ESSAYS ON FIXED-INCOME SECURITIES

BY

Ilona Shiller

A Thesis

Submitted to the Faculty of Graduate Studies
In Partial Fulfillment of the Requirements for the Degree of

DOCTOR OF PHILOSOPHY

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0-494-08800-1

Your file Votre référence
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Our file Notre référence
ISBN:

THE UNIVERSITY OF MANITOBA

FACULTY OF GRADUATE STUDIES

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A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University of Manitoba in partial fulfillment of the requirements of the degree

of

DOCTOR OF PHILOSOPHY

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Abstract

Governments' budget surpluses in the 1990's in western countries led to lower financing needs of these governments. The reduced issuance of government fixed-income securities has led to an increased interest in alternative investments such as corporate bonds. Corporate bondholders face risks such as default, callability, and liquidity risks. This thesis studies the significance of accounting for these risks in the pricing and hedging aspects of fixed-income securities. In addition, this thesis develops a theoretically consistent pricing model for Treasury Inflation Protected Securities, and examines their effective elasticity.

In the first essay, we apply risk-neutral valuation to price inflation-protected bonds, paying close attention to an option embedded in these bonds that guarantees that bondholders are not affected by deflation. We also consider the elasticity of inflation-protected bonds with respect to the real rate and the nominal rate, and compare it to Macaulay duration. A numerical simulation demonstrates that ignoring the embedded option is costly. The validity of our model is also supported by an empirical test that confirms its properties with respect to TIPS yield spreads.

We derive a model for the valuation of government bonds subject to liquidation risk in the second essay. We show that the value of a government bond depends on the term structures of the conditional probability of liquidation facing the portfolio manager, the bid-ask spread, and the riskless rate. We derive an expression for the elasticity of a government bond adjusted for the risk of liquidation with respect to shifts in the riskless term structure of interest rates.

Recent structural models for valuing callable corporate bonds show that both callability and default options have important implications for the interest-rate sensitivity of yield spreads and the bond duration. Special attention is given to the interaction between these two risks in the third essay. We test the main implications of these models. We also examine the impact of both risks and their interaction on the effective duration of corporate bonds. We find evidence supporting the main predictions of the theory.

Overall, this thesis establishes the importance of taking into account the considered risk factors in pricing and managing fixed-income portfolios.

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Acknowledgements

I would like to acknowledge the help of many people during my time as a graduate student. First, I would like to thank my supervisor Dr. Gady Jacoby for guidance and direction. Although strict, he was also patient, and his advice was invaluable. Thank you a lot for introducing me to the world of research and guiding my first steps.

Thanks are also in order to Dr. Wayne Simpson for giving me many helpful comments and suggestions and giving me a lot of technical advice over the passage of my time in this program. He was always willing to help me with all of my data problems and statistical questions. His advice was relevant and timely; and it helped me improve my papers again, and again.

I am also grateful to Dr. Charles Mossman who has spent a lot of time reading my thesis, correcting my mistakes, and raising many interesting questions. I have benefited from discussions with him over the course of my study.

I am indebted to Dr. Jean Helwege for taking time to read my thesis, and for giving me many relevant comments that helped me improve my papers. Her comments will undoubtedly help me improve the quality of my papers in the future as well. I deeply appreciate the time taken to clarify some of the ambiguous statements in the thesis and to personally attend my defence.

I offer my deepest gratitude to my family members for believing in me. Your continuous support means a lot.

Lastly, I would like to thank the financial supporters of my doctoral studies. Namely, the Asper School of Business, Dr. Gady Jacoby through his research funds, SIGPAC, the Graduate Student Association, and the Faculty of Graduate Studies. The financial assistance provided made it possible for me to carry through the program and attend conferences as well as doctoral consortiums.

Ilona Shiller

CHAPTER 1

Introduction

In recent years, the market for fixed-income instruments has experienced substantial growth. This swell in market value of debt instruments motivated a fresh upsurge in fixed-income research.[1] Government budget surpluses in the 1990's in western countries led to lower financing needs of these governments. The reduced issuance of government fixed-income securities has led to an increased interest in alternative investments such as corporate bonds. Corporate bondholders face risks such as default, callability, and liquidity risks. This thesis studies the significance of accounting for these risks in the pricing and hedging aspects of fixed-income securities.

On the other hand, the higher demand for corporate bonds, which serve as substitute for government bonds, also poses a challenge to bond portfolio managers, pension funds, and insurance companies in the way they measure interest-rate risk. These managers need to estimate the sensitivity of the bond price to a change in interest rates. This sensitivity can be quantified using a measure called duration, which shows the magnitude of the percentage change in the market value of the bond following a given change in interest rate.

[1] The relative importance of fixed-income markets has also increased in recent years. For example, Mishkin and Eakins (2005) report that, as of the end of 2004, the market value of debt in the U.S. amounted to $35.1 trillion, whereas the market value of equities amounted only to $15.9 trillion.

The issue of duration has been occupying the minds of academic researchers for years. The concept of duration allows practitioners to immunize bond investments from the risk of adverse changes in interest rates, and to speculate on the direction of movements in interest rates with better knowledge of the level of risk.

Macaulay (1938) introduces the concept of duration as the weighted average maturity of a bond that takes into account the amount and timing of coupon payments. Assuming a flat term structure of interest rates, Hicks (1939) shows that duration is the bond-price elasticity with respect to shifts in the term structure. Similar concepts, though for different applications, are developed by Samuelson (1945) and Redington (1952). Samuelson (1945) analyzes the sensitivity of a financial institution's net worth to changes in interest rates. He concludes that when the duration of liabilities is greater than the duration of assets, positive interest-rate shifts are desirable. Redington (1952) shows that, at the point where durations of assets and liabilities are equalized, the impact of shifts in interest rates is neutralized. Redington is the first to introduce the concept of immunization.

Fisher and Weil (1971) build on previous work, but relax the assumption of a flat term structure while maintaining that interest rate changes are the same for different maturities. They show that fixed-income portfolios can be immunized against interest rate risk by keeping the duration of these portfolios equal to the investment planning horizon. Following Fisher and Weil's work the concept of duration became popular among portfolio managers as an immunization tool.

Since then, a vast body of work was written on duration. A number of studies show that Macaulay duration works well for liquid government bonds (see, for example, Bierwag, Kaufman, Schweitzer, and Toevs, 1981; Bierwag, Kaufman, and Toevs, 1982b; and Bierwag and Roberts, 1990). However, duration may be inappropriate for corporate bonds or other bonds influenced by risks that may cause their effective duration to deviate from Macaulay duration.

Macauly duration is exposed to risks such as inflation (see Roll, 2004), default (see Sarkar and Hong, 2004; Jacoby and Roberts, 2003; Acharya and Carpenter, 2002; Fooladi, Roberts, and Skinner, 1997; Babbel, Merrill, and Panning, 1997; Chance, 1990; and Bierwag and Kaufman, 1988), illiquidity (see Longstaff, Mithal, and Neis, 2004; Houweling, Mentink, and Vorst, 2003; Duffie and Ziegler, 2003; and Sarig and Warga, 1989), callability (see Sarkar and Hong, 2004; Jacoby and Roberts, 2003; Acharya and Carpenter, 2002; Kihn, 1994; Brooks and Attinger, 1992; and Dunetz and Mahoney, 1988), and other risks.[2]

Due to these frictions, the yield of such bonds exceeds that of a benchmark fixed-income security. This yield differential, or spread, is a sum of yield premiums required due to such risks. For example, besides interest-rate risk, corporate bondholders face the risk of default, callability and illiquidity, and these risks frequently interrelate with each other. Thus, one has to adjust duration for these risks in order to achieve an effective immunization strategy.

All of the above listed sources of risk may potentially alter the amount of cash flows received by the bondholder. Also, all of the above risks, with the exception of inflation risk, may alter the timing of cash flows. Therefore, these risks may have a significant impact on duration.

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[2] For example, foreign-exchange risk (see, for example, Chow, Lee, and Solt, 1997; and Dym, 1991, 1992). Other risks may also include those originating from other bond features such as purchase and sinking funds, adjustability, convertability, extendibility, and exchangeability provisions.

This thesis contributes three studies to the extant fixed-income literature, addressing the impact of some of these risks on duration. Following the trend of shrinking government debt in the 1990's, there has been increased interest in finding alternatives for these instruments. Such alternatives include corporate bonds, foreign bonds, and other government bonds, like inflation-protected securities. The studies in this thesis deal with inflation, illiquidity, default, and callability risks. Chapter 2 and 3 are concerned with pricing, interest-rate elasticity, and implications for management of government bonds. Chapter 4 examines the interest-rate elasticity of the call spread and that of the default spread, while accounting for the interaction between both spreads for corporate bonds.

Chapter 2 studies aspects related to the dynamic pricing and hedging of inflation-protected government bonds. In this essay, risk-neutral valuation is applied to price inflation-protected bonds, paying close attention to an option embedded in these bonds that guaranties that bondholders are not affected by deflation. We also derive the elasticity and the yield spread formulas for these bonds. In the same essay, we examine the determinants of the dynamic pricing of inflation-adjusted bonds, with a focus on the implications of our theoretical model. A simulation and empirical test confirm the theoretical implications of our model.

In Chapter 3 we value a government bond under the risk of liquidation facing a representative bondholder. The lower marketability in the market for fixed-income securities, the higher bid-ask spreads on securities being liquidated. Duffie and Ziegler (2003) show that when positions must be liquidated, bid-ask spreads widen. They suggest that a potential strategy would be to leave enough cash as contingency funds so that liquidation of illiquid positions can be timed. Assuming that no contingency funds are left aside to liquidate illiquid positions, we examine the effect on the price elasticity of government bonds with respect to interest rates. We derive the liquidation risk-adjusted elasticity for government bonds and show that the value of the bond depends on the term structure of interest rates, liquidation probabilities, and the magnitude of bid-ask spreads.

Chapter 4 empirically studies corporate default and call spread changes. Previous empirical research in this area studied the two sources of risk independently. We analyze not only the impact of both default and call risks on duration of callable corporate bonds, but also take into account the impact of the interaction between them on duration. We test the implications of the model using estimated zero-coupon spot rates. The dataset used in this study includes both fixed-price and make-whole callable defaultable bonds. This allows us to highlight the similarities and differences in the risk-adjustment required for estimating duration of these two types of callable bonds.

We find that the duration of callable corporate bonds is significantly affected by cross terms, arising due to the interaction between default and call risks. The findings suggest that in order to have an effective risk-adjustment for duration, one must account for these effects when estimating the yield spread elasticity under default and call risks. Chapter 5 concludes the thesis.

CHAPTER 2

Understanding Inflation-Protected Bonds

2.1 Introduction

Treasury inflation-protected securities were first issued in the U.S. fixed-income market in January 1997. These U.S. bonds are commonly called Treasury Inflation Protected Securities (TIPS). Their introduction directed academic and practitioner interest to these instruments. Sacks and Elsasser (2002) report that the U.S. Treasury had issued close to $130 billion in TIPS, with maturity dates ranging from 2002 to 2032.[3]

A TIPS bond pays a constant coupon rate that applies to principal that is fully adjusted to inflation based on a Consumer Price Index (CPI).[4] Thus, coupons will be adjusted upwards in case of inflation, and downwards following deflation. The treatment of the principal payment is somewhat different. The inflation-protection scheme guaranties that the value of the inflation-adjusted principal is never below its original value. This means that, at redemption, bondholders receive a principal

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[3] The U.S. Treasury issued the first inflation-protected bond in January 1997. The U.S. Treasury sold approximately $7 billion 10-year notes at the first auction and joined a number of other countries that already used inflation-indexed debt. These countries include the U.K., Israel, Sweden, Canada, Australia, and New Zealand. Wilcox (1998) reports that, as of mid 1997, the U.K. issued the highest aggregate volume of inflation-indexed debt (71.1 billion U.S. dollars). Israel ranked second (27.9 billion U.S. dollars), then the U.S. (15.0 billion U.S. dollars), Sweden (5.7 billion U.S. dollars), Canada (4.3 billion U.S. dollars), Australia (2.7 billion U.S. dollars), and New Zealand (0.1 billion U.S. dollars).

[4] Note that TIPS are adjusted for the CPI with a two to three month lag. Wilcox (1998) argues that TIPS bondholders still face a small but certain amount of inflation risk due to this lag. Since this exposure is trivial, in the current paper we assume that the inflation adjustment is contemporaneous.

payment higher than the original principal if, in total, the inflation rate is positive over the life of the bond. On the other hand, if in total the CPI decreases over the life of the bond (deflation), then bondholders receive the original (fixed) principal amount.

In terms of pricing, the inflation-adjustment scheme implies that, in nominal terms, a long position in a pure-discount inflation-protected bond is equivalent to an unadjusted (nominal) Treasury bond and a long position in a European call option written on a fully adjusted (real) pure-discount riskless bond. This implies that, if in total the CPI increases over the life of the bond, at maturity the bondholder will exercise the call option and swap the nominal bond for the upward-adjusted principal of the real bond. On the other hand, in case of deflation over the life of the bond, the call will expire worthless leaving the bondholder with the unadjusted principal payment of the nominal bond.

Alternatively, in real terms a long position in a pure-discount inflation-protected bond is equivalent to a fully adjusted (real) riskless bond and a long position in a European put option written on a real pure-discount riskless bond. This implies that, in case of inflation over the life of the bond, at maturity the put option will expire worthless, leaving the bondholder with the fully adjusted principal of the real bond. On the other hand, in case of deflation over the life of the bond, the bondholder will exercise the put option and swap the real bond for the unadjusted principal payment of the nominal bond.

Most research in this area assumes, explicitly or implicitly, that the value of the put option that offers protection against deflation is trivial. Given the recent inflation history in most major economies, experiencing deflation in total over the life of a bond appears to be an unlikely event. However, the more recent experience of Japan puts the validity of this assumption in question. Even for the U.S. market, Richard Roll claims that deflation "... no longer seems such an unlikely event" (Roll (2004, p. 31)). For example, in Israel, which ranked second in terms of aggregate volume of inflation-indexed debt (27.9 billion U.S. dollars), the prolonged recession in the beginning of the new millennium implies that one cannot ignore the value of the embedded option.

In this chapter we use Option Pricing Theory (following Black and Scholes, 1973) to model the value of an inflation-adjusted TIPS, accounting for the value of the embedded option. We analyze the determinants of the price of TIPS and provide an appropriate bond elasticity measure. In light of the greater potential for deflation, this chapter provides an improved pricing model for inflation-protected bonds. This enhances our understanding of inflation-protected bonds and provides bond portfolio managers with adequate tools for dealing with TIPS. To the best of our knowledge, this is the only attempt to price TIPS while considering the embedded option. A numerical simulation, based on a sample of Treasury bonds, demonstrates that the embedded option is nontrivial.

Two papers in the existing literature look at the issue of pricing inflation-protected bonds. Jarrow and Yildirim (2003) use the foreign currency analogy of Jarrow and Turnbull (1998) and Heath-Jarrow-Morton (1992) model to price the evolution of real and nominal zero-coupon bonds. They confirm the validity of their model by testing its hedging performance. Brown and Schafer (1994) fit the Cox-Ingersol-Ross (CIR) (1985) model to the real term structure obtained from British government index-linked bond prices. They find that the CIR model approximates the shape of the estimated real term structure closely. Both of these papers do not address the embedded option.

The remainder of this chapter is organized as follows. In Section 2.2 we derive the value of an inflation-protected bond under deterministic and stochastic interest rate environments. This section also examines the comparative statics of the bond-pricing equation with respect to variables of interest and demonstrates that our model is well behaved. In Section 2.3 we derive and compare "nominal" and "real" duration measures for inflation-protected bonds. Section 2.4 derives an equation for the yield spread between yields on nominal Treasury securities and yields on inflation-protected securities. It also examines the comparative statics for the yield spread. In Section 2.5 we calibrate our model based on the parameters estimated from the universe of TIPS, and test its empirical implications. Section 2.6 validates the model with an empirical test and the last section concludes this chapter.

2.2 The Value of an Inflation-Protected Bond

Before we proceed to price an inflation-protected bond, we highlight a few observations. First, the price of the TIPS calculated with respect to nominal cash flows and discounted with the nominal interest rate equals the price of the TIPS calculated with respect to real cash flows and discounted with the real interest rate. Being protected against deflation, when we use real rates, a pure-discount inflation-protected bond is equivalent to a portfolio of a fully adjusted (real) pure-discount bond and a European put option written on the value of a fully adjusted (real) bond with an expiration equal to the bond maturity and an exercise price equal to the face value of the bond. Alternatively, when we use nominal rates, a pure-discount inflation-protected bond is equivalent to a portfolio of an unprotected (nominal) pure-discount bond and a European call option written on the value of a fully adjusted (real) bond with exercise price equal to the face value of the bond. Below we adopt the latter interpretation in order to price a pure-discount inflation-protected bond. Note that, using put-call parity, one can easily show that both approaches must produce the same price.[5]

The value of a zero-coupon inflation-protected bond with maturity at time T and an initial face value of F dollars at time t ($t<T$) is given by:

$$B_{TIPS,t} = \max[B_t^*, B_t],$$

Where $B_t^* = Fe^{-r(T-t)} = FP(r,t,T)$, and $B_t = Fe^{-i(T-t)} = FP(i,t,T)$. Also, r is the continuously compounded real riskless rate and i is the continuously compounded nominal riskless rate. Rearranging the above equation we get:

$$B_{TIPS,t} = \max[B_t^* - B_t, 0] + B_t.$$

[5] Note that the protection against deflation provided by coupon-bearing TIPS applies only to the face value paid at maturity, not to the coupon payments. Thus, one cannot price coupon-bearing TIPS by individually pricing strips and aggregating these prices to obtain the value of the inflation-protected coupon bond. In the analysis below we price pure-discount inflation-protected bonds. To price the coupons, one can strip the bond and treat each coupon as pure-discount real bond.

Thus, the payoff of a zero-coupon inflation-protected bond at its maturity (T) is given by:

$$B_{TIPS,T} = \max[Fe^{\pi(T-t)} - F, 0] + F.$$

Where π is the continuously compounded stochastic inflation. Assuming a one-dollar face value for this bond (F=1), we write:

$$B_{TIPS,T} = \max[e^{\pi(T-t)} - 1, 0] + 1.$$

Thus, an inflation-protected bond is equivalent to a portfolio of an unprotected (nominal) bond (paying \$1 at maturity) and a call option written on the value of a fully adjusted (real) bond, denoted by B^*_t (with a stochastic value of $e^{\pi(T-t)}$ at time T) with an exercise price of \$1.

We solve for the current value of the inflation-protected bond under two separate sets of assumptions. We first solve under the assumption that the nominal and real interest rates are deterministic. Note that this does not preclude the inflation rate from being stochastic. Recall that under the continuous-time version of the Fisher relation, we have: $i = r + \mathrm{E}[\pi]$. Therefore, even if the nominal and real rates are deterministic, the difference between them is the expected value of the *future* inflation rate (which is still stochastic). Alternatively, we solve for the value of the inflation-protected bond allowing for stochastic interest rates.

2.2.a The Value of an Inflation-Protected Bond under Deterministic Interest Rates

Recall that the value of an inflation-protected bond is equivalent to the value of a portfolio consisting of a corresponding unadjusted riskless bond and a call option on the value of the real bond with an exercise price of \$1. The payoff of a zero-coupon inflation-protected bond at its maturity (T) is given by:

$$B_{TIPS,T} = \max[e^{\pi(T-t)}-1, 0] + 1.$$

Using risk-neutral valuation, we show that the time-t value of the inflation-protected bond carrying an unadjusted face value of F is given by:[6]

$$B_{TIPS,t} = Fe^{-i(T-t)}N(-d_2) + Fe^{-r(T-t)}N(d_1), \qquad (2.1)$$

where $d_1 = \dfrac{(i - r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = \dfrac{(i - r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$.

2.2.b The Value of an Inflation-Protected Bond under Stochastic Interest Rates

In this section we solve for the value of the inflation-protected bond allowing for stochastic interest rates. Specifically, we assume that $P(r,t,T)$ and $P(i,t,T)$ follow geometric Brownian motion processes. Following Merton (1973), and analogous to Chance's (1990) model for the valuation of defaultable bonds, we can express the

[6] See derivation in the Appendix.

value of the zero-coupon inflation-protected bond in a stochastic interest rate environment as:

$$B_{TIPS,t} = FP(i,T)N(-d_2) + FP(r,T)N(d_1), \tag{2.2}$$

where: $d_1 = \dfrac{\ln P(r,T) - \ln P(i,T) + (\sigma^2/2)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$, and:

$$\sigma^2 T = \int_0^T \{\sigma_{p(r)}(t) + \sigma_{p(i)}(t) - 2\rho\sigma_{p(r)}\sigma_{p(i)}(t)\}dt, \ \rho = corr(i,r).$$

Although equation (2.2) is derived under a more realistic case, where interest rates are stochastic, in the remainder of this chapter we will focus on the model in equation (2.1). This is mainly because it leads to more tractable results when we examine the comparative statics of the model. Note that, unlike model (2.1), model (2.2) has the correlation between the nominal rate and the real rate as one of the price determinants. Below we show that the relationship between the two rates is an important factor also under model (2.1).

2.2.c Comparative Statics

In our analysis below, we show that many of the results depend on the sign of partial derivatives based on the Fisher relation. Signing these derivatives is an empirical, rather than a theoretical, question. Recall that the continuous-time version of the Fisher effect is given by: $i = r + \pi^*$, where π^* is the expected continuously compounded inflation rate. Theoretically, the Fisher effect implies that the relationship between nominal interest rates and expected inflation rates is one-to-one, while the movements in real interest rates and expected inflation rates are largely independent. This theoretical result has been criticized (see, for example, Van Horne, 2001). In practice, the assumed independence between the real interest rate and the inflation rate does not hold. The relationship between the two will vary across time and across countries, depending on the implemented monetary policy.

Voluminous empirical research exists in the extant literature testing this relationship in different countries and different time periods (for a good discussion see Van Horne, 2000). Recent U.S. studies by Roll (2004), Evans (1998), Crowder and Hoffman (1996), and Crowder (2003) generally agree that: (i) the relationship between changes in expected inflation rates and changes in nominal rates is positive (see, for example, Jarrow and Yildirim, 2003; Crowder, 2003; McCulloch and Kochin, 2000; and Crowder and Hoffman, 1996); (ii) the relationship between changes in expected inflation rates and changes in real rates is positive (see Roll, 2004; Evans, 1998); and (iii) the relationship between changes in nominal and real rates is positive (see Jarrow and Yildirim, 2003; McCulloch and Kochin, 2000).

These recent empirical results are in agreement with the monetary policy of the U.S. Federal Reserve Bank since the early 1990s. The main policy objective of the Federal Reserve is to achieve price stability to attain maximum sustainable economic growth. To achieve this goal, when inflation expectations decline and the expected growth in the economy decelerates, the Federal Reserve will try to ease the monetary policy and stimulate the economy by lowering real interest rates. On the other hand, when inflation expectations rise and the expected growth in the economy accelerates, the Federal Reserve will try to tighten the monetary policy by raising real rates. With this in mind, we now proceed with the comparative statics for the bond pricing equation (2.1) for inflation-protected securities.

Proposition 1 *As the face value of the inflation-protected bond increases, the value of this bond rises.*

Proof: All proofs of propositions are in Appendix B.

Proposition 2 *As the volatility of the return on the fully adjusted (real) bond increases, the value of the inflation-protected bond increases.*

Since the fully adjusted (real) bond is the underlying asset for the call option, a higher volatility of its return implies a higher probability that the call will be in-the-money. In other words, a higher volatility implies that there is a higher chance of deflation and, therefore, the protection against deflation (provided by the call option) becomes more valuable.

Proposition 3 *As the nominal interest rate increases, the value of the inflation-protected bond can increase or decrease, depending on the monetary policy practiced.*

To see the intuition behind this result, consider the derivative of the value of the inflation-protected security with respect to the nominal rate:

$$\frac{dB_{TIPS,t}}{di} = -\tau\left[\frac{dr}{di}Fe^{-r\tau}N(d_1) + Fe^{-i\tau}N(-d_2)\right] \lesseqgtr 0\,.$$

To sign this derivative, one has to take a closer look at the term: $\frac{dr}{di}$. Rearranging the Fisher relation, we get: $r = i - \pi^*$. Thus, we have: $\frac{dr}{di} = \left(1 - \frac{d\pi^*}{di}\right)$. Recall that recent empirical evidence suggests that the relationship between changes in expected inflation rates and changes in nominal rates, $\frac{d\pi^*}{di}$, is positive but lower than one (see Jarrow and Yildirim, 2003; Crowder, 2003; McCulloch and Kochin, 2000; and Crowder and Hoffman, 1996). Also recall that the empirical relation between changes in nominal and real rates is positive (see Jarrow and Yildirim, 2003; McCulloch and Kochin, 2000). This implies that empirically $\frac{dr}{di}$ is positive. This is consistent with the current monetary policy practiced by the U.S. Federal Reserve.

Thus, under the current monetary policy, the value of the inflation-protected bond decreases as the nominal interest rate increases. This is due to: (i) the direct negative impact of the nominal interest rate shift on the value of the bond (this is captured by the second term in the square brackets in the above derivative); and (ii) the indirect negative impact of the nominal interest rate increase on the value of the bond through its impact on the real interest rate (this is captured by the first term in the square brackets in the above derivative).

Proposition 4 *As the real interest rate increases, the value of the inflation-protected bond can increase or decrease, depending on the monetary policy practiced.*

To see the intuition behind this result, consider the derivative of the value of the inflation-protected security with respect to the real rate:

$$\frac{dB_{TIPS,t}}{dr} = -\tau\left[Fe^{-r\tau}N(d_1) + \frac{di}{dr}Fe^{-i\tau}N(-d_2)\right] \lesseqgtr 0.$$

To sign this derivative, one has to examine: $\frac{di}{dr}$. Based on the Fisher relation, we get: $\frac{di}{dr} = \left(1 + \frac{d\pi^*}{dr}\right)$. Recent empirical evidence suggests that the relation between changes in expected inflation rates and changes in real rates is positive (see Roll, 2004; and Evans, 1998). Recall that the U.S. Federal Reserve uses the real rate to control expectations of future inflation. This means that empirically we have: $\frac{di}{dr} > 0$, and the value of the inflation-protected bond decreases in the real rate.

Under the current monetary policy in the U.S., the value of the inflation-protected bond decreases as the real interest rate increases. This is due to: (i) the direct negative impact of the real interest rate increase on the value of the bond (this is captured by the second term in the square brackets in the above derivative); and (ii) the indirect negative impact of the real interest rate increase on the value of the bond through its impact on the nominal interest rate (this is captured by the first term in the square brackets in the above derivative).

Proposition 5 *As the expected inflation rate increases, the value of the inflation-protected bond can increase or decrease, depending on the monetary policy practiced.*

Once again, to see the intuition behind this proposition, consider the derivative of the value of the inflation-protected security with respect to the expected inflation rate:

$$\frac{dB_{TIPS,t}}{d\pi *} = -\tau\left[\frac{dr}{d\pi *}Fe^{-r\tau}N(d_1) + \frac{di}{d\pi *}Fe^{-i\tau}N(-d_2)\right] \lessgtr 0\,. \tag{2.3}$$

To sign this derivative, we have to examine: $\frac{di}{d\pi *}$ and $\frac{dr}{d\pi *}$. Based on the Fisher relation, we get: $\frac{di}{d\pi *} = \left(\frac{dr}{d\pi *} + 1\right)$. Given this result, we rewrite the above derivative as:

$$\frac{dB_{TIPS,t}}{d\pi *} = -\tau\left[\frac{dr}{d\pi *}B_{TIPS,t} + Fe^{-i\tau}N(-d_2)\right] \lessgtr 0\,.$$

This result shows that the relationship between the value of the inflation-protected bond and expected inflation depends on the sign of $\frac{dr}{d\pi^*}$. As previously noted, when expectations for inflation decline (increase) and the growth in the economy decelerates (accelerates), the Federal Reserve stimulates (tightens) the economy by lowering (increasing) real interest rates. Given recent empirical evidence and the monetary policy applied by the Federal Reserve, this relationship is currently positive in the U.S.

This means that under the current monetary policy in the U.S., the value of the inflation-protected bond decreases as the expected inflation rate increases. This is due to: (i) the direct positive impact an increase in the expected inflation rate has on the nominal rate, that in turn has a negative impact on the value of the bond (this is captured by the first term in the square brackets in equation (2.3)); and (ii) the positive impact an increase in the expected inflation rate has on the real rate, that in turn has a negative impact on the value of the bond (this is captured by the second term in the square brackets in equation (2.3)).

Next, we consider the impact of a potential change in the Federal Reserve's policy on our results. If the Federal Reserve decides to pursue a neutral policy with respect to changes in expected inflation, the value of $\frac{dr}{d\pi^*}$ will be zero and the value of the inflation-protected bond will still decline with higher expected inflation rate. Note that it does not make sense for the Federal Reserve to apply a policy that stimulates (slows down) the economy when there are expectations for inflation (deflation). Therefore, $\frac{dr}{d\pi^*}$ will not take a negative sign.

Proposition 6 *As the maturity of the inflation-protected bond increases, its value can increase or decrease.*

To explain the intuition behind this proposition, consider the derivative of the value of the inflation-protected security with respect to the bond's time to maturity:

$$\frac{dB_{TIPS,t}}{d\tau} = Fe^{-r\tau}N'(d_1)\frac{\sigma}{2\sqrt{\tau}} - rFe^{-r\tau}N(d_1) - iFe^{-i\tau}N(-d_2) \leq\geq 0 .$$

There are two opposing effects of the time to maturity on the value of the bond. On one hand, as τ increases the value of the inflation-protected bond drops due to the time value of money impact. On the other hand, as τ increases the value of the call option embedded in the inflation-protected bond increases.

2.3 Duration for Inflation-Protected Bonds

The practice of bond portfolio managers, using duration as a measure of risk or as an immunization tool, is to calculate the elasticity of the inflation-protected bond with respect to its own (real) yield to maturity. By doing so, they implicitly assume that the value of the embedded put option is trivial, and therefore the value of the inflation-protected bond is equal to the value of the fully adjusted (real) bond: $B_t^* = Fe^{-r\tau}$. This duration measure is the bond's standard Macaulay duration, which is equal to the time to maturity of the bond for a pure-discount inflation-protected bond (τ). When deflation is a feasible state, Macaulay duration may be a biased estimator for the elasticity of the inflation-protected bond. If a portfolio manager uses nominal benchmarks, such as a Lehman Bond Index, to evaluate the performance of an inflation-protected bond, then s/he needs to use a duration measure calculated with respect to the nominal interest rate. Below we show that when the embedded call option is taken into consideration, the effective duration of an inflation-protected bond is different than its Macaulay duration. We consider the elasticity with respect to both the nominal rate (i) and the real rate (r).

2.3.a "Nominal" Duration

We apply the standard price-elasticity definition of duration on bond-pricing equation (2.1), when the nominal interest rate, i, is continuously compounded: $D_i = -\frac{1}{B_{TIPS,t}} \frac{\partial B_{TIPS,t}}{\partial i}$. This yields the following "nominal" duration:[7]

$$D_i = \tau\left(1 - \frac{\partial \pi^*}{\partial i} \frac{Fe^{-r\tau} N(d_1)}{B_{TIPS,t}}\right). \tag{2.4}$$

Recall that a pure-discount inflation-protected bond is equivalent to a portfolio of an unprotected (nominal) pure-discount bond and the embedded call option. Given this equivalence, one can show that the nominal duration of the inflation-protected

[7] See Appendix C for the derivations of both duration measures.

bond in equation (2.4) is equal to the weighted average of the durations of the unprotected bond and that of the embedded call option on the real bond.[8]

As noted previously, recent empirical evidence for the U.S. suggests that the relationship between changes in the expected inflation rate and changes in the nominal rate is positive. This means that empirically $\frac{\partial \pi^*}{\partial i}$ is positive. Given this evidence and the duration in equation (2.4), one may expect the nominal duration of a U.S. TIPS (D_i) to be lower than that of an unprotected nominal Treasury bond (τ). This is due to the protection provided to the TIPS holder against loss resulting from inflation risk that unprotected bondholders face. Roll (2004) confirms this result empirically. He shows that the effective nominal duration is lower for inflation-protected bonds than that for comparable Treasury bonds.

Equation (2.4) also shows that the nominal duration of the inflation-protected security converges to the duration of the nominal Treasury security when the embedded call option is deep out-of-the-money, then the option's delta, $N(d_1)$, approaches zero. In this case a shift in the nominal rate will have a trivial effect on the value of the embedded call.

[8] Garman (1985) shows that an option's elasticity is a measure of its interest-rate sensitivity. Similar to our paper, Chance (1990) uses this approach to model the duration of the limited-liability option imbedded in a corporate bond.

2.3.b "Real" Duration

We apply the standard price-elasticity definition of duration on bond-pricing equation (2.1), when the real interest rate, r, is continuously compounded: $D_r = -\frac{1}{B_{TIPS,t}} \frac{\partial B_{TIPS,t}}{\partial r}$. This yields the following "real" duration:

$$D_r = \tau \left(1 + \frac{\partial \pi^*}{\partial r} \frac{Fe^{-r\tau} N(-d_2)}{B_{TIPS,t}} \right). \qquad (2.5)$$

When one uses real rates, a pure-discount inflation-protected bond is equivalent to a portfolio of a fully adjusted (real) pure-discount bond and an embedded European put option written on the value of a fully adjusted (real) bond. Given this equivalence, one can show that the real duration of the inflation-protected bond in equation (2.5) is equal to the weighted average of the durations of the fully adjusted (real) bond and that of the embedded put option on the real bond.

As noted previously, empirical evidence for the U.S. suggests that the relationship between changes in expected inflation rates and changes in real rates is positive. In other words, empirically $\frac{\partial \pi^*}{\partial r}$ is positive. In the context of equation (2.5), this evidence means that the real duration of a U.S. TIPS (D_r) is higher than that of a fully adjusted (real) Treasury bond (τ). This is due to the added sensitivity of the embedded put option.

Equation (2.5) also shows that the real duration of the inflation-protected security approaches the duration of the real Treasury security when the embedded put option is deep out-of-the-money and that the probability of exercising the option, $N(-d_2)$, is close to zero. In this case, a shift in the real rate will have a marginal effect on the value of the embedded put.

2.3.c Comparing the Two Duration Measures

The relative size of the duration measures derived above is in agreement with Wilcox (1998) who argues that nominal interest rate shocks will have a small effect on the prices of inflation-protected bonds, but a significant effect on the prices of nominal bonds. At the same time, real interest rate shocks will have a considerable effect on the prices of inflation-protected bonds, but a lower effect on the prices of nominal bonds. This is consistent with our nominal duration being lower than Macaulay duration and our real duration being greater than Macaulay duration.

2.4 Yield Spreads for Inflation-Protected Bonds

Bond portfolio managers often use yield spreads to price inflation protected bonds. A yield spread for a protected bond is given by the difference between the (nominal) yield to maturity on an unprotected bond (i) and the (real) yield to maturity on the protected bond. In this section, we derive this yield spread based on our bond-pricing equation (2.1), and contrast it with its common interpretation as the anticipated inflation rate. We then proceed to study its determinants.

2.4.a Yield Spread vs. Anticipated Inflation

We first derive the yield to maturity on the inflation-protected bond, $y(\tau)$, which must satisfy:

$$B_{TIPS,t} = Fe^{-y(\tau)\tau}. \tag{2.6}$$

This yield is the real internal rate of return that the investor will realize by holding the inflation-protected security to maturity. Equation (2.6) together with bond-pricing equation (2.1) implies the following expression for the yield to maturity on the inflation-protected bond: $y(\tau) = -\frac{1}{\tau}\ln\left(e^{-i\tau}N(-d_2) + e^{-r\tau}N(d_1)\right)$. Since the yield obtained for holding a nominal Treasury bond is simply i, the yield spread between nominal bond's rate and the rate acquired by holding an inflation-protected bond is given by:

$$\begin{aligned} S(\tau) &= i - y(\tau) \\ &= \frac{1}{\tau}\tau\ln e^{i} + \frac{1}{\tau}\ln\left(Fe^{-i\tau}N(-d_2) + Fe^{-r\tau}N(d_1)\right) \\ &= \frac{1}{\tau}\ln\left(N(-d_2) + e^{(i-r)\tau}N(d_1)\right) \end{aligned} \tag{2.7}$$

This yield spread is the premium that bondholders demand for the expected losses on the unprotected nominal debt related to inflation risk. Note that when the embedded call is deep in-the-money ($N(d_2)=1 \Rightarrow N(-d_2)=0$), the option will be exercised with certainty ($N(d_1)=1$) and the bondholder will give up the nominal bond in return for the fully adjusted (real) bond. Under this special case we have: $y(\tau) = r$, the yield to maturity on the protected bond is exactly the real interest rate, and the yield spread would equal the expected inflation rate: $S(\tau) = \pi^*$. Numerous researchers in this area assume (explicitly or implicitly) that the call option is always in-the-money, and then proceed to use the above yield spread as an estimator for the anticipated inflation rate (see Roll, 2004; Sack, 2000; McCulloch and Kochin, 2000; and Woodward, 1990). As previously mentioned, equation (2.7) demonstrates that this is an unbiased estimator only under a special case where the put option is deep out-of-the-money (and the prospects of deflation are slim). When this is not the case, equation (2.7), together with observed market yield spreads, can be very useful for calculating implied expected inflation rates.

2.4.b Comparative Statics for the Yield Spread

Equation (2.7) implies that the yield spread between yields on nominal Treasury bonds and yields on inflation-protected securities is a function of: (i) the volatility of the return of the real bond (σ); (ii) the expected inflation rate (π^*), and (iii) the time to maturity (τ). Alternatively, we can replace (ii) with the nominal rate (i) and the real rate (r). Below, we examine the relationship between the yield spread and these parameters.

Proposition 7 *As the volatility of the return on the real bond increases, the yield spread widens.*

Proof: All proofs of propositions are in Appendix B.

Intuitively, higher real-rate volatility implies a higher value for the call option embedded in the inflation-protected security. This will make the nominal bond less desirable for potential bondholders, who will demand a higher premium.

Proposition 8 *As the nominal interest rate increases, the yield spread can widen or shrink.*

To see the intuition behind this result, consider the derivative of the yield spread with respect to the nominal rate:

$$\frac{dS(\tau)}{di} = \frac{d\pi^*}{di}\left(\frac{e^{(i-r)\tau}N(d_1)}{N(-d_2)+e^{(i-r)\tau}N(d_1)}\right) \lessgtr 0.$$

To sign this derivative, one has to take a closer look at the term: $\dfrac{d\pi^*}{di}$. Recall that recent empirical evidence suggests that the relationship between changes in expected inflation rates and changes in nominal rates, $\frac{d\pi^*}{di}$, is positive. Recall that this implies that $\frac{dr}{di}$ is empirically positive, and it is consistent with the current monetary policy practiced by the U.S. Federal Reserve. Thus, under the current monetary policy in the U.S., the yield spread increases as the nominal interest rate increases. This is due to the indirect positive impact of the nominal interest rate increase on the yield spread through its impact on the expected inflation rate.

Proposition 9 *As the real interest rate increases, the yield spread can widen or shrink.*

Consider the derivative of the yield spread with respect to the real rate:

$$\frac{dS(\tau)}{dr} = \frac{d\pi^*}{dr}\left(\frac{e^{(i-r)\tau}N(d_1)}{N(-d_2)+e^{(i-r)\tau}N(d_1)}\right) \lesseqgtr 0 .$$

To sign this derivative, we need to determine the sign of: $\frac{d\pi^*}{dr}$. Since recent empirical evidence suggests that $\frac{\partial \pi^*}{\partial r}$ is positive under the current monetary policy in the U.S., one should expect the yield spread to increase as the real interest rate increases. This is due to the indirect positive impact of the real interest rate increase on the yield spread through its impact on the expected inflation rate.

Proposition 10 *As the expected inflation rate increases, the yield spread widens.*

Intuitively, higher expected inflation implies a higher value for the inflation-protected security. This will make the nominal bond less desirable for potential bondholders, who will demand a higher premium.

Proposition 11 *As the maturity of the inflation-protected bond increases, the yield spread can widen or shrink.*

Consider the derivative of the yield spread with respect to the time to maturity:

$$\frac{dS(\tau)}{d\tau} = \left[-\frac{1}{\tau^2} \ln\left(N(-d_2) + e^{\pi^* \tau} N(d_1) \right) \right] + \left[\frac{1}{\tau} \left(\frac{\pi^* e^{\pi^* \tau} N(d_1) + e^{\pi^* \tau} N'(d_1) \dfrac{\sigma}{2\sqrt{\tau}}}{N(-d_2) + e^{\pi^* \tau} N(d_1)} \right) \right] \lesseqgtr 0 .$$

In general, the term structure of yield spreads for inflation-protected bonds can take any shape. This is due to two opposite effects of the time to maturity on the spread. First, a longer maturity implies a higher value of the call option embedded in the inflation-protected bond. This is the standard Black and Scholes (1973) result, that the probability of the option being exercised increases with its time to expiry (this is captured by the term in the second square brackets in the above derivative). Second, the time to maturity has a direct impact on the spread. A longer maturity implies a lower yield spread (this is captured by the term in the first square brackets in the above derivative).

2.5 Numerical Simulation

Given recent inflationary history in western countries, one may argue that the value of the embedded option, designed to protect against deflation, is trivial. In this section we run a simulation based on parameter values observed in the real world in order to estimate the magnitude of the error, resulting from ignoring the embedded option. We use a sample of daily returns on Treasury inflation-protected securities, obtained from the U.S. Department of Treasury. We then calculate the annualized historical volatility of TIPS returns with: $\sigma = \sqrt{252} \times \sqrt{\frac{1}{N-1} \times \sum_{t=1}^{N} (R_{ti} - \overline{R})^2}$, where R_t is the daily return on the inflation-protected bond, $\overline{R}$ is the mean of daily returns, and N is the number of observations in the sample.

Panel A of Table 2.1 reports that the annualized return volatility on inflation-protected bond return data ranges between 2.55% to 11.12%.[9] Thus, in the ensuing numerical simulation, we allow the volatility to range between 0% and 15%. For the continuously compounded nominal and real interest rates, we use the averages of the 5-year daily Treasury nominal and real spot rates for August 2004, taken from the

[9] Our volatility estimates of daily returns on existing inflation-protected bonds are consistent with the estimates obtained by Roll (2004). He uses slightly different samples of daily inflation-protected bond return data, but obtains approximately equivalent annualized standard deviations in the range of 1.08% to 10.70%. Note that, optimally, one needs to use real bonds (unprotected for deflation) rather than TIPS bonds. Unfortunately, only TIPS are issued by the U.S. Treasury. Thus, we use the TIPS volatility to estimate the volatility on real bond returns.

United States Department of Treasury website (based on daily estimated yield curves using a cubic spline model). These rates are 3.47% and 1.12%, respectively.

Table 2.1: Inflation-Protected Bonds and their Characteristics

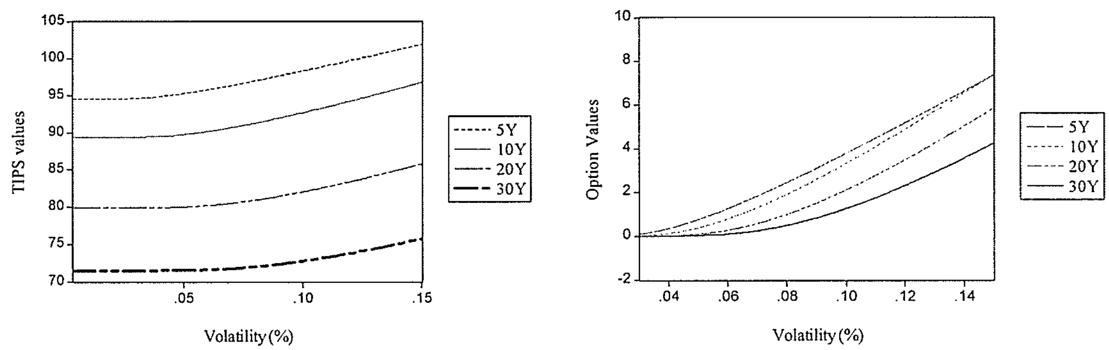
Panel A of this table reports the bond-specific characteristics of the Treasury inflation-protected securities in our sample. The annualized historical mean return and its associated annualized historical volatility of each inflation-protected bond over the specified sample period are also reported in Panel A. We calculate the annualized mean return with: $\mu = 252 \times \left(\frac{1}{N} \sum_{t=1}^{N} R_t \right)$. The annualized historical volatility of TIPS returns is calculated with:

$\sigma = \sqrt{252} \times \sqrt{\frac{1}{N-1} \times \sum_{t=1}^{N} (R_t - \overline{R})^2}$, where R_t is the daily return on the inflation-protected bond, $\overline{R}$ is the mean of daily returns, and N is the number of observations in the sample. Panel B of this table reports average daily nominal and real spot rates for maturities of 5, 7, 10, and 20 years, calculated for August 2004, taken from the United States Department of Treasury website (based on daily estimated yield curves).

Panel A: TIPS Data								
Coupon Rate	Issue Date	Maturity Date	Years to Maturity	Daily Data Begin	Daily Data End	Number of Observations	Mean Return	Standard Dev. of Returns
3.375	01/01/1997	15/01/2007	10	30/01/1997	24/02/2004	1836	1.24%	2.55%
3.625	01/01/1998	15/01/2008	10	09/01/1998	07/07/2004	1692	1.29%	3.40%
3.875	01/01/1999	15/01/2009	10	07/01/1999	07/07/2004	1433	1.64%	3.54%
4.25	01/01/2000	15/01/2010	10	13/01/2000	07/07/2004	1168	1.99%	4.08%
3.5	01/01/2001	15/01/2011	10	10/01/2001	07/07/2004	908	3.21%	5.02%
3.375	01/01/2002	15/01/2012	10	10/01/2002	07/07/2004	649	4.38%	5.88%
3	01/07/2002	15/07/2012	10	12/07/2002	07/07/2004	518	4.01%	6.66%
1.875	01/07/2003	15/07/2013	10	10/07/2003	07/07/2004	259	0.23%	8.00%
2	15/01/2004	15/01/2014	10	09/01/2004	07/07/2004	128	-1.16%	7.31%
3.625	01/04/1998	15/04/2028	30	10/04/1998	07/07/2004	1623	3.38%	7.50%
3.875	01/04/1999	15/04/2029	30	09/04/1999	07/07/2004	1362	3.95%	7.88%
3.375	01/10/2001	15/04/2032	30.5	11/10/2001	07/07/2004	713	8.20%	11.12%
Mean							2.70%	6.08%

Panel B: Current Spot Rates				
Average daily Treasury real spot rates for 08:2004	1.12 (5Y)	1.51 (7Y)	1.86 (10Y)	2.23 (20Y)
Average daily Treasury nominal spot rates for 08:2004	3.47 (5Y)	3.90 (7Y)	4.28 (10Y)	5.07 (20Y)

Panel A of Figure 2.1 illustrates the calculated model values of an inflation-protected bond as a function of the return volatility of the fully adjusted (real) bond for different maturities. The results show a positive TIPS value – real bond return volatility relation. This is consistent with Proposition 2. Recall that, since the fully adjusted (real) bond is the underlying asset for the embedded option, a higher volatility of its return implies a higher probability of deflation and, therefore, the protection against deflation provided by the option becomes more valuable. The figure also shows that, everything else being equal, as the maturity of the inflation-protected bond increases, its value decreases due to the time value of money.



Panel A **Panel B**

Figure 2.1: The Value of the Embedded Option

This numerical exercise assumes continuously compounded nominal and real interest rates of 3.47 and 1.12%, respectively. These rates are the averages of the 5-year daily Treasury nominal and real spot rates for August 2004, taken from the United States Department of Treasury website (based on estimated yield curves). The assumed volatility of the fully adjusted (real) bond ranges between 0% and 15%, whereas the face value is maintained at \$100. We simulate values for 5, 10, 20, and 30-year bonds. In Panel A we plot the calculated model values of an inflation-protected bond as a function of the return volatility for different maturities. In Panel B we plot the error of ignoring the embedded option as a function of the return volatility for different maturities.

In Panel B of Figure 2.1 we compare calculated model values of the inflation-protected bond with the value of a comparable real bond. The latter ignores the embedded option, and is calculated by discounting the face value of the bond with the real interest rate. We then calculate the error of ignoring the embedded option as the difference between the model price and the price of the real bond. Note that this difference gives the value of the European put option written on a real pure-discount riskless bond. For a volatility ranging from 0% to 15%, the pricing error for the 5-year (30-year) inflation-protected bond is between 0 and \$8.10 (\$4.26) for a \$100 face value bond. However, for the 6.08% average return volatility of the sampled TIPS, the pricing error for the 5-year (30-year) inflation-protected bond amounts to \$1.31 (\$0.10) for a \$100 face value bond. Thus, the magnitude of the simulated error is economically significant. It is evident that the pricing error is smaller for longer maturity inflation-protected bonds. This result is intuitive. The probability that, in total, the CPI will decrease over the life of the bond (deflation) is higher over shorter horizons (shorter maturity).

2.6 Empirical Test

Equation (2.7) of our model shows that the yield spread between the yield on a nominal Treasury bond and the yield on an inflation-protected bond is a function of: (i) the volatility of the return of the real bond (σ); (ii) the nominal rate, (i) the real rate (r), and (iii) the time to maturity (τ). Our comparative statics predict that changes in the yield spread are positively related to changes in the volatility of returns on real bonds, and under the current monetary policy in the U.S., spreads are positively related to changes in both nominal and real rates.

In this section, we propose a simple way of simultaneously estimating the volatility embedded in real bond returns using the generalized autoregressive conditional heteroskedasticity (GARCH) framework introduced by Bollerslev (1986) and test the implications of the comparative statics results based on equation (2.7).

To this end, we estimate a bivariate GARCH model. In this section, we test the relationship between yield spread changes and changes in the estimated volatility and a proxy for the nominal rate.

2.6.a The Data

We construct a daily index of inflation-protected bond yields, using daily time-series data of yields on existing TIPS. Specifically, we use yield data of the first five TIP bonds described in the Panel A of Table 2.1. We restrict our sample to these five bonds so that we have a number of daily observations that is adequate for the GARCH model. We also use the same TIPS to construct a bond-price index for which we calculate daily returns. In the absence of pure real bonds in the U.S., we use the yield and return on this TIPS index as a proxy for the fully-adjusted bond yield and return index. To obtain the reference rate for the spreads, we construct a daily nominal Treasury yield index that has a weighted average maturity matching the maturity of the TIPS index. As the average maturity of the TIPS index drops from 8.02 years to 4.53 years over the sample period, we construct the nominal index by taking a weighted average of the 7 and 10-year, 5 and 7-year, and 3 and 5-year daily constant maturity Treasury nominal yields available from the Federal Reserve Bank of St. Louis. The sample extends from January 10, 2001 to July 7, 2004.[10]

To avoid the multicollinearity that exists due to the high correlation between changes in nominal yields and changes in real yields, we focus our empirical analysis

[10] Note that there are no data available for the first bond in Table 1 beyond February 24, 2004. For the period between this date and July 7, 2004, we use the other four bonds to construct our index.

on the relation between changes in spreads and changes in nominal rates and the volatility on the return of the real bond. We use changes in 3-month Treasury bill rates to proxy for changes in nominal rates, which is an independent variable in our analysis.

We calculate the augmented Dickey-Fuller (ADF) and the Phillips-Perron (PP) statistics to test for the null of the unit root against the alternative of AR (1) process, and our results show that all considered time series in the level form are integrated of order one, whereas all considered series in the first differenced form are stationary. The time series of return levels on TIPS are also stationary.

Almost all our time series exhibit significant serial dependence in the residuals. We look at the estimates of the Ljung-Box statistics for changes and squared changes in TIPS prices, 3-month Treasury bill rates, real rates, and yield spreads. We find a significant serial correlation in changes and squared changes of all variables, with a larger statistic for squared changes. This indicates volatility clustering as well as time-varying volatility. Results of these tests are reported in Table 2.2 below.

Table 2.2: Results of Ljung-Box and Unit Root Tests

Q (4), Q (12), sq, Q (4), sq, Q (12) are the Ljung-Box test statistics at lag *(m)*, which tests the null hypothesis that all autocorrelation coefficients are simultaneously equal to zero up to *(m)* lags; it is asymptotically distributed as χ^2_m with m degrees of freedom. *Q (4)* and *Q (12)* offer the Ljung-Box test statistics for standardized residuals, whereas *sq, Q (4)* and *sq, Q (12)* – for squared standardized residuals. ADF is the augmented Dickey-Fuller unit root test statistic, whereas PP is the Phillips-Perron test statistic.

Table 2.2: Results of Ljung-Box and Unit Root Tests

Variable	*Q (4)*	*Q (12)*	*sq, Q (4)*	*sq, Q (12)*	*ADF*	*PP*
r	3589.000	10528.000	3589.300	10515.000	-0.975	-1.015
rc	7.763	20.504	43.177	105.660	-28.043	-28.048
i	3549.000	10196.000	3502.300	9840.200	-4.715	-4.671
ic	35.595	63.978	101.540	250.640	-24.939	-25.497
s	3482.300	9856.700	3488.100	9858.400	-1.412	-1.260
sc	9.885	18.764	10.191	39.618	-31.089	-31.350
ret	17.533	26.105	24.601	54.101	-26.602	-26.779

r stands for real rates in levels; *i* stands for nominal rates in levels; *ret* stands for the return on real bonds; *s* stands for yield spreads in levels; and _*c* stands for the associated _ variable in first differences.

To further demonstrate the time-varying volatility of yield spreads, nominal yields and real-bond returns, we plot these variables over time in Figure 2.2.

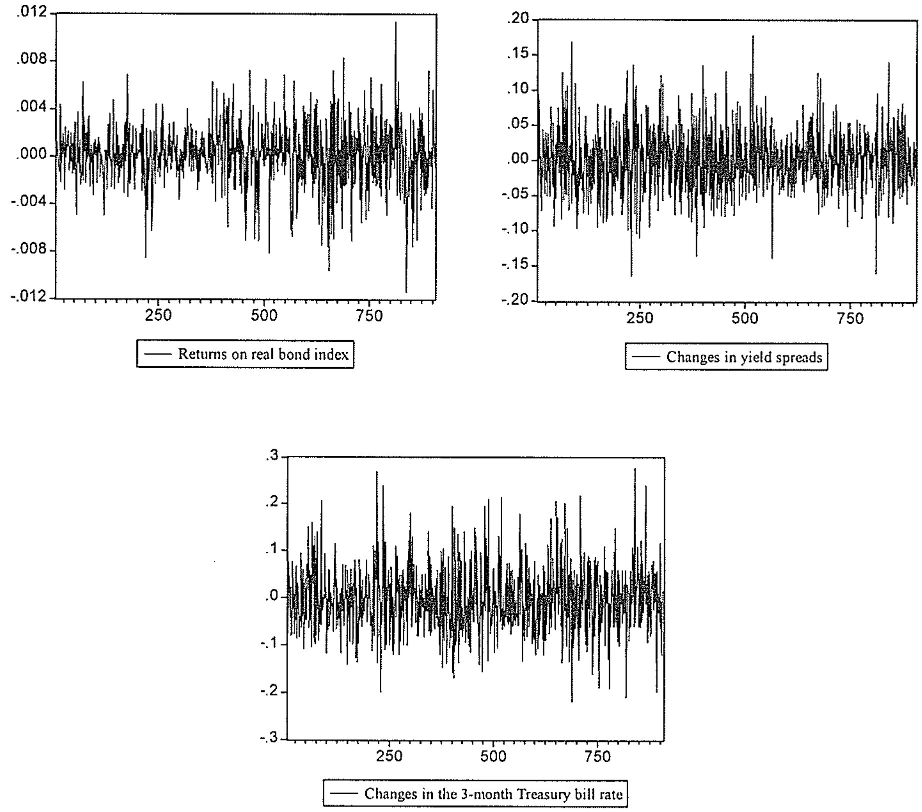


Figure 2.2: Daily Changes in the 3-month Treasury Bill rate, Yield Spread, and Daily Returns on TIPS

The dataset covers the period from January 10, 2001 to July 7, 2004. Observations are reported in decimal points.

Figure 2.2 exhibits volatility clustering for all variables that enter our bivariate GARCH model. This implies that periods of high volatility are followed by periods of low volatility.

2.6.b Methodology and Results

We estimate the unrestricted multivariate GARCH model introduced by Baba, Engle, Kraft, and Kroner (1987) (BEKK). We specify mean equations as follows:

$$R_t = \mu_1 + \lambda_1 \cdot R_{t-1} + \lambda_2 \cdot R_{t-2} + \varepsilon_{1t} \tag{2.8}$$

$$\Delta S_t = \mu_2 + \gamma_1 \Delta I_t + \gamma_2 \Delta \sigma_{R,t} + \varepsilon_{2t} \tag{2.9}$$

The first mean equation is for daily returns on the constructed TIPS index, R_t, which proxies for the return on the real bond. We include lagged returns to the second degree. We chose the second degree by minimizing the Akaike information criterion calculated for the simultaneously estimated model. The dependent variable in the second equation, ΔS_t, is the daily change in the yield spread, defined as the difference between the daily yield on our TIPS index and the daily yield on a matching maturity nominal Treasury index. The choice of the independent variables is consistent with our theoretical expression for the yield spread. ΔI_t is the daily changes in the 3-month Treasury bill yield, which proxies for changes in nominal rates. Our variance specification is: $H_t = \varpi'\varpi + \beta' H_{t-1}\beta + \alpha'\varepsilon_{t-1}\varepsilon'_{t-1}\alpha \;\forall\, i,j$, where ϖ represents the upper triangular coefficient matrix, α - represents a matrix of ARCH coefficients, and β represents a matrix of GARCH coefficients. We estimate equations (2.8) and (2.9) simultaneously using a maximum likelihood method (Marquardt algorithm).

The results of the bivariate GARCH model are reported in Table 2.3. We report the estimates of equations (2.8) and (2.9).

Table 2.3: Results of the Empirical Test

This table reports the results of the following bivariate GARCH model:

$$R_t = \mu_1 + \lambda_1 \cdot R_{t-1} + \lambda_2 \cdot R_{t-2} + \varepsilon_{1t}$$

$$\Delta S_t = \mu_2 + \gamma_1 \Delta I_t + \gamma_2 \Delta \sigma_{R,t} + \varepsilon_{2t}$$

where $\varepsilon_{1t}, \varepsilon_{2t} \sim N(0, H_t)$, and $H_t = \varpi'\varpi + \beta' H_{t-1}\beta + \alpha'\varepsilon_{t-1}\varepsilon'_{t-1}\alpha \; \forall \, i,j$, ϖ is the upper triangular coefficient matrix, α is a matrix of ARCH coefficients, and β is a matrix of GARCH coefficients. The model is estimated simultaneously using a maximum likelihood method (Marquardt algorithm). The table also reports the Ljung-Box test statistics at lag 12, denoted by $Q_j(12)$ for standardized residuals and by $Q_j^2(12)$ for squared standardized residuals. The Q_j statistic with j = 1 (j = 2) refers to the standardized residuals obtained from the first (second) equation in the bivariate model. The Q_j statistic with j = 12 refers to the standardized residuals obtained from both equations.

Table 2.3: Results of the Empirical Test						
Sample period: 10/01/2001 to 07/07/2004						
Bivariate GARCH Model Coefficients	*GARCH with σ = st deviation*			*GARCH with σ = variance*		
	Coefficent	**St. Error**	***t* statistic**	**Coefficent**	**St. Error**	***t* statistic**
μ_1	0.00016	0.00007	2.20612	0.00014	0.00008	1.78627
λ_1	0.18291	0.03338	5.48016	0.18676	0.03377	5.53064
λ_2	0.01474	0.02933	0.50239	0.01949	0.02998	0.64997
μ_2	0.00184	0.00298	0.61725	0.00165	0.0024	0.68796
γ_1	0.25707	0.03692	6.96212	0.25984	0.0366	7.09976
γ_2	15.88326	0.07458	212.9691	1545.585	8.50289	181.7718
Ljung-Box test statistics	*Q statistics*	*p value*		*Q statistics*	*p value*	
$Q_1(12)$	6.223	0.904		6.489	0.889	
$Q_2(12)$	11.839	0.459		11.836	0.459	
$Q_1^2(12)$	19.726	0.072		19.91	0.069	
$Q_2^2(12)$	16.04	0.189		16.233	0.181	
$Q_{12}^2(12)$	30.919	0.002		4.706	0.967	

The estimated Ljung-Box test statistics for standardized residuals and squared standardized residuals show that there is no remaining serial dependence for the results provided in Panel A of Table 2.3. We report these statistics for 12 lags. This demonstrates that our mean, and variance equations are specified correctly and that the bivariate GARCH model represents the dynamic patterns of our dependent variables well. We present results for the GARCH model for two alternative cases; we first represent volatility by the standard deviation and, then, by the variance of real returns. In both cases, the yield spread is statistically significantly and positively related to changes in nominal rates, and also to changes in the real return volatility. We re-estimate the model for the period ranging from January 10, 2001 to December 31, 2002. This period is characterized by higher expectations for deflation (see Roll, 2004). The results for this period are similar to the results obtained for the entire sample. [11]

These results show that the data agrees with the comparative statics of our theoretical model. We confirm our theoretical conjectures by finding that changes in the yield spread are positively related to changes in the volatility of returns on real bonds. We also find that changes in the yield spread are positively related to changes in nominal rates – a finding consistent with the current monetary policy practiced in the U.S. Thus, our empirical test shows that both variables are priced in the market. [12]

---

[11] Note that for this sub period we only have 512 observations. This makes the results of the bivariate GARCH model less robust relative to the results reported for the entire sample period. The results for this sub period are available upon request.

[12] Note that there may be other variables influencing the estimated relationship. For example, TIPS are less liquid than nominal Treasuries, and therefore the yield spread will be lower due to the illiquidity premium associated with TIPS. However, with the increase of TIPS issued over the years and the improved marketability this issue becomes less problematic.

2.7 Summary and Conclusions

Most research on inflation-protected securities assumes, explicitly or implicitly, that the value of the put option that offers protection against deflation is trivial. At the present time, in many countries deflation is not an unlikely event. In the current chapter we use Option Pricing Theory to model the value of an inflation-adjusted TIPS, accounting for the value of the embedded option. We examine several determinants of TIPS and provide an adjusted elasticity measure. Our model predicts that the pricing of TIPS is determined by the volatility of the return of a real bond, the nominal rate, the real rate, the expected inflation rate, and the time to maturity. We examine the relationship between the price of an inflation-protected security and these parameters. We then proceed to model the yield spread for inflation-protected bonds. It is common for researchers to estimate expected inflation as the spread between the nominal Treasury bond yield and the yield on TIPS. We show that this spread is an unbiased estimator of inflation expectations only when the embedded option is deep out of the money. A numerical simulation, based on a sample of Treasury bonds, demonstrates that ignoring the embedded option is costly. The validity of our model is also strengthened by an empirical test that confirms its properties with respect to TIPS yield spreads.

To the best of our knowledge, this is the only attempt to price TIPS while considering the embedded option. We note that the protection against deflation provided by coupon-bearing TIPS applies only to the face value paid at maturity, not to the coupon payments. Thus, one cannot price coupon-bearing TIPS by individually pricing strips and aggregating these prices to obtain the value of the inflation-protected coupon bond. Since in our model we price pure-discount inflation-protected bonds, one can view this study as a first step in this line of research. Future research should focus on the pricing of coupon inflation-protected bonds.

Appendix A: Derivation of the Bond-Pricing Equation and the Yield Spread Equation

Note that, since we need to discount expected nominal cash flows, we use the nominal riskless rate (i) under risk-neutral valuation. Recall that the payoff of a zero-coupon inflation-protected bond at its maturity (T) is given by:

$$B_{TIPS,T} = \max[e^{\pi(T-t)} - 1, 0] + 1.$$

Under risk-neutral valuation, the time-t value of the call option written on the real bond is given by:

$$C(B_t^*, t, T) = e^{-i(T-t)} E_q\left(\max\left[B_T^* - 1, 0\right]\right) = e^{-i(T-t)} E_q\left(\max\left[e^{\pi(T-t)} - 1, 0\right]\right).$$

To calculate the value of the call, we assume that the stochastic value of the real bond, B^*, is well described under the risk neutral measure Q, by the following Geometric Brownian motion process:

$$dB^* = B^* i dt + B^* \sigma dZ.$$

Let $G = \ln B^*$. Applying Ito's Lemma, we get:

$$dG = \left[\frac{dG}{dB^*} B^* i + \frac{dG}{dt} + \frac{1}{2}\frac{d^2G}{dB^{*2}} \sigma^2 B^{*2}\right] dt + \frac{dG}{dB^*} B^* \sigma dZ,$$

and:

$$d\ln B^* = \left[\frac{d\ln B^*}{dB^*}B^*i + \frac{d\ln B^*}{dt} + \frac{1}{2}\frac{d^2\ln B^*}{dB^{*2}}\sigma^2 B^{*2}\right]dt + \frac{d\ln B^*}{dB^*}B^*\sigma dZ. \quad \text{(A.1)}$$

Taking the derivatives in the above stochastic differential equation, we get:

$$\frac{d\ln B^*}{dB^*} = \frac{1}{B^*}, \frac{d\ln B^*}{dt} = 0, \text{ and } \frac{d^2\ln B^*}{dB^{*2}} = -\frac{1}{B^{*2}}.$$

Substituting these values back into equation (A.1) we get:

$$d\ln B^* = (i - \frac{\sigma^2}{2})dt + \sigma dZ.$$

This implies the following risk-neutral distribution for the continuously compounded return on the real bond:

$$\ln\left[\frac{B^*_T}{B^*_t}\right] \sim N\left[\left(i - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right].$$

Or we can write that:

$$\ln B^*{}_T \sim N\left[\ln B^* + (i - \frac{\sigma^2}{2})(T-t), \sigma^2(T-t)\right].$$

Noting that: $\ln B^* = -r(T\text{-}t)$, we get:

$$\ln B^*{}_T \sim N\left[(i - r - \frac{\sigma^2}{2})(T-t), \sigma^2(T-t)\right].$$

Following Black and Scholes (1973), the time-t value of the call option is given by:

$$C(B_t^*, t, T) = e^{-r(T-t)}N(d_1) - e^{-i(T-t)}N(d_2),$$

where $d_1 = \dfrac{(i - r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = \dfrac{(i - r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$.

Recall that the value of an inflation-protected bond is equivalent to the value of a portfolio consisting of a corresponding unadjusted riskless bond and the above call option. Given the value of the call, the time-t value of the inflation-protected bond is given by:

$$B_{TIPS,t} = e^{-i(T-t)} + \left[e^{-r(T-t)}N(d_1) - e^{-i(T-t)}N(d_2)\right].$$

Rearranging, we get:

$$B_{TIPS,t} = e^{-i(T-t)}N(-d_2) + e^{-r(T-t)}N(d_1).$$

For a bond with an initial face value of F, we get the inflation-protected bond-pricing equation:

$$B_{TIPS,t} = Fe^{-i(T-t)}N(-d_2) + Fe^{-r(T-t)}N(d_1). \qquad (2.1)$$

The inflation-protected bond's yield to maturity ($r(\tau)$) must satisfy:

$$B_{TIPS,t} = Fe^{-r(\tau)\tau} \qquad (A.2)$$

This yield is given by the real return that the investor will realize by holding the inflation-protected security to maturity. Manipulating equation (A.2), we find that the yield to maturity for holding an inflation-protected bond is simply:

$$r(\tau) = -\frac{1}{\tau}\ln\left(\frac{B_{TIPS,t}}{F}\right), \text{ where } B_{TIPS,t} = Fe^{-i\tau}N(-d_2) + Fe^{-r\tau}N(d_1) \qquad (A.3)$$

Manipulating (A.3) further, we get:

$$r(\tau) = -\frac{1}{\tau}\ln\left(e^{-i\tau}N(-d_2) + e^{-r\tau}N(d_1)\right) \qquad (A.4)$$

Since the yield obtained for holding a nominal Treasury bond is simply i, the yield spread between nominal bond's rate and the rate acquired by holding an inflation-protected bond is given by:

$$i - r(\tau) = \frac{1}{\tau}\tau \ln e^{i} + \frac{1}{\tau}\ln\left(e^{-i\tau}N(-d_2) + e^{-r\tau}N(d_1)\right) = \frac{1}{\tau}\left[\ln e^{i\tau} + \ln\left(e^{-i\tau}N(-d_2) + e^{-r\tau}N(d_1)\right)\right] =$$
$$= \frac{1}{\tau}\ln\left[\left(e^{-i\tau}N(-d_2) + e^{-r\tau}N(d_1)\right) \times e^{i\tau}\right] = \frac{1}{\tau}\ln\left(N(-d_2) + e^{(i-r)\tau}N(d_1)\right) =$$
$$= \frac{1}{\tau}\ln\left(N(-d_2) + e^{\pi^{*}\tau}N(d_1)\right) \tag{2.7}$$

This yield spread between the yield on the nominal Treasury bond and inflation-protected bond represents the inflation premium incorporated into the yield realized on nominal Treasury bonds. This inflation premium provides the best available estimate on the anticipated levels of inflation.

Appendix B: Proofs of Propositions

Proof of Proposition 1: Differentiating the value of the inflation-protected security in equation (1) with respect to the face value, we find a positive relationship between the two:

$$\frac{dB_{TIPS,t}}{dF} = e^{-i\tau}N(-d_2) + e^{-r\tau}N(d_1) > 0, \text{ where } \tau = T\text{-}t. \qquad \text{Q.E.D.}$$

Proof of Proposition 2: Differentiating the value of the inflation-protected security with respect to the volatility of the return on the fully adjusted (real) bond, we get:

$$\begin{aligned}
\frac{dB_{TIPS,t}}{d\sigma} &= Fe^{-r\tau}\frac{\partial N(d_1)}{\partial\sigma} - Fe^{-i\tau}\frac{\partial N(d_2)}{\partial\sigma} \\
&= Fe^{-r\tau}N'(d_1)\frac{\partial d_1}{\partial\sigma} - Fe^{-i\tau}N'(d_2)\frac{\partial d_2}{\partial\sigma} \\
&= Fe^{-r\tau}N'(d_1)\frac{\partial d_1}{\partial\sigma} - Fe^{-i\tau}N'(d_1)e^{-(r-i)\tau}\frac{\partial d_2}{\partial\sigma} \\
&= Fe^{-r\tau}N'(d_1)\left[\frac{\partial d_1}{\partial\sigma} - \frac{\partial d_2}{\partial\sigma}\right] \\
&= Fe^{-r\tau}N'(d_1)\sqrt{\tau} > 0.
\end{aligned}$$

Q.E.D.

Proof of Proposition 3: Differentiating the value of the inflation-protected security with respect to the nominal rate, we get:

$$
\begin{aligned}
\frac{dB_{TIPS,t}}{di} &= -\tau Fe^{-i\tau} - \tau\frac{dr}{di}Fe^{-r\tau}N(d_1) + Fe^{-r\tau}\frac{\partial N(d_1)}{\partial i} + \tau Fe^{-i\tau}N(d_2) - Fe^{-i\tau}\frac{\partial N(d_2)}{\partial i} \\
&= -\tau Fe^{-i\tau} - \tau\frac{dr}{di}Fe^{-r\tau}N(d_1) + Fe^{-r\tau}N'(d_1)\frac{\partial d_1}{\partial i} + \tau Fe^{-i\tau}N(d_2) - Fe^{-i\tau}N'(d_2)\frac{\partial d_2}{\partial i} \\
&= -\tau Fe^{-i\tau} - \tau\frac{dr}{di}Fe^{-r\tau}N(d_1) + Fe^{-r\tau}N'(d_1)\frac{\partial d_1}{\partial i} + \tau Fe^{-i\tau}N(d_2) - Fe^{-i\tau}N'(d_1)e^{(i-r)\tau}\frac{\partial d_2}{\partial i} \\
&= -\tau Fe^{-i\tau} - \tau\frac{dr}{di}Fe^{-r\tau}N(d_1) + \tau Fe^{-i\tau}N(d_2) \\
&= -\tau\left[\frac{dr}{di}Fe^{-r\tau}N(d_1) + Fe^{-i\tau}N(-d_2)\right] \lesseqgtr 0.
\end{aligned}
$$

Q.E.D.

Proof of Proposition 4: Similar to the differentiation for Proposition 3, we get:

$$
\frac{dB_{TIPS,t}}{dr} = -\tau\left[\frac{di}{dr}Fe^{-i\tau}N(-d_2) + Fe^{-r\tau}N(d_1)\right] \lesseqgtr 0.
$$

Q.E.D.

Proof of Proposition 5: The expected inflation rate is given by: $\pi^* = i - r$. Differentiating the value of the inflation-protected security with respect to π^*, we get:

$$
\begin{aligned}
\frac{dB_{TIPS,t}}{d\pi^*} &= -\tau\frac{di}{d\pi^*}Fe^{-i\tau}N(-d_2) - Fe^{-i\tau}\frac{\partial N(d_2)}{\partial \pi^*} - \tau\frac{dr}{d\pi^*}Fe^{-r\tau}N(d_1) + Fe^{-r\tau}\frac{\partial N(d_1)}{\partial \pi^*} \\
&= -\tau\frac{di}{d\pi^*}Fe^{-i\tau}N(-d_2) - Fe^{-i\tau}N'(d_2)\frac{\partial d_2}{\partial \pi^*} - \tau\frac{dr}{d\pi^*}Fe^{-r\tau}N(d_1) + Fe^{-r\tau}N'(d_1)\frac{\partial d_1}{\partial \pi^*} \\
&= -\tau\frac{di}{d\pi^*}Fe^{-i\tau}N(-d_2) - Fe^{-i\tau}N'(d_1)e^{(i-r)\tau}\frac{\partial d_1}{\partial \pi^*} - \tau\frac{dr}{d\pi^*}Fe^{-r\tau}N(d_1) + Fe^{-r\tau}N'(d_1)\frac{\partial d_1}{\partial \pi^*} \\
&= -\tau\left[\frac{di}{d\pi^*}Fe^{-i\tau}N(-d_2) + \frac{dr}{d\pi^*}Fe^{-r\tau}N(d_1)\right] \lesseqgtr 0.
\end{aligned}
$$

Q.E.D.

Proof of Proposition 6: Differentiating the value of the inflation-protected security with respect to τ, we get:

$$
\begin{aligned}
\frac{dB_{TIPS,t}}{d\tau} &= -iFe^{-i\tau} - rFe^{-r\tau}N(d_1) + Fe^{-r\tau}\frac{\partial N(d_1)}{\partial \tau} + iFe^{-i\tau}N(d_2) - Fe^{-i\tau}\frac{\partial N(d_2)}{\partial \tau} \\
&= -iFe^{-i\tau} - rFe^{-r\tau}N(d_1) + Fe^{-r\tau}N'(d_1)\frac{\partial d_1}{\partial \tau} + iFe^{-i\tau}N(d_2) - Fe^{-i\tau}N'(d_2)\frac{\partial d_2}{\partial \tau} \\
&= -iFe^{-i\tau} - rFe^{-r\tau}N(d_1) + Fe^{-r\tau}N'(d_1)\frac{\partial d_1}{\partial \tau} + iFe^{-i\tau}N(d_2) - Fe^{-i\tau}N'(d_1)e^{(i-r)\tau}\frac{\partial d_2}{\partial \tau} \\
&= -iFe^{-i\tau} - rFe^{-r\tau}N(d_1) + iFe^{-i\tau}N(d_2) + Fe^{-r\tau}N'(d_1)\left[\frac{\partial d_1}{\partial \tau} - \frac{\partial d_2}{\partial \tau}\right] \\
&= Fe^{-r\tau}N'(d_1)\left[\frac{\partial d_1}{\partial \tau} - \frac{\partial d_2}{\partial \tau}\right] - rFe^{-r\tau}N(d_1) - iFe^{-i\tau}N(-d_2) \\
&= Fe^{-r\tau}N'(d_1) - rFe^{-r\tau}N(d_1) - iFe^{-i\tau}N(-d_2) \lesseqgtr 0.
\end{aligned}
$$

Q.E.D.

Proof of Proposition 7: Differentiating the yield spread with respect to σ, we get:

$$\frac{dS(\tau)}{d\sigma}=\frac{1}{\tau}\left(\frac{1}{N(-d_2)+e^{(i-r)\tau}N(d_1)}\right)\left[e^{-(i-r)\tau}\frac{\partial N(d_1)}{\partial\sigma}-\frac{\partial N(d_2)}{\partial\sigma}\right]$$

$$=\frac{1}{\tau}\left(\frac{1}{N(-d_2)+e^{(i-r)\tau}N(d_1)}\right)\left[e^{-(i-r)\tau}N'(d_1)\frac{\partial d_1}{\partial\sigma}-N'(d_2)\frac{\partial d_2}{\partial\sigma}\right]$$

$$=\frac{1}{\tau}\left(\frac{1}{N(-d_2)+e^{(i-r)\tau}N(d_1)}\right)\left[e^{-(i-r)\tau}N'(d_1)\frac{\partial d_1}{\partial\sigma}-N'(d_1)e^{-(r-i)\tau}\frac{\partial d_2}{\partial\sigma}\right]$$

$$=\frac{1}{\tau}\left(\frac{1}{N(-d_2)+e^{(i-r)\tau}N(d_1)}\right)\left[e^{-(i-r)\tau}N'(d_1)\left(\frac{\partial d_1}{\partial\sigma}-\frac{\partial d_2}{\partial\sigma}\right)\right]$$

$$=\frac{1}{\sqrt{\tau}}\left(\frac{e^{-(i-r)\tau}N'(d_1)}{N(-d_2)+e^{(i-r)\tau}N(d_1)}\right)>0.$$

Q.E.D.

Proof of Proposition 8: Differentiating the yield spread with respect to i, we get:

$$\frac{dS(\tau)}{di}=\frac{1}{\tau}\left(\frac{1}{N(-d_2)+e^{(i-r)\tau}N(d_1)}\right)\left[\tau\left(1-\frac{dr}{di}\right)e^{(i-r)\tau}N(d_1)+e^{(i-r)\tau}\frac{\partial N(d_1)}{\partial i}-\frac{\partial N(d_2)}{\partial i}\right]$$

$$=\frac{1}{\tau}\left(\frac{1}{N(-d_2)+e^{(i-r)\tau}N(d_1)}\right)\left[\tau\frac{d\pi^*}{di}e^{(i-r)\tau}N(d_1)+e^{(i-r)\tau}N'(d_1)\frac{\partial d_1}{\partial i}-N'(d_2)\frac{\partial d_2}{\partial i}\right]$$

$$=\frac{1}{\tau}\left(\frac{1}{N(-d_2)+e^{(i-r)\tau}N(d_1)}\right)\left[\tau\frac{d\pi^*}{di}e^{(i-r)\tau}N(d_1)+e^{(i-r)\tau}N'(d_1)\frac{\partial d_1}{\partial i}-N'(d_1)e^{-(r-i)\tau}\frac{\partial d_2}{\partial i}\right]$$

$$=\frac{\frac{d\pi^*}{di}e^{(i-r)\tau}N(d_1)}{N(-d_2)+e^{(i-r)\tau}N(d_1)}\lesseqgtr 0.$$

Q.E.D.

Proof of Proposition 9: Similar to the differentiation for Proposition 8, we get:

$$\frac{dS(\tau)}{dr} = \frac{d\pi^*}{dr}\left(\frac{e^{(i-r)\tau}N(d_1)}{N(-d_2)+e^{(i-r)\tau}N(d_1)}\right) \lesseqgtr 0.$$

Q.E.D.

Proof of Proposition 10: Differentiating the yield spread with respect to π^*, we get:

$$\begin{aligned}
\frac{dS(\tau)}{d\pi^*} &= \frac{1}{\tau}\left(\frac{1}{N(-d_2)+e^{\pi^*\tau}N(d_1)}\right)\left[\tau e^{\pi^*\tau}N(d_1)+e^{\pi^*\tau}\frac{\partial N(d_1)}{\partial \pi^*}-\frac{\partial N(d_2)}{\partial \pi^*}\right] \\
&= \frac{1}{\tau}\left(\frac{1}{N(-d_2)+e^{\pi^*\tau}N(d_1)}\right)\left[\tau e^{\pi^*\tau}N(d_1)+e^{\pi^*\tau}N'(d_1)\frac{\partial d_1}{\partial \pi^*}-N'(d_2)\frac{\partial d_2}{\partial \pi^*}\right] \\
&= \frac{1}{\tau}\left(\frac{1}{N(-d_2)+e^{\pi^*\tau}N(d_1)}\right)\left[\tau e^{\pi^*\tau}N(d_1)+e^{\pi^*\tau}N'(d_1)\frac{\partial d_1}{\partial \pi^*}-N'(d_1)e^{-(r-i)\tau}\frac{\partial d_2}{\partial \pi^*}\right] \\
&= \frac{e^{\pi^*\tau}N(d_1)}{N(-d_2)+e^{\pi^*\tau}N(d_1)} > 0.
\end{aligned}$$

Q.E.D.

Proof of Proposition 11: Differentiating the yield spread with respect to τ we get:

$$\frac{dS(\tau)}{d\tau} = \left[-\frac{1}{\tau^2}\ln\left(N(-d_2)+e^{\pi^*\tau}N(d_1)\right)\right] + \left[\frac{1}{\tau}\left(\frac{\pi^* e^{\pi^*\tau}N(d_1)+e^{\pi^*\tau}\frac{\partial N(d_1)}{\partial\tau}-\frac{\partial N(d_2)}{\partial\tau}}{N(-d_2)+e^{\pi^*\tau}N(d_1)}\right)\right]$$

$$= \left[-\frac{1}{\tau^2}\ln\left(N(-d_2)+e^{\pi^*\tau}N(d_1)\right)\right] + \left[\frac{1}{\tau}\left(\frac{\pi^* e^{\pi^*\tau}N(d_1)+e^{\pi^*\tau}N'(d_1)\frac{\partial d_1}{\partial\tau}-N'(d_2)\frac{\partial d_2}{\partial\tau}}{N(-d_2)+e^{\pi^*\tau}N(d_1)}\right)\right]$$

$$= \left[-\frac{1}{\tau^2}\ln\left(N(-d_2)+e^{\pi^*\tau}N(d_1)\right)\right] + \left[\frac{1}{\tau}\left(\frac{\pi^* e^{\pi^*\tau}N(d_1)+e^{\pi^*\tau}N'(d_1)\left(\frac{\partial d_1}{\partial\tau}-\frac{\partial d_2}{\partial\tau}\right)}{N(-d_2)+e^{\pi^*\tau}N(d_1)}\right)\right]$$

$$= \left[-\frac{1}{\tau^2}\ln\left(N(-d_2)+e^{\pi^*\tau}N(d_1)\right)\right] + \left[\frac{1}{\tau}\left(\frac{\pi^* e^{\pi^*\tau}N(d_1)+e^{\pi^*\tau}N'(d_1)\frac{\sigma}{2\sqrt{\tau}}}{N(-d_2)+e^{\pi^*\tau}N(d_1)}\right)\right] \lesseqgtr 0.$$

Q.E.D.

Appendix C: Derivations of the Duration Measures

The Nominal Duration

Applying the standard price-elasticity definition of duration to bond-pricing equation (1), we get:

$$
\begin{aligned}
D_i &= -\frac{1}{B_{TIPS,t}} \frac{\partial B_{TIPS,t}}{\partial i} \\
&= \frac{1}{B_{TIPS,t}} \tau \left[\frac{dr}{di} F e^{-r\tau} N(d_1) + F e^{-i\tau} N(-d_2) \right] \\
&= \frac{1}{B_{TIPS,t}} \tau \left[\left(1 - \frac{d\pi^*}{di} \right) F e^{-r\tau} N(d_1) + F e^{-i\tau} N(-d_2) \right] \\
&= \frac{\tau B_{TIPS,t} - \tau \dfrac{\partial \pi^*}{\partial i} F e^{-r\tau} N(d_1)}{B_{TIPS,t}} \\
&= \tau \left(1 - \frac{\partial \pi^*}{\partial i} \frac{F e^{-r\tau} N(d_1)}{B_{TIPS,t}} \right).
\end{aligned}
$$

The Real Duration

Applying the standard price-elasticity definition of duration to bond-pricing equation (1), we get:

$$
\begin{aligned}
D_r &= -\frac{1}{B_{TIPS,t}} \frac{\partial B_{TIPS,t}}{\partial r} \\
&= \frac{1}{B_{TIPS,t}} \tau \left[Fe^{-r\tau} N(d_1) + \frac{di}{dr} Fe^{-i\tau} N(-d_2) \right] \\
&= \frac{1}{B_{TIPS,t}} \tau \left[Fe^{-r\tau} N(d_1) + \left(1 + \frac{d\pi^*}{dr}\right) Fe^{-i\tau} N(-d_2) \right] \\
&= \frac{\tau B_{TIPS,t} + \tau \dfrac{\partial \pi^*}{\partial r} Fe^{-r\tau} N(-d_2)}{B_{TIPS,t}} \\
&= \tau \left(1 + \frac{\partial \pi^*}{\partial r} \frac{Fe^{-r\tau} N(-d_2)}{B_{TIPS,t}} \right).
\end{aligned}
$$

CHAPTER 3

Bond Elasticity Under Liquidation Risk

3.1 Introduction

Corporate fixed-income markets are illiquid (Warga, 1991). This problem of thin corporate bond trading is amplified by the trend of shrinking government debt. This continuing development will undoubtedly generate great interest in the liquidity of the government-bond market.

Numerous researchers empirically document the existence and importance of a liquidity effect in bond valuation. For example, Amihud and Mendelson (1991) and Kamara (1994) demonstrate that highly liquid Treasury Bills trade at a premium relative to less liquid Treasury Notes. More studies demonstrating the impact of liquidity include Delianedis and Geske (2001), Elton et al (2001), Pages (2001), and Houweling et al (2002).

Theoretical work on the impact of liquidity on asset prices includes Amihud and Mendelson (1986), and Jacoby, Fowler, and Gottesman (2000). Huang (2002) models the motivation for trade by a random arrival time of liquidation. The similarity of his model and the model derived in the current research is that Huang also models an investor facing an exogenously determined liquidity shock. In his model, liquidity shocks occur in a continuous-time overlapping-generations economy with two consol bonds, one liquid and one illiquid. He finds that the required liquidity premium depends on the stochastic nature of the investor's holding horizon.

In this chapter, we follow Jacoby's (2003) certainty equivalent approach to model a risk-adjusted duration. While Jacoby's model adjusts duration to default risk, our model generates the bond pricing equation and an expression for the term structure of liquidity premia for a default-free government bond[13] with a risk of liquidation.

3.2 Model

Consider a coupon-bearing, noncallable government bond with a face value of F dollars and T-years to maturity. A potential bondholder faces exogenous random liquidity shocks, at which time s/he is forced to sell the bond for its bid price. An example for such an investor is a bond portfolio manager, who is facing exogenous liquidation shocks originated from his clients' personal consumption shocks. We assume that the government bond market is dominated by these portfolio managers, and that they have homogeneous beliefs regarding the process governing the arrival of the liquidation shocks. For simplicity, we assume that liquidity shocks occur an instant prior to the bond coupon payment. This means that the bondholder liquidates the bond an instant prior to receiving a cash coupon, and that bid-ask spread costs apply to the pending coupon payment as well[14].

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[13] In this chapter we consider a default-free government riskless bond to be able to isolate the liquidation risk from the risk of default.

[14] Economically, a rational bondholder will wait one instant and liquidate the bond after receiving the cash coupon payment. However, this leads to intractable results. Our assumption is equivalent to the

If the investor (bond portfolio manager) liquidates his/her position with probability of p_t at any time (t) prior to maturity, $1 \leq t \leq T$, then s/he will receive the sum of the accrued coupon payments (C_t) and the market value of the bond excluding accrued coupon (V_t) an instant prior to the time when the coupon is due. In this layout, the probability of not liquidating ($1-p_t$) is the conditional probability of no liquidity shock at t. This means that the unconditional probability of not liquidating up to t is $\prod_{l=1}^{t}(1-p_l)$, whereas the unconditional probability of liquidating at t is $p_t\prod_{l=1}^{t-1}(1-p_l)$. If the bond is liquidated at time t (with probability p_t), the bondholder will receive the amount of $(C_t+V_t)(1-S_t)$, where S_t is the proportional full bid-ask spread on the bond at t given by $S_t=\dfrac{V_{at}-V_{bt}}{V_{mt}}$. V_{at} and V_{bt} denote the ask and bid prices prevailing at time t, respectively, whereas V_{mt} denotes the mid-price given by: $V_{mt}=\dfrac{V_{at}+V_{bt}}{2}$. The continuously compounded percentage full spread is given by $s_t=-\ln(1-S_t)$. Thus, the bid price the investor receives if liquidating at time t is given by $(C_t+V_t)e^{-s_t}$.

We also assume that all investors are risk-averse and maximize their expected logarithmic utility function with an exogenously given level of wealth. As Jacoby (2003) shows for corporate bonds, the logarithmic utility function allows us to derive a tractable analytic expression for the bond pricing equation[15]. Following the methodology of Jacoby (2003), our representative investor will be inclined to

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bondholder liquidating the bond at any time between two coupons. For tractability purposes, we chose the liquidation to occur right before the coupon day.

[15] Jacoby (2003) derives a duration model for defaultable corporate bonds.

substitute the risky value of the bond at any t $(C_t + V_t)$ with the present value of the certainty equivalent amount received at $t\text{-}1$ $(PVCE_{t-1})$[16]. We can express this mathematically as:

$$PVCE_{t-1} = A_t(C_t + \mathrm{V}_t), \tag{3.1}$$

where A_t is the present value certainty equivalent factor with respect to both liquidation risk and uncertainty associated with the term structure of the interest rates.

Our risk-averse representative bondholder will be indifferent between the present value of a risky value promised by the bond subject to liquidity risk at t $(PV(C_t + \mathrm{V}_t))$, and a present value certainty equivalent amount received at $t\text{-}1$ $(PVCE_{t-1})$. This implies the following equation:

$$U(PVCE_{t-1}) = E_{t-1}U\{PV(C_t + \mathrm{V}_t)\}, \tag{3.2}$$

where E_{t-1} is the expectation operator at $t\text{-}1$ for the representative bondholder with respect to both liquidation and interest rate risks.

Assuming that the representative bondholder has a logarithmic utility function we rewrite equation (3.2) as:

[16] Note that in our setup, the portfolio manager observes a subjective probability distribution of liquidation for his/her clients. This requires us to assume that our portfolio manager is the representative bondholder. Thus, this portfolio manager's personal valuation of the bond at t will be equal to the bond's market value at t (V_t).

$$\ln(PVCE_{t-1}) = E_{t-1}\left[\ln\{PV(C_t + \mathrm{V}_t)\}\right] \tag{3.3}$$

The following relation follows from equation (3.3):

$$\ln(PVCE_{t-1}) = \ln\left\{(C_t + V_t)\exp\left(-\left[p_t s_t + \mu_{(t-1,t)}\right]\right)\right\}, \tag{3.4}$$

where, $\mu_{(v,\varphi)} = \int_0^\infty f(r_{(v,\varphi)})r_{(v,\varphi)}dr_{(v,\varphi)}$ is the time 0 expected riskless future spot rate of a perfectly liquid (zero bid-ask spread) riskless bond for a period (v,φ); $r_{(v,\varphi)}$ is the uncertain spot (perfectly liquid) riskless rate of interest for the holding period (v,φ); and $f(r_{(v,\varphi)})$ is the probability distribution function of $r_{(v,\varphi)}$.

Equations (3.1) and (3.4) yield the certainty equivalent factor of our representative bondholder as:

$$A_t = \exp\left(-\left[p_t s_t + \mu_{(t-1,t)}\right]\right) \tag{3.5}$$

Substituting this expression for the present value certainty equivalent factor into equation (3.1), we get:

$$V_{t-1} = (C_t + V_t)\exp\left(-\left[p_t s_t + \mu_{(t-1,t)}\right]\right) \tag{3.6}$$

3.3 The Bond Pricing Equation

Following Jacoby (2003), we use backward induction to derive the representative bondholder's bond pricing equation. To this end we assume that all future expectation operators are known at any point in time. One period prior to maturity, the investor will set his/her valuation of the bond equal to:

$$V_{T-1} = A_T(C_T + V_T) = (C_T + F)\exp\left(-\mu_{(T-1,T)}\right) \qquad (3.7)$$

since V_T equals to the face value of the bond at maturity[17]. The value of the bond two periods prior to maturity will be:

$$V_{T-2} = A_{T-1}(C_{T-1} + V_{T-1}) \qquad (3.8)$$

If we substitute for V_{T-1} from equation (3.7) into equation (3.8), we get:

$$\begin{aligned} V_{T-2} &= A_{T-1}\left(C_{T-1} + C_T\exp\left(-\mu_{(T-1,T)}\right) + F\exp\left(-\mu_{(T-1,T)}\right)\right) \\ &= A_{T-1}C_{T-1} + A_{T-1}C_T\exp\left(-\mu_{(T-1,T)}\right) + A_{T-1}F\exp\left(-\mu_{(T-1,T)}\right) \end{aligned}$$

The value of the bond three periods prior to maturity is respectively:

[17] Since the bond will be redeemed at its maturity, if the bondholder did not liquidate it previously, our certainty equivalent operator in equation (3.5) at maturity is only related to interest rate risks and is given by: $A_T = \exp\left(-\mu_{(T-1,T)}\right)$.

$$V_{T-3} = A_{T-2}\left(C_{T-2} + V_{T-2}\right) = A_{T-2}\left(C_{T-2} + A_{T-1}C_{T-1} + A_{T-1}(C_T + F)\exp\left(-\mu_{(T-1,T)}\right)\right)$$
$$= A_{T-2}C_{T-2} + (A_{T-2}A_{T-1})C_{T-1} + (A_{T-2}A_{T-1})(C_T + F)\exp\left(-\mu_{(T-1,T)}\right)$$

This means that the market value of the bond at time 0 is given by:

$$V_0 = \sum_{t=1}^{T-1}\left(\prod_{k=1}^{t} A_k\right)C_t + \left(\prod_{k=1}^{T-1} A_k\right)\left(C_T + F\right)\exp\left(-\mu_{(T-1,T)}\right) \tag{3.9}$$

Substituting for the certainty equivalent factor from equation (3.5) into equation (3.9), we obtain the bond pricing equation for a risk-averse bondholder having a logarithmic utility function:

$$\begin{aligned} V_0 = &\sum_{t=1}^{T-1} \exp\left(-\sum_{k=1}^{t}[p_k s_k + \mu_{(k-1,k)}]\right)C_t + \\ &+\exp\left(-\sum_{k=1}^{T-1}[p_k s_k + \mu_{(k-1,k)}]\right)\left(C_T + F\right)\exp\left(-\mu_{(T-1,T)}\right) \end{aligned} \tag{3.10}$$

In the absence of expected arbitrage opportunities, we have $\sum_{k=1}^{t}\mu_{(k-1,k)} = t\mu_{(0,t)}$. Thus, our bond pricing equation with respect to the term structure of spot interest rates (free of liquidity risk) is given by:

$$\begin{aligned} V_0 = &\sum_{t=1}^{T-1} \exp\left((-\sum_{k=1}^{t} p_k s_k)\exp(-t\mu_{(0,t)})\right)C_t + \\ &+\exp\left((-\sum_{k=1}^{T-1} p_k s_k)\exp(-(T-1)\mu_{(0,T-1)}\right)\left(C_T + F\right)\exp\left(-\mu_{(T-1,T)}\right) \end{aligned} \tag{3.11}$$

Denoting the expected yield on the bond, subject to liquidity shocks of our representative bondholder by $Y_{(t-1,t)}$ for a holding period of *(t-1,t)*, we can write the time 0 bond value as:

$$V_0 = \sum_{t=1}^{T-1}\left(\prod_{k=1}^{t} \exp\left[-Y_{(k-1,k)}\right]\right)C_t + \\ + \exp\left(\prod_{k=1}^{T-1} \exp\left[-Y_{(k-1,k)}\right]\right)\left(C_T + F\right)\exp\left(-\mu_{(T-1,T)}\right) \tag{3.12}$$

If the bondholder's valuation of equation (3.12) is identical to his/her certainty equivalent valuation of equation (3.11) at any t, then the following equality should hold:

$$\exp\left[-Y_{(t-1,t)}\right] = \exp\left(-\left\lfloor p_t s_t + \mu_{(t-1,t)}\right\rfloor\right).$$

Which we can re-express as:

$$Y_{(t-1,t)} = \left\lfloor p_t s_t + \mu_{(t-1,t)}\right\rfloor \tag{3.13}$$

Equation (3.13), represents the expected yield demanded by the bondholder for holding the bond over the time period *(t-1,t)*. It implies that the bondholder demands a liquidation risk premium equal to the expected cost of liquidation given by $p_t s_t$. Let us examine the components of equation (3.13) more closely. The second term on the right-hand side of equation (3.13) is the expected riskless yield demanded by the bondholder for holding a perfectly liquid riskless bond for the holding period of *(t-1,t)*. The first term on the right-hand side of equation (3.13), is the premium to compensate the bondholder for the expected loss from the short position taken in a put option corresponding to liquidation at *t*. This is also the premium the bondholder demands for the expected cost of the exogenous liquidation risk.

The expected yield spread between our bond and a perfectly liquid bond (i.e., the liquidation risk premium), is given by: $LP_{(t-1,t)} = Y_{(t-1,t)} - \mu_{(t-1,t)}$. From equation (3.13), the liquidation risk premium is the expected liquidation cost given by the proportional half bid-ask spread[18].

$$LP_{(t-1,t)} = \left[p_t s_t + \mu_{(t-1,t)}\right] - \mu_{(t-1,t)} = p_t s_t. \tag{3.14}$$

To find the spot term structure of the bond yield spreads, we re-write equation (3.12) using a more convenient notation:

[18] Note that when the probability of liquidation is zero, the bondholder demands no liquidation risk premium for holding the government bond, therefore $LP_{(t-1,t)}\big|_{p_t=0} = 0$. When the liquidation is certain, $(p_t = 1)$, then the risk premium that the bondholder demands is the half-spread cost: $LP_{(t-1,t)}\big|_{p_t=1} = s_t$. Or, expressed in an exponential form: $\exp\left[-LP_{(t-1,t)}\right]_{p_t=1} = \exp\left[-s_t\right]$

$$
\begin{aligned}
V_0 &= \sum_{t=1}^{T-1}\left(-\sum_{k=1}^{t} Y_{(k-1,k)}\right)C_t + \exp\left(-\sum_{k=1}^{T-1} Y_{(k-1,k)}\right)(C_T + F)\exp\left(-\mu_{(T-1,T)}\right) \\
&= \sum_{t=1}^{T-1} \exp\left(-tY_{(0,t)}\right)C_t + \exp\left(-(T-1)Y_{(0,T-1)}\right)(C_T + F)\exp\left(-\mu_{(T-1,T)}\right)
\end{aligned}
\tag{3.15}
$$

If the bondholder's certainty equivalent valuation of the bond in equation (3.11) is consistent with the valuation in equation (3.15) at any time *(t)*, then we have the following equality:

$$
\exp\left(-tY_{(0,t)}\right) = \sum_{t=1}^{T-1} \exp\left(-\sum_{k=1}^{t} p_k s_k\right)\exp\left(-t\mu_{(0,t)}\right)
$$

This implies the following expression for the government bond's t-period spot rate:

$$
Y_{(0,t)} = \mu_{(0,t)} + \frac{1}{t}\left(\sum_{k=1}^{t} p_k s_k\right) \tag{3.16}
$$

Denoting $LP_{(0,t)} = Y_{(0,t)} - \mu_{(0,t)}$ as the liquidation risk premium demanded for a *(0,t)* expected holding period, we find:

$$
LP_{(0,t)} = \frac{\sum_{k=1}^{t} LP_{(k-1,k)}}{t} = \frac{1}{t}\left(\sum_{k=1}^{t} p_k s_k\right) \tag{3.17}
$$

Therefore, the annual liquidation risk premium demanded by the bondholder for holding the government bond until t is the average premium demanded for holding the bond at every year until t.

Copeland and Galai (1983) characterize the bid-ask spread as a straddle position. This straddle is a combination of a long call option to buy at the asking price and a long put option to sell at the bid price. In our context, a long position in a perfectly liquid government bond (with zero bid-ask spread), is equivalent to a long position in a corresponding illiquid government bond plus a long position in a series of European put options with the illiquid government bond as the underlying asset. Each put option in this series corresponds to a specific possible liquidation date, conditional on the bondholder's decision to keep that bond until that time. These put options provide immediacy to the bondholder who can liquidate the illiquid bond if necessary. Since the value of the put options is positive, the value of a perfectly liquid bond must exceed the value of the illiquid bond. This result is in agreement with the existence of our liquidation premium and with empirical results presented by Amihud and Mendelson (1991) and Kamara (1994) who demonstrate that highly liquid Treasury Bills trade at a premium relative to less illiquid Treasury Notes.

The relationship between the liquidation probability and the liquidation risk premium is positive, which is not a surprise. Since the probability of liquidation is the bondholder's probability of facing a liquidity cost (the full spread), the liquidation premium will increase with higher liquidation probability as long as the bid-ask spread of the bond is positive (the bond is not perfectly liquid). Also, the relationship between the bid-ask spread and the liquidity premium is positive. This last result is supported by empirical evidence presented by Chen, Lesmond, and Wei (2002), who show that liquidity costs influence bond yield spreads.

3.5 An Elasticity Measure for Government Bonds Subject to Liquidation Risk

In this section we derive a risk-adjusted elasticity measure for government bonds subject to liquidation risk. In Appendix B we show that this liquidation risk-adjusted elasticity measure is an ex-ante immunizing elasticity. The bond value in equation (3.11) implies the following liquidation risk-adjusted elasticity measure:

$$E = \frac{\sum_{t=1}^{T-1}\left(t + \sum_{k=1}^{t}\frac{\partial p_k}{\partial \mu_{(k-1,k)}}s_k + \frac{\partial s_k}{\partial \mu_{(k-1,k)}}p_k\right)\exp\left(-\sum_{k=1}^{t} p_k s_k\right)\exp\left(-t\mu_{(0,t)}\right)C_t}{V_0} + \frac{\left(T + \sum_{k=1}^{T-1}\frac{\partial p_k}{\partial \mu_{(k-1,k)}}s_k + \frac{\partial s_k}{\partial \mu_{(k-1,k)}}p_k\right)e^{\left(-\sum_{k=1}^{T-1} p_k s_k\right)}e^{\left(-(T-1)\mu_{(0,T-1)}\right)}(C_T+F)e^{\left(-\mu_{(T-1,T)}\right)}}{V_0} \tag{3.18}$$

To derive equation (3.18) we use the standard formula for elasticity:

$$E = -\frac{1}{V_0}\left[\sum_{t=1}^{T-1}\left(\frac{\partial V_0}{\partial \mu_{(0,t)}} + \sum_{k=1}^{t}\frac{\partial V_0}{\partial \mu_{(k-1,k)}} + \frac{\partial V_0}{\partial \mu_{(T-1,T)}}\right)\right].$$

Thus, the bond's elasticity depends on the how shifts in the term structure affect the liquidation probabilities and the bond's bid-ask spread.

Substituting for the liquidation risk-adjusted yield from equation (3.19) into the equation (3.18), we get:

$$E = \frac{\sum_{t=1}^{T-1} t \exp\left\{-tY_{(0,t)}\right\} C_t + T \exp\left\{-(T-1)Y_{(0,T-1)}\right\}(C_T + F)\exp\left(-\mu_{(T-1,T)}\right)}{V_0} +$$
$$+ \frac{\sum_{t=1}^{T-1}\left(\sum_{k=1}^{t} \frac{\partial p_k}{\partial \mu_{(k-1,k)}} s_k + \frac{\partial s_k}{\partial \mu_{(k-1,k)}} p_k\right)\exp\left\{-tY_{(0,t)}\right\} C_t}{V_0} +$$
$$+ \frac{\left(\sum_{k=1}^{T-1} \frac{\partial p_k}{\partial \mu_{(k-1,k)}} s_k + \frac{\partial s_k}{\partial \mu_{(k-1,k)}} p_k\right)\exp\left\{-(T-1)Y_{(0,T-1)}\right\}(C_T + F)\exp\left(-\mu_{(T-1,T)}\right)}{V_0} =$$
$$= \left[\sum_{t=1}^{T-1} t \frac{PV(C_t)}{V_0} + T \frac{PV(C_T + F)\exp(-\mu_{(T-1,T)})}{V_0}\right] +$$
$$+ \left[\sum_{t=1}^{T-1}\left(\sum_{k=1}^{t} \frac{\partial p_k}{\partial \mu_{(k-1,k)}} s_k + \frac{\partial s_k}{\partial \mu_{(k-1,k)}} p_k\right)\frac{PV(C_t)}{V_0}\right] +$$
$$+ \left[\left(\sum_{k=1}^{T-1} \frac{\partial p_k}{\partial \mu_{(k-1,k)}} s_k + \frac{\partial s_k}{\partial \mu_{(k-1,k)}} p_k\right)\frac{PV(C_T + F)\exp\left(-\mu_{(T-1,T)}\right)}{V_0}\right] \tag{3.19}$$

The first term in the squared brackets in equation (3.19) is the elasticity of a perfectly liquid government bond, with respect to changes in the term structure of interest rates unadjusted for liquidation risk. The remaining terms in equation (3.19) represent the liquidation risk adjustment for an illiquid bond. In general, the above derivation demonstrates that the liquidation risk-adjusted elasticity is different than the unadjusted elasticity for government bonds. Provided that our assumptions are satisfied, the unadjusted elasticity will fail to immunize against shifts in the term structure of interest rates, unless the probability of a liquidity shock is zero or the bond is perfectly liquid.

For a government bond with a zero probability of liquidation, we have: $\sum_{k=1}^{t}\frac{\partial p_k}{\partial \mu_{(k-1,k)}}s_k+\frac{\partial s_k}{\partial \mu_{(k-1,k)}}p_k=0.$ From equation (3.16) we find that $Y_{(0,t)}=\mu_{(0,t)}$, and the risk-adjusted elasticity measure becomes:

$$E=\frac{\sum_{t=1}^{T-1}t\exp\left\{-t\mu_{(0,t)}\right\}C_t}{V_0}+\frac{T\exp\left\{-(T-1)\mu_{(0,T-1)}\right\}(C_T+F)\exp\left(-\mu_{(T-1,T)}\right)}{V_0} \quad (3.24)$$

This demonstrates that the unadjusted elasticity is a special case of our liquidation risk-adjusted elasticity in equation (3.19) and, therefore, will fail to immunize against shifts in the term structure.

Assuming that the term structure is flat, the value of the bondholder's bond position based on bond pricing equation (3.11) will be at the global minimum when the bondholder's investment horizon is equated to the elasticity measure in equation (3.18). (In Appendix B to this chapter we show that our liquidity-adjusted elasticity measure is an ex-ante immunizing elasticity).

3.6 Model Implications

This section examines the nature of the relationship between the liquidation risk-adjusted elasticity and the ordinary elasticity (Macaulay, 1938) for a government bond. Given that our model is correct, it is possible to find an adjustment factor that converts Macaulay's ordinary elasticity measure to our liquidation risk-adjusted elasticity. If one fails to adjust for liquidity shocks, the result is an inefficient immunization strategy. This may lead to considerable losses for bond portfolio managers, who utilize active management techniques, making our liquidation risk-adjusted elasticity of great importance to bond portfolio managers.

3.6.a The Liquidation Risk-Adjusted Elasticity vs. Macaulay's Standard Elasticity

In this section we assume a flat term structure of interest rates. The yield of a perfectly liquid riskless bond is given by r, and the yield of an illiquid riskless bond is given by y. The liquidation risk-adjusted elasticity of a government bond is given by: $E_a = -\frac{1}{V}\frac{\partial V}{\partial r}$. In Appendix C we show that:

$$E_a = E_m \frac{\partial y}{\partial r} \tag{3.21}$$

where $E_m = -\frac{1}{V}\frac{\partial V}{\partial y}$ is Macaulay's standard elasticity.

Referring back to equation (3.16), the yield to maturity of a T-year coupon bond subject to liquidation is:

$$y = r + p_T s_T \tag{3.22}$$

where p_T is the annual conditional probability of liquidation for the T-year coupon bond. Equations (3.21) and (3.22) imply:

$$E_a = E_m \left(1 + \frac{\partial p_T}{\partial r} s_T + \frac{\partial s_T}{\partial r} p_T \right) \tag{3.23}$$

Let $e_T = \left(1 + \frac{\partial p_T}{\partial r} s_T + \frac{\partial s_T}{\partial r} p_T \right)$, then the standard Macaulay elasticity measure can be adjusted for liquidation risk by simply multiplying it by e_T. Equation (3.23) demonstrates that our liquidation risk-adjusted elasticity may differ from Macaulay's standard elasticity depending on the magnitude of e_T.

3.6.b Liquidity-Adjusted Convexity

Our liquidation risk-adjusted elasticity measure suffers from the same weakness as Macaulay's standard elasticity. Both approximate the sensitivity of the bond under consideration to very small changes in the term structure of interest rates. Therefore, by neglecting larger changes in interest rates, the elasticity measure becomes substantially biased. This bias can be minimized by considering convexity. In Appendix D we show that the relationship between changes in the price of the illiquid government bond, subject to liquidation risk, and changes in the term structure of interest rates can be approximated by:

$$\frac{\Delta V}{V} = -E_a \Delta r + \frac{1}{2} C_a (\Delta r)^2 \tag{3.24}$$

where E_a is the liquidation risk-adjusted elasticity of the considered bond, $C_a = \frac{1}{V}\frac{\partial^2 V}{\partial r^2} = C_m e^2 T$ is the liquidation risk-adjusted convexity of the government bond subject to liquidation risk, and $C_m = \frac{1}{V}\frac{\partial^2 V}{\partial y^2}$ is Macaulay's standard elasticity measure.

In accordance with our derivations, the liquidation risk-adjusted convexity will differ from Macaulay's standard convexity measure. This result has two implications: first, failing to adjust for convexity of government bonds subject to liquidation will result in an inefficient approximation of government bond's sensitivity with respect to changes in the term structure of interest rates, which is implied by the elasticity measure alone. Second, that convexity analysis of government bonds with positive probability of liquidation needs to account for the risk of liquidity shocks.

3.7 Summary and Conclusions

This chapter develops a valuation model for a government bond subject to liquidation risk. This model provides expressions for the term structures of yields and liquidation risk premiums. Particularly, our model shows that the value of an illiquid government bond subject to liquidation depends on the evolution of the following parameters through time: the (conditional) probability of liquidation, the riskless term structure, and the proportional bid-ask spread.

In our model, the bondholder demands a liquidation risk premium for an illiquid government bond. Everything else being equal, the liquidation risk premium increases with both the probability of liquidation and the bond's bid-ask spread.

Deriving the liquidation risk-adjusted elasticity of the illiquid government bond, we find that using Macaulay's standard elasticity measure may lead to an inefficient immunization strategy on the part of bond portfolio managers. We provide a formula for liquidity-adjusted elasticity and prove that it is an ex-ante immunizing elasticity. Our liquidity-adjusted elasticity is the sum of two components: the first component is the standard elasticity of a perfectly liquid government bond, and the second is a liquidation risk adjustment. Finally, we also provide a liquidation risk-adjusted formula for the convexity of an illiquid government bond. Our results demonstrate that the assessment and valuation of bonds must consider dimensions other than interest-rate risk. More specifically, one must consider variables proxying for the liquidity shock risk for government bonds facing a typical large bond portfolio manager.

Appendix A. The Derivation of Equation (3.4) from Equation (3.3).

Equation (3.3) for a representative investor with a logarithmic utility function is:

$$\ln(PVCE_{t-1}) = E_{t-1}\left[\ln\left(PV\left(C_t + V_t\right)\right)\right] \tag{3.3}$$

Taking into account the probabilities for the representative investor, with respect to the liquidation shocks, the above equation can also be expressed as:

$$\begin{aligned} \ln(PVCE_{t-1}) = (1-p_t)E_{t-1}\left[\ln\left((C_t + V_t)\exp\left\{-r_{(t-1,t)}\right\}\right)\right] + \\ p_t E_{t-1}\left[\ln\left((C_t + V_t)\exp\left\{-\left(s_t + r_{(t-1,t)}\right)\right\}\right)\right] \end{aligned} \tag{A.1}$$

where E_{t-1} is the expectation operator of a representative investor with regard to the relevant uncertainty. Interpreting equation (A.1), we see that the first expectation operator defines the expected utility of the present value of the risky cash flow received at t if the investor does not liquidate the bond. The second expectation operator of (A.1) defines the expected utility of the present value of the risky cash flow received at t if the investor does liquidate the bond at t. The cash flow received if the investor liquidates the bond will be equal to the spread-adjusted cash flow received otherwise.

We can also re-write (A.1) as:

$$\ln\left(PVCE_{t-1}\right)=(1-p_t)E_{t-1}\left[\ln\left(C_t+V_t\right)-r_{(t-1,t)}\right]+ \\ p_t E_{t-1}\left[\ln\left(C_t+V_t\right)-s_t-r_{(t-1,t)}\right] \tag{A.2}$$

Let $f(r_{(v,\varphi)})$ denote the probability distribution function of the uncertain spot risk-free interest rates, beginning at time *(v)* and ending at time *(φ)* (thus, $\varphi \geq v$). If we incorporate the probability distribution of spot risk-free rates into equation (A.2) and integrate resulting expression with respect to the risk-free rate (r), then we can re-write (A.2) as:

$$\ln\left(PVCE_{t-1}\right)=(1-p_t)\int_0^{\infty} f(r_{(t-1,t)})\left[\ln\left(C_t+V_t\right)-r_{(t-1,t)}\right]dr_{(t-1,t)}+ \\ p_t\int_0^{\infty} f(r_{(t-1,t)})\left[\ln\left(C_t+V_t\right)-s_t-r_{(t-1,t)}\right]dr_{(t-1,t)} \tag{A.3}$$

Now, let us denote $\mu_{(v,\varphi)}=\int_0^{\infty} f(r_{(v,\varphi)})r_{(v,\varphi)}dr_{(v,\varphi)}$ as the expected future spot risk-free rate, and since $\int_0^{\infty} f(r_{(v,\varphi)})dr_{(v,\varphi)}=1$, (A.3) becomes:

$$\ln\left(PVCE_{t-1}\right)=(1-p_t)\left[\ln\left(C_t+V_t\right)-\mu_{(t-1,t)}\right]+p_t\left[\ln\left(C_t+V_t\right)-s_t-\mu_{(t-1,t)}\right] \text{ or,}$$

$$\ln\left(PVCE_{t-1}\right)=(1-p_t)\ln\left\{\left(C_t+V_t\right)\exp\left(-\left[p_t\left\{s_t+\mu_{(t-1,t)}\right\}+(1-p_t)\mu_{(t-1,t)}\right]\right)\right\} \\ =\ln\left\{\left(C_t+V_t\right)\exp\left(-\left[p_t s_t+\mu_{(t-1,t)}\right]\right)\right\} \tag{3.4}$$

Appendix B. Proof that the Liquidity-Adjusted Elasticity is an Ex-ante Immunizing Elasticity

We illustrate that our liquidation risk-adjusted elasticity measure is immunizing under the assumption of a flat term structure. Suppose that (r) denotes the yield to maturity on a riskless bond, corresponding to the underlying government bond subject to liquidation risk. Thus equation (3.11) becomes:

$$V_0 = \sum_{t=1}^{T-1} \exp\left((-r[t]) - \sum_{k=1}^{t} p_k s_k \right) C_t + \exp\left((-r[T]) - \sum_{k=1}^{T-1} p_k s_k \right) (C_T + F) \qquad \text{(B.1)}$$

When we suppose that the bondholder has an investment horizon of (H) periods and that the riskless yield to maturity changes to ($r*$) right after purchase, the bondholder's certainty equivalent value of her/his position at the end of the investment horizon is:

$$\begin{aligned} V_H(r^*) &= V_0(r^*)e^{Hr*} \\ &= \sum_{t=1}^{T-1} \exp\left(\left(Hr^* - r^*[t]\right) - \sum_{k=1}^{t} p_k s_k \right) C_t + \qquad \text{(B.2)} \\ &\quad + \exp\left(\left(Hr^* - r^*[T]\right) - \sum_{k=1}^{T-1} p_k s_k \right) (C_T + F) \end{aligned}$$

We now find the optimal investment horizon such that if the riskless yield will change, the bondholder's position is at least the projected amount before any changes in the yield to maturity. We then find a global minimum for $V_H(r^*)$ with respect to r^* in order to find the optimal horizon. When the first derivative of $V_H(r^*)$ with respect to r^*, $V'_H(r^*)$, is zero, and the second derivative, $V''_H(r^*)$, is positive, then $V_H(r^*)$ is convex and at the global minimum. Calculating the first derivative, we find:

$$\begin{aligned} V'_H(r^*) = \sum_{t=1}^{T-1}\left(H - t - \left\{ \sum_{k=1}^{t} \frac{\partial p_k}{\partial r^*} s_k + \frac{\partial s_k}{\partial r^*} p_k \right\}\right) e^{\left(\left(Hr^* - r^*[t]\right) - \sum_{k=1}^{t} p_k s_k\right)} C_t \; + \\ + \left(H - T - \left\{ \sum_{k=1}^{T-1} \frac{\partial p_k}{\partial r^*} s_k + \frac{\partial s_k}{\partial r^*} p_k \right\}\right) e^{\left(\left(Hr^* - r^*[T]\right) - \sum_{k=1}^{T-1} p_k s_k\right)} (C_T + F) \end{aligned} \tag{B.3}$$

The second derivative of equation (B.2) with respect to (r^*) is given by:

$$\begin{aligned} V''_H(r^*) = \sum_{t=1}^{T-1}\left(H - t - \left\{ \sum_{k=1}^{t} \frac{\partial p_k}{\partial r^*} s_k + \frac{\partial s_k}{\partial r^*} p_k \right\}\right)^2 e^{\left(\left(Hr^* - r^*[t]\right) - \sum_{k=1}^{t} p_k s_k\right)} C_t \; + \\ + \left(H - T - \left\{ \sum_{k=1}^{T-1} \frac{\partial p_k}{\partial r^*} s_k + \frac{\partial s_k}{\partial r^*} p_k \right\}\right)^2 e^{\left(\left(Hr^* - r^*[T]\right) - \sum_{k=1}^{T-1} p_k s_k\right)} (C_T + F) \end{aligned} \tag{B.4}$$

It is obvious that the second derivative is always strictly positive for the considered bond. Even though it is possible for $H = t + \left\{ \sum_{k=1}^{t} \frac{\partial p_k}{\partial r^*} s_k + \frac{\partial s_k}{\partial r^*} p_k \right\}$, we expect that the squared terms in equation (B.4) are positive most of the time. Equation (B.2) is, thus, strictly convex and a global minimum. Now we find the equation for the bondholder's optimal investment horizon at this minimum (H^*) by equating (B.3) to zero and solving for H^*:

$$H^* = \frac{\sum_{t=1}^{T-1}\left[t+\left\{\sum_{k=1}^{t}\frac{\partial p_k}{\partial r^*}s_k+\frac{\partial s_k}{\partial r^*}p_k\right\}\right]\exp\left(\left(-r^*[t]\right)-\Sigma_{k=1}^{t}p_k s_k\right)C_t}{V_0(r^*)}+ \\ +\frac{\left[T+\left\{\sum_{k=1}^{T-1}\frac{\partial p_k}{\partial r^*}s_k+\frac{\partial s_k}{\partial r^*}p_k\right\}\right]\exp\left(\left(-r^*[T]\right)-\Sigma_{k=1}^{T-1}p_k s_k\right)(C_T+F)}{V_0(r^*)} \tag{B.5}$$

This expression is exactly our liquidation risk-adjusted elasticity (E) in equation (3.19), evaluated at (r^*). If the bondholder sets his investment horizon (H) equal to (E), which is the global minimum of $V_E(r^*)$ with respect to r^*, then the bondholder's position will not devalue as a result of changes in term structure. The derivations of this Appendix, thus, show that the liquidation risk-adjusted elasticity measure developed in Section 3.5 is an ex-ante immunizing elasticity.

Appendix C. The Relationship between our Liquidity-Adjusted Elasticity and Macaulay's Elasticity

When we let *f(.)* be the function that relates the price of the government bond subject to liquidation to its yield, and to other factors (for a review of factors - see Section 3.5), the bond's standard elasticity, where the yield is the reference rate, is as follows:

$$E_m = -\frac{1}{V}\frac{\partial f}{\partial y} \tag{C.1}$$

However our derivations above show that:

$$\frac{\partial V}{\partial r} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \tag{C.2}$$

If we multiply both sides of the previous equation by $\left(-\frac{1}{V}\right)$, we find:

$$-\frac{1}{V}\frac{\partial V}{\partial r} = -\frac{1}{V}\frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \tag{C.3}$$

Thus, equations (C.3) and (C.1) in conjunction demonstrate that:

$$E_a = E_m \frac{\partial y}{\partial r} \tag{C.4}$$

Thus, our liquidation risk-adjusted elasticity is a multiple of Macaulay's elasticity and an additional term, adjusting for liquidation risk.

Appendix D. The Government Bond's Liquidity-Adjusted Convexity

Utilizing Taylor series expansion, we can express the change in our bond's price as:

$$\Delta V = \frac{\partial V}{\partial r}\Delta r + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}(\Delta r)^2 \tag{D.1}$$

Using $\left(\frac{1}{V}\right)$ as the transformation and applying it to the equation above, we find:

$$\frac{\Delta V}{V} = \left(\frac{1}{V}\frac{\partial V}{\partial r}\right)\Delta r + \frac{1}{2}\left(\frac{1}{V}\frac{\partial^2 V}{\partial r^2}\right)(\Delta r)^2 \tag{D.2}$$

Defining elasticity as $E_a = -\frac{1}{V}\frac{\partial V}{\partial r}$ and liquidation risk-adjusted convexity with respect to riskless rate as $C_a = \frac{1}{V}\frac{\partial^2 V}{\partial r^2}$, and substituting these expressions into equation (D.2), we then find: $\frac{\Delta V}{V} = -E_a \Delta r + \frac{1}{2}C_a(\Delta r)^2$. Since the bond's convexity with respect to its own yield equals $C_m = \frac{1}{V}\frac{\partial^2 V}{\partial y^2}$, the following equality emerges:

$$C_a = \frac{1}{V}\frac{\partial^2 V}{\partial r^2} = \left(\frac{1}{V}\frac{\partial^2 V}{\partial y^2}\right)\left(\frac{\partial y}{\partial r}\right)^2 \tag{D.3}$$

CHAPTER 4

The Effective Elasticity of Callable Corporate Bonds

4.1 Introduction

The fixed-income literature is rich with theoretical models for the valuation of corporate bonds. This is especially the case for structural models. These models are based on an asset-based default process, where default is triggered once the stochastic value of the firm's assets hits a default barrier (see for example: Merton, 1974; Black and Cox, 1976; Longstaff and Schwartz, 1995). Many structural models following that of Merton (1974) model a bond with a safety covenant that allows default to take place prior to the bond maturity. Default is triggered once the value of the firm's assets hits the value of some stipulated deterministic or stochastic threshold default level (e.g., Black and Cox, 1976; Longstaff and Schwartz, 1995; and Briys and Varenne, 1997). Other structural models, such as that of Kim, Ramaswamy, and Sundaresan (1993), allow for a cash flow based default process instead.[19]

However, most corporate bonds issued in the U.S. are callable. Berndt (2004) reports that as of April 2003, 60 percent of the corporate bonds in her U.S. bond database are callable. Similarly, Jacoby and Liao (2004) report that close to 56 percent of their sample of Canadian corporate bonds for December 1999 are callable. This implies that majority of corporate bonds are subject to both default and call risks.

[19] Longstaff and Schwartz (1995) allow for both asset and cash flow based default processes.

A bondholder of a callable corporate bond is facing a positive probability that, in the event of default or call, promised cash flows will be altered both in terms of size and timing. This has strong implications not only on the pricing of these bonds, but also on their elasticity (duration).

Despite the popularity of callable bonds, theoretical work in this area is scarce. Dunetz and Mahoney (1988) and Brooks and Attinger (1992) use option-pricing theory to price and model the duration for callable bonds. However, both models focus on pricing default-free bonds. Kihn (1994) considers both default and call risks. He notes that a callable and defaultable bond is equivalent to a portfolio consisting of a long position in a host riskless and non-callable bond, a short position in a call option written on the bond issue, and a short position in a limited liability (put) option written on the firm's assets. Kihn derives an elasticity measure, but he fails to provide an analytical expression for duration.

Kihn's work demonstrates that, when considered separately, both default and call risks shorten duration. Focusing on low-grade bonds, he claims that one should consider the interaction between the two types of risks. This is because heightened default risk results in depressed bond prices, influencing the moneyness of the call provision and lowering the risk that the bond will be called. Thus, this interaction will increase the duration of the bond.

Building on the implications of Kihn's work and developments in structural modeling, Acharya and Carpenter (2002) derive an analytical expression for the value and duration of a callable corporate bond. They pay careful attention to modelling the dynamic interaction between the two sources of risk. To this end, Acharya and

Carpenter endogenize the call and default decisions. To the best of our knowledge, this is the first theoretical paper that addresses the importance of the interaction between the corporate default and call risks.

Acharya and Carpenter (2002) model the defaultable and callable bond as a portfolio with a long position in a default-free, noncallable host bond and a short position in a call option written on this host bond. Since the exercise price of the call option is the provisional call price, and that of the default option is the value of the firm's assets, the exercise price of the option to call or default is the minimum of the two exercise prices.

Focusing on the interaction between call and default, they note that the probability of default for a callable and defaultable bond is lower than that associated with a noncallable corporate bond. Similarly, the probability of a call for a callable corporate bond is lower than that of a riskless callable bond. Intuitively, this is because the existence of one option delays the exercise of the other. For example, a higher default risk provides an incentive to the issuing company to wait longer before calling the bond, resulting in a lower probability of a call.

This has important implications to both bond pricing (represented by yield spread components) and duration. For yield spreads, the call spread on a riskless callable bond has to be higher than that of a defaultable and callable bond because of the higher call probability associated with the former. Furthermore, the credit spread of a noncallable and defaultable bond will be higher than that of a callable corporate bond due to the lower default probability associated with the latter. For duration, when considered separately, both default and call shorten duration. However, the duration of a risk-free callable bond has to be shorter than that of a callable corporate bond due to the lower call probability associated with the latter. In addition, the duration of a noncallable corporate bond will be shorter than that of a callable corporate bond because of the lower probability of default associated with the latter.

Sarkar and Hong (2004) also derive a structural model for pricing a fixed-price callable and defaultable bond. They focus on the pricing of a perpetual bond. In their model, the call risk reduces the default risk of a callable bond by reducing the value of the default trigger. Similar to Acharya and Carpenter's (2002) model, this implies that the decision of the firm to declare bankruptcy is delayed and the duration of the callable defaultable bond becomes longer than that of a noncallable bond. They present empirical support for a longer duration of fixed-price callable bonds as compared with noncallable bonds for speculative-grade bonds.

Acharya and Carpenter (2002) and Jacoby and Roberts (2003) show that the yield spread – riskless rate relation mirrors the relation between the bond's default- and call-adjusted duration and its Macaulay duration. This is because duration is a measure of interest-rate elasticity for the bond price, and at the same time the spread – rate relation is a measure of interest-rate elasticity of the spread.

Structural models for noncallable corporate bonds predict that credit spreads are negatively related to the riskless rate (see Longstaff and Schwartz, 1995; Merton, 1974). Ceteris paribus, under risk-neutral valuation, an increase in the riskless rate implies a higher drift for the value of the firm's assets relative to the default threshold, a lower risk-neutral default probability, and a lower credit spread.[20] This is also true

[20] This prediction of structural models inspired empirical research aiming at testing it (e.g., Avramov et al., 2004; Collin-Dufresne et al., 2001; Longstaff and and Schwartz, 1995; Duffee, 1998).

in the Acharya and Carpenter's model. However, their model also considers callability. Ceteris paribus, they show that an increase in the riskless rate will result in a lower yield spread due to the call option.

This result is in agreement with Jacoby and Roberts (2003), who argue that when interest rates are higher the firm will find it less profitable to call the bond. Jacoby and Roberts assume no interaction between default and call risks. Recall that in Acharya and Carpenter (2002) the existence of one option reduces the probability of exercise for the other. This means that, ceteris paribus, following an increase in the riskless rate, the risk-neutral probability of exercising each of these options separately increases. However, this probability increase for one option will ultimately result in a lower probability for the other due to the interaction between the options, and in a lower yield spread. Thus, the interaction term may offset the negative spread – rate relation resulting from the default and call options when considered separately.

In recent years U.S. and Canadian firms issue bonds with a new type of a call option, called the make-whole call provision in the U.S. and the Canada call provision in Canada. Unlike the old standard call provision, for which the call price is fixed, the make-whole call has a call price that rises if Government of Canada rates drop. Under the make-whole call provision the call price is set as the maximum of par value or the present value of the bond's remaining cash flows calculated using the yield of a riskless bond (with equal maturity) plus a preset premium, called the make-whole premium.

Mann and Powers (2003) report that the first bond with a make-whole call provision was issued in the U.S. in 1995, and most newly issued callable bonds carry this type of option. Kaplan (1998) and Jacoby and Roberts (2003) document that vast majority of investment-grade callable corporate bonds in Canada carry this provision since 1987. Jacoby and Liao (2004) show that the proportion of make-whole call bonds in a sample of Canadian corporate bonds sharply increased from close to 23 percent in January 1993 to almost 48 percent in December 1999. At the same time the proportion of bonds carrying the standard call has rapidly decreased.

The increased predominance of make-whole bonds within the universe of callable corporate bonds in North America, called for a new theory for pricing these bonds. Powers and Tsyplakov (2004) step up to the task with a structural model for pricing bonds carrying a make-whole call option. However, they do not provide the expressions for duration and do not conduct analysis similar to that of Acharya and Carpenter (2002).

Jacoby and Liao (2004) show that the make-whole call option is in-the-money when the bond yield spread is lower than the make-whole premium, which is pre-specified in the bond indenture. Thus, changes in the riskless rate will influence the value of the make-whole option through their impact on the bond yield spread. This means that the dynamics of the interest-rate elasticity of yield spread is still important for bonds carrying this option, including the interaction between the default and credit spreads.

In this chapter we test the main implications of the Acharya and Carpenter (2002) and Sarkar and Hong (2004) theoretical models. In particular, we test the interest-rate elasticity of the call spread and that of the default spread, allowing for interaction between both spreads. We also examine the impact of both risks and their interaction on the effective duration of corporate bonds. In our tests we examine both bonds carrying a standard call option and those carrying a make-whole option. We find evidence supporting the predictions of the two models.

The remainder of this chapter is organized as follows. In Section 4.2 we discuss the data and the methodology used to estimate the components of the yield spread. We discuss the impact of default and call risks, and their interaction, on duration in Section 4.3. In Section 4.4 we test the predictions of Acharya and Carpenter (2002) and Sarkar and Hong (2004) models and discus the results. Section 4.5 concludes this chapter.

4.2 The Data

4.2.a Description of the Data

In this study we use two databases. We retrieve corporate bond data from the Jacoby and Roberts Canadian Corporate Bond (JRCCB) Database. We also use data on government bonds from the Jacoby and Roberts Canadian Government Bond (JRCGB) Database. Both databases consist of month-end data. Monthly yields and prices in the JRCCB and the JRCGB databases are taken from the Financial Post Bond Prices. The JRCCB database also reports data from the Financial Post Corporate Bond Record. Among other features, this includes coupon rate, date of issuance, maturity, and bond optionality features, such as call, put, convertibility, exchangeability, extendibility, and sinking fund.

For callable bonds, the JRCCB database reports specific call feature, such as an identifier for a fixed-price vs. make-whole call provision. For bonds with a fixed-price call option, data on the call-protection period and call price is also provided. For bonds carrying a make-whole call provision, the make-whole premium is reported. The JRCCB database also reports monthly Dominion Bond Rating Service (DBRS) ratings. We limit our sample to Canadian-dollar denominated bonds issued by Canadian companies. We eliminate floating-rate bonds and bonds that contain any option other than a call option.[21]

Except for monthly Government of Canada bond yield and price data, the JRCGB database provides details on coupon rate, date of issuance, and maturity. All bonds in our sample pay coupons semiannually. Our sample consists of monthly bond data covering the January 1991 to December 2000 10-year period.

The spreads calculated in this study are based on zero curves estimated using the Nelson and Siegel (1987) procedure. Unfortunately we find that, except for A-rated bonds, there is not sufficient data for corporate bonds of other ratings to yield reliable estimates of the zero curves. This is reflected in the low number of bonds of a given rating within each month, as well as in the tight maturity spectrum covered by these bonds. Thus, we limit our sample to A-rated bonds.

Also, in agreement with the findings of Jacoby and Liao (2004), we find that the number of fixed-price (make-whole) callable A-rated bonds steadily decreases (increases) throughout the sample period. While there is sufficient number of noncallable and make-whole callable A-rated bonds for the entire sample period (120

[21] All bonds with sinking funds, purchase funds, retractability and convertibility provisions are excluded from the analysis.

monthly observations), there are enough fixed-price callable bonds only up to December 1997 (84 monthly observations). This problem does not exist for Government of Canada bonds. Thus, for comparative purposes the analysis in this chapter focuses on the January 1991 to December 1997 period. For noncallable and make-whole callable bonds we also provide the results for the January 1991 to December 2000 period.

4.2.b Estimating the Components of the Yield spread

Jacoby and Liao (2004) indicate that Canadian bond data have several important advantages over U.S. bond data. Unlike U.S. data, data from the JRCCB database allow controlling for factors that may bias the estimation of the yield spread and its components. One of these factors is the tax differential between Treasury and corporate bonds in the U.S. Elton, Gruber, Agrawal, and Mann (2001) stress that U.S. corporate bonds are taxed at the federal and state levels, whereas Treasury bonds are taxed only at the federal level. They find that a large proportion of corporate bond yield spreads is attributed to this tax differential. Unlike U.S. data, government and corporate bonds in Canada are taxed at the same rate.

Another issue related to taxation is the tax-timing option that is valuable in the U.S. bond market (see Jordan and Jordan, 1991). Prisman, Roberts, and Tian (1996) show that the value of the tax-timing option is trivial in the Canadian bond market. Thus, the ability to control for the tax-timing option and for the differential tax treatment gives us a unique advantage in empirical research over that conducted with U.S. data.

As previously mentioned, we estimate zero curves for A-rated corporate and government bonds using the Nelson and Siegel (1987) methodology (see Appendix A). This procedure is computationally more efficient relative to that offered by Svensson (1994), yet flexible enough in its ability to generate different shapes of the term structure, including an increasing, decreasing, humped-shaped, S-shaped, and a U-shaped term structure.[22]

Since our analysis is based on the estimated zero curves we are also able to control for the coupon bias described by Duffee (1998). Since corporate bonds tend to have higher coupons as compared to government bonds, they have lower interest-rate elasticity. This induces a negative relationship between yield spreads and government yield of the same maturity but with different coupon rates. Thus, the ability to control for the above-mentioned potential biases allows us to gain clearer understanding of the yield spread elasticity, and the elasticity of its components.

Callable and noncallable spot rates for government and A-rated corporate bonds are derived from the estimated zero curves using noncallable and callable (but devoid of other options) bonds for every month over the January 1991 to December 2000 period. Specifically, we derive zero-coupon yields for samples of Government of Canada, and different types of corporate bonds. These include noncallable, fixed-

[22] In their paper on U.S. corporate bonds, Elton et al. (2001) apply both the McCulloch cubic spline method and the Nelson and Siegel procedure and find very similar results throughout their analysis. See Green and Odegaard (1997) and Dahlquist and Svensson (1996) for further comparison of the Nelson and Siegel procedure with other methodologies.

price callable, and make-whole callable corporate bonds. The estimated term structure of zero-coupon government interest rates is presented in Figure 4.1.

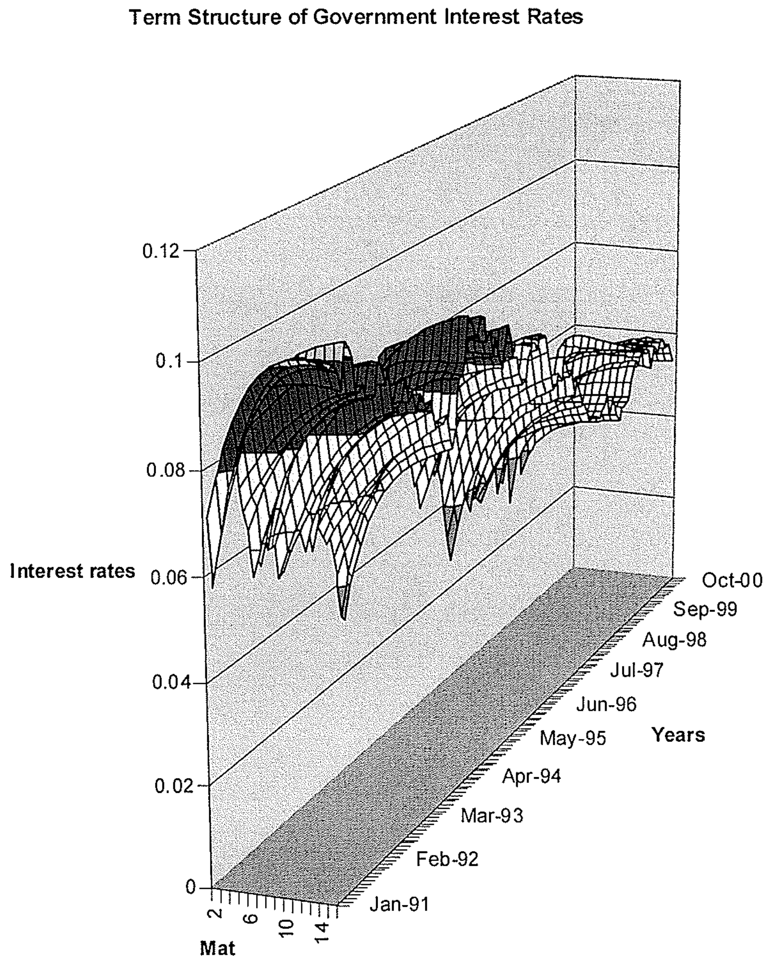

Figure 4.1: The Term Structure of Government Interest Rates

The monthly term structure of Government of Canada rates is estimated using the Nelson and Siegel model. The dataset covers the period from January, 1991 to December, 2000.

We estimate the default spread for A-rated bonds as the difference between the zero-coupon yield of a noncallable corporate bond and the zero-coupon yield of a government bond with the same term to maturity. The fixed-price call spread is calculated as the difference between zero-coupon yields on fixed-price callable and noncallable bonds with equal time to maturity. Similarly, the make-whole call spread is represented by the difference in zero-coupon yields on make-whole callable and noncallable bonds with equal time to maturity.

Table 4.1 provides descriptive statistics for the estimated default spreads, fixed-price and make-whole call spreads over the sample period from January 1991 to December 1997. We find the following empirical patterns for our default spread data: (1) at the mean, the term structure of default spreads for A-rated bonds is humped shaped; (2) default spread volatility for A-rated bonds monotonically decreases as a function of maturity.

Structural models predict that credit spreads rise with higher leverage. The higher the leverage, the put option is more likely to be in-the-money and the probability of default is higher. This means a higher credit spread. Structural models also predict that the credit spread declines with an increasing value of the firm's assets. We control for the leverage effect by stratifying our data into rating groups. Firms that are highly rated by the rating agency are expected to have lower leverage ratios, whereas speculative-grade issuers are expected to have significantly higher leverage ratios. The Canadian market for junk bonds is virtually nonexistent.[23] Effectively, Canadian A-rated bonds are medium-rated bonds. Thus, the first

[23] For example, examining the transition of corporate bonds across DBRS rating categories for the period extending from January 1990 to December 2000, we find that bonds with rating of BB or lower account for approximately 2.29% of all rated corporate bonds in the database.

empirical pattern observed for default spreads is in agreement with structural models (see Lee, 1981; Briys and Varenne, 1997) that predict a humped-shaped term structure for medium-rated bonds of medium-leveraged firms.

Table 4.1

Descriptive Statistics for the Estimated Default Spreads (Premiums), Fixed-Price Call Spreads (Premiums) and Make-Whole Call Spreads (Premiums).

This table presents descriptive statistics for default spreads (premiums), estimated monthly as the difference in the zero-coupon yield rates applicable to a noncallable bond and that of a government bond with equal time to maturity across the period from January 1991 to December 1997 (In Panel B of this table we report results across the entire period from January 1991 to December 2000). The first column reports average values of derived government spot rates. The second, and third columns report mean and volatility values of default spreads for the A-rated rating group. This table also presents descriptive statistics for fixed-price and make-whole call spreads (premiums), estimated monthly as the difference in the zero-coupon yield rates applicable to a fixed-price or make-whole callable bond and that of a noncallable bond with equal time to maturity. The fourth, and fifth columns report mean and volatility values of standard call spreads for the A-rated rating group. Mean and volatility values of make-whole call spreads are reported in columns six, and seven respectively. In Panel B, we report descriptive statistics for default spreads (premiums) and make-whole spreads (premiums) across the entire sample period from January 1991 to December 2000.

Maturity	Treasuries	Credit Spreads		Standard Call Spreads		Make-Whole Spreads	
		Mean	Volatility	Mean	Volatility	Mean	Volatility
			Panel A: 1991-1997 (N=84)				
3	5.68064	0.47893	0.87087	2.06242	1.71577	0.44801	1.52376
4	6.05782	0.69377	0.73262	1.47830	1.29779	0.43499	1.52376
5	6.34803	0.79451	0.65615	1.04719	1.18243	0.33759	1.22334
6	6.57878	0.83297	0.59514	0.73646	1.10056	0.24188	1.00040
7	6.76555	0.83833	0.53756	0.50690	1.03217	0.16396	0.83514
8	6.91824	0.82712	0.48187	0.33826	0.97669	0.10388	0.70962
9	7.04371	0.80879	0.42965	0.21605	0.93300	0.05821	0.61285
10	7.14695	0.78884	0.38332	0.12275	0.90044	0.02333	0.53827
11	7.23175	0.77039	0.34544	0.06380	0.87181	-0.03748	0.48178
12	7.30106	0.75519	0.31831	0.02750	0.84721	-0.02529	0.44055
13	7.35721	0.74413	0.30347	0.00950	0.82560	-0.04295	0.41225
14	7.40211	0.73759	0.30114	0.00643	0.80709	-0.05790	0.39473
15	7.43732	0.73562	0.31000	0.01568	0.79284	-0.07097	0.38589

	Default Spreads		Make-Whole Spreads	
Maturity	Mean	Volatility	Mean	Volatility
Panel B: 1991-2000 (N=120)				
3	0.48913	0.75178	0.51241	2.20944
4	0.67339	0.63382	0.49227	1.92227
5	0.76844	0.56916	0.40819	1.66582
6	0.81391	0.51703	0.32541	1.46288
7	0.83232	0.46867	0.25590	1.29717
8	0.83650	0.42396	0.19974	1.15684
9	0.83388	0.38484	0.15456	1.03523
10	0.82883	0.35347	0.11780	0.92856
11	0.82393	0.35347	0.08735	0.83465
12	0.82066	0.33149	0.06186	0.75209
13	0.81982	0.31965	0.03934	0.67997
14	0.82184	0.31763	0.01972	0.61762
15	0.82685	0.32418	0.00210	0.56456

Panel A of Table 4.1 also provides descriptive statistics for the estimated fixed-price and make-whole call spreads over the sample period from January 1991 to December 1997. We observe the following empirical patterns: (1) in general, fixed-price and make-whole mean call spreads for A-rated firms monotonically decrease with higher maturity; (2) on average, the volatility of fixed-price and make-whole call spreads is at least twice as great as the volatility levels observed for default spreads; (3) call spread volatility monotonically decreases as a function of maturity for both types of call provisions; (4) the magnitude of standard-call spreads is more than double in size as compared with the magnitude of make-whole spreads.[24]

The finding of a downward-sloping term structure for fixed-price call spreads is intuitive and consistent with findings of King (2000). Fixed-price call spreads tend to be lower within the call-protected period relative to higher call spreads within the period with no call protection. During the call-protected period the issuing company

[24] The patterns for the January 1991 to December 2000 period are similar (see Panel B of Table 4.1).

cannot exercise the call. This period normally extends to at least five years after issuance. Our samples of fixed-price callable bonds include a higher proportion of call-protected bonds. The proportion of call-protected A-rated bonds with the attached fixed-price call provision is depicted in Figure 4.2. On average, approximately 62% of all considered bonds, within the A-rated category and for the entire sample period, are call-protected.[25]

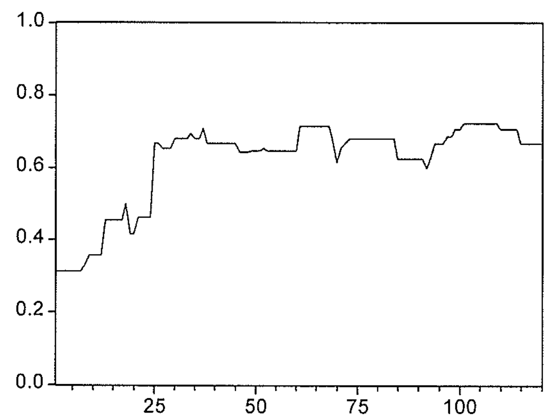

Figure 4.2: Estimated Proportion of Call-Protected Fixed-Price Callable Bonds

In Table 4.2 we report the average and the standard deviation of the number of call-protected and not call-protected bonds within the fixed-price callable corporate bond universe for different maturity groups: short-term, medium-term, and long-term.

[25] The JRCCB database reports whether a bond is call protected as a binary variable (0 if not protected and 1 if protected). To proxy the proportion of callable bonds in the call-protected period, we calculate the average of this binary variable for each month in our sample period.

Table 4.2

Call-protection of Fixed-Price Callable Corporate Bonds

This Table reports the average and the standard deviation of the number of call-protected and not call-protected bonds within the fixed-price callable corporate bond universe for different maturity groups: short-term (bonds with remaining term to maturity of lower than 5 years), medium-term (bonds with remaining term to maturity of between 5 and 10 years), and long-term (bonds with remaining term to maturity of greater than 10 years).

Sample: 01/1990 - 12/1997 (N=84)				
	Not Call-Protected		*Call-Protected*	
Maturity Groups	*Number*	*St. Dev*	*Number*	*St. Dev*
Short-term (< 5 years)	1.6310	2.0929	0.4286	0.4978
Medium-term (5-10 years)	4.4167	1.5770	5.8810	3.7041
Long-term (> 10 years)	2.7381	1.6363	8.2619	2.3903

The results reported in Table 4.2 show that shorter-maturity fixed-price callable bonds are more often past their call-protection period. On average, the number of call-protected bonds is higher than the number of unprotected bonds within the medium-term maturity group. The data also support the fact that longer-maturity bonds are mostly call-protected. This reinforces our claim that longer-maturity bonds, which tend to be call-protected, will have lower call spreads than shorter bonds without the call protection.

The downward-sloping term structure of make-whole spreads is also consistent with the literature. Powers and Tsyplakov (2004) show that, at issuance, a make-whole option is set to be out-of-the-money. As they get closer to maturity, these bonds get deeper in-the-money. Thus, the make-whole call option is unlikely to be in-the-money for bonds with long maturities, but is often in-the-money for medium maturity bonds. This explains the downward-sloping shape of the term structure of make-whole spreads.

The results indicate that default spreads dominate call spreads in magnitude.

However, call spreads are more volatile relative to default spreads. Surprisingly, we also find that, on average, make-whole call spread values for the A-rated debt are negative (but close to zero) for maturities between 11 to 15 years for the January 1991 to December 1997 period.[26] We can offer two explanations for this finding. First of all, Mann and Powers (2003) argue that primary benefits of attaching the make-whole provision to corporate bonds may not be profit-oriented. They suggest, based on results of the conducted survey of chief executive officers, that firms may attach the make-whole provision because they want to enjoy the flexibility of deciding to retire 100% of the entire debt issue in order to avoid costs associated with expensive tender offers.

Jacoby and Stangeland (2004) and Powers and Tsyplakov (2004) suggest that this attachment also allows the management of the firm to avoid costs associated with restructuring or elimination of the problematic covenant. For the bondholder, this could mean that a call will result in a financial advantage rather than a financial damage. This could ultimately result in a negative make-whole spread.

A second explanation for the negative make-whole spreads is related to liquidity. We proxy for the marketability of the different types of debt with the market value of outstanding debt. The marketability of A-rated make-whole bonds is significantly greater than of noncallable bonds of the same rating. This is depicted in Figure 4.3.

[26] For the January 1991 to December 2000 period, mean spreads are always positive for these maturities, but also close to zero (see Panel B of Table 4.1).

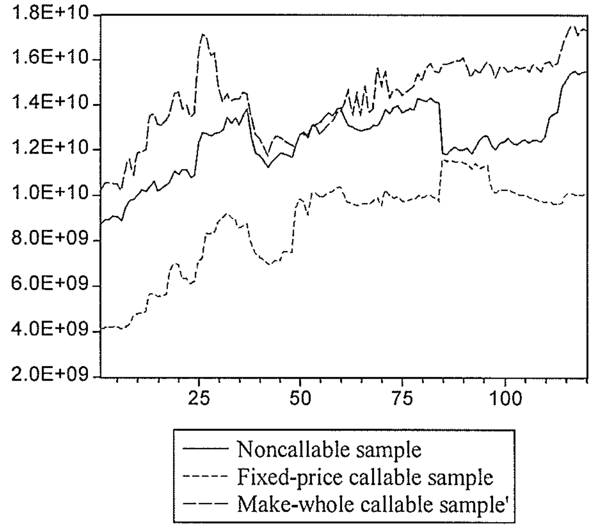

Figure 4.3: The Marketability of

A-rated Corporate Debt

The higher liquidity of make-whole bonds versus that of noncallable bonds may account for the negative make-whole call values. Our definition of spreads ignores issues related to liquidity. However, an illiquidity premium is a component of the yield spread. Since make-whole bonds are more marketable relative to noncallable bonds, one may expect their illiquidity premium to be lower. Recall that we construct make-whole call spread by taking the difference between zero-coupon yields on make-whole callable and noncallable bonds with equal time to maturity. Thus, by construction, our make-whole call spread includes an illiquidity-premium differential, given by the difference between the illiquidity premium associated with make-whole bonds and that associated with noncallable bonds. Given our estimates for marketability, it is reasonable that this liquidity premium is higher for the less liquid noncallable bonds relative to that of make-whole bonds. This implies that the above illiquidity-premium differential, which is a component of our make-whole spread, is negative. This could cause the negative make-whole spread for the longer end of the term structure.

4.3 Duration Measure Under Default and Call Risks

The issue of duration has been occupying the minds of academic researchers for years. This is due to the fact that this concept allows practitioners to immunize bond portfolios against the risk of unfavourable shifts in interest rates and to successfully speculate on the direction of interest-rate movements. For the immunization to be effective, one has to use the duration measure properly adjusted for default and call risks (see Jacoby and Roberts, 2003). The theoretical work of Acharya and Carpenter (2002) and Sarkar and Hong (2004) stresses the importance of modeling the call risk by paying attention to its relationship not only to changes in the riskless term structure, but also to changes in the firm's default risk.

In this section, following Jacoby and Roberts (2003), we use the elasticity definition of duration to consider the influence of default and call risks on duration, when these risks are considered separately or jointly (i.e., the interaction between the two risks is taken into account). In contrast to their analysis, which assumes that the interaction between default and call risks is trivial, we offer an analysis that not only separates the affect of changing interest rates on default risk from that on a call risk, but also considers the affect of the default and call risks together. Thus, our analysis reflects the predictions of the work of Acharya and Carpenter (2002) and Sarkar and Hong (2004) properly.

Jacoby and Roberts (2003) show that the ratio of the default- and call-adjusted duration to its Macaulay counterpart is equal to the sensitivity of yield on the callable corporate bond (Y^c) with respect to changes in the continuously compounded riskless rate (r): $\frac{\partial Y^c}{\partial r}$. They examine this derivative by considering the impact of the riskless rate on the probability of default and the probability of a call, but their work does not consider the interaction between them.

We alter the Jacoby and Roberts (2003) framework, and define the yield on a callable pure-discount corporate bond (Y^C) as the sum of: the yield on a corresponding pure-discount government bond (r); the default (credit) spread (*DS*); and the call spread (*CS*). Following Acharya and Carpenter (2002) and Sarkar and Hong (2004), the default spread is a function of the riskless rate and the call spread (among other parameters): $DS=DS\ (CS,r)$. Similarly, the call spread is a function of the riskless rate and the default spread (among other parameters): $CS=CS\ (DS,r)$. Thus, the yield on this bond is given by:

$$Y^c = r + DS(CS,r) + CS(DS,r)\,. \tag{4.1}$$

We examine the relationship between Macaulay duration (D_m) and the default- and call-adjusted duration (D_a) for our sample of A-rated Canadian corporate bonds. Recall that Jacoby and Roberts (2003) show that $\frac{D_a}{D_m} = \frac{\partial Y^c}{\partial r}$. We adjust their analysis taking into consideration the interaction between default and call risks. Thus, given our definition of the yield in equation (4.1), the above ratio is given by:

$$\frac{D_a}{D_m} = \frac{\partial Y^c}{\partial r} = 1 + \frac{\partial DS}{\partial r} + \frac{\partial CS}{\partial r} + \frac{\partial DS}{\partial CS}\frac{\partial CS}{\partial r} + \frac{\partial CS}{\partial DS}\frac{\partial DS}{\partial r} . \tag{4.2}$$

Comparing equation (4.2) to the relationship derived by Jacoby and Roberts, we can see that we now have two extra terms. These are the last two terms on the right-hand side of equation (4.2). To simplify the analysis, we now take a closer look at the signs of the four terms on the right-hand side of the above equation. Proponents of the structural approach to bond pricing assert that the sign of $\frac{\partial DS}{\partial r}$ is negative. Under risk neutrality, asset returns are expected to grow at the riskless rate. Thus a higher riskless rate implies a higher expected future value for the firm's assets, and a lower probability of default. This ultimately means a lower default spread. This negative relation is also in line with the recent findings of empirical studies in the area (see Avramov, Jostova, and Philipov, 2004; Collin-Dufresne, Goldstein, and Martin, 2001; Duffee, 1998; and Longstaff and Schwartz, 1995).

Jacoby and Roberts (2003) argue that the sign of $\frac{\partial CS}{\partial r}$ for fixed-price callable bonds is always negative. This is because a higher riskless rate implies a lower probability of a call, and, therefore, results in a lower call spread. For make-whole bonds, Jacoby and Roberts argue that the effect of changing interest rates is neutralized by the yield on the make-whole bond floating in the same direction as the Government of Canada rate. In contrast, we argue that the sign of the relationship between the make-whole call spread and the riskless rate is also negative. First, lower interest rates set by the central bank may lead to favourable borrowing terms and better opportunities for the typical firm, resulting in ameliorated credit rating for the firm and a higher probability of a call (see Jacoby and Liao, 2004; and Mann and Powers, 2003). Second, since the activity of the central bank ensures that the riskless rate is adjusted once the economic-growth indicators or inflation growth change, higher Government of Canada interest rates may indicate higher expectations for growth and higher expected recovery rates. This may realign the conflict of interests between bondholders and stockholders and eliminate the need to call back the debt in pursuit of higher returns on investments. Thus, we can expect a negative relationship between make-whole call spread and the riskless rate.

Previous empirical research has failed to consider the interaction between the default and call risks in estimating duration. This interaction is reflected by the higher-order cross terms in equation (4.2): $\frac{\partial CS}{\partial DS}\frac{\partial DS}{\partial r}$ and $\frac{\partial DS}{\partial CS}\frac{\partial CS}{\partial r}$. We argue that these interaction terms may not be trivial. Following Acharya and Carpenter (2002) and Sarkar and Hong (2004), we expect the sign of the relationship between default and call spreads to be negative for fixed-price bonds. The same prediction holds for make-whole bonds: a higher probability of default implies a higher default spread and a higher yield spread. Jacoby and Liao (2004) show that the make-whole call option is out-of-the-money when the bond yield spread is higher than the pre-specified make-whole premium. Thus, a higher yield spread will result in a lower probability of a call and a lower call spread. Hence, the negative default spread – call spread relation for make-whole bonds.

Taking into consideration our expectations with respect to the signs of the partial derivatives discussed above, we can re-write the relationship between the risk-adjusted duration of a default-prone callable bond and its Macaulay duration as follows:

$$\frac{D_a}{D_m} = 1 + \underset{(-)}{\frac{\partial DS}{\partial r}} + \underset{(-)}{\frac{\partial CS}{\partial r}} + \underbrace{\underset{(-)}{\frac{\partial DS}{\partial CS}} \underset{(-)}{\frac{\partial CS}{\partial r}}}_{(+)} + \underbrace{\underset{(-)}{\frac{\partial CS}{\partial DS}} \underset{(-)}{\frac{\partial DS}{\partial r}}}_{(+)} . \tag{4.3}$$

Equation (4.3) implies that the risk-adjusted duration of default-prone callable bonds can be either shorter or longer than the unadjusted Macaulay duration. The relative magnitude of the effective duration depends on whether the direct effects of a shift in a riskless rate on default and call spreads dominate or subordinate the indirect effects. Equation (4.3) also agrees analytically with the proposition of Acharya and Carpenter (2002) that the duration of a default-prone bond increases with a higher risk of a call and that of a callable bond increases with a higher risk of default. When $\left|\frac{\partial DS}{\partial r} + \frac{\partial CS}{\partial r}\right| = \left|\frac{\partial DS}{\partial CS}\frac{\partial CS}{\partial r} + \frac{\partial CS}{\partial DS}\frac{\partial DS}{\partial r}\right|$, or when all partials are trivial, the duration of a default-prone callable bond is equal to that of a riskless bond.

Our approach to the study of the empirical and economic significance of the error introduced by considering the Macaulay duration for callable defaultable bonds is different from previous work. First, we separate the impact of default from that of a call on duration using estimated spot curves to obtain curves for call spreads and for default spreads. Second, in our analysis we consider the impact of the interaction between the default and call risks on the risk-adjusted duration. Third, our analysis covers two types of callable bonds: fixed-price and make-whole callable bonds. In the next section we estimate the sign of the interaction terms in equation (4.3) and their significance. We then proceed to empirically estimate the ratio of the risk-adjusted duration to its unadjusted counterpart.

4.4 Test Methodology and Regression Results

4.4.a Estimating the Signs of the Interaction Terms

We are interested in simultaneously testing the response of the default and call spreads to shifts in the riskless rate and in finding whether the interaction between default and call risks is economically and statistically significant. We estimate two regressions, using heteroscedasticity and autocorrelation consistent covariances of Newey and West (1987). The first regression's dependent variable is default spread, while the second regression's dependent variable is call spread.[27]

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[27] We estimate OLS regressions due to the fact that three-stage least squares (3SLS) procedure is not applicable. Instrumental variables that are highly correlated with default (call) terms but uncorrelated with error terms cannot be found. This means that we cannot account for prospective correlation between error terms and regressors by introducing an appropriate instrument (at least one excluded exogenous variable) for each endogenous regressor. We find below, however, that there is little evidence of bicausality, and hence endogeneity bias, between default and call spreads using Granger causality tests.

First, we conduct Granger causality tests in a vector autoregression framework to make sure that there is no bilateral causality running from the default spread to the call spread and vice versa. To this end, we select the optimal lag length for our vector autoregressions using the Schwartz Information Criterion (SIC), which is best suited for sample sizes of less than 120 observations.[28] We find that it is optimal to include up to 2 lags of each variable depending on maturity. Then we run Granger causality tests to test if past information is important to predict contemporaneous default (call) spread changes. Results of these tests for fixed-price (make-whole) callable bonds are reported in Panel A (B) of Table 4.3.

[28] In majority of cases the optimal lag length selected by the Akaike and Hannan-Quinn Information criteria coincides with that selected by the SIC.

Table 4.3

Results of Granger Causality Tests based on VAR analysis.

$$\begin{cases} \Delta DS_{it} = \gamma_i + \sum\limits_{v=1}^{n} \delta_{v,i} \Delta CS_{i[t-v]} + \sum\limits_{v=1}^{n} \varpi_{v,i} \Delta DS_{i[t-v]} + \varepsilon_{it} \\ \Delta CS_{it} = \chi_i + \sum\limits_{v=1}^{n} \beta_{v,i} \Delta CS_{i[t-v]} + \sum\limits_{v=1}^{n} \psi_{v,i} \Delta DS_{i[t-v]} + \xi_{it} \end{cases}$$

where ΔDS_{it} and ΔCS_{it} represent changes in default premiums, and call premiums, respectively. Vector autoregressions are estimated for the sample period t, t = 01:1991-12:1997 (N=84) and maturity group i, i= 3, 4,…, 15. $\delta_{v,i}$ and $\beta_{v,i}$ are coefficients associated with lagged changes in call spread changes, and $\varpi_{v,i}$ and $\psi_{v,i}$ are coefficients associated with lagged changes in default spreads, γ_i and χ_i are intercepts, whereas ε_{it} and ξ_{it} represent error terms. The optimal lag length v is selected using the Schwartz Information criterion (SIC), which is best suited for sample sizes of less than 120 observations. Most of the times the optimal lag length selected by Akaike and Hannan-Quinn Information criteria coincided with that selected by considering the SIC. Results of Granger Causality tests show that call spread changes do not cause default spread changes for both types of call provision. However, default spread changes do cause fixed-price call spread changes at medium maturities.

Panel A: Credit Spread Changes and Fixed-Price Spread Changes

Null	ΔCS doesn't cause ΔDS			ΔDS doesn't cause ΔCS		
	Lags	*F-statisitc*	*Decision*	*F-statisitc*	*Decision*	*N*
2	1	0.00310	Accept	0.10976	Accept	84
3	1	0.19741	Accept	0.59941	Accept	84
4	1	0.00151	Accept	2.96622	Accept	84
5	1	0.24482	Accept	3.9508**	Reject	84
6	1	0.43927	Accept	4.72189**	Reject	84
7	1	0.41494	Accept	5.35288**	Reject	84
8	1	0.23385	Accept	5.62766**	Reject	84
9	1	0.05750	Accept	5.39046**	Reject	84
10	2	1.47208	Accept	3.88319**	Reject	84
11	2	2.09774	Accept	3.23718**	Reject	84
12	2	2.53869	Accept	2.29454	Accept	84
13	2	2.61255	Accept	1.36179	Accept	84
14	2	2.36940	Accept	0.68817	Accept	84
15	2	2.02800	Accept	0.32486	Accept	84

Panel B: Credit Spread Changes and Make-Whole Spread Changes

Null	ΔCS doesn't cause ΔDS			ΔDS doesn't cause ΔCS		
	Lags	*F-statisitc*	*Decision*	*F-statisitc*	*Decision*	*N*
3	1	0.09281	Accept	0.21813	Accept	84
4	1	0.40659	Accept	0.60897	Accept	84
5	1	1.34006	Accept	0.76538	Accept	84
6	1	1.97559	Accept	0.69384	Accept	84
7	1	2.25304	Accept	0.53799	Accept	84
8	1	2.27458	Accept	0.38577	Accept	84
9	1	2.11273	Accept	0.26411	Accept	84
10	1	1.79934	Accept	0.17269	Accept	84
11	1	1.35960	Accept	0.10321	Accept	84
12	2	2.71633	Accept	0.45711	Accept	84
13	2	2.15614	Accept	0.76909	Accept	84
14	2	1.65001	Accept	1.14166	Accept	84
15	2	1.27895	Accept	1.55723	Accept	84

In none of our results do we find evidence that default spread changes and make-whole call spread changes are mutually causal (or bicausal). Results of these tests for fixed-price callable defaultable bonds differ, depending on maturity. Evidence of unidirectional causality from default spread changes to fixed-price call spread changes is present for maturities from 5 to 11 years at the 95% confidence level. However, default spread changes and fixed-price call spread changes exhibit independence at all other maturities. Because our cross terms are not both endogenous, we have more confidence in using OLS regression analysis to test relationships we are interested in.

We estimate the following OLS regression models for A-rated bonds for the maturity group i, $i = 3, 4, \ldots, 15$ years, and the sample period t, t = 01:1991-12:1997 (N=84):

$$\Delta DS_{it} = \gamma_i + \delta_i \Delta r_t + \varpi_i \Delta CS_{it} + \varepsilon_{it} \quad (4.4)$$

$$\Delta CS_{it} = \nu_i + \beta_i \Delta r_t + \psi_i \Delta DS_{it} + \xi_{it} \quad (4.5)$$

where

ΔDS_{it} and ΔCS_{it} = monthly changes in default and call spreads, respectively;

Δr_t = monthly changes in the spot Government of Canada yield corresponding to the maturity of default- and call premiums;

δ_i and β_i = coefficients associated with changes in interest rates, which capture direct effects of shifts in the riskless rate on default and call spreads, respectively;

ϖ_i and ψ_i = coefficients associated with changes in the interaction terms, which capture indirect effects of shifts in the riskless rate on corporate default and call spreads;

γ_i and ν_i = intercepts; and

ε_{it} and ξ_{it} = regression error terms.

The results for regression models (4.4) and (4.5) include a first order autoregressive term, which is always significant. In addition to the estimates of the regression parameters and the adjusted R^2, Table 4.4 reports the results of the Wald χ^2 tests, which tests the restriction that either ϖ_i or ψ_i, which are coefficients associated with the interaction terms ($\frac{\partial DS}{\partial CS}$ and $\frac{\partial CS}{\partial DS}$), is equal to zero, and the associated probability for this statistic.

Table 4.4

A Test of the Significance of the Government of

Canada Interest Rate and the Interaction Term for Default- and

Call-Spread Changes of A-rated Defaultable Callable Bonds.

$$\Delta DS_{it} = \gamma_i + \delta_i \Delta r_t + \varpi_i \Delta CS_{it} + \varepsilon_{it}$$
$$\Delta CS_{it} = \nu_i + \beta_i \Delta r_t + \psi_i \Delta DS_{it} + \xi_{it}$$

where ΔDS_{it} and ΔCS_{it} represent changes in default premiums, and call premiums, respectively. OLS regressions are estimated for each maturity i, $i = 3, 4, \ldots, 15$, and sample period t, t = 01:1991-12:1997 (N=84). Δr_t are changes in the zero-coupon Government of Canada yields corresponding to the maturity of default- or call premiums, δ_i and β_i are coefficients associated with changes in interest rates, which capture the direct effect of shifts in the riskless rate on corporate default-prone callable yields through the effects on default and call risks, ϖ_i and ψ_i are coefficients associated with changes in cross-terms, which capture the indirect effect of shifts in the riskless rate on corporate default-prone callable yields through the interaction between default and call risks, γ_i and ν_i are intercepts, whereas ε_{it} and ξ_{it} represent error terms. Statistical significance at the .01, .05, and .1 levels are denoted by ***, **, and *, respectively. Heteroscedasticity and autocorrelation consistent covariances of Newey and West (1987) are used. Each regression includes a first order autoregressive term, which is significant. In addition to the estimates of the regression parameters and the adjusted R^2, the table reports the following:

DW reports the Durbin-Watson statistic for each reported regression model,
Wald χ^2 tests the restriction that either ϖ_i or ψ_i is zero,
Probability gives an estimate of the associated probability level of the Wald test statistic, and
N is the number of monthly observations.

Table 4.4: Continued

	Panel A: Default Spread Changes								Panel B: Fixed-Price Spread Changes						
	Intercept	Δr	ΔCS	Adj R^2	DW	Wald χ^2	Prob		Intercept	Δr	ΔCS	Adj R^2	DW	Wald χ^2	Prob
3	-0.02899	-0.27372**	-0.17609***	0.32929	1.91598	18.821	0.000	3	-0.02349	-0.40393	-1.06366***	0.35867	1.96955	17.316	0.000
4	-0.02327	-0.32378***	-0.25141***	0.42054	2.01539	33.158	0.000	4	-0.01655	-0.51296**	-1.21930***	0.38647	2.10089	34.791	0.000
5	-0.02120	-0.33909***	-0.21164***	0.38012	2.03338	23.268	0.000	5	-0.01040	-0.56476**	-1.13339***	0.32503	2.12657	25.217	0.000
6	-0.02005	-0.33870***	-0.18185***	0.34612	2.04276	16.781	0.000	6	-0.00988	-0.62952**	-1.03339***	0.27886	2.14346	18.718	0.000
7	-0.01906	-0.32555***	-0.15596***	0.31637	2.05467	12.264	0.001	7	-0.01223	-0.68565**	-0.94620***	0.24225	2.15849	14.326	0.000
8	-0.01821	-0.30663***	-0.13001***	0.28822	2.07345	8.812	0.004	8	-0.01515	-0.71835***	-0.87305***	0.20870	2.16728	10.971	0.001
9	-0.01763	-0.28937***	-0.10555***	0.26366	2.09952	6.268	0.014	9	-0.01712	-0.71850***	-0.81385***	0.17647	2.16582	8.400	0.005
10	-0.01754	-0.27880***	-0.08156**	0.24168	2.13362	4.129	0.046	10	-0.01767	-0.69290***	-0.75212***	0.13806	2.17233	6.267	0.014
11	-0.01801	-0.27997***	-0.06623**	0.23172	2.16003	3.029	0.086	11	-0.01656	-0.63754**	-0.71327***	0.10658	2.15994	5.007	0.028
12	-0.01904	-0.29147***	-0.05755**	0.23263	2.17374	2.477	0.120	12	-0.01441	-0.56811**	-0.68605**	0.07894	2.14089	4.286	0.042
13	-0.02048	-0.31004***	-0.05495*	0.24346	2.17128	2.442	0.132	13	-0.01184	-0.49603**	-0.66759**	0.05705	2.11648	4.006	0.049
14	-0.02214	-0.33154***	-0.05759*	0.25986	2.15564	2.442	0.122	14	-0.00941	-0.42997*	-0.65477**	0.04262	2.08804	4.107	0.046
15	-0.02384	-0.35252***	-0.06464**	0.27633	2.13280	2.783	0.099	15	-0.00752	-0.37486*	-0.64604**	0.03674	2.05821	4.556	0.036

Table 4.4: Continued

	Panel A: Default Spread Changes								Panel B: Make-Whole Spread Changes						
	Intercept	Δr	ΔCS	Adj R^2	DW	*Wald* χ^2	*Prob*		Intercept	Δr	ΔDS	Adj R^2	DW	*Wald* χ^2	*Prob*
3	-0.03156	-0.23399**	-0.11400***	0.25943	1.935	9.222	0.003	3	-0.04587	-0.2546178	-0.90081***	0.166441	2.176	8.804	0.004
4	-0.02717	-0.23426**	-0.14845***	0.30387	1.981	13.566	0.000	4	-0.03092	-0.0656737	-1.00088***	0.263407	2.116	13.571	0.000
5	-0.02410	-0.23346**	-0.18948***	0.34714	1.984	18.149	0.000	5	-0.02246	-0.0427968	-0.99929***	0.330018	2.065	17.995	0.000
6	-0.02132	-0.22365**	-0.22539***	0.38043	1.981	22.342	0.000	6	-0.01804	-0.0610764	-0.98738***	0.374759	2.035	21.866	0.000
7	-0.01907	-0.21397**	-0.25145***	0.40180	1.986	25.608	0.000	7	-0.01602	-0.0912797	-0.97819***	0.403878	2.025	24.790	0.000
8	-0.01764	-0.21084**	-0.26637***	0.41172	2.000	27.516	0.000	8	-0.01555	-0.1260903	-0.96941***	0.419656	2.029	26.405	0.000
9	-0.01711	-0.21655**	-0.27101***	0.41172	2.021	27.890	0.000	9	-0.01613	-0.1631972	-0.95735***	0.423225	2.041	26.611	0.000
10	-0.01743	-0.23070***	-0.26777***	0.40462	2.045	26.898	0.000	10	-0.01741	-0.2009457	-0.93898***	0.415942	2.055	25.635	0.000
11	-0.01844	-0.25164***	-0.26033***	0.39498	2.064	25.037	0.000	11	-0.01908	-0.23745*	-0.91272***	0.400091	2.068	23.981	0.000
12	-0.01998	-0.27730***	-0.25339***	0.38859	2.071	23.014	0.000	12	-0.02088	-0.27097**	-0.87966***	0.379315	2.074	22.298	0.000
13	-0.02187	-0.30573***	-0.25214***	0.39031	2.060	21.582	0.000	13	-0.02267	-0.30080**	-0.84454***	0.358649	2.070	21.233	0.000
14	-0.02399	-0.33536***	-0.26122***	0.40164	2.033	21.378	0.000	14	-0.02444	-0.32754***	-0.81437***	0.343784	2.056	21.304	0.000
15	-0.02624*	-0.36525***	-0.28320***	0.42099	1.995	22.809	0.000	15	-0.02621	-0.35244***	-0.79457***	0.339645	2.033	22.847	0.000

For fixed-price bonds (Panel A), we find that the sign of the estimator for δ_i (estimating $\frac{\partial DS}{\partial r}$) is always significantly negative. The finding of a negative relationship between the default spread and interest rates is not new. Xie, Liu, and Wu (2002) also find that the default-adjusted duration is shorter than the Macaulay duration because they estimate the sign of $\frac{\partial DS}{\partial r}$ to be negative.

The sign of the estimator for β_i is also significantly negative for all maturities, except for the 3-year term. This finding is consistent with our expectations. The interest-rate sensitivity of changes in fixed-price call spreads reaches its peak at medium maturities and decreases after that. This result is intuitive. For short maturities, the likelihood of exercising the fixed-price call option is low since these bonds are closer to maturity. Fixed-price callable bonds of long maturities are mostly call protected. Thus, the call spread changes for short and long maturities are relatively inelastic to changes in interest rates. Fixed-price callable bonds of medium maturities are more likely to be called and are therefore more sensitive in spread to shifts in the riskless rate. For bonds carrying a make-whole option (Panel B), we find that the relationship between changes in make-whole call spreads and changes in interest rates is always negative. This negative relationship is statistically insignificant at short and medium maturities, but statistically significant at maturities greater than 10 years. This is consistent with the assertion that the likelihood of the make-whole call option being exercised increases with a longer maturity. This is because during a longer time horizon (term to maturity) there is a greater potential for growth of the considered company. This means a higher probability of a call, and a higher interest-rate sensitivity.

As we predicted, we find that the signs of the interaction terms, $\frac{\partial DS}{\partial CS}$ and $\frac{\partial CS}{\partial DS}$, for which ϖ_i and ψ_i are proxies, are all negative and statistically significant.[29] For each regression model we test the single-equation restriction of no relationship between default and fixed-price (make-whole) call spreads. We find that the hypothesis of zero coefficients associated with the interaction terms between changes in default and call spreads can be rejected with the 99 (at least 87) percent confidence level for A-rated fixed-price callable bonds with maturities from 3 to 9 (10 to 15) years. For A-rated make-whole bonds, we find that the interaction terms are highly statistically significant for all maturities.

The results for the interaction terms provide strong support for the theoretical models of Acharya and Carpenter (2002) and Sarkar and Hong (2004). These models predict that the sign of the relationship between default and call spreads to be negative for fixed-price bonds. This is because as the probability of default increases (default spread increases), calling the bond is less likely (call spread decreases), and vice versa.

4.4.b The Impact of Default and Call Risks on Duration

Similar to Fons (1990) and Jacoby and Roberts (2003), we now directly estimate the ratio between the risk-adjusted duration for callable corporate bonds and its Macaulay counterpart. We are able to directly estimate the impact of $\frac{\partial DS}{\partial r}$ on estimating duration using spot noncallable corporate bond yields. Direct estimation of the ratio of the default-adjusted or the default- and call-adjusted duration to the riskless duration implies the following relationship: $\frac{D_a}{D_m} = \frac{\partial Y^c}{\partial r}$. Thus, to estimate the significance of the default adjustment we need to

[29] Please note that the estimates for the interaction terms provide us with the same information irrespectively of whether we are examining $\frac{\partial DS}{\partial CS}$, or $\frac{\partial CS}{\partial DS}$.

regress changes in the spot noncallable corporate yield on changes in spot Government of Canada matching yield. To estimate the importance of both the default and the call adjustments we need to regress changes in the spot callable corporate yield on changes in spot Government of Canada matching yield.

Thus, we estimate the following model for A-rated bonds for the maturity $i = 3, 4, \ldots,$ 15 years and the sample period t, t = 01:1991-12:1997 (N=84), and 01:1991-12:2000 (N=120):

$$\Delta y_{it}^{k} = \gamma_i + \delta_i \Delta r_t + \varepsilon_{it}, \tag{4.6}$$

where

Δy_{it}^{k} = monthly changes in the spot corporate yield of type k, k = noncallable, callable;

Δr_t = monthly changes in the spot Government of Canada yield corresponding to the maturity of the corporate yield;

δ_i = the slope coefficient that measures the ratio between the risk-adjusted duration and its Macaulay counterpart ($\frac{D_a}{D_m}$);

γ_i = intercept; and

ε_{it}, = error term.

The residuals from equation (4.6) follow a first order autoregressive process, and therefore a first-order error term is included in every regression, which turns out to be significant. The above regression model is estimated using ordinary least squares. The one-tailed t-statistic of the hypothesis that the slope coefficient, that measures the ratio between the risk-adjusted duration to its Macaulay counterpart, is lower than one is reported. We also report the Wald χ^2 test statistic that tests the hypothesis that the same slope coefficient is equal to unity and its associated probability. The results of the two tests agree.

Since we have a sample of noncallable corporate bonds, we can distinguish between the adjustment required for the efficient immunization for noncallable and callable bonds. The elasticity between yield spreads and interest rates for noncallable bonds depends only on the default-spread term, which is expected to be negative: $\frac{D_a}{D_m} = 1 + \underset{(-)}{\frac{\partial DS}{\partial r}}$.

We discuss results of applying regression model (4.6) for noncallable corporate spot yields first. These results are given in Panel A of Table 4.5.[30] Consistent with our expectations, the default-adjusted duration is shorter than Macaulay duration for A-rated bonds across the entire range of maturities. The one-tailed *t*-statistic of the hypothesis that the slope coefficient is lower than one is statistically significant at the 1 percent level. This finding confirms the prediction of the model of Acharya and Carpenter (2002) that default risk by itself shortens duration. It is also in agreement with the empirical results of Jacoby and Roberts (2003) for Canadian corporate bond indices. The real issue is what happens to duration when both risks of calling and defaulting are present. We look at these results next.

[30] Table 4.4 also reports the estimation results for the January 1991 to December 2000 period when the data is available (Panel A for noncallable and Panel C for make-whole callable bonds). Qualitatively, these results are the same as those of the January 1991 to December 1997 period. For the sake of comparison with the results for the fixed-price callable bonds, in our discussion we focus on the latter sample period.

Table 4.5

Estimation of the Relationship Between the Default- and Call-Adjusted Duration and its Corresponding Macaulay Duration.

We estimate the following regression model:

$$\Delta y_{it}^{k} = \gamma_i + \delta_i \Delta r_t + \varepsilon_{it},$$

where Δy_{it}^{k} are monthly changes in the spot corporate yield of type k, where k stands for noncallable or callable bonds, maturity group i and the sample period t; i includes maturities 3, 4, ..., 15, whereas t can be represented by the following sample periods: 01:1991-12:1997 (N=84), and 01:1991-12:2000 (N=120); Δr_t are changes in the zero-coupon Government of Canada yield corresponding to the maturity of the noncallable or callable yield Δy_{it}^{k} with the rating of i, δ_i is the slope coefficient which represents the relationship between the default- and call-adjusted duration of series i and its Macaulay duration. γ_i is the intercept, and ε_i is the error term. Statistical significance at the .01, .05, and .1 levels are denoted by ***, **, and *, respectively. Heteroscedasticity and autocorrelation consistent covariances of Newey and West (1987) are used. Each regression includes a first order autoregressive term, which is significant. In addition to the estimates of the regression parameters and the adjusted R^2, the table reports the following:
DW reports the Durbin-Watson statistic,
F-statistic of the regression tests that all coefficients are simultaneously equal to zero (except the intercept),
SE is the standard error of the slope estimator, and
$t(\delta<1)$ is the t-statistic used to test the null hypothesis that the slope coefficient is smaller or greater than unity. The hypothesis that is tested is stated in the next column,
Wald χ^2 tests the restriction that the slope coefficient equals to unity, and
N is the number of monthly observations.

$\Delta y_{it}^{noncallable} = \gamma_i + \delta_i \, \Delta r_t + \varepsilon_i$

Panel A: Noncallable yields

Regression Model: $\Delta y_{it}^{noncallable} = \gamma_i + \delta_i \Delta r_t + \varepsilon_i$

	Intercept	δ_i	*Adj* R^2	*DW*	*F-stat*	*SE*	*t(d<1)*	*Wald* χ^2	*Probability*	*N*
3	-0.02977	0.76541***	0.32515	1.91915	20.51330	0.12700	1.84712***	3.41195	0.06847	84
4	-0.02659	0.73550***	0.28248	2.01280	16.94437	0.12901	2.05020***	4.20344	0.04366	84
5	-0.02434	0.72424***	0.26747	2.05875	15.78779	0.12201	2.26011***	5.10818	0.02657	84
6	-0.02197	0.73418***	0.27716	2.09095	16.52907	0.11477	2.31609***	5.36408	0.02315	84
7	-0.01974	0.75131***	0.29961	2.12038	18.32531	0.10902	2.28121***	5.20412	0.02523	84
8	-0.01805	0.76508***	0.32777	2.14941	20.74709	0.10247	2.29264***	5.25605	0.02453	84
9	-0.01712	0.77055***	0.35801	2.17739	23.58519	0.09392	2.44296***	5.96816	0.01680	84
10	-0.01700	0.76641***	0.38774	2.20130	26.64871	0.08395	2.78247***	7.74222	0.00674	84
11	-0.01761	0.75339***	0.41398	2.21606	29.60989	0.07399	3.33286***	11.10835	0.00131	84
12	-0.01881	0.73329***	0.43296	2.21643	31.92370	0.06571	4.05897***	16.47492	0.00012	84
13	-0.02042	0.70864***	0.44093	2.20069	32.94231	0.06048	4.81747***	23.20798	0.00001	84
14	-0.02227*	0.68220***	0.43582	2.17285	32.28581	0.05871	5.41287***	29.29823	0.00000	84
15	-0.02421*	0.65618***	0.41871	2.14017	30.17296	0.05962	5.76672***	33.25547	0.00000	84
	Intercept	δ_i	*Adj* R^2	*DW*	*F-stat*	*SE*	*t(d<1)*	*Wald* χ^2	*Probability*	*N*
3	-0.01485	0.69482***	0.31930	1.92300	28.44067	0.1189015	2.56666***	6.58773	0.01030	120
4	-0.01434	0.69391***	0.29105	2.00340	25.01690	0.1102948	2.77520***	7.70170	0.00550	120
5	-0.01311	0.69770***	0.28093	2.04448	23.85524	0.1006834	3.00253***	9.01516	0.00270	120
6	-0.01122	0.71475***	0.29129	2.07512	25.04430	0.0947506	3.01056***	9.06350	0.00260	120
7	-0.00917	0.73468***	0.31292	2.10399	27.64242	0.0919507	2.88541***	8.32561	0.00390	120
8	-0.00743	0.74892***	0.33954	2.13196	31.07463	0.0884253	2.83935***	8.06190	0.00450	120
9	-0.00620	0.75367***	0.36767	2.15727	35.01428	0.0823147	2.99249***	8.95498	0.00280	120
10	-0.00556	0.74848***	0.39476	2.17651	39.15593	0.0740539	3.39649***	11.53615	0.00070	120
11	-0.00545	0.73468***	0.41811	2.18540	43.03473	0.0650688	4.07753***	16.62626	0.00000	120
12	-0.00577	0.7145***	0.43447	2.18083	45.94271	0.0570105	5.00780***	25.07796	0.00000	120
13	-0.00641	0.69067***	0.44058	2.16342	47.07317	0.0513427	6.02480***	36.29822	0.00000	120
14	-0.00723	0.66598***	0.43457	2.13781	45.96189	0.0486883	6.86040***	47.06503	0.00000	120
15	-0.00814	0.64247***	0.41732	2.10973	42.89735	0.0485167	7.36910***	54.30417	0.00000	120

Panel B: Make-Whole Yields										
Regression Model: $\Delta y_{it}^{callable} = \gamma_i + \delta_i \Delta r_t + \varepsilon_i$										
	Intercept	*δι*	*Adj R^2*	*DW*	*F-stat*	*SE*	*t(d<1)*	*Wald χ^2*	*Probability*	*N*
3	-0.04875884	0.72360**	0.11426	2.17632	6.22451	0.37229	0.74245	0.55123	0.46002	84
4	-0.03089846	0.93456***	0.18887	2.11639	10.43034	0.34091	0.19196	0.03685	0.84826	84
5	-0.02247748	0.95701***	0.23041	2.06446	13.12574	0.29841	0.14407	0.02076	0.88581	84
6	-0.01831784	0.93560***	0.25615	2.03318	14.94638	0.25911	0.24853	0.06177	0.80437	84
7	-0.01644987	0.90341***	0.27531	2.02128	16.38632	0.22391	0.43141	0.18611	0.66735	84
8	-0.01609551	0.86693***	0.29130	2.02298	17.64651	0.19260	0.69090	0.47734	0.49165	84
9	-0.01685524	0.82734***	0.30451	2.03226	18.73227	0.16515	1.04547	1.09300	0.29899	84
10	-0.01844345	0.78526***	0.31433	2.04330	19.56655	0.14165	1.51595*	2.29812	0.13352	84
11	-0.02061352	0.74170***	0.32012	2.05070	20.06918	0.12216	2.11444**	4.47085	0.03763	84
12	-0.02314462	0.69796***	0.32164	2.05024	20.20266	0.10662	2.83296***	8.02568	0.00585	84
13	-0.02584527	0.65542***	0.31917	2.03975	19.98613	0.09483	3.63359***	13.20301	0.00050	84
14	-0.02855899	0.61537***	0.31337	2.01933	19.48365	0.08648	4.44775***	19.78247	0.00000	84
15	-0.03116848	0.57883***	0.30508	1.99085	18.77973	0.08111	5.19230***	26.96003	0.00000	84
	Intercept	*δι*	*Adj R^2*	*DW*	*F-stat*	*SE*	*t(d<1)*	*Wald χ^2*	*Probability*	*N*
3	-0.01567	0.47976***	0.17926	2.23808	13.77720	0.39075	1.33138*	1.77258	0.18310	120
4	-0.00982	0.66468***	0.20872	2.23252	16.43077	0.36383	0.92164	0.84942	0.35670	120
5	-0.00518	0.75664***	0.22657	2.23589	18.13677	0.29217	0.83293	0.69377	0.40490	120
6	-0.00130	0.81620***	0.23861	2.24051	19.33309	0.22473	0.81786	0.66889	0.41340	120
7	0.00136	0.84409***	0.24762	2.24414	20.25302	0.17918	0.87011	0.75709	0.38420	120
8	0.00245	0.83879***	0.25422	2.24637	20.94175	0.15155	1.06377	1.13161	0.28740	120
9	0.00204	0.80558***	0.25856	2.24692	21.40088	0.13170	1.47626*	2.17934	0.13990	120
10	0.00049	0.75373***	0.26082	2.24510	21.64228	0.11592	2.12459**	4.51387	0.03360	120
11	-0.00177	0.69275***	0.26129	2.24000	21.69247	0.10569	2.90703***	8.45083	0.00360	120
12	-0.00432	0.63031***	0.26026	2.23069	21.58222	0.10224	3.61590***	13.07473	0.00030	120
13	-0.00688	0.57169***	0.25796	2.21637	21.33677	0.10389	4.12267***	16.99639	0.00000	120
14	-0.00924	0.51998***	0.25450	2.19633	20.97076	0.10754	4.46385***	19.92594	0.00000	120
15	-0.01130	0.47665***	0.24988	2.16998	20.48777	0.11072	4.72689***	22.34462	0.00000	120

Panel C: Fixed-Price yields

Regression Model: $\Delta y_{it}^{callable} = \gamma_i + \delta_i \Delta r_t + \varepsilon_i$

	Intercept	δ_t	*Adj* R^2	*DW*	*F-stat*	*SE*	$t(d<1)$	*Wald* χ^2	*Probability*	*N*
3	-0.02153	0.61325**	0.16945	1.97227	9.26293	0.31347	1.23378	1.52222	0.21730	84
4	-0.01019	0.55661**	0.08880	2.11428	4.94699	0.27714	1.59988*	2.55961	0.10960	84
5	-0.00697	0.47637**	0.08721	2.13450	4.86960	0.27035	1.93686**	3.75143	0.05280	84
6	-0.00911	0.38037*	0.07872	2.14578	4.46046	0.26252	2.36034***	5.57121	0.01830	84
7	-0.01334	0.29924	0.06488	2.15407	3.80982	0.24769	2.82920***	8.00437	0.00470	84
8	-0.01752	0.24750	0.05015	2.15587	3.13821	0.22945	3.27955***	10.75542	0.00100	84
9	-0.02036	0.23223	0.03730	2.14899	2.56936	0.21011	3.65413***	13.35268	0.00030	84
10	-0.02190	0.23942*	0.02108	2.15317	1.87214	0.19070	3.98828***	15.90639	0.00010	84
11	-0.02157	0.27969**	0.01268	2.14138	1.51997	0.17205	4.18663***	17.52787	0.00000	84
12	-0.02027	0.33384**	0.00858	2.12529	1.35031	0.15551	4.28360***	18.34923	0.00000	84
13	-0.01863	0.39067***	0.00922	2.10527	1.37702	0.14201	4.29078***	18.41075	0.00000	84
14	-0.01715	0.44198***	0.01462	2.08190	1.60071	0.13192	4.23000***	17.89288	0.00000	84
15	-0.01619	0.48357***	0.02388	2.05728	1.99076	0.12549	4.11531***	16.93573	0.00000	84

Panel B of Table 4.5 reports the results of applying regression model (4.6) to make-whole spot yields. Given the results obtained from regression model (4.5) for make-whole bonds, we expect that $\frac{\partial CS}{\partial r}$ will be trivial for A-rated make-whole callable bonds with short and medium maturities. This implies that the make-whole call risk term and its associated cross-term are irrelevant in estimating duration for these bonds, and: $\frac{D_a}{D_m} = 1 + \underset{(-)}{\frac{\partial DS}{\partial r}} + \underset{(+)}{\frac{\partial CS}{\partial DS}\frac{\partial DS}{\partial r}}$. The findings show that the ratio of the risk-adjusted duration to Macaulay duration is indistinguishable from unity for bonds with less than 10 years to maturity. Therefore, it must be that the absolute value of the negative $\frac{\partial DS}{\partial r}$ term (as estimated with the noncallable yields) offsets the positive value of the $\frac{\partial CS}{\partial DS}\frac{\partial DS}{\partial r}$ interaction term, so that the risk-adjusted duration is equivalent to Macaulay duration.

Given the results obtained from regression model (4.5) for make-whole bonds, we also expect that $\frac{\partial CS}{\partial r}$ will be negative for A-rated make-whole callable bonds with maturities longer than 10 years. And since we find that the ratio of the risk-adjusted duration to Macaulay duration is significantly lower than one for these maturities, the absolute value of direct effects should dominate the value of indirect (interaction) effects.

For A-rated fixed-price callable bonds the results of regression model (4.5) indicate that $\frac{\partial CS}{\partial r}$ is significantly negative for all maturities greater than three years. The results in Panel C of Table 4.5 show that the risk-adjusted duration is significantly shorter than Macaulay duration for all maturities but for the three-year term. The implication of this finding is that the absolute value of direct effects, $\left|\frac{\partial DS}{\partial r}+\frac{\partial CS}{\partial r}\right|$, dominates the absolute value of indirect effects, $\left|\frac{\partial DS}{\partial CS}\frac{\partial CS}{\partial r}+\frac{\partial CS}{\partial DS}\frac{\partial DS}{\partial r}\right|$.

Our findings imply that one need to account not only for the impact of default and call risks on duration, but also for the impact of the interaction between these two risks for one to successfully immunize corporate bond portfolios. We find that the risk-adjustment required for estimating duration of callable corporate bonds differs across the maturity spectrum. For make-whole callable A-rated bonds with short and medium maturities, Macaulay duration serves as a good approximation of the risk-adjusted duration. This is also the case for fixed-price bonds with shorter maturity. Our results for A-rated fixed-price callable bonds with medium and long maturities and make-whole callable bonds with long maturities indicate that the risk-adjusted duration is shorter than the Macaulay duration. This is due to the fact that the absolute value of the direct effects on duration exceeds that of the indirect effects, associated with the interaction terms between default and call risks.

Comparing the estimated ratio across the different types of bonds also sheds a light on the relative importance of the direct callability term and that of the interaction terms. For both fixed-price and make-whole bonds in all cases where the risk-adjusted duration is found significantly lower than Macaulay duration, the estimated ratio is always lower than that estimated for noncallable bonds. This means that that the direct impact of the call option's negative interest-rate sensitivity, $\frac{\partial CS}{\partial r}$, always dominates the interaction terms, and shortens the risk-adjusted duration.

Recall that for make-whole bonds with less than 10 years to maturity we found insignificant interest-rate sensitivity for the call option ($\frac{\partial CS}{\partial r}=0$). However, the ratio for these maturities of the make-whole bonds estimated in regression model (4.6) is (in general) higher than that estimated for the noncallable bonds. Note that noncallable A-rated bonds are only subject to default risk, while make-whole callable a-rated bonds are subject to both default and call risks. Thus, the introduction of a call option lengthens the risk-adjusted duration due to the positive interaction term associated with default, $\frac{\partial CS}{\partial DS}\frac{\partial DS}{\partial r}$.

This provides some evidence in support of the predictions of the theoretical models of Acharya and Carpenter (2002) and Sarkar and Hong (2004) with respect to duration. They note that the probability of default for a callable and defaultable bond is lower than that associated with a noncallable corporate bond. This is because the existence of one option delays the exercise of the other. Thus, for short- and medium-maturity A-rated bonds, the introduction of a call option lengthens duration (relative to noncallable, but still defaultable, A-rated bonds). Finally, it should be noted that the results obtained for the noncallable bonds clearly indicate that the default term alone significantly shortens duration.

4.5. Summary and Conclusions

Acharya and Carpenter (2002) model the value of a defaultable and callable bond. Focusing on the interaction between the call and default options, they note that the existence of one option delays the exercise of the other. This has strong empirical implications to duration and to the determinants of yield spread changes. For duration, when considered separately, both default and call shorten duration.

However, when the interaction between the two risks is also accounted for, it is shown to lengthen duration. Since duration is equivalent to the interest-rate elasticity of the bond, the above analysis is also important for the interest-rate sensitivity of the bond yield spread. Sarkar and Hong (2004) also derive a structural model for pricing a fixed-price callable and defaultable bond, and their model has similar implications to that of Acharya and Carpenter (2002).

In this chapter we test the main implications of these theoretical models. In particular, we test the interest-rate elasticity of the call spread and that of the default spread, allowing for interaction between both spreads. We also examine the impact of both risks and their interaction on the effective duration of corporate bonds. In our tests we examine both bonds carrying a standard (fixed-price) call option and those carrying the newer and more popular make-whole option.

We find evidence supporting the predictions of the two models. First, our data shows a statistically significant interaction between default spreads and call spreads. Second, as the theory predicts, we show that, when considered separately, both default and call risks shorten duration for fixed-price callable bonds. For short- and medium-term bonds carrying a make-whole option, we find a trivial direct effect of the call option. However, the interaction term seem to lengthen the effective duration for these bonds. This is in agreement with the theory.

The implication of our findings is that portfolio managers of callable corporate bonds, who use duration as an immunization tool or practice active rate anticipation strategies, must pay attention to the maturity of the bond. For example, it is clear that no risk-adjustment is necessary when one manages A-rated make-whole callable bonds with short and medium maturities. However, it is important to consider not only the impacts of default and call risks on estimating duration, but also the interaction between these variables when one manages callable debt of all other maturities.

The models of Acharya and Carpenter (2002) and Sarkar and Hong (2004) focus on pricing fixed-price callable bonds. However, the make-whole call provision became the standard for callable bonds in recent years, both in the U.S. and Canada. Powers and Tsyplakov (2004) derive a structural model for pricing make-whole callable bonds. Their model does not allow the study of duration similar to that of Acharya and Carpenter (2002) and Sarkar and Hong (2004). Given the popularity of make-whole bonds, a possible direction for future research is the introduction of a structural bond-pricing model that allows studying the duration and yield-spread elasticity of these bonds.

Appendix A: Estimating Zero-coupon Curves with the Nelson and Siegel (NS) Model.

The extraction of the zero-coupon yield curve from coupon bonds makes it possible to not only price interest rate contingent claims, but also to empirically test term structure models.

The observed price of the bond, P_j, is assumed to be different from the price of a bond determined using the NS model, $\hat{P}_j$. The model price is derived from the following theoretical equation of the bond's present value:

$$\hat{P}_j = c_j \sum_{i=1}^{Mj} B(t,\Omega,\varpi_{ji}) + 100 \times B(t,\Omega,\varpi_{ji}),\ j = 1,\ldots,N, \text{ where}$$

$B(t,\Omega(\varpi_{ji})$ is the discount function for the applicable jth bond, Ω is a vector of parameters, and ϖ_{ji} is the coupon payment date $i = 1,2,3, \ldots,M$.

For the purpose of fitting the term structure, all coupon payments are adjusted for accrued interest payments. Accrued interest is calculated as the fraction of the coupon C: $A_t = a_t C$, where $a_t = (1 - n_t / 365)$ and n_t is the number of days since the last coupon payment. The difference between the observed and theoretical prices is captured by the assumed to be normally distributed zero mean errors ε_j.

To determine the estimates of the parameter function Ω, we have to minimize the following maximum likelihood quadratic representation:

$$\hat{\Omega} = \arg\min_{\Omega} \sum_{j=1}^{N} \left[\hat{P}_j - P_j \right]^2$$

NS model the implied forward rate curve explicitly. They choose a parsimonious functional form, which allows the forward rate curve to take various shapes. They show that the solution to the second order linear differential equation with two identical real roots is the instantaneous forward rate:

$$f(m,\Omega) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_1}\right), \qquad \text{(A.1)}$$

where Ω incorporates the following parameters $(\beta_0, \beta_1, \beta_2, \tau_1)$, where m stands for the time to maturity.

Integrating equation (A1) on the interval *[0,m]* and diving by m, NS obtain the following relationship between the spot rates and maturity:

$$s(m,\Omega) = \beta_0 + (\beta_1 + \beta_2) \frac{1 - \exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}} + \beta_2 \exp\left(-\frac{m}{\tau_1}\right) \qquad \text{(A.2)}$$

As m goes to infinity, the limit of $s(m,\Omega)$ approaches β_0. In other words, it is a long-term interest rate (in that the limit forward and spot rates coincide). As m goes to zero, the limit of $s(m,\Omega)$ approaches $(\beta_0 + \beta_1)$. This sum can be interpreted as the instantaneous interest rate. This implies that $(-\beta_1)$ can be construed as the spread between the long-term and short-term interest rates.

The asymptotic properties of the NS approach are:

- β_0 is supposed to be positive to reflect the asymptotic value of the instantaneous forward rate curve; especially for long maturities, both forward and spot curves tend towards the asymptote as the term to maturity approaches infinity.

- β_1 is the speed of convergence of the curve to its long-term trend. This parameter determines the starting (or short-term value) of the curve in terms of deviations from the asymptotic value. If this coefficient is positive, then the curve will have a negative slope. The summation of the first two coefficients in the equation above (A.2) determines the value of both curves at zero maturity.

- β_2 is a scale parameter and represents the magnitude of the hump.

- τ_1 is a scale parameter and reflects the position of the hump.

The last two parameters have to be positive for the model to be easily interpretable. In particular, the sign of the "β_2" parameter determines the shape of the estimated yield curve (positive coefficient meaning downward-sloping curve and negative coefficient meaning upward-sloping yield curve). The "β_2" parameter is the original NS time constant parameter, or rate at which dependent variables decay to zero.

For the purpose of fitting yield curves, in their original paper NS have parameterized the equation (A.2) in the following form:

$$s(m,\Omega) = a + \beta \times \frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\left(\frac{m}{\tau_1}\right)} + c \times \exp\left(-\frac{m}{\tau_1}\right) \tag{A.3}$$

For any value of τ it is now possible to calculate sample values of other regressors. Utilizing the grid search for all provisional values of τ, the best fitting values for τ, a, b, and c are extracted.

CHAPTER 5

Conclusion

This thesis contains three essays on fixed-income securities. In the first essay we apply risk-neutral valuation to price inflation-protected bonds. There is a long history of inflation-protected bonds issued by governments worldwide. These include Israel, the U.K., Canada, Australia, New Zealand, and the U.S. Treasury Inflation-Protected Securities (TIPS) were first issued in the U.S. in 1997. Brynjolfsson (2001) notes that, as of February 2000, the U.S. market for TIPS amounted to $97 billion in market value outstanding, and was expected to grow by $16 billion annually. Therefore, it is important to price TIPS while considering the embedded option in these bonds in order to be theoretically consistent.

We show that there is an option embedded in TIPS. In nominal terms, TIPS includes a call option on the value of a real bond, which is adjusted to increases and decreases in the general price level. In real terms, TIPS includes a put option on the value of the real bond. The embedded option provides the investor with protection against erosion in the bond face value in case of deflation. We derive the nominal and real duration (elasticity) for TIPS. We argue that when one needs to compare the performance of TIPS with that of nominal Treasury bonds, one should use a duration measure, calculated with respect to changes in nominal interest rates for TIPS.

We also define the TIPS yield spread as the difference in yields on a nominal bond and TIPS and contrast it with its common interpretation as the expected inflation rate. We show that this yield spread is the expected inflation rate only when the put option is deeply out of the money. Then, the yield on TIPS is approximately the real rate, and the yield spread equals the expected inflation rate. However, in recent years deflation was not such an unlikely event (Roll, 2004). The comparative statics in this essay make us better equipped to understanding pricing and price elasticity of TIPS in a deflationary environment. An empirical test confirms the main predictions of our model.

In the second essay we derive a model to value government bonds under the risk of liquidation. The lower marketability of government bonds implies higher bid-ask spreads, especially for those positions that must be liquidated with immediacy. We show that the value of the government bond for a bondholder facing exogenous liquidation shocks depends on the term structures of the conditional probabilities of liquidation, the bid-ask spreads, and the riskless rates.

We show that the expected yield demanded by the bondholder incorporates a liquidation risk premium, which is given by the expected cost of liquidation. A risk-adjusted elasticity measure for bonds subject to liquidation risk is derived. This elasticity measure is shown to be an ex-ante immunizing elasticity, dependent upon changes in the liquidation probabilities and the bond's bid-ask spread as a result of changes in interest rates. We also provide a liquidation risk-adjusted formula for the convexity of a government bond with less than perfect liquidity.

In the third essay, we provide evidence highlighting the importance of using a proper default- and call-adjusted elasticity measure when immunizing bond price changes to changes in interest rates in the presence of both default and callability risks. We obtain corporate bond default and call spreads, derived from estimated zero curves, for callable and noncallable bonds. We then study the interest-rate elasticity of the call spread and that of the default spread, while accounting for the interaction between both spreads. The sample of analyzed callable bonds includes both fixed-price callable and the newer make-whole callable bonds. We find evidence supporting the predictions of the recent structural models for pricing callable corporate bonds.

First, we confirm that it is incorrect to model the process for the credit risk separately from the processes for the call risk. This is due to the strong interaction between both risks. Second, we show that, when considered separately, both default and call risks shorten duration for fixed-price callable bonds. This is consistent with the theory. We show that the impact of shifts in the riskless term structure on default spread is always negative, implying that default shortens the effective duration of noncallable bonds relative to government bonds.

The effect of callability on effective duration of callable corporate bonds differs depending on the type of the call provision and the maturity of the bond issue. For short- and medium-term bonds carrying a make-whole option, we find that the effect of a call option is trivial. However, the interaction term seems to lengthen the effective duration for these bonds. Fixed-price callable bonds of all maturities exhibit effective duration coefficients that are shorter than their noncallable counterparts. This implies that the influence of the call risk on duration dominates that associated with the interaction between the call and default risks.

Overall, this thesis extends a warning to bond portfolio managers, calling them to account for underlying risks like default, callability, illiquidity, and inflation, that affect the bond-price elasticity with respect to shifts in interest rates. Failing to adjust duration for these risks is not only ineffective, but may also be quite costly when utilizing an immunization or a rate-anticipation strategy.

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