f

$Q \downarrow \qquad\qquad \downarrow Q$ $\quad Q(f)\,(U') =_{def} f^{-1}(U')$, $Q(f)(d(a') = d(h(a'))\ (\forall a' \in A')$

$Q(X) \xleftarrow{(Q(f))} Q(X')$

$g\uparrow \qquad\qquad \uparrow g'$ $\quad g(E)$ (resp. $g'(E')$) an open subbasis of X (resp. X')

$E \xleftarrow{(r)} E'$

$t\downarrow \qquad\qquad \downarrow t$ $\quad t(e')$ is an indeterminate corresponding to $g(e')$

$K \xleftarrow{(h_o)} K' = \kappa[t(e'): e' \in E']$, $h_0(t(e')) =_{def} t(r(e'))$

$\eta\downarrow \qquad\qquad \downarrow\eta'$ $\eta'(t(e')) = T(e')$, $T(e')(x') = 1\ (x' \in g'(e'))$, $0\ (x' \notin g'(e'))$

$A \xleftarrow{(h)} A' = \kappa[T(e'): e' \in E']$, $h \circ \eta' = \eta \circ h_0$

$v_x\downarrow \qquad\qquad \downarrow v'_{x'}$ $\quad v'_{x'}(T(e')) =_{def} t(e')(x')$

$\Pi_{x\in X}A(x) \leftarrow \Pi_{x'\in X'}A'(x') : \Pi_{x\in X}h_x,$ $\qquad A(x) = \kappa[t(e): e \in E, x \in g(e)]$

$\downarrow \qquad\qquad \downarrow$

$\Pi_{x\in X}A[a\#b](x) \leftarrow \Pi_{x'\in X'}A'[a'\#b'](x')$

$\downarrow \qquad\qquad \downarrow$

$\Pi_{x\in X}M(A)(x) \leftarrow \Pi_{x'\in X'}M(A')(x')$

$\downarrow \qquad\qquad \downarrow$

$\Pi_{x\in X}A^*(x) \longleftarrow \Pi_{x'\in X'}A'^*(x')$, $A^*(x)$ is the field of fractions of $A(x)$

$v\downarrow \qquad\qquad \downarrow v'$ $\quad v = (v_p)$, $v_p: A(y) \to \mathbf{Z}$, $p = (y, x) \in \sigma(X)$

$\mathbf{Z} \qquad\qquad \mathbf{Z}.$

Section 2. Some applications.

Theorem (2.2.1). — Every spectral space and morphism is in the image of Spec (In other words, Spec: $\mathscr{C} \to \mathscr{S}$ is a full functor). ([Hoc,1], Th.6)

[Pf] Let X, X' be spectral spaces, f: X $\to$ X' a spectral map and id_X, $id_{X'}$ are identity maps of X and X' respectively. Then $\mathscr{R} = \{X, X'; id_X, id_{X'}, f\}$ is a subcategory of $\mathscr{S}$. Define G: $\mathscr{R} \to \mathscr{I}$ thus: $G(X) = (X, Q(X) \times Q(X'), g_X)$, where g_X: $Q(X) \times Q(X') \to Q(X)$ defined by $g_X(U, U') = U$, for $U \in Q(X)$ and $U' \in Q(X')$. $G(X') = (X', Q(X'), g_{X'})$, where $g_{X'} = id_{Q(X')}$. Let r: $Q(X') \to Q(X) \times Q(X')$ defined by $r(V') = (f^{-1}(V'), V')$ for $V' \in Q(X')$. Let $G(f) = (f, r)$, $G(id_X) = id_{G(X)}$ and $G(id_{X'}) = id_{G(X')}$. Then obviously $g_X(Q(X) \times Q(X')) = Q(X)$ is an open basis of X and $g_Y(Q(X')) = Q(X')$ is an open basis of X', so that G(X), $G(X') \in Obj(\mathscr{I})$. It is not difficult to show that r is injective and $Q(f) \circ g_{X'} = g_X \circ r$, so that (f, r) is a morphism in the category $\mathscr{I}$. Therefore G is a space preserving functor. Hence Spec is invertible on $\mathscr{R}$, and so the functor Spec is a full functor from $\mathscr{C}$ to $\mathscr{S}$. ■

Proposition (2.2.2). — On the following subcategories $\mathscr{R}_i$ of $\mathscr{S}$, Spec is invertible.

(i) The subcategory $\mathscr{R}_0$ of all spectral spaces and surjective spectral maps.

(ii) For each spectral space X, the subcategory $\mathscr{R}_1$ of $\mathscr{S}$ consisting of spectral subspaces and inclusions of these. ([Hoc,1], Th.6)

[Pf] (i): Let X, X' $\in$ Obj($\mathscr{R}_0$) and f: X $\to$ X' be the surjective spectral map. Then Q(f) is injective. For let U', V' $\in$ Q(X') such that Q(f)(U') = Q(f)(V'), then we obtain U' = f(f $^{-1}$(U')) = f(Q(f)(U')) = f(Q(f)(V')) = f(f $^{-1}$(V')) = V' by the surjectivity of f.

Define G_0: $\mathscr{R}_0 \to \mathscr{S}$ by G_0(X) = (X, Q(X), $id_{Q(X)}$), G_0(X') = (X', Q(X'), $id_{QX')}$). Then evidently G_0(X), G_0(X') $\in$ Obj($\mathscr{S}$). Since obviously $id_{Q(X)} \circ$ Q(f) = Q(f) $\circ$ $id_{Q(X')}$, G_0(f) = (f, Q(f)) is a morphism in $\mathscr{S}$. Hence G_0 is a space preserving functor from $\mathscr{R}_0$ to $\mathscr{S}$. Therefore Spec is invertible on $\mathscr{R}_0$.

(ii): Let Y $\in$ Obj($\mathscr{R}_1$), and f: Y $\to$ X the inclusion map. Define G_1(Y) = (Y, Q(X), Q(f)), G_1(X) = (X, Q(X), $id_{Q(X)}$), and G_1(f) = (f, $id_{Q(X)}$). For any U $\in$ Q(Y), U $\subseteq$ Y $\subseteq$ X, so Q(f)$^{-1}$(U) = {V $\in$ Q(X): Q(f)(V) = f $^{-1}$(V) = U} $\neq \varnothing$ since U $\in$ Q(f)$^{-1}$(U). Hence Q(f) is surjective, and Q(f)(Q(X)) = Q(Y) is an open basis of Y consisting of quasi-compact open subsets of Y. Therefore G_1(Y) $\in$ Obj($\mathscr{S}$). Obviously G_1(X) $\in$ Obj($\mathscr{S}$). Since Q(f) $\circ$ $id_{Q(X)}$ = Q(f) $\circ$ $id_{Q(X)}$, hence G_1(f) is a morphism in $\mathscr{S}$. So G_1 is a space preserving functor from $\mathscr{R}_1$ to $\mathscr{S}$ and so Spec is invertible on $\mathscr{R}_1$. ■

Proposition (2.2.3). — The functor Spec is not invertible on the following subcategories $\mathscr{R}_i$ of $\mathscr{S}$, i = 2, 3, 4, 5.

(i) Subcategory $\mathscr{R}_2$ containing a spectral space X, a one-point space P, and maps f: P $\to$ X, f ': X $\to$ P such that f(P) is not a closed point in X.

(ii) Subcategory $\mathscr{R}_3$ containing a spectral space Y with generic point, a family {X} of spectral spaces with generic point such that the cardinalities of the spaces in the family are not bounded, and for each X, a map from Y to X which preserves generic points.

(iii) Subcategory $\mathscr{R}_4$ containing a one-point space P, spectral spaces X, Y with generic points, a map taking P to the generic point of Y, and two distinct maps from Y to X preserving generic points.

(iv) For any spectral space X containing a point x_0 such that Card(Cl{x_0}) ≥ 3, the subcategory $\mathscr{R}_5$ of $\mathscr{S}$ whose objects are spectral subspaces of X and whose morphisms are embeddings (as opposed to inclusions) of these into others. ([Hoc,1], Prop.3)

[Pf] We prove these assertions in the following manner: Assume the existence of a space preserving functor G_i from the subcategories $\mathscr{R}_i$ to $\mathscr{I}$. So LG_i is a space preserving functor from $\mathscr{R}_i$ to $\mathscr{C}$. Since the functor from $\mathscr{C}$ to the full subcategory $\mathscr{N}$ of $\mathscr{C}$ whose objects are the reduced rings which carries every commutative ring to its reduction is space preserving, we obtain a fuctor T_i from $\mathscr{R}_i$ to $\mathscr{N}$. Since every commutative ring and its reduction have the same prime spectrum, the functor from $\mathscr{R}_i$ to $\mathscr{N}$ is space preserving and inverts Spec. We show that this leads to a contradiction.

(i): Since P is a one-point space, $f' \circ f = id_P$ and $T_2(P)$ is a field. Now $id_{T_2(P)} = T_2(id_P) = T_2(f' \circ f) = T_2(f) \circ T_2(f')$, so that $T_2(f): T_2(X) \to T_2(P)$ is a surjective homomorphism. Identifying X with $\mathrm{Spec}(T_2(X))$ and P with $\mathrm{Spec}(T_2(P))$, we can identify $\mathrm{Spec}\,(T_2(f))$ with f. Then by (1.1.15.1), $f(P) = \mathrm{Spec}(T_2(f))(\mathrm{Spec}(T_2(P))) = {}^aT_2(f)(\mathrm{Spec}(T_2(P)))$ is a closed subset of $X = \mathrm{Spec}(T_2(X))$, a contradiction.

(ii): Let $f: Y \to X$ be a morphism preserving generic point, $T_3: \mathscr{R}_3 \to \mathscr{N}$ the functor inverting Spec on $\mathscr{R}_3$. Since Y and X are T_0 spaces there is one and only one generic point (say) y_0 (resp. x_0) of Y (resp. X). Then $f(y_0) = x_0$ by hypothesis. On the other hand for any $h \in Y$, $y \in \mathrm{Cl}\{y_0\} = V(j_{y_0})$ (where j_{y_0} is the prime ideal corresponding to y_0), so that $j_{y_0} = \cap\{j_y \in T_3(Y) : y \in Y\} = \mathrm{rad}(0) = \{0\}$ since $T_3(Y) \in \mathrm{Obj}(\mathscr{N})$ is reduced. Therefore $T_3(f)^{-1}(0) = \{0\}$, that is, $T_3(f)$ is injective. But since X can be taken with arbitrary cardinality, $T_3(X)$ can have arbitrary large cardinality, a contradiction.

(iii): Let $f_Y: P \to Y$ be a map which sends P to the generic point y_0 of Y. Let $f, f': Y \to X$ be maps preserving generic points and $f \neq f'$. Let x_0 be the generic point of X. Then $(f \circ f_Y)(P) = f(y_0) = x_0 = f'(x_0) = (f' \circ f_Y)(P)$, thus $f \circ f_Y = f' \circ f_Y$. Hence $T_4(f_Y) \circ T_4(f) = T_4(f \circ f_Y) = T_4(f' \circ f_Y) = T_4(f_Y) \circ T_4(f')$. As a special case of (ii), $T_4(f_Y)$ is injective. Therefore $T_4(f) = T_4(f')$ and thus $f = (\mathrm{Spec} \circ T_4)(f) = (\mathrm{Spec} \circ T_4)(f') = f'$, a contradiction.

(iv): We can reduce this case to the case that $X = \{x, x', x''\}$, where $x', x'' \in \mathrm{Cl}\{x\}$. Let $Y = \{x, x'\}$, $f: Y \to X$ the inclusion map, $f': Y \to X$ the map defined by f '(x) = x, f '(x') = x" and f_p: {x} $\rightarrow$ X defined by f(x) = x. Then obviously, f $\neq$ f '. By applying (iii) we have a contradiction f = f '. ■

Remark (2.2.4.1). — Our next object is to characterize the topological spaces underlying preschemes.

At this point refer to (2.1.13.1) about inverting Spec on subcategories of $\mathscr{S}$. We also note an additional remark which would be useful in the proof of the characterization theorem (2.2.10).

Remark (2.2.4.2). — Let G be a space presersving functor from a subcategory $\mathscr{R}$ of $\mathscr{S}$ to $\mathscr{S}$. Then for any object X of $\mathscr{R}$, the corresponding ring LG(X) has the following property:

Let b $\in$ LG(X) be such that b does not vanish at any point of X, then b is invertible in LG(X).

[Pf] Since b does not vanish at any point of X, d(b) = Df(b) = X = D(1). And so 1#b = 1/b. For any p = (y, x) $\in$ σ(X), $v_p(1(y))$ is clearly 0 and $v_p(b(y)) = 0$ as whenever x $\in$ Cl{y}, x $\in$ X = d(b). So that $v_p(1(y)) \geq v_p(b(y))$. Thus by (2.1.20), 1#b = 1/b induces a v-extension of MHG(X) = ((X, {LG(X)(x)}, LG(X), v). But MHG(X) is maximal with respect to v-extensions, so 1/b $\in$ FMHG(X) = LG(X). ■

Definition (2.2.5). — A topological space is callled *locally spectral* if it has a cover by open spectral subspaces. ([Hoc,1], p.58)

Proposition (2.2.6). — A locally spectral space X is T_0 and its open spectral subspaces form an open basis. ([Hoc,1], p.58)

[Pf] Let $\mathbb{B}$ be the family of all open spectral subspaces of X. Let U be open in X and $x \in U$. Since $\mathbb{B}$ is a cover of X, there is $Y \in \mathbb{B}$ such that $x \in Y$. Then $U \cap Y$ is open relative to Y and $x \in U \cap Y$. Since Y is spectral, it has an open basis consisting of quasi-compact open subsets which are spectral (because they are patches, cf. (1.2.6)). So there is an open spectral subspace Z of Y such that $x \in Z \subseteq U \cap Y \subseteq U$ (note that $Z \in \mathbb{B}$). ∎

Note (2.2.7). — Let X be a topological space. A *presheaf* F on X consists of:

(i) For any open subset U of X, F assigns a set F(U).

(ii) For all pairs of open sets $U_1 \subseteq U_2$ there exists a map, called *restriction*, $res_{U_2,U_1} : F(U_2) \to F(U_1)$ with the following axioms being satisfied:

(a) $res_{U,U} = id_{F(U)}$.

(b) If $U_1 \subseteq U_2 \subseteq U_3$, then $res_{U_3,U_1} = res_{U_2,U_1} \circ res_{U_3,U_2}$. ([Mum], I, sec.4, Def.1)

Let F_1, F_2 be presheaves on X, a *map* (of sheaves) $\varphi: F_1 \to F_2$ is a collection of maps $\varphi(U): F_1(U) \to F_2(U)$ for each open U such that if $U \subseteq V$, $\varphi(U) \circ res_{1V,U} = res_{2V,U} \circ \varphi(V)$, where res_1 and res_2 are the restriction maps of F_1 and F_2 respectively. ([Mum], I, sec.4, Def.2)

A presheaf F is called a *sheaf* if for every collection $\{U_i\}$ of open sets in X with $U = \cup\{U_i\}$, the map $\prod res_{U,U_i}: F(U) \to \prod_i F(U_i)$ is injective and its image is the set on which $\prod res_{U_i,U_i \cap U_j}: \prod_i F(U_i) \to \prod_{i,j} F(U_i \cap U_j)$ and $\prod_j res_{U_j,U_i \cap U_j}: \prod_j F(U_j) \to \prod_{i,j} F(U_i \cap U_j)$ agree.

We may write $\Gamma(U, F)$ for F(U), and call it the *set of sections of F over U.* $\Gamma(X, F)$ is called the set of *global sections of F.*

Let F be a sheaf on X, and $x \in X$. The collection of all F(U), where each U is open and containing x, is a direct system ordered by inclusion and we can form F_x = dir.lim.F(U), which is called the *stalk of F at x.* ([Mum], I, sec.4, Example (1))

Let F_0 be a presheaf on X. Then there is a sheaf F and a map f: $F_0 \to F$ such that if g: $F_0 \to F'$ is any map with F' being a sheaf, there is a unique map h: $F \to F'$ such that $g = h \circ f$. We call this process a *sheafification* of the presheaf F_0 and F *the sheaf associated to the presheaf* F_0. ([Mum], I, sec.4, Example (2))

Suppose for all U, F(U) is a group (resp. ring, field, etc.) and each restriction map is a group (resp. ring, field, etc.) homomorphism. Then F is called a *sheaf of groups (resp. rings, fields, etc.)*. In this case F_x is a group (resp. ring, field, etc.). ([Mum], I, sec.4, p.37)

Let X be a toplogical space, $\mathbb{B}$ an open basis of X. Then we call F' the *presheaf on* $\mathbb{B}$ *with values in* $\mathscr{C}$, a family of objects $F'(U) \in Obj(\mathscr{C})$ for each U

$\in \mathbb{B}$ and a family of morphisms (in $\mathscr{C}$) $\mathrm{res}_{V,U}$: $F'(V) \to F'(U)$ defined for any $U, V \in \mathbb{B}$, $U \subseteq V$, with $\mathrm{res}_{U,U} = \mathrm{id}_{F'(U)}$ and $\mathrm{res}_{W,U} = \mathrm{res}_{V,U} \circ \mathrm{res}_{W,V}$ if $U, V, W \in \mathbb{B}$, $U \subseteq V \subseteq W$. Then we can associate to F' a presheaf F_0 on X in the following way:

For any open subset U of X, $F_0(U) = \mathrm{inv.lim.}_V F'(V)$, where V belongs to the subset $\{V \in \mathbb{B}: V \subseteq U\}$ of $\mathbb{B}$ ordered by inclusion. If $U \in \mathbb{B}$ then the canonical morphism $F_0(U) \to F'(U)$ is an isomorphism. ([Gro-Die], (0,3.2.1)).

Definition (2.2.8). — A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ in X. X is called the *underlying space* of the ringed space $(X,\mathcal{O}_X)$ and $\mathcal{O}_X$ is called its *structure sheaf.* A ringed space $(X, \mathcal{O}_X)$ is called a *locally ringed space* if for each point $x \in X$ the stalk $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ is a local ring. When there is no confusion $(X, \mathcal{O}_X)$ is written as X briefly.

A *morphism of ringed spaces* $(f, f^{\#})$: $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ (occasionally written as f) is a pair of continuous map $f: X \to Y$ and a map $f^{\#}: \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings onY, where $f_*\mathcal{O}_X$ is the sheaf on Y defined thus: for any open set V contained in Y, $f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$. A *morphism* of ringed spaces $(f, f^{\#})$ is said to be an *isomorphism* if f is a homeomorphism and $f^{\#}$ is an isomorphism of sheaves. A *morphism $(f, f^{\#})$ of locally ringed spaces* is a morphism of ringed spaces such that for each point $x \in X$, $f_x^{\#}: \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring homomorphism, i.e., $(f_x^{\#})^{-1}(M_x) \subseteq M_{f(x)}$, where M_x

(resp. $M_{f(x)}$) is the maximal ideal of the local ring $\mathcal{O}_{X,x}$ (resp. $\mathcal{O}_{Y,f(x)}$). ([Har], II, sec.3, Def., p.72)

Definition (2.2.9). — An *affine scheme* is a locally ringed space (X, $\mathcal{O}_X$) which is isomorphic to (Spec(A), $\mathcal{O}_{\mathrm{Spec}(A)}$) for some ring A. A *prescheme* is a locally ringed space (X, $\mathcal{O}_X$) in which every point has an open neighbourhood U such that the topological space U, together with the restricted sheaf $\mathcal{O}_X | U$ is an affine scheme.

A *morphism of preschemes* (f, $f^{\#}$) is that of locally ringed spaces.

An *open subscheme* (U, $\mathcal{O}_U$) of a prescheme (X, $\mathcal{O}_X$) is such that U is an open subset of X and structure sheaf $\mathcal{O}_U$ is isomorphic to the restriction $\mathcal{O}_X | U$ of the structure sheaf of X. An *open immersion* is a morphism f: X $\to$ Y, which induces an isomorphism of (X, $\mathcal{O}_X$) with an open subscheme of (Y, $\mathcal{O}_Y$). A *closed subscheme* (Y, $\mathcal{O}_Y$) of a prescheme (X, $\mathcal{O}_X$) is a prescheme, together with a morphism (i, $i^{\#}$): Y $\to$ X, where Y is a closed subset of X and i is the inclusion map, and the induced map $i^{\#}$: $\mathcal{O}_X \to i_*\mathcal{O}_Y$ of sheaves on X is surjective. A *closed immersion* is a morphism f: (X, $\mathcal{O}_X$) $\to$ (Y, $\mathcal{O}_Y$) which induces an isomorphism of (Y, $\mathcal{O}_Y$) onto a closed subscheme of (X, $\mathcal{O}_X$). ([Har], II, sec.3, Def., p.85).

Let f: X $\to$ Y be a morphism of preschemes. The *diagonal morphism* is the unique morphism Δ: X $\to$ X $\times_Y$ X whose composition with both projections is identity map of X, $p_1 \circ \Delta = p_2 \circ \Delta = id_X$. We say that the morphism is *separated* if the diagonal morphism Δ is a closed immersion. In that case we also say *X is separated over Y*. A prescheme X is said to be a *scheme* if it is separated over Spec($\mathbb{Z}$). ([Har], II, sec.4, p.96)

Now we are ready to state and prove the main theorem of this section.

Theorem (2.2.10). — A topological space is the underlying space of a prescheme iff it is locally spectral. ([Hoc,1], Th.9)

[Pf] From the definition of the prescheme, it is clear that the underlying space of a prescheme is locally spectral.

Conversely, let X be a locally spectral space. By (2.2.6), it is T_0, and the set $\mathbf{B}$ of all the open spectral subspaces of X is an open basis of X. We construct in what follows a prescheme structure (X, $\mathcal{O}_X$) on X and show that for any Y $\in \mathbf{B}$, (Y, $\mathcal{O}_X | Y$) is an affine scheme and thus (X, $\mathcal{O}_X$) is a prescheme. The proof is long and technical so we break it into several steps. Let κ be an arbitrary field and {t(U): U $\in \mathbf{B}$} a family of algebraically independent indeterminates over κ (disjoint from κ) indexed by $\mathbf{B}$. Let $\Omega = \kappa(t(U)\colon U \in \mathbf{B})$. We propose to construct a prescheme structure on X over κ, then sections are functions with values in Ω.

Step 1. Consider the subcategory $\mathscr{R}$ of $\mathscr{S}$ defined as follows: Obj($\mathscr{R}$) = $\mathbf{B}$ and morphisms are inclusion maps (then $\mathscr{R}$ is clearly a subcategory of $\mathscr{S}$). Let Y $\in \mathbf{B}$, E(Y) = {U $\in \mathbf{B}$: U $\cap$ Y $\in \mathbf{B}$} and g_Y: E(Y) $\to$ Q(Y) defined by $g_Y(U)$ = U $\cap$ Y. This map is well-defined since U $\cap$ Y $\in \mathbf{B}$ by the definition of E(Y),

$= U \cap Y$. This map is well-defined since $U \cap Y \in \mathbb{B}$ by the definition of E(Y), and so $U \cap Y$ is quasi-compact open in Y. We show that g_Y is the retraction of E(Y) over Q(Y). For let $U \in Q(Y)$, then $U \cap Y = U$ is quasi-compact open in the open spectral subspace Y of X, so that $U \cap Y = U$ is an open spectral subspace of X, thus $U \in E(Y)$ and $(g_Y \mid Q(Y))(U) = U \cap Y = U$. This shows that $g_Y \mid Q(Y) = id_{Q(Y)}$. Therefore g_Y is the retraction of E(Y) over Q(Y). Note that g_Y is thus surjective and $\{U \in \mathbb{B}: U \in Q(Y)\} \subseteq Q(Y)$. So $g_Y(E(Y))$ is an open basis of Y consisting of quasi-compact open subsets of Y. Therefore we can define $G(Y) = (Y, E(Y), g_Y) \in Obj(\mathscr{R})$.

Let $Y' \in \mathbb{B}$ and $f: Y \to Y'$ the inclusion spectral map. Then clearly, $Q(Y) \subseteq Q(Y')$ and $Q(f): Q(Y') \to Q(Y)$ is the retraction of Q(Y') over Q(Y). As G(Y), $G(Y') = (Y', E(Y'), g_{Y'})$ is an object of $\mathscr{I}$. We observe that $E(Y') \subseteq E(Y)$. For let $U' \in E(Y')$, then $U', U' \cap Y' \in \mathbb{B}$. Since f is spectral, $U' \cap Y = U' \cap Y' \cap Y = f^{-1}(U' \cap Y')$ is quasi-compact open in Y. Thus $U' \cap Y$ is an open spectral subspace in Y and so is in X. Therefore $U' \cap Y \in \mathbb{B}$ and $U' \in E(Y)$.

Let $r_f: E(Y') \to E(Y)$ be the inclusion map. Obviously $g_Y \circ r_f = Q(f) \circ g_{Y'}$, so that $g(f) = (f, r_f)$ is a morphism in $\mathscr{I}$ from G(Y) to G(Y') corresponding to f and thus $G: \mathscr{R} \to \mathscr{I}$ is a space preserving functor.

Step 2. Let Ω be as in the beginning of the proof, and $Y \in Obj(\mathscr{R})$. (Note that if $U \in E(Y)$ then $U \cap Y \in \mathbb{B}$.) Let $FHG(Y) = \kappa[T_Y(U): U \in E(Y)]$, where

Then for any $U \in E(Y)$, $T_Y(U)$ is a function on Y with values in Ω. Thus by (2.1.27), $LG = FMHG$ is a space preserving functor from $\mathscr{R}$ to $\mathscr{C}$ which inverts Spec on $\mathscr{R}$.

Let $Y' \in Obj(\mathscr{R})$ be such that $Y \subseteq Y'$. We show that $T_Y(U)$ is the restriction to Y of a function of the form $T_{Y'}(U')$ in $FHG(Y') = \kappa[T_{Y'}(U'): U' \in E(Y')]$. Since $U \in E(Y)$, $g_Y(U) = U \cap Y \in Q(Y) \subseteq Q(Y')$. Since $g_{Y'}$ is onto, it follows that there exists $U' \in E(Y')$ such that $g_{Y'}(U') = g_Y(U)$. Consider the function $T_{Y'}(U')$ and take $y \in Y$. If $y \in g_Y(U) = g_{Y'}(U')$, then we obtain $T_Y(U)(y) = t(U \cap Y)$ and $T_{Y'}(U')(y) = t(U' \cap Y') = t(U \cap Y) = T_Y(U)(y)$. And if $y \notin g_Y(U)$, then $T_Y(U)(y) = 0 = T_{Y'}(U')(y)$. Therefore $T_{Y'}(U') \mid Y = T_Y(U)$. As a consequence of this observation, for any $a \in FHG(Y)$, there exists an element $a' \in FHG(Y')$ such that $a' \mid Y = a$, regarding a (resp. a') as a function from Y (resp. Y') into Ω.

Step 3. By (2.1.26.1) and (2.1.27), the functor $LG: \mathscr{R} \to \mathscr{C}$ inverts Spec on $\mathscr{R}$. The functor thus defines a presheaf structure on the basis $\mathbb{B}$ of X, which, by [Gro-Die],(0,3.2.1), can be extended to a sheaf structure $\mathscr{O}_X$ on X because by the remark in Step 2, elements of $LG(Y)$, $Y \in \mathbb{B}$, can be regarded as functions on Y with values in Ω. Thus if V is an open subset of X and $a \in \Omega^V$, then $a \in \Gamma(Y, \mathscr{O}_X)$ iff there is a covering $\{V_i \in \mathbb{B}: i \in I\}$ of V such that for any i, $a \mid V_i \in LG(V_i)$. (If V is quasi-compact open in X, the above covering can be taken to be a finite covering. We also note that for any $Y \in \mathbb{B}$, $LG(Y)$

can be taken to be a finite covering. We also note that for any $Y \in \mathbf{B}$, LG(Y) $\subseteq \Gamma(Y, \mathcal{O}_X)$. So from Step 2 it follows that the restriction homomorphism from $\Gamma(Y', \mathcal{O}_X)$ to $\Gamma(Y, \mathcal{O}_X)$ carries FHG(Y') onto FHG(Y).

Convention. Let $Y \subseteq Y'$. Note that elements of $\Gamma(Y, \mathcal{O}_X)$ (resp. $\Gamma(Y', \mathcal{O}_X)$) are functions from Y (resp. Y') to Ω . For any element $a \in Y$ extend a to a function from Y' to Ω by setting $a \mid Y' - Y \equiv 0$. Let us denote the extended function also by a. Notice however a may not belong to $\Gamma(Y, \mathcal{O}_X)$. In what follows we adopt this convention. Let $a \in \Gamma(Y, \mathcal{O}_X)$, $a' \in \Gamma(Y', \mathcal{O}_X)$. By aa' we mean the function from Y' to Ω defined by extending a to a function from Y' to Ω in the above manner and taking its product with a'.

Step 4. Since for any $Y \in \mathbf{B}$, Y is spectral, (Y, LG(Y)) is an affine spring. We assert that for any $Y \in \mathbf{B}$, $(Y, \Gamma(Y, \mathcal{O}_X))$ is an affine spring, or equivalently, $(Y, \{\Gamma_y\}, \Gamma(Y, \mathcal{O}_X))$ is an affine spring, where $\Gamma_y = \{a(y): a \in \Gamma(Y,\mathcal{O}_X)\}$ for any $y \in Y$. In order to show that it is a spring, we need to prove that (i) for any $a \in \Gamma(Y, \mathcal{O}_X)$, $d(a) = \{ y \in Y: a(y) \neq 0\}$ is quasi-compact open in Y and (ii) $\{d(a): a \in \Gamma(Y, \mathcal{O}_X)\}$ is an open basis of Y.

By Step 3 above we have a finite open covering $\{V_i \in \mathbf{B}: i = 1,\dots,n\}$ of Y such that for any i, $a \mid V_i \in LG(V_i)$. Since V_i is a spectral space, V_i = $LG(V_i)$ and so $d(a \mid V_i) = D(a \mid V_i)$ is quasi-compact open inY. So the finite union of them $d(a) = \cup\{d(a \mid V_i): i = 1,\dots,n\}$ is quasi-compact open in Y. For (ii), observing that Y is spectral, we know that $\{d(a): a \in LG(Y)\}$ forms an open basis of Y. Since $LG(Y) \subseteq \Gamma(Y, \mathcal{O}_X)$, $d(a)$: $a \in \Gamma(Y, \mathcal{O}_X)\}$ also forms an open basis of Y.

Next, we show that $(Y, \Gamma(Y, \mathcal{O}_X))$ is a simple spring. Let $a \in \Gamma(Y, \mathcal{O}_X)$. We note that for any $y \in Y$, $a(y) \in \Omega$. Since $a \in \Gamma(Y, \mathcal{O}_X)$, there is a finite covering $\{V_i \in \mathbf{B}: i \in I\}$ of Y such that $a \mid V_i \in LG(V_i)$. For each i, $LG(V_i)$ is simple, so the image of $a \mid V_i$ is finite subset of Ω. Therefore $(Y, \Gamma(Y, \mathcal{O}_X))$ is simple.

Finally we prove that $(Y, \Gamma(Y, \mathcal{O}_X))$ satisfies the condition (**) of (2.1.11). Let $a, a' \in \Gamma(Y, \mathcal{O}_X)$ and $d(a) \subseteq d(a')$. By the definition of $\Gamma(Y, \mathcal{O}_X)$, there exist coverings $\{V_i \in \mathbf{B}: i = 1,\dots,n\}$ and $\{U_j \in \mathbf{B}: j = 1,\dots,m\}$ of Y such that $a \mid V_i \in LG(V_i)$ for any i, $a' \mid V_j \in LG(U_j)$ for any j, and $Y = \cup_i \cup_j \{V_i \cap U_j: i = 1,\dots,n, j = 1,\dots,m\}$. For this finer covering, $a \mid V_i \cap U_j$, $a' \mid V_i \cap U_j \in LG(V_i \cap U_j)$ and $d(a \mid V_i \cap U_j) \subseteq d(a' \mid V_i \cap U_j)$. More generally, given any finite number of elements $a_1,\dots, a_n \in \Gamma(Y, \mathcal{O}_X)$ we may assume that there is a suitably fine finite covering $\{V_i \in \mathbf{B}: i = 1,\dots,m\}$ of Y such that $a_j \mid V_i$, $a'_j \mid V_i \in LG(V_i)$ and $d(a_j \mid V_i) \subseteq d(a'_j \mid V_i)$ for $i = 1,\dots,m$, $j = 1,\dots,n$. Let $\{Y_i \in \mathbf{B}: i = 1,\dots,n\}$ be such a fine covering of Y. Then $a \mid Y_i$, $a' \mid Y_i \in LG(Y_i)$ and $d(a \mid Y_i) \subseteq d(a' \mid Y_i)$ for all i. Hence for each i we can choose an integer $N_i > 0$ such that $(a \mid Y_i)^n \#(a' \mid Y_i) \in LG(Y_i)$ for all $n \geq N_i$. Thus for some n, $a^n \# a' \in \Gamma(Y, \mathcal{O}_X)$ and so $a \in rad(a')$ in the ring $\Gamma(Y, \mathcal{O}_X)$. Therefore the condition (**) of (2.1.11) holds. Since $(Y, \Gamma(Y, \mathcal{O}_X))$ is simple, $(Y, \Gamma(Y,\mathcal{O}_X))$ is an affine spring.

This completes the proof of Step 4, and so we are allowed to use z and V (also d and D) interchangeably.

Step 5. In order to verify that $(Y, \mathcal{O}_X | Y)$ is an affine scheme, we have to show that for any $b \in \Gamma(Y, \mathcal{O}_X)$, $b \neq 0$, $\Gamma(Y, \mathcal{O}_X)_b$ is isomorphic to $\Gamma(D(b), \mathcal{O}_X)$. (Recall that $D(b) = d(b) = \{y \in Y: a(y) \neq 0\}$.)

(i) Since $b \in \Gamma(Y,\mathcal{O}_X)$, $b \neq 0$, $b | D(b) \in \Gamma(D(b), \mathcal{O}_X)$ does not vanish at any point of D(b). So 1/b makes sense on D(b). We claim that $1/b \in \Gamma(D(b), \mathcal{O}_X)$. There exists a finite cover $\{V_i \in \mathbf{B}: i = 1,\ldots,n\}$ of D(b) such that $b | V_i \in LG(V_i)$ for any i. So $b | V_i$ does not vanish at any point of V_i. By (2.2.6), for any i, $1/b \in LG(V_i)$, and so by the definition of $\Gamma(D(b), \mathcal{O}_X)$, $1/b \in \Gamma(D(b), \mathcal{O}_X)$.

(ii) Let f be the restriction map of the sheaf $\mathcal{O}_X$, $f = res_{Y,D(b)}: \Gamma(Y, \mathcal{O}_X) \to \Gamma(D(b), \mathcal{O}_X)$ then f extends uniquely to a homomorphism $h: \Gamma(Y, \mathcal{O}_X)_b \to \Gamma(D(b), \mathcal{O}_X)$ with $f = h \circ g$, where $g: \Gamma(Y, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_X)_b$ is the canonical homomorphism defined by $b(a) = a/1$, because b is invertible in $\Gamma(D(b), \mathcal{O}_X)$ (cf. (1.1.16.1)).

We verify that this homomorphism is an isomorphism. First we show that h is injective. Let a/b^n, a'/b^m be two elements of $\Gamma(Y, \mathcal{O}_X)_b$ such that $h(a/b^n) = h(a'/b^m)$. Then for any $y \in D(b)$, $(a/b^n)(y) = (a'/b^m)(y)$ in $\Gamma(Y, \mathcal{O}_X)_{j_y}$. Thus there exists $c \notin j_y$ such that $c(b^m a - b^n a') = 0$ in $\Gamma(Y, \mathcal{O}_X)$. Let $J = Ann(b^m a - b^n a')$, then $c \in J - j_y$, that is, $J \not\subseteq j_y$ for every $y \in D(b)$. So that $V(J) \cap D(b) = \varnothing$, hence $b \in rad(J)$. Therefore there is an integer $k \geq 0$ such that $b^k \in J$, and so $b^k(b^m - b^n a') = 0$. Thus $a/b^n = a'/b^m$ in $\Gamma(D(b), \mathcal{O}_X)$. Therefore h is injective (which can be viewed as restriction of functions (cf. Step 2). So it remains to verify that h is surjective.

(iii) Let $a \in \Gamma(D(b), \mathcal{O}_X)$. We are going obtain an integer $n \geq 0$ such that $b^n a \in \Gamma(Y, \mathcal{O}_X)$ (cf. Convention at the end of Step 3). For then there is $c \in \Gamma(Y, \mathcal{O}_X)$ such $b^n a = c$, so that $f(c)f(b^n)^{-1}$ since f(b) is invertible in $\Gamma(D(b), \mathcal{O}_X)$. Since g(b) is invertible in $\Gamma(Y, \mathcal{O}_X)_b$ and its inverse is 1/b, where 1/b designates the equivalence class which contains 1/b. So that from the injectivity of h and $f = g \circ h$ it follows that $a = h(a/1) = f(a) = f(c)f(b^n)^{-1} = h(g(c))h(g(b^n)^{-1}) = h(g(c)g(b^n)^{-1}) = h((c/1)(1/b^n)) = h(c/b^n)$. Therefore a is the image of c/b^n by h, and this proves that h is surjective. So we are done if we show how to obtain such an integer.

Now there exists a sufficiently fine finite covering $\{D(b_i): i \in I\}$ of D(b) such that $b_i \in LG(Y) \subseteq \Gamma(Y, \mathcal{O}_X)$ and $a \mid D(b_i) \in LG(D(b_i))$. Let us denote by a_i the restriction of a to $D(b_i)$, and we regard b_i as an element of $LG(D(b_i))$ because b_i vanishes only outside of $D(b_i)$. By (2.2.4), $LG(D(b_i)) \cong LG(D(b_i))_{b_i}$, so there is an integer $m(i) \geq 0$ such that $b_i^{m(i)} a_i \in LG(D(b_i))$ for any i.

We show that for every $i \in I$, there exists an integer $n(i) \geq 0$ such that $b_i^{n(i)} a_i \in LG(Y)$ (cf. Convention at the end of Step 3). Let $A_i = \{x \in LG(D(b_i)): b_i^n x \in LG(Y)$ for some integer $n \geq 0\}$. Then clearly A_i is a subring of $LG(D(b_i))$. We show that $A_i = LG(D(b_i))$.

For the sake of convenience, we drop the subscript i till the end of the proof in (iii). We observe that $A \supseteq FHG(D(b))$. For this it suffices to prove that for any $U \in E(D(b))$, $T_{D(b)}(U) \in A$. By the note at the end of Step 2, there is $U' \in E(Y)$ such that $T_{Y'}(U') \mid D(b) = T_{D(b)}(U)$. Then $bT_{D(b)}(U)$, regarded as a function on Y by Convention in Step 3, is easily verified to be equal to $bT_Y(U')$. Since $FHG(Y) \subseteq LG(Y), T_Y(U') \in LG(Y)$ and $b \in LG(Y)$, so $bT_Y(U') \in LG(Y)$. Hence $bT_{D(b)} \in LG(Y)$, that is, $T_{D(b)}(U) \in A$.

By (2.1.22), LG(D(b)) (resp. LG(Y)) is the underlying ring of the maximal v-extension of HG(D(b)) (resp. HGY)). So that every element x of LG(D(b)) is of the form a#a' for some a, $a' \in FHG(D(b)) \subseteq A$. Thus by the definition of A, there is an integer $n \geq 0$ such that $b^n a, b^n a' \in LG(Y)$. As a#a' is defined, $D(a) \subseteq D(a')$. So $D(b^{m+n}a) = D(b) \cap D(a) \subseteq D(b) \cap D(a') = D(b^n a')$, thus $b^{m+n}\#b^n a'$ has meaning for every integer $m \geq 0$. It is easy to see that for any integer n, $b^m(a\#a') = b^{m+n}a\#b^n a'$. Since $b^n a' \in LG(Y) = FMHG(Y)$, by (I_2) of (2.1.17), there is an integer $n \geq 0$ such that $v_p(b(y)^n a'(y)) < m$ for any $p \in \sigma(Y)$.

We claim that $(b^{m+n}a, b^n a')$ satisfies the condition (i) of (2.1.20), that is, we verify that for any $p = (y, x) \in \sigma(Y)$ with $y \in D(b^{m+n}a)$, $v_p(b^{m+n}a(y)) \geq v_p(b^n a'(y))$, and that equality holds only if $x \in D(b^n a')$.

Now $v_p(b^{m+n}a(y)) - v_p(b^n a'(y)) = (m + n)v_p(b(y)) + v_p(a(y)) - nv_p(b(y)) - v_p(a'(y)) = mv_p(b(y)) + v_p(a(y)) - v_p(a'(y))$. So if $x \in V(b)$, that is, if $v_p(b(y)) > 0$, then $v_p(b^{m+n}a(y)) \geq m + n \geq m > v_p(b^n a'(y))$. So we have $v_p(b^{m+n}a(y)) > v_p(b^n a'(y))$. If $x \in D(b)$, then $y \in D(b)$ because $x \in Cl\{y\}$ and $y \in D(b)$ implies that $x \in V(b)$. Thus we have only one case left to examine, namely, $p = (y, x) \in D(b) \times D(b)$. Since $x \in D(b)$, $v_p(b(y)) = 0$. But $a\#a' \in LG(D(b))$ implies that $a\#a'$ induces a v-extension. So by (2.1.20), $v_p(a(y)) - v_p(a'(y)) \geq 0$ equality holds only if $x \in D(a')$. Therefore $v_p(b^{m+n}a(y)) \geq v_p(b^n a'(y))$ and equality holds only if $x \in D(b) \cap D(a') = D(b^n a'(y))$. This shows that $b^m(a\#a') = b^{m+n}a\#b^n a' \in LG(Y) \subseteq \Gamma(Y, \mathcal{O}_X)$. Consequently, $x = a\#a' \in A$, and $A = LG(D(b))$. From now on we retain the subscript i.

(iv) By the preceding result, for each i, there is an integer $n(i) \geq 0$ such that $b_i^{n(i)}a_i \in LG(Y) \subseteq \Gamma(Y, \mathcal{O}_X)$. Let $N \geq \max\{n(i): i \in I\}$ and $b_i^N = c_i$. Then $c_i \in \Gamma(Y, \mathcal{O}_X)$, $c_i a_i = c_i a$ on $D(b_i)$ for each i. Since $D(b) = \cup\{D(b_i): i \in I\}$, $V(b) = \cap\{V(b_i): i \in I\}$. By an argument similar to one in the proof of (2.1.11), there is an element c in the ideal (c_i) generated by $\{c_i\}$ in $\Gamma(Y, \mathcal{O}_X)$ such that $V(c) = \cap\{V(c_i): i \in I\}$. Since (c_i) is an ideal, $ca \in (c_i) \subseteq \Gamma(Y, \mathcal{O}_X)$. It is easy to see that $V(c) = V(b)$, so that $b \in rad(c)$, that is, $b^n \in (c)$ for some $n \in \mathbb{N}$. Therefore $b^n a \in (c) \subseteq (c_i) \subseteq \Gamma(Y, \mathcal{O}_X)$. This is what we wanted in order to conclude that h is surjective (cf. the beginning of (iii)). Consequently h is an isomorphism, and $(Y, \mathcal{O}_X)$ is an affine scheme. ■

Definition (2.2.11). — A topological space is said to be *semispectral* if the intersection of any two quasi-compact open subsets is quasi-compact. A continuous map f of semispectral spaces will be called *spectral* if the preimage by f of every IQO set is IQO. ([Hoc.1], pp.55,56)

Remark (2.2.12). — It is immediately seen that a topological space X is semispectral iff all the quasi-compact open sets are IQO. If the range space of a continuous function f can be covered by quasi-compact open sets then f is spectral iff preimage of every quasi-copact open set is IQO. Note also that the category $\mathscr{S}$ of spectral spaces and spectral maps is a full subcategory of the category $\mathscr{U}$ of semispectral spaces and spectral maps, by observing that if the space is quasi-compact then IQO sets are quasi-compact open sets.

Proposition (2.2.13). — The following conditions on a topological space X are equivalent:

(i) X is the underlying space of some scheme.

(ii) X is locally spectral and semispectral.

(iii) X is homeomorphic with an open subspace of a spectral space.

(iv) X is the underlying space of an open subscheme of an affine scheme. ([Hoc.1], Prop.16)

[Pf] (iii) $\Rightarrow$ (iv): Let X be homeomorphic with an open subspace U of a spectral space Y. By the proof of (2.2.10), (U, $\{\Gamma_y\}$, $\Gamma(U, \mathcal{O}_Y)$) is an affine spring and therefore (U, $\Gamma(U, \mathcal{O}_Y)$) is an affine scheme.

(iv) $\Rightarrow$ (i): Obvious.

(i) $\Rightarrow$ (ii): Since the set of open affine subsets is an open basis of the topology of the underlying space of a prescheme (cf. [Gro-Die], (I,2.1.3), X is locally spectral because an open affine subset V of X is an open spectral subspace. The intersection of two open affine subsets is an open affine subset, so that the intersection of two quasi-compact open subsets of X is quasi-compact open. For every quasi-compact open subset of X, it is a finite union of open affine subsets. Thus X is semispectral.

(ii) $\Rightarrow$ (iii): Consider the spectral embedding $e: X \to W^V$ in the proof of (1.3.4). Let $\{U_i: i \in I\}$ be a cover of X consisting of open spectral subspaces of X. Since e is an embedding, for any $i \in I$, $e(U_i)$ is an open subspace of W^V, and $e(X) = \cup\{e(U_i): i \in I\}$ is an open subspace of W^V. This completes the proof. ■

Chapter 3. Minimal spectral spaces.

Section 1. Minimal prime spectra.

In this section we define the minimal prime spectrum Min(A) of a commutative ring with unity and study its properties. We discuss Min(A) for two special types of rings A in (3.1.5) and in (3.1.6), (3.1.7) and (3.1.8).

Recall that the nilradical of a ring is the set of all the nilpotent elements, and it is the intersection of all prime ideals (it is called the *prime radical* of A in [Lam]). A ring is called *semiprime* if its nilradical is 0. A ring is called *reduced* if it has no nonzero nilpotents. We know that for commutative rings these two notions coincide, so we adopt the term "reduced" to indicate this type of ring.

Definition (3.1.1). — Let A be any commutative ring with or without unity. By the *minimal prime spectrum Min(A) of A* we mean the set of all minimal prime ideals of A equipped with the Zariski topology. If A has unity then Min(A) could be identified with a subspace of Spec(A). We define for any subset E of A, $V_m(E) = V(E) \cap \mathrm{Min}(A)$, and $D_m(E) = D(E) \cap \mathrm{Min}(A)$.

Proposition (3.1.1.1). — A prime ideal P of a ring A is minimal iff for each $x \in P$ there exists $a \in A - P$ and an integer $k > 0$ such that $ax^k = 0$. ([Kis,1], Lem.3.1)

[Pf] Suppose that P is minimal. Then $S = \{ax^k : k \in N, a \in A - P\}$ is a multiplicative subset of A. If $0 \notin S$, then by Zorn's Lemma there exists a prime ideal P' disjoint from S. Since $x \notin P'$, P' is properly contained in P. This contradicts the assumption that P is minimal prime, so $0 \in S$ and the condition is satisfied.

Conversely, let $Q \subsetneq P$ be another prime ideal, and $x \in P - Q$. Then there exist an element $a \in A - P \subsetneq A - Q$ and an integer $k > 0$ such that $ax^k = 0$. But A is reduced, so $(ax)^k = a^{k-1}ax^k = 0$. This implies $a \in Q$ or $x \in Q$, which is a contradiction. So P is minimal prime. ■

Proposition (3.1.2). — Min(A) is a Hausdorff space which has a basis for open sets consisting of clopen subsets. ([Hen-Jer], Cor.2.4; [Kis,1], Th.3.2)

[Pf] We know that $\{D(f): f \in A\}$ forms an open basis of Spec(A). We are going to show that $D_m(f)$ is closed in Min(A). Let P be a minimal prime ideal of A such that $[P] \notin D_m(f)$, then $f \in P$ and by (3.1.1.1), there exist an element $t \notin P$ and a positive integer i such that $tf^i = 0$. Hence $\varnothing = D_m(0) = D_m(tf^i) = D_m(t) \cap D_m(f)$ and $[P] \in D_m(t) \subseteq V_m(f)$, that is, $V_m(f)$ is open, therefore $D_m(f)$ is closed. To show that Min(A) is Hausdorff is easy because it is T_0 and $D_m(f)$ is clopen for any $f \in A$. ■

As immediate corollaries to (3.1.1.1), we have the following two characterization of a minimal prime ideal of a reduced ring:

Corollary (3.1.2.2). — A prime ideal P of a reduced ring A is minimal iff for any $x \in P$, $Ann(x) \not\subseteq P$.

[Pf] If P is minimal and x is an element of P, then by (3.1.1.1), $ax^k = 0$, for some $a \in A - P$ and some integer $k > 0$. So $ax = 0$ since A is reduced. Then $a \in \mathrm{Ann}(x)$.

Conversely, let $x \in P$. If there exists $a \in \mathrm{Ann}(x) - P$, the condition of (3.1.1.1) is satisfied by taking $k = 1$. ■

Corollary (3.1.2.3). — A prime ideal P of a reduced ring A is minimal iff every finitely generated ideal I in P, $\mathrm{Ann}(I) \not\subseteq P$. ([Art-Mar], Lemma, p.80).

[Pf] The condition of (3.1.2.3) obviously implies that of (3.1.2.2). Conversely, let $a_1, \ldots, a_n \in P$ generate I, and for each $i = 1, \ldots, n$, choose $b_i \in \mathrm{Ann}(a_i) - P$; then $b = b_1 \ldots b_n \in \mathrm{Ann}(I) - P$. ■

Note (3.1.2.4). — Thus if A is reduced then for any finitely generated ideal I, $\mathrm{Ann}(I) = 0$ iff $V_m(I) = \varnothing$.

Proposition (3.1.3). — Min(A) is a subspace of Spec(A) with respect to the patch topology. ([Art-Mar-Mor], p.95)

[Pf] Recall that $\{D_m(f) : f \in A\}$ is an open subbasis of the subspace Min(A) of Spec(A) consisting of clopen subsets (cf. (3.1.2)), $\mathrm{Min}(A) = D_m(1)$ is a patch in Spec(A). ■

We next discuss Min(A), where A is a direct product of domains.

Definition (3.1.4). — Let $\gamma(I)$ be the set of ultrafilters on a set I. We call the *canonical space of ultrafilters on I* the topological space $\gamma(I)$ with the topology whose closed sets are of the form $V_\gamma(Y) = \{\mathscr{U} \in \gamma(I) : Y \in \mathscr{U}\}$ where $Y \subseteq I$.

Proposition (3.1.5). — Let $\{A_i: i \in I\}$ be a family of domains none of which has characteristic 2 and let $A = \prod\{A_i: i \in I\}$. Then Min(A) is homeomorphic to the canonical space of ultrafilters $\gamma(I)$ on I. ([Pic], Remark, p.27)

[Pf] For any $a \in A$, define $S(a) = \{i \in I: pr_i(a) = 0\}$. Then obviously $S(a) \cap S(b) \subseteq S(a + b)$ and $S(a) \cup S(b) = S(ab)$. For each minimal prime ideal j_x of A, let $\psi(x) = \{S(a): a \in j_x\}$, and for every ultrafilter $\mathscr{U}$ on I, let $\varphi(\mathscr{U}) = \{a \in A: S(a) \in \mathscr{U}\}$. We show that $\psi(x)$ is an ultrafilter on I, $\varphi(\mathscr{U})$ is a minimal prime ideal of A. Furthermore the two maps $\varphi: \gamma(I) \to \mathrm{Spec}(A)$ and $\psi: \mathrm{Spec}(A) \to \gamma(I)$ are homeomorphisms inverse to each other.

(i) $\psi(x) = \{S(a): a \in j_x\}$ is an ultrafilter: Let $a \in j_x$, $U \subseteq I$ and $S(a) \subseteq U$. Define an element $c \in A$ as follows: $pr_i(c) = 1$ if $i \notin U - S(a)$ and 0 if $i \in U - S(a)$. Then $S(c) = U - S(a)$ and $S(ac) = S(a) \cup (c) = S(a) \cup (U - S(a)) = U$. Since j_x is an ideal and $a \in j_x$, $ac \in j_x$. Thus $S(a) \subseteq U$ implies $U = S(ac) \in \psi(x)$. Let a, b $\in j_x$ and $S(a), S(b) \in \psi(x)$. Let $J = \{i \in I: i \in S(a+b) - S(a) \cap S(b)\}$. Then for $J = \varnothing$, $S(a) \cap S(b) = S(a + b)$. So $S(a) \cap S(b) \in \psi(x)$. If $J \neq \varnothing$, for every $i \in J$, $pr_j(a + b) = 0$, $pr_j(a) \neq 0$ and $pr_j(b) \neq 0$. Note that $pr_j(a) = - pr_j(b) \neq 0$. Define an element c as $pr_i(c) = 1$ if $i \notin J$ and $- 1$ if $i \in J$. Then since a, b $\in j_x$, we have $bc \in j_x$ and $a + bc \in j_x$. Now $S(a) \cap S(b) \subseteq S(a) \cap (S(b) \cup S(c)) = S(a) \cap S(bc) \subseteq S(a + bc)$. Conversely let $i \in S(a + bc)$. Then if $i \in J$, $pr_i(a + b) = pr_i(a) + pr_i(b) = 0$ and $0 = pr_i(a + bc) = pr_i(a) + pr_i(b)pr_i(c) = pr_i(a) - pr_i(b)$. So if the characteristic of A_i is not 2, $pr_i(a) = pr_i(b) = 0$. On the other hand if $i \notin$

J, then $i \notin S(a + b)$ or $i \in S(a) \cap S(b)$. Since $i \in S(a + b)$, $i \in S(a) \cap S(b)$. Therefore in every case $i \in S(a + bc)$ implies $i \in S(a) \cap S(b)$. This shows that $S(a + bc) = S(a) \cap S(b)$, and so $S(a) \cap S(b) = S(a + bc) \in \psi(x)$. Thus $\psi(x)$ is closed for finite intersection. Next we show $\varnothing \notin \psi(x)$, in other words, $S(a) \neq \varnothing$ for every $a \in j_x$, that is, for every $a \in j_x$ there is $i \in I$ such that $pr_i(a) = 0$. Suppose not, that is, suppose for every $i \in I$, $pr_i(a) \neq 0$. Since x is a minimal prime ideal and $a \in j_x$, there exist $b \in A - j_x$ and an integer $k > 0$ such that $ba^k = 0$ (cf. (3.1.1.1)). Thus for each $i \in I$, $pr_i(ba^k) = pr_i(b)pr_i(a)^k = 0$. Since each A_i is a domain, $pr_i(b) = 0$ or $pr_i(a) = 0$. By supposition, $pr_i(a) \neq 0$ for each i, so $pr_i(b) = 0$ for each i, that is, $b = 0$. But then $0 \in A - j_x$, a contradiction. Consesquently, for any $a \in j_x$, there is $i \in I$ such that $pr_i(a) = 0$, so $S(a) \neq \varnothing$ for every $a \in j_x$. Hence $\varnothing \notin \psi(x)$. These show that $\psi(x)$ is a filter.

Furtheremore, let $a, b \in A$ and $S(a) \cup S(b) \in \psi(x)$. Then $S(ab) = S(a) \cup S(b) \in \psi(x)$ so $ab \in j_x$. Since j_x is a prime ideal, $a \in j_x$ or $b \in j_x$. So that $S(a) \in \psi(x)$ or $S(b) \in \psi(x)$. Therefore $\psi(x)$ is an ultrafilter on I. ∎

(ii) $\varphi(\mathscr{Z}) = \{a \in A: S(a) \in \mathscr{Z}\}$ is a prime ideal of A: Let $a, b \in \varphi(\mathscr{Z})$, then $S(a - b) \supseteq S(a) \cap S(b) \in \mathscr{Z}$. Since $\mathscr{Z}$ is a filter, $S(a - b) \in \mathscr{Z}$, so $a - b \in \varphi(\mathscr{Z})$. Let $a \in A$, $b \in \varphi(\mathscr{Z})$, then $S(ab) = S(a) \cup S(b)$ since for any i, A_i is a domain. Now $S(a) \cup S(b) \supseteq S(b) \in \psi(x)$ implies $ab \in \varphi(\mathscr{Z})$, because $\mathscr{Z}$ is a filter. Let $cd \in \varphi(\mathscr{Z})$, then $S(c) \cup S(d) = S(cd) \in \mathscr{Z}$. Since $\mathscr{Z}$ is an ultrafilter, we have $S(c) \in \mathscr{Z}$ or $S(d) \in \mathscr{Z}$, that is, $c \in \varphi(\mathscr{Z})$ or $d \in \varphi(\mathscr{Z})$. This shows that $\varphi(\mathscr{Z}) \in \mathrm{Spec}(A)$. Let $a \in \varphi(\mathscr{Z})$,

$a \neq 0$. Then $S(a) \subsetneq I$. Define $c \in A$ such that $pr_i(c) = 1$ if $i \in S(a)$ and 0 if $i \notin S(a)$. Then $S(c) \cap S(a) = \emptyset$. Since $\mathscr{U}$ is an ultrafilter, we have $S(c) \notin \mathscr{U}$, that is, $c \notin \varphi(\mathscr{U})$. Therefore for any $a \in \varphi(\mathscr{U})$, there is $c \in A - \varphi(\mathscr{U})$ such that $ac = 0$, so that $\varphi(\mathscr{U})$ is a minimal prime ideal.

So far we have established two functions $\varphi: \gamma(I) \to \mathrm{Min}(A)$ and $\psi: \mathrm{Min}(A) \to \gamma(I)$.

(iii) φ and ψ are bijective and inverse to each other: Let $x \in \mathrm{Min}(A)$ and $a \in \varphi(\psi(x))$. Then $S(a) \in \psi(x)$ and so $a \in j_x$. Thus $\varphi(\psi(x)) \subseteq j_x$ as an ideal, but j_x and $\varphi(\psi(x))$ are minimal prime so we have $\varphi(\psi(x)) = x$. On the other hand, let $U \in \psi(\varphi(\mathscr{U}))$. Then there is $a \in \varphi(\mathscr{U})$ such that $U = S(a)$. So $S(a) \in \mathscr{U}$, that is, $\psi(\varphi(\mathscr{U})) \subseteq \mathscr{U}$. But $\psi(\varphi(\mathscr{U}))$ is an ultrafilter, so $\psi(\varphi(\mathscr{U})) = \mathscr{U}$.

(iv) φ and ψ are homeomorphisms:

At first note that $V_\gamma(U) = \{\mathscr{U} \in \gamma(I): U \in \mathscr{U}\}$, $U \subseteq I$ (resp. $V_m(E) = \{x \in \mathrm{Min}(A): E \subseteq j_x\}$, $E \subseteq A$) are the closed sets of $\gamma(I)$ (resp. $\mathrm{Min}(A)$). Let $U \subseteq I$. Then there is $a \in A$ such that $U = S(a)$. (For example, set $pr_i(a) = 0$ if $i \in U$ and 1 if $i \notin U$.) Since φ is bijective, $S(a) = U \in \mathscr{U}$ is equivalent to $a \in \varphi(\mathscr{U})$, so that $V_\gamma(U) = \{\mathscr{U} \in \gamma(I): a \in \varphi(\mathscr{U})\}$. Hence $\varphi(V_\gamma(U)) = \{\varphi(\mathscr{U}) \in \mathrm{Min}(A): a \in \varphi(\mathscr{U})\} = V_m(a)$. So that φ is a closed map and $\psi = \varphi^{-1}$ is continuous. Let $E \subseteq A$ and $U = \cap\{S(a): a \in E\}$. Then $V_m(E) = \{x \in \mathrm{Min}(A): E \subseteq j_x\}$. Since $E \subseteq j_x$ is equivalent to $U = \cap\{S(a): a \in E\} \in \psi(x)$, $\psi(V_m(E)) = \{\psi(x) \in \gamma(I): U \in \psi(x)\} = V_\gamma(U)$.

Therefore ψ is a closed map and $\varphi = \psi^{-1}$ is continuous. Consequently, φ and ψ are homeomorphisms.■

Definition (3.1.6). — For any ring A, denote by *Max(A)* the subspace of Spec(A) consisting of all the maximal ideals of A.

A topological space is called a *Stone space* if it is T_1, quasi-compact and has an open basis consisting of clopen sets.

Proposition (3.1.7). — Let A be a ring, X = Spec(A) and $j: A \to R = \prod\{A^*(x): x \in X\}$ be the canonical homomorphism, where $A^*(x)$ is the residue field of A at j_x (note that it is also the field of fractions of A/j_x). Then Spec(R) = Min(R) = Max(R).

[Pf] Let X = Spec(A), $P(x) = \{0\} \times \prod\{A^*(y): y \in X - \{x\}\}$ and pr_x be the projection map $R \to A^*(x)$. It is obvious that P(x) is a prime ideal of R, and if $x \neq y$, then $P(x) \neq P(y)$. Since $R/P(x) = A^*(x)$ for any $x \in X$, P(x) is a maximal ideal.

Now take an arbitrary prime ideal P of R. Then $j^{-1}(P)$ is a prime ideal of A, say $j^{-1}(P) = j_{x_0}$, $x_0 \in X$. Take an element $\alpha \in P$, then $pr_{x_0}(\alpha) \in A^*(x_0)$, so $pr_{x_0}(\alpha) = (b + j_{x_0})/(c + j_{x_0})$ for some $b, c \in A$, $c \notin j_{x_0}$ since $A^*(x_0)$ is the field of fractions of A/j_{x_0}. Now $j(c) \in P$, and P is a (prime) ideal of R. So $j(c)\alpha \in P$. But $pr_{x_0}(j(c)\alpha) = b + j_{x_0}$, so that $b + j_{x_0} \in j^{-1}(P) = j_{x_0}$. Thus $pr_{x_0}(\alpha) = 0 \in A^*(x_0)$. So every prime ideal of R is contained in some P(x). Consequently, $\{P(x): x \in X\}$ is the set of all maximal ideal of R.

Let P be any minimal prime ideal of R. Since $\{P(x): x \in X\}$ are the maximal ideals of R, P is contained in some $P(x_0)$, $x_0 \in X$. If P is properly contaied in $P(x_0)$, choose $\delta \in P(x_0) - P$, and ε such that $pr_{x_0}(\varepsilon) = 1$, $pr_y(\varepsilon) = 0$ for $y \neq x_0$. Note that $\varepsilon \notin P(x_0)$ and so $\varepsilon \notin P$. But $\varepsilon\delta = 0 \in P$. This contradicts the assumption that P is prime. So that $P = P(x_0)$, and so P(x) is a minimal prime ideal of R for any x in X.

Consequently any prime ideal of R is of the form P(x) for some x in X, and Spec(R) = Min(R) = Max(R). ■

Proposition (3.1.8). — Let A, X and R be as above. Then

(i) Spec(R) $\rightarrow$ Spec(A) is surjective;

(ii) Spec(R) is a Stone space;

(iii) γ(Spec(A)) is homeomorphic with Spec(R). ([Pic,2], Rem.2, p.27)

[Pf] (i): For any x in X, we have $j_x = j^{-1}(P(x))$ (cf. the proof of (3.1.7)). Thus ${}^{a}j$: Spec(R) $\rightarrow$ X = Spec(A) is a surjective map.

(ii): Since Spec(R) = Max(R), by (3.1.7), any prime ideal of R is maximal, that is, every point of Spec(R) is closed by (1.1.10). So Spec(R) is T_1. For any ring, its prime spectrum is quasi-compact (cf.(1.1.13)), in particular, Spec(R) is quasi-compact.

By (3.1.2), we know that Min(R) is a Hausdorff space which has a base for open sets consisting of clopen subsets. By (3.1.7) we have Spec(R) = Min(R). Therefore Spec(R) is a Stone space.

(iii) We proved in (3.1.5) that $\gamma(\mathrm{Spec}(A))$ is homeomorphic to $\mathrm{Min}(R_0)$, where $R_0 = \Pi_{x \in X} A(x)$, $A(x) = A/j_x$, and $X = \mathrm{Spec}(A)$. So it suffices to show that $\mathrm{Min}(R_0)$ is homeomorphic to $\mathrm{Spec}(R)$. Let P be any minimal prime ideal of R_0. We claim that there is at most one $x \in X$ such that $pr_x(P) \neq A_x$. Suppose there are $x, y \in X$, $x \neq y$ such that $pr_x(P)$ and $pr_y(P)$ are properly contained in A_x and A_y respectively. Then $pr_x(P)$ and $pr_y(P)$ are proper prime ideals of A_x and A_y respectively. Define $a \in R_0$ by $pr_z(a) = 1$ if $z = x$ and 0 otherwise, and $b \in R_0$ by $pr_z(b) = 1$ if $y = z$ and 0 otherwise. Then $a, b \notin P$ as $pr_x(a) = 1 \notin pr_x(P)$ and $pr_y(b) = 1 \notin pr_y(P)$ because $pr_x(P)$ and $pr_y(P)$ are proper prime ideals of A_x and A_y respectively. Yet $ab = 0$, which contradicts the hypothesis that P is a prime ideal. So there exists at most one $x \in X$ such that $pr_x(P) \neq A_x$. Since $pr_x(P)$ is a minimal prime ideal of a domain $A(x)$, $pr_x(P) = \{0\}$. Clearly, $P \subseteq P(x)$, but by the above claim, no ideal properly contained in $P(x)$ can be prime. So it follows that $P = P(x)$.

Let g be the inclusion map of R_0 to R. Then we have ${}^a g$: $\mathrm{Spec}(R) \to \mathrm{Spec}(R_0)$. We know from the proof of (3.1.7) that every element of Spec(R) is of the form $\{0\} \times \Pi\{A^*(y) : y \in X - \{x\}\}$ for some $x \in X$. Therefore ${}^a g$ maps Spec(R) bijectively to $\mathrm{Min}(R_0)$. Now it is routine to verify that ${}^a g$ is indeed a homeomorphism. ■

Section 2. Minimal spectral spaces.

Definition (3.2.1). — A topological space is called a *minimal spectral space* (or *minspectral space*) if it is homeomorphic to the minimal prime spectrum Min(R) of a commutative ring R (with or without unity) (It will be shown in (3.2.6) that Min(R) is homeomorphic to Min(A) for some ring A with unity). By an *m-subbasis* (resp. *m-basis*) $\mathscr{B}$ for a Hausdorff space X we mean a subbasis (resp. basis) for the open sets such that each subset of $\mathscr{B}$ with (FIP) intersects. We call a subbasis $\mathscr{B}$ *full* if $\varnothing$, X $\in \mathscr{B}$ and $\mathscr{B}$ is closed under finite union and intersection. ([Hoc,2], p.749)

Proposition (3.2.2). — $\mathscr{B}$ is an m-subbasis iff it is an open subbasis and at the same time a closed subbasis of a quasi-compact topology on X. ([Hoc,2], p.749)

[Pf] The topology taking $\mathscr{B}$ as a subbasis for closed sets as well as that for open sets is quasi-compact as any basic open set is a finite intersection of the members of $\mathscr{B}$, and each subfamily of $\mathscr{B}$ with (FIP) intersects (cf. [Kel], Th.6, p.139). Conversely, if the condition is satisfied, each member of $\mathscr{B}$ is clopen, so the quasi-compactness implies the additional condition that $\mathscr{B}$ be an m-subbasis. ■

Proposition (3.2.3). — The following conditions on an open basis $\mathscr{B}$ of a Hausdorff space X are equivalent:

(1) $\mathscr{B}$ is a full m-basis.

(2) There is a commutative ring A with unity and an embedding $f: X \to \mathrm{Spec}(A)$ such that $f(X) = \mathrm{Min}(A)$ and h induces a bijection of $\mathscr{B}$ onto $\mathscr{C} = \{U \cap \mathrm{Min}(A): U \in Q(\mathrm{Spec}(A))\}$. ([Hoc,2], Prop.1)

[Pf] (1) $\Rightarrow$ (2): Let $W = \{0, 1\}$ be the topological space with the set of open sets $\{\varnothing, \{0\}, W\}$ as in (1.2.10). Let W_B a copy of W for each $B \in \mathscr{B}$. Let $f_B: X \to W_B$ be defined by $f_B(x) = 0$ if $x \in B$ and $f_B(x) = 1$ if $x \notin B$, then f_B is continuous for any $B \in \mathscr{B}$ and we get a continuous (evaluation) map $f = \prod\{f_B: B \in \mathscr{B}\}: X \to P = \prod\{W_B: B \in \mathscr{B}\}$. Let us topologize P with the product toplogy of W_B's and denote by P_d the topological space P endowed with the product topology obtained by letting each W_B have the discrete topology (which coincides with the patch topology on W_B). Let Y be the closure of $f(X)$ in P_d and topologize Y as a subspace of P. Then by (1.2.10), Y is spectral, that is, there is a commutative ring A with unity such that Y is homeomorphic to Spec(A). Let $Y_0 = \mathrm{Min}(A) \subseteq \mathrm{Spec}(A)$.

Since $\mathscr{B}$ is an open basis of a Hausdorff space X, the maps f_B separate points and closed sets, and hence f is an embedding of X into P (cf. [Kel], p.116). We show that $f(X) = Y_0$. We observe that for any $y \in Y$ there is $x \in X$ such that $y \in \mathrm{Cl}_Y\{f(x)\}$. For each $y \in Y$, let $\mathscr{B}_y = \{B \in \mathscr{B}: f_B(y) = 0\}$, then the sets of the form $U(\mathscr{F}) = \{z \in P: f_F(z) = 0 \text{ for any } F \in \mathscr{F}\}$, where $\mathscr{F}$ is a finite subset of $\mathscr{B}_y$, form a basis of neighbourhoods of y in P. In order to prove this observation it suffices to show that the intersection $\cap\{U(\mathscr{F}) \cap f(X): \mathscr{F}$ is a finite subset of $\mathscr{B}_y\}$ is nonempty. For then any neighbourhood of y contains the point f(x) in the intersection, and this is equivalent to $\cap\{f^{-1}(U(\mathscr{F}): \mathscr{F}$ is a finite subset of $\mathscr{B}_y\} \neq \varnothing$ as x is in this inersection. But since $f^{-1}(U(\mathscr{F}) = \cap\{F: F \in \mathscr{F}\}$, $\mathscr{F} \subseteq \mathscr{B}_y \subseteq \mathscr{B}$ and a family of sets in $\mathscr{B}$ with (FIP) intersects, we need only show that if $\mathscr{F}_1,\ldots,\mathscr{F}_k \subseteq \mathscr{B}_y$ then $J = \cap\{\cap\{F: F \in \mathscr{F}_i\}: i = 1,\ldots,k\} \neq \varnothing$. Now y is in the closure of f(X) in P_d and the set $\cap(\cap\{\mathscr{F}_i: i = 1,\ldots,k\}) = \{z \in P: f_F(z) = 0$ for any $F \in \cup\{\mathscr{F}_i: i = 1,\ldots,k\}\}$ is an open neighbourhood of y in P_d, so that it meets f(X). This means $\cap(\cap\{F: F \in \mathscr{F}_i\}: i = 1,\ldots,k\}) \neq \varnothing$, which is what we wanted to show. Thus for any $y \in Y$ there exists $x \in X$ such that $y \in Cl\{f(x)\}$.

We can now prove that $f(X) = Y_0$. Let $x \in X$. Then $f(x) \in Y = Spec(A)$. So there exists a minimal prime ideal j_y of A such that $j_y \subseteq j_{f(x)}$, that is, $f(x) \in Cl\{y\}$. By the observation above, there exists $x' \in X$ such that $y \in Cl_Y\{f(x')\}$. Since f is an embedding of X into Y (onto f(X)), $x \in Cl_X\{x'\}$, and so $x = x'$ because X is Hausdorff. Thus $f(x) = f(x') = y$. Hence $f(X) \subseteq Y_0 = Min(A)$.

Conversely, let $y \in Y_0$. Then $y \in Y$. So by the above observation, $y \in Cl_Y\{f(x)\}$ for some $x \in X$. Since y is a minimal prime ideal, this implies that $y = f(x) \in f(X)$. Consequently, $f(X) = Y_0$. It follows by a routine verification that f induces a homeomorphism of X onto Y.

It remains to show that f establishes a bijective correspondence between $\mathscr{B}$ and the set $\{U \cap Y_0: U \in Q(Y)\}$. We note that for any $B \in \mathscr{B}$, $f(B) = \{y \in Y_0:$

$f_B(y) = 0\} = (U_B \cap Y) \cap Y_0$, where $U_B = \{z \in P{:}f_B(z) = 0\}$, is quasi-compact open in P, and $U_B \cap Y$ is quasi-compact open in Y. Now every quasi-compact open subset of Y is a finite union of the sets which have the form $U_{B_1} \cap ... \cap U_{B_k} \cap Y$, since the sets of the form constitute an open basis of Y. The inverse image of any quasi-compact open subset of Y is, therefore, a finite union of finite intersesctions of $B_i \in \mathscr{B}$, and is in $\mathscr{B}$ since $\mathscr{B}$ is full.

(2) ⇒ (1): Let A be a ring. We show that Min(A) has a full m-basis. Since Q(Spec(A)) is a full basis for Spec(A), $\mathscr{C} = \{U \cap \mathrm{Min}(A){:}\ U \in Q(\mathrm{Spec}(A))\}$ is a full basis for Min(A). Let $\mathscr{C}' = \{U_i \cap \mathrm{Min}(A){:}\ U_i \in Q(\mathrm{Spec}(A), i \in I\} \subseteq \mathscr{C}$ has (FIP). By (1.2.4), the patch topology is compact, so $\cap\mathscr{C}' \neq \varnothing$. Let $p \in \cap\mathscr{C}$, $p \in$ Spec(A). Then there is $p' \in$ Min(A) such that $j_{p'} \subseteq j_p$, that is, $p \in \mathrm{Cl}\{p'\}$. For each $i \in I$, $p \in U_i$, so that $p' \in U_i$. Therefore $p' \in \cap\mathscr{C}'$. This shows that $\mathscr{C}$ is an m-basis of Min(A). ■

In order to prove our main characterization theorem of this chapter (Th. (3.2.6) below) we need some preliminaries.

Definition (3.2.4). — If $\mathscr{B}$ is an open subbasis of X, we call the topology on X which has $\mathscr{B}$ as a subbasis for its closed sets the *dual topology* on X determined by $\mathscr{B}$.

Note (3.2.4.1). — We note that any subbasis $\mathscr{B}$ generates a least full basis containing it, consisting $\varnothing$, X, and the finite unions of finite intersections of sets in $\mathscr{B}$. This full basis and $\mathscr{B}$ obviously determine the same dual topology.

Proposition (3.2.5). — Let X be a Hausdorff space. Then

(a) If $\mathscr{B}$ is an m-basis, the full basis $\mathscr{B}$ generated by $\mathscr{B}$ is an m-basis.

(b) If $\mathscr{B}$ is an m-subbasis, any subset $\mathscr{B}'$ of $\mathscr{B}$ which is a subbasis is an m-subbasis.

(c) If $\mathscr{B}$ is an m-subbasis (resp. m-basis, full m-basis) for X, and $Y \subseteq X$ is closed in the dual topology determined by $\mathscr{B}$, then $\{B \cap Y: B \in \mathscr{B}\}$ is an m-subbasis (resp. m-basis, full m-basis) for Y.

(d) If $\mathscr{B}$ is an m-basis for X and $U \subseteq X$ is open, then $\{B \in \mathscr{B} : B \subseteq U\}$ is an m-basis for U.

(e) If $\mathscr{B}$ is an m-subbasis for X, each set in $\mathscr{B}$ is clopen. ([Hoc,2], Prop.2)

[Pf] (a): Let $\mathscr{B}$ be an m-basis of X and $\mathscr{B}'$ the full basis generated by $\mathscr{B}$. By (3.2.4.1), $\mathscr{B}$, $\mathscr{B}'$ have the same dual topology. Since $\mathscr{B}$ is an m-basis for X, by Alexander Subbasis Theorem ([Kel], Th.6, p.139), the dual topology determined by $\mathscr{B}$ is quasi-compact. So the dual topology determined by $\mathscr{B}'$ is quasi-compact. Since members of $\mathscr{B}'$ form a subbasis of closed sets for this quasi-compact topology, it follows that $\mathscr{B}'$ is an m-basis.

(b): We notice that any subfamily of $\mathscr{B}'$ is a subfamily of $\mathscr{B}$, and so the result follows.

(c): It suffices to show in the case of m-subbasis. By observing that Y is quasi-compact in the inherited dual topology determined by $\{B \cap Y: B \in \mathscr{B}\}$, we have the assertion by (3.2.2).

(d): Since U is open, for any $B \in \mathscr{B}$, $B = B \cap U$ is equivalent to $B \subseteq U$, and $\{B \cap U: B \in \mathscr{B}\}$ is obviously an m-basis for U.

(e): It suffices to show $Cl_X B \subseteq B$ for any $B \in \mathscr{B}$. Let $p \in Cl_X B$, and $\mathscr{B}_p = \{C \in \mathscr{B}: p \in C\}$. For each finite subset $\{C_1,\dots,C_k\}$ of $\mathscr{B}_p$, $(C_1 \cap \dots \cap C_k) \cap B \neq \emptyset$ since $C_1 \cap \dots \cap C_k$ is a neighbourhood of p. Hence $\mathscr{B}_p \cup \{B\}$ has (FIP), and $\cap\{C : C \in \mathscr{B}_p\} \cap B \neq \emptyset$; but $\cap\{C: C \in \mathscr{B}_p\} = \{p\}$ since X is Hausdorff. Therefore $p \in B$. ■

Theorem (3.2.6). — The following conditions on a Hausdorff space X are equivalent:

(a) X is minspectral, that is, X is homeomorphic to the minimal prime spectrum Min(R) for some commutative ring R (not necessarily with unity).

(b) X is homeomorphic to the minimal prime spectrum for some commutative ring A with unity.

(c) X has an m-subbasis.

(d) X has a full m-basis. ([Hoc,2], Th.1)

[Pf] (b) $\Leftrightarrow$ (d): This follows from (3.2.3).

(c) $\Rightarrow$ (d): By (3.2.4.2) and (3.2.5), we have this implication.

(d) $\Rightarrow$ (c), (b) $\Rightarrow$ (a): Obvious.

(a) $\Rightarrow$ (b): Assume (a) and suppose R has no unity element. Then we can extend R to a ring A with unity in the standard way by considering $A = R \times \mathbb{Z}$ defining addition componentwise, and multiplication on A by the rule $(a, m)(b, n) = (ab + mb + na, mn)$ and identifying R with $R \times \{0\}$. With this identification, it is clear that R is a prime ideal of A. Since $\{0\}$ is the only minimal prime ideal of $\mathbb{Z}$, it follows that R is a minimal prime ideal of A and also there is no minimal prime ideal of A which properly contains R. Therefore $\mathrm{Min}(R) = \mathrm{Min}(A) - \{R\}$. By (3.1.2), Min(A) is Hausdorff. Thus Min(R) is open in Min(A), and so by (3.2.3) and (d) of (3.2.5), Min(R) has a full m-basis. This shows that (a) implies (d). Since (b) is equivalent to (d), it follows that (a) implies (b). ■

We prove the following applications of these results.

Proposition (3.2.7). — A locally compact totally disconnected Hausdorff space X is minspectral. ([Hoc,2], Prop.3)

[Pf] Let $\mathscr{B}$ be the family of all open compact sets. Since X is locally compact Hausdorff and totally disconnected, $\mathscr{B}$ is a basis for X. Since each member of $\mathscr{B}$ is compact, if $\mathscr{F}$ is a subfamily of $\mathscr{B}$ with (FIP), then clearly $\cap\mathscr{F} \neq \varnothing$. Therefore $\mathscr{B}$ is an m-basis for X. Thus by (3.2.6), it follows that X is minspectral. ■

Proposition (3.2.8). — Arbitrary products and arbitrary topological union of minspectral spaces are minspectral. Open subspaces of minspectral spaces are minspectral. ([Hoc,2], Prop.5)

[Pf] Product: Let $\{X_\lambda: \lambda \in \Lambda\}$ be a family of minspectral spaces with an m-basis for each λ. Then subsets of $X = \prod\{X_\lambda: \lambda \in \Lambda\}$ of the form $B_{\lambda_1} \times \ldots \times \prod\{X_\lambda: \lambda \notin \{\lambda_1, \ldots, \lambda_k\}\}$ with $B_{\lambda_i} \in \mathscr{B}_{\lambda i}$ for each i, form a subbasis $\mathscr{B}$ of open sets for the product topology. As $\mathscr{B}_\lambda$ is an m-basis for each X_λ, the dual topology is quasi-compact for each λ. So by Tychonov's Theorem, the dual topology of the product topology on X is quasi-compact. So it follows that $\mathscr{B}$ is an m-basis for X. By (3.2.6), it follows that $X = \prod\{X_\lambda; \lambda \in \Lambda\}$ is a minspectral space.

Union: It is obvious that the union of m-basis for the various spaces in the union forms an m-basis for the union of the minspectral spaces.

Open subspace: This is precisely (d) of (3.2.5). ∎

Chapter 4. Maximal spectral spaces.

Section 1. Maximal ideal spectra.

Definition (4.1.1). — The set of maximal ideals of a ring A with the spectral topology is called the *maximal ideal spectrum* of A, and denoted by Max(A).

Define for any subset E of A, $V_M(E) = \mathrm{Max}(A) \cap V(E) = \{M \in \mathrm{Max}(A): E \subseteq M\}$, and $D_M(E) = \mathrm{Max}(A) - V_M(E)$.

Recall that the Jacobson radical (or radical in [Lam]) of A is the intersection of all the maximal ideals of A.

Proposition (4.1.2). — The maximal ideal space of a ring A, Max(A), is T_1.

[Pf] We know that every singleton subset of Spec(A) consisting of a maximal ideal is closed in Spec(A), by (1.1.10). ■

Proposition (4.1.3). — Let Y be a subspace of the prime spectrum Spec(A) of a ring A. If Y contains all maximal ideals of A, then Y is quasi-compact. In particular, Max(A) is quasi-compact. ([Lam], chap.2, sec.5, Prop.1)

[Pf] Let $\{J_i: i \in I\}$ be a family of ideals of A such that $Y = \cup\{D(J_i): i \in I\} = D(\Sigma\{J_i: i \in I\})$ (cf. (1.1.2), (iv)). Then $\Sigma\{J_i: i \in I\}$, the ideal generated by the family $\{J_i: i \in I\}$, is contained in no maximal ideals, thus it contains 1. But then $1 \in \Sigma\{J_i: i \in F\}$ for some finite subset F of I. Hence $Y = D(\Sigma\{J_i: i \in F\} = \cup\{D(J_i): i \in F\}$, that is, Y is quasi-compact. ■

Definition (4.1.4). — A ring A is called *regular* (in the sense of von Neumann) if for any element a of A, there exists an element x of A such that $x^2a = x$, $a^2x = a$.

Proposition (4.1.5). — Let A be a reduced ring. Then A is regular iff every prime ideal of A is maximal, that is, iff Max(A) = Spec(A).

[Pf] Suppose A is regular. Let P be any prime ideal of A and a an element which doesn't belong to P. Then there is $x \in A$ such that $a^2x = a$, that is, $a(ax - 1) = 0 \in P$. Thus $ax - 1 \in P$, and so $1 \in P + ax$, hence P is maximal.

Conversely, suppose Max(A) = Spec(A). If a = 0 or 1, the existence of such an element x is obvious (just take a = x). Assume $a \in A - \{0, 1\}$. If $D(a) = \{[P] \in \mathrm{Spec}(A): a \notin P\} = \varnothing = D(0)$, then $a^n = 0$ for some $n \in \mathbf{N}$ by (1.1.6), and we have a = 0 since A is reduced. So $D(a) \neq \varnothing$, and there exists a prime ideal P which does not contain a. By hypothesis, P is also maximal, hence P + aA = A. So there is an element x in A such that $1 - ax \in P$. Note that this x does not belong to P, for otherwise, $ax \in P$, which induces a contradiction that $1 \in P$. Now consider $S^{-1}A$, the ring of fractions of A with respect to S = A - P. If $(x/s)^n = 0$ for some x/s in $S^{-1}A$ and $n \in \mathbf{N}$, then there exists an element t in S such that $tx^n = 0$, thus $(tx)^n = 0$. Since A is reduced, tx = 0. This implies that x/s = 0, and so $S^{-1}A$ is reduced. Now x/s = 0 = 0/s, and so sx = 0. We now observe that if $(x/s)^n = 0$ for some n, then sx = 0. By (1.1.16), $\mathrm{Spec}(S^{-1}A)$ is homeomorphic to $Y = \{[P'] \in \mathrm{Spec}(A): P' \cap S = \varnothing\}$. Since every prime ideal of Spec(A) is maximal and $S = A - P$, $Y = \{[P]\}$. Thus we can obtain $\mathrm{Spec}(S^{-1}A) = \{[S^{-1}P]\}$, and $S^{-1}P = \mathrm{rad}(0)$. Therefore $a \in A - P = S$ and $1 - ax \in P$ imply $(1 - ax)/a \in S^{-1}A$, and so $((1 - ax)/a)^n = 0$ for some $n \in \mathbb{N}$. Then by the preceding observation, $a(1 - ax) = 0$ (and also $x(1 - ax) = 0$, since $x \in A - P = S$), which shows that A is regular. ■

Definition (4.1.6). — A subset E of a topological space X is called *locally closed* if for any point x of E, there exists an open neighbourhood U of X such that $E \cap U$ is closed in U.

A subset X_0 of X is said to be *very dense* if for any nonempty locally closed subset E of X, $E \cap X_0$ is nonempty.

Proposition (4.1.7). — Let L be a subset of a topological space X. Then the following conditions are equivalent:

(a) L is locally closed;

(b) L is open in $\mathrm{Cl}_X L$;

(c) L is an intersection of an open subset U and closed subset F of X.

[Pf] (b) $\Rightarrow$ (c): L is the intersection of $\mathrm{Cl}_X L$ and an open subset U of X.

(c) $\Rightarrow$ (a): For any point x of L, U is an open neighbourhood of x in X and $L = U \cap F$ is closed in U.

(a) $\Rightarrow$ (b): By definition, for any $x \in L$, there exists an open neighbourhood U of x such that $U \cap L$ is closed in U. Thus $U \cap L = \mathrm{Cl}_U L = U \cap \mathrm{Cl}_X L$, that is, x is in the interior of L. Therefore L is open in $\mathrm{Cl}_X L$. ■

Proposition (4.1.8).— Let A be a ring, then the following are equivalent:

(a) Max(A) is locally closed in Spec(A);

(b) Max(A) is closed in Spec(A);

(c) Max(A) = V(J), where J is the Jacobson radical of A;

(d) A/J is a regular ring.

On the other hand if any of the above is verified, Max(A) is a Hausdorff space. ([Fon-Mar], Lem.3.2)

[Pf] Let X = Spec(A) and X_0 = Max(A).

(a) $\Rightarrow$ (b): Suppose X_0 is locally closed. Then there is an open set U such that $X_0 = U \cap Cl_X X_0$. If X_0 is not closed, then there is a (proper) prime ideal j_x of A such that $x \in ClX_0 - X_0$. Now $x \notin U$ because if $x \in U$, $x \in U \cap ClX_0 = X_0$, so we have a contradiction. Since j_x is proper prime, there is a maximal ideal j_z of A such that $j_x \subseteq j_z$. Therfore $z \in V(j_x) = Cl\{x\} \subseteq X - U$ because $x \in X - U$ and X - U is closed. Thus $z \notin U$ yet $z \in X_0$, this contradicts $X_0 \subseteq U$. Consequently $ClX_0 = X_0$ and thus X_0 is closed.

(b) $\Rightarrow$ (c): If X_0 is closed in X, there is a radical ideal J' of A such that X_0 = V(J'). This means all maximal ideals contain J', so that $J = \cap\{j_x : x \in X_0\}$ = I(V(J')) = rad(J') = J'. Therefore X_0 = V(J).

(c) $\Rightarrow$ (a): If X_0 = V(J), then X_0 is closed in X, so it is locally closed in X.

(c) $\Leftrightarrow$ (d): Spec(A/J) is homeomorphic to the subspace of X consisting of all prime ideals of A containing J, which is V(J). So if V(J) = X_0, V(J) is homeomorphic to Max(A/J). Thus Max(A/J) $\cong$ Spec(A/J), that is A/J is regular by (4.1.5). Conversely, if A/J is regular, Spec(A/J) $\cong$ Max(A/J) again by (4.1.5). Since spec(A/J) $\cong$ V(J) is closed in X and Max(A/J) $\cong$ Max(A) = X_0, X_0 is closed in X.

Finally suppose X_0 = Max(A) is closed in X = Spec(A). Let x, y $\in X_0$, x $\neq$ y. Since con(X) is Hausdorff, there exist disjoint open neighbourhoods U(x) and U(y) of x and y respectively. Since X_0 is closed in X, X_0 - U(x) is closed in con(X). Furthermore X_0 - U(x) $\subseteq X_0$, so that X_0 - U(x) is closed under specialization, and so by (1.2.7), X_0 - U(x) is closed in X. Similarly X_0 - U(y) is closed in X. Hence $X_0 \cap$ U(x) and $X_0 \cap$ U(y) are disjoint open neighbourhoods of x and y in X_0. Therefore X_0 is Hausdorff. ■

Section 2. Maximal spectral spaces.

Definition (4.2.1). — A topological space is called a *maximal spectral space* (or *M-spectral space*) if it is homeomorphic to Max(A) for some ring A.

Our characterization of M-spectral spaces is the following proposition.

Proposition (4.2.2). — A topological space X is M-spectral iff X is T_1 and quasi-compact. ([Hoc,1], Prop.11)

[Pf] $\Rightarrow$: We have proved this implication in (4.1.2) and (4.1.3).

$\Leftarrow$: Suppose X is T_1 and quasi-compact. Let V be the set of continuous maps from X to W = {0,1}. Let f be the evaluation map from X to W^V, and X' the closure of f(X) with respect to the patch topology on W^V. Topologize X' with the relative product topology of spectral topology. Then W^V is a spectral space by (1.2.9.1), and X' is a patch in W^V, so X' is spectral, that is, there exists a ring A such that X' = Spec(A). Since f is an embedding by the proof of (1.3.6), f(X) is homeomorpic to X. So f(X) is T_1 and quasi-compact in X'. Since f(X) is T_1, every singleton subset of f(X) is closed. Conversely let a be a closed point of X'. We claim $a \in f(X)$. Let $\{U_\lambda(a): \lambda \in \Lambda\}$ be the set of all quasi-compact open neighbourhoods of a. Then $U_\lambda(a) \cap f(X) \neq \varnothing$ for every λ in Λ. So $\{U_\lambda(a) \cap f(X)\}$ has (FIP) and since f(X) is quasi-compact $\cap\{U_\lambda(a) \cap f(X)\} \neq \varnothing$. Let $x_0 \in \cap\{U_\lambda(a) \cap f(X)\}$. Then every neighbourhood $U_\lambda(a)$ of a contains x_0, that is, $a \in \mathrm{Cl}\{x_0\}$. Since $x_0 \in f(X)$, $\mathrm{Cl}\{x_0\} = x_0$ and $a = x_0 \in f(X)$. Therefore $f(X)$ is precisely the set of all closed points of $X' = \mathrm{Spec}(A)$. So $f(X) = \mathrm{Max}(A)$. ■

Chapter 5. Summary of further results.

The area of study on associating with commutative rings with unity topological spaces which arrises as topologies on various subsets of the prime spectrum is a very wide and active field. In this thesis we discussed the topological spaces, Spec(A), Min(A) and Max(A). For reasons of length, we did not include Jac(A), the Jacobson spectrum and Gold(A), the Goldmen spectrum. In this chapter we define these spectra and present in summary form the principal results.

Section 1. Jacobson spectra and Goldman spectra of a ring.

Definition (5.1.1). — A prime ideal of A is called a *Jacobson ideal* or *J-ideal* if it is an intersection of maximal ideals of A. Let *Jac(A)* denote the subspace of Spec(A) consisting of all the Jacobson ideals of A. Jac(A) is called the *Jacobson ideal spectrum* of A. ([Fon-Mar], p.743)

Definition (5.1.2). — A prime ideal of A is called a *Goldman ideal* or *G-ideal* if it is the contraction of a maximal ideal of A[x], the polynomial ring in one indeterminate over A. Let *Gold(A)* denote the subspace of Spec(A) consisting of all the G-ideals of A. Gold(A) is called the *Goldman ideal spectrum* of A. ([Pic,1]), p.73)

Note (5.1.2.1). — Every prime ideal of A is an intersection of G-ideals of A (cf. the proof of [Pic,1], Prop.1).

Definition. (5.1.3). — A topological space is called *sobre* if every closed irreducible subset of X has a unique generic point. ([Gro-Die], (0,2.1.1))

Definition (5.1.4). — Let X be a topological space. Let ${}^{s}X$ be the family of all closed irreducible subsets of X topologized with the following topology: Let $\mathscr{U}$ be the family of all open subsets of X. Let $V \in \mathscr{U}$. Let us denote by $V^{\sim}$ all elements S of ${}^{s}X$ (which are closed irreducible subsets of X) such that $S \cap V \neq \varnothing$. Let $\mathscr{U}^{\sim} = \{V^{\sim} : V \in \mathscr{U}\}$. Then $\mathscr{U}^{\sim}$ has the required properties of a family of subsets to be the family of open subsets of a topology on ${}^{s}X$. We denote this topological space also by ${}^{s}X$ and call it the *associated sobre space* of X. ([Gro-Die], (0,2.9.1), (0,2.9.2))

Definition (5.1.5). — A topological space X is called *j-spectral* if it has the properties (i) - (iv) listed below: (i) X is quasi-compact, (ii) X is T_0, (iii) every closed irreducible subsets has a generic point, and (iv) every closed irreducible set is the closure of the set of its closed points. [Wie-Wie], p.139)

Proposition (5.1.6). — (a) $\mathrm{Jac}(A) \subseteq \mathrm{Gold}(A) \subseteq \mathrm{Spec}(A)$.

(b) Jac(A), Gold(A) are quasi-compact subspaces of Spec(A).

(c) Max(A) is very dense in Jac(A) (for the definition of "very dense" cf. (4.1.6)). Gold(A) is very dense in Spec(A).

(d) $\mathrm{Jac}(A) = {}^{s}\mathrm{Max}(A)$ and $\mathrm{Spec}(A) = {}^{s}\mathrm{Gold}(A)$. ([Fon-Mar], pp.743-744)

(e) A topological space X is j-spectral iff $X = \mathrm{Jac}(A)$ for some ring A. ([Wie-Wie], Prop.1)

Definition (5.1.7). — A ring is called a *Jacobson ring* if every prime ideal is a Jacobson ideal.

Definition (5.1.8). — A topological space is called a *Jacobson space* if the subset of all closed points of X is very dense in X. ([Gro-Die], (0,2.8.1))

Proposition (5.1.9). — For a ring A the following are equivalent:

(a) A is a Jacobson ring.

(b) Spec(A) is a Jacobson space. ([Gro-Die], (I,6.4.1))

(c) Jac(A) = Spec(A).

(d) Max(A) = Gold(A).

(e) Gold(A) is T_1. ([Fon-Mar], sec.1, Prop.2)

For details of the proof, refer to the papers [Fon-Mar], [Pic,1] and [Wie-Wie].

Section 2. Outline of quasi-compactness of minimal prime spectra.

All the rings in this section are reduced commutative rings with unity unless otherwise specified.

We know that Spec(A), Max(A) are quasi-compact spaces. From the section 1 above, Jac(A) and Gold(A) are also quasi-compact. Mewborn gave an example of a reduced commutative ring A with unity such that Min(A) is not quasi-compact (cf. [Mew], Lem. 3.2). Thus the question of the quasi-compactness of Min(A) is significant. Since Min(A) is Hausdorff (cf. (3.1.2), quasi-compactness of Min(A) is equivalent to compactness.

Various conditions for the compactness of Min(A) are known. These are summarized in the following (5.2.1), (5.2.2) and (5.2.3).

Theorem (5.2.1). — For a ring A, the following are equivalent:

(a) Min(A) is compact and A satisfies the annihilator condition (A is said to satisfy the annihilator condition if for any $a, b \in A$, there exists $c \in A$ such that $\mathrm{Ann}(c) = \mathrm{Ann}(a) \cap \mathrm{Ann}(b)$.).

(b) Min(A) is compact and $\{V_m(a): a \in A\}$ is an open basis for Min(A).

(c) For any $a \in A$, there is $b \in A$ such that $\mathrm{Ann}(\mathrm{Ann}(b)) = \mathrm{Ann}(a)$. ([Hen-Jer], Th. 3.4)

Henriksen-Jerison observed in the above paper that given $a \in A$ there is a finite subset $\{b_1, \ldots, b_n\}$ of A such that $V(\mathrm{Ann}(a)) = V(b_1) \cap \ldots \cap V(b_n)$ to be a necessary condition for the compactness of Min(A). Then they raised the question whether this condition is sufficient as well. [Que], [Mew] and others answered this affirmatively and proved the following results.

Theorem (5.2.2). — For a ring A the following are equivalent:

(a) Min(A) is compact.

(b) $\prod\{A_P: P \in \mathrm{Min}(A)\}$ is a flat A-module.

(c) The injective envelope $I_A(A)$ of A is a flat A-module.

(d) For any $a \in A$, there exists a finitely generated ideal I of A such that $aI = \mathrm{Ann}(Aa + I) = 0$. ([Que], Prop. 3, 4)

Theorem (5.2.3). — For a ring A, the following are equivalent:

(a) Min(A) is compact.

(b) The complete ring of quotients Q(A) of A is a flat A-module.

(c) For any $x \in \mathrm{Spec}(Q(A))$ ($= \mathrm{Max}(Q(A))$), the intersection $j_x \cap A$ is a minimal prime ideal of A.

(d) For any $a \in A$ and $U = \{x \in \mathrm{Spec}(Q(A)): a \notin j_x \cap A\}$, there exists a finitely generated ideal I of A such that $\mathrm{Spec}(Q(A)) - U = \{x \in \mathrm{Spec}(Q(A)): I \not\subseteq j_x \cap A\}$. ([Mew], Th. 3.1)

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