Simulation of Fuzzy Dynamic Systems with Multiple Fuzzy Parameters and Initial Conditions

by

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Abstract

Under some conditions in real world, precise parameters and/or initial values of dynamic systems are hard to be determined. Fuzzy Differential Equation (FDE) is a powerful tool to model dynamical systems with the uncertainty of impreciseness. This thesis presents the first numerical solution for Fuzzy Differential Equations with multiple fuzzy parameters and initial Values (FDEPIV) problems. Previous approaches for solving the FDEs only focused on FDEs with single fuzzy condition. In this thesis, we applied the proper fuzzy arithmetic on Runge-Kutta method for solving the FDEPIV problems with multiple fuzzy parameters and initial conditions. Furthermore, comparing with directly applying the extension principle in solving FDEPIV, the complexity of the proposed method is much lower, and parallelization of the proposed algorithm is feasible. Numerical examples of the FDEPIV problems are presented to demonstrate the effectiveness of the proposed method.

Acknowledgements

I wishes to express my great gratitude to my advisor, Prof. Dr. Wai-keung Fung, who was abundantly helpful and offered invaluable assistance and guidance. This thesis would not have been possible without the support of him. I also want to thank my parents and my girl friend Siyang Lei for supporting and encouraging me to pursue this degree.

— Taiming Zhang

Nomenclature

μ	Membership grade of fuzzy set or fuzzy number, which the range is [0,1]
$\tilde{a}$	Fuzzy set or fuzzy number
$\mathbb{R}$	Real Space
$\mathbb{R}^n$	n-dimensional Real number space
E^n	Fuzzy set space
D	Hausdorff distance
L	Lipschitz Condition
C_k	Power set which contains all nonempty compact subsets of K
$t[a,b,c]$	Triangular fuzzy number, the *support* is $[a,c]$ and the *core* is b
$g[a,b,c]$	Gaussian Fuzzy Number
$q[a,b,c]$	Quadratic Fuzzy Number
$e[a,b,c]$	Exponential Fuzzy Number
$\tilde{a}^\alpha$	α is the α-cut of $\tilde{a}$, and the membership grade of the α-cut is $\mu = \alpha$
$\underline{a}$	Lower bound of fuzzy number $\tilde{a}$ at a particular membership grade
$\overline{a}$	Upper bound of fuzzy number $\tilde{a}$ at a particular membership grade
$\dot{y}, \frac{dy}{dt}$	Time derivative of y
FDE	Fuzzy Differential Equation
FIVP	Fuzzy Initial Value Problem
FDEPIV	Fuzzy Differential Equation with multiple fuzzy Parameters and Initial Values
KVL	Kirchhoff's Voltage Law
VAR	Volt Ampere Relationship

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Chapter 1

Introduction

Dynamic systems are ubiquitous. Inverted pendulums, biped walking robots, nuclear reactors, power system networks, missiles, biological neurons, flying birds, population dynamics of multiple species in a closed ecosystem and earth quakes and tsunamis are just few examples of dynamic systems found in human world and nature. In order to understand and analyze the working mechanisms of these systems, mathematical models of the interested systems are constructed to simulate their behaviors under the influence of external inputs. Dynamic systems are usually modeled by differential equations with various parameters and initial or boundary conditions. For instance, weights of the mass and the length of the rod are examples of the parameters of a differential equation that models an inverted pendulum.

Precise values of parameters and initial conditions of a given differential equation must be known in order to obtain the accurate behavior of the corresponding dynamic systems. However, it is hard to obtain exact values of parameters or initial conditions in these system in real life. Vague estimates of these system parameters are usually obtained. In order to incorporate uncertainty in behavior analysis of real world dynamic systems, the notion of fuzzy differential equations (FDE), in which parameters or initial conditions of the systems are represented by fuzzy numbers, was proposed in the 80's [23]. The fuzzy sets of the solution of fuzzy differential equations show all possible solution trajectories of the target systems with different degrees of membership.

1.1 Motivations

Various formulations and computational schemes have been proposed to obtain solution of fuzzy differential equations and they have limitations like ever-expanding support in fuzzy solution trajectory in time evolution and computational intractability of the solution with multiple fuzzy parameters. Moreover, previous approaches usually focused on FDEs with single fuzzy condition (fuzzy initial value or parameter) only. The limitation of these works is that they could not handle FDEs with multiple fuzzy conditions. Although in [22] [12], the authors gave theroetical investigations on solving FDEs with multiple fuzzy initial values and parameters, these approaches are not practical in reality as they could not provide a computational method for solving FDEs. Theoretically, it is possible to apply the extension principle to solve the FDEs with multiple fuzzy conditions. However, the complexity in using the extension principle is high and intractable. Under this condition, the necessity of a practical method in solving FDE with multiple parameters and/or initial values problems is needed for simulation of fuzzy dynamic systems.

1.2 Overview of the proposed approach

In this thesis, the proper fuzzy arithmetic with Runge-Kutta method is employed for solving Fuzzy Differential Equations with fuzzy Parameters and Initial Values (FDEPIV) problems. Application of proper fuzzy arithmetic in fuzzifying common numerical solution schemes for ordinary differential equations(ODE) is proposed. The proposed method can guarantee convergent support of the fuzzy solution trajectories. In particular, this thesis focuses on fuzzifying the Runge-Kutta scheme for solving FDEs with multiple fuzzy parameters and initial conditions. The proposed method not only overcomes the shortcomings of existing methods mentioned in Section 1.2, but also provides an efficient numerical scheme for solving nonlinear, higher order fuzzy differential equations with multiple fuzzy parameters and fuzzy initial conditions. Moreover, parallelization of the proposed methods is feasible.

1.3 Contribution

The proposed method for solving FDEPIV applies proper fuzzy arithmetic with Runge-Kutta method, which does not need the analytical solution of the non-fuzzy counterpart, and provides solution with high accuracy. This method eliminates the overestimation of using standard fuzzy arithmetic in solving the FDEPIV problems. Moreover, it can solve nonlinear and higher order FDEPIV problems. To the best of the author's knowledge, this thesis proposes the first numerical scheme for higher order, nonlinear differential equations with multiple fuzzy parameters and initial conditions.

1.4 Thesis Organization

In Chapter 2, the background of fuzzy sets theory is provided, which includes the extension principle, standard fuzzy arithmetic and the proper fuzzy arithmetic. In Chapter 3, fuzzy differential equations and some preliminary analysis on FDEs are introduced. Moreover, the technical challenges in solving FDEPIV problems are discussed. In Chapter 4, a new method on solving the FDEPIV problems using proper fuzzy arithmetic is proposed. The examples and comparisons between new method and other approaches are illustrated in Chapter 5. Finally, the conclusion and the future works of the thesis are presented in Chapter 6.

Chapter 2

Basics of Fuzzy Sets

Professor Lotfi A.Zadeh in 1965 first proposed the notion of fuzzy sets for representation of concept of "vagueness" and "impreciseness" in [42]. Fuzziness occurs when the extent of information is not sharply defined. In classical set theory, the membership of elements in a set is binary. An element either belongs or does not belong to the set. In contrast, fuzzy set permits a gradual assessment of membership of elements in a set, which can be considered as a generalization of a classical set or a crisp set. Nowdays, fuzzy sets theory has been developing in a high speed to becoming a well-established tools for uncertainty handling, and enormous progress has been made. Branches of fuzzy set theory have emerged, like fuzzy logic, fuzzy modeling, fuzzy expert system, fuzzy control, fuzzy arithmetic and fuzzy differential equations, and it is widely applied in engineering and science. In this chapter, we will introduce basic notions of fuzzy sets, which are important in this thesis.

2.1 Fuzzy sets

In the real world, there is information that can be expressed precisely, such as *'The retire age in Canada is 65'*: a person who is aged 65, can retire from their work place and enjoy the pension from the society or company; an international student, whose TOFEL test score is over 80 is qualified to enroll in the University of Manitoba. People whose age are 65 and older are all in the set $A : \{qualified\ retired\ person\ in\ Canada\}$ and whose test scores are higher than 80 are in the set $B : \{qualified\ person\ enrolled\ in\ U\ of\ M\}$.

However, some notions do not have precise measurement standard, as *'Tom is an old man'*, the interpretation of '*old*' is fuzzy. If Tom is 65 years old, we can say 'Tom is *a little bit old*'. If Tom is 90 years old, we can say he is '*pretty much old*'. If a person's TOFEL score is 80 we say the score is *fair good*. If a person's score is 100, we can say his score is *excellent*. The elements in the sets $C : \{old\ men\}$ and $D : \{good\ TOFEL\ scores\}$ have a degree of the membership in the sets. The phrases '*a little bit old*', '*pretty much old*', '*fair good*' and '*excellent*' describe the degree of elements in the corresponding sets. In fact, the sets A, B are classical or crisp sets examples, and C, D are fuzzy sets examples. The following subsections presents concepts of fuzzy sets that are important in this thesis.

2.1.1 Universal set

In set theory, a universal set is a set which contains all objects, including itself [10]. The universal set X is a nonempty set consisting of all possible elements x of relevance in a particular context. The characteristic function $\mu_X(x)$ of universal set X is given by:

$$\mu_X(x) = 1,\ \forall x \in X. \tag{2.1}$$

2.1.2 Membership function

The main difference between classical sets and fuzzy sets is their characteristic function (in fuzzy sets we name it as membership function). Assume that a classical set A includes all the possible elements x, and X is the continuous and uncountable universal set. The classical set A can be written as:

$$A \ = \ \{x \in X \mid \text{for } x \text{ satisfies } A(x)\} \tag{2.2}$$

where $A(x)$ denotes a property of an element x that belongs to the set A. For example, in Section 2.1, the property $A(x)$ for set A : $\{$*retired person in Canada*$\}$ is '*x is equal to or over 65*'. For any element $x \in X$ satisfying $A(x)$, the characteristic function is as expressed

$$\mu_A(x) = \begin{cases} 1 \quad , \quad x \text{ satisfies } A(x) \\ 0 \quad , \quad \text{otherwise} \end{cases} \quad , \ x \in X \ . \tag{2.3}$$

In the classical set theory, the characteristic function $\mu_A(x)$ of equation 2.3 is binary and has only values 0 and 1. In contrast, the charactiristic function or membership function of a fuzzy set for fuzzy sets $\tilde{A}$ is generalized as follows: the membership grade of a fuzzy set takes a value in the continuous interval [0,1], instead of two distinct values, 0 or 1.

$$\mu_{\tilde{A}} \ : \ X \ \rightarrow \ [0,1] \tag{2.4}$$

The fuzzy set $\tilde{A}$ can be written as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \ | x \in X \ , \mu_{\tilde{A}}(x) \in [0,1]\}. \tag{2.5}$$

A fuzzy set can be classified as a continuous fuzzy set or a discrete fuzzy set. For continuous fuzzy set, it is represented in form of (2.5). The discrete fuzzy set is represented in the form as follows:

$$\tilde{A} = \big(\mu_{\tilde{A}}(x_1)/x_1, \mu_{\tilde{A}}(x_2)/x_2, ..., \mu_{\tilde{A}}(x_n)/x_n\big), \ x_i \in X, \ x_i = 1, 2, 3, ..., n. \tag{2.6}$$

The following example illustrates the difference between the classical sets and fuzzy sets.

Example 2.1.1. A basketball team in U of M is recruiting people now. We denote a set A as $\{$*Players' heights requirement in U of M basketball team*$\}$. The minimum height requirement of being enrolled into University of Manitoba basketball team is 180cm. If a student whose height is over 180cm, we say that he satisfies the minimum height requirement to become a basketball player in U of M. Even if he is just 179.5cm tall, the basketball team will not accept this person. Fig.1.1

depicts the perception of '*tallness*' of a person using fuzzy sets. If a Tom's height is 175cm, which is between 170-180 cm we can say that he is '*a little bit tall*' and his degree of '*tallness*' is '*medium*'. In other words, the height of Tom is one element in the fuzzy set *Medium*, with membership grade of 1. It is also an element in fuzzy set *Tall*, with the membership grade of 0.5.

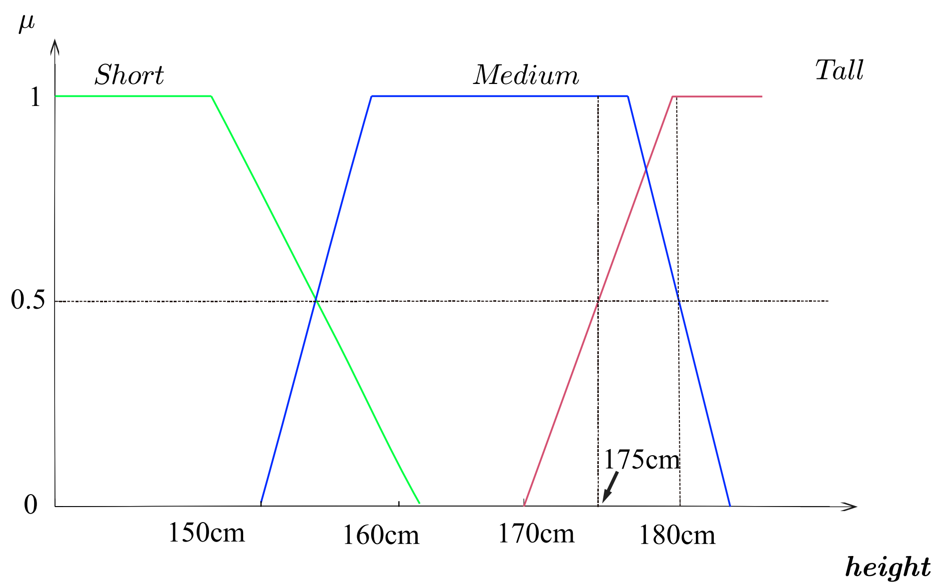


Fig. 2.1: Comparison of classical set and fuzzy set

2.1.3 α-cuts of Fuzzy Sets

Given a universal set x, and a fuzzy set $\tilde{A}$, $\tilde{A} \subseteq X$, the α-cut of set $\tilde{A}$ is denoted as $cut_{\alpha}(\tilde{A}) = A_{\alpha}$, A_{α} is a crisp set of all elements $x \in X$ that belongs to fuzzy set $\tilde{A}$ with the membership grade $\alpha \in [0, 1]$ or higher. Specifically, when $u_{\tilde{A}}(x) = 0$, the corresponding α-cut is called the support of the fuzzy set $\widetilde{A}$, or $\text{supp}(\tilde{A})$. When $u_{\tilde{A}}(x) = 1$, the α-cut is called the core of the fuzzy set $\tilde{A}$, or core($\tilde{A}$). They are defined as follows.

$$\text{cut}_{\alpha}(\tilde{A}) = A_{\alpha} = \{x \in X|\ \mu_{\tilde{A}}(x) \geq \alpha\} \tag{2.7}$$

and

$$\text{cut}_{\alpha=0}(\tilde{A}) = \text{supp}(\tilde{A}) \tag{2.8}$$

$$\text{cut}_{\alpha=1}(\tilde{A}) = \text{core}(\tilde{A}) \tag{2.9}$$

$$\alpha_1 < \alpha_2 \Rightarrow \text{cut}_{\alpha_2}(\tilde{A}) \subset \text{cut}_{\alpha_1}(\tilde{A}) \tag{2.10}$$

The definition of $\alpha - cuts$ is illustrated in Fig.2.2: In particular, α-cut of the fuzzy set in Fig.2.2

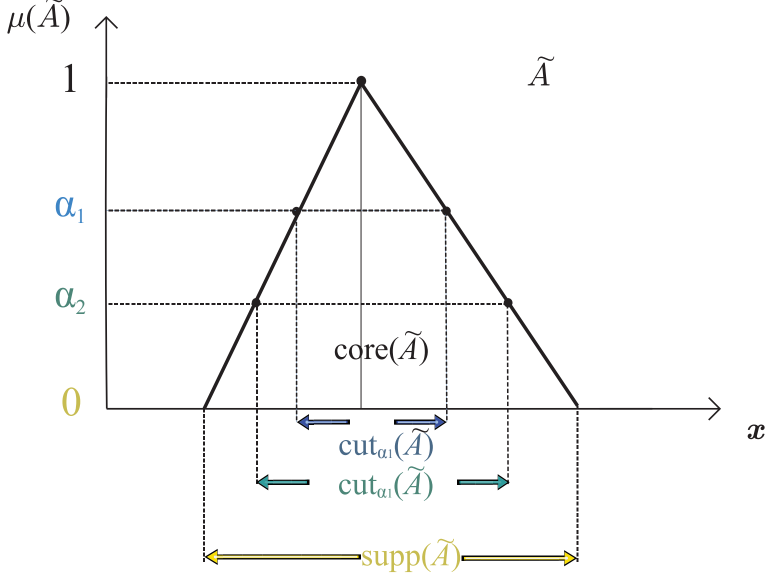


Fig. 2.2: The illustration for *core support* and α-cuts of fuzzy set $\tilde{A}$.

is a closed interval $[\underline{a}, \overline{a}]$, where $\underline{a}$,$\overline{a}$ are the lower and upper bounds of the interval respectively.

2.1.4 Convexity of Fuzzy Sets

The convexity of a fuzzy set is defined as follow:

Definition 2.1.2. A fuzzy set $\tilde{A}$ is convex if and only if all possible α-cuts in the fuzzy set are convex. If every element $u \in \text{cut}_{\alpha}(\tilde{A})$ and $v \in \text{cut}_{\alpha}(\tilde{A})$, for every membership grade $\alpha \in [0, 1]$, we have

$$u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)],\ \forall\ \lambda \in [0,1] \tag{2.11}$$

Example 2.1.3. Two fuzzy set $\tilde{A}$ and $\tilde{B}$ are given by their membership function $\mu(\tilde{A})$ and $\mu(\tilde{B})$ as show in Fig.2.3 and 2.4 respectively. For fuzzy set $\tilde{A}$, every possible α-cut is convex. In fuzzy set $\tilde{B}$ at least one α-cut is not convex.

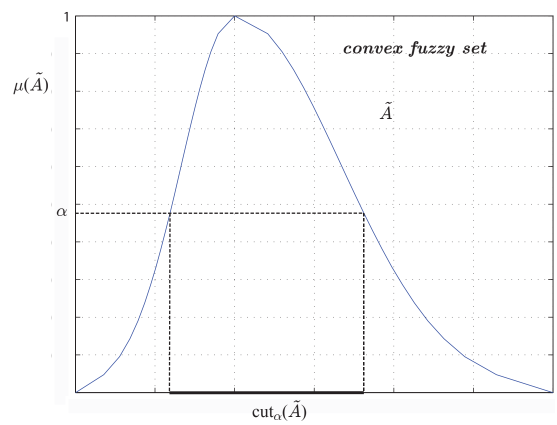


Fig. 2.3: Convex fuzzy set

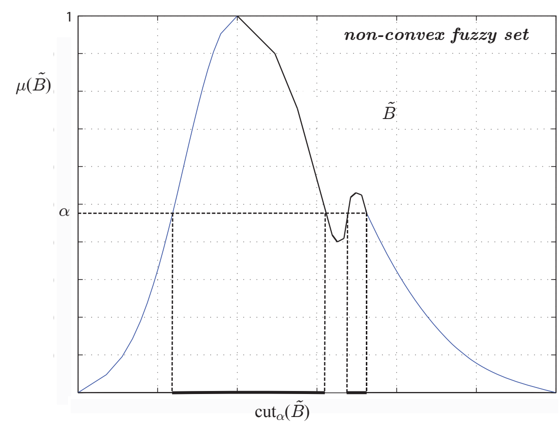


Fig. 2.4: Non-convex fuzzy set

2.1.5 Fuzzy numbers and its properties

A fuzzy number is a special case of a convex fuzzy set[19], which is convex and normalized. A fuzzy number has following properties:

Definition 2.1.4. For a fuzzy set whose membership function is $u : \mathbb{R} \to I = [0,1]$, and $u(x)$ has the following properties:

i $u(x)$ is upper semi-continuous;

ii $u(x)$ is bounded subset of $\mathbb{R}$;

iii $u(x)$ is fuzzy convex, ie. $u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)]$ for any $\lambda \in [0,1]$;

iv $u(x) \geq \alpha$ is compact subset of $\mathbb{R}$ for all α-cuts, $\alpha \in I$;

v There are real number $a, b, c \in \mathbb{R}$, and $0 \leq a \leq b \leq c$ for:

1. $u(x)$ is monotonic increasing on $[a, b]$,
2. $u(x)$ is monotonic decreasing on $[b, c]$,
3. $u(b) = 1$.

2.1.6 Types of fuzzy numbers

As listed in (v) of Definition 2.1.4, the membership function can be divided into left and right sides. Based on this observation, there are standard types of fuzzy numbers [19] discussed in this thesis:

Triangular Fuzzy Numbers

Consider $\tilde{p}$ is a triangular fuzzy number, we denote it as:

$$\tilde{p} = t[a, b, c], \text{ with } a \leq b \leq c, \ a, b, c \in \mathbb{R} \tag{2.12}$$

with membership function:

$$\mu_{\tilde{p}}(x) = \begin{cases} 0 & for \quad x \leq a, \\ 1 + \dfrac{x-b}{b-a} & for \quad a < x < b, \\ 1 - \dfrac{x-b}{c-b} & for \quad b \leq x < c, \\ 0 & for \quad x \geq c. \end{cases} \qquad \forall x \in \mathbb{R} \tag{2.13}$$

Gaussian Fuzzy Numbers

We denote the fuzzy number $\tilde{p}$ as a gaussian fuzzy number:

$$\tilde{p} = g[a, b, c], \text{ with } a \leq b \leq c, \ a, b, c \in \mathbb{R} \tag{2.14}$$

with membership function

$$\mu_{\tilde{p}}(x) = \begin{cases} 0 & for \quad x \leq a, \\ e^{\dfrac{-(x-b)^2}{2(\beta_1)^2}} & for \quad a < x < b \ , (\beta_1 = \dfrac{b-a}{3}), \\ e^{\dfrac{-(x-b)^2}{2(\beta_2)^2}} & for \quad b \leq x < c \ , (\beta_2 = \dfrac{c-b}{3}), \\ 0 & for \quad x \geq c. \end{cases} \qquad \forall x \in \mathbb{R} \tag{2.15}$$

Quadratic Fuzzy Numbers

We denote the fuzzy number $\tilde{p}$ as a quadratic fuzzy number:

$$\tilde{p} = q[a, b, c], \text{ with } a \leq b \leq c, \ a, b, c \in \mathbb{R} \tag{2.16}$$

with membership function

$$\mu_{\tilde{p}}(x) = \begin{cases} 0 & for \quad x \leq a, \\ 1 - \dfrac{-(x-b)^2}{\gamma_1^2} & for \quad a < x < b \ , (\gamma_1 = b-a), \\ 1 - \dfrac{-(x-b)^2}{\gamma_2^2} & for \quad b \leq x < c \ , (\gamma_2 = c-b), \\ 0 & for \quad x \geq c. \end{cases} \qquad \forall x \in \mathbb{R} \tag{2.17}$$

Exponential Fuzzy Numbers

We denote the fuzzy number $\tilde{p}$ as an exponential fuzzy number:

$$\tilde{p} = e[a, b, c], \text{ with } a \leq b \leq c, \ a, b, c \in \mathbb{R} \tag{2.18}$$

with membership function

$$\mu_{\tilde{p}}(x) = \begin{cases} 0 & for \quad x \leq a, \\ e^{\dfrac{-(x-b)}{\delta_1}} & for \quad a < x < b \ , (\delta_1 = \dfrac{b-a}{4.5}), \\ e^{\dfrac{-(x-b)}{\delta_2}} & for \quad b \leq x < c \ , (\delta_2 = \dfrac{c-b}{4.5}), \\ 0 & for \quad x \geq c. \end{cases} \qquad \forall x \in \mathbb{R} \tag{2.19}$$

Fig.2.5,2.6,2.7 and 2.8 show the membership functions of each type of fuzzy numbers.

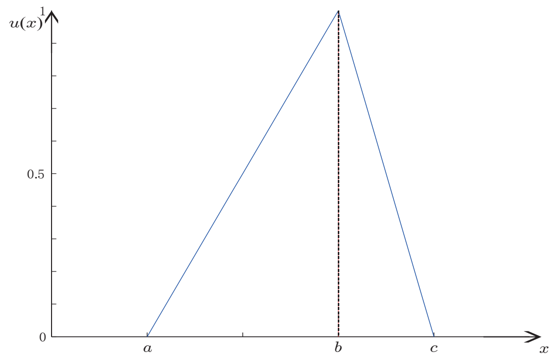


Fig. 2.5: Triangular fuzzy number

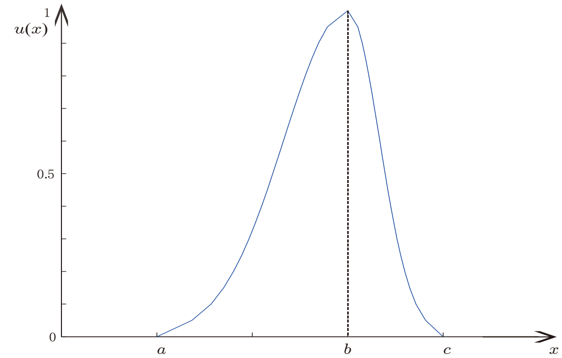


Fig. 2.6: Gaussian fuzzy number

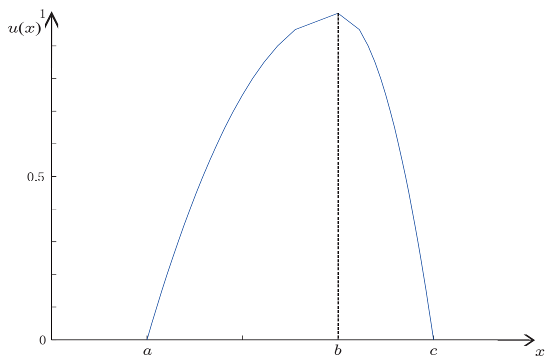


Fig. 2.7: Quadratic fuzzy number

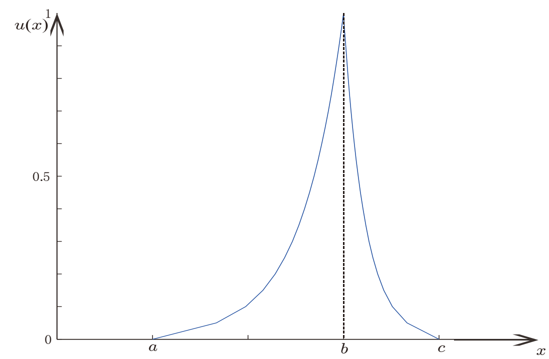


Fig. 2.8: Exponential fuzzy number

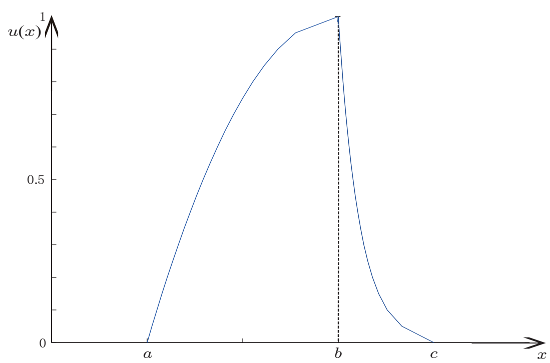


Fig. 2.9: Other types fuzzy number

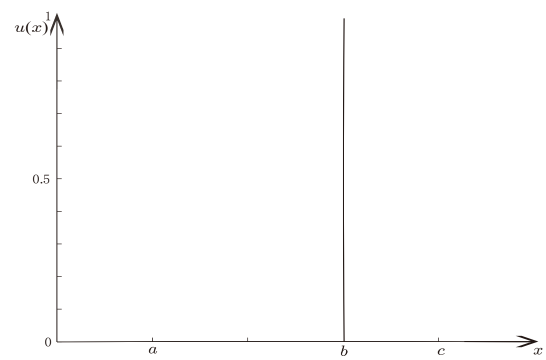


Fig. 2.10: Fuzzy singleton (crisp value)

Other types of fuzzy numbers

As long as a fuzzy set satisfies the conditions in Definition 2.1.4, it is called a fuzzy number. See Fig. 2.9.

Example 2.1.5. We can define a fuzzy number that satisfies the following membership grade:

$$\mu_{\tilde{p}}(x) = \begin{cases} 0 & for \quad x \leq a, \\ 1 - \dfrac{-(x-b)^2}{\omega_1^2} & for \quad a < x < b \ , (\omega_1 = b - a), \\ e^{\dfrac{-(x-b)}{\omega_2}} & for \quad b < x < c \ , (\omega_2 = \dfrac{c-b}{4.5}), \\ 0 & for \quad x \geq c. \end{cases} \qquad \forall x \in \mathbb{R} \tag{2.20}$$

Fuzzy Singletons

When $a = b = c$, the fuzzy number is reduced to a fuzzy singleton, or a crisp number, which is illustrated in Fig.2.10. In this thesis, we only consider the first 4 types of fuzzy numbers.

2.2 Extension Principle

The Extension principle is introduced by Zadeh in 1965 [42], and it provides an universal way of extending mappings or functions from the crisp domain to the fuzzy domain

Definition 2.2.1. Suppose that function f is a mapping $X \to Y$, $\tilde{A}$ and $\tilde{B}$ are fuzzy sets, X is the universe of discourse of $\tilde{A}$, Y is the universe of discourse of $\tilde{B}$. $\tilde{A} \subseteq X$ and $\tilde{B} \subseteq Y$ respectively, $\tilde{A} = \big(\mu_{\tilde{A}}(x_1)/x_1, \mu_{\tilde{A}}(x_2)/x_2,\ \mu_{\tilde{A}}(x_3)/x_3, \ldots, \mu_{\tilde{A}}(x_m)/x_m\big),\ x_i \in X,\ i = 1, 2, 3, \ldots, m.$ Thus we have,

$$\widetilde{B} = f(\tilde{A}) \tag{2.21}$$

Consequently we have $\tilde{B} = \big(\mu_{\tilde{A}}(x_1)/y_1,\ \mu_{\tilde{A}}(x_2)/y_2,\ \mu_{\tilde{A}}(x_3)/y_3,\ \ldots,\ \mu_{\tilde{A}}(x_m)/y_m\big)$, $f(x_i) = y_i$, $y_i \in Y$,$m = 1, 2, 3, \ldots, m$. If there exist $y_j = y_k = \xi$, $j \neq k$ $\mu_{\tilde{A}}(x_j) \neq \mu_{\tilde{A}}(x_k)$, we take

$\mu_{\tilde{B}}(y=\xi) = \sup(\mu_{\tilde{A}}(x_j), \mu_{\tilde{A}}(x_k))$. We denote this as:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup \mu_{\tilde{A}}(x) & for \quad y = f(x), \\ 0 & for \quad y \neq f(x). \end{cases} \tag{2.22}$$

Example 2.2.2. $\tilde{A} = (0.1/-1, 0.5/0.5, 1/1, 0.5/2, 0.1/3), \tilde{B} = f(\tilde{A}) = \tilde{A}^2$.

$\tilde{B} = (0.1/1, 0.5/0.25, 1/1, 0.25/4, 0.1/9)$, The output 1 can be obtained from input 1 or -1, which have different membership grade, 0.1 and 1 respectively. According to the extension principle, we take the maximum of them, so the fuzzy set $\tilde{B} = (0.5/0.25, 1/1, 0.25/4, 0.1/9)$.

Definition 2.2.3. In general, if we have fuzzy sets $\tilde{A}_i$ and crisp sets X_i, where X_i is the universal set of $\tilde{A}_i$ ($\tilde{A}_i \subseteq X_i,\ for\ i = 1, 2, ..., n.$), defined by membership functions $\mu_{\tilde{A}_1}, \mu_{\tilde{A}_2}, \mu_{\tilde{A}_3}, \dots, \mu_{\tilde{A}_n}$ respectively. Define a function f:

$$f : X_1 \times X_2 \times \ldots \times X_n \to Y \tag{2.23}$$

The membership function for $\tilde{B} \subseteq Y$ is then given as:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\{\min\{\mu_{\tilde{A}_1}(x_{1,j}), \mu_{\tilde{A}_2}(x_{2,j}), \dots, \mu_{\tilde{A}_n}(x_{n,j})\}\} & for \quad y = f(x_{1,j}, x_{2,j}, \dots, x_{n,j}), \\ 0 & for \quad y \neq f(x_{1,j}, x_{2,j}, \dots, x_{n,j}). \end{cases} \tag{2.24}$$

with $x_{i,j} \in X_i$, $i, j = 1, 2, 3, \dots, n$.

Example 2.2.4. $\tilde{A}_1 = (0.1/1, 0.5/2, 1/3, 0.5/4, 0.1/5), \tilde{A}_2 = (0.1/2, 0.5/4, 1/6, 0.5/7, 0.1/8)$. Define $\tilde{B} = \tilde{A}_1 + \tilde{A}_2$. Suppose $x_{1,i} \in \tilde{A}_1, x_{2,i} \in \tilde{A}_2, i = 1, 2, 3, 4, 5$.

$$\begin{cases} x_{1,1} = 1 & , & x_{1,2} = 2 & , & x_{1,3} = 3 & , & x_{1,4} = 4 & , & x_{1,5} = 5 \\ \mu_{\tilde{A}_1}(x_{1,1}) = 0.1 & , & \mu_{\tilde{A}_1}(x_{1,2}) = 0.5 & , & \mu_{\tilde{A}_1}(x_{1,3}) = 1 & , & \mu_{\tilde{A}_1}(x_{1,4}) = 0.5 & , & \mu_{\tilde{A}_1}(x_{1,5}) = 0.1 \end{cases} \tag{2.25}$$

$$\begin{cases} x_{2,1} = 2 & , & x_{2,2} = 4 & , & x_{2,3} = 6 & , & x_{2,4} = 7 & , & x_{2,5} = 8 \\ \mu_{\tilde{A}_2}(x_{2,1}) = 0.1 & , & \mu_{\tilde{A}_2}(x_{2,2}) = 0.5 & , & \mu_{\tilde{A}_2}(x_{2,3}) = 1 & , & \mu_{\tilde{A}_2}(x_{2,4}) = 0.5 & , & \mu_{\tilde{A}_2}(x_{2,5}) = 0.1 \end{cases} \tag{2.26}$$

By using the extension principle, we have the following table:

$\mu_{\tilde{B}}(y)/y$	$\mu_{\tilde{A}_1}(x_{1,1})/x_{1,1}$	$\mu_{\tilde{A}_1}(x_{1,2})/x_{1,2}$	$\mu_{\tilde{A}_1}(x_{1,3})/x_{1,3}$	$\mu_{\tilde{A}_1}(x_{1,4})/x_{1,4}$	$\mu_{\tilde{A}_1}(x_{1,5})/x_{1,5}$
$\mu_{\tilde{A}_2}(x_{2,1})/x_{2,1}$	0.1/3	0.1/4	0.1/5	0.1/6	0.1/7
$\mu_{\tilde{A}_2}(x_{2,2})/x_{2,2}$	0.1/5	0.5/6	0.5/7	0.5/8	0.1/9
$\mu_{\tilde{A}_2}(x_{2,3})/x_{2,3}$	0.1/7	0.5/8	1/9	0.5/10	0.1/11
$\mu_{\tilde{A}_2}(x_{2,4})/x_{2,4}$	0.1/8	0.5/9	0.5/10	0.5/11	0.1/12
$\mu_{\tilde{A}_2}(x_{2,5})/x_{2,5}$	0.1/9	0.1/10	0.1/11	0.1/12	0.1/13

(2.27)

y	membership grades	max
3	0.1	0.1
4	0.1	0.2
5	0.1	0.2
6	0.1, 0.5	0.4
7	0.1, 0.5	0.4
8	0.5, 0.1	0.5
9	0.1, 1, 0.5	1
10	0.5, 0.1	0.5
11	0.1, 0.5	0.5
12	0.1	0.1
13	0.1	0.1

(2.28)

and we have $\tilde{B} = (0.1/3, 0.1/4, 0.1/5, 0.5/6, 0.5/7, 0.5/8, 1/9, 0.5/10, 0.5/11, 0.1/12, 0.1/13)$.

2.3 Standard Fuzzy arithmetic

Standard fuzzy arithmetic operation is based on the Extension Principle. Interested readers may refer to [19] for details of standard fuzzy arithmetic.

2.3.1 Fuzzy sets operations using the Extension Principle

Consider the Example 2.2.4 with fuzzy sets $\tilde{A}_1$ and $\tilde{A}_2$ are fuzzy numbers, $\tilde{A}_1 = t[0.9, 3, 5.1]$ and $\tilde{A}_2 = t[1.9, 6, 8.1]$. For two crisp values $x_1 = 3$ from $\tilde{A}_1$, $\mu_{\tilde{A}_1}(x_1 = 3) = 1$ and $x_2 = 6$ from $\tilde{A}_2$, $\mu_{\tilde{A}_2}(x_2 = 6) = 1$, $y = x_1 + x_2 = 9$, the membership grade is given by $\min\{\mu_{\tilde{A}_1}(x_1 = 3),\ \mu_{\tilde{A}_2}(x_2 = 6)\} = 1$. However, the result $y = x_1 + x_2 = 9$ can be obtianed from $x_1 = 5$, $\mu_{\tilde{A}_1}(x_1 = 5) = 0.1$, $x_2 = 4$, $\mu_{\tilde{A}_2}(x_2 = 4) = 0.5$, the membership grade of y is given by $\min\{\mu_{\tilde{A}_1}(x_1 = 5),\ \mu_{\tilde{A}_2}(x_2 = 4)\} = 0.5$. According to the Definition 2.1.4, a fuzzy number is a continuous fuzzy set. There is an infinite number of combinations of x_1 and x_2 , which lead to the same result y. This makes fuzzy arithmetic using the Extension Principle difficult.

2.3.2 Standard fuzzy arithmetic

Directly applying the extension principle for the operation of fuzzy numbers is not practical in the computational context. Approximated solution of fuzzy arithmetic is given by discretizing the continuous membership grade range $[0, 1]$ and then performing interval arithmetic on the $\alpha - cuts$ of input parameters [17].

Definition 2.3.1. For any two fuzzy numbers $\tilde{a}$ $\tilde{b}$, at any membership grade $\mu(\tilde{a}) = \mu(\tilde{b}) = \alpha$,$\alpha \in [0, 1]$, we have

- $[\underline{a}, \overline{a}]^\alpha + [\underline{b}, \overline{b}]^\alpha = [(\underline{a} + \underline{b}), (\overline{b} + \overline{b})]^\alpha$
- $[\underline{a}, \overline{a}]^\alpha - [\underline{b}, \overline{b}]^\alpha = [(\underline{a} - \overline{b}), (\overline{a} - \underline{b})]^\alpha$
- $[\underline{a}, \overline{a}]^\alpha \cdot [\underline{b}, \overline{b}]^\alpha = [\min\big(\underline{a}\overline{b}, \underline{a}\underline{b}, \overline{a}\underline{b}, \underline{a}\overline{b}\big), \max\big(\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\big)]^\alpha$
- $[\underline{a}, \overline{a}]^\alpha \ / \ [\underline{b}, \overline{b}]^\alpha = [\min\big(\frac{\underline{a}^\alpha}{\underline{b}}, \frac{\underline{\underline{a}}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}, \frac{\overline{a}}{\overline{\overline{b}}}\big), \max\big(\frac{\underline{a}}{\underline{b}}, \frac{\underline{\underline{a}}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}, \frac{\overline{a}}{\overline{\overline{b}}}\big)]^\alpha$

where $\underline{a} < \overline{a}$ and $\underline{b} < \overline{b}$. Moreover, $\underline{a}^\alpha$ $\underline{b}^\alpha$ denote as the lower bound, and $\overline{a}^\alpha$ $\overline{b}^\alpha$ denote as the upper bound of α-cuts of $[\underline{a}, \overline{a}]^\alpha$ and $[\underline{b}, \overline{b}]^\alpha$ respectively, at membership grade α, $\alpha \in [0, 1]$. If $\alpha = 1$, $\underline{a} = \overline{a} = a$, $\underline{b} = \overline{b} = b$.

Example 2.3.2. Suppose $\tilde{x}_1 = t[1,2,4]^\dagger$, $\tilde{x}_2 = g[3,5,6]^\dagger$.

1. For the membership grade $\alpha = 0$,

   $\tilde{x}_1^{\alpha=0} + \tilde{x}_2^{\alpha=0} = \text{supp}(\tilde{x}_1) + \text{supp}(\tilde{x}_2) = [1,4] + [3,6] = [4,10].$

2. For the membership grade $\alpha = 1$,

   $\tilde{x}_1^{\alpha=1} = \text{core}(\tilde{x}_1) = 2,\ \tilde{x}_2^{\alpha=1} = \text{core}(\tilde{x}_2) = 5,\ \text{core}(\tilde{x}_1+\tilde{x}_2) = 7,$ the result for $\tilde{x}_1+\tilde{x}_2$ is $[4,7,10]$ ††.

3. For the membership grade $\alpha = 0.1$

   $\tilde{x}_1^{\alpha=0.1} = [1.1, 3.8],\ \tilde{x}_2^{\alpha=0.1} = [3.5, 5.7],\ [\tilde{x}_1 + \tilde{x}_2]^{\alpha=0.1} = [4.6, 9.5].$

4. For the membership grade $\alpha = 0.5$

   $\tilde{x}_1^{\alpha=0.5} = [1.5, 3],\ \tilde{x}_2^{\alpha=0.5} = [4.2, 5.4],\ [\tilde{x}_1 + \tilde{x}_2]^{\alpha=0.5} = [5.7, 8.4].$

The resultant and fuzzy number is denoted in Fig.2.11.

Example 2.3.3. $\tilde{x}_1 = q[1,2,3]^\dagger$, $\tilde{x}_2 = e[3,5,6]^\dagger$.

1. For the membership grade $\alpha = 0$, $\text{supp}(\frac{\tilde{x}_2^{\alpha=0}}{\tilde{x}_1^{\alpha=0}}) = \frac{[3,6]}{[1,3]} = [1,6]$.

2. For the membership grade $\alpha = 1$, $\tilde{x}_1^{\alpha=1} = \text{core}(x_1) = 2,\ \tilde{x}_2^{\alpha=1} = \text{core}(x_2) = 5,\ \text{core}(\frac{\tilde{x}_2^{\alpha=1}}{\tilde{x}_1^{\alpha=1}}) =$ 2.5.

The result for $\tilde{x}_1 + \tilde{x}_2$ is $[1, 2.5, 6]$ ††. The resultant fuzzy number is denoted in Fig.2.12.

---

† t denotes triangular fuzzy number, g denotes gaussian fuzzy number, q denotes quadratic fuzzy number, e denotes exponential fuzzy number.

†† The resultant fuzzy set may not follow the standard types described before, as Triangular or Gaussian fuzzy numbers. This representation only lists the core and support of the fuzzy number, but not the left side and right side membership functions. These functions may not follow standard forms.

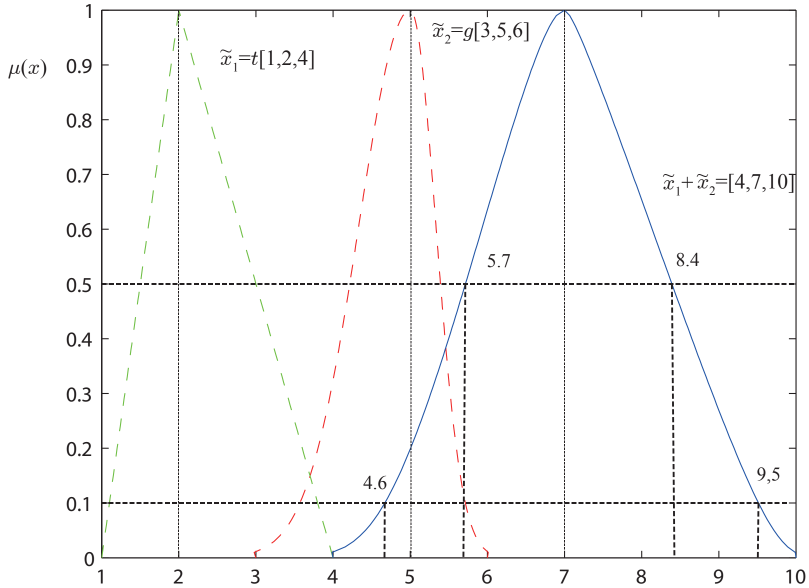


Fig. 2.11: Example of Fuzzy Arithmetic operation: addition

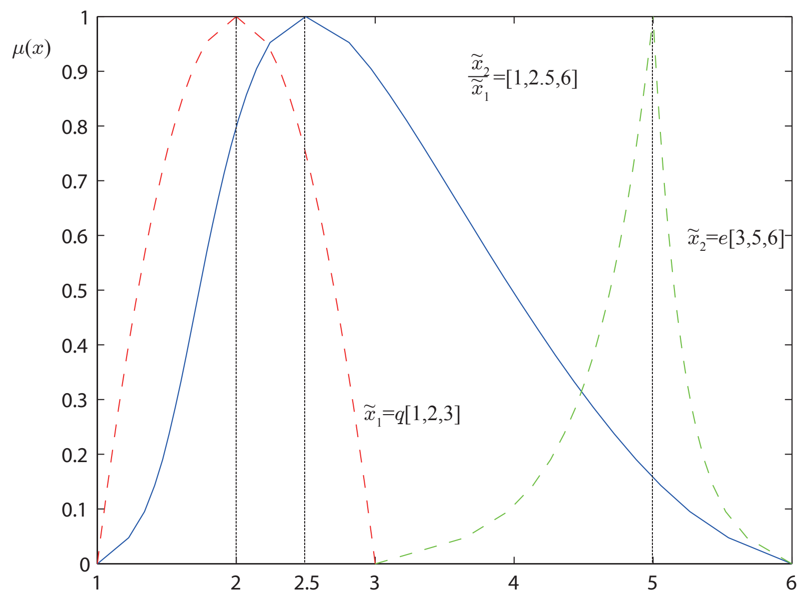


Fig. 2.12: Example of Fuzzy Arithmetic operation: division

2.3.2.1 Drawbacks of the Standard Fuzzy Arithmetic

When we perform with the arithmetic operations of fuzzy numbers, standard fuzzy arithmetic is based on interval arithmetic, which may overestimate the interval at each α-cut of the resultant fuzzy set.

Example 2.3.4. Considering the fuzzy polynomial function:

$$f(\tilde{x}) = \tilde{x}^2 - \tilde{x} \tag{2.29}$$

Let $\tilde{x} = t[2, 2.5, 3]$. By directly applying the fuzzy arithmetic we can get:

$$\begin{aligned} \text{support}: &\quad [2,3]^2 - [2,3] = [4,9] - [2,3] = [1,7], \\ \text{core}: &\quad 2.5^2 - 2.5 = 3.75. \end{aligned} \tag{2.30}$$

The result is [1,3.75,7]. Consider the same function in factored form:

$$f(\tilde{x}) = \tilde{x}^2 - \tilde{x} = \tilde{x}(\tilde{x} - 1) \tag{2.31}$$

using the standard fuzzy arithmetic we have:

$$\begin{aligned} \text{support}: &\quad [2,3]([2,3] - 1) = [2,3][1,2] = [2,6], \\ \text{core}: &\quad 2.5(2.5 - 1) = 3.75. \end{aligned} \tag{2.32}$$

In fact, the function f is monotonic increasing in $[2, 3]$, the local minimum value for $\tilde{x} = t[2, 2.5, 3]$ in function f is $f(2) = 2^2 - 2 = 2$, the local maximum value is $f(3) = 3^2 - 3 = 6$. The second result of using the factored form which will lead the exact result. However, not all the fuzzy function written as other forms using the standard fuzzy arithmetic can get the exact result. For example:

Example 2.3.5. Considering the fuzzy polynomial function:

$$f(\tilde{x}) = \tilde{x}^3 - 2\tilde{x}^2 - 3x \tag{2.33}$$

$\tilde{x} = t[2, 2.5, 3]$, directly using the standard fuzzy arithmetic we can get:

$$\begin{aligned}
\text{support}: \quad & [2,3]^3 - 2[2,3]^2 - 3[2,3] \\
= \quad & [8,27] - [4,6] - [6,9] \\
= \quad & [2,23] - [6,9] \\
= \quad & [-7,17]. \\
\text{core}: \quad & 2.5^3 - 12.5 - 7.5 = -4.375.
\end{aligned} \tag{2.34}$$

Consider the function as a factored form:

$$f(\tilde{x}) = \tilde{x}^3 - 2\tilde{x}^2 - 3x = \tilde{x}(\tilde{x} - 3)(\tilde{x} + 1) \tag{2.35}$$

using the standard fuzzy arithmetic we can get:

$$\begin{aligned}
\text{support}: \quad & [2,3] \cdot ([2,3] - 3)([2,3] + 1) \\
= \quad & [2,3] \cdot [-1,0][3,4] \\
= \quad & [-3,0] \cdot [3,4] \\
= \quad & [-12,0] \\
\text{core}: \quad & 2.5 \cdot (2.5 - 3) \cdot (2.5 + 1) \\
= \quad & 2.5 \cdot (-0.5) \cdot (3.5) \\
= \quad & -4.375.
\end{aligned} \tag{2.36}$$

Express the function in Horner's form:

$$f(\tilde{x}) = \tilde{x}^3 - 2\tilde{x}^2 - 3x = \tilde{x}\big((\tilde{x} - 2)\tilde{x} - 3\big) \tag{2.37}$$

using the standard fuzzy arithmetic we can get:

$$\begin{aligned}
\text{support}: \quad & [2,3] \cdot \big(([2,3] - 2) \cdot [2,3] - 3\big) \\
= \quad & [2,3] \cdot ([0,1] \cdot [2,3] - 3) \\
= \quad & [2,3] \cdot [-3,0] \\
= \quad & [-9,0]. \\
\text{core}: \quad & 2.5 \cdot \big((2.5 - 2) \cdot 2.5 - 3\big) = -4.375.
\end{aligned} \tag{2.38}$$

In fact, the function f is monotonic increasing in the interval $[2,3]$, the local minimum value for $\tilde{x} = t[2, 2.5, 3]$ in function f is $f(2) = 2^3 - 2 \cdot 2^2 - 3 \cdot 2 = -6$, the local maximum value is $f(3) = 3^3 - 2 \cdot 3^2 - 3 \cdot 3 = 0$. The exact result is [-6,-4.375,0]. Comparing with the 3 results, apparently, the supports of three results in using standard fuzzy arithmetic are larger than the exact result, in which the size of α-cuts are overestimated. Proper fuzzy arithmetic [18] can alleviate the overestimation problem in resultant α-cuts of fuzzy set under arithmetic operations.

2.4 Proper Fuzzy Arithmetic

Proper fuzzy arithmetic is proposed in [18]. (Interested readers may refer to [18] for the proof of overestimation avoidance by proper fuzzy arithmetic.)

Definition 2.4.1. If we have n fuzzy numbers: $\tilde{x}_i, i = 1, 2, 3, \ldots, n$, each of which has a membership function $\mu_{\tilde{x}_i}(x)$, where $x \in \tilde{x}_i$. Assume that the result of operating these n fuzzy numbers is $\tilde{X}$, for each α-cut of $\mu(\tilde{X}) \geq \alpha \in [0,1]$ we have following properties:

$$[X]^\alpha = [\underline{X}, \overline{X}]^\alpha,$$

$$\underline{X}^\alpha = \min(C),$$

$$\overline{X}^\alpha = \max(C).$$

where C is all possible combinations of operating the crisp values of upper bound and lower bound of α-cuts of all fuzzy numbers at membership grade $\mu(\tilde{X}_i) \geq \alpha \in [0,1]$. The number of combinations is 2^n.

The following example shows how *all possible combinations of upper and lower bounds of α-cuts* are generated:

Example 2.4.2. Assume that we have a fuzzy function:

$$f(x_1 x_2 x_3) = \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 - \frac{\tilde{x}_1 \tilde{x}_2}{\tilde{x}_3} \tag{2.39}$$

with $\tilde{x}_1 = t[-1, x_1, 2], \tilde{x}_2 = t[1, x_2, 3], \tilde{x}_3 = t[2, x_3, 3], (-1 < x_1 < 2), (1 < x_2, 3), (2 < x_3 < 3)$. At the membership grade $\mu = 0$ we have the support of the result, by using the standard fuzzy arithmetic, we have

$$\begin{aligned} \text{support}: \quad & [-1,2]\cdot[1,3]\cdot[2,3] - \frac{[-1,2]\cdot[1,3]}{[2,3]} \\ = \quad & [-3,6]\cdot[2,3] - \frac{[-3,6]}{[2,3]} \\ = \quad & [-9,18] - [-1.5,3] \\ = \quad & [-12,19.5]. \end{aligned} \tag{2.40}$$

Actually if we directly apply the extension principle, the result is [-8,16] †.

If the proper fuzzy arithmetic is applied, we can construct this matrix using the upper bounds and lower bounds of α-cuts of fuzzy sets $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{x}_3$ at $\mu = 0$:

$$\begin{pmatrix} -1 & -1 & -1 & -1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 1 & 1 & 3 & 3 \\ 2 & 3 & 2 & 3 & 2 & 3 & 2 & 3 \end{pmatrix} \tag{2.41}$$

For $n = 3$ (3 fuzzy numbers), we have $2^n = 8$ combinations which located in every column in (2.41), the set of these combination is:

$$\big\{\{-1,1,2\},\{-1,1,3\},\{-1,3,2\},\{-1,3,3\},\{2,1,2\},\{2,1,3\},\{2,3,2\},\{2,3,3\}\big\}. \tag{2.42}$$

and we have :

$$\begin{cases} (-1\cdot 1\cdot 2) - (\dfrac{-1\cdot 1}{2}) & = \quad -1\frac{1}{2}; \\ (-1\cdot 1\cdot 3) - (\dfrac{-1\cdot 1}{3}) & = \quad -2\frac{2}{3}, \\ (-1\cdot 3\cdot 2) - (\dfrac{-1\cdot 3}{2}) & = \quad -4\frac{1}{2}, \\ (-1\cdot 3\cdot 3) - (\dfrac{-1\cdot 3}{3}) & = \quad -8 \ (minimum), \\ (2\cdot 1\cdot 2) - (\dfrac{2\cdot 1}{2}) & = \quad 3, \\ (2\cdot 1\cdot 3) - (\dfrac{2\cdot 1}{3}) & = \quad 5\frac{1}{3}, \\ (2\cdot 3\cdot 2) - (\dfrac{2\cdot 3}{2}) & = \quad 9, \\ (2\cdot 3\cdot 3) - (\dfrac{2\cdot 3}{3}) & = \quad 16 \ (maximum). \end{cases} \tag{2.43}$$

†Calculation is not shown due to the complexity. It is calculated in the Appendix A.

At $\mu = 0$, the lower bound of α-cuts of $\tilde{x}_1, \tilde{x}_2$ and $\tilde{x}_3$ are $-1, 1, 2$ respectively; the upper bound of α-cuts of $\tilde{x}_1, \tilde{x}_2$ and $\tilde{x}_3$ are $2, 3, 3$ respectively. The support of the resultant fuzzy set is [-8,16]. Through the results comparison we can find that the proper fuzzy arithmetic can overcome the over estimation by using the standard fuzzy arithmetic.

2.4.1 Modified Proper Fuzzy Arithmetic

If the function is monotonic, the local minimum or maximum of a α-cut interval of the resultant fuzzy set is at the lower or upper bounds. When a fuzzy function is non-monotonic, the local minimum or maximam of the functions f are located within the range of α-cuts of the fuzzy numbers, at membership grade value $\mu \in [0, 1]$. We discretize the corresponding α-cut interval in to multiple subintervals for calculation to locate the local minimum or maximum.†

Example 2.4.3. Consider the fuzzy polynomial function in Example 2.3.4:

$$f(\tilde{x}) = \tilde{x}^2 - \tilde{x}$$

with $\tilde{x} = t[0, 1, 3]$, which is triangular fuzzy number. At membership grade $\mu = 0$, by directly using the fuzzy arithmetic the support of the resultant fuzzy set is,

$$[0, 3]^2 - [0, 3] = [0, 9] - [0, 3] = [-3, 9].$$

by using proper fuzzy arithmetic, we have the result:

$$f([0, 3]) = [0, 6].$$

Actually, from the Fig.2.13 we can see that, the actual exact range is [-0.25, 6]. The reason for this error is that the local minimum of the function f is located at $x = 0.5$, $0.5 \in [0, 3]$. Due to this phenomenon, we can discretize the fuzzy number at membership grade $\mu = 0$ into l subintervals, and use proper fuzzy arithmetic to divide each subintervals to get the local minimum and maximum crisp values within the range. The larger the value m is, the more accurate the result is. The example is illustrated in Fig.2.14.

†The reason why using this method instead of other optimization methods is explained in Section 4.4.3

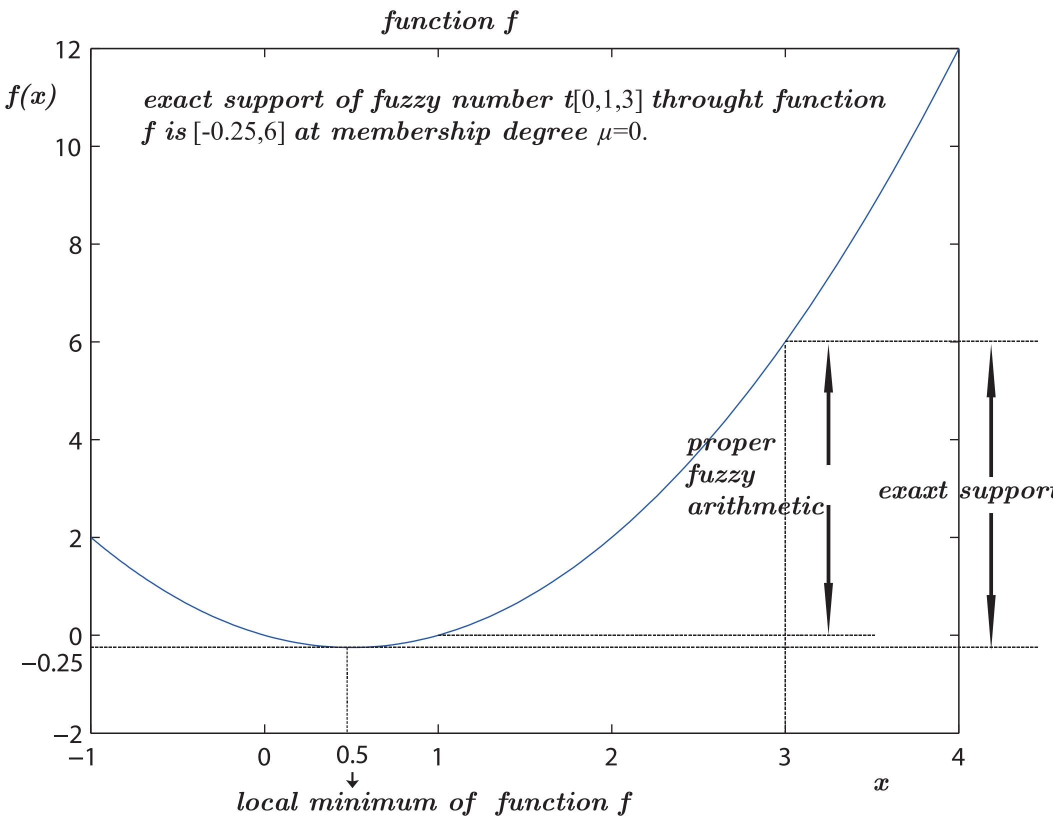


Fig. 2.13: Range of the function

The minimum value of the function located at $x = 0.5$ at the lower bound of the interval

The exact support of the result is [-0.25,6]

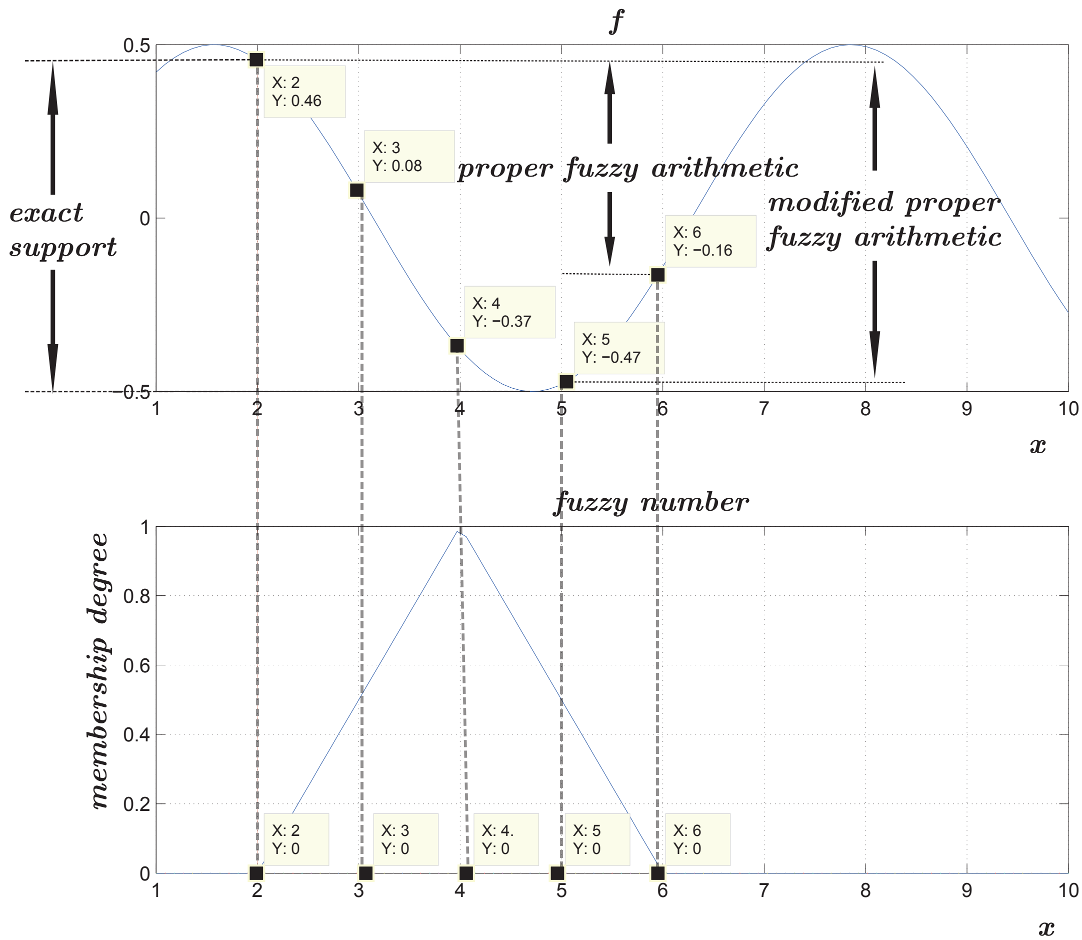


Fig. 2.14: Discretization of the fuzzy number

Example 2.4.4. As the Fig.2.14 shown, $f = 0.5sin(x)$, $x = t[2, 4, 6]$. At membership grade $\mu = 0$, the valley value of f is located in the interval as the figure shown. We discretize the fuzzy number at membership grade $\alpha = 0$ into 3 pairs: [2,6],[3,5],[4,4], and calculate each of them to find the minimum and maximum, in order to get optimal boundary values at $\mu = 0$. For the next membership grade, the same procedure is repeated.

In this chapter, we investigated the background of fuzzy sets, discussed the extension principle and standard fuzzy arithmetic. Finally, proper fuzzy arithmetic is introduced.

Chapter 3

Fuzzy Differential Equations

Dynamic systems in real world are modeled by ordinary differential equations(ODE) as:

$$\dot{x} = f(t, x), \quad x(t_0) = x_0. \tag{3.1}$$

or a system of differential equations, for the study of their behaviors. However, it is hard to obtain precise parameters of the model of a dynamic system in reality. Moreover, it is difficult to get accurate analysis of behaviors of the systems, which will make decision making difficult due to the uncertainty. One type of uncertainty in real world modeling is impreciseness or vagueness. If the initial value or the parameters in the ODE is uncertain, the model of the dynamic system is uncertain.

Based on the fuzzy sets theory, the calculus of fuzzy functions is investigated. In [34] the authors first developed the concept of fuzzy functions. Based on that, the concept of fuzzy derivative was first introduced by Chang and Zadeh [9]. Later Dubois and Prade [14] defined fuzzy derivative using the extension principle. Based on the above notions, the ODEs with imprecise information is investigated. Differential equations with the fuzzy initial values or parameters are named as Fuzzy differential equations (FDEs). In 1987, Kaleva [23] initially developed the theory of FDEs.

FDEs are usually employed for the purpose of modeling dynamic systems with uncertain parameters in science and engineering. Most of the problems in science and engineering require the solution of a FDE which is characterized by fuzzy parameters or/and initial conditions, and is named as a fuzzy initial value problem (FIVP) or fuzzy cauchy problem. The FDE and the initial value problem (cauchy problem) were rigorously investigated by Kaleva [23] and Seikkala [37]. In the last few years, many works have been reported in theoretical and applied research in FDE based on Hukuhara derivative(H-derivative). The H-derivative of a fuzzy-number-valued function was introduced in [34]. Under this condition, mainly existence and uniqueness theorems for the solution of a fuzzy differential equation are considered (e.g.[23] [37] [38]). However, in some cases this approach suffers disadvantages since the diameter of the support of the solution $x(t)$ of an FDE is unbounded as time t progresses, which has been pointed out in [11] [13] [6] and [40] . Under this condition, generalized differentiability theorem is introduced in [3] [4], which allow us to overcome the above mentioned shortcomings.

Another approach in modeling uncertain dynamic systems is to treat fuzzy differential equation as a system of differential inclusions (see e.g.[22] [12]), where they introduce a more general definition of derivative for fuzzy functions enlarging the class of differentiable fuzzy mappings by considering fuzzy lateral H-derivatives.

In some other approaches, the extension principle is directly applied to solve FDEs, like in [30]. However, in paper [39] the authors show an counter-example to prove the result in paper [30] was not valid if the FDE is non-monotonic. The aim this paper was to find an explicit solution of the FDE.

3.1 Fuzzy Derivatives

In this section, we will review some definitions and properties of fuzzy differential equation. Recall the Definition 2.1.4 in Chapter 2, we have:

Definition 3.1.1. Denote that E^n is the space of all fuzzy subsets u of $\mathbb{R}^n$ satisfying the Definition 2.1.4: For $u, v \in E^n$, $\lambda \in \mathbb{R}$, we can have $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda [u]^\alpha$, $\forall \alpha \in [0,1]$, where '$[\ ]^\alpha$' denotes the α-cuts.

Definition 3.1.2. Let $f : T \to E^n$, T is the time span, $T \subset \mathbb{R}^+$, its α-cut ($\alpha \in [0,1]$) is written as follow:

$$[f(t)]^\alpha = [\underline{f}(t), \overline{f}(t)]^\alpha. \tag{3.2}$$

where $\underline{f}(t)^\alpha$ denotes the lower bound, and $\overline{f}(t)^\alpha$ denotes the upper bound of the α-cut.

The metric structure is given by the Hausdorff distance:

$$D(u,v) = sup\ \max\{|\underline{u}(x) - \underline{v}(x)|, |\overline{u}(x) - \overline{v}(x)|\}, u, v \in E^n. \tag{3.3}$$

Theorem 3.1.3. (E^n, D) is a complete metric space.

Proof. See the reference in [40]. □

And we have the following properties:

$$D(u+w, v+w) = D(u,v),\ \forall u,v,w \in E^n \tag{3.4}$$

$$D(ku, kv) = |k|D(u,v),\ \forall k \in \mathbb{R} \setminus \{0\}, u, v \in E^n \tag{3.5}$$

$$D(u+v, w+e) \le D(u,w) + D(v,e),\ \forall u,v,w,e \in E^n \tag{3.6}$$

Some basic definitions of H-derivatives were described in Definition 3.1.4 and 3.1.5.

Definition 3.1.4. Let $x, y \in E^n$. If there exist $z \in E^n$ such that $x = y + z$, then z is called the H-difference of x, y and it is denoted by $x \ominus y$. [20]

Note that $x \ominus y \neq x + (-1)y = x - y$.

Definition 3.1.5. Let $f : T \to E^n$ is differentiable at $t_0 \in T$, if there exist a $f'(t_0) \in E^n$, it is defined as,

$$\lim_{h \to 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0). \tag{3.7}$$

as in [35].

Definition 3.1.6. If f is differentiable, then the multi-valued mapping f_α is H-differentiable for all $\alpha \in [0,1]$ and

$$Df_\alpha = [f'(t)]^\alpha \tag{3.8}$$

where f_α denotes the α-cuts of f, Df_α denotes the H-derivative of f_α [23].

Example 3.1.7. Consider a FDE problem

$$\dot{x} = ax,\ \tilde{x}_0 = t[-1, 0, 1],\ a \in \mathbb{R},\ \tilde{x}_0 \in E^n \tag{3.9}$$

From the Definition 3.1.2, we have:

$$[x(t)]^\alpha = [\underline{x}(t), \overline{x}(t)]^\alpha,\ \text{for } \alpha \in [0, 1],\ \text{and } t \in [0, T] \tag{3.10}$$

From the H-derivative, we have:

$$[\dot{x}]^\alpha = [\underline{\dot{x}}, \overline{\dot{x}}]^\alpha,\ \text{for } \alpha \in [0, 1],\ \text{and } t \in [0, T] \tag{3.11}$$

By applying the extension principle, we have:

$$[ax(t)]^\alpha = [\min\{a\underline{x}(t)^\alpha, a\overline{x}(t)^\alpha\}, \max\{a\underline{x}(t)^\alpha, a\overline{x}(t)^\alpha\}] \tag{3.12}$$

which will lead to:

$$\underline{\dot{x}} = \min[a\underline{x}(t)^\alpha, a\overline{x}(t)^\alpha],\ \overline{\dot{x}} = \max[a\underline{x}(t)^\alpha, a\overline{x}(t)^\alpha]. \tag{3.13}$$

Now consider $a = -1$, we have FDE:

$$\dot{x} = -x,\ \tilde{x}_0 = t[-1, 0, 1],\ -1 \in \mathbb{R},\ \tilde{x}_0 \in E^n \tag{3.14}$$

Equation (3.13) becomes:

$$\underline{\dot{x}}^\alpha = -\overline{x}^\alpha, \overline{\dot{x}}^\alpha = -\underline{x}^\alpha \tag{3.15}$$

Assume $\lambda_1 = \underline{x}^\alpha, \lambda_2 = \overline{x}^\alpha$, we have:

$$\begin{cases} \dot{\lambda}_1 &= -\lambda_2 \\ \dot{\lambda}_2 &= -\lambda_1 \end{cases} \Rightarrow \begin{cases} \ddot{\lambda}_1 &= -\dot{\lambda}_2 = \lambda_1 \\ \ddot{\lambda}_2 &= -\dot{\lambda}_1 = \lambda_2 \end{cases} \tag{3.16}$$

Apparently, the differential equation $\ddot{x} = x$ has solution $x = k_1 e^t + k_2 e^{-t}$. With the initial value $\tilde{x}_0 = t[-1, 0, 1]$ with α-cuts $[x_0]^\alpha = [\underline{x_0}, \overline{x_0}]^\alpha = [\alpha - 1, 1 - \alpha]$, we can get the solution for λ_1 and λ_2:

$$\begin{cases} \lambda_1 &= \frac{1}{2}[\alpha - 1 - (1 - \alpha)]e^t + \frac{1}{2}[\alpha - 1 + (1 - \alpha)]e^{-t}, \\ \lambda_2 &= \frac{1}{2}[1 - \alpha - (\alpha - 1)]e^t + \frac{1}{2}[1 - \alpha + (\alpha - 1)]e^{-t}. \end{cases} \tag{3.17}$$

We can obtain $\lambda_1 = \underline{x}(t)^\alpha = (\alpha - 1)e^t$, $\lambda_2 = \overline{x}(t)^\alpha = (1 - \alpha)e^t$, which $\lambda_1 = \underline{x}(t)^\alpha$ and $\lambda_2 = \overline{x}(t)^\alpha$ are left side and right side membership function of the solution respectively. As time $t \to +\infty$, the $\mathrm{diam}(x(t)) \to +\infty$ The support of the solution keeps increasing as time t increases.

To overcome the inconvenience, we define generalized H-derivatives:

Definition 3.1.8. Let $f : T \to E^n$ is differentiable at $t_0 \in T$, if there exist a $f'(t_0) \in E^n$ and meet the following limits:

i

$$\lim_{h \to 0^+} \frac{f(t_0+h) \ominus f(t_0)}{h} = \lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0-h)}{h} = f'(t_0). \tag{3.18}$$

ii

$$\lim_{h \to 0^-} \frac{f(t_0+h) \ominus f(t_0)}{h} = \lim_{h \to 0^-} \frac{f(t_0) \ominus f(t_0-h)}{h} = f'(t_0). \tag{3.19}$$

as defined in [3].

Theorem 3.1.9. [8] Let $f : T \to E^n$ be a function and denote

$$[f(t)]^\alpha = [\underline{f}_\alpha(t), \overline{f}_\alpha(t)] \tag{3.20}$$

for each $\alpha \in [0, 1]$. Then:

i if x is differentiable in the form (i) of Definition 3.1.8, then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and

$$[f'(t)]^\alpha = [\underline{f}'_\alpha(t), \overline{f}'_\alpha(t)] \tag{3.21}$$

ii if x is differentiable in the form (ii) of Definition 3.1.8, then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and

$$[f'(t)]^\alpha = [\overline{f}'_\alpha(t), \underline{f}'_\alpha(t)] \tag{3.22}$$

Let us consider the fuzzy differential equations with initial value problem (FIVP):

$$x'(t) = f(t, x(t)),\ t \in T,\ x(0) = \tilde{x_0} \tag{3.23}$$

where $f : T \to E^n$ is a continuous fuzzy mapping and $\tilde{x_0}$ is a fuzzy number. Let the α-cut $[x(t)]^\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha]$. If $x(t)$ satisfies (i) of Theorem 3.1.9, define $[D_1x(t)]^\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$ and

(3.23) is translated into following set of ordinary differential equations:

$$\begin{cases} \underline{x}'(t) &= \underline{f}_\alpha(t, \underline{x}_\alpha, \overline{x}_\alpha) = F(t, \underline{x}, \overline{x}), \\ \overline{x}'(t) &= \overline{f}_\alpha(t, \underline{x}_\alpha, \overline{x}_\alpha) = G(t, \underline{x}, \overline{x}). \end{cases} \tag{3.24}$$

We call it as (1)-solution.

If $x(t)$ satisfies (ii) of Theorem 3.1.9, define $[D_2x(t)]^\alpha = [\overline{x}_\alpha(t), \underline{x}_\alpha(t)]$ and (3.23) is translated into following set of ordinary differential equations:

$$\begin{cases} \underline{x}'(t) &= \overline{f}_\alpha(t, \underline{x}_\alpha, \overline{x}_\alpha) = G(t, \underline{x}, \overline{x}), \\ \overline{x}'(t) &= \underline{f}_\alpha(t, \underline{x}_\alpha, \overline{x}_\alpha) = F(t, \underline{x}, \overline{x}). \end{cases} \tag{3.25}$$

We call it as (2)-solution.

In [8], the authors proved that the solution of $[\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$ of (3.24) are valid α-cuts of a fuzzy number-valued function. If $[\underline{x}'_\alpha(t), \overline{x}'_\alpha(t)]$ are valid α-cuts of a fuzzy valued function, by the Stacking theorem [23] , it is possible to get the (1)-solution(3.24) of FIVP (3.23). In the meantime, we can proceed in a similar way with system (3.25) for (2)-solution. Details of the proof is given in [33].

Theorem 3.1.10. If f satisfies a Lipschitz condition on an interval T then f is uniformly continuous on T.

If f satisfies Lipschitz condition on an interval T if there exists $M > 0$ such that

$$|f(x_1) - f(x_2)| \leq M|x_1 - x_2| \tag{3.26}$$

for all $x_1, x_2 \in T$ and $x_1 \neq x_2$. Notice that if $x_1 = x_2$ then $f(x_1) = f(x_2)$ and Lipschitz is automatically satisfied for any M.

Proof. For any $\epsilon > 0$ we choose $\delta < \frac{\epsilon}{M}$ so that for all $x_1, x_2 \in T$ that satisfy $|x_1 - x_2| < \delta$ we have

$$|f(x_1) - f(x_2)| \leq M|x_1 - x_2| < M\delta < \epsilon$$

□

Thus, consider the problem (3.23) we have:

Theorem 3.1.11. $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are equicontinuous, there exists $L > 0$:

$$\begin{cases} |\underline{f}_\alpha(t, x_1, y_1) - \overline{f}_\alpha(t, x_2, y_2)| \leq L \max\{|x_1 - x_2|, |y_1 - y_2|\} \\ |\overline{f}_\alpha(t, x_1, y_1) - \underline{f}_\alpha(t, x_2, y_2)| \leq L \max\{|x_1 - x_2|, |y_1 - y_2|\} \end{cases}, \ \forall \alpha \in [0, 1]. \tag{3.27}$$

(3.23) and (3.24) are equivalent under the condition (i) of Theorem 3.1.9, (3.23) and (3.25) are equivalent under the condition (ii) of Theorem 3.1.9.

Proof. See [8] and [2]. □

3.2 Numerical Methods for Fuzzy Initial Value Problem

This section presents traditional numerical methods for FDE solution using Runge-Kutta method.

3.2.1 Runge-Kutta Method

Given an ODE,

$$\frac{dy}{dt} = f(t, y) \tag{3.28}$$

The Runge-Kutta solution is given as,

$$\begin{aligned} y_{n+1} &= y_n + h \sum_{i=1}^{s} b_i k_i \\ k_i &= f\left(t_n + c_i h, y_n + h \sum_{j=1}^{i-1} a_{ij} k_j\right). \end{aligned} \tag{3.29}$$

where h is the time step, a_{ij}, b_i, c_i are coefficients listed in the table.

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \dots & a_{1s} \\ c_2 & a_{21} & a_{22} & \dots & a_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & a_{s2} & \dots & a_{ss} \\ \hline & b_1 & b_2 & \dots & b_s \end{array} \tag{3.30}$$

In this chapter, we consider the RK-2 numerical method for the FDEs in Section 3.2.2:

$$\begin{array}{c|cc} 0 & & \\ \theta & \theta & \\ \hline & 1 - \frac{1}{2\theta} & \frac{1}{2\theta} \end{array} \tag{3.31}$$

In this thesis, we apply the Dormand-Prince coefficients (RK-5) [16] in numerical examples presented in Chapter 5:

$$
\begin{array}{c|ccccccc}
0 & & & & & & & \\
1/5 & 1/5 & & & & & & \\
3/10 & 3/40 & 9/40 & & & & & \\
4/5 & 44/45 & -56/15 & 32/9 & & & & \\
8/9 & 19372/6561 & -25360/2187 & 64448/6561 & -212/729 & & & \\
1 & 9017/3168 & -355/33 & 46732/5247 & 49/176 & -5103/18656 & & \\
1 & 35/384 & 0 & 500/1113 & 125/192 & -2187/6784 & 11/84 & \\
\hline
 & 35/384 & 0 & 500/1113 & 125/192 & -2187/6784 & 11/84 & 0
\end{array}
\tag{3.32}
$$

Among other RK methods, Dormand-Prince coefficients usually minimize the error of the fifth-order solution, which is constructed so that the fourth-order solution has a small error[16]. For this reason, the Dormand-Prince method is more suitable in the higher-order solution.

In this section we present Runge-Kutta (RK) method for solving FDE

$$\dot{x}(t) = f(t, x), \; x(t_0) = \tilde{x}_0, \tag{3.33}$$

$f : T \times E^n \to E^n, \; \tilde{x}_0 \in E^n$. The existence theorem for FDE is stated as follows [15]:

Theorem 3.2.1. Suppose the equation (3.33) considered under generalized differentiability of Definition 3.1.8, which has solution:

$$x(0) = x_0, \quad x_{n+1}(t) = x_0 + \int_{t_0}^{T} f(s, x_n(s))ds \tag{3.34}$$

and

$$x(0) = x_0, \quad x_{n+1}(t) = x_0 - \int_{t_0}^{T} -f(s, x_n(s))ds \tag{3.35}$$

converge to the Definition 3.1.8 of the limits (i) and (ii) separately.

Proof. See [38] and [4] □

3.2.2 Existing approach in solving FDEs using the Runge-Kutta method

Assume that $Y = [\underline{Y}, \overline{Y}]$ is the exact solution and $y = [\underline{y}, \overline{y}]$ be the approximated solution using the RK method, which was introduced in [7] of the α-cuts of the FDE problem(3.33). Consider the time interval $[t_0, T]$ in the solution, we have $t_0 < t_1 < t_2 < ... < t_N = T$, $t_i = t_0 + ih, h = \frac{T-t_0}{N}$, so $[Y(t_n)]^\alpha = [\underline{Y_n}(t), \overline{Y_n}(t)]^\alpha$ and $[y(t_n)]^\alpha = [\underline{y_n}(t), \overline{y_n}(t)]^\alpha$. Consider $[Y_1(t_n)]^\alpha = [\underline{Y_1}(t_n), \overline{Y_1}(t_n)]^\alpha$ and $[y_1(t_n)]^\alpha = [\underline{y_1}(t_n), \overline{y_1}(t_n)]^\alpha$ satisfy equation(3.24); $[Y_2(t_n)]^\alpha = [\underline{Y_2}(t_n), \overline{Y_2}(t_n)]^\alpha$ and $[y_2(t_n)]^\alpha = [\underline{y_2}(t_n), \overline{y_2}(t_n)]^\alpha$ satisfy equation(3.25). We have to ensure the the approximated solution is convergent to the exact solution. At each instant time t_i, $0 \leq n \leq N$, the solutions for exact and approximation are denoted as $[Y_n(t)]^\alpha$ and $[y_n(t)]^\alpha$. Based on the RK-2 coefficient, $\forall \alpha \in [0, 1]$we have :

$$\begin{cases}
\underline{y1}_\alpha(t_{n+1}) &= \underline{y1}_\alpha(t_n) + (1 - \frac{1}{2\theta})hF(t_n, \underline{y1}_\alpha(t_n), \overline{y1}_\alpha(t_n)) + \frac{1}{2\theta}hF(t_n + \theta h, \underline{z1}^\alpha_{n+1}, \overline{z1}^\alpha_{n+1}), \\
\overline{y1}_\alpha(t_{n+1}) &= \overline{y1}_\alpha(t_n) + (1 - \frac{1}{2\theta})hG(t_n, \underline{y1}_\alpha(t_n), \overline{y1}_\alpha(t_n)) + \frac{1}{2\theta}hG(t_n + \theta h, \underline{z1}^\alpha_{n+1}, \overline{z1}^\alpha_{n+1}), \\
\underline{z1}^\alpha_{n+1} &= \underline{y1}_\alpha(t_n) + \theta hF(t_n, \underline{y1}_\alpha(t_n), \overline{y1}_\alpha(t_n)), \\
\overline{z1}^\alpha_{n+1} &= \overline{y1}_\alpha(t_n) + \theta hG(t_n, \underline{y1}_\alpha(t_n), \overline{y1}_\alpha(t_n)), \\
\underline{y2}_\alpha(t_{n+1}) &= \underline{y2}_\alpha(t_n) + (1 - \frac{1}{2\theta})hF(t_n, \underline{y2}_\alpha(t_n), \overline{y2}_\alpha(t_n)) + \frac{1}{2\theta}hF(t_n + \theta h, \underline{z2}^\alpha_{n+1}, \overline{z2}^\alpha_{n+1}), \\
\overline{y2}_\alpha(t_{n+1}) &= \overline{y2}_\alpha(t_n) + (1 - \frac{1}{2\theta})hG(t_n, \underline{y2}_\alpha(t_n), \overline{y2}_\alpha(t_n)) + \frac{1}{2\theta}hG(t_n + \theta h, \underline{z2}^\alpha_{n+1}, \overline{z2}^\alpha_{n+1}), \\
\underline{z2}^\alpha_{n+1} &= \underline{y2}_\alpha(t_n) + \theta hF(t_n, \underline{y2}_\alpha(t_n), \overline{y2}_\alpha(t_n)), \\
\overline{z2}^\alpha_{n+1} &= \overline{y2}_\alpha(t_n) + \theta hG(t_n, \underline{y2}_\alpha(t_n), \overline{y2}_\alpha(t_n)).
\end{cases} \tag{3.36}$$

which satisfy the conditions (i) and (ii) in Definition 3.1.8.

The following theorem shows that the generalized Runge-Kutta approximation pointwisely converges to the exact solutions. Let $F(t, u, v)$ and $G(t, u, v)$ be the functions F and G of equations (3.36), where u and v are constants and $u \leq v$. The domain where F and G are defined is :
$K = \{(t, u, v)|0 \leq t \leq A, 0 \leq v \leq \infty, 0 \leq u \leq v\}$.

Theorem 3.2.2. Let $F(t, u, v)$ and $G(t, u, v)$ belong to C_K † and let the partial derivatives of F and G be bounded over K. Then for any $\alpha \in [0, 1]$ the generalized Runge-Kutta approximation of

†C_K: Power set which contains all nonempty compact subsets of K

equations (3.36) converge to the exact solution $[\underline{Y}^{\alpha}, \overline{Y}^{\alpha}]$ uniformly in it.

Proof. The details of the proof can be found in [15] [31] [28] [1] □

3.3 Fuzzy Differential Equations with Fuzzy Parameters and Initial Conditions

In the previous section, we discussed the FDEs with fuzzy initial values. Fuzzy differential equations with fuzzy parameters is discussed in this section. Assume we have a FDE with a single parameter $\tilde{p}$ and fuzzy initial value $\tilde{x}_0$:

$$\begin{cases} \dot{x}(t) &= f(t, x(t), \tilde{p}) \\ x(t_0) &= \tilde{x}_0 \end{cases} \quad \tilde{p}, \tilde{x}_0 \in E^n. \tag{3.37}$$

Normally, the membership grade of $\tilde{p}$ is discretized into m subintervals, there are m α-cuts (excluding core of fuzzy parameter), the left and right side membership functions of the fuzzy parameters will be discretized into m subintervals either. Here we only need to consider $2m+1$ boundary values from m α-cuts and the core. As the Fig.3.1 shown, each α-cut is an interval, which includes several elements with the same membership grade. If we find one element A in α-cut at membership grade a_i, we can always find an boundary value B of α-cut at membership grade a_j, $1 \geq a_j > a_i$. Therefore, if the number of discretization is m, we need only $2m+1$ α-cuts boundary values (for core, there is only one value). Consider these elements as the parameter of the original FDE, we have $2m+1$ transformed FDEs with the same fuzzy initial value. We have some bounded conditions for functions f of problem (3.37):

Proposition 3.3.1. For the new $2m+1$ FDEs created from (3.37), P_i is the discretized element, $i = 1, 2, ...2m+1$. as $t \to 0$, the support of $|f(t, x(t), p) - f(t, x(t), P_i)|$ will not goes to infinity. ($p \in \mathbb{R}$ is the core of fuzzy number $\tilde{p}$)

Using the extension principle is a directly brute force method for the solution of (3.37), which enumerate all combinations of possible parameter values, computations are repeated. If we have

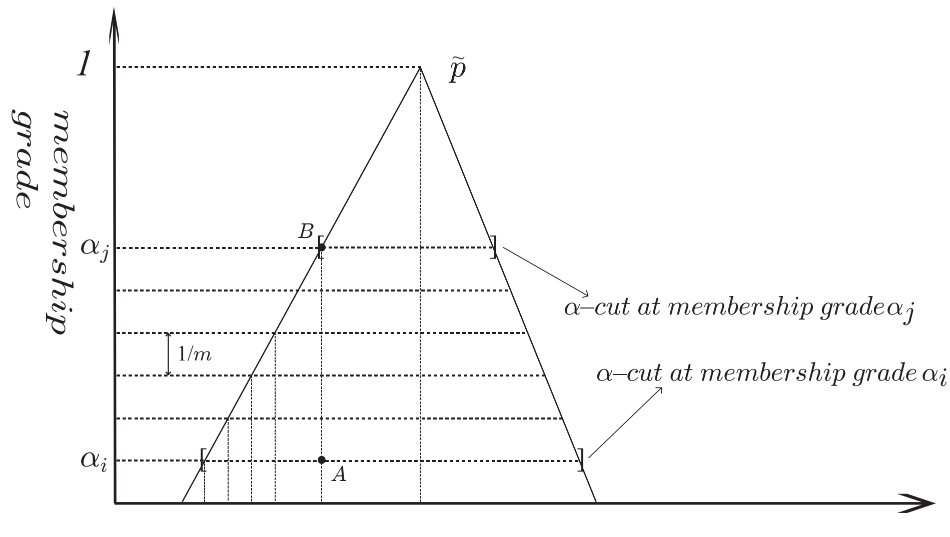


Fig. 3.1: Discretization of fuzzy parameter $\tilde{p}$

multiple fuzzy parameters, the process of calculating the solution is complicated. Suppose there are k fuzzy parameters within the FDE, discretize each parameter into m subintervals regarding the membership grade $\alpha \in [0,1]$, we have $(2m+1)^k$ transformed FDEs need to be solved. According to the previous approaches, to the best of our knowledge we do not have a practical method to solve such FDE.

Few works were reported on research of FDE with multiple fuzzy parameters. In[36], the authors T. Rzezuchowski and J. Wasowski only give preliminary study about FDE with multiple fuzzy parameters:

Let $f : T(\text{time dimension}) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous function and $p_1, p_2, ...p_m \in \mathbb{R}^n$ are parameters, $\tilde{x}_0 \in \mathbb{R}^n$ We consider the initial value problem with parameters:

$$\begin{cases} \dot{x}(t) &= f(t, x(t), \tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_m), \\ x(t_0) &= \tilde{x}_0. \end{cases} \tag{3.38}$$

Assume that the parameters $\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_m \in E^n$ and initial values $x_0 \in E^n$. The functions f will satisfy the following conditions:

Definition 3.3.2. The function f should satisfy:

i There is an integrable functions $L : T \to \mathbb{R}_+$, for all $x, y \in \mathbb{R}^n$ and $p_i \in \mathbb{R}$

$$\|f(t, x, p_1, \ldots, p_n) - f(t, y, p_1, \ldots, p_n)\| \leq L(t)\|x - y\| \ ;$$

ii f is continuous for every fixed $x \in \mathbb{R}^n$;

iii f is convergent;

iv There is an integrable functions $L : T \to \mathbb{R}_+$

$$\|f(t, x, p_1, \ldots, p_n)\| \leq L(t)\|x\| + \psi(t) \ ;$$

Note that $\psi(t)$ is an integrable function.

Proof. See [36]. □

Assume there is a function F which satisfies: $F^\alpha(t, x) = f(t, x^\alpha, p_1^\alpha, ..., p_n^\alpha)$. We have the following differential inclusions:

$$\dot{x} \in F^\alpha(t, x) \quad x(t_0) \in [x_0]^\alpha. \tag{3.39}$$

[21] [36] and [40] emphasized on fuzzy differential inclusions, which is a general method of FDE with multiple fuzzy parameters and initial values. However, no analytical or numerical solution scheme was given. In the next chapter, computational algorithms for solving the FDEs with multi-fuzzy parameters and initial values is proposed.

In this chapter, the theory of fuzzy derivatives were briefly reviewed, including H-derivative and fuzzy differential equations. The concept generalized H-derivative is introduced. Moreover, the properties of FDEs with multiple fuzzy parameters and initial values is discussed.

Chapter 4

FDE Numerical Solution with multi-fuzzy parameters and initial values

In this thesis, we propose to combine the RK-5 (Dormand - Prince coefficients [16]) and the proper fuzzy arithmetic to solve the FDE with multiple fuzzy parameters and fuzzy initial values. In this thesis we name the Fuzzy Differential Equation with multiple fuzzy Parameters and fuzzy Initial Values for short, as FDEPIV.

4.1 Using Proper Fuzzy Arithmetic for Solving FDE

For example, a fuzzy function is given as:

$$f = (\tilde{a}x_0)^{\tilde{b}} \quad \tilde{a}, \tilde{b} \in E^n,\ x_0 \in \mathbb{R}. \tag{4.1}$$

Suppose at the membership grade $\alpha \in [0,1]$, the lower and upper bound of α-cuts of a, b are $\underline{a}, \overline{a}, \underline{b}, \overline{b}, (\underline{a} < \overline{a}, \underline{b} < \overline{b},\ \ \underline{a}, \overline{a}, \underline{b}, \overline{b} \in \mathbb{R})$ repectivley (see Fig.4.1). By using the proper fuzzy arithmetic, we have the solutions set X :

$$Y = \left((x_0\underline{a})^{\underline{b}}, (x_0\underline{a})^{\overline{b}}, (x_0\overline{a})^{\underline{b}}, (x_0\overline{a})^{\overline{b}}\right), X^{\alpha} = [\min Y, \max Y]. \tag{4.2}$$

The number of the elements in the set Y is 2^2 (x_0 is a crisp value, see Fig.4.2). If the function involves n fuzzy parameters, at membership grade α, we have 2^n elements in set Y (see Fig.4.3). If x_0 is a fuzzy number, we have $[x_0]^{\alpha} = [\underline{x}_0^{\alpha}, \overline{x}_0^{\alpha}]$ at membership grade $\alpha \in [0, 1]$, substitute the boundary values into the function, we get a set (as the set Y in (4.1)) which includes $2 \cdot 2^n$ elements. Especially, if the functions f is non-monotonic, recall the modified proper fuzzy arithmetic method discussed in Section 2.4.1, we have $2p \cdot 2^n$ elements in the solution, where p is the number of discretization pairs † of each α-cut.

The advantage of using the proper arithmetic method to solve the fuzzy function f with several

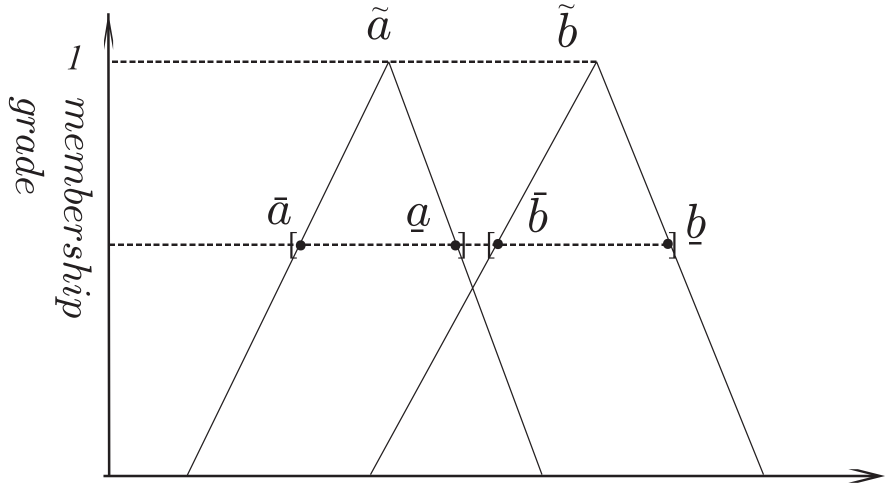


Fig. 4.1: α-cuts at corresponding membership grade

fuzzy parameters, especially if the function is non-monotonic, is that we can efficiently avoid the over estimation of the solution as applying the standard fuzzy arithmetic. In solving the FDEPIVs, we apply the Runge-Kutta method for every transformed FDE with proper fuzzy arithmetic method, the most approximated result can be get from. If any transformed FDE does not satisfy the

†The core overlapped in computation.

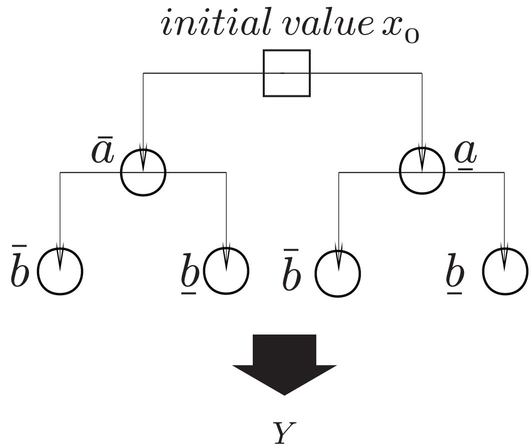


Fig. 4.2: 2 fuzzy parameter in function f (The crisp initial value needs 2^2 calculations)

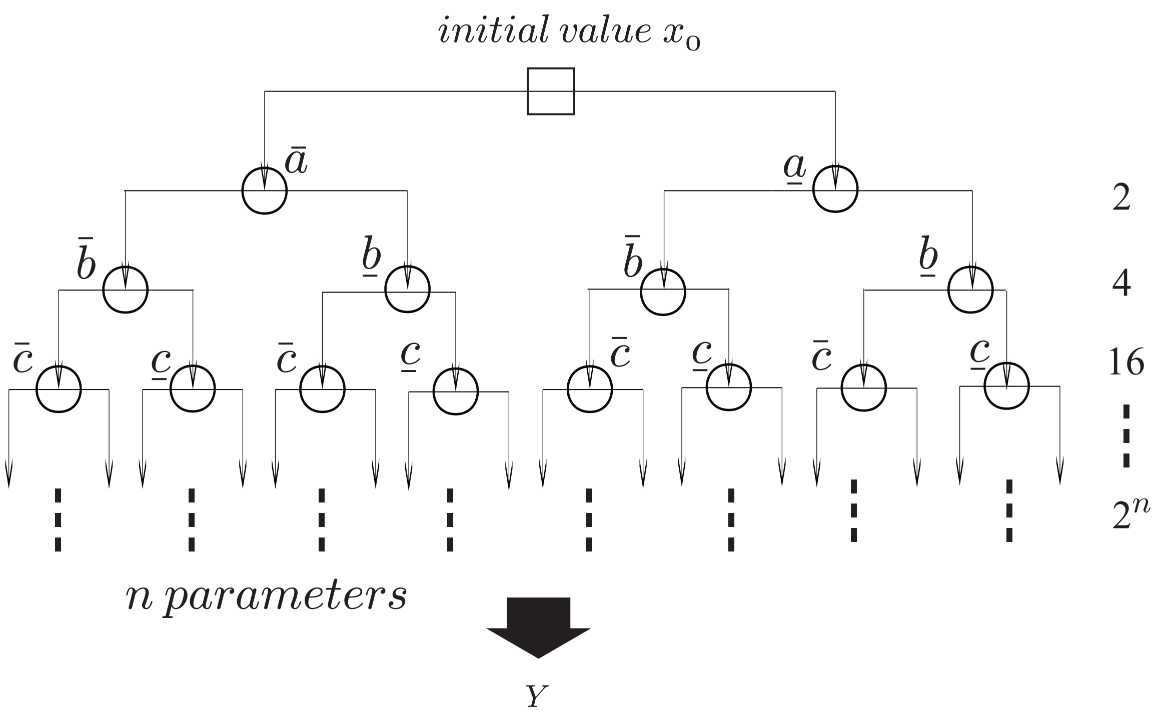


Fig. 4.3: n parameter in function f (The crisp initial value needs 2^n calculations, if the number of fuzzy parameters is n)

Proposition 3.3.1, the trajectory support of the solution will keep expanding.

4.2 Higher Order FDEs

Under the generalized H-derivative, there usually exists two solutions of first order fuzzy initial value problems [5] and four solutions of second order initial value problems [25].In this paper we need only cite two conditions of them, which is discussed in Chapter 3. The higher order FDE problems were investigated in paper [25] [32] [27] [26]. Normally, the previous papers study in the higher order FDE will consider the 4 kinds of H-differentiable conditions, in this paper, we translate high order FDE into multi-dimensional first order system, like multi-dimensional FDEs system.Suppose we have a second order FDE:

$$\ddot{x} + \tilde{p}_1\dot{x} + \tilde{p}_2 x + \tilde{p}_3 = 0,\ \dot{x}(t_0) = \tilde{a}_1, x(t_0) = \tilde{a}_2 \tag{4.3}$$

we can easily transform the FDE into 2 first order FDEs:

$$\begin{cases} \dot{x} & = y \\ \dot{y} & = -\tilde{p}_1 y - \tilde{p}_2 x - \tilde{p}_3 \end{cases}, \quad y(t_0) = \tilde{a}_1, x(t_0) = \tilde{a}_2. \tag{4.4}$$

Or in matrix form, define

$$\begin{aligned} \xi &= \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \\ \dot{\xi} &= \begin{bmatrix} 0 & 1 \\ -\tilde{p}_2 & -\tilde{p}_1 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ -\tilde{p}_3 \end{bmatrix}, \\ x &= \begin{bmatrix} 1 & 0 \end{bmatrix} \xi. \end{aligned} \tag{4.5}$$

4.3 Computational methods

Consider we have a FDEPIV with k fuzzy parameters and l initial values ($k, l \in \mathbb{N}$), three approaches are presented in the following three sections.

4.3.1 By the Extension Principle

Suppose a FDEPIV with initial values $\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_l$, and parameters $\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_k$. If we directly using the extension principle to solve the problem, we have to transform the FDEPIV into multiple FDEs with different parameters (crisp) which include the same fuzzy initial values. Suppose the discretized number of membership grade is m, if we have k parameters, we need to compute $(2 \cdot m + 1)^k$ (2 boundary values of α-cut, one for core) times for just one combination of crisp initial values. This is due to the characteristic of the extension principle: if there are k fuzzy numbers for calculation, the discretized membership grade number is m, we need to process all possible combinations of each α-cuts crisp values regarding to their membership grade. Additionally, we have l fuzzy initial values, therefore there will be in total $2 \cdot (2 \cdot m + 1)^{k+l} = (4 \cdot m + 2)^{k+l}$ ordinary differential equations needed to be solved in time span T. The algorithm analysis is showed in following table [†]. The difficulty in using the extension principle for solving FDEPIV is that, the process of calculation is tedious, it is hard to derive a general program to solve all kinds of FDEs problems. On the other hand, the extension principle is the fundamental calculation rule for the operation for the fuzzy sets, in solving the FDE problems, it is hard to directly using it practically due to the 'curse of dimensionality'.

[†]Due to the high complexity of the extension principle method, we only consider one fuzzy parameter and one fuzzy initial value as in the algorithm table.

Algorithm 1 Calculate FDEPIV using Extension principle

equal the fuzzy parameters and initial values into m intervals of length $\triangle\mu = \frac{1}{m}$

initialize (time step h, *processing time span*)

Define μ-cut of parameters as $[\underline{b}, \overline{b}]^{\mu_p}$, which has 2 crisp number combinations as parameters at local membership grade in the equation

for $\mu_{parameters}$=$0 : \triangle\mu : 1$ **do**

reset the process time $t \to 0$

for *crisp parameters values combinations* $= 1 : 2$ **do**

reset $\mu_{initial} = 0$

for $\mu_{initial} = 0 : \triangle\mu : 1$ **do**

reset the process time $t \to 0$

Define μ-cut of initial value $[\underline{a}, \overline{a}]^{\mu_i}$, which has 2 crisp initial values combinations at current membership grade

for *crisp initial values combinations* $= 1 : 2$ **do**

reset the process time $t \to 0$

while $t <$ *processing time span* **do**

calculate the function using RK-5 method, derive each crisp initial values into the equation get a series values, store into a set R

$t = t + h$(time step)

end while

end for

end for

end for

find the minimum and maximum values at each time step t at local membership grade.

end for

create a membership grade matrix

4.3.2 By Using the Numerical Method in Chapter 3.2.2

Recall the preliminaries in Chapter 3.2.2, if we utilize the previous approach about Runge-Kutta method on solving FDE for solving the FDEPIV. To the best of our knowledge, there is no paper discussed the issue yet. The current approaches on solving the FDEs by using the numerical methods are all based on the exact solution can be derived from FDEs. However, it is hard to get an exact solutions in most cases in the real world. Therefore, the previous approach discussed in Chapter 3.2.2 is not practical for solving FDEPIV problems.

4.3.3 Our approach

In order to overcome the shortcomings of existing methods, we propose to solve the FDEPIV problems using the proper fuzzy arithmetic with Runge-Kutta 5 method. The procedure of utilizing the FDE solver is summarized in algorithm table.

Suppose we have a FDEPIV problem, the α-cut of the solution is $[x(t_i)]^\alpha$ at time t_i at membership grade $\alpha \in [0, 1]$. We can apply the golden ratio search method (bisection search method or other mathematical optimization methods) to locate the minimum and maximum of the solution at t_i at membership grade $\alpha \in [0, 1]$. However, if the system is nonlinear, the boundary values of $[x(t_i + \triangle t)]^\alpha$ for next time step may not be obtained from the boundary values $[x(t_i)]^\alpha$. As shown in Fig.4.4, points A and B are the boundary values of $x(t_i)$ at membership grade α, $\alpha \in [0, 1]$. If we consider A and B as the initial values for next time step $t_i + \triangle t$, we get point C from A and D from B. However, the actual lower bound and upper bound of $x(t_i + \triangle t)$ are F and E respectively. Therefore, if we apply golden ratio search method or other mathematical optimization methods to locate the boundary values at t_i, which may not lead the boundary values at $t_i + \triangle t$.

Under this condition, we apply the modified proper fuzzy arithmetic, which discussed in Section 2.4.1. We do not consider the $x(t_i)$ as initial value for next time step calculation. Instead, we consider the transformed crisp values form the fuzzy initial values as initial values, in order to get a solutions set. From the solutions set, we can obtain the boundary values at each membership

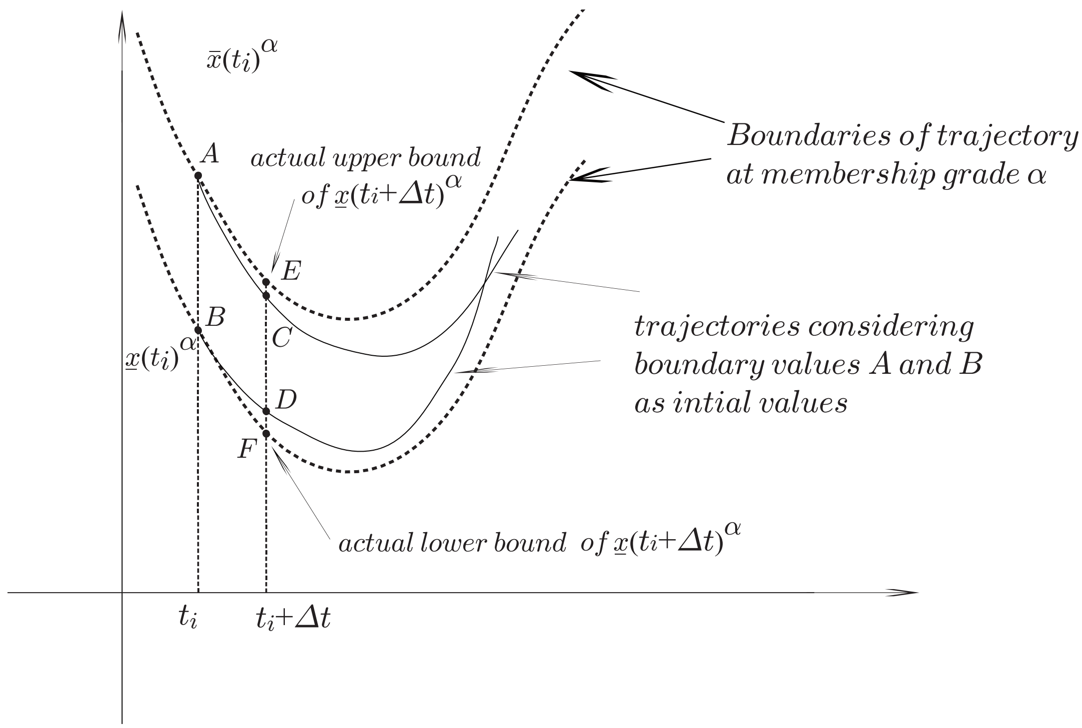


Fig. 4.4: The boundary values of $[x(t_i+\triangle t)]^{\alpha}$ (points E and F) are not obtained from the previous boundary values of $[x(t_i)]^{\alpha}$ (point A and B)

grades for each time step, in order to get the solution trajectory of the FDEPIV problem. The advantages of using the proposed FDE solver can be summarized:

1. Eliminate the overestimation of using the standard fuzzy arithmetic in calculating the FDE-PIV problem,

2. Has less complexity than using the extension principle,

3. Can solve higher order and nonlinear FDEPIV problems,

4. Be able to parallelize the computation †.

The algorithm is shown below. We have a initial values Matrix $l \times 2^l$ and a parameters Matrix $k \times 2^k$. Additionally, we use the modified proper fuzzy arithmetic, there will be c pairs subintervals for corresponding membership grade of initial values. Therefore, there will be $2 \cdot 2^{k+l} \cdot m \cdot c$ ordinary

†Will be discussed in Chapter 6

differential equations needed to be solved in time span T. On the other hand, the number of ODEs solved using the Extension Principle is $(4\cdot m+2)^{k+l}$ (se Section 4.4.1). Since $m \gg 2$, the complexity of the proposed method is significantly lower than that of the Extension Principle.

In this chapter, a new method in solving the FDEPIV is proposed. Comparing with other methods, advantages and contributions are discussed as well.

Algorithm 2 Calculate FDEPIV using proper fuzzy arithmetic with RK-5 Part 1

```
Divide the membership grade range μ ∈ [0,1] of the fuzzy parameters and initial values into m
    equal intervals △μ = 1/m
initialization (time step h, processingtimespan)
for μ=0 : △μ : 1 do
    create a proper fuzzy arithmetic Matrix with parameter boundary values, denote as M_p
    discretize initial values into c pairs
    for 1 → c pairs do
                                        ▷ The algorithm continues in next page.
```

Algorithm 3 Calculate FDEPIV using proper fuzzy arithmetic with RK-5 Part 2

create a initial value fuzzy arithmetic Matrix with current pair as boundary values, denote as M_i
use k as indexing values for initial value Matrix M_i, $M_i(k, ;)$ gives the k-th row of M_i
use l as indexing values for parameters Matrix M_p, $M_p(l, ;)$ gives the k-th row of M_p
for The crisp values from $1 \to\ l$ **do**
  for The crisp values from $1 \to\ k$ **do**
    reset $t \to 0$
    while $t < endtime$ **do**
      calculate the function using RK-5 method, derive the crisp values into the equation get a series values, as a set R
      $t = t + h$
    end while
    find the maximum and minimum of R at each time step as the solution for current membership grade μ
  end for
end for
reset the Matrix M_p and M_i
end for
end for
create a membership grade matrix

Chapter 5

Numerical Examples

In this chapter, numerical examples are given to illustrate the advantages of the proposed FDE solver. The proposed FDE solver is implemented using MATLAB.

5.1 Comparison with solving FDE using the Extension principle and the proposed method

Consider the fuzzy initial value problem:

$$\begin{cases} \dot{X} &= -X(t) \\ X(0) &= \tilde{C} \end{cases}, \quad \tilde{C} \in E^n. \tag{5.1}$$

Its crisp version is:

$$\dot{x}(t) = -x(t),\ x(0) = c,$$

which possesses the exact solution

$$x(t,c) = ce^{-t}.$$

Note that $x(t,c)$ is continuous in $c \in \mathbb{R}$ for each $t \geq 0$ fixed. We apply the extension principle to $x(t,c)$ in relation to c, for each $t \geq 0$ fixed. Then we obtain the unique fuzzy solution $X(t) = \hat{x}(t,C), C \in \mathbb{R}$ of problem (5.1) for any initial fuzzy number C, which is given by

$$X(t) = Ce^{-t},\ t \geq 0. \tag{5.2}$$

Assume we have a fuzzy initial value problem:

$$\begin{cases} \dot{X} &= -X(t), \\ X(0) &= \tilde{4}. \end{cases} \tag{5.3}$$

$\tilde{4} = t[2, 4, 5] \in E^n$. According to the result in (5.2), we have:

$$X(t) = \tilde{4}e^{-t},\ t \geq 0. \tag{5.4}$$

Discretize the membership grade $\mu \in [0, 1]$, we can get the result as following ($t : 0 \to 1$, time step h=0.1):

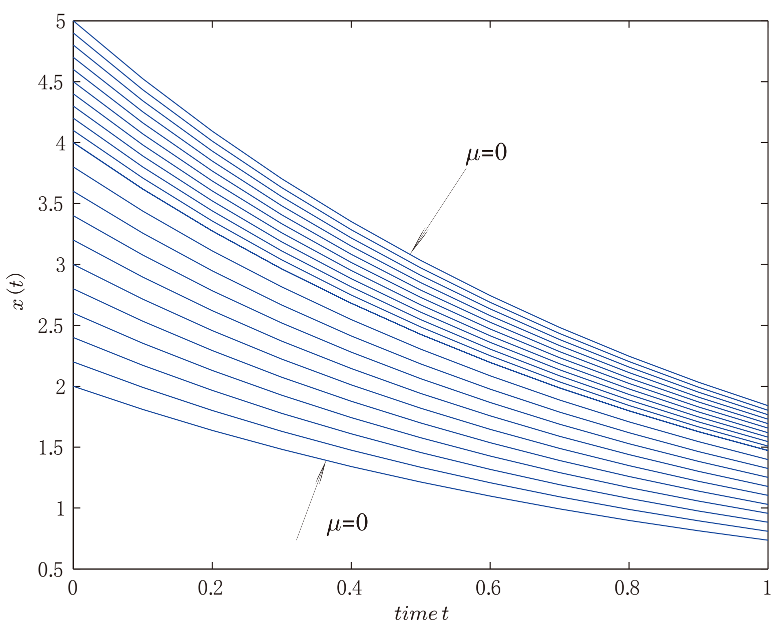


Fig. 5.1: The solution of problem (5.5)

A comparison between the exact and the approximate solutions at $t = 0.1$ and the error of using the FDE solver is shown in the following tables and Fig. 5.1.

μ	$\underline{x}$	$\underline{X}$	error
0	1.80967483607192	1.80967483666667	5.94747584514721e-10
0.1	1.99064231967911	1.99064232033333	6.54222231943891e-10
0.2	2.17160980328630	2.17160980400000	7.13697101417665e-10
0.3	2.35257728689350	2.35257728766667	7.73171748846835e-10
0.4	2.53354477050069	2.53354477133333	8.32646840365214e-10
0.5	2.71451225410788	2.71451225500000	8.92121043705174e-10
0.6	2.89547973771507	2.89547973866667	9.51596135223554e-10
0.7	3.07644722132226	3.07644722233333	1.01107122674193e-09
0.8	3.25741470492945	3.25741470600000	1.07054587417110e-09
0.9	3.43838218853665	3.43838218966667	1.13002007751106e-09
1	3.61934967214384	3.61934967333333	1.18949516902944e-09

μ	$\overline{x}$	$\overline{X}$	error
0	4.52418709017980	4.52418709166667	1.48686929435371e-09
0.1	4.43370334837620	4.43370334983333	1.45713130450531e-09
0.2	4.34321960657261	4.34321960800000	1.42739420283533e-09
0.3	4.25273586476901	4.25273586616667	1.39765710116535e-09
0.4	4.16225212296541	4.16225212433333	1.36791999949537e-09
0.5	4.07176838116182	4.07176838250000	1.33818200964697e-09
0.6	3.98128463935822	3.98128464066667	1.30844446388778e-09
0.7	3.89080089755463	3.89080089883333	1.27870736221780e-09
0.8	3.80031715575103	3.80031715700000	1.24897026054782e-09
0.9	3.70983341394743	3.70983341516667	1.21923227069942e-09
1	3.61934967214384	3.61934967333333	1.18949516902944e-09

In the two tables the $\underline{x}$ and $\overline{x}$ are lower bound and upper bound of exact solution, $\underline{X}$ and $\overline{X}$ are lower bound and upper bound of approximated solution. Fig.5.2 shows the fuzzy set of the solution

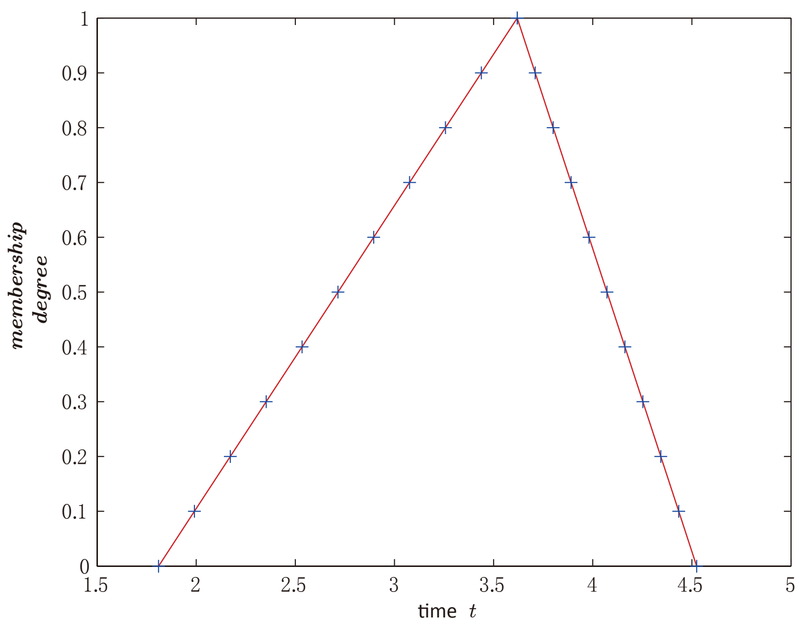


Fig. 5.2: Solid line:exact solution (+):FDE solver

at time step $h = 0.1s$. The two solutions match perfectly.

5.2 Solving the Example 3.1.7 in proposed method

Assume we have a fuzzy initial value problem:

$$\begin{cases} \dot{X} &= -X(t), \\ X(0) &= \tilde{0}. \end{cases} \tag{5.5}$$

$\tilde{4} = t[-1, 0, 1] \in E^n$. Applying the proposed method, we can obtain the trajectory of the solution in Fig.5.2 and Fig.5.2. Fig.5.2 shows the mesh and surface plot of trajectories of $X(t)$ in different membership grade as time t increase.

Fig.5.2 shows trajectories of $X(t)$ in the form of fuzzy numbers in three dimensions and two dimensions as time t increase respectively.

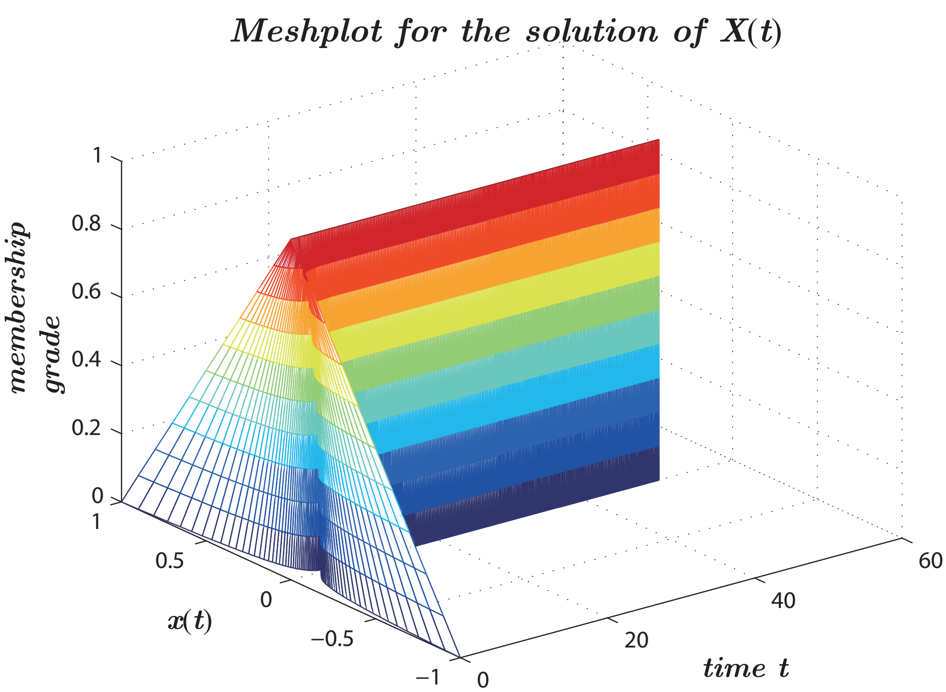
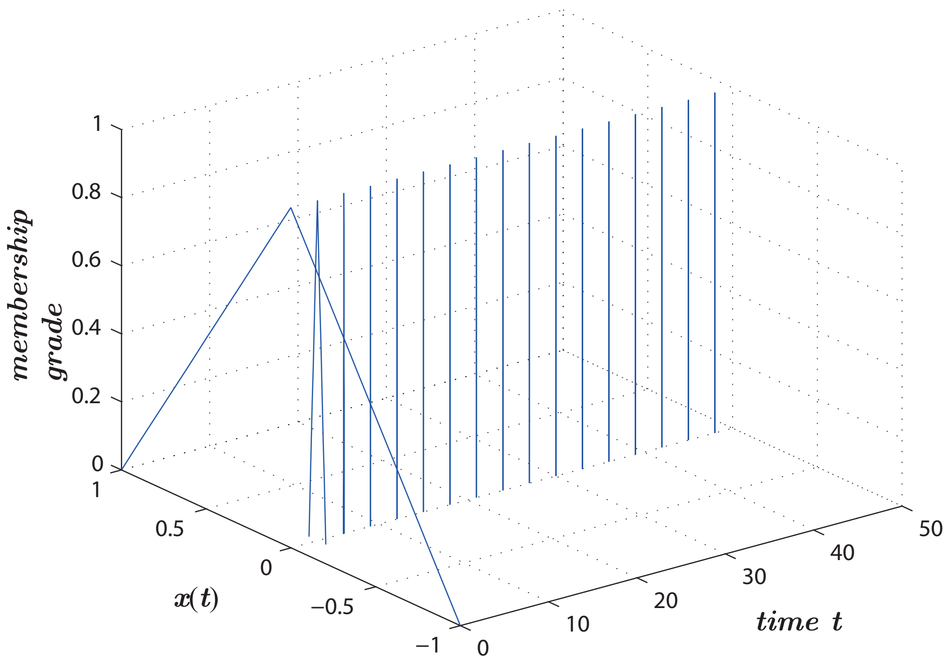


Fig. 5.3: $X(t)$ in Mesh plot



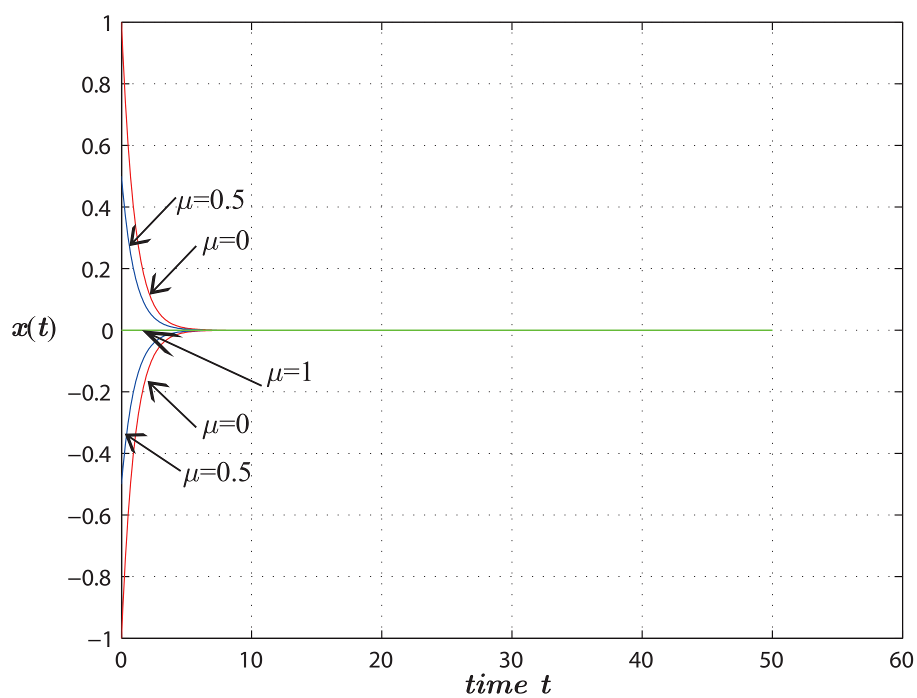


Fig. 5.4: $X(t)$ in 3D and 2D plot

5.3 Zero input response for RLC electric circuit

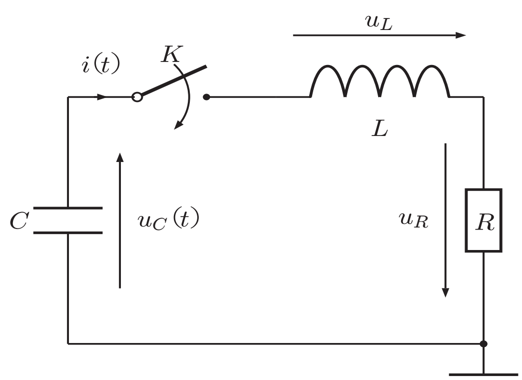


Fig. 5.5: The zero input response circuit

The RLC electrical circuit can be modeled as:

$$KVL \quad : u_L + u_R + u_C = 0, \tag{5.6}$$

$$VAR \quad : i_L = C\frac{du_C}{dt}, u_L = L\frac{di_L}{dt}, u_R = Ri_L. \tag{5.7}$$

From equation (1) and (2), we can get the second-order system:

$$LC\frac{d^2u_C}{dt^2} + RC\frac{du_C}{dt} + u_C = 0 \ (t \geq 0_+) \tag{5.8}$$

at time $t = 0$, the switch is turn on and we have the following initial values:

$$u_C(0) = 0 \ , i(0) = 1. \tag{5.9}$$

Assume all the parameters and initial values are fuzzy numbers, and are listed in Table (5.3). We use the MATLAB FDE solver in Appendix Bto solve the fuzzy differential equation, and get the trajectories for $u_C(t)$ and $i(t)$ (time $t : 0 \to 20s$).

In Fig.5.8, the red lines (inner lines) are the boundaries of the trajectories at membership grade $\mu = 1$; the blue lines (middle lines) are the boundaries of the trajectories at membership grade $\mu = 0.5$; the green lines (outer lines) are the boundaries of the trajectories when the membership

Table. 5.1: Initial values and parameters of the RLC circuit

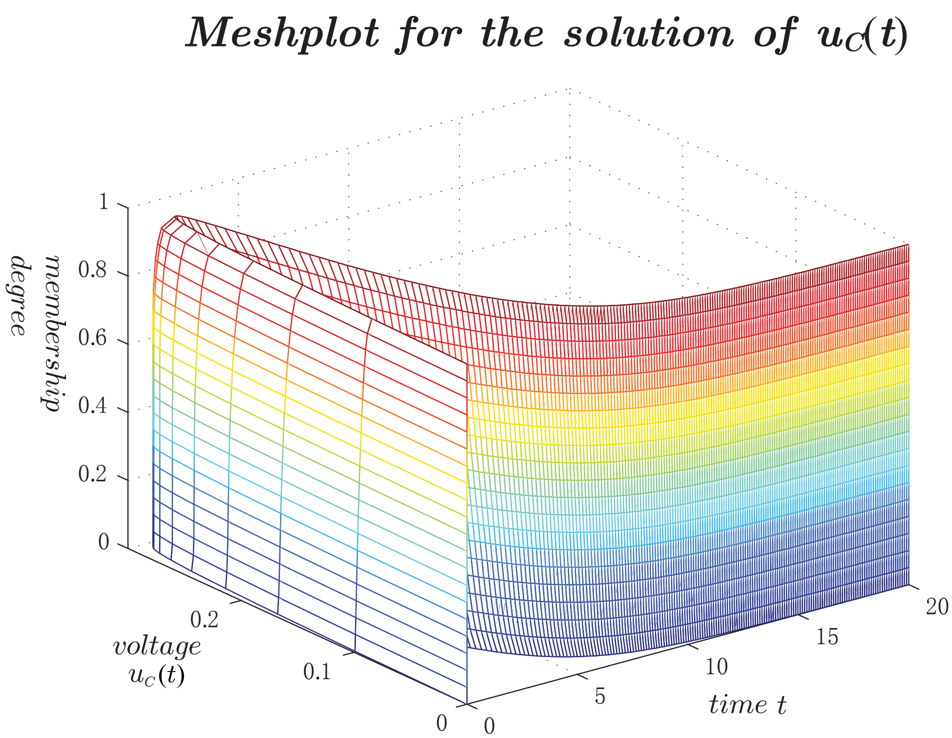
Elements	Description	Core	Support	Type of Fuzzy number	Matlab code
L	inductance	1	[0.8 1.1]	Triangular	1
R	resistance	3	[2.8 3.4]	Gaussian	2
C	capacitance	1	[0.9 1.2]	Exponential	4
$u_C(0)$	initial voltage of C	0	0	Singleton(crisp number)	N/A
$i(0)$	initial current of L	1	[0.9 1.2]	Exponential	3

grade is $\mu = 0$.

The Fig.5.3 shows the mesh and surface plots of trajectories of voltage u_c and current $i(t)$ in different membership grade as time t increase.

The Fig.5.7 shows the trajectories of voltage u_c and current $i(t)$ in the form of fuzzy numbers as time t increase.

The Fig.5.8 shows the trajectories of voltage u_c and current $i(t)$ in the form of different membership grade at 0,0.5 and 1.



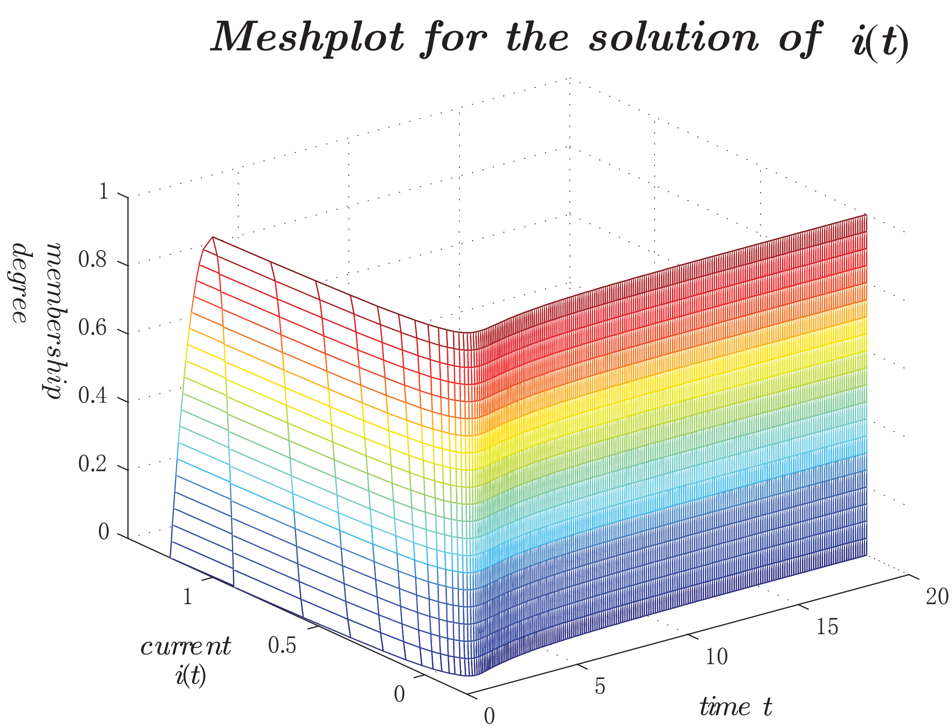


Fig. 5.6: u_C and $i(t)$ in Mesh plot

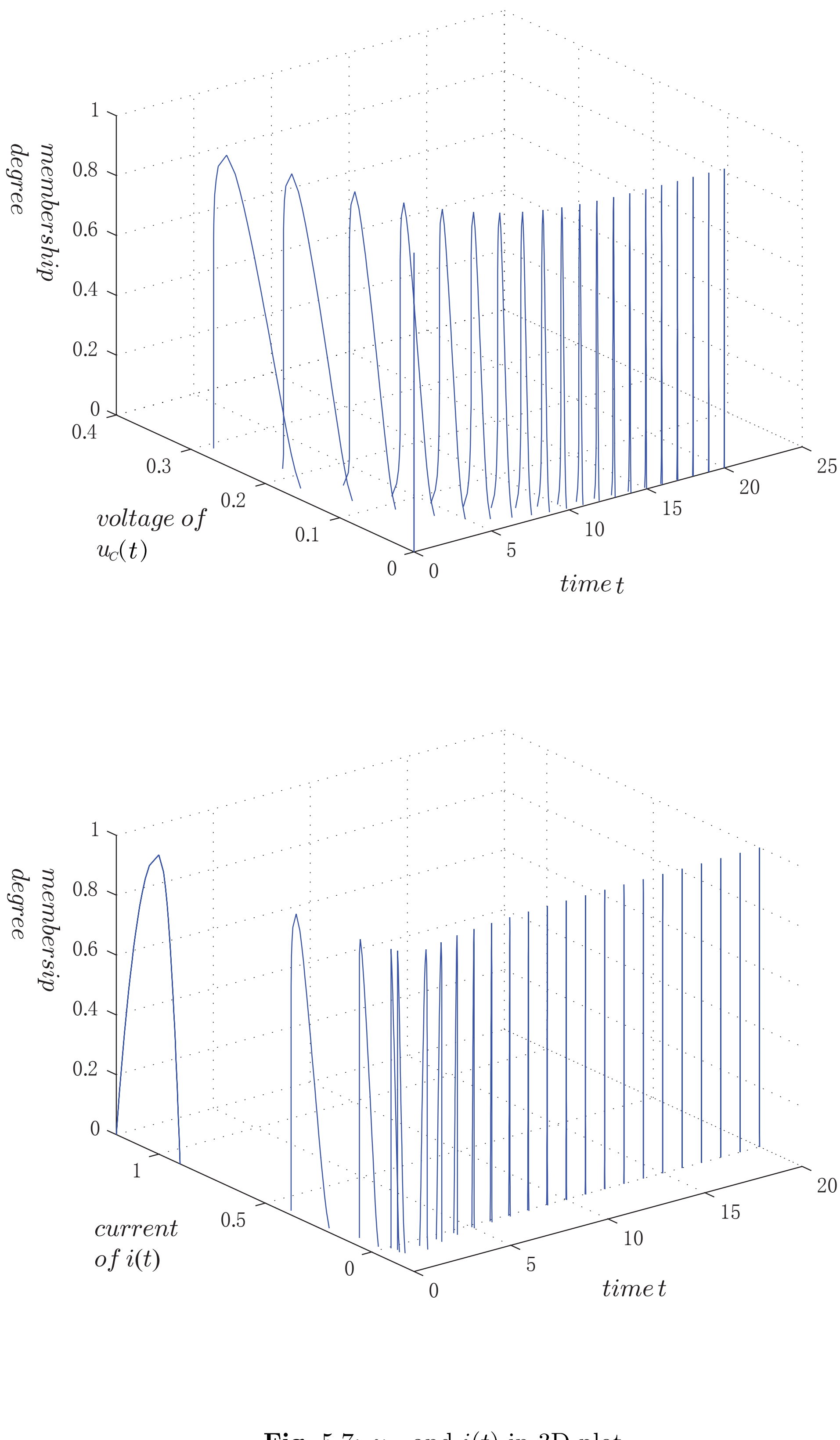
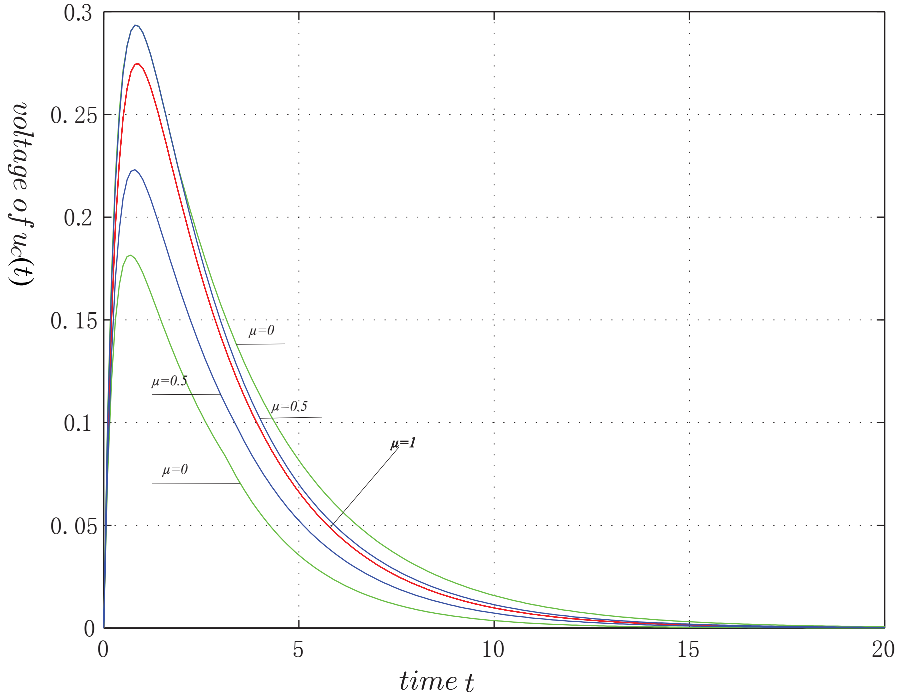


Fig. 5.7: u_C and $i(t)$ in 3D plot



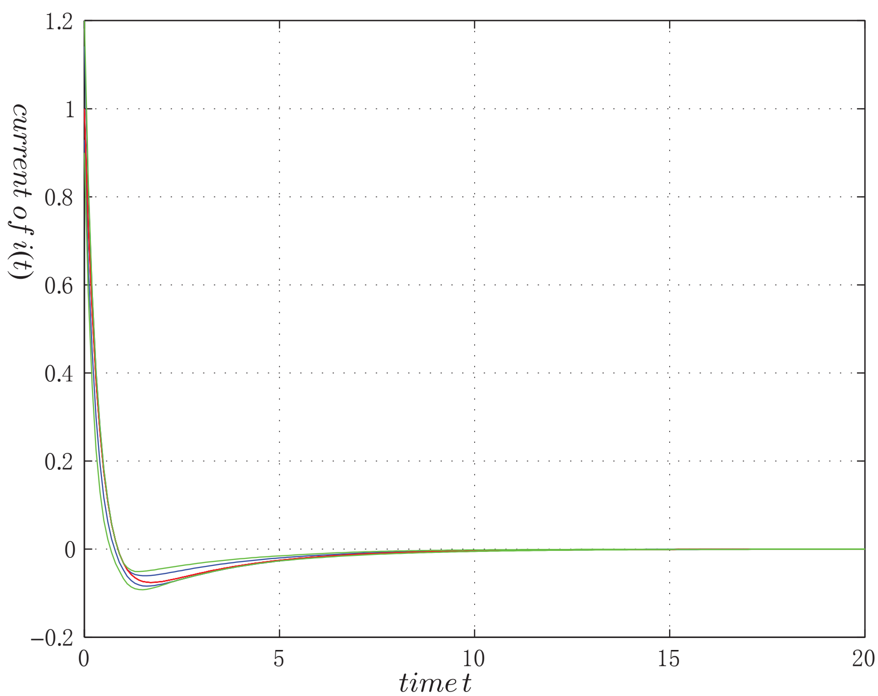


Fig. 5.8: u_C and $i(t)$ in 2D plot

5.4 A second-order Non-Linear System: The Pendulum

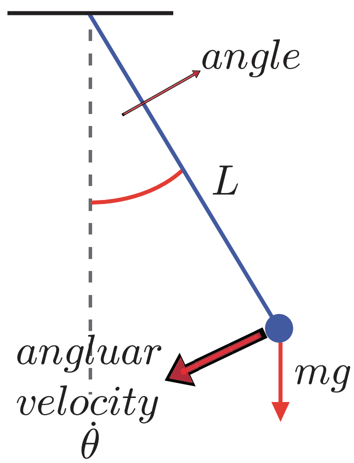


Fig. 5.9: The Pendulum System

The dynamic model of the pendulum is given as,

$$\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + k^2 sin(\theta) = 0 \tag{5.10}$$

with $k^2 = \frac{g}{L}$. The constant c is a measure of the amount of friction or air resistance. We convert this to a system by setting

$$x = \theta, y = \dot{\theta}. \tag{5.11}$$

therefore,

$$\dot{y} = -cy - \frac{g}{L} sinx \tag{5.12}$$

Or in matrix form, define

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{5.13}$$

$$\begin{aligned} \dot{\xi} &= \begin{bmatrix} 0 & 1 \\ -c & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ -\frac{g}{L} \end{bmatrix} sin\xi, \\ x &= \begin{bmatrix} 1 & 0 \end{bmatrix} \xi. \end{aligned} \tag{5.14}$$

with initial condition:

$$\text{initial angle position} : \xi = p_1 \tag{5.15}$$

$$\text{initial angle velocity} : x = p_2 \tag{5.16}$$

Which p_1 and p_2 are initial values. Assume all the parameters and initial values are fuzzy numbers, and they are listed in Table (5.2):

Table. 5.2: Initial values and parameters of the The Pendulum System

Elements	Description	Core	Support	Type of Fuzzy number	Matlab code
c	air friction	0.2	[0.18 0.24]	Quadratic	3
g	gravity	9.8	[9.7 10][†]	Gaussian	2
L	length of the rod	9.8	[9.6 10.2]	Exponential	4
$\theta(0)$	initial angle position	0	0	Singleton(crisp number)	N/A
$\dot{\theta}x(0)$	initial angular velocity	1	[0.5 1.2]	Triangular	1

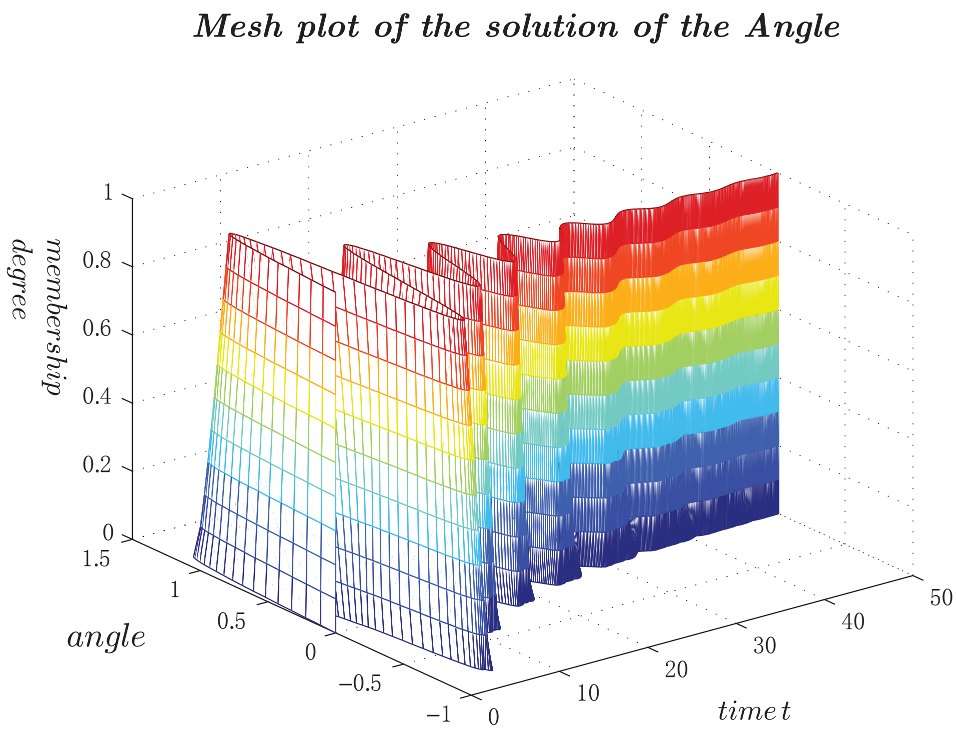
[†] In Fig.5.10, the red lines (inner lines) are the boundaries of the trajectories when the membership grade is $\mu = 1$; the blue lines (middle lines) are the boundaries of the trajectories when the membership grade is $\mu = 0.5$; the green line (outer lines) are the boundaries of the trajectories when membership grade is $\mu = 0$.

The Fig.5.10 shows the mesh and surface plots of trajectories of angle and angle velocity in different membership grade as time t increase, which display the surfaces in three dimensions.

The Fig.5.11 shows trajectories of angle and angle velocity in the form of fuzzy numbers as time t increase.

The Fig.5.12 shows trajectories of angle and angle velocity in the form of different membership grade 0,0.5 and 1, in 2-D.

[†]Here we consider the gravity is uncertain due to unknown of the local altitude.



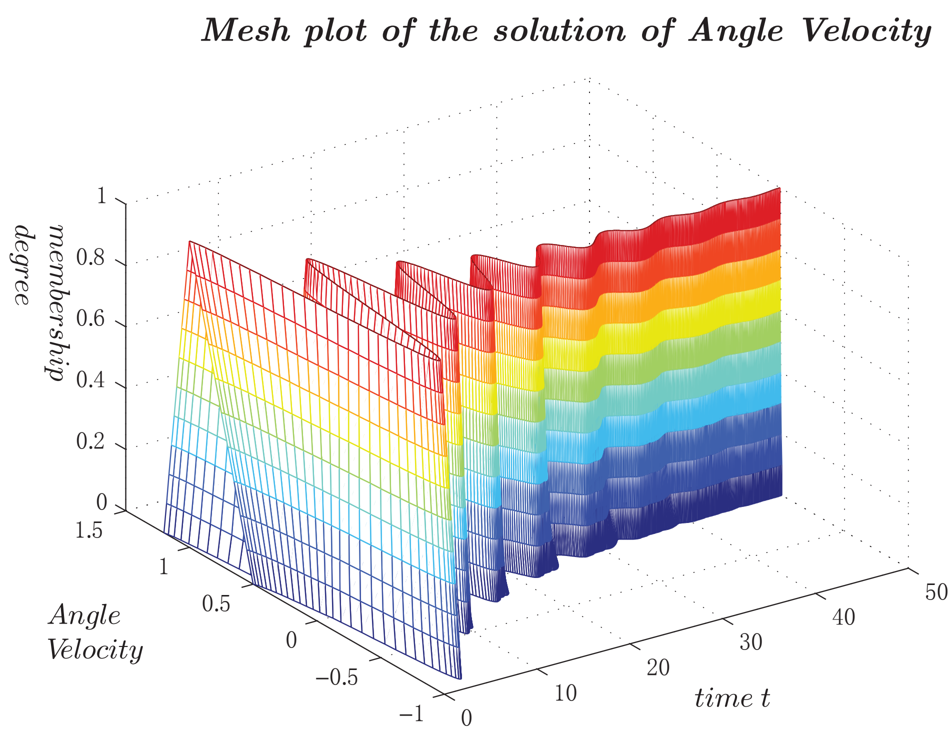


Fig. 5.10: Angle and Angle velocity in Meshplot

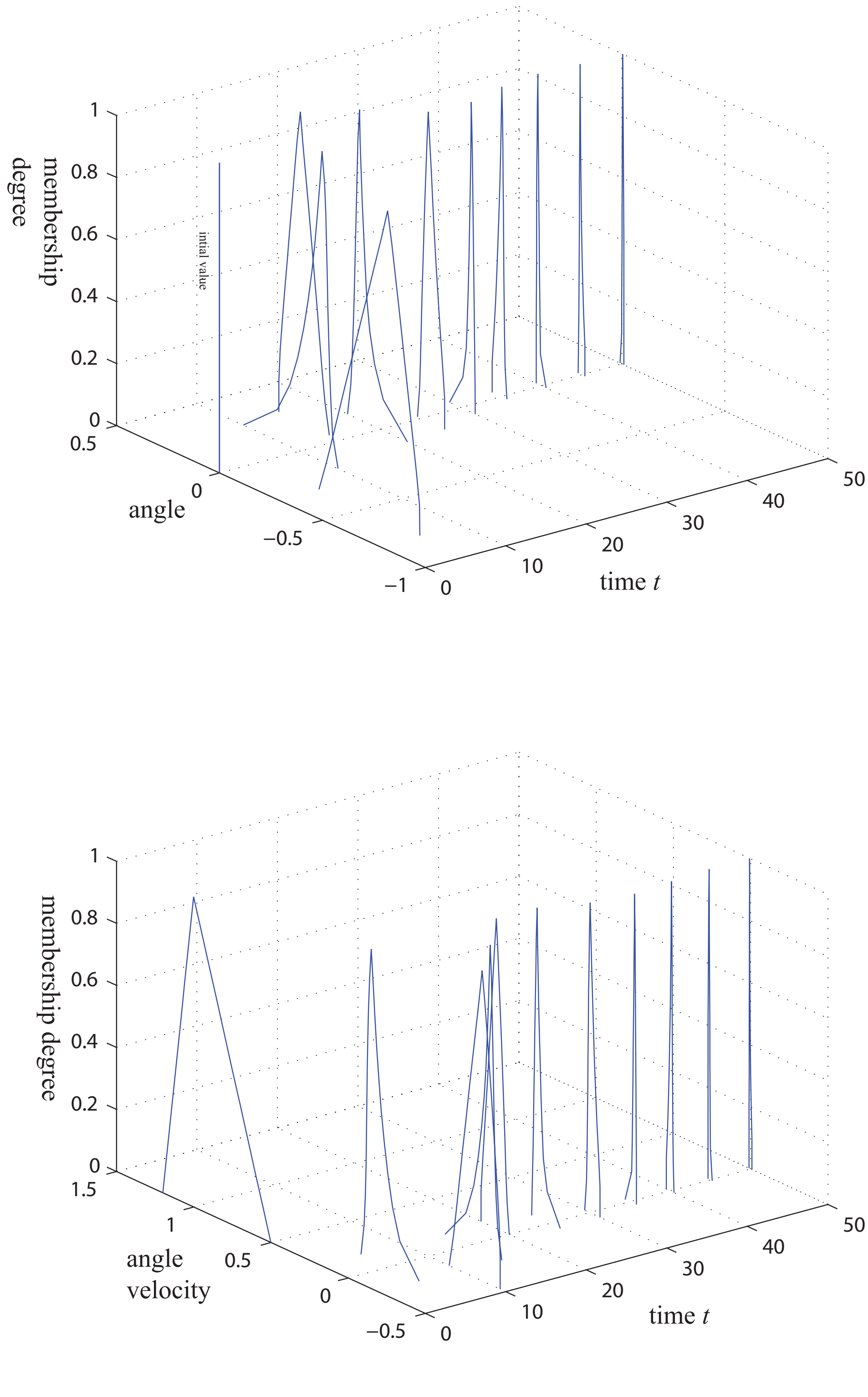
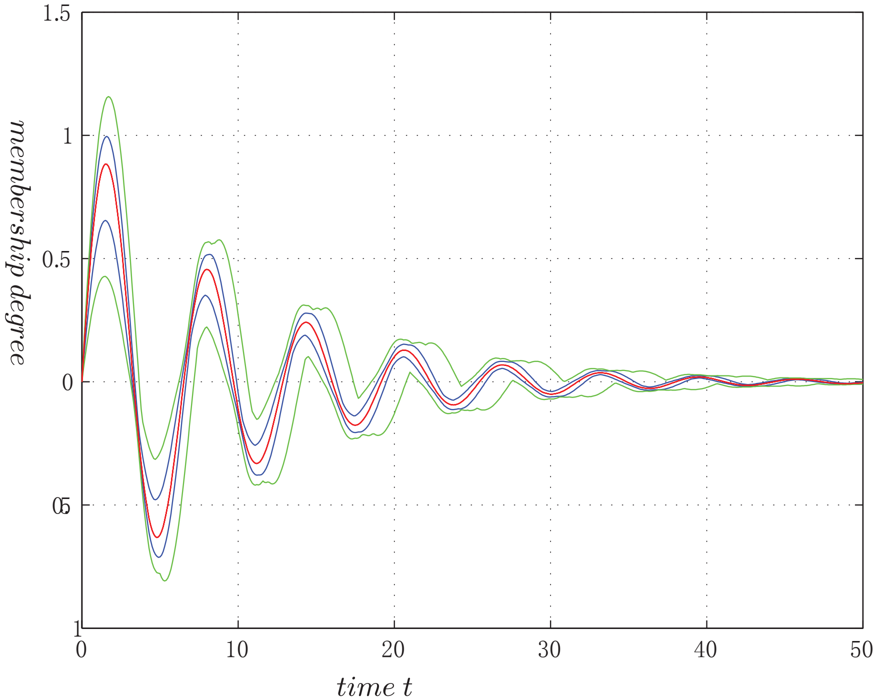


Fig. 5.11: Angle and Angle velocity in 3D plot



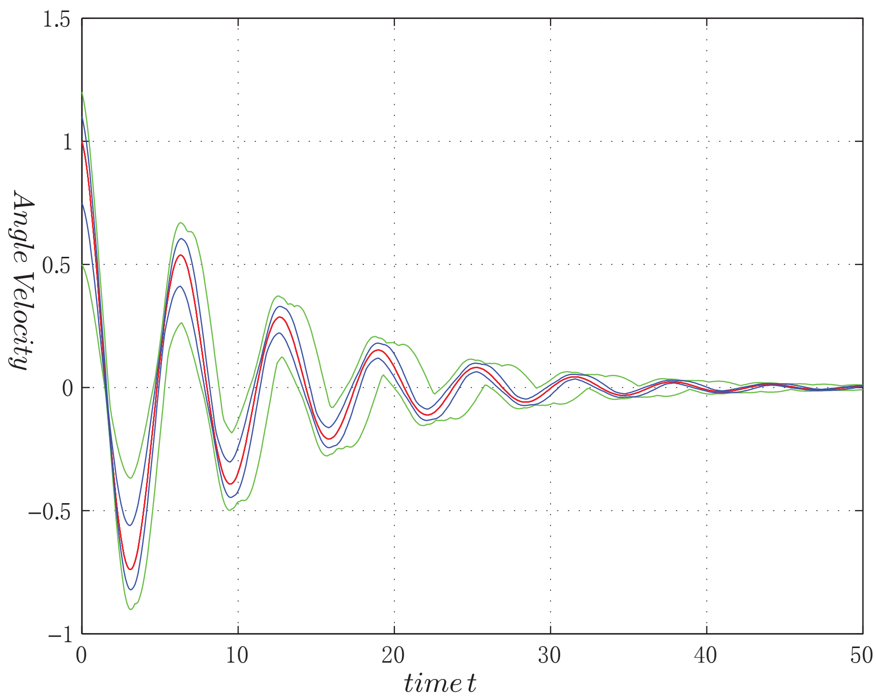


Fig. 5.12: Angle velocity in 2D

5.5 Van Der Pol oscillator as a modified Chua's circuit

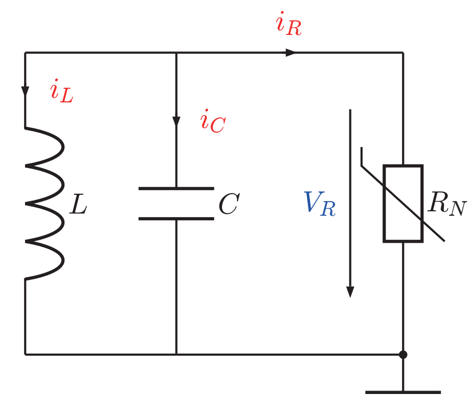


Fig. 5.13: The Van Der Pol oscillator circuit

The Van Der Pol oscillator is a fundamental example in nonlinear oscillation theory [24]. It was used by Van Der Pol in the 1920s to study oscillations in vacuum tube circuits [41]. The dynamical equation of the Van Der Pol oscillator is given below.

$$\ddot{x} - \omega(x - x^2)\dot{x} + x = 0 \tag{5.17}$$

We employ the circuit from [29] and it is modified as in Fig.5.13:

Kirchoff's Current Law:

$$\begin{aligned} i_C + i_C + i_R &= 0 \\ \Rightarrow \frac{di_L}{dt} + \frac{di_C}{dt} + \frac{di_R}{dt} &= 0 \\ \Rightarrow \frac{v_R}{L} + C\frac{d^2 v_R}{dt^2} + \frac{di_R}{dt} &= 0 \\ \Rightarrow \frac{v_R}{L} + C\frac{d^2 v_R}{dt^2} + \frac{di_R}{dv_R}\frac{dv_R}{dt} &= 0 \end{aligned} \tag{5.18}$$

let us rescale the time variable as [24]:

$$\tau \equiv \frac{t}{\sqrt{LC}} \tag{5.19}$$

Table. 5.3: Initial values and parameters of the Van Der Pol circuit

Elements	Description	Value	Support	Type of Fuzzy number	Matlab code
L	inductance	1	[0.3 1.6]	Triangular	1
R	resistance	3	[1.5 4]	Exponential	3
C	capacitance	1	[0.3 1.9]	Exponential	4
$i_R(0)$	initial voltage of R_N	2	[1.1 3.3]	Gaussian	2
$v_R(0)$	initial current of R_N	1	[0.5 1.9]	Triangular	1

Substitute (5.19) in (5.18):

$$\begin{aligned}\frac{C}{LC}\frac{d^2v_R}{d\tau^2}+\frac{1}{\sqrt{LC}}\frac{di_R}{dv_R}\frac{dv_R}{d_\tau}+\frac{v_R(\tau)}{L}&=0\\ \Rightarrow\frac{d^2v_R}{dt^2}+\frac{L}{\sqrt{LC}}\frac{di_R}{dv_R}\frac{dv_R}{dt}+\frac{v_R}{L}&=0\\ \Rightarrow\ddot{v_R}+\omega\frac{di_R}{dv_R}\dot{v_R}+v_R=0,\omega&=\sqrt{\frac{L}{C}}\end{aligned}\tag{5.20}$$

Assume the current of nonlinear resistor is:

$$i_R=\frac{-v_R+\frac{v_R^3}{3}}{R},R_N=R\tag{5.21}$$

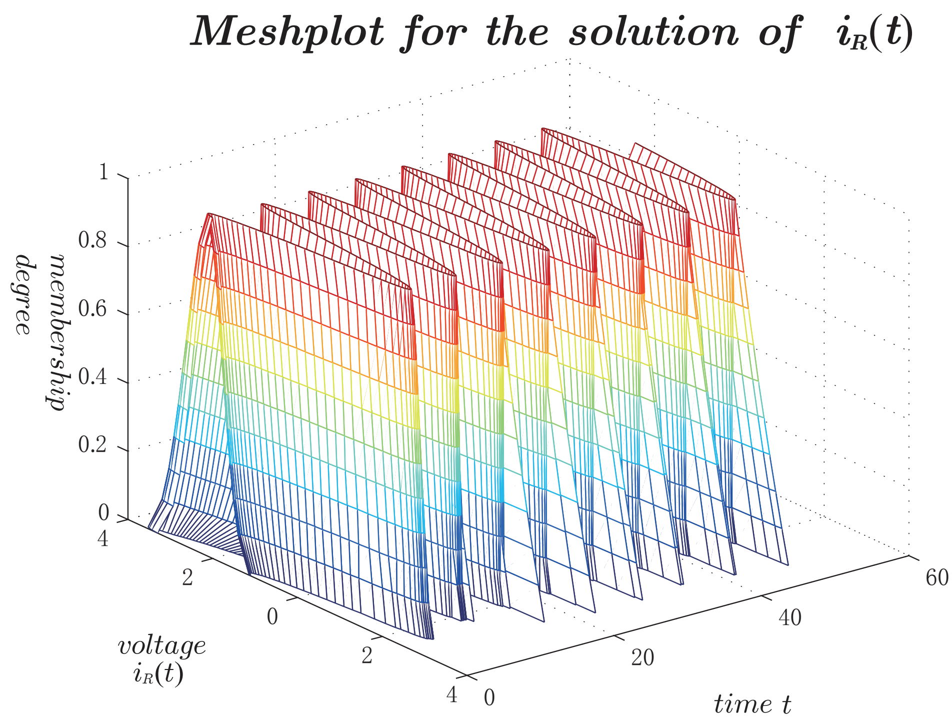
Comparing (5.20) to (5.17):

$$\begin{aligned}\ddot{v_R}+\omega\frac{d}{dv_R}(-v_R+\frac{v_R^3}{3})\frac{\dot{v_R}}{R}+v_R&=0\\ \Rightarrow\ddot{v_R}-\varphi(1-v_R^2)\dot{v_R}+v_R=0,\varphi&=\frac{1}{R}\sqrt{\frac{L}{C}}\end{aligned}\tag{5.22}$$

We convert this system by setting

$$x=v_R,y=\dot{x}=i_R\tag{5.23}$$

With initial condition and parameters: In Fig.5.16, the red line (inner lines) shows the boundaries of the trajectories at membership membership grade $\mu=1$; the blue lines (middle lines) are the boundaries of the trajectories when the membership grade is $\mu=0.5$; the green line (outer lines) are the boundaries of the trajectories when the membership grade is $\mu=0$.



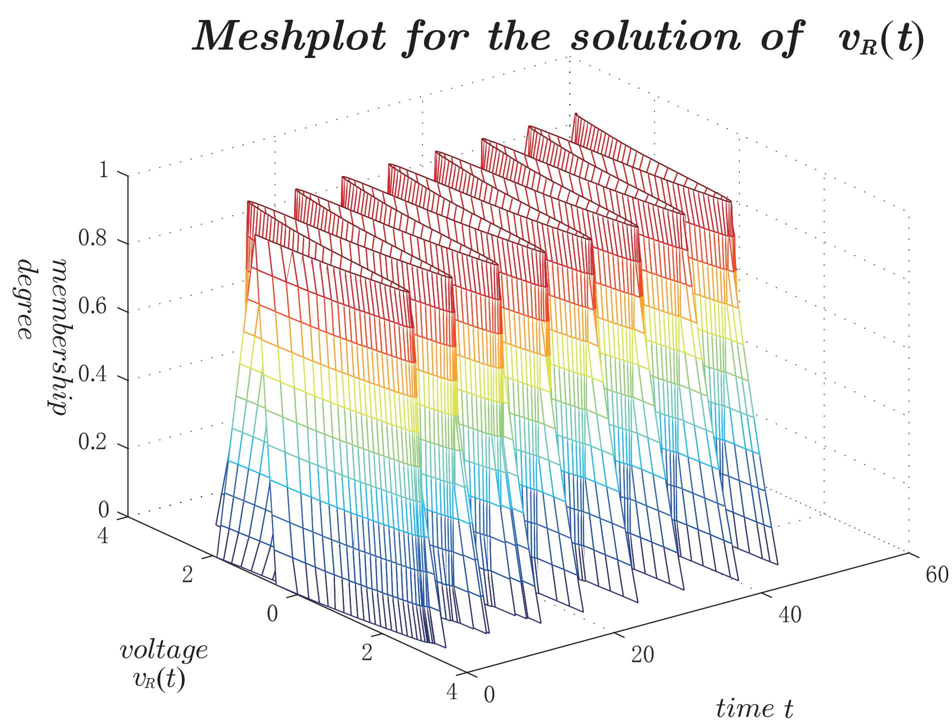


Fig. 5.14: i_R and v_R in Mesh plot

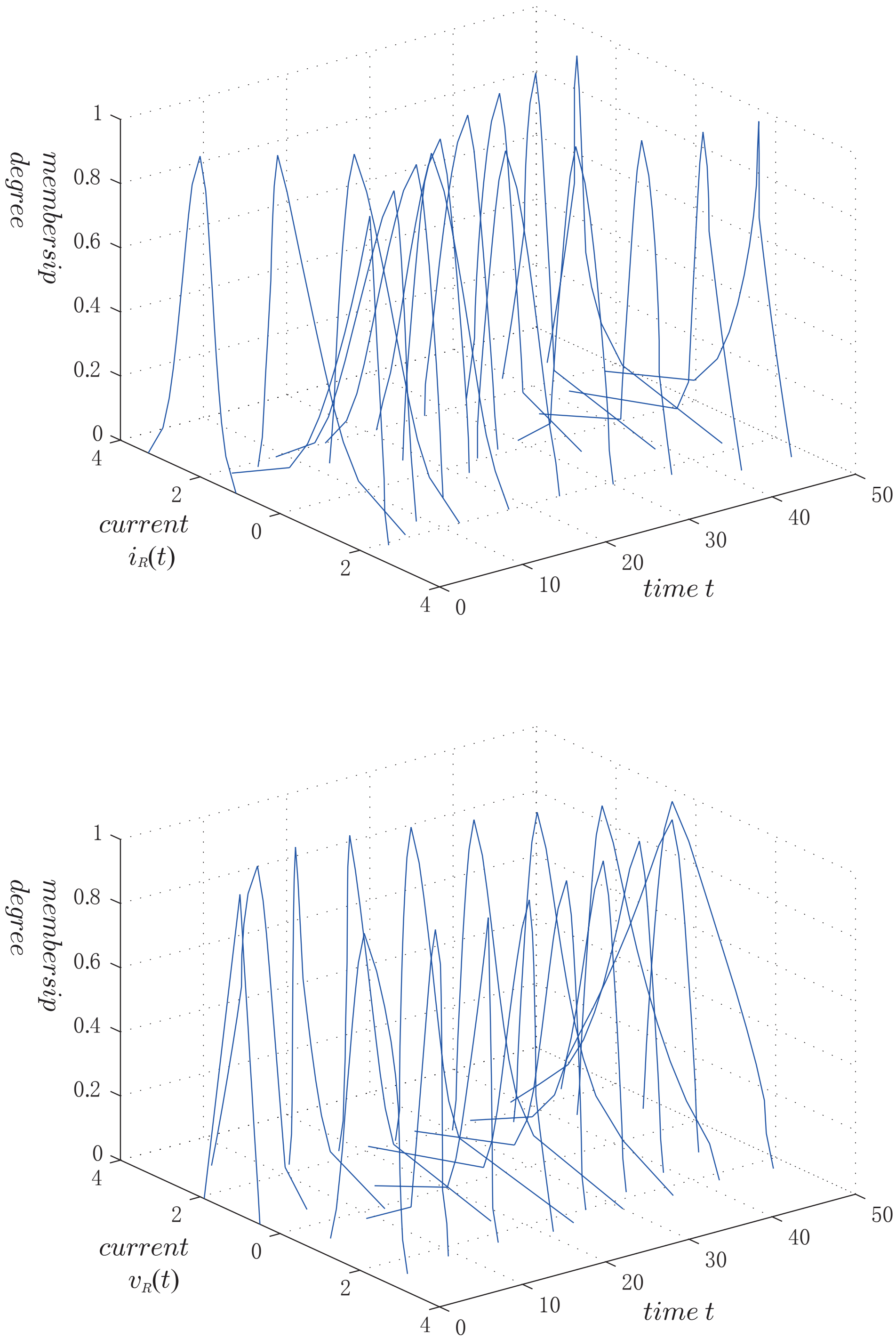


Fig. 5.15: I_R and v_R in 3D plot

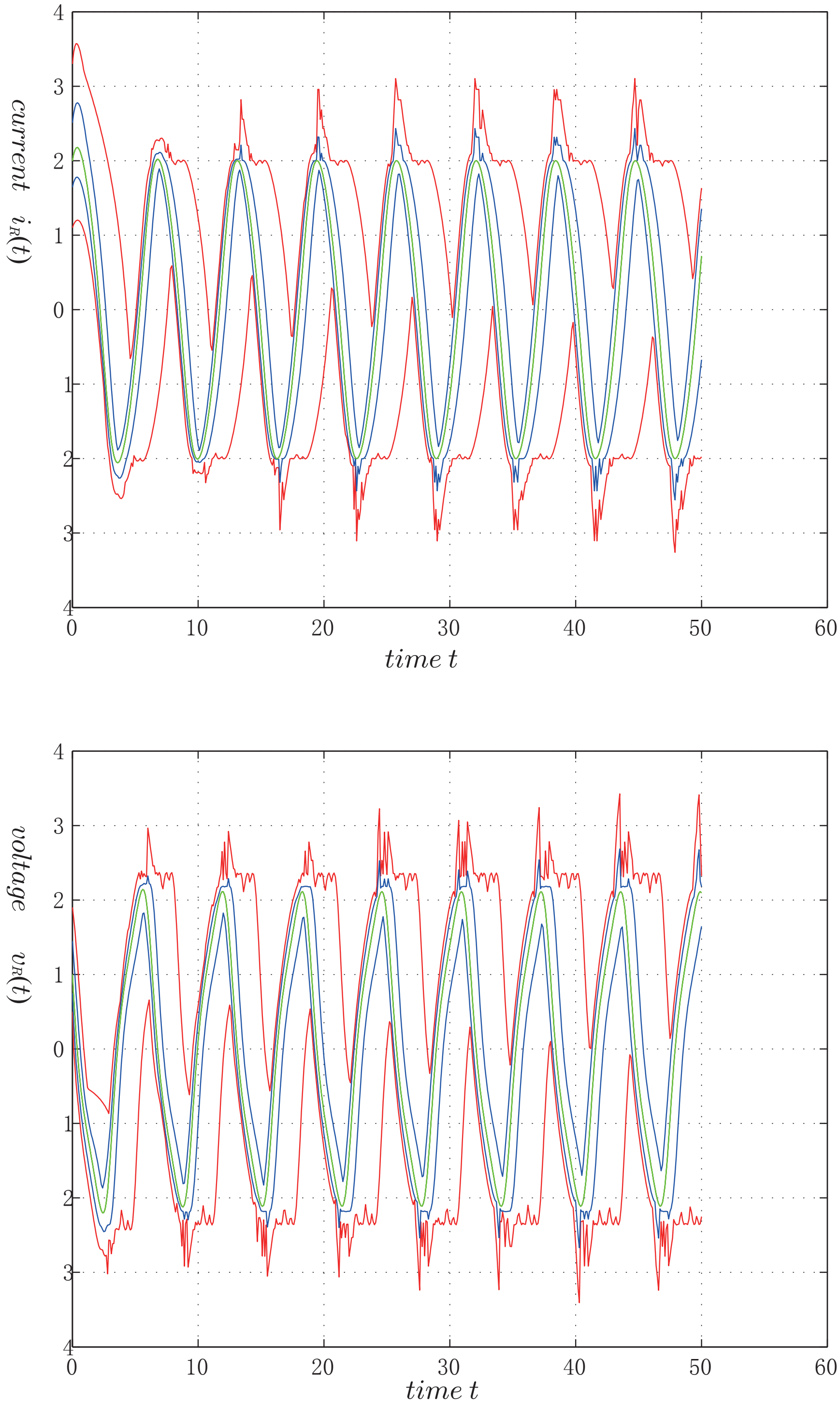


Fig. 5.16: i_R and u_R in 2D plot

The Fig.5.14 shows the mesh and surface plots of trajectories of current $i_R(t)$ and voltage $v_R(t)$ in different membership grade as time t increase. The Fig.5.15 shows trajectories of current $i_R(t)$ and voltage $v_R(t)$ in the form of fuzzy numbers in three dimensions as time t increase. The Fig.5.16 shows trajectories of current $i_R(t)$ and voltage $v_R(t)$ in the form of different membership grade at 0,0.5 and 1, in 2-D.

In this chapter, five examples to demonstrate the effectiveness of the proposed new method in solving the FDEPIV problems were given.

Chapter 6

Conclusions

In this thesis we studied the numerical solution of Fuzzy Differential Equations with multiple fuzzy parameters and fuzzy initial Values (FDEPIV). In particular, this thesis proposed to apply proper fuzzy arithmetic in fuzzifying Runge-Kutta method for the solution of FDEPIV problems. In the previous approaches, researchers either only considered the FDE with fuzzy initial values, or not given a computational method for solving FDEPIV. One of the most powerful tool of calculating the fuzzy numbers is standard fuzzy arithmetic, which is based on the extension principle. In order to avoid the drawback of the standard fuzzy arithmetic, we utilize the proper fuzzy arithmetic, in generating more accurate solutions in solving FDEPIV problems numerically. The proposed method, to the best of our knowledge, is the first numerical method that solve the FDEs with multiple fuzzy parameters and initial values. One limitation of the proposed method is high algorithmic complexity.

Future work for solving the FDEPIV will focus on developing a method to reduce the calculation time. More approaches will be investigated, such as parallelization of the proposed method. Recalling the subsection 2.3.1 in Chapter 2, we investigated using the extension principle for fuzzy sets calculation will lead to an infinite number of combinations of input values of x_1 and x_2 that produce the same output of y. According to this limitation, we utilize the fuzzy arithmetic for the fuzzy sets operations. When applying the fuzzy arithmetic we discretize the membership grade into m intervals, each α-cut is independent in the calculation process. Due to this characteristic, computation at each membership grade of the resultant fuzzy set can be executed in parallel, in order to achieve high efficiency of the proposed method.

In Chapter 4, we established a model in nonlinear FDEPIV problem. Some nonlinear system will become chaotic if their parameters reach critical values. For example: Verhulst Predator-Prey Relationship mathematical model has a parameter, r, which is the intrinsic growth rate.

$$x_{n+1} = rx_n(1 - x_n) \tag{6.1}$$

If r changes from $2.6 \to 3.6$ the dynamic behavior of the system changes drastically.

- For r=2.6 and $x_0 = 0.2$, the system has a single equilibrium point $x \approx 0.61538$ (Fig.6.1),
- For r=3.1 and $x_0 = 0.2$, the system has a 2-Points steady state at $a_1 \approx 0.5578$ and $a_2 \approx 0.7679$ (Fig.6.2),
- For r=3.5 and $x_0 = 0.2$, the system has a 4-Point steady state at $b_1 \approx 0.3828$ $b_2 \approx 0.5011$ $b_3 \approx 0.0.8270$ $b_4 \approx 0.8750$ (Fig.6.3),
- For r=3.6 and $x_0 = 0.2$, the system has a $2n$-Point periodic steady state with an even number of points (Fig.6.4).

As the value of r increases further, the system becomes chaotic. 3.6 is the critical value of the system.

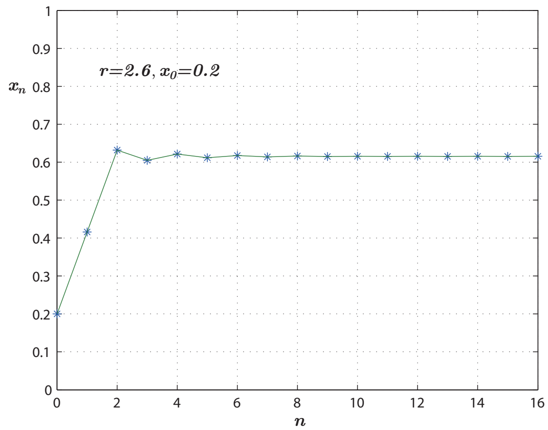


Fig. 6.1: Fixed Point Stability

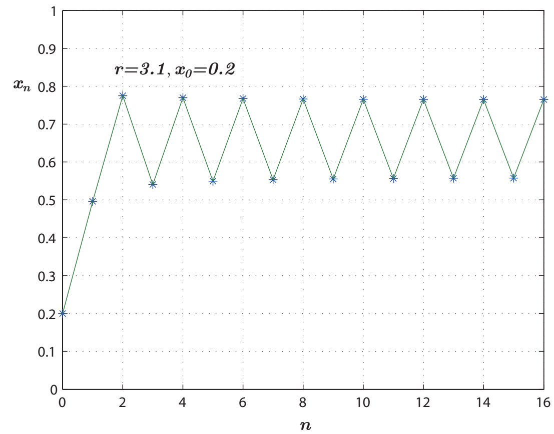


Fig. 6.2: 2-Points Steady State

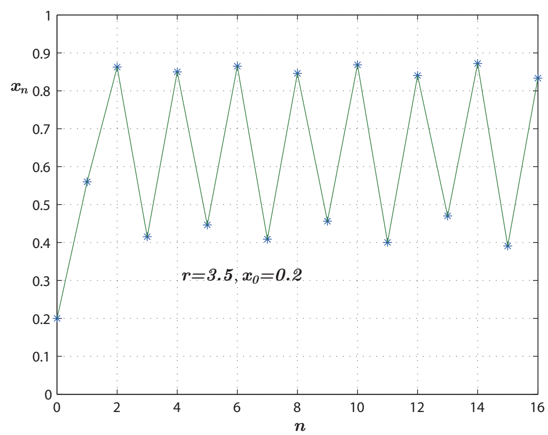


Fig. 6.3: 4-Point Steady State

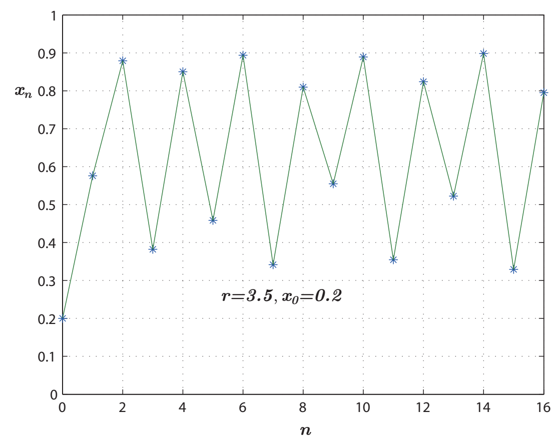


Fig. 6.4: $2n$-Point Steady State

Chaotic system is sensitive to initial conditions. The fifth example in Chapter 5 of using the FDEPIV solver ignore the factor of chaos of the system. In the future work, we may study the behavior of fuzzy chaotic system with parameter around its critical values. For example, if a chaotic system with a fuzzy parameter or initial value $\tilde{p}$ (see Fig.6.5).

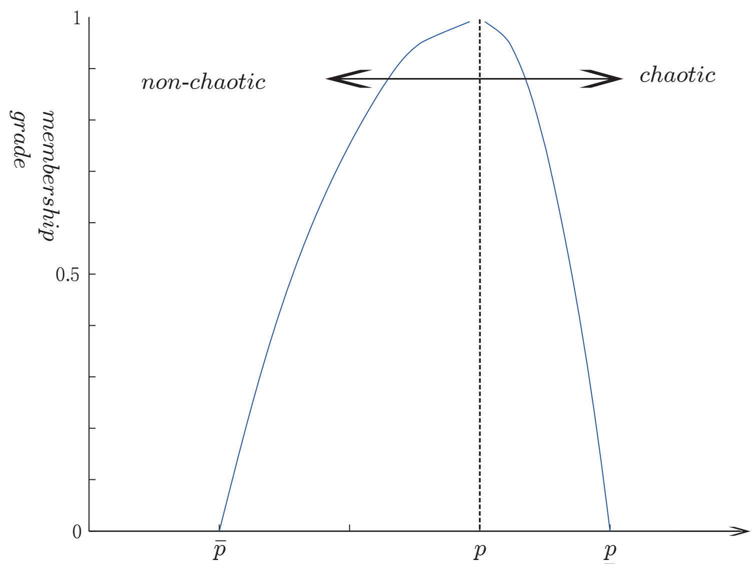


Fig. 6.5: non-chaotic → chaotic

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Appendix A

Appendix

Applying the Extension Principle for solving the Example 2.4.2

The original function f needs 3 multiple multiplications, 1 division and 1 subtraction. Due to the extension principle is complex for calculation, we re-write the function in factored form:

$$f = x_1 x_2 (x_3 - \frac{1}{x_3}) \tag{A.1}$$

Therefore, the function in factored form needs 2 multiple multiplications, 1 division and 1 subtraction. For $x_1 x_2$: we can follow the Example 2.2.4, $\tilde{x}_1 = (0/-1, 0.5/0, 1/1, 0.5/1.5, 0/2), \tilde{x}_2 = (0/1, 0.5/1.5, 1/2, 0.5/2.5, 0/3)$. Where $x_{1,i} \in \tilde{x}_1, x_{2,i} \in \tilde{x}_2, i = 1, 2, 3, 4, 5$.

$$\begin{cases} x_{1,1} = -1 & , & x_{1,2} = 0 & , & x_{1,3} = 1 & , & x_{1,4} = 1.5 & , & x_{1,5} = 2 \\ \mu_{\tilde{x}_1}(x_{1,1}) = 0 & , & \mu_{\tilde{x}_1}(x_{1,2}) = 0.5 & , & \mu_{\tilde{x}_1}(x_{1,3}) = 1 & , & \mu_{\tilde{x}_1}(x_{1,4}) = 0.5 & , & \mu_{\tilde{x}_1}(x_{1,5}) = 0 \end{cases} \tag{A.2}$$

$$\begin{cases} x_{2,1} = 1 & , & x_{2,2} = 1.5 & , & x_{2,3} = 2 & , & x_{2,4} = 2.5 & , & x_{2,5} = 3 \\ \mu_{\tilde{x}_2}(x_{2,1}) = 0 & , & \mu_{\tilde{x}_2}(x_{2,2}) = 0.5 & , & \mu_{\tilde{x}_2}(x_{2,3}) = 1 & , & \mu_{\tilde{x}_2}(x_{2,4}) = 0.5 & , & \mu_{\tilde{x}_2}(x_{2,5}) = 0 \end{cases} \tag{A.3}$$

By using the extension principle, we have the following table:

$\mu(x_1x_2)/x_1x_2$	$\mu_{\tilde{x}_1}(x_{1,1})/x_{1,1}$	$\mu_{\tilde{x}_1}(x_{1,2})/x_{1,2}$	$\mu_{\tilde{x}_1}(x_{1,3})/x_{1,3}$	$\mu_{\tilde{x}_1}(x_{1,4})/x_{1,4}$	$\mu_{\tilde{x}_1}(x_{1,5})/x_{1,5}$
$\mu_{\tilde{x}_2}(x_{2,1})/x_{2,1}$	$0/-1$	$0/0$	$0/1$	$0/1.5$	$0/2$
$\mu_{\tilde{x}_2}(x_{2,2})/x_{2,2}$	$0/-1.5$	$0.5/0$	$0.5/1.5$	$0.5/2.25$	$0/3$
$\mu_{\tilde{x}_2}(x_{2,3})/x_{2,3}$	$0/-2$	$0.5/0$	$1/2$	$0.5/3$	$0/4$
$\mu_{\tilde{x}_2}(x_{2,4})/x_{2,4}$	$0/-2.5$	$0.5/0$	$0.5/2.5$	$0.5/3.75$	$0/5$
$\mu_{\tilde{x}_2}(x_{2,5})/x_{2,5}$	$0/-3$	$0/0$	$0/3$	$0/4.5$	$0/6$

(A.4)

For $x_3 - \frac{1}{x_3}$, set $y = \frac{1}{x_3}$,

$\tilde{x}_3 = (0/2, 0.5/2.25, 1/2.5, 0.5/2.75.5, 0/3), \tilde{y} = (0/0.5, 0.5/0.44.5, 1/0.4, 0.5/0.36, 0/33)$. Where $x_{3,i} \in \tilde{x}_3, y_i \in \tilde{y}, i = 1, 2, 3, 4, 5$.

$$\begin{cases} x_{3,1} = 2 \quad , \quad x_{3,2} = 2.25 \quad , \quad x_{3,3} = 2.5 \quad , \quad x_{3,4} = 2.75 \quad , \quad x_{3,5} = 3 \\ \mu_{\tilde{x}_3}(x_{3,1}) = 0 \quad , \quad \mu_{\tilde{x}_3}(x_{3,2}) = 0.5 \quad , \quad \mu_{\tilde{x}_3}(x_{3,3}) = 1 \quad , \quad \mu_{\tilde{x}_3}(x_{3,4}) = 0.5 \quad , \quad \mu_{\tilde{x}_3}(x_{3,5}) = 0 \end{cases} \tag{A.5}$$

$$\begin{cases} y_1 = 0.5 \quad , \quad y_2 = 0.44 \quad , \quad y_3 = 0.4 \quad , \quad y_4 = 0.36 \quad , \quad y_5 = 0.33 \\ \mu_{\tilde{y}}(y_1) = 0 \quad , \quad \mu_{\tilde{y}}(y_2) = 0.5 \quad , \quad \mu_{\tilde{y}}(y_3) = 1 \quad , \quad \mu_{\tilde{y}}(y4) = 0.5 \quad , \quad \mu_{\tilde{y}}(y_5) = 0 \end{cases} \tag{A.6}$$

$\mu(x_3 - \frac{1}{x_3})/(x_3 - \frac{1}{x_3})$	$\mu_{\tilde{x}_3}(x_{3,1})/x_{3,1}$	$\mu_{\tilde{x}_3}(x_{3,2})/x_{3,2}$	$\mu_{\tilde{x}_3}(x_{3,3})/x_{3,3}$	$\mu_{\tilde{x}_3}(x_{3,4})/x_{3,4}$	$\mu_{\tilde{x}_3}(x_{3,5})/x_{3,5}$
$\mu_{\tilde{y}}(y_1)/y_1$	$0/1.5$	$0/1.75$	$0/2$	$0/2.25$	$0/2.5$
$\mu_{\tilde{y}}(y_2)/y_2$	$0/1.56$	$0.5/1.81$	$0.5/2.06$	$0.5/2.31$	$0/2.56$
$\mu_{\tilde{y}}(y_3)/y_3$	$0/1.6$	$0.5/1.85$	$1/2.1$	$0.5/2.35$	$0/2.6$
$\mu_{\tilde{y}}(y_4)/y_4$	$0/1.64$	$0.5/1.89$	$0.5/2.14$	$0.5/2.39$	$0/2.64$
$\mu_{\tilde{y}}(y_5)/y_5$	$0/1.67$	$0/1.92$	$0/2.17$	$0/2.42$	$0/2.67$

(A.7)

From A.4 and A.7, we can find that the two result for $a_1 = x_1x_2$ and $a_2 = x_3 - \frac{1}{x_3}$, when membership grade $\mu = 0$, $\mathrm{supp}(a_1) = [-3, 6]$, $\mathrm{supp}(a_2) = [1.5, 2.67]$; when membership grade $\mu = 0.5$, $a_1^{\alpha=0.5} = [0, 3.75]$, $a_2^{\alpha=0.5} = [1.81, 2.39]$; $\mathrm{core}(a_1) = 2$, $\mathrm{core}(a_2) = 2.1$. The function $f = a_1a_2$. Repeat the procedure of very beginning: $\tilde{a}_1 = (0/-3, 0.5/0, 1/2, 0.5/3.75, 0/6), \tilde{a}_2 =$

$(0/1.5, 0.5/1.81, 1/2.1, 0.5/2.39, 0/2.67)$. Where $a_{1,i} \in \tilde{a}_1, a_{2,i} \in \tilde{a}_2, i = 1,2,3,4,5$. By using the extension principle, we have the following table:

$\mu(a_1a_2)/a_1a_2$	$\mu_{\tilde{a}_1}(a_{1,1})/a_{1,1}$	$\mu_{\tilde{a}_1}(a_{1,2})/a_{1,2}$	$\mu_{\tilde{a}_1}(a_{1,3})/a_{1,3}$	$\mu_{\tilde{a}_1}(a_{1,4})/a_{1,4}$	$\mu_{\tilde{a}_1}(a_{1,5})/a_{1,5}$
$\mu_{\tilde{a}_2}(a_{2,1})/a_{2,1}$	$0/-4.5$	$0/0$	$0/3$	$0/5.625$	$0/9$
$\mu_{\tilde{a}_2}(a_{2,2})/a_{2,2}$	$0/-5.43$	$0.5/0$	$0.5/3.62$	$0.5/6.79$	$0/10.86$
$\mu_{\tilde{a}_2}(a_{2,3})/a_{2,3}$	$0/-6.3$	$0.5/0$	$1/4.2$	$0.5/7.88$	$0/12.6$
$\mu_{\tilde{a}_2}(a_{2,4})/a_{2,4}$	$0/-7.17$	$0.5/0$	$0.5/4.78$	$0.5/8.96$	$0/14.34$
$\mu_{\tilde{a}_2}(a_{2,5})/a_{2,5}$	$0/-8$	$0/0$	$0/5.34$	$0/10$	$0/16$

(A.8)

Therefore, $\text{supp}(a_1a_2) = [-8, 16]$, $[a_1a_2]^{\alpha=0.5} = [0, 8.96]$, $\text{core}(a_1a_2) = 4.2$.

Appendix B

Appendix

Usage of FDE solver

The MATLAB code for using this FDE solver is shown in Listing B.1.

Listing B.1: MATLAB Code

```
[left,right,tout,u] = xxpro45(FUN,tspan,miu,h,number_para,varargin)
% Here 'left' and 'right' are the boundaries matrix of solutions at differernt
% a-cuts, 'tout' is the time period matrix, 'u' is the membership degree matrix. On
% RHS, 'FUN' is the fuzzy differential equation, 'tspan' is the time
% period, 'miu' is the value of discrete constant space of membership
% degrees, 'h' is time step, 'number_para' is the number of all parameters,
% and 'varargin' is the input values for initial values and parameters, for
% example, if a fuzzy initial value is a triangluar fuzzy number p1 with
% support [2 5], and membership degree u=1, p1 = 3, so the input can be
% written as [2 3 5 1], the last '1' is the type of fuzzy number which is
% triangular. Here is the list of the fuzzy number type:
% 1. Triangular
% 2. Gaussian
```

```
% 3. Quadratic
% 4. Exponential
```

Back to Section 2, the m file of the circuit is shown in Listing B.2

Listing B.2: MATLAB Code of FUN

```
function dy = lcr(t,y,l,c,r)
dy(1) = y(2);
dy(2) = -(r/l)*y(2)-(1/(c*l))*y(1)
```

For example: if we try to solve the model in Section 2, with the fuzzy numbers described in 5.3, we can use the solver as following: [left,right,tout,u] = fdesolver(@fde,[0 20],0.1,0.1,3,0,[0.9 1 1.2 3],[0.8 1 1.1 1],[0.8 3 3.4 2],[0.9 1 1.2 4])