The University of Manitoba

Modal Analysis of Planar Waveguide Junctions

by

C. W. Hasselfield

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ABSTRACT

Planar waveguide discontinuities are studied using the modal analysis method of transverse field matching. The derivation and the methods of calculations are revised from Wexler's method. Several types of junctions with one and two dimensional steps are solved. A new analytic expression is used here to help fit the singularities which may be present in either the electric or magnetic field. A study is made of the uses and limitations of the method in dealing with thin structures, particularily irises.

ACKNOWLEDGEMENT

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1. INTRODUCTION

Step discontinuities in waveguides have been solved by Wexler (1), Clarricoats and Slinn (2), and Cole, Nagelberg, and Nagel (3) using modal analysis. Among the types of discontinuities that this method is applicable to are semi-infinite E or H-plane steps, rectangular bifurcations of any thickness, thick or thin rectangular irises, irises or steps in circular waveguides, and multiple guide problems.

The method is not without numerical difficulties, particularily in very thin irises and bifurcations. For these problems convergence is a problem and in zero thickness structures the solution does not converge at all. It will be shown here that the field singularities found in steps are dealt with very poorly in the modal analysis solution.

In this work convergence for the H-plane step is investigated and the H-plane iris is studied as a combination of two H-plane steps. The method provides good results for all configurations tested, except for zero thickness structures. An indication of the number of modes required to give a desired accuracy for a given thickness of iris is given here. A solution for the thin iris is attempted as an example of the possible difficulties that arise. Convergence for an iris is found to be

slower as the thickness decreases and in the limit a zero thickness iris was found not to converge at all. A look at the field plots reveals the position and nature of the singularity in the magnetic field. From these plots the attempted field match of the numerical solution is displayed.

The E-plane step is studied next in much the same manner as the H-plane step. Convergence for this problem is found to be similar to the H-plane step but with the method used, twice as many modes are required for the same type of problem. The numerical difficulties of the thin iris are again shown since they are more apparent in this case than they were with the H-plane iris. The two dimensional simultaneous E and H-plane step and iris are investigated next but here a lack of exact solutions makes comparisons difficult. There are however approximate solutions which indicate that the solutions are correct. Convergence to the solution is slower, due to the two dimensional mode types required. Only approximate results can be obtained without setting up very large systems of linear equations.

Difficulties due to the singularity are partially resolved by using a combination of the analytic solution near the edge and the normal waveguide modes on the H-plane step. It is found that faster convergence and better field matching is obtained with this combination method. The number of calculations is increased however since the analytic solution must be made orthogonal to the normal waveguide modes for the modal analysis method to be

applied.

A new notation simpler than Wexler's is presented here with an alteration allowing any mode incident, not just the dominant first one.

2. THE MODAL ANALYSIS FORMULATION

Any discontinuity problem in a reciprocal waveguide, that has a series of planar junctions such as an iris, can be solved by modal analysis. Each junction is treated separately, taking into account scattering from neighboring junctions. The fields on either side of the junction are expressed as a series of normal waveguide modes and these fields are equated across the junction. The technique for solving for the unknown series coefficients is given here and the expression for the interaction of the planar junctions is presented.

2.1 Fields at the Junction

Top and end views of a representative junction problem are given in Fig. 2.1. An incident mode, numbered r with coefficient a_r, impinges upon junction m in the z direction. Let the waveguide section on the generator side of junction m be denoted $A^{(m)}$ and the waveguide section on the load side be denoted $B^{(m)}$. Using this notation on a multiguide section, the dimensions of section $A^{(m+1)}$ equal section $B^{(m)}$. The fields at junction m, on the generator side are expanded in terms of the normal waveguide

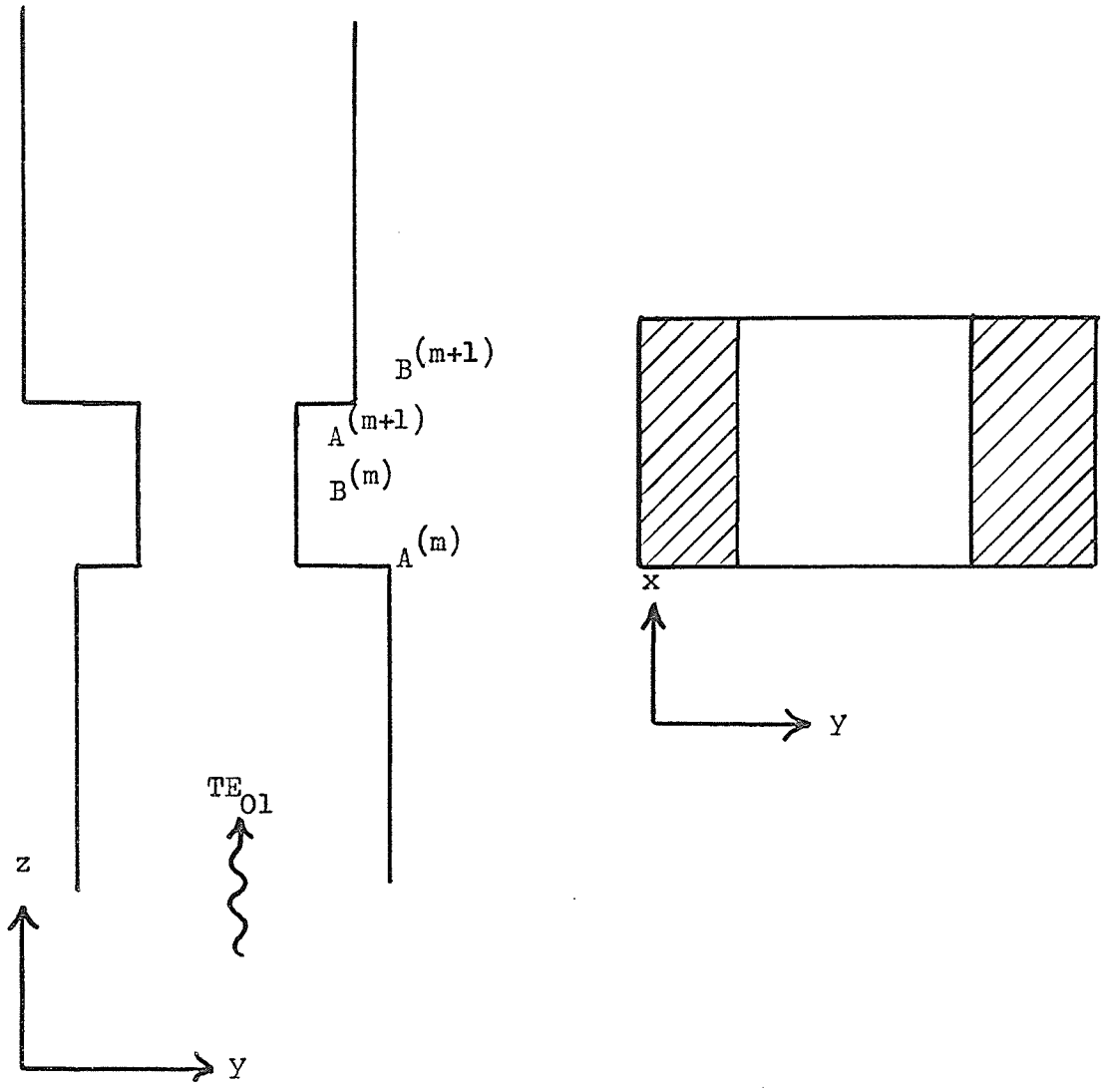

Fig. 2.1 Top and end view of a discontinuity in a waveguide

modes in $A^{(m)}$ and are given by Wexler(4) as

$$\overline{E} = (1 + \rho_r)\ a_r \overline{e}_{ar} + \sum_{\substack{i=1 \\ i \neq r}}^{\infty} a_i \overline{e}_{ai} \tag{2.1}$$

and

$$\overline{H} = (1 - \rho_r)\ a_r \overline{h}_{ar} + \sum_{\substack{i=1 \\ i \neq r}}^{\infty} a_i \overline{h}_{ai} \tag{2.2}$$

The transverse vector mode functions $\overline{e}_{ak}$ and $\overline{h}_{ak}$ for guide $A^{(m)}$ are known in advance. The corresponding mode coefficients for the functions are a_r and ρ_r and the total transverse electric and magnetic fields are $\overline{E}$ and $\overline{H}$. If γ_i is the z-dependent propagation constant of the i-th mode in $A^{(m)}$, then any mode may be referred a distance ℓ towards the source by multiplying by $e^{\gamma_i \ell}$.

The same field may be expressed on the load side of junction m by a series of modes of waveguide section $B^{(m)}$ as follows:

$$\overline{E} = \sum_{j=1}^{\infty} b_j\ (\overline{e}_{bj} + \sum_{k=1}^{\infty} s_{jk}\ \overline{e}_{bk}) \tag{2.3}$$

and

$$\overline{H} = \sum_{j=1}^{\infty} b_j\ (\overline{h}_{bj} - \sum_{k=1}^{\infty} s_{jk}\ \overline{h}_{bk}) \tag{2.4}$$

The scattering coefficients s_{jk} of (2.3) and (2.4) account for the contribution from the j-th mode incident on junction m+1 from junction m and scattered back to junction m as the k-th mode. This scattering matrix S, contains information about the propagation of the modes involved between junctions m and m+1, and also the scattering properties of junction m+1.

For this reason the last junction nearest the load must be solved first, with the assumption that it is followed by a simple or known termination, such as a short or a matched load. Any junction numbered m falls into two general classes: boundary reduction where section $A^{(m)}$ is larger than section $B^{(m)}$; and boundary enlargement where section $A^{(m)}$ is smaller than section $B^{(m)}$.

2.2 Solution of the Forward Scattered Field and ρ

For the solution of junction m, the scattering matrix S, due to the cumulative effect of junctions in the direction of the load, is known. The derivation given by Wexler (4), generalized to allow the r-th mode incident on a junction, rather than only the first mode, is shown in appendix A.2. The results are

$$\rho_r \delta_{nr} + \sigma \sum_{j=1}^{N} \left(\frac{b_j}{a_r} \sum_{\substack{i=1 \\ i \neq r}}^{M} \frac{\delta_{ji} + \sigma \sum_{k=1}^{N} s_{jk} \delta_{ki} \delta_{ni}}{\alpha_i} \right) + \sigma \left(\frac{b_m}{a_r} - \sigma \sum_{j=1}^{N} \frac{b_j}{a_r} s_{jn} \right) \beta_n = \sigma \delta_{nr} \qquad (2.5)$$

$$n = 1, 2, 3, \ldots, N$$

and

$$\rho_r \alpha_r - \sigma \sum_{j=1}^{N} \frac{b_j}{a_r} (\delta_{jr} + \sigma \sum_{k=1}^{N} s_{jk} \delta_{kr}) = -\sigma\alpha_r \tag{2.6}$$

where

$$\delta_{ij} = \int_b \overline{e}_{bj} \times \overline{h}_{ai} \cdot \tilde{u}_z \, ds \tag{2.7}$$

$$\alpha_i = \int_a \overline{e}_{ai} \times \overline{h}_{ai} \cdot \tilde{u}_z \, ds \tag{2.8}$$

$$\beta_j = \int_b \overline{e}_{bj} \times \overline{h}_{ai} \cdot \tilde{u}_z \, ds \tag{2.9}$$

and

$$\sigma = 1 \tag{2.10}$$

For the boundary enlargement problem, (2.7) and (2.10) are replaced by

$$\delta_{ij} = \int_a \overline{e}_{ai} \times \overline{h}_{bj} \cdot \tilde{u}_z \, ds \tag{2.11}$$

and

$$\sigma = -1 \tag{2.12}$$

The series in (2.5) and (2.6) have been truncated to M and N modes in guide $A^{(m)}$ and $B^{(m)}$ respectively. Since (2.5) supplies N linear complex equations and (2.6) supplies one more, the set can be solved yielding the coefficients ρ_r, $\frac{b_1}{a_r}$, $\frac{b_2}{a_r}$, ..., $\frac{b_N}{a_r}$.

2.3 Back Scattering and the S Matrix

Using the solved coefficients, the back scattered mode coefficients $\frac{a_i}{a_r}$ are computed by

$$\frac{a_i}{a_r} = \frac{\sigma \sum_{j=1}^{N} \frac{b_j}{a_r} (\delta_{ji} + \sigma \sum_{k=1}^{N} s_{jk} \delta_{ki})}{\alpha_i} \qquad (2.13)$$

which completes the analysis of junction m.

The scattering matrix S for the next junction, numbered m-1, at a distance ℓ towards the source, can now be assembled. For each mode r incident on junction m, there are M scattering coefficients for junction m-1 which are given by

$$s_{ri} = \begin{cases} \frac{a_i}{a_r} e^{-(\gamma_i + \gamma_r)\ell} & i \neq r \\ \rho_r e^{-2\gamma_r \ell} & i = r \end{cases} \qquad i = 1, 2, \ldots, M \qquad (2.14)$$

where the a_i are for junction m. There is a term of the form (2.14) for each of the M modes incident on junction m, thus generating M^2 scattering coefficients.

When power enters a system of connected waveguides with discontinuities, the incident modes are scattered at junction 1 and propagated, according to the waveguide dimensions, to

junction 2 as an infinite number of forward scattered modes. At junction 2 this infinite set of modes are again scattered individually, some propagating back to junction 1 and others propagating to junction 3. In this way the properties of a junction are dependent on the junctions further down the guide.

3 SOLUTION OF THE H-PLANE STEP

Approximations for various H-plane step configurations with known accuracy limits are given by Marcuvitz (5) for ranges of frequency and step width.

The types of modes excited and the calculation using them are given here for the H-plane step. Integrations involving them are done numerically rather than by using analytic results. The solution of the step is given and the transverse field plots show the degree of field match. The corresponding boundary enlargement problem is studied in order that the H-plane iris can be solved.

3.1 Mode Types Excited

For an incident TE_{0i} mode on the simple junction of Fig. 3.1, the only type of mode which can be excited are of the TE_{oN} type. If the discontinuity is symmetrical about an x direction symmetry line, then if i is odd, N is odd and if i is even, N is even.

Assuming TE_{01} incident the expression for the transverse E

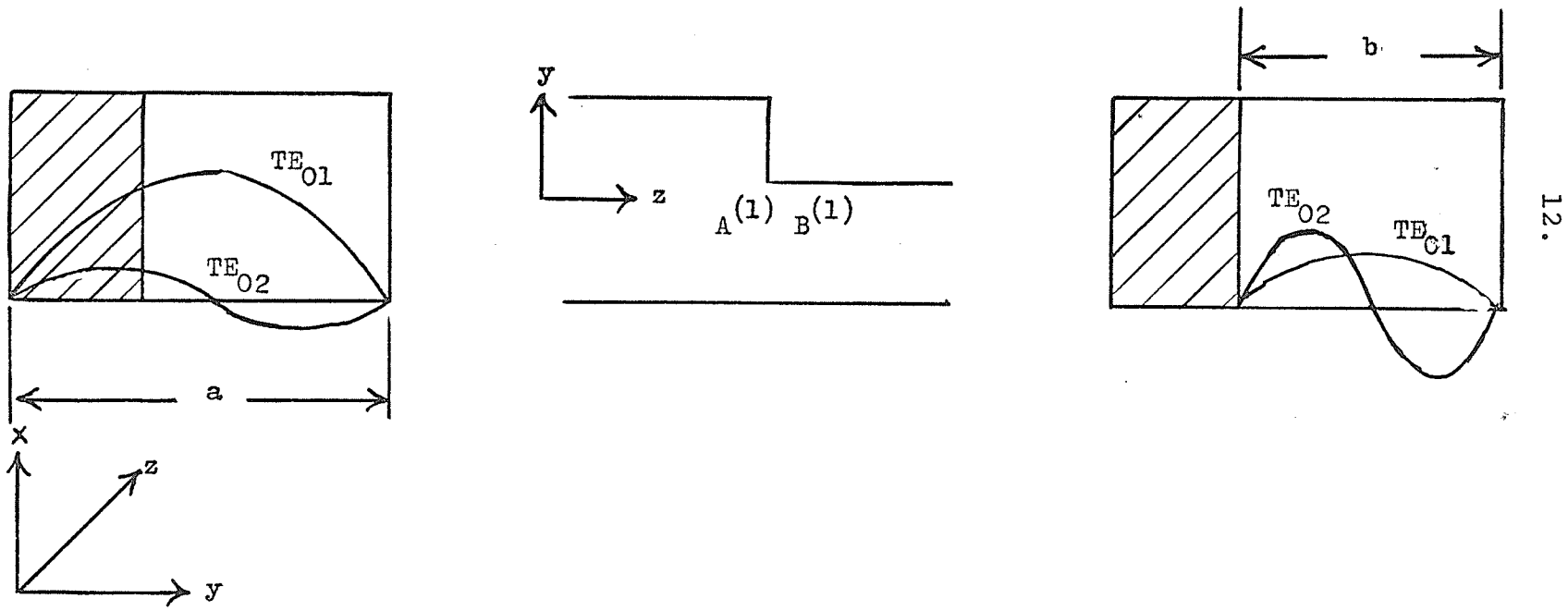

Fig. 3.1 End and top views of junction showing magnitude of electric field on opposite sides of a junction

and H fields for the i-th mode in guide $A^{(1)}$ are

$$\bar{e}_{ai} = \frac{j\omega\mu}{k_{ci}^2} k_{yi} \sin k_{yi} y \, \tilde{u}_x \tag{3.1}$$

$$\bar{h}_{ai} = \frac{\gamma_i}{k_{ci}^2} k_{yi} \sin k_{yi} y \, \tilde{u}_y \tag{3.2}$$

where

$$k_{yi} = \frac{i\,\pi}{a} \tag{3.3}$$

$$\gamma_i^2 = k_{yi}^2 - \omega^2 \mu \varepsilon \tag{3.4}$$

and

$$k_{ci}^2 = k_{yi}^2 \tag{3.5}$$

and the frequency ω is in radians/sec.

Similar expressions hold for the i-th mode of $B^{(1)}$ but here

$$k_{yi} = \frac{i\,\pi}{b} \tag{3.6}$$

3.2 Integral Calculations

The integrals of (2.7-2.9) and (2.11) are calculated numerically rather than analytically as done by Wexler (5). Even though analytic expressions for α_i and β_i could be obtained for the two dimensional steps discussed later, a general expression for δ_{ij} would be practically impossible for these configurations.

The integration routine used was a modified version of the

IBM scientific subroutine package program QATR which uses mesh halving with Simpson's trapezoidal rule and Richardson's extrapolation using two sucessive steps. By calculating the difference between each step a measure of the accuracy can be obtained. The integrands of (2.7) and (2.11) behaved in such a way that the rounding error did not decrease monotonically for every mesh halving. All calculations were done in single precision. For this reason checks had to be built in to be sure that rounding error was indeed larger than discretization error. In all test cases the integration results came to within five significant figures of the exact answer for all mode types used. For the higher order modes used, the discretization error became more of a problem and hence more mesh halving was necessary and thus rounding error limited the accuracy of the results. For the maximum mode order used here, ie. 30, no accuracy problems were encountered.

3.3 Semi-Infinite H-Plane Reduction

As a test case for the program, the step in Fig. 3.2 was studied. With an incident TE_{01} mode, the mode types excited are TE_{on} with n = 1, 3, 5, ... All calculations are done using standard X-band guides, with no loss of generality, as all examples can be scaled.

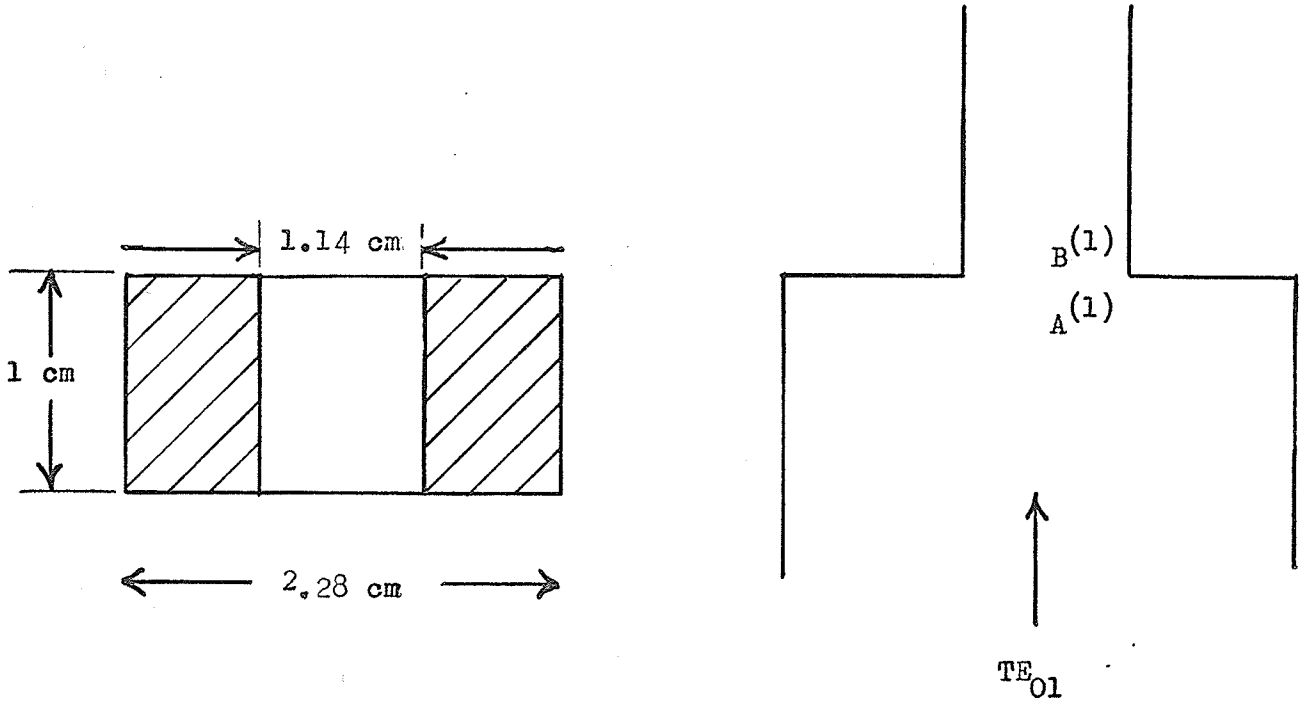

Fig. 3.2 H-plane step. End and top view

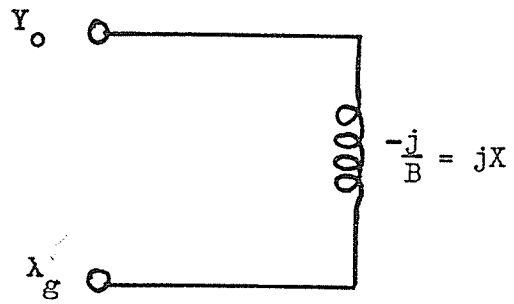

Fig. 3.3 Equivalent circuit of H-plane step of Fig. 3.2
Y_o is guide admittance and λ_g is guide wavelength

		From Modal Analysis		
f(GHz)	Marcuvitz	M=N=3	M=N=10	M=N=15
8	-4.46	-4.211	-4.430	-4.441
9	-3.13	-2.934	-3.093	-3.101
10	-2.28	-2.198	-2.323	-2.330
11	-1.77	-1.665	-1.788	-1.773
12	-1.17	-1.207	-1.292	-1.290

Table 3.1 Equivalent circuit for H-plane step. TE_{03} mode propagate at 19.65 GHz. and cutoff for TE_{01} mode in section $B^{(1)}$ is 13.1 GHz.

Convergence for this type of junction was quite rapid and the error due to the number of modes used decreased with increasing M and N. For M = N = 3, 10, and 15 the calculated normalized impedence of the equivalent circuit of Fig. 3.3 given by Marcuvitz (7) are compared in Table 3.1. Marcuvitz estimates the error to be less than 1%. The small real part of the admittance is due to the rounding error of the equation solution and should in theory be exactly zero, that is, it has a reactive load even for finite M and N, since the guide is lossless and is in cutoff after the junction.

3.4 Transverse Field Plots

The total field expressions of (2.1) to (2.4) can be used to produce plots of the E and H fields on the generator and load side of the junction. For a frequency of 10 GHz, the E and H magnitudes are shown in Fig. 3.4 for the three modal series truncations.

The E and H fields in guide $B^{(1)}$ are defined to be zero from the equation to the left of the conductor wall at y = 0.5715 cm. and in both guides the fields are symmetric about the center line at y = 1.143 cm.

The E field shows a progressively better match within the guide as the number of modes increases. The H field however is

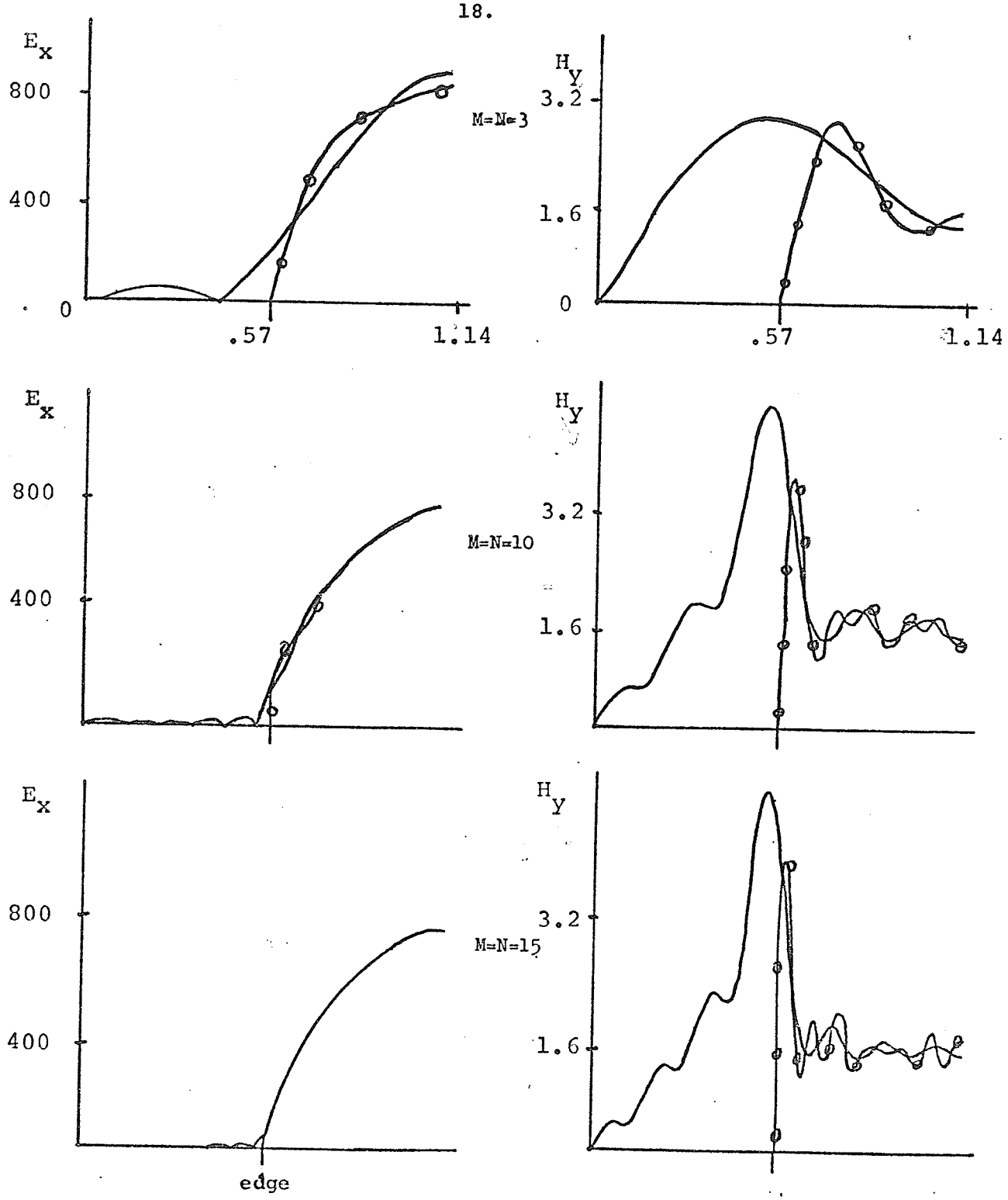

Fig. 3.4 Magnitudes of field match electric fields on right magnetic on left. Solid curve field in section A, circled curve section B

not as good a match. There is a singularity at the metal edge, the field going to infinity. It can be seen that the H field in guide $A^{(1)}$ rises on the face of the edge while in guide $B^{(1)}$ the field rises inside the guide. As the number of modes increases the peaks become higher and closer to the edge. The H field in $B^{(1)}$ has particular trouble since the mode functions that make up the total are each zero at the edge but the sum at the edge should be infinite in the limit. Note that the H field in $A^{(1)}$ on the face of the edge is not zero as is the E field.

3.5 Boundary Enlargement

As a prelude to solving the thick iris, the semi-infinite H-plane enlargement of Fig 3.5 was studied. Guide $A^{(1)}$ was excited by a TE_{01} mode at a frequency of 16 GHz., so that it was not in cutoff. The resulting normalized input admittance for a matched termination on $B^{(1)}$ and M = N = 10 was

$$y = 1.1810 - j0.3999 \qquad (3.7)$$

Here the real part results from power flowing down guide $B^{(1)}$ and has unit magnitude with respect to the characteristic guide admittance in $B^{(1)}$. The equivalent circuit, shown in Fig. 3.6, omits the shift of reference plane for guide $B^{(1)}$ since it is

0.002 cm towards the generator as calculated by Marcuvitz (8) and is negligible.

A check of the solution can be obtained by finding the input admittance of the same junction excited from the opposite side and terminated in a matched load. The configuration is shown in Fig. 3.2 but the frequency was kept at 16 GHz., since cutoff for this section is at 13.1 GHz. The normalized input admittance using M = N = 10 for this reduction problem was

$$y = 0.8381 - j.03325 \tag{3.8}$$

This circuit is shown in Fig. 3.7. The real part of (3.8) is the matched load seen by the junction and is thus $\frac{Y_o'}{Y_o}$ for the reduction. The real part in (3.7) is again the load on the junction and is $\frac{Y_o'}{Y_o}$ for a boundary enlargement. Since the real part of (3.7) and (3.8) are not quite inverses, an indication of the error can be obtained. The reduction admittance ratio $\frac{Y_o'}{Y_o}$ which is the real part of (3.7) can be checked by an expression from Marcuvitz (9). The impedence ratio is given as

$$\frac{Z_o'}{Z_o} = \frac{Y_o}{Y_o'} = 1.2 \tag{3.9}$$

The inverse of this is 0.834 and the limit of error is given as 10%. This inverse however compares favorably with the real part of (3.8) as does the ratio in (3.9) compare with the real part of (3.7). Thus the imaginary part divided by the real part in (3.7)

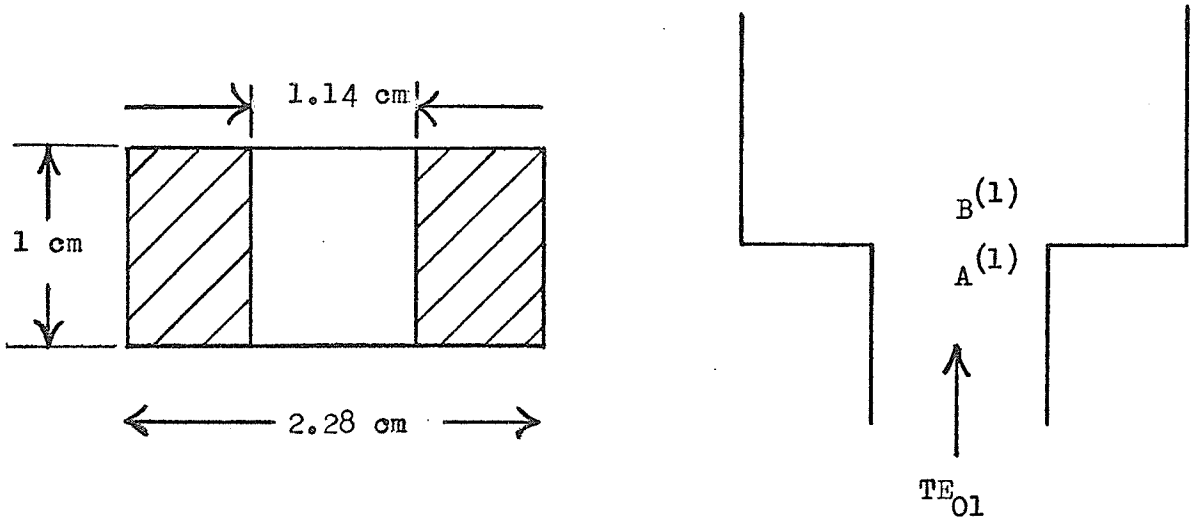

Fig. 3.5 H-plane step end and top views

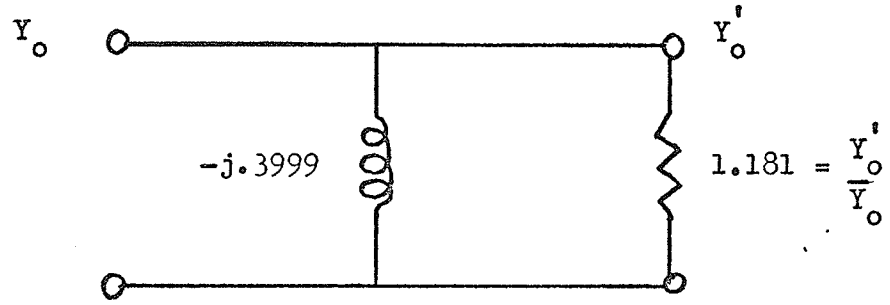

Fig. 3.6 Equivalent circuit of H-plane step showing normalized admittance with respect to guide A. The admittance in B is Y_o'.

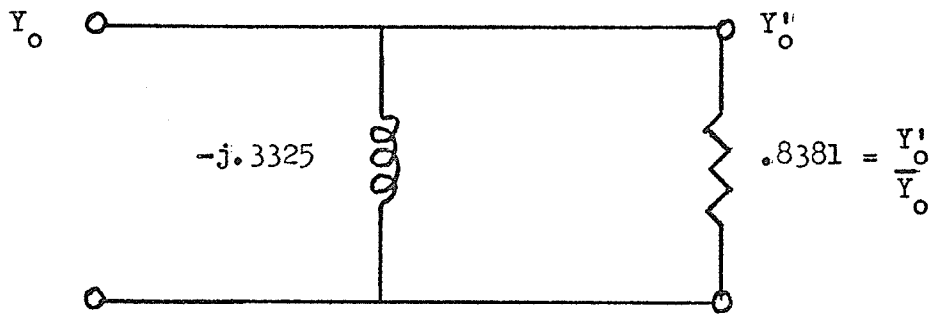

Fig. 3.7 Equivalent circuit of H-plane step f = 16 GHz.

gives the imaginary part of (3.8) and vice versa since (3.7) can be written as

$$y = \frac{Y_o'}{Y_o} - j\frac{B}{Y_o}$$

and

$$\frac{\frac{B}{Y_o}}{\frac{Y_o'}{Y_o}} = \frac{B}{Y_o'} \qquad (3.10)$$

The configuration of Fig. 3.5 was tested for f = 10 GHz with the wave impinging on the discontinuity in cutoff. The normalized input admittance was

$$y = 0.5786 + j0.6655 \qquad (3.11)$$

There are no available results to compare with. Here the normalized admittance Y_o is a negative imaginary number and the characteristic impedence ratio is

$$\frac{Y_o'}{Y_o} = j\ o\ .\ 6655 \qquad (3.12)$$

with Y_o' real. The real part of the admittance represents an inductance which can be seen if it is normalized with respect to guide $B^{(1)}$.

$$\frac{B}{Y_o} = \frac{.5715}{\frac{Y_o'}{Y_o}} = \frac{0.5786}{j\ .\ 06655} = -\ j\ .\ 867 \qquad (3.13)$$

The enlargement studied here will be used next in the study of irises.

3.6 Irises

The thick symmetrical iris of Fig. 3.8 was studied by considering it to be two coupled discontinuities of the type of Sections 3.3 and 3.5. A π equivalent circuit of the iris at the reference planes T, as given by Marcuvitz (10), is shown in Fig. 3.9. Note that Y_{12} is defined to be positive here. Due to the symmetry of the iris there are only two parameters to be determined for the equivalent circuit. Since the circuit is lossless, an open and a short circuit at the load plane T will completely determine Y_{11} and Y_{12}.

For a given load, the discontinuity at the load side is solved first and with M = N = 10 the back scattering coefficients for the ten lowest modes incident on the junction are found. Then these are transformed through the distance ℓ and the load side discontinuity with an incident TE_{01} mode is solved.

For the first junction, the s_{jk} terms are set to correspond to the type of load required. For a short circuit at the load side T

$$s_{jj} = - e^{-(\lambda g/2)(2\gamma_j)}$$

and

$$s_{jk} = 0 \quad , \quad j \neq k \tag{3.14}$$

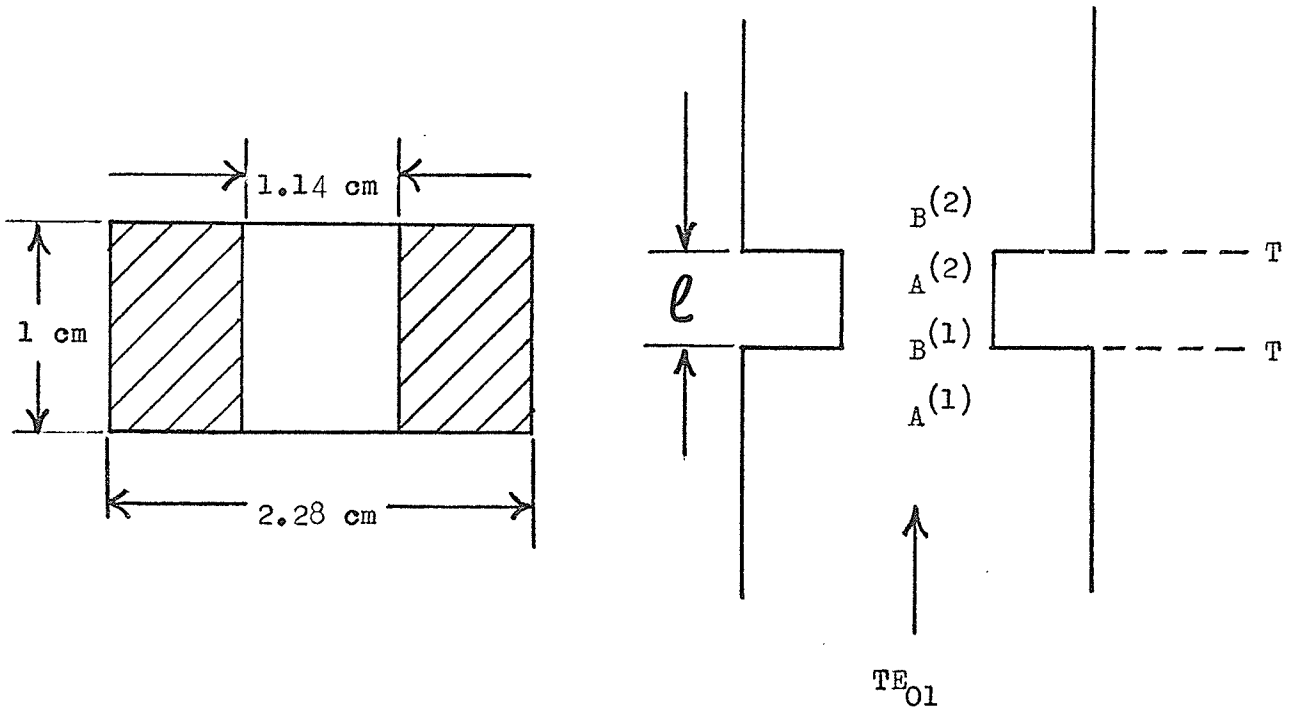

Fig. 3.8 Symmetrical iris, end and top view

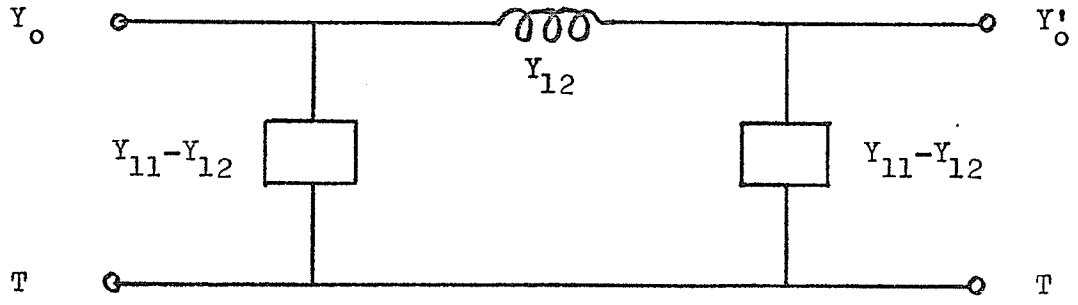

Fig. 3.9 Equivalent circuit of thick iris of Fig. 3.8

and for an open circuit

$$s_{jj} = - e^{-(\lambda g/4)(2\ \gamma_j)}$$

and

$$s_{jk} = 0 \quad , \quad j \neq k \qquad (3.15)$$

Approximate results are given by Marcuvitz (11) but no maximum limit is set on the error. For ℓ = 2 cm. and f = 10 GHz, the value given for $Y_{11}-Y_{12}$ and Y_{12} are j0.434 and -j0.594 respectively. The resulting values from modal analysis are j0.401 and -j0.560.

In order to approximate a zero thickness iris the thickness ℓ is set small but not exactly zero. The difficulty is that the slow convergence for thin vanes, as pointed out by Wexler (12), apply here also and so the thickness of the iris was reduced until numerical problems arose. Then tests were run on the thinnest case that gave no numerical problems.

It was found that an iris with thickness 0.03 cm. gave an equivalent circuit of Fig. 3.10 with negligibly small upper arms values, while still giving good convergence in modal analysis. A thickness of half that did not converge. Note that the reference planes are at T, not at the center of the iris as in Marcuvitz (13), so that the upper arms are thus inductive rather than capacitive. Table 3.2 gives a comparison of the thin iris circuit values for f = 10 GHz and ℓ = 0.03 cm. The upper-arm values have converged to j0.019 and Z_{12} is converging to a value that is within 10% of the zero thickness iris. Any attempt to

converging to a value that is within 10% of the zero thickness iris. Any attempt to decrease the thickness resulted in more and more modes required for convergence.

An extension of the modal analysis method is presented by Masterman, Clarricoats and Hannaford (15) for the solution of zero thickness irises. It involves writing one set of field functions in the iris that must match the normal waveguide modes just outside the iris. They use normal waveguide mode functions but with no impedence defined between the E and H fields, that is, seperate series for the electric and magnetic fields. Their results are accurate for only one ratio of functions in the aperture, to functions outside the aperture. The value of this ratio must be determined experimentally and an empirical rule is given to approximate this ratio. The justification for this method is that the seperate series for the magnetic field allows it to indepently approximate the field. The expressions for the fields outside the aperture are not independent and thus can give no better approximation. The sucess of the method may be due to the optimum M/N ratio rather than the aperture functions. It is noted that the aperture functions are already orthogonal to each other, thus making the method simpler than in Chapter six.

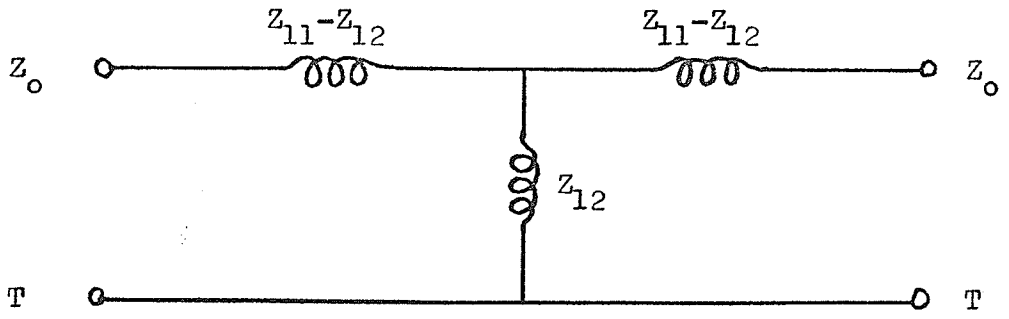

Fig. 3.10 Equivalent circuit of thin iris of Fig. 3.3

Iris of		$Z_{11}-Z_{12}$	Z_{12}
zero thickness Marcuvitz		0	j.637
Modal	M=N=8	j.019	j.615
	M=N=10	j.019	j.603
	M=N=12	j.019	j.596

Table 3.2 Equivalent circuit parameter values of Fig. 3.10 Thickness was .03 cm. Values are normalized to the guide admittance.

4 SOLUTION OF THE E-PLANE STEP

As a parallel to the study of the H-plane step, the E-plane step is investigated with similar configurations as laid out in the previous chapter. An exact expression for the impedence is available for the change of height which gives another independent check of the results. Limitations are indicated for the degree of accuracy for steps and irises.

4.1 Mode Types Excited

The change of height, or semi-infinite E-plane step, is shown with the co-ordinate system given in Fig. 4.1. If there is an incident TE_{01} mode in guide $A^{(1)}$, then because of the symmetry in the y direction, the mode types excited are transverse electric to x, TE_{xml}, or LSE_{ml} modes. If the discontinuity is symmetrical about a center line in the y direction, then m is even only. These TE_{xml} modes can also be expressed as the sum of TE_{ml} and TM_{ml} modes for each m. To keep the program general the ordinary TE_{ml} and TM_{ml} modes are used rather than using the TE_x modes.

The transverse E and H field functions for a TE_{ml} and TM_{ml}

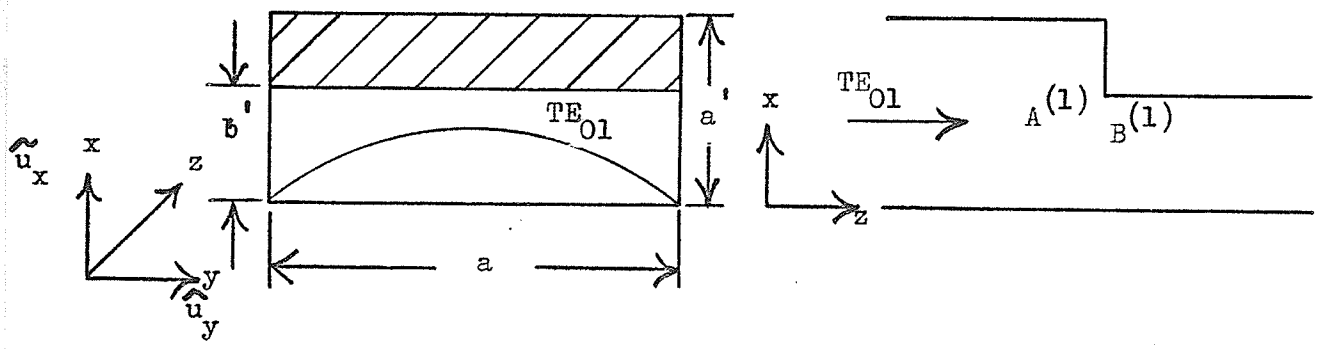

Fig. 4.1 E-plane step with incident TE_{01} mode

mode are then the following:

TE_{ml} modes

$$\bar{e}_{ai} = \frac{j\omega\mu}{k_{cml}^2} (k_{yi}\cos k_{xm}x \sin k_{yl}y\tilde{u}_x - k_{ym}\sin k_{xm}x \cos k_{yl}y\tilde{u}_y) \quad (4.1)$$

$$\bar{h}_{ai} = \frac{\gamma_{ml}}{k_{cml}^2} (-k_{xm}\sin k_{xm}x \cos k_{yl}y\tilde{u}_x - k_{yl}\cos k_{xm}x \sin k_{yl}y\tilde{u}_y) \quad (4.2)$$

and TM_{ml} modes

$$\bar{e}_{aj} = \frac{\gamma_{ml}}{k_{cml}^2} (k_{xm}\cos k_{xm}x \sin k_{yl}y\tilde{u}_x - k_{yl}\sin k_{xm}x \cos k_{yl}y\tilde{u}_y) \quad (4.3)$$

$$\bar{h}_{aj} = \frac{j\omega\varepsilon}{k_{cml}^2} (k_{yl}\sin k_{xm}x \cos k_{yl}y\tilde{u}_x - k_{xm}\cos k_{xm}x \sin k_{yl}y\tilde{u}_y) \quad (4.4)$$

where

$$k_{xm} = \frac{m\pi}{a'} \qquad k_{yl} = \frac{\pi}{a}$$

$$\gamma_{ml}^2 = k_{xm}^2 + k_{yl}^2 - \omega^2\mu\varepsilon \quad (4.5)$$

$$k_{cml}^2 = k_{xm}^2 + k_{yl}^2$$

Similar modes are excited in guide B$^{(1)}$ but with

$$k_{xm} = \frac{m\pi}{b'} \quad (4.6)$$

In guide B$^{(1)}$ there is also a TE_{01} mode.

4.2 Integral Calculations

Since the modes are two dimensional functions, a two

dimensional integration is now required. However the modes described in (4.1-4.4) are always products of x and y functions, and thus the double integration can be broken into the the sum of the product of two one-dimensional integrations. The result is that the number of function evaluations is proportional to the sum of the grid evaluations in each co-ordinate direction rather than the product. This simplication can be used in most rectangular and circular problems. The integrations are then calculated as described in Sec. 3.2 since now the functions involved are exactly the same.

4.3 Semi-infinite Symmetrical E-plane Reduction

Modal analysis was applied to the E-plane step of Fig. 4.2 to obtain the reflection coefficient of the TE_{01} incident mode. The mode types excited here are TE_{m1} and TM_{m1} (m = 2, 4, 6,...) in guide $_A{}^{(1)}$ and TE_{01} plus TE_{m1} and TM_{m1} (m = 2, 4, 6,...) in guide $_B{}^{(1)}$.

Convergence for this step was not as rapid as with the H-plane step. In addition there are twice as many modes required to get the same mode order since both TE and TM modes are used. In spite of this the equivalent circuit obtained, using only the ten lowest order modes, agrees well with standard references.

The circuit of Fig. 4.3 has a capacitive value within 3% of Marcuvitz (16) which is within the error limit given in the reference for f = 10 GHz.

The characteristic impedence ratio is given by the exact expression

$$\frac{Y'_o}{Y_o} = \frac{b}{b'} \qquad (4.7)$$

The conductance for the matched load case, of Fig. 4.2, is 1.5 mhos which is exactly the value of the impedence ratio

$$\frac{b}{b'} = 1.5 \qquad (4.8)$$

This indicates that the real part has converged faster than the imaginary part.

4.4 Boundary Enlargement

As in Chapter 3, the boundary enlargement problem must be solved before most multi-junction problems are handled. The step of Fig. 4.4 was solved using the ten lowest modes again. The resulting equivalent circuit with the load corresponding to a matched load is shown in Fig. 4.5.

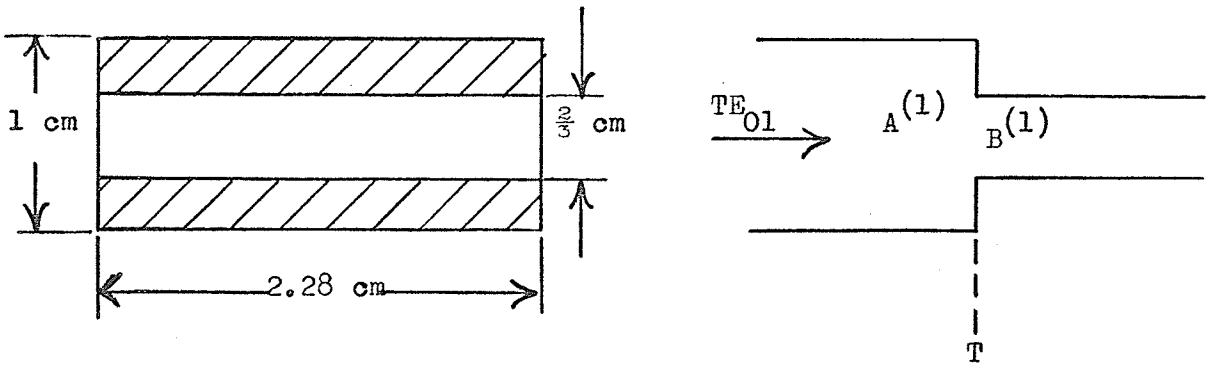

Fig. 4.2 Symmetrical E-plane step. End and side view of reduction

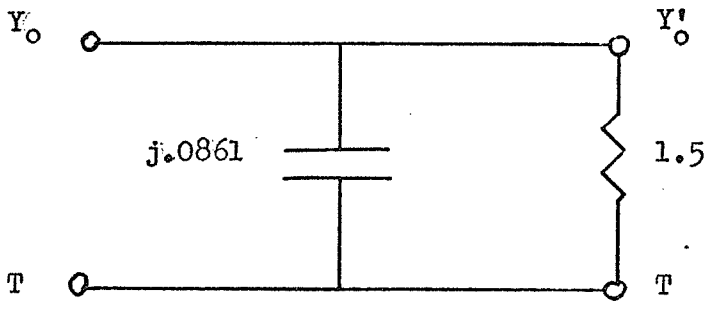

Fig. 4.3 Equivalent circuit of E-plane step with load

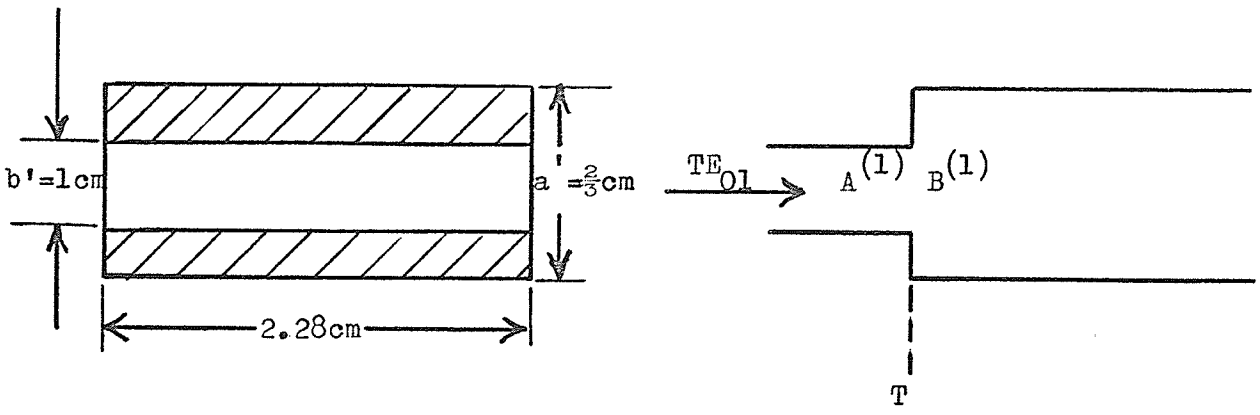

Fig. 4.4 E-plane step enlargement. End and side view

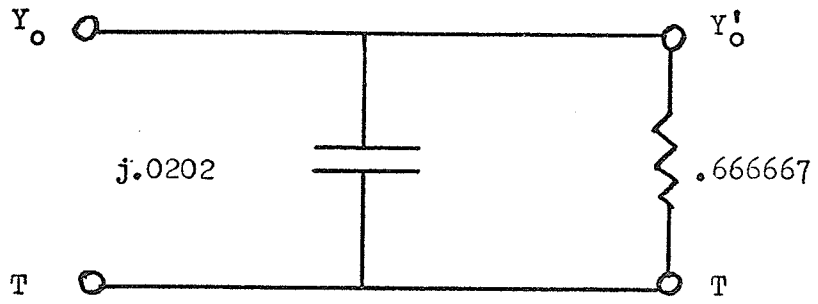

Fig. 4.5 Equivalent circuit of E-plane enlargement with matched load

4.5 Irises

The symmetrical iris shown in Fig. 4.6 is studied as in Section 3.6 where it is considered as two coupled simple discontinuities. The equivalent circuit is given in Fig. 4.7. Due to the symmetry of the iris about a transverse plane, there are again only two parameters to be determined, Y_{12} and $Y_{11} - Y_{12}$. An open and short circuit termination are placed at the load reference plane T and the reflection coefficient of the unit incident TE_{01} mode is found for each case. The s_{jk} terms are determined as in (3.13) and (3.14).

For M = N = 12 the equivalent circuit obtained is compared to the circuit from Marcuvitz (17) in Fig. 4.8. The difference in circuit values is well within the upper limit of error as stated by Marcuvitz.

In an attempt to approximate the zero thickness iris, the T equivalent circuit is found rather than the π for comparison purposes. In this case convergence is more of a problem than with the inductive iris. The circuit values for M = N = 16 are obtained. The circuit given by Marcuvitz (18) is shown with the circuit obtained by modal analysis for ℓ = 0.03 cm. in Fig. 4.9. The values of the upper arms are calculated by modal analysis to be zero so are not shown. The lower arm member agrees to within

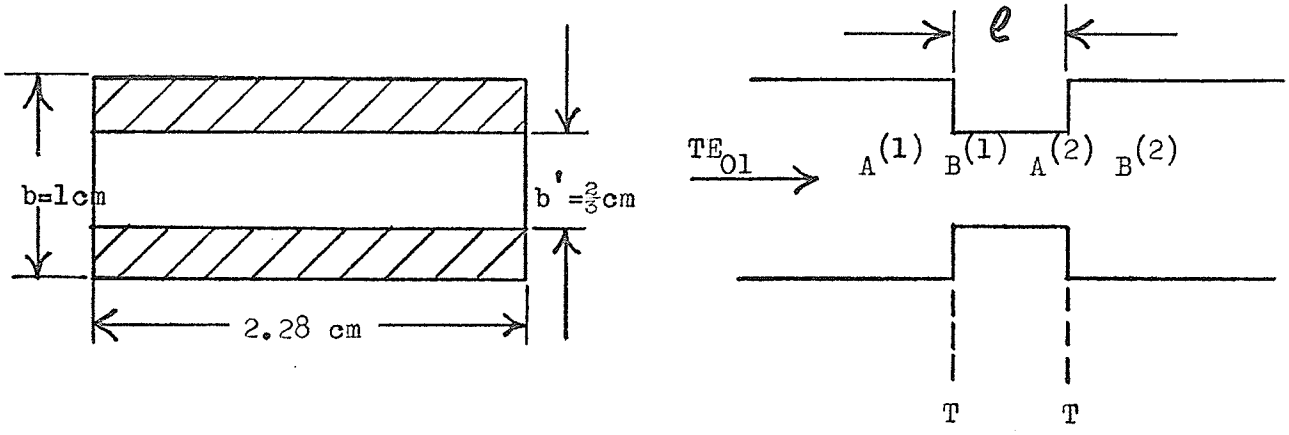

Fig. 4.6 Thick symmetrical iris, end and side view

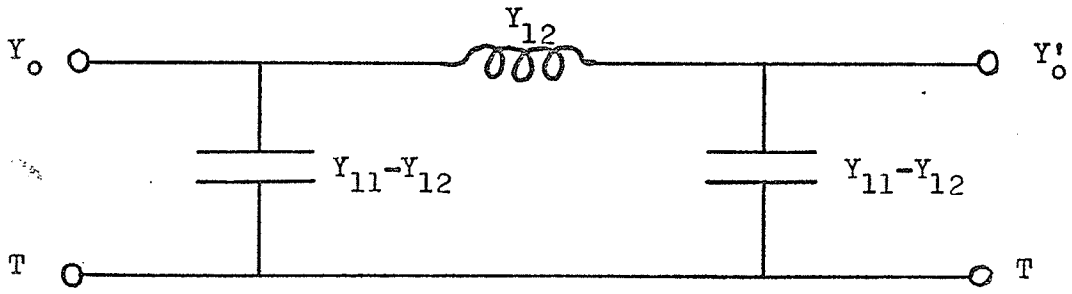

Fig 4.7 Equivalent circuit of thick capacitive iris

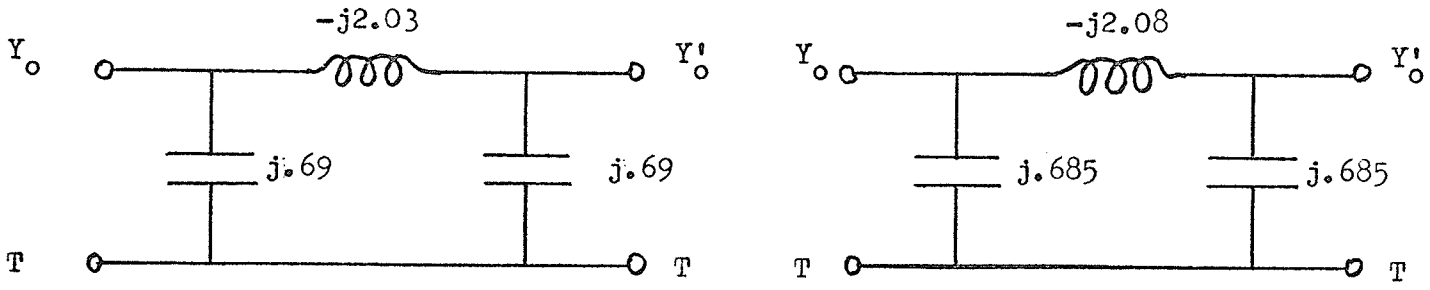

Fig. 4.8 Equivalent circuits from Marcuvitz (17) and modal analysis on right. Thick iris case.

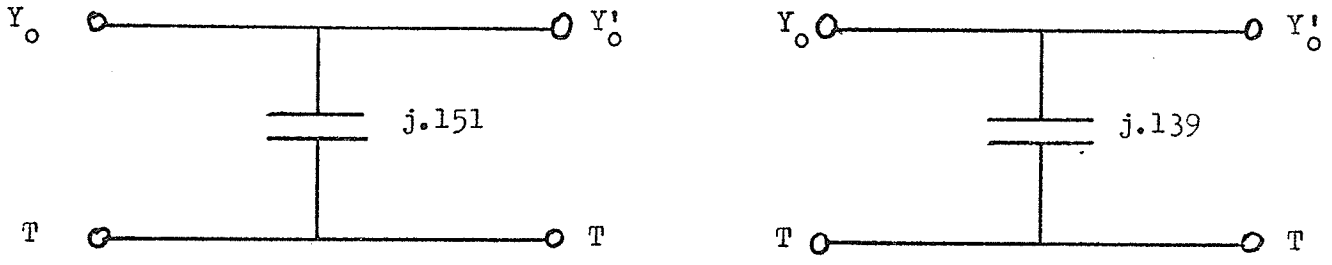

Fig. 4.9 Equivalent circuits from Marcuvitz (17) and modal analysis on right. Thin iris case.

be zero so are not shown. The lower arm member agrees to within 8%.

5 TWO DIMENSIONAL STEP

The combined E and H-plane step is of considerable interest but there are no general analytic techniques available to solve it. The iris composed of two such steps is often used in guides since it can be made to resonate at a particular frequency by a suitable choice of dimensions. Comparison is made with an empirical rule given for the construction of resonant windows. It is shown that it is feasible for the modal analysis method to give the dimensions of a window.

5.1 Mode Types Excited

The two dimensional semi-infinite step is shown in Fig. 5.1. With an incident TE_{01} mode in the z direction, the modes excited on both sides of the junction are TE_{mn} with m = 0, 1, 2,... and n = 1, 2, 3,... and when $m \neq 0$ TM_{mn}. If there is symmetry about a central axial y-z plane then n is odd. The transverse E and H field patterns for an i-th TE_{mn} and a j-th TM_{mn} mode in guide $A^{(1)}$ are the following:

TE_{mn} modes

$$\bar{e}_{ai} = \frac{j\omega\mu}{k^2_{cmn}} (k_{ym}\cos k_{xm}x \sin k_{yn}y\tilde{u}_x - k_{xm}\sin k_{xm}x \cos k_{yn}y\tilde{u}_y) \quad (5.1)$$

$$\bar{h}_{ai} = \frac{\gamma mn}{k^2_{cmn}} (-k_{xm}\sin k_{xm}x \cos k_{yn}y\tilde{u}_x - k_{yn}\cos k_{xm}x \sin k_{yn}y\tilde{u}_y) \quad (5.2)$$

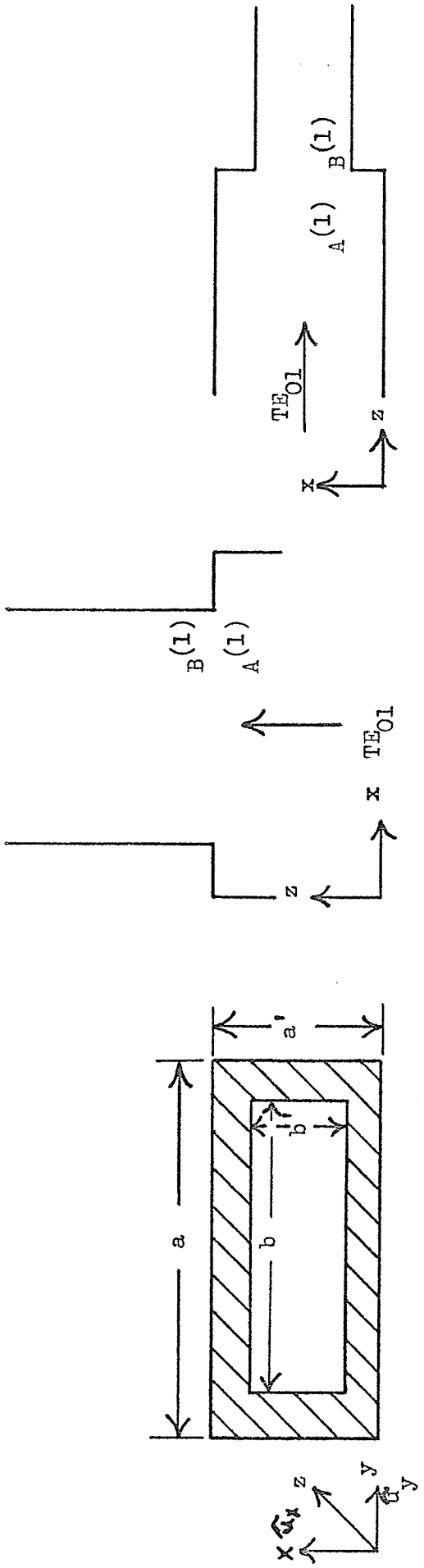

Fig. 5.1 Two dimensional step. End, top and side view

and TM_{mn} modes

$$\bar{e}_{aj} = \frac{\gamma_{mn}}{k^2_{cmn}} (k_{xm}\cos k_{xm}x \sin k_{yn}y\tilde{u}_x + k_{yn}\sin k_{xm}x \cos k_{yn}y\tilde{u}_y) \quad (5.3)$$

$$\bar{h}_{aj} = \frac{j\omega\varepsilon}{k^2_{cmn}} (k_{yn}\sin k_{xm}x \cos k_{yn}y\tilde{u}_x - k_{xm}\cos k_{xm}x \sin k_{yn}y\tilde{u}_y) \quad (5.4)$$

where

$$k_{xm} = \frac{m\pi}{a'} \qquad k_{yn} = \frac{n\pi}{a}$$

$$\gamma^2_{mn} = k^2_{xm} + k^2_{yn} - \omega^2\mu\varepsilon \quad (5.5)$$

$$k^2_{cmn} = k^2_{xm} + k^2_{yn}$$

Similar mode field patterns exist in guide B [1] with

$$k_{xm} = \frac{m\pi}{b'} \qquad k_{yn} = \frac{n\pi}{b} \quad (5.6)$$

Since each mode has two independent variables describing it, m and n, and for each m and n not zero there are two resulting modes TE_{mn} and TM_{mn}, an ordering system has to be applied. The system used was to call TE_{01} the first mode and increment each m and n to a predetermined m_{max} and n_{max} including the corresponding TM mode if it exists. For example if $m_{max} = 4$ and $n_{max} = 3$ and the junction is completely symmetrical, then m is even and n is odd. The resulting 8 modes are TE_{01}, TE_{03}, TE_{21}, TM_{21}, TE_{41}, TM_{41}, TE_{43}, TM_{43} By a suitable choice of m_{max} and n_{max} the mode types can be chosen to cater for more or less severe discontinuities in either the E-plane or the H-plane.

5.2 Integral Calculations

The mode types are essentially the same as in Chapter 4, that is, they are two dimensional functions. The double integrations can again be broken into the product of two one dimensional integrals thus saving considerable calculations.

5.3 Semi-infinite Symmetrical Step

The step of Fig. 5.1 is solved for symmetrical cases only with an incident TE_{01} mode. Here the number of modes required to get a good solution is just less than the square of the number of modes needed for a one dimensional step. For this step the number of modes used was 18 with $m_{max} = 6$ and $n_{max} = 5$. Since b = 1.5 and b' = 0.8, the severity of the step was approximately equal in both directions, hence $m_{max} \simeq n_{max}$. For f = 13 GHz. the resulting input admittance is as indicated in Fig. 5.2. This agrees to within 20% of a very approximate method given by Bandler (19) for the same problem. It is interesting to run the same problem with b = 2.0, b' = 0.7 and f = 13 GHz. The resulting equivalent circuit is shown in Fig. 5.3. Because the steps are relatively small, Bandler's (19) approximation method is more accurate and the results agree to within 2%.

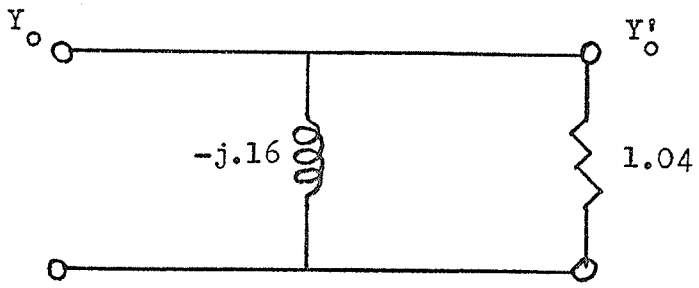

Fig. 5.2 Two dimensional step equivalent circuit with load b = 1.5 cm and b' = .8 cm.

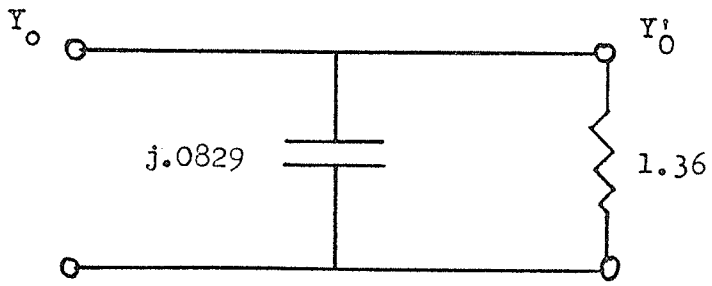

Fig. 5.3 Two dimensional step equivalent circuit with load b = 2 cm and b' = .7 cm

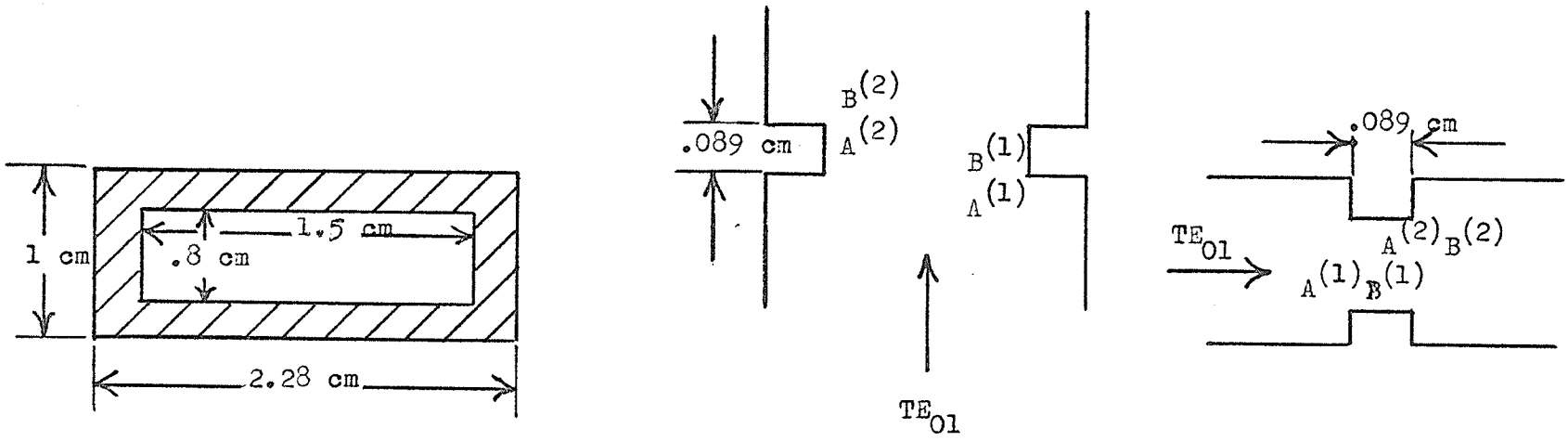

Fig. 5.4 Symmetrical window in waveguide. End, top and side view

These two examples show that this type of step can be either capacitive or inductive depending on the sizes b and b' and also the frequency. The step can be made to resonate at any frequency by the proper choice of parameters. Under this condition the capacitive and inductive components of the step are exactly equal. This leaves no imaginary component.

5.4 Irises

The symmetrical iris or window of Fig. 5.4 is studied by using the single step results of Sec. 5.4. The resonant properties of this particular window are described by Walker (20) and the VSWR is plotted against λ_o, the free space wavelength, in Fig. 5.5. These are experimental results and are thus for a very slightly lossy system and hence the VSWR does not go to exactly one. For modal analysis results the imaginary parts must exactly cancel and so small errors in determining each part of the impedence, ie. capacitance or inductance, lead to large percentage errors in the resulting resonant frequency.

The resulting input impedance of the window is not constant with the number of modes used but jumps around when the frequency is near resonance. Three VSWR curves corresponding to M = N = 16, 18, and 20 are given in Fig. 5.5. The minima for the curves are centered about the experimental minimum thus indicating that modal analysis can give relatively close answers.

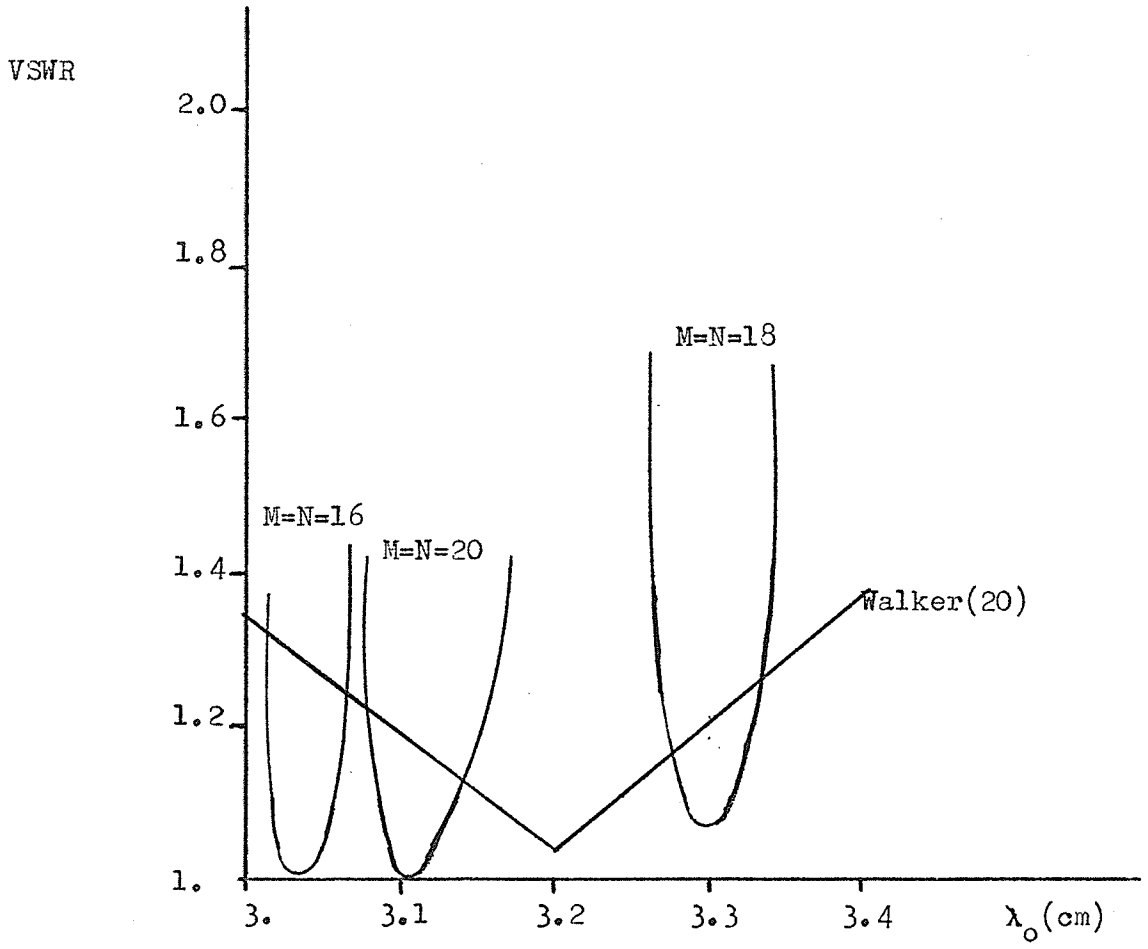

Fig. 5.5 Standing wave ratio of resonant window

6. FIELD SINGULARITIES

The results of the previous chapters have indicated that convergence is rather slow for all configurations when accuracy is desired. The outstanding aspect about the field plots of Chapter 3 is that the H-field goes to infinity near the edge of the discontinuity. A great many modes must be taken into account to accurately reproduce the field. In this chapter a term is introduced to help create the field singularity.

6.1 Numerical Treatment of the Singularity

Fox and Sankar (22) solved a screw propellor problem using finite differences by first ignoring a singularity which was known to occur at a point. The solution showed considerable disturbances at the singularity point and so an alternative method was sought. They composed a special analytic function which was valid only in the neighborhood of the singularity and was completely known in form except for three magnitude coefficients. This function satisfies the differential equation and boundary conditions in the neighborhood of the singularity

point.

The finite difference technique was applied to the problem again; this time leaving out a small region surrounding the singularity. Then the constants of the special expression were solved using three finite difference points. The special function was then used to define the field in the neighborhood of the singularity. This technique provides accurate answers to the problem since they were able to make a comparison with a known analytic solution.

A modification of this technique will be attempted to help the modal analysis method converge to a solution.

6.2 E-Wave on Wedge in Free Space

A wedge in free space is considered next since it will be used in the treatment of waveguide steps. If the right-angled wedge in Fig. 6.1 is in the path of a plane E-wave travelling in the -y direction then the magnetic field near the edge goes to infinity. An E-wave is a polarized plane wave with electric field in the z-direction. There is an edge condition that requiries the energy density to be integrable at the edge. While the magnetic field goes to infinity at the edge,

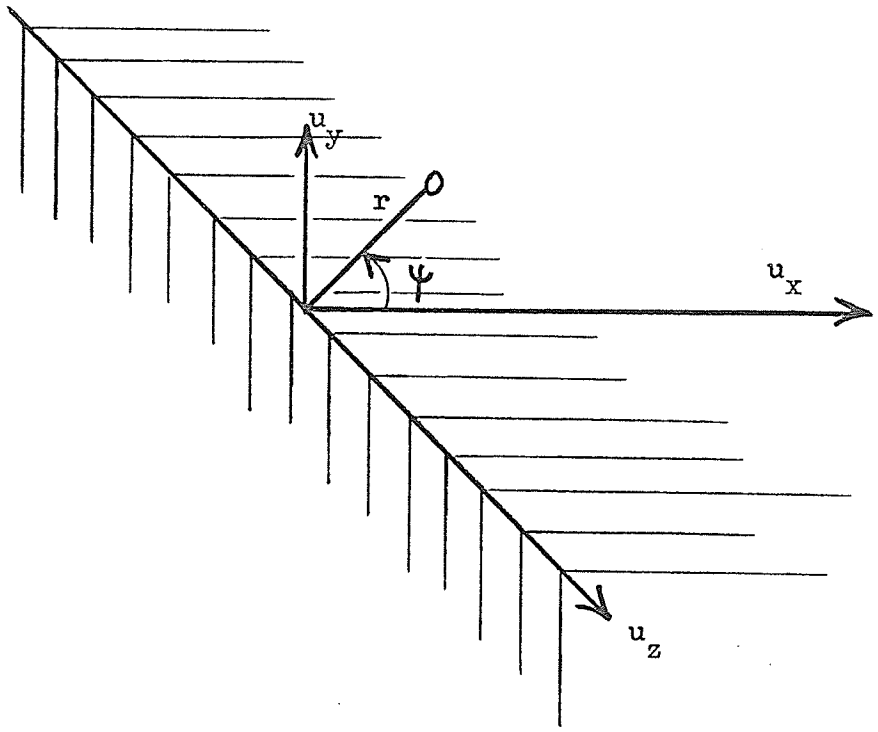

Fig. 6.1 Right angled wedge in free space.

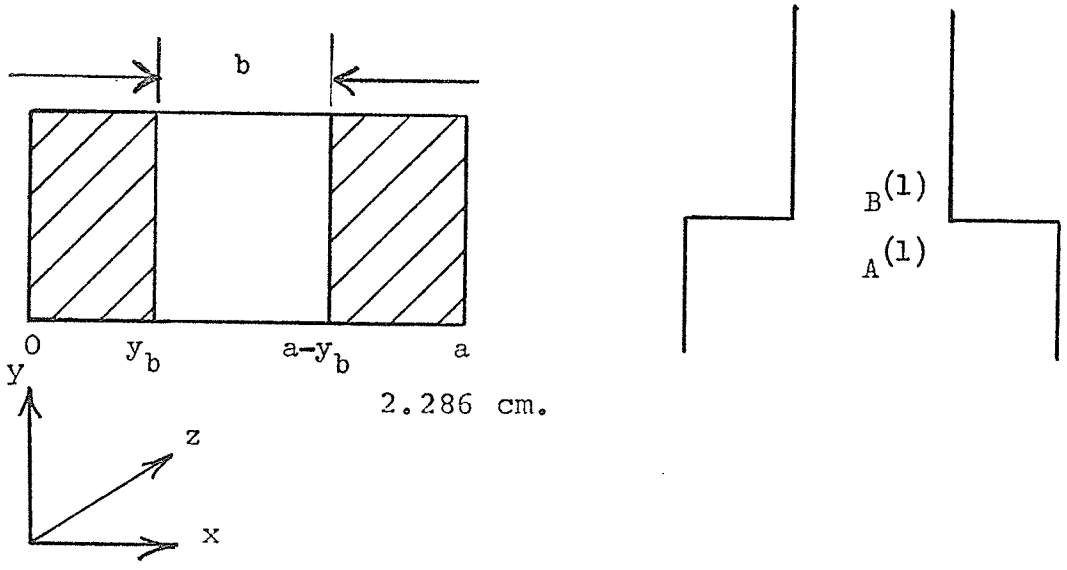

Fig. 6.2 H-plane step end and top view

the electric field remains finite in the vicinity of the edge and vanishes right at the edge. If r-ψ are the polar co-ordinates centered on the wedge edge, the fields in the vicinity of the edge are given by Van Bladel (21) as

$$E_z = A\, r^{2/3} \sin \tfrac{2}{3}\psi \qquad (6.1)$$

and

$$H_t = -\frac{2}{3}\frac{A}{j\omega\mu}\frac{1}{r^{1/3}}\left(\tilde{u}_x \cos \psi/3 + \tilde{u}_y \sin \psi/3\right)$$

These expressions are valid only in free space and are accurate only near the wedge, but they can be used in a stepped waveguide to describe fields very near the edge.

6.3 Special Mode Formulation

The solution of the semi-infinite H-plane step of Ch. 3 is considered again. The step of Fig. 6.2 has two corners in the same orientation as that of Fig. 6.1 with $\psi = 0$ or $\psi = 180°$. By re-writing the electric field part of (6.1) and changing the co-ordinate system to that of the waveguide in Fig (6.2), the E-field near the edge in guide $A^{(1)}$ is represented by a special function E_{Axs}

$$E_{Axs} = \begin{cases} 3/2\ \sqrt{3} j\omega\mu\ A\ (y - y_b)^{2/3} & y_b < y < y_b + \delta \\ 3/2\ \sqrt{3} j\omega\mu\ A\ (a - y_b - y)^{2/3} & a - y_b - \delta < y < a - y_b \\ 0 & \text{Elsewhere} \end{cases} \qquad (6.3)$$

The field in this function is defined to be zero if it is not within an arbitrary distance δ of the edge and when it is tangential to the metal. In this study δ was chosen to be 0.1 cm. for the X-band guide of 2.23 cm. The corresponding H-field special function definition is then

$$H_{Ays} = \begin{cases} \dfrac{A}{(y-y_b)^{\frac{1}{3}}} & y_b \leq y \leq y_b + \delta \\ \dfrac{A}{(a-y_b-y)^{\frac{1}{3}}} & a - y_b - \delta \leq y \leq a - y_b \\ \dfrac{2}{\sqrt{3}} \dfrac{A}{(y-y_b)^{\frac{1}{3}}} & y_b - \delta \leq y \quad y_b \\ \dfrac{2}{\sqrt{3}} \dfrac{A}{(y-a+y_b)^{\frac{1}{3}}} & a - y_b \leq y \leq a - y_b + \delta \end{cases} \tag{6.4}$$

There is a component of the magnetic field tangential to the conductor wall for $\psi = 0°$. These two functions E_{Axs} and H_{Ays} will be treated as one special mode to be used with the other normal waveguide modes.

Any function that is used in the modal analysis technique must be linearly independent of the other functions to be fitted. This is necessary to ensure that the integration of (A.2.1) goes to zero. For this problem the functions E_{sx} and H_{ys} were made orthogonal to the TE_{on} normal waveguide modes by using the Gram-Schmidt procedure to eliminate the mode components in the

special function. This ensures that all functions used are linearly independent since orthogonality guarantees linear independence. If M normal waveguide modes in guide $A^{(1)}$ are used, then the new orthogonalized special functions are

$$E'_{Axs} = E_{Axs} - \frac{2}{a} \sum_{i=1}^{M} \sin k_{ai} y \int_a \sin k_{ai} y \, E_{Axs} \, dy \qquad (6.5)$$

and

$$H'_{Ays} = H_{Ays} - \frac{2}{a} \sum_{i=1}^{M} \sin k_{ai} y \int_a \sin k_{ai} y \, H_{Ays} \, dy \qquad (6.6)$$

6.4 Methods of Integrating With Singularities

The integrations in (6.5) and (6.6) are not given in integral tables and thus must be evaluated numerically as described in Sec 3.2. An added complication is that the integrand in (6.6) goes to infinity at two places at the edge along the y-axis while the integration along the axis is finite. A direct numerical technique fails since the derivitive also goes to infinity at these two places and a finite number of evaluations of the integrand near the edge do not give the resulting integral. The contribution to the integral very near the singularity goes to zero and thus this point may be omitted. The region omitted was arbitrarily chosen as the smallest distance that the word length

of the computer used would allow. The region omitted was 10^{-5} cm. wide and using the Simpson's rule integration technique, results were obtained.

There are however other numerical methods which remove the singularities from the domain of the integrand. Of the four methods suggested by Froberg (23), the substitution method is applicable here. The integrations are carried out only in the vicinity of the edge and so part of the integral of (6.6) becomes

$$\frac{2}{\sqrt{3}} \int_{y_b-\delta}^{y_b} \frac{\sin k_{ai}\, y}{(y-y_b)^{\frac{1}{3}}}\, dy = \frac{-2}{\sqrt{3}} \int_{.1}^{0} \frac{\sin k_{ai}\, (y_b-y)}{y^{\frac{1}{3}}}\, dy \qquad (6.7)$$

Let $y=t^3$ in (6.7) and we have

$$\frac{-2}{\sqrt{3}} \int_{\sqrt[3]{.1}}^{0} \sin[k_{ai}\, (y_b-t^3)]t\, dt \qquad (6.8)$$

The integrand of (6.7) has a singularity while (6.8) has none and the derivitives are regular also. The numerical integration technique was then applied to (6.8) to arrive at the results of Table 6.1.

Special functions are defined for the junction in guide B(1), but here both the electric and magnetic field exist only for $\psi = 180^{o}$ in the plane of the junction. The special fields are given as

$$E_{Bxs} = \begin{cases} \frac{3}{2}\sqrt{3} j\omega\mu\ A\, y^{2/3} & y \leq \delta \\ \frac{3}{2}\sqrt{3} j\omega\mu\ A(b-y)^{2/3} & y \geq b-\delta \\ 0 & \text{Elsewhere} \end{cases} \qquad (6.10)$$

and

$$H_{Bys} = \begin{cases} \frac{A}{y^{\frac{1}{3}}} & y \leqslant \delta \\ \\ \frac{A}{(b-y)^{\frac{1}{3}}} & y \geqslant b - \delta \end{cases} \tag{6.11}$$

As with the functions in the $A^{(1)}$ guide, these special functions have to be made orthogonal to the $B^{(1)}$ guide normal waveguide modes. This is done by

$$E'_{Bxs} = E_{Bxs} - \frac{2}{b} \sum_{i=1}^{N} \sin k_{bi} y \int_{b} \sin k_{bi} y \, E_{bxs} \, dy \tag{6.12}$$

$$H'_{Bys} = H_{Bys} - \frac{2}{b} \sum_{i=1}^{N} \sin k_{bi} y \int_{b} \sin k_{bi} y \, H_{bys} \, dy \tag{6.13}$$

where N is the number of normal modes used in waveguide $B^{(1)}$.

If the number of normal modes is nine, and one special mode is used, then the nine coefficients of the $\sin(k_{ai}y)$ and $\sin(k_{bi}y)$ terms are given in Table 6.1 for the special functions in the form

$$E'_{Axs} = E_{Axs} - \sum_{i=1}^{M} E_{Ai} \sin k_{ai} y \tag{6.14}$$

$$H'_{Ays} = H_{Ays} - \sum_{i=1}^{M} H_{Ai} \sin k_{ai} y \tag{6.15}$$

$$E'_{Bxs} = E_{Bxs} - \sum_{i=1}^{N} E_{Bi} \sin k_{ai} y \tag{6.16}$$

$$H'_{Bys} = H_{Bys} - \sum_{i=1}^{N} H_{Bi} \sin k_{ai} y \tag{6.17}$$

These functions can be used in the modal analysis method since

i	E_{Ai}	H_{Ai}	E_{Bi}	H_{Bi}
1	.01729	.8651	.07715	.12373
2	.01132	.8525	.02184	.35454
3	-.02088	-.7944	.03231	.53853
4	-.00399	-.7874	.03754	.65443
5	.02152	.6802	.03699	.69359
6	-.00343	.6714	.03127	.66103
7	-.01923	-.5308	.02192	.57363
8	.00968	-.5237	.01109	.45599
9	.01457	.3697	.00099	.33487

Table 6.1 Coefficients to orthogonalize special functions from (6.14) to (6.17). For wedge.

they are now linearly independent of the normal waveguide modes.

6.5 Modified Solution of H-Plane Step

The H-plane step is solved using nine regular wave-guide modes and the functions (6.14)-(6.15) and (6.16)-(6.17) as one mode each. The first step is to calculate the integrals of (1.7) to (1.9) which involve these special functions. The method of calculation was essentially the same as described in Sec. 6.4. Using these values plus the regular integrals, a solution to the step is obtained by the modal analysis method. The results are compared with the results for M = N = 10 and 15 for f = 10 GHz. using normal waveguide modes. Two results from Table 3.1 are summarized here along with the special function solution.

$$Y = -j2.323 \quad M = N = 10$$

$$Y = -j2.330 \quad M = N = 15$$

$$Y = -j2.328 \quad \text{special } M = N = 9$$

The magnitudes of the H-field at the junction are plotted for this solution in Fig. 6.3. For comparison purposes the corresponding plot of the magnetic field match from Fig. 3.4 for M = N = 15 is shown above it. As can be seen the field match near the metal edge is very close as compared to the normal mode method. There is however a mismatch in field values where the special function starts on each side of the metal edge. The effect of the special function on the electric field is small and

there is no difference in the already near perfect electric field match and for this reason it is not plotted.

Assuming that the normal solution for M = N = 15 is more accurate than the normal solution for M = N = 10, one can see a slight improvement using the special mode formulation. The result does not, however, appear as accurate as the M = N = 15 case. To clarify the potential of the special mode method to give improved results, tests were run on the semi-infinite symmetrical bifurcation for which an exact solution is known.

6.6 Modified Solution of the H-Plane Bifurcation

The semi-infinite H-Plane bifurcation shown in Fig. 6.4 was solved by Wexler(24), using modal analysis. The solution for a large number of modes compared favorably with the exact solution given by Marcuvitz(25).

To derive a special function needed for this solution, consider an incident E-wave on the thin edge with electric field parallel to the edge in Fig. 6.5. The solutions for the fields near the edge are again given by Van Bladel(26) as

$$E_z = A\, r^{\frac{1}{2}} \sin \psi/2$$

$$H_t = -\frac{A}{2j\omega\mu r^{\frac{1}{2}}} (u_x \cos \psi/2 + u_y \sin \psi/2) \qquad (6.18)$$

These equations are quite similar to (6.1), since here also the

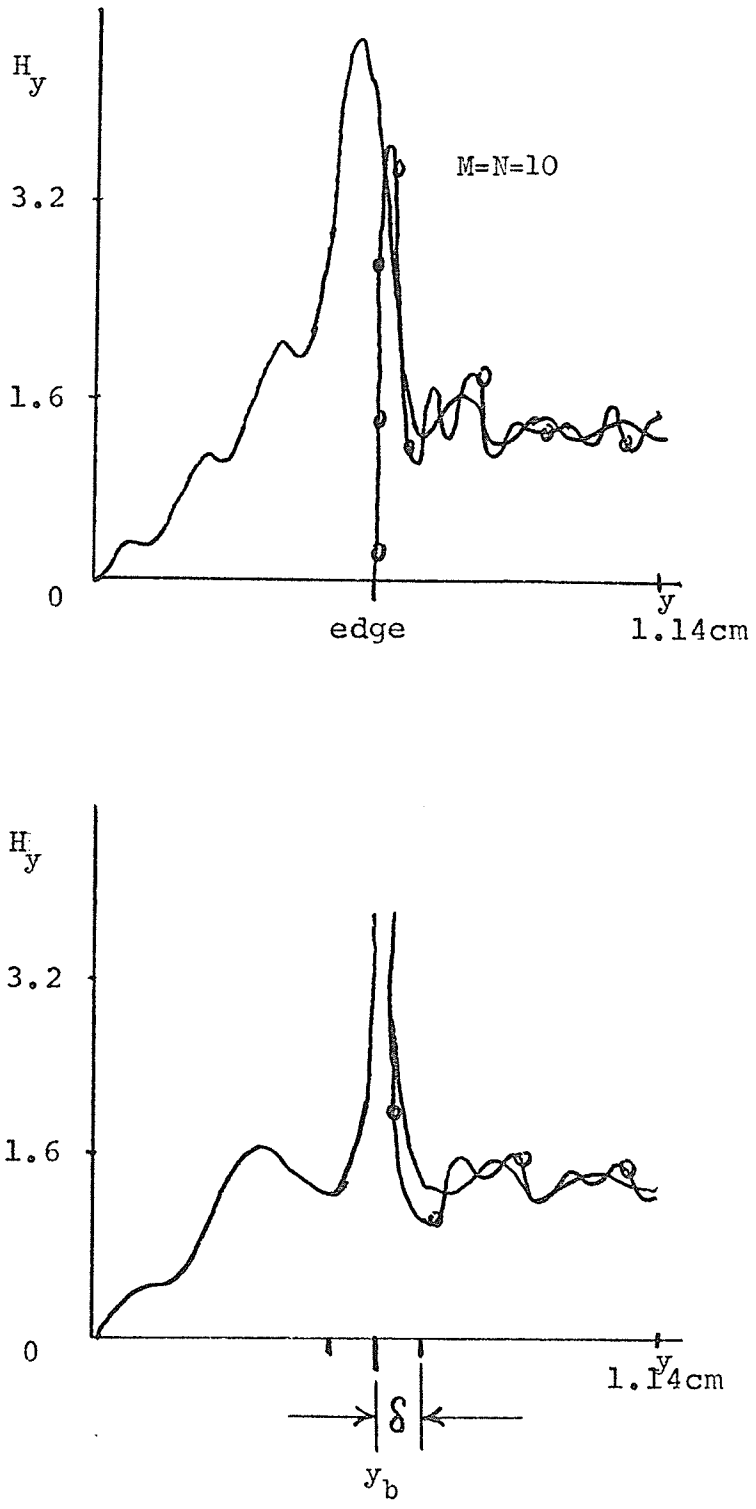

Fig. 6.3 Magnitude of magnetic field showing match with modal method above and special function method below

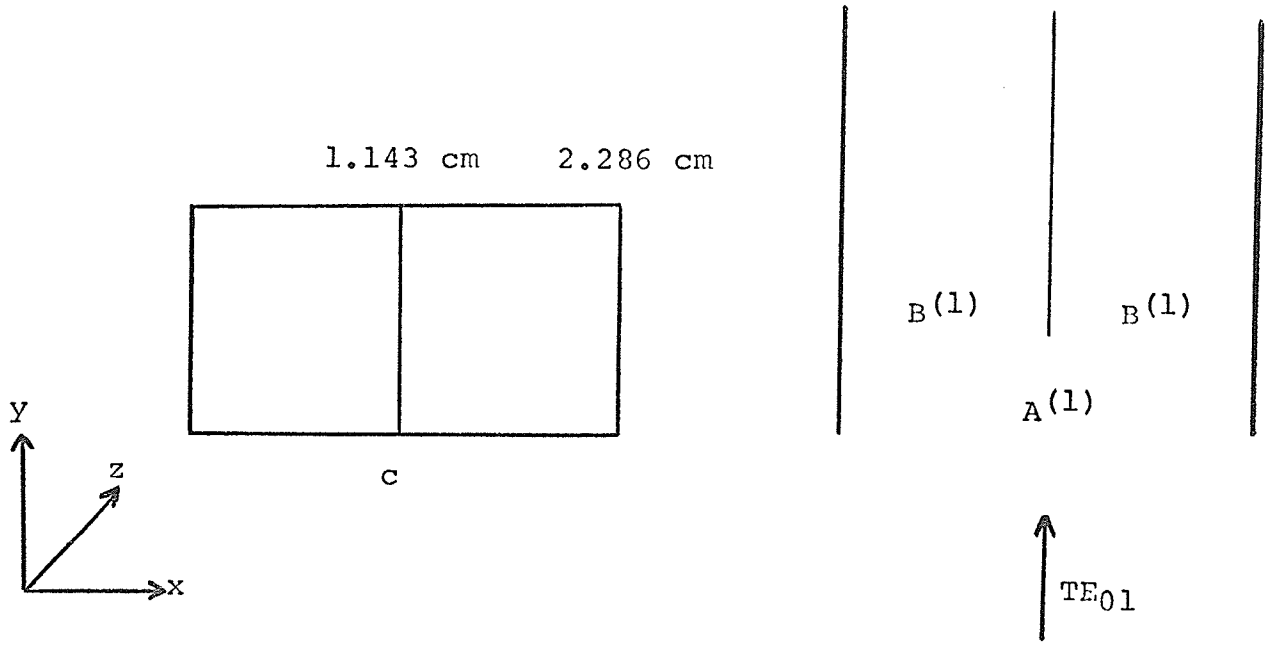

Fig. 6.4 H-Plane bifurcation, end and top view. Guide B is in cutoff.

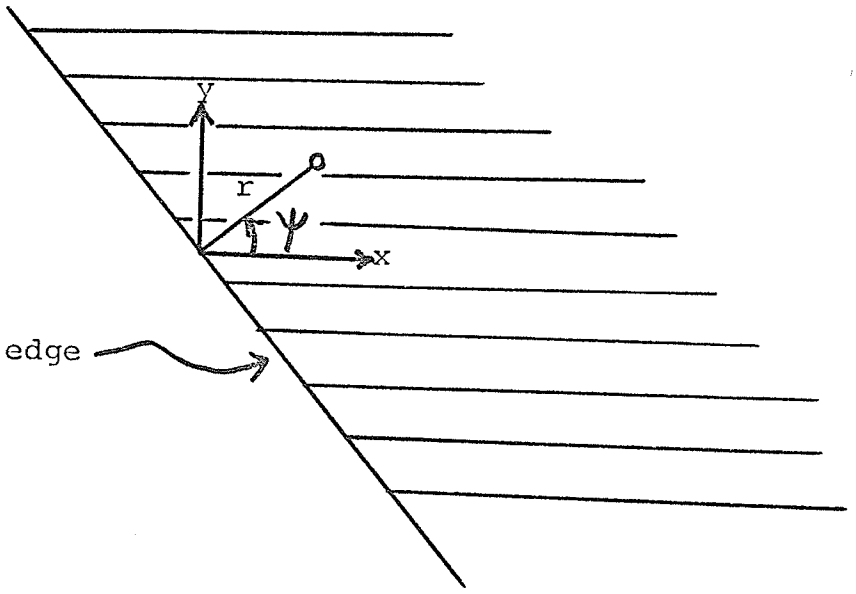

Fig. 6.5 Thin edge with coordinate system.

H-field must go to infinity at the edge and the E-field goes to zero at the edge.

By re-arranging (6.18) and changing the co-ordinate system to that of Fig. 6.4, the functions used in the waveguide become

$$\begin{array}{ll} 2j\omega\mu\ A(c-y)^{\frac{1}{2}} & c-\delta \leqslant y \leqslant c \\ 0 & \text{Elsewhere} \end{array} \qquad (6.19)$$

and

$$\begin{array}{ll} \dfrac{A}{(c-y)^{\frac{1}{2}}} & c-\delta \leqslant y \leqslant c \\ 0 & \text{elsewhere} \end{array} \qquad (6.20)$$

These correspond to the special functions for the wedge which were given in (6.3) and (6.4). The angle Ψ was taken as -90° and since the guide is completely symmetrical about the center c, the function is shown only for the left half of the guide. Similar expressions exist for the right hand side but are not used since all calculations are done on the left side only and then multiplied by two.

The corresponding functions used in guide $B^{(1)}$ are exactely the same as (6.19) and (6.20) since the bifurcation has zero width.

The functions in (6.19) and (6.20) are orthogonalized to the normal waveguide modes in exactely the same way as (6.3) and (6.4) were done by using (6.5) and (6.6). The procedure for $B^{(1)}$ guide is done by using (6.12) and (6.13), but substitute E_{Axs} for E_{Bxs}, and H_{Ays} for H_{Bys} as defined in (6.19) and (6.20).

Table 6.2 gives the resulting coefficients as defined in (6.14) to (6.17). It was pointed out by Wexler(27), that for TE_{01} exectation, the normal modes in guide $A^{(1)}$ are odd while the modes in $B^{(1)}$ are both odd and even with the same mode form defined over both $B^{(1)}$ sections.

The functions (6.14) to (6.17) are used as one mode in each guide with the summation taken over the number of normal modes used in each guide. Results were obtained for four and nine normal waveguide modes, to give a total of five and ten terms respectively.

The exact solution for the distance d of the shift of a short circuit equivalent circuit, from the plane of the discontinuity, is given by Marcuvitz(25) and is computed by

$$\frac{2\pi d}{\lambda_g} = \frac{2a}{\lambda_g}(1 + 1n\tfrac{1}{2}) - S_2\left(\frac{2a}{\lambda_g};\ 1,\ 0\right) + 2S_1\left(\frac{a}{\lambda_g};\ \tfrac{1}{2},\ 0\right) \qquad (6.21)$$

where

$$S_N(x;\ \alpha, 0) = \sum_{n=N}^{\infty} \sin^{-1}\frac{x}{\sqrt{n^2 - \alpha^2}} - \frac{x}{n} \qquad (6.22)$$

i	E_{Ai}	H_{Ai}	E_{Bi}	H_{Bi}
1	.03674	1.1046	.01208	.20170
2	-.03556	-1.0880	-.02366	-.39691
3	.03325	1.0555	.03426	.57945
4	-.02992	-1.0085	-.04344	-.74376
5	.02574	.94892	.05087	.88511
6	-.02089	-.87912	-.05627	-.99992
7	.01561	.80192	.05948	1.08568
8	-.01012	-.72030	-.06044	-1.14133
9	.00468	.63736	.05923	1.16718

Table 6.2 Coefficients to orthogonalize special function using (6.14) to (6.17) for thin edge.

and

$$\lambda_g = \frac{\lambda}{\sqrt{1 - \left(\frac{\lambda}{2a}\right)^2}}$$

where λ is the free space wavelength.

For a frequency corresponding to $\frac{a}{\lambda} = .7$ the normalized input admittance at the junction was calculated from (6.21) to be $y = 2.4175$. This value differs by 0.002 from the value given by Wexler(28). The infinite summation in (6.22) was terminated when the terms made no difference to the answer.

The solutions using the modal analysis method with the special function are summarized in Table 6.3 along with the solution using modal analysis with normal modes only.

The improvement using the special function for five terms is substantial but the improvement using ten terms is only a little better than using ten normal modes.

One would expect nearly perfect results by using the special function along with normal modes, since the function space spanned by the set used should be nearly complete as far as representing the fields. One improvement may be realized by using a function that represents the singularity across the whole guide width, rather than in the neighborhood of the singularity only. This would require using a power series rather than the first term which dominates near the discontinuity.

no. of terms	normal modes only	with special function
5	-j2.363	-j2.389
10	-j2.396	-j2.401
20	-j2.410	
40	-j2.415	
Exact	-j2.4175	

Table 6.3 Normalized admittance at the bifurcation. Frequency corresponds to $\frac{a}{\lambda} = .7$ and guide width a = 2.286 cm.

This would however bring up the complication that the electric field at the guide wall would not be zero as is required by the boundary conditions. However the normal modes may combine to approximate a zero field. This would add computations but in fact may save time in the end since it may permitt fewer modes.

These examples demonstrate that using non-orthogonal functions is possible and can give improved results. Perhaps some other functions could be used to give better results.

7. DISCUSSION AND CONCLUSIONS

The modal analysis formulation presented by Wexler apparently converges for any reasonable M/N ratio. The method converged for all step sizes attempted as opposed to the other modal formulation of Clarricoates and Slinn (2), which converge for only one ratio for a given geometry.

The use of a scattering matrix here allows for the solution of multi-junction problems while other formulations given do not allow complicated back scattering from discontinuities on the load side of the junction being solved.

The more accurate Simpson's 1/3 rule (23) could be used for all the integrations if the number of strips required for the desired accuracy were known beforehand. This is not usually the case but when a configuration is used over again many times with only minor changes in parameters, the 1/3 rule would give slightly faster integral calculations.

Transverse field plots generally show how a solution has converged but it is not possible to assess how accurate the resulting reflection coefficient is. Usually a good indication can be obtained by running tests on a junction, varying the number of modes used and noting the change in the reflection coefficent ρ. The error is likely to be less than the difference

of the two cases with the highest modes.

The same applies to irises providing they are not too thin. To determine the number of modes required for a relatively thin iris is much harder, since the sequence formed by increasing the number of modes used converges very slowly. There seems to be no way to determine the accuracy of the results obtained without comparison to a known solution.

For configurations which have excited modes that are two dimensional functions, the test for convergence must include jumps in modes used that include modes from the next group along an axis. For example if TE_{87} was the highest mode for a trial run then the next highest mode number to use must include the TE_{98} mode.

The formulation given for the field singularity appears to give improved results for the same number of modes. The method used was quite cumbersome and it was not easy to run test cases varying the number of modes used. A real test of its worth would be in the two dimensional step where convergence was a real problem. This would necessitate the inclusion of a term for the electric field since for the E-plane step the electric field has a singularity at the edge.

APPENDIX

A.1 Program Description

The program consists of a main calling program which calls nine subroutines which in turn use eleven function subprograms. The calling sequence is given in Fig A.1 and a description of each subprogram follows.

The main program declares many of the variables used throughout the program and are thus stored in common. The parameters of the first junction to be solved are read in. Waveguide parameters are calculated which are necessary for the placement of short and open circuit terminations, and then the scattering matrix for the first junction to be solved is assembled. Subroutine CNTRL is then called and data for the next junction is read. Then the reflection coefficient is printed for the previous junction and these last three steps are repeated until all junctions are solved.

Subroutine CNTRL is simply a sequence of calling statements which call MATS, SIMCOM, SLVEQ, and BASC in that order.

Subroutine MATS is a routine which computes all the integrals and scattering matrices required by the program. It also calls GAM to get mode types and characteristics. MATS also

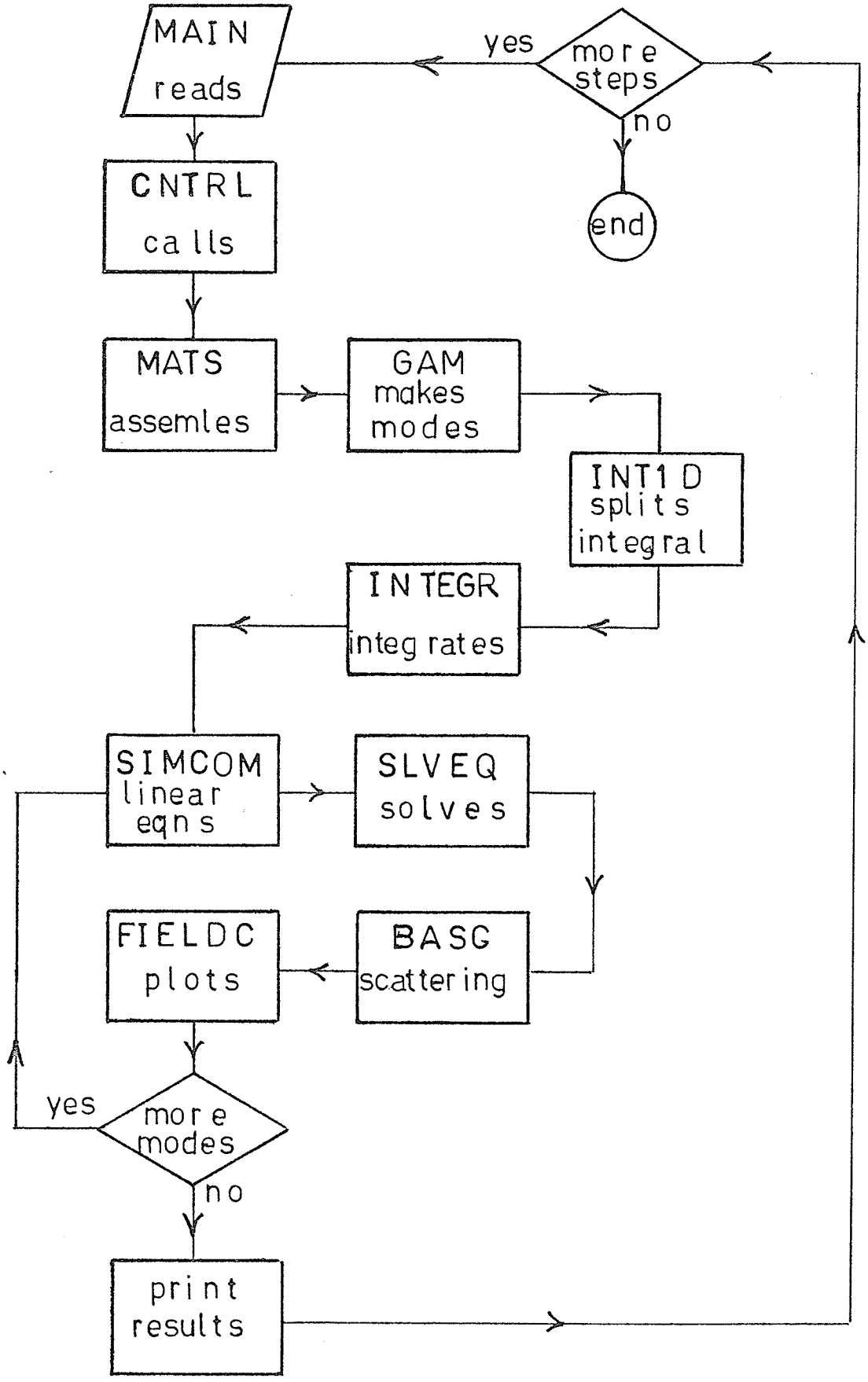

Fig. A1 Flow chart of modal analysis program

calls routines TF for field plots, and INT1D to carry out the integrations.

Subroutine GAM generates the necessary modes and their propagation constants for particular junction geometrys.

TF is used, when desired, to produce a map of the field value contributions of each mode used. It is used in conjunction with subroutine FIELDC to sum up the actual fields after the coeffieients are known. Both these subroutines are used only for plotting purposes and are not necessary for the junction solution.

Subroutine INT1D is a routine which breaks two dimensional integrations into one dimensional integrations. It calls the routine INTEGR to perform the actual integrations of the functions F1, F2, F3, ..., F8.

SIMCOM takes the results of the integrations from MATS and produces a set of complex linear equations according to (1.5 - 1.12). These equations are solved by subroutine SOLVEQ which uses complex elimination with partial pivoting. The solution vector is printed out and next subroutine BASC is called. This routine calculates the back scatterings coefficients according to (1.13). If field plots are desired then FIELDC is called next.

```
0001            REAL LAMBDA,LENGTH,KXA(30),KYA(30),KCASC(30),KSC,MU/12.56637E - 9/      CC1C
0002            INTEGER R,RMIN,RMAX,FJCN,XSYM,YSYM                                      0020
CC03            DATA C/29.97925/,TWCPI/6.283185/,EPSLCN/8.8542E - 14/                   CC3C
0004            INTEGER SHORT/'SC'/,CPEN/'CC'/,MATCH/'ML'/                              C040
0005            COMPLEX A(30),AVEC(30,30),B(30),CD(30,30),COEF(31,31),RHO,S(30,30)      0C50
               &,VEC(31),Y,GAMA(30)                                                     CC60
0006            EQUIVALENCE (RHO,AVEC(1,1))                                             C070
0007            INTEGER*2 MODEA(3,30)                                                   0080
0008            COMMON A,B,CD,S,COEF,VEC,AVEC,M,N,NP1,R,RMIN,RMAX,CLASS,FJCN,JCN        CC9C
0009            COMMON /MATCOM/GAMA,MODEA,KXA,KYA,KCASC,OMEGA,KSC,XA,YA,XB,YB,APRM      0100
               &,ADPRM,BPRM,BDPRM,SML,XSYM,YSYM,MPM,MDPM                                C11C
CC1C            COMMON /FNCTN/I,J,YYB,YBDPRM                                            C120
0011      10    READ (5,20) XA,YA,APRM,ADPRM,XB,YB,BPRM,BDPRM,LENGTH,CLASS,LOAD,        0130
               &NJCNS,F,M,N,MPM,MDPM,XSYM,YSYM,FJCN                                     C14C
0012            IFL=F                                                                   C15C
               C-----SET UP INTEGER FREQUENCY IN GHZ.                                   0160
CC13            DO 120 IF=IFL,IFL                                                       C17C
               C      F=IF                                                             C18C
0014            REWIND 4                                                                0190
               C-----R IS THE INCIDENT MODE NUMBER                                      C2CC
0015            RMIN=1                                                                  0210
0016            RMAX=1                                                                  0220
CC17            IF (NJCNS .GT. 1) RMAX=M                                                C230
0018      20    FORMAT (4F10.1/4F1C.1/2F1C.1,3X,A2,I5,F10.1/7I5)                        C240
CC19            LAMBDA=C/F                                                              C250
CC2C            WRITE (6,30) XA,YA,APRM,ADPRM,XB,YB,BPRM,BDPRM,LENGTH,CLASS,LOAD,       C26C
               &NJCNS,F,M,N,MPM,MDPM,XSYM,YSYM,FJCN                                     0270
0021      30    FORMAT ('CXA ='F8.4,4X,'YA ='F8.4,4X,4HA' =F8.4,4X,4HA" =F8.4,4X,'      C280
               &XB ='F8.4,4X,'YB ='F8.4,4X,4HB' =F8.4,4X,4HB" ='F8.4/' LEN ='F7.3,      C29C
               &5X,'CLASS='F4.1,5X,'LOAD='A2,I8,' JUNCTICNS'5X,'F='F6.2,5X,5X,'M =      0300
               &'I3,5X,'N ='I3,5X'MP'='I2/' MP''='I2,5X,'XSYM='I1,5X,'YSYM='I1,5X       C31C
               &'FJCN='I1)                                                              C320
               C-----SET ANGULAR FREQUENCY                                              0330
CC22            OMEGA=TWCPI*F*1.E9                                                      C34C
               C-----CALCULATE GUIDE WAVELENGTHS                                        0350
0023            SUA=1. - (LAMBDA/2./ADPRM)**2                                           C360
CC24            SUB=1. - (LAMBDA/2./BDPRM)**2                                           C37C
0025            GLAMA=LAMBDA/SIGN(SQRT(ABS(SUA)),SUA)                                   C380
0026            GLAMB=LAMBDA/SIGN(SQRT(ABS(SUB)),SUB)                                   C390
CC27            WRITE (6,4C) GLAMA,GLAMB,LAMBDA                                         C400
0028      40    FORMAT (' LAMBDAG A='G12.5,'      LAMBDAG B='G12.5,' LAMBDA='G12.5      0410
               &)                                                                       C420
0029            KSC=OMEGA*OMEGA*ML*EPSLCN                                               C43C
0030            NP1=N + 1                                                               0440
0031            SML=LENGTH                                                              C450
               C-----DETERMINE LOAD TYPE FOR FIRST JUNCTION SOLVEC                      C46C
               C-----AND WHERE TO PLACE SHORT                                           0470
0032            IF (LOAD .EQ. CPEN) SML=GLAMB/4.                                        C48C
0033            IF (LOAD .EQ. SHORT) SML=GLAMB/2.                                       C49C
0034            IF (LOAD .NE. MATCH) PRINT 5C,SML                                       0500
CC35      5C    FORMAT (' SHORT AT'G15.5,'CM.')                                         C51C
               C-----SET SCATTERING MATRIX TO C FOR MATCHED LOAD AND -1 ON DIAGONAL      0520
               C-----FOR SHORT                                                          0530
CC36            DO 7C I=1,N                                                             C54C
0037            DO 6C J=1,N                                                             0550
0038      60    AVEC(I,J)=(0.,0.)                                                       0560
CC39            IF (LOAD .NE. MATCH) AVEC(I,I)=( - 1.,0.)                               C57C
0040      70    CONTINUE                                                                0580
0041            DC 11C JCN=1,NJCNS                                                      C590
               C-----SOLVE EACH JUNCTION IN LOOP                                        0600
```

```
0042            CALL CNTRL                                                      C610
0043            Y=(1. - RHO)/(1. + RHO)                                         0620
C044            WRITE (6,80) RHC,Y                                              C630
0045      80    FORMAT ('CRHC ='2G16.8,5X,'Y ='2G16.8)                          C640
0046            IF (NJCNS .EQ. 1) GO TO 11C                                     0650
          C-----SET PARAMETERS IN FIRST JUNCTION                                C660
0047            RMAX=1                                                          C67C
C048            XB=XA                                                           C680
CC49            YB=YA                                                           C690
C05C            BPRM=APRM                                                       C7CC
0051            BCPRM=ACPRM                                                     0710
          C-----READ PARAMETERS FCR NEXT JUNCTION                               C720
0052            READ (5,1CC,END=120) XA,YA,APRM,ACPRM,SML,CLASS                 C730
0053            PRINT 90,XA,YA,APRM,ADPRM,SML,CLASS                             0740
C054      90    FCRMAT ('CXA',T16,'YA',T32,'APRM',T48,'ACPRM',T64,'LENGTH',T8C,'CL C750
                &ASS'/6G16.6)                                                   C760
0055      100   FCRMAT (4F10.1/2F10.1)                                          C770
C056      11C   CCNTINUE                                                       C780
CC57      12C   CONTINUE                                                       C790
0058      130   CALL EXIT                                                       08C0
C059            END                                                             C810
```

```
0001            SUBROUTINE CNTRL                                                CC10
      C-----CALLS SUBROUTINES FOR EACH INCIDENT MODE                            CC20
CCC2            REAL LAMBDA                                                     CC30
0003            INTEGER R,RMIN,RMAX,FJCN                                        0040
0004            COMPLEX A(30),AVEC(30,30),B(30),CD(30,30),COEF(31,31),PRHO,S(30,30  CC50
               &),VEC(31),Y,D                                                   CC60
0005            COMMON A,B,CD,S,CCEF,VEC,AVEC,M,N,NP1,R,RMIN,RMAX,CLASS,FJCN,JCN  0070
CCC6            CALL MATS                                                       CCEC
CCC7            DO 10 R=RMIN,RMAX                                               CC90
0008            CALL SIMCOM                                                     0100
0009            CALL SLVEC(CCEF,VEC,NP1,31,D,6)                                 C110
C010      10    CALL BASC                                                       C120
0011            PRINT 20,(VEC(I),I=2,NP1)                                       0130
CC12      20    FORMAT (//(' BJ'2G16.8))                                        C140
0013            PRINT 30,(AVEC(I,1),I=1,M)                                      0150
0014      30    FORMAT (//(' AI'2G16.8))                                        C160
CC15            BJ=CABS(VEC(2))                                                 C170
0016            AI=CABS(AVEC(1,1))                                              C180
0017            PRINT 40,BJ,AI                                                  0190
0018      40    FORMAT (//' |B1|='G15.7,'      |RHO|='G15.7)                   C200
0019            IF (FJCN .NE. C) CALL FIELDC(VEC,AVEC,M - 1,N)                  0210
      C-----M-1 FOR SPECIAL MODE ONLY                                           0220
C020            RETURN                                                          C230
0021            END                                                             0240
```

```
0001            SUBROUTINE MATS                                                 0010
0002            COMPLEX A(30),AVEC(30,30),B(30),CC(30,30),COEF(31,31),S(30,30),VEC  0020
               &(31),GAMA(30),GAMB(30),DUMMY,GA,GL                              0030
0003            INTEGER*2 MODEA(3,30),MODEB(3,30),TE/'TE'/,TM/'TM'/              0040
0004            EXTERNAL F1,F2,F3,F4,F5,F6,F7,F8                                0050
0005            EXTERNAL FSE,FSH,FD                                             0060
0006            REAL KSQ,KXA(30),KXB(30),KYA(30),KYB(30),KCASQ(30),KCBSQ(30),INTID  0070
               &,MU/12.56637E - 9/,INT                                          0080
0007            DATA PI/3.141593/,EPSLON/8.8542E - 14/                          0090
0008            INTEGER MM,MMP1,NN,NNP1,XSYM,YSYM,SPACE(4)                      0100
0009            COMMON A,B,CC,S,COEF,VEC,AVEC,M,N,SPACE,CLASS,FJCN              0110
0010            COMMON /MATCOM/GAMA,MODEA,KXA,KYA,KCASQ,OMEGA,KSQ,XA,YA,XB,YB,APRM  0120
               &,ADPRM,BPRM,BDPRM,SML,XSYM,YSYM,MPMAX,MDPMAX                    0130
0011            COMMON /FNCTN/I,J                                               0140
0012            COMMON /GB/KXB,KYB,KCBSQ                                        0150
0013            COMPLEX CEXP                                                    0160
      C....PRECEDING CARD FOR WATFIV ONLY                                       0170
0014            READ 10,NN                                                      0180
      C-----SET MM=M FOR SPECIAL MODE FORMULATION ONLY                          0190
0015      10    FORMAT (I2)                                                     0200
      C-----SET UP MODE TYPES FOR THE TWO GUIDES AT THE JUNCTION                0210
0016            CALL GAM(MPMAX,MDPMAX,XSYM,YSYM,KSQ,APRM,ADPRM,GAMA,MODEA,NN)   0220
0017            CALL GAM(MPMAX,MDPMAX,XSYM,YSYM,KSQ,BPRM,BDPRM,GAMB,MODEB,NN)   0230
      C-----CHECK TO SEE HOW MANY MODES GENERATED                                0240
0018            IF (M .NE. NN) PRINT 20,NN                                      0250
0019      20    FORMAT (' # MODES='I2)                                          0260
0020            DO 50 J=1,M                                                     0270
0021            I=J                                                             0280
      C-----CALCULATE MODE PATTERNS FOR BOTH GUIDES                             0290
0022            KXA(J)=MODEA(1,J)/APRM*PI                                       0300
0023            KYA(J)=MODEA(2,J)/ADPRM*PI                                      0310
0024            KCASQ(J)=KXA(J)*KXA(J) + KYA(J)*KYA(J)                          0320
0025            KXB(J)=MODEB(1,J)/BPRM*PI                                       0330
0026            KYB(J)=MODEB(2,J)/BDPRM*PI                                      0340
0027            KCBSQ(J)=KXB(J)*KXB(J) + KYB(J)*KYB(J)                          0350
0028            IF (MODEB(3,J) .EQ. TM) GO TO 40                                0360
      C-----CALCULATE INTEGREATIOS                                              0370
0029            AP=KXA(J)*INTID(F1,F1,F2,F2,XA,XA + APRM,YA,YA + ADPRM)*KXA(J)  0380
0030            ADP=KYA(J)*INTID(F3,F3,F4,F4,XA,XA + APRM,YA,YA + ADPRM)*KYA(J) 0390
0031            A(J)=(0.,1.)*OMEGA/KCASQ(J)*MU/KCASQ(J)*GAMA(J)*(AP + ADP)      0400
0032            BP=KXB(J)*INTID(F5,F5,F6,F6,XB,XB + BPRM,YB,YB + BDPRM)*KXB(J)  0410
0033            BDP=KYB(J)*INTID(F7,F7,F8,F8,XB,XB + BPRM,YB,YB + BDPRM)*KYB(J) 0420
0034            B(J)=(0.,1.)*OMEGA/KCBSQ(J)*MU/KCBSQ(J)*GAMB(J)*(BP + BDP)      0430
      C-----SET UP FIELD PATTERNS IN A GUIDE                                    0440
      C-----E FIELD IN A                                                        0450
0035            IF (FJCN .NE. 0 .AND. J .LT. NM) CALL TF((0.,1.)*OMEGA*MU*KYA(J)/  0460
               &KCASQ(J),XA,YA,F3,F4)                                           0470
      C-----H FIELD IN A                                                        0480
0036            IF (FJCN .NE. 0 .AND. J .LT. NM) CALL TF((1.,0.)*GAMA(J)/KCASQ(J)*  0490
               &KYA(J),XA,YA,F3,F4)                                             0500
0037            GO TO 30                                                        0510
0038            IF (FJCN .NE. 0 .AND. J .EQ. NM) CALL TF((0.,1.)*OMEGA*MU*1.5*SQRT  0520
               &(3.),XA,YA,FD,FSE)                                              0530
0039            IF (FJCN .NE. 0 .AND. J .EQ. NM) CALL TF((1.,0.),XA,XA,FD,FSH)  0540
      C-----E FIELD IN B                                                        0550
0040      30    IF (FJCN .NE. 0 .AND. J .LT. NM) CALL TF((0.,1.)*OMEGA*MU*KYB(J)/  0560
               &KCBSQ(J),XB,YB,F7,F8)                                           0570
      C-----H FIELD IN B                                                        0580
0041            IF (FJCN .NE. 0 .AND. J .LT. NM) CALL TF((1.,0.)*GAMB(J)/KCBSQ(J)*  0590
               &KYB(J),XB,YB,F7,F8)                                             0600
```

```
0042            GO TO 5C                                                        0610
0043            IF (FJCN .NE. 0 .AND. J .EQ. NM) CALL TF((C.,1.)*OMEGA*MU*1.5*SQRT 0620
               &(3.),XB,YB,FC,FSE)                                              0630
0044            IF (FJCN .NE. C .AND. J .EQ. NM) CALL TF((1.,C.),XB,XB,FC,FSH)  0640
CC45            GO TC 50                                                        0650
C046      40    DP=KXB(J)*INT1D(F7,F7,F8,F8,XB,XB + BPRM,YB,YB + BDPRM)*KXB(J)  0660
C047            DDP=KYB(J)*INT1C(F5,F5,F6,F6,XB,XB + BPRM,YB,YB + BDPRM)*KYB(J) 0670
C048            B(J)=(C.,1.)*CMEGA/KCBSC(J)*EPSLCN/KCBSC(J)*GAMB(J)*(DP + DDP)  0680
0049            AP=KXA(J)*INT1D(F3,F3,F4,F4,XA,XA + APRM,YA,YA + ADPRM)*KXA(J)  0690
CC5C            ADP=KYA(J)*INT1C(F1,F1,F2,F2,XA,XA + APRM,YA,YA + ADPRM)*KYA(J) 0700
0051            A(J)=(C.,1.)*CMEGA/KCASC(J)*EPSLCN/KCASC(J)*GAMA(J)*(AP + ADP)  0710
          C     PRINT 20,DP,DDP                                                 0720
CC52            IF (FJCN .NE. 0) CALL TF(GAMB(J)*KXB(J)/KCBSC(J),XB,YB,F7,F8)   0730
C053      50    CONTINUE                                                        0740
0054            IF (N .LT. NM) GC TO 7C                                         0750
          C-----READ IN PARAMETERS FCR SPECIAL MCDE FCRMULATICN                 0760
0055            READ 6C,A(10),B(1C)                                             0770
0056      60    FCRMAT (8F10.1)                                                 0780
0057            DUMMY=(C.,1.5)*CMEGA*MU*1.73205                                 0790
          C-----REPLACE THE NEXT MODE BY SPECIAL MODE                           0800
CC58            A(10)=A(1C)*DUMMY                                               0810
C059            B(1C)=B(1C)*DUMMY                                               0820
0060      70    CONTINUE                                                        0830
C061            PRINT 80,(J,A(J),B(J),J=1,N)                                    0840
0062      80    FORMAT (//(' AB'I2,1X,4G16.8))                                  0850
          C-----CALCULATE THE CROSS INTEGRAL                                    0860
0063            DO 14C I=1,N                                                    0870
0064            DO 14C J=1,N                                                    0880
0065            IF (CLASS .EQ. - 1) GO TO 90                                    0890
C066            XL=XB                                                           0900
0067            XU=XB + BPRM                                                    0910
0068            YL=YB                                                           0920
C069            YU=YB + BDPRM                                                   0930
0070            IF (MODEA(3,I) .EQ. TM) GO TC 12C                               0940
C071            IF (MCDEB(3,J) .EQ. TM) GO TO 110                               0950
C072            GO TO 1CC                                                       0960
0073      90    XL=XA                                                           0970
CC74            XU=XA + APRM                                                    0980
CC75            YL=YA                                                           0990
C076            YU=YA + ADPRM                                                   1000
C077            IF (MCDEB(3,J) .EQ. TM) GC TC 120                               1010
0078            IF (MODEA(3,I) .EQ. TM) GC TC 11C                               1020
0079      100   GA=GAMA(I)                                                      1030
CC8C            IF (CLASS .EQ. - 1.) GA=GAMB(J)                                 1040
0081            CD(I,J)=(C.,1.)*OMEGA/KCASQ(I)*ML/KCBSC(J)*GA*(KXA(I)*INT1C(F1,F5, 1050
               &F2,F6,XL,XU,YL,YU)*KXB(J) + KYA(I)*INT1D(F3,F7,F4,F8,XL,XL,YL,YU)* 1060
               &KYB(J))                                                         1070
0082            GO TO 140                                                       1080
0083      11C   IF (CLASS .EQ. 1.) INT=(( - KXA(I))*INT1D(F1,F5,F2,F6,XL,XU,YL,YU) 1090
               &*KYB(J) + KYA(I)*INT1D(F3,F7,F4,F8,XL,XU,YL,YU)*KXB(J))         1100
0084            IF (CLASS .EQ. - 1.) INT= - (KXA(I)*INT1D(F3,F7,F4,F8,XL,XU,YL,YU) 1110
               &*KYB(J) + KYA(I)*INT1C(F1,F5,F2,F6,XL,XU,YL,YU)*KXB(J))         1120
0085            CD(I,J)=GAMA(I)/KCASC(I)*GAMB(J)/KCBSC(J)*INT                   1130
0086            GC TC 140                                                       1140
0087      120   IF (MCDEB(3,J) .EQ. TM .AND. CLASS .EQ. 1.) GC TO 130          1150
0088            IF (MODEA(3,I) .EQ. TM .AND. CLASS .EQ. - 1) GC TO 130          1160
C089            IF (CLASS .EC. 1.) INT=( - KXA(I)*INT1D(F3,F7,F4,F8,XL,XU,YL,YL)* 1170
               &KYB(J) + KYA(I)*INT1D(F1,F5,F2,F6,XL,XL,YL,YU)*KXB(J))          1180
0090            IF (CLASS .EQ. - 1.) INT=(( - KXA(I))*INT1D(F1,F5,F2,F6,XL,XU,YL, 1190
               &YU)*KYB(J) - KYA(I)*INT1C(F3,F7,F4,F8,XL,XU,YL,YU)*KXB(J))      1200
```

```
0091            CD(I,J)=KSQ/(KCASQ(I)*KCBSQ(J))*INT                                  1210
0092            GO TO 140                                                            1220
0093      130   GA=GAMB(J)                                                           1230
0094            IF (CLASS .EQ. - 1.) GA=GAMA(I)                                      1240
0095            CD(I,J)=(0.,1.)*OMEGA/KCASQ(I)*EPSLON/KCBSQ(J)*GA*(KXA(I)*INT1C(F3   1250
               &,F7,F4,F8,XL,XU,YL,YU)*KXB(J) + KYA(I)*INT1D(F1,F5,F2,F6,XL,XL,YL,   1260
               &YU)*KYB(J))                                                          1270
0096      140   CONTINUE                                                             1280
0097            IF (M .LT. NM) GO TO 160                                             1290
          C-----READ IN PARAMETER FOR SPECIAL MODE                                   1300
0098            READ 60,A10                                                          1310
0099            READ 60,(CD(I,10),I=1,9),(CD(10,J),J=1,9)                            1320
0100            DO 150 I=1,3                                                         1330
          C-----REPLACE NEXT MODE WITH VALUES FOR SPECIAL MODES                      1340
0101            CD(I,10)=CD(I,10)*DUMMY*GAMA(I)/KCASQ(I)*KYA(I)                      1350
0102      150   CD(10,I)=CD(10,I)*(0., - 1.)*OMEGA*MU/KCBSQ(I)*KYB(I)                1360
0103            CD(10,10)=DUMMY*A10                                                  1370
0104      160   CONTINUE                                                             1380
0105      170   FORMAT (//(' CD',2I3,1X,2G16.8))                                     1390
          C-----CALCULATE SCATTERING MATRIX FOR NEXT JUNCTION TO BE SOLVED           1400
0106            DO 180 I=1,N                                                         1410
0107            DO 180 J=1,N                                                         1420
0108            GL= - SML*(GAMB(I) + GAMB(J))                                        1430
          C     IF(ABS(GL) .GT. 1.E10) GL=-1.E10                                     1440
0109      180   S(I,J)=CEXP(GL)*AVEC(J,I)                                            1450
0110            RETURN                                                               1460
0111            END                                                                  1470
```

```
0001            SUBROUTINE GAM(MPM,MDPM,XSYM,YSYM,KSQ,APRM,ADPRM,GAMM,MODE,N)          C010
0002            REAL KSQ,KX,KY                                                        C020
0003            INTEGER XSYM,YSYM,YYSYM                                               0030
0004            INTEGER*2 TE/'TE'/,TM/'TM'/,MODE(3,30)                                C040
0005            COMPLEX GAMM(30)                                                      C050
0006            COMPLEX CSQRT,CMPLX                                                   0060
0007            DATA PI/3.141593/                                                     C070
0008            YYSYM=YSYM                                                            C080
0009            IF (YSYM .EQ. C) YYSYM=1                                              0090
0010            N=0                                                                   C100
0011      10    IF (XSYM .NE. 0) GO TO 20                                             C110
0012            MPRM=0                                                                0120
0013            GO TO 30                                                              C130
0014      20    MS=XSYM                                                               C140
0015            MMPM=MPM + XSYM                                                       0150
0016            DO 100 MMPRM=MS,MMPM,XSYM                                             C160
                C-----GENERATES MODES WITH X VARYING                                  C170
0017            MPRM=MMPRM - XSYM                                                     0180
0018      30    KX=MPRM/APRM*PI                                                       C190
0019      40    NS=1                                                                  C200
0020            DO 80 MDPRMS=NS,MDPM,YYSYM                                            0210
                C-----GENERATES MODES WITH Y VARYING                                  C220
0021            MDPRM=MDPRMS                                                          0230
0022            IF (MPRM .EQ. 0 .AND. MDPRM .EQ. 0) GO TO 80                          C240
0023      50    KY=MDPRM/ADPRM*PI                                                     C250
0024            N=N + 1                                                               0260
                C-----CALCULATES PROPOGATION CONSTANT                                 C270
0025      60    GAMM(N)=CSQRT(CMPLX(KX*KX + KY*KY - KSQ,0.))                          C280
                C-----SET UP CODES FOR TE MODE TYPE                                   0290
0026            MODE(1,N)=MPRM                                                        C300
0027            MODE(2,N)=MDPRM                                                       0310
0028      70    MODE(3,N)=TE                                                          0320
0029            IF (N .GE. 30) GO TO 110                                              C330
                C-----CHECK FOR POSSIBILITY OF TM MODE TYPE                           0340
0030            IF (MPRM .EQ. 0 .OR. MDPRM .EQ. 0) GO TO 80                           C350
0031            N=N + 1                                                               C360
                C-----GAMMA IS SAME FOR TM MODE                                       0370
0032            GAMM(N)=GAMM(N - 1)                                                   C380
0033            MODE(1,N)=MPRM                                                        0390
0034            MODE(2,N)=MDPRM                                                       0400
0035            MODE(3,N)=TM                                                          C410
0036            IF (YSYM .EQ. 2 .AND. MDPRM .EQ. 0) MDPRM= - 1                        C420
0037            IF (N .GE. 30) GO TO 110                                              C430
0038      80    CONTINUE                                                              C440
                C-----ALL MODE ARE GENERATED IF ONLY TEON TYPE                        C450
0039      90    IF (XSYM .EQ. 0) GO TO 110                                            C460
0040     100    CONTINUE                                                              C470
0041     110    CONTINUE                                                              0480
0042            PRINT 120,(MODE(3,I),MODE(1,I),MODE(2,I),GAMM(I),I=1,N)               C490
0043     120    FORMAT (//(' 'A2,I2,1X,I2,'    GAM='2G15.7))                          C500
0044            RETURN                                                                C510
0045            END                                                                   C520
```

```
0001            SUBROUTINE SIMCOM                                                   0010
                C-----ASSEMBLES INTEGRALS INTO SET OF EQUATIONS                     0020
0002            INTEGER R,RMIN,RMAX                                                 0030
0003            REAL LAMBDA                                                         0040
0004            COMPLEX A(30),AVEC(30,30),B(30),CD(30,30),COEF(31,31),S(30,30),VEC  0050
               &(31),SUM,PRD                                                        0060
0005            COMMON A,B,CD,S,COEF,VEC,AVEC,M,KAPN,KAPNP1,R,RMIN,RMAX,CLASS       0070
0006            EQUIVALENCE (CLASS,SIGN)                                            0080
0007            COEF(1,1)=A(R)                                                      0090
0008            VEC(1)= - SIGN*A(R)                                                 0100
0009            DO 30 J=1,KAPN                                                      0110
0010            SUM=0                                                               0120
0011            DO 10 K=1,KAPN                                                      0130
0012      10    SUM=SUM + CD(R,K)*S(J,K)                                            0140
0013      20    COEF(1,J + 1)= - SIGN*CD(R,J) - SUM                                 0150
0014      30    CONTINUE                                                            0160
0015            DO 40 N=1,KAPN                                                      0170
0016            NP1=N + 1                                                           0180
0017            COEF(NP1,1)=CD(R,N)                                                 0190
0018      40    VEC(NP1)=SIGN*CD(R,N)                                               0200
0019            DO 110 N=1,KAPN                                                     0210
0020            NP1=N + 1                                                           0220
0021            DO 90 J=1,KAPN                                                      0230
0022            SUM=0                                                               0240
0023            DO 80 I=1,M                                                         0250
0024            IF (I .EQ. R) GO TO 80                                              0260
0025      50    PRD=0.                                                              0270
0026            DO 60 K=1,KAPN                                                      0280
0027      60    PRD=PRD + CD(I,K)*S(J,K)                                            0290
0028      70    SUM=SUM + CD(I,N)*(SIGN*PRD + CD(I,J))/A(I)                         0300
0029      80    CONTINUE                                                            0310
0030      90    COEF(NP1,J + 1)=SIGN*SUM                                            0320
0031            COEF(NP1,NP1)=SIGN*B(N) + COEF(NP1,NP1)                             0330
0032            DO 100 J=1,KAPN                                                     0340
0033            JP1=J + 1                                                           0350
0034      100   COEF(NP1,JP1)=COEF(NP1,JP1) - B(N)*S(J,N)                           0360
0035      110   CONTINUE                                                            0370
0036            RETURN                                                              0380
0037            END                                                                 0390
```

```
0001            SUBRCUTINE SLVEC(A,B,N,NDIM,C,LNIT)                     0C10
      C-----SOLVES A SET CF LINEAR ECUATICNS                            CC20
0002            INTEGER LNIT                                            0030
0003            COMPLEX A(NDIM,NDIM),B(N),BIGA,HCLD,D                   CC40
C004      10    NM1=N - 1                                               CC50
0005            D=1.                                                    0060
0006            DO 80 K=1,NM1                                           CC70
C007            BIGA=A(K,K)                                             CC80
0008            J=K                                                     0090
0009            KP1=K + 1                                               C100
CC1C            DC 2C I=KP1,N                                           C110
0011            IF (CABS(BIGA) .GE. CABS(A(I,K))) GO TC 20              0120
0012            BIGA=A(I,K)                                             0130
CC13            J=I                                                     C140
0014      20    CONTINUE                                                0150
0015            D=D*BIGA                                                C160
0016            IF (CABS(D) .EQ. C.) GC TC 100                          C170
0017            IF (J .LE. K) GO TO 40                                  0180
CC18            DO 30 I=K,N                                             C190
CC19            HOLD=A(K,I)                                             02C0
0020            A(K,I)=A(J,I)/BIGA                                      0210
CC21      30    A(J,I)=HCLD                                             C220
0022            HOLD=B(K)                                               0230
0023            B(K)=B(J)/BIGA                                          0240
CC24            B(J)=HCLD                                               C250
CC25            GO TO 6C                                                0260
0026      40    DO 50 I=KP1,N                                           C270
0C27      50    A(K,I)=A(K,I)/BIGA                                      0280
CC28            B(K)=B(K)/BIGA                                          C290
0029      60    DO 80 I=KP1,N                                           C3C0
CC3C            DO 7C J=KP1,N                                           C310
0031      70    A(I,J)=A(I,J) - A(I,K)*A(K,J)                           0320
0032      80    B(I)=B(I) - A(I,K)*B(K)                                 0330
CC33            D=D*A(N,N)                                              C340
0034            IF (CABS(D) .EQ. C.) GO TO 1CC                          0350
0035            B(N)=B(N)/A(N,N)                                        0360
0036            DO 9C I=1,NM1                                           C370
0037            K=N - I                                                 0380
0038            KP1=K + 1                                               C390
CC39            DO 9C J=KP1,N                                           C4C0
0040      90    B(K)=B(K) - A(K,J)*B(J)                                 0410
0041            RETURN                                                  0420
0042      10C   WRITE (LNIT,11C)                                        0430
0043      11C   FORMAT ('OSINGULAR SET OF ECUATICNS')                   0440
CC44            RETURN                                                  C450
CC45            END                                                     0460
```

```
0001          FUNCTION F1(X)                                                    0010
      C-----DEFINES FUNCTIONS FOR EACH INTEGRATION                              0020
0002          COMPLEX A(30),B(30),CD(30,30),S(30,30),COEF,GAMB(30),GAMA(30)     0030
0003          INTEGER*2 MODEA(3,30)                                             0040
0004          REAL KXA(30),KYA(30),KCASQ(30),KXB(30),KYB(30),KCBSQ(30),KSQ      0050
0005          COMMON A,B,CD,S,COEF,GAMB                                         0060
0006          COMMON /MATCOM/GAMA,MODEA,KXA,KYA,KCASQ,OMEGA,KSQ,XA,YA,XB,YB     0070
0007          COMMON /FNCTN/I,J                                                 0080
0008          COMMON /GB/KXB,KYB,KCBSQ                                          0090
      C-----F1 - F4 FOR A GUIDE                                                 0100
0009          F1=SIN(KXA(I)*(X - XA))                                           0110
0010          RETURN                                                            0120
0011          ENTRY F2(Y)                                                       0130
0012          F2=COS(KYA(I)*(Y - YA))                                           0140
0013          RETURN                                                            0150
0014          ENTRY F3(X)                                                       0160
0015          F3=COS(KXA(I)*(X - XA))                                           0170
0016          RETURN                                                            0180
0017          ENTRY F4(Y)                                                       0190
0018          F4=SIN(KYA(I)*(Y - YA))                                           0200
0019          RETURN                                                            0210
0020          ENTRY F5(X)                                                       0220
      C-----F5 - F8 FOR B GUIDE                                                 0230
0021          F5=SIN(KXB(J)*(X - XB))                                           0240
0022          RETURN                                                            0250
0023          ENTRY F6(Y)                                                       0260
0024          F6=COS(KYB(J)*(Y - YB))                                           0270
0025          RETURN                                                            0280
0026          ENTRY F7(X)                                                       0290
0027          F7=COS(KXB(J)*(X - XB))                                           0300
0028          RETURN                                                            0310
0029          ENTRY F8(Y)                                                       0320
0030          F8=SIN(KYB(J)*(Y - YB))                                           0330
0031          RETURN                                                            0340
0032          END                                                               0350
```

```
0001            SUBROUTINE BASC                                                 0010
      C-----CALCULATES BACK SCATTERING CORFFICIENTS                              0020
0002            INTEGER R,RMIN,RMAX                                             0030
0003            REAL LAMBDA                                                     0040
0004            COMPLEX A(30),AVEC(30,30),B(30),CD(30,30),COEF(31,31),S(30,30),VEC  0050
               &(31),SUM1,SUM2,SUM3                                             0060
0005            COMMON A,B,CC,S,CCEF,VEC,AVEC,M,N,NP1,R,RMIN,RMAX,CLASS         0070
      C-----SUM IS NEGATIVE FOR ENLARGEMENT                                     0080
CC06            EQUIVALENCE (CLASS,SIGN)                                        0090
      C-----RHO IS FIRST UNKNOWN                                                0100
CC07            AVEC(R,R)=VEC(1)                                                0110
CC08            DO 30 I=1,M                                                     0120
CC09            IF (I .EQ. R) GO TO 30                                          0130
0010            SUM1=0.                                                         0140
0011            DO 20 J=1,N                                                     0150
0012            SUM2=0.                                                         0160
0013            DO 10 K=1,N                                                     0170
0014       10   SUM2=SUM2 + CD(I,K)*S(J,K)                                      0180
0015       20   SUM1=SUM1 + VEC(J + 1)*(SIGN*SUM2 + CD(I,J))                    0190
0016            AVEC(I,R)=SIGN*SUM1/A(I)                                        0200
0017       30   CONTINUE                                                        0210
0018            RETURN                                                          0220
0019            END                                                             0230
```

```
0001            SUBROUTINE TF(CON,R2,R1,FX,FY)                          0010
          C-----TOTAL UP FIELD FOR EACH NORMAL WAVEGUIDE MODE            0020
0002            REAL Y(120),DELY/.01/,L(2),U(2),CENTER(2)/1.143,.5/     0030
0003            COMPLEX CON,FEY(120)                                    0040
0004            L(1)=R1                                                 0050
0005            L(2)=R2                                                 0060
0006            DO 40 J=1,1                                             0070
0007            Y(1)=0.                                                 0080
          C-----CALCULATE FIELD IN X AND Y DIRECTIONS                   0090
0008            IF (J .EQ. 1) FEY(1)=CON*FX(.5)*FY(Y(1))                0100
0009            IF (J .EQ. 2) FEY(1)=CON*FX(0.)*FY(1.15)                0110
0010            IF (L(J) .GT. C.) FEY(1)=C.                             0120
0011            DO 20 I=2,120                                           0130
0012            Y(I)=Y(I - 1) + DELY                                    0140
          C-----CALCULATE FIELD ONLY TO CENTERLINE                      0150
0013            IF (Y(I) .GT. CENTER(J)) GO TO 30                       0160
          C-----FIELD IS C IF Y IS LESS THAN ORIGIN                     0170
0014            IF (Y(I) .GE. L(J)) GO TO 10                            0180
0015            FEY(I)=(0.,0.)                                          0190
0016            GO TO 20                                                0200
0017      10    IF (J .EQ. 1) FEY(I)=CON*FX(.5)*FY(Y(I))                0210
0018            IF (J .EQ. 2) FEY(I)=CON*FX(Y(I))*FY(1.15)              0220
0019      20    CONTINUE                                                0230
0020      30    I=I - 1                                                 0240
          C-----WRITE THE MODE PATTERN TEMPORARILY ON DISK              0250
0021            WRITE (4) I,(FEY(K),K=1,I)                              0260
0022      40    CONTINUE                                                0270
0023            RETURN                                                  0280
0024            END                                                     0290
```

```
0001            SUBROUTINE FIELDD(VEC,AVEC,M,N)                                 0010
0002            COMPLEX VEC(31),AVEC(30,30),FEY(120),FIELD(120,4)               0020
0003            COMMON /BUFF/FEY                                                0030
0004            INTEGER NPTS(4)                                                 0040
          C-----SIGN IS NEGATIVE FOR H FIELD                                    0050
0005            REAL SIGN(4)/1.,1., - 1., - 1./                                 0060
          C-----PLACE RECORD AT THE FIRST MODE                                  0070
0006            REWIND 4                                                        0080
0007            DO 20 J=1,3,2                                                   0090
          C-----READ FIRST MODE FOR E FIELD THEN H FIELD IN A                   0100
0008            READ (4) I,(FEY(K),K=1,I)                                       0110
0009            NPTS(J)=I                                                       0120
          C     PRINT 200,(FEY(K),K=1,I)                                        0130
0010      10    FORMAT (' FEY'/(12G11.3))                                       0140
          C-----CALCULATE FIELD PATTERN DUE TO FIRST MODE                       0150
0011            DO 20 K=1,I                                                     0160
0012      20    FIELD(K,J)=(1. + SIGN(J)*AVEC(1,1))*FEY(K)                      0170
0013            DO 30 MODE=2,M                                                  0180
          C-----DUMMY READ TO SKIP FIELD IN GUIDE B                             0190
0014            READ (4) I,(FEY(K),K=1,I)                                       0200
0015            READ (4) I,(FEY(K),K=1,I)                                       0210
0016            DO 30 J=1,3,2                                                   0220
          C-----READ NEXT E AND H PATTERN FOR GUIDE A                           0230
0017            READ (4) I,(FEY(K),K=1,I)                                       0240
          C     PRINT 200,(FEY(K),K=1,I)                                        0250
0018            DO 30 K=1,I                                                     0260
          C-----ADD THEM TO FIELD                                               0270
0019      30    FIELD(K,J)=FIELD(K,J) + AVEC(MODE,1)*FEY(K)*SIGN(J)             0280
0020            DO 50 J=1,3,2                                                   0290
0021            I=NPTS(J)                                                       0300
          C-----PRINT OUT TOTAL FIELD PATTERNS FOR GUIDE A                      0310
0022            PRINT 40,(FIELD(K,J),K=1,I)                                     0320
0023      40    FORMAT (//' EAX FIELD'/(1X,8G15.7))                             0330
          C-----WRITE TOTAL FIELD PATTERNS ON DISK                              0340
0024            WRITE (8) I,(FIELD(K,J),K=1,I)                                  0350
0025      50    CONTINUE                                                        0360
          C-----POSITION TO FIRST RECORD ON DISK AGAIN                          0370
0026            REWIND 4                                                        0380
0027            DO 60 K=1,120                                                   0390
0028            DO 60 J=1,3,2                                                   0400
0029      60    FIELD(K,J)=0.                                                   0410
0030            DO 70 MODE=2,N                                                  0420
          C-----TWO DUMMY READS TO SKIP FIELDS IN A                             0430
0031            READ (4) I,(FEY(K),K=1,I)                                       0440
0032            READ (4) I,(FEY(K),K=1,I)                                       0450
0033            DO 70 J=1,3,2                                                   0460
          C-----READ E AND H FIELDS IN GUIDE B                                  0470
0034            READ (4) I,(FEY(K),K=1,I)                                       0480
0035            NPTS(J)=I                                                       0490
0036            DO 70 K=1,I                                                     0500
          C-----SUM UP TOTAL FIELD PATTERN IN GUIDE B                           0510
0037      70    FIELD(K,J)=FIELD(K,J) + VEC(MODE)*FEY(K)                        0520
0038            DO 80 J=1,3,2                                                   0530
0039            I=NPTS(J)                                                       0540
          C-----WRITE TOTAL FIELD PATTERNS ON DISK                              0550
0040            WRITE (8) I,(FIELD(K,J),K=1,I)                                  0560
          C-----PRINT OUT TOTAL FIELD PATTERNS E AND H FIELD GUIDEB             0570
0041      80    PRINT 90,(FIELD(K,J),K=1,I)                                     0580
0042      90    FORMAT (//' EBX FIELD'/(1X,8G15.7))                             0590
0043            RETURN                                                          0600
```

```
0001            SUBROUTINE INTEGR(FX,GX,XL,XU,EPS,Y)                    0010
0002            LOGICAL TWO                                             0020
0003            XD=XU - XL                                              0030
0004            H=XD/100                                                0040
0005            N=XD/H + .5                                             0050
0006            N=N/2*2                                                 0060
0007            H=XD/N                                                  0070
0008            NM1=N - 1                                               0080
0009            SUM=0                                                   0090
0010            TWO= .FALSE.                                            0100
0011            DO 10 I=1,NM1                                           0110
0012            X=I*H + XL                                              0120
0013            MULT=4                                                  0130
0014            IF (TWO) MULT=2                                         0140
0015            TWO= .NOT. TWO                                          0150
0016       10   SUM=SUM + MULT*FX(X)*GX(X)                              0160
0017            SIMP=(SUM + FX(XU)*GX(XU) + FX(XL)*GX(XL))*H/3          0170
0018            Y=SIMP                                                  0180
0019            RETURN                                                  0190
0020            END                                                     0200
```

A.2 Modal Equation Derivation

This derivation is given by Wexler (1) and is repeated here with a slight generalization for the incident mode.

Assuming an orthogonal set of modes in each guide section described in (2.1) to (2.4), then for $i \neq m$

$$\int_a \bar{e}_{ai} \times \bar{h}_{am} \cdot \tilde{u}_z \, ds = 0 \qquad (A.2.1)$$

Equate (2.1) and (2.3) and take the cross product of each term with $\bar{h}_{am}$ and integrate over the cross section of $A^{(m)}$. For simplicity of notation let

$$\alpha_i = \int_a \bar{e}_{ai} \times \bar{h}_{ai} \cdot \tilde{u}_z \, ds \qquad (A.2.2)$$

and

$$\delta_{ji} = \int_b \bar{e}_{bj} \times \bar{h}_{ai} \cdot \tilde{u}_z \, ds \qquad (A.2.3)$$

we then have

$$(1 + \rho_r) \, a_r \, \alpha_r = \sum_{j=1}^{\infty} b_j \, (\delta_{jr} + \sum_{k=1}^{\infty} s_{jk} \, \delta_{kn}) \quad m = r \qquad (A.2.4)$$

and

$$a_i \, \alpha_i = \sum_{j=1}^{\infty} b_j \, (\delta_{ji} + \sum_{k=1}^{\infty} s_{jk} \, \delta_{ki}) \quad m \neq r \qquad (A.2.5)$$

Equate (2.2) and (2.4) and take the cross product of each term with $\bar{e}_{bn}$ and integrate over the cross section of guide $B^{(m)}$. Using the orthogonality condition

$$\int_b \bar{e}_{bn} \times \bar{h}_{bj} \cdot \tilde{u}_z \, ds \qquad (A.2.6)$$

and the notation

$$\beta_j = \int_b \bar{e}_{bj} \times \bar{h}_{bj} \cdot \tilde{u}_z \, ds$$

we have

$$(1 - \rho_r)\, a_r\, \delta_{nr} - \sum_{\substack{i=1 \\ i \neq r}}^{\infty} a_i\, \delta_{ni} = (b_n - \sum_{j=1}^{\infty} b_j\, s_{jn})\, \beta_n \qquad \text{(A.2.7)}$$

using continuity of the magnetic field.

Substituting (A.2.5) into (A.2.7) and re-arranging we get

$$\rho_r\, \delta_{nr} + \sum_{j=1}^{N} \frac{b_j}{a_r} \sum_{\substack{i=1 \\ i \neq r}}^{M} \frac{\delta_{ji} + \sum_{k=1}^{N} s_{jk}\, \delta_{ki}\, \delta_{ni}}{\alpha_i} \qquad \text{(A.2.8)}$$

and from (A.2.4) we get

$$\rho_r\, \alpha_r - \sum_{j=1}^{N} \frac{b_j}{a_r} (\delta_{jr} + \sum_{k=1}^{N} s_{jk}\, \delta_{kr}) = -\alpha_r \qquad \text{(A.2.9)}$$

The equations were truncated at M and N modes in waveguides A[1] and B respectively. The boundary enlargement problem results in similar equations but some integrations are over different limits and there are are some sign changes. For completeness the general equations are given in (2.5) to (2.12).

Equation (A.2.5) can be re-written to give $\frac{a_i}{a_r}$ and is given in (2.13).

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