THREE DIMENSIONAL VECTOR GEOMETRY

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ABSTRACT

The central problem discussed is the coordinatisation of three-dimensional affine geometry. Our study begins with the presentation and development of an axiom system for affine geometry. Vectors are introduced into the geometry as equivalence classes of ordered pairs of points. Scalars are defined as functions of vectors onto vectors which in essence preserve parallelism and fix the zero vector.

It is proved that the scalars, under suitably defined operations, form a division ring and the vectors form an abelian group under addition. We next show that the vectors form a vector space of dimension three over the scalars. We further outline a proof that, given a vector space of dimension three over a division ring, under suitable definitions of point, line, plane, incidence, and parallelism, the geometry of the vector space is three-dimensional affine, in the sense of our axiom system. The central problem is thus completely solved.

Axioms of order are then introduced into the system and the notion of direction is discussed with regard to classes of parallel lines and to vectors. Real affine geometry is characterized as an ordered affine geometry where the order relation is analagous to that of the real numbers. Lastly, a metric is introduced into real affine geometry and the usual characterization of vectors in terms of length and direction is obtained.

In developing affine geometry in preparation for a study of vectors, we consider various Desarquesian configurations and prove that the corresponding configuration theorems are valid. In particular, the affine little Desarques Theorem is used to prove that the relation defining vectors is an equivalence relation. The central affine Desarques Theorem is used in the main existence theorem for scalars.

CHAPTER I - INTRODUCTION

In the present chapter, we shall outline the scope of this work and compare its approach with that of the literature on the subject.

Chapter II is devoted entirely to the study of a set of nine axioms for three-dimensional affine geometry. In order to obtain a convenient notation for the point of intersection of two lines, we have chosen to regard lines as nothing more than distinguished subsets of points. The affine theory developed is entirely equivalent to that obtained by regarding lines as entities associated with points by an incidence relation $I \subseteq \mathcal{P} \times \mathcal{L}$, for the element-set inclusion symbol '$\in$' plays essentially the same role in our presentation as 'I' does in the more common one.

In line with the above comments, a plane is defined to be a point set of a special type, and it is shown that any three noncollinear points determine a unique plane. Parallelism of lines is defined as an equivalence relation satisfying Playfair's equivalent of Euclid's Parallel postulate. From this, it is proved as Cor. 2.4.6 that two lines are parallel if and only if they are coplanar and do not intersect. Chapter II concludes with a study of finite affine spaces and several counting arguments to compute, for example, the total number of points in a plane or in an affine space.

On referring to D. Hilbert's Grundlagen der Geometrie, we see that Chapter II essentially covers the so-called "Axioms of Connection" and "axiom of parallel". We note that in our system, Euclid's defining characteristic of a plane, namely that it contain all points of a line containing two points of the plane, is obtained as Theorem 2.3. Furthermore, parallelism of planes is defined and proved to be an equivalence relation satisfying an analogue of Euclid's parallel postulate. As indicated in E. Moise, Elementary Geometry from an Advanced Standpoint, this result is usually proved in a metric geometry with the aid of congruence postulates.

Chapter III is concerned with the definition of vectors in a three dimensional affine space. Although similar studies have been made in two dimensions, the specific details for the three-dimensional case do not appear to have been worked out elsewhere. Our approach owes much to G. Papy, Géométrie Affine Plane et Nombres Réels. One important difference between Papy's work and the present study is that, unlike Papy, we do not suppose initially that we are dealing with an ordered infinite geometry. Another is that three-dimensionality allows us to prove, via Desarques' Theorem, that vectors are well-defined. Restricted to two dimensions, Papy postulates well-definedness of vectors. Specifically, we are indebted to Papy for the definition of the relation '$\square$'.

Similar definitions of vectors are to be found in K. Reidemeister, Grundlagen der Geometrie and also in notes from a seminar given by Prof. G. Pickert at Justus Liebig University in Giessen in 1963.

In Chapter IV, we adopt Dr. Pickert's functional definition of multiplication of a vector by a scalar. Theorem 4.3 states that any non zero scalar is a one-to-one mapping of the vectors onto themselves. Corollary 4.8 states that the vectors form a vector space of dimension three over the division ring of scalars, thus indicating the connection of our definition of vectors with the common algebraic definition of a vector space.

Definitions similar to, but not identical with, Prof. Pickert's definition of scalars are found in the literature. In Geometric Algebra, E. Artin studies dilatations, defined as mappings of points into points such that if $P \neq Q$ and ℓ is the unique line through f (P) and parallel to PQ, then f (Q) lies on ℓ. Artin proves that dilatations are determined uniquely by the images of two distinct points and that all nonconstant dilatations are one to one onto functions. In <u>Grundlagen der Elementarmathematik</u>, H. Lenz studies <u>dehnungen</u>, mappings f of points onto points such that either f (A) = f (B) or f (A) f (B) $\parallel$ AB. Such a mapping is called a <u>vektor</u> if it is either the identity mapping or fixed point free. Lenz proves that any <u>dehnung</u> which maps two distinct points into the same point is a constant function. Moreover, if O is a fixed point of the <u>dehnung</u> f and P $\neq$ O, then f (P) lies on OP. Lenz proves the existence theorem that given O, A $\neq$ O, B on OA, there is a unique <u>dehnung</u> f such that f (O) = O, f (A) = B; moreover, given any points A, B, there is exactly one <u>vektor</u> g such that g (A) = B.

In terms of the definitions and notation adopted in the present work, if the vector space is represented at any point P, then any multiplication by a scalar f may be associated with the mapping f^{*} defined as follows. For any point Q, let $\underline{v} \in \mathcal{V}$ be such that (P, Q) $\in \underline{v}$ and let R $\in \mathcal{P}$ be such that (P, R) $\in$ f ($\underline{v}$). Define f^{*} (Q) = R. It is readily shown from the results of this thesis that f^{*} is a function. Moreover, f^{*} is a <u>dehnung</u>, but clearly not a <u>vektor</u>. Similarly, given a representation of the vector space at a point P, any non-constant <u>dehnung</u> which is not a <u>vektor</u>, may be interpreted as a scalar, with the aid of Desarques' Theorems. The proof consists in representing the vector space at the unique fixed point of the <u>dehnung</u> and arguing as in Theorems 4.1, 4.2.

With regard to dilatations, we note that if (A, B) $\square$ (C, D) or if (A, B) $\uparrow$ (C, D), then there exists a unique dilatation f such that f (A) = C and f (B) = D, provided A $\neq$ B.

Chapter IV concludes with a comprehensive study of non-degenerate Desarquesian configurations. The author is not aware of any similar study elsewhere in the literature.

In Chapter V, axioms of order are introduced into an affine geometry and direction is defined in terms of equivalence classes of certain types of orders on lines of a parallel class under parallel projection. Positiveness of scalars is defined, and from this an order relation on the scalars is induced so that the scalars form an ordered division ring. Theorems 5. 12 and 5. 13 are the so-called "plane-separation axiom" and "space-separation axiom" respectively. In the book by E. Moise referred to above, these results are each taken as axioms. As a corollary to Th. 5. 12, we prove the classical axiom of Pasch. Adopting Papy's vector definition of mid-point, we prove the existence of unique mid-points and generalize and solve the analagous problem of n-division points.

The proof of Theorem 5. 2 is a simplification of that found in H. Lenz, Grundlagen der Elementarmathematik. Lenz uses axioms of order equivalent to those in Chapter V.

The reader interested in seeing other approaches to order defined in terms of postulates of betweenness and the axiom of Pasch is referred to the works of D. Hilbert and E. Moise mentioned above. An alternative approach to order in which axioms for order of a one dimensional subgroup of vectors are used is to be found in Papy's book.

Chapter VI serves as a transition from ordered affine geometry to metric geometry. By means of Corollary 4. 8 and Theorems 5. 12, 5. 13, we define an orientation of our vector space as an equivalence class of bases, where the relevant equivalence relation is defined with the aid of a ternary "box product" function from the vector space into the scalars. The classical scalar product is used to define the length of a vector. The traditional geometric definition of a vector as a quantity uniquely determined by its length and direction is then obtained as Theorem 6.3. Finally, with the aid of well-known results from linear algebra, the class of metric-preserving linear transformations of the vector space onto itself whose matrix representation has positive determinant is mapped into the real numbers from 0 to π.

The numbering system employed in this work is illustrated in the following table:

Axiom 2.1	First axiom introduced in Chapter II.
Definition 2.1	First definition in Chapter II.
Proposition 3.1	First proposition in Chapter III.
Theorem 2.1	First theorem in Chapter II.
Lemma 2.4.1	First lemma leading to the proof of Th. 2.4.
Corollary 2.1.3	Third corollary to Th. 2.1.

II Axiomatic Study of Affine Spaces

An affine space $\mathcal{Q}$ is a quadruple $(\mathcal{P}, \mathcal{L}, \parallel, \in)$ where $\mathcal{P}$ is a set of elements called "points", $\mathcal{L}$ is a collection of subsets of $\mathcal{P}$ called "lines", and $\in$ and $\parallel$ are undefined relations. The statement "$\ell \parallel m$" may be read "The line ℓ is parallel to the line m." The statement "$P \in \ell$" may be read "The point P lies on the line ℓ" or "The line ℓ passes through the point P." Its negation will be abbreviated as "$P \notin \ell$". We shall understand all statements of the form "$P \notin \ell$" or $P \in \ell$" to imply that P is a point of $\mathcal{Q}$ and ℓ is a line of $\mathcal{Q}$. Lastly, "$P_1, P_2 \in \ell_1, \ell_2$" will be taken to mean that $P_i \in \ell_j$ $(i, j = 1, 2)$.

Axiom 2.1 $P_1, P_2 \in \ell_1, \ell_2 \Rightarrow P_1 = P_2$ or $\ell_1 = \ell_2$ (or both).

Notice that this is simply a concise way of saying that any two distinct points (resp. lines) determine at most one line (resp. point).

Axiom 2.2 $P_1 \neq P_2 \Rightarrow$ there exists ℓ such that $P_1, P_2 \in \ell$.

In view of Ax. 2.1 - 2, it is clear that any two distinct points P_1 and P_2 determine exactly one line which we may denote indifferently by either "P_1P_2" or "P_2P_1". The expression "P_1P_2" will be used only in the event $P_1 \neq P_2$.

Axiom 2.3 Each line passes through at least two distinct points.

Axiom 2.4 Parallelism among lines is an equivalence relation.

We denote the parallelism of any lines ℓ and m by "ℓ // m" or "m // ℓ".

Axiom 2.5 $P \notin \ell \Rightarrow$ there exists exactly one line m such that $P \in m$ and ℓ // m.

The reader will recognize Ax. 2.5 as Playfair's equivalent of the famous parallel postulate of Euclid.

Theorem 2.1 $\ell \neq m, \ell // m \Rightarrow \ell \cap m = \emptyset$ (the null set)

Proof Suppose $Q \in \ell$, m. By Ax. 2.1, 3, there exist points R and S such that $R \in \ell$, $S \in m$, $R \notin m$, $S \notin \ell$, and $R \neq S$. R and S determine the unique line RS. If $Q \in RS$, then we have Q, $R \in \ell$, RS. As $Q \in m$ and $R \notin m$, we may infer $Q \neq R$, whence Ax. 2.1 implies $\ell = RS$. Interchanging the roles of R and S gives $m = RS$ and so $\ell = m$, a contradiction. Thus $Q \notin RS$.

Consequently, there is a unique line $n \,//\, RS$ such that $Q \in n$, by Ax. 2.5. Now, Ax. 2.3 gives a point $T \in n$ such that $T \neq Q$. If $T \in m, \ell$, then since $T \neq Q$, Ax. 2.1 implies $\ell = m$, a contradiction. Assuming for definiteness that $T \notin m$, we have a unique line λ through T such that $\lambda \,//\, m$.

Suppose $Q \in \lambda$. Since $Q \neq T$ and $Q, T \in \lambda$, we may conclude that $m \parallel \lambda = QT = n \parallel RS$ by Ax. 2.1, 2. We now have $\ell \parallel m \parallel n \parallel RS$ by Ax. 2.4. Further $\ell \neq m$, $RS \neq \ell$, and $RS \neq m$ as $Q \notin RS$. Through Q, m and ℓ are distinct lines parallel to RS, a contradiction to Ax. 2.5. Thus $Q \notin \lambda$. However, $Q \notin \lambda$, $Q \in \ell \cap m$, and $\ell \parallel m \parallel \lambda \Rightarrow \ell = m$ by Ax. 2.5. As this contradicts the hypotheses, no Q satisfies $Q \in \ell, m$. Thus, $\ell \cap m = \emptyset$, as required.

Alternate proof of Th. 2.1:[1]

Let ℓ, m, Q be as in the preceding proof. By Ax. 2.4, 5, since $\ell \neq m$, all lines parallel to ℓ pass through Q.

Now let n be any line through Q other than ℓ or m and let T be any point of n other than Q. By Ax. 2.1, 2, $T \notin \ell$ since $\ell \neq n$. It follows from Ax. 2.5 that there exists a unique line p through T and parallel to ℓ. By our earlier comments $Q \in p$. As $T, Q \in n, p$ and $T \neq Q$, Ax. 2.1 $\Rightarrow$ $n = p$. Hence all lines through Q are parallel to ℓ.

As $Q \notin RS$, there is a unique line q through Q and parallel to RS. Since $Q \in q$, it follows that $q \parallel \ell$. Ax. 2.4 implies that $\ell \parallel RS$, whence $Q \in RS$, a contradiction. Thus $\ell \cap m = \emptyset$.

Corollary 2.1.1 For any point P and line ℓ, there exists exactly one line m such that $P \in m$ and $\ell \parallel m$.

The proof is immediate from Ax. 2.5 and Th. 2.1.

[1] Proof due to Dr. W. J. Jonsson.

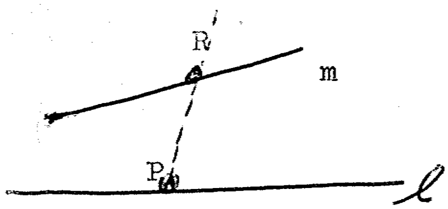
Corollary 2.1.2 All points are contained in the same number of lines. Moreover, this number is the number of parallel classes in $\mathcal{L}$.

The proof follows from the preceding corollary.

Corollary 2.1.3 If two distinct lines exist, then not all lines are parallel.

Proof: Suppose that ℓ and m are two distinct lines. By Ax. 2.1, 3 there exist distinct points R and P such that $P \in \ell$, $P \notin m$, $R \in m$, and $R \notin \ell$. If $RP \parallel \ell$, then Cor. 2.1.1 $\Rightarrow R \in \ell$, a contradiction. Thus, $RP \nparallel \ell$.

Axiom 2.6 If P, Q, and R are distinct non-collinear points such that $P \in \ell \parallel QR$ and $R \in m \parallel PQ$, then there exists a point $S \in \ell \cap m$.



It is easy to show that this familiar process of "completing the parallelogram" leads to a unique S. By Ax. 2.1, it suffices to show $\ell \neq m$. If $\ell = m$, then Ax. 2.1, 2 $\Rightarrow \ell = m = PR \parallel PQ$. However, Th. 2.1 $\Rightarrow PR = PQ$, a contradiction to the non-collinearity of P, Q, and R. Consequently, S is unique.

It is also clear that Ax. 2.3, 4, 6 and Th. 2.1 $\Rightarrow$ parallel lines have the same number of points.

We are now in a position to state the following definitions.

Definition 2.1 If ℓ_1 and ℓ_2 are distinct lines intersecting in a point P, then the plane $\Pi\,\ell_1, \ell_2$ is defined as $\{x \mid x \in (m_1 \cap m_2),\ m_i \parallel \ell_i\ (i - 1, 2),\ m_i \cap \ell_j \neq \emptyset \text{ for } i \neq j\}$.

Definition 2.2 ℓ is a line of the plane $\Pi\,\ell_1, \ell_2$ if and only if there exist distinct points P and Q such that $P, Q \in (\ell \cap \Pi\,\ell_1, \ell_2)$.

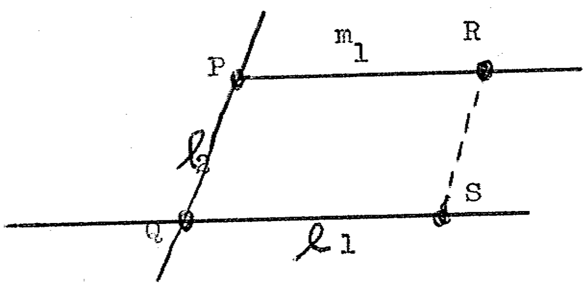
We remark that $P \in \Pi\,\ell_1, \ell_2$ as one need only let $m_i = \ell_i$ $(i = 1, 2)$. Also any other point of $\ell_1 \cup \ell_2$ is in $\Pi\,\ell_1, \ell_2$ by Cor. 2.1.1. Ax. 2.3 then guarantees that ℓ_1 and ℓ_2 are each lines of $\Pi\,\ell_1, \ell_2$. The next theorem extends these comments.

<u>Theorem 2.2</u> (i) With the notation of Def. 2.1, $m_1 \cap m_2$ contains exactly one point. (ii) $\ell_2 \cap m_1 \neq \emptyset$, $\ell_1 \| m_1 \Rightarrow m_1$ is a line of $\Pi\, \ell_1, \ell_2$.

<u>Proof</u> (i) Unless $m_1 = \ell_1$ or $m_2 = \ell_2$, Ax. 2.6 and the comments immediately following it $\Rightarrow m_1 \cap m_2 = \{S\}$ where S is a unique point.

If $m_i = \ell_i$ ($i = 1$ or 2), then $m_1 \cap m_2$ is either $m_1 \cap \ell_2$, $\ell_1 \cap m_2$, or $\ell_1 \cap \ell_2$. Using Ax. 2.1, 4 and $\ell_1 \nparallel \ell_2$, we may readily show that these sets each contain exactly one point.

(ii) Suppose $\ell_1 \cap \ell_2 = \{Q\}$, $m_1 \| \ell_1$ and $m_1 \cap \ell_2 = \{P\}$. By Ax. 2.3, there exists a point $R \neq P$ such that $R \in m_1$. Now by our comments after Def. 2.2, we may assume $\ell_1 \neq m_1$. Hence, by Th. 2.1, $m_1 \cap \ell_1 = \emptyset$ and so P, Q, and R are non-collinear. Then Ax. 2.6 allows us to infer that the line through R parallel to ℓ_2 intersects ℓ_1 in a unique point S. By Def. 2.1, we may conclude that $R \in \Pi\, \ell_1, \ell_2$. Since $P \in \Pi\, \ell_1, \ell_2$ by our comments after Def. 2.2, Def. 2.2 $\Rightarrow m_1$ is a line $\Pi\, \ell_1, \ell_2$.



<u>Axiom 2.7</u> If ℓ and m are any lines of any plane such that $\ell \nparallel m$, then there exists a point P such that $P \in (\ell \cap m)$.

As $\ell \nparallel m \Rightarrow \ell \neq m$, Ax. 2.1 $\Rightarrow$ P is unique.

<u>Theorem 2.3</u> If ℓ is a line of the plane $\Pi\, \ell_1, \ell_2$, then for any point P, $P \in \ell \Rightarrow P \in \Pi\, \ell_1, \ell_2$.

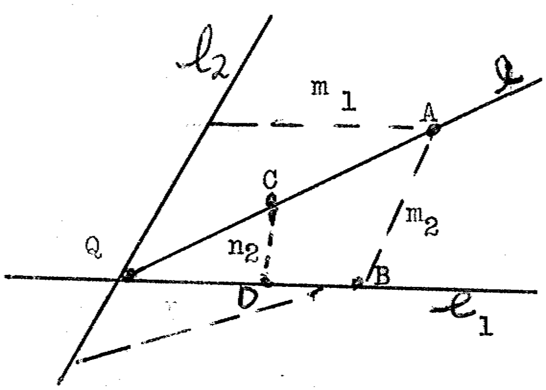
<u>Proof</u>: If $\ell \| \ell_1$, then Ax. 2.4 and Th. 2.1 imply $\ell \nparallel \ell_2$. As ℓ_2 is a line of $\Pi\, \ell_1, \ell_2$, we may infer that $\ell \cap \ell_2 \neq \emptyset$. If $\ell \cap \ell_2 = \{T\}$, then $T \in \Pi$, as $T \in \ell_2$. For any point R of ℓ such that $R \neq T$, we may apply the

proof of Th. 2.2 (ii) to show that $R \in \Pi\,\ell_1, \ell_2$.

Since $\Pi\,\ell_1, \ell_2 = \Pi\,\ell_2, \ell_1$, the above argument takes care of the case $\ell \parallel \ell_2$ as well.

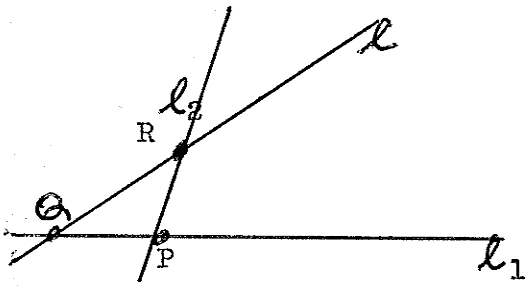
We may now assume $\ell \nparallel \ell_i$ $(i = 1, 2)$. By Ax. 2.7, there exist points Q and R such that $\ell \cap \ell_1 = \{Q\}$, and $\ell \cap \ell_2 = \{R\}$. Depending on whether $Q = R$, we have two cases to consider.

If $Q = R$, then $\ell \cap \ell_2 = \{Q\}$, by Ax. 2.1. As ℓ is a line of $\Pi\,\ell_1, \ell_2$, there exists a point $A \in (\ell \cap \Pi)$ such that $A \neq Q$. Let m_1, m_2 be as in Def. 2.1 as applied to A.



If $B \in (\ell_1 \cap m_2)$, then Ax. 2.6 as applied to Q, A, and B allows us to conclude that the line through B parallel to AQ intersects ℓ_2. Hence, $B \in \Pi\,\ell_2, \ell$. Since $Q \in \Pi\,\ell_2, \ell$ and $A \neq Q \Rightarrow A \notin \ell_2 \Rightarrow \ell_2 \cap m_2 = \emptyset \Rightarrow B \notin \ell_2 \Rightarrow Q \neq B$, it follows that ℓ_1 is a line of $\Pi\,\ell_2, \ell$. Now $Q \in \Pi\,\ell_1, \ell_2$. For any point $C \neq Q$ of ℓ, the line through C parallel to ℓ_2 is a line of $\Pi\,\ell_2, \ell$ by Th. 2.2 (ii). This line (n_2 in the figure) is not parallel to ℓ_1, and so Ax. 2.7 $\Rightarrow (n_2 \cap \ell_1)$ contains a single point D. As $Q \notin n_2$, the points Q, C, and D are non-collinear, whence Ax. 2.6 $\Rightarrow$ the line through C and parallel to ℓ_1 intersects ℓ_2. By Def. 2.1, it follows that $C \in \Pi\,\ell_1, \ell_2$ and so the theorem is proved if $Q = R$.

However, if $Q \neq R$, then suppose $\ell_1 \cap \ell_2 = \{P\}$. It is clear that $Q, P \in \pi\,\ell_2, \ell$ and $Q \neq P$. Thus, ℓ_1 is a line of $\pi\,\ell_2, \ell$ and the proof may now be completed as in the previous case with the use of Ax. 2.6.



We note that an immediate consequence of Th. 2.3 is that the point P of Ax. 2.7 is in any plane of which ℓ and m are each lines.

<u>Axiom 2.8</u> If π_1 and π_2 are any planes and P any point such that $P \in (\pi_1 \cap \pi_2)$, then there is at least one line ℓ such that ℓ is a line of both π_1 and π_2.

We shall see that Ax. 2.8 is the principal reason that $\mathbb{Q}$ is three-dimensional.

<u>Axiom 2.9</u> There exist (at least) four distinct non-collinear, non-coplanar points.

From this particular existence axiom, many interesting results follow. In particular, Cor. 2.1.3 $\Rightarrow$ not all lines are parallel. Th. 2.3 implies not all lines lie in the same plane. Ax. 2.9 clearly implies that every point lies on at least three distinct lines. Consequently, non-collinear points must be distinct. Furthermore, non-coplanar lines must be distinct, as any line PQ lies in a plane determined by it and PQ and PR, where R is a point $\notin$ PQ, the existance of R being guaranteed by Ax. 2.9.

<u>Lemma 2.4.1</u> If ℓ_3 is a line of $\pi\,\ell_1, \ell_2$ such that $\ell_2 \parallel \ell_3$, then $\pi\,\ell_1, \ell_3 = \pi\,\ell_1, \ell_2$.

<u>Proof</u> Since $\ell_1 \nparallel \ell_2$, Ax. 2.4 $\Rightarrow \ell_1 \nparallel \ell_3$ and Ax. 2.7 $\Rightarrow \ell_1$ intersects ℓ_3 in a unique point, P. Thus $\pi\,\ell_1, \ell_3$ is defined.

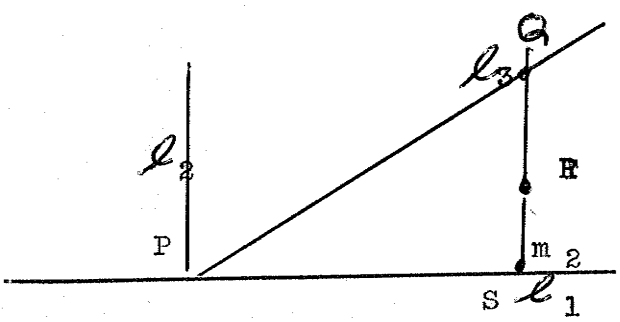
Without loss of generality, $\ell_2 \nparallel \ell_3$. Let R be any point of $\overline{\pi}\,\ell_1, \ell_2$ such that $R \notin (\ell_1 \cup \ell_3)$. Suppose that the line m_2 through R parallel to ℓ_2 intersects ℓ_1 in the point T. (T exists since $R \in \overline{\pi}\,\ell_1, \ell_2$). As $P \notin m_2$ by Th. 2.1, it follows from Ax. 2.6 as applied to R, T, and P that the line through R parallel to ℓ_1 intersects ℓ_3. Thus $R \in \overline{\pi}\,\ell_1, \ell_3$. It is clear that all points of $\overline{\pi}\,\ell_1, \ell_2$ that are in $(\ell_1 \cup \ell_3)$ are also in $\overline{\pi}\,\ell_1, \ell_3$. Consequently, $\overline{\pi}\,\ell_1, \ell_2 \subseteq \overline{\pi}\,\ell_1, \ell_3$.

Now, Th. 2.2 (ii) $\Rightarrow \ell_2$ is a line of $\overline{\pi}\,\ell_1, \ell_3$. Repeating the above argument, we have $\overline{\pi}\,\ell_1, \ell_3 \subseteq \overline{\pi}\,\ell_1, \ell_2$. Hence, $\overline{\pi}\,\ell_1, \ell_2 = \overline{\pi}\,\ell_1, \ell_3$.

<u>Lemma 2.4.2</u> If ℓ_3 is a line of $\overline{\pi}\,\ell_1, \ell_2$ such that $\ell_3 \cap \ell_1 = \ell_3 \cap \ell_2 = \ell_1 \cap \ell_2$, then $\overline{\pi}\,\ell_2, \ell_3 = \overline{\pi}\,\ell_1, \ell_2$.

<u>Proof</u> Since $\ell_1 \cap \ell_3$ contains a single point, $\overline{\pi}\,\ell_1, \ell_3$ is defined.

Let R be any point of $\overline{\pi}\,\ell_1, \ell_2$ such that $R \notin (\ell_2 \cup \ell_3)$. The line through R parallel to ℓ_2 (denoted m_2 in the figure) intersects ℓ_1 since $R \in \overline{\pi}\,\ell_1, \ell_2$. Suppose $m_2 \cap \ell_1 = \{S\}$. If $R = S$, then m_2 is a line of $\overline{\pi}\,\ell_1, \ell_2$ by Th. 2.2(ii). If $R \neq S$, then m_2 is a line of $\overline{\pi}\,\ell_1, \ell_2$ by Def. 2.2. Since $\ell_2 \cap \ell_3$ contains a single point, Ax. 2.3, 4 and Th. 2.1 $\Rightarrow \ell_2 \nparallel \ell_3 \Rightarrow \ell_1 \nparallel \ell_3$. Consequently, Ax. 2.7 $\Rightarrow \ell_1 \cap \ell_3 \neq \emptyset$. If $\ell_2 \cap \ell_3 = \{P\}$ and $m_2 \cap \ell_3 = \{Q\}$, then Ax. 2.1 $\Rightarrow Q \notin \ell_2$ since $\ell_2 \nparallel \ell_3$. **Hence**, Ax. 2.6 applied to P, Q, and R im-

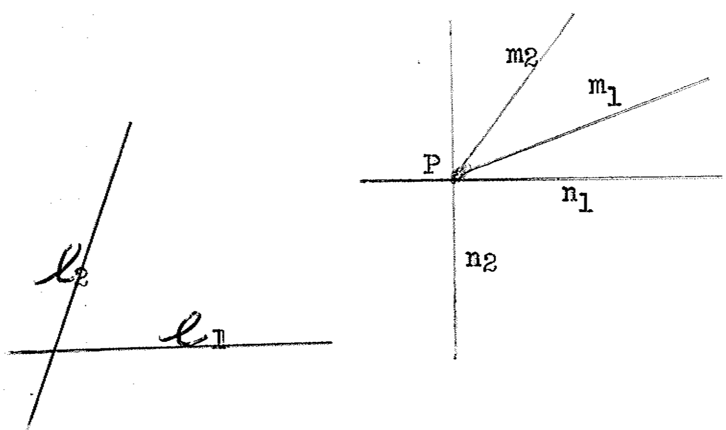


plies that the line through R parallel to ℓ_3 intersects ℓ_2. According to Def. 2.1, we have shown $R \in \overline{\pi}\,\ell_2, \ell_3$. As in the previous lemma, $\overline{\pi}\,\ell_1, \ell_2 \subseteq \overline{\pi}\,\ell_2, \ell_3$.

By the reasoning of the previous lemma, the proof will be complete if we show that ℓ_1 is a line of $\overline{\pi}\,\ell_2, \ell_3$. With the above notation, it clearly suffices to show that $S \in \overline{\pi}\,\ell_2, \ell_3$. We already have $S \in m_2 \parallel \ell_2$ such that $m_2 \cap \ell_3 = \{Q\} \neq \emptyset$. It only remains to prove that the line through S parallel to ℓ_3 intersects ℓ_2. As above, this follows by Ax. 2.6.

<u>Theorem 2.4</u> If m_1 and m_2 are any two distinct intersecting lines of $\overline{\pi}\,\ell_1, \ell_2$, then $\overline{\pi}\,\ell_1, \ell_2 = \overline{\pi}\,m_1, m_2$.

<u>Proof</u> Suppose that $m_1 \cap m_2 = \{P\}$ and $P \in n_i \parallel \ell_i$ $(i = 1, 2)$. By Th. 2.3, $P \in \overline{\pi}\,\ell_1, \ell_2$ and so $n_2 \cap \ell_1 \neq \emptyset$, $n_1 \cap \ell_2 \neq \emptyset$. Then either by Th. 2.2



(ii) or Def. 2.2, n_1 and n_2 are lines of $\overline{\pi}\,\ell_1, \ell_2$. Lemma 2.4.1 implies $\overline{\pi}\,\ell_1, \ell_2 = \overline{\pi}\,\ell_1, n_2 = \overline{\pi}\,n_2, \ell_1 = \overline{\pi}\,n_2, n_1$. From the hypotheses, it follows that m_1 and m_2 are each lines of $\overline{\pi}\,n_1, n_2$. Without loss of generality, $n_i \not\parallel m_j$ $(i, j = 1, 2)$. Then Lemma 2.4.2 implies $\overline{\pi}\,n_2, n_1 = \overline{\pi}\,m_1, n_2 = \overline{\pi}\,m_2, m_1$. Thus $\overline{\pi}\,\ell_1, \ell_2 = \overline{\pi}\,m_1, m_2$ and the proof is complete.

<u>Corollary 2.4.1</u> If P_1, P_2 and P_3 are non-collinear points, then there exists a unique plane $\overline{\pi}$ such that $P_i \in \overline{\pi}$ $(i = 1, 2, 3)$.

Proof (i) There exists at least one such $\overline{\pi}$, namely $\overline{\pi}\ P_1P_2,\ P_2P_3$. To verify this one need only guarantee $P_i \neq P_j$ whenever $i \neq j$. This follows from our comments preceding Lemma 2.4.1.

(ii) We must now show that there is at most one such $\overline{\pi}$. This follows from Def. 2.2 and Th. 2.4 as any such $\overline{\pi}$ equals $\overline{\pi}\, P_1P_2,\ P_2P_3$.

Corollary 2.4.2 For any point P_1 and line ℓ such that $P_1 \notin \ell$, there exists exactly one plane $\overline{\pi}$ such that $P_1 \in \overline{\pi}$ and ℓ is a line of $\overline{\pi}$.

Proof (i) By Ax. 2.3, there exist distinct points $P_i \in \ell$ ($i = 2, 3$). Since $P_1 \notin \ell$, we have $P_1 P_2$ and P_3 non-collinear. The preceding corollary implies that there exists a plane $\overline{\pi}^*$ such that $P_i \in \overline{\pi}^*$ ($i = 1, 2, 3$). Since $P_2 \neq P_3$, Def. 2.1 $\Rightarrow \ell$ is a line of $\overline{\pi}^*$. Hence $\overline{\pi}^*$ is a satisfactory $\overline{\pi}$.

(ii) We now show that there is at most one such $\overline{\pi}$, for any satisfactory $\overline{\pi}$ must contain P_i ($i = 1, 2, 3$) by Th. 2.3 and so by Cor. 2.4.1, $\overline{\pi}$ is unique.

Corollary 2.4.3 Any two distinct planes intersect in either the null set or a line.

Proof Suppose two distinct planes have a non-empty intersection. By Ax. 2.8, that intersection contains a line ℓ. However, no point not on ℓ can be in both planes, for such a point, together with ℓ, determines a unique plane, by Cor. 2.4.2. Of course, the intersection of two distinct planes may be empty.

Corollary 2.4.4 For any two planes $\overline{\pi}_1$ and $\overline{\pi}_2$, $\overline{\pi}_1 = \overline{\pi}_2 \Leftrightarrow$ (for any line ℓ, ℓ is a line of $\overline{\pi}_1 \Leftrightarrow \ell$ is a line of $\overline{\pi}_2$).

Proof (i) If $\overline{\pi}_1 = \overline{\pi}_2$, then $\overline{\pi}_1$ and $\overline{\pi}_2$ contain the same points which by Ax. 2.1, 2 determine the same lines satisfying Def. 2.2.

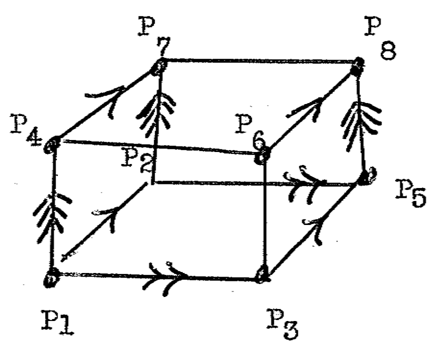
(ii) If $\Pi_1 \neq \Pi_2$, then there are two cases to consider.

If $\Pi_1 \cap \Pi_2 = \emptyset$, then if $\Pi_1 \equiv \Pi\ell_1, \ell_2$, it is clear that ℓ_1 is a line of Π_1, but not of Π_2.

Otherwise, $\Pi_1 \cap \Pi_2 \neq \emptyset \Rightarrow \Pi_1 \cap \Pi_2 = m$ for some line m, by the preceding corollary. As any plane contains three non-collinear points, we may suppose that P_1, P_2, and P_3 are points such that $P_i \in m$ (i.= 1, 2), $P_3 \notin m$, and $P_i \in \Pi_1$ (i = 1, 2, 3). Now, Cor 2.4.1 implies that P_i (i = 1, 2, 3) uniquely determine a plane, which must be Π_1. Since $\Pi_1 \neq \Pi_2$ by assumption and $P_i \in \Pi_2$ (i = 1, 2), it follows that $P_3 \notin \Pi_2$, whence P_1P_3 (or indeed P_2P_3) is a line of Π_1, but not of Π_2.

<u>Corollary 2.4.5</u> There exist at least eight distinct points.[1]

<u>Proof</u> By Ax. 2.9 and Cor. 2.4.2, there exist four distinct non-coplanar points P_i (i = 1, 2, 3, 4), no three of which are collinear. Ax. 2.6 allows us to complete the parallelogram of P_1, P_2, and P_3 with the point P_5. By the reasoning of Th. 2.2 (ii), P_5 lies in the plane determined by P_i(i = 1, 2, 3). Furthermore, P_5 is distinct from P_1, P_2, P_3, and P_4, since $P_3P_5 \cap P_1P_2 = P_2P_5 \cap P_1P_3 = \emptyset$ and P_i (i = 1, 2, 3, 4) are non-coplanar.



By precisely the same argument, there exists a point P_6 in the plane of P_i (i = 1, 3, 4) such that P_1, P_2, P_3, P_4, P_5, and P_6 are all distinct. Similarly, P_i (i = 1, 2, 4) lead to P_7 and P_i (i = 3, 5, 6) lead to P_8 as indicated in the figure.

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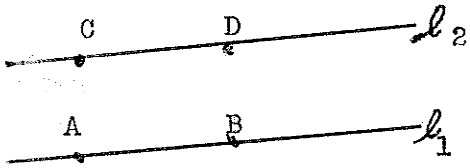
[1] Essentially the same treatment of Cor. 2.4.5 and Th. 2.5 is to be found in [8], which was published after the final draft of this thesis.

Corollary 2.4.6 Any two distinct lines ℓ_1 and ℓ_2 are parallel if and only if they are coplanar and do not intersect.

Proof (i) Suppose $\ell_1 \neq \ell_2$ and $\ell_1 \parallel \ell_2$. By Th. 2.1, $\ell_1 \cap \ell_2 = \emptyset$. Ax. 2.3 $\Rightarrow$ distinct points A, B, C, and D exist such that A, B $\in \ell_1$, and C, D $\in \ell_2$. By Cor. 2.4.1, we know that A, B, and C determine a unique plane π CA, AB. Th. 2.2 (ii) implies that ℓ_2 is a line of π CA, AB. However,

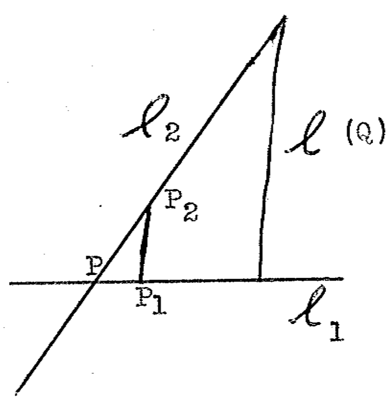


Def. 2.2 $\Rightarrow \ell_1$ is a line of π CA, AB. Thus ℓ_1 and ℓ_2 are coplanar and the plane in which they both lie is, by Cor. 2.4.1, unique.

Coplanar, non-intersecting lines are parallel, by Ax. 2.7.

Corollary 2.4.7 If ℓ_1 and ℓ_2 are any distinct intersecting lines, then they have the same number of points.

Proof Suppose $\ell_1 \cap \ell_2 = \langle P \rangle$. By Ax. 2.1-3, there exist points P_i ($i = 1, 2$) such that $P \neq P_i$ ($i = 1, 2$) and $P_i \in \ell_i$ ($i = 1, 2$). Now it follows by Ax. 2.1, 2 that P_1P_2 is a well-determined line of $\pi\ \ell_1, \ell_2$. Furthermore, $P_1P_2 \neq \ell_2$, for $P_1P_2 = \ell_2 \Rightarrow P_1P_2 \cap \ell_1 = \ell_2 \cap \ell_1 \Rightarrow P = P_1$, a contradiction. Thus Th. 2.1 implies $P_1P_2 \nparallel \ell_2$ and we may show similarly that $P_1P_2 \nparallel \ell_1$.



For any point Q of ℓ_1, Cor. 2.1.1 implies that there exists a unique line $\ell(Q)$ such that $\ell(Q) \parallel P_1P_2$. Since $\ell(Q)$ is a line of $\pi\ \ell_1, \ell_2$ such that $\ell(Q) \nparallel \ell_2$, it follows from Ax. 2.7 that $\ell_2 \cap \ell(Q)$ is a single point $f(Q)$. Furthermore, the same argument shows that $T \in \ell_2 \Rightarrow$ there exists $V \in \ell$ such that $f(V) = T$.

Now, for any point W of ℓ_2, $W = f(P_3) = f(P_4)$ implies that $P_3 = P_4$

since the line through W parallel to P_1P_2 intersects ℓ_1 in a single point, as $P_1P_2 \not\parallel \ell_1$. Thus f is a one-to-one mapping from ℓ_1 onto ℓ_2 and the proof is complete.

We note that instead of showing $T \in \ell_2 \Rightarrow$ there exists $V \in \ell$ such that f (V) = T, one could exhibit functions from ℓ_1 (resp. ℓ_2) into ℓ_2 (into ℓ_1) and then establish the corollary by appealing to a famous result of Bernstein and Schroeder.

Corollary 2.4.8 All lines have the same number of points.

Proof Suppose ℓ_1 and ℓ_2 are two non-intersecting lines (distinct by Ax. 2.3). Where P is an arbitrary point of ℓ_1, let ℓ_3 be the line through P parallel to ℓ_2. If $\ell_1 = \ell_3$, then it is trivial that ℓ_1 and ℓ_3 have the same number of points. Otherwise, the same result follows from the preceding corollary.

However our comments after Ax. 2.6 imply that ℓ_2 and ℓ_3 have the same number of points. Thus, the same is true of ℓ_1 and ℓ_2.

By Ax. 2.2, 9 it follows that at least one line, say m, exists. Now every line either fails to intersect m, intersects m in only one point, or is identical with m. The above comments, along with Cor. 2.4.7, establish that all lines have the same number of points as m, and so, have the same number of points.

We note that Cor. 2.1.2 implies that all points lie on the same number of lines. The following result will also be useful.

Corollary 2.4.9 Let π be any plane and P any point of π. If every line contains exactly n points, then P lies on exactly n + 1 lines of π.

Proof Let ℓ be any line of π such that $P \notin \ell$. By Ax. 2.7, for any line m of π such that $P \in$ m, either $\ell \parallel$ m or ℓ intersects m in a unique point. If

Q and R are distinct points of ℓ, then $PQ \neq PR$, since $PQ = PR \Rightarrow P \in QR = \ell$, a contradiction. Thus there is a one-to-one mapping from the points of ℓ onto the lines of Π through P and not parallel to ℓ. As there is exactly one line of Π through P parallel to ℓ, the number of lines of Π through P is $n+1$.

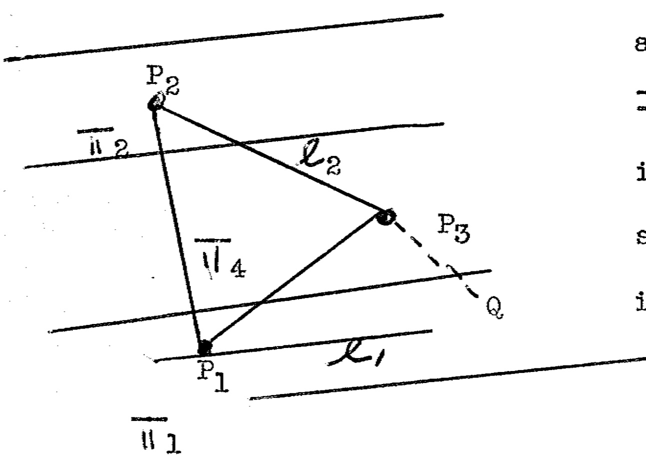
<u>Definition 2.3</u> Any two distinct planes having an empty intersection are said to be <u>parallel</u>. For the sake of completeness, we define any plane to be parallel to itself and any line ℓ to be parallel to any plane Π if $(\ell \cap \Pi)$ does not consist of a single point.

<u>Lemma 2.5.1</u> For any planes Π_i $(i = 1, 2, 3)$, if $\Pi_2 \cap \Pi_3 = \emptyset$ and if $\Pi_1 \nparallel \Pi_j$ $(j = 2,3)$, then $\Pi_1 \cap \Pi_2 \parallel \Pi_1 \cap \Pi_3$.

<u>Proof</u> As $\Pi_1 \nparallel \Pi_j$ $(j = 2,3)$, it follows by Cor. 2.4.3 that $\Pi_1 \cap \Pi_2$ and $\Pi_1 \cap \Pi_3$ are each lines. Since $\Pi_2 \cap \Pi_3 = \emptyset$, these lines do not intersect. However, they are coplanar (in Π_1). Thus Cor. 2.4.6 implies that they are parallel.

<u>Lemma 2.5.2</u> For any planes Π_i $(i = 1, 2)$ and lines ℓ_i $(i = 1, 2)$, if $\Pi_1 \cap \Pi_2 = \emptyset$, ℓ_1 is a line of Π_1, $\ell_1 \parallel \ell_2$, and $\ell_2 \cap \Pi_2 \neq \emptyset$, then ℓ_2 is a line of Π_2.

<u>Proof</u> Suppose that ℓ_2 is not a line of Π_2. Then $\ell_2 \cap \Pi_2$ contains exactly one point, say P_2. As $\Pi_2 \cap \Pi_1 = \emptyset$, Th. 2.3 implies that $\ell_2 \cap \Pi_1$ either is empty or contains a single point. We shall show that each of these alternatives is untenable.

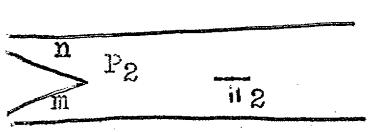


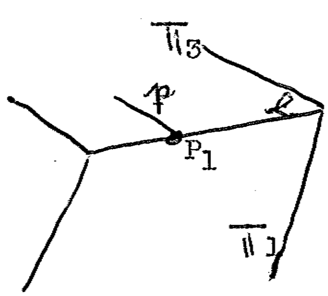
If $\ell_2 \cap \pi_1$ is not empty, suppose that it contains the single point Q. As $P_2 \in \pi_2$ and ℓ_1 is a line of π_1, Th. 2.1 and $\pi_1 \cap \pi_2 = \emptyset$ imply $P_2 \notin \ell_1$. Thus $\ell_1 \neq \ell_2$ and Th. 2.1 $\Rightarrow$ $Q \notin \ell_1$. By Cors. 2.4.2, 6, it follows that the plane containing ℓ_1 and ℓ_2 is determined by ℓ_1 and Q. By the preceding lemma, this plane intersects π_1 in ℓ_1, and π_2 in a line through P_2 and parallel to ℓ_1. Since Ax. 2.5 implies that this line is ℓ_2, it follows by Th. 2.3 that $Q \in (\pi_2 \cap \pi_1) = \emptyset$, a contradiction.

If $\ell_2 \cap \pi_1 = \emptyset$, let P_3 be any point of ℓ_2 other than P_2 and let P_1 be any point of ℓ_1. As P_i (i = 1, 2, 3) are non-collinear, Cor. 2.4.1 implies that they determine a unique plane, π_4. It is clear that $P_1 \in (\pi_4 \cap \pi_1) \parallel (\pi_4 \cap \pi_2)$. Now ℓ_2 and $(\pi_4 \cap \pi_1)$ are coplanar (in π_4), but do not intersect, as $\ell_2 \cap \pi_1 = \emptyset$ by assumption. Hence Cor. 2.4.6 $\Rightarrow$ $\ell_2 \parallel (\pi_4 \cap \pi_1)$ and Cor. 2.1.1 $\Rightarrow$ $\pi_4 \cap \pi_1 = \ell_1$. However, $P_2 \in (\pi_4 \cap \pi_2) \parallel (\pi_4 \cap \pi_1) = \ell_1$ implies that $\pi_4 \cap \pi_2 = \ell_2$, again by Cor. 2.1.1. This contradicts the assumption that ℓ_2 is not a line of π_2, and so the lemma is established.

Theorem 2.5 Parallelism of planes is an equivalence relation.

Proof The only non-trivial part of this theorem concerns the transitivity of $\parallel$. Let π_i (i = 1, 2, 3) be any planes such that $\pi_2 \cap \pi_j = \emptyset$ (j = 1, 3). If $\pi_1 \nparallel \pi_3$, then $\pi_1 \cap \pi_3$ is a line ℓ. Let P_2 be any point of π_2 and m the line through P_2 parallel to ℓ. Let n be any line of π_2 through P_2 such that $n \neq m$. (We have not claimed that m is a line of π_2. Although Lemma 2.5.2 implies that it is, the existence of n follows simply from the fact that every plane contains at least three non-collinear points.)



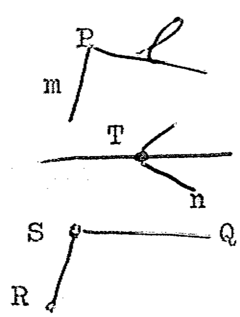


Let P_1 be any point of ℓ and p, the line through P_1 and parallel to n. As $p \parallel n \neq m \parallel \ell$, Ax. 2.4 and Th. 2.1 imply $p \nparallel \ell$ and so $p \neq \ell$. However, the immediately preceding lemma implies that p is a line of both π_3 and π_1. Since $p \neq \ell$ and ℓ is a line of both π_3 and π_1, we may infer by Th. 2.4 that $\pi_1 = \pi\ell, p = \pi_3$, a contradiction to the assumption $\pi_1 \nparallel \pi_3$; thus $\pi_1 \parallel \pi_3$ and the proof is complete. With regard to the conditions of Lemma 2.5.1, we note that Th. 2.5, $\pi_2 \cap \pi_3 = \emptyset$, and $\pi_1 \nparallel \pi_2 \Rightarrow \pi_1 \nparallel \pi_3$. In other words, we may infer that if any plane π_1 intersects another plane π_2 in a line ℓ_1, then π_1 intersects any plane parallel to π_2 in a line ℓ_2 such that $\ell_1 \parallel \ell_2$.

<u>Corollary 2.5.1</u> Given any plane π and point P, there exists exactly one plane π^* containing P and parallel to π.

<u>Proof</u> (i) If $P \in \pi$, then π is clearly a satisfactory π^*. Moreover, no other plane π^{**} could suffice, as Th. 2.5 and Def. 2.3 imply $\pi^* = \pi^{**}$.

(ii) Suppose $P \notin \pi$, where π may be defined as $\pi\, RS, SQ$. By Ax. 2.5, there exist lines ℓ and m such that $P \in (\ell \cap m)$, $\ell \parallel SQ$, and $m \parallel RS$. The non-collinearity of Q, R, and S, along with Cor. 2.1.1, implies $\ell \neq m$. We now proceed to show that $\pi\ell, m$ (which exists by virtue of the foregoing remarks) is a satisfactory π^*.



If $\pi\ell, m \nparallel \pi$, then $\pi\ell, m$ and π intersect in a line, say s. As s cannot be parallel to both ℓ and m since $\ell \neq m$, we may suppose without loss of generality that $s \nparallel \ell$. Where T is any point of s, the line n through T parallel to ℓ is distinct from s and lies in both π and $\pi\ell, m$ by Th. 2.2 (ii), Ax. 2.4 and Th. 2.4.

It follows by Th. 2.4 that $\overline{\pi} = \overline{\pi}\, s, n = \overline{\pi}\, \ell, m$, a contradiction to the assumption $\overline{\pi}\, \ell, m \nparallel \overline{\pi}$. Thus $\overline{\pi}\, \ell, m$ is a satisfactory $\overline{\pi}^{x}$.

Furthermore no other $\overline{\pi}^{xx}$ suffices by Th. 2.5 and the argument of part (i) of this proof.

<u>Corollary 2.5.2</u> There exist at least twenty-eight distinct lines.

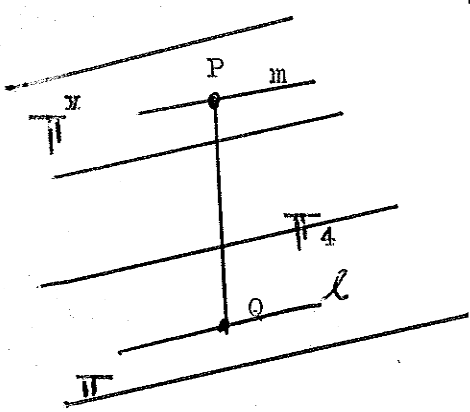
<u>Proof</u> We continue the argument of Cor. 2.4.5. There are clearly at most twenty-eight distinct lines determined by taking combinations of P_i (i = 1-8) two at a time as repetitions may occur. We now proceed to show that repetitions do not occur.

The argument of the preceding corollary proves that the plane containing P_i (i = 1, 2, 3, 5) is parallel to, but distinct from, the plane containing P_j (j = 4, 6, 7, 8). It follows that every line of the form P_iP_j (i, j = 1, 2, 3, 5; $i \neq j$) is distinct from every line of the form P_iP_j (i, j = 4, 6, 7, 8; $i \neq j$). Furthermore, no three of P_i (i = 1, 2, 3, 5 or 4,6,7,8) are collinear, by the considerations of Cor. 2.4.5.

It only remains to prove that all lines of the form P_iP_j (i = 1, 2, 3, 5; j = 4, 6, 7, 8) are distinct from each other and from each of the previously enumerated lines. This follows since any such P_iP_j intersects the planes of the P_i and the P_j in one point each, by virtue of Th. 2.3.

<u>Corollary 2.5.3</u> Let P and Q be any points and $\overline{\pi}$ any plane such that $P \notin \overline{\pi}$, $Q \in \overline{\pi}$. Then the plane through P parallel to $\overline{\pi}$ contains all the lines through P parallel to lines of $\overline{\pi}$ through Q.

<u>Proof:</u> By Cor. 2.5.1, the plane through P parallel to $\overline{\pi}$ exists and is unique. Denote this plane by $\overline{\pi}^{x}$. Let ℓ be any line of $\overline{\pi}$ such that $Q \in \ell$.

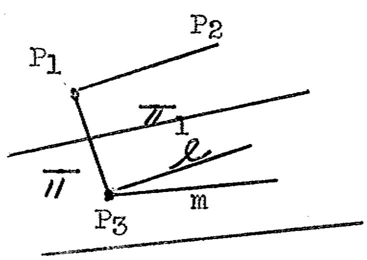


Further let m be the line through P parallel to ℓ. We proceed to show that m is a line of $\overline{\pi}^{x}$. By Cor. 2.4.6, ℓ and m determine a plane, which by Cor. 2.4.1 is unique. If this plane is denoted by $\overline{\pi}_4$,

$(\pi^* \cap \pi_4) \parallel (\pi_4 \cap \pi) = \ell$, by Lemma 2.5.1. As $P \in (\pi^* \cap \pi_4)$, Cor. 2.1.1 implies that $\pi^* \cap \pi_4 = m$ and so m is a line of π^*.

<u>Corollary 2.5.4</u> For any points P_i $(i = 1, 2, 3)$ andplane π, $P_3 \in \pi$, $P_1P_2 \cap \pi = \emptyset \Rightarrow P_1P_2$ is a line of the plane through P_1 and parallel to π.

<u>Proof</u> Let ℓ be the line through P_3 parallel to P_1P_2. By virtue of the preceding corollary, it suffices to show that ℓ is a line of π. By Cor. 2.4.6, ℓ and P_1P_2 determine a plane, which by Cor. 2.4.1, is unique. If this plane is denoted by π_1, then $\pi_1 \cap \pi$ is a line m through P_3. As $m \cap P_1P_2 = \emptyset$ and m and P_1P_2 are lines of π_1, Cor. 2.4.6 $\Rightarrow m \parallel P_1P_2$, whence by Ax. 2.5, $\ell = m$. Since $\pi_1 \cap \pi = \ell$, ℓ is a line of π and the proof is complete.



<u>Corollary 2.5.5</u> For any line ℓ and plane π, ℓ if parallel to π if and only if there exists a line m of π such that $\ell \parallel m$.

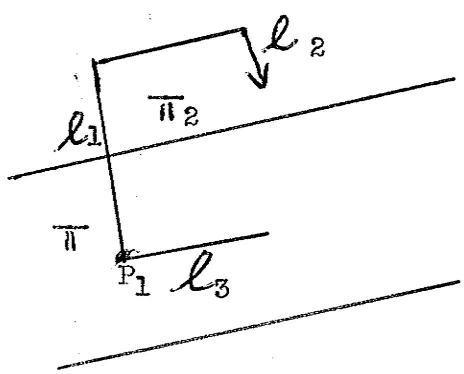
<u>Proof</u> It is clear that the only non-trivial case is ℓ not a line of π.

Now, if ℓ is parallel to π, $\ell \cap \pi = \emptyset$ and so the preceding result implies that ℓ is a line of a plane parallel to, but distinct from, π. Hence, Cor. 2.5.3 $\Rightarrow$ through any point of π, there exists a line m of π such that $\ell \parallel m$. Thus there exists at least one satisfactory m. (Indeed, there are obviously many more).

Conversely, if there exists a line m of π such that $\ell \parallel m$, then Cor. 2.5.3 $\Rightarrow \ell$ lies in a plane parallel to, but distinct from, π. Thus $\ell \cap \pi = \emptyset$, whence ℓ is parallel to π.

<u>Corollary 2.5.6</u> For any distinct lines ℓ_1 and ℓ_2 and plane π, if $\ell_1 \parallel \ell_2$ and $\ell_1 \cap \pi$ contains exactly one point, then $\ell_2 \cap \pi$ contains exactly one point.

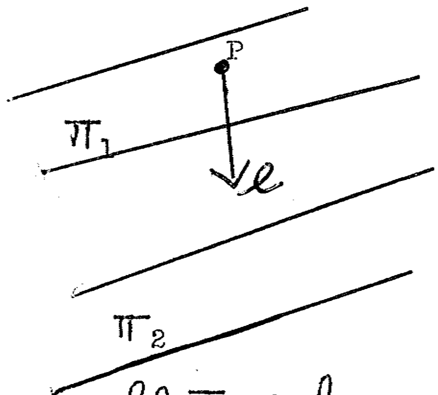
<u>Proof</u> As $\ell_1 \cap \pi$ contains a single point, say P_1, then π_2, the plane of ℓ_1 and ℓ_2, intersects π in a line ℓ_3 such that $P_1 \in \ell_3$.



Now if $\pi \cap \ell_2 = \emptyset$, then ℓ_2 and ℓ_3 are coplanar (in π_2) but do not intersect. Hence, by Cor. 2.4.6, $\ell_2 \parallel \ell_3$. As $\ell_1 \parallel \ell_2$ by assumption, Ax. 2.4 $\Rightarrow$ $\ell_1 \parallel \ell_3$. Since $P_1 \in (\ell_1 \cap \ell_3)$, Th. 2.1 $\Rightarrow \ell_1 = \ell_3$. This, however contradicts the assumption that $\ell_1 \cap \pi = \{P_1\}$, because of Ax. 2.3. Thus $\ell_2 \cap \pi \neq \emptyset$. On the other hand, $\ell_2 \cap \pi$ cannot contain more than one point. The plane determined by ℓ_1 and ℓ_2 is the plane determined by ℓ_2 and P_1 since $\ell_1 \cap \ell_2 = \emptyset \Rightarrow P_1 \notin \ell_2$, by Th. 2.1. If $\ell_2 \cap \pi = \ell_2$ then $\pi_2 = \pi$, a contradiction to the assumption that $\pi \cap \pi_2 = \ell_3 \neq \pi$.

<u>Corollary 2.5.7</u> If π_i $(i = 1, 2)$ are any planes and ℓ any line such that $\pi_1 \cap \pi_2 = \emptyset$ and $\ell \cap \pi_1$ contains exactly one point P, then $\ell \cap \pi_2$ contains exactly one point.

<u>Proof</u> If $\ell \cap \pi_2 = \emptyset$, then Cor. 2.5.4 implies that ℓ lies in a plane parallel to π_2. It follows from Cor. 2.5.1 that ℓ lies in π_1, as π_1 is the unique plane through P parallel to π_2. This however contradicts Def. 2.2 and the assumption about $\ell \cap \pi_1$. Consequently, $\ell \cap \pi_2 \neq \emptyset$.



If $\ell \cap \pi_2 = \ell$, it follows that $\ell \cap \pi_1 = \emptyset$ (since $\pi_2 \cap \pi_1 = \emptyset$), a contradiction. Hence $\ell \cap \pi_2$ contains exactly one point, as it is clear from Th. 2.3 that all other possibilities lead to contradiction.

We note that the last two corollaries show that if a plane is pierced by one of two parallel lines then it is pierced by the other and that if a line pierces one of two parallel planes, thenit pierces the other. These facts are important in those special cases where one wishes to introduce the notion of the normal to a plane.

<u>Corollary 2.5.8</u> If π is any plane and L is any equivalence class of parallel lines, then π and L have the same number of elements.

<u>Proof</u> (a) Let ℓ be any element of L and suppose there exists a point P such that $\ell \cap \pi = \langle P \rangle$. Then by Cor. 2.5.6, for any line m, if $m \in L$, then $m \cap \pi$ contains exactly one point. It is clear from Th. 2.1 that ℓ, $m \in L$, $\ell \neq m \Rightarrow \ell \cap \pi \neq m \cap \pi$. Thus we have displayed a one-to-one mapping from L into π. This mapping is clearly onto π as well; that is, Ax. 2.5 $\Rightarrow$ for any point Q of π, there exists $q \in L$ such that $Q \in q$.

(b) Let π_1 and π_2 be any distinct planes. (Such planes exist by Cor. 2.5.1.) If P_1 and P_2 are any points such that $P_i \in \pi_j \Leftrightarrow i = j$, then the line P_1P_2 clearly pierces both π_1 and π_2, by virtue of Th. 2.3.

By case (a), it follows that Π_1 and Π_2 contain the same number of points.

(c) Let ℓ be any element of L. By the above, we may assume without loss of generality that $\ell \cap \Pi$ does not contain a unique element. There are now two possibilities. Either $\ell \cap \Pi = \ell$ or $\ell \cap \Pi = \emptyset$. If $\ell \cap \Pi = \emptyset$, then by Cor. 2.5.4, ℓ lies in a plane Π_1 such that $\Pi_1 \parallel \Pi$. Since case (b) $\Rightarrow \Pi_1$ and Π have the same number of points, it clearly suffices to suppose that $\ell \cap \Pi = \ell$.

If P is any point of ℓ, R is any point of Π not in ℓ and S is any point $\notin \Pi$ then Cor. 2.5.1 $\Rightarrow$ P, R, and S determine a unique plane Π^*, which, since $S \notin \ell$ and $\ell \in \Pi$, is not identical with Π. As case (b) shows that Π^* and Π have the same number of elements, we need only prove that the same is true of L and Π^*. By case (a), however, it is enough to establish that ℓ pierces Π^*. Finally, we know ℓ pierces Π^* since $P \in (\ell \cap \Pi^*)$ and since $\ell \subseteq (\ell \cap \Pi^*) \Rightarrow \ell \subseteq \Pi^* \Rightarrow \Pi = \Pi^*$ (by Cor. 2.4.2 as applied to ℓ and $R \notin \ell$), a contradiction. Thus the proof is complete.

We note that Cor. 2.5.8 proves that (1) all planes have the same number of points and (2) all equivalence classes of parallel lines have the same number of lines, by allowing Π and then L to vary. The next result provides two ways of determining this number.

For the remainder of the corollaries of Th. 2.5, let n denote the number of points on any line.

Corollary 2.5.9 Every plane contains exactly n^2 points.

Proofs (1) Let Π be defined as $\Pi_{\ell, m}$. Then Def. 2.1 and Cor. 2.1.1 imply that for every point P of Π, there exists a unique line (of Π) through P parallel to ℓ. As there are n such lines, each containing n points, Π contains exactly n^2 points.

(2) By Ax. 2.6, if $\ell \cap m = \langle P \rangle$ and Q, R any points such that $P \in \ell$, $Q \in m$, and $Q \neq P \neq R$, then P, Q, and R determine a unique point T of Π. Now

Th. 2.1 implies that P, Q, and R determine the same point as P, Q^1, and R^1 if and only if $Q = Q^1$ and $R = R^1$. Thus, excluding the points of ℓ and m, π contains $(n - 1)^2$ points. Since ℓ and m have P in common, the number of points in π is $(n - 1)^2 + 2n - 1 = n^2$, as above.

We have immediately

<u>Corollary 2.5.10</u> Every equivalence class of parallel lines contains exactly n^2 lines.

<u>Corollary 2.5.11</u> There exist exactly $n^2 + n + 1$ lines through any point.

<u>Proof</u> Let P be any point and π any plane such that $P \in \pi$. Now there are $n + 1$ lines of π through P, by Cor. 2.4.9. By Def. 2.2, Cors. 2.5.1, 4, and Th. 2.3, if π^* is any plane other than π such that $\pi^* \parallel \pi$, then any line ℓ through P not in π intersects π^* in a single point Q. Since $\pi \cap \pi^* = \emptyset$, there is a one-to-one mapping from the lines through P not in π onto the points of π^*. Thus the number of such lines is n^2, by Cor. 2.5.9. Consequently, the number of lines through P is $n^2 + n + 1$, as claimed.

<u>Corollary 2.5.12</u> Every equivalence class of parallel planes contains exactly n planes.

<u>Proof</u> Let ℓ be any line and π any plane such that ℓ pierces π. Then, through distinct points of ℓ, there are distinct planes parallel to π. Furthermore, ℓ pierces every plane parallel to π (otherwise, if $\ell \cap \pi^* = \emptyset$ where $\pi^* \parallel \pi$, then Cor. 2.5.4 implies ℓ is a line of π, a contradiction), whence there is a one-to-one function from the points of ℓ onto the set of planes parallel to π. Thus, the result follows.

Since Corollary 2.5.1 states that every point of $\mathcal{X}$ lies on exactly one plane parallel to a fixed plane, by considering plane π^* parallel to a fixed plane π, each π^* through a different point of any fixed line ℓ piercing

$\overline{\pi}$, we may prove.

Corollary 2.5.13 $\mathcal{P}$ contains exactly n^3 points.

Corollary 2.5.14 For any plane $\overline{\pi}$, $\overline{\pi}$ contains exactly $n^2 + n$ lines.

Proof If $\overline{\pi}$ is defined as $\overline{\pi}_{\ell, m}$, then there exist n lines of $\overline{\pi}$ other than ℓ through every point of ℓ and by Axs. 2.1, 2, these are all distinct. Furthermore, by Ax. 2.7 and Cor. 2.1.1, there is a one-to-one mapping from the points of m onto the lines of $\overline{\pi}$ that are parallel to ℓ. Thus there are n such lines and $\overline{\pi}$ contains $n(n) + n = n^2 + n$ lines in all.

Corollary 2.5.12 There exist exactly $n^4 + n^3 + n^2$ lines in $\mathcal{L}$.

Proof Let $\overline{\pi}$ be any plane. Then the number of lines lying in some plane parallel to $\overline{\pi}$ is $n(n^2 + n)$. Any other line pierces $\overline{\pi}$ (by Cor. 2.5.4) and so is determined by one point in $\overline{\pi}$ and one other point in some fixed plane $\overline{\pi}^* \parallel \overline{\pi}$, $\overline{\pi}^* \cap \overline{\pi} = \emptyset$. As each plane contains n^2 points, there are $(n^2)(n^2)$ such lines and so $(n^2)(n^2) + n(n^2 + n) = n^4 + n^3 + n^2$ lines in all.

From Cors. 2.5.10, 15, we have immediately

Corollary 2.5.16. There exist exactly $\dfrac{n^4 + n^3 + n^2}{n^2} = n^2 + n + 1$ equivalence classes of parallel lines.

Corollary 2.5.17 For any line ℓ, there exist exactly $n + 1$ planes containing ℓ.

Proof By Cor. 2.4.2, any plane containing ℓ is determined by a point not on ℓ. Since any such plane contains $n^2 - n$ points not on ℓ, by considering the total number of ways of associating ℓ with a point not on ℓ, we shall determine any plane through ℓ exactly $n^2 - n$ times. Since there exist exactly $n^3 - n$ points of $\mathcal{P}$ not on ℓ, the number of planes through ℓ is

$$\frac{n^3 - n}{n^2 - n} = n + 1.$$

Corollary 2.5.18 For any point P, there exist exactly $n^2 + n + 1$ planes through P.

Proof Let π be any plane containing P. Corresponding to each of the $n + 1$ lines ℓ of π through P, there exist n planes (other than π) containing ℓ. By Cors. 2.4.3, 9, there exist $(n+1)\ n + 1 = n^2 + n + 1$ planes through P.

Corollary 2.5.19 There exist exactly $n^3 + n^2 + n$ planes.

Proof Let π be any plane. The number of planes (other than π) which intersect π is the product of the number of planes other than π on a line ℓ of π and the number of lines in π; that is $n\ (n^2 + n) = n^3 + n^2$. As Cor. 2.5.12 $\Rightarrow$ there exist exactly n planes parallel to n, it follows that there exist exactly $n^3 + n^2 + n$ planes in $\mathcal{Q}$.

From Cor. 2.5.12, 19, we have immediately

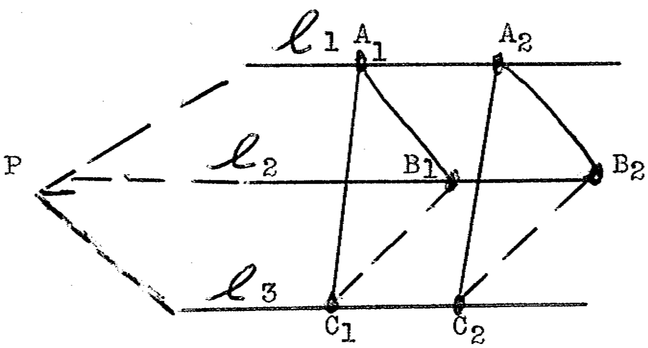
Cor. 2.5.20 There exist exactly $n^2 + n + 1$ equivalence classes of parallel planes.

Remark Let us return once again to Corollary 2.4.5. By Corollary 2.5.1, the planes determined by P_i $(i = 1, 3, 4)$ and P_j $(j = 2,5,7)$ are parallel and distinct. Thus $P_4P_6 \cap P_7P_8 = \emptyset$, by virtue of Th. 2.3. However Th. 2.2 (ii) implies that P_k $(k = 4, 6, 7, 8)$ are coplanar. Thus, Cor. 2.4.6 allows us to infer that $P_4P_6 \parallel P_7P_8$. In other words, P_ℓ $(\ell = 4, 6, 7)$ and P_m $(m = 3,5,6)$ both lead to the same point, P_8, on completion of the respective parallelograms.

By essentially the same argument, one may show that $P_4P_3 \parallel P_7P_5$. This observation leads us to consider the following important results originally proved by Desarques.

Theorem 2.6A Let ℓ_i $(i = 1, 2, 3)$ be any distinct lines such that there exists a point $P \in (\ell_1 \cap \ell_2 \cap \ell_3)$. Further, let A_i, B_i, C_i $(i = 1, 2)$ be any points other than P such that $A_1, A_2 \in \ell_1$, $B_1, B_2 \in \ell_2$, and $C_1, C_2 \in \ell_3$. Then if $A_1C_1 \parallel A_2C_2$ and $A_1B_1 \parallel A_2B_2$, it follows that $B_1C_1 \parallel B_2C_2$.

Proof (a) If $A_1 = A_2$, then $A_1B_1 \parallel A_2B_2 \Rightarrow A_1B_1 \parallel A_1B_2$. By Th. 2.1, it follows that $A_1B_1 = A_1B_2$. Denoting the common value of A_1B_1 and A_1B_2 by "m", we see that $m \not\parallel \ell_2$, for $m \parallel \ell_2 \Rightarrow m = \ell_2$ (by Th. 2.1) $\Rightarrow A_1 \in \ell_2$ (by definition of m) $\Rightarrow \ell_1 = \ell_2$ (by Ax. 2.1, as $A_1 \neq P$), which contradicts the assumption that ℓ_i (i = 1, 2, 3) are distinct.

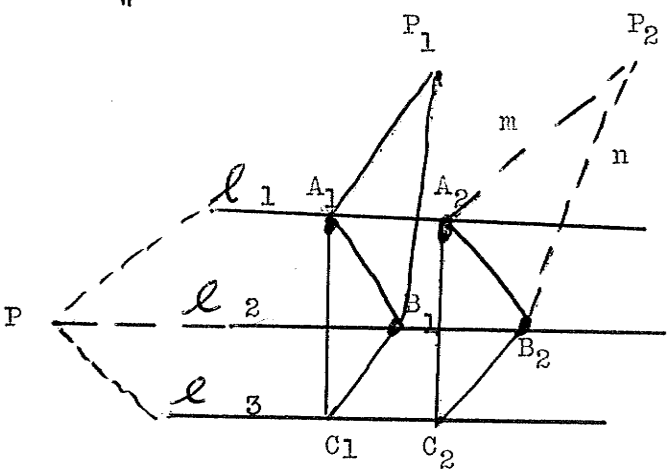


Since $m \not\parallel \ell_2$, we have that $m \neq \ell_2$, whence Ax. 2.2 implies that $m \cap \ell_2$ contains at most one point. Consequently, $B_1 = B_2$. In general, it is clear that $A_1 = A_2 \Leftrightarrow B_1 = B_2 \Leftrightarrow C_1 = C_2$. Then, $A_1 = A_2 \Rightarrow B_1C_1 = B_2C_2$ and so the required result follows by Ax. 2.4. Henceforth, we may assume $A_1 \neq A_2$, $B_1 \neq B_2$, and $C_1 \neq C_2$.

If A_1, B_1, and C_1 are collinear, then $A_1B_1 = A_1C_1 \Rightarrow A_2B_2 = A_2C_2$, by Ax. 2.5. As a result, A_2, B_2 and C_2 are also collinear. Similarly, it is clear that the collinearity of A_2, B_2, and C_2 implies the collinearity of A_1, B_1, and C_1. Now, if A_1, B_1, and C_1 are collinear, $B_1C_1 = A_1C_1 \parallel A_2C_2 = B_2C_2$, proving the theorem in this case. Hencefore, we may assume that A_i, B_i, and C_i (i = 1, 2) are each non-collinear.

It is clear from Cor. 2.4.1 and Cor. 2.5.1 that the planes $\overline{\pi}_{B_1A_1, C_1A_1}$ and $\overline{\pi}_{B_2A_2, C_2A_2}$ are parallel. If their intersection is the empty set, then $B_1C_1 \cap B_2C_2 = \emptyset$. However, B_iC_i (i = 1, 2) are coplanar (in $\overline{\pi}_{\ell_2, \ell_3}$) by Th. 2.3 and Def. 2.2. Then Cor. 2.4.6 implies that $B_1C_1 \parallel B_2C_2$, the required result. Henceforth we may assume $\overline{\pi}_{B_1A_1, C_1A_1} = \overline{\pi}_{B_2A_2, C_2A_2}$, which common value we denote by "$\overline{\pi}$".

(b) Let P_1 be any point $\notin \Pi$. (P_1 exists by Ax. 2.9.) By Cor. 2.1.1, there exist unique lines m and n such that $A_2 \in m \parallel A_1P_1$ and $B_2 \in n \parallel B_1P_1$. Now, Cor. 2.5.1 implies that the planes determined by m, A_2B_2 and n, A_2B_2 respectively are each parallel to $\Pi\ P_1A_1, B_1A_1$, whence $\Pi\ m, A_2B_2 = \Pi\ n, A_2B_2$, which common value we denote by "Π_1". Now m and n are not parallel, for $m \parallel n \Rightarrow A_1P_1 \parallel B_1P_1 \Rightarrow A_1P_1 = B_1P_1$ (by Th. 2.1) $\Rightarrow P_1 \in \Pi$ (as $A_1 \neq B_1$) by Th. 2.3, a contradiction. Since m and n are lines of Π_1 such that $m \nparallel n$, Ax. 2.7 implies that there exists a unique point $P_2 \in (m \cap n)$. It is clear from Cor. 2.5.6 that $P_2 \notin \Pi$.



Now, Th. 2.3 and Cor. 2.4.1 imply that the plane $\Pi\ A_iP_i, A_iC_i$ ($i = 1, 2$) exist and Cor. 2.5.1 implies that they are parallel. Furthermore, they are distinct, for the assumption that they are identical, along with Cor. 2.4.1, implies $P_1 \in \Pi$, a contradiction.

Consider the line P_1P. If $P_1P \parallel m$, then Ax. 2.5 implies $P_1P \parallel P_1A_1$ and Th. 2.1 $\Rightarrow P_1P = P_1A_1$. However, as $P_1 \notin \Pi$, we know from Th. 2.3 that $(P_1P \cap \Pi)$ contains at most one point. Thus, $P_1P = P_1A_1 \Rightarrow P = A_1$, a contradiction to our assumptions. As a result, $P_1P \nparallel m$. Since ℓ_1 and P_1 determine a unique plane $\Pi\ \ell_1, P_1A_1$ (by Cor. 2.4.2), it follows from Th. 2.2 (ii) that P_1P and m are coplanar. Consequently, as $P_1P \nparallel m$, it follows from Ax. 2.7 that there exists a unique point $Q \in (PP_1 \cap m)$. We now proceed to show $Q = P_2$.

It is clear that $Q \neq A_2$, for if $Q = A_2$, then Th. 2.3 implies that $P_1 \in \pi$, a contradiction. By a previous case, since $P \in (P_1 Q \cap A_1A_2 \cap B_1B_2)$, $A_1B_1 \parallel A_2B_2$, $A_1P_1 \parallel A_2 Q = m$, and planes $\pi\ A_1 P_1, P_1B_1$ and $\pi\ m, n$ are distinct, it follows that $P_1 B_1 \parallel Q B_2$. Now Ax. 2.4 $\Rightarrow Q B_2 \parallel n$, whence by Cor. 2.1.1, $QB_2 = n$. Since $P_2, Q \in m, n$ and $m \neq n$ (for $m \nparallel n \Rightarrow m \neq n$), we have from Ax. 2.1 that $Q = P_2$.

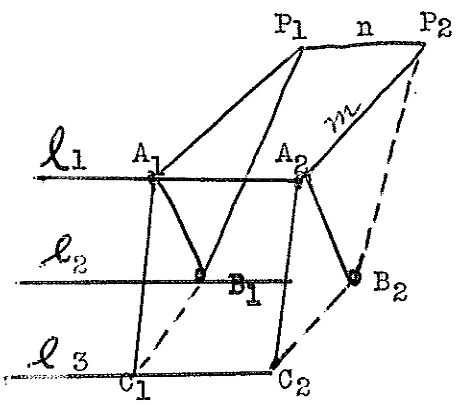
We have now shown that $P \in (P_1P_2 \cap A_1A_2 \cap C_1C_2)$, $A_1P_1 \parallel A_2P_2$, $A_1C_1 \parallel A_2C_2$, and π_1 is parallel to, but distinct from $\pi\ A_1C_1, A_1P_1$. By a previous case, it follows that $C_1P_1 \parallel C_2P_2$. Finally, it is clear that $\pi\ P_1B_1, B_1C_1$ is parallel to, but distinct from, $\pi\ B_2C_2, n$. Since $P \in (P_1 P_2 \cap B_1 B_2 \cap C_1C_2)$, $P_1B_1 \parallel P_2B_2$ and $C_1P_1 \parallel C_2P_2$, we get $B_1C_1 \parallel B_2C_2$: Q.E.D.

<u>Theorem 2.6B</u> Let ℓ_i $(i = 1, 2, 3)$ be any distinct lines such that $\ell_1 \parallel \ell_2 \parallel \ell_3$. Further, let A_i, B_i, C_i $(i = 1, 2)$ be any points such that $A_1, A_2, \in \ell_1$, $B_1, B_2 \in \ell_2$, and $C_1, C_2 \in \ell_3$. Then if $A_1C_1 \parallel A_2C_2$ and $A_1B_1 \parallel A_2B_2$, it follows that $B_1C_1 \parallel B_2C_2$.

<u>Proof</u> (a) The argument of part (a) of the preceding theorem holds almost entirely in this case as well. The only exceptions are the proofs that $A_1 = A_2 \Rightarrow B_1 = B_2$ and that B_iC_i $(i = 1, 2)$ are coplanar. In the case now under consideration, $A_1 = A_2 \Rightarrow A_1B_1 \parallel A_1B_2 \Rightarrow A_1B_1 = A_1B_2$ (by Th. 2.1) and since $A_1 \notin \ell_2$, $A_1B_1 \neq \ell_2 \Rightarrow B_1 = B_2$ since $B_1, B_2 \in A_1B_1, \ell_2$. Furthermore, B_iC_i $(i = 1, 2)$ are coplanar in virtue of Cor. 2.4.6. We may therefore assume henceforth that $A_1 \neq A_2$, $B_1 \neq B_2$, $C_1 \neq C_2$, A_i, B_i and C_i $(i = 1, 2)$ are non-collinear, and $\pi\ A_1B_1, A_1C_1 = \pi\ A_2B_2, A_2C_2$, which common value we denote by "π".

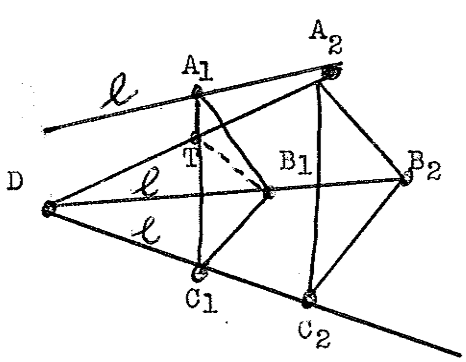
(b) Let P_1 be any point $\notin \pi$. (P_1 exists by Ax. 2.9.) By Cor.

2.1.1, there exist unique lines m and n such that $A_2 \in m \parallel A_1P_1$ and $P_1 \in n \parallel \ell_1$. It follows from Ax. 2.6 that there exists exactly one point $P_2 \in (m \cap n)$. Now $\overline{\pi}$ A_1P_1, A_1B_1 and $\overline{\pi}$ A_2B_2, m are parallel by virtue of Cor. 2.5.1 and distinct since any assumption to the contrary implies by Cor. 2.4.1 that $P_1 \in \overline{\pi}$, a contradiction. Thus the case considered in part (a) allows us to conclude that $P_1B_1 \parallel P_2B_2$. A similar argument shows $\overline{\pi}$ A_1P_1, A_1C_1 is parallel to, but distinct from, $\overline{\pi}$ A_2P_2, A_2C_2, whence $P_1C_1 \parallel P_2C_2$. Now the planes $\overline{\pi}$ B_1C_1, B_1P_1 and $\overline{\pi}$ B_2C_2, B_2P_2 are parallel and distinct, again by a similar argument, whence $B_1C_1 \parallel B_2C_2$: Q.E.D.



Theorem 2.6C Let ℓ_i $(i = 1, 2, 3)$ be any distinct lines and A_i, B_i, C_i $(i = 1, 2)$ be distinct points such that $A_1, A_2 \in \ell_1$, $B_1, B_2 \in \ell_2$ and $C_1, C_2 \in \ell_3$. If $A_1B_1 \parallel A_2B_2$, $A_1C_1 \parallel A_2C_2$, $B_1C_1 \parallel B_2C_2$, and there exists a unique point $D \in (B_1B_2 \cap C_1C_2)$, then if $A_1 \neq A_2$ and $C_1 \notin A_1B_1$, it follows that $D \in A_1A_2$.

Proof It is clear we may assume $D \neq A_2$, for $D = A_2 \Rightarrow D \in A_1A_2$. Furthermore, as $C_1 \neq C_2$, either $D \neq C_1$ or $D \neq C_2$. For definiteness, we may suppose $D \neq C_1$. Furthermore, $A_2 \notin DC_1$ for $A_2 \in DC_1 = C_1C_2 \Rightarrow A_1C_1 = A_2C_2 = \ell_3$ (by Th. 2.1) $\Rightarrow \ell_3 = A_1A_2 = \ell_1$, a contradiction. By Cor. 2.4.1 and Th. 2.3, the points D, A_2, C_1 and C_2 de-



termine a unique plane π such that DA_2 is a line of π.

Now, $DA_2 \nparallel A_1C_1$, for $DA_2 \parallel A_1C_1 \Rightarrow DA_2 \parallel A_2C_2$ (by Ax. 2.4) $\Rightarrow DA_2 = A_2C_2$ (by Th. 2.1). If $D = C_2$, then $D \neq B_2$ as the hypotheses state $C_2 \neq B_2$. The line through C_1 parallel to B_2C_2 is the line through C_1 parallel to B_2D, that is parallel to ℓ_2. By Th. 2.1, the line through C_1 parallel to ℓ_2 is ℓ_2. This implies that B_1, B_2, and C_1 are collinear. Since $B_1 \neq B_2$ it follows that B_1, B_2, D, C_1, and C_2 are collinear whence by Ax. 2.1, $\ell_2 = \ell_3$, a contradiction. Thus $D \neq C_2$.

Since $DA_2 = A_2C_2$, it follows that $A_2 \in \ell_3 = DC_2$. As A_1 lies on the line through C_1 parallel to A_2C_2, it follows that $A_1 \in \ell_3$. Since $A_1 \neq A_2$ by hypothesis, Ax. 2.1 $\Rightarrow \ell_2 = \ell_3$, a contradiction to the assumptions. Thus $DA_2 \nparallel A_1C_1$, although both these lines are coplanar (in π). It follows from Cor. 2.4.6 that there is exactly one point, say T, common to DA_2 and A_1C_1.

Since $D \in (DA_2 \cap \ell_2 \cap \ell_3)$, $B_1C_1 \parallel B_2C_2$ and $A_1C_1 \parallel A_2C_2$, it follows from Th. 2.6A that $B_1T \parallel B_2A_2$. As $A_1B_1 \parallel A_2B_2$ by hypothesis, Ax. 2.5 $\Rightarrow B_1T \parallel A_1B_1$ and Th. 2.1 $\Rightarrow B_1T = A_1B_1$. Consequently, $A_1, T \in A_1C_1, A_1B_1$. As we are assuming $C_1 \notin A_1B_1$, Ax. 21, 2 imply $A_1C_1 \neq A_1B_1$, whence Ax. 2.1 implies $A_1 = T$. As a result $A_1 = T \in DA_2$, whence $D \in A_1A_2$: Q.E.D.

<u>Theorem 2.6D</u> If ℓ_i $(i = 1, 2, 3)$ are distinct lines and A_i, B_i, and C_i $(i = 1, 2)$ points such that $A_1, A_2 \in \ell_1$, $B_1, B_2 \in \ell_2$, $C_1, C_2 \in \ell_3$, $A_1B_1 \parallel A_2B_2$, $A_1C_1 \parallel A_2C_2$, $B_1C_1 \parallel B_2C_2$, and $B_1B_2 \parallel C_1C_2$ then if $A_1 \neq A_2$, $A_1A_2 \parallel B_1B_2$.

<u>Proof</u>: In order to verify that A_i, B_i, C_i $(i = 1, 2)$ be distinct, it suffices to show that $A_1 \neq B_2$, $A_1 \neq C_2$, $A_2 \neq B_1$, and $A_2 \neq C_1$. As all these cases are essentially the same, we shall consider only the first one.

If $A_1 = B_2$, then $\ell_2 = B_1B_2 = B_1A_1 \parallel B_2A_2 \Rightarrow B_1B_2 = B_2A_2$. Consequently, A_i, B_j $(i, j = 1, 2)$ are all collinear, whence Ax. 2.1, 2 imply $\ell_1 = \ell_2$, a contradiction. By our above remarks, all of A_i, B_i, C_i $(i = 1, 2)$ are distinct.

As $A_1B_1 \parallel A_2B_2$, it follows from Th. 2.2 (ii) and Th. 2.3 that A_1A_2 and B_1B_2 are coplanar. It is clear that the theorem is proved if we show $A_1A_2 \cap B_1B_2 = \emptyset$. If there exists a point $P \in (A_1A_2 \cap B_1B_2)$, then Th. 2.6C implies $P \in (C_1C_2)$, whence $B_1B_2 \cap C_1C_2 \neq \emptyset$. Since $B_1B_2 \parallel C_1C_2$, it follows from Th. 2.1 that $B_1B_2 = C_1C_2$. However Ax. 2.1, 2 imply $B_1B_2 = \ell_2$ and $C_1C_2 = \ell_3$, whence $\ell_2 = \ell_3$, a contradiction to the hypotheses. Hence $A_1A_2 \cap B_1B_2 = \emptyset$: Q.E.D.

III Vectors: Definition and Elementary Properties

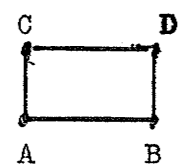
Crucial to the following discussion is the notion of an <u>ordered pair of points</u>. Where $\mathcal{Q} = \{\mathcal{P}, \mathcal{L}, \parallel, \in\}$, we define the set of ordered point pairs to be the Cartesian product $\mathcal{P} \times \mathcal{P}$. By definition, we shall say that the ordered point pair (P, Q) is equal to the ordered point pair (R, T) iff both P = R and Q = T. With regard to the ordered point pair (P, Q), we shall refer to P (resp. Q) as its <u>initial</u> (resp. <u>terminal</u>) <u>point</u>. It is clear that equality of ordered point pairs is an equivalence relation.

<u>Definition 3.1</u> We say that the ordered point pair, (A, B) is <u>related by a parallelogram</u> to the ordered point pair (C, D) and write "(A, B) $\square$ (C, D)" if and only if at least one of the following conditions obtains:

(1) A = B and C = D   A C / B D

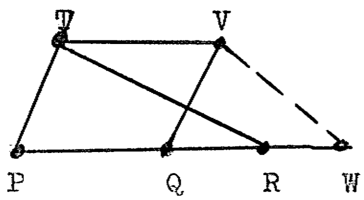
(2) A = C and B = D   A B / C D

(3) $AB \neq CD$, $AB \parallel CD$, and $AC \parallel BD$



Obviously the "parallelogram" in the above definition refers to case (3), which is, in a sense, the only non-trivial (non-degenerate) one. We remark that in case (3), all the points, A, B, C, and D are distinct, by virtue of our convention about expressions of the form "P_1P_2" and by an application of Th. 2.1.

We note in passing that $\square$ is both reflexive and symmetric. However, the transitivity of $\square$ is not obviously true in $\mathcal{Q}$. If the enumeration of points and lines in Cor. 2.5.2 does in fact exhaust $\mathcal{Q}$, the reader may verify that $\square$ is transitive. If, on the other hand, there exists a line ℓ containing distinct points P, Q, and R and if there exist distinct points T and V not on ℓ such that $(P, Q) \square (T, V)$, then by Cor. 2.4.1, Th. 2.2(ii), Ax. 2.5, and Ax. 2.7, there exists a point $W \in \ell$ such that $TR // VW$. According to Def. 3.1, this means that $(T, V) \square (R, W)$, although $Q \neq P \neq R \Rightarrow (P, Q) \not\square (R, W)$. By virtue of Ax. 2.9, we are guaranteed the existence of P, Q, and T, and consequently, that of V as well (by Ax. 2.6). Thus, it is clear that $\square$ is transitive (and hence an equivalence relation) if and only if every line contains exactly two points. Since we shall see that affine spaces with more than two points per line are of considerable interest, $\square$ is unsatisfactory as an equivalence relation on $\mathcal{P} \times \mathcal{P}$.



Definition 3.2 We say that the ordered point pair (A, B) is equipollent to the ordered point pair (C, D) and write "$(A,B) \uparrow (C,D)$" if and only if there exists an ordered point pair (E, F) such that both $(A, B) \square (E, F)$ and $(E, F) \square (C, D)$.

It is clear that the symmetry and reflexivity of $\square$ imply that $\uparrow$ also has these properties. We shall shortly demonstrate that, unlike $\square$, $\uparrow$ is transitive in all affine spaces, regardless of the number of points per line in the space. Indeed, Th. 3.2 will justify our introduction of $\uparrow$.

Proposition 3.1 $(A,B) \square (C,D) \Rightarrow (A,B) \uparrow (C,D)$.

Proof $(A,B) \square (A,B)$ by Def. 3.1, (2), whence by Def. 3.2, $(A,B) \square (C,D) \Rightarrow (A,B) \square (A,B)$ and $(A,B) \square (C,D) \Rightarrow (A,B) \uparrow (C,D)$: Q.E.D.

In anticipation of later developments, it is convenient to introduce the notation $\underline{0}$ for the set $\{ (P,P) \mid P \in \mathcal{P} \}$.

<u>Proposition 3.2</u> (i) $(A,B) \sqsupset (C,D)$ and (either $A = B$ or $C = D$) $\Rightarrow$ $(A,C) = (B,D)$.

(ii) $(A,B) \uparrow (C,D)$ and (either $A = B$ or $C = D$) $\Rightarrow$ $(A,C) = (B,D)$.

<u>Proof</u> (i) If $A = B$, then we may conclude $C = D$, by Def. 3.1, (1)(2). Similarly, $C = D \Rightarrow A = B$. By the definition of equality of ordered point pairs, $(A,C) = (B,D)$. Consequently, $X \in \underline{0} \Rightarrow (y \sqsupset x \Leftrightarrow y \in \underline{0})$.

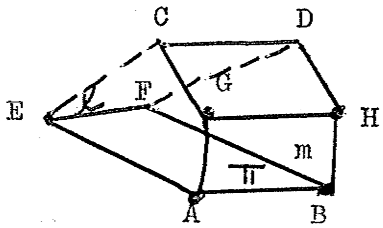
(ii) By Def. 3.2, there exists an ordered point pair (E,F) such that $(A,B) \sqsupset (E,F)$ and $(E,F) \sqsupset (C,D)$. If $A = B$, then $E = F$, by part (i). By repeating the argument, $C = D$. Hence $A = B \Rightarrow C = D$. Similarly $C = D \Rightarrow A = B$. As above, it follows that $(A,C) = (B,D)$. Consequently, $x \in \underline{0} \Rightarrow (y \uparrow x \Leftrightarrow y \in \underline{0})$.

<u>Theorem 3.1</u> For any points, A, B, C, and D such that $(A,B) \uparrow (C,D)$ and $AB \not\approx CD$ and any plane $\overline{\Pi}$ containing A, B, C, and D, there exists points E and F such that $E \not\in \overline{\Pi}$, $F \not\in \overline{\Pi}$, $(A,B) \sqsupset (E,F)$ and $(E,F) \sqsupset (C,D)$.

<u>Proof</u> By Def. 3.2, there exists an ordered point pair (G,H) such that $(A,B) \sqsupset (G,H)$ and $(G,H) \sqsupset (C,D)$. If $G \not\in \overline{\Pi}$, then since $A \neq B$ and $(A,B) \sqsupset (G,H)$, it follows from Def. 3.1 that $AB \parallel GH$. As $A, B \in \overline{\Pi}$, Th. 2.3 $\Rightarrow$ all points of AB lie in $\overline{\Pi}$, whence $G \notin \overline{\Pi} \Rightarrow G \notin AB \Rightarrow AB \neq GH$ (by Th. 2.1). Thus, Cors. 2.4.1, 6 imply that there is a unique plane $\overline{\Pi}^{*}$ containing AB and GH. Since $AB \not\approx GH$, Th. 2.1 $\Rightarrow AB \cap GH = \emptyset$, whence $H \not\in AB$. As A, B, and H uniquely determine $\overline{\Pi}^{*}$ by virtue of Cor. 2.4.1, $H \in \overline{\Pi} \Rightarrow \overline{\Pi} = \overline{\Pi}^{*} \Rightarrow G \in \overline{\Pi}$, a contradiction. Consequently, $H \not\in \overline{\Pi}$ and so if $G \not\in \overline{\Pi}$, one need only let $(E,F) = (G,H)$ in order to prove the theorem. As a result, we may henceforth assume without loss of generality that $G \in \overline{\Pi}$. Under this assumption, the above argument shows $H \in \overline{\Pi}$, on interchanging the roles of

G. and H.

By Axiom 2.10, there exists a point $E \notin \pi$. Consequently, $E \neq A \in \pi$, and the line EA exists (by Ax. 2.1, 2). By Cor. 2.1.1, there exist unique lines ℓ and m such that $E \in \ell$ // AB and $B \in m$ // EA. Now $\ell \nparallel m$, for if ℓ // m, Ax. 2.4 $\Rightarrow$ EA // AB $\Rightarrow$ EA = AB, by Th. 2.1. Hence by Th. 2.3, EA = AB $\Rightarrow$ $E \in AB \Rightarrow E \in \pi$, a contradiction. However, since ℓ // AB, Th. 2.2 (ii) implies that ℓ and m are coplanar, whence Ax. 2.7 implies that there exists a unique point $F \in \ell \cap m$.



It is clear from the above construction that (A,B) $\square$ (E,F) according to Def. 3.1, (3). Now the argument of the first paragraph of this proof shows that $F \notin \pi$. If CD = GH, then Def. 3.1 implies (C,D) = (G,H) and CA // DB since CD = GH $\Rightarrow$ AB $\neq$ GH $\Rightarrow$ (A,B) $\square$ (G,H) according to Def. 3.1 (3). Similarly, AB = GH implies AC // DB. If GH is distinct from both CD and AB then Def. 3.1, (3) implies CG //DH, GA // HB, $CD \cap GH = GH \cap AB = AB \cap CD = \emptyset$, whence by Th. 2.6B, CA // DB. Thus we have CA // DB in all cases.

As EF = ℓ // AB // CD (by Ax. 2.4 and Def. 3.1), EA// FB = m, and CA // DB, it follows from Th. 2.6B that EC // FD. In conjunction with ℓ // CD, this implies that (E,F) $\square$ (C,D) by virtue of Def. 3.1(3). As we have already shown (A,B) $\square$ (E,F) the proof is complete.

Corollary 3.1.1 (A,B) $\uparrow$ (C,D), AB $\neq$ CD $\Rightarrow$ (A,B) $\square$ (C,D).

Proof With the notation of the preceding theorem, there exist points E and F such that (A,B) $\square$ (E,F), (E,F) $\square$ (C,D), and AB $\neq$ EF $\neq$ CD $\neq$ AB. Since A $\neq$ B, Prop. 3.2 implies E $\neq$ F. As AB // EF // CD, EC // FD and EA // FB, it follows from Th. 2.6.B that CA //DB. In conjunction with AB //CD, this implies (A,B) $\square$ (C,D), the required result.

Corollary 3.1.2 $(A,B) \uparrow (C,D) \Rightarrow (A,B) \square (C,D)$ or $AB = CD$.

Proof If $AB \neq CD$, the result follows from the preceding corollary. The only other possibilities are $A = B$ and $AB = CD$. If $A = B$, it follows from Prop. 3.2 that $(A,B) \square (C,D)$. Lastly, if $AB = CD$, there is nothing to prove.

Corollary 3.1.3 If $(A,B) \uparrow (C,D)$, then (1) $A = C \Leftrightarrow B = D$ and (2) $B = C$ or $A = D \Rightarrow AB = CD$ or $A = B$.

Proof (1) If $(A,B) \uparrow (C,D)$ and $A = C$, then either $(A,B) \square (A,D)$ or $AB = AD$, by virtue of the last corollary. If $(A,B) \square (A,D)$, it follows from Def. 3.1 that either $A = B$ and $A = D$ or $B = D$. In either of these cases, we have $B = D$. If $AB = AD$ then, as there exists an ordered point pair (E,F) such that $(A,B) \square (E,F)$ and $(E,F) \square (A,D)$, it follows that either $AB = EF$, in which case $(A,B) = (E,F) = (A,D) \Rightarrow B = D$, or $AB \neq EF$, $AB \parallel EF$, $AE \parallel BF$ and $EA \parallel FD$. The last two equations imply, via Th. 2.1, that $BF = FD$. If $F \in AB$ then since $AB \parallel EF$, Th. 2.1 $\Rightarrow AB = EF$, a contradiction. Consequently $F \notin AB$ and so $BF \neq AB$. Finally, $B, D \in AB, BF \Rightarrow B = D$, by Ax. 2.1. We have proved that $A = C \Rightarrow B = D$. The proof that $B = D \Rightarrow A = C$ is similar and hence omitted.

(2) Let $(A,B) \uparrow (C,D)$, $B = C$, and $A \neq B$. By Prop. 3.2 (ii), we have $C \neq D$. Hence AB and CD exist and we need only show $AB = CD$.

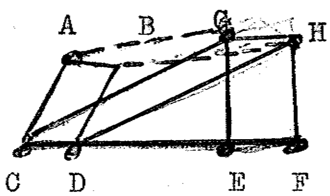
If $AB \neq CD$, then Cor. 3.1.2 implies $(A,B) \square (C,D)$. Under these conditions, $AB \parallel CD$ and $AC \parallel BD$ by Def. 3.1. Since we are supposing $B = C$, Th. 2.1 implies $AB = CD$, a contradiction. Thus, with the assumption $B = C$, we have shown $AB = CD$, as required. The case $A = D$ is handled similarly.

We now proceed to show that $\uparrow$ is transitive. In virtue of Prop. 3.2(ii), it suffices to prove the following result.

Theorem 3.2 If (A,B), (C,D), and (E,F) are any ordered point pairs not in $\underline{0}$ such that $(A,B) \uparrow (C,D)$ and $(C,D) \uparrow (E,F)$, then $(A,B) \uparrow (E,F)$.

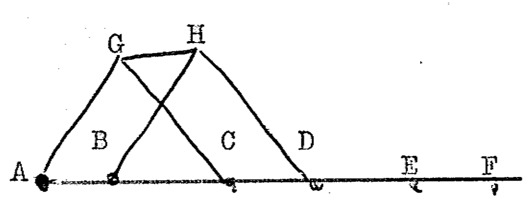
<u>Proof</u> (i) If $AB \neq CD$ and $CD \neq EF$, then by two applications of Cor. 3.1.2, Def. 3.2 implies $(A,B) \uparrow (E,F)$, as required.

(ii) We consider the remaining cases where exactly two of AB, CD, and EF are distinct. It suffices to prove the theorem on the assumption $AB \neq CD = EF$. Then, as $AB \neq CD$, we have from Cor. 3.1.2 that $(A,B) \sqsupset (C,D)$ and so, by Def. 3.1, $AB \parallel CD$ and $AC \parallel BD$. It follows from Th. 2.2 (ii) that A, B, C, D, E, and F determine a unique plane, say π. By Th. 3.1, there exists $(G,H) \notin \underline{0}$ such that $G \notin \pi$, $H \notin \pi$, $(C,D) \sqsupset (G,H)$ and $(G,H) \sqsupset (E,F)$. Thus $CD \parallel GH$, $CG \parallel DH$ and $EG \parallel FH$.



In particular, we have that $CD \neq AB \neq GH \neq CD$, $AC \parallel BD$ and $CG \parallel DH$. It follows from Th. 2.6B that $AG \parallel BH$. In conjunction with $AB \parallel GH$ (which follows from Ax. 2.4), this implies $(A,B) \sqsupset (G,H)$. Since $(G,H) \sqsupset (E,F)$, Def. 3.2 allows us to conclude $(A,B) \uparrow (E,F)$, as required.

(iii) Suppose $AB = CD = EF$. Without loss of generality, $(A,B) \neq (C,D) \neq (E,F)$. By Def. 3.1 there exists $(G,H) \notin \underline{0}$ such that $GH \neq AB$, $(A,B) \sqsupset (G,H)$ and $(G,H) \sqsupset (C,D)$. By Prop. 3.1, $(G.H) \uparrow (C,D)$, and so the preceding case implies $(G.H) \uparrow (E,F)$.



Since $GH \neq AB = EF$, Cor. 3.1.1 implies that $(G,H) \sqsupset (E,F)$. In conjunction with $(A,B) \sqsupset (G,H)$, this implies that $(A,B) \uparrow (E,F)$.

The preceding theorem completes the demonstration that $\uparrow$ is a symmetric, reflexive, and transitive relation, that is, an equivalence relation. By a well known result, it follows that the relation $\uparrow$ induces a partitioning of the set of all ordered point pairs into mutually exclusive and exhaustive nonempty subsets called "equivalence classes". This observation motivates the following definition.

Definition 3.3 Any subset T of the set $\mathcal{P} \times \mathcal{P}$ of all ordered point pairs is said to be a vector if and only if T is identical with exactly one of the equivalence classes corresponding to $\uparrow$ in $\mathcal{P} \times \mathcal{P}$. A single vector will be denoted by an underlined lower case letter, for example, $\underline{v}$. The set of all vectors will be denoted by $\mathcal{V}$.

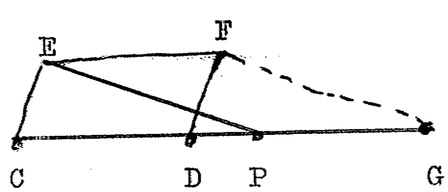
We shall now prove that $\mathcal{V}$ has a unique representation at every point $P \in \mathcal{P}$.

Theorem 3.3 For any vector $\underline{v}$ and any point P, there exist unique points Q and R such that $(P,Q) \in \underline{v}$ and $(R, P) \in \underline{v}$.

Proof We shall establish that there is a unique point Q such that $(P,Q) \in \underline{v}$. The proof that a unique point R exists such that $(R, P) \in \underline{v}$ is similar and hence omitted. It is clear from Prop. 3.2(ii) that we may assume $\underline{v} \cap \underline{0} = \emptyset$.

There cannot exist two distinct satisfactory Q, for $(P,Q_1) \uparrow (P, Q_2) \Rightarrow Q_1 = Q_2$, by Cor. 3.1.3. Thus it suffices to demonstrate the existence of at least one satisfactory Q.

If $(A,B) \in \underline{v}$ and $P \notin AB$, then Ax. 2.6 guarantees the existence of a point T such that $PT \parallel AB$ and $PA \parallel TB$. By Def. 3.1, these relations imply that $(A,B) \square (P,T)$ and so by Prop. 3.1, $(A,B) \uparrow (P,T)$. Thus $(P,T) \in \underline{v}$ and T is a satisfactory Q.



If $(C,D) \in \underline{v}$ and $P \in CD$, let E be any point $\notin CD$. Then Ax. 2.6 guarantees the existence of a point F such that $(C,D) \square (E,F)$. Since $E \notin CD$, it is clear from Def. 3.1 and Th. 2.1 that

E, F, and P are distinct. Consequently, Ax. 2.6 and Cor. 2.1.1 guarantee the existence of a point $G \in CD$ such that $EP \parallel FG$. As $EF \parallel PG = CD$, we have $(E,F) \square (P,G)$ and so $(E,F) \uparrow (P,G)$. In virtue of the preceding paragraph and Th. 3.2, these relations imply $(C,D) \uparrow (P,G)$, whence $(P,G) \in \underline{V}$, with G serving as Q.

We now discuss the first of the vector operations we shall consider, that of vector addition.

<u>Definition 3.4</u> Let $\underline{U}$ and $\underline{V}$ be any vectors (possibly $\underline{U} = \underline{V}$) and P any point. Let Q and R be the unique points such that $(P,Q) \in \underline{U}$ and $(Q,R) \in \underline{V}$. We define the <u>sum of $\underline{U}$ and $\underline{V}$</u> to be the unique vector $\underline{W}$ containing (P,R) and denote $\underline{W}$ by $\underline{U}+\underline{V}$.

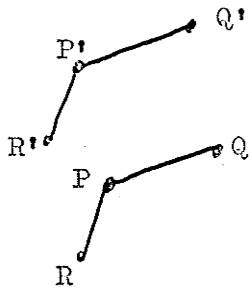
<u>Theorem 3.4</u> The sum of any two vectors $\underline{U}$ and $\underline{V}$ is well defined, that is, independent of the point P of Def. 3.4.

<u>Proof</u> Let P, Q, and R be any points such that $(P,Q) \in \underline{U}$ and $(Q,R) \in \underline{V}$. It is clear from Prop. 3.2(ii) that we may assume $P \neq Q \neq R$.

(i) Suppose $R \in PQ$. Let P' be any point such that $P' \notin PQ$. By Th. 3.3, there exist points Q' and R' such that $(P', Q') \in \underline{U}$, $(Q', R') \in \underline{V}$, and $P' \neq Q' \neq R'$. As $(P,Q) \uparrow (P',Q')$ and $PQ \neq P'Q'$, we have from Cor. 3.1.1 that $(P,Q) \square (P',Q')$. By Def. 3.1, it follows that $PQ \parallel P'Q'$, and $PP' \parallel QQ'$. Thus Th. 2.1 implies that $PQ \cap P'Q' = \emptyset$, whence in particular $Q' \notin QR = PQ$. Consequently, $Q'R' \neq QR$ and, as above, we get $QR \parallel Q'R'$ and $QQ' \parallel RR'$. Since $P'Q' \parallel PQ = QR \parallel Q'R'$, Th. 2.1 implies that P', Q', and R' are collinear. Applications of Ax. 2.4 give $PR \parallel P'R'$ and $PP' \parallel RR'$ whence Def. 3.1 $\Rightarrow (P, R) \square (P', R')$ and Prop. 3.1 $\Rightarrow (P, R) \uparrow (P', R')$.

If P'' is any point of PQ, then the above argument shows that $\underline{U}+\underline{V}$ calculated from P'' equals $\underline{U}+\underline{V}$ calculated from P, since each equals $\underline{U}+\underline{V}$ calculated from P'.

(ii) If P, Q, and R are not collinear then, by Cor. 2.4.1, they are contained in a unique plane, say Π. If P' is any point such that P' $\notin \Pi$, there exist unique points Q' and R' such that $(P',Q') \in \underline{u}$, $(Q',R') \in \underline{v}$ and $P' \neq Q' \neq R'$. As Th. 2.3 $\Rightarrow$ $P'Q \not\subset \Pi$, it follows that $P' \notin PQ$ and so $P'Q' \neq PQ$. As in case (i), we get $PQ \parallel P'Q'$ and $P'P \parallel Q'Q$. Similarly $PR \parallel P'R'$ and $P'P \parallel R'R$. By Ax. 2.4 and Th. 2.6B, it follows that $Q'R' \parallel QR$ and $R'R \parallel Q'Q$. Thus, Def. 3.1 $\Rightarrow$ $(P,R) \square (P',R')$ and $(P,R) \uparrow (P',R')$.



If P'' is any point of Π, then $\underline{u}+\underline{v}$ calculated from P'' equals $\underline{u}+\underline{v}$ calculated from P, since each equals $\underline{u}+\underline{v}$ calculated from P'.

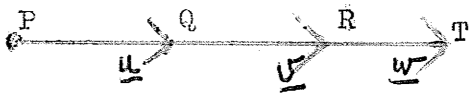
The reader will realize that the argument in case (i) of Th. 3.4 establishes the following important result.

Prop. 3.3 Let $\underline{u}$ and $\underline{v}$ be any vectors and A and D any points. Let B, C, E, and F be the unique points such that $(A,B) \in \underline{u}$, $(B,C) \in \underline{v}$, $(D,E) \in \underline{u}$, and $(E,F) \in \underline{v}$. Then A, B, and C are collinear if and only if D, E, and F are collinear.

If, in the notation of Prop. 3.3, it happens that A, B, and C are collinear, we say that the vectors $\underline{u}$ and $\underline{v}$ are collinear or parallel and write $\underline{u} \parallel \underline{v}$. Parallelism of vectors is clearly reflexive and symmetric. In the following proposition, we prove that collinearity of vectors is also a transitive relation in a special subset of $\mathcal{V}$.

Proposition 3.4 If $\underline{u}$, $\underline{v}$ and $\underline{w}$ be any three vectors, each of which intersects $\underline{0}$ in $\emptyset$, then $\underline{u} \parallel \underline{v}, \underline{v} \parallel \underline{w} \Rightarrow \underline{u} \parallel \underline{w}$.

Proof By virtue of the preceding result, it suffices to consider the representation of $\mathcal{V}$ at any point P. Let Q, R, and T be the unique points guaranteed by Th. 3.3 such that $(P,Q)\in\underline{u}$, $(Q,R)\in\underline{v}$, and $(R,T)\in\underline{w}$.



From the definition of the collinearity of vectors, P and T are each collinear with Q and R. As $\underline{v}\cap\underline{0}=\emptyset$, $Q\neq R$ and so by Axs. 2.1, 2 we conclude that P, Q, R, and T are collinear.

Let X be the unique point such that $(Q,X)\in\underline{w}$, that is, such that $(Q,X)\uparrow(R,T)$. As $\underline{w}\cap\underline{0}=\emptyset$, $R\neq T$ and so by Prop. 3.2 (ii), $Q\neq X$. It follows from Cor. 3.1.2 that either $QX = RT$ or $(Q,X)\,\square\,(R,T)$. Since $R\neq T$, $Q\neq R$, and $Q\in RT$, Def. 3.1 and Th. 2.1 imply $(Q,X)\,\not\square\,(R,T)$. Thus $QX = RT$.

Since $\underline{u}\cap\underline{0}=\emptyset$, $P\neq Q$ and Axs. 2.1, 2 imply $PQ = RT$. Thus $PQ = QX$, whence P, Q, and X are collinear. This proves that $\underline{u}$ and $\underline{w}$ are collinear, as claimed.

<u>Proposition 3.5</u> For any vectors $\underline{u}$, $\underline{v}$, and $\underline{w}$, if $\underline{u}\parallel\underline{v}$, $\underline{u}\nparallel\underline{w}$, and $\underline{v}\cap\underline{0}=\emptyset$, then $\underline{u}\nparallel\underline{v}+\underline{w}$.

<u>Proof</u> Since $\underline{0}$ is collinear with every vector, it follows that both $\underline{v}$ and $\underline{w}$ intersect $\underline{0}$ in $\emptyset$. By Prop. 3.4, we may conclude that $\underline{v}\nparallel\underline{w}$. Assume $\underline{u}\parallel\underline{v}+\underline{w}$.

Let P be any point and Q, R, and S the uniquely determined points such that $(P,Q)\in\underline{u}$, $(Q,R)\in\underline{v}$ and $(R,S)\in\underline{w}$. Thus $(Q,S)\in\underline{v}+\underline{w}$. Since $\underline{v}$ and $\underline{w}$ each intersect $\underline{0}$ in $\emptyset$, $Q\neq R\neq S$. Furthermore, $Q\neq S$, for $Q = S \Rightarrow (R,Q)\in\underline{w}\Rightarrow\underline{v}\parallel\underline{w}$, a contradiction. Let T be the unique point such that $(Q,T)\in\underline{w}$. As $Q\neq R\neq S\neq Q$, the argument of Prop. 3.4 proves $QT = RS$. Since $\underline{u}\parallel\underline{v}$, $PQ = RS$, whence $PQ = QT$ and $\underline{u}\parallel\underline{w}$, a contradiction. Consequently, $\underline{u}\nparallel\underline{v}+\underline{w}$, as claimed.

<u>Proposition 3.6</u> For any vectors $\underline{u}$ and $\underline{v}$ and any point P, if $(P,Q)\in\underline{u}$ and $(P,R)\in\underline{v}$, then $\underline{u}\parallel\underline{v}$ if and only if, P,Q, and R are collinear.

Proof: Suppose $\underline{U} \parallel \underline{V}$. If $(Q,T) \in \underline{V}$, then P, Q, and T are collinear. Without loss of generality, $R \ddagger P \ddagger Q \ddagger T$ and $Q \ddagger R$. Since $(P,R) \uparrow (Q,T)$, we get from Th. 2.1, Def. 3.1, and Cor. 3.1.2 that PR = QT, as in the proof of Prop. 3.4. Thus $Q \in PR$ and P, Q, and R are collinear.

Conversely, if P, Q, and R are collinear and if $(Q,T) \in \underline{V}$ then $(P,R) \uparrow (Q,T)$. Without loss of generality, $T \ddagger Q \ddagger P \ddagger R$ and the above argument proves PR = QT. Thus P, Q, and T are collinear and $\underline{U} \parallel \underline{V}$. This completes the proof.

It will now be shown that $\mathcal{V}$ forms an abelian group under addition. More precisely, we have the following theorem.

Theorem 3.5 (i) For all vectors $\underline{U}$, $\underline{V}$, and $\underline{W}$, it follows that $\underline{U} + (\underline{V} + \underline{W}) = (\underline{U} + \underline{V}) + \underline{W}$.

(ii) There is a unique vector $\underline{0}^*$ such that, for all $\underline{V} \in \mathcal{V}$, $\underline{0}^* + \underline{V} = \underline{V} + \underline{0}^* = \underline{V}$.

(iii) For each vector $\underline{V}$, there is a unique vector $\underline{V}'$ (often denoted $-\underline{V}$) such that $\underline{V} + \underline{V}' = \underline{V}' + \underline{V} = \underline{0}$.

(iv) For all vectors $\underline{U}$ and $\underline{W}$, there is a unique vector $\underline{X}$ (often denoted $\underline{W} - \underline{U}$) such that $\underline{X} + \underline{U} = \underline{W}$.

(v) For all vectors $\underline{U}$ and $\underline{V}$, $\underline{U} + \underline{V} = \underline{V} + \underline{U}$.

Proof (i) There exist points, P, Q, R, and S such that $(P,Q) \in \underline{U}$, $(Q,R) \in \underline{V}$, and $(R,S) \in \underline{W}$. Since $(P,R) \in (\underline{U} + \underline{V})$, we have that $(P, S) \in (\underline{U} + \underline{V}) + \underline{W}$. However, since $(Q,S) \in \underline{V} + \underline{W}$, we also have that $(P, S) \in \underline{U} + (\underline{V} + \underline{W})$. As any ordered point pair belongs to exactly one vector, it must be that $(\underline{U} + \underline{V}) + \underline{W} = \underline{U} + (\underline{V} + \underline{W})$.

(ii) Let $\underline{0}^*$ be a vector such that for all $\underline{V} \in \mathcal{V}$, $\underline{V} + \underline{0}^* = \underline{0}^* + \underline{V} = \underline{V}$. For any points P and Q, there is a unique vector $\underline{V}$ such that $(P,Q) \in \underline{V}$. If $(Q,S) \in \underline{0}^*$, then $(P,S) \in \underline{V} + \underline{0}^* = \underline{V}$ and Th. 3.3 implies that Q = S.

Since $(S,S) \in \underline{0}^*$, Prop. 3.2 (ii) implies that there is at most one satisfactory $\underline{0}^*$, namely the vector $\underline{0}$ defined before as $\{ (P,P) \mid P \in \mathcal{P} \}$.

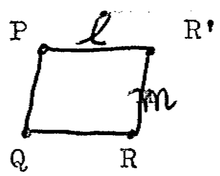
Let $\underline{w}$ be an arbitrary vector and A any point. There is a unique point B such that $(A,B) \in \underline{w}$. Since $(A,A) \in \underline{0}'$ and $(B,B) \in \underline{w}'$, the reasoning of case (i) allows us to conclude $\underline{0}' + \underline{w} = \underline{w} + \underline{0}' = \underline{w}$.

(iii) For any vector $\underline{v}$ and any point P, there is a point Q such that $(P, Q) \in \underline{v}$. If $\underline{v}'$ is defined as the vector containing (Q,P), then Th. 3.4 $\Rightarrow \underline{v} + \underline{v}' = \underline{v}' + \underline{v} = \underline{0}$. The uniqueness of $\underline{v}'$ follows from the unique representation of $\underline{0}$, for if $(P,Q) \in \underline{v}$, $(Q,R) \in \underline{v}'$, and $\underline{v} + \underline{v}' = \underline{0}$ then (P,R) and (P,P) are each in $\underline{0}$, whence Th. 3.3 $\Rightarrow$ R = P and $\underline{v}'$ is unique.

(iv) If $\underline{X} + \underline{u} = \underline{w}$, then $\underline{X} = \underline{X} + \underline{0} = \underline{X} + (\underline{u} + \underline{u}') = (\underline{X} + \underline{u}) + (\underline{u}') = \underline{w} + \underline{u}'$. Thus there is at most one satisfactory value of $\underline{X}$. The verification that $(\underline{w} + \underline{u}') + \underline{u} = \underline{w}$ is straightforward and shows that a unique satisfactory $\underline{X}$ exists.

(v) By (ii), we may assume that $\underline{u} \neq \underline{0} \neq \underline{v}$.

Assume $\underline{u} \nparallel \underline{v}$. Let P be any point and Q and R points such that $(P,Q) \in \underline{u}$ and $(Q,R) \in \underline{v}$. Then by Ax. 2.5, 6, there exist unique lines ℓ and m and unique point R' such that $P \in \ell \parallel QR$, $R \in m \parallel PQ$, and $\ell \cap m = \{R'\}$. By



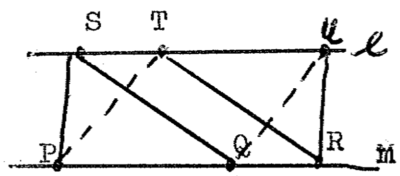
Def. 3.1, $(P,R') \square (Q,R)$ and $(P,Q) \square (R',R)$ whence $(P,R') \uparrow (Q,R)$ and $(P,Q) \uparrow (R',R)$. Consequently, $(P,R) \in \underline{v} + \underline{u}, \underline{u} + \underline{v}$ and so $\underline{v} + \underline{u} = \underline{u} + \underline{v}$.

If $\underline{u} \parallel \underline{v}$, let $\underline{w}$ be any vector $\nparallel \underline{u}$. Then $\underline{u} \nparallel \underline{v} + \underline{w}$, by Prop. 3.5. Using what has already been proved for non-collinear vectors, we find $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w}) = (\underline{v} + \underline{w}) + \underline{u} = \underline{v} + (\underline{w} + \underline{u}) = \underline{v} + (\underline{u} + \underline{w}) = (\underline{v} +$

$\underline{u}$) + $\underline{w}$. Adding $\underline{w}'$ to the extreme members of this equation, we get $\underline{u} + \underline{v} = \underline{v} + \underline{u}$, as required.

Since $[\underline{u} + (-\underline{v})] + \underline{v} = \underline{u} + [-\underline{v} + \underline{v}] = \underline{u} + \underline{0} = \underline{u}$, it is clear that $\underline{u} + (-\underline{v}) = \underline{u} - \underline{v}$. It is also clear that $-\underline{0} = \underline{0}$.

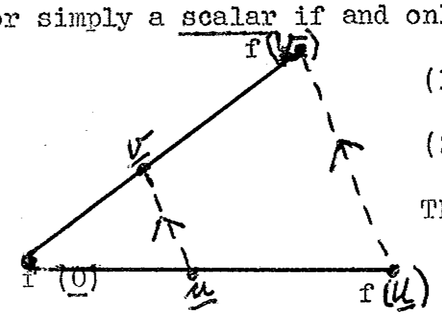
Remark The commutativity of vector addition is equivalent to the minor theorem of Pappns: If ℓ and m are distinct parallel lines with S, T, and U distinct points of ℓ and P, Q, and R distinct points of ℓ, then any two of $SQ \parallel TR$, $TP \parallel UQ$, and $SP \parallel UR$ imply the third.



IV Scalars and Coordinatisation

Intuitively multiplication by a scalar is a function which multiplies the length of any line segment representing any vector by a fixed number, without changing the vector's line of action. Using a theorem from Euclidean geometry, we may make this notion more precise. It is well known that if ABC is a triangle, B' a point on AB, and C' a point on AC, then BC $\parallel$ B'C' if and only if AB' : AB :: AC' : AC. This is the principal motivation for the following definition.

Definition 4.1 A function f from $\mathcal{V}$ into $\mathcal{V}$ is a multiplication by a scalar, or simply a scalar if and only if



(1) $f(\underline{0}) = \underline{0}$ and

(2) $f(\underline{v}) - f(\underline{u}) \parallel \underline{v} - \underline{u}$, for all $\underline{u}, \underline{v} \in \mathcal{V}$.

The set of scalars will be denoted by Σ.

Proposition 4.1 For all scalars f and all vectors $\underline{v}$, $f(\underline{v})$ and $\underline{v}$ are collinear.

Proof Let $\underline{u} = \underline{0}$ in Def. 4.1, (2). Then since $f(\underline{0}) = (\underline{0})$, Th. 3.5 (ii) implies that $f(\underline{v}) = f(\underline{v}) + \underline{0} = f(\underline{v}) + (-\underline{0}) = (\underline{v}) - \underline{0} = f(\underline{v}) - f(\underline{u}) \parallel \underline{v} - \underline{u} = \underline{v} - \underline{0} = \underline{v} + (-\underline{0}) = \underline{v} + \underline{0} = \underline{v}$. Thus $f(\underline{v}) \parallel \underline{v}$.

Proposition 4.2 (i) There exists a unique scalar 0 such that $0(\underline{v}) = \underline{0}$ for all $\underline{v} \in \mathcal{V}$.

(ii) There exists a unique scalar 1 such that $1(\underline{v}) = \underline{v}$ for all $\underline{v} \in \mathcal{V}$.

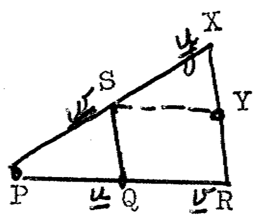
Proof (i) Let α be a function from $\mathcal{V}$ into $\mathcal{V}$ such that $\alpha : \underline{v} \to \underline{0}$, all $\underline{v} \in \mathcal{V}$. Certainly, there is exactly one such function. It remains only to verify that α is a scalar. By the definition of α, $\alpha(\underline{0}) = \underline{0}$. For any vectors $\underline{u}$ and $\underline{v}$, we have $\alpha(\underline{u}) - \alpha(\underline{v}) = \underline{0} - \underline{0} = \underline{0} + (-0) = \underline{0} + \underline{0} = \underline{0} \parallel \underline{u} - \underline{v}$, since $\underline{0}$ is collinear with every vector. Thus α satisfies Def. 4.1 and so is a scalar.

(ii) The proof for (ii) is similar to the above and hence omitted.

It is convenient to have an alternative form of the definition of scalars in terms of parallelism of lines and parallelism of vectors.

Proposition 4.3 For any vectors $\underline{u}$, $\underline{v}$, $\underline{w}$ and $\underline{z}$, and any points P, Q, R, S, and T, and any scalar f, if $\underline{u} \parallel \underline{v} \neq \underline{0}$, $\underline{u} \nparallel \underline{w}$, $(P,Q) \in \underline{u}$, $(P,R) \in \underline{v}$, $(P, S) \in \underline{w}$, $(P, T) \in \underline{z}$, and $\underline{v} = f(\underline{u})$, then $\underline{z} = f(\underline{w})$ if and only if $T \in PS$ and $SQ \parallel TR$.

Proof: Let $\underline{y} = f(\underline{w})$ and $(P,X) \in \underline{y}$. Then $(Q,S) \in \underline{w} - \underline{u}$ and $(R,X) \in f(\underline{w}) - f(\underline{u})$. Let Y be the unique point such that $(Q,S) \uparrow (R,Y)$. Since $\underline{0} \neq \underline{u} \nparallel \underline{w} \neq \underline{0}$, it follows that $Q \neq S$, whence Defs. 3.1, 2 and Ax. 2.4 imply $QS \parallel RY$. Since $\underline{w} - \underline{u} \parallel f(\underline{w}) - f(\underline{u})$, Prop. 3.6 implies that R, Y, and X are collinear. Since $QS \parallel RY = RX$, Ax. 2.4 implies $SQ \parallel XR$.



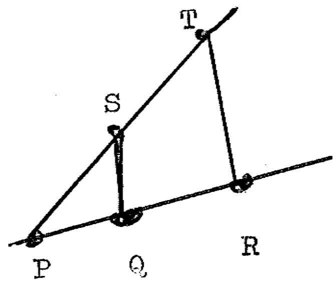
Conversely, if $(P,T) \in \underline{z}$, $T \in PS$, and $SQ \parallel TR$, it follows that $\underline{z} = f(\underline{w})$. For if $(P,V) \in f(\underline{w})$ then $V, T \in PS$, VR by what has already been proved. However, $PS \neq VR$ for $PS = VR \Rightarrow PS \parallel SQ \Rightarrow PS = SQ \Rightarrow \underline{u} \parallel \underline{w}$, a contradiction. Thus by Ax. 2.1, $V = T$ and the proof is complete.

We now prove an important theorem about the existence of scalars.

Theorem 4.1 $\underline{0} \neq \underline{v} \parallel \underline{u} \neq \underline{v}, \underline{0} \Rightarrow$ there exists a scalar f such that $\underline{v} = f(\underline{u})$.

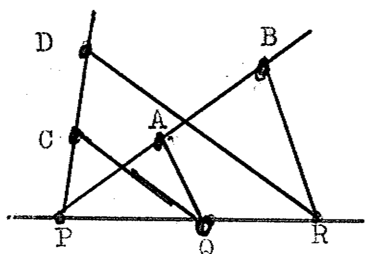
Proof We shall define a mapping f from $\mathcal{V}$ into $\mathcal{V}$ such that $\underline{v} = f(\underline{u})$ and then show that f is indeed a scalar.

For any $\underline{\alpha} \in \mathcal{V}$, if $\underline{\alpha} \nparallel \underline{u}$, then supposing that $(P,Q) \in \underline{u}$, $(P,R) \in \underline{v}$, and $(P,S) \in \underline{\alpha}$, it follows from Prop. 3.6 that the line through R parallel to SQ intersects PS in a unique point T (for nonparallel coplanar



lines have a unique common point). Define (f$\underline{\alpha}$) to be the vector containing (P,T).

Setting aside for the moment the question of the images under f of vectors collinear with $\underline{u}$, we show f ($\underline{\beta}$) - f ($\underline{\gamma}$) $\parallel$ $\underline{\beta}-\underline{\gamma}$ for all other $\underline{\beta}, \underline{\gamma} \in \mathcal{V}$. If $\underline{\beta} \parallel \underline{\gamma}$, the result is trivial, by virtue of Prop. 4.1.

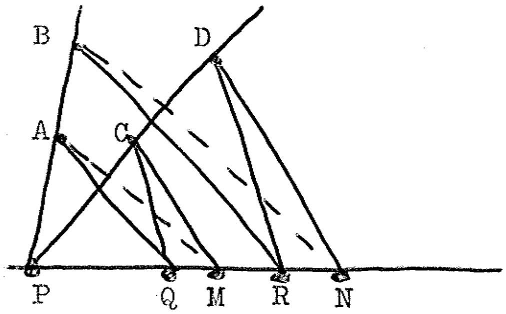


If $\underline{\beta} \nparallel \underline{\gamma}$, suppose (P,Q) $\in \underline{u}$, (P,R) $\in \underline{v}$, (P,A) $\in \underline{\beta}$, (P,B) $\in$ f ($\underline{\beta}$), (P,C) $\in \underline{\gamma}$, and (P,D) $\in$ f ($\underline{\gamma}$). It follows from Th.2.6A that CA $\parallel$ DB. Now (C,A) $\in \underline{\beta}-\underline{\gamma}$ and (D,B) $\in$ f ($\underline{\beta}$) - f ($\underline{\gamma}$). If (D,F) $\uparrow$ (C,A), then $\underline{\beta} \nparallel \underline{\gamma}$ $\Rightarrow$ C $\neq$ A $\Rightarrow$ DF $\parallel$ CA (from Def. 3.1, 2 and Ax. 2.4) $\Rightarrow$ DF = DB (from Th. 21) $\Rightarrow$ D, F, and B are collinear $\Rightarrow$ $\underline{\beta}-\underline{\gamma} \parallel$ f ($\underline{\beta}$) - f ($\underline{\gamma}$), by Prop. 3.6.

It now remains to define the image under f of vectors collinear with $\underline{u}$ and show that, with regard to them, f still satisfies Def. 4.1. It suffices to show that if a vector $\underline{c} \nparallel \underline{u}$ is employed to determine f ($\underline{\alpha}$) for all $\underline{0} \neq \underline{\alpha} \parallel \underline{u}$, the result is the same as if one were to use $\underline{w} \nparallel \underline{c}, \underline{u}$ instead of $\underline{c}$. (Clearly, for all vectors $\underline{w}$ and $\underline{c}$, we must define f ($\underline{0}$) = $\underline{0}$.)

Suppose (P,Q) $\in \underline{u}$, (P,R) $\in \underline{v}$, (P,A) $\in \underline{w}$, (P,B) $\in$ f ($\underline{w}$), (P,C) $\in \underline{c}$, (P,D) $\in$ f ($\underline{c}$), (P,M) $\in \underline{\alpha}$ and (P,N) $\in$ f ($\underline{\alpha}$) where f ($\underline{\alpha}$) is calculated by means of f ($\underline{c}$). By Prop. 4.3, it suffices to show that AM $\parallel$ BN.

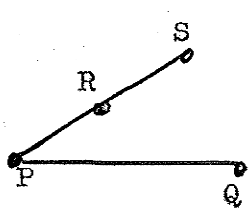
By our previous comments, CA $\parallel$ DB and by construction, CM $\parallel$ DN. One may



verify that the conditions we have imposed on $\underline{u}$ and $\underline{v}$ imply $P \neq A, B, C, D, M, N$. Thus, by Th. 2.6A, we get $AM \parallel BN$, which proves the theorem.

Remark The proof of Th. 4.1 establishes not only the existence of a scalar f but also the uniqueness of f, under the stated conditions. Now if we omit the condition $\underline{u} \neq \underline{v}$, then, given $\underline{u} = \underline{v}$, it is clear that a satisfactory f, namely 1, exists. In order to show that 1 is the only scalar that fixes the non-zero vector $\underline{u}$, consider any vector $\underline{w} \nparallel \underline{u}$ and let $\underline{u} = f(\underline{u})$. We need only show that $f(\underline{w}) = \underline{w}$. If $(P,Q) \in \underline{u}$, $(P,R) \in \underline{w}$, and $(P,S) \in f(\underline{w})$, then Prop. 4.3 $\Rightarrow S \in PR$ and $QR \parallel SR$, whence $QR = SR$ and $S, R \in PS, QS$. As in the proof of Prop. 4.3, $S = R$ and $\underline{w} = f(\underline{w})$.

Suppose we omit the condition $\underline{v} \neq \underline{0}$ in Th. 4.1. Certainly, if $\underline{v} = \underline{0}$, there exists a scalar f, namely 0, such that $\underline{0} = f(\underline{u})$. In order to show that 0 is the only scalar that sends the non-zero vector $\underline{u}$ into $\underline{0}$, consider any vector $\underline{w} \nparallel \underline{u}$ and let $\underline{0} = f(\underline{u})$. We need only show that $\underline{0} = f(\underline{w})$. Let $(P,Q) \in \underline{u}$, $(P,R) \in \underline{w}$ and $(P,S) \in f(\underline{w})$. It is clear that $(Q,R) \in \underline{w} - \underline{u}$, $(P,S) \in f(\underline{w}) - f(\underline{u})$, and $\underline{w} - \underline{u} \parallel f(\underline{w}) - f(\underline{u})$. If $(P, S) \uparrow (Q,T)$ and $P \neq S$, then by the argument of Prop. 4.3, $PS \parallel QT$. Furthermore, Q, R, and T are collinear since $\underline{w} - \underline{u} \parallel f(\underline{w}) - f(\underline{u})$. Now, $R \neq T$ for $R = T \Rightarrow (P,S) \uparrow (Q,R) \Rightarrow PS \parallel QR$



$\Rightarrow Q \in PR \Rightarrow \underline{u} \parallel \underline{w}$, a contradiction. Consequently, $QT = RT \parallel PS = PR$ $\Rightarrow Q \in PR$, again a contradiction. Thus $S = P$, $(P,P) \in f(\underline{w})$ and $f(\underline{w}) = \underline{0}$. The above comments establish the following theorem and corollaries.

Theorem 4.2 $\underline{v} \parallel \underline{u} \neq \underline{0} \Rightarrow$ there exists a unique scalar β such that $\underline{v} = \beta(\underline{u})$.

Cor. 4.2.1 For any scalar f and any non-zero vector $\underline{v}$, $f(\underline{v}) = \underline{v} \Rightarrow$ $f = 1$.

Cor. 4.2.2 For any scalar f and any vector $\underline{v}$, $f(\underline{v}) = \underline{0} \Rightarrow f = 0$ or $\underline{v} = \underline{0}$ (or both).

Cor. 4.2.3 There exists a one-to-one correspondence between Σ and the points on any line of $\mathcal{L}$.

Proof By Cor. 2.4.8, it suffices to prove the result for any line ℓ of $\mathcal{L}$. Let P and Q be any distinct points of ℓ and $\underline{u}$, the non zero vector containing (P,Q). Associate with every point R of ℓ, the unique scalar f such that if $(P,R) \in \underline{w}$ then $\underline{w} = f(\underline{u})$. Since scalars are functions, Th. 3.3.implies that no scalar is associated with more than one point. Furthermore, for any scalar g, $g(\underline{u}) \parallel \underline{u}$ and there exists a point $T \in \ell$ such that the image of T is g under the mapping described above. As we have displayed a one-to-one mapping from the points of ℓ onto the set of scalars , the proof is complete.

We shall now show that any non zero scalar is a one-to-one mapping from $\mathcal{V}$ onto $\mathcal{V}$. It suffices to prove the following result.

Theorem 4.3 For any non zero scalar f and non zero vector $\underline{u}$, there is a unique (non zero) vector $\underline{w} \parallel \underline{u}$ such that $\underline{u} = f(\underline{w})$.

Proof Let $\underline{v}$ be any vector such that $\underline{v} \nparallel \underline{u}$. Let P, Q, R, and S be points such that $(P,Q) \in \underline{v}$, $(P,R) \in f(\underline{v})$, and $(P,S) \in \underline{u}$. If ℓ is the line through

Q parallel to RS, then $\ell \nparallel PS$ since $\ell \parallel PS \Rightarrow PS \parallel RS \Rightarrow PS = RS \Rightarrow R \in PS$ $\Rightarrow R = P$ (since $PQ \nparallel PS$) $\Rightarrow f(\underline{v}) = \underline{0} \Rightarrow f = 0$ or $\underline{v} = \underline{0}$, a contradiction. Since ℓ and PS both lie in $\Pi_{SR,SP}$, there is a unique point T common to ℓ and PS. If $\underline{z}$ is the vector containing (P,T) we assert $f(\underline{z}) = \underline{u}$. This follows from Prop. 4.3 as $P \neq T$. ($P = T \Rightarrow \ell \parallel PR \parallel RS \Rightarrow PQ = PR = RS \Rightarrow S \in PQ \Rightarrow \underline{u} \parallel \underline{v}$, a contradiction.)

We have shown that one satisfactory $\underline{w}$ exists. We now show that only one exists, for let $(P, U) \in \underline{w}$ and $f(\underline{w}) = \underline{u}$. Then by Prop. 4.3, $RS \parallel QU$ and $U = T$ where T is the uniquely determined point of the preceding paragraph. Thus $\underline{w}$ is uniquely determined as the vector containing (P,T).

We are now in a position to define addition and multiplication of scalars.

Definition 4.2 For any $f, g \in \Sigma$ and any $\underline{v} \in \mathcal{V}$, the sum and product of f and g, denoted by "$f+g$" and "fg" respectively are defined by

(1) $(f+g)\underline{v} = f(\underline{v}) + g(\underline{v})$ and

(2) $(fg)\underline{v} = f(g(\underline{v}))$

Proposition 4.4 Σ is closed under the operations of addition and multiplication.

Proof Let f and g be any functions satisfying Def. 4.1. It suffices to show that $f+g$ and fg also satisfy Def. 4.1.

For any $\underline{u}, \underline{v} \in \mathcal{V}$, $(f+g)\underline{u} - (f+g)\underline{v} = f(\underline{u}) + g(\underline{u}) - (f(\underline{v}) + g(\underline{v})) = f(\underline{u}) + g(\underline{u}) - f(\underline{v}) - g(\underline{v}) = f(\underline{u}) - f(\underline{v}) + g(\underline{u}) - g(\underline{v})$, where we have used the readily proved fact that $-(f(\underline{v}) + g(\underline{v})) = -f(\underline{v}) - g(\underline{v})$. Since $f(\underline{u}) - f(\underline{v})$ and $g(\underline{u}) - g(\underline{v})$ are each collinear with $\underline{u} - \underline{v}$, let P be any point and Q, R, and S the unique points such that $(P, Q) \in \underline{u} - \underline{v}$, $(P, R) \in f(\underline{u}) - f(\underline{v})$, and $(P,S) \in g(\underline{u}) - g(\underline{v})$. By

Prop. 3.6 and Axs. 2.1, 2, and 9, it follows that P, Q, R, and S are collinear, whence $f(\underline{u}) - f(\underline{v}) \parallel g(\underline{u}) - g(\underline{v})$. If T is the unique point such that $(R, T) \in g(\underline{u}) - g(\underline{v})$, it follows from the definition of collinearity of vectors that P, R, and T are collinear. As $(P,T) \in f(\underline{u}) - f(\underline{v}) + g(\underline{u}) + g(\underline{v}) = (f+g)\underline{u} - (f+g)\underline{v}$, the collinearity of P, T, and Q implies $\underline{u} - \underline{v} \parallel (f+g)\underline{u} - (f+g)\underline{v}$, by Prop. 3.6. Since $(f+g)\,\underline{0} = f(\underline{0}) + g(\underline{0}) = \underline{0}+\underline{0} = \underline{0}$, $f+g$ is a scalar.

Similarly, for any $\underline{u}, \underline{v} \in \mathcal{V}$, $(fg)\underline{u} - (fg)\underline{v} = f(g(\underline{u})) - f(g(\underline{v})) \parallel g(\underline{u}) - g(\underline{v}) \parallel \underline{u} - \underline{v}$, since $g(\underline{u}), g(\underline{v}) \in \mathcal{V}$. Since $\underline{0} \parallel \underline{a}$ for all $\underline{a} \in \mathcal{V}$, without loss of generality, $f \neq 0 \neq g$ and $\underline{u} \neq \underline{v}$, whence Prop. 3.4 $\Rightarrow$ $(fg)\underline{u} - (fg)\underline{v} \parallel \underline{u} - \underline{v}$. Finally, $(fg)\,\underline{0} = f(g(\underline{0})) = f(\underline{0}) = \underline{0}$ and so fg is also a scalar.

<u>Theorem 4.4</u> For all scalars f and g,

(i) $fg = 0,\ f \neq 0 \Rightarrow g = 0$; $gf = 0,\ f \neq 0 \Rightarrow g = 0$.

(ii) $f + 0 = 0 + f = f$.

(iii) $g \neq 0 \Rightarrow$ there exists a unique $t \in \Sigma$ such that $f = tg$.

(iv) $f(gh) = (fg)h$; $(fg)^{-1} = g^{-1}f^{-1}$, if either side exists.

(v) For all $h \in \Sigma$, $(g+h)f = gf + hf$.

(vi) $f + (g+h) = (f+g) + h$.

(vii) $f + g = g + f$.

(viii) there exists a unique scalar $h \in \Sigma$ such that $f + h = h + f = 0$.

<u>Proof</u> (i) Suppose $fg = 0 \neq f$. For arbitrary non zero vector $\underline{v}$, Cor. 4.2.2 $\Rightarrow \underline{0} = (fg)\underline{v} = f(g(\underline{v})) \Rightarrow \underline{0} = g(\underline{v}) \Rightarrow g = 0$.

Similarly, if $gf = 0 \neq f$, then for arbitrary non zero vector $\underline{v}$, $f(\underline{v}) \neq \underline{0}$ and $\underline{0} = (gf)\underline{v} = g(f(\underline{v})) \Rightarrow g = 0$.

(ii) Let $\underline{v}$ be arbitrary non zero vector. Then we have $(f+0)\underline{v} = f(\underline{v}) + 0(\underline{v}) = f(\underline{v}) + \underline{0} = f(\underline{v})$. Similarly, $(0+f)\underline{v} = f(\underline{v})$. By Th.

4.2, $\underline{v} \neq \underline{0} \Rightarrow 0 + f = f = f + 0$.

(iii) Let $\underline{v}$ be arbitrary non zero vector. By Prop. 3.4 and Cor. 2.4.2, $f(\underline{v}) \parallel g(\underline{v})$ unless $f(\underline{v}) = \underline{0}$. Since $\underline{0} \parallel \underline{a}$, for all $\underline{a} \in \mathcal{V}$, it follows that $f(\underline{v}) \parallel g(v) \neq \underline{0}$. By Th. 4.2, there exists a unique scalar h such that $f(\underline{v}) = h(g(\underline{v})) = (hg)\underline{v}$. From Prop. 4.4 and Th. 4.2, we get $f = hg$, whence h is a satisfactory t. Conversely, $f = tg \Rightarrow tg(\underline{v}) = hg(\underline{v}) \Rightarrow t = h$, since $g(\underline{v}) \neq \underline{0}$. Thus h is unique such that $f = hg$.

In particular, we denote the unique scalar h corresponding to $g \neq 0$ such that $1 = hg$ by "g^{-1}" and call h "<u>the multiplicative inverse of g</u>".

(iv) Let $\underline{v}$ be arbitrary non-zero vector. Then $(f(gh))\underline{v} = f((gh)\underline{v}) = f(g(h(\underline{v}))) = (fg)(h(\underline{v})) = ((fg)h)\underline{v}$, whence Th. 4.2 implies $(fg)h = f(gh)$.

By (i), if f^{-1} and g^{-1} exist, then $f \neq 0 \neq g$, $(fg) \neq 0$ and $(fg)^{-1}$ exists and, of course, is unique. By the result of the preceding paragraph $(g^{-1}f^{-1})(fg) = (g^{-1}(f^{-1}f))g = (g^{-1}(1))g = g^{-1}g = 1$, whence $g^{-1}f^{-1} = (fg)^{-1}$, as claimed. By letting $f = g = 1$, we get $1^{-1}1^{-1} = 1^{-1} \Rightarrow 1^{-1} = 1^{-1}1 = 1^{-1}(1^{-1}1) = (1^{-1}1^{-1})1 = 1^{-1}1 = 1$; that is, $1^{-1} = 1$, whence $(1^{-1})^{-1} = 1^{-1} = 1$. In general, the substitution $f = g^{-1}$ gives $g^{-1}g = 1 = 1^{-1} = (g^{-1}g)^{-1} = g^{-1}(g^{-1})^{-1}$, so that multiplication of the extreme members on the left by $(g^{-1})^{-1}$ gives $g = (g^{-1})^{-1}$; that is, $gg^{-1} = (g^{-1})^{-1}g^{-1} = 1$. Note that $(g^{-1})^{-1}$ exists, since $g^{-1} = 0 \Rightarrow 1 = g^{-1}g = 0g = 0$, a contradiction to the existence of two distinct points.

The above remarks show that $g \neq 0 \Rightarrow$ there exists a unique $t \in \Sigma$ such that $f = gt$. For $t = g^{-1}f$ clearly suffices, and $gt = gs \Rightarrow t = 1t = (g^{-1}g)t = g^{-1}(gt) = g^{-1}(gs) = (g^{-1}g)s = 1s = s$.

(v) For any non zero vector $\underline{v}$, $((g+h)f)\underline{v} = (g+h)(f(\underline{v})) = g(f(\underline{v})) + h(f(\underline{v})) = (gf)\underline{v} + (hf)\underline{v} = (gf + hf)\underline{v}$ and the conclusion follows from Th. 4.2

(vi) For any non zero vector $\underline{v}$, Th. 3.5 (i) implies that $((f+g)+h)\underline{v} = (f+g)\underline{v} + h(\underline{v}) = (f(\underline{v})+g(\underline{v}))+h(\underline{v}) = f(\underline{v}) + (g(\underline{v}) + h(\underline{v})) = f(\underline{v}) + (g+h)\underline{v} = (f+(g+h))\underline{v}$, and Th. 4.2 implies $(f+g)+h = f+(g+h)$.

(vii) For any non zero vector $\underline{v}$, Th. 3.5 (v) implies that $(f+g)\underline{v} = f(\underline{v}) + g(\underline{v}) = g(\underline{v}) + f(\underline{v}) = (g+f)\underline{v}$, and the conclusion follows as in (v) and (vi).

(viii) For any non zero vector $\underline{v}$, Th. 3.5 (iii) implies that there is a unique vector $\underline{u}$ such that $f(\underline{v}) + \underline{u} = \underline{0}$. By the definition of vector addition, it follows that $\underline{u} \parallel f(\underline{v})$. As $f(\underline{v}) \parallel \underline{v}$ by Prop. 4.1, Prop. 3.4 $\Rightarrow \underline{u} \parallel \underline{v}$, unless $f(\underline{v}) = \underline{0}$. If $f(\underline{v}) = \underline{0}$ then $f(\underline{v}) + \underline{u} = \underline{0} \Rightarrow \underline{u} = f(\underline{v}) - \underline{0} = f(\underline{v}) + (-\underline{0}) = \underline{0} + (-\underline{0}) = -0 = \underline{0} \Rightarrow \underline{u} \parallel \underline{v}$. Thus, $\underline{u} \parallel \underline{v} \neq \underline{0}$ in all possible cases.

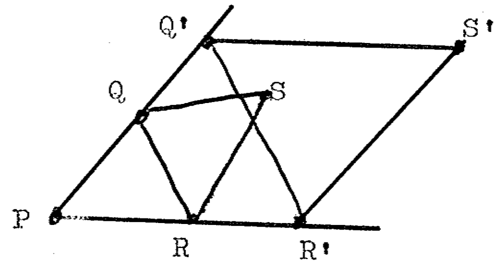
By Th. 4.2, there exists a unique scalar g such that $\underline{u} = g(\underline{v})$. Then $(f+g)\underline{v} = f(\underline{v}) + g(\underline{v}) = f(\underline{v}) + \underline{u} = \underline{0} = 0(\underline{v})$ and, as in (v), $f+g = 0$. From (vii), we get $g+f = 0$. Furthermore if $f+g = f+h$, then $g = 0+g = (g+f)+g = g+(f+g) = g+(f+h) = (g+f)+h = 0+h = h$, so that $0 = f+k = k+f \Leftrightarrow k = g$. We denote g by "$-f$" and call g "<u>the additive inverse of f</u>". By analogy with vectors, we define $f-g$ as $f+(-g)$.

<u>Theorem 4.5</u> For all scalars f and all $\underline{u}, \underline{v} \in V$, it follows that $f(\underline{u}+\underline{v}) = f(\underline{u}) + f(\underline{v})$.

<u>Proof</u>: (1) If $\underline{u} = \underline{0}$, $\underline{v} = \underline{0}$, $f = 0$, or $f = 1$, the theorem is trivial. Henceforth, we assume $\underline{u} \neq \underline{0} \neq \underline{v}$, $f \neq 0$, and $f \neq 1$.

(2) Suppose $\underline{u} \nparallel \underline{v}$, and let P be any point. Let R, R', Q, and Q' be the unique points such that $(P,R) \in \underline{u}$, $(P,R') \in f(\underline{u})$, $(P,Q) \in \underline{v}$, and (P,Q')

$\in f(\underline{v})$. Since $\underline{u} \nparallel \underline{v}$, it is clear that P, Q, R, P', Q', and R' are all distinct.



Let the line through Q (resp. Q') parallel to PR intersect the line through R (resp. R') in the unique point S (resp. S'), by Ax. 2.6. Since $(Q,S) \square (P,R)$ by construction, Prop. 3.1 and Def. 3.4 imply $(P,S) \in \underline{v} + \underline{u}$, and similarly $(P, S') \in f(\underline{v}) + f(\underline{u})$. Since Th. 2.1 $\Rightarrow S \neq S'$, it follows from Prop. 4.3 and Th. 2.6C that $P \in SS'$; thus $RS \parallel R'S' \Rightarrow f(\underline{v}+\underline{u}) = f(\underline{v}) + f(\underline{u})$, as claimed.

(3) If $\underline{u} \parallel \underline{v}$, let $\underline{w}$ be any vector such that $\underline{u} \nparallel \underline{w}$. Since Prop. 3.5 $\Rightarrow \underline{u} \nparallel \underline{v} + \underline{w}$, the preceding case implies that $f(\underline{u} + (\underline{v} + \underline{w})) = f(\underline{u}) + f(\underline{v}+\underline{w}) = f(\underline{u}) + (f(\underline{v}) + f(\underline{w})) = (f(\underline{u}) + f(\underline{v})) + f(\underline{w})$. If $\underline{u} + \underline{v} \neq \underline{0}$ then if $\underline{w} \parallel \underline{u} + \underline{v}$, there exist scalars g and h such that $\underline{w} = g(\underline{u}+\underline{v}) = g(\underline{u} + h(\underline{u})) = g(\underline{u}) + g(h(\underline{u})) = g(\underline{u}) + (gh)\underline{u} = (g+gh)\underline{u}$, so that Props. 4.1, 4 $\Rightarrow \underline{u} \parallel \underline{w}$, a contradiction. Thus, if $\underline{u}+\underline{v} \neq \underline{0}$, then $f(\underline{u} + (\underline{v} + \underline{w})) = f((\underline{u}+\underline{v}) + \underline{w}) = f(\underline{u}+\underline{v}) + f(\underline{w})$. By Th. 3.5 (iv), it follows that $f(\underline{u}) + f(\underline{v}) = f(\underline{u} + \underline{v})$.

If $\underline{u} + \underline{v} = 0$ then $\underline{u} = -\underline{v}$ and $f(\underline{u}+\underline{v}) = f(\underline{0}) = \underline{0}$. To complete the proof, we need only show that $f(-\underline{v}) + f(\underline{v}) = \underline{0}$. Since for all $\underline{Z} \in \mathcal{V}$, $\underline{Z} + (-\underline{Z}) = \underline{0} = 0\,\underline{Z} = (1 + (-1))\,\underline{Z} = 1\,\underline{Z} + (-1)\,\underline{Z} = \underline{Z} + (-1)\,\underline{Z}$, Th. 3.5 (iv) $\Rightarrow -\underline{Z} = (-1)\,\underline{Z}$. Thus, it suffices to prove that $f((-1)\underline{v}) + f(\underline{v}) = \underline{0}$. However since $f(1) = f$, Th. 4.4 (v) implies $f((-1)\underline{v}) + f(\underline{v}) = f((-1)\underline{v}) + f(1\underline{v}) = (f(-1))\underline{v} + (f1)\underline{v} = (f(-1+1))\underline{v} = (f\,0)\underline{v} = f(0\,\underline{v}) = f(\underline{0})$, and the proof is complete.

Corollary 4.5.1 For all scalars f, g, and h, it follows that $f\ (g + h) = fg + f\ h$.

Proof Let $\underline{v}$ be any non zero vector. The conclusion follows from Th. 4.2 since Th. 4.5 implies $(f\ (g + h))\ \underline{v} = f\ ((g + h)\ \underline{v}) = f\ (g(\underline{v}) + h\ (\underline{v})) = f\ (g(\underline{v})) + f\ (h\ (\underline{v})) = (f\ g)\ \underline{v} + (f\ h)\ \underline{v} = (f\ g + f\ h)\ \underline{v}$.

Corollary 4.5.2 For all scalars f and g, $-f = (-1)\ f = f\ (-1)$, $-(f + g) = -g - f$ and $-(-f) = f$.

Proof By Th. 4.4 (v), (viii), we have $0 = (1 + (-1))\ f = 1f + (-1)\ f = f + (-1)\ f = f + (-f) \Rightarrow (-1)\ f = -f$. Similarly by Th. 4.4 (viii) and Cor. 4.5.1, one may show $f\ (-1) = -f$.

Since $(-g - f) + (f + g) = (-g + (-f + f)) + g = (-g + 0) + g = -g + g = 0$, the uniqueness of $-(f + g)$ implies $-(f + g) = -g - f$.

Similarly, we may conclude that $-(-f) = f$, since $-(-f) + (-f) = (-1) \cdot (-f) + 1\ (-f) = (-1 + 1)(-f) = 0\ (-f) = 0$.

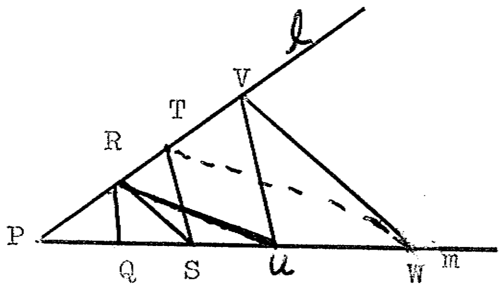
Our next corollary simply restates the results of Th. 4.4 and Cor. 4.5.1.

Cor. 4.5.3 $(\Sigma, +, \cdot)$ is a division ring.

We have shown that the scalars 0, 1, and -1 commute with all scalars under the operation of multiplication. In general, we shall see that multiplication of scalars need not be commutative. (If it were, then Th. 4.4 (v) $\Rightarrow$ Cor. 4.5.1, without the aid of Th. 4.4.) If one postulates that multiplication of scalars is indeed commutative, then Th. 4.4 and Cor. 4.5.1 imply that the set Σ of scalars forms a field under the operations of addition and multiplication. In the next chapter, we shall indicate why such an assumption is useful.

Remark In view of the preceding comments, it is clear that the commutativity of multiplication of scalars is equivalent to the major theorem of Pappus:

If ℓ and m are any distinct lines intersecting in a point P, if Q, S, U, W, R, T, and V are any points distinct from each other and also from P, if Q, S, U, W $\in \ell$, if R, T, W $\in$ m, and if RQ $\parallel$ TS $\parallel$ VU, then RS $\parallel$ VW if and only if RU $\parallel$ TW.



It is well known that the major theorem of Pappus cannot be proved on the basis of the axioms we have introduced thus far, as there exist non-commutative division rings which may be used as models for Σ.

One well known example is that of the quaternions $\{Q, +, \cdot\}$ where $Q = \{(X_1, X_2, X_3, X_4) \mid X_i \text{ real}\}$, $(X_1, X_2, X_3, X_4) + (y_1, y_2, y_3, y_4) = (X_1 + y_1, X_2 + y_2, X_3 + y_3, X_4 + y_4)$ and $(X_1, X_2, X_3, X_4) \cdot (y_1, y_2, y_3, y_4) =$
$(X_1 y_1 - X_2 y_2 - X_3 y_3 - X_4 y_4, X_1 y_2 + X_2 y_1 + X_3 y_4 - X_4 y_3, X_1 y_3 + X_3 y_1 + X_4 y_2 - X_2 y_4, X_1 y_4 + X_4 y_1 + X_2 y_3 - X_3 y_2)$.

We shall now show that in Euclidean geometry, with which the reader is likely familiar, $fg = gf$, for all scalar multiplications f and g.

Let $\underline{w}, \underline{v} \in \mathcal{V}$ be such that $\underline{w} \nparallel \underline{v}$. Suppose $(P,Q) \in \underline{v}$, $(P,R) \in f(\underline{v})$, $(P,S) \in g(\underline{v})$, $(P,T) \in \underline{w}$, $(P,U) \in f(\underline{w})$ and $(P,V) \in g(\underline{w})$. If the line from V parallel to TR intersects PQ in X, then clearly $(P,X) \in (gf)\underline{v}$. As $\underline{v} \neq \underline{0}$, to show $fg = gf$, it suffices to establish that TS $\parallel$ U X.

Without loss of generality, $f \neq 0 \neq g$, and so P is distinct from X, V, U, and S. By the theorem quoted in the first paragraph of this chapter, $\frac{PR}{PX} = \frac{PT}{PV}$ and $\frac{PR}{PS} = \frac{PU}{PV}$. Thus PX . PT = P$U$ PS (= PR . PV) and $\frac{PT}{PU} = \frac{PS}{PX}$, whence T$U$ $\parallel$ U X and the proof is complete.

Vector Equations of Lines and Planes

The theory of scalars allows us to represent equations of lines and planes in very compact form and will also furnish us with a proof that $\mathcal{V}$ forms a vector space of dimension three over the division ring Σ.

Definition 4.3 The position vector of any point Q with respect to a fixed point P called the "origin" is the vector containing (P,Q).

Lemma 4.6 Given any line ℓ through any origin P and given any point Q with position vector $\underline{q}$ with respect to P, $Q \in \ell \Leftrightarrow \underline{q} = f(\underline{v})$ where $f \in \Sigma$ and $\underline{v}$ is the position vector (with respect to P) of a fixed point $R \in \ell$ such that $R \neq P$.

Proof (i) Suppose $Q \in \ell$. Since $\underline{q} \parallel \underline{v} \neq \underline{0}$ by Prop. 3.2 (ii), the existence of a unique satisfactory f is guaranteed by Th. 4.2.

(ii) If $\underline{q} = f(\underline{v})$, then Prop. 4.1 implies $f(\underline{v}) \parallel \underline{v}$ and Prop. 3.6 implies P, Q, and R are collinear. Thus $R \neq P \Rightarrow Q \in \ell$, by Axs. 2.1, 2.

Theorem 4.6 Suppose that ℓ is a line determined by (distinct) points Q and R, with position vectors $\underline{q}$ and $\underline{r}$ respectively, such that ℓ does not contain the origin P. Then any point T, with position vector $\underline{t}$, lies on ℓ if and only if there exists a scalar f such that $\underline{t} = f(\underline{r}) + (1 - f)(\underline{q})$.

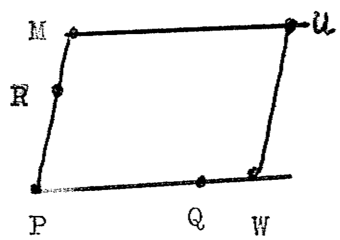
Proof (i) Suppose $T \in \ell$. By considering the representation of $\mathcal{V}$ at Q, we may infer from the preceding lemma that a unique scalar f exists such that $\underline{t} - \underline{q} = f(\underline{r}) + f(-\underline{q}) = f(\underline{r}) - f(\underline{q})$, by Cor. 4.5.2. Hence $\underline{t} = f(\underline{r}) - f(\underline{q}) + \underline{q} = f(\underline{r}) + (1 - f)\underline{q}$, as claimed.

(ii) If $\underline{t} = f(\underline{r}) + (1 - f)\underline{q}$, then we may reverse the steps of (i) to obtain $\underline{t} - \underline{q} = f(\underline{r} - \underline{q})$, whence $\underline{t} - \underline{q} \parallel \underline{r} - \underline{q}$, by Prop. 4.1. By considering the representation of $\mathcal{V}$ at Q, we may infer from Prop. 3.6 that Q, T, and R are collinear. Thus, as above, $T \in \ell$.

We remark that the vector equation of a line given in Th. 4.6 is completely general, for $\underline{q} = f(\underline{v}) \Rightarrow \underline{q} = f(\underline{v}) + (1-f)(\underline{0})$, and if $\underline{q} = f(\underline{v}) + (1-f)\underline{w}$ where $\underline{w} = h(\underline{v})$, then $\underline{q} = (f + h - fh)\underline{v}$.

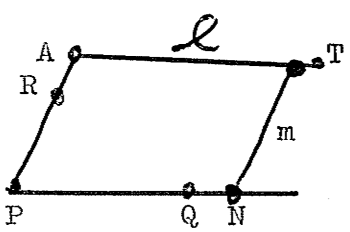
Lemma 4.7 If the plane Π is determined by the origin P and the points Q and R, with position vectors $\underline{q}$ and $\underline{r}$ respectively, then the point T (with position vector $\underline{t}$) is in Π if and only if there exist scalars f and g such that $\underline{t} = f(\underline{q}) + g(\underline{r})$.

Proof (i) Let $\underline{t} = f(\underline{q}) + g(\underline{r})$, $(P,W) \in f(\underline{q})$, and $(P,M) \in g(\underline{r})$.



Let the line through M parallel to PQ intersect the line through W parallel to PR at U. It follows from Def. 2.1 that $U \in \Pi$. Since Prop. 3.1 $\Rightarrow (P,U) \in f(\underline{q}) + g(\underline{r})$, the unique representation of V at P implies $U = T$. Thus $T \in \Pi$.

(ii) Suppose $T \in \Pi$. By Cor. 2.4.1, there exist lines ℓ and m such that $\ell \parallel PQ$, $m \parallel PR$, $T \in \ell$, $T \in m$, $\ell \cap PR \neq \emptyset$, and $m \cap PR \neq \emptyset$. It follows from Th. 2.2 (i) that $\ell \cap PR$ and $m \cap PR$ each consist of single points, say A and N respectively. Now, Th. 4.2 $\Rightarrow$ there exist unique scalars f and g such that



that $\underline{a} = g(\underline{r})$ and $\underline{n} = f(\underline{q})$. By Prop. 3.1, $(N,T) \uparrow (P,A)$, whence $(N,T) \in g(\underline{r})$. Thus, by Def. 3.4, $\underline{t} = f(\underline{q}) + g(\underline{r})$, as required. Since $\underline{q} \nparallel \underline{r}$, it follows that f and g are uniquely determined, for $t = f_1(\underline{q}) + g_1(\underline{r}) = f_2(\underline{q}) + g_2(\underline{r}) \Rightarrow (f_1 - f_2)\,\underline{q} \parallel (g_1 - g_2)\,\underline{r} \Rightarrow \underline{q} \parallel \underline{r}$, a contradiction unless $f_1 - f_2 = g_1 - g_2 = 0$.

Theorem 4.7 If the plane Π is determined by the points R, S, and T, the origin $P \notin \Pi$, and Q is any point, then $Q \in \Pi \Leftrightarrow$ there exist scalars f and g such that $\underline{q} = f(\underline{r}) + g(\underline{s}) + (1 - f - g)\,\underline{t}$.

Proof (i) If $\underline{q} = f(\underline{r}) + g(\underline{s}) + (1 - f - g)\,\underline{t}$, then $\underline{q} - \underline{t} = f(\underline{r} - \underline{t}) + g(\underline{s} - \underline{t})$. Interpreting this statement in terms of the representation of $\mathcal{V}$ at T, we may infer from the preceding lemma that Q lies in the plane of R, S, and T; that is, $Q \in \pi$.

(ii) If $Q \in \pi$, then by representing $\mathcal{V}$ at T, we may use Lemma 4.7 to establish the existence of unique scalars f and g such that $\underline{q} - \underline{t} = f(\underline{r} - \underline{t}) + g(\underline{s} - \underline{t})$, whence the stated result follows.

As with Th. 4.6, the vector equation of Th. 4.7 may be shown to be completely general.

Definition 4.4 The vectors $\underline{u}$, $\underline{v}$, and $\underline{w}$ are said to be coplanar iff for any point P, $(P,Q) \in \underline{u}$, $(P,R) \in \underline{v}$, $(P,S) \in \underline{w} \Rightarrow$ P, Q, R, and S are co-planar.

Notice that, for all vectors $\underline{u}$ and $\underline{v}$, it follows that $\underline{0}$, $\underline{u}$, and $\underline{v}$ are coplanar. Furthermore, for all vectors $\underline{u}$ and $\underline{v}$ and all scalars f, it also follows that $\underline{u}$, $f(\underline{u})$, and $\underline{v}$ are coplanar.

Proposition 4.5 The vectors $\underline{u}$, $\underline{v}$, and $\underline{w}$ are coplanar iff they satisfy Def. 4.4 for (at least) one point P.

Proof Suppose $(P,Q) \in \underline{u}$, $(P,R) \in \underline{v}$, $(P,S) \in \underline{w}$ and P, Q, R, and S are co-planar. Now if, P, Q, R, and S fail to determine a unique plane, then $\underline{u}$, $\underline{v}$, and $\underline{w}$ are parallel, and hence satisfy Def. 4.4 at any point. Otherwise, let P, Q, R, and S determine the unique plane π. It suffices to show $P' \notin \pi \Rightarrow \underline{u}, \underline{v}, \underline{w}$ satisfy Def. 4.4 at P'. Without loss of generality, none of $\underline{u}$, $\underline{v}$, $\underline{w}$ is $\underline{0}$ or a scalar multiple of another vector in the set $\{\underline{u}, \underline{v}, \underline{w}\}$. Since P, Q, R, S $\in \pi$, Lemma 4.7 $\Rightarrow$ there exist scalars f and g such that $\underline{u} = f(\underline{v}) + g(\underline{w})$. If $(P', Q') \in \underline{u}$, $(P', R') \in \underline{v}$ and $(P', S') \in \underline{w}$, the last equation implies P', Q', R', and S' are coplanar.

Theorem 4.8 Let P be any point and $\underline{u}$, $\underline{v}$, and $\underline{w}$ any **noncoplanar** vectors such that $(P,Q) \in \underline{u}$, $(P,R) \in \underline{v}$, and $(P,S) \in \underline{w}$. For any vector $\underline{x}$, there exist unique scalars α, β, and γ such that $\underline{x} = \alpha\underline{u} + \beta\underline{v} + \gamma\underline{w}$.

Proof (i) There is at most one such ordered triple (α, β, γ). For if $\alpha_1\underline{u} + \beta_1\underline{v} + \gamma_1\underline{w} = \alpha_2\underline{u} + \beta_2\underline{v} + \gamma_2\underline{w}$ where, say, $\alpha_1 \neq \alpha_2$, we find $(\alpha_1 - \alpha_2)\underline{u} = (\beta_2 - \beta_1)\underline{v} + (\gamma_2 - \gamma_1)\underline{w}$. Now $\alpha_1 \neq \alpha_2 \Rightarrow \alpha_1 - \alpha_2 \neq 0 \Rightarrow (\alpha_1 - \alpha_2)^{-1}$ exists $\Rightarrow \underline{u} = (\alpha_1 - \alpha_2)^{-1}(\beta_2 - \beta_1)\underline{v} + (\alpha_2 - \alpha_1)^{-1}(\gamma_2 - \gamma_1)\underline{w}$. Using Lemma 4.7, we may conclude that P, Q, R, and S are coplanar, a contradiction to the fact that $\underline{u}$, $\underline{v}$, and $\underline{w}$ are noncoplanar.

(ii) We must now exhibit the existence of (α, β, γ). If $\underline{x} \parallel \underline{w} (\neq \underline{0})$, then Th. 4.2 $\Rightarrow$ there exists f such that $\underline{x} = f(\underline{w})$. In this case, we need only let $(\alpha, \beta, \gamma) = (0, 0, f)$. Otherwise, if $(P, T) \in \underline{x}$, then P, T, and S determine a plane distinct from that of P, R, and Q, since P, Q, R, and S are noncoplanar. These planes intersect in a line ℓ containing P. Let $W \in \ell$ such that $W \neq P$. Then if $(P,W) \in \underline{y}$, $\underline{y} \neq \underline{0}$, and by Lemma 4.7, there exist scalars a, b, c, and d such that $\underline{y} = a\underline{w} + b\underline{x} = c\underline{u} + d\underline{v}$. Now P, Q, R, and S are not coplanar and $\underline{y} \neq \underline{0} \Rightarrow$ $b \neq 0 \Rightarrow b^{-1}$ exists $\Rightarrow \underline{x} = (b^{-1}c)\underline{u} + (b^{-1}d)\underline{v} + (-b^{-1}a)\underline{w}$. This equation displays appropriate values of α, β, and γ.

Corollary 4.8 $\mathcal{V}$ forms a vector space of dimension three over the division ring Σ.

Proof The proof is immediate, from the uniqueness of α, β, γ in the preceding theorem.

Comment For a fixed noncoplanar set of vectors $\underline{u}$, $\underline{v}$, and $\underline{w}$, each $\underline{x} \in \mathcal{V}$ may be associated with the unique triple (α, β, γ) such that $\underline{x} = \alpha\underline{u} + \beta\underline{v} + \gamma\underline{w}$. (α, β, γ) is called the affine coordinate of $\underline{x}$ with respect to $(\underline{u}, \underline{v}, \underline{w})$. By mapping $\underline{x}$ into (α, β, γ), we may define a function from $\mathcal{V}$ into $\Sigma \times \Sigma \times \Sigma$. Since vector addition is well defined, this mapping is both one-to-one and on to.

Let $\underline{X}$ be as above and $f \in \Sigma$. Then $\underline{X} \longleftrightarrow (\alpha, \beta, \gamma)$ and $f(\underline{X}) \longleftrightarrow (f\alpha, f\beta, f\gamma)$, by Th. 4.5. Furthermore if $\underline{y} \longleftrightarrow (\delta, \epsilon, n)$, then $\underline{X} + \underline{y} \longleftrightarrow (\alpha + \delta, \beta + \epsilon, \gamma + n)$ by Th. 3.5 (i), (v). Thus $\mathscr{V}$ is isomorphic to the vector space $\Sigma \times \Sigma \times \Sigma$ under Σ in which addition of vectors (i.e., elements of $\Sigma \times \Sigma \times \Sigma$) is defined componentwise and multiplication of vectors by a scalar is defined distributively. This observation explains why some authors define a vector as an ordered triple (n - tuple) of entities called "scalars". We shall see in Chapter V that the isomorphism of $\mathscr{V}$ and $\Sigma \times \Sigma \times \Sigma$ is very important and useful.

Given a vector space V of dimension three over a division ring D, we shall indicate how one may define various geometric concepts so that each of Axs. 2.1 - 9 is verified, i.e. so that the defined geometry is three-dimensional affine in the sense of Chapter II.

Let $\mathcal{a}' = \{\mathcal{P}', \mathcal{L}', \|', \in'\}$ be defined as follows. $\mathcal{P} = \{v \mid v \in V\}$ is the set of points and $\mathcal{L}$ the set of lines, where a line ℓ is $\{au + (1 - a)v \mid a \in D\}$ for fixed distinct $u, v \in V$. We define $P \in' \ell \Longleftrightarrow P \in \ell$ to mean that P lies on, or is contained in, ℓ. The line $\ell = \{au + (1 - a)v\}$ is said to parallel to the line $m = \{bw + (1 - b)Z\}$ if there exists $d \in D$ such that $v - u = d(Z - W)$. The parallelism of ℓ to m is denoted by $\ell \|' m$. A plane then is $\{au + bv + (1 - a - b)w \mid a, b \in D\}$ for fixed distinct $u, v, w \in V$ such that none of u, v, w lies on a line containing the other two.

We note first that parallelism is well defined. Suppose $\ell = \{au + (1 - a)v\} \|' m = \{bw + (1 - b)Z\}$ such that $v - u = d(Z - W)$. If we may also represent ℓ as $\{a'u' + (1 - a')v'\}$ and m as $\{bw' + (1 - b')Z'\}$, then there exist a'', a''', b'', b''' $\in$ D such that $u = a''u' + (1 - a'')v'$, $v = a'''u' + (1 - a''')v'$, $W = b''w' + (1 - b'')Z'$, and $Z = b'''w' +$

$(1 - b''')Z'$. Since $V - U = d\,(Z - W)$, it is readily verified that $V' - U' = (a'' - a''')^{-1} d\,(b'' - b''')(Z' - W')$ unless $a'' - a''' = 0$. However $a'' - a''' = 0 \Rightarrow a'' = a''' \Rightarrow U = V$, a contradiction. Since $(a'' - a''')^{-1} d\,(b'' - b''') \in D$, parallelism is well defined.

Note that Ax. 2.2 follows immediately from the definition of $\mathcal{L}'$, as $0 \neq 1$. For the same reason, the line $\ell = \{aU + (1 - a)V\}$ contains the distinct points U, V. Thus Ax. 2.3 has been verified.

It is a well known result from the algebraic theory of vector spaces that $0V = 0$, for all $V \in V$. Suppose the line $\ell = \{aU + (1 - a)V\}$ is parallel to $m = (bW + (1 - b)Z\}$; i.e., $\exists\, d \in D$ such that $V - U = d \cdot (Z - W)$. By the above remark, $d = 0 \Rightarrow V - U = 0 \Rightarrow V = U$, a contradiction. Hence $Z - W = d^{-1}(V - U)$, $d^{-1} \in D$, whence $m \parallel' \ell$ and $\parallel'$ is symmetric. The reflexivity and transitivity of $\parallel'$ are immediate. This verifies Ax. 2.4.

Suppose $\ell = \{aU + (1 - a)V\} \in \mathcal{L}'$ and $W \in \mathcal{P}'$. Since $U \neq V$, $W \neq W - (U - V)$ and so $W \in' m = \{aw + (1 - a)(w - (U - V))\} \parallel' \ell$. If $n \in \mathcal{L}'$ is such that $W \in' n = \{bt + (1 - b)Z\} \parallel' \ell$, there exists c, $d \in D$ such that $W = ct + (1 - c)Z$ and $Z - t = d\,(V - U)$. By the remarks of the preceding paragraph, $d \neq 0$ and so $W - (U - V) = ct + (1 - c)Z + d^{-1} \cdot (Z - t) = (c - d^{-1})t + (1 - (c - d^{-1}))Z \in n$. As $W \in' n$, Ax. 2.1 $\Rightarrow m = n$ whence Ax. 2.5 follows, once Ax. 2.1 is proved.

If the line $\ell = \{aU + (1 - a)V\}$ contains distinct points $x = \alpha U + (1 - \alpha)V$ and $y = \beta U + (1 - \beta)V$, we prove $\ell = m = \{bx + (1 - b)y\}$. For $m \subset \ell$; it suffices then to show $U, V \in m$. As $x \neq y$, $\alpha \neq \beta$, and so $x - y = (\alpha - \beta)U + (\beta - \alpha)V$, $(\beta - \alpha)^{-1}(x - y) = V - U$. Thus $V = x - \alpha(U - V) = x + \alpha(\beta - \alpha)^{-1}(x - y) = bx + (1 - b)y \in m$, with $b = 1 + \alpha(\beta - \alpha)^{-1}$. Similarly $U \in m$, and so Ax. 2.1 is proved.

Let u, v, w be distinct non collinear points, ℓ' the unique line containing v, w and m' the unique line containing u, v. By the remarks of the second last paragraph, the unique line through u parallel to ℓ' is $\ell = \{au + (1 - a)(u - (v - w))\}$. Similarly, the unique line through w parallel to m' is $m = \{bw + (1 - b)(w - (u - v))\}$. To verify Ax. 2.6, it suffices to display a solution for a and b in the equation $au + (1 - a)(u - (v - w)) = bw + (1 - b)(w - (u - v))$. Such a solution is $a = 0$, $b = 1 + 1$.

We shall now verify Ax. 2.7. Let $\Pi = \{au + bv\}$ and ℓ, m non-parallel lines of Π. Thus there exist $v_1, v_2 \in \ell \cap \Pi$ and there exist $v_3, v_4 \in m \cap \Pi$. Furthermore, there exist a_i, b_i ($i = 1 - 4$) $\in D$ such that $v_i = a_i u + b_i v$ ($i = 1 - 4$). Since $\ell \nparallel' m$, for all $d \in D$, $v_2 - v_1 \neq d(v_4 - v_3)$.

Consider the equation $av_1 + (1 - a)v_2 = bv_3 + (1 - b)v_4$ in the unknowns a, b. Substituting for v_i ($i = 1 - 4$), we see that it suffices to solve the equations

$$a(a_1 - a_2) + b(a_4 - a_3) = a_4 - a_2$$

and $$a(b_1 - b_2) + b(b_4 - b_3) = b_4 - b_2,$$ obtained by equating corresponding coefficients of u and v. This system of equations has a solution for a, b, unless the left column rank of the matrix

$$\begin{pmatrix} a_1 - a_2 & a_4 - a_3 \\ b_1 - b_2 & b_4 - b_3 \end{pmatrix}$$ is less than two.

However, if the left column rank is less than two, the vector space of column vectors is of dimensionality at most one, a contradiction to the fact that for all $d \in D$, $v_2 - v_1 \neq d(v_4 - v_3)$.

If the above problem is altered by letting $\Pi = \{au + bv + (1 - a - b)w\}$

$= \{ a(u - w) + b(v - w) + w \}$, then the non-parallel lines ℓ and m of π intersect. For the sets $\ell' = \{ v - w \mid v \in \ell \}$ and $m' = \{ x - w \mid x \in m \}$ are clearly nonparallel lines of $\{ y - w \mid y \in \pi \}$, which is evidently a plane. By the previous case, $\ell' \cap m' \neq \emptyset$, whence $\ell \cap m \neq \emptyset$. This proves Ax. 2.7.

If $\pi = \{ au + bv \}$, then it is clear that π is closed under addition; that is $x \in \pi, y \in \pi \Rightarrow x + y \in \pi$. Thus π is a subspace of V of dimensionality two, for if there exists $d \in D$ such that $u = dv$, then 0, u, and v are collinear, a contradiction to the definition of a plane. Similarly, $\pi^{*} = \{ ft + gw \}$ is also a subspace of V of dimensionality two. It is well known from the algebraic theory of vector spaces that

$$\text{dimension } (\pi_1) + \text{dimension } (\pi_2) - \text{dimension } (\pi_1 + \pi_2) = \text{dimension } (\pi_1 \cap \pi_2).$$

Since dimension $(\pi_1 \cap \pi_2) \leqq$ dimension (V) = 3, it follows that dimension $(\pi_1 \cap \pi_2) \geqq 1$. Hence, any two planes intersecting at $0 \in P'$ have at least one line in common.

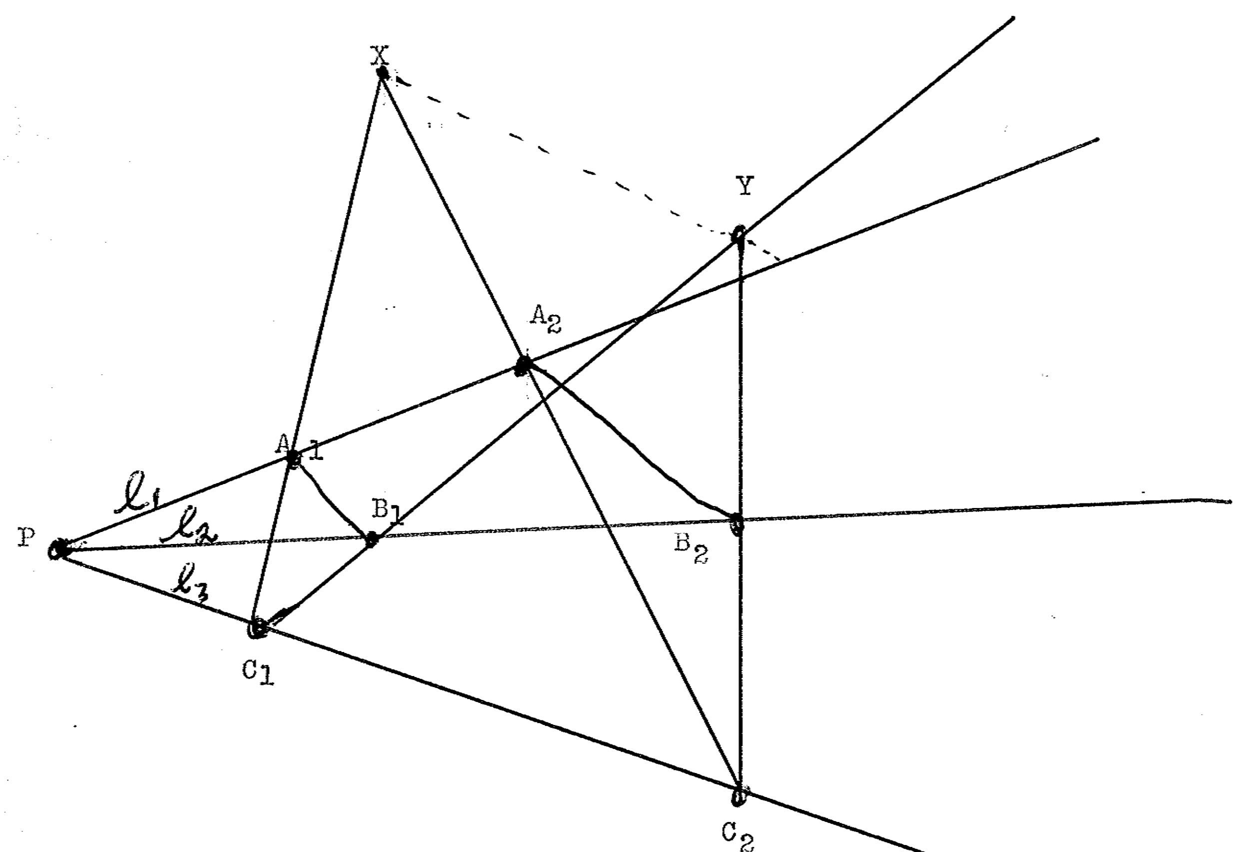
By the same device used to prove Ax. 2.7 in the general case, one may now prove Ax. 2.8, namely that any two intersecting planes have at least one line in common. The details of the argument are left to the reader.

Finally, as V is isomorphic to D X D X D, it is easily checked that (0, 0, 0), (1, 0, 0), (0, 1, 0), and (0, 0, 1) are four non-coplanar points. This verifies Ax. 2.9 and completes the demonstration.

We shall now apply the theory of scalars to the study of Desarquesian configurations. In Chapter II, four results of interest were established (Th. 2.6 A - D). We now investigate eight additional cases (Th. 4.9 A - H).

<u>Theorem 4.9A</u> Let ℓ_1, ℓ_2, and ℓ_3 be any distinct lines and P, A_i, B_i, C_i ($i = 1, 2$) be any points such that $A_i \in \ell_1$, $B_i \in \ell_2$, and $C_i \in \ell_3$ ($i = 1, 2$),

$A_i \neq B_i \neq C_i \neq A_i$ ($i = 1, 2$), $P \neq A_1$, $P \neq B_1$, and $P \neq C_1$. Then if uniquely determined distinct points X and Y exist such that $A_1C_1 \cap A_2C_2 = \{X\}$ and $B_1C_1 \cap B_2C_2 = \{Y\}$, and if $A_1B_1 \parallel A_2B_2$, then $B_1A_1 \parallel XY$.



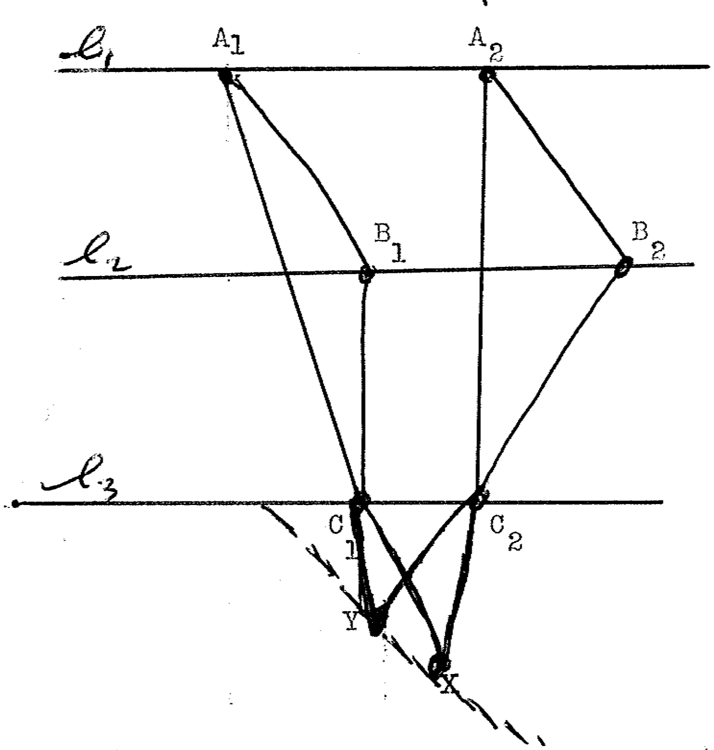
<u>Proof</u> Let $(P, A_1) \in \underline{u}$, $(P, C_1) \in \underline{v}$, and $(P, B_1) \in \underline{w}$. From Prop. 3.2, (ii), it follows that $\underline{u}$, $\underline{v}$, and $\underline{w}$ are each non zero. Thus Th. 4.2 implies that there exist unique scalars f and g such that $(P, A_2) \in f(\underline{u})$ and $(P, C_2) \in g(\underline{v})$. By Prop. 4.3, $A_1B_1 \parallel A_2B_2 \Rightarrow (P, B_2) \in f(\underline{w})$. Furthermore, $f \neq g$ since $f = g \Rightarrow A_1C_1 \parallel A_2C_2$ by Prop. 4.3, a contradiction to the existence of a unique point X such that $X \in A_1C_1 \cap A_2C_2$. Thus $f - g \neq 0$ and $(f - g)^{-1}$ exists.

Since $A_1C_1 \cap A_2C_2 = \{X\}$, Th. 4.6 $\Rightarrow$ there exist unique scalars α

and β such that $\underline{X} = \alpha \underline{u} + (1-\alpha)\underline{v} = \beta f(\underline{u}) + (1-\beta) g(\underline{v})$. Similarly, there exist unique scalars γ and δ such that $\underline{Y} = \gamma \underline{v} + (1-\gamma)\underline{w} = \delta g(\underline{v}) + (1-\delta) f(\underline{w})$. Since $\underline{u} \nparallel \underline{v}$, the first of these equations implies $\alpha = \beta f$ and $1-\alpha = (1-\beta) g$. Similarly, $\underline{v} \nparallel \underline{w} \Rightarrow \gamma = \delta g$ and $1-\gamma = (1-\delta) f$. Thus $(1-\beta) g = 1-\alpha = 1-\beta f = g - \beta g \Rightarrow 1 - g = \beta f - \beta g = \beta(f-g) \Rightarrow \beta = (1-g)(f-g)^{-1}$. Consequently, $1-\beta = (f-g)(f-g)^{-1} - (1-g)(f-g)^{-1} = (f-g-1+g)(f-g)^{-1} = (f-1)(f-g)^{-1}$. Thus $\underline{X} = (1-g)(f-g)^{-1} f(\underline{u}) + (f-1)(f-g)^{-1} g(\underline{v})$. Similarly, $\underline{Y} = (1-f)(g-f)^{-1} g(\underline{v}) + (g-1)(g-f)^{-1} f(\underline{w})$.

It is straightforward to verify that $\underline{X} - \underline{Y} = (1-g) f (f-g)^{-1}(\underline{u} - \underline{w})$. Thus Prop. 4.1 $\Rightarrow \underline{X}-\underline{Y} \parallel \underline{u}-\underline{w}$. Since $(X, Y) \in (\underline{X}-\underline{Y})$ and $(B_1, A_1) \in (\underline{u}-\underline{w})$, if $(B_1, D) \in (\underline{X}-\underline{Y})$, then Prop. 3.6 $\Rightarrow B_1$, D, and A_1 are collinear. However, $(B_1, D)\uparrow (X, Y)$, $X \neq Y \Rightarrow B_1D \parallel XY$. Thus $B_1D = B_1A_1$ (by Axs. 2.1, 2), whence $B_1A_1 \parallel XY$. Q.E.D.

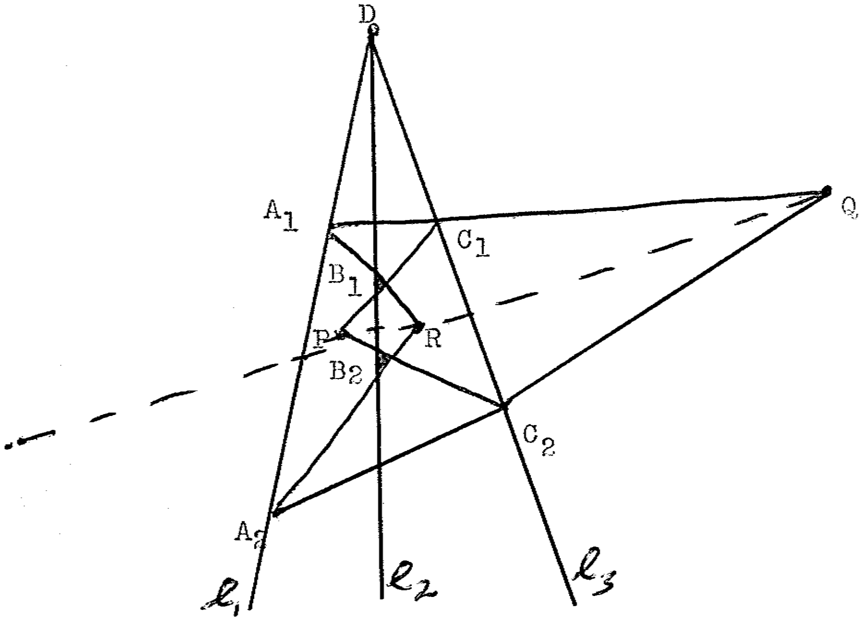
Theorem 4.9B Let ℓ_1, ℓ_2, and ℓ_3 be any three distinct parallel lines and A_i, B_i, C_i $(i = 1, 2)$ any distinct points such that $A_i \in \ell_1$, $B_i \in \ell_2$, and $C_i \in \ell_3$ $(i = 1, 2)$. Then if uniquely determined points X and Y exist such that $A_1C_1 \cap A_2C_2 = \{X\}$ and $B_1C_1 \cap B_2C_2 = \{Y\}$, and if $A_1B_1 \parallel A_2B_2$, then $A_1B_1 \parallel XY$.



<u>Proof</u> Represent $\mathcal{V}$ at any origin $P \in \ell_1$. By the conditions of the theorem, $(A_1, A_2) \sqcap (B_1, B_2) \Rightarrow (A_1, A_2) \uparrow (B_1, B_2)$ (by Prop. 3.1) $\Rightarrow \underline{a}_2 - \underline{a}_1 = \underline{b}_2 - \underline{b}_1 \neq \underline{0}$. By Th. 4.2, $A_1B_1 \parallel A_2B_2 \Rightarrow$ there exists unique scalars h and g such that $\underline{c}_2 - \underline{c}_1 = g(\underline{a}_2 - \underline{a}_1)$ and $\underline{b}_1 - \underline{a}_1 = h(\underline{b}_2 - \underline{a}_2)$. As in the preceding theorem, there exist unique scalars α, β, γ, and δ such that $\underline{X} = \alpha \underline{a}_1 + (1 - \alpha)\underline{c}_1 = \beta \underline{a}_2 + (1 - \beta)\underline{c}_2$ and $\underline{y} = \gamma \underline{b}_1 + (1 - \gamma)\underline{c}_1 = \delta \underline{b}_2 + (1 - \delta)\underline{c}_2$. Thus $\underline{X} = \alpha \underline{a}_1 + (1 - \alpha)\underline{c}_1 = \beta \underline{a}_2 + (1 - \beta)(\underline{c}_1 + g(\underline{a}_2 - \underline{a}_1))$, whence by the method of the preceding theorem, $\alpha = (\beta - 1)g$, $0 = \beta + (1 - \beta)g$, and $1 - \alpha = 1 - \beta$. (We need only show that no point other than P of PC_1 can lie on ℓ_1. This is clear since any assumption to the contrary implies $\ell_1 = \ell_3$ (by Axs. 2.1, 2, Th. 2.1), a contradiction.)

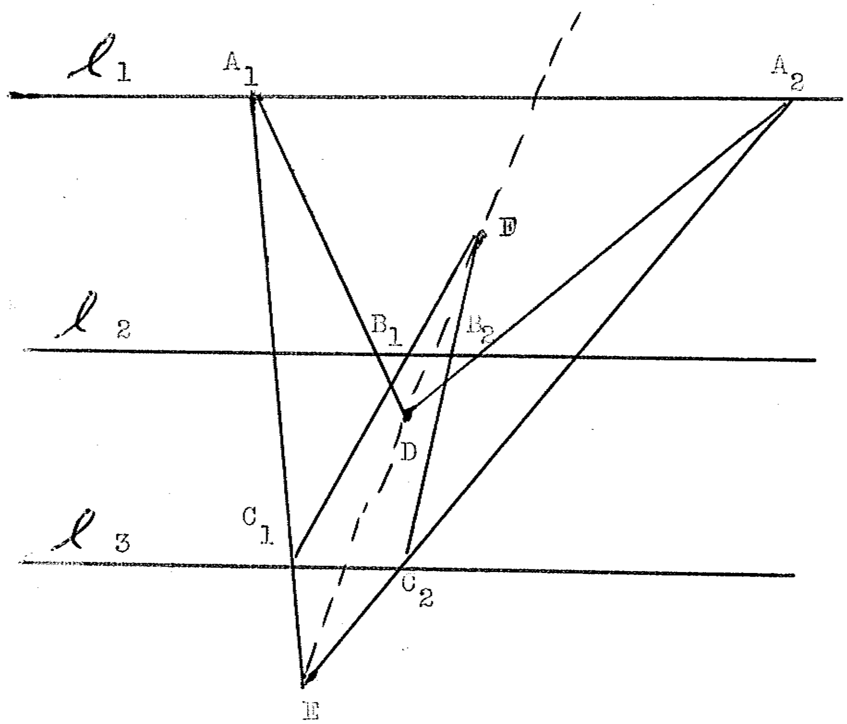
It may be readily shown that $\beta = \alpha = g(g - 1)^{-1}$ and so $\underline{X} = g(g - 1)^{-1}\underline{a}_1 + (-1)(g - 1)^{-1}\underline{c}_1$. Similarly, $\underline{y} = g(g - 1)^{-1}\underline{b}_1 + (-1)(g - 1)^{-1}\underline{c}_1$. Thus $\underline{y} - \underline{X} = g(g - 1)^{-1}(\underline{b}_1 - \underline{a}_1)$. Now, $g \neq 1$ since there is a unique $X \in A_1C_1 \cap A_2C_2$, and $g \neq 0$ since $A_1 \neq A_2$. Thus by the argument of Th. 4.9A, we find $B_1A_1 \parallel XY$. <u>Q.E.D.</u>

<u>Theorem 4.9C</u> Let ℓ_1, ℓ_2, and ℓ_3 be any distinct lines and P, A_i, B_i, C_i $(i = 1, 2)$ be any distinct points such that $A_i \in \ell_1$, $B_i \in \ell_2$, $C_i \in \ell_3$ $(i = 1, 2)$ and $\ell_1 \cap \ell_2 \cap \ell_3 = \{P\}$. Suppose that there exist distinct points P, Q, and R such that $B_1C_1 \cap B_2C_2 = \{P\}$, $C_1A_1 \cap C_2A_2 = \{Q\}$, and $A_1B_1 \cap A_2B_2 = \{R\}$. If there exists a point D such that $A_1A_2 \cap B_1B_2 \cap C_1C_2 = \{D\}$, then P, Q, and R are collinear.



<u>Proof</u> Since $A_1A_2 \cap B_1B_2 \cap C_1C_2 = \{D\}$, Th. 4.6 implies that there exist scalars α, β, γ, and δ such that $\underline{d} = \alpha\underline{a}_1 + (1-\alpha)\underline{a}_2 = \beta\underline{b}_1 + (1-\beta)\underline{b}_2 = \gamma\underline{c}_1 + (1-\gamma)\underline{c}_2$. It follows from Th. 4.6, the existence of D, and the distinctness of ℓ_1, ℓ_2, and ℓ_3 that $\underline{p} = (\beta-\gamma)^{-1}(\beta\underline{b}_1 - \gamma\underline{c}_1)$, $\underline{q} = (\gamma-\alpha)^{-1}(\gamma\underline{c}_1 - \alpha\underline{a}_1)$, and $\underline{r} = (\alpha-\beta)^{-1}(\alpha\underline{a}_1 - \beta\underline{b}_1)$. Thus $(\beta-\gamma)\underline{p} + (\gamma-\alpha)\underline{q} + (\alpha-\beta)\underline{r} = \underline{0}$ and $\underline{p} = (\beta-\gamma)^{-1}(\alpha-\gamma)\underline{q} + (\beta-\gamma)^{-1}(\beta-\alpha)\underline{r}$. Now $(\beta-\gamma)^{-1}(\alpha-\gamma) + (\beta-\gamma)^{-1}(\beta-\alpha) = (\beta-\gamma)^{-1}(\alpha-\gamma+\beta-\alpha) = (\beta-\gamma)^{-1}(\beta-\gamma) = 1$, whence Th. 4.6 implies P, Q, and R are collinear. Q.E.D.

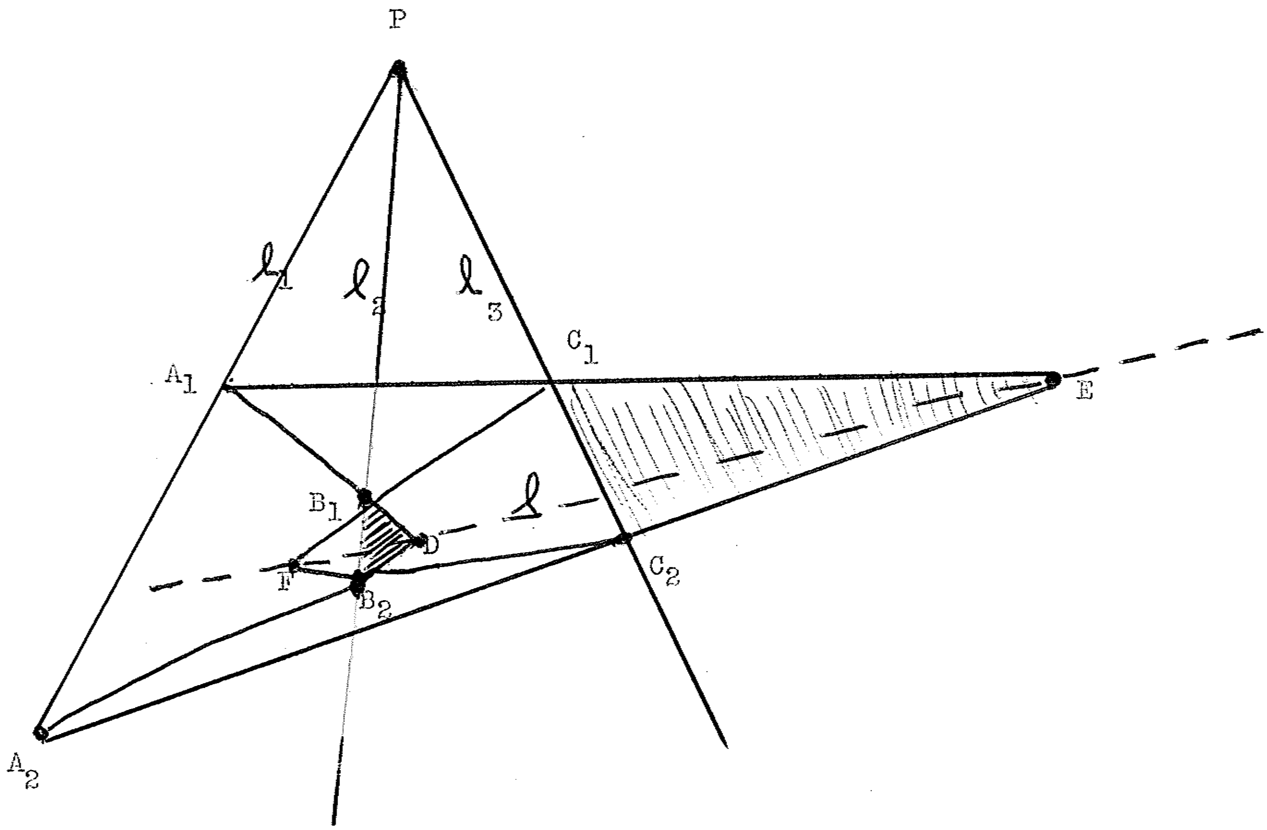
<u>Theorem 4.9D</u> Let ℓ_1, ℓ_2, and ℓ_3 be any three distinct parallel lines and A_i, B_i, C_i $(i = 1, 2)$ any distinct points such that $A_i \in \ell_1$, $B_i \in \ell_2$, and $C_i \in \ell_3$ $(i = 1, 2)$. Then if there exist distinct points D, E, and F such that $A_1B_1 \cap A_2B_2 = \{D\}$, $A_1C_1 \cap A_2C_2 = \{E\}$ and $B_1C_1 \cap B_2C_2 = \{F\}$, then D, E, and F are collinear.



Proof Since $\ell_1 \parallel \ell_2 \parallel \ell_3$ and $C_1 \neq C_2$, it follows as in Th. 4.9B that there exist scalars ρ and σ such that $\underline{a}_2 - \underline{a}_1 = \rho(\underline{c}_2 - \underline{c}_1)$ and $\underline{b}_2 - \underline{b}_1 = \sigma \cdot (\underline{c}_2 - \underline{c}_1)$. Now $\rho \neq 1 \neq \sigma$ by virtue of the existence of E and F. Furthermore, $1 \neq \rho\sigma^{-1}$ since $1 = \rho\sigma^{-1} \Rightarrow \rho = \sigma \Rightarrow \underline{a}_2 - \underline{a}_1 = \underline{b}_2 - \underline{b}_1$, which contradicts the existence of D. Thus $\underline{d} = (1 - \rho\sigma^{-1})^{-1}(\underline{a}_1 - \rho\sigma^{-1}\underline{b}_1) = (1 - \rho\sigma^{-1})^{-1} \cdot (\underline{a}_2 - \rho\sigma^{-1}\underline{b}_2)$, $\underline{e} = (1 - \rho)^{-1}(\underline{a}_2 - \rho\underline{c}_2) = (1 - \rho)^{-1}(\underline{a}_1 - \rho\underline{c}_1)$ and $\underline{f} = (1 - \sigma)^{-1}(\underline{b}_2 - \sigma\underline{c}_2) = (1 - \sigma)^{-1}(b_1 - \sigma c_1)$. It is readily verified that $\rho^{-1}(1 - \rho\sigma^{-1})\underline{d} + \rho^{-1}(\rho - 1)\underline{e} + \sigma^{-1}(1 - \sigma)\underline{f} = \underline{0}$. Since $\rho^{-1}(1 - \rho\sigma^{-1})^{-1}$ is non zero, we may multiply this equation by its inverse and obtain $d = (\rho^{-1}(1 - \rho\sigma^{-1}))^{-1}\rho^{-1}(1 - \rho)\underline{e} + (\rho^{-1}(1 - \rho\sigma^{-1}))^{-1}\sigma^{-1} \cdot (\sigma - 1)\underline{f} = \underline{0}$. By our comments in the proof of Th. 4.4 (iv), the sum of the coefficients of $\underline{e}$ and $\underline{f}$ is $(1 - \rho\sigma^{-1})\rho\rho^{-1}(1 - \rho) + (1 - \rho\sigma^{-1})^{-1}\rho\sigma^{-1}(\rho^{-1}) = (1 - \rho\sigma^{-1})(1 - \rho) + (1 - \rho\sigma^{-1})(\rho - \rho\sigma^{-1}) = (1 - \rho\sigma^{-1})^{-1}(1 - \rho\sigma^{-1}) = 1$. Thus $4.6 \Rightarrow$ D, E, and F are collinear. Q.E.D.

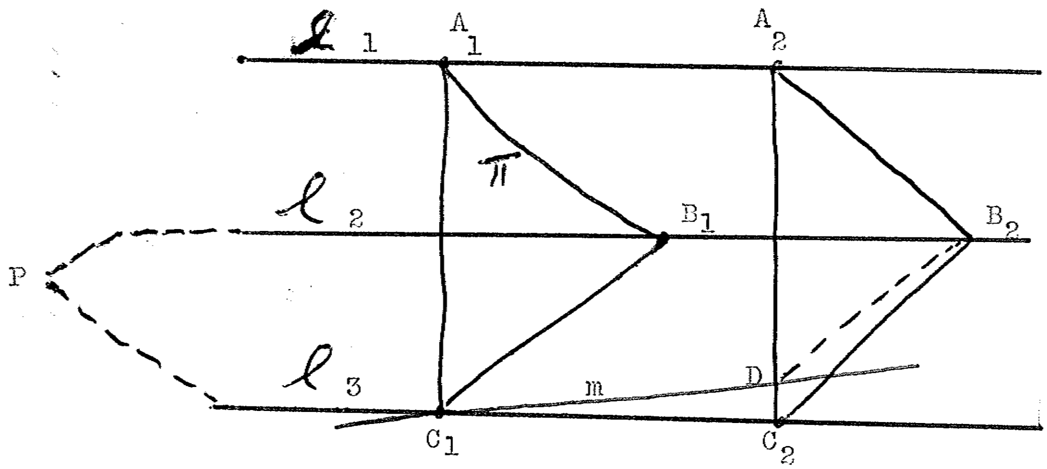
Theorem 4.9E Let ℓ_1, ℓ_2, and ℓ_3 be distinct lines and A_i, B_i, C_i ($i = 1, 2$) distinct points such that $A_i \in \ell_1$, $B_i \in \ell_2$, and $C_i \in \ell_3$ ($i = 1, 2$).

Let D, E, and F be points of a unique line $\ell \neq A_iB_i$ (i = 1, 2) such that $A_1B_1 \cap A_2B_2 = \{D\}$, $A_1C_1 \cap A_2C_2 = \{E\}$, and $B_1C_1 \cap B_2C_2 = \{F\}$. If there exists a point $P \neq A_i$ (i = 1, 2) such that $\ell_2 \cap \ell_3 = \{P\}$, and if D, E, F, B_i, C_i (i = 1, 2) are all distinct then $P \in \ell_1$.



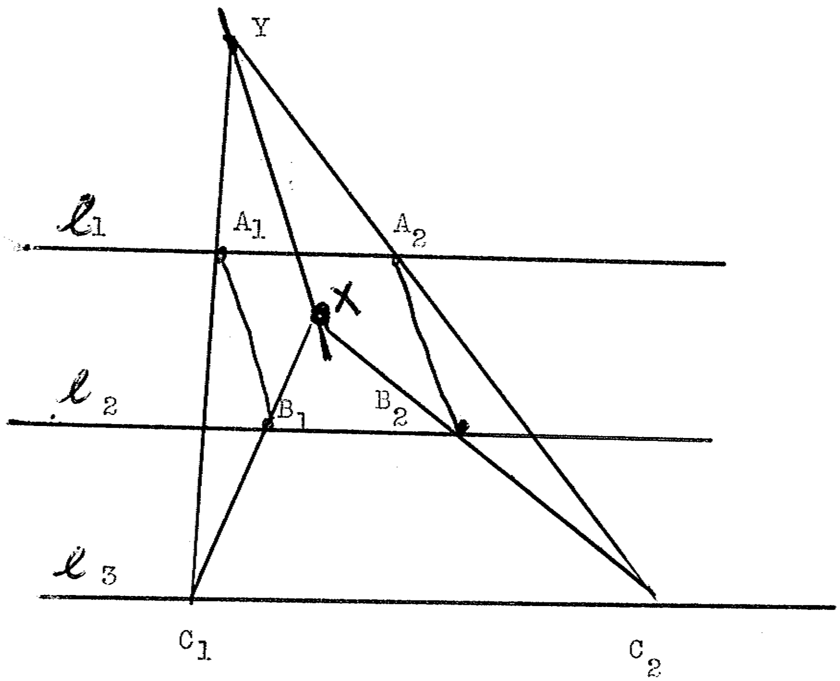
Proof It is straightforward to verify that Th. 4.9C may be applied to the lines $B_i C_i$ (i = 1, 2) and ℓ to obtain the collinearity of A_1, A_2, and P. Since $A_1 \neq A_2$, Ax. 2.1 implies $P \in \ell_1$. Q.E.D.

Theorem 4.9F Let ℓ_1, ℓ_2, and ℓ_3 be distinct lines and A_i, B_i, C_i (i = 1, 2) distinct points such that $A_i \in \ell_1$, $B_i \in \ell_2$, and $C_i \in \ell_3$ (i = 1, 2). Further, suppose $A_1C_1 \parallel A_2C_2$, $A_1B_1 \parallel A_2B_2$, $B_1C_1 \parallel B_2C_2$, $\ell_1 \parallel \ell_2$, and $C_1 \notin A_1B_1$. Then $\ell_1 \parallel \ell_3$.



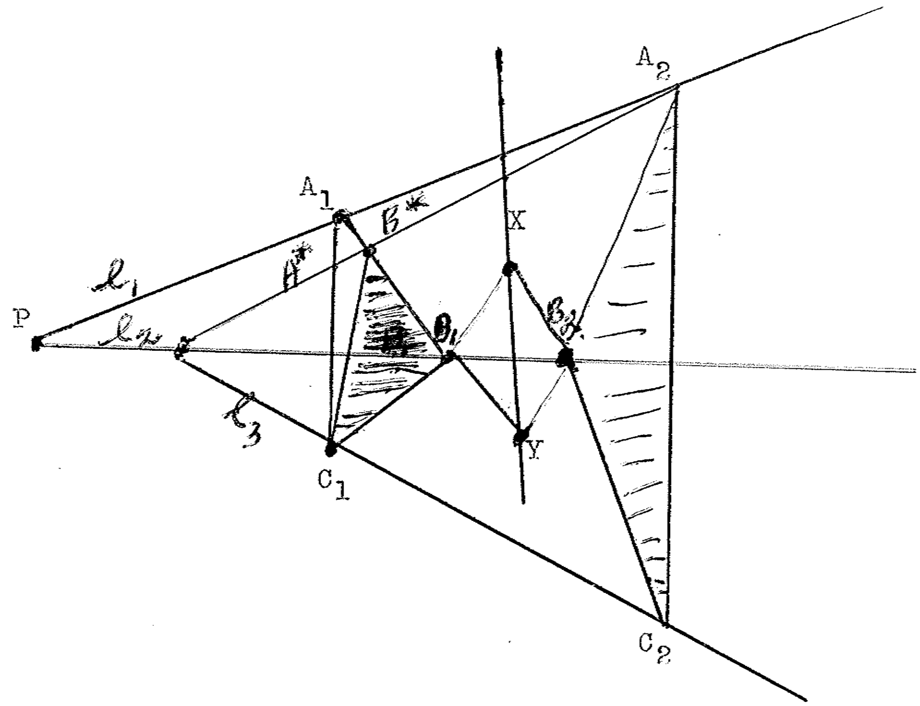
<u>Proof</u> Let the plane determined by ℓ_2 and A_1 be π. If ℓ_3 lies in a plane π^x parallel to π, then there is a line m of π^x through C_1 parallel to ℓ_1, by Cor. 2.5.3. By Ax. 2.6, A_2C_2 intersects m in a point D, since π^x is the plane determined by A_1, A_2, C_1 (by Cor. 2.5.1 and Ax. 2.7). By Th. 2.6B, it follows that $B_2D \parallel B_1C_1$, whence Cor. 2.1.1 and Ax. 2.4 imply $B_2D = B_2C_2$. Thus $D \in B_2C_2$. If $B_2 \notin \pi^x$, then Th. 2.3 $\Rightarrow$ $D = C_2 \Rightarrow m = C_1D$ (since $C_1 \neq C_2$) $\Rightarrow m = \ell_3 \Rightarrow \ell_3 \parallel \ell_1$. Otherwise, ℓ_i (i = 1, 2, 3) are coplanar. If ℓ_2 intersects ℓ_3 in some point P, then Th. 2.6C implies $P \in \ell_1 \Rightarrow \ell_1 \cap \ell_2 \neq \emptyset$, a contradiction to Th. 2.1. Thus $\ell_1 \cap \ell_3 = \emptyset$ and Cor. 2.4.6 $\Rightarrow \ell_1 \parallel \ell_3$. Q.E.D.

<u>Theorem 4.9G</u> Let ℓ_1, ℓ_2, and ℓ_3 be distinct lines and A_i, B_i, and C_i (i = 1, 2) distinct points such that $A_i \in \ell_1$, $B_i \in \ell_2$, $C_i \in \ell_3$, and $C_i \notin A_iB_i$ (i = 1, 2). Suppose x and Y are distinct points such that $C_1B_1 \cap C_2B_2 = \{x\}$, $C_1A_1 \cap C_2A_2 = \{Y\}$, and $xY \parallel A_1B_1 \parallel A_2B_2$. If $\ell_1 \parallel \ell_2$, then $\ell_1 \parallel \ell_3$.



Proof: It suffices to verify that Th. 4.9B may be applied to the lines A_iB_i $(i = 1, 2)$ and xY to obtain $C_1C_2 \parallel A_1A_2 \parallel B_1B_2 \Rightarrow \ell_3 \parallel \ell_1$. We need only show that A_iB_i $(i = 1, 2)$ and xY are distinct. Now $A_1B_1 = A_2B_2 \Rightarrow A_1, A_2 \in \ell_1, A_2B_2$. Since $\ell_1 \cap \ell_2 = \emptyset$ and $B_2 \in \ell_2$, clearly $B_2 \notin \ell_1$, whence $\ell_1 \neq A_2B_2$. By Ax. 2.1, $A_1 = A_2$, a contradiction; thus, $A_1B_1 \neq A_2B_2$. Furthermore, $A_iB_i \neq xY$, for $A_iB_i = xY$ $(i = 1 \text{ or } 2) \Rightarrow C_iA_i \cap xY = \{A_i\} = \{Y\} \Rightarrow A_i = Y$. ($C_iA_i \nparallel xY$, for $C_iA_i \parallel xY \Rightarrow C_iA_i \parallel B_iA_i \Rightarrow C_iA_i = B_iA_i$, a contradiction since $C_i \notin A_iB_i$.) However, $A_i = Y \Rightarrow A_i, A_j$ $(j \neq i) \in \ell_1, YC_j$. Since $A_i \neq A_j$, Ax. 2.1 $\Rightarrow \ell_1 = YC_j$. Similarly, $B_iC_i \cap xY = \{B_i\} = \{x\} \Rightarrow B_i = x$ and $\ell_2 = xC_j$. However $\ell_1 \cap \ell_2 = \emptyset \neq YC_j \cap xC_j$. Thus $A_iB_i \neq xY$, and the proof is complete.

Theorem 4.9H Let ℓ_1, ℓ_2, and ℓ_3 be distinct lines and P, A_i, B_i, C_i $(i = 1, 2)$ distinct points such that $\ell_1 \cap \ell_2 = \{P\}$, $C_i \notin A_iB_i$ $(i = 1, 2)$, $C_1 \notin (\ell_1 \cup \ell_2)$ and $C_2 \notin \ell_2$. Suppose that there exist distinct points x and Y such that $C_1B_1 \cap C_2B_2 = \{x\}$, $A_1B_1 \cap A_2B_2 = \{Y\}$ and $xY \parallel A_1C_1 \parallel A_2C_2$. Then $P \in \ell_3$.



Proof

Case 1 Assume P, A_1, B_1, and C_1 are not coplanar. Let $(P, A_1) \in \underline{u}$, $(P, B_1) \in \underline{v}$, $(P, C_1) \in \underline{w}$, and $(P, C_2) \in \underline{x}$. By Th. 4.2, there exist scalars f and g such that $(P, A_2) \in f(\underline{u})$ and $(P, B_2) \in g(\underline{v})$. Now $A_1C_1 \parallel A_2C_2 \Rightarrow \underline{u} - \underline{w} \parallel f(\underline{u}) - \underline{x}$ by Ax. 2.4, Prop. 3.4 and Defs. 3.1, 2. Thus Th. 4.2 $\Rightarrow$ there exists a scalar h such that $\underline{x} - f(\underline{u}) = h(\underline{u} - \underline{w})$; that is, $\underline{x} = (f + h)\underline{u} - h(\underline{w})$.

Since [illegible] $= C_1B_1 \cap C_2B_2$, there exist scalars α and β such that $\underline{e} = \alpha\underline{w} + (1 - \alpha)\underline{v} = \beta\underline{x} + (1 - \beta)g(\underline{v}) = \beta((f + h)\underline{u} - h(\underline{w})) + (1 - \beta)g(\underline{v})$. Since $\underline{u}$, $\underline{v}$, and $\underline{w}$ are noncoplanar, we may equate corresponding coefficients. Consequently, $\alpha = -\beta h$, $0 = \beta(f + h)$, and $1 - \alpha = (1 - \beta)g$. If $\beta = 0$, then $\alpha = 0$ and $\underline{e} = \underline{v} = g(\underline{v})$. However, $\underline{v} \neq \underline{0} \Rightarrow g = 1 \Rightarrow B_1 = B_2$, a contradiction. Thus $\beta \neq 0$, whence $f + h = 0$.

Substitution in the expression for $\underline{x}$ gives $\underline{x} = f(\underline{v})$; that is, P is collinear with C_1 and C_2. Since $C_1 \neq C_2$, $\ell_3 = C_1C_2$ and so $P \in \ell_3$.

<u>Case 2</u> Assume P, A_1, B_1, and C_1 are coplanar. Since $\ell_1 \nparallel \ell_2$, it follows that $\ell_2 \nparallel \ell_3$, by Th. 4.9C. By Ax. 2.7, there exists a point R (without loss of generality $R \neq P$) such that $\ell_2 \cap \ell_3 = \{R\}$. Since $P \neq A_2$ and $\ell_1 \cap \ell_2 = \{P\}$, it is clear that $A_2 \notin \ell_2$, whence $R \neq A_2$. Let RA_2 intersect A_1C_1 and A_1B_1 respectively in the points A^* and B^*. Furthermore, notice that A^* and B^* are uniquely determined, for if $A_1C_1 = RA_2$ then $C_1 \in A_1A_2 = \ell_1$, a contradiction. Similarly, $A_1B_1 = RA_2 \Rightarrow A_1A_2 = \ell_1 = \ell_2 = B_1R$ (a contradiction) unless $R = B_1$. However if $R = B_1$, then RA_2 intersects A_1B_1 only in B_1, for any assumption to the contrary implies $B_1 \in \ell_2$, $A_1A_2 = \ell_1 \Rightarrow P = B_1$, a contradiction.

Since $A_2 \notin \ell_2$, it is clear that $RA_2 \neq \ell_2$. Moreover, $RA_2 \neq \ell_3$, as $RA_2 = \ell_3 = A_2C_2 = C_1C_2 \parallel A_1C_1 \Rightarrow C_1C_2 = A_1C_1 = A_2C_2 \Rightarrow \ell_1 = \ell_3$, a contradiction. Furthermore, $B^* \neq B_1$ and $B^* \neq C_1$, since $B^* = B_1 \Rightarrow \ell_1 = A_2B_1 \Rightarrow B_1 = P$, a contradiction and $B^* = C_1 \Rightarrow A_1C_1 = A_1B_1$, a contradiction since $C_1 \notin A_1B_1$. Now, R is distinct from B^*, C_1, and B_1 since $A_2 \notin \ell_2$, $\ell_2 \neq \ell_3$, and $R = B_1 \Rightarrow RA_2 \cap A_1B_1 = \{B_1\} = \{B^*\}$ from above, contradicting $B_1 \neq B^*$. Finally R is distinct from C_2 since $C_2 \notin \ell_2$, and $R \neq B_2$ since $R = B_2 \Rightarrow RA_2 = A_2B_2 = \ell_2 \Rightarrow A_2 \in \ell_1 \cap \ell_2 \Rightarrow A_2 = P$, a contradiction to the hypotheses. Thus we may apply Th. 4.9C to show that $B^*C_1 \nparallel A_2C_2 \Rightarrow XY \cap A_2C_2 \neq \emptyset \Rightarrow XY = A_2C_2 \Rightarrow$ (since $C_2 \notin A_2B_2$) $X = B_2 = Y$, a contradiction. Hence $B^*, A_1 \in A_1C_1, A_1B_1 \Rightarrow B^* = A_1$. Therefore $\{R\} = A_1A_2 \cap \ell_2 = \ell_1 \cap \ell_2 = \{P\}$, whence $R = P$. Q.E.D.

Consider the correspondence

Theorem 2.6A $\longleftrightarrow$ Theorem 2.6C

Theorem 2.6B $\longleftrightarrow$ Theorem 2.6D

Theorem 4.9A $\longleftrightarrow$ Theorem 4.9H

Theorem 4.9B $\longleftrightarrow$ Theorem 4.9G

Theorem 4.9C $\longleftrightarrow$ Theorem 4.9E

Theorem 4.9D $\longleftrightarrow$ Theorem 4.9F

This illustrates the six important cases of Desarques' Theorem and their converses where, in the sense of projective geometry, Desarques' Theorem may be taken to say that any two triangles in perspective from a point are in perspective from a line.

V ORDER AND DIRECTION

We begin our discussion of order by clarifying some intuitive notions.

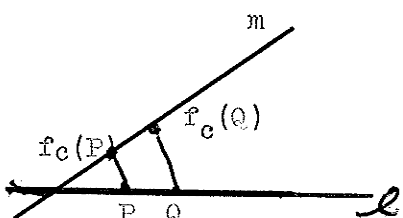
Definition 5.1 An order on a set S is a binary relation $<$ such that (1) Exactly one of $P<Q$, $P=Q$, $Q<P$ can hold for $P, Q \in S$ and (2) $P, Q, R \in S \Rightarrow (P<Q,\ Q<R \Rightarrow P<R)$. The pair $(S,<)$ is called an ordered set. $P<Q$ may be read as "P precedes Q".

It is easily seen that the "opposite" ordering $>$ also satisfies Def. 5.1, where we define $P>Q \Leftrightarrow Q<P$ whenever $P, Q \in S$.

Definition 5.2 f is a (strictly) monotonic function from the ordered set $(S,<)$ into the ordered set $(S',<')$ if and only if either (1) $a, b \in S \Rightarrow (a<b \Rightarrow f(a) <' f(b))$ or (2) $a, b \in S \Rightarrow (a<b \Rightarrow f(b) <' f(a))$.

In Def. 5.2, the orders on S and S' were represented by different symbols. In general, we shall use '$<$' for an order on any set. No ambiguity need result, as the context will make the meaning of '$<$' clear.

Definition 5.3 A parallel projection from a line ℓ to a line m is a mapping f_c of the points of ℓ into the points of m such that $P \neq f_c(P) \Rightarrow Pf_c(P) \in C$ for all $P \in \ell$, where C is a (fixed) equivalence class of parallel lines such that $\ell \notin C$ and $m \notin C$.



Where context permits, we shall write 'f' for f_c'.

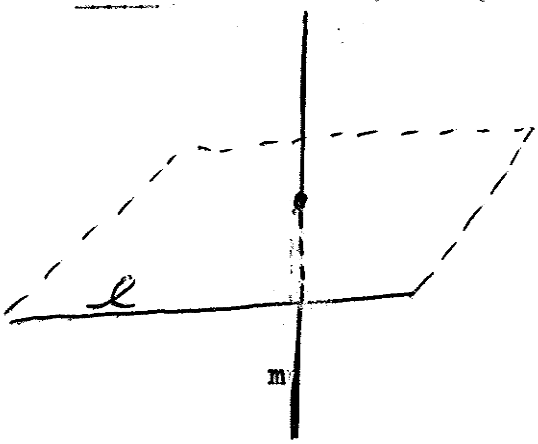
Proposition 5.1 If f_c is a parallel projection from a line ℓ to a line m and P is a point such that $P \in \ell \cap m$, then $f(P) = P$.

Proof If $P \neq f_c(P) \in m$, then Axs. 2.1, 2 and Def. 5.3 $\Rightarrow m = Pf(P) \in C$, a contradiction to Def. 5.3. Thus $P = f_c(P)$.

We note that Prop. 5.1 implies there exists exactly one parallel projection from a line ℓ into itself, namely the identity transformation $f(P) = P$ for all $P \in \ell$.

<u>Proposition 5.2</u> If ℓ and m are noncoplanar lines, then there exists no parallel projection from ℓ to m.

<u>Proof</u> Let P and Q be any distinct points of ℓ. If the theorem is false,



there exists a parallel projection f from ℓ to m. As ℓ and m are noncoplanar, Def. 2.1 implies $\ell \cap m = \emptyset$. If f (P) = f (Q), then Cor. 2.1.1 $\Rightarrow$ P f (P) = Q f (Q) $\Rightarrow$ P, Q $\in \ell$, P f (P) $\Rightarrow$ P = Q, by Ax. 2.1 since $\ell \cap m = \emptyset \Rightarrow$ f (P) $\notin \ell \Rightarrow \ell \neq$ P f (P). As P $\neq$ Q by hypothesis, it follows that f (P) $\neq$ f (Q). However, Th. 2.2 (ii) and Th. 2.3 $\Rightarrow$ f (Q) lies in the plane π (unique by Cor. 2.4.1) determined by P, Q, and f (P). Since f (P) $\neq$ f (Q), it follows from Def. 2.1 that ℓ and m are each lines of π, contradicting the hypothesis that ℓ and m are noncoplanar. Thus no parallel projection f from ℓ to m exists.

<u>Proposition 5.3</u> If f_c is a parallel projection from a line ℓ to a line m, then f_c is a one-to-one onto mapping.

<u>Proof</u> Suppose there exists a point P $\in \ell \cap$ m. By Prop. 5.1, P = f (P). Moreover, P = f (W), W $\neq$ P $\Rightarrow \ell \neq$ m, ℓ = W P = W f (W) $\in$ C, a contradiction. Let Q be any point on m. If Q $\notin \ell$, then the line through Q and in C intersects ℓ in exactly one point. As this point is not on m, its image must be Q. Thus there is at least one point which maps into Q under f. Furthermore, there cannot be more than one as Q = f (R) = f (T) $\Rightarrow$ QR $\parallel$ QT (since Q $\notin \ell$, QR and QT exist) $\Rightarrow$ QR = QT by Th. 2.1 and Cor. 2.1.1. If R $\neq$ T, then Q $\in$ RT = ℓ, a contradiction. Consequently R = T, and the uniqueness of the pre-image of Q is established; that is, f is one-to-one from ℓ onto m.

Proposition 5.4 If ℓ and m are any distinct coplanar lines, then there exists a different parallel projection from ℓ to m for each arbitrary assignment of an image f_c (P) $\notin$ m to any fixed point $P \in \ell$, $P \notin m$.

Proof It suffices to show that each choice of f (P) determines a unique function.

Let a particular f (P) be assigned and let Q be any point other than P of ℓ such that $Q \notin m$. If one were to assign $R = f(Q) \in m$, then $QR \in C$ $\Rightarrow QR \parallel Pf(P) \nparallel m$. By Ax. 2.4, $QR \nparallel m$, whence Ax. 2.7 $\Rightarrow$ there exists a unique point $T \in QR \cap m$. Thus R = T and so f can be defined in at most one way, given f (P).

Consider $$f(Q) = \begin{cases} Q, & Q \in \ell \cap m \\ m \cap t, & Q \in \ell,\ Q \notin m,\ Q \in t,\ t \in \{h : h \parallel Pf(P)\} \end{cases}$$

It is readily verified that f is a parallel projection from ℓ to m. Consequently f can be uniquely defined as a parallel projection, given f (P), and the proof is complete.

In a trivial manner, the lines of an affine space with only two points on every line may be ordered. As this "geometry of the sheared sugar cube" and other finite affine geometries have been thoroughly investigated (Cor. 2.4.5, Cor. 2.5.2, Remark following Cor. 2.5.20; Cor. 2.4.9, Cors. 2.5.9 - 20), it will now be of interest to investigate affine spaces characterized by non finite n, the number of points on a line. Our assumptions are as follows.

Axiom 5.1 For each $\ell \in \mathcal{L}$ there exist (at least one) order $<_\ell$ on ℓ such that, for any $m \in \mathcal{L}$ and for any parallel projection f from ℓ to m, either

1) $P <_\ell Q \Rightarrow f(P) = f(Q)$ or $f(P) <_m f(Q)$, for all $P, Q \in \ell$ or

2) $P <_\ell Q \Rightarrow f(P) = f(Q)$ or $f(P) >_m f(Q)$, for all $P, Q \in \ell$.

Henceforth, we shall use the symbol '$<_\ell$' only to refer to those $<$ of ℓ whose existence is guaranteed by Ax. 5.1. Unless it is explicitly stated to the contrary, all orderings referred to and all occurences of '$<$' are supposed of this type.

It is immediate from Ax. 5.1, Prop. 5.3 and Def. 5.1 that parallel projection is a <u>strictly</u> monotonic function. Moreover, parallel projection is a one-one onto function and so invertible. Consequently, in Ax. 5.1, we may replace 'from ℓ to m' and '$<_\ell$, $<_m$' by 'from m to ℓ' and '$<_m$, $<_\ell$' respectively, whenever the alternative forms are more convenient.

<u>Axiom 5.2</u> There exists $m \in \mathcal{L}$ containing (at least) three distinct points.

It follows from Cor. 2.4.8 that every $\ell \in \mathcal{L}$ contains at least three distinct points.

Our next result indicates that the affine space now under consideration is not characterized by finite n.

<u>Theorem 5.1</u> For all points P, and lines ℓ,

(i) $P \in \ell \Rightarrow$ there exists a point $Q \in \ell$ such that $P <_\ell Q$

(ii) $P \in \ell \Rightarrow$ there exists a point $R \in \ell$ such that $R <_\ell P$

(iii) ℓ contains infinitely many points.

<u>Proof</u> Note that (iii) follows immediately from either (i) or (ii), by Ax. 5.1. We shall prove (i) and (ii) simultaneously.

Consider any line m parallel to, but distinct from, ℓ and P_1, Q_1, R_1 any three distinct points of m ordered so that $R_1 <_m P_1 <_m Q_1$. By Th. 2.1 and Prop. 5.4, there exists a parallel projection from m to ℓ such that $f(P_1) = P$. If $f(Q_1) = Q$ and $f(R_1) = R$, then our remark following the statement of Ax. 5.1 implies that either $R <_\ell P <_\ell Q$ or $Q <_\ell P <_\ell R$. In either case, (i) and (ii) are proved.

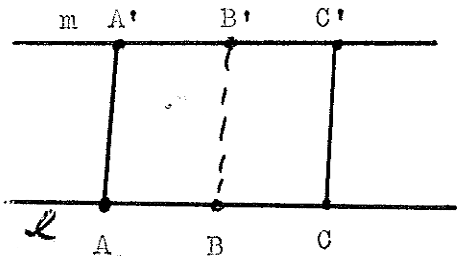
<u>Corollary 5.1.1</u> There exist infinitely many scalars.

<u>Proof</u> The result follows immediately from Cor. 4.2.3 and Th. 5.1.

__Theorem 5.2__ For all points A and C and lines ℓ, $A, C \in \ell$, $A <_\ell C \Rightarrow$ there exists a point $B \in \ell$ such that $A <_\ell B <_\ell C$.

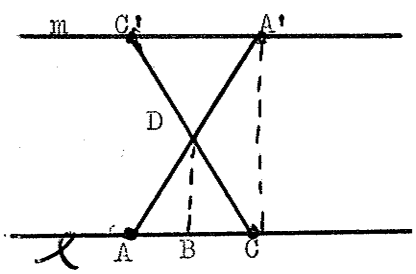
__Proof__ Let m be any line parallel to, but distinct from, ℓ and A', B', and C' any points of m such that $A' <_m B' <_m C'$.

__Case 1__ Suppose $A'A \parallel C'C$. Then the parallel projection from ℓ to m such that f (A') = A and f (C') = C defines a point $B \in \ell$ satisfying f (B') = B. By Ax. 5.1 either $A <_\ell B <_\ell C$ or $C <_\ell B <_\ell A$. However, the latter condition contradicts $A <_\ell C$ (since $C <_\ell B$, $B <_\ell A \Rightarrow C <_\ell A \Rightarrow A \not<_\ell C$). Thus $A <_\ell B <_\ell C$.

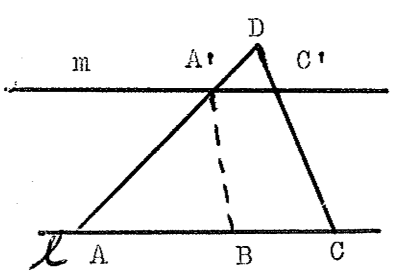


__Case 2__ Suppose $A'A \nparallel C'C$. By Ax. 2.7 and Prop. 5.2, there is a unique point D common to A'A and C'C. We may assume without loss of generality that $A < A'$.

If $A < D < A'$, then the parallel projection from A A' to ℓ using lines parallel to A'C gives a point $B \in \ell$ as the image of D. By Ax. 5.1, it follows that either $A <_\ell B <_\ell C$ or $C <_\ell B <_\ell A$. The argument concludes as in Case 1.



It suffices to consider $A < A' < D$ as the remaining possibility. Let the parallel projection g from A'A to ℓ using lines parallel to C'C define g (A') = B. As above $A < A' < D$ and $A <_\ell C$ imply $A <_\ell B <_\ell C$, as required.



In order to simplify our notation, we make the following definition.

__Definition 5.4__ We say that the point B lies __between__ the point A and C and write

"$A \mid B \mid C$" if and only if for every ordering $<_\ell$ of any line ℓ containing both A and C, either $A <_\ell B <_\ell C$ or $C <_\ell B <_\ell A$.

<u>Lemma 5.5</u> For any points A, B, and C and any line ℓ, if for some ordering $<_\ell$ compatible with Ax. 5.1, either $A <_\ell B <_\ell C$ or $C <_\ell B <_\ell A$, then $A \mid B \mid C$.

<u>Proof</u> Let m be any line parallel to, but distinct from, AB. Without loss of generality, we may assume $A <_\ell B <_\ell C$. If f is any parallel projection from AB to m, then Ax. 5.1 implies that either $f(A) <_m f(B) <_m f(C)$ or $f(C) <_m f(B) <_m f(A)$. Without loss of generality, we may consider $f(A) <_m f(B) <_m f(C)$. Now, it is clear that Prop. 5.3 $\Rightarrow$ the function f^{-1} is a parallel projection from m to AB. Consequently, for any ordering $<'_\ell$ of ℓ satisfying Ax. 5.1, either $f^{-1}(f(A)) <'_\ell f^{-1}(f(B)) <'_\ell f^{-1}(f(C))$ or $f^{-1}(f(C)) <'_\ell f^{-1}(f(B)) <'_\ell f^{-1}(f(A))$; that is, either $A <'_\ell B <'_\ell C$ or $C <'_\ell B <'_\ell A$. Thus $A \mid B \mid C$.

<u>Proposition 5.5</u> (i) For all points A, B, and C, $A \mid B \mid C \Leftrightarrow C \mid B \mid A$.

(ii) For any distinct collinear points A, B, and C exactly one lies between the remaining two.

(iii) For any distinct collinear points A', B', C', and D', there exist exactly two distinct functions f mapping $\{A', B', C', D'\}$ onto $\{A', B', C', D'\}$ such that both $f(A') \mid f(B') \mid f(C')$ and $f(B') \mid f(C') \mid f(D')$.

(iv) If f is a parallel projection from any line ℓ to any (coplanar) line m and A, B, and C are any points of ℓ, then $A \mid B \mid C \Leftrightarrow f(A) \mid f(B) \mid f(C)$.

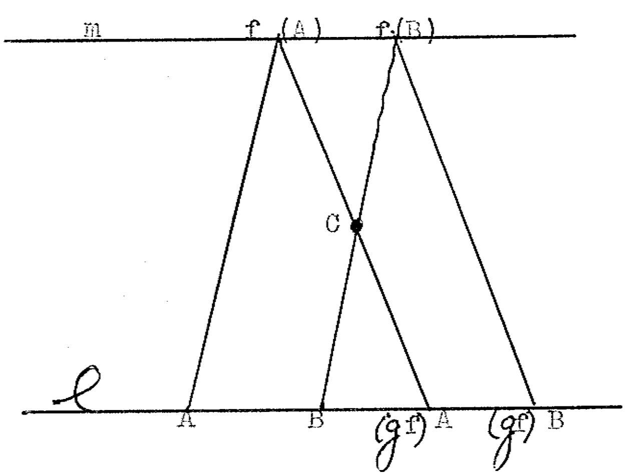
Motivation for the definition of vectors given in Chapter II is provided in part by the next theorem.

<u>Theorem 5.3</u> Let ℓ and m be any lines such that $\ell \parallel m$. If f and g are any parallel projections satisfying $f : \ell \to m$ and $g : m \to \ell$, then the compo-

site function $g\,f : \ell \to \ell$ preserves all orderings $<_\ell$ of ℓ.

Proof We wish to show that $A, B \in \ell$ and $A <_\ell B \Rightarrow (gf)\,A <_\ell (gf)\,B$. Without loss of generality, we may assume $\ell \neq m$, for if $\ell = m$, then Prop. 5.1 $\Rightarrow gf$ is the order-preserving identity transformation, $g\,f : \ell \to \ell$. Since g and f are strictly monotonic (parallel projections), their composite function $g\,f$ is also strictly monotonic and so either preserves or reverses $<_\ell$ on ℓ. Thus it suffices to test one ordered point pair $(A,B) \notin \underline{0}$ of $\ell \times \ell$ in order to verify the theorem. As $A = g\,f\,(A) \Rightarrow g\,f$ is the (order-preserving) identity transformation, $g\,f : \ell \to \ell$, by Prop. 5.4, we may assume without loss of generality that $A <_\ell (gf)\,A$. Since we need only test one $(A, B) \notin \underline{0}$, it follows from Th. 5.2 that we may assume $A <_\ell B <_\ell gf\,(A)$.

If $A\,f\,(A) \parallel f\,(A)\,g\,f\,(A)$, then Cor. 2.1.1 $\Rightarrow A\,f\,(A) = f\,(A)\,g\,f\,(A) \Rightarrow A, (gf)\,A \in \ell, A\,f\,(A)$. Moreover, Cor. 2.1.1 $\Rightarrow f\,(A) \notin \ell \Rightarrow A\,f\,(A) \neq \ell \Rightarrow A = (gf)\,A$, by Ax. 2.1. As we are supposing $A <_\ell (gf)\,A$, it follows that $A\,f\,(A) \nparallel f\,(A)\,gf\,(A)$. By Cor. 2.4.6 and Ax. 2.7, there exists a unique point $C \in B\,f\,(B) \cap f\,(A)\,(gf)\,A$. Since $A \,|\, B \,|\, (gf)\,A$, Ax. 5.1 $\Rightarrow (gf)\,A \,|\, C \,|\, f\,(A)$, by considering the parallel projection from ℓ to $f\,(A)\,(gf)\,A$ which maps A onto $f\,(A)$. Similarly, the parallel projection from $f\,(A)\,(gf)\,A$ to $B\,f\,(B)$ by lines parallel to ℓ gives $B \,|\, C \,|\, f\,(B)$.



Finally, the parallel projection from $B\,f\,(B)$ to ℓ by lines parallel to $f\,(B)\,(gf)\,B$ gives $B \,|\, (gf)\,A \,|\, (gf)\,B$. As $A <_\ell B <_\ell (gf)\,A$, we may conclude $(gf)\,A <_\ell (gf)\,B$. Q.E.D.

<u>Corollary 5.3.1</u> For any ordering $\prec_\ell$ and any distinct points A and B, if A $\prec_\ell$ B and if the betweenness relationships that exist among the points of ℓ are given, then the order relationships under $\prec_\ell$ of any points of ℓ may be determined.

<u>Proof</u> If A $|$ B $|$ C, then ($A \prec_\ell B \prec_\ell C$ or $C \prec_\ell B \prec_\ell A$), $A \prec_\ell B \Rightarrow A \prec_\ell B \prec_\ell C$; that is, $A \prec_\ell C$ and $B \prec_\ell C$. If A $|$ C $|$ B, then ($A \prec_\ell C \prec_\ell B$ or $B \prec_\ell C \prec_\ell A$), $A \prec_\ell B \Rightarrow A \prec_\ell C \prec_\ell B$, since $B \prec_\ell C \prec_\ell A$, $A \prec_\ell B \Rightarrow B \prec_\ell A$, $A \prec_\ell B \Rightarrow B \prec_\ell B$, a contradiction. Thus $A \prec_\ell C$ and $C \prec_\ell B$. Similarly, C $|$ A $|$ B $\Rightarrow C \prec_\ell A$, $C \prec_\ell B$. Since A, B, and C are distinct, Prop. 5.5 (i), (ii) imply that the three cases considered above exhaust all possibilities.

<u>Corollary 5.3.2</u> If A, B are distinct points of a line ℓ then the stipulation $A \prec_\ell B$ determines exactly one ordering of ℓ of the type guaranteed by Ax. 5.1. Thus there are exactly two such orderings on every line.

<u>Proof</u> We may assume that the betweeness relationships that exist among points of ℓ are given. If there is an order $\prec_\ell$ satisfying Def. 5.1 such that $A \prec_\ell B$, then Cor. 5.3.1 $\Rightarrow$ the stipulation $A \prec B$ uniquely determines $\prec_\ell$. However, at least one order $\prec_\ell^*$ exists satisfying Def. 5.1. Since $_\ell^*\!\succ$ also satisfies Def. 5.1 and since either $A \prec_\ell^* B$ or $A \; _\ell^*\!\succ B$, there exists in fact, some order $\prec$ on ℓ compatible with Def. 5.1 such that $A \prec B$. By the above comments, this order is uniquely determined by the stipulation that $A \prec B$. By reversing the roles of A and B in the above argument, one gets a similar result with respect to the stipulation that $B \prec A$. Since Def. 5.1 $\Rightarrow$ (not both $A \prec B$ and $B \prec A$), the proof is complete.

Corollary 5.3.3 Let $\underline{a}$ be any non zero vector. Let ℓ be any line and P some point of ℓ such that $(P,R) \in \underline{a} \Rightarrow PR = \ell$. Let S and T be any distinct points of ℓ and U and V the unique points such that $(S, U) \in \underline{a}$ and $(T,V) \in \underline{a}$. Then (i) U , $V \in \ell$ and

(ii) For any ordering $<_\ell$ of ℓ, $S <_\ell T \Rightarrow U <_\ell V$.

Proof Since $(P, R) \uparrow (S, U) \notin \underline{0}$, it follows from Defs. 3.1, 2, Ax. 2.4 and Th. 2.1 that $PR = SU$. Thus $U \in \ell$ and similarly $V \in \ell$. Now, it is readily verified that, for all points, $A, B \in \ell$, $(S, U) \sqsupset (A, B) \Rightarrow (A, B) \not\sqsupset (T, V)$ since $S \neq T$. Thus there exist a line $m \neq \ell$ and points $C, D \in m$ such that $m \parallel \ell$, $(S, U) \sqsupset (C, D)$ and $(C, D) \sqsupset (T, V)$. It is evident that $S \to C$ defines a parallel projection f from ℓ to m such that $U \to D$. Similarly, $C \to T$ defines a parallel projecting from m to ℓ such that $D \to V$. As $S <_\ell T$, Th. 5.3 implies $U <_\ell V$ since $g f : S \to T, U \to V$.

Corollary 5.3.4 $1 + 1 \neq 0$

Proof Let P and Q be any distinct points and R the unique point such that $(P, Q) \uparrow (Q, R)$. By Cor. 3.1.3, $R \in PQ$. If $<_\ell$ is the ordering of PQ such that $P <_\ell Q$, then Cor. 5.3.3 implies $Q <_\ell R$. Since $<_\ell$ is transitive, $P <_\ell R$; that is $P \neq R$. If $\underline{v}$ is the vector containing (P, Q), then $(P, R) \in \underline{v} + \underline{v} = 1\underline{v} + 1\underline{v} = (1 + 1)\underline{v}$ and $(P, P) \in 0\underline{v}$. As $P \neq R$, it follows from Th. 3.3 that $1 + 1 \neq 0$.

Corollary 5.3.5 For all positive integers n, $1 + 1 + \dots + 1$ (n times) $\neq$ 0.

Proof The proof proceeds via mathematical induction on n. As the theorem has been verified for n = 1, 2, suppose $1 + 1 + \dots + 1$ (k times) $\neq$ 0 where k is any positive integer other than 1. Let P and Q be any distinct points and let $\underline{v}$ be the vector containing (P, Q). Then if $<_\ell$ is the ordering of PQ such that $P <_\ell Q$, consider the points R and S where $(P, R) \in \underbrace{(1 + \ldots + 1)}_{k \text{ times}} \mathcal{V}$ and $(R, S) \in \mathcal{V}$. As we may assume as part of our induction hypothesis that $P <_\ell Q <_\ell R$, it follows from Th. 5.3 that $P <_\ell R \Rightarrow Q <_\ell S$. Thus $P <_\ell Q$, $Q <_\ell S \Rightarrow P <_\ell S \Rightarrow P \neq S$. Since $(P, S) \in \underbrace{(1 + \ldots + 1)}_{(k+1) \text{ times}} \mathcal{V}$, we have, by the argument of the preceding corollary, $\underbrace{(1 + \ldots + 1)}_{k+1} \neq 0$. This completes the proof.

The result of Cor. 5.3.5 is often expressed by stating that Σ is of <u>characteristic zero</u>.

Let Z^+ be the set of positive integers. Define the relation ψ from Z^+ into Σ as follows: $\psi(1) = 1$, $\psi(n + 1) = \psi(n) + 1$ for all positive integers other than 1. It is possible to show by mathematical induction that ψ is a (single-valued) one-to-one function. If the range of ψ is A, then A is isomorphic to Z^+. For, it is readily shown by mathematical induction that $\psi(m + n) = \psi(m) + \psi(n)$, for all positive integers m, n. One may then show that $\psi(m\,n) = \psi(m)\psi(n)$. In particular, since $m\,n = n\,m$, it follows that $\psi(m)\psi(n) = \psi(n)\psi(m)$; that is, multiplication of scalars is commutative in A.

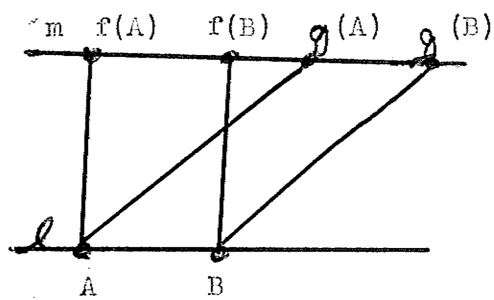
Similarly, Σ has a subset $B \supset A$ such that B is isomorphic to the set of all integers.

<u>Definition 5.5</u> Let ℓ and m be any parallel lines and $<_\ell$ an order on ℓ of the type guaranteed by Ax. 5.1. The <u>ordering $<_m$ of m corresponding to $<_\ell$</u> is the ordering of m such that, under any projection f from ℓ to m, $A, B \in \ell$ $A <_\ell B \Rightarrow f(A) <_m f(B)$.

<u>Theorem 5.4</u> Let ℓ, m, and n be any parallel lines with $<_\ell$ a fixed ordering of ℓ. Then $<_m$ and $<_n$ are well defined and $<_n$ is the ordering of n corresponding to $<_m$ on m .

<u>Proof</u> If $\ell = m$, then the only parallel projection from ℓ to m is the identity transformation and $<_m$ may be uniquely chosen as $<_\ell$, since $A <_\ell B \Rightarrow$ not $(A >_\ell B)$. If $\ell \neq m$, then let A and B be any points of ℓ such that $A <_\ell B$. Since any parallel projection is one-to-one, if f is any definite parallel projection from ℓ to m, Cor. 5.3.2 implies that there is a unique ordering $<^{*}$ on m such that $f(A) <^{*} f(B)$. Now, Ax. 5.1 implies that $<^{*}$ is independent of the particular A and B we have considered.

Moreover, we claim that $<^{*}$ is independent of the parallel projection f. For, let A and B be the definite points considered above, g any parallel projection from ℓ to m, and $<^{**}$ the ordering of m such that $g(A) <^{**} g(B)$. As f^{-1} is a parallel projection from m to ℓ, it follows from Th. 5.3 that $f(A) <^{*} f(B) \Rightarrow g(f^{-1}(f(A))) <^{*} g(f^{-1}(f(B)))$; that is, $g(A) <^{*} g(B)$. Since $g(A) <^{**} g(B)$ by assumption, Cor. 5.3.2 $\Rightarrow\ <^{*} = <^{**}$, whence the claim is proved. Thus $<_m$ and (similarly) $<_n$ are well defined.



We shall now show that $<_n$ is the ordering of n corresponding to $<_m$ on m. By the first remark of this proof, we may assume $\ell \neq m \neq n$. Furthermore, we may assume $\ell \neq n$ without loss of generality. For if f is any parallel projection from $\ell = n$ to m and A and B are points of ℓ such that $A <_\ell B$, then $f(A) <_m f(B) \Rightarrow f^{-1}(f(A)) <_{*} f^{-1}(f(B))$; that is, $A <_{*} B$, where $<_{*}$ is the order of $\ell = n$ corresponding to $<_m$ on m. As $A <_\ell B$, Cor. 5.3.2 implies $<_\ell = <_{*}$. Since $<_\ell = <_n$, we have $<_n = <_{*}$, the required result. Thus it remains to consider the case ℓ, m, and n distinct. Under these conditions, if $\prec_{XY}$ is the ordering of n corresponding to $\prec_m$ on m, it is an immediate consequence of Th. 2.6B that $\prec_{YX} = \prec_n$. This completes the proof.

It is clear that Th. 5.4 implies that the orderings of lines of any given parallel class are separated into two nonempty equivalence classes under parallel projection such that two orderings, $\prec_1$ of ℓ_1 and $\prec_2$ of ℓ_2 are in the same equivalence class if and only if $\prec_1$ is the ordering of ℓ_1 corresponding to $\prec_2$ on ℓ_2 .

Definition 5.6 An equivalence class of orderings on lines of some parallel class under parallel projection is called a direction. Two directions D_1 and D_2 are said to be opposite if and only if there exists an ordering $\prec \in D_1$ such that $\succ \in D_2$.

Note that any two orderings are in the same direction only if they are orderings of parallel lines.

From our preceding remarks, it is readily seen that corresponding to any direction D_1 containing an ordering $\prec$, there exists a unique direction D_2 (namely, the direction containing $\succ$) such that D_1 and D_2 are opposite directions. Since for all orders $\prec$, the order opposite to the order opposite to $\prec$ is $\prec$, it follows that D_1 and D_2 are opposite directions if and only if D_2 and D_1 are opposite directions. Thus, we may refer unambiguously to the directioh opposite to a given direction D. In particular, for all directions D, the direction opposite to the direction opposite to D is D itself.

Definition 5.7 For any $(A,B) \not\triangleq \underline{0}$, if $\prec$ is the ordering of AB such that $A \prec B$, then the unique direction containing $\prec$ is said to be the direction determined by (A,B).

Theorem 5.5 For any (A,B), $(C,D) \not\triangleq \underline{0}$, $(A,B) \upuparrows (C,D) \Rightarrow$ (A,B) and (C,D) determine the same direction.

Proof If $(A,B) \sqsupset (C,D)$ and $AB \neq CD$, then Def. 3.1 $\Rightarrow AB \parallel CD$ and $AC \parallel BD$. Thus there exists a parallel projection f from AB to CD such that $f(A) = C$ and $f(B) = D$. By Th. 5.4, (A,B) and (C,D) determine the same direction. Otherwise, Cor. 3.1.2 $\Rightarrow AB = CD$. If $(A,B) = (C,D)$, the theorem is trivial. If $(A,B) \neq (C,D)$, then Def. 3.1 and Th. 2.1 $\Rightarrow$ there exists points E, F and a line $m \neq AB$ such that $E, F \in m$, $(A,B) \sqsupset (E,F)$ and $(E,F) \sqsupset (C,D)$. By the first case considered, (A,B) and (E,F) determine the same direction, as do (E,F) and (C,D). Thus, (A,B) and (C,D) determine the same direction.

Remark By the previous theorem, for any $\underline{v} \in \mathcal{V}$ such that $\underline{v} \neq \underline{0}$, we may refer to the unique direction determined by any $(A,B) \in \underline{0}$, $(A,B) \underline{v}$ as the direction of $\underline{v}$. The vector $\underline{0}$ is not assigned a direction. For any direction D, it is clear from Th. 5.1 (ii) and Th. 3.3 that there exist infinitely many vectors with direction $\underline{v}$. Furthermore, we have now shown the manner in which our definition of vectors compares with the more common approach of defining vectors as sense-preserving rigid parallel displacements.

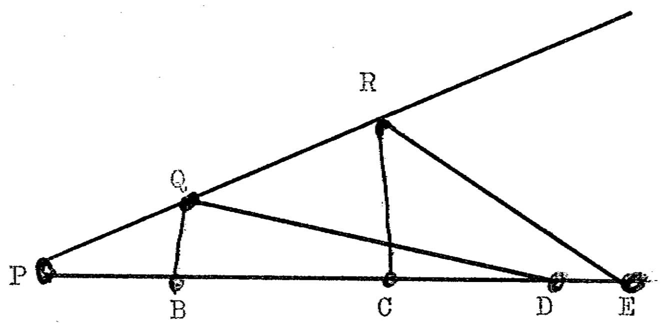
We now relate the theory of ordered affine spaces to that of scalars.

Definition 5.8 Let ℓ be any line, $<$ any order of ℓ of the type postulated in Ax. 5.1, and P any point of ℓ. Then we define the half-lines of ℓ with respect to P to be the sets $\{ Q \mid Q \in \ell,\ Q < P \text{ or } Q = P \}$ and $\{ Q \mid Q \in \ell,\ P < Q \text{ or } Q = P \}$.

Definition 5.9 A non zero scalar f is positive if and only if for any points P, Q, and R and non zero vector $\underline{v}$ such that $(P, Q) \in \underline{v}$ and $(P, R) \in f(\underline{v})$, it follows that Q and R lie on some half-line relative to P.

Lemma 5.6 Let P, Q, R, B, and C be points, f a scalar, and $\underline{w}$ and $\underline{v}$ non zero vectors such that $(P, Q) \in \underline{v}$, $(P, R) \in f\,(\underline{v})$, $(P, B) \in \underline{w}$, $(P, C) \in f\,(\underline{w})$, and Q and R lie on some halfline relative to P. Then B and C also lie on some halfline at C.

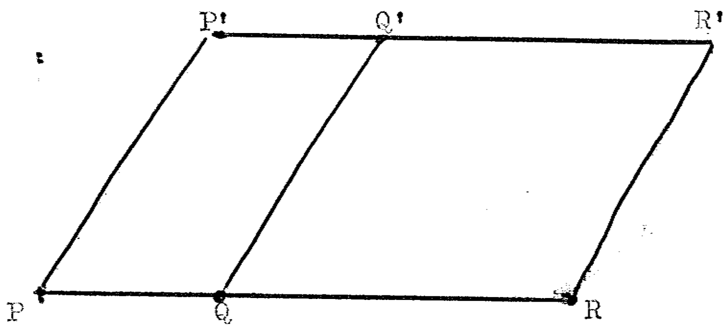
Proof Suppose $\underline{w} \nparallel \underline{v}$. If $P \mid Q \mid R$, then Prop 4.3 implies $QB \parallel RC$, and Prop. 5.5 (iv) implies $P \mid B \mid C$ by considering the parallel projection from



PR to PB by lines parallel to QB. Consequently, B and C lie on exactly one half - line relative to P. Similarly, $P \mid R \mid Q \Rightarrow P \mid C \mid B \Rightarrow$ C and B lie on exactly one half-line relative to P. Moreover, $R = Q \Rightarrow f = 1$ (by Cor. 4.2.1) $\Rightarrow B = C \Rightarrow$ B and C lie on exactly one half line (corresponding to PB) relative to P. Thus it remains only to consider the case of a non zero vector $\underline{u} \parallel \underline{v}$ such that $(P,D) \in \underline{u}$ and $(P,E) \in f\,(\underline{u})$. Since we have already shown Q and R lie on some half line re lative to P and since $\underline{w} \nparallel \underline{u}$ (by Prop. 3.4), the above argument shows that D and E also lie on some half line relative to P.

Proposition 5.6 For any scalar f, if there exists a point P such that for any points Q and R and any vector $\underline{v}$, $(P, Q) \in \underline{v}$, $(P, R) \in f\,(\underline{v}) \Rightarrow$ Q and R lie on some half line relative to P, then f is positive.

Proof Let $\underline{v}$ be any non zero vector and Q and R points such that $(P, Q) \in \underline{v}$ and $(P, R) \in f\,(\underline{v})$. Let P' be any point not on PQ and Q' and R' points such that $(P', Q') \in \underline{v}$ and $(P', R') \in f\,(\underline{v}$. Then by the property of f at P, we have



one of $\left\{\begin{array}{l} P|Q|R \Rightarrow P'|Q'|R' \\ P|R|Q \Rightarrow P'|R'|Q' \\ R = Q \Rightarrow f = 1 \Rightarrow Q' = R' \\ R = P \Rightarrow f = 0 \Rightarrow P' = R' \end{array}\right\} \Rightarrow$ Q' and R' lie on some half line relative to P'

by arguments similar of those of the preceding lemma. By Lemma 5.6, we may infer that all points P'' not on PQ have the stated property of P. However, for arbitrary point T, there exists a non zero vector $\underline{w}$ such that for any point R, $(P, R) \in \underline{w} \Rightarrow T \notin PR$. Thus all points have the stated property of P; that is, f is positive.

<u>Remark</u> Lemma 5.6 and Prop. 5.6 establish that a scalar f is positive if and only if there exist points P, Q, and R and a non zero vector $\underline{v}$ such that $(P, Q) \in \underline{v}$, $(P,R) \in f\,(\underline{v})$ and Q and R lie on some half line relative to P.

<u>Definition 5.10</u> Any non-positive, non-zero scalar is said to be <u>negative</u>.

The scalars have now been separated into three mutually exclusive and exhaustive sets : that of positive scalars, that of negative scalars, and the set $\{0\}$. It is clear from Defs.5.8 - 10 that we may think of positive scalars as order preserving, negative scalars as order reversing, and the zero scalar as order destroying.

The following theorem justifies the use of the words "positive" and "negative" in referring to scalars.

<u>Theorem 5.6</u> For any scalars f and g,

f positive, g positive $\Rightarrow$ $f + g$ positive, fg positive f positive, g negative $\Rightarrow$ f g negative

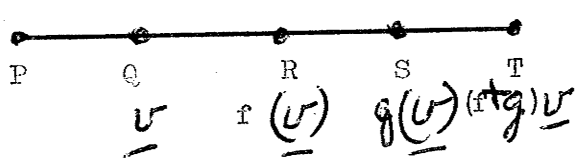
f negative, g negative $\Rightarrow$ $f + g$ negative, fg positive

f negative, g positive $\Rightarrow$ f g negative

<u>Proof</u> (i) If f, g are each positive scalars, $\underline{0} \neq \underline{v} \in \mathcal{V}$, $(P, Q) \in \underline{v}$, $(P, R) \in g(\underline{v})$ and $(P, S) \in (fg)\underline{v}$, then Q and R lie on some half line (of a line ℓ) relative to P. Similarly, R and S lie on some half line (of a line m) at P. If P = R, then Cor. 4.2.2 $\Rightarrow g = 0 \Rightarrow (fg)\underline{v} = f(g(\underline{v})) = f(\underline{0}) = \underline{0} \Rightarrow P = S \Rightarrow$ Q and S lie on some half line relative to P, by Ax. 2.2. Since Th. 4.4 (i) implies $fg \neq 0$, we have that fg is positive, by the remark following the proof of Prop. 5.6. If $P \neq R$, then ℓ and m are uniquely determined lines which, by Ax. 2.1, are the same. Thus Q and S (and R) lie on some half line relative to P, whence fg is positive.

(ii) If f and g are any negative and positive scalars respectively, $\underline{0} \neq \underline{v} \in \mathcal{V}$, $(P, Q) \in \underline{v}$, $(P, R) \in g(\underline{v})$, and $(P, S) \in (fg)\underline{v}$, then S and R do not lie on a half line relative to P, while R and Q do. Consequently, Q and S do not lie on a half line relative to P and fg is negative. Similarly, gf is negative.

(iii) If f and g are any negative scalars, $\underline{0} \neq \underline{v} \in \mathcal{V}$, $(P, Q) \in \underline{v}$, $(P, R) \in g(\underline{v})$ and $(P, S) \in (fg)\underline{v}$, then Q and R are not on a half line relative to P, and neither are R and S. Since every line has but two half lines relative to P and Props. 3.6, 4.1 $\Rightarrow$ P, Q, R, and S are collinear, it follows that Q and S lie on some half line relative to P. Thus fg is positive.



(iv) Let f and g be any positive scalars such that $(P,Q) \in \underline{v}$, $(P,R) \in f(\underline{v})$, $(P,S) \in g(v)$ and $(P,T) \in (f+g)\underline{v}$. In order to compute $(f+g)\underline{v}$, we employ Def. 5.9 to find that Q and R are on a half line relative to P, as are Q and S. It follows from Cor. 5.3.3 that Q and T are on some half line relative to P. For if $P < R$ (resp. $R < P$), then $S < T$ (resp. $T < S$). Furthermore, $P < Q$ (resp. $Q < P$) and $P < S$ (resp. $S < P$). Thus we have $P < S < T$ (resp. $T < S < P$) $\Rightarrow P < T$ (resp. $T < P$). Since $P < Q$ (resp. $Q < P$), it is indeed true that Q and T lie on some half line relative at P, whence $f+g$ is positive.

(v) By a method analagous to that of (iv), one may prove that the sum of any two negative scalars is itself negative. For, with the notation of (iv), we have R and Q are not on a half line relative to P, and neither are S and Q. Since P, Q, R, and S are collinear, R and S lie on some half line relative to P. By Theorem 5.6, if $P < R$ (resp. $R < P$) then $S < T$ (resp. $T < S$). Moreover, $Q < P$ (resp. $P < Q$) and $P < S$ (resp. $S < P$). Thus $Q < P < S < T$ (resp. $T < S < P < Q$) $\Rightarrow Q < P < T$ (resp. $T < P < Q$) $\Rightarrow$ Q and T do not lie on a half line relative to P. This completes the proof.

<u>Corollary 5.6.1</u> For any non zero scalar f, ff is positive.

<u>Proof</u> Either f is positive or f is negative. In the former case, the proof is immediate from Th. 5.7. If f is negative, we assert that $-f$ is positive, for if $-f = 0$, then $f = --f = -0 = 0$ (which is not positive) and if $-f$ were negative, then $0 = f + (-f)$, as the sum of two negative scalars is itself negative, a contradiction. Thus $-f$ is positive, and by the first case considered, $(-f)(-f)$ is also positive. Since $(-f)(-f) = ((-1)f)\,((-1)f)$ $= (-1)(f((-1)f)) = (-1)((f(-1))f) = (-1)(((-1)f)f) = (-1)((-1)(ff)) = -((-1)(ff))$ $= --(ff) = ff$, the proof is complete.

In proving Cor. 5.6.1, we have also proved

Corollary 5.6.2 For any scalar f, f is positive $\Leftrightarrow$ -f is negative.

Corollary 5.6.3 1 is positive.

Proof Since $1 \neq 0$, if 1 is not positive, then 1 is negative. By the preceding corollary, as $1 = -(-1)$, we have that -1 is positive. By Th. 5.7, $(-1)(-1)$ is positive. However $(-1)(-1) = 1$ since $(-1)(-1) + (-1) = (-1)\cdot(-1) + (1)(-1) = (-1 + 1)(-1) = 0\,(-1) = 0$. Thus 1 is positive, a contradiction. This completes the proof.

Cor. 5.6.4 For all positive integers n, $\underbrace{1 + \ldots + 1}_{n \text{ times}}$ is positive.

The trivial proof by induction may safely be left to the reader.

Corollary 5.6.5 For any non zero scalar f, f pos (resp. neg) $\Rightarrow f^{-1}$ pos (resp. neg.)

Proof As 1 is positive, so must $f^{-1} f$ be positive. Since $f^{-1} \neq 0$, the conclusion follows from Th. 5.6.

Corollary 5.6.6 For any directions D_1 and D_2, there are exactly as many vectors with direction D_1 as vectors with direction D_2.

Proof Without loss of generality $D_1 \neq D_2$. If D_1 and D_2 are opposite directions, then the result is obvious; for corresponding to every vector $\underline{v}$ with direction D_1, there is a vector $-\underline{v}$, with direction D_2. If D_1 and D_2 are not opposite directions, then they contain orderings of lines of distinct parallel classes, C_1 and C_2.

Let P be any point and ℓ_i the unique line of C_i $(i = 1, 2)$ through P. Now there are exactly as many vectors with direction D_i as there are points (other than P) on the half line relative to P whose corresponding line is in C_i, by Th. 3.3 and Th. 5.5. This latter number is simply the number of positive (or indeed the number of negative) scalars, by Th. 4.2. As this number is the same for (that is, functionally independent of) D_i, the conclusion follows.

In proving Cor. 5.6.6 we have also proved

Corollary 5.6.7 The number of positive scalars is the same as the number of negative scalars which common value is the same as the number of vectors with any given direction and also the same as the number of points on any half line.

Definition 5.11 For any scalars f and g, we say that f precedes g and write $f \prec g$ if and only if $g + (-f)$ is positive.

Proposition 5.7 For any scalars f, g, and h, (i) $f \prec g, g \prec h \Rightarrow f \prec h$

(ii) $f \prec g \Rightarrow f + h \prec g + h$

(iii) $f \prec g,\ 0 \prec h \Rightarrow f h \prec g h,\ h f \prec h g$

(iv) $f \prec g,\ h \prec 0 \Rightarrow g h \prec f h,\ h g \prec h f$

Proof (i) By Th. 5.6, $h + (-f) = (g + (-f)) + (h + (-g)) \Rightarrow h + (-f)$ is positive $\Rightarrow f \prec h$.

(ii) Since $(g + h) + (-(f + h)) = (g + h) + ((-h) + (-f)) = g + (-f)$, it follows that $f \prec g \Rightarrow g + (-f)$ is positive $\Rightarrow (g + h) + (-(f + h))$ is positive $\Rightarrow f + h \prec g + h$.

(iii) Since $(g h) + (-(fh)) = g h + ((-1)(f h)) = g h + (((-1)f)h) = g h + ((-f) h) = (g + (-f)) h$ and both $g + (-f)$ and h are positive, it follows from Th. 5.6 that $(g h) + (-(f h))$ is positive; that is, $f h \prec g h$. Similarly $h f \prec h g$.

(iv) The proof of (iv) may be patterned after that of (iii).

Proposition 5.8 For any scalars f and g such that $f \prec g$, there exists at least one scalar h such that $f \prec h$ and $h \prec g$.

Proof Let $h = (f + g)(1 + 1)^{-1}$. Now, $h + (-f)$

$$= (f + g)(1+1)^{-1} + (-f)\ 1 = (f + g)(1+1)^{-1} + (-f)((1+1)(1+1)^{-1})$$

$$= ((f + g) + (-f)(1+1))\ (1+1)^{-1} = ((f + g) + (-f-f))\ (1+1)^{-1}$$

$$= (g + (-f))\ (1 + 1)^{-1}.$$

By Cors. 5.6.4, 5, Def. 5.11 and Th. 5.6, it follows that $h + (-f)$ is positive. Thus $f \prec h$ and one shows similarly that $h \prec g$.

Recall that Σ contains subsets A, B which are isomorphic (with respect to addition and multiplication) to the sets of positive integers and integers respectively. It is now evident that these isomorphisms persist with respect to order.

Let R be the set of rational numbers. The elements of R are equivalence classes of ordered pairs (m, n) of integers for which $n \neq 0$, such that (m, n) is in the same rational number as (p, q) if and only if $mq = pn$. Denote the rational number containing (m, n) by $[m, n]$. Where ψ is the function considered in the remark following the proof of Cor. 5.3.5, define the relation ϕ from R into Σ as follows:

$$\phi([m, n]) = \begin{cases} \psi(m)\ (\psi(n))^{-1} & \text{if } m \text{ and } n \text{ are each positive,} \\ -\psi(-m)\ (\psi(n))^{-1} & \text{if } m \text{ is negative and } n \text{ is positive,} \\ 0 & \text{if } m = 0. \end{cases}$$

It is now possible to show that ϕ is a (single-valued) one-to-one function from R onto a subset D of Σ, $B \subset D$, such that D is isomorphic to R (with respect to addition, multiplication, and order). This result is also a direct consequence of the well-known theorem that, for any division ring F of characteristic zero whose identity element for multiplication generates a sub division-ring G of F, G is isomorphic to R (with respect to addition, multiplication, and order.)

Proposition 5.9 For any scalars f and g, exactly one of the following holds:

(1) $f = g$

(2) $f < g$

(3) $g < f$

Proof Since 0 is not positive, it is clear that (1) cannot hold when (2) or (3) does. Moreover (2) and (3) cannot hold together, for $f < g$, $g < f \Rightarrow 0 = (g + (-f)) + (f + (-g))$ is positive, a contradiction.

We need only show that $f \neq g$, $g \not< f \Rightarrow f < g$. Now, if $f \neq g$ and $f \not< g$, then $f + (-g)$ is neither positive nor zero and hence is negative. As 0 is not negative, Th. 5.6 $\Rightarrow -(f + (-g))$ is positive. Thus $-(f + (-g)) = -(-g) + (-f) = g\ (-f)$ is positive and so $f < g$.

Preparatory to stating an important theorem about Σ, we make the following definition.

An ordered field is a field in which an order relation $<$ is defined satisfying the conditions of Props. 5.7, 9.

Definition 5.12 Any nonempty set S in an ordered division ring $G = (F, +, \cdot, <)$ is bounded above if and only if there exists an element f in F such that for all $x \in S$, either $x = f$ or $x < f$. Any such f is called an upper bound of S. If there exists an upper bound g of S such that for all upper bounds h of S, either $g = h$ or $g < h$, then g is called a least upper bound of S. G is said to be complete if and only if every nonempty set S of F has a least upper bound.

It is well known that any complete ordered field is isomorphic to the real number system. We have proved that Σ is a division ring (Cor. 4.5.3) and ordered (Props. 5.7, 9). Consequently, we have the following important result.

<u>Theorem 5.7</u> The ordered division ring Σ of scalars is isomorphic to the real number system if and only if

(1) $f, g \in \Sigma \Rightarrow f g = g f$

and (2) Σ is complete.

We conclude this chapter by showing that some classical results in Euclidean geometry are also valid in the ordered affine space under consideration.

<u>Definition 5.13</u> Let P and Q be any points. Then any point R such that $(P,R) \uparrow (R,Q)$ is called a <u>midpoint</u> of P and Q.

It is immediate from Defs. 3.1, 2 that any midpoint of P and Q is also a midpoint of Q and P.

<u>Theorem 5.8</u> If P and Q are any points, then there exists a unique point R such that R is a midpoint of P and Q.

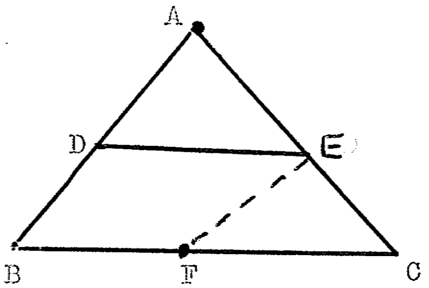
<u>Proof</u> (i) Let P = Q. If there exists a point R such that $(P,R) \uparrow (R,Q)$ and $\underline{W}$ is the vector containing (P,R), then $(P,R) \uparrow (R,P) \Rightarrow \underline{W} = -\underline{W}$ $\Rightarrow (1+1)\underline{W} = 1\underline{W} + 1\underline{W} = \underline{W} + \underline{W} = -\underline{W} + \underline{W} = \underline{0} \Rightarrow \underline{W} = (1+1)^{-1}(\underline{0}) = \underline{0}$. Since $(P,R) \in \underline{0}$, Th. 3.3 implies P = R; thus, P and Q have at most one midpoint. As it is clear that $(P,P) \uparrow (P,P)$, we have that P is a midpoint of P and Q, whence P is the unique midpoint of P and Q.

(ii) Let $P \neq Q$ and $\underline{V}$ be the (non zero) vector containing (P,Q). By Th. 4.3, there exists unique $\underline{W} \in \mathcal{V}$ such that $\underline{V} = (1+1)\underline{W}$. If T is the unique point such that $(P,T) \in \underline{W}$, then we need only let T = R. For, $\underline{V} = \underline{W} + \underline{W} \Rightarrow (T,Q) \in \underline{W}$ (by Th. 3.5 (ii)-(v)) $\Rightarrow (P,T) \uparrow (T,Q)$.

It is clear that no value for R other than T suffices. For if $(P,R) \uparrow (R,Q)$ and $(P,R) \in \underline{U}$, then $\underline{W} = (1+1)\underline{U} = (1+1)\underline{W} \Rightarrow \underline{U} = (1+1)^{-1}((1+1)\underline{U}) = (1+1)^{-1}((1+1)\underline{W}) = \underline{W} \Rightarrow R = T$, by Th. 3.3.

Thus, it is unambiguous to refer to the midpoint of P and Q. Using this notion, we now prove the celebrated theorem of Thales and its converse.

<u>Theorem 5.9</u> If A, B, and C are any three noncollinear points and D is the mid point of A and B, then for any point $E \in AC$, E is the midpoint of A and C if and only if $DE \parallel BC$. Moreover, if E is the midpoint of A and C and $\underline{u}$ and $\underline{v}$ are vectors such that $(B,C) \in \underline{u}$ and $(D,E) \in \underline{v}$, then $\underline{u} = (1+1)\underline{v}$.



<u>Proof</u> (i) Let E be any point such that $DE \parallel BC$. Then if $(A,D) \in \underline{d}$, $(A,B) \in \underline{b}$, $(A,E) \in \underline{e}$ and $(A,C) \in \underline{c}$, it follows from Prop. 4.3 that $\underline{b} = (1+1)\underline{d}$ $\Rightarrow \underline{c} = (1+1)\underline{e} = \underline{e} + \underline{e}$. If $(E,C) \in \underline{g}$, then $\underline{c} = \underline{e} + \underline{e} = \underline{e} + \underline{g} \Rightarrow$ $\underline{e} = \underline{g} \Rightarrow (A,E) \uparrow (E,C) \Rightarrow$ E is the midpoint of A and C.

(ii) Suppose E is the midpoint of A and C and $DE \nparallel BC$. Then if ℓ is the line through D parallel to BC, it follows from Th. 2.3, Cor. 2.4.1, Ax. 2.7, and part (i), that the intersection of ℓ and AC contains only the midpoint of A and C, namely E. By Ax. 2.1, $\ell =$ DE, a contradiction. Thus $DE \parallel BC$.

(iii) Let D and E be the midpoints of A, B and A, C respectively. Let the line through E parallel to AB intersect BC in the point F. Let $(B,C) \in \underline{u}$, $(D,E) \in \underline{v}$, and $(B,F) \in \underline{w}$. Then by part (i), $\underline{u} = (1+1)\underline{w}$. Since $(D,E) \square (B,F)$ by construction, we have $(D,E) \uparrow (B,F) \Rightarrow \underline{u} = \underline{w} + \underline{w} = \underline{v} + \underline{v} = (1+1)\underline{v}$. Q.E.D.

We shall now generalize the preceding results.

<u>Definition 5.14</u> Let P and Q be any points and n any positive integer. The $(n-1)$ tuple $(P_1, \ldots, P_{n-1})$ is called an <u>n-division</u> of P and Q if and only if $(P,P_1) \uparrow (P_{n-1}, Q) \uparrow (P_i, P_{i+1}) : i = 1, 2, \ldots, n-2$.

It is immediate that if $(P_1, P_2, \ldots, P_{n-1})$ is an n-division of P and Q, then $(P_{n-1}, P_{n-2}, \ldots, P_2, P_1)$ is an n-division of Q and P.

Theorem 5.10 Let P and Q be any points and n any positive integer. Then there exists a unique n-division of P and Q.

Proof (i) We shall prove that there exists at most one n-division of P and Q.

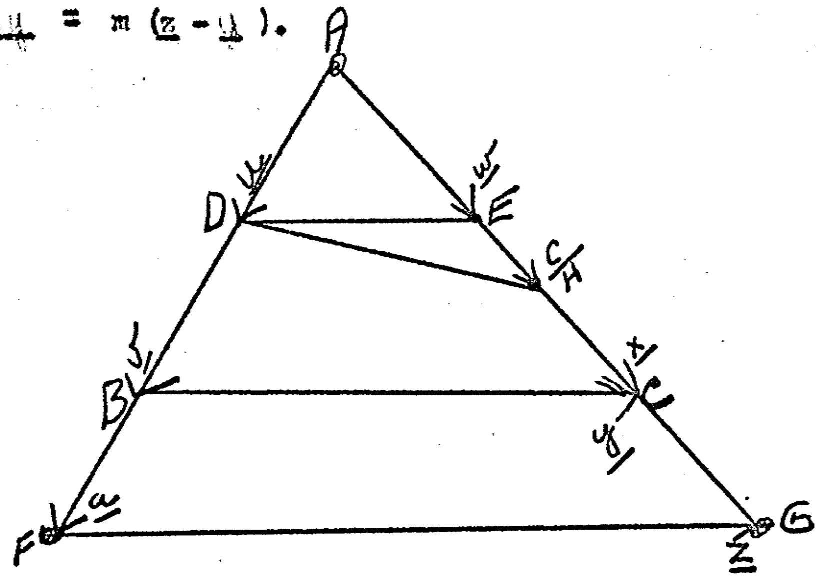
Suppose $P = Q$. Then if $(P_1, \ldots, P_{n-1})$ is an n-division of P and Q, let $\underline{w}$ be the vector containing (P,P_1). By the definition of an n division, it follows that $(P,Q) \in n\underline{w}$. Since $Q = P$, $n\underline{w} = \underline{0}$ and Cor. 4.2.2 and Cor. 5.3.5 imply $\underline{w} = \underline{0}$, whence Th. 3.3 gives $P = P_1$. By Prop. 3.2 (ii) and an obvious mathematical induction on i, we have $P_i = P$: $i = 1, 2, \ldots, n-1$. Consequently, at most one n-division of P and Q exists.

If $P \neq Q$, let $(P_1, \ldots, P_{n-1})$ be an n-division of P and Q, $\underline{u}$ the vector containing (P, P_1) and $\underline{v}$ $(\neq \underline{0})$ the vector containing (P,Q). As above, it follows that $\underline{v} = n\,\underline{u}$. Since $1 + \ldots + 1$ (n times) $\neq 0$ and $\underline{v} \neq \underline{0}$, Th. 4.3 implies that $\underline{v}$ uniquely determines $\underline{u}$, whence Th. 3.3 implies P_1 is uniquely determined. Similarly, by mathematical induction, one shows that $(P_1, \ldots, P_{n-1})$ is uniquely determined.

(ii) It remains to find an n-division of P and Q. If $P = Q$, it is evident that $P_i = P$: $i = 1, 2, \ldots, n-1$ is satisfactory. If $P \neq Q$ and $\underline{w}$ is the (non zero) vector containing (P,Q), let $\underline{u}$ be the unique vector such that $\underline{w} = n\,\underline{u}$.

If $P_1, \ldots, P_{n-1}$ are the points such that $(P, P_1) \in \underline{u}$, $(P_i, P_{i+1}) \in \underline{u}$: $i = 1, 2, \ldots, n-1$, then it is immediate that $(P_1, \ldots, P_{n-1})$ is an n-division of P and Q.

Theorem 5.11 Let A, B, and C be three noncollinear points, D and E any points such that $D \in AB$ and $E \in AC$, $\underline{u}$, $\underline{v}$, $\underline{w}$, and $\underline{x}$ the vectors containing (A, D), (A,B), (A,E), and (A,C) respectively, m and n any positive integers, and $\underline{a}$ a vector such that $\underline{a} = n\,\underline{u} = m\,\underline{v}$. Then $n\,\underline{w} = m\,\underline{x}$ if and only if DE $\parallel$ BC. If DE $\parallel$ BC, and $\underline{y}$ and $\underline{z}$ are the vectors such that $(D,E) \in \underline{y}$ and $(B,C) \in \underline{z}$, then $n\,\underline{y} = m\,(\underline{z} - \underline{y})$.



Proof Let F, G, and H be the unique points and $\underline{b}$ the vector such that $(A,F) \in \underline{a}$, $(A,G) \in m\,\underline{x} = \underline{b}$, and the vector $\underline{c}$ containing (A,H) is such that $n\,\underline{c} = \underline{b}$. By Prop. 4.3, DH $\parallel$ FG and BC $\parallel$ FG. By Ax. 2.4, we have DH $\parallel$ BC. Now DE $\parallel$ BC $\Longleftrightarrow$ E = H since the line through D parallel to BC intersects AC in exactly one point; for all the lines being considered are coplanar and A, B, C noncollinear $\Rightarrow \underline{v} \neq \underline{0} \Rightarrow m\,\underline{v} \neq \underline{0} \Rightarrow \underline{u} \neq \underline{0} \Rightarrow A \neq D \Rightarrow D \notin AC$ (by Ax. 2.1). This proves that $n\,\underline{w} = n\,\underline{x}$ if and only if DE $\parallel$ BC. The final conclusion of the theorem follows by an argument analogous to that of Th. 5.9 (iii).

Remark Let ℓ_i (i = 1, 2, 3) be distinct lines intersecting in a point P. Let P_i (i = 1 - 6) be points distinct from one another and also from P such that $P_{2i-1}, P_{2i} \in \ell_i$ $(i = 1,2)$, $P_1P_3 \parallel P_2P_4$, and P_3P_6. If there exist positive integers m and n such that $m\,\overrightarrow{PP_3} = n\,\overrightarrow{PP_4}$, than $P_1P_5 \parallel P_2P_6$ (where $\overrightarrow{PP_3}$ is the vector containing (P,P_3)). For $m\,\overrightarrow{PP_3} = n\,\overrightarrow{PP_4} \Longrightarrow$

$m\,\overrightarrow{PP_1} = n\,\overrightarrow{PP_2}$, $m\,\overrightarrow{PP_5} = n\,\overrightarrow{PP_6} \Longrightarrow$

$P_1P_5 \parallel P_2P_6$, by Th. 5.11. Thus the generalized form of Thales' theorem allows us to infer one of Desargues' theorems, under certain special conditions (namely, that m and n be "positive integral" scalars).

Definition 5.15 An ordered set $(S, \prec)$ is convex if and only if $P, Q \in S \Rightarrow (P \prec R \prec Q$ or $Q \prec R \prec P \Rightarrow R \in S)$.

Theorem 5.12 Given a line ℓ and a plane π such that $\ell \subset \pi$ the set of points of π not on ℓ can be expressed as the union of two nonempty sets called half planes such that (1) each half plane is convex and (2) for all points P and Q in distinct half planes, there exists a point $R \in \ell$ such that $P \mid R \mid Q$.

Define the linary relation $\#$ on $\pi - \ell$ as follows.

For all $P, Q \in \pi - \ell$, $P \# Q \Leftrightarrow$ at least one of

(1) $P = Q$

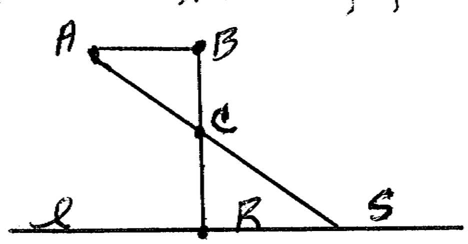
(2) $PQ \parallel \ell$

(3) there exists a point $R \in PQ \cap \ell$ such that $P \mid Q \mid R$ or $Q \mid P \mid R$

holds.

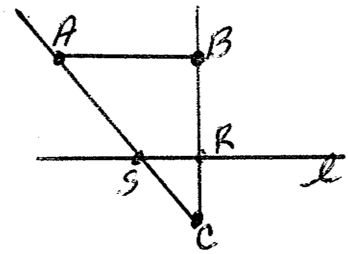
We now proceed to show that $\#$ is an equivalence relation on $\pi - \ell$.

Proof (i) Suppose A, B, and C are distinct points of $\pi - \ell$ such that $AB \parallel \ell \nparallel BC$. Let R and S be the unique points such that $\{R\} = BC \cap \ell$ and $\{S\} = AC \cap \ell$. If $B \mid C \mid R$, then the parallel projection from BC to AC with lines parallel to ℓ gives $A \mid C \mid S$, by Prop. 5.5 (iv).



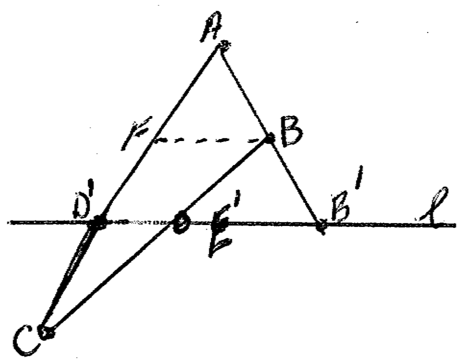
(ii) Let A, B, and C be distinct points of $\pi - \ell$ such that there exist unique points R and S satisfying $\{R\} = BC \cap \ell$ and $\{S\} = AC$

$\cap \ell$. If $B \mid R \mid C$ and $AB \parallel \ell$, then the parallel projection from BC to AC with lines parallel to AB gives $A \mid S \mid C$.

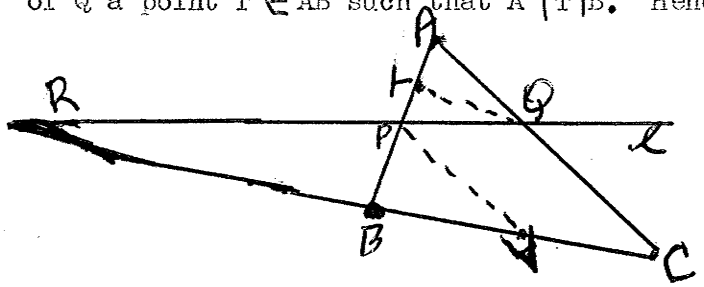


(iii) Let A, B, and C be distinct points of $\pi - \ell$ such that $AB \parallel \ell$ and $BC \parallel \ell$. Then Ax. 2.4 $\Rightarrow AC \parallel \ell$.

(iv) Let, A, B, and C be distinct points of $\pi - \ell$ such that there exists unique points B', D', and E' satisfying $AB \cap \ell = \{B'\}$, $AC \cap \ell = \{D'\}$, $BC \cap \ell = \{E'\}$, $A \mid B \mid B'$, and $A \mid D' \mid C$. We shall prove $B \mid E' \mid C$. For $A \mid B \mid B' \Rightarrow$ there exists $F \in AC$ such that $A \mid F \mid D'$, by considering the parallel projection from AB to AC by lines parallel to ℓ. As $A \mid D' \mid C$, it follows that $F \mid D' \mid C$. Parallel projection from AC to BC by lines parallel to ℓ gives $B \mid E' \mid C$.



(v) Let A, B, and C be distinct points of $\pi - \ell$ such that there exist unique points P, Q, and R satisfying $\{P\} = AB \cap \ell$, $\{Q\} = AC \cap \ell$, and $\{R\} = BC \cap \ell$. Suppose that $A \mid P \mid B$ and $A \mid Q \mid C$. If P, Q, and R are not all distinct, then A, B, and C are collinear and it is clear that $B \mid C \mid R$. Henceforth we assume P, Q, and R are all distinct. Then the parallel projection from AC to AB by lines parallel to BC gives as the image of Q a point $T \in AB$ such that $A \mid T \mid B$. Hence, either $P \mid T \mid B$ or $T \mid P \mid B$.



Now the parallel projection from AB to PQ with lines parallel to BC implies $P|Q|R$ or $Q|P|R$, since either $P|T|B$ or $T|P|B$. Then the parallel projection from AB to BC by lines parallel to AC yields a point $V \in BC$ as the image of P such that $B|V|C$. Thus the parallel projection from PR to BR by lines parallel to QC gives $V|C|R$ or $C|V|R$. It is clear that $B|V|C$ and $V|C|R \Rightarrow B|C|R$. If $B|V|C$ and $C|V|R$, we claim $C|B|R$ or $B|C|R$, since $B|R|C$ contradicts (iv), as applied to the line AB and the points $C, R,$ and Q.

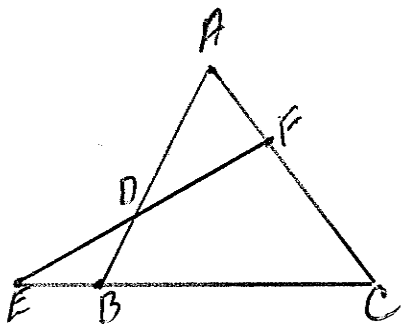
(vi) Let A, B, and C be any distinct points of $\pi \sim \ell$ and B', A', and C' the unique points such that $AB \cap \ell = \{B'\}$, $AC \cap \ell = \{C'\}$, $BC \cap \ell = \{A'\}$, $A|B|B'$ and $A|C|C'$. If $B|A'|C$, then we have a contradiction to (v).

It is immediate from (i) to (vi) that $\mathcal{H}$ is an equivalence relation on $\pi - \ell$, with exactly two equivalence classes (half planes) which are evidently convex. Furthermore neither half plane is empty, for if $P \notin \ell$, $Q \in \ell$, and $P < Q$, then there exists a point $T \in PQ$ by Th. 5.1 such that $Q < T$, that is, $P|Q|T \Rightarrow P$ and T are in distinct half planes $\Rightarrow$ neither half plane is empty. This completes the proof.

<u>Corollary 5.12.1</u> Let ℓ be any line and π any plane such that $\ell \subset \pi$. If H_1 and H_2 are the half-planes of π relative to ℓ, then H_1 and H_2 contain the same number of points.

<u>Proof</u> Let P be any point of ℓ and Q any point of $\pi - \ell$. Corresponding to any point $R \in \pi - \ell$, there is exactly one line $m \parallel PQ$ such that $R \in m$. It follows from Lemma 2.4.1 and Ax. 2.7 that $m \cap \ell$ contains exactly one point. Thus, for any line $n \parallel PQ$, if S, T, and W are any distinct points of n such that $n \cap \ell = \{W\}$; then S and T lie in the same half plane H_1 or H_2 if and only if S and T lie on a half line relative to W. Since any two half lines contain the same number of points, the fact that each $R \in \pi \sim \ell$ lies on exactly one line parallel to PQ implies the required result.

Corollary 5.12.2 Let A, B, and C be any noncollinear points of a plane π and ℓ a line of π containing none of A, B, C. If there exist points D, E, F such that $D \in \ell \cap AB$, $E \in \ell \cap BC$ and $F \in \ell \cap AC$, then



(1) $A|D|B$, $B|E|C \Rightarrow A|C|F$ or $C|A|F$

and (2) ($A|B|D$ or $B|A|D$), ($B|C|E$ or $C|B|E$) $\Rightarrow A|C|F$ or $C|A|F$.

Proof (1) Since $A|D|B$, it follows that A and B lie in distinct half planes relative to ℓ. Similarly, $B|E|C \Rightarrow$ B and C lie in distinct half planes relative to ℓ. As there exist but two half planes of π relative to ℓ, it follows that some one of them contains both A and C. Thus F does not lie between A and C.

(2) As in the previous case, A and B lie in some half plane of π relative to ℓ, and so do B and C. Since B lies in exactly one half plane of π relative to ℓ, it follows that A and C lie in the same half planes, whence it cannot be the case that $A|F|C$.

Corollary 5.12.3 Let A, B, and C be any noncollinear points of a plane π and ℓ a line of π such that $B \notin \ell$. If there exists a point D such that $A|D|C$ and $\ell \cap AC = \{D\}$, then there exists a point E such that either $A|E|B$ or $B|E|C$.

Proof Since $A|D|C$, it follows that $A \neq D \neq C$, whence $A \notin \ell$ and $C \notin \ell$. If there does not exist a point E with the stated properties, then A and B lie in some half plane of π relative to ℓ, and so do B and C. Thus the same is true of A and C, a contradiction to the fact that $A|D|C$.

The reader will recognize Cor. 5.12.3 as the famous axiom of Pasch.

With the aid of Th. 5.12 and its corollaries, one may define the interior of the angle between the half lines PQ and QR in the plane π PQ, QR as the intersection of the half plane determined by PQ and containing R with the half plane determined by QR and containing P. It is clear that the definition is independent of P and R. One may then prove that if D is in the interior of the angle between the half lines BA and AC, then there exists a point E such that $B|E|C$ and not $(D|A|E)$. It is then immediate that the diagonals of a convex quadrilateral intersect one another.

For details, the reader is referred to [4], Chapter 4.

<u>Theorem 5.13</u> Let π be any plane. Let $\mathcal{R}$ be the relation such that, for any points $P, Q \in \mathcal{P} - \pi$, $P \mathcal{R} Q \Leftrightarrow$ at least one of

(1) $P = Q$

(2) $PQ \cap \pi = \emptyset$

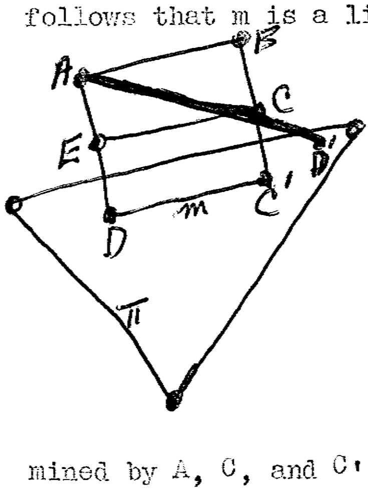
(3) There exists a point $R \in PQ \cap \pi$ such that $P|Q|R$ or $Q|P|R$ holds.

Then $\mathcal{R}$ is an equivalence relation on $\mathcal{P} - \pi$.

<u>Proof</u> Let A, B, and C be distinct points of $\mathcal{P} - \pi$.

(i) If $AB \cap \pi = \emptyset$ and there exist points C' and D' of π such that $B|C|C'$ and $AC \cap \pi = \{D'\}$, we shall show $A|C|D'$.

Let m be the line through C' parallel to AB. By Cors. 2.5.1, 3, 4, it follows that m is a line of π. Let D be the point of m which is on the line

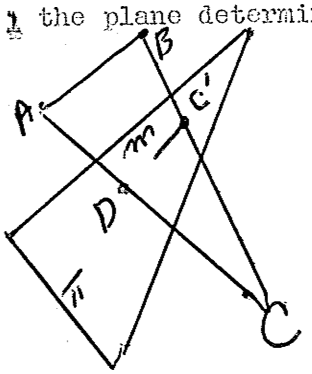


through A parallel to BC; let E be the point of AD which is on the line through C parallel to m. Since $B|C|C'$, it follows from Prop. 5.5 (iv) that $A|E|D$. Th. 2.2 (i) and Ax. 2.7 imply that AC intersects m in a unique point, for m and AC are nonparallel lines of the plane determined by A, C, and C'. As $AC \cap \pi = \{D'\}$, we may infer that $m \cap AC = \{D'\}$. By considering the parallel projection from AD to AD' with lines parallel to m, we find $A|E|D \Rightarrow A|C|D'$.

(ii) If $AB \cap \pi = \emptyset$ and there exists a point $C' \in \pi$ such that $B|C'|C$, then we show there exists a point $D \in \pi$ such that $A|D|C$.

If AC is parallel to π, then Cor. 2.5.1 $\Rightarrow$ the plane π^{*} parallel to π is determined uniquely by A, B, and C. As $A \notin \pi$, we have $\pi^{*} \cap \pi = \emptyset \Rightarrow BC \cap \pi = \emptyset$, a contradiction to the existence of C'. Thus $AC \cap \pi \neq \emptyset$.

Let D be the point in $AC \cap \pi$, m the line through C' parallel to AB, and π_1 the plane determined by A, B, and C. By Th. 2.2 (i), AD and m each lie in π_1 but are not parallel since $C \notin AB$. As Ax. 2.7 $\Rightarrow$ $AD \cap m$ contains exactly one point and $AD \cap \pi = \{D\}$, it follows that $D \in m$. Since $A \notin BC$, we have $C' \neq D$, whence Axs. 2.1, 2 imply $C'D = m$. By considering the parallel projection from BC to AC with lines parallel to m, we find $B|C'|C \Rightarrow A|D|C$.



(iii) If $AB \cap \pi = BC \cap \pi = \emptyset$, we shall show $AC \cap \pi = \emptyset$. By Cor. 2.5.1, AB and BC lie in π_1, the plane through B parallel to π. Without loss of generality, $AB \neq BC$, whence π_1 is uniquely determined by A, B, and C. Since $AC \subset \pi_1$ and $A \notin \pi \Rightarrow \pi_1 \cap \pi = \emptyset$, it follows that $AC \cap \pi = \emptyset$.

(iv) If there exist points B', C', and D of π such that $A|B'|B$ and $A|C'|C$, we may infer that $B|C|D$ or $C|B|D$. For $B'C'$ and BC each lie in the plane of A, B, and C (which is unique since $C \in AB \Rightarrow B' = C' = D$, a trivial case) and are not parallel since $BC \cap \pi \neq \emptyset$. It follows from Ax. 2.7, Th. 2.3 that $B'C' \cap BC = \{D\}$. Thus either $B|C|D$ or $C|B|D$, by Cor. 5.12.2.

(v) If there exist points B' and C' of π such that $A|B'|B$ and $A|C|C'$, then we shall show there exists a point $D \in \pi$ such that $B|D|C$. For if $BC \parallel$

π, then $A \mid B' \mid B \Rightarrow A \mid C' \mid C$, by (ii), a contradiction. Let D be the unique point such that $BC \cap \pi = \{D\}$. Since the plane of A, B, and C intersects π in precisely a line, it follows that $D \in B'C'$, for the hypotheses exclude the possibility that A, B, C collinear. We may infer $B \mid D \mid C$ from Cor. 5.12.2.

(vi) If there exist points B', C', and D of π such that $A \mid B \mid B'$, $A \mid C \mid C'$ and $D \in BC$, then $D \in B'C'$ by the argument of (v). It follows from Cor. 5.12.1 that either $C' \mid B' \mid D$ or $B' \mid C' \mid D$.

Thus $\mathcal{R}_\pi$ is transitive. The reflexivity and symmetry of $\mathcal{R}_\pi$ are evident.

For a given plane π, we call the two equivalence classes of the relation $\mathcal{R}_\pi$ the <u>half spaces relative to π</u>. It is evident from the definition of $\mathcal{R}_\pi$ that half spaces are convex sets. It may be shown, as in Th. 5.12, that half spaces are non-empty. With the aid of Cor. 2.5.6, one may use the method of Cor. 5.12.1 to prove that, for a fixed plane π, the corresponding half spaces contain the same number of points.

VI TRIPLE VECTOR PRODUCT, BASES, AND METRIC GEOMETRY

Definition 6.1 We define a triple vector product (or "box product") as as a function from $\mathcal{V} \times \mathcal{V} \times \mathcal{V}$ into Σ satisfying the following conditions for all vectors $\underline{v}_i$ and scalars f:

(1) $[\underline{v}_1, \underline{v}_2, \underline{v}_3] = -[\underline{v}_1, \underline{v}_3, \underline{v}_2] = [\underline{v}_3, \underline{v}_1, \underline{v}_2]$,

(2) $[\underline{v}_1 + f\,\underline{v}_2, \underline{v}_3, \underline{v}_4] = [\underline{v}_1, \underline{v}_3, \underline{v}_4] + f\,[\underline{v}_2, \underline{v}_3, \underline{v}_4]$,

and (3) $[\underline{i}, \underline{j}, \underline{k}] = 1$ where $\underline{i}$, $\underline{j}$, and $\underline{k}$ are three fixed noncoplanar vectors (which, by Th. 4.8, span $\mathcal{V}$ under Σ).

Proposition 6.1 (i) $\underline{c}, \underline{d} \in \mathcal{V} \Rightarrow [\underline{0}, \underline{c}, \underline{d}] = 0$

(ii) $\underline{b}, \underline{c}, \underline{d} \in \mathcal{V}, f \in \Sigma \Rightarrow [f\,\underline{b}, \underline{c}, \underline{d}] = f\,[\underline{b}, \underline{c}, \underline{d}]$

(iii) $\underline{a}, \underline{b} \in \mathcal{V} \Rightarrow [\underline{b}, \underline{a}, \underline{a}] = 0$

(iv) $\underline{d}_k \in \mathcal{V} \Rightarrow$ any interchange of $\underline{d}_i$ and $\underline{d}_j$, $i \neq j$, in $[\underline{d}_1, \underline{d}_2, \underline{d}_3]$ changes a box product into $-[\underline{d}_1, \underline{d}_2, \underline{d}_3]$.

Proof (i) On substituting $f = 1$, $\underline{v}_2 = \underline{0}$ into Def. 6.1(2), we find $[\underline{v}_1, \underline{v}_3, \underline{v}_4] = [\underline{v}_1, \underline{v}_3, \underline{v}_4] + [\underline{0}, \underline{v}_3, \underline{v}_4]$. By adding $-[\underline{v}_1, \underline{v}_3, \underline{v}_4]$ to both sides of this equation and setting $\underline{v}_3 = \underline{a}$, $\underline{v}_4 = \underline{d}$, we get $[\underline{0}, \underline{c}, \underline{d}] = 0$.

(ii) Substitute $\underline{v}_1 = \underline{0}$, $\underline{v}_2 = \underline{b}$, $\underline{v}_3 = \underline{c}$, $\underline{v}_4 = d$ into Def. 6.1 (2) and use (i).

(iii) By Def. 6.1 (1) we have $[\underline{b}, \underline{a}, \underline{a}] = -[\underline{b}, \underline{a}\ \underline{a}] \Rightarrow (1+1)\cdot [\underline{b}, \underline{a}, \underline{a}] = 0 \Rightarrow [\underline{b}, \underline{a}, \underline{a}] = (1+1)^{-1}\,0 = 0$.

(iv) Proof is left to the reader.

The reader is asked to supply the long, but straight-forward, computation required to prove the following results.

Proposition 6.2 For all $a_{ij} \in \Sigma$ and $\underline{v}_i \in \mathcal{V}$ (i, j = 1, 2, 3), it follows that $\left[\sum_{j=1}^{3} a_{1j}\,\underline{v}_1, \sum_{j=1}^{3} a_{2j}\,\underline{v}_2, \sum_{j=1}^{3} a_{3j}\,\underline{v}_3\right]$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} [\underline{v}_1, \underline{v}_2, \underline{v}_3] = (\det a_{ij}) [\underline{v}_1, \underline{v}_2, \underline{v}_3].$$

Corollary 6.2 Let $\underline{v}_1$, $\underline{v}_2$, $\underline{v}_3$ be any vectors and $a_{\ell m}$ the unique scalars such that $\underline{v}_n = a_{n1}\underline{i} + a_{n2}\underline{j} + a_{n3}\underline{k}$, $(n = 1, 2, 3)$. Then $[\underline{v}_1, \underline{v}_2, \underline{v}_3] = \det (a_{\ell m})$.

Definition 6.2 The triple $(\underline{u}, \underline{v}, \underline{w})$ is a basis for $\mathcal{V}$ if and only if $\underline{x} \in \mathcal{V} \Rightarrow$ there exist $f, g, h \in \Sigma$ such that $\underline{x} = f\underline{u} + g\underline{v} + h\underline{w}$.

It follows from Th. 3.5 that $(\underline{u}, \underline{v}, \underline{w})$ is a basis for $\mathcal{V} \Rightarrow$ so are $(\underline{u}, \underline{w}, \underline{v})$, $(\underline{w}, \underline{v}, \underline{u})$, $(\underline{w}, \underline{u}, \underline{v})$, $(\underline{v}, \underline{u}, \underline{w})$ and $(\underline{v}, \underline{w}, \underline{u})$. An immediate consequence of Cor. 4.8 is that $(\underline{u}, \underline{v}, \underline{w})$ is a basis for $\mathcal{V} \Leftrightarrow \underline{u}$, $\underline{v}$, $\underline{w}$ are not coplanar.

Theorem 6.1 Let $\underline{v}_1, \underline{v}_2, \underline{v}_3 \in \mathcal{V}$ and $a_{\ell m}$ the unique scalars such that $\underline{v}_n = a_{n1}\underline{i} + a_{n2}\underline{j} + a_{n3}\underline{k}$, $(n = 1, 2, 3)$. Then $(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ is a basis for $\mathcal{V}$ if and only if $\det (a_{\ell m}) \neq 0$.

Proof We have $[\underline{v}_1, \underline{v}_2, \underline{v}_3] = \det (a_{\ell m})$ from Cor. 6.2.

(i) If $(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ is a basis for $\mathcal{V}$, then there exist i_p, j_p, k_p $(p = 1, 2, 3) \in \Sigma$ such that $\underline{i} = \sum_{q=1}^{3} i_q \underline{v}_q$, $\underline{j} = \sum_{q=1}^{3} j_q \underline{v}_q$ and $\underline{k} = \sum_{q=1}^{3} k_q \underline{v}_q$. By Prop. 6.2 and Def. 6.1 (3), we have $0 \neq 1 = [\underline{i}, \underline{j}, \underline{k}] = \begin{vmatrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{vmatrix} [\underline{v}_1, \underline{v}_2, \underline{v}_3]$

$$= \begin{vmatrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{vmatrix} (\det a_{\ell m}).$$ By Prop. 4.2 (i), we have $(\det a_{\ell m}) \neq 0$.

(ii) Suppose $(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ is not a basis. Then by Th. 4.8, it follows that $\underline{v}_1$, $\underline{v}_2$, and $\underline{v}_3$ are coplanar. Without loss of generality, there exist $a, b, c \in \Sigma$, $a \neq 0$, $a\underline{v}_1 + b\underline{v}_2 + c\underline{v}_3 = 0$. Letting $-a^{-1} b = d$ and $-a^{-1} c =$

f, we find det $a_{\ell m} = [\underline{v}_1, \underline{v}_2, \underline{v}_3]$

$= [b\underline{v}_2 + c\underline{v}_3, \underline{v}_2, \underline{v}_3] = b[\underline{v}_2, \underline{v}_2, \underline{v}_3] + c[\underline{v}_2, \underline{v}_2, \underline{v}_3]$

$= b\,0 + c\,0 = 0 + 0 = 0$, by Prop. 6.1 (iii), (iv).

<u>Corollary 6.1</u> Let $f \in \Sigma$, $\underline{u}, \underline{v} \in \mathcal{V}$ and $\underline{u} \nparallel \underline{v}$. Then there exist infinitely many $\underline{w} \in \mathcal{V}$ such that $[\underline{u}, \underline{v}, \underline{w}] = f$.

<u>Proof</u> If $f = 0$, let $\underline{w} = a(\underline{u} + \underline{v})$ where a is any scalar. If $f \neq 0$, then Ax. 2.9, Cor. 4.8, and Th. 6.1 there exists $\underline{x} \in \mathcal{V}$ such that $[\underline{u}, \underline{v}, \underline{x}] = g$ for some $0 \neq g \in \Sigma$. We need only let $\underline{w} = f g^{-1} \underline{x} + \underline{b}\ (\underline{u} + \underline{v})$ for any $b \in \Sigma$.

<u>Theorem 6.2</u> Let $\underline{u}, \underline{v}, \underline{w} \in \mathcal{V}$ be noncoplanar, P, Q, R, and S be points such that $(P,Q) \in \underline{u}$, $(P,R) \in \underline{v}$, and $(P,S) \in \underline{w}$, and π the unique plane containing P, Q, and R. Let $\underline{x} \in \mathcal{V}$ such that $\underline{u}$, $\underline{v}$, and $\underline{x}$ are noncoplanar, and T the point such that $(P,T) \in \underline{x}$. Then $0 < [\underline{u}, \underline{v}, \underline{w}][\underline{u}, \underline{v}, \underline{x}] \Leftrightarrow$ S and T are in the same half space relative to π.

<u>Proof</u> If a, b, $c \in \Sigma$ are such that $\underline{x} = a\underline{u} + b\underline{v} + c\underline{w}$, then $0 < [\underline{u}, \underline{v}, \underline{w}][\underline{u}, \underline{v}, \underline{x}] = [\underline{u}, \underline{v}, \underline{w}][\underline{u}, \underline{v}, a\underline{u} + b\underline{v} + c\underline{w}] = c[\underline{u}, \underline{v}, \underline{w}][\underline{u}, \underline{v}, \underline{w}] \Leftrightarrow 0 < c$, by Th. 5.6 and Cor. 5.6.1. We claim $0 < c \Leftrightarrow$ T and S are in the same half space relative to π.

Suppose $a = b = 0$. Then $\underline{x} = c\underline{w}$ with $c \neq 0$. If $0 < c$, then S and T lie on a half line relative to P, so that either $P|S|T$, $P|T|S$, or $S = T$. By Th. 5.13, it follows that S and T lie in the same half space relative to π. If $c < 0$, then S and T do not lie on a half line relative to P, and so $S|P|T$. Thus S and T lie in different half spaces relative to π and the theorem is proved, if $a = b = 0$.

Let us remove the restriction $a = b = 0$. The required conclusion follows from the above argument, by virtue of Def. 3.4, Cor. 3.1.1, Cor. 2.5.1 and Th. 5.13 which imply that the addition of $(a\underline{u} + b\underline{v})$ to $c\underline{w}$ concerns only transformations which preserve the half space.

In view of the preceding result, we make the following definition.

Definition 6.3 Any two bases $(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ and $(\underline{w}_1, \underline{w}_2, \underline{w}_3)$ of $\mathcal{V}$ have the same orientation if and only if $0 < [\underline{v}_1, \underline{v}_2, \underline{v}_3][\underline{w}_1, \underline{w}_2, \underline{w}_3]$.

Let B be the set of bases of $\mathcal{V}$ and R the relation defined on B such that $b_1 \, R \, b_2 \Leftrightarrow b_1$ and b_2 have the same orientation, for $b_1, b_2 \in B$. By Cors. 5.6.1, 5 and Th. 5.6, one may readily show that R is an equivalence relation, whose equivalence classes we may define to be the orientations of $\mathcal{V}$. We note that Def. 6.1 (1) and Cor. 5.6.1 $\Rightarrow$ $(\underline{u}, \underline{v}, \underline{w})$, $(\underline{v}, \underline{w}, \underline{u})$ and $(\underline{w}, \underline{u}, \underline{v})$ are all in the same orientation, for any basis $(\underline{u}, \underline{v}, \underline{w})$.

Remark In the remainder of this chapter, we shall assume

(1) $hk = kh$, for all $h, k \in \Sigma$

and (2) $0 \leq f \in \Sigma \Rightarrow$ there exists a unique $g \geq 0, g \in \Sigma$ such that $f = gg$. We denote g by $(f)^{\frac{1}{2}}$.

Definition 6.4 For any vectors $\underline{u} = u_1 \underline{i} + u_2 \underline{j} + u_3 \underline{k}$ and $\underline{v} = v_1 \underline{i} + v_2 \underline{j} + v_3 \underline{k}$, the inner product of $\underline{u}$ by $\underline{v}$ is defined as $\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$. In particular, if $\underline{u} = \underline{v}$, then we define the length of $\underline{u}$ to be $|\underline{u}| = (\underline{u} \cdot \underline{u})^{\frac{1}{2}}$.

It is immediate that $(f\underline{u}) \cdot \underline{v} = f(\underline{u} \cdot \underline{v})$, $(\underline{u} \cdot \underline{v}) = (\underline{v} \cdot \underline{u})$, and $(\underline{u} + \underline{v}) \cdot \underline{w} = \underline{u} \cdot \underline{w} + \underline{v} \cdot \underline{w}$ for all $f \in \Sigma$, $\underline{u}, \underline{v}, \underline{w} \in \mathcal{V}$. Moreover, $(\underline{u} \cdot \underline{u}) > 0$ for all non zero $\underline{u} \in \mathcal{V}$ and so $|\underline{u}|$ is well defined.

Theorem 6.3 For any non zero $\ell \in \Sigma$ and any direction D, there exists exactly one vector $\underline{v} \in \mathcal{V}$ with length ℓ and direction D.

Proof By the remark following Th. 5.5, there exists a non zero vector $\underline{w}$ with direction D. The required vector $\underline{v}$ clearly must be of the form $k\underline{w}$, $k \in \Sigma$. It is apparent that $k = \frac{\ell}{|\underline{w}|}$ is the unique scalar such that $|k\underline{w}| = \ell$.

<u>Definition 6.5</u> For any P, Q $\in \mathcal{P}$, let $\underline{v} \in \mathcal{V}$ be the unique vector containing (P,Q). The <u>length from P to Q</u> is defined to be $|\underline{v}|$ and is denoted by $\overline{PQ}$.

Since $(-f)(-f) = ff$ for all $f \in \Sigma$, one may readily prove, for all P, Q $\in \mathcal{P}$, that

(i) $\overline{PQ} = \overline{QP}$

and (ii) $\overline{PQ} > 0$ unless P = Q, in which case $\overline{PQ} = 0$

<u>Definition 6.6</u> For any $f \in \Sigma$, the <u>absolute value of f</u>, denoted by $|f|$, is defined as the non negative member of the pair (f, -f).

It follows from Prop. 5.9 that $|f|$ is well defined.

Consider any vectors $\underline{u} = u_1 \underline{i} + u_2 \underline{j} + u_3 \underline{k}$ and $\underline{v} = v_1\underline{i} + v_2 \underline{j} + v_3 \underline{k}$. For any $t \in \Sigma$, $\sum_{i=1}^{3} (u_i + t v_i)^2 \geq 0$. If $\sum_{i=1}^{3} v_i^2 = 0$, then $\underline{v} = \underline{0}$ and $|\underline{u} \cdot \underline{v}| = |\underline{u}||\underline{v}| = 0$. If $\sum_{i=1}^{3} v_i^2 \neq 0$, then $0 \leq \left(t + \frac{\sum u_i v_i}{\sum v_i^2}\right)^2 + \left(\frac{\sum u_i^2}{\sum v_i^2} - \left(\frac{\sum u_i v_i}{\sum v_i^2}\right)^2\right)$. Setting $t = -\frac{\sum u_i v_i}{\sum v_I^2}$, we get $|\underline{u} \cdot \underline{v}| < |\underline{u}|$ $|\underline{v}|$. This proves,

<u>Proposition 6.3</u> $\underline{u}, \underline{v} \in \mathcal{V} \Rightarrow |\underline{u} \cdot \underline{v}| \leq |\underline{u}||\underline{v}|$ (Cauchy - Schwarz inequality).

As $|\underline{u} + \underline{v}|^2 = (\underline{u} + \underline{v}) \cdot (\underline{u} + \underline{v}) = (\underline{u} \cdot \underline{u}) + 2(\underline{u} \cdot \underline{v}) + (\underline{v} \cdot \underline{v}) = |\underline{u}|^2 + 2(\underline{u} \cdot \underline{v}) + |\underline{v}|^2 \leq |\underline{u}|^2 + 2|\underline{u} \cdot \underline{v}| + |\underline{v}|^2$, Prop. 6.3 $\Rightarrow |\underline{u} + \underline{v}|^2 \leq |\underline{u}|^2 + 2|\underline{u}||\underline{v}| + |\underline{v}|^2 = \{|\underline{u}| + |\underline{v}|\}^2$. Since $0 \leq |\underline{u} + \underline{v}|, |\underline{u}| + |\underline{v}|$, the order properties of scalars imply $|\underline{u} + \underline{v}| \leq |\underline{u}| + |\underline{v}|$, for any assumption to the contrary brings an immediate contradiction to Prop. 5.7, (i), (iii).

The argument of the preceding paragraph implies that $\overline{PR} \leq \overline{PQ} + \overline{QR}$ for all P, Q, R $\in \mathcal{P}$, for we need only let $\underline{u}, \underline{v}$ be the vectors containing (P,Q), (Q,R) respectively. Thus, using the standard terminology, we have proved

Theorem 6.4 (i) For any $\underline{u}, \underline{v} \in \mathcal{V}$, the triangle inequality holds; i.e., $|\underline{u} + \underline{v}| \leq |\underline{u}| + |\underline{v}|$.

(ii) The function $m: P \times P \to \Sigma$ defined by $m(P,Q) = \overline{PQ}$ is a metric for P.

Definition 6.7 An isometry f is a linear transformation from $\mathcal{V}$ into $\mathcal{V}$ such that $\underline{v} \in \mathcal{V} \Rightarrow |f(\underline{v})| = |\underline{v}|$.

Proposition 6.4 Let $f: \mathcal{V} \to \mathcal{V}$ be linear, $f(\underline{i}) = i_1\underline{i} + i_2\underline{j} + i_3\underline{k}$, $f(\underline{j}) = j_1\underline{i} + j_2\underline{j} + j_3\underline{k}$ and $f(\underline{k}) = k_1\underline{i} + k_2\underline{j} + k_3\underline{k}$. Then for any $a, b, c \in \Sigma$, $f(a\underline{i} + b\underline{j} + c\underline{k}) = (ai_1 + bj_1 + ck_1)\underline{i} + (ai_2 + bj_2 + ck_2)\underline{j} + (ai_3 + bj_3 + ck_3)\underline{k}$.

Proof The result is immediate, from the linearity of f.

Remark Using the notation of Prop. 6.4, note that

$$f: \begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} i_1 & j_1 & k_1 \\ i_2 & j_2 & k_2 \\ i_3 & j_3 & k_3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

By letting (a, b, c) be (1, 0, 0(, (0, 1,0), and (0, 0, 1) in succession, it is readily seen that there is only one matrix A such that $f: \begin{pmatrix} a \\ b \\ c \end{pmatrix} \to A \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

This matrix A is said to represent f with respect to $\{\underline{i}, \underline{j}, \underline{k}\}$.

If f is an isometry then $(a, b, c) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a, b, c) A^T A \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ where A^T is the transpose of A.

By letting a, b, c vary over Σ, one may readily prove that the identity matrix $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A^T A$. Since the determinant of a matrix product is the product of the matrices' own determinants, $I = \det(I) = \det(A^T) \det(A) = \det(A) \det(A) = (\det(A))^2$, whence $\det A = \pm 1$. The fact that $A^T A = I$

may be interpreted to mean that A is orthogonal.

If B represents f with respect to a basis $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ of $\mathcal{V}$, then one may verify that $B = N A N^{-1}$, where the $t^{\underline{th}}$ column of N consists of the affine coordinates of $\underline{v}_t$ with respect to $\{\underline{i}, \underline{j}, \underline{k}\}$. Thus det (B) = det (N) det (A) det (N^{-1}) = det (A) det (N) det (N^{-1}) = det (A) det (NN^{-1}) = det (A) det (I) = det A. These comments prove

Theorem 6.5 All isometries are on to. Moreover, the class of isometries may be partitioned into two mutually exclusive and exhaustive subfamilies: those representable by matrices with determinant +1 and those representable by matrices with determinant -1.

Definition 6.8 An isometry representable by a matrix with determinant +1 is said to be a rotation.

It is well known that every entry in a matrix A representing a rotation Λ is equal to its cofactor in the corresponding determinant. Using this fact, one may prove that unless Λ is the identity map, Λ fixes a one-dimensional subspace of vectors. Consequently, A has at least one eigenvalue equal to 1.

If Σ is isomorphic to the field of real numbers, then it is well known that A orthogonal $\Rightarrow$ A orthogonally similar to a matrix of the form

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$ The eigenvalues of M are 1, $\cos\theta \pm i\sin\theta$. Since any matrix similar to A has these same eigenvalues, we may uniquely assign to Λ a real number θ between 0 and π as follows.

Let Λ be represented with respect to $\{\underline{i}, \underline{j}, \underline{k}\}$ by the matrix N with eigenvalues 1, $a \pm ib$. Without loss of generality, $b \geq 0$. By the theory of trigonometric functions, there exists a unique θ such that $0 \leq \theta \leq \pi$, $\cos\theta = a$, and $\sin\theta = b$. We assign θ to Λ and call θ the angle of the rotation Λ.

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