ANALYSIS OF STOCHASTICALLY EXCITED NONLINEAR SYSTEMS - THEORY AND APPLICATION

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by

Donald Joseph Gussin

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ABSTRACT

A new method of analyzing a nonlinear feedback system which has a single zero-memory nonlinearity followed by a low-pass element in the forward loop is discussed in detail. Excitation is by wideband, stationary, ergodic noise with a Gaussian probability distribution. This new method consists of an analysis by open-loop considerations of a sampled system equivalent to the given one.

The results of the application of this method to a particular system are found to compare very closely to those yielded by an application of the Booton technique and to those obtained in simulation of the system.

PREFACE

The object of this thesis is the investigation of a proposed method of analysis of certain nonlinear feedback systems excited by noise, the method first being described in a paper by Henry and Schultheiss, published in 1962. This method consists of replacing the given system by an equivalent sampled one, and calculating, by open-loop considerations, the probability of transition from a hypothesized state to any other state in one sampling interval. The closed-loop response is then found by the solution of an integral equation, achieved by the use of a digital computer.

The investigation consists of a detailed discussion of the application of the new method. A new procedure to shorten the calculation time and increase the accuracy of calculation is described, followed by the analysis of an example, using a cubic term for the nonlinear portion of the system. A method of choosing the sampling interval to make valid the replacement of the continuous system by the sampled one is also discussed.

The results are evaluated by a comparison with the results of a study of an analogue simulation of the system and with those results obtained from the Booton technique of analysis which consists of approximating the nonlinearity by an equivalent linear term.

This evaluation is discussed in Chapter 5, where possibilities for future work with this method are outlined.

After completion of the writing of this thesis, two errors were found to have been included.

The first involves the equations of Bright concerning the conditional transition probability function. This expression, being Gaussian, is a function of the variance of the process being described, whereas Bright used the mean-squared value of that process, which is the variance plus the square of the mean value. The appropriate correction was made in the computer program (effected by setting Z2, Z4, and Z6 equal to zero), and the set of data considered in the text was presented, the correct error curves being displayed in Graph 5.

The second error involves a misinterpretation of the significance of the analogue simulation of the sampled system. Since the choice of sampling interval discussed in the text is only useful in making the Schultheiss technique predict the same response as that of the continuous system, and since a sampled system of the type considered (with non-zero sampling interval) will not yield the same response as the corresponding continuous system, it can be seen that the comparison of the Schultheiss prediction and the analogue simulation response, on the one hand, to the sampled system response, on the other hand, is not relevant, and should be ignored. All other comparisons are unaffected by this change.

The author is very grateful to Prof. R. A. Johnson, who provided invaluable insight into the problem and numerous suggestions regarding its solution.

The author would also like to thank the National Research Council of Canada for the Research Grant 4584 which enabled his work to continue uninterrupted.

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CHAPTER I

INTRODUCTION

The problem of obtaining an exact solution for the response of a nonlinear feedback system to a given excitation has been the subject of much research during the past several years, such a solution being, so far, unattainable. There are, however, several methods for obtaining an approximate solution, possibly the most widely used of which has been the quasi-linearization technique described by Booton (1); this method, in spite of its wide acceptance, has not been justified mathematically, but was only suggested as a possible procedure which could give acceptable results in many cases. This lack of rigorous derivation curtails the usefulness of the procedure since no estimate of error is possible, and does not contribute to progress toward obtaining an exact solution.

In 1962, a new method of predicting the response of a nonlinear feedback system to a given stochastic excitation was described by Henry and Schultheiss (2). This method was derived using probability theory in a rigorous proof and had all assumptions and restrictions clearly available, thus providing a possible step toward the goal of an exact treatment of such systems.

This thesis deals with the Schultheiss technique, both in principle and in practical application to a particular system (with the aid of a digital computer), along with a comparison of the results so obtained with both those produced by the Booton technique and those observed in a simulation of the system on an analogue computer.

CHAPTER II

THE SCHULTHEISS TECHNIQUE

The Schultheiss technique is applicable to a unity feedback system of the form shown in Fig. 2-1, excited by a stationary, ergodic, wideband Gaussian process. N is a zero-memory nonlinearity, and H_2 a low-pass element with a sufficiently low upper-cutoff frequency that its response is approximately Gaussian even though the output of the nonlinearity may be grossly non-Gaussian (3).

2.1 BASIC ASSUMPTION - THE MARKOV PROCESS

In order to apply the technique, the given system must be approximated by the sampled system shown in Fig. 2-2.

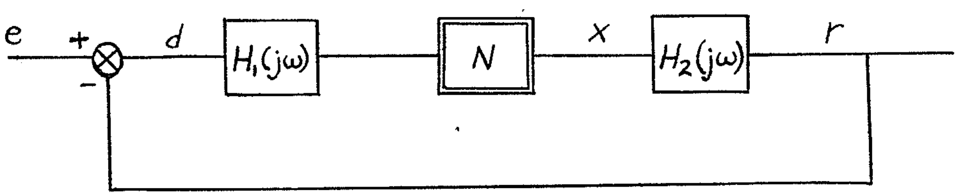

FIG. 2-1

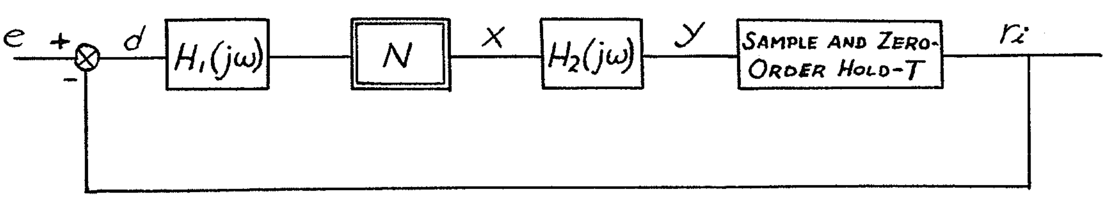

FIG. 2-2

The response r_i is then a sequence of constants, each lasting for one sampling interval, T, and having the value of the signal at the input to the sample and hold circuit at the preceding sampling instant. It is essential to the method that each value in this sequence be completely specified subject only to the hypothesis of the value preceding it; such a sequence is known as a Markov chain, and this Markov property must be proved for any system considered before an attempt is made to apply the method.

2-2 THE CONDITIONAL TRANSITION PROBABILITY FUNCTION

If the response of the system between the i^{th} and $(i+1)$th sampling instants (referred to hereinafter as the i^{th} sampling interval) is written r_i, then r_{i+1} may be written as

$$r_{i+1} = r_i + \Delta_1 y_i + \Delta_o y_i \qquad \text{2-1}$$

where $\Delta_1 y_i$ and $\Delta_o y_i$ are the changes in the value of y during the i^{th} sampling interval due to the excitation e and feedback r_i, respectively. That the choice of origin in time is arbitrary is assured by the assumption of stationariness of the input process.

If e is sufficiently wideband, $\Delta_1 y_i$ will be independent of r_i; in other words, the sampling interval must be long compared to the correlation time of the excitation, so that r_{i+1} is independent of the value of e at the i^{th} sampling instant. (This is the requirement that the response be a Markov process.) If this is true, then a specification of r_i will permit calculation of the stochastic properties of r_{i+1}, which relationship is expressed in the conditional probability function of r_{i+1} subject only to the hypothesis of r_i, written $p_c(r_{i+1}|r_i)$. This calculation may be carried out more easily by assuming a Gaussian distribution for y -- an approximately correct assumption even for violently nonlinear N provided that H_2 is sufficiently low-pass (3).

The probability distribution of the response may be obtained from the well-known probability integral equation (5):

$$p(r_{i+1}) = \int_{-\infty}^{\infty} p_c(r_{i+1}/r_i)\, p(r_i)\, dr_i \qquad 2\text{-}2$$

where p(x) denotes the probability density function of x (the absolute probability of x).

It should be noted at this time that the sampling interval must be small with respect to the correlation time of the response, because if this were not so, the probability of the response at any given sampling instant would not depend upon the value of the output at the previous sampling instant. (In most cases, however, the correlation time of the response would depend upon the length of the sampling interval.) Also, in order to make the sampled approximation valid, the sampling interval must be small with respect to the system response time.

The combination of the restrictions on the sampling interval T may be expressed in the relationships:

$$T_i < T < T_o \qquad \text{and} \qquad T < T_r$$

where T_i and T_o are some measure of the correlation times of the excitation and response processes, respectively, and T_r is the system response time.* Note that this requires that the response have a much longer correlation time than the excitation, which means that the bandwidth of the excitation must be much greater than the upper cutoff frequency of the system.

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* The method of specifying the correlation times quantitatively will determine how much greater T and T_o must be than T_i and T, respectively.

Equation 2-1 may be rewritten as

$$r_{i+1} = r_i + \Delta y_i$$

in which Δy_i is defined as the change in y during the i^{th} sampling interval. It may now be seen that a specification of r_i will determine the probability of r_{i+1} if the stochastic properties of Δy_i can be found; these latter properties are available from open-loop considerations during any one sampling interval since the system is representable as an open loop excited by the given input process less a hypothesized constant r_i (which excitations are uncorrelated by virtue of the assumption that the input is wideband).

Equation 2-2 is in general too complex to be solved by analytical methods, and special techniques must be used.

2-3 THE SCHULTHEISS SOLUTION OF THE PROBABILITY INTEGRAL EQUATION

Equation 2-2 was solved by Schultheiss by quantizing the response r_i into a finite number, N, of states, each of width Δr, and operating on the resulting conditional probability transition function.

By considering the probability of transition from an initial state j in the i^{th} sampling interval, j_i, to a subsequent state k in the $(i+1)^{th}$ interval, k_{i+1}, the transition probability, p_{jk}, is given by:

$$p_{jk} = p_c(k_{i+1} | j_i) \ .$$

The transition probability matrix may be written from the conditional transition probability function:

$$[p_{jk}] \equiv \begin{bmatrix} p_{11} & p_{12} & - - & p_{1N} \\ p_{21} & p_{22} & - - & p_{2N} \\ - & & & - \\ - & & & - \\ p_{N1} & p_{N2} & - - & p_{NN} \end{bmatrix} .$$

Then, if the probability of transition from the j^{th} to the k^{th} states in n sampling intervals be denoted by p^n_{jk}, since the response is a Markov process, it may be shown that the absolute probability of the k^{th} state, $p(k_{i+1})$ is given by the expression (4,pp. 19-22; 5,pp. 356-7):

$$p(k_{i+1}) = \lim_{n \to \infty} (p_{jk})^n .$$

Then, for a transition probability matrix $[p_{jk}]$ as above, it may be shown that

$$[p^n_{jk}] = [p_{jk}]^n .$$

Thus the computational method employed by Schultheiss consists of successive multiplications of the transition probability matrix until the rows are identical, i.e., until further multiplications produce no change in the matrix product. Then any row represents the absolute probability density of the response r (in quantized form). This multiplication may be carried out on a digital computer.

2-4 A REFINEMENT BY BRIGHT

The Schultheiss technique of solution of the transition probability equation was refined by Bright (4), who considered only one row of the matrix multiplying the transition probability matrix, in order to conserve space in the computer memory, thus yielding greater accuracy by permitting a larger number of quantized states for a given memory size. An arbitrary row matrix (initial vector) was chosen for the initial multiplier, and multiplication with the transition probability matrix was carried out until the row matrix was unchanged by further multiplications. That this method is consistent with the original will now be shown.

At any stage of computation with the Schultheiss technique, multiplication by the transition probability matrix consists of pre-multiplying that matrix by another matrix; the multiplication is carried out for one row of the latter matrix at a time, each of which is approaching, independently of every other, a common value. Thus, any one row may be considered. The Schultheiss technique has the possible advantage of greater speed of computation, since the matrix multiplications may be carried out by matrix squaring, thus yielding more multiplications per operation; however, this advantage may be outweighed by the longer time necessary for each multiplication, since N rows must be handled. Which method would be faster would be determined by how closely the initial vector could be made to approximate the final distribution.

It is of value to note that the Schultheiss technique is in reality an iterative solution of equation 2-2. That this is so will now be demonstrated.

Consider any row matrix $[pk]$ representing an arbitrary initial vector in the Bright refinement, and let this pre-multiply the transition probability matrix $[p_{jk}]$ thusly:

$$\begin{bmatrix} q_1 & q_2 & --- & q_N \end{bmatrix} \equiv \begin{bmatrix} p_1 & p_2 & --- & p_N \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & --- & p_{1N} \\ p_{21} & p_{22} & --- & p_{2N} \\ - & & & \\ - & & & \\ p_{N1} & p_{N2} & --- & p_{NN} \end{bmatrix} \qquad 2\text{-}3$$

in which p_m and q_m represent the probability of occurrence of the m^{th} quantized state during the i^{th} and $(i+1)^{th}$ sampling intervals, respectively. Note that q_m can be expressed in terms of the probability of the "centre" of the k^{th} state, thus:

$$q_k = q(r_k) \cdot \Delta r_k$$

where r_k is the "centre" of the k^{th} state, it being understood that k refers to a state during the $(i+1)^{th}$ sampling interval. Then, writing equation 2-3 in summation form yields:

$$q(r_k) \equiv \sum_{j=1}^{N} p_c(r_k | j_i)\, p_j \quad .$$

Similarly, replacing p_j by an expression in terms of the probability of the centre of the j^{th} state yields:

$$q(r_k) = \sum_{j=1}^{N} p_c(r_k | r_j)\, p(r_j)\, \Delta r_j \quad .$$

Letting $N \rightarrow \infty$ and $r_j \rightarrow 0$ (i.e., eliminating the quantized approximation) produces the expression:

$$q(r_k) = \int_{-\infty}^{\infty} p_c(r_k | r_j)\, p(r_j)\, drj \ .$$

The notation concerning the quantized states may now be dropped, writing instead, r_i for a value of r during the i^{th} sampling interval:

$$q(r_{i+1}) = \int_{-\infty}^{\infty} p_c(r_{i+1} | r_i)\, p(r_i)\, dr_i \quad . \qquad 2\text{-}4$$

The solution is obtained when $[q_k]$ is identically equal to $[p_k]$; i.e., the equality:

$$p(r_{i+1}) = \int_{-\infty}^{\infty} p_c(r_{i+1} | r_i)\, p(r_i)\, dr_i$$

is satisfied, which is identically equation 2-2, the equation whose solution was required.

The iterative solution of equation 2-2, then, consists of assuming a distribution for $p(r_i)$, which is the initial vector in the Bright refinement, and each row in the transition probability matrix in the Schultheiss technique. This distribution is substituted into equation 2-2, which is then numerically integrated over a finite range to obtain $p(r_{i+1})$. This new distribution is then taken to be the new initial vector, and another integration is carried out. The correct solution is that distribution $p(r_i)$ which repeats as $p(r_{i+1})$.

That the p and q functions in equation 2-4 are identical when the correct distribution for p is considered may be seen by noting that since the excitation is a stationary process, and since the system being considered is time-invariant, the probability of occurrence of a given value of r must be the same, at all instants in time. In other words, $p(r_{i+1})$ must be identically equal to $p(r_i)$.

Thus it can be seen that the Schultheiss method of solution of the probability integral equation is equivalent to a solution by iteration, which is a completely general method in that it imposes no further restrictions on the functions being considered. In future work, it may be found necessary to shorten the sampling interval, thus allowing dependence between the values of excitation at successive sampling instants; this would produce a non-Markov response and a probability integral different from that of equation 2-2. The derivation by Schultheiss of the method of solution of equation 2-2 assumed the Markov property, so that this method of solution is not applicable to the modified probability integral equation in the new case. The accompanying proof that this solution is equivalent to one by iteration provides a method of solution, and illustrates that nothing is gained by use of the Schultheiss method.

2-5 A FURTHER REFINEMENT

If the response r is assumed Gaussian, and if its mean value u is known, it can be seen that a solution for p(u) will specify the entire distribution. Thus if the Bright initial vector $p_1(r_k)$ is chosen to be some arbitrary Gaussian distribution with mean value u, it may be multiplied by the column of the transition probability matrix $\left[p_{jk}\right]$ corresponding to $r_j = u$; this will yield an intermediate value q_1 of the probability of the $r = u$ state:

$$q_1 \equiv \begin{bmatrix} p_1 & p_2 & - - - & p_N \end{bmatrix} \begin{bmatrix} p_{1u} \\ p_{2u} \\ - \\ - \\ p_{Nu} \end{bmatrix}$$

A comparison of q_1 with $p_1(u)$ will indicate whether the assumed distribution p_1 was correct. If q_1 does not equal p(u), the variance σ_2^2 of the Gaussian distribution $p_2(r_k)$ for which $p_2(u) = q_1$ may be found from the relation: *

$$\sigma_2 = \frac{1}{\sqrt{2\pi}\, q_1}$$

The Gaussian distribution with the mean value u and variance σ_2^2 may then be computed and used as a new initial vector.

As in the preceding section, this procedure may be continued until the value of the probability of the centre state in the trial vector equals the value of the probability of the centre state produced by the multiplication. It may be possible by inspection of trial data, to determine a functional relationship between successive values of σ^2 and the variance of the actual result when it is finally reached, so that the desired result may be obtained with very few multiplications in similar computations.

This simplified procedure greatly increases the speed with which computations may be performed, since only the centre column of the transition probability matrix need be calculated; furthermore, many more quantized states may be used (a virtually unlimited number), so that error due to this quantization may be made negligible. Using this method, solution time was reduced to less than five minutes per set of data, compared to the forty-five minutes required with the Bright technique.

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* This is found from the Gaussian density function

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[- \frac{(x-u)^2}{2\sigma^2} \right]$$

with x = u.

2-6 SUMMARY

The Schultheiss technique, then, may be applied to a continuous nonlinear feedback system with a single zero-memory nonlinearity in cascade with a low-pass element in the forward loop. This system is approximated by a sampled system, with a sampling interval assumed to be long with respect to the excitation correlation time and short with respect to the system response time.

The low-pass element is assumed to have a sufficiently low upper-cutoff frequency so that its output is approximately Gaussian. With these restrictions and approximations, the conditional transition probability function of the response may be written from open-loop considerations, and the closed-loop response found by the iterative solution of an integral equation.

CHAPTER III

AN EXAMPLE

As an example, the Schultheiss technique was applied to a system of the form shown in Figure 3-1, excited by a stationary ergodic Gaussian process with a (double-sided) power spectrum assumed to be of the form

$$W(\omega) = \frac{2\beta\sigma^2}{\beta^2 + \omega^2} \quad \text{watts/rad/sec.} \qquad 3\text{-}1$$

In this chapter, the linear sampled approximation (Figure 3-2, B = 0) will be analyzed in detail by the Schultheiss technique, and, for purposes of comparison, the continuous system of Figure 3-1 will be analyzed by the Booton technique. A method of choosing the sampling interval to make the approximation valid will be discussed.

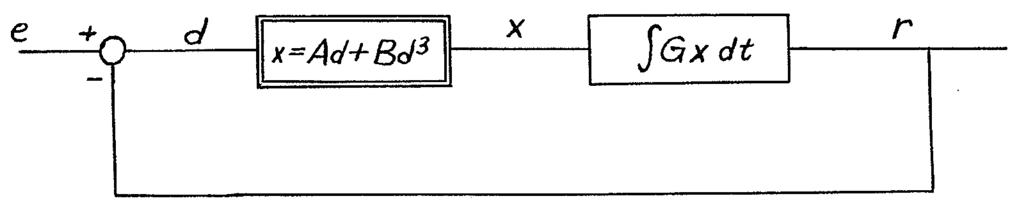


FIG. 3-1

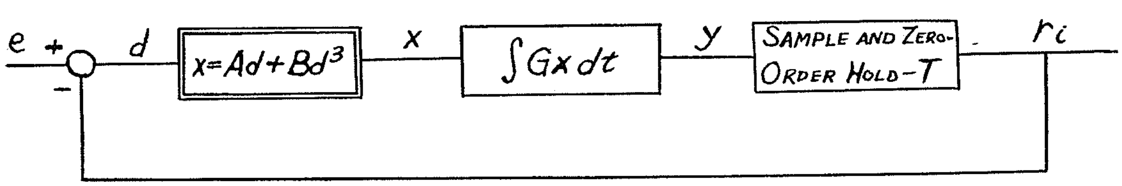


FIG. 3-2

3-1 THE SCHULTHEISS ANALYSIS OF THE SAMPLED SYSTEM

In order to display the Schultheiss technique, the response of the system to the excitation described will be derived; both the linear case, for which the probability integral, equation 2-2, may be solved explicitly, and the nonlinear case, for which a digital computer must be used, will be discussed.

3-1-1 The Linear Case

Before the Schultheiss technique is applied, it is necessary to show that the response of the system has the Markov property.

Considering equation 2-1, it may be seen that for the system under consideration, r_{i+1} may be written as

$$r_{i+1} = r_i + \Delta_1 y_i + \int_0^T AG\, r_i\, dt \qquad 3\text{-}1$$

and

$$r_{i+1} = (1 + AGT)r_i + \Delta_1 y_i \quad .$$

Assuming that T is sufficiently long so that $\Delta_1 y_i$ is independent of the value of the excitation at the i^{th} sampling instant, it may be seen that r_{i+1} is completely specified by a hypothesis of r_i, and is calculable since the stochastic properties of $\Delta_1 y_i$ are available from the statistics of the excitation. Thus the response is shown to be Markov.

To apply the Schultheiss technique, consider equation 3-1. It is clear that the function $p_c(r_{i+1}|r_i)$ is specified only by the stochastic properties of the term $\Delta_1 y_i$. Since the excitation is assumed ergodic, the ensemble average of the system response over many sampling intervals may be considered in place of the time average. Since the response has a zero mean value and is Gaussian, it is completely specified by its mean-square value, which will then specify the statistics of $\Delta_1 y_i$, and thus those of $p_c(r_{i+1}|r_i)$.

Applying harmonic analysis techniques (3), it may be shown that (see Appendix A) the mean-square change is given by the expression:

$$\overline{(\Delta_1 y_i)^2} = \frac{2\sigma^2 (AG)^2 \, T}{\beta} \left[1 - \frac{1 - e^{-\beta T}}{\beta T} \right]$$

The conditional probability transition function may then be written:

$$p_c(r_{i+1} | r_i) = \frac{1}{2\pi \overline{(\Delta_1 y_i)^2}} \exp\left[- \frac{[r_{i+1} - (1 - AGT) r_i]^2}{2 \overline{(\Delta_1 y_i)^2}} \right] \qquad 3\text{-}2$$

Substitution of equation 3-2 into the probability integral equation 2-2, along with the Gaussian probability distribution of the excitation, yields an expression which may be solved explicitly (see (4), pp. 25-6 and appendix B, for the method of solution), the result being the expression:

$$\sigma_o^2 = \frac{AG\sigma^2}{\beta(1 - \frac{AGT}{2})} \left[1 - \frac{1 - e^{-\beta T}}{\beta T} \right] \qquad 3\text{-}3$$

where σ_o^2 is the variance of the response of the linear sampled system of Figure 3-2 excited by the assumed process; this expression will be exact provided that T is sufficiently large so that values of the excitation at successive sampling instants are uncorrelated.

3-1-2 The Nonlinear Case

The expressions for $\overline{\Delta_1 y_i}$ and $\overline{(\Delta_1 y_i)^2}$ have been calculated (4); the results are quoted for reference in Appendix B. The solution of the resulting probability integral equation was obtained by the use of an I.B.M. 1620 digital computer; the program used is presented in Appendix C, and the results of the computations presented and discussed in Chapter 5.

3-2 THE BOOTON ANALYSIS OF THE CONTINUOUS SYSTEM

The Booton treatment of the continuous system involves the quasi-linearization of the nonlinearity and a straightforward analysis of the resulting linear system by the harmonic analysis method. This procedure was carried out by Bright (4), and his results are quoted here.

For the system of Figure 3-1, the variance of the response is given by the expression:

$$\sigma_o^2 = \frac{K_{eq}\sigma^2}{\beta + K_{eq}} \qquad \text{3-4(a)}$$

in which K_{eq}, the equivalent linear gain of the nonlinear element, is given by the equation:

$$K_{eq} = AG(1 + 3\frac{B}{A}\sigma^2) \qquad \text{3-4(b)}$$

3-3 OPTIMUM CHOICE OF THE SAMPLING INTERVAL

By equating the expressions for the response yielded by the Booton analysis of the linear continuous system and the Schultheiss analysis of the linear sampled system, a value of sampling interval is specified which will make the Schultheiss technique yield the same results as that of Booton, which, in the linear case, yields the exact response of the physical system.

Setting the right-hand side of equations 3-3 and 3-4(a) equal, one obtains the expression:

$$\frac{AG\sigma^2}{\beta(1-\frac{AGT}{2})}\left[1 - \frac{1-e^{-\beta T}}{\beta T}\right] = \frac{AG\sigma^2}{\beta + AG} \qquad \text{3-5}$$

By assuming T large (which must be true in order to satisfy the Schultheiss restriction on the sampling interval), equation 3-5 may be solved explicitly for T. The result is easily obtained as the expression:

$$T = \frac{1}{\beta}\left[\sqrt{3+\frac{2\beta}{AG}} - 1\right] \qquad 3\text{-}6$$

This value of T, used in the Schultheiss analysis of the linear sampled system, will thus yield the same results as the Booton analysis of the continuous system. For the nonlinear case, the term 'AG' in equation 3-6 may be replaced by K_{eq} to obtain an approximate value for the correct T to be used.

It should be noted that the result of the Schultheiss analysis, equation 3-3, predicts a zero value of response for zero sampling interval, a behaviour which is not predicted by conventional analysis of sampled systems (8). This anomaly is due to the violation of the assumption that T is large compared to the correlation time of the excitation.

3-4 SUMMARY

The approximation of the continuous system by a sampled one is only valid for a careful choice of T. The requirement in the Schultheiss technique that the sampling interval be large complicates that selection when using that method of analysis; however, it was possible to make such a choice analytically in the example considered.

CHAPTER IV

THE ANALOGUE AND DIGITAL COMPUTER STUDIES

The nonlinear feedback system described in Chapter III was simulated on a Pace TR-48 analogue computer. The methods of measurement of the excitation and response processes, and the simulation of the system, along with some considerations of the digital computer study, are discussed in this chapter.

4-1 THE FREQUENCY SPECTRUM OF THE GENERATOR

The frequency spectrum of the noise generator was measured in two parts: high and low frequency components were measured by separate techniques, the dividing frequency being chosen, somewhat arbitrarily, as ten cycles per second.

4-1-1 High Frequency Components

The amplitude of a noise power spectrum at a frequency f_o may be measured by passing the noise through a bandpass filter centred at f_o and measuring the variance of the output process. However, if the spectrum is not flat in the vicinity of f_o, it becomes difficult to estimate the amplitude of the spectrum at that point. Also, it is difficult to obtain a filter with a narrow passband, variable centre frequency, and constant (or at least accurately calculable) bandwidth.

These difficulties were overcome in the measurement of the noise generator spectrum by utilizing the heterodyning system depicted in Figure 4-1.

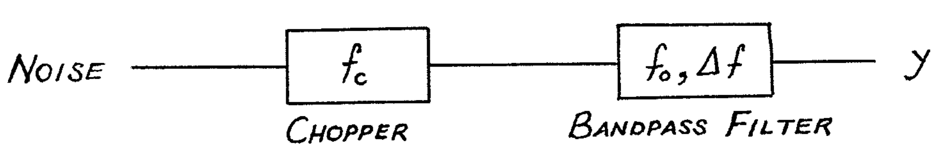


FIG. 4-1

The mean-square value of the output of the filter is proportional to the value of the component of the power spectrum of the chopped noise centred at $f_c - f_o$, the lower sidebands being chosen to avoid interference from the harmonics of the chopped signal. Tuning to various frequencies is easily accomplished by varying f_c.

The filter used was of the form shown in Figure 4-2, and placed in the feedback loop of an operational amplifier to achieve a bandpass configuration. The bandwidth Δf was adjusted by means of the resistance R, the value used in the measurements being 0.5 cycles per second (between -3db points) at a centre frequency f_o of approximately 1500 cps. The filter-amplifier transfer function is shown in Graph 1.

It was found that the filter characteristics were very sensitive to changes in the resistance of the inductor; to avoid this difficulty, the filter was placed in a thermally-insulated chamber.

Because the output of the filter was a slowly-varying random signal, it was impossible to measure its mean-square value with any available meter. However, for Gaussian noise, F(t), it may be shown (6) that the r.m.s. value e is given by the expression

$$e = \sqrt{\frac{\pi}{2}}\,\frac{1}{T}\int_0^T |F(t)|\;dt$$

for large T.

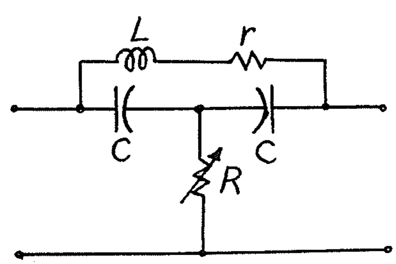
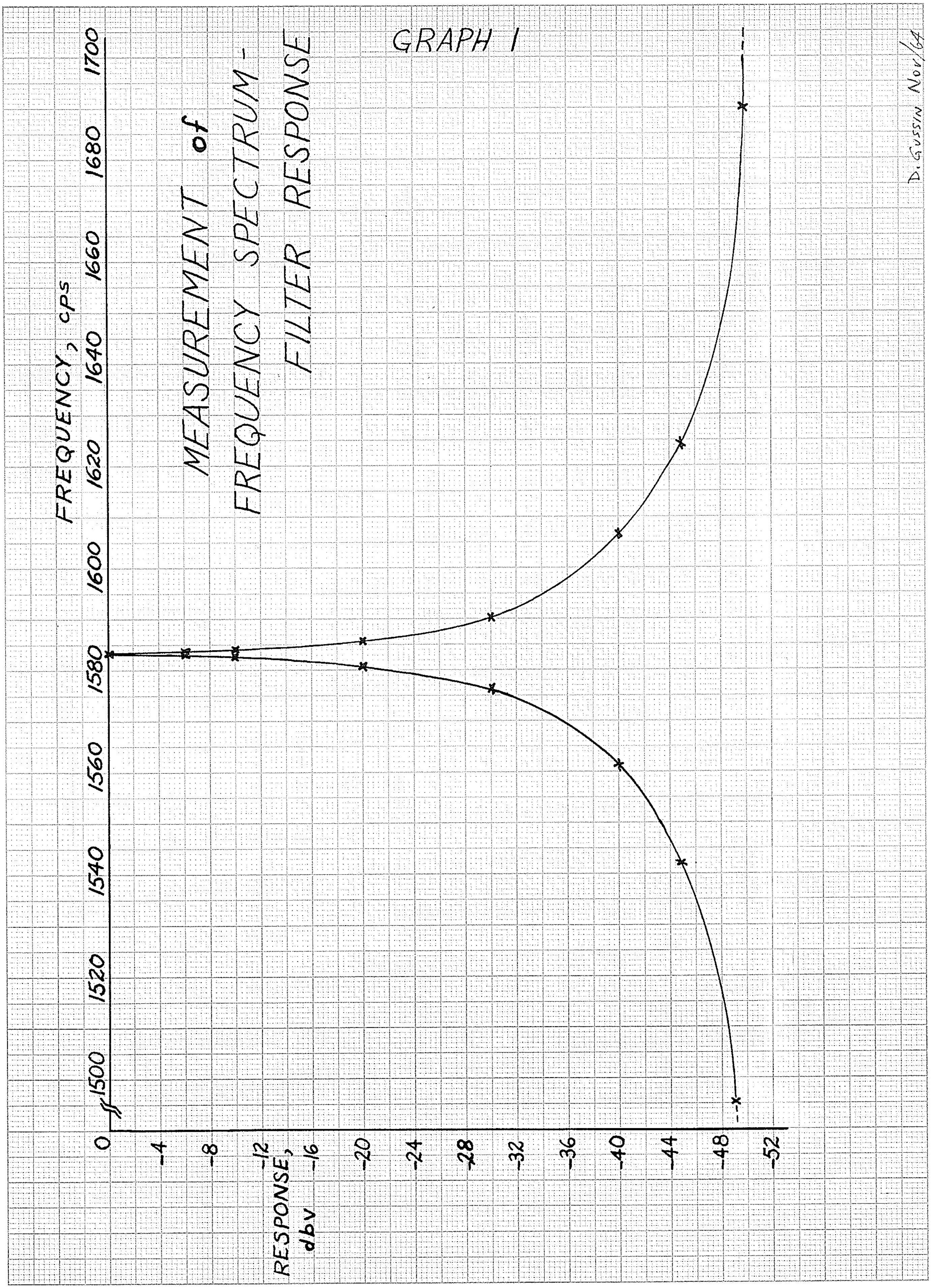


FIG. 4-2



limeters to the Centimeter

Since a linearly filtered Gaussian wave remains Gaussian in nature, the method just described may be used to determine the r.m.s. value of the filter output. The circuit used to effect this measurement is shown in Figure 4-3.

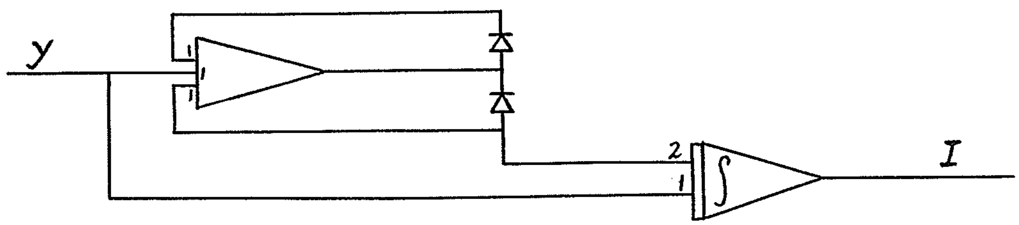


FIG. 4-3

As a check, the generator noise was also measured with a commercial version of the system shown in Figure 4-1.[*] Since the readout mechanism of this instrument was a quickly-responding meter, which was useless for noise measurements, the meter signal (which was a slowly-varying D.C. voltage) was integrated over a long period of time.

The results of the two methods of measurement were equal to within 3%.

4-1-2 Low Frequency Components

It was not possible to measure the spectrum of the noise below five cycles per second by the above method, since the fundamental component of the chopper wave became significant in the filter response, and since the filter bandwidth became significant with respect to the frequency of the noise component being examined. However, the spectrum of the noise for these low frequencies was approximately flat, as can be seen from Graph 2, p.21A, and was of sufficiently low frequency to enable the use of a filter simulated on the computer (6).

---

* General Radio Corp., Wave Analyzer Type 736-A.

The transfer function of the filter used is given by the expression:

$$\frac{X}{W}(S) = Y(S) = \frac{2\zeta\, s/\omega_o}{(s/\omega_o)^2 + 2\,\zeta\, s/\omega_o + 1}$$

which was simulated by the circuit shown in Figure 4-4.

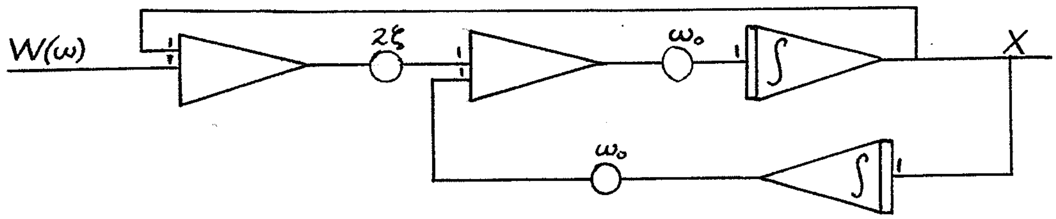


FIG. 4-4

Excitation of this filter by noise with a power spectrum of the form

$$W(\omega) = \frac{N_o}{1 + \frac{\omega^2}{\beta^2}} \qquad 4\text{-}1$$

produces noise with a mean-square value $\overline{x^2}$ given by the expression (6):

$$X^2 = N_o\zeta\omega_o\left[\frac{\zeta\omega_o/\beta + 1}{(\omega_o/\beta)^4 + 4\zeta(\omega_o/\beta)^3 + 2(2\zeta^2+1)(\omega_o/\beta)^2 + 4\zeta\,\omega_o/\beta + 1}\right] \qquad 4\text{-}2(a)$$

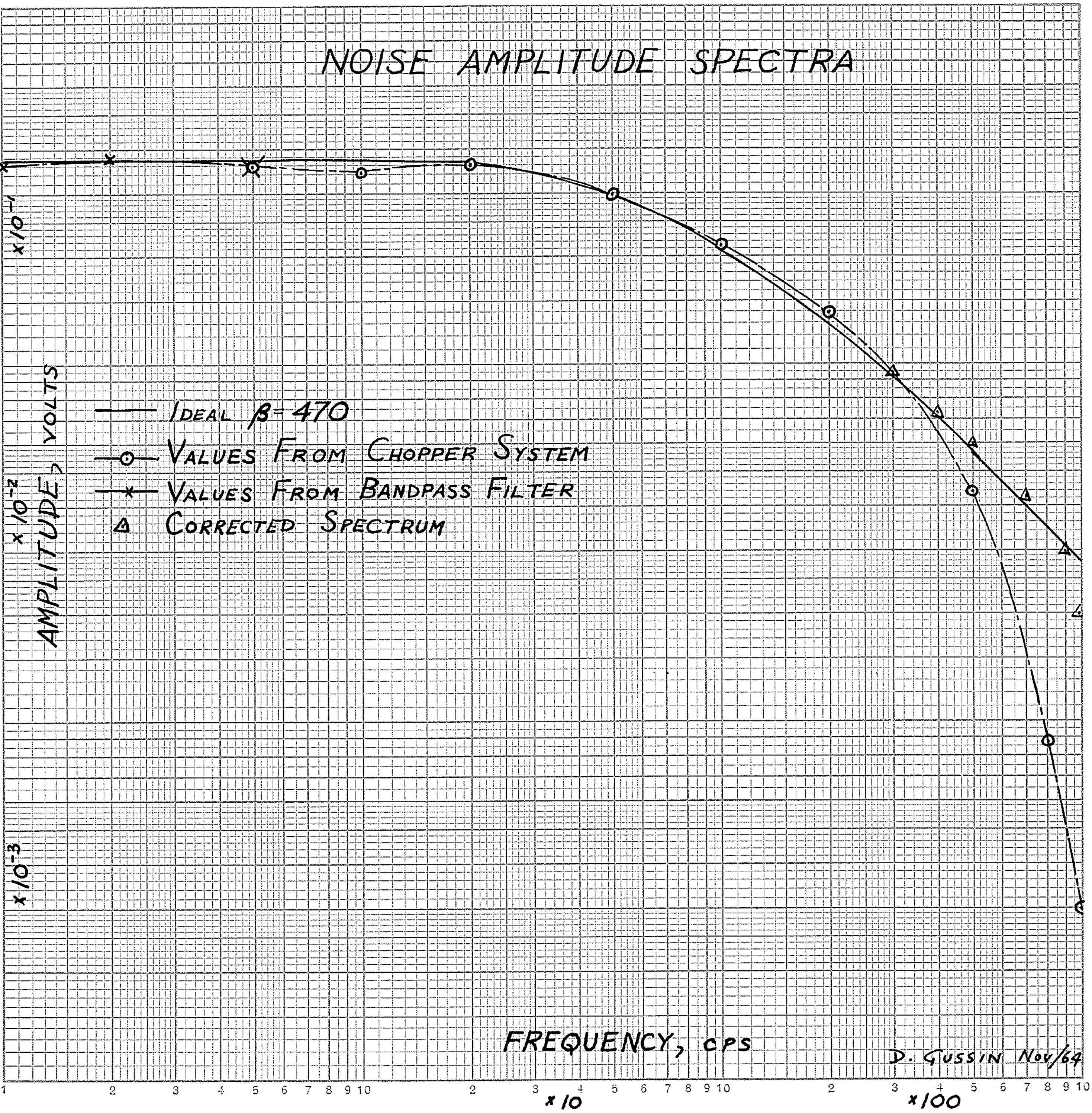
which, for $\omega_o \ll \beta$, reduces to the simple relationship:

$$\overline{X^2} = N_o\zeta\omega_o \quad . \qquad 4\text{-}2(b)$$

The mean-square value of the response of this filter was measured by squaring and integrating.

The amplitude spectrum of the noise generator is displayed in Graph 2, from which it may be seen that the spectral density assumed in the analysis of the system (equation 3-1), was available only up to a frequency of 300 cycles per second; in order to correct the high frequency portion, the noise was filtered, obtaining a spectrum which was equal to the assumed spectrum, to within 5%, up to a frequency

GRAPH 2


NOISE AMPLITUDE SPECTRA
IDEAL β=470
VALUES FROM CHOPPER SYSTEM
VALUES FROM BANDPASS FILTER
CORRECTED SPECTRUM
AMPLITUDE, VOLTS
x 10^-1
x 10^-2
x 10^-3
FREQUENCY, CPS
x 10
x 100
1 2 3 4 5 6 7 8 9 10
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of 1000 cycles per second; above this frequency correction was not possible. However, owing to the low-pass nature of the system being studied, it is felt by the author that the absence of these high frequencies produced negligible error in the system measurements. The filtered generator output and the ideal (assumed) amplitude spectra, are also displayed in Graph 2.

4-2 THE VARIANCE OF THE EXCITATION

During the measurements on the system being studied, the variance of the excitation was determined by the use of the filter shown in Figure 4-4, the output of which was squared and integrated. This procedure measured the spectral level, N_o, of the excitation about a given low frequency, which was chosen to be five radians per second. Error in the determination of the variance of the excitation, due to the absence of the frequencies above 1000 cycles per second, was thus eliminated.

The damping coefficient ζ had to be chosen large enough to eliminate the detection of any ripple in the low frequency part of the spectrum, but small enough so that the spectrum was still essentially flat over the pass band of the filter (to make the correction term in brackets in equation 4-2(a) as close to unity as possible); a value of 0.200 was chosen, necessitating a correction term of 1.005.

4-3 PROBABILITY DISTRIBUTION MEASUREMENT

The noise generator was found to have a slowly-varying D.C. component which had to be removed before any measurements could be made. To achieve this, a high-pass filter of the form shown in

Figure 4-5 was used to remove the D.C. component in the corrected spectrum. The lower cutoff frequency of this filter, ω_c, was chosen

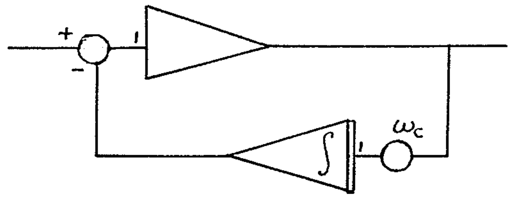


FIG. 4-5

to be 0.005 radians per second.

The assumption that the excitation and response processes be Gaussian was necessary for the Schultheiss analysis technique to be applicable. This property was measured by means of the circuit shown in Figure 4-6.

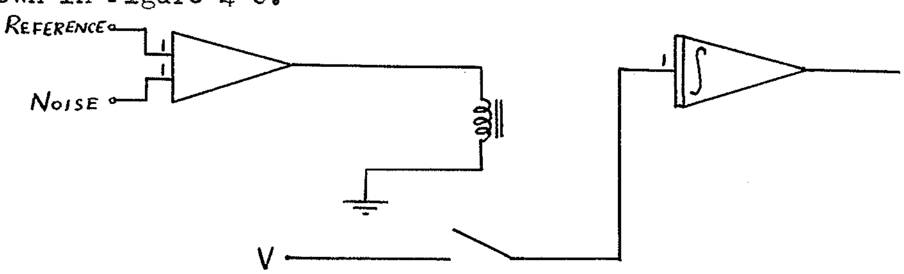


FIG. 4-6

The comparator utilized had a response time of approximately five microseconds, and its hysteresis loop width was less than 0.01 volts. Since the noise being measured had a lower cutoff period of more than 15 milliseconds, it can be seen that negligible error was incurred in this measurement due to the use of the comparator.

It was found by use of this technique that the excitation and response processes were Gaussian to within the confidence limits of the measurement process, for all values of the system being considered. (See Appendix D for the probability distribution of these processes).

4-4 ESTIMATE OF CONFIDENCE LIMITS ON MEASUREMENTS

Since the processes being considered were stochastic, with components at arbitrarily low frequencies, an infinite measurement time would have been necessary in order to determine their values exactly; since only a finite time was used, there is an uncertainty in the measured values. An approximate estimate of this uncertainty will now be made.

4-4-1 Bandpass Filters

A precise estimate of the confidence limits for the measurements using the filter shown in Figure 4-2 was not obtained. However, it may be shown that the uncertainty of measured values obtained by the use of a bandpass filter followed by a squarer and integrator is approximately the same for several different sharply selective filters (6). Since only an approximate estimate is necessary, the confidence limits for an ideal bandpass filter will be used to estimate those of the filter used for the spectrum measurement.*

For an ideal bandpass filter of bandwidth Δf, if a measurement is made of a random Gaussian process with a flat power spectrum in the passband of the filter, there is a 95% probability that this measured value is within p% of its real value for a measurement time T given by the expression (6):

$$T = \frac{4 \times 10^4}{p^2 \Delta f} \quad \text{seconds} . \qquad 4\text{-}2$$

* The limits for this type of filter are the largest of the types considered in the quoted reference; thus it is expected that the limits quoted for the measurements made will be modest.

In the determination of the spectrum, approximately 3000 seconds was used for each measurement; since the bandwidth of the filter was 0.5 cycles per second, there is a 95% probability that the measured values lie within approximately 5% of the real value.

With the filter used in the determination of the variance of the excitation, it may be shown (6) that the approximate measurement time T required is given by the expression:

$$T = \frac{4 \times 10^4}{\pi p^2 \Delta f} \quad \text{seconds.}$$

where the bandwidth Δf may be defined as the frequency difference between half-power points.

In the determination of the excitation variance, the bandwidth of the filter used was 0.5 cycles per second, yielding a 95% confidence limit of 3%.

4-4-2 Squaring and Integrating

For Gaussian noise with a spectrum of the form given in equation 3-1, the time T required for a 95% probability of obtaining a measured value within p% of the actual value is given by the expression:

$$T = \frac{4 \times 10^4}{\pi \beta p^2} \quad \text{seconds.}$$

In the analogue computer study, this method was used to determine the variance of the response, which had a different bandwidth β for each of the measurements made, due to the different equivalent gain values for the nonlinear element. This bandwidth had a minimum value of one radian per second and a maximum of approximately 16 radians per second; the time of measurement was 2000 seconds for all points considered except the linear cases, for which the time was 6000 seconds. The resulting confidence limits are displayed in Graph 4, p.29B, and are all less than 2%.

4-4-3 Probability Distribution Measurements

By examination of the data taken in the determination of the probability distribution of the noise, it was found that, for the measurement time used, there was a 95% probability of the measured probability density values being within 2% of the measured values.

4-5 SIMULATION OF THE SAMPLE AND CLAMP CIRCUIT

The sample and clamp circuit was simulated by the circuit shown in Figure 4-7.

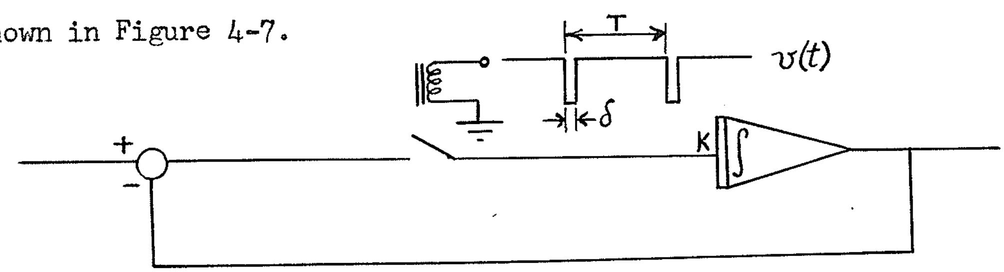


FIG. 4-7

If K is made large compared to the bandwidth of the excitation, the system response (with the comparator switch closed) will equal the excitation. If the switch is opened at any time, the response will remain at its value at the instant that the switch was opened. Thus if the switch is closed for a short time, δ seconds, and then opened, and this action repeated every T seconds, the circuit will approximate a sample and zero-order hold circuit.

In the circuit used in the simulation, K was set to be 2000, so that a "closed" time, δ, of approximately one millisecond was necessary to allow the response to become equal to the excitation during that "closed" interval. This "closed" time is small compared to the sampling intervals used (which were all greater than ten milliseconds), so that the approximation to an ideal sample and clamp circuit was good.

4-6 CHOICE OF CONSTANTS

From equations 3-3 and 3-4, and from the equations in Appendix B, it can be seen that A, the gain of the linear portion of the nonlinear element, is a scale factor, so that it may be set equal to unity with no loss of generality. Since the bandwidth of the available noise generator was approximately 500 radians per second, the gain G of the integrator was made unity, so that the response of the system would be Gaussian for all nonlinearities considered (B was adjusted successively to 0, 1, 3, and 5).

4-7 DIGITAL COMPUTER CONSIDERATIONS

In the solution of equation 2-2, an estimate of the response, σ_o^2, was made by inserting the Booton equivalent gain for the nonlinear element into equation 3-3. The limits of the numerical integration were $\pm 4\sigma_o$, so that the area under the normal curve considered was greater than 99.99% of the total area. Examples were computed for the number of quantized states, N, equal to 51, 101, and 201, from which it was found that use of 51 states (which was the number used in all computations) yielded results which differed from those obtained using 201 states by less than .01%. In fact, by careful choice of the integration limits, 21 or 31 states could have been used with negligible error.

It was noted in the calculations that the sequence of σ^2 values, s_1, s_2, s_3, generated by two successive multiplications of a trial vector (with variance s_1) with the column of the transition probability matrix corresponding to the mean ($r = 0$) state, bore a definite experimental relationship to each other and to the ultimately computed σ^2; this relationship is given by the expression:

$$\frac{s_2 - s_1}{\sigma^2 - s_1} = \frac{s_3 - s_2}{\sigma^2 - s_2}$$

from which a forecast of σ^2 may be made:

$$\sigma^2 = \frac{s_2^2 - s_1 s_3}{2s_2 - s_1 - s_3} \quad .$$

Thus a guess of the final (true) σ_0^2 (s_1 in the above equations) will lead to a very close estimate with only two multiplications.

4-8 SUMMARY

In the simulation of the system, measurements and corrections were made to ensure that the noise excitation and response were of the assumed form. The parameters of the system were chosen according to the restrictions laid down by the available equipment.

Checks were made during the digital computations to ensure that the numerical integration was taken over sufficiently wide limits, and use was made of a convergence property to bring the computations to a rapid solution.

CHAPTER V

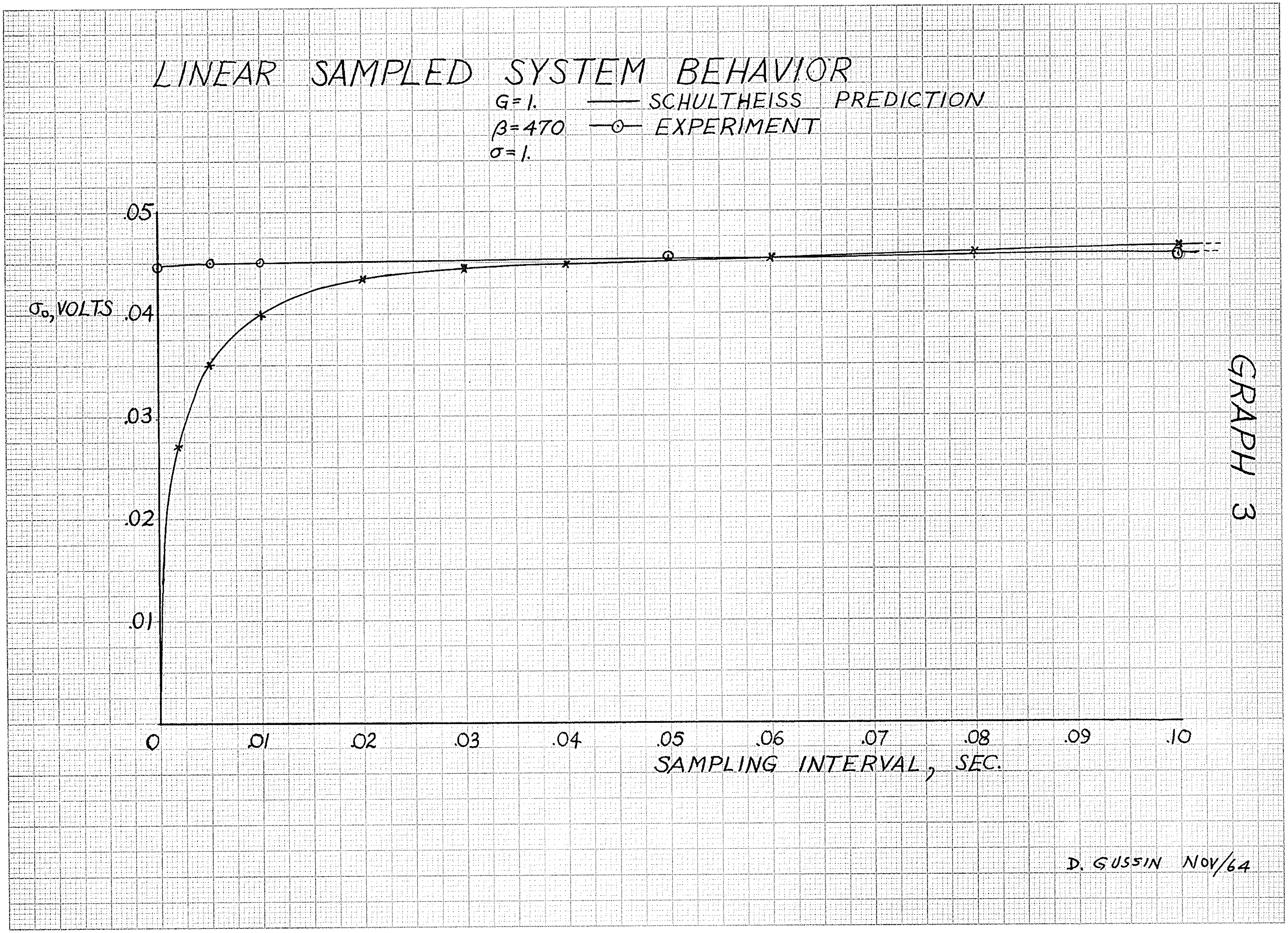
DISCUSSION OF RESULTS AND CONCLUSIONS

The results of this investigation will be discussed as follows: first, the Schultheiss analysis of the sampled system will be examined, and then the approximation of the continuous system by the sampled one will be discussed. The results of the Schultheiss computations compared to the analogue computer results of the study of the continuous system, followed by a comparison of these values with the results of the Booton analysis of the continuous system, will complete the discussion. Conclusions based on these results and on the material discussed in Chapter II will be presented.

5-1 THE SCHULTHEISS ANALYSIS OF THE SAMPLED SYSTEM

From Graph 3, it can be seen that the Schultheiss analysis yields results which agree with the analogue computer results for the sampled system for all except small sampling intervals. For these values of T, as has been pointed out, successive values of the excitation are not uncorrelated, which is a violation of an assumption in the Schultheiss analysis.

Graph 4 displays the behaviour of the example used for various nonlinearities, the sampling interval chosen according to equation 3-6 to make the approximation of the continuous system by the sampled system valid. Inspection of this graph shows that there is excellent agreement between the Schultheiss analysis and the behaviour of the sampled system in the linear case; for the nonlinear cases, the computer analysis yields results which are within the confidence



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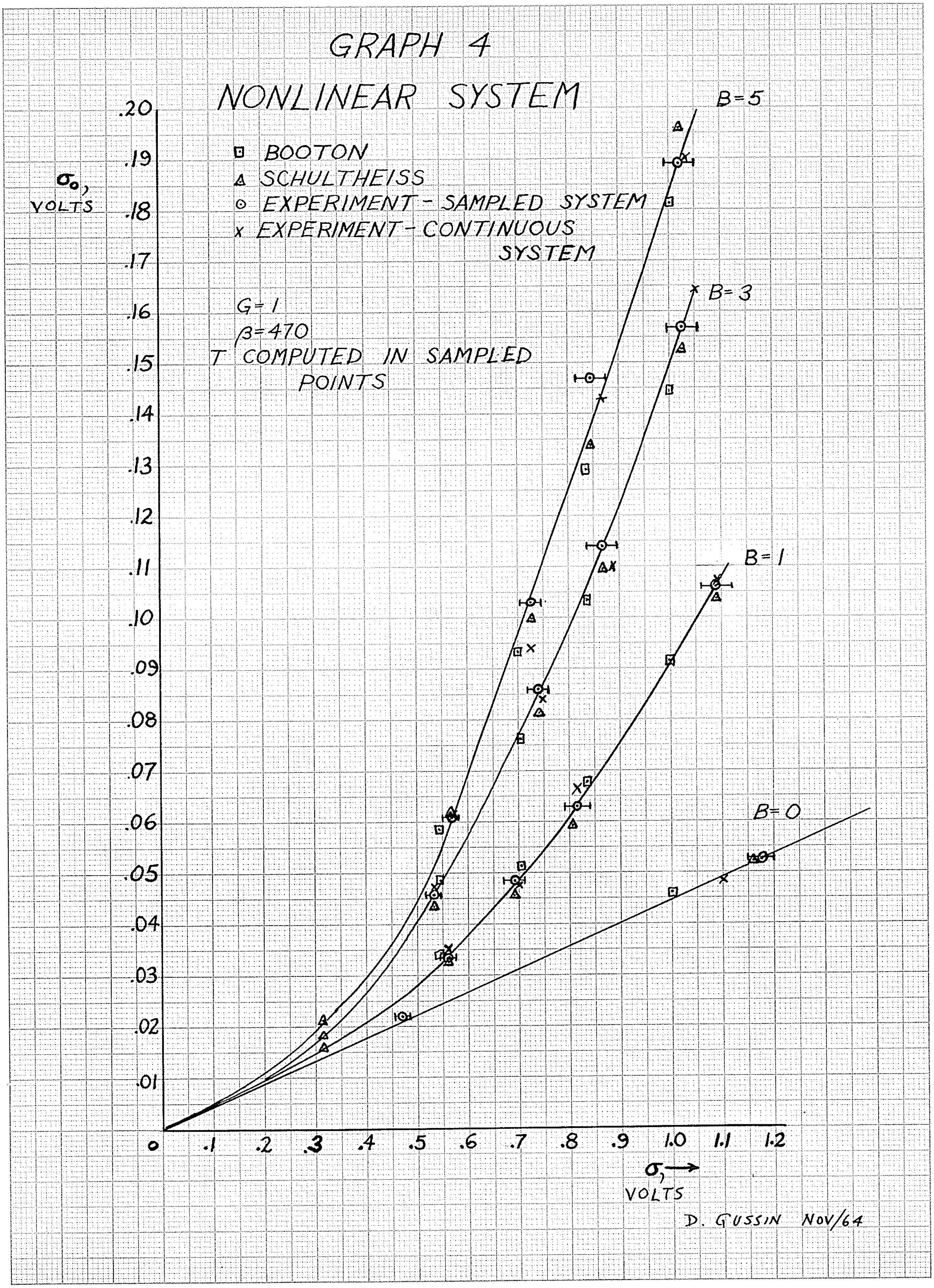
GRAPH 4
NONLINEAR SYSTEM
□ BOOTON
△ SCHULTHEISS
⊙ EXPERIMENT - SAMPLED SYSTEM
x EXPERIMENT - CONTINUOUS SYSTEM
G = 1
β = 470
T COMPUTED IN SAMPLED POINTS
B = 5
B = 3
B = 1
B = 0
σ₀, VOLTS
σᵢ, VOLTS
.20
.19
.18
.17
.16
.15
.14
.13
.12
.11
.10
.09
.08
.07
.06
.05
.04
.03
.02
.01
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
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limits of the experimental measurements in nearly all cases. However, the computed values are almost all less than the measured values, which may be due to a slight error in the Schultheiss analysis or to experimental error, arising from stray noise in the system, slight deviations from the assumed excitation spectrum, or from error in the diode approximations to the multipliers used in the cubic term of the simulated system and in the measurement techniques.

5-2 THE ANALOGUE COMPUTER STUDY OF THE SAMPLED AND CONTINUOUS SYSTEMS

A comparison of the results of the analogue study of the sampled and continuous systems will indicate the validity of the choice of T. It was found that this choice was critical for large nonlinearities --- a 10% change in T produced a 10% change in the sampled system output. However, from Graph 4 it can be seen that the sampled system behaviour was the same as the continuous system behaviour, within confidence limits, for most points. This would indicate that the use of equation 3-6 yields the best choice of sampling interval possible, in spite of the approximation of the forward gain by the Booton equivalent gain.

A single value of T could have been used for all sampled points in Graph 4, but a difference between sampled and continuous system behaviour of up to 50-100% would have occurred, depending on how T was chosen. The smallest maximum difference for the conditions considered in Graph 4 is about 15-20%, for all possible T.

5-3 SCHULTHEISS AND BOOTON RESULTS COMPARED TO THE ANALOGUE COMPUTER STUDY OF THE CONTINUOUS SYSTEM

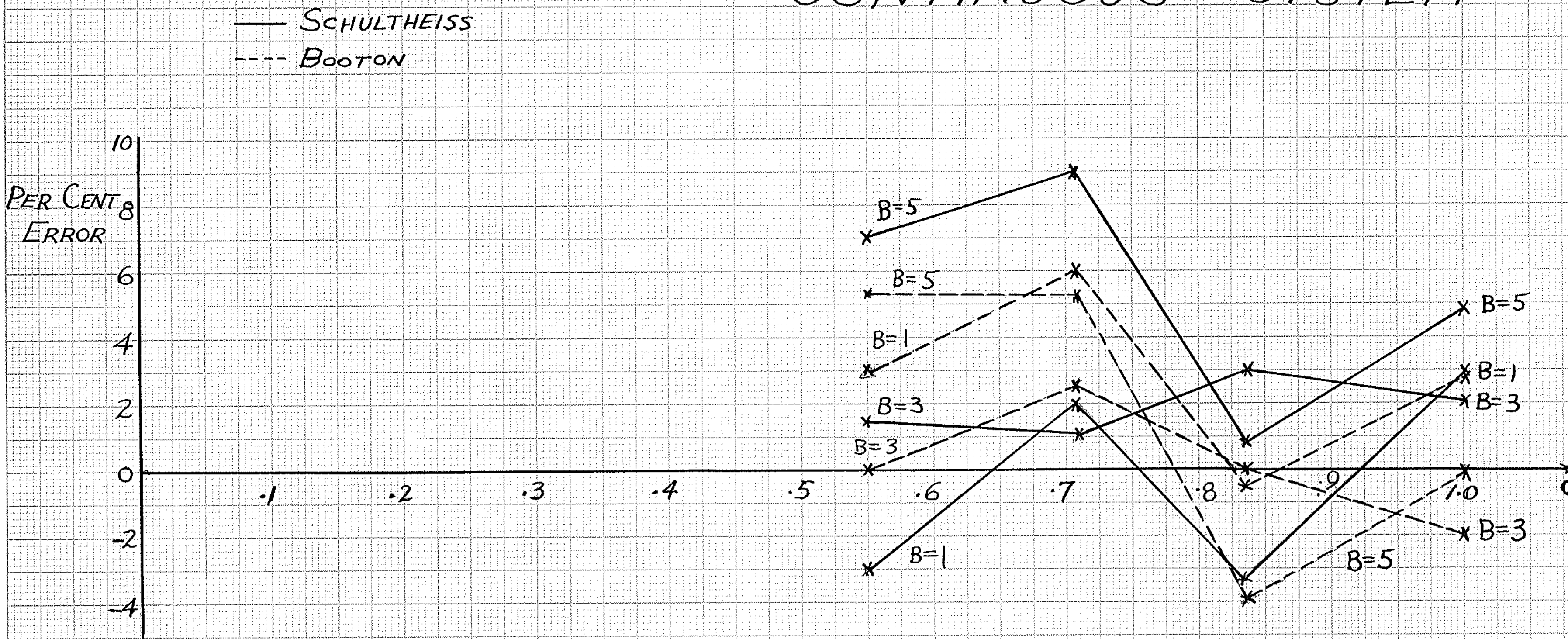
Since the purpose of the application of the Schultheiss method was to analyze the continuous system, a direct comparison of the Schultheiss results with those obtained from the analogue computer is in order. To give an idea of the quality of the new analytical method, the Booton results will also be studied.

Graph 4 displays the behaviour of the continuous system as determined from the analogue computer, and the results of the Schultheiss and Booton analyses; from this, and from Graph 5, it can be seen that the Schultheiss technique yields results with a slightly larger error than the Booton method. The error in almost all cases, however, is within the confidence limits of the measurement techniques.

5-4 CONCLUSIONS AND FUTURE INVESTIGATION

The Schultheiss technique was applied to a given sampled system, the results comparing favorably with those obtained in an analogue computer study of that system, provided that the sampling interval was large compared to the correlation time of the excitation. Furthermore, an approximate method of choice of sampling interval, using the Schultheiss equation for the linear system with the Booton quasi-linearized gain for the nonlinear element, made the sampled system approximate the behaviour of the corresponding continuous system, so that, with that choice of sampling interval, the Schultheiss technique was applicable to the continuous system. A

GRAPH 5 - ERROR BETWEEN PREDICTED AND MEASURED PERFORMANCE OF CONTINUOUS SYSTEM



D. GUSSIN

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comparison of the results yielded by the Schultheiss and Booton techniques with the results of the analogue computer study of the continuous system showed that the results for both methods were equal to the analogue computer results to within the confidence limits of the measurement techniques.

The Schultheiss method of analysis is exact when applied to a sampled system, provided that the sampling interval is large compared to the correlation time of the excitation. When applied to a continuous system, the only source of error is seen to be the choice of sampling interval to make the sampled and continuous systems behave identically.

From the considerations in Chapter II, it can be seen that the Schultheiss technique is equally applicable to noise having other probability distributions than the normal, provided only that the form of the probability density function is known in terms of the mean and r.m.s. values of noise at the input to the nonlinearity, and of the excitation and response processes. Use of power series approximations to the probability density functions of these processes may render an analytical solution of the non-linear problem possible.

The main drawback in the application of the Schultheiss technique is the necessity of choosing a sampling interval so that the sampled system is a reasonable approximation to the continuous one being analyzed. From Chapter II, it may be seen that the solution of the probability integral equation (equation 2-2) is a general one, not requiring any restrictions on the functions p and p_c. Thus the conditional transition probability function does not have to be Markov, so that if, in its derivation, allowance be made for statistical dependence between values of the excitation at successive sampling instants, the sampling interval T could be made arbitrarily small, and the approximation of the continuous system by the sampled one would not be necessary.

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APPENDIX A

CALCULATION OF THE OPEN-LOOP RESPONSE

Excitation of the system has a power spectrum of the form:

$$F(\omega) = \frac{2\beta\sigma^2}{\beta^2+\omega^2} . \qquad \text{A-1}$$

Integration over a period T may be written as a Fourier transform expression:

$$H(j\omega) = \frac{AG}{j\omega}\left(1 - e^{-j\omega T}\right) . \qquad \text{A-2}$$

The open-loop output power spectrum can be found from the relationship:

$$G(\omega) = F(\omega) \left| H(j\omega) \right|^2 \qquad \text{A-3}$$

in which, the last factor for this example may be written as:

$$\left| H(j\omega) \right|^2 = \frac{2(AG)^2}{\omega^2}\left(1 - \cos \omega T\right) . \qquad \text{A-4}$$

Substitution of equations A-1 and A-4 into equation A-3 produces:

$$G(\omega) = \frac{2\beta\sigma^2}{\beta^2+\omega^2} \cdot \frac{2(AG)^2}{\omega^2}\left(1 - \cos \omega T\right) . \qquad \text{A-5}$$

Note that this expression is continuous and finite for all real ω.

The variance, σ_1, of the output can be found from the expression:

$$\sigma_1^2 = R(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)\, d\omega , \qquad \text{A-6}$$

In which $R(\tau)$ is the autocorrelation function of the output. Thus the response is given by the expression:

$$\sigma_1^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4\beta\sigma^2 (AG)^2}{\omega^2(\beta^2 + \omega^2)}\left(1 - \cos\omega T\right) d\omega . \qquad \text{A-7}$$

Rewriting the integrand in terms of $x = \omega T$ produces:

$$\sigma_1^2 = \frac{2\beta\sigma^2 (AG)^2 T^3}{\pi} \int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2\left[(\beta T)^2 + x^2\right]} dx \qquad \text{A-8}$$

$$= \frac{2\beta\sigma^2 (AG)^2 T^3}{\pi (\beta T)^2} \int_{-\infty}^{\infty} \left[\frac{1 - \cos x}{x^2} - \frac{1}{(\beta T)^2 + x^2} + \frac{\cos x}{(\beta T)^2 + x^2}\right] dx$$

The first term in the integrand may be integrated by parts, the second is a standard form, and the third may be integrated by complex variable theory (9). The solution is given by the expression:

$$\sigma_1^2 = \frac{2\sigma^2 (AG)^2 T^3}{\beta}\left[1 - \frac{1 - e^{-\beta T}}{\beta T}\right]. \qquad \text{A-9}$$

Since the excitation is ergodic, the above expression also gives the mean-square change of the response during any sampling interval. This is the desired result.

APPENDIX B

$\overline{\Delta y_i}$ AND $\overline{(\Delta y_i)^2}$ FOR THE NONLINEAR CASE

$$\overline{\Delta y_i} = -(AG)T\left[\frac{B}{A}r_i^2 + \left(1 + 3\frac{B}{A}\sigma^2\right)\right]r_i$$

$$\overline{(\Delta y_i)^2} \doteq s_6\, r_i + s_4\, r_i + s_2\, r_i + s_0$$

where r_i is the hypothesized state, and

$$s_4 \doteq 2(AG)^2\, T^2 \frac{B}{A}\left(1 + 3\frac{B}{A}\sigma^2\right) + \frac{1}{\beta^2}\left[18(AG)^2\left(\frac{B}{A}\right)^2\sigma^4\left(e^{-\beta T} + \beta T - 1\right)\right]$$

$$s_2 \doteq (AG)^2\, T^2\left(1 + 3\frac{B}{A}\sigma^2\right)^2 + \frac{1}{\beta^2}\left[12(AG)^2\,\frac{B}{A}\sigma^2\left(1 + 2\frac{B}{A}\sigma^2\right)\left(e^{-\beta T} + \beta T - 1\right) + 18(AG)^2\left(\frac{B}{A}\right)^2\sigma^2\left(e^{-2\beta T} + 2\beta T - 1\right)\right]$$

$$s_0 \doteq \frac{1}{\beta^2}\left[4(AG)^2\left(\frac{B}{A}\right)^2\sigma^6\left(e^{-3\beta T} + 3\beta T - 1\right) + 2(AG)^2\,\sigma^2\left(9\left(\frac{B}{A}\right)^2\sigma^4 + 4\frac{B}{A}\sigma^2 + 1\right)\left(e^{-\beta T} + \beta T - 1\right)\right]$$

$$s_6 \doteq (AG)^2\left(\frac{B}{A}\right)^2 T^2$$

APPENDIX C

COMPUTER PROGRAM

```
C   PROGRAM USING T COMPUTED TO GIVE SAME RESULTS AS CONTINUOUS SYSTEM.
C   N IS THE NUMBER OF QUANTIZED STATES IN THE NUMERICAL INTEGRATION.
C   N MUST BE ODD, LESS THAN 210.
C   BE IS BETA, B IS THE NONLINEAR COEFFICIENT, T THE SAMPLING INTERVAL.
C   SIG IS SIGMA**2 INPUT, SIGO IS SIGMA**2 OUTPUT.
C   SENSE SWITCH 1 ON FOR N=101, OFF FOR N=51.
C   SENSE SWITCH 2 ON TO COMPUTE T, OFF TO ACCEPT T.
C   SENSE SWITCH 3 ON FOR TYPED DATA INPUT, OFF FOR CARDS.
      DIMENSION P(210),Y(210),Q(4)
    1 FORMAT(18HBOOTON ESTIMATE ISF11.7,14H VOLTS-SQUARED)
    2 FORMAT(21HSAMPLING INTERVAL IS F6.4,4H SEC)
    3 FORMAT(F6.3)
    4 FORMAT(2HB=F6.3,5H SIG=F6.3,6H SIGO=F11.7)
    5 IF(SENSESWITCH1)6,7
    6 N=101
      GOTO8
    7 N=51
    8 IF(SENSESWITCH3)9,11
    9 ACCEPT3,B,SIG
      IF(SENSESWITCH2)13,10
   10 ACCEPT3,T
      GOTO13
   11 READ3,B,SIG
      IF(SENSESWITCH2)13,12
   12 READ3,T
   13 NC=(N+1)/2
      L=NC-1
      XN=N-1
      G=1.
      A=1.
      BE=470.
C   XKEQ IS BOOTON EQUIVALENT LINEAR GAIN.
      XKEQ=G*(A+3.*B*SIG)
      IF(SENSESWITCH2)14,15
   14 T=(SQRTF(3.+(2.*BE)/XKEQ)-1.)/BE
C   SIGE IS ESTIMATE OF FINAL OUTPUT.
   15 SIGE=XKEQ*SIG*(1.+(EXPF(-BE*T)-1.)/(BE*T))/(BE-.5*BE*XKEQ*T)
      PRINT2,T
      PRINT1,SIGE
      SIGO=SIGE
      AMAX=4.*SQRTF(SIGE)
      DEL=2.*AMAX/XN
      A1=G*T*BZ
      A2=G*T*(3.*B*SIG+A)Z
      B2=9.*G*G*B*B*SIG**2*(EXPF(-2.*BE*T)+2.*BE*T-1.)/(BE**2)
      B3=12.*G*G*B*B*SIG**3*(EXPF(-3.*BE*T)+3.*BE*T-1.)/(9.*BE**2)
      Y1=2.*G*G*(EXPF(-BE*T)+BE*T-1.)/(BE**2)
      X0=Y1*(9.*B*B*SIG**3+4.*A*B*SIG*SIG+A*A*SIG)Z
```

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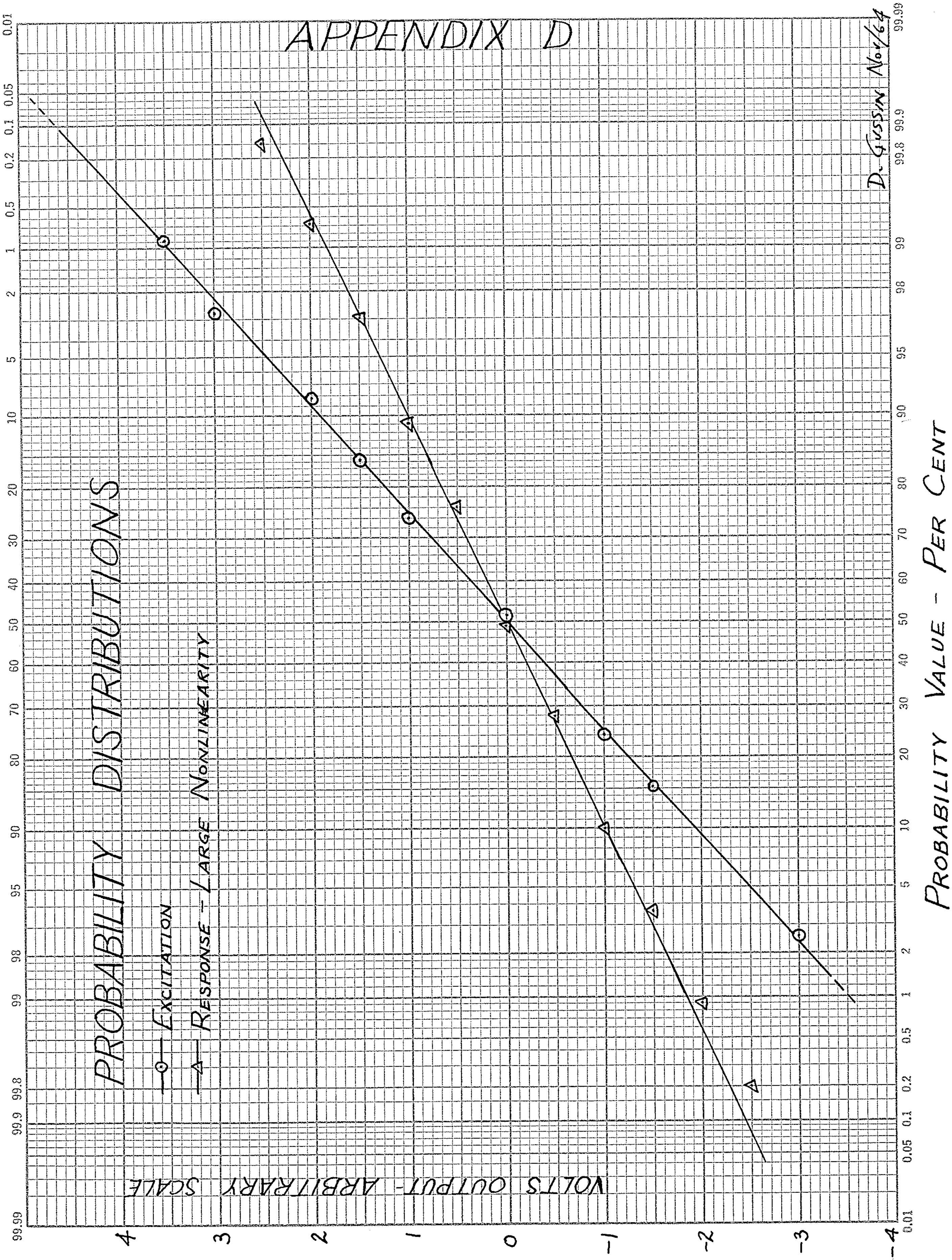
      X2=Y1*(12.*B*B*SIG*SIG+6.*A*B*SIG)Z
      X4=Y1*(9.*B*B*SIG*SIG)Z
      Y2=G*G*T*TZ
      Z2=Y2*(9.*B*B*SIG*SIG+6.*A*B*SIG+A*A)Z
      Z4=Y2*(6.*B*B*SIG+2.*A*B)Z
      Z6=Y2*B*BZ
      S0=B3+X0Z
      S2=X2+Z2+B2Z
      S4=Z4+X4Z
      S6=Z6Z
      SAVE=AMAX
      AMAX=AMAX+DEL
C   K IS ROW NUMBER OF CENTER COLUMN OF MATRIX.
      DO 16 K=1,NC
      AMAX=AMAX-DEL
      XMEA=ABSF(A1*AMAX**3+(A2-1.)*AMAX)
      XMEAS=S6*AMAX**6+S4*AMAX**4+S2*AMAX**2+S0
      P(K)=DEL*(EXPF(-(XMEA**2)/(2.*XMEAS)))/(2.5066283*SQRTF(XMEAS))
      MM=N+1-K
   16 P(MM)=P(K)
C    COLUMN MATRIX COMPUTED
C
C   GENERATE THREE TERMS IN SEQUENCE OF SIG'S USING SAX AS FIRST TERM.
      SAX=SIGO
   17 JJ=0
   18 JJ=JJ+1
      X=SAVE
      DENOM=2.5066283*(SQRTF(SIGO))
      DO 19 I=2,NC
      X=X-DEL
      Y(I)=DEL*(EXPF(-(X**2)/(2.*SIGO)))/DENOM
      MM=N+1-I
   19 Y(MM)=Y(I)
      S=0.
      DO 20 I=2,L
   20 S=S+Y(I)
      Y(1)=(1.-2.*S-Y(NC))/2.
      Y(N)=Y(1)
      C=0.
      DO 21 I=1,L
   21 C=C+Y(I)*P(I)
      C=2.*C+Y(NC)*P(NC)
      Q(JJ)=1./(6.2831853*((C/DEL)**2))
      SIGO=Q(JJ)
      IF(JJ-2)18,22,23
C   D IS ESTIMATE OF FINAL SIGMA**2 FROM CONVERGENCE PROPERTY.
   22 D=ABSF((Q(1)**2-SAX*Q(2))/(SAX+Q(2)-2.*Q(1)))
C   CHECK ESTIMATE BY COMPUTING A NEW SIGMA**2 FROM IT AND
C   COMPARING THE TWO SIGMAS FOR EQUALITY WITHIN .5 PERCENT.
      SIGO=D
      GOTO18
C   IF ESTIMATE IS GOOD, PRINT RESULT AND ACCEPT NEW DATA.
C   IF ESTIMATE IS NO GOOD, USE IT AS INITIAL SIG IN A NEW SEQUENCE.
   23 IF(ABSF(D-Q(3))-Q(3)/1000.)25,25,24
```

```
   24 SIGO=Q(3)
C  IF SIGMA**2 IS APPROACHING ZERO, ACCEPT NEW DATA.
      IF(SIGO-(.1E-05))5,5,17
   25 PRINT4,B,SIG,Q(3)
      GOTO5
      END
```


APPENDIX D
PROBABILITY DISTRIBUTIONS
—⊙— EXCITATION
—△— RESPONSE - LARGE NONLINEARITY
VOLTS OUTPUT - ARBITRARY SCALE
PROBABILITY VALUE - PER CENT
D. Gussin Nov/64