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TOPICS IN BALANCED INCOMPLETE BLOCK DESIGNS

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ROGER A. KINGSLEY

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By: Roger Alfred Kingsley

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To Nancy, Andrew, Paul, and Michael

ABSTRACT

In this thesis, geometric methods of constructing symmetric balanced incomplete block designs with parameters (36, 15, 6), (45, 12, 3), and (56, 11, 2) are described. Other methods of constructing designs with the same parameters are surveyed. Methods of testing designs for isomorphism are given, and these methods applied to the designs which we have listed. An appendix is included, which contains computer programs for construction of incidence matrices for the designs, programs for conducting isomorphism tests, listings of the block designs, and the results of the isomorphism tests.

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Chapter 1 - Introduction

The object of this thesis is twofold. First, we shall present some geometric constructions for balanced incomplete block designs. Second, we shall describe some methods of testing different designs for isomorphism, and apply these methods to several sets of known designs. In this chapter, we define balanced incomplete block designs and give some elementary results here as well.

The material in this chapter is elementary, and may be found in a variety of sources. In particular, Hall [18] and Ryser [30] discuss both BIBD's and finite geometries. Stanton [37] provides an excellent source for some of the relevant aspects of Combinatorial Theory.

§1.1 - Balanced Incomplete Block Designs

A balanced incomplete block design (BIBD) with parameters (v, b, r, k, λ) is a pair of sets,

$$\mathcal{V} = \{V_1, V_2, \ldots, V_v\} ,$$

and $$\mathcal{B} = \{B_1, B_2, \ldots, B_b\} ,$$

called, respectively, varieties and blocks, and an incidence relation between varieties and blocks such that

(i) each variety is incident with r blocks,

(ii) each block is incident with k varieties, and

(iii) every pair of distinct varieties is mutually incident with λ blocks.

From these definitions, it is immediately apparent that $v \geq k \geq 0$ and $b \geq r \geq \lambda \geq 0$. It is usual, in order that trivial cases may be avoided, to demand that these inequalities be sharpened to $v > k > 1$, and $b > r > \lambda > 0$. We are led immediately to ask whether a block design exists for any parameters (v, b, r, k, λ) satisfying the specified inequalities. That some further restrictions are necessary is seen from the following well-known theorem.

Theorem 1.1.1. If a balanced incomplete block design exists with parameters (v, b, r, k, λ), then

(i) $vr = bk$, and

(ii) $\lambda(v-1) = r(k-1)$.

Proof: (i) We shall count the number, x, of pairs (V_i, B_j) of a variety and a block which are incident. First, for a fixed variety V_i, there are r blocks B_j incident with V_i. As there are v varieties, we have $x = vr$. Similarly, by counting varieties incident with a fixed block, we obtain $x = bk$. Hence $vr = bk$.

(ii) In this case, we count the number, x, of pairs (V_i, B_j) of a variety and a block which are incident, and such that B_j is also incident with a fixed variety, V_1. First, let V_i be any variety other than V_1. There are λ blocks B_j incident with both V_i and V_1. As there are v-1 choices for V_i, we have $x = \lambda(v-1)$. On the other hand, let B_j be a fixed block incident with V_1. There are k varieties incident with B_j, one of which is V_1. Thus, from B_j, we may select V_i in k-1 ways. As there are r choices for B_j, we have $x = r(k-1)$. Hence $\lambda(v-1) = r(k-1)$. ☒

There are other restrictions on the parameters. Before we obtain these, we introduce the idea of an incidence matrix for any relation on finite sets. Let us take the finite sets

$$\mathcal{V} = \{V_1, \ldots, V_v\}$$

and $\mathcal{B} = \{B_1, \ldots, B_b\}$, and let a relation from $\mathcal{V}$ to $\mathcal{B}$ be given. Let $A = (a_{ij})$ be the $v \times b$ matrix given by

$$a_{ij} = \begin{cases} 1 & \text{if } V_i \text{ and } B_j \text{ are incident,} \\ 0 & \text{otherwise.} \end{cases}$$

We say that A is the <u>incidence matrix</u> for the relation. (If $\mathcal{V} = \mathcal{B}$, we shall always take $V_i = B_i$, $i = 1, \ldots, v$.) Obviously, the matrix A is dependent upon the order in which we number the varieties and the blocks, different numberings leading to different matrices. We shall return to this problem later, when we discuss isomorphism of designs.

Throughout this thesis, we shall let I_n denote the $n \times n$ identity matrix, and J_{mn} denote the $m \times n$ matrix (α_{ij}) given by

$$\alpha_{ij} = 1 , \qquad i = 1, \ldots, m \atop j = 1, \ldots, n$$

Whenever the context makes their values clear, we shall omit the subscripts.

Theorem 1.1.2.

(i) $AA^T = \lambda J + (r-\lambda)I$

(ii) $J_{1v}A = kJ_{1b}$

Proof: (i) The element b_{ij} in $B = AA^T$ is the inner product of row i and row j of A. These rows correspond to varieties V_i and V_j in the design, and a typical term in the inner product will be 1 if the varieties are both incident with the appropriate block, 0 otherwise. Thus b_{ij} counts the number of blocks with which V_i and V_j are incident. For $i = j$, this value is r, and for $i \neq j$, the value is λ. Thus, B contains r on its diagonal, λ elsewhere.

(ii) $J_{1v}A$ computes the column sums of A. Each column represents a block, and contains k 1's. ⊠

Theorem 1.1.3. If a $v \times b$ matrix A exists which contains only 0's and 1's, and which satisfies (i) and (ii) of the previous theorem, then there is a BIBD with parameters (v, b, r, k, λ).

We can use Theorem 1.1.2 to obtain another restriction on the parameters, known as Fisher's inequality.

Theorem 1.1.4. $b \geq v$ and $r \geq k$.

Proof: Since $vr = bk$, the two inequalities are equivalent.

Now, AA^T contains r on its diagonal, λ elsewhere, and is $v \times v$. It is easy to compute

$$\begin{aligned} \det (AA^T) &= (r-\lambda)^{v-1}(\lambda(v-1) + r) \\ &= (r-\lambda)^{v-1}(r(k-1) + r) \\ &= rk(r-\lambda)^{v-1} . \end{aligned}$$

As we have demanded that all parameters be positive and that $r > \lambda$, we see that $\det (AA^T) > 0$. Thus, AA^T is nonsingular, and consequently, $v = \text{rank } (AA^T) \le \text{rank } (A) \le \min (v, b)$. Thus $v \le b$. ⊠

Thus we see that, if $b > v$, A^T can not be the matrix of a BIBD. We shall now prove that, in the case $b = v$, the exact opposite is true: A^T does yield a BIBD.

If a balanced incomplete block design has $v = b$, we shall call the design symmetric.

Theorem 1.1.5. For a symmetric BIBD,

(i) $r = k$,

(ii) $|\det A| = k(k-\lambda)^{\frac{v-1}{2}}$, and

(iii) $A^T A = AA^T$.

Proof: (i) We have $vr = bk$ and $v = b$. Hence $r = k$.

(ii) Since A is square, det A exists, and $\det A = \det A^T$. Thus,

$$\begin{aligned} (\det A)^2 &= \det A \det A^T \\ &= \det (AA^T) = rk(r-\lambda)^{v-1} \\ &= k^2(k-\lambda)^{v-1} . \end{aligned}$$

Hence $|\det A| = k(k-\lambda)^{\frac{v-1}{2}}$.

(iii) Let $J = J_{vv}$. Let $k-\lambda = n$. We have $AA^T = nI + \lambda J = B$, and $JA = kJ$. It is clear that we also have $AJ = kJ$. By (ii), A is non-singular. From $AJ = kJ$, we derive $J = A^{-1}(kJ)$ or $A^{-1}J = k^{-1}J$.

Now
$$\begin{aligned} A^T &= A^{-1}AA^T \\ &= A^{-1}(nI + \lambda J) \\ &= nA^{-1} + \lambda A^{-1}J \\ &= nA^{-1} + \lambda k^{-1}J \end{aligned}$$

Thus
$$\begin{aligned} A^TA &= nA^{-1}A + \lambda k^{-1}JA \\ &= nI + \lambda k^{-1}kJ \\ &= nI + \lambda J = B . \end{aligned}$$ ⊠

We usually write (v, k, λ) rather than (v, v, k, k, λ) for the parameters of a symmetric BIBD.

Corollary 1.1.6. If A is the incidence matrix of a symmetric BIBD with parameters (v, k, λ), then A^T is also the incidence matrix of a symmetric BIBD with parameters (v, k, λ).

The design given by A^T is called the dual design of that given by A.

Suppose that a design with parameters (v, b, r, k, λ) has an incidence matrix A. We noted earlier that the matrix A depends upon the order in which varieties and blocks are numbered. Clearly, renumbering varieties is equivalent to permuting the rows of A, and renumbering blocks is equivalent to permuting columns. Clearly, any one of the family of matrices obtained from A by row and column permutations will serve as an incidence matrix for the design.

Let D and D' be two designs with the same parameters (v, b, r, k, λ). Let the varieties and blocks of D be $\mathcal{V}$ and $\mathcal{B}$, and let the varieties and blocks of D' be $\mathcal{V}'$ and $\mathcal{B}'$. We will say that D and D' are isomorphic if and only if there exists a pair of bijections

$$f: \mathcal{V} \to \mathcal{V}'$$
$$\text{and} \quad g: \mathcal{B} \to \mathcal{B}'$$

such that V is incident with B (in D) if and only if f(V) is incident with g(B) (in D').

Lemma 1.1.7. Let D and D' be two BIBD's with the same parameters, and let A and A' be incidence matrices for the designs. Then D is isomorphic to D' if and only if A' can be obtained from A by row and column permutations.

Lemma 1.1.8. If a symmetric design is given, then its dual design is unique up to isomorphism.

There are very strong conditions which must be satisfied by the parameters (v, k, λ) for a symmetric design.

Theorem 1.1.9. If a symmetric BIBD has parameters (v, k, λ), then

(i) $\lambda(v-1) = k(k-1)$

(ii) If $v \equiv 0(2)$, then $n = k-\lambda$ is a perfect square.

Proof: (i). We have $\lambda(v-1) = r(k-1)$ and $r = k$. Our result is immediate.

(ii) Let A be the incidence matrix for our design.

We know that det A is an integer, and have computed

$$|\det A| = k\, n^{(v-1)/2} .$$

Thus, when v is even, v-1 is odd, and n must be a perfect square. ☒

The second part of Theorem 1.1.9 is half of the well-known Chowla-Ryser Theorem. The other half of this theorem (c.f. Ryser [28]) deals with the case $v \equiv 1(2)$ and asserts

Theorem 1.1.10. If a symmetric BIBD has parameters (v, k, λ), $v \equiv 1(2)$, then the equation

$$z^2 = nx^2 + (-1)^{(v-1)/2} \lambda\, y^2$$

has a non-trivial solution in integers. (Again, $n = k-\lambda$.)

We note that, at the present time, no counter-example has been obtained for the hypothesis that the conditions of Theorems 1.1.9 and 1.1.10 are sufficient to guarantee the existence of a (v, k, λ) design.

Lemma 1.1.11. The parameter sets (36, 15, 6), (45, 12, 3), and (56, 11, 2) all satisfy the conditions of Theorem 1.1.9 or 1.1.10.

Proof: In each case, $\lambda(v-1) = k(k-1)$ holds, and $n = k-\lambda = 9$. For (36, 15, 6) and (56, 11, 2), we are done when we note that n is a perfect square. For (45, 12, 3), Theorem 1.1.10 must be applied, and we see that $(x, y, z) = (1, 0, 3)$ is a non-trivial solution to the Diophantine equation. ☒

We shall construct designs with these parameters.

There is one method of presenting BIBD's which allows a concise description of the design, and a short method of testing that the purported design exists. This is the construction by means of a group difference set.

A <u>group difference set</u> (or <u>difference set</u>) with parameters (v, k, λ) is a k-subset $\{g_1, g_2, \ldots, g_k\}$ of an abelian group G of order v such that if d is a non-zero element of G, there are exactly λ ordered pairs (g_i, g_j) such that $d = g_i - g_j$.

A simple example is afforded by {1, 2, 4} in the group of integers modulo 7. This is a (7, 3, 1) difference set. We shall allow one abuse of standard mathematical notation, which is best illustrated by an example. It is well known that a (16, 6, 2) difference set is given by {(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 0, 0), (0, 0, 1, 1)} in the group $C_2 \oplus C_2 \oplus C_2 \oplus C_2$. We shall abbreviate this by saying that {1000, 0100, 0010, 0001, 1100, 0011 mod (2222)} is a (16, 6, 2) difference set. This notation tacitly assumes that the abelian group G is the direct sum of cyclic groups of orders < 10.

<u>Theorem 1.1.12.</u> If a (v, k, λ) difference set exists, then there is a symmetric BIBD with parameters (v, k, λ).

<u>Proof</u>: Let G be an abelian group of order v, and let K be the k-subset of G which yields a (v, k, λ) difference set. Let $\mathcal{V} = \{Vg \mid g \in G\}$ and $\mathcal{B} = \{Bg \mid g \in G\}$ be two

v-sets of formal symbols. We define an incidence relation

$$Vg \text{ is incident with } Bg'$$

if and only if $g - g' \in K$. We now prove that we have defined a (v, k, λ) BIBD.

Let $K = \{g_1, \ldots, g_k\}$. Fix a variety Vg. Now, there are exactly k choices for g' such that $g - g' \in K$, namely,

$$g' = g - g_i \ , \quad 1 \leq i \leq k \ .$$

Thus, Vg is incident with exactly k blocks. Similarly, Bg' is incident with k varieties. Finally, let Vg and Vg' be two distinct varieties. These varieties occur together in Bg" if and only if

$$g - g'' = a \in K$$

$$\text{and} \qquad g' - g'' = b \in K \ .$$

Subtracting, we have

$$g - g' = a - b \ .$$

Now, the difference $d = g - g'$ is fixed, and non-zero. Hence, there are exactly λ distinct values for g". ☒

We make one comment: if we define incidence by "Vg is incident with Bg' if and only if $g + g' \in K$ ", the theorem remains valid. The two designs thus obtained are isomorphic (cf. Theorem 4.2.4).

§1.2 - Projective Geometry

In this section, we define a projective geometry, and present a family of explicit examples of projective geometries. While this family will not exhaust all projective geometries, it is sufficiently large for our purposes.

A projective geometry is a system composed of two disjoint sets, P and L, called points and lines respectively, and a relation $I \subseteq P \times L$ called incidence, which satisfies certain axioms. Before presenting the axioms, we introduce some optional terminology for the incidence relation: if the point P and the line ℓ are incident, we may also say "P is on ℓ", "ℓ passes through P", or even "ℓ contains P". If there is a point P incident with two distinct lines k and ℓ, we will say that "k meets ℓ" or, more explicitly, that "k meets ℓ in P". The axioms which our objects must satisfy are three in number:

(A1) There are at least four distinct points, and no three of these are incident with a single line.

(A2) Every pair of distinct points is contained in exactly one line. (The points A and B lie on a line denoted by AB.)

(A3) If A, B, C, D are four distinct points, and if AB meets CD, then AC meets BD.

(A4) Every line is incident with at least three distinct points.

Our definition makes it clear that a line and the set of points on that line are distinct entities. We shall generally not distinguish between them, allowing a certain amount of set terminology as a convenience.

Projective geometries may be derived from vector spaces; so we present a minimal amount of linear algebra here, without proof.

Let F be a field, and let V be a vector space over F. Underscored Roman letters denote vectors.

i) If $X \subseteq V$, $S(X)$ is the smallest subspace of containing X.

ii) If U_1 and U_2 are subspaces of V, then $U_1 + U_2 = S(U_1 \cup U_2)$.

iii) $\dim U_1 + \dim U_2 = \dim (U_1 + U_2) + \dim (U_1 \cap U_2)$

iv) $S(X) + S(Y) = S(X \cup Y)$

<u>Theorem 1.2.1</u>. Let V be a vector space of dimension 3 or greater over F. If we call 1-dimensional subspaces, points, and 2-dimensional subspaces, lines, and if we define incidence by containment, then we have a projective geometry.

<u>Proof</u>: Since V has dimension at least 3, there are at least 3 independent vectors in V . Let $\underline{e}_1, \underline{e}_2, \underline{e}_3$ be independent. Let $\underline{f} = \underline{e}_1 + \underline{e}_2 + \underline{e}_3$. It is clear that $S(\underline{e}_1)$, $S(\underline{e}_2)$, $S(\underline{e}_3)$ and $S(\underline{f})$ are 4 distinct points, and that any 3 of the points generate $S(\underline{e}_1, \underline{e}_2, \underline{e}_3)$, which has dimension 3. Thus, no 3 of the points lie on a single line, and (A1) holds. Secondly, the points $S(\underline{u})$ and $S(\underline{v})$ are distinct if and only if $\{\underline{u}, \underline{v}\}$ is an independent set. In this case, $S(\underline{u}, \underline{v})$ is the unique line containing $S(\underline{u})$ and $S(\underline{v})$.

This verifies (A2). Third, let $S(\underline{u}_i)$, $i = 1, \ldots, 4$, be distinct points. If $S(\underline{u}_1, \underline{u}_2)$ meets $S(\underline{u}_3, \underline{u}_4)$, their intersection has dimension ≥ 1, and $\dim S(\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4) \leq 3$. If, on the other hand, $S(\underline{u}_1, \underline{u}_3)$ and $S(\underline{u}_2, \underline{u}_4)$ do not meet, $\dim S(\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4)$ is 4, a contradiction. Thus, (A3) holds. Finally, to see (A4), we note that the line $S(\underline{u}, \underline{v})$ is incident with the points $S(\underline{u})$, $S(\underline{v})$, and $S(\underline{u} + \underline{v})$. Thus, we have a projective geometry. ⊠

If V is the vector space F^{n+1} $(n \geq 2)$, we denote the geometry of Theorem 1.2.1 by $PG(n, F)$. If, further, $F = GF(s)$ where s is a prime power, we will write $PG(n, s)$.

In the vector space F^{n+1}, every vector is an $(n+1)$-tuple. A point in the corresponding geometry is a 1-dimensional subspace, and this is the set of all scalar multiples of any non-zero vector in the subspace. We agree that any non-zero vector shall serve as coördinates for its corresponding point. From a set of coördinates for a particular point may be obtained <u>other</u> coördinates for the <u>same</u> point, by multiplying by any non-zero scalar. In particular, in $PG(n, s)$, every point has $(s-1)$ distinct coördinate sets. These are called <u>homogeneous coördinates</u>.

In $PG(n, F)$, an <u>r-flat</u> is the set of points in an $(r+1)$-dimensional subspace of the underlying space F^{n+1}. This has meaning for $-1 \leq r \leq n$, although a (-1)-flat is the null set. In this terminology, the points are 0-flats, the lines are 1-flats, and there is exactly one n-flat, namely the set of all points in the geometry. For PG(n, s) , we can count various quantities associated with r-flats.

Theorem 1.2.2. In PG(n, s) , any r-flat contains $\frac{s^{r+1} - 1}{s - 1}$ points.

Proof: We are, in fact, counting the number of points in an (r+1)-dimensional subspace. There are s^{r+1} vectors in the subspace, of which one is the zero vector. The remaining vectors determine each point s-1 times, whence the number of points is $\frac{s^{r+1} - 1}{s - 1}$. ☒

Corollary 1.2.3. The number of points in PG(n, s) is $(s^{n+1}-1)/(s-1)$.

Corollary 1.2.4. The number of points on any line (1-flat) is s+1.

As we shall be particularly interested in planes (2-flats), we note

Corollary 1.2.5. PG(2, s) contains $1 + s + s^2$ points.

There are "dual" results for corollaries 1.2.4 and 1.2.5 in the sense of Hall [17].

Theorem 1.2.6. In PG(2, s), any two lines meet.

Proof: Let $S(\underline{u}_1, \underline{u}_2)$ and $S(\underline{u}_3, \underline{u}_4)$ be lines. Since the underlying vector space has dimension 3, then $\dim S(\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4) \leq 3$, whence $\dim (s(\underline{u}_1, \underline{u}_2) \cap S(\underline{u}_3, \underline{u}_4)) \geq 1$, and there is at least a point in common. ☒

<u>Corollary 1.2.7</u>. Through every point pass $s+1$ lines.

<u>Proof</u>: Let P be a point, and let ℓ be a line not through P. Every line through P meets ℓ in a point, and every point of ℓ determines a line through P. Thus, the number of lines through P is equal to the number of points on ℓ, namely $s+1$. ☒

<u>Corollary 1.2.8</u>. There are $1 + s + s^2$ lines in $PG(2, s)$.

<u>Lemma 1.2.9</u>. $PG(2, s)$ is a symmetric BIBD with parameters $(s^2 + s + 1, s + 1, 1)$.

Chapter 2 - A Design from a Finite Plane

§2.1 - Arcs and Ovals

In this section, we discuss the existence of sets of points in a finite plane, no three collinear. Upper bounds on the size of such sets are obtained, and the number of such sets counted for certain values.

The results 2.1.1 - 2.1.8 are originally due to Bose [7]. The geometric proofs come from Quist [29]. The counting theorems 2.1.9 - 2.1.14 are due to the author.

Consider the finite projective plane $PG(2, s)$, where s is a prime power. We say that a set of points is an <u>arc</u> if no three of the points are collinear. More specifically, an arc containing m points is an <u>m-arc</u>.

Two numbers associated with the arcs of $PG(2, s)$ are of interest. First, we should like to know the greatest integer $m = m(s)$ for which an m-arc exists. Second, we should like to know how many k-arcs there are, for $1 \le k \le m(s)$. We shall call this number $n(k, s)$. Finally, an $m(s)$-arc is termed an <u>oval</u>.

The number $m(s)$ has been studied in a more general context by Quist [29] and others. Let s be a prime power, and V the vector space of dimension r over $GF(s)$. Then $m(t, r, s)$ is used to denote the cardinality of the largest set of vectors in V with the property that any t are independent. It is evident that our quantity $m(s)$ is, in fact, $m(3, 3, s)$.

Theorem 2.1.1. $m(s) \leq s+2$

Proof: Let A be an m-arc in $PG(2, s)$, and let P be any point of A. There are $s+1$ lines through A, each of which may contain at most one further point of A. Thus

$$m \leq 1 + (s+1) = s+2 \qquad \boxtimes$$

Theorem 2.1.2. $s+1 \leq m(s)$

Proof: Let

$A = \{(0, 0, 1), (0, 1, 0)\} \cup \{(1, \alpha, \alpha^{-1}) | \alpha \in GF(s) \text{ and } \alpha \neq 0\}$

A is clearly an $(s+1)$-set, and we need only test whether A is an arc. We may test any three points of A by the determinant test. There are four cases.

Case (i) Consider the points $(1, \alpha, \alpha^{-1})$, $(0, 1, 0)$, $(0, 0, 1)$. We have

$$\begin{vmatrix} 1 & \alpha & \alpha^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0 ,$$

so the points are not collinear.

Case (ii) Consider the points $(1, \alpha, \alpha^{-1})$, $(1, \beta, \beta^{-1})$, $(0, 1, 0)$, where $\alpha \neq \beta$. We have

$$\begin{vmatrix} 1 & \alpha & \alpha^{-1} \\ 1 & \beta & \beta^{-1} \\ 0 & 1 & 0 \end{vmatrix} = \alpha^{-1} - \beta^{-1} \neq 0 .$$

Again, the points are not collinear.

Case (iii) Consider the points $(1, \alpha, \alpha^{-1})$, $(1, \beta, \beta^{-1})$, $(0, 0, 1)$, where $\alpha \neq \beta$. We have

$$\begin{vmatrix} 1 & \alpha & \alpha^{-1} \\ 1 & \beta & \beta^{-1} \\ 0 & 0 & 1 \end{vmatrix} = \beta - \alpha \neq 0 .$$

Again, the points are not collinear.

Case (iv) Consider the points $(1, \alpha, \alpha^{-1})$, $(1, \beta, \beta^{-1})$, $(1, \gamma, \gamma^{-1})$, where α, β, γ are all distinct. We have

$$\begin{vmatrix} 1 & \alpha & \alpha^{-1} \\ 1 & \beta & \beta^{-1} \\ 1 & \gamma & \gamma^{-1} \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha\beta\gamma)^{-1} ,$$

which is non-zero, as each of its factors is non-zero. Again, the points are not collinear.

As this includes all possible cases of three points from A, we conculde that A is an $(s+1)$-arc, and that

$$s+1 \leq m(s) . \qquad \boxtimes$$

Before we proceed to determine which of $s+1$ or $s+2$ is the correct value for $m(s)$, we shall define some additional terms. Let A be an arc, and ℓ any line. As no three points of A are collinear, the line ℓ meets A in 0, 1, or 2 points. We shall say that ℓ is a <u>skew line</u>, a tangent, or a secant to A, according as the number is 0, 1, or 2.

Theorem 2.1.3. An $(s+1)$-arc has a unique tangent at every point.

Proof: Let P be a point of the $(s+1)$-arc A. There are $s+1$ lines through P, s of which are secants (one for each remaining point of A). This leaves one and only one line through P, which forms a tangent. ⊠

We note that, if an $(s+2)$-arc exists, each of its $(s+1)$-subsets is an $(s+1)$-arc.

Theorem 2.1.4. An $(s+1)$-arc may be extended to an $(s+2)$-arc in at most one way. This extension is possible if and only if the tangents of the $(s+1)$-arc are concurrent.

Proof: Suppose the tangents to an $(s+1)$-arc, A, concur in the point P. Then $A' = A \cup \{P\}$ is an $(s+2)$-set. Now, there are $s+1$ tangents to A, all passing through P. Thus, no secants of A pass through P, and no three points of A' may be collinear, so A' is an $(s+2)$-arc.

Second, suppose $A' = \{P\} \cup A$ is an $(s+2)$-arc. The lines from P to points of A are clearly the tangents to A; so these tangents concur in P. As there is at most one point P in which all tangents may concur, there is at most one extension of A to an $(s+2)$-arc. ⊠

Theorem 2.1.5. If s is even, the tangents to any $(s+1)$-arc are concurrent, and the arc can be extended to an $(s+2)$-arc.

Proof: Let A be an $(s+1)$-arc, and let P and Q be distinct points of A. The line PQ is a secant. Let X be any point on the line PQ. There are a number, say n, of secants through X. There are then $2n$ points of A lying on secants through X. However, $s+1$ is odd, so there is at least one point, Y, which is in A, not on a secant through X. This means that the line XY is a tangent. Thus, through every point on a secant passes at least one tangent. There are $s+1$ points on any secant, and $s+1$ tangents. Thus, through every point on a secant passes exactly one tangent. If P is a point where any two tangents meet, there are no secants through P, whence the $(s+1)$ lines joining P to points of A are all the tangents to A, and $A \cup \{P\}$ is an $(s+2)$-arc. ☒

Theorem 2.1.6. An $(s+2)$-arc has no tangents.

Proof: Let A be an $(s+2)$-arc, and let P be a point on A. Through P pass $s+1$ secants to A, one for each further point of A. This accounts for all the lines through P, so there is no tangent at P. ☒

Theorem 2.1.7. If s is odd, $m(s) \leq s+1$.

Proof: Let A be an $(s+2)$-arc, and let P be a point not on A. The lines through P consist of secants and skew lines, as theorem 2.1.6 ensures no tangents. Thus, the number of points on A is even, being twice the number of secants through P. This contradicts the fact that $s+2$ is odd; thus, an $(s+2)$-arc cannot exist, and $m(s) \leq s+1$. ☒

Gathering these results together, we have

Theorem 2.1.8. $m(s) = \begin{cases} s+1 , & s \text{ odd} \\ s+2 , & s \text{ even} \end{cases}$

Proof: For s odd, theorems 2.1.2 and 2.1.7 yield $m(s) = s+1$.

For s even, theorems 2.1.1 and 2.1.5 yield $m(s) = s+2$. ☒

Henceforth, we use the term "oval" to denote an $m(s)$-arc in $PG(2, s)$.

Theorem 2.1.9. $n(1, s) = s^2 + s + 1$

Proof: Evidently, $n(1, s)$ is the number of points in $PG(2, s)$. ☒

Before proceeding to determine $n(k, s)$ for values of $k \geq 2$, we introduce some further functions. Let $t(k, s)$ be the number of ordered sets of k distinct points which form a k-arc (more briefly, an ordered k-arc).

Lemma 2.1.10. $t(k, s) = k! \; n(k, s)$.

Proof: There are $n(k, s)$ k-arcs. Each gives rise to $k!$ ordered k-arcs, one for each permutation of the vertices. As these are all distinct, and comprise all ordered k-arcs, we have our result. ☒

Let $e(k, s)$ be the number of points which may not be chosen to augment a k-arc into a (k+1)-arc, and let $f(k, s)$ be the number of points which may be chosen. Evidently, $e(k, s) + f(k, s) = s^2 + s + 1$. As we shall see, these numbers do not depend on the particular k-arc chosen, provided that k is sufficiently small.

Lemma 2.1.11. If $f(k, s)$ is invariant, then

$t(k+1, s) = t(k, s) \times f(k, s)$.

Proof: An ordered (k+1)-arc can be obtained uniquely by first selecting an ordered k-arc, and then augmenting this to a (k+1)-arc. The first step can be performed in $t(k, s)$ ways, and, assuming that $f(k, s)$ is invariant, each ordered k-arc so chosen may be extended in $f(k, s)$ ways. Thus

$t(k+1, s) = t(k, s) \times f(k, s)$. ⊠

Corollary 2.1.12. $n(k+1, s) = \dfrac{n(k, s) \times f(k, s)}{k+1}$, provided $f(k, s)$ is invariant.

Proof: Substitute the result of lemma 2.1.10 for both $t(k, s)$ and $t(k+1, s)$ in the equation of lemma 2.1.11. The result is immediate. ⊠

Theorem 2.1.13. If $k \le 5$, the quantity $e(k, s)$ is invariant, and is given by

$$\begin{aligned} e(0, s) &= 0 \\ e(1, s) &= 1 \\ e(2, s) &= s+1 \\ e(e, s) &= 3s \\ e(4, s) &= 6s - 5 \\ e(5, s) &= 10s - 20 \end{aligned}$$

Proof: For $k = 1$, it is evident that we may choose any point other than the one already taken.

Now, let the k-arc be $A = \{P_1, \ldots, P_k\}$, $k \ge 2$.

We have to count the points on the secants of A. There are $\binom{k}{2}$ secants, each containing $s+1$ points. However, the quantity $(s+1)\binom{k}{2}$ counts several points more than once. The points of A are each counted $k-1$ times, so we must subtract $k(k-2)$ as a correction for this fact. There is also the problem that, for $k \geq 4$, there are "diagonal" points, that is, points lying on 2 or more secants, but not part of A. It is here that the condition $k \leq 5$ appears. We note that a point not in A can lie on at most 2 secants, for otherwise, $k \geq 6$. Now, each diagonal point is determined by 2 secants, which are in turn determined by any 4 points of A. Each set of 4 points of A determines 3 diagonal points, whence the number of diagonal points is $3\binom{k}{4}$. Each diagonal point is counted twice; so we must further subtract $3\binom{k}{4}$ as a correction. This yields

$$e(k, s) = \binom{k}{2}s + \binom{k}{2} - k(k-2) - 3\binom{k}{4},$$

and computing these quantities for $k = 2, 3, 4, 5$, gives the stated results. $\boxtimes$

Corollary 2.1.14.

$$\begin{aligned}
f(0, s) &= s^2 + s + 1 \\
f(1, s) &= s(s+1) \\
f(2, s) &= s^2 \\
f(3, s) &= (s-1)^2 \\
f(4, s) &= (s-2)(s-3) \\
f(5, s) &= s^2 - 9s + 21
\end{aligned}$$

We note the consistency of these formulae with earlier theorems. When s is even, we should have $f(s+1, s) = 1$ and $f(s+2, s) = 0$; for s odd, we should have $f(s+1, s) = 0$.

These are exactly the results obtained here for small values of s, namely

$$f(3, 2) = 1,$$
$$f(4, 2) = 0,$$
$$f(4, 3) = 0,$$
$$\text{and}\quad f(5, 4) = 1.$$

We conclude this section by tabulating $f(k, s)$ and $n(k, s)$ for $s = 2, 3, 4, 5$, and $1 \le k \le m(s)$.

Table 2.1.1 -- f(k, s)

k \ s	2	3	4	5
0	7	13	21	31
1	6	12	20	30
2	4	9	16	25
3	1	4	9	16
4	0	0	2	6
5			1	1
6			0	0

Table 2.1.2 -- n(k, s)

k \ s	2	3	4	5
1	7	13	21	31
2	21	78	210	465
3	28	234	1120	3875
4	7	234	2520	15500
5			1008	18600
6			168	3100

§2.2 - The Ovals of PG(2, 4)

We now fix our attention on PG(2, 4), the finite plane of 21 points and lines. According to Theorem 2.1.8, an oval is a 6-arc, and the number of ovals is 168, as indicated in Table 2.1.2. The results of this section all come from Edge [15].

Several other quantities will be of value. First, let $f(k)$ be the number of ovals containing a fixed k-arc. Let $g(k)$ be the number of ovals meeting a fixed oval exactly in a specified k-arc. Let $h(k)$ be the number of ovals meeting a specified oval in exactly k points. Evidently, f, g, and h are defined for $0 \leq k \leq 6$. The values of f, g, and h are given in

Table 2.2.1

k	0	1	2	3	4	5	6
f(k)	168	48	12	3	1	1	1
g(k)	10	12	3	2	0	0	1
h(k)	10	72	45	40	0	0	1

We now establish these values.

<u>Theorem 2.2.1</u>. Any quadrangle (4-arc) is contained in exactly one oval.

<u>Proof</u>: Referring to Table 2.1.1, we see that $f(4, 4) = 2$. This implies that there are only 2 points available for extending a quadrangle, and so at most one 6-arc. That we may use both points to obtain an oval is a consequence of Theorem 2.1.5, or

of the fact that $f(5, 4) = 1$. ⊠

Equivalently, we may state

Corollary 2.2.2. If 2 ovals have 4 or more points in common, they are coincident.

Hence

Corollary 2.2.3. The values given in Table 2.2.1 are correct for $4 \le k \le 6$.

If we denote by $F(k, m, s)$ the number of m-arcs containing a fixed k-arc (for $0 \le k \le m \le 6$), in $PG(2, s)$, we have

Theorem 2.2.4. $$F(k, m, s) = \frac{\pi_{i=k}^{m-1} f(i, s)}{(m-k)!}$$

where we understand that $F(k, k, s) = 1$.

Proof: Evidently, $(m-k)!F(k, m, s)$ counts the number of ordered sets which may be used to extend a k-arc to an m-arc, and this quantity is exactly $\pi_{i=k}^{m-1} f(i, s)$. ⊠

Corollary 2.2.5. The values of $f(k)$ are as asserted in Table 2.2.1.

Proof: $f(k) = F(k, 6, 4)$.

Lemma 2.2.6. $h(k) = \binom{6}{k} g(k)$

Proof: Each oval contains $\binom{6}{k}$ k-arcs.

Theorem 2.2.7. The values of $g(k)$ and $h(k)$ given in Table 2.2.1 are correct.

Proof: Let H be a fixed oval, and let A be a 3-arc contained in H . A is contained in $f(3) = 3$ ovals. One of these is H , and the other two have at most 3 points in common with H , by Corollary 2.2.2. Thus, these two ovals have exactly A in common with H , and $g(3) = 2$. Now let A be a fixed 2-arc in H . There are $f(2) = 12$ ovals containing A . Of these 12, one is H . Also, there are 4 ways to choose a third point of H , and, for each of these cases, there are $g(3) = 2$ ovals meeting H in 3 points. Thus 8 ovals of the 12 meet H in 3 points. Excluding these, we see that $g(2) = 12 - 1 - 8 = 3$. Similarly, we see that

$$\begin{aligned} g(1) &= f(1) - 1 - \binom{5}{2} g(3) - \binom{5}{1} g(2) \\ &= 48 - 1 - 20 - 15 = 12 , \end{aligned}$$

and

$$\begin{aligned} g(0) &= f(0) - 1 - h(1) - h(2) - h(3) \\ &= 168 - 1 - 72 - 45 - 40 \\ &= 10 . \end{aligned}$$

☒

Theorem 2.2.8. In PG(2, s) , an (s+1)-arc has $\binom{s+1}{2}$ secants, (s+1) tangents, and $\binom{s}{2}$ skew lines. An (s+2)-arc has $\binom{s+2}{2}$ secants, 0 tangents, and $\binom{s}{2}$ skew lines.

Proof: The number of secants for a k-arc is $\binom{k}{2}$. By Theorems 2.1.3 and 2.1.6, (s+1)-arcs and (s+2)-arcs have, respectively, s+1 and 0 tangents. However,

$$\binom{s+1}{2} + (s+1) = \frac{s^2 + 3s + 2}{2} = \binom{s+2}{2}$$

so, in both cases, the number of skew lines is

$$s^2 + s + 1 - \frac{s^2 + 3s + 2}{2} = \binom{s}{2} . \qquad \boxtimes$$

Corollary 2.2.9. An oval (in PG(2, 4)) has 6 skew lines.

Let us use the term hexagram for the figure dual to a 6-arc, that is, for 6 lines, no 3 of which are concurrent.

Theorem 2.2.10. The 6 lines skew to an oval form a hexagram.

Proof: Let P be a point where two skew lines meet. There are always 3 secants through P (because there are no tangents). Thus, if 3 skew lines were concurrent in P, we should have 6 lines through P, whereas there are in fact only $s+1 = 5$ such lines. $\boxtimes$

We can use the hexagram skew to a fixed oval, A, to define all the ovals disjoint from A.

Theorem 2.2.11. If the hexagram skew to an oval A be partitioned into two triangles, the vertices of these triangles form an oval disjoint from A. Conversely, every oval disjoint from A arises in this way.

Proof: As the 6 lines skew to A form a hexagram, any partition of the lines into two triangles does, in fact, yield 6 distinct vertices, for, otherwise, we should have four of the lines concurrent. Let the vertices of the triangle be P_1, P_2, P_3, and P_4, P_5, P_6. If some 3 of these points were collinear, they would have to be divided as 2 from one triangle, one from the other. (If all 3 were from one triangle, they would, in fact, be the same point, and 3 of the skew lines would then concur in that point).

Assume now that the three collinear points are P_1, P_2, and P_4. In this case, P_4 lies on the lines P_1P_2, P_4P_5, and P_4P_6, all of which are lines of the hexagram. Thus we may conclude that the vertices of the triangles form an oval, and, as these points all lie on lines skew to A, this oval is disjoint from A.

That this accounts for all ovals disjoint from A may be seen by counting. There are $\frac{1}{2}\binom{6}{3} = 10$ ways of partitioning the hexagram into triangles, and hence, there are 10 ovals disjoint from A which arise out of this construction. The total number of ovals disjoint from A is $g(0) = h(0) = 10$; hence, all have been constructed. ☒

<u>Theorem 2.2.12.</u> Let A, B, and C be three distinct ovals such that $A \cap B = A \cap C = \emptyset$. Then B and C have two points in common.

<u>Proof</u>. Let the hexagram of lines skew to A be $\{a, b, c, d, e, f\}$. Then B and C arise from partitions of the hexagram into triangles. Let these partitions be, respectively, abc/def and abd/cef. It is immediately evident that the points ab and ef are common to both B and C, and additional common points would violate the hexagram property. ☒

Corollary 2.2.13. There do not exist three mutually disjoint ovals.

We recall that, by Theorem 2.2.1, an oval is determined by any 4 of its points. This fact permits us to describe a geometric method of finding the 2 extending points, when given a 4-arc.

Theorem 2.2.14. If A is a quadrangle (4-arc), the diagonal points of A are collinear. This line contains 5 points. The two points which are not diagonal points are the points which extend A to an oval.

Proof: That the diagonal points are collinear follows from the fact that $s = 4$ is a power of 2. The number 5, of points on this line, is, of course, $s+1$. Removing the 3 diagonal points leaves exactly 2 points remaining. Let the points of the 4-arc be $P_1, \ldots, P_4$, and let the two constructed points be X and Y. That no 3 of these six points are collinear follows by considering two cases.

First, we can not have P_i, X, and Y collinear, for this would mean that P_i is on the line of diagonal points, and, hence, infact, is a diagonal point. Secondly, we can not have P_i, P_j, and one of X or Y collinear, for P_iP_j meets the line of diagonal points in one of the diagonal points, not in X or Y.

Thus, $\{P_1, \ldots, P_4, X, Y\}$ is a 6-arc or oval; by Theorem 2.2.1, this is the unique oval containing P_1, P_2, P_3, P_4. ☒

Theorem 2.2.15. Let A, B, C be ovals such that $|A \cap B| = 2$ and $|A \cap C| = 0$. Then $|B \cap C| = 0$ or 2.

Proof: Let $A \cap B = \{P, Q\}$. Then $A = \{P, Q\} + q_0$, where q_0 is a quadrangle. The line PQ contains three other points, D_1, D_2, and D_3, which are, by Theorem 2.2.14, the diagonal points of q_0. Now, the sides of q_0 passing through D_1 contain 4 points not associated with q_0. These form a quadrangle q_1. It is clear that $A_1 = q_1 + \{P,Q\}$ is a 6-arc. If 3 points of A_1 are collinear, they can not all come from q_1 (which is a 4-arc), nor can they be P, Q and one from q_1 (else the latter point would be one of D_1, D_2, D_3). This leaves the possibility that the points are P and two points of q_1. The two from q_1 can not lie on a side of q_0 through D_1, because this line does not meet P. Thus, we have $q_1 = \{X_1, Y_1, X_2, Y_2\}$ where X_1X_2 and Y_1Y_2 meet in D_1, are sides of q_0, and the points X_1, Y_2, and P are collinear. By Theorem 2.1.6, X_1Y_2P is a secant of A, and meets A in another point, R. R can not be Q; so R is one of the points of q_0. Hence $RD_1X_1Y_2$ is a side of both q_0 and q_1, and contains another point of q_0; thus it contains 3 points of A - a contradiction. Consequently, A_1 is an oval, as likewise are the sets A_2 and A_3 determined analogously from the diagonal points D_2 and D_3 via quadrangles q_2 and q_3.

It is clear that the quadrangles q_i are mutually disjoint (and exhaust the 16 points of our plane not on PQ).

Since $g(2) = 3$, A_1, A_2, and A_3 are exactly the 3 ovals meeting A in $\{P,Q\}$. Hence B is one of A_1, A_2, A_3.

Consider now the 5 lines through D_1. One is PQ, and two more are common sides of q_0 and q_1. Hence, the other two lines are common sides of q_2 and q_3. Likewise, common sides of q_0, q_2 and q_1, q_3 pass through D_2. For D_3, the pairing is q_0, q_3 and q_1, q_2.

Now consider the hexagram of lines skew to A. These 6 lines each meet PQ, but not in either of the points P or Q. Thus, they must pass, in pairs, through each of D_1, D_2, and D_3. The two lines through D_1 do not meet q_0 and thus can not meet q_1. Thus, they must be the lines forming sides of q_2 and q_3. Similar results hold for the lines through D_2 and D_3.

By Theorem 2.2.11, C is an oval obtained by partitioning the hexagram into triangles. There are two ways in which this can happen. In the one case, the sides of one triangle contain each of D_1, D_2, and D_3. Then the sides of the other triangle do likewise. Consider the vertex on the lines through D_1 and D_2. The line through D_1 contains points from q_2 and q_3. The line through D_2 contains points from q_1 and q_3. Thus, this vertex lies in q_3. But this happens for both triangles similar results occur for D_1 and D_3 and for D_2 and D_3. Hence C meets two points from each of q_1, q_2, and q_3, and hence from A_1, A_2, and A_3.

But one of these is B , so $|B \cap C| = 0$. (This accounts for 4 of the possible ways of fixing C disjoint from A.)

In the other case (which accounts for the other 6 choices for C), the first triangle has two sides meeting in, say D_1 , and one passing through, say D_3 . The second triangle then has two sides through D_2 and one through D_3 . From the first triangle, we get D_1 and two points of q_2 . From the second, we get D_2 and two points of q_1 . Thus $|C \cap D_1| = |C \cap D_2| = 2$ and

$$|C \cap D_3| = 0 .$$

Again, B is one of A_1, A_2, A_3 . This accounts for all the cases. ☒

Corollary 2.2.16. If $|A \cap B| = 2$, there are exactly two ovals C such that $|A \cap C| = |B \cap C| = 0$.

Proof: Using the notation of Theorem 2.2.15, we may assume $B = A_1$. C must be obtained by partitioning the hexagram of lines skew to A so that the lines through D_3 are separated, and the lines through D_1, D_2, respectively, not separated. This can happen in only 2 ways. ☒

§2.3 - The Block Design (56,11,2)

We now see how a (56,11,2) block design may be obtained from the geometric properties of ovals in PG(2,4) . The results of this section are due to the author.

We now fix our attention on any one oval, A .

Let $a = \{X| \ |A \cap X| \equiv 0 \ (2)$, and X is an oval$\}$

<u>Theorem 2.3.1</u>. $|a| = 56$

<u>Proof:</u> Evidently
$$\begin{aligned}|a| &= h(0) + h(2) + h(4) + h(6)\\ &= 10 + 45 + 0 + 1\\ &= 56\end{aligned}$$

<u>Theorem 2.3.2</u>. If $X \varepsilon a$ and $X \cap Y = \emptyset$ then $Y \varepsilon a$.

<u>Proof</u>: There are clearly 4 cases depending on $|X \cap A|$. First, if $|X \cap A| = 6$, $X = A$ and the result is trivial. Second, if $|X \cap A| = 2$ and $|X \cap Y| = 0$, then $|A \cap Y| = 0$ or 2 , by Theorem 2.2.15. In either case, $Y \varepsilon A$. Thirdly, if $|X \cap A| = 0$ and $|X \cap Y| = 0$, then $|Y \cap A| = 2$ by Theorem 2.2.12. Finally, $|X \cap A| = 4$ is not possible, by Corollary 2.2.2.

☒

Now, define a relation $\equiv$ on A by

$$X \equiv Y \quad \text{if} \quad |X \cap Y| = 0 \text{ , or } X = Y \text{ .}$$

Let $[X] = \{Y | X \equiv Y\}$.

<u>Lemma 2.3.3</u>. The relation $\equiv$ is symmetric, and so is the incidence matrix for $\equiv$.

<u>Theorem 2.3.4</u>. $|[X]| = 11$

<u>Proof</u>:
$$\begin{aligned}|[X]| &= h(0) + 1\\ &= 10 + 1\\ &= 11\end{aligned}$$

☒

<u>Theorem 2.3.5</u>. If $X, Y \varepsilon a$ and $X \neq Y$, $|[X] \cap [Y]| = 2$.

<u>Proof</u>: There are two cases, $|X \cap Y| = 0$ or 2 .

If $|X \cap Y| = 0$, then $\{X,Y\} \subseteq [X] \cap [Y]$

By Corollary 2.2.13, there are no other ovals disjoint from X and Y;

so $|[X] \cap [Y]| = |\{X,Y\}| = 2$.

If $|X \cap Y| = 2$, there are exactly two ovals disjoint from both, and these comprise $|[X] \cap [Y]|$.

Corollary 2.3.6. The incidence matrix, M, for $\equiv$, yields a symmetric balanced incomplete block design which has parameters $v = b = 56$, $r = k = 11$, $\lambda = 2$. The design is self-dual.

Proof: $v = b = 56$ expresses the size of M, and hence the dimensions of the incidence matrix.

Now, $r = k = 11$ is the result that each block has 11 elements, together with the fact that M is symmetric.

Finally, $\lambda = 2$ is obtained from Theorem 2.3.5, again using the symmetry of M. ⊠

Chapter 3 - Two Cubic Surface Block Designs

In this chapter, we present the structural geometry of the double-six configuration, and indicate its relation to lines on a cubic surface in 3-space. From this, we derive symmetric balanced incomplete block designs with parameters (36, 15, 6) and (45, 12, 3) .

For this chapter, we confine our attention to PG(3, R) , real projective 3-space. Before proceeding with a discussion of the double-six, we summarize properties of ruled quadric surfaces, and consider related topics. The theory in the next section is presented without proofs. (cf. Baker [3].)

§3.1 - Preliminary Topics

A ruled quadric surface is a set of points whose homogeneous co-ordinates, after proper choice of axes have been made, satisfy

$$x_1^2 + x_2^2 = x_3^2 + x_4^2 \quad .$$

In linear algebraic terms, a ruled quadric surface is the set of points determined by

$$Q(x_1, x_2, x_3, x_4) = 0 \ ,$$

where Q is a quadratic form of rank 4 and signature 0 . The properties of ruled quadric surfaces are enumerated in the following sequence of theorems.

Theorem 3.1.1. Through every point on the surface, there pass exactly two (distinct) lines lying wholly within the surface.

Theorem 3.1.2. The lines of ruling may be partitioned into two families with the property that two lines from the same family never meet, but two lines from distinct families always meet. Consequently, the lines of ruling through a fixed point comprise one from each of the two families.

Theorem 3.1.3. If two distinct quadric surfaces have two intersecting lines as common points, then their remaining common points lie in a single plane.

Theorem 3.1.4. A ruled quadric surface is completely determined when three distinct lines from one of its families have been given.

We shall return to the ideas of theorem 3.1.4 shortly. A set X of lines is called skew if no two of the lines meet. A line ℓ is called a transversal of a set of lines X if ℓ meets each member of X . With this terminology, we can rephrase Theorem 3.1.2 as

The lines on a ruled quadric surface are partitioned into two skew sets. Each line of one set is a transversal of the other set.

We now summarize the nature of transversals for small finite sets of lines.

Theorem 3.1.5. A set of two skew lines has a unique transversal through every point not on one of the lines.

Theorem 3.1.6. A set of three skew lines has a unique transversal through any point on one of the lines.

According to Theorem 3.1.4, a set of three skew lines is contained in at most one ruled quadric surface. We now see that such a set is, in fact, contained in exactly one ruled quadric surface.

Theorem 3.1.7. The set of points on the family of transversals to a set of three skew lines forms a ruled quadric surface. The transversals are one of the skew sets of lines on the surface, and the three given lines are in the other skew set.

Theorem 3.1.8. A set of four skew lines having one transversal has, in general, exactly one more.

Proof: We give the proof of this theorem mainly to indicate conditions under which it fails to hold. Let three of the four lines be fixed. They determine a unique ruled quadric surface, and lie in one of the skew families. The transversal lies in the second family. There are several possibilities for the fourth line. First, it may lie on the surface, as part of the first skew family. In this case, the four have an infinitude of transversals (all of the second family). Second, it may be a tangent. In this case, there is no further transversal. Finally, if neither of the first two cases holds, the fourth line will meet the surface in two distinct points, one of which is already known, namely, its point of intersection with the given transversal. From Theorems 3.1.1 and 3.1.2, we see that there is a line of the second skew family through the other point of intersection with the surface. This is, then, a second common transversal to the set of four skew lines, and no further transversal is possible unless the four lines lie wholly in the quadric surface determined by any three of them. ⊠

§3.2 - The Double Six Configuration

The theorems in this section are to be found in Baker [3] and Henderson [2] . The notation for the double-six is due to Schläfli [31] , and additional historical material may be found in Sylvester [39].

A double-six or Schläfli double-six is a set of twelve distinct lines $\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$ with the intersection properties

1) a_i meets b_j for $i \neq j$
2) there are no other intersections
3) no three of the lines concur.

Lemma 3.2.1. A set of 12 lines satisfying properties 1) and 3) above and having one other intersection, lies in a plane, in which case all lines meet; or in a ruled quadric, with the a_i lines in one skew family, and the b_j in the other.

Proof: There are two cases to consider, namely the case that a_1 meets a_2, or that a_1 meets b_1. In the first case, a_1 and a_2 define a plane π. All the lines b_3, b_4, b_5, b_6 lie in this plane, as they meet a_1 and a_2. Then, a_3, a_4, a_5, and a_6 lie in this plane, as each meets three of the b-lines. Finally, this places b_1 and b_2 in the plane.

In the second case, consider the ruled quadric surface generated by a_1, a_2, a_3. Let A be the family of rulings including a_1, a_2, a_3. Let B be the other family. Now, b_1, b_4, b_5, b_6 are transversals of a_1, a_2, a_3. Thus $b_1, b_4, b_5, b_6 \in B$. Finally, b_2 and $b_3 \in B$; so all of the lines lie on a ruled quadric. ⊠

It is by no means obvious that double-six configurations actually exist. This is guaranteed by the following theorem, which migh appropriately be termed the "Fundamental Theorem of Double-Sixes".

Theorem 3.2.2. Let lines $a_1, \ldots, a_5$ and b_6 be given, so that b_6 is a transversal of $a_1, \ldots, a_5$. If the lines $a_1, \ldots, a_5$ are placed so that each four of them determine uniquely another common transversal (as in Theorem 3.1.8), then there is a unique double-six including the given lines.

Proof: There are unique lines b_1, b_2, b_3, b_4, and b_5 such that each meets four of $a_1, \ldots, a_5$ (all except its like-numbered mate). It remains to be seen that b_1, b_2, b_3, b_4, and b_5 have a common transversal. Any four of the latter have a transversal not yet noted. For example, b_1, b_2, b_3, and b_4 all are met by a_5 and a unique other line, ℓ. Likewise, b_1, b_2, b_3, and b_5 have a_4 and another line m as transversals. If we can show that ℓ and m coincide, then, denoting this single line as a_6, we have constructed a complete double-six in a unique way. Alternatively, we may show that ℓ is completely determined by the lines a_1, a_2, a_3, b_1, b_2, b_3, and b_6. We should then conclude that ℓ and m coincide.

Consider the nine points P_{12}, P_{21}, P_{13}, P_{31}, P_{23}, P_{32}, P_{16}, P_{26}, P_{36}, where P_{ij} is the intersection of a_i with b_j. Now, consider the two quadric surfaces generated by a_2, a_3, a_5 and b_2, b_3, b_4. These two surfaces share lines a_5 and b_4 (which meet one another) and so, by Theorem 3.1.3, their remaining common points lie in a single plane. P_{23} and P_{32} are such common points, as is P_{16}. These three points thus determine the plane. Finally, let X be the intersection of b_1 and ℓ. X is common to both quadrics, and so lies in the plane of P_{16}, P_{23}, P_{32}, where this plane is met by b_1. The same argument will place the intersection of b_1 and m in this plane, again at the point X. Thus, ℓ, m and b_1 concur in a single point, X. Likewise, ℓ, m, and b_3 concur in a point Z (in plane P_{36}, P_{12}, P_{21}). Thus, ℓ and m are identical, and yield the final line, a_6, for the double-six configuration. ☒

Having been given a double-six configuration, we may derive further lines from it. For example, since a_1 meets b_2, they define a plane π_{12}. Likewise, a_2 and b_1 define another plane π_{21}. The intersection of these two planes is a line which we may denote either by c_{12} or c_{21}. This method allows us to derive $\binom{6}{2} = 15$ lines c_{ij}. We state now, but reserve for later proof, a fundamental theorem.

Theorem 3.2.3 The 27-line configuration of 12 lines in a double-six and its 15 derived lines form a closed system, in the sense that, finding a second double-six amongst the 27 lines, and deriving 15 new lines from this, we are led to the same totality of 27 lines as before.

We call this configuration a complete double-six. It is connected with the set of lines on a general cubic surface; we indicate this connection by stating without proof several theorems.

Theorem 3.2.4. A general cubic surface contains exactly twenty-seven lines.

Theorem 3.2.5. The twenty-seven lines on a cubic surface form a complete double-six.

Theorem 3.2.6. There is exactly one cubic surface containing a complete double-six.

Thus we see that the study of the complete double-six is exactly the study of lines upon a general cubic surface. While we have not proved this connection, we shall feel free to use terminology which arises naturally therefrom. Our results do not in any way depend on this connection with cubic surfaces.

§3.3 - The Structure of the Complete Double-Six

The theorems 3.3.1 - 3.3.10 which establish the structure of the complete double-six come from Baker[3], and Henderson [20]. The combinatorial results 3.3.11 - 3.3.14, whereby the (36, 15, 6) design is obtained, are due to the author.

Theorem 3.3.1. The 27 lines of a complete double-six have the following intersection properties:

1) a_i meets b_j if $i \neq j$;
2) a_i meets $c_{ij} = c_{ji}$;
3) b_j meets $c_{ij} = c_{ji}$;
4) c_{ij} meets c_{mn} if $\{i, j\} \cap \{m, n\} = \emptyset$;
5) there are no further intersections.

Proof: The first property follows from the definition of the double-six, and the second and third properties follow directly from the definition of the derived lines c_{ij}.

We utilize the proof of Theorem 3.2.2 to show the dual of property (4). As in that theorem, let P_{ij} be the point of intersection of the lines a_i and b_j.

Now, let $d_{ij} = d_{ji}$ be the line joining P_{ij} to P_{ji}. Noting that we may identify the point X of Theorem 3.2.2 as P_{61}, we have that P_{16}, P_{61}, P_{23}, and P_{32} are all coplanar. Thus, d_{16} meets d_{23}. The dual result is that c_{16} meets c_{23}. This case is general.

Finally, we show that no further intersections can occur. There are two cases to consider. We may have a derived line meeting a line of the original double-six, or we may have two derived lines meeting. As an example of the first case, suppose that a_1 meets c_{23}. Then a_1, b_2, and c_{23} all meet, and hence lie in the plane π_{12}. Likewise, the lines a_1, b_3, and c_{23} all meet, and thus lie in plane π_{13}. These two planes, containing both a_1 and c_{23}, must coincide; thus, b_2 meets b_3 and we have violated a condition on the original double-six.

In the second case, suppose that c_{12} meets c_{13}. Now, b_2 and b_3 meet both these lines, and consequently lie in the plane determined by c_{12} and c_{13}. This means that b_2 and b_3 meet, again a violation of the double-six requirements. ☒

This theorem is the foundation upon which all of our structural results rest; taken as a starting point, it leads to the block designs without regard to the reality of the complete double-six.

Theorem 3.3.2. Every line meets exactly 10 others.

Proof: Lines a_1 and c_{12} are general cases. Line a_1 meets b_2 b_3, b_4, b_5, b_6, and c_{12}, c_{13}, c_{14}, c_{15}, c_{16}, and no others. Line c_{12} meets a_1, a_2, b_1, b_2, and c_{34}, c_{35}, c_{36}, c_{45}, c_{46}, c_{56}, and no others. ⊠

Theorem 3.3.3. There are 216 pairs of skew lines, and 135 pairs of lines which meet.

Proof: Each line meets 10 others, and is skew to 16. Thus, the number of ordered skew pairs is 27×16, and the number of unordered pairs is $\frac{1}{2} \times 27 \times 16 = 216$. Similarly, the number of unordered pairs of lines which meet is $\frac{1}{2} \times 27 \times 10 = 135$. ⊠

Theorem 3.3.4. A pair of skew lines has exactly 5 transversals.

Proof: We consider cases. Lines a_1 and a_2 are skew, and meet each of b_3,,b_4, b_5, b_6, and c_{12}. Lines a_1 and b_1 are skew, and meet each of c_{12}, c_{13}, c_{14}, c_{15}, c_{16}. Lines a_1 and c_{23} are skew, and meet b_2, b_3, c_{14}, c_{15}, c_{16}. Finally, lines c_{12} and c_{13} are skew and meet a_1, b_1, c_{45}, c_{46}, c_{56}. This accounts for all possible cases. ⊠

In our original double-six, we say that a_i and b_i are mated. It is clear that the property of being mated is intrinsic to the double-six, and is not dependent on the way in which we assign names to the twelve lines. Let us use the notation

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix}$$

to assert that the set $\{a_1, \ldots, a_6, b_1, \ldots, b_6\}$ forms a double-six.

Theorem 3.3.5. There are at least 36 double-sixes.

Proof: We provide a list. There is the original double-six. Now

$$\begin{pmatrix} a_1 & a_2 & a_3 & c_{56} & c_{46} & c_{45} \\ c_{23} & c_{13} & c_{12} & b_4 & b_5 & b_6 \end{pmatrix}$$

is a double-six, and is characteristic of $\binom{6}{3} = 20$ examples. Likewise

$$\begin{pmatrix} a_1 & b_1 & c_{23} & c_{24} & c_{25} & c_{26} \\ a_2 & b_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{pmatrix}$$

is a double-six, characteristic of $\binom{6}{2} = 15$ cases.

Thus, at least $36 = 1 + 20 + 15$ distinct double-sixes exist. ☒

Theorem 3.3.6. A given pair of skew lines can occur as mates in at most one double-six.

Proof: Given two lines, we require ten more. There are ten lines meeting the first given line, five of which also meet the second. These five cannot be used. Likewise, of the ten lines meeting the second, five also meet the first, and so cannot be used. This gives us exactly ten lines from which to choose ten, so we can construct the double-six in at most one way. ☒

Corollary 3.3.7. There are exactly 36 double-sixes.

Proof: We have 216 pairs of skew lines. Each can occur as mates in at most one double-six, and each double-six accounts for 6 mated pairs. Since $216 = 36 \times 6$, we have accounted for all the pairs, so no further double-sixes are possible. ⊠

Theorem 3.3.8. For a given pair of lines which meet, there is exactly one further line meeting both of the given lines.

Proof: We illustrate by cases. Lines a_1 and b_2 meet c_{12}. Lines a_1 and c_{12} meet line b_2. Lines c_{12} and c_{34} meet line c_{56}. These account for all possible cases. ⊠

Sets of three lines, each two meeting the third, are called tritangent planes. This terminology arises from the relation with cubic surfaces in an obvious way.

Corollary 3.3.9. There are exactly 45 tritangent planes.

Proof: There are 135 pairs of intersecting lines. Each tritangent plane contains 3 of these pairs, so the number of tritangent planes is $1/3 \times 135 = 45$. ⊠

We are now in a position to prove

Theorem 3.2.3. The complete double-six is a closed figure, in the sense that from any double-six amongst the 27 lines, the 15 derived lines are the remaining 15 lines of the figure.

Proof: We show the truth of this theorem for the two double-sixes of theorem 3.3.5. Having picked a double-six, we will rename its lines $A_1, \ldots, A_6 B_1, \ldots, B_6$ and let C_{ij} be the 15 derived lines. We have to show that the derived lines are lines of the original configuration.

In the first case, let

$$\begin{array}{ll} A_1 = a_1 & B_1 = c_{23} \\ A_2 = a_2 & B_2 = c_{13} \\ A_3 = a_3 & B_3 = c_{12} \\ A_4 = c_{56} & B_4 = b_4 \\ A_5 = c_{46} & B_5 = b_5 \\ A_6 = c_{45} & B_6 = b_6 \end{array}$$

The results we require can then be obtained from Theorem 3.3.7. For example, C_{12} lies in the plane formed by a_1 and c_{13}. C_{12} also lies in the plane formed by a_2 and c_{23}. Now b_3 is the unique line common to both planes. Thus, $C_{12} = b_3$. Similarly, $C_{13} = b_2$, $C_{23} = b_1$, and $C_{45} = a_6$, $C_{46} = a_5$, $C_{56} = a_4$. Now C_{14} lies in the plane determined by a_1 and b_4. C_{14} also lies in the plane determined by c_{56} and c_{23}. The single line common to both planes is c_{14}. Thus, $C_{14} = c_{14}$, and similarly, $C_{ij} = c_{ij}$ for $i = 1, 2, 3$ and $j = 4, 5, 6$.

In the second case, let

$$\begin{array}{ll} A_1 = a_1 & B_1 = a_2 \\ A_2 = b_1 & B_2 = b_2 \\ A_3 = c_{23} & B_3 = c_{13} \\ A_4 = c_{24} & B_4 = c_{14} \\ A_5 = c_{25} & B_5 = c_{15} \\ A_6 = c_{26} & B_6 = c_{16} \end{array}$$

It is immediate that $C_{12} = c_{12}$. Now, C_{13} lies in the plane formed by a_1 and c_{13}, and also in the plane formed by a_2 and c_{23}. Thus $C_{13} = b_3$. Similarly, $C_{14} = b_4$, $C_{15} = b_5$, $C_{16} = b_6$, and $C_{23} = a_3$, $C_{24} = a_4$, $C_{25} = a_5$, $C_{26} = a_6$. Finally, $C_{34} = c_{56}$, $C_{35} = c_{46}$, $C_{36} = c_{45}$, $C_{45} = c_{36}$, $C_{46} = c_{35}$, and $C_{56} = c_{34}$. ⊠

For the double-six

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix},$$

we say that $\{a_1, \ldots, a_6\}$ and $\{b_1, \ldots, b_6\}$ are the <u>rows</u> of the double-six. It is evident that the partition of the twelve lines into two rows is determined without reference to the labelling of the lines.

<u>Theorem 3.3. 10</u>. Two distinct double-sixes are related in one of two ways:

i) each row of the first shares one line element with each row of the second, or

ii) one row of the first shares three lines with a row of the second, and shares no lines with the other; the second row of the first shares respectively 0 and 3 lines with the rows of the second.

<u>Proof</u>: We may take one of the double-sixes to be

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix}$$

and the other to be

either

$$\begin{pmatrix} a_1 & b_1 & c_{23} & c_{24} & c_{25} & c_{26} \\ a_2 & b_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{pmatrix}$$

or

$$\begin{pmatrix} a_1 & a_2 & a_3 & c_{56} & c_{46} & c_{45} \\ c_{23} & c_{13} & c_{12} & b_4 & b_5 & b_6 \end{pmatrix} .$$

The results follow by inspection. ☒

Now, let A be the set of double-sixes. Define a relation, $\equiv$, on A by $X \equiv Y$ if and only if the rows of X and Y share one line each.

<u>Lemma 3.3.11</u>. The relation $\equiv$ is irreflexive and symmetric.

<u>Theorem 3.3.12</u>. If $X \equiv Y$, there are exactly 6 double-sixes, Z, such that $X \equiv Z$ and $Y \equiv Z$.

<u>Proof</u>: We may suppose that

$$X = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix}$$

and

$$Y = \begin{pmatrix} a_1 & b_1 & c_{23} & c_{24} & c_{25} & c_{26} \\ a_2 & b_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{pmatrix} .$$

Let (i, j, k, ℓ, m, n) be a permutation of $(1, \ldots, 6)$. Then the most general double-six related to X is

$$D_{ij} = \begin{pmatrix} a_i & b_i & c_{jk} & c_{j\ell} & c_{jm} & c_{jn} \\ a_j & b_j & c_{ik} & c_{i\ell} & c_{im} & c_{in} \end{pmatrix}$$

and we see immediately that

$$Y \equiv D_{ij} \quad \text{if and only if} \quad 3 \le i, j \le 6 \text{ .}$$

There are $\binom{4}{2} = 6$ such cases. $\boxtimes$

Theorem 3.3.13. If $X \not\equiv Y$ and $X \neq Y$, there are exactly 6 Z such that

$$X \equiv Z \quad \text{and} \quad Y \equiv Z \text{ .}$$

Proof: We may assume

$$X = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix}$$

$$Y = \begin{pmatrix} a_1 & a_2 & a_3 & c_{56} & c_{46} & c_{45} \\ c_{23} & c_{13} & c_{12} & b_4 & b_5 & b_6 \end{pmatrix} .$$

Letting D_{ij} be the same as in the previous theorem, we find that

$$Y \equiv D_{ij} \quad \text{if and only if} \quad 1 \le i, j \le 3 \quad \text{or} \quad 4 \le i, j \le 6 \text{ .}$$

This yields $\binom{3}{2} + \binom{3}{2} = 3 + 3 = 6$ cases. $\boxtimes$

Theorem 3.3.14. The incidence matrix for $\equiv$ yields a (36, 15, 6) block design.

Proof: Since there are 36 double-sixes, $v = b = 36$. We have noted that the relation $\equiv$ is symmetric. With the count given in Theorem 3.3.5, we see that $r = k = 15$.

Finally, Theorems 3.3.12 and 3.3.13 together imply $\lambda = 6$. $\boxtimes$

§3.4 - The Tritangent Planes

In this section, we develop the (45, 12, 3) design from the 45 tritangent planes. Theorems 3.4.1 - 3.4.4 come from Baker [3] and Henderson [20]. Theorems 3.4.5 and 3.4.6 are due to the author.

Theorem 3.4.1. Every line lies in five tritangent planes.

Proof: According to Theorem 3.3.2, a fixed line meets 10 other lines. Then, by Theorem 3.3.8, they must pair off, to yield 5 planes. ⊠

Theorem 3.4.2. Each tritangent plane shares one line with 12 others.

Proof: A given tritangent plane is composed of three lines. Each line lies in five planes, one of which is the plane under consideration. Thus, the total number of planes sharing one line with the fixed plane is $3 \times (5 - 1) = 12$. ⊠

Now, let A be the set of tritangent planes, and define a relation $\equiv$, on A by

$$X \equiv Y \quad \text{if and only if} \quad X \text{ and } Y \text{ share one line.}$$

Theorem 3.4.3. If $X \equiv Y$, then there are exactly 3 tritangent planes Z such that

$$X \equiv Z \quad \text{and} \quad Y \equiv Z \, .$$

Proof: Let ℓ be the line shared by X and Y. There are five planes including ℓ, by Theorem 3.4.1. Two of these are X and Y. The other three serve as choices for Z. Now suppose that $X = \{\ell, m_1, m_2\}$, $Y = \{\ell, n_1, n_2\}$, and $Z = \{m_1, n_1, p\}$. Then ℓ, m_1, and n_1 meet in pairs, and so they form a tritangent plane. This means that X = Y. Thus, the only choices for Z are the 3 additional planes through ℓ. ☒

Theorem 3.4.4. Let $X = \{\ell_1, \ell_2, \ell_3\}$ and $Y = \{m_1, m_2, m_3\}$ be two planes with no line in common. Then, by renumbering if necessary, we may assert that ℓ_i meets m_j if and only if $i = j$.

Proof: We may choose $X = \{a_1, b_2, c_{12}\}$. The proof follows by cases. For example, if $Y = \{a_2, b_3, c_{23}\}$, then

a_1 meets b_3,

b_2 meets c_{23}, and

c_{12} meets a_2.

If $Y = \{c_{23}, c_{14}, c_{56}\}$,

a_1 meets c_{14},

b_2 meets c_{23},

c_{12} meets c_{56}.

If $Y = \{a_3, b_4, c_{34}\}$,

a_1 meets b_4,

b_2 meets a_3,

c_{12} meets c_{34}.

All other cases follow similarly. ☒

Theorem 3.4.5. If $X \not\equiv Y$ and $X \neq Y$, then there are exactly three Z such that $X \equiv Z$ and $Y \equiv Z$.

Proof: Let $X = \{\ell_1, \ell_2, \ell_3\}$ and $Y = \{m_1, m_2, m_3\}$ where ℓ_i meets m_i, with no other intersections. Now ℓ_i and m_i meet, and thus form a tritangent plane with some further line n_i. This gives 3 planes Z. No other possibilities exist, for otherwise we should have ℓ_i meeting m_j for $i \neq j$. ⊠

Theorem 3.4.6. The incidence matrix for $\equiv$ yields a (45, 12, 3) block design.

Proof: $v = b = 45$, because of the number of tritangent planes. Noting that $\equiv$ is symmetric, we see that Theorem 3.4.2 gives $r = k = 12$. Finally, $\lambda = 3$ is a consequence of Theorems 3.4.3 and 3.4.5. ⊠

Chapter 4 - Isomorphism of Designs

In this chapter, we discuss two general ways of testing whether designs with the same parameters are isomorphic. Both methods of testing were conceived by the author, although Wallis [44] and Newman [28] have discussed the first method as an isomorphism test as well.

§4.1 - The Smith Normal Form

In this section, theorems 4.1.1 - 4.1.16 are taken from Stanton [37]. Theorems 4.1.17 - 4.1.23 are from Wallis [44].

We recall that we have used the term incidence matrix in the following way: given a balanced incomplete block design with parameters (v, b, r, k, λ), let the varieties be $\{V_1, V_2, \ldots, V_v\}$ and let the blocks be $\{B_1, B_2, \ldots, B_b\}$, where each set is numbered in arbitrary fashion. Then define the $v \times b$ matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 \text{ if } V_i \in B_j, \\ \\ 0 \text{ if } V_i \notin B_j. \end{cases}$$

The basic properties of A are

(i) the column sums of A are all k;

(ii) the matrix AA^T has r on the main diagonal and λ elsewhere.

Conversely, if a matrix A satisfies conditions (i) and

(ii) above is given, then the reverse procedure yields a balanced incomplete block design with parameters (v, b, r, k, λ).

Before proceeding with our discussion of block designs, we consider some properties of permutation matrices. A permutation matrix is a square matrix consisting of 0's and 1's such that each row and each column contains exactly one 1. The basic theorem on permutation matrices is

Theorem 4.1.1. The set of $n \times n$ permutation matrices forms a group isomorphic to S_n (the symmetric group on n symbols).

Proof: Let α be a permutation from S_n. Define the $n \times n$ matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 & \text{if} \quad j = \alpha(i) \\ 0 & \text{if} \quad j \neq \alpha(i) . \end{cases}$$

It is clear that A is a permutation matrix, and that the relation $\alpha \to A$ is invertible. We need only show that matrix multiplication is equivalent to group multiplication in S_n. To this end, let β be another permutation in S_n, and $B = (b_{ij})$ be the corresponding matrix. Now $\delta = \alpha\beta$ means $\delta(i) = \beta(\alpha(i))$. Thus

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} .$$

We know that $a_{ik} = 0$ unless $k = \alpha(i)$, and then $a_{ik} = 1$. Thus,

$$c_{ij} = b_{\alpha(i)\,j}.$$

By the definition of b_{ij},

$$c_{ij} = \begin{cases} 1 & \text{if} \quad j = \beta(\alpha(i)) = \delta(i) \, , \\ 0 & \text{if} \quad j \neq \beta(\alpha(i)) = \delta(i) \, . \end{cases}$$

Thus C is the matrix corresponding to $\delta = \alpha\beta$. ☒

Corollary 4.1.2. $A^{-1} = A^T$ for permutation matrices.

Corollary 4.1.3. If A is a permutation matrix, then $\det A = \pm 1$.

Proof: Let $\det A = a$. We have $I = AA^{-1} = AA^T$, and thus $1 = \det I = \det A \det A^T = a^2$. Hence $a = \pm 1$. ☒

A matrix A with integer entries and determinant ±1 will be called an integer non-singular matrix, or INS-matrix. Thus we obtain

Corollary 4.1.4. Permutation matrices are INS-matrices.

The utility of permutation matrices in connection with block designs arises from the well-known

Lemma 4.1.5. If P is a permutation matrix, and if A is any matrix, then PA is a matrix which can be obtained from A by row permutations. Conversely, any row permutation of A can be realized in this way, and such correspondence is biunique.

A similar result holds for column permutations, but the permutation matrix appears on the right.

Let A and B be two matrices of the same size. We will say that A is permutation isomorphic to B if B can be obtained from A by use of row and column permutations. In view of Lemma 4.1.5, this is equivalent to asserting that permutation matrices P and Q (of appropriate sizes) exist, such that B = PAQ.

It is easy to check the following result.

Lemma 4.1.6. Permutation isomorphism is an equivalence relation.

Lemma 4.1.7. If A is the incidence matrix for a balanced incomplete block design with parameters (v, b, r, k, λ), and if B is permutation-isomorphic to A, then B is also the incidence matrix for a balanced incomplete block design with parameters (v, b, r, k, λ).

We note that two balanced incomplete block designs with parameters (v, b, r, k, λ) are isomorphic if their incidence matrices are permutation-isomorphic.

A canonical form for permutation-isomorphism is not known. As a result, algorithms for testing permutation-isomorphism of matrices generally take the form of merely testing row and column permutations of one matrix against the other. However, there are generalizations of permutation-isomorphism for which canonical forms are known, and these can be used to obtain negative results. For example, we might consider ordinary equivalence of matrices. Two matrices A and B are called equivalent if there exist non-singular matrices P and Q such that B = PAQ. This relation is an equivalence relation, and a canonical form is known. Every matrix is equivalent to a matrix containing r 1's on the main diagonal, and 0's elsewhere. The quantity r is invariant, and is called the rank of the matrix. Thus, we see that two matrices are equivalent if and only if they have the same dimensions and rank. By Theorem 4.1.3, permutation matrices are non-singular, hence, if A is permutation-isomorphic to B, then A is equivalent to B. Equivalently, if A is not equivalent to B, then A is not permutation-isomorphic to B. This does not lead to any useful results, since, if A is the incidence matrix of a (v, b, r, k, λ) block design, the proof of Fisher's inequality $(v \leq b)$ yields incidentally the fact that A has rank v. Thus, all incidence matrices for balanced incomplete block designs with v varieties are equivalent. This can not possibly help us. The technique, however, can be applied in another case.

In what follows, we are essentially considering equivalence of matrices whose entries lie in a commutative ring with cancellation. The traditional application of these results is to matrices with polynomial entries for the purpose of determining invariant factors. We shall state and prove our results with respect to the ring of integers.

Let A and B be $n \times n$ matrices. We say that A is integer-equivalent to B if there exist two INS matrices P and Q such that $B = PAQ$. One immediately obtains

Lemma 4.1.8. Integer equivalence is an equivalence relation.

An elementary integer operation on a matrix A is one of the following:

i) interchanging two rows of A,

ii) multiplying a row of A by -1,

iii) adding an integer multiple of one row to another row;

or any similar column operation.

Lemma 4.1.9. Now, let an elementary row operation E be specified. Then there is a unique INS matrix P such that $E(A) = PA$, that is, elementary row operations can be performed by premultiplication by a suitable matrix. Dually, elementary column operations can be performed by a post-multiplication such as $E(A) = AP$. In either case, the desired matrix is given by $P = E(I)$.

Corollary 4.1.10. If B is obtained from A by a sequence of elementary operations, then A and B are integer equivalent.

Now, let A be a fixed $n \times n$ matrix with integer entries. Let $e_k = e_k(A)$ be the greatest common divisor of all $k \times k$ minors of A, $1 \le k \le r = \text{rank } A$. We shall also set $e_0 = 1$ and $e_k = 0$ for $r < k \le n$. We obtain

Lemma 4.1.11. (i) $e_k > 0$, $0 \le k \le r$.

(ii) $e_n = |\det A|$.

(iii) $e_1 = \gcd$ (all a_{ij}).

(iv) $e_{k-1} | e_k$, $1 \le k \le r$.

We now define $d_k = e_k/e_{k-1}$, $1 \le k \le r$ and $d_k = 0$, $r < k \le n$. We call $\{d_1, \ldots, d_n\}$ the invariant factors of A. Further, let $S = S(A) = \text{diag}(d_1, \ldots, d_n)$.

We call S the <u>Smith Normal Form</u> of A. In terms of the d_i, the previous Lemma becomes

<u>Lemma 4.1.12.</u> (i) $d_k > 0$, $1 \le k \le r$.

(ii) $e_k = d_1 \dots d_k$, $1 \le k \le n$.

(iii) $d_1 = \gcd$ (all a_{ij}).

<u>Lemma 4.1.13.</u> If B is obtained from A by an elementary operation, then $S(B) = S(A)$.

<u>Lemma 4.1.14.</u> If $d_1, \dots, d_n$ are integers such that

(i) $d_k > 0$, $1 \le k \le r$;

(ii) $d_k = 0$, $r < k \le n$;

(iii) $d_{k-1} | d_k$, $1 < k \le r$;

and if $D = \operatorname{diag}(d_1, \dots, d_k)$, then $S(D) = D$.

<u>Theorem 4.1.15.</u> Let A be any square matrix with integer entries. Then, there exists a matrix D which can be obtained from A by elementary operations and which satisfies the requirements of Lemma 4.1.4.

<u>Proof</u>: Let r = rank A. If r = 0, we are finished. Otherwise, let d_1 be the greatest common divisor of the elements of A. Now, let a_{ij} be the non-zero element which is smallest in absolute value. Place this element in position (1, 1) by row and column permutations, and multiply row 1 by -1 if $a_{ij} < 0$. There are now two cases, according as $a_{11} = d_1$ or $a_{11} > d_1$. In the first case, by adding appropriate multiples of row 1 to other rows, we may reduce all entries in column 1 except a_{11} to zeroes. Similarly, the same pro-

cedure may clear row 1 to 0's, and we have A converted to

$$\left(\begin{array}{c|c} d_1 & 0 \\ \hline 0 & A' \end{array}\right),$$

where d_1 divides every element of A'.

In the second case, we show that we can replace a_{11} by a smaller number. There are two subcases, according as a_{11} does not divide every element of row 1 and column 1 evenly, or does. In the first subcase, let $a_{ij} = q\, a_{11} + r$, where $0 < r < a_{11}$. Then, subtracting q times column 1 from column j produces a matrix containing r, and we have accomplished a reduction. In the second subcase, we may clear all of row 1 and column 1 to 0's (except, of course, a_{11}). As $a_{11} > d_1$, and, by Lemma 4.1.13, d_1 is the greatest common divisor of the entries in the matrix, there is an entry a_{ij} such that $a_{11} \nmid a_{ij}$. Adding row i to row 1 leaves a_{11} unaltered, and places the value a_{ij} into position (1, j). We may accomplish a reduction as in the first subcase. After repeating this process a finite number of times, we eventually find that $a_{11} = d_1$, and the first case applies. Thus, A may always be reduced to

$$\left(\begin{array}{c|c} d_1 & 0 \\ \hline 0 & A' \end{array}\right),$$

where d_1 divides every element of A'.

Now, let d_2 be the greatest common divisor of A'. Clearly $d_1 | d_2$. Evidently, repeating this procedure yields

$$\left(\begin{array}{ccc|c} d_1 & & & \\ & \ddots & & 0 \\ & & d_r & \\ \hline & 0 & & 0 \end{array}\right) .$$

⊠

Corollary 4.1.15. If $D = \text{diag}\,(d_1, \ldots, d_k) = S(A)$, then $d_{k-1} | d_k$, $1 < k \leq r$.

Corollary 4.1.16. A is integer-equivalent to B if and only if $S(A) = S(B)$, that is, the Smith Normal Form is a canonical form for integer equivalence.

Lemma 4.1.17. If A is permutation isomorphic to B, then A is integer equivalent to B.

In terms of BIBD's, this result may be stated as:

If A and B are the incidence matrices for isomorphic designs, then $S(A) = S(B)$.

or

If $S(A) \neq S(B)$, then A and B represent different designs.

Unlike the rank, the Smith Normal Form will serve to distinguish among designs with the same parameters. We now present some results concerning the invariant factors of the incidence matrix of a balanced incomplete block design. In the following, J is a matrix of appropriate dimensions, all of whose entries are 1.

<u>Theorem 4.1.18.</u> Let A be the incidence matrix of a (v, k, λ) block design, and let $n = k - \lambda$. Let the invariant factors of A be $\{a_1, \ldots, a_v\}$. Then $a_{v-1} = n$, and $a_v = nk/(n, k)$.

<u>Proof</u>: Let $B = kA^T - \lambda J$.

Since $$AA^T = nI + \lambda J$$

and $$AJ = kJ ,$$

then $$\begin{aligned} AB &= kAA^T - \lambda AJ \\ &= knI + k\lambda J - k\lambda J \\ &= knI . \end{aligned}$$

Let $D = \text{diag}\,(a_1, \ldots, a_v)$ be the Smith Normal Form of A, and let P and Q be the INS matrices such that $D = PAQ$. Then $Q^{-1}BP^{-1} = knD^{-1}$. Let $b_1, \ldots, b_v$ be the invariant factors of B. The matrix $Q^{-1}BP^{-1} = knD^{-1}$ has integer entries, is diagonal, and every element on the diagonal divides the <u>preceding</u> one. Thus, knD^{-1} is the Smith Normal Form of B with its entries in reversed order, that is

$$a_i\, b_{v+1-i} = kn .$$

The matrix B contains $n = k - \lambda$ wherever A^T contains a 1, and B contains $-\lambda$ wherever A^T contains 0. Since both situations occur,

$$\begin{aligned} b_1 &= \gcd\,(n, \lambda) \\ &= \gcd\,(k, n) , \end{aligned}$$

and

$$a_v = kn/(k, n) .$$

Likewise, to obtain a_{v-1} , we must show $b_2 = k$. We know that $b_1 b_2$ is the gcd of all 2×2 submatrices of B. There are $2^4 = 16$ possible 2×2 submatrices. In the cases where all entries are the same (n or $-\lambda$), the determinant is 0. If there is one n and three $-\lambda$'s, the determinant is $\pm\lambda k$. If there are two n's and two $-\lambda$'s, the determinant is either 0 or $\pm(n^2 - \lambda^2) = \pm k(n-\lambda)$. Finally, three n's and one $-\lambda$ yield $\pm kn$. Thus

$$\begin{aligned} b_1 b_2 &= \gcd(k\lambda,\ k(n-\lambda),\ kn) \\ &= k \gcd(\lambda,\ n-\lambda,\ n) \\ &= k \gcd(\lambda,\ n) = kb_1 \ . \end{aligned}$$

Thus $b_2 = k$, and $a_{v-1} = n$. ⊠

This theorem is very powerful, for it may greatly reduce the amount of work required to calculate the invariant factors of the matrix at a balanced incomplete block design.

Theorem 4.1.19. If $n = p^2$, the invariant factors for a (v, k, λ) block design are completely determined once the number of 1's in the Smith Normal Form is known.

Proof: Let the invariant factors be $a_1, \ldots, a_v$. We know that $a_v = nk/(n, k)$ and $a_{v-1} = n = p^2$. From the fact that

$$a_i \mid a_{v-1} \quad , \quad i \le v-1 \ ,$$

we must have

$$a_i = 1,\ p,\ \text{or}\ p^2 \ , \qquad i \le v-1 \ .$$

We also know

$$|\det A| = k\ n^{\frac{1}{2}(v-1)} = kp^{v-1} \ ,$$

and
$$a_1 a_2 \ldots a_v = |\det A| \ .$$

Now
$$(n, k) = p^a \ , \quad 0 \le a \le 2 \ .$$

Hence

$$a_1 a_2 \cdots a_{v-1} = \frac{k\,p^{v-1}}{k\,p^2} \times p^a$$

$$= p^{v+a-3} \ .$$

Let 1, p, and p^2 occur respectively α, β, and γ times in $a_1, \ldots, a_{v-1}$.

Then

$$\alpha + \beta + \gamma = v - 1 \ ,$$

$$\beta + 2\gamma = v + a - 3 \ .$$

As v, a, and α are all known quantities, these equations have a unique solution for β and γ, namely

$$\beta = v + 1 - a - 2\alpha$$

$$\gamma = a - 2 + \alpha \ . \qquad \boxtimes$$

We shall say that the <u>Smith class</u> of a balanced incomplete block design is the number of 1's among the invariant factors. For each of the block designs which we have constructed, we have n = 9. Thus,

<u>Corollary 4.1.20</u>. The invariant factors of a block design with parameters (36, 15, 6), (45, 12, 3), or (56, 11, 2) are completely determined by its Smith class.

<u>Theorem 4.1.21</u>. If two block designs with the same parameters lie in different Smith classes, they are not isomorphic.

We shall close this section by stating the following theorem without proof.

<u>Theorem 4.1.22</u>. Let A be an n × n matrix of 0's and 1's. Then the number of 1's among the invariant factors of A is at least

$1 + [\log_2 n]$, provided A is non-singular.

We can combine Theorem 4.1.22 and the condition $\beta = v + 1 - a - 2\alpha \geq 0$ of Theorem 4.1.19 to obtain bounds on the Smith classes of our designs.

Theorem 4.1.23. Let α be the Smith class of a (v, k, λ) block design.

If $(v, k, \lambda) = (36, 15, 6)$, $6 \leq \alpha \leq 18$.

If $(v, k, \lambda) = (45, 12, 3)$, $6 \leq \alpha \leq 22$.

If $(v, k, \lambda) = (56, 11, 2)$, $6 \leq \alpha \leq 28$.

§4.2 - Generalized Intersection Numbers

It is well-known that, if A be the incidence matrix for a (v, k, λ) block design, then A^T is likewise the matrix of a (v, k, λ) design, called the dual design. The next results, 4.2.1 - 4.2.4, are due to the author.

Lemma 4.2.1. If A is permutation isomorphic to B, then A^T is permutation isomorphic to B^T.

We shall say that A is self-dual if A is permutation isomorphic to A^T. We shall say that A is strongly self-dual if A is permutation isomorphic to a symmetric matrix.

Lemma 4.2.2. If A is strongly self-dual, then A is self-dual.

Proof: Let $S = PAQ$, where S is symmetric, and P and Q

are permutation matrices. Then $S^T = Q^T A^T P^T = S$. Thus $PAQ = Q^T A^T P^T$, or $A^T = (QP)A(QP)$. ⊠

Lemma 4.2.3. A is strongly self-dual if and only if there is a permutation matrix P such that $A^T = PAP$.

It is known (c.f. Marshall Hall [16]) that a symmetric block design need not be self-dual. It is an open question whether a design may be self-dual without being strongly self-dual, and we shall regard this as possible.

Theorem 4.2.4. A balanced incomplete block design arising from an abelian group difference set is strongly self-dual.

Proof: The customary construction of the incidence matrix does not yield the result directly. However, if a row-permutation is introduced which interchanges the rows corresponding to g and $-g$, the result is a symmetric matrix, since the condition $g_j - g_i \in D$ then becomes $g_j + g_i \in D$. ⊠

It is apparent that $S(A) = S(A^T)$, so testing invariant factors is not a final determination of isomorphism. The test described in this section can sometimes distinguish a design from its dual. We shall describe our quantities first in terms of a non-symmetric design.

Let $A = (a_{ij})_{v \times b}$ be the incidence matrix of a (v, b, r, k, λ) block design. We define the intersection numbers of the first kind. (c.f. Stanton and Sprott [36])

$$r_X = \sum_{j=1}^{b} \Pi_{i\in X}\, a_{ij} \text{ , where } X \text{ is a subset of the}$$

rows, and dually,

$$k_X = \sum_{i=1}^{v} \Pi_{j\in X}\, a_{ij} \text{ , where } X \text{ is subset of the columns.}$$

Lemma 4.2.5.

(i) $k_\emptyset = v$.

(ii) $r_\emptyset = b$.

If X is a singleton,

(iii) $r_X = r$, and

(iv) $k_X = k$.

If X is a doubleton,

(v) $r_X = \lambda$.

These numbers clearly are intrinsic to a design, and not dependent upon a particular incidence matrix. They are not particularly convenient to use for classifying designs, as the number of values involved, respectively 2^v and 2^b , are large. They are called intersection numbers because, for example, $k_{\{1,2,3\}}$ is the number of elements in $B_1 \cap B_2 \cap B_3$.

We now define intersection numbers of the second kind, which will also serve as invariants, and are fewer in number. These definitions and the results 4.2.6 - 4.2.11 come from Dulmage and Mendelsohn [14], and were established independently by the author.

R^n_m = the number of X such that X is an n-subset of varieties, and $r_X = m$.

Dually,

K^n_m = the number of X such that X is an n-subset of blocks and $k_X = m$.

Thus, R^3_2 counts the number of unordered triples of varieties which occur together exactly twice.

Theorem 4.2.6. (i)
$$\begin{aligned} R^0_m &= 1 \text{ if } m = b , \\ &= 0 \text{ if } m \neq b . \\ K^0_m &= 1 \text{ if } m = v , \\ &= 0 \text{ if } m \neq v . \end{aligned}$$

(ii)
$$\begin{aligned} R^1_m &= v \text{ if } m = r , \\ &= 0 \text{ if } m \neq r . \\ K^1_m &= b \text{ if } m = k , \\ &= 0 \text{ if } m \neq k . \end{aligned}$$

(iii)
$$\begin{aligned} R^2_m &= \binom{v}{2} \text{ if } m = \lambda , \\ &= 0 \text{ if } m \neq \lambda . \end{aligned}$$

Proof: (i) There is one 0-subset, namely, the empty set. Our values follow from $r_\emptyset = b$, $k_\emptyset = v$.

(ii) There are v 1-sets of varieties, and each contains r elements. Hence, we obtain the values of R^1_m . K^1_m is obtained in dual fashion.

(iii) In similar fashion, there are $\binom{v}{2}$ 2-sets of varieties, and each 2-set occurs in λ blocks. ☒

Theorem 4.2.7.

(i) For $n > k$,

$$R^n_m = \begin{cases} \binom{v}{n} & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases}$$

For $n > r$,

$$K^n_m = \begin{cases} \binom{b}{n} & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases}$$

(ii) For $n \geq 1$ and $m > r$, $R^n_m = 0$.

For $n \geq 1$ and $m > k$, $K^n_m = 0$.

Proof: (i) If $n > k$, let X be an n-set of varieties. Clearly $r_X = 0$, since every term in the sum for r_X contains a factor of 0 . The dual result holds for blocks.

(ii) These conditions reflect the fact that a set of blocks can share at most the k varieties in any one of them, and the dual condition for varieties.

Theorem 4.2.8. If $n \geq 2$ and $m > \lambda$, then $R^n_m = 0$. For a symmetric design, the same results hold for K^n_m .

Let N_{st} be the number of $s \times t$ minors consisting of all ones in the incidence matrix A . For $s = 0$, we shall take this to mean

$$N_{0t} = \binom{b}{t} ,$$

and for $t = 0$

$$N_{s0} = \binom{v}{s} .$$

Theorem 4.2.9.

$$N_{st} = \sum_{m \geq t} \binom{m}{t} R_m^s$$

$$= \sum_{m \geq s} \binom{m}{s} K_m^t$$

Proof: Fix the integer m. There are R_m^s ways to choose s rows so that there are exactly m columns consisting entirely of 1's. From the m columns, we may now make a selection in $\binom{m}{t}$ ways. Summed over all values of m, we obtain

$$N_{st} = \Sigma_m \binom{m}{t} R_m^s \ .$$

A similar argument applied to columns yields the dual result. ☒

Corollary 4.2.10.

$$\text{(i)} \quad \Sigma_m R_m^n = \binom{v}{n} \ ,$$

$$\text{(ii)} \quad \Sigma_m K_m^n = \binom{b}{n} \ ,$$

$$\text{(iii)} \quad \Sigma_m mR_m^n = b \binom{k}{n} \ ,$$

$$\text{(iv)} \quad \Sigma_m mK_m^n = v \binom{r}{n}$$

$$\text{(v)} \quad \Sigma_m \binom{m}{2} K_m^n = \binom{v}{2} \binom{\lambda}{n} \ .$$

Proof: (i) is the assertion $N_{n0} = \binom{v}{n}$.

Dually, (ii) is the result that $N_{0n} = \binom{b}{n}$.

(iii) We require N_{n1}. Since there are b ways of picking the column, and $\binom{k}{n}$ ways of picking the n rows, we obtain

$$N_{n1} = b \binom{k}{n} \ .$$

(iv) is dual to (iii). Hence

$$N_{1n} = v \binom{r}{n} .$$

(v) We require N_{2n} . There are $\binom{v}{2}$ ways of selecting two rows. This gives λ columns from which n must be selected. Thus

$$N_{2n} = \binom{v}{n}\binom{\lambda}{n} .$$ ⊠

Corollary 4.2.11. For a symmetric design,

$$\Sigma_m \binom{m}{2} R_m^n = \binom{v}{2} \binom{\lambda}{n} .$$

The utility of considering intersection numbers of the second kind in testing isomorphism in balanced incomplete block designs arises from the following observation of the author.

Theorem 4.2.12. The intersection numbers of the second kind and the numbers N_{st} are invariant under row and column permutations of the incidence matrix A .

Equivalently, we obtain

Theorem 4.2.13. If two designs differ in any intersection number of the second kind, they are not isomorphic.

We shall employ the results of this chapter to give computer tests for isomorphism of designs. Our results are discussed in Chapter Five.

Chapter 5 - Algorithmic Techniques in Block Designs

In this chapter we discuss a variety of topics. First, we describe computer programs for constructing incidence matrices for the three designs developed in Chapters Two and Three. Second, we survey the literature for other occurrences of designs with the parameters (36, 15, 6), (45, 12, 3), and (56, 11, 2), and provide a tabulation of these designs. Third, we apply the theory of invariant factors and of generalized intersection numbers as described in Chapter Four to determine nearly completely the isomorphisms which exist among the known designs.

§5.1 - Computer Construction of Incidence Matrices

§5.1.1 - The (56, 11, 2) Design

The plane PG(2, 4) is an example of a (21, 5, 1) block design. It is well-known that the difference set

$$K = \{0, 1, 4, 14, 16 \text{ modulo } 21\}$$

describes PG(2, 4), and it is this description which we use in our computer program; PG(2, 4) is specified by a set

$\{P_0, P_1, \ldots, P_{20}\}$ of points, a set $\{L_0, L_1, \ldots, L_{20}\}$ of lines, and the incidence rule

$$P_x \text{ is on } L_y \text{ if and only if } x + y \in K ,$$

the addition being performed modulo 21.

<u>Lemma 5.1.1</u>. There are two unique functions f and g with range $\{1, \ldots, 20\}$ and domain K such that

$$n \equiv f(n) - g(n) \qquad (\text{modulo } 21).$$

<u>Proof</u>: This is just another way of saying that K forms a $(21, 5, 1)$ difference set. ⊠

In any computer program for $PG(2, 4)$ we shall need the function $h(x,y)$ which produces the subscript z of the line L_z which passes through P_x and P_y. We note that, because of the form in which we have stated the incidence rule, the function h also computes the subscript for the point common to a pair of lines.

<u>Lemma 5.1.2</u>. $h(x,y) \equiv f(x - y) - x$ (modulo 21)

$$\equiv g(x - y) - y \quad (\text{modulo } 21).$$

<u>Proof</u>: Let L_z be the line through P_x and P_y. Then we have

$$x + z = d_1 \in K , \text{ and}$$
$$y + z = d_2 \in K .$$

Subtracting, we obtain $x - y = d_1 - d_2$. As d_1 and d_2 are both in K, they must be the unique components of the difference $x - y$. Then $d_1 = f(x - y)$ and $d_2 = g(x - y)$. Our results are immediate. ⊠

To illustrate, we shall compute the three ovals through three fixed points (cf. Table 2.2.1). We begin by forming a table of the differences of pairs in K . We then use these differences to tabulate the functions f and g .

Table 5.1.1 - Differences (x - y) from K

x \ y	0	1	4	14	16
0	-	20	17	7	5
1	1	-	18	8	6
4	4	3	-	11	9
14	14	13	10	-	19
16	16	15	12	2	-

Table 5.1.2 - f(x) and g(x)

x	1	2	3	4	5	6	7	8	9	10
f(x)	1	16	4	4	0	1	0	1	4	14
g(x)	0	14	1	0	16	16	14	14	16	4

x	11	12	13	14	15	16	17	18	19	20
f(x)	4	16	14	14	16	16	0	1	14	0
g(x)	14	4	1	0	1	0	4	4	16	1

We start by selecting P_0, P_1, and P_2 . We have $h(1,0) = 0$, $h(2,0) = 14$, and $h(2,1) = 20$. Thus, we see that the points are not collinear, but rather form a triangle whose sides are L_0, L_{14}, and L_{20} . Tabulating the points on these three lines, we get

$$L_0 = \{P_0, P_1, P_4, P_{14}, P_{16}\},$$

$$L_{14} = \{P_0, P_2, P_7, P_8, P_{11}\} \quad \text{and}$$

$$L_{20} = \{P_1, P_2, P_5, P_{15}, P_{17}\}.$$

None of the twelve points in this list may be used to augment our triangle to a quadrangle. P_3 is an allowable point. Computing again, we obtain $h(3,0) = 1$, $h(3,1) = 13$, $h(3,2) = 19$. Thus, we have a complete quadrangle P_0, P_1, P_2, P_3, whose sides, written in opposite pairs, are L_0 and L_{19}; L_{13} and L_{14}; and L_1 and L_{20}. Using h again, we get $h(19,0) = 16$, $h(14,13) = 8$, and $h(20,1) = 15$. The diagonal points for the quadrangle are thus P_8, P_{15}, and P_{16}. As we have $h(15,8) = h(16,8) = h(16,15) = 6$, we see that L_6 are P_8, P_{10}, P_{15}, P_{16}, and P_{19}. Excluding the three diagonal points, we find that $\{P_0, P_1, P_2, P_3, P_{10}, P_{19}\}$ is an oval. Similarly, selecting P_6 to complete a quadrangle gives rise to the oval $\{P_0, P_1, P_2, P_6, P_{12}, P_{13}\}$. Finally, choosing P_9 yields $\{P_0, P_1, P_2, P_9, P_{18}, P_{20}\}$. Denote these three ovals respectively by A, B, and C.

The program for computing the incidence matrix begins by computing all quadrangles. For each quadrangle, the diagonal points, the diagonal line, and finally, the two non-diagonal points on the diagonal line are computed. These two points complete the quadrangle to an oval. As each oval is obtained, it is compared against the three known ovals A, B, and C. The computed oval is placed into one of three lists, depending on which one of A, B, or C it meets an even number of times.

After all of the quadrangles have been computed, each of the lists contains 56 ovals.

The program then computes the incidence matrix for the (56, 11, 2) design directly from the definition given in §2.3. Only those ovals which meet A an even number of times are used. The rows and columns of the incidence matrix correspond to these ovals in the order in which they arise during the first stage of the program. This happens to be the lexicographic order which arises from our numbering scheme for the points of PG(2,4).

Finally, the incidence matrix, with an appropriate heading, is written onto an external storage medium. The matrix will be examined later to verify that a BIBD has been produced, and to calculate some of the invariants associated with the design (cf. §5.3). The program is listed on pages A.1 - A.3, and the design on pages B.12 - B.13.

§5.1.2 - The Double-Six Design

When dealing with the cubic surface designs, we will use the integers 1, 2, ..., 27 to denote the lines.

Lemma 5.1.3. The function

$$c(i,j) = i \times (11 - i)/2 + j + 6$$

maps $\{(i,j) \mid 1 \leq i < j \leq 6\}$ biuniquely onto $\{13, \ldots, 27\}$.

We then adopt the representation

$a_i \rightarrow i$

$b_j \rightarrow 6 + j$, and

$$c_{ij} \rightarrow c(i,j), \text{ where } i < j .$$

The starting point for the program is Theorem 3.3.5. We prepare a list of the 36 double-sixes as enumerated there. First, the original double-six

$$\begin{Bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{Bmatrix}$$

is recorded. Next, the 20 double-sixes characterized by

$$\begin{Bmatrix} a_1 & a_2 & a_3 & c_{56} & c_{46} & c_{45} \\ c_{23} & c_{13} & c_{12} & b_4 & b_5 & b_6 \end{Bmatrix}$$

are recorded, and finally, the 15 double-sixes characterized by

$$\begin{Bmatrix} a_1 & b_1 & c_{23} & c_{24} & c_{25} & c_{26} \\ a_2 & b_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{Bmatrix}$$

are recorded.

After the list of double-sixes is complete, the incidence matrix for the (36, 15, 6) design is computed. The definition of $\equiv$ given in §3.3 is used. As in the case of the (56, 11, 2) design, the incidence matrix is then stored for later use. The program is listed on pages A.4 - A.6, and the design on page B.5.

§5.1.3 The (45, 12, 3) Design

In this program, we begin by constructing the incidence matrix for the lines of the complete double-six, according to Theorem 3.3.1. We then consider all $\binom{27}{2}$

unordered pairs of lines. Whenever the lines meet, we find the third line which meets both, and enter the three into a table of the tritangent planes. Finally, we compute and save the incidence matrix for the (45, 12, 3) design, using the definition of ≡ from §3.4. The program is listed on pages A.7–A.8, and the design on page B.10.

§5.2 - Other Constructions for the Designs

In this section, we tabulate additional descriptions of the designs with parameters (36, 15, 6), (45, 12, 3), and (56, 11, 2). If the presentation is made by a difference set, the difference set is included here. Otherwise, only a brief comment as to the nature of the derivation is included. Programs to compute incidence matrices for all of the designs listed here may be found in an appendix. The presentation of this section is chronological.

In 1962, K. Takeuchi [40, 41] gave difference sets for both (36, 15, 6) and (45, 12, 3). We shall refer to these designs as KT36 and KT45 . The difference sets are

{000, 010, 020, 100, 110, 120, 001, 101,
201, 002, 112, 222, 003, 123, 213 modulo 334}, and
{001, 011, 021, 002, 102, 202,
003, 113, 223, 004, 124, 214 modulo 335}.

A program was written to produce an incidence matrix from a difference set in any Abelian group which is the direct sum of three cyclic groups. This program appears on page A.9, and the designs are listed on pages B.3 and B.8.

In 1962, Shrikhande and Singh [34] show that BIBD's can be derived from two-class association schemes. We describe their method in detail, as we shall return to it ultimately in settling the (45, 12, 3) isomorphisms. Although association schemes may be defined with a larger number of classes, we shall only discuss the two-class case here.

A <u>two-class association scheme on v varieties</u> is given when

(i) any (unordered) pair of varieties is said to be either first or second associates; and

(ii) each variety has n_1 first associates and n_2 second associates; and

(iii) if an (ordered) pair of distinct varieties are i-th associates, then there are p^i_{jk} varieties which jth associates of the first and kth associates of the second variety, for $1 \leq i, j, k \leq 2$, the numbers p^i_{jk} being independent of the choice of varieties.

<u>Theorem 5.2.1</u>. (Shrikhande and Singh [34])

(i) $p^i_{jk} = p^i_{kj} \qquad 1 \leq i, j, k \leq 2$

(ii) $n_1 + n_2 = v-1$

(iii) $p^i_{j1} + p^i_{j2} = \begin{array}{ll} n_j & \text{if } i \neq j \\ n_{j-1} & \text{if } i = j . \end{array}$

The main theorem of Shrikhande and Singh [34] is

<u>Theorem 5.2.2</u>. If there is a two-class association scheme with parameters v, n_1, n_2, and p^i_{jk}, $1 \leq i, j, k \leq 2$ such that $p^1_{11} = p^2_{11}$, then there is a strongly-self-dual BIBD with

parameters $v, k = n_1, \lambda = p_{11}^1 = p_{11}^2$.

Proof: Define blocks B_i, $1 \le i \le v$ so that block B_i contains all first associates of V_i . Each block contains n_i varieties. The intersection of distinct blocks B_i and B_j contains p_{11}^1 or p_{11}^2 varieties, according as V_i and V_j are first or second associates. As these numbers are equal, two distinct blocks intersect in exactly $\lambda = p_{11}^1 = p_{11}^2$ elements. Consequently, our incidence structure is the dual of a symmetric BIBD with parameters v, $k = n_1$, λ, and hence a BIBD of those parameters. That the BIBD is strongly self-dual follows from the fact that $V_i \varepsilon B_j$ if and only if V_i and V_j are first associates if and only if $V_j \varepsilon B_i$. ⊠

Corollary 5.2.3. If a two-class association scheme has $p_{22}^1 = p_{22}^2$, then there is a strongly-self-dual BIBD with parameters $v, k = n_2, \lambda = p_{22}^1 = p_{22}^2$.

In describing a two-class association scheme, it is customary to write the parameters p_{jk}^i as matrices

$$P_i = \begin{pmatrix} p_{11}^i & p_{12}^i \\ p_{21}^i & p_{22}^i \end{pmatrix} \quad . \quad = 1, 2 .$$

In 1954, Bose, Clatworthy, and Shrikhande [9] presented a two-class association scheme with $v = 45$, $n_1 = 12$, $n_2 = 32$, and $P_1 = \begin{pmatrix} 3 & 8 \\ 8 & 21 \end{pmatrix}$, $P_2 = \begin{pmatrix} 3 & 9 \\ 9 & 22 \end{pmatrix}$.
Theorem 5.2.2 applies to yield a (45, 12, 3) design, which we denote by SS45.

A program was written to convert certain two-class association schemes into (45, 12, 3) designs. This programme appears on page A.10, and the associated design on page B.11.

Shrikhande and Singh refer to the following theorem of Bose and Shimamoto [8]:

Theorem 5.2.4: If there exists a set of $(t - 2)$ mutually orthogonal Latin squares of side $2t$, then there exists a two-class association scheme with $v = 4t^2$, $n_1 = t(2t - 1)$, and $p^1_{11} = p^2_{11} = t(t - 1)$.

Of course, the other parameters are also mentioned, but we omit them here as they are not necessary for our purposes. They may be easily deduced from theorem 5.2.2. An immediate corollary of theorems 5.2.2 and 5.2.4 is:

Theorem 5.2.5: If there exists a set of $(t - 2)$ mutually orthogonal Latin squares of side $2t$, then there exists a balance incomplete block design with parameters $v = 4t^2$, $k = t(2t - 1)$, and $\lambda = t(t - 1)$.

Shrikhande and Singh state this result, which has been obtained independently by the author. It is immediate that the conditions of the theorem hold for $t = 3$, and this yields a (36, 15, 6) design. We shall return to this design when we discuss the work of Blackwelder below.

In 1963, a (36, 15, 6) design was printed in [16]. The design, attributed to P. K. Menon, we shall call PM36. It is given as

{11, 12, 14, 16, 21, 24, 26, 35, 41, 42, 46,
53, 61, 62, 64 modulo 66}.

A program for converting (6,6) difference sets into (36, 15, 6) designs was written. This program appears on page A.11, and the design on page B.4.

In 1966, Bose and Chakravarti [10] constructed a two-class association scheme from the points on a cubic surface in PG(3,4). Using the method of theorem 5.2.2 they derive a (45, 12, 3) design which we call RB45. Using the same program as for SS45, we obtain the design listed on page B.9.

In 1969, J. Wallis [40] presented both a (36, 15, 6) and a (45, 12, 3) by listing the incidence matrices. The latter design is given incorrectly, and should be corrected as follows: change the 9×9 minor in rows 37 - 45 and columns 27 - 36

$$\text{from} \quad \begin{pmatrix} I & M & L \\ I & M & L \\ I & M & L \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} I & L & M \\ I & L & M \\ I & L & M \end{pmatrix} .$$

This should be regarded as a correction to a transcription error in the source text. We shall refer to these designs as JW36 and JW45 respectively. Programs for converting the abbreviated descriptions in the source to incidence matrices are given on pages A.12 - A.15, and the designs are listed on pages B.2 and B.7.

In 1969, W. Blackwelder [5,6] presented the construction of theorem 5.2.2. From the two-class association scheme of Bose, Clatworthy and Shrikhande he derived a (45, 12, 3) design, which we shall call WB45. He also obtained a (36, 15, 6) design from a single 6×6 Latin square, tacitly using theorems 5.2.2, 5.2.4, and 5.2.5. We shall call this design WB36.

The author has shown that the design WB36 may be realized as a difference set. To see this, we first explain Blackwelder's construction. Let $A = (a_{ij})$ be the 6×6 Latin square defined by $a_{ij} \equiv i + j \pmod 6$, for $0 \leq i, j \leq 5$.

Let the varieties consist of the set of ordered pairs $\{(i,j) \mid 0 \leq i,j \leq 5\}$.

Two distinct ordered pairs (i,j) and (k,l) are first associates if

i) $i = k$, or

ii) $j = l$, or

iii) $a_{ij} = a_{kl}$.

Theorem 5.2.6: The design WB36 is isomorphic to the design given by the difference set $\{01,02,03,04,05,10,20,30,40,50,$ $15,24,33,42,55$ modulo $66\}$.

Proof: We have an immediate identification of the varieties with our group elements. We now interpret the three original conditions on varieties as group conditions on the difference $d = (i,j) - (k,l) = (i - k, j - l)$. The first and second conditions are equivalent respectively to $d = (0,x)$ and $d = (x,0)$, where, in either case, $x \neq 0$. If the third case applies, we have

$$a_{ij} = a_{kl}, \text{ whence}$$
$$i + j \equiv k + l \pmod 6, \text{ and}$$
$$i - k \equiv -(j - l) \pmod 6.$$

That is to say, $d = (x,6-x)$ for some $x \neq 0$. This argument can be reversed as well. We see then that each of the three Latin square conditions accounts for five of the fifteen group difference elements which we have specified. Thus we have exactly the noted difference set. ☒

The difference set was used with the program cited for PM36, and the resulting design is listed on page B.6.

In 1970, Hall, Lane, and Wales [19] presented both a (45, 12, 3) and a (56, 11, 2). We shall call these designs MH45 and MH56. MH45 is derived from a cubic surface in PG(3,4), and MH56 from ovals in PG(2,4). In both cases, the development is via permutation groups, not geometry.†

In 1971, E. Spence [35] showed that a (36, 15, 6) design may be derived from a Hadamard matrix of side 36. A program for performing this construction is listed on page A.16, and the design, which we call ES36, is on page B.1.

Finally, we shall use the names RK36, RK45, and RK56 to designate the designs derived in Chapters Two and Three of this thesis. We see that we have six designs with parameters (36, 15, 6): ES36, JW36, KT36, PM36, RK36, and WB36; we have seven designs with parameters (45, 12, 3): JW45, KT45, MH45, RB45, RK45, SS45, and WB45; and we have two designs with parameters (56, 11, 2): MH56 and RK56.

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† In a private communication, A. Rudvalis states that he has obtained a (56, 11, 2) and a (36, 15, 6), and that Hall has obtained a (36, 15, 6). These designs have all been obtained by group-theoretic methods. Rudvalis states that his (56, 11, 2) is isomorphic with that of Hall (MH56), but that the (36, 15, 6) designs are not isomorphic. We will not deal with these designs in this thesis.

§5.3 - Isomorphisms among the Designs

For the designs with parameters (56, 11, 2), we have noted that both are derived from the ovals of PG(2,4), although no details of the derivation for MH56 are given. However, Jonsson [22] gives the geometric derivation of MH56, and this work shows that the designs are the same. Thus, in the case of parameters (56, 11, 2), we have one design.

In the (45, 12, 3) case, there are two obvious isomorphisms among the seven designs, namely MH45 with RB45, and SS45 with WB45. The possible number of distinct designs is thereby reduced to five. We shall see that there are at least three non-isomorphic designs among the five.

Among the six (36, 15, 6) designs, there are no obvious isomorphisms. We shall see that there are at least four non-isomorphic designs among the six.

We employ a computer program to calculate some of the invariants described in Chapter Four. First, the intersection numbers R_m^3 $(0 \leq m \leq \lambda)$ are computed. (For the remainder of this section, we shall drop the superscript, and write R_m.) During the computation of these quantities, it is convenient to test that the row (variety) restrictions for a BIBD are satisfied, and this is done. After these values have been calculated, the incidence matrix is transposed, and the process repeated. If the incidence matrix is symmetric, or if the design is known to be strongly self-dual (as, for example, is the case when the design arises from a difference set), the latter step is supressed. Finally, the Smith class is computed for each design. In fact, this last calculation is not a full computation of the invariant factors. We continue reduction of the incidence matrix, and consequent counting of 1's among the invariant factors, for as long as the matrix contains an entry of ±1. When all of these have been exhausted, the remaining entries are tested for divisibility by three, as this is the indication that we have completely determined the invariant factors. It is entirely possible that we might not reach this stage. However, in practice, it turns out that ES36 is the only design which does not run to completion, and other factors make it unnecessary to pursue the calculation in this case. The program is listed on pages A.17 - A.21.

The matrices WB45, MH45, and MH56 were not tested, as they are known to be isomorphic to others among the designs. In every case, the design and its dual turned out to have the same row (variety) structure. We might note that the numbers R_m for the (56, 11, 2) design can be calculated without computation. We demonstrate this, and provide check values for the other cases in

Theorem 5.3.1. (i) For the (36, 15, 6) design, we have

$$\Sigma_{m=0}^{6} R_m = 7140,$$

$$\Sigma_{m=1}^{6} mR_m = 16380, \text{ and}$$

$$\Sigma_{m=2}^{6} \binom{m}{2} R_m = 12700.$$

(ii) For the (45, 12, 3) designs, we have

$\Sigma_{m=0}^{3} R_m = 14200,$

$\Sigma_{m=1}^{3} mR_m = 9900,$ and

$\Sigma_{m=2}^{3} \binom{m}{2} R_m = 990.$

(iii) For the (56, 11, 2) designs,

$R_0 = 18480,$

$R_1 = 9240,$ and

$R_2 = 0.$

<u>Proof</u>: Put n = 3 in (i) and (ii) of Theorem 4.2.10, and also in Corollary 4.2.11. In the first two cases, this gives the stated results. In the case of (56, 11, 2), we get $R_2 = 0$ from Corollary 4.2.11, and the other two conditions enable us to solve for R_0 and R_1. We could also have had $R_2 = 0$ as a consequence of Theorem 2.2.13. ⊠

Table 5.3.1 - Invariants of the (36, 15, 6) Designs

DESIGN	R_0	R_1	R_2	R_3	R_4	R_5	R_6	SMITH CLASS
ES36	0	846	3753	2142	396	0	3	≥16
JW36	144	468	3996	2232	288	0	12	12
KT36	72	648	3888	2196	324	0	12	14
PM36	144	468	3996	2232	288	0	12	12
RK36	0	540	4320	2160	0	0	120	15
WB36	144	468	3996	2232	288	0	12	12

Table 5.3.2 - Invariants of the (45, 12, 3) Designs

DESIGN	R_0	R_1	R_2	R_3	SMITH CLASS
JW45	5220	8100	810	60	18
KT45	5220	8100	810	60	17
RB45	5040	8640	270	240	15
RK45	5040	8640	270	240	15
SS45	5040	8640	270	240	15

The design RK56 has a Smith class of 20.

We can now draw some conclusions, based on Tables 5.3.1 and 5.3.2. First, the (36, 15, 6) designs ES36, KT36, and RK36 are all distinct. Second, the designs JW36, PM36, and WB36 are distinct from the first group of three designs, but, of this group, any or all of the designs may be isomorphic. Thus, there are at least four, and possibly as many as six, non-isomorphic designs with parameters (36, 15, 6).

We also see that the designs JW45 and KT45 are distinct. Among the designs RB45, RK45, and SS45, which are all distinct from the first two designs, we may again have some or all isomorphic.

The possible isomorphisms among JW36, PM36, and WB36 remain unresolved at the current time, although the fact that the latter two arise from (6,6) difference sets suggests that there may be an isomorphism here at least. The (36, 15, 6) designs of Hall and Rudvalis remain outside our classification as well.

§5.4 - Isomorphism of RK45, RB45, SS45

We now complete the discussion of isomorphism among the designs RK45, RB45, and SS45. We shall see, in fact, that the designs are all isomorphic, and obtain explicit permutations to demonstrate this.

As noted earlier, the designs RB45 and SS45 are both obtained from two-class association schemes. The association schemes are presented in identical fashion: 27 groups of 5 varieties each are given, the varieties in each group being mutually first associates; each variety occurs equally often (3 times), and a pair of varieties occurs together at most once. The program cited for creating the designs RB45 and SS45 is designed to process association schemes of exactly this form.

We note now that our design RK45 can also be described in this format: for each line on the cubic surface we note as a group the five tritangent planes which contain that line. The association schemes for the three designs are listed in Table 5.4.1. The lines of RK45 occur in the order implied by the representation noted in §5.1.2, immediately following lemma 5.1.3. The numbers assigned to the tritangent planes are the row and column numbers that apply to the incidence matrix for RK45. These numbering schemes are displayed explicitly in tables 5.4.2 and 5.4.3.

We now utilize table 5.4.1 to construct explicit isomorphisms between RK45 and RB45, and between RK45 and SS45.

[illegible]	RK45					RB[illegible]					[illegible]				
1:	1	2	3	4	5	1	10	17	[illegible]	[illegible]	1	2	3	[illegible]	[illegible]
2:	6	7	8	9	10	4	13	17	28	39	1	[illegible]	7	[illegible]	[illegible]
3:	11	12	13	14	15	7	10	[illegible]	[illegible]	[illegible]	1	10	[illegible]	[illegible]	[illegible]
4:	16	17	18	19	20	1	2	27	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	17
5:	21	22	23	24	25	4	14	19	[illegible]	[illegible]	[illegible]	[illegible]	19	20	[illegible]
6:	26	27	28	29	30	7	11	[illegible]	31	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
7:	6	11	16	21	26	1	36	37	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
8:	1	12	17	22	27	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	30	[illegible]	[illegible]	[illegible]
9:	2	7	18	23	28	7	12	19	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
10:	3	8	13	24	29	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	39	40	[illegible]
11:	4	9	14	19	30	5	[illegible]	[illegible]	29	37	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
12:	5	10	15	20	25	[illegible]	10	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
13:	1	6	31	32	33	2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
14:	2	11	34	35	36	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
15:	3	16	37	38	39	8	11	[illegible]	[illegible]	[illegible]	[illegible]	10	[illegible]	[illegible]	[illegible]
16:	4	21	40	41	42	[illegible]	36	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
17:	5	26	43	44	45	5	15	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
18:	7	12	37	40	43	[illegible]	12	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
19:	8	17	34	41	44	3	16	19	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
20:	9	22	35	38	45	6	[illegible]	19	[illegible]	[illegible]	10	14	22	[illegible]	[illegible]
21:	10	27	36	39	42	9	10	[illegible]	27	[illegible]	10	[illegible]	[illegible]	[illegible]	[illegible]
22:	13	18	31	42	45	3	[illegible]	23	30	[illegible]	11	[illegible]	[illegible]	[illegible]	[illegible]
23:	14	23	32	39	44	6	14	[illegible]	30	[illegible]	11	[illegible]	[illegible]	[illegible]	40
24:	15	28	33	38	41	9	11	[illegible]	[illegible]	[illegible]	[illegible]	16	26	[illegible]	[illegible]
25:	19	24	33	36	43	3	36	39	40	[illegible]	[illegible]	[illegible]	[illegible]	37	[illegible]
26:	20	29	32	35	40	6	15	[illegible]	33	[illegible]	[illegible]	17	[illegible]	[illegible]	39
27:	25	30	31	34	37	9	12	25	29	42	13	[illegible]	[illegible]	[illegible]	45

TABLE 5.4.1 - ASSOCIATION SCHEMES

FOR THE (45, 12, 3) DESIGNS

For the remainder of this discussion, we presume that the isomorphisms exist, and that the problem is to identify them. We also presume that the isomorphisms may be recognized by treating the association schemes for RB45 and SS45 as line/plane incidence schemes analogous to the cubic surface scheme. Henceforth we shall use the terms line and plane with respect to the latter two designs in the way suggested by our presumptions. Of course, we say that two lines of a scheme are incident if they are incident with a common plane.

We deal first with the question of isomorphism between RK45 and RB45. We attempt to find a double-six among the lines of RB45. Theorem 3.3.2 gives us the key to our search: find a set of five skew lines with a common transversal, and a double-six should appear. We begin by selecting the first line of RB45, and identifying it as line a_1 in the Schläfli notation. We immediately see that the ten lines 2 and 3, 4 and 7, 10 and 19, 14 and 15, and 26 and 27 all meet line a_1 ; the lines have been written in incident pairs, the incidences arising out of planes 17, 1, 16, 21, and 25 respectively. Accordingly, we assign names b_2, b_3, b_4, b_5 and b_6 respectively to lines 2, 4, 10, 14, and 26. This has immediate implications; lines 3, 7, 19, 15, and 27 must be respectively lines c_{12}, c_{13}, c_{14}, c_{15}, and c_{16}; planes 17, 1, 16, 21, 25 in RB45 must be planes 1, 2, 3, 4, 5 respectively in RK45. We should now be able to complete our naming of the lines in RB45 according to the Schläfli scheme, following the outline of theorem 3.3.2.

Every 4-subset of the lines $\{b_2, b_3, b_4, b_5, b_6\}$ must have two transversals, a_1 and another, the latter being a_i, where b_i has been excluded from the 4-subset. Moreover, there must be a transversal to all of $\{a_2, a_3, a_4, a_5, a_6\}$ and it must be skew to b_i, $i = 2, \ldots, 6$. This line is, of course, b_1. Having established these names, we can proceed to name all other lines according to theorem 3.3.1. Of course, we get immediate identification of our planes.

Proceeding along these lines, we find that line 13 is a second transversal of $\{b_2, b_3, b_4, b_5\}$, so we name this line a_6. Unfortunately, we now have problems:

i) a_6 and b_6 meet (in plane 33),

ii) none of the other 4-subsets has a second transversal, and

iii) there is no line skew to $b_2, b_3, b_4,$ b_5 and b_6.

All of these problems are consistent with a reversal of the roles of the two lines b_6 and c_{16}; we try again, this time identifying line 27 as b_6, (and line 26 as c_{16}). This does not force a change in any of the other names assigned. Proceeding as before, we find that lines 18, 5, 11, and 21 are respectively a_5, a_4, a_3, and a_2. Finally, we discover that line 12 is b_1, and we do in fact have a double-six. Continuing to name lines according to theorem 3.3.2, we are led eventually to a complete double-six in the twenty-seven lines; as no further problems arise, we have

<u>Theorem 5.4.1</u>. The designs RK45 and RB45 are isomorphic.

A similar argument shows that the lines

$$\begin{pmatrix} 1 & 20 & 13 & 14 & 16 & 18 \\ 21 & 2 & 4 & 6 & 8 & 10 \end{pmatrix}$$

form a double-six in SS45, and again the naming of the 15 auxiliary lines can be completed without problem. Thus

<u>Theorem 5.4.2</u>. The designs RK45 and SS45 are isomorphic.

The actual isomorphisms are summarized in tables 5.4.2 and 5.4.3.

We may now conclude that there are three distinct (45, 12, 3) designs, characterized by JW45, KT45, and RK45.

SCHLÄFLI	*RK45*	*RB45*	*SS45*
A1	1	1	1
A2	2	21	20
A3	3	11	13
A4	4	5	14
A5	5	18	16
A6	6	13	18
B1	7	12	21
B2	8	2	[illegible]
B3	9	4	4
B4	10	10	[illegible]
B5	11	14	[illegible]
B6	12	27	10
C12	13	3	3
C13	14	7	[illegible]
C14	15	19	7
C15	16	15	9
C16	17	26	11
C23	18	20	12
C24	19	8	15
C25	20	25	17
C26	21	24	[illegible]
C34	22	6	[illegible]
C35	23	17	[illegible]
C36	24	22	[illegible]
C45	25	23	27
C46	26	16	[illegible]
C56	27	9	[illegible]

TABLE 5.4.2 - *ISOMORPHISM OF LINES*

SCHLÄFLI			RK45	RB45	SS45
A1	B2	C12	1	17	1
A1	B3	C13	2	1	2
A1	B4	C14	3	16	3
A1	B5	C15	4	21	4
A1	B6	C16	5	25	[illegible]
A2	B1	C12	6	10	10
A2	B3	C23	7	27	[illegible]
A2	B4	C24	8	23	[illegible]
A2	B5	C25	9	40	[illegible]
A2	B6	C26	10	9	[illegible]
A3	B1	C13	11	37	[illegible]
A3	B2	C23	12	13	[illegible]
A3	B4	C34	13	18	16
A3	B5	C35	14	5	21
A3	B6	C36	15	29	33
A4	B1	C14	16	20	[illegible]
A4	B2	C24	17	4	7
A4	B3	C34	18	31	[illegible]
A4	B5	C45	19	14	30
A4	B6	C46	20	42	41
A5	B1	C15	21	8	38
A5	B2	C25	22	39	6
A5	B3	C35	23	35	[illegible]
A5	B4	C45	24	22	[illegible]
A5	B6	C56	25	12	[illegible]
A6	B1	C16	26	33	44
A6	B2	C26	27	26	9
A6	B3	C36	28	28	17
A6	B4	C46	29	2	25
A6	B5	C56	30	32	22
C12	C34	C56	31	7	11
C12	C35	C46	32	43	12
C12	C36	C45	33	30	13
C13	C24	C56	34	45	19
C13	C25	C46	35	36	[illegible]
C13	C26	C45	36	41	31
C14	C23	C56	37	19	39
C14	C25	C36	38	3	27
C14	C26	C35	39	24	[illegible]
C15	C23	C46	40	38	37
C15	C24	C36	41	34	35
C15	C26	C34	42	11	34
C16	C23	C45	43	6	[illegible]
C16	C24	C35	44	15	40
C16	C25	C34	45	44	42

TABLE 5.4.3 -ISOMORPHISM OF THE PLANES

APPENDIX A

—

COMPUTER PROGRAMS

```
$JOB   WATFIV   ROGER,NOWARN,NOEXT
      INTEGER FUNCTION THRU (M,N)
C
C     THE FUNCTION THRU PRODUCES THE NUMBER OF THE LINE THROUGH
C     THE TWO POINTS WHOSE NUMBERS ARE GIVEN AS ARGUMENTS.  IT IS
C     A SELF-DUAL FUNCTION.  IT IS BASED ON THE DIFFERENCE SET
C     NOTATION FOR PG(2,4) OBTAINED FROM THE SET (0,1,4,14,16).
C
      INTEGER IDT(20)/1,16,4,4,0,1,0,1,4,14,4,16,2*14,2*16,0,1,14,0/
      THRU=MOD(IDT(MOD(M-N+21,21))-M+21,21)
      RETURN
      END
C     ARRAYS AND INITIALIZATION
C

      IMPLICIT INTEGER (A-Z)
      REAL SQRT
      INTEGER P(6),KKK(3)/3*1/,HEX(56,6,3)
      EQUIVALENCE (P,P1),(P(2),P2),(P(3),P3),(P(4),P4),(P(5),P5),
     X(P(6),P6)
      INTEGER HEXAD(56,6)
      EQUIVALENCE (HEXAD,HEX)
      INTEGER*2 MATRIX(56,56)/3136*0/
C     GET THE HEXADS OF PG(2,4)
C
C     OBTAIN THE QUADRANGLE WITH VERTICES P1, P2, P3, P4.
C     SIMULTANEOUSLY OBTAIN ITS SIDES:  L1=P1.P2, ETC..
C     VERIFY THAT THE SIDES ARE ALL DISTINCT.
C
      DO 11 I1=1,21
      P1=I1-1
      DO 12 I2=I1,21
      IF(I2.EQ.I1)GO TO 12
      P2=I2-1
      L1=THRU(P2,P1)
      DO 13 I3=I2,21
      IF(I3.EQ.I2)GO TO 13
      P3=I3-1
      L2=THRU(P3,P1)
      IF (L2.EQ.L1)GO TO 13
      L3=THRU(P3,P2)
      IF (L3.EQ.L1.OR.L3.EQ.L2) GO TO 13
      DO 14 I4=I3,21
      IF (I4.EQ.I3) GO TO 14
      P4=I4-1
      L4=THRU(P4,P1)
      IF(L4.EQ.L1.OR.L4.EQ.L2.OR.L4.EQ.L3)GO TO 14
      L5=THRU(P4,P2)
      IF (L5.EQ.L1.OR.L5.EQ.L2.OR.L5.EQ.L3.OR.L5.EQ.L4) GO TO 14
      L6=THRU(P4,P3)
      IF(L6.EQ.L1.OR.L6.EQ.L2.OR.L6.EQ.L3.OR.L6.EQ.L4.OR.L6.EQ.L5)
     XGOTO14
```

```
C
C     GET THE DIAGONAL POINTS AND THE LINE THRU THEM.
C
      D1=THRU(L1,L6)
      D2=THRU(L2,L5)
      D3=THRU(L3,L4)
      DL=THRU(D1,D2)
C
C     FIND THE TWO NON-DIAGONAL POINTS ON THE DIAGONAL LINE.
C     THESE POINTS COMPLETE THE HEXAD.  THESE POINTS ARE FOUND BY
C     FINDING THE EQUATIONS X + Y = U AND X * Y = V, AND SOLVING.
C
      DL1=MOD(21-DL,21)
      DL2=MOD(22-DL,21)
      DL3=MOD(25-DL,21)
      DL4=MOD(35-DL,21)
      DL5=MOD(37-DL,21)
      U=DL1+DL2+DL3+DL4+DL5-(D1+D2+D3)
      V=(DL1+1)*(DL2+1)*(DL3+1)*(DL4+1)*(DL5+1)
      V=V/((D1+1)*(D2+1)*(D3+1))-U-1
      V=SQRT(.1+(U*U-4*V))
      P5=(U-V)/2
      IF(P5.LE.P4) GO TO 14
      P6=(U+V)/2
C     THE HEXAD BELONGS IN ONE OF THREE CLASSES, ACCORDING AS
C     IT MEETS (0,1,2,3,10,19) OR
C              (0,1,2,6,12,13) OR
C              (0,1,2,9,18,20) AN EVEN NUMBER OF TIMES.
C     DETERMINE WHICH CLASS THIS HEXAD BELONGS TO.
C
      A=B=D=0
      DO 20 I=1,6
      U=P(I)
      IF (U.EQ.0.OR.U.EQ.1.OR.U.EQ.2) D=D+1
      IF (U.EQ.3.OR.U.EQ.10.OR.U.EQ.19) A=A+1
      IF (U.EQ.6.OR.U.EQ.12.OR.U.EQ.13) B=B+1
   20 CONTINUE
      T=3
      IF (A+D.EQ.(A+D)/2*2)T=1
      IF (B+D.EQ.(B+D)/2*2) T=2
      KT=KKK(T)
      DO 21 I=1,6
   21 HEX(KT,I,T)=P(I)
      KKK(T)=KT+1
   14 CONTINUE
   13 CONTINUE
   12 CONTINUE
   11 CONTINUE
```

```
C
C      COMPUTE THE BLOCK DESIGN OBTAINED FROM THE HEXADS.
C      SIMULTANEOUSLY COMPUTE THE INCIDENCE MATRIX FOR THE DESIGN.
C      THE INCIDENCE MATRIX IS THEORETICALLY SYMMETRIC.
C
       DO 30 B=1,56
       MATRIX(B,B)=1
       DO 31 BCH=1,56
       IF (B.EQ.BCH) GO TO 31
       DO 32 I=1,6
       DO 32 J=1,6
       IF (HEXAD(B,I).EQ.HEXAD(BCH,J)) GO TO 31
32     CONTINUE
       MATRIX(B,BCH)=1
31     CONTINUE
30     CONTINUE
       PUNCH,'56 11  2 BLOCK DESIGN DUE TO R. A. KINGSLEY AND R. G. STAN
      *TON'
       PUNCH 44444,MATRIX
44444 FORMAT (56I1)
       PRINT 88888
88888 FORMAT('1')
1000   STOP
       END

$ENTRY
```

```
$JOB   WATFIV    ROGER,NOWARN,NOEXT
C
C        THIS PROGRAMME COMPUTES THE INCIDENCE
C        MATRIX FOR RK36 FROM THE DOUBLE-SIXES
C        AMONG THE LINES ON A CUBIC SURFACE
C
         IMPLICIT INTEGER (A-Z)
         INTEGER D(36,12),FLAG(6),LINE(15)
         INTEGER*2 MA(36,36)/1296*0/
         DATA (D(1,J),J=1,12)/1,2,3,4,5,6,7,8,9,10,11,12/
         C(I,J)=MINO(I,J)*(11-MINO(I,J))/2+MAXO(I,J)+6
C
C        COMPUTE THE 20 DOUBLE-SIXES OF TYPE 1
C
         DX=2
         DO 1 I=1,4
         JB=I+1
         DO 1 J=JB,5
         KB=J+1
         DO 1 K=KB,6
         DO 2 V=1,6
    2    FLAG(V)=0
         FLAG(I)=1
         FLAG(J)=1
         FLAG(K)=1
         N=0
    3    N=N+1
         IF(FLAG(N).EQ.1)GO TO 3
         L=N
    4    N=N+1
         IF(FLAG(N).EQ.1)GO TO 4
         M=N
    5    N=N+1
         IF(FLAG(N).EQ.1)GO TO 5
         D(DX,1)=I
         D(DX,2)=J
         D(DX,3)=K
         D(DX,4)=C(M,N)
         D(DX,5)=C(L,N)
         D(DX,6)=C(L,M)
         D(DX,7)=C(J,K)
         D(DX,8)=C(I,K)
         D(DX,9)=C(I,J)
         D(DX,10)=L+6
         D(DX,11)=M+6
         D(DX,12)=N+6
    1    DX=DX+1
```

```
C
C     COMPUTE THE 15 DOUBLE-SIXES OF TYPE 2
C
      DO 6 I=1,5
      JB=I+1
      DO 6 J=JB,6
      DO 7 V=1,6
    7 FLAG(V)=0
      FLAG(I)=1
      FLAG(J)=1
      N=0
    8 N=N+1
      IF(FLAG(N).EQ.1)GO TO 8
      K=N
    9 N=N+1
      IF(FLAG(N).EQ.1)GO TO 9
      L=N
   10 N=N+1
      IF(FLAG(N).EQ.1)GO TO 10
      M=N
   11 N=N+1
      IF(FLAG(N).EQ.1)GO TO 11
      D(DX,1)=I
      D(DX,2)=I+6
      D(DX,3)=C(J,K)
      D(DX,4)=C(J,L)
      D(DX,5)=C(J,M)
      D(DX,6)=C(J,N)
      D(DX,7)=J
      D(DX,8)=J+6
      D(DX,9)=C(I,K)
      D(DX,10)=C(I,L)
      D(DX,11)=C(I,M)
      D(DX,12)=C(I,N)
    6 DX=DX+1
```

```
C
C     COMPUTE THE INCIDENCE MATRIX
C
      DO 12 I=1,35
      JB=I+1
      DO 12 J=JB,36
      KT=0
      DO 13 K=1,6
      LL=D(I,K)
      DO 13 L=1,6
13    IF(LL.EQ.D(J,L))KT=KT+1
      IF(KT.NE.1) GO TO 12
      MA(I,J)=1
      MA(J,I)=1
12    CONTINUE
      PUNCH,'36 15  6 BLOCK DESIGN DUE TO R. A. KINGSLEY AND R. G. STAN
     *TON'
      PUNCH 40,MA
40    FORMAT (36I1)
      PRINT 11121
11121 FORMAT('1')
100   STOP
      END

$ENTRY
```

```
$JOB   WATFIV    ROGER,HOWARD,NOEXT
C
C       THIS ROUTINE COMPUTES THE DESIGN RK45
C       BASED ON THE TRITANGENT PLANES
C       TO A CUBIC SURFACE
C
C       MEET IS A FUNCTION WHICH COMPUTES THE INCIDENCE MATRIX
C       FOR THE RULINGS ON A CUBIC SURFACE
C
        INTEGER FUNCTION MEET(X,Y)
        IMPLICIT INTEGER (A-Z)
        INTEGER T(15,2)/5*1,4*2,3*3,2*4,5,2,3,4,5,6,3,4,5,6,4,5,6,5,6,6/
        A=MIN0(X,Y)
        B=MAX0(X,Y)
        MEET=-1
        IF(A.LT.1.OR.A.EQ.B.OR.B.GT.27)RETURN
        MEET=0
        IF(B.LE.6)RETURN
        IF(B.GT.12)GO TO 13
        IF(A.GT.6)RETURN
        IF(B-A.EQ.6)RETURN
        MEET=1
        RETURN
 13     B1=T(B-12,1)
        B2=T(B-12,2)
        IF(A.GT.12)GO TO 12
        A1=1+MOD(A-1,6)
        IF(A1.EQ.B1.OR.A1.EQ.B2)MEET=1
        RETURN
 12     A1=T(A-12,1)
        A2=T(A-12,2)
        MEET=1
        IF(A1.EQ.B1.OR.A1.EQ.B2.OR.A2.EQ.B1.OR.A2.EQ.B2)MEET=0
        RETURN
        END
C
C       COPY THE FUNCTION MEET INTO A TABLE T FOR SPEED
C

        IMPLICIT INTEGER (A-Z)
        INTEGER PLANE(45,3),T(27,27)
        INTEGER*2 M(45,45)/2025*0/
        INTEGER LINE(12)
        DO 1 I=1,26
        JB=I+1
        DO 1 J=JB,27
        K=MEET(I,J)
        T(I,J)=K
 1      T(J,I)=K
```

```
C
C       COMPUTE THE 45 TRITANGENT PLANES
C
        N=1
        DO 2 I=1,25
        JB=I+1
        DO 2 J=JB,26
        IF(1.NE.T(I,J))GO TO 2
        KB=J+1
        DO 3 K=KB,27
        IF(1.EQ.T(I,K).AND.1.EQ.T(J,K))GO TO 4
 3      CONTINUE
        GO TO 2
 4      PLANE(N,1)=I
        PLANE(N,2)=J
        PLANE(N,3)=K
        N=N+1
 2      CONTINUE
C
C       COMPUTE THE INCIDENCE MATRIX
C       FOR THE DESIGN
C
        DO 12 I=1,44
        JB=I+1
        DO 12 J=JB,45
        DO 14 K=1,3
        MM=PLANE(I,K)
        DO 14 L=1,3
        IF(MM.EQ.PLANE(J,L))GO TO 21
 14     CONTINUE
        GO TO 12
 21     M(I,J)=1
        M(J,I)=1
 12     CONTINUE
        PUNCH,'45 12  3 BLOCK DESIGN DUE TO R. A. KINGSLEY AND R. G. STAN
       *TON'
        PUNCH 40,M
 40     FORMAT (45I1)
        PRINT 22233
 22233  FORMAT('1')
 100    STOP
        END

$ENTRY
```

```
$JOB  WATFIV    ROGER,NOWARN,NOEXT
C
C       THIS ROUTINE COMPUTES AN INCIDENCE MATRIX
C       FROM A DIFFERENCE SET GIVEN IN A GROUP
C       WHICH IS THE DIRECT SUM OF 3 CYCLIC GROUPS
C
        IMPLICIT INTEGER (A-Z)
        INTEGER*2 M(60,60)
        INTEGER DA(20),DB(20),DC(20)
        CHARACTER HDR*80
  333   READ 42,HDR,V,K,L
  42    FORMAT(A80,T1,3I3)
        IF(V.EQ.0)GO TO 30
        PUNCH 43,HDR
        PRINT 43,HDR
  43    FORMAT(A80)
        DO 999 IB=1,V
        DO 999 IV=1,V
  999   M(IV,IB)=0
        READ,MA,MB,MC
        MBC=MB*MC
        READ 41,(DA(I),DB(I),DC(I),I=1,K)
  41    FORMAT (80I1)
        NA=0
        NB=0
        NC=0
        DO 13 IB=1,V
        DO 12 I=1,K
        XA=MOD(NA+DA(I),MA)
        XB=MOD(NB+DB(I),MB)
        XC=MOD(NC+DC(I),MC)
        IV=1+XC+MC*XB+MBC*XA
  12    M(IV,IB)=1
        NC=NC+1
        IF(NC.LT.MC) GO TO 13
        NC=0
        NB=NB+1
        IF (NB.LT.MB) GO TO 13
        NB=0
        NA=NA+1
  13    CONTINUE
        DO 11 IV=1,V
  11    PUNCH 40,(M(IV,IB),IB=1,V)
  40    FORMAT (80I1)
  10    GO TO 333
  30    PRINT 44
  44    FORMAT('1')
        STOP
        END

$ENTRY
 12   3 BLOCK DESIGN DUE TO K. TAKEUCHI
 15   6 BLOCK DESIGN DUE TO K. TAKEUCHI
```

```
$JOB    WATFIV    ROGER,NOWARN,NOEXT
C
C        THIS PROGRAM CONVERTS A TWO-CLASS ASSOCIATION
C        SCHEME INTO THE INCIDENCE MATRIX FOR
C        A (45,12,3) BIBD BY THE METHOD OF
C        SHRIKHANDE AND SINGH
C
         INTEGER*2 M(45,45),A,B,C,D,E
         CHARACTER*80 HDR
   1     READ 41,HDR
  41     FORMAT(A80)
         IF(HDR.EQ.' ') GO TO 30
         PRINT 41,HDR
         PUNCH 41,HDR
         DO 33 I=1,45
         DO 33 J=1,45
  33     M(I,J)=0
         DO 12 I=1,27
         READ,A,B,C,D,E
         M(A,B)=1
         M(A,C)=1
         M(A,D)=1
         M(A,E)=1
         M(B,A)=1
         M(B,C)=1
         M(B,D)=1
         M(B,E)=1
         M(C,A)=1
         M(C,B)=1
         M(C,D)=1
         M(C,E)=1
         M(D,A)=1
         M(D,B)=1
         M(D,C)=1
         M(D,E)=1
         M(E,A)=1
         M(E,B)=1
         M(E,C)=1
  12     M(E,D)=1
         PUNCH 40,M
  40     FORMAT (45I1)
         GO TO 1
  30       PRINT 44
  44     FORMAT('1')
  10     STOP
         END

$ENTRY
12   3 BLOCK DESIGN DUE TO SHRIKHANDE AND SINGH
12   3 BLOCK DESIGN DUE TO R. C. BOSE
```

```
      $JOB   WATFIV    ROGER,NOWARN,NOEXT
      C
      C       THIS PROGRAM COMPUTES THE INCIDENCE
      C       MATRIX FOR A (36,15,6) BLOCK DESIGN
      C       FROM A (6,6) DIFFERENCE SET
      C
              IMPLICIT INTEGER (A-Z)
              INTEGER*2 M(36,36)
              INTEGER DA(20),DB(20)
              CHARACTER*80 HDR
      1       READ 40,HDR
              IF(HDR.EQ.' ') GO TO 30
              PRINT 40,HDR
      40      FORMAT(A80)
              PUNCH 40,HDR
              READ 41,(DA(I),DB(I),I=1,15)
      41      FORMAT (80I1)
              NA=0
              NB=0
              DO 13 IB=1,36
              DO 12 I=1,15
              XA=MOD(NA+DA(I),6)
              XB=MOD(NB+DB(I),6)
              IV=1+XB+6*XA
      12      M(IV,IB)=1
              NB=NB+1
              IF(NB.LT.6) GO TO 13
              NB=0
              NA=NA+1
      13      CONTINUE
              PUNCH 48,M
      48      FORMAT(36I1)
              GO TO 1
      30      PRINT 50
      50      FORMAT('1')
              STOP
              END

      $ENTRY
15   6 BLOCK DESIGN DUE TO P. K. MENON
15   6 BLOCK DESIGN DUE TO W. BLACKWELDER
```

```
C
C        THIS ROUTINE COMPUTES THE INCIDENCE MATRIX
C        FOR THE DESIGN JW36
C
         INTEGER*2  M(36,36)/1296*0/
         CHARACTER Z(12,12),XX
         READ 40,Z
   40    FORMAT (12A1)
         PRINT 41,Z
   41    FORMAT (1X,12A1)
         DO 1 I=1,12
         IB=3*(I-1)+1
         DO 1 J=1,12
         JB=3*(J-1)+1
         XX=Z(J,I)
         IF (XX.NE.'I') GO TO 2
         M(IB,JB)=1
         M(IB+1,JB+1)=1
         M(IB+2,JB+2)=1
         GO TO 1
    2    IF (XX.NE.'L') GO TO 3
         M(IB,JB+1)=1
         M(IB+1,JB+2)=1
         M(IB+2,JB)=1
         GO TO 1
    3    IF (XX.NE.'M') GO TO 4
         M(IB,JB+2)=1
         M(IB+1,JB)=1
         M(IB+2,JB+1)=1
         GO TO 1
    4    IF (XX.NE.'J') GO TO 5
         IT=IB+2
         JT=JB+2
         DO 6 IX=IB,IT
         DO 6 JX=JB,JT
    6    M(IX,JX)=1
         GO TO 1
    5    IF (XX.EQ.' ') GO TO 1
         PRINT,'ERROR',I,J,XX
         STOP
    1    CONTINUE
```

```
      PUNCH,'36 15  6 BLOCK DESIGN DUE TO J. WALLIS'
      PUNCH 42,M
42    FORMAT(36I1)
      PRINT 43
43    FORMAT('1')
      STOP
      END
```

```
        $ENTRY
JIIIIIIIII
JLLLMMMIII
 MMMLLLIII
L JJILMIML
LJ JLMIMLI
LJJ MILLIM
MIML JJILM
MMLIJ JLMI
MLIMJJ MIL
IILMIML JJ
ILMIMLIJ J
IMILLIMJJ
```

```
C
C     THIS PROGRAM COMPUTES THE INCIDENCE MATRIX
C     FOR THE BLOCK DESIGN JW45
C
      INTEGER*2 M(45,45)/2025*0/
      CHARACTER Z(15),XX,ZZ*15
      EQUIVALENCE (Z,ZZ)
      IB=-2
      DO 1 I=1,15
      IB=IB+3
      READ,ZZ
      PRINT,ZZ
      JB=-2
      DO 1 J=1,15
      JB=JB+3
      XX=Z(J)
      IF(XX.NE.'I')GOTO2
      M(IB,JB)=1
      M(IB+1,JB+1)=1
      M(IB+2,JB+2)=1
      GO TO 1
2     IF(XX.NE.'L')GO TO 3
      M(IB,JB+1)=1
      M(IB+1,JB+2)=1
      M(IB+2,JB)=1
      GO TO 1
3     IF(XX.NE.'M')GOTO4
      M(IB,JB+2)=1
      M(IB+1,JB)=1
      M(IB+2,JB+1)=1
      GO TO 1
4     IF(XX.NE.'J')GO TO 5
      IT=IB+2
      JT=JB+2
      DO 6 IX=IB,IT
      DO 6 JX=JB,JT
6     M(IX,JX)=1
      GO TO 1
5     IF(XX.EQ.' ')GO TO 1
      PRINT,'ERROR',I,J,XX
      STOP
1     CONTINUE
```

```
      C
      C      CONTINUATION OF THE PROGRAM TO COMPUTE
      C      THE INCIDENCE MATRIX FOR JW45
      C
             PUNCH,'45 12  3 BLOCK DESIGN DUE TO J. WALLIS'
             PUNCH 44,M
         44  FORMAT(45I1)
             PRINT 45
         45  FORMAT('1')
             STOP
             END

       $ENTRY
    IIIIIIIII
J   LLLMMMIII
 J  MMMLLLIII
  J ILMIMLILM
   JLMIMLIILM
    JMILLLIMILM
LMIMLJ     IML
MIMLI J    IML
LLIM  J    IML
MLILMIMLJ
ILMIIML J
MMILIML  J
IIII    ILMJ
MLLL    ILM J
LMMM    ILM  J
```

```
$JOB    WATFIV    ROGER,NOWARN,NOEXT
C
C       THIS PROGRAM COMPUTES THE INCIDENCE MATRIX FOR ES36
C
        IMPLICIT INTEGER (A-Z)
        INTEGER*2 M(36,36)/1296*0/
        INTEGER A(9)/1,3*0,1,1,3*0/
        INTEGER B(9)/1,0,0,1,0,0,1,0,0/
        INTEGER C(9)/1,0,1,4*0,1,0/
        INTEGER D(9)/1,1,6*0,1/
        PRINT 1
1       FORMAT('1')
        DO 12 IA=1,9
        IB=IA+9
        IC=IA+18
        ID=IA+27
        DO 12 JA=1,9
        JB=JA+9
        JC=JA+18
        JD=JA+27
        P=1+MOD(9+JA-IA,9)
        X=A(P)
        M(IA,JA)=1-X
        M(IB,JB)=X
        M(IC,JC)=X
        M(ID,JD)=X
        X=B(P)
        M(IA,JB)=X
        M(IB,JA)=X
        M(IC,JD)=1-X
        M(ID,JC)=X
        X=C(P)
        M(IA,JC)=X
        M(IB,JD)=X
        M(IC,JA)=X
        M(ID,JB)=1-X
        X=D(P)
        M(IA,JD)=X
        M(IB,JC)=1-X
        M(IC,JB)=X
12      M(ID,JA)=X
        PUNCH,'36 15  6 BLOCK DESIGN DUE TO E. SPENCE'
        DO 14 I=1,36
14      PUNCH 41,(M(I,J),J=1,36)
41      FORMAT (36I1)
10      STOP
        END

$ENTRY
```

```
$JOB    WATFIV     ROGER,NOWARN,NOEXT,NOCHECK
C
C        THE FOLLOWING IS THE MAIN ROUTINE FOR
C        LISTING BLOCK DESIGNS, COMPUTING
C        INTERSECTION NUMBERS, AND COMPUTING
C        SMITH NORMAL FORM
C
         IMPLICIT INTEGER(A-Z)
         INTEGER M(64,64)/4096*0/
         REAL*8 P(64)
         CHARACTER FLAG,HDR*62,FMT*28(1),FFF(5),FBF*2
         EQUIVALENCE(FMT,FFF),(FFF(5),FBF)
         DATA FMT/'(1X,10(''R'',I1,''='',I5,''     ''))'/
         INTEGER KT(10)
         READ(1,41)V,K,L,FLAG,HDR
 41      FORMAT(3I3,A1,A62)
         WRITE(2,40)HDR
         PRINT 40,HDR
 40      FORMAT('1',A62)
         WRITE(2,42)V,K,L
         PRINT 42,V,K,L
         PRINT,'FLAG=',FLAG
 42      FORMAT(' WITH PARAMETERS V=',I2,', K=',I2,', LAMBDA=',I2/)
         DO 1 I=1,V
 1       READ(1,43)(M(I,J),J=1,V)
 43      FORMAT(64I1)
         CALL PRINT(M,V,K)
         CALL PACK(M,P,V)
         CALL TRIPLES(P,V,K,L,KT,ERR)
         IF(ERR.NE.0) STOP
         LL=L+1
         WRITE(2,*)'THE TRIPLE STRUCTURE OF THE DESIGN IS'
         WRITE(FBF,47)LL
 47      FORMAT(I2)
         WRITE(2,FMT)(I-1,KT(I),I=1,LL)
         IF(FLAG.EQ.'S') GO TO 30
         CALL TRANS(M,V,FLAG)
         IF(FLAG.EQ.'S') GO TO 30
         CALL PACK(M,P,V)
         CALL TRIPLES(P,V,K,L,KT,ERR)
         IF(ERR.NE.0)STOP
         WRITE(2,*)'AND THE TRIPLE STRUCTURE OF THE DUAL DESIGN IS'
         WRITE(2,FMT)(I-1,KT(I),I=1,LL)
         GO TO 31
 30      WRITE(2,*)'DESIGN IS (STRONGLY) SELF-DUAL'
 31      CLASS=0
         PRIME=3
         CALL SMITHA(M,V,CLASS,PRIME)
         WRITE(2,*)'SMITH CLASS=',CLASS
         PRINT,'PRIME=',PRIME
         STOP
         END
```

```
      SUBROUTINE TRIPLES(P,V,KK,L,C,ERR)
C
C     THIS IS THE ROUTINE WHICH VALICATES THE
C     INCIDENCE MATRIX, AND CCMPUTES THE ROW
C     INTERSECTION NUMBERS
C
      IMPLICIT INTEGER (A-Z)
      REAL*8 P(64),A,B,CC,SA
      INTEGER C(10)
      COMMON/W/A,B,CC,N
      ERR=0
      DO3I=1,10
    3 C(I)=0
      V1=V-1
      V2=V-2
      DO 1 I=1,V2
      J=0
      A=SA=P(I)
      CALL BITS
      IF(N.NE.KK) GO TO 30
      IX=I+1
      DO 1 J=IX,V1
      B=P(J)
      CC=SA
      CALL DAND
      IF(N.NE.L) GO TO 30
      CC=A
      JX=J+1
      DO 1 K=JX,V
      B=P(K)
      CALL DAND
      N=N+1
    1 C(N)=C(N)+1
      GO TO 31
   30 ERR=100*I+J
   31 PRINT,'ERR=',ERR
      RETURN
      END
```

```
      SUBROUTINE CAND
C
C     THIS ROUTINE COMPUTES THE INNER PRODUCT
C     OF TWO ROWS OF THE INCIDENCE MATRIX, USING
C     BOOLEAN (BIT STRING) OPERATIONS
C
      EQUIVALENCE(Z,L)
      COMMON/W/X,Y,A,B,C,D,N
      X=AND(C,A)
      Y=AND(D,B)
      ENTRY BITS
      N=0
      Z=X
    1 IF(L.EQ.0)GO TO 2
      N=N+1
      Z=AND(L,L-1)
      GO TO 1
    2 Z=Y
    3 IF(L.EQ.0)RETURN
      N=N+1
      Z=AND(L,L-1)
      GO TO 3
      END
```

```
      SUBROUTINE TRANS(M,V,F)
C
C     THIS ROUTINE TRANSPOSES THE INCIDENCE
C     MATRIX, AND SIMULTANEOUSLY CHECKS TO
C     SEE IF THE MATRIX IS SYMMETRIC (IN WHICH
C     CASE, THE DESIGN IS STRONGLY SELF-DUAL
C
      INTEGER V,M(64,64)
      CHARACTER F
      PRINT,'STARTING TO TRANSPOSE'
      F='S'
      MM=V-1
      DO 1 I=1,MM
      N=I+1
      DO 1 J=N,V
      MU=M(I,J)
      ML=M(J,I)
      IF(MU.EQ.ML)GO TO 1
      M(I,J)=ML
      M(J,I)=MU
      F='N'
    1 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SMITHA(M,V,N,P)
C
C     THIS ROUTINE CALCULATES THE SMITH
C     NORMAL FORM OF THE INCIDENCE MATRIX
C
      IMPLICIT INTEGER (A-Z)
      INTEGER M(64,64),A(64)
    1 DO 3 I=1,V
      DO 3 J=1,V
      MM=M(I,J)+2
      GO TO (2,3,2),MM
    3 CONTINUE
      DO 4 I=1,V
      DO 4 J=1,V
      IF(0.NE.MOD(M(I,J),P)) GO TO 30
    4 CONTINUE
      RETURN
   30 P=0
      RETURN
    2 DO 5 JJ=1,V
    5 A(JJ)=M(I,JJ)
      N=N+1
      IX=1
      DO 6 II=1,V
      IF(I.EQ.II) GO TO 6
      L=M(II,J)
      IF(MM.EQ.3)L=-L
      JX=1
      DO 7 JJ=1,V
      IF(JJ.EQ.J) GO TO 7
      M(IX,JX)=M(II,JJ)+L*A(JJ)
      JX=JX+1
    7 CONTINUE
      IX=IX+1
    6 CONTINUE
      V=V-1
      IF(V.GT.0) GO TO 1
      RETURN
      END
```

```
      SUBROUTINE PACK(M,P,V)
C
C     THIS ROUTINE PACKS THE INCIDENCE MATRIX
C     INTO BOOLEAN (BIT-STRING) FORM FOR
C     PROCESSING BY 'TRIPLES'
C
      IMPLICIT INTEGER (A-Z)
      INTEGER M(64,64)
      REAL*8 P(64),D,X*4(2)
      EQUIVALENCE(D,X,IY),(X(2),IZ)
      DO 11 I=1,V
      IY=0
      IZ=0
      MM=1
      DO 2 J=1,32
      IF(M(I,33-J).EQ.1)IY=IY+MM
    2 MM=MM+MM
      JS=65-V
      MM=2**(JS-1)
      DO 3 J=JS,32
      IF(M(I,65-J).EQ.1)IZ=IZ+MM
    3 MM=MM+MM
   11 P(I)=D
      RETURN
      END
```

```
      SUBROUTINE PRINT(M,V,K)
C
C     THIS ROUTINE PRINTS THE BIBD
C     BY LISTING THE VARIETIES WHICH
C     OCCUR IN EACH BLOCK; THE NUMBER
C     AT THE HEAD OF EACH ROW IS THE
C     BLOCK NUMBER, AND THE OTHER NUMBERS
C     ARE THE VARIETIES INCIDENT WITH THAT
C     BLOCK
C
      IMPLICIT INTEGER(A-Z)
      INTEGER M(64,64),LINE(30)
      DO 1 J=1,V
      LX=0
      DO 2 I=1,V
      IF(0.EQ.M(I,J))GO TO 2
      LX=LX+1
      LINE(LX)=I
    2 CONTINUE
    1 WRITE(2,40)J,(LINE(I),I=1,LX)
   40 FORMAT(1X,I3,':',30I4/(5X,30I4))
      RETURN
      END
```

APPENDIX B

LISTINGS OF DESIGNS

BLOCK DESIGN DUE TO E. SPENCE
WITH PARAMETERS V=36, K=15, LAMBDA= 6

1:	2	3	4	7	8	9	10	13	16	19	21	26	28	29	36
2:	1	3	4	5	8	9	11	14	17	20	22	27	28	29	30
3:	1	2	4	5	6	9	12	15	18	19	21	23	29	30	31
4:	1	2	3	5	6	7	10	13	16	20	22	24	30	31	32
5:	2	3	4	6	7	8	11	14	17	21	23	25	31	32	33
6:	3	4	5	7	8	9	12	15	18	22	24	26	32	33	34
7:	1	4	5	6	8	9	10	13	16	23	25	27	33	34	35
8:	1	2	5	6	7	9	11	14	17	19	24	26	34	35	36
9:	1	2	3	6	7	8	12	15	18	20	25	27	28	35	36
10:	1	4	7	10	14	15	19	20	27	29	31	32	33	34	36
11:	2	5	8	11	15	16	19	20	21	28	30	32	33	34	35
12:	3	6	9	12	16	17	20	21	22	29	31	33	34	35	36
13:	1	4	7	13	17	19	21	22	23	28	30	32	34	35	36
14:	2	5	8	10	14	18	22	23	24	28	29	31	33	35	36
15:	3	6	9	10	11	15	23	24	25	28	29	30	32	34	36
16:	1	4	7	11	12	16	24	25	26	28	29	30	31	33	35
17:	2	5	8	12	13	17	25	26	27	29	30	31	32	34	36
18:	3	6	9	13	14	18	19	26	27	28	30	31	32	33	35
19:	1	3	8	12	13	14	15	16	17	19	23	24	28	31	34
20:	2	4	9	13	14	15	16	17	18	20	24	25	29	32	35
21:	1	3	5	10	14	15	16	17	18	21	25	26	30	33	36
22:	2	4	6	10	11	15	16	17	18	22	26	27	28	31	34
23:	3	5	7	10	11	12	16	17	18	19	23	27	29	32	35
24:	4	6	8	10	11	12	13	17	18	19	20	24	30	33	36
25:	5	7	9	10	11	12	13	14	18	20	21	25	28	31	34
26:	1	6	8	10	11	12	13	14	15	21	22	26	29	32	35
27:	2	7	9	11	12	13	14	15	16	22	23	27	30	33	36
28:	1	2	9	10	12	17	20	21	23	24	26	27	28	32	33
29:	1	2	3	11	13	18	19	21	22	24	25	27	29	33	34
30:	2	3	4	10	12	14	19	20	22	23	25	26	30	34	35
31:	3	4	5	11	13	15	20	21	23	24	26	27	31	35	36
32:	4	5	6	12	14	16	19	21	22	24	25	27	28	32	36
33:	5	6	7	13	15	17	19	20	22	23	25	26	28	29	33
34:	6	7	8	14	16	18	20	21	23	24	26	27	29	30	34
35:	7	8	9	10	15	17	19	21	22	24	25	27	30	31	35
36:	1	8	9	11	16	18	19	20	22	23	25	26	31	32	36

THE TRIPLE STRUCTURE OF THE DESIGN IS
R0= 0 R1= 846 R2= 3753 R3= 2142 R4= 396 R5= 0
R6= 3
AND THE TRIPLE STRUCTURE OF THE DUAL DESIGN IS
R0= 0 R1= 846 R2= 3753 R3= 2142 R4= 396 R5= 0
R6= 3
SMITH CLASS= 16

BLOCK DESIGN DUE TO J. WALLIS
WITH PARAMETERS V=36, K=15, LAMBDA= 6

1:	4	5	6	7	8	9	10	13	16	19	22	25	28	31	34
2:	4	5	6	7	8	9	11	14	17	20	23	26	29	32	35
3:	4	5	6	7	8	9	12	15	18	21	24	27	30	33	36
4:	1	2	3	7	8	9	11	14	17	21	24	27	28	31	34
5:	1	2	3	7	8	9	12	15	18	19	22	25	29	32	35
6:	1	2	3	7	8	9	10	13	16	20	23	26	30	33	36
7:	1	2	3	4	5	6	12	15	18	20	23	26	28	31	34
8:	1	2	3	4	5	6	10	13	16	21	24	27	29	32	35
9:	1	2	3	4	5	6	11	14	17	19	22	25	30	33	36
10:	1	6	8	13	14	15	16	17	18	19	23	27	28	33	35
11:	2	4	9	13	14	15	16	17	18	20	24	25	29	31	36
12:	3	5	7	13	14	15	16	17	18	21	22	26	30	32	34
13:	1	6	8	10	11	12	16	17	18	20	24	25	30	32	34
14:	2	4	9	10	11	12	16	17	18	21	22	26	28	33	35
15:	3	5	7	10	11	12	16	17	18	19	23	27	29	31	36
16:	1	6	8	10	11	12	13	14	15	21	22	26	29	31	36
17:	2	4	9	10	11	12	13	14	15	19	23	27	30	32	34
18:	3	5	7	10	11	12	13	14	15	20	24	25	28	33	35
19:	1	5	9	10	15	17	22	23	24	25	26	27	28	32	36
20:	2	6	7	11	13	18	22	23	24	25	26	27	29	33	34
21:	3	4	8	12	14	16	22	23	24	25	26	27	30	31	35
22:	1	5	9	12	14	16	19	20	21	25	26	27	29	33	34
23:	2	6	7	10	15	17	19	20	21	25	26	27	30	31	35
24:	3	4	8	11	13	18	19	20	21	25	26	27	28	32	36
25:	1	5	9	11	13	18	19	20	21	22	23	24	30	31	35
26:	2	6	7	12	14	16	19	20	21	22	23	24	28	32	36
27:	3	4	8	10	15	17	19	20	21	22	23	24	29	33	34
28:	1	4	7	10	14	18	19	24	26	31	32	33	34	35	36
29:	2	5	8	11	15	16	20	22	27	31	32	33	34	35	36
30:	3	6	9	12	13	17	21	23	25	31	32	33	34	35	36
31:	1	4	7	11	15	16	21	23	25	28	29	30	34	35	36
32:	2	5	8	12	13	17	19	24	26	28	29	30	34	35	36
33:	3	6	9	10	14	18	20	22	27	28	29	30	34	35	36
34:	1	4	7	12	13	17	20	22	27	28	29	30	31	32	33
35:	2	5	8	10	14	18	21	23	25	28	29	30	31	32	33
36:	3	6	9	11	15	16	19	24	26	28	29	30	31	32	33

THE TRIPLE STRUCTURE OF THE DESIGN IS
R0= 144 R1= 468 R2= 3996 R3= 2232 R4= 288 R5= 0
R6= 12
DESIGN IS (STRONGLY) SELF-DUAL
SMITH CLASS= 12

BLOCK DESIGN DUE TO K. TAKEUCHI
WITH PARAMETERS V=36, K=15, LAMBDA= 6

1:	1	2	3	4	5	9	13	14	17	19	21	24	26	32	35
2:	1	2	3	4	6	10	14	15	18	20	21	22	27	29	36
3:	1	2	3	4	7	11	15	16	17	19	22	23	28	30	33
4:	1	2	3	4	8	12	13	16	18	20	23	24	25	31	34
5:	1	5	6	7	8	9	13	16	17	18	21	23	27	30	36
6:	2	5	6	7	8	10	13	14	18	19	22	24	28	31	33
7:	3	5	6	7	8	11	14	15	19	20	21	23	25	32	34
8:	4	5	6	7	8	12	15	16	17	20	22	24	26	29	35
9:	1	5	9	10	11	12	13	15	17	20	21	22	28	31	34
10:	2	6	9	10	11	12	14	16	17	18	22	23	25	32	35
11:	3	7	9	10	11	12	13	15	18	19	23	24	26	29	36
12:	4	8	9	10	11	12	14	16	19	20	21	24	27	30	33
13:	2	8	11	13	14	15	16	17	21	25	26	29	31	33	36
14:	3	5	12	13	14	15	16	18	22	26	27	30	32	33	34
15:	4	6	9	13	14	15	16	19	23	27	28	29	31	34	35
16:	1	7	10	13	14	15	16	20	24	25	28	30	32	35	36
17:	3	6	12	13	17	18	19	20	21	25	28	29	30	33	35
18:	4	7	9	14	17	18	19	20	22	25	26	30	31	34	36
19:	1	8	10	15	17	18	19	20	23	26	27	31	32	33	35
20:	2	5	11	16	17	18	19	20	24	27	28	29	32	34	36
21:	4	7	10	13	17	21	22	23	24	25	27	29	32	33	34
22:	1	8	11	14	18	21	22	23	24	26	28	29	30	34	35
23:	2	5	12	15	19	21	22	23	24	25	27	30	31	35	36
24:	3	6	9	16	20	21	22	23	24	26	28	31	32	33	36
25:	1	2	5	7	9	12	14	20	23	25	26	27	28	29	33
26:	2	3	6	8	9	10	15	17	24	25	26	27	28	30	34
27:	3	4	5	7	10	11	16	18	21	25	26	27	28	31	35
28:	1	4	6	8	11	12	13	19	22	25	26	27	28	32	36
29:	1	4	5	6	9	11	15	18	24	25	29	30	31	32	33
30:	1	2	6	7	10	12	16	19	21	26	29	30	31	32	34
31:	2	3	7	8	9	11	13	20	22	27	29	30	31	32	35
32:	3	4	5	8	10	12	14	17	23	28	29	30	31	32	36
33:	1	3	5	8	9	10	16	19	22	25	29	33	34	35	36
34:	2	4	5	6	10	11	13	20	23	26	30	33	34	35	36
35:	1	3	6	7	11	12	14	17	24	27	31	33	34	35	36
36:	2	4	7	8	9	12	15	18	21	28	32	33	34	35	36

THE TRIPLE STRUCTURE OF THE DESIGN IS
R0= 72 R1= 648 R2= 3888 R3= 2196 R4= 324 R5= 0
R6= 12
AND THE TRIPLE STRUCTURE OF THE DUAL DESIGN IS
R0= 72 R1= 648 R2= 3888 R3= 2196 R4= 324 R5= 0
R6= 12
SMITH CLASS= 14

BLOCK DESIGN DUE TO P. K. MENON
WITH PARAMETERS V=36, K=15, LAMBDA= 6

1:	2	3	5	7	8	9	11	13	14	17	24	25	26	27	34
2:	3	4	6	8	9	10	12	14	15	18	19	26	27	28	35
3:	1	4	5	7	9	10	11	13	15	16	20	27	28	29	36
4:	2	5	6	8	10	11	12	14	16	17	21	28	29	30	31
5:	1	3	6	7	9	11	12	15	17	18	22	25	29	30	32
6:	1	2	4	7	8	10	12	13	16	18	23	25	26	30	33
7:	4	8	9	11	13	14	15	17	19	20	23	30	31	32	33
8:	5	9	10	12	14	15	16	18	20	21	24	25	32	33	34
9:	6	7	10	11	13	15	16	17	19	21	22	26	33	34	35
10:	1	8	11	12	14	16	17	18	20	22	23	27	34	35	36
11:	2	7	9	12	13	15	17	18	21	23	24	28	31	35	36
12:	3	7	8	10	13	14	16	18	19	22	24	29	31	32	36
13:	1	2	3	10	14	15	17	19	20	21	23	25	26	29	36
14:	2	3	4	11	15	16	18	20	21	22	24	26	27	30	31
15:	3	4	5	12	13	16	17	19	21	22	23	25	27	28	32
16:	4	5	6	7	14	17	18	20	22	23	24	26	28	29	33
17:	1	5	6	8	13	15	18	19	21	23	24	27	29	30	34
18:	1	2	6	9	13	14	16	19	20	22	24	25	28	30	35
19:	6	7	8	9	16	20	21	23	25	26	27	29	31	32	35
20:	1	8	9	10	17	21	22	24	26	27	28	30	32	33	36
21:	2	9	10	11	18	19	22	23	25	27	28	29	31	33	34
22:	3	10	11	12	13	20	23	24	26	28	29	30	32	34	35
23:	4	7	11	12	14	19	21	24	25	27	29	30	33	35	36
24:	5	7	8	12	15	19	20	22	25	26	28	30	31	34	36
25:	1	2	5	12	13	14	15	22	26	27	29	31	32	33	35
26:	2	3	6	7	14	15	16	23	27	28	30	32	33	34	36
27:	1	3	4	8	15	16	17	24	25	28	29	31	33	34	35
28:	2	4	5	9	16	17	18	19	26	29	30	32	34	35	36
29:	3	5	6	10	13	17	18	20	25	27	30	31	33	35	36
30:	1	4	6	11	13	14	18	21	25	26	28	31	32	34	36
31:	1	2	3	5	7	8	11	18	19	20	21	28	32	33	35
32:	2	3	4	6	8	9	12	13	20	21	22	29	33	34	36
33:	1	3	4	5	7	9	10	14	21	22	23	30	31	34	35
34:	2	4	5	6	8	10	11	15	22	23	24	25	32	35	36
35:	1	3	5	6	9	11	12	16	19	23	24	26	31	33	36
36:	1	2	4	6	7	10	12	17	19	20	24	27	31	32	34

THE TRIPLE STRUCTURE OF THE DESIGN IS
R0= 144 R1= 468 R2= 3996 R3= 2232 R4= 288 R5= 0
R6= 12
AND THE TRIPLE STRUCTURE OF THE DUAL DESIGN IS
R0= 144 R1= 468 R2= 3996 R3= 2232 R4= 288 R5= 0
R6= 12
SMITH CLASS= 12

BLOCK DESIGN DUE TO R. A. KINGSLEY AND R. G. STANTON
WITH PARAMETERS V=36, K=15, LAMBDA= 6

1:	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
2:	9	10	11	15	16	17	18	19	20	22	23	27	34	35	36
3:	7	8	11	13	14	17	18	19	21	22	24	28	32	33	36
4:	6	8	10	12	14	16	18	20	21	22	25	29	31	33	35
5:	6	7	9	12	13	15	19	20	21	22	26	30	31	32	34
6:	4	5	11	13	14	15	16	20	21	23	24	29	30	31	36
7:	3	5	10	12	14	15	17	19	21	23	25	28	30	32	35
8:	3	4	9	12	13	16	17	18	21	23	26	28	29	33	34
9:	2	5	8	12	13	16	17	19	20	24	25	27	30	33	34
10:	2	4	7	12	14	15	17	18	20	24	26	27	29	32	35
11:	2	3	6	13	14	15	16	18	19	25	26	27	28	31	36
12:	4	5	7	8	9	10	17	20	21	25	26	27	28	31	36
13:	3	5	6	8	9	11	16	19	21	24	26	27	29	32	35
14:	3	4	6	7	10	11	15	18	21	24	25	27	30	33	34
15:	2	5	6	7	10	11	14	19	20	23	26	28	29	33	34
16:	2	4	6	8	9	11	13	18	20	23	25	28	30	32	35
17:	2	3	7	8	9	10	12	18	19	23	24	29	30	31	36
18:	2	3	4	8	10	11	14	16	17	22	26	30	31	32	34
19:	2	3	5	7	9	11	13	15	17	22	25	29	31	33	35
20:	2	4	5	6	9	10	12	15	16	22	24	28	32	33	36
21:	3	4	5	6	7	8	12	13	14	22	23	27	34	35	36
22:	1	2	3	4	5	18	19	20	21	31	32	33	34	35	36
23:	1	2	6	7	8	15	16	17	21	28	29	30	34	35	36
24:	1	3	6	9	10	13	14	17	20	27	29	30	32	33	36
25:	1	4	7	9	11	12	14	16	19	27	28	30	31	33	35
26:	1	5	8	10	11	12	13	15	18	27	28	29	31	32	34
27:	1	2	9	10	11	12	13	14	21	24	25	26	34	35	36
28:	1	3	7	8	11	12	15	16	20	23	25	26	32	33	36
29:	1	4	6	8	10	13	15	17	19	23	24	26	31	33	35
30:	1	5	6	7	9	14	16	17	18	23	24	25	31	32	34
31:	1	4	5	6	11	12	17	18	19	22	25	26	29	30	36
32:	1	3	5	7	10	13	16	18	20	22	24	26	28	30	35
33:	1	3	4	8	9	14	15	19	20	22	24	25	28	29	34
34:	1	2	5	8	9	14	15	18	21	22	23	26	27	30	33
35:	1	2	4	7	10	13	16	19	21	22	23	25	27	29	32
36:	1	2	3	6	11	12	17	20	21	22	23	24	27	28	31

THE TRIPLE STRUCTURE OF THE DESIGN IS
R0= 0 R1= 540 R2= 4320 R3= 2160 R4= 0 R5= 0
R6= 120
DESIGN IS (STRONGLY) SELF-DUAL
SMITH CLASS= 15

BLOCK DESIGN DUE TO W. BLACKWELDER
WITH PARAMETERS V=36, K=15, LAMBDA= 6

1:	2	3	4	5	6	7	12	13	17	19	22	25	27	31	32
2:	1	3	4	5	6	7	8	14	18	20	23	26	28	32	33
3:	1	2	4	5	6	8	9	13	15	21	24	27	29	33	34
4:	1	2	3	5	6	9	10	14	16	19	22	28	30	34	35
5:	1	2	3	4	6	10	11	15	17	20	23	25	29	35	36
6:	1	2	3	4	5	11	12	16	18	21	24	26	30	31	36
7:	1	2	8	9	10	11	12	13	18	19	23	25	28	31	33
8:	2	3	7	9	10	11	12	13	14	20	24	26	29	32	34
9:	3	4	7	8	10	11	12	14	15	19	21	27	30	33	35
10:	4	5	7	8	9	11	12	15	16	20	22	25	28	34	36
11:	5	6	7	8	9	10	12	16	17	21	23	26	29	31	35
12:	1	6	7	8	9	10	11	17	18	22	24	27	30	32	36
13:	1	3	7	8	14	15	16	17	18	19	24	25	29	31	34
14:	2	4	8	9	13	15	16	17	18	19	20	26	30	32	35
15:	3	5	9	10	13	14	16	17	18	20	21	25	27	33	36
16:	4	6	10	11	13	14	15	17	18	21	22	26	28	31	34
17:	1	5	11	12	13	14	15	16	18	22	23	27	29	32	35
18:	2	6	7	12	13	14	15	16	17	23	24	28	30	33	36
19:	1	4	7	9	13	14	20	21	22	23	24	25	30	31	35
20:	2	5	8	10	14	15	19	21	22	23	24	25	26	32	36
21:	3	6	9	11	15	16	19	20	22	23	24	26	27	31	33
22:	1	4	10	12	16	17	19	20	21	23	24	27	28	32	34
23:	2	5	7	11	17	18	19	20	21	22	24	28	29	33	35
24:	3	6	8	12	13	18	19	20	21	22	23	29	30	34	36
25:	1	5	7	10	13	15	19	20	26	27	28	29	30	31	36
26:	2	6	8	11	14	16	20	21	25	27	28	29	30	31	32
27:	1	3	9	12	15	17	21	22	25	26	28	29	30	32	33
28:	2	4	7	10	16	18	22	23	25	26	27	29	30	33	34
29:	3	5	8	11	13	17	23	24	25	26	27	28	30	34	35
30:	4	6	9	12	14	18	19	24	25	26	27	28	29	35	36
31:	1	6	7	11	13	16	19	21	25	26	32	33	34	35	36
32:	1	2	8	12	14	17	20	22	26	27	31	33	34	35	36
33:	2	3	7	9	15	18	21	23	27	28	31	32	34	35	36
34:	3	4	8	10	13	16	22	24	28	29	31	32	33	35	36
35:	4	5	9	11	14	17	19	23	29	30	31	32	33	34	36
36:	5	6	10	12	15	18	20	24	25	30	31	32	33	34	35

THE TRIPLE STRUCTURE OF THE DESIGN IS
R0= 144 R1= 468 R2= 3996 R3= 2232 R4= 288 R5= 0
R6= 12
DESIGN IS (STRONGLY) SELF-DUAL
SMITH CLASS= 12

BLOCK DESIGN DUE TO J. WALLIS
WITH PARAMETERS V=45, K=12, LAMBDA= 3

1:	1	2	3	19	22	25	28	31	34	37	40	43
2:	1	2	3	20	23	26	29	32	35	38	41	44
3:	1	2	3	21	24	27	30	33	36	39	42	45
4:	4	5	6	20	23	26	30	33	36	37	40	43
5:	4	5	6	21	24	27	28	31	34	38	41	44
6:	4	5	6	19	22	25	29	32	35	39	42	45
7:	7	8	9	21	24	27	29	32	35	37	40	43
8:	7	8	9	19	22	25	30	33	36	38	41	44
9:	7	8	9	20	23	26	28	31	34	39	42	45
10:	10	11	12	19	23	27	28	33	35	37	41	45
11:	10	11	12	20	24	25	29	31	36	38	42	43
12:	10	11	12	21	22	26	30	32	34	39	40	44
13:	13	14	15	20	24	25	30	32	34	37	41	45
14:	13	14	15	21	22	26	28	33	35	38	42	43
15:	13	14	15	19	23	27	29	31	36	39	40	44
16:	16	17	18	21	22	26	29	31	36	37	41	45
17:	16	17	18	19	23	27	30	32	34	38	42	43
18:	16	17	18	20	24	25	28	33	35	39	40	44
19:	1	5	9	10	15	17	19	20	21	37	42	44
20:	2	6	7	11	13	18	19	20	21	38	40	45
21:	3	4	8	12	14	16	19	20	21	39	41	43
22:	2	6	7	12	14	16	22	23	24	37	42	44
23:	3	4	8	10	15	17	22	23	24	38	40	45
24:	1	5	9	11	13	18	22	23	24	39	41	43
25:	3	4	8	11	13	18	25	26	27	37	42	44
26:	1	5	9	12	14	16	25	26	27	38	40	45
27:	2	6	7	10	15	17	25	26	27	39	41	43
28:	1	6	8	10	14	18	19	24	26	28	29	30
29:	2	4	9	11	15	16	20	22	27	28	29	30
30:	3	5	7	12	13	17	21	23	25	28	29	30
31:	3	5	7	11	15	16	19	24	26	31	32	33
32:	1	6	8	12	13	17	20	22	27	31	32	33
33:	2	4	9	10	14	18	21	23	25	31	32	33
34:	2	4	9	12	13	17	19	24	26	34	35	36
35:	3	5	7	10	14	18	20	22	27	34	35	36
36:	1	5	8	11	15	16	21	23	25	34	35	36
37:	1	4	7	10	13	16	28	32	36	37	38	39
38:	2	5	8	11	14	17	29	33	34	37	38	39
39:	3	6	9	12	15	18	30	31	35	37	38	39
40:	3	6	9	11	14	17	28	32	36	40	41	42
41:	1	4	7	12	15	18	29	33	34	40	41	42
42:	2	5	8	10	13	16	30	31	35	40	41	42
43:	2	5	8	12	15	18	28	32	36	43	44	45
44:	3	6	9	10	13	16	29	33	34	43	44	45
45:	1	4	7	11	14	17	30	31	35	43	44	45

THE TRIPLE STRUCTURE OF THE DESIGN IS
R0= 5220 R1= 8100 R2= 810 R3= 60
AND THE TRIPLE STRUCTURE OF THE DUAL DESIGN IS
R0= 5220 R1= 8100 R2= 810 R3= 60
SMITH CLASS= 18

BLOCK DESIGN DUE TO K. TAKEUCHI
WITH PARAMETERS V=45, K=12, LAMBDA= 3

1:	2	3	4	5	7	12	18	24	30	33	40	44
2:	1	3	4	5	8	13	19	25	26	34	36	45
3:	1	2	4	5	9	14	20	21	27	35	37	41
4:	1	2	3	5	10	15	16	22	28	31	38.	42
5:	1	2	3	4	6	11	17	23	29	32	39	43
6:	2	7	8	9	10	12	20	23	29	34	38	45
7:	3	6	8	9	10	13	16	24	30	35	39	41
8:	4	6	7	9	10	14	17	25	26	31	40	42
9:	5	6	7	8	10	15	18	21	27	32	36	43
10:	1	6	7	8	9	11	19	22	28	33	37	44
11:	2	7	12	13	14	15	19	25	28	35	39	43
12:	3	8	11	13	14	15	20	21	29	31	40	44
13:	4	9	11	12	14	15	16	22	30	32	36	45
14:	5	10	11	12	13	15	17	23	26	33	37	41
15:	1	6	11	12	13	14	18	24	27	34	38	42
16:	3	10	14	17	18	19	20	22	27	33	39	45
17:	4	6	15	16	18	19	20	23	28	34	40	41
18:	5	7	11	16	17	19	20	24	29	35	36	42
19:	1	8	12	16	17	18	20	25	30	31	37	43
20:	2	9	13	16	17	18	19	21	26	32	38	44
21:	4	8	15	17	22	23	24	25	27	35	38	44
22:	5	9	11	18	21	23	24	25	28	31	39	45
23:	1	10	12	19	21	22	24	25	29	32	40	41
24:	2	6	13	20	21	22	23	25	30	33	36	42
25:	3	7	14	16	21	22	23	24	26	34	37	43
26:	5	9	13	17	22	27	28	29	30	34	40	43
27:	1	10	14	18	23	26	28	29	30	35	36	44
28:	2	6	15	19	24	26	27	29	30	31	37	45
29:	3	7	11	20	25	26	27	28	30	32	38	41
30:	4	8	12	16	21	26	27	28	29	33	39	42
31:	3	9	15	18	25	29	32	33	34	35	37	42
32:	4	10	11	19	21	30	31	33	34	35	38	43
33:	5	6	12	20	22	26	31	32	34	35	39	44
34:	1	7	13	16	23	27	31	32	33	35	40	45
35:	2	8	14	17	24	28	31	32	33	34	36	41
36:	5	8	14	19	23	30	32	37	38	39	40	42
37:	1	9	15	20	24	26	33	36	38	39	40	43
38:	2	10	11	16	25	27	34	36	37	39	40	44
39:	3	6	12	17	21	28	35	36	37	38	40	45
40:	4	7	13	18	22	29	31	36	37	38	39	41
41:	4	10	13	20	24	28	32	37	42	43	44	45
42:	5	6	14	16	25	29	33	38	41	43	44	45
43:	1	7	15	17	21	30	34	39	41	42	44	45
44:	2	8	11	18	22	26	35	40	41	42	43	45
45:	3	9	12	19	23	27	31	36	41	42	43	44

THE TRIPLE STRUCTURE OF THE DESIGN IS
R0= 5220 R1= 8100 R2= 810 R3= 60
AND THE TRIPLE STRUCTURE OF THE DUAL DESIGN IS
R0= 5220 R1= 8100 R2= 810 R3= 60
SMITH CLASS= 17

BLOCK DESIGN DUE TO R. C. BOSE
WITH PARAMETERS V=45, K=12, LAMBDA= 3

1:	16	17	21	25	26	27	31	35	36	37	41	45
2:	16	18	22	23	26	28	32	33	36	38	42	43
3:	16	19	20	24	26	29	30	34	36	39	40	44
4:	13	14	15	17	20	23	28	31	34	39	42	45
5:	13	14	15	18	21	24	29	32	35	37	40	43
6:	13	14	15	19	22	25	27	30	33	38	41	44
7:	10	11	12	17	18	19	30	31	32	43	44	45
8:	10	11	12	20	21	22	33	34	35	37	38	39
9:	10	11	12	23	24	25	27	28	29	40	41	42
10:	7	8	9	17	20	23	27	30	33	37	40	43
11:	7	8	9	18	21	24	28	31	34	38	41	44
12:	7	8	9	19	22	25	29	32	35	39	42	45
13:	4	5	6	17	18	19	27	28	29	37	38	39
14:	4	5	6	20	21	22	30	31	32	40	41	42
15:	4	5	6	23	24	25	33	34	35	43	44	45
16:	1	2	3	17	18	19	20	21	22	23	24	25
17:	1	4	7	10	13	16	21	25	28	30	39	43
18:	2	5	7	11	13	16	22	23	29	31	37	44
19:	3	6	7	12	13	16	20	24	27	32	38	45
20:	3	4	8	10	14	16	19	24	31	33	37	42
21:	1	5	8	11	14	16	17	25	32	34	38	40
22:	2	6	8	12	14	16	18	23	30	35	39	41
23:	2	4	9	10	15	16	18	22	27	34	40	45
24:	3	5	9	11	15	16	19	20	28	35	41	43
25:	1	6	9	12	15	16	17	21	29	33	42	44
26:	1	2	3	27	28	29	30	31	32	33	34	35
27:	1	6	9	10	13	19	23	26	31	35	38	40
28:	2	4	9	11	13	17	24	26	32	33	39	41
29:	3	5	9	12	13	18	25	26	30	34	37	42
30:	3	6	7	10	14	17	22	26	29	34	41	43
31:	1	4	7	11	14	18	20	26	27	35	42	44
32:	2	5	7	12	14	19	21	26	28	33	40	45
33:	2	6	8	10	15	20	25	26	28	32	37	44
34:	3	4	8	11	15	21	23	26	29	30	38	45
35:	1	5	8	12	15	22	24	26	27	31	39	43
36:	1	2	3	37	38	39	40	41	42	43	44	45
37:	1	5	8	10	13	18	20	29	33	36	41	45
38:	2	6	8	11	13	19	21	27	34	36	42	43
39:	3	4	8	12	13	17	22	28	35	36	40	44
40:	3	5	9	10	14	21	23	27	32	36	39	44
41:	1	6	9	11	14	22	24	28	30	36	37	45
42:	2	4	9	12	14	20	25	29	31	36	38	43
43:	2	5	7	10	15	17	24	30	35	36	38	42
44:	3	6	7	11	15	18	25	31	33	36	39	40
45:	1	4	7	12	15	19	23	32	34	36	37	41

THE TRIPLE STRUCTURE OF THE DESIGN IS
R0= 5040 R1= 8640 R2= 270 R3= 240
DESIGN IS (STRONGLY) SELF-DUAL
SMITH CLASS= 15

BLOCK DESIGN DUE TO R. A. KINGSLEY AND R. G. STANTON
WITH PARAMETERS V=45, K=12, LAMBDA= 3

1:	2	3	4	5	6	12	17	22	27	31	32	33
2:	1	3	4	5	7	11	18	23	28	34	35	36
3:	1	2	4	5	8	13	16	24	29	37	38	39
4:	1	2	3	5	9	14	19	21	30	40	41	42
5:	1	2	3	4	10	15	20	25	26	43	44	45
6:	1	7	8	9	10	11	16	21	26	31	32	33
7:	2	6	8	9	10	12	18	23	28	37	40	43
8:	3	6	7	9	10	13	17	24	29	34	41	44
9:	4	6	7	8	10	14	19	22	30	35	38	45
10:	5	6	7	8	9	15	20	25	27	36	39	42
11:	2	6	12	13	14	15	16	21	26	34	35	36
12:	1	7	11	13	14	15	17	22	27	37	40	43
13:	3	8	11	12	14	15	18	24	29	31	42	45
14:	4	9	11	12	13	15	19	23	30	32	39	44
15:	5	10	11	12	13	14	20	25	28	33	38	41
16:	3	6	11	17	18	19	20	21	26	37	38	39
17:	1	8	12	16	18	19	20	22	27	34	41	44
18:	2	7	13	16	17	19	20	23	28	31	42	45
19:	4	9	14	16	17	18	20	24	30	33	36	43
20:	5	10	15	16	17	18	19	25	29	32	35	40
21:	4	6	11	16	22	23	24	25	26	40	41	42
22:	1	9	12	17	21	23	24	25	27	35	38	45
23:	2	7	14	18	21	22	24	25	28	32	39	44
24:	3	8	13	19	21	22	23	25	29	33	36	43
25:	5	10	15	20	21	22	23	24	30	31	34	37
26:	5	6	11	16	21	27	28	29	30	43	44	45
27:	1	10	12	17	22	26	28	29	30	36	39	42
28:	2	7	15	18	23	26	27	29	30	33	38	41
29:	3	8	13	20	24	26	27	28	30	32	35	40
30:	4	9	14	19	25	26	27	28	29	31	34	37
31:	1	6	13	18	25	30	32	33	34	37	42	45
32:	1	6	14	20	23	29	31	33	35	39	40	44
33:	1	6	15	19	24	28	31	32	36	38	41	43
34:	2	8	11	17	25	30	31	35	36	37	41	44
35:	2	9	11	20	22	29	32	34	36	38	40	45
36:	2	10	11	19	24	27	33	34	35	39	42	43
37:	3	7	12	16	25	30	31	34	38	39	40	43
38:	3	9	15	16	22	28	33	35	37	39	41	45
39:	3	10	14	16	23	27	32	36	37	38	42	44
40:	4	7	12	20	21	29	32	35	37	41	42	43
41:	4	8	15	17	21	28	33	34	38	40	42	44
42:	4	10	13	18	21	27	31	36	39	40	41	45
43:	5	7	12	19	24	26	33	36	37	40	44	45
44:	5	8	14	17	23	26	32	34	39	41	43	45
45:	5	9	13	18	22	26	31	35	38	42	43	44

THE TRIPLE STRUCTURE OF THE DESIGN IS
R0= 5040   R1= 8640   R2= 270   R3= 240
DESIGN IS (STRONGLY) SELF-DUAL
SMITH CLASS= 15

BLOCK DESIGN DUE TO S. S. SHRIKHANDE AND N. K. SINGH
WITH PARAMETERS V=45, K=12, LAMBDA= 3

1:	2	3	4	5	6	7	8	9	10	11	12	13
2:	1	3	4	5	14	15	16	17	18	19	20	21
3:	1	2	4	5	22	23	24	25	26	27	28	29
4:	1	2	3	5	30	31	32	33	34	35	36	37
5:	1	2	3	4	38	39	40	41	42	43	44	45
6:	1	7	8	9	14	18	23	29	31	37	39	45
7:	1	6	8	9	15	19	22	28	33	35	41	43
8:	1	6	7	9	16	20	24	27	30	36	40	42
9:	1	6	7	8	17	21	25	26	32	34	38	44
10:	1	11	12	13	14	18	22	28	30	36	38	44
11:	1	10	12	13	15	19	23	29	32	34	40	42
12:	1	10	11	13	16	20	25	26	31	37	41	43
13:	1	10	11	12	17	21	24	27	33	35	39	45
14:	2	6	10	15	16	17	22	29	30	37	38	45
15:	2	7	11	14	16	17	23	28	33	34	41	42
16:	2	8	12	14	15	17	24	26	31	36	40	43
17:	2	9	13	14	15	16	25	27	32	35	39	44
18:	2	6	10	19	20	21	23	28	31	36	39	44
19:	2	7	11	18	20	21	22	29	32	35	40	43
20:	2	8	12	18	19	21	25	27	30	37	41	42
21:	2	9	13	18	19	20	24	26	33	34	38	45
22:	3	7	10	14	19	23	24	25	30	35	38	43
23:	3	6	11	15	18	22	24	25	31	34	39	42
24:	3	8	13	16	21	22	23	25	33	36	40	45
25:	3	9	12	17	20	22	23	24	32	37	41	44
26:	3	9	12	16	21	27	28	29	31	34	38	43
27:	3	8	13	17	20	26	28	29	30	35	39	42
28:	3	7	10	15	18	26	27	29	33	36	41	44
29:	3	6	11	14	19	26	27	28	32	37	40	45
30:	4	8	10	14	20	22	27	31	32	33	38	42
31:	4	6	12	16	18	23	26	30	32	33	39	43
32:	4	9	11	17	19	25	29	30	31	33	40	44
33:	4	7	13	15	21	24	28	30	31	32	41	45
34:	4	9	11	15	21	23	26	35	36	37	38	42
35:	4	7	13	17	19	22	27	34	36	37	39	43
36:	4	8	10	16	18	24	28	34	35	37	40	44
37:	4	6	12	14	20	25	29	34	35	36	41	45
38:	5	9	10	14	21	22	26	30	34	39	40	41
39:	5	6	13	17	18	23	27	31	35	38	40	41
40:	5	8	11	16	19	24	29	32	36	38	39	41
41:	5	7	12	15	20	25	28	33	37	38	39	40
42:	5	8	11	15	20	23	27	30	34	43	44	45
43:	5	7	12	16	19	22	26	31	35	42	44	45
44:	5	9	10	17	18	25	28	32	36	42	43	45
45:	5	6	13	14	21	24	29	33	37	42	43	44

THE TRIPLE STRUCTURE OF THE DESIGN IS
R0= 5040 R1= 8640 R2= 270 R3= 240
DESIGN IS (STRONGLY) SELF-DUAL
SMITH CLASS= 15

BLOCK DESIGN DUE TO R. A. KINGSLEY AND R. G. STANTON
WITH PARAMETERS V=56, K=11, LAMBDA= 2

1:	1	44	45	47	48	49	50	51	53	54	56
2:	2	20	34	35	36	41	42	43	45	46	48
3:	3	30	31	32	35	38	43	44	47	52	55
4:	4	29	32	33	37	38	39	40	46	50	56
5:	5	21	25	26	28	39	42	43	51	55	56
6:	6	21	23	24	27	38	40	41	45	54	55
7:	7	20	22	23	25	40	42	46	47	49	52
8:	8	19	25	27	28	33	34	35	50	52	54
9:	9	17	18	24	28	34	37	46	47	53	55
10:	10	19	23	24	26	32	36	37	48	51	52
11:	11	18	19	22	27	31	36	40	43	53	56
12:	12	18	21	22	26	30	35	37	41	49	50
13:	13	17	22	24	25	29	30	36	39	44	54
14:	14	18	20	23	28	29	31	33	41	44	51
15:	15	17	20	21	27	31	32	34	39	48	49
16:	16	17	19	20	26	30	33	38	42	45	53
17:	9	13	15	16	17	40	41	43	50	51	52
18:	9	11	12	14	18	38	39	42	48	52	54
19:	8	10	11	16	19	39	41	44	46	49	55
20:	7	14	15	16	20	35	36	37	54	55	56
21:	5	6	12	15	21	33	36	44	46	52	53
22:	7	11	12	13	22	32	33	34	45	51	55
23:	6	7	10	14	23	30	34	39	43	50	53
24:	6	9	10	13	24	31	33	35	42	49	56
25:	5	7	8	13	25	31	37	38	41	48	53
26:	5	10	12	16	26	29	31	34	40	47	54
27:	6	8	11	15	27	29	30	37	42	47	51
28:	5	8	9	14	28	30	32	36	40	45	49

29:	2	4	13	14	26	27	29	49	52	53	55
30:	3	12	13	16	23	27	28	30	46	48	56
31:	3	11	14	15	24	25	26	31	45	46	50
32:	3	4	10	15	22	28	32	41	42	53	54
33:	4	8	14	16	21	22	24	33	43	47	48
34:	2	8	9	15	22	23	26	34	38	44	56
35:	2	3	8	12	20	24	35	39	40	51	53
36:	2	10	11	13	20	21	28	36	38	47	50
37:	4	9	10	12	20	25	27	37	43	44	45
38:	3	4	6	16	18	25	34	36	38	49	51
39:	4	5	13	15	18	19	23	35	39	45	47
40:	4	6	7	11	17	26	28	35	40	44	48
41:	2	6	12	14	17	19	25	32	41	47	56
42:	2	5	7	16	18	24	27	32	42	44	50
43:	2	3	5	11	17	23	33	37	43	49	54
44:	1	3	13	14	19	21	34	37	40	42	44
45:	1	2	6	16	22	28	31	37	39	45	52
46:	2	4	7	9	19	21	30	31	46	51	54
47:	1	3	7	9	26	27	33	36	39	41	47
48:	1	2	10	15	18	25	30	33	40	48	55
49:	1	7	12	15	19	24	28	29	38	43	49
50:	1	4	8	12	17	23	31	36	42	50	55
51:	1	5	10	14	17	22	27	35	38	46	51
52:	3	7	8	10	17	18	21	29	45	52	56
53:	1	9	11	16	21	23	25	29	32	35	53
54:	1	6	8	13	18	20	26	32	43	46	54
55:	3	5	6	9	19	20	22	29	48	50	55
56:	1	4	5	11	20	24	30	34	41	52	56

THE TRIPLE STRUCTURE OF THE DESIGN IS
R0=13480 R1= 9240 R2= 0
DESIGN IS (STRONGLY) SELF-DUAL
SMITH CLASS= 20

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