ESTIMATION OF LOCATION AND SCALE PARAMETERS OF THE EXPONENTIAL DISTRIBUTION WITH APPLICATIONS TO LIFE TESTING

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ABSTRACT

In this thesis we consider the exponential distribution involving location and scale parameters both assumed unknown. The problem of estimation of the parameters involved and the reliability function is attempted in case of a full sample and also in case of single stage right censored sample. The methods employed are minimum variance unbiased estimation, maximum likelihood and best linear unbiased estimates based on components of order statistics.

Comparisons are made between the various estimates on the basis of bias, variance and mean squared error and generalized mean squared error in case of simultaneous estimation. On the basis of generalized mean squared error we compare estimates based on full samples with those based on single stage right censored samples. An interesting result is the rather appreciable increase in the generalized mean squared error for even slight censoring.

Chapter I is introductory and gives a brief survey of literature. In Chapter II and III we present detailed derivation of estimators. These chapters also contain generalizations of results due to Pugh (1963) in case of known location but unknown scale parameters. In the last chapter we consider application of these results to life testing experiments.

CHAPTER I

INTRODUCTION

A statistical analyst is often confronted with the task of understanding the problem being presented, relating this to known theory and sometimes developing appropriate theory to fit the problem. When the problem is related to life testing experiments, the analyst now has at his disposal a large amount of recently developed theory and expanding industrial interest in life testing offers yet more to come.

It is a characteristic feature of most life tests that they give rise to ordered observations. For example should machine breakdown time be analyzed, the time of breakdown for any x_i item tested would occur in such a way that $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ where $x_{(i)}$ is a random value taken by any of n machines in the sample. We may consider any $x_{(i)}$ value as the time to failure.

The choice of the distribution for the failure time is sometimes very difficult and can include such distributions as the log-normal, normal, uniform and exponential. For illustrative examples we refer to Herd (1956). Estimating procedures for gamma and rectangular distributions have been given by Patil and Wani (1966). The model following Weibull distribution has recently been considered by such researchers as Harter and Moore (1965), Cohen (1965) and Johns Jr. and Lieberman (1966).

In this thesis we will consider a model for life testing based on the exponential distribution with location and scale parameters. It is our intention to estimate these parameters, as well as a useful function of these parameters, the reliability function. The problem of estimation from full and censored samples dates back as early as 1897 and for a historical background the reader is referred to Herd (1956).

When working with a full sample case the individual failure times are available for all observations, and consequently the analysis is rather straightforward. However, censored samples may give rise to various theoretical problems.

A random sample of size n is drawn from the exponential population, where the data becomes available such that the smallest observation comes first, the second smallest comes second, until the largest observation is finally obtained. Clearly we can discontinue experimentation well before all n items have failed. Such censoring is often essential in life testing experiments due to limitations of time, item availability and economic considerations. Suppose in our above experiment we continue to run the experiment until only r (fixed) items out of n have failed. This is known as type II right censored sample and the data here is typically represented by $x_{(1)}, x_{(2)}, \ldots, x_{(r)}$ where (n-r) items have survived beyond $x_{(r)}$. Another type of censoring that naturally occurs is if instead of fixing the number of items failed, we fix the time until which the experiment will last. In this case the data is typically represented by $x_{(1)}, x_{(2)}, \ldots, x_{(k)}$, k of the items having failed before time T (fixed) and (n-k) items having survived beyond time T. This type of censoring is known as type I censoring. Epstein (1960a) gives an excellent discussion and also a numerical example of the two types of censoring mentioned above for the case of known location parameter. In a later paper Epstein (1960 b) gives point and confidence interval estimates from life test data involving both types of censoring for known location parameter. Although theoretically distinct, the differences in estimators for the two cases above are negligible if the sample size is large (Plackett (1959)).

In the preceeding paragraph we have considered life testing experiments where the total sample size n is known. This knowledge of the number of censored observations, (n-r) where r is fixed and (n-k) where k is a random variable distinguishes the notion of a censored sample from that of a truncated sample in which for a sample of unknown overall size, all $x_i \leq T$ are observed. The distinction between truncated and censored samples is well illustrated by Deemer and Votaw (1955) where these authors calculate the maximum likelihood estimate (MLE) of the scale parameter for the exponential case assuming the location parameter to be zero. In the truncated exponential model Holla (1967) has obtained a minimum variance unbiased estimate (MVUE) of Reliability when the point of truncation is known for the known location parameter case. We shall obtain in chapter three a uniform minimum variance unbiased estimate (UMVUE) of Reliability when we have single stage type II right censored sample for the two parameter exponential model.

Censoring where each x_i has known different truncation point T_i has been discussed by Bartholomew (1957). He obtains MLE of scale parameter for known location and also considers a numerical example. A much more detailed discussion of this MLE and its asymptotic properties are discussed by Bartholomew (1963).

In many practical situations, the initial censoring results only in withdrawal of a portion of survivors. Further censoring may then occur at later stages and we speak of progressively censored samples. This often results when a compromise between the need for more rapid testing and desire to include some extreme life spans is made (Cohen (1963)). Progressively censored samples where times of censoring and the number of items removed are the result of random causes have been considered by Sampford (1952). Progressively censored samples, when fixed number of observations k_i are removed at random times, have been considered by Herd (1956) who refers to them as multiple censored samples. We consider a numerical example in the last chapter where data are results of response time studies involving animals, and we resort to Herd's estimators for the two parameter exponential model.

In this thesis we develop estimates for both full and single stage censored samples considering the two parameter exponential model. As a first reduction of data, we determine sufficient statistics for the location and scale parameter by using factorization criteria for sufficient statistics. Since they are complete sufficient statistics [Epstein and Sobel (1954)] we derive Uniformly Minimum Variance Unbiased Estimates (UMVUE) by using well known Rao-Blackwell, Lehmann-Scheffé Theorem [Rao (1966)]. For Best Linear Unbiased Estimation (BLUE) based on components of order statistics [Sarhan and Greenburg (1962)], we use the standard least squares theory together with Gauss Markoff Theorem [Rao (1966)]. We also consider MLE and an expression for bias of the MLE of reliability is obtained. This is a slight generalization of results due to Pugh (1963) for unknown scale and known location parameter. An appropriate list of references is included. However for an extensive bibliography for two parameter exponential distribution in general we refer to Saleh (1964).

Feigl and Zelen (1965) have considered the applications of life testing experiment techniques in some areas of cancer research. In the last chapter we also consider some new applications of theory developed in previous chapters to life testing experiments in the area of cancer research.

CHAPTER II

Estimating Reliability Functions from Full Sample

Let $x_1, x_2, \ldots, x_n$ be a random sample from a continuous distribution whose p.d.f. is given by

$$f(x;\ \theta, a) = \begin{cases} \frac{1}{\theta} \exp\left[-\frac{(x-a)}{\theta} \right] & x \geq a,\ \theta > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

It is desired to estimate "a" the location parameter and "θ" the scale parameter which determine the underlying distribution. We will be considering in this chapter various procedures for estimating "a" and "θ" as well as $R(t|a', \theta')$, a function of a', θ', estimates for "a" and "θ" respectively, when a complete sample is available. In any estimation procedure, we always use the sufficient statistic if one is available. We will therefore first obtain a vector valued sufficient statistic for the vector valued parameter

$$\tilde{\theta} = [a, \theta], \quad a \in R_1,\ \theta > 0.$$

Derivation of Sufficient Statistics:

We first consider the order statistics $(x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ where $a \leq x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)} < \infty$. Then as is well known, [Hogg and Craig (1959)], the joint p.d.f. of $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ is given by

$$g(x_{(1)}, \ldots, x_{(n)};\ a, \theta) = n! \prod_{i=1}^{n} \left(\frac{1}{\theta}\right) \exp\left[-\frac{x_{(i)} - a}{\theta} \right] \tag{2.2}$$

for $a \leq x_{(1)} \leq \cdots \leq x_{(n)} < \infty$ and zero elsewhere.

Define the indicator function $c(a,b)$ such that

$$c(a,b) = \begin{cases} 1 & \text{if } a \geq b \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

Then we can write the joint p.d.f. of $x_{(i)}$, $i = 1,2,\ldots,n$, as

$$H(x_{(1)},\ldots,x_{(n)} \,;\, a,\theta) = \frac{n!}{\theta^n} \exp\left[- \sum_{i=1}^{n} \frac{(x_{(i)}-a)}{\theta} \right] .$$

$$\prod_{j=2}^{n} c(x_{(j)}, x_{(j-1)})\, c(x_{(1)}, a)$$

and by factorization criteria it is clear that the order statistics $x_{(1)},x_{(2)},\ldots,x_{(n)}$ are sufficient. We consider further reduction and show that

$$u = x_{(1)},\ v = \sum_{i=2}^{n} \left(x_{(i)} - x_{(1)}\right),$$

are jointly sufficient for $\left[a,\theta\right]$. The joint p.d.f. of the order statistics can be written as

$$H(x_{(1)},\ldots,x_{(n)} \mid a,\theta) = \frac{n}{\theta} \exp\left[- \frac{n}{\theta} (x_{(1)} - a)\right] c(x_{(1)},a).$$

$$\frac{(n-1)!}{\theta^{n-1}} \exp\left[- \sum_{j=2}^{n} \frac{(x_{(j)} - x_{(1)})}{\theta}\right] \cdot \prod_{j=2}^{n} c(x_{(j)},x_{(j-1)}) \tag{2.4}$$

$$= H_1(x_{(1)} \mid a,\theta).\ H(x_{(2)},\ldots,x_{(n)} \mid x_{(1)},\theta)$$

where $H_1(x_{(1)} \mid a,\theta)$ is the p.d.f. of $x_{(1)}$ and $H(x_{(2)},\ldots,x_{(n)} \mid x_{(1)},\theta)$ is the conditional p.d.f. of $x_{(2)},\ldots,x_{(n)}$ given $x_{(1)}$.

Define $Z_1 = x_{(1)}$, $Z_{(i)} = x_{(i)} - x_{(1)}$ for $i = 2,\ldots,n$. Then $\theta \leq Z_{(2)} \leq \cdots \leq Z_{(n)} < \infty$ and the Jacobian of the transformation gives $|J| = 1$. Then (2.4) can be written as

$$H(x_{(1)},\ldots,x_{(n)} | a,\theta) = \frac{n}{\theta} \exp\left[- \frac{n}{\theta} (Z_1 - a)\right] c(Z_1,a) \frac{(n-1)!}{\theta^{n-1}} \cdot$$

$$\exp\left[- \sum_{2}^{n} \frac{Z_{(i)}}{\theta}\right] c(Z_{(2)},0)\ldots c(Z_{(n)}, Z_{(n-1)}) \tag{2.5}$$

This shows that $Z_{(2)},\ldots,Z_{(n)}$ given Z_1 are order statistics from a sample of size (n-1) from a distribution as in (2.1) with $a = 0$, $\theta = \theta$. We now define $v = \sum_{i=2}^{n} Z_{(i)}$, $u = Z_{(1)}$, and we have that joint p.d.f. of (u,v) is given by

$$g(u,v) = \frac{n}{\theta} \exp\left[- \frac{n}{\theta} (u-a)\right] c(u,a) \cdot g_1 (v|u) \tag{2.6}$$

Since $Z_{(i)}$, $i = 2,\ldots,n$ are order statistics from a Gamma distribution* with $\alpha = 1$, $\beta = \theta$, the unordered Z_i for $i = 2,\ldots,n$ are distributed as Gamma r.v. with $\alpha = 1$, $\beta = \theta$. Using moment-generating function technique, then

$$M_{Z_i}(t) = (1 - \theta T)^{-1} \quad \text{for } i = 2,3,\ldots,n. \text{ Then}$$

$$v = \sum_{i=2}^{n} Z_{(i)} = \sum_{i=2}^{n} Z_i \quad \text{has m.g.f. given by}$$

$$M_v(t) = (1 - \theta T)^{-(n-1)}$$ which is the m.g.f. of Gamma distribution with $\alpha = (n-1)$ and $\beta = \theta$. Thus the conditional p.d.f. of v given u is

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* $g(z\ ;\alpha, \beta) = \frac{1}{\Gamma(\alpha)} \exp\left[-z/\beta\right] z^{(\alpha-1)} \frac{1}{\beta^{\alpha}}$, where $0 \leq z < \infty$.

$$g_1(v|u) = \frac{1}{\Gamma(n-1)} \exp\left[-\frac{v}{\theta}\right] \left(\frac{v}{\theta}\right)^{n-2} \frac{1}{\theta}\, c(v,0) \qquad (2.7)$$

and therefore the joint p.d.f. of (u,v) is given by,

$$g(u,v) = \frac{n}{\theta} \exp\left[-\frac{n(u-a)}{\theta}\right] c(u,a) \cdot \frac{1}{\Gamma(n-1)} \exp\left[-\frac{v}{\theta}\right] \left(\frac{v}{\theta}\right)^{n-2} \frac{1}{\theta}\, c(v,0)$$

$$(2.8)$$

Referring to (2.4) we see that it can be written in the form

$$H(x_{(1)},\ldots,x_{(n)} \mid a,\theta) = g(u,v) \left[\frac{\Gamma(n)\,\Gamma(n-1)}{\left[\sum_{i=2}^{n} (x_{(i)}-x_{(1)})\right]^{n-2}} \cdot \frac{\prod_{j=2}^{n} c(x_{(j)} - x_{(j-1)},0)}{c(\sum_{2}^{n} (x_{(i)} - x_{(1)}),0)} \right]$$

The likelihood of the sample $(x_1,\ldots,x_n)$ can be written as

$$L(x_1,\ldots,x_n|a,\theta) = \frac{1}{\theta^n} \exp\left[-\frac{\sum_{1}^{n} (x_i-a)}{\theta}\right] \prod_{i=1}^{n} c(x_i,a)$$

$$= H(x_{(1)},\ldots,x_{(n)}|a,\theta) \cdot T_1(x_{(1)},\ldots,x_{(n)})$$

$$= g(u,v) \cdot T_2(x_{(1)},\ldots,x_{(n)}) \qquad (2.9)$$

and by factorization criteria, u,v are jointly sufficient for $\tilde{\theta} = [a,\theta]$. These joint sufficient statistics are known to be complete and for the proof we refer to Epstein and Sobel (1954).

We now consider three methods of estimation of parameters (a,θ): (i) The method of uniformly minimum variance unbiased estimation (UMVUE); (ii) The method of best linear unbiased estimation using linear functions of components of order statistics (BLUE); and (iii) The method of maximum likelihood estimation (MLE).

(i) UMVUE for "a" and "θ"

We have shown that a joint sufficient statistics for $\tilde{\theta}$ exists and from Epstein and Sobel (1953), we know that it is complete. We know, Rao (1966), that if a complete sufficient statistic T exists, then any function g(T) is a UMVUE of its expected value $\psi(\tilde{\theta}) = E[g(T)]$. We thus limit our search of estimators to functions of complete sufficient statistics. This technique is a well known application of Rao-Blackwell, Lehman-Scheffé theorem for which we refer to Rao (1966). Thus given $\psi(\tilde{\theta})$ UMVUE could be obtained by trial and error method by searching for a function g(T) such that $E[g(T)] = \psi(\tilde{\theta})$. Otherwise we start with any unbiased estimator $T_i(\underline{x})$ such that $E[T_i(\underline{x})] = \psi(\tilde{\theta})$. Then we take its conditional expectation, given the sufficient statistics u,v. By the theorem this expectation $E[E[T_i(\underline{x})]|u,v]$ is UMVUE. Since u,v are jointly sufficient then a^*, θ^* the UMVUE of a,θ respectively will be functions of u and v.

We wish to estimate θ. Define $T_1'(u,v) = v$. Now we see that $E[v] = E[E[v|u]]$ where

$$E[v|u] = \emptyset(u) = \int_0^\infty \frac{1}{\Gamma(n-1)} \exp\left[-\frac{v}{\theta}\right] \left(\frac{v}{\theta}\right)^{n-1} dv = (n-1)\theta$$

which is independent of u. Thus $E[\emptyset(u)] = (n-1)\theta = E(v)$ and by Rao-Blackwell, Lehman-Scheffé theorem, UMVUE of θ is $T_1(u,v) = \frac{v}{n-1}$.

Now define $T_2'(u,v) = u$. Again

$$E[u] = \int_a^\infty u\left(\frac{n}{\theta}\right) \exp\left[-\frac{n}{\theta}(u-a)\right] du = a + \frac{\theta}{n}.$$

Now,

$$E\left[\frac{u-T_1(u,v)}{n}\right] = E[u] - \frac{1}{n} E\left[\frac{v}{n-1}\right] = a + \frac{\theta}{n} - \frac{\theta}{n} = a.$$

Then, $x_{(1)} - \frac{v}{n(n-1)} = T_2(u,v)$ is UMVUE of a.

Now, $a^* = x_{(1)} - \frac{v}{n(n-1)} = x_{(1)} - \frac{n(\bar{x}-x_{(1)})}{n(n-1)} = \frac{n\,x_{(1)} - \bar{x}}{n-1}$

and $\theta^* = \frac{v}{n-1} = \frac{\sum_2^n (x_{(i)} - x_{(1)})}{n-1} = \frac{n}{n-1}(\bar{x} - x_{(1)})$ and are UMVUE for a and θ respectively.

(ii) BLUE Based on Components of Order Statistics

Considering the order statistics $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ and defining

$$Z_{(r)} = \frac{(x_{(r)} - a)}{\theta}$$

we have the p.d.f.

$$f(Z_{(r)}) = \frac{n!}{(n-r)!(r-1)!} [1 - \exp(-Z_{(r)})]^{r-1} \exp[-(n-r+1)Z_{(r)}] \tag{2.10}$$

for $Z_{(r)} > 0$. We wish to calculate $E[Z_{(r)}]$. We define

$$W_{(r)} = \exp[-Z_{(r)}] = W_r$$

$$\frac{n!}{(n-r)!(r-1)!} = \frac{\Gamma(a+b)}{\Gamma(a)\ \Gamma(b)} = \frac{1}{\beta(a,b)} \quad \text{for } a = n-r+1, b = r$$

and we rewrite (2.10) as

$$g(W_r) = \frac{1}{\beta(a,b)} W_r^{a-1} (1-W_r)^{b-1} \quad \text{for } 0 \leq W_r \leq 1 \tag{2.11}$$

zero elsewhere. Then W_r is clearly distributed as a beta variable. We want to obtain $E[Z_{(r)}]$ which is $E[-\ln W_r]$ where

$$\beta(a,b) = \int_0^1 W_r^{a-1} (1-W_r)^{b-1}\, dW_r$$

and differentiation under the integral sign being valid [Cramer (1946)], then

$$\frac{\partial}{\partial a} \beta(a,b) = \int_0^1 \ln(W_r)\, W_r^{a-1} (1-W_r)^{b-1}\, dW_r$$

so that

$$E[-\ln W_r] = -E[\ln W_r] = -\frac{1}{\beta(a,b)} \frac{\partial}{\partial a} \beta(a,b)$$

$$= -\frac{\partial}{\partial a} \ln \beta(a,b)$$

$$= \frac{\partial}{\partial a} \ln \Gamma(a+b) - \frac{\partial}{\partial a} \ln \Gamma(a) \tag{2.12}$$

which by the well known properties of digamma functions [Pairman (1919)] reduces to the form

$$E[Z_{(r)}] = E[-\ln W_r] = \sum_{i=1}^{r} (n-i+1)^{-1} \tag{2.13}$$

which is the result stated by Sarhan (1954).

Similarly Sarham has stated that

$$\mathrm{var}(Z_{(r)}) = \underset{r<s}{\mathrm{cov}}\,(Z_{(r)},Z_{(s)}) = \sum_{i=1}^{r} (n-i+1)^{-2} \tag{2.14}$$

This result can also be verified by differentiation under the integral sign technique, then using properties of trigamma functions [Pairman (1919)]. Tables of numerical values of $E[Z_{(r)}]$ and $\mathrm{var}[Z_{(r)}]$ for samples of size 1(1) 10 are available [Sarhan and Greenburg (1958)].

From the definition of $Z_{(r)}$ and using the results (2.13) and (2.14) then

$$\left.\begin{aligned} E[x_{(r)}] &= a+\theta \sum_{i=1}^{r} (n-i+1)^{-1} \\ V[x_{(r)}] &= \theta^2 \sum_{i=1}^{r} (n-i+1)^{-2} = \underset{r<s}{\mathrm{cov}}\,(x_{(r)},x_{(s)}) = \nu_r \end{aligned}\right\} \tag{2.15}$$

It is now possible, with the use of the observations from a r.s. of size n, and the results of (2.15) to calculate the BLUE based on components of o.s. for "a" and "θ" if we make use of the powerful "Gauss-Markoff Theorem".

Theorem: Let x be a $k \times 1$ vector of observations, $\tilde{\theta}$ be a $s \times 1$ vector of unknown parameters $(s \leq K)$; let A be a known $K \times s$ matrix of rank s, and let W be a known positive definite $K \times K$ matrix. Assume that $E[x] = A\theta$ and $V[x] = \sigma^2 W$ where σ^2 is a known or unknown real number. Then $T = A'W^{-1}A)^{-1} A'W^{-1}X$ is the BLUE of $\tilde{\theta}$ and $V[T] = A'W^{-1}A)^{-1} \sigma^2$.

For the full sample case $(K = n)$ we define

$$X = \begin{pmatrix} x_{(1)} \\ x_{(2)} \\ \vdots \\ x_{(n)} \end{pmatrix}_{n\times 1} ; \quad \tilde{\theta} = \begin{pmatrix} a \\ \theta \end{pmatrix}_{2\times 1} ; \quad A = \begin{pmatrix} 1, & n^{-1} \\ 1 & \sum_{1}^{2} (n-i+1)^{-1} \\ \vdots & \\ 1 & \sum_{1}^{n} (n-i+1)^{-1} \end{pmatrix}_{n\times 2}$$

and

$$A' = \begin{pmatrix} 1 & 1 & \cdot & \cdot\cdot & 1 \\ n^{-1} & \sum_{1}^{2} (n-i+1)^{-1} & , & , & \sum_{1}^{n} (n-i+1)^{-1} \end{pmatrix}.$$

Note that we also can find variance of each $x_{(i)}$ term so that a $n \times n$ positive definite matrix W can be found such that $V[x] = W\theta^2$ (where $\theta^2 = \sigma^2$). Now since $v_r = V[X_{(r)}]$ by definition then

$$W = \begin{pmatrix} v_1 & v_1 & & \cdots & v_1 \\ v_1 & v_3 & & \cdots & v_2 \\ v_1 & v_2 & v_3 & \cdots & v_n \end{pmatrix}_{n\times n}$$

and to find W^{-1} we augment W with the identity matrix $I_{n\times n}$ then perform row operations on $[W:I]_{n\times 2n}$ until we have the form $[I:Z]_{n\times 2n}$ where Z is our required W^{-1} matrix. Then we obtain the result

$$W^{-1} = \begin{pmatrix} n^2+(n-1)^2 & -(n-1)^2 & 0 & 0 & \cdots & 0 \\ -(n-1)^2 & (n-1)^2+(n-2)^2 & -(n-2)^2 & 0 & \cdots & 0 \\ 0 & -(n-2)^2 & (n-2)^2+(n-3)^2 & & -(n-3)^2 & \cdots 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & \cdot & \cdot & & -1 & 1 \end{pmatrix}_{n\times n}$$

Then also, $A'\,W^{-1} = \begin{bmatrix} n^2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ (2.16)

$$A'W^{-1}A = \begin{bmatrix} n^2 & n \\ n & n \end{bmatrix} = n \begin{bmatrix} n & 1 \\ 1 & 1 \end{bmatrix}$$

$$(A'W^{-1}A)^{-1} = \frac{1}{n(n-1)} \begin{bmatrix} 1 & -1 \\ -1 & n \end{bmatrix} \qquad (2.17)$$

and combining (2.16) and (2.17) we have,

$$[A'W^{-1}A]^{-1}\ A'W^{-1} = \frac{1}{n(n-1)} \begin{bmatrix} (n^2-1) & -1 & -1 & \cdots -1 \\ -n(n-1) & n & n & \cdots\ n \end{bmatrix}$$

From G.M. Theorem we obtain $T = [A'W^{-1}A]^{-1}\ A'W^{-1}X$ which is least squares estimate of $\tilde{\theta}$ = BLUE. Then by premultiplying [X] by $[A'W^{-1}\ A]^{-1}\ A'W^{-1}$ we have

$$T = \frac{1}{n(n-1)} \begin{bmatrix} (n^2-1) & -1 & \cdots & -1 \\ -n(n-1) & n & \cdots & n \end{bmatrix} \begin{bmatrix} x_{(1)} \\ x_{(2)} \\ \vdots \\ x_{(n)} \end{bmatrix}$$

$$= \frac{1}{n(n-1)} \begin{bmatrix} (n^2-1)\, x_{(1)} - \sum_{i=2}^{n} x_{(i)} \\ -n^2 x_{(1)} + n \sum_{i=1}^{n} x_{(i)} \end{bmatrix} = \frac{1}{n-1} \begin{bmatrix} n \quad x_{(1)} - \bar{x} \\ n \quad \bar{x} - nx_{(1)} \end{bmatrix}$$

Thus for sample size n, the BLUE are

$$\frac{(nx_{(1)} - \bar{x})}{(n-1)} \quad \text{and} \quad \frac{n(\bar{x}-x_{(1)})}{n-1} \quad \text{for "a" and "}\theta\text{" respectively.}$$

The theorem also enables the calculation of variance covariance matrix where we readily see that

$$V[a^{**}] = \frac{1}{n(n-1)}\,\theta^2 \; ; \; V[\theta^{**}] = \frac{1}{(n-1)}\,\theta^2 \; ; \; \mathrm{cov}(a^{**},\theta^{**}) = \frac{-1}{n(n-1)}\,\theta^2 .$$

Note here that BLUE and UMVUE noted in (i) are the same. This is not surprising since both are functions of u,v each of which is a linear function of components of order statistics.

<u>(iii) MLE of "a" and "θ".</u>

The MLE of (a,θ) is that value of a,θ in the parameter space which, for given values of x_i, maximize the likelihood function. Then the likelihood function is

$$L(x_1,\ldots,x_n \mid a,\theta) = \left(\frac{1}{\theta}\right)^n \exp\left[- \frac{\sum_1^n (x_i - a)}{\theta} \right] \prod_{i=1}^{n} c(x_i,a) .$$

Since MLE is a function of sufficient statistics u,v then from (2.9) we can consider only the joint p.d.f. of u,v. The MLE of (a,θ) is that value of (a,θ) in the parameter space which, given u,v maximizes $\ell n\, g(u,v|a,\theta)$. We then have u,v fixed and maximize a function of two variables a,θ for $a \in R$, and $\theta > 0$. Thus for $u > a$ and u,v fixed

$$\ell n\, g(u,v|a,\theta) = -n\, \ell n\, \theta - \frac{v}{\theta} - \frac{n}{\theta}(u-a) + c$$

where c does not depend on either 'a' or 'θ'. Now define

$$K = c - n\, \ell n\, \theta - \frac{v}{\theta}$$

which is independent of 'a'. For fixed $\theta = \theta_o$, $\ell n\, g$ is maximized for $a = u$. That is $K - \frac{n}{\theta_o}(u-a)$ is largest when $(u-a)$ is smallest, or when $a = u = x_{(1)}$ which is independent of θ_o. Therefore $\hat{a} = u = x_{(1)}$ for all $\theta > 0$.

Consider now $\ell n\, g(u,v|a,\theta)$ when $a = \hat{a}$. Then

$$\ell n\, g(u,v|\hat{a},\theta) \geq \ell n\, g(u,v|a,\theta)$$

for $a \in R_1$, $\theta > 0$. To obtain $\hat{\theta}$ we need to maximize $\ell n\, g(u,v|\hat{a},\theta)$.

$$\ell n\, g(u,v|\hat{a},\theta) = -n\ell n\theta - \frac{v}{\theta} + \ell n\, c$$

Taking the derivative w.r.t. θ and setting $\frac{\partial}{\partial\theta}\ell n\, g(u,v|\hat{a},\theta)$ to zero, we obtain $-\frac{n}{\theta} = -\frac{v}{\theta^2}$. Clearly then $\hat{\theta} = \frac{v}{n}$ for we have

$$\ell n\, g(u,v|\hat{a},\hat{\theta}) \geq \ell n\, g(u,v|\hat{a},\theta).$$

From our earlier definition of u,v we have that $x_{(1)}$ and $\bar{x} - x_{(1)}$ are MLE for 'a' and 'θ' respectively.

We can see that the MLE's are linear functions of the x_i's. Sarhan (1954) indicates that BLUE's are in agreement with these MLE's. An intuitive illustration of this agreement can be seen if we consider the expression $\hat{\theta} = \bar{x} - \hat{a}$. If we now replace $\hat{a}$ by a^* we have

$$\hat{\hat{\theta}} = \bar{x} - \frac{n(x_{(1)} - \bar{x})}{n-1} = \frac{n(\bar{x} - x_{(1)})}{n-1} = \theta^* .$$

Mean Squared Error of Estimates in (i), (ii) and (iii).

Clearly, UMVUE and BLUE are unbiased with variances which can easily be obtained from linearity property of $E[\psi(x_{(i)})]$ using the results that $\text{var}(\bar{x}) = \frac{\theta^2}{n} = \text{var}(x_{(1)})$ and $\text{cov}(x_{(1)}\ \bar{x}) = \frac{\theta^2}{n^2}$. Similarly, the variances of u,v can be obtained directly from the matrix $[A'W^{-1}A]^{-1}$ in (ii). These respective variances, as well as those of MLE's are reported in table 2.1. Although the MLE's have smaller variances, they are biased and the extent of bias is included in mean square error (MSE) terms. The bias is calculated as

$$\text{Bias} = |E[\hat{\theta}] - \theta| \text{ and } |E[\hat{a}] - a| \text{ for } \hat{\theta} \text{ and } \hat{a} \text{ respectively.}$$

The extent of bias is negligeable for large sample size but on comparing ratios of MSE we see that MSE for $\hat{\theta}$ will always be lower than that for θ^* although they tend to near equality as $n \to \infty$. On the other hand, MSE for $\hat{a}$ will be approximately double that of a^*; i.e. Ratio of MSE $[\hat{a}:a^*] = [2\frac{n}{n-1}]$

Table 2.1. Comparison of Bias, Variance, and Mean Squared Error

	a	Var	bias	MSE	θ	Var	bias	MSE
UMVUE and BLUE	$\frac{n\,x_{(1)} - \bar{x}}{(n-1)}$	$\frac{\theta^2}{n(n-1)}$	nil	$\frac{\theta^2}{n(n-1)}$	$\frac{n(\bar{x}-x_{(1)})}{(n-1)}$	$\frac{\theta^2}{n-1}$	nil	$\frac{\theta^2}{(n-1)}$
MLE	$x_{(1)}$	$\frac{\theta^2}{n^2}$	$\frac{\theta}{n}$	$\frac{2\theta^2}{n^2}$	$\bar{x}-x_{(1)}$	$\frac{(n-1)\theta^2}{n^2}$	$\frac{\theta}{n}$	$\frac{\theta^2}{n}$

In order to determine which set of estimators to use relative to precision, we shall speak of the <u>Generalized mean squared error</u> (GMSE) of (a^*, θ^*) and $(\hat{a},\hat{\theta})$. Since (a^*, θ^*) are unbiased then MSE = Variance and by Rao (1966), the GMSE is the value of the determinant

$$\begin{vmatrix} \operatorname{var}(a^*) & \operatorname{cov}(a^*\ \theta^*) \\ \operatorname{cov}(a^*\ \theta^*) & \operatorname{var}(\theta^*) \end{vmatrix} \qquad \text{and}$$

$\operatorname{cov}(a^*\ \theta^*)$ can be evaluated linearly from $E\,[\psi(x_{(1)},\dots,x_{(n)})]$. Thus, by definition

$$\operatorname{cov}(a^*\ \theta^*) = E\left[\frac{n\,x_{(1)} - \bar{x}}{n-1} \cdot \frac{n(\bar{x} - x_{(1)})}{n-1}\right] - a\theta$$

$$= \frac{n(n+1)}{(n-1)^2} E[\bar{x}\,x_{(1)}] - \frac{n^2}{(n-1)^2} E[x_{(1)}^{\ 2}] - \frac{n}{(n-1)^2} E[\bar{x}^2]$$

and it can easily be shown that $E[\bar{x}\,x_{(1)}] = \frac{\theta^2}{n^2} + (a + \frac{\theta}{n})(a+\theta)$.

$$E[\bar{x}^2] = \frac{\theta^2}{n} + (a+\theta)^2 \quad \text{and} \quad E[x_{(1)}{}^2] = (a + \frac{\theta}{n})^2 + \frac{\theta^2}{n^2} \quad \text{so that}$$

$$\text{cov}(a^*,\theta^*) = -\frac{\theta^2}{n(n-1)} + \frac{n}{(n-1)^2}\left[\frac{n^2 a\theta + a\theta - 2na\theta}{n}\right] - a\theta$$

$$= -\frac{\theta^2}{n(n-1)}$$ which can also be obtained from $(A'W^{-1}A)^{-1}$ in (ii)

Then

$$\text{GMSE}^* = \begin{vmatrix} \frac{\theta^2}{n(n-1)} & \frac{-\theta^2}{n(n-1)} \\ \frac{-\theta^2}{n(n-1)} & \frac{\theta^2}{(n-1)} \end{vmatrix} = \frac{\theta^4}{n^2(n-1)}$$

Similarly, for GMSE of $(\hat{a},\hat{\theta})$ we evaluate the determinante

$$\begin{vmatrix} E[(\hat{a}-a)^2] & E[(\hat{a}-a)(\hat{\theta}-\theta)] \\ E[(\hat{a}-a)(\hat{\theta}-\theta)] & E[(\hat{\theta}-\theta)^2] \end{vmatrix}$$ where

$$E[(\hat{a}-a)(\hat{\theta}-\theta)] = E[(x_{(1)} - a)(\bar{x} - x_{(1)} - \theta)]$$

$$= E[x_{(1)}\bar{x} - x_{(1)}^2 - \theta x_{(1)} - \bar{x}a + ax_{(1)} + a\theta]$$

$$= E[x_{(1)}\bar{x}] - E[x_{(1)}{}^2] + (a-\theta)E[x_{(1)}] - aE(\bar{x}) + a\theta$$

$$= (a + \frac{\theta}{n})(a - \frac{\theta}{n}) - a^2 = -\frac{\theta^2}{n^2}$$

so that $\widehat{GMSE} = \begin{vmatrix} 2\,\dfrac{\theta^2}{n^2} & -\dfrac{\theta^2}{n^2} \\ -\dfrac{\theta^2}{n^2} & \dfrac{\theta^2}{n} \end{vmatrix} = \dfrac{(2n-1)\theta^4}{n^4}$

Comparing the values of GMSE (a^*, θ^*) and GMSE $(\hat{a},\hat{\theta})$ then we see that

$$\frac{\theta^4}{n^2(n-1)} \qquad \frac{(2n-1)\theta^4}{n^4} = \frac{n^2}{(n-1)\ (2n-1)} \qquad \text{and}$$

for $n = 2$, Ratio is > 1

for $n \geq 3$, Ratio is < 1 .

On the basis of smaller mean square error (better precision), then for a sample of size 3 or greater we would prefer a^*, θ^* to $\hat{a},\hat{\theta}$ and for samples of size 2 we would prefer $\hat{a},\hat{\theta}$ to a^*, θ^*.

<u>UMVUE of Reliability of Mission Time t</u>

The exponential distribution is particularly useful when applied to life testing procedures. One such application is when one wishes to know the probability of an item surviving at least up to a given time 't' which Basu (1964) calls the mission time. Then the reliability at time 't' of a system whose life follows the probability law $f(x|a,\theta)$ given by (2.1) is expressed as

$$R(t) = Pr[\ x \geq t] = 1 - Pr[\ x < t\ ] \tag{2.18}$$

$$= \int_t^\infty f(x|a,\theta)dx$$

Here it is assumed that $t > a$, for if $t \leq a$, $R(t) = 1$. In order to find a UMVUE for $R(t)$ we shall follow the method of Patil and Wani (1966) and more specifically Basu (1964).

Consider a sample of size 'n' made up of two subsamples of size (n-1) and one. For subsample of size (n-1) we define

$$v^* = \sum_{i=2}^{n-1} (x_{(i)} - x_{(1)}), \quad u^* = x_{(1)}$$

where v^* and u^* are jointly sufficient and complete for $\tilde{\theta}$ as shown earlier for sample of size $n' = n-1$. We can then write

$$g(u^*,v^* \mid a,\theta) = \frac{(n-1)}{\theta} \exp\left[-(n-1)\frac{(u^*-a)}{\theta}\right] c(u^*,a)\cdot g_1(v^* \mid u^*)$$

where

$$g_1(v^* \mid u^*) = \frac{1}{\Gamma(n-2)} \exp\left[-\frac{v^*}{\theta}\right] \left(\frac{v^*}{\theta}\right)^{n-3} \frac{1}{\theta}\, c(v^*,0).$$

Therefore,

$$g(u^*,v^* \mid a,\theta) = \frac{(n-1)}{\theta} \exp\left[-\frac{(n-1)}{\theta}(u^*-a)\right] c(u^*,a)\cdot \frac{1}{\Gamma(n-1)}$$

$$\exp\left[-\frac{v^*}{\theta}\right] \left(\frac{v^*}{\theta}\right)^{n-3} \frac{1}{\theta}\, c(v^*,0) \qquad (2.19)$$

Let ξ be an additional independent observation, then

$$g(\xi, u^*,v^* \mid a,\theta) = \frac{1}{\theta} \exp\left[-\frac{(\xi-a)}{\theta}\right] c(\xi,u^*) \cdot g(u^*,v^* \mid a,\theta)$$

Consider the situation in which $\xi \geq u^*$

$$v = \sum_{i=2}^{n} (x_{(i)} - x_{(1)}) = \sum_{2}^{n-1} x_{(i)} - (n-2)x_{(1)} + \xi - x_{(1)}$$

$$= v^* + \xi - x_{(1)}$$

and $$u = u^* \, c(\xi, u^*)$$

so that, after transformation $(v^*, u^*) \rightarrow (v, u)$ then (2.19) can be rewritten as

$$g(\xi, u, v | a, \theta) = \frac{1}{\theta} \exp\left[-\frac{(\xi - a)}{\theta}\right] c(\xi, u) \cdot \frac{(n-1)}{\theta} \, .$$

$$\exp\left[-\frac{(n-1)\ (u-a)}{\theta}\right] c(u,a) \cdot \frac{1}{\Gamma(n-2)} \exp\left[-\frac{(v-\xi+x_{(1)})}{\theta}\right] \cdot \frac{1}{\theta} c(v,0).$$

$$\left(\frac{v-\xi+x_{(1)}}{\theta}\right)^{n-3}$$

Therefore, $$g(\xi, a, \theta | u, v) = g(\xi, u, v | a, \theta) / g_1(u, v, | a, \theta)$$

$$= \frac{1}{n} \qquad \text{if} \quad u = \xi$$

$$= \frac{(n-1)}{n} (n-2) \left(\frac{1}{v}\right)^{n-2} (v+u-\xi)^{n-3} \quad \text{for } u < \xi < u+v$$

$$= 0 \qquad \text{otherwise.} \qquad (2.20)$$

Expression (2.20) is in agreement with equation (8) of Laurent (1963) if we write $nY = v$.

We now define $h(\xi) = 1$ if ξ belongs to interval (t, ∞)

$= 0$ otherwise.

Then $h(\xi)$ is an unbiased estimate of $R(t)$. Thus UMVUE as given by Rao Blackwell-Lehman Scheffé theorem is

$$E\,[H(\xi) \mid g(\xi, a, \theta \mid u, v)]$$

where u, v are jointly sufficient and complete for (a, θ). Therefore

$$\begin{aligned} R^*(t) &= \int_t^\infty g(\xi, a, \theta \mid u, v)\,d\xi \\ &= 1 \quad \text{when } t < x_{(1)} \\ &= \int_t^{v+u} \frac{(n-1)(n-2)}{2} \left(\frac{1}{v}\right)^{n-2} (v+u-\xi)^{n-3}\, d\xi \\ &= 0 \quad \text{otherwise}. \end{aligned}$$

Then for t belonging to $(x_{(1)}, u+v)$ we have

$$R^*(t) = \left(\frac{n-1}{n}\right) \left(\frac{v+u-t}{v}\right)^{n-2} = \left(1 - \frac{1}{n}\right) \left[1 - \frac{t - x_{(1)}}{v}\right]^{n-2} \tag{2.21}$$

If we now again write $v = nY$ in (2.21) we have

$$\begin{aligned} R^*(t) &= 1 \quad \text{if } t < x_{(1)} \\ &= \left(1 - \frac{1}{n}\right) \left[1 - \frac{t - x_{(1)}}{nY}\right]^{n-2} \quad \text{if } x_{(1)} \leq t \leq x_{(1)} + nY \\ &= 0 \quad \text{if } t > x_{(1)} + nY \end{aligned} \tag{2.22}$$

which is the same result obtained by Laurent in equation (10).

BLUE and MLE of R(t) for Mission Time T

We now wish to obtain BLUE for R(t) if it exists. A linear estimate of components of order statistics is of the form $\Sigma\, \ell_{(i)} x_{(i)}$ and linear unbiased estimate would be one for which

$$E\left[\sum_{i=1}^{n} \ell_{(i)} x_{(i)}\right] = \exp\left[-\frac{t-a}{\theta}\right]$$

but

$$E\left[\sum_{i=1}^{n} \ell_{(i)} x_{(i)}\right] = \sum_{i=1}^{n} \ell_{(i)}\,(a+\nu\theta) \neq \exp\left[-\frac{t-a}{\theta}\right]$$

so that R(t) is not estimable from linear function of components of order statistics. Therefore it follows that the BLUE based on the components of order statistics does not exist for R(t).

We saw earlier (2.18) that $R(t) = \int_t^{\infty} f(x \mid a,\theta)dx$ and that MLE is that estimator which maximizes the likelihood of $f(x|a,\theta)$ for given u,v. Then since $f(x \mid \hat{a},\hat{\theta}) \geq (x|a,\theta)$ it is clear that MLE of R(t) will be $\int_t^{\infty} f(x| \hat{a},\hat{\theta})dx$ so that

$$\begin{aligned} \hat{R}(t) &= \int_t^{\infty} \frac{1}{\theta} \exp\left[-\frac{x-\hat{a}}{\hat{\theta}}\right]dx && \text{given } \hat{a},\hat{\theta},t \\ &= 1 && \text{for } t < x_{(1)} \\ &= \exp\left[-\frac{t-\hat{a}}{\hat{\theta}}\right] && \text{for } t \geq x_{(1)} \end{aligned}$$

which, if expressed in terms of u,v becomes

$$\begin{aligned} \hat{R}(t) &= 1 && \text{for } t < u \\ &= \exp\left[-\frac{n}{v}(t-u)\right] && \text{for } t \geq u \end{aligned} \qquad (2.23)$$

The comparison between MLE and UMVUE is rather difficult for the reason that the bias and the mean squared error for MLE are very difficult to obtain. Even though UMVUE has bias zero, its variance is also difficult to compute. We illustrate this for the bias of the MLE.

$$E\,[\psi(u,v)] = \iint \psi(u,v)\; g(u,v|a,\theta)\; dudv \qquad \text{for } t \geq a \tag{2.24}$$

$$= \iint_{u\geq t\geq a;v\geq 0} 1 \quad g(u,v|a,\theta)\; dudv + \iint_{a\leq u\leq t;v\geq 0} \exp\,[-\tfrac{n}{v}(t-u)]\,g(u,v|a,\theta)dudv$$

$$= \Pr\,[u \geq t] + \iint_{v\geq 0,\, a\leq u\leq t} \exp \quad [-n(t-u)/v]\,g(u,v|a,\theta)dudv$$

and $\quad \Pr\;[u \geq t] = \frac{n}{\theta} \exp\,[-\frac{n}{\theta}(u-a)]\; du$

$$= \exp\,[-\frac{n}{\theta}(t-a)]$$

So (2.24) becomes

$$E[\hat{R}(t)] = \exp\,[-\frac{n}{\theta}(t-a)] + \int_o^\infty\!\!\int_a^t \frac{n}{\theta^2\Gamma(n-1)} \exp\,[-\frac{v}{\theta}]\;(\frac{v}{\theta})^{n-2}\cdot$$

$$\exp\,[-\frac{n}{\theta}(u-a)]\;\exp\,[-\frac{n}{v}(t-u)]\;dudv \tag{2.25}$$

$$= \exp\,[-\frac{n}{\theta}(t-a)] + \emptyset \quad \text{where } \emptyset \text{ must be calculated.}$$

Now $\quad \emptyset = \dfrac{n}{\theta^2\,\Gamma(n-1)} \displaystyle\int_o^\infty \exp\,[-\frac{v}{\theta}]\;(\frac{v}{\theta})^{n-2} \cdot \int_a^t \exp\,[-\frac{n}{\theta}(u-a) + \frac{n}{v}(t-u)]dudv$

$$= \frac{1}{\Gamma(n-1)} \int_0^\infty \exp[-\frac{v}{\theta}](\frac{v}{\theta})^{n-1} (\frac{1}{v-\theta}) \exp[-\frac{n}{\theta v}(\theta t - va)] \cdot$$

$$\left\{\exp[-\frac{n(v-\theta)a}{\theta v}] - \exp[-\frac{n(v-\theta)t}{\theta v}]\right\} dv$$

$$= \frac{1}{\Gamma(n-1)} \int_0^\infty \frac{1}{(v-\theta)} (\frac{v}{\theta})^{n-1} \exp[-\frac{v}{\theta}] \exp[-\frac{n}{v}(t-a)] dv$$

$$- \frac{1}{\Gamma(n-1)} \exp[-\frac{n}{\theta}(t-a)] \int_0^\infty (\frac{v}{\theta})^{n-1} \exp[-\frac{v}{\theta}] \cdot \frac{1}{(v-\theta)} dv$$

and for $w = v/\theta$,

$$\emptyset = \frac{1}{\Gamma(n-1)} \int_0^\infty \frac{1}{w-1} w^{n-1} \exp[-w]\left\{\exp[-\frac{n}{\theta w}(t-a)] - \exp[-\frac{n}{\theta}(t-a)]\right\} dw$$

and for $R = \exp[-\frac{(t-a)}{\theta}]$, $\emptyset$ becomes

$$\emptyset = \frac{1}{\Gamma(n-1)} \int_0^\infty \frac{1}{w-1} w^{n-1} \exp[-w] (R^{n/w} - R^n) dw \qquad (2.26)$$

For $w = 1$, $v = \theta$ and $\emptyset' = \frac{R^n n}{\Gamma(n-1)\theta^2} (t-a) \exp[-1]$ when $\frac{v}{\theta}$ is set equal to 1 in (2.25). Since $\theta > 0$, then $\emptyset$ is defined at $w = 1$ and $E[\hat{R}(t)]$ becomes

$$E[\hat{R}(t)] = R^n + \frac{1}{\Gamma(n-1)} \lim_{\epsilon \to 0} \int_o^{1-\epsilon} \frac{1}{w-1} w^{n-1} \exp[-w] \, (R^{n/w} - R^n) \, dw$$

$$+ \frac{1}{\Gamma(n-1)} \lim_{\epsilon \to 0} \int_{1+\epsilon}^{\infty} \frac{1}{w-1} w^{n-1} \exp[-w] \, (R^{n/w} - R^n) \, dw$$

$$= R^n + \frac{1}{\Gamma(n-1)} \int_o^{\infty} \frac{1}{w-1} w^{n-1} \exp[-w] \, (R^{n/w} - R^n) \, dw \qquad (2.27)$$

When 'a' is known, say equal to zero, Pugh (1963) showed that the bias $\hat{R}(t)$ can be expressed as a Bessel function. In the case where 'a' is unknown it appears that we cannot express bias in terms of Bessel or other known standard functions. However, the actual evaluation of the bias could be carried out by computer iteration.

It can be seen from (2.26) that as n increases, R decreases and bias becomes assymptotically zero.

CHAPTER III

ESTIMATING RELIABILITY FUNCTIONS FROM CENSORED SAMPLES

Let $x_1, x_2, \ldots, x_n$ be a r.s. from the distribution (2.1) of previous chapter. We define a condition where from this sample, only the "r" smallest values of x_i are available $(i = 1, \ldots, n)$ where $1 \leq r \leq n$. This is a single point censored sample (from the right) where $(n - r)x_i$'s are not available. It is desired to estimate "a" and "θ" the location and scale parameters respectively which determine the underlying distribution. We will consider in this chapter various procedures for estimating "a" and "θ" as well as $R(t|a',\theta')$ from the censored sample defined above. We first wish to obtain a vector valued sufficient statistic for $\Theta = (a,\theta)$, $a \in R$, $\theta > 0$, if one exists.

Derivation of Sufficient Statistics

Consider the likelihood of the experiment which yields $x_1, \ldots, x_r, (n-r)x_i \geq x_{(r)}$. This is given by

$$L(x_1, \ldots, x_r,\ (n-r)x_i \geq x_{(r)}|a,\theta) = \left(\frac{1}{\theta}\right)^r \exp\left[-\sum_{i=1}^{r} \frac{(x_i - a)}{\theta}\right] .$$

$$\Pr\ [(n-r)x_i \geq x_r\ |\ a,\theta] \qquad (3.1)$$

where

$$\Pr\ [(n-r)x_i \geq x_r | a,\theta] = \binom{n}{r}\left[\int_{x_{(r)}}^{\infty} \frac{1}{\theta} \exp\left[-\frac{x_i - a}{\theta}\right] dx_i\right]^{n-r}$$

$$= \binom{n}{r} \exp\left[-(n-r)\frac{(x_{(r)} - a)}{\theta}\right]$$

so 3.1 becomes

$$L(x_1,\dots,x_r,\ (n-r)x_i \geq x_r \mid a,\theta) = (\tfrac{1}{\theta})^r \binom{n}{r} \exp\left[-\sum_{i=1}^{r} \frac{(x_i - a)}{\theta}\right].$$

$$\exp\left[-\ (n-r)\ \frac{(x_{(r)} - a)}{\theta}\right] \qquad (3.2)$$

Consider now the order statistics $x_{(1)},\dots,x_{(r)},\dots,x_{(n)}$ where $a \leq x_{(1)} \leq \dots \leq x_{(r)} \leq x_{(r+1)} \leq \dots \leq x_{(n)} < \infty$. Now from (2.2) it is known [Epstein and Sobel (1954)] that from a sample of size n, given a,θ,

$$g(x_{(1)},\dots,x_{(r)} \mid a,\theta) = n!\ (\tfrac{1}{\theta})^r \exp\left[-\sum_{1}^{r} \frac{x_{(i)} - a}{\theta}\right].$$

$$\int\cdots\int \exp\left[-\sum_{r+1}^{n} \frac{(x_{(i)} - a)}{\theta}\right] dx_{r+1} \cdots dx_n .$$

$$= \frac{n!}{(n-r)!}\ (\tfrac{1}{\theta})^r \exp\left[-\sum_{i=1}^{r} \frac{x_{(i)} - a}{\theta}\right] .$$

$$\exp\left[-\ \frac{(n-r)}{\theta}\ (x_{(r)} - a)\right] \qquad (3.3)$$

for $a \leq x_{(1)} \leq \dots \leq x_{(r)} < \infty$, zero elsewhere. Thus 3.2 can be expressed as

$L(x_1,\dots,x_r,\ (n-r)x_i \geq x_r \mid a,\theta) = g(x_{(1)},\dots,x_{(r)}, \mid a,\theta).\ r!$ and by factorization theorem, the order statistics $x_{(1)},\dots,x_{(r)}$ are sufficient for $\tilde{\theta}$.

We now define indicator function $c(a,b)$ as in (2.3) and $u = x_{(1)}$, $v = \sum_{i=2}^{r} (x_{(i)} - x_{(1)}) + (n-r)\ (x_{(r)} - x_{(1)})$. We now can rewrite 3.3 as

$$g(x_{(1)},\dots,x_{(r)},| a,\theta) = \frac{n}{\theta} \exp\left[-\frac{n}{\theta}(x_{(1)}-a)\right] c(x_{(1)},a) \quad (\frac{1}{\theta})^{r-1} \quad .$$

$$\exp\left[-\frac{\sum_2^r x_{(i)} + (n-r)\, x_{(r)} - (n-1)\, x_{(1)}}{\theta}\right]$$

$$\cdot \quad \frac{(n-1)!}{(n-r)!} \prod_{i=2}^{r} c(x_{(i)}, x_{(i-1)})$$

$$= g(u) \quad (\frac{1}{\theta})^{r-1} \quad \exp\left[-\frac{v}{\theta}\right] c(v,o) \frac{(n-1)!}{(n-r)!} \qquad (3.4)$$

We wish to show that u,v are jointly sufficient for (a,θ) where $g(u,v)$ is defined as in 2.6.

Define $x_{(i)} - x_{(1)} = z_i$ for $i = 2,\dots,r$ and we see that $z_1, z_2, \dots, z_r$ become the new order statistics. Let

$v = \sum_{i=2}^{r} z_i + (n-r)z_r$. Now we define $W_i = (n-i+1)\,(z_i - z_{i-1})$, for $i = 2,\dots,r$ such that

$$v = \sum_{i=2}^{r} W_i = \sum_{2}^{r} z_i + (n-r)z_r \; .$$

If we make above transformation, where Jacobian is given by

$|J| = (\frac{1}{n}) \quad (\frac{1}{n-1}) \dots (\frac{1}{n-r+1})$ for $W_i = \psi(z_i)$ then 3.4 becomes

$$g(x_{(1)},\dots,x_{(r)} | a,\theta) = g(u) \quad (\frac{1}{\theta})^{r-1} \exp\left[-\sum_{i=2}^{r} \frac{W_i}{\theta}\right] \qquad (3.5)$$

and W_i are i.i.d.r.v. from exponential distribution, given $x_{(1)} = u$. Thus v is the sum of (r-1) i.i.d.r.v. and by m.g.f. technique,

$$g(v|u,a,\theta) = \frac{1}{\Gamma(r-1)} \left(\frac{1}{\theta}\right)^{r-1} \exp\left[-\frac{v}{\theta}\right] v^{r-2} \quad c(v,0) \tag{3.6}$$

$$g(v\ u,|a,\theta) = \frac{n}{\theta} \exp\left[-\frac{n}{\theta}(u-a)\right] \ c(u,a) \ \frac{1}{\Gamma(r-1)} \cdot \frac{1}{\theta^{r-1}} \cdot$$

$$\exp\left[-\frac{v}{\theta}\right] v^{r-2}\ c(v,0). \tag{3.7}$$

So we can express (3.5) as $g(u,v\,|\,a,\theta) \quad \frac{(n-1)}{(n-r)!} \quad v^{r-2}$.

Thus the likelihood of $x_1,\ldots,x_r$ from a sample of size n can be written as

$$L(x_1,\ldots,x_r,\ (n-r)x_i \geq x_{(r)}|\,a,\theta) = g(x_{(1)},\ldots,x_{(r)}|\,a,\theta)T_1\ (x_{(1)},\ldots,x_{(r)})$$

$$= g(u,v|\,a,\theta)T_2\ (x_{(2)},\ldots,x_{(r)})$$

and by factorization criterion, u,v are jointly sufficient for $\tilde{\theta}$. Furthermore u,v are known to be complete and for proof we refer to Epstein and Sobel (1954).

(i) <u>UMVUE for "a" and "θ"</u>

As in previous chapter we use the results of Rao (1966) and from the R.B.L.S. Theorem we start with an unbiased estimator $T_i(\underline{x}_i)$ such that $E\left[T_i(\underline{x}_i)\ |\ u,v\right] = \psi(\tilde{\theta})$ where $\underline{x}_i = (x_1,x_2,\ldots,x_r)$ and u,v are jointly sufficient and complete.

We wish to estimate θ. Let $T_1^1\ (u,v) = v$. Then

$$E[T_1^1(u,v) \mid u,v] = \int_0^\infty v \frac{1}{\Gamma(r-1)} \left(\frac{1}{\theta}\right)^{r-1} v^{r-2} \exp\left[-\frac{v}{\theta}\right] dv$$

$$= E[v] = E[E(v \mid u)] = (r-1)\theta$$

which is independent of u; thus $\emptyset(u) = E[v \mid u]$, then $E[\emptyset(u)] = \theta(r-1)$. Then by our theorem, unbiased estimate of θ is $T_1(u,v) = T_1^1(u,v) / (r-1) = \frac{v}{(r-1)}$

and is UMVUE for θ.

Now define $T_2^1(u,v) = u$ where

$$E[u] = \int_a^\infty u \frac{n}{\theta} \exp\left[-\frac{n}{\theta}(u-a)\right] du = a + \frac{\theta}{n}$$

and clearly $u - \frac{T_1(u,v)}{n}$ has expectation given by

$$E[U - \frac{T_1(u,v)}{n}] = E[u] - E[\frac{T_1(u,v)}{n}] = a$$

Thus $x_{(1)} - \frac{v}{n(r-1)} = T_2(u,v)$ is UMVUE for "a". Then

$$x_{(1)} - \frac{\sum_2^r (x_{(i)} - x_{(1)}) + (n-r)\ (x_{(r)} - x_{(1)})}{n(r-1)}$$

and $\frac{\sum_2^r (x_{(i)} - x_{(1)}) + (n-r)\ (x_{(r)} - x_{(1)})}{(r-1)}$ are UMVUE for "a" and

"θ" respectively. These results are in agreement with those of Epstein (1960).

(ii) B.L.U.E. based on components of order statistics

Consider the r smallest $x_{(i)}$'s from a sample of size n such that $a \leq x_{(1)} \leq \cdots \leq x_{(r)} <$ where $1 \leq r \leq n$. Then previous results (2.14) and (2.15) are still valid and we again use the Gauss-Markov theorem. Thus from the censored sample above we define

$$x = \begin{pmatrix} x_{(1)} \\ x_{(2)} \\ \vdots \\ x_{(r)} \end{pmatrix}_{r \times 1} ; \quad \tilde{\theta} = \begin{pmatrix} a \\ \theta \end{pmatrix}_{2 \times 1} ; \quad A = \begin{pmatrix} 1 & 1/n \\ 1 & \sum_1^2 (n-i+1)^{-1} \\ \vdots & \\ 1 & \sum_1^r (n-i+1)^{-1} \end{pmatrix}_{r \times 2}$$

and

$$A' = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1/n & \sum_1^2 (n-i+1)^{-1} & \cdots & \sum_1^r (n-i+1)^{-1} \end{pmatrix}_{r \times 2}$$

and we have an $r \times r$ positive definite matrix W

where

$$W = \begin{pmatrix} V_{11} & V_{12} & \cdots & V_{1r} \\ \vdots & \ddots & & \vdots \\ V_{r1} & \cdots\cdots & & V_{rr} \end{pmatrix}$$

where $V_{ij} = \text{cov}\,(x_{(i)}\ x_{(j)}),\ i \leq j$ and $V[X] = W\,\theta^2$.

Thus to find W^{-1} we augment W with identity matrix I_{rxr} then perform row operations on $W:I_{rx2r}$ until we have the form $I:Z_{rx2r}$ where Z is the required W^{-1} matrix. We then obtain

$$
W^{-1} = \begin{pmatrix}
n^2 + (n-1)^2 & -(n-1)^2 & 0 & 0 & \cdots & 0 \\
-(n-1)^2 & (n-1)^2+(n-2)^2 & -(n-2)^2 & 0 & \cdots & 0 \\
0 & -(n-2)^2 & (n-2)^2+(n-3)^2 & -(n-3)^2 & \cdots & 0 \\
\vdots & & & & & \vdots \\
0 & & & & -(n-r+1)^2 & (n-r+1)^2
\end{pmatrix}
$$

$$
A'W^{-1} = \begin{pmatrix}
n^2 & 0 & 0 & \cdots & 0 \\
1 & 1 & 1 & \cdots & (n-r+1)
\end{pmatrix}
$$

$$
A'W^{-1}A = \begin{pmatrix}
n^2 & n \\
n & r
\end{pmatrix}
$$

$$
(A'W^{-1}A)^{-1} = \frac{1}{r-1} \begin{bmatrix}
r/n^2 & -1/n \\
-1/n & 1
\end{bmatrix} \tag{3.8}
$$

Therefore $(A'W^{-1}A)^{-1} \ A'W^{-1} = \dfrac{1}{r-1}$.

$$
\begin{bmatrix}
(r-1/n), & -1/n \ , & -1/n \ , \ldots, & -1/n, & -(n-r+1)/n \\
-(n+1), & 1, & 1 \ , \ldots \ , & 1 \ , & (n-r+1)
\end{bmatrix} \tag{3.9}
$$

Premultiplying X by (3.9) gives us T, the least squares estimate of $\tilde{\theta}$

$$T = \frac{1}{(r-1)} \begin{bmatrix} r-1/n, & -1/n, & -1/n, & \ldots, & -1/n, & -(n-r+1)/n \\ \\ \\ -(n+1), & 1, & 1, & \ldots, & 1, & (n-r+1) \end{bmatrix} \begin{bmatrix} x_{(1)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{(r)} \end{bmatrix}$$

which is BLUE of $\begin{bmatrix} a \\ \theta \end{bmatrix}$. Therefore BLUE of "a" is

$$a^{**} = \frac{1}{r-1} \left[(r-1/n)x_{(1)} - \frac{1}{n} \sum_{2}^{r-1} x_{(i)} - \frac{(n-r+1)}{n} x_{(r)} \right]$$

$$= \frac{1}{r-1} \left[rx_{(1)} - \frac{\sum_{1}^{r} x_{(i)} + (n-r)x_{(r)}}{n} \right]$$

$$= x_{(1)} - \frac{\sum_{2}^{r} (x_{(i)} - x_{(1)}) + (n-r)\,(x_{(r)} - x_{(1)})}{n(r-1)}$$

and BLUE for "θ" is

$$\theta^{**} = \frac{1}{r-1} \left[-(n+1)x_{(1)} + \sum_{2}^{r-1} x_{(i)} + (n-r+1)x_{(r)} \right]$$

$$= \frac{\sum_{1}^{r} x_{(i)} + (n-r)x_{(r)} - nx_{(1)}}{r-1}$$

$$= \frac{\sum_{2}^{r} (x_{(i)} - x_{(1)}) + (n-r)\,(x_{(r)} - x_{(1)})}{r-1}$$

Again we note that a^{**}, θ^{**} are identical to UMVUE for a, θ respectively. These results are in agreement with Sarhan (1955).

From (3.8) we see also that

$$\mathrm{Var}\,(a^{**}) = \frac{r\,\theta^2}{(r-1)n^2} \quad ; \quad \mathrm{Var}(\theta^{**}) = \frac{\theta^2}{(r-1)} \quad ; \quad \mathrm{Cov}\,(a^{**},\theta^{**}) = -\frac{\theta^2}{n(r-1)} \; .$$

(iii) <u>M.L.E. of "a" and "θ"</u>

Since M.L.E. is a function of the sufficient statistics u,v, then from (3.6) it is clear that maximizing $L(x_1,\ldots,x_r,\ (n-r)x_i \geq x_{(r)} \mid a,\theta)$ is equivalent to maximizing $g(u,v \mid a,\theta)$.

The M.L.E. of (a,θ) is that value of (a,θ) in the parameter space $a \in R_1$, $\theta > 0$ which, given u,v maximizes $\ell n\ g$. Then for $u \geq a$

$$\ell n\ g\ (u,v \mid a,\theta) = \ell n\ (\tfrac{n}{\theta}) - \tfrac{n}{\theta}\ (u-a) - (r-1)\ \ell n\ \theta - (\tfrac{v}{\theta}) + C$$

where C does not depend on 'a' or 'θ'. We define

$$K = C - \frac{v}{\theta} - (r-1)\ \ell n\ \theta + \ell n\ (\tfrac{n}{\theta}) \; .$$

Then for fixed $\theta = \theta_o$, $K_{\theta=\theta_o} = K_o$ and we have

$$\ell n\ g\ (u,v \mid a,\theta_o) = K_o - \tfrac{n}{\theta}(u-a) \; .$$

Clearly $\ell n\ g(\text{for } \theta = \theta_o)$ as a function of 'a' alone is maximized for $\hat{a} = u = x_{(1)}$, and $\hat{a}$ is independent of θ_o so that $\hat{a} = x_{(1)}$ for any $\theta = \theta_o > 0$.

Consider now $\ell n\ g(u,v|a,\theta)$ when $a = \hat{a}$. Then

$$\ell n\ g(u,v|\hat{a},\theta) \geq \ell n\ g(u,v|a,\theta)$$

for $a \in R_1$, $\theta > 0$. We shall now obtain $\hat{\theta}$ by maximizing $\ell n\ g(u,v|\hat{a},\theta)$

$$\ell n\ g(u,v|\hat{a},\theta) = C - r\ \ell n\ \theta - \frac{v}{\theta}$$

where C does not depend on θ, and $\ell n\ g(u,v|\hat{a},\theta)$ is continuous for θ, so

$$\frac{\partial}{\partial\theta}\ \ell n\ g(u,v|\hat{a},\theta) = \frac{v}{\theta^2} - \frac{r}{\theta}$$

and setting the above derivative to zero, then clearly $\frac{v}{r}$ maximizes

$$\ell n\ g(u,v|\hat{a},\theta).$$

That is,

$$\ell n\ g(u,v|\hat{a},\hat{\theta}) \geq \ell n\ g(u,v|\hat{a},\theta)$$

for all $\theta > 0$ and $\hat{\theta} = \frac{v}{r}$ is the MLE of θ. From our previous definition of u,v we have $x_{(1)}$ and

$$\frac{\sum_{2}^{r} (x_{(i)} - x_{(1)}) + (n-r)\ (x_{(r)} - x_{(1)})}{r}$$

as the MLE for 'a' and 'θ' respectively.

Mean Squared Error of Estimates in (i), (ii) and (iii).

Since both UMVUE and BLUE are unbiased estimates then the precision is expressed in terms of their associated variances. However the MLE are not unbiased and precision must be expressed in terms of variance and bias. The bias tends to zero for large n, and the actual values are summarized in Table 3.1.

Table 3.1

Comparison of Bias, Variance, Mean Square Error

	a	Var	bias	MSE	θ	Var	bias	MSE
UMVUE(*) or BLUE	$x_{(1)} - \frac{V_c}{n(r-1)}$	$\frac{r\,\theta^2}{n^2(r-1)}$	nil	$\frac{r\,\theta^2}{n^2(r-1)}$	$\frac{V_c}{(r-1)}$	$\frac{\theta^2}{(r-1)}$	nil	$\frac{\theta^2}{(r-1)}$
MLE(^)	$x_{(1)}$	$\frac{\theta^2}{n^2}$	$\frac{\theta}{n}$	$\frac{2\theta^2}{n^2}$	$\frac{V_c}{r}$	$\frac{(r-1)\theta^2}{r^2}$	$\frac{\theta}{r}$	$\frac{\theta^2}{r}$
	$2 \leq r \leq n$;	$V_c = \sum_{2}^{r} (x_{(i)} - x_{(1)}) + (n-r)(x_{(r)} - x_{(1)})$						

We calculate Generalized Mean Square Error (GMSE) for $\tilde{\theta}^*$ and $\hat{\tilde{\theta}}$ respectively. By definition

$$\text{GMSE}\ (\tilde{\theta}^*) = \begin{vmatrix} \text{Var}\ (a^*) & \text{Cov}\ (a^*,\theta^*) \\ \text{Cov}\ (a^*,\theta^*) & \text{Var}\ (\theta^*) \end{vmatrix} = \begin{vmatrix} \frac{r\theta^2}{(r-1)n^2} & \frac{-\theta^2}{n(r-1)} \\ \frac{-\theta^2}{n(r-1)} & \frac{\theta^2}{(r-1)} \end{vmatrix}$$

$$= \frac{\theta^4}{(r-1)^2} \left[\frac{r}{n^2} - \frac{1}{n^2} \right] = \frac{\theta^4}{n^2(r-1)} \qquad (3.10)$$

When we compare (3.10) with that of full sample case, then define efficiency of censored to full sample as

$$\text{Eff.} = \text{G.M.S.E.}\ (a^*, \theta^*)\ \text{full} \Big/ \text{G.M.S.E.}\ (a^*,\theta^*)\ \text{censored} \ .100\%$$

$$= \frac{\theta^4}{n^2(n-1)} \cdot \frac{n^2(r-1)}{\theta^4} = \frac{(r-1)}{(n-1)}\ 100\%$$

and as $r \to n$, efficiency approaches 100%. Efficiency is quite sensitive to number of available observations.

Similarly we now calculate G.M.S.E. $(\hat{\tilde{\theta}})$ given by

$$\text{G.M.S.E.}(\hat{\tilde{\theta}}) = \begin{vmatrix} E\,[(\hat{a}-a)^2] & E[(\hat{a}-a)\ (\hat{\theta}-\theta)] \\ E\,[(\hat{a}-a)\ (\hat{\theta}-\theta)] & E[(\hat{\theta}-\theta)^2] \end{vmatrix} \quad \text{where}$$

$$E[(\hat{a}-a)\ (\hat{\theta}-\theta)] = E[(u-a)\ (\frac{v_c}{r} - \theta)] = E\,[\frac{uv_c}{r}] - \theta E[u] - \frac{a}{r}E[v] + a\theta$$

$$= \frac{1}{r}E[uv_c] - \theta\ (a+\theta/n) - a\theta(r-1)/r + a\theta$$

$$= \frac{(r-1)\ \theta}{r}\ (a + \frac{\theta}{n}) - a\theta - \frac{\theta^2}{n} + \frac{a\theta}{r}$$

$$= -\frac{\theta^2}{2n}$$

and

$$\text{G.M.S.E.}\ (\hat{\tilde{\theta}}) = \begin{vmatrix} \frac{2\theta^2}{n^2} & -\frac{\theta^2}{rn} \\ -\frac{\theta^2}{rn} & \frac{\theta^2}{r} \end{vmatrix} = \frac{\theta^4\ (2r-1)}{n^2r^2} \qquad (3.11)$$

We compare (3.11) with the G.M.S.E. in full sample case and define efficiency of censored to full sample for M.L.E. estimates as

$$\text{EFF.} = \left\{ \text{G.M.S.E. } (\hat{a},\hat{\theta}) \text{ full} \Big/ \text{G.M.S.E. } (\hat{a},\hat{\theta}) \text{ censored} \right\} \cdot 100\%$$

$$= \frac{(2n-1)\,\theta^4}{n^4} \cdot \frac{n^2\, r^2}{\theta^4 (2r-1)} = \frac{(2n-1)\, r^2}{(2r-1)\, n^2} \cdot 100\%$$

and as $r \to n$, efficiency approaches 100%. Although $\tilde{\hat{\theta}}$ is sensitive to censoring it is not as sensitive as $\tilde{\theta}^*$ with respect to G.M.S.E. (see table 3.2)

Table 3.2

Comparison of Efficiencies, w.r.t. Censoring

N = 10; r =	2	5	8	10
EFF.% $(\hat{a},\hat{\theta})$	25.3	52.7	81.2	100
Eff.% (a^*,θ^*)	11.1	44.5	78.0	100

We denote the level of efficiency by α. Then using $\alpha = \frac{r-1}{n-1}$ for $0 < \alpha \leq 1$ relative to GMSE of UMVUE $\tilde{\theta}^*$ we can use the formula

$$r = [(n-1)\alpha + 1] + 1 \qquad (3.12)$$

where [u] implies the greatest integer contained in u . From (3.12) we see that for $\alpha = .90$ and $n = 20$, we need at least $r = [19(.9) + 1] + 1 = 19$ observations. If only 15 observations be available, then our estimating procedure would at most be 73.6% as efficient as if we had all 20 observations.

If we define α relative to GMSE of MLE $\hat{\tilde{\theta}}$ then

$$\alpha = \frac{(2n-1)\ r^2}{(2r-1)\ n^2} \qquad \text{for} \quad 0 < \alpha \leq 1 \qquad \text{and}$$

$$r = \frac{n^2\alpha \pm \sqrt{n^2\alpha\ (n^2\alpha - (2n-1))}}{(2n-1)} \ ; \qquad \text{for} \quad n^2\alpha \geq 2n-1$$

$$r = \left[\frac{n^2\alpha + \sqrt{n^2\alpha\ (n^2\alpha - 2n + 1)}}{(2n-1)}\right] + 1 \tag{3.13}$$

From (3.13) it can be shown that for $\alpha = .90$, $n = 20$, we require at least 18 observations.

We now compare GMSE (censored) for (a^*, θ^*) w.r.t. $(\hat{a}, \hat{\theta})$.

Therefore $$\text{G.M.S.E.}_c\ \tilde{\theta}^* \Big/ \text{G.M.S.E.}_c\ \hat{\tilde{\theta}} = \frac{\theta^4}{n^2(r-1)} \cdot \frac{n^2\ r^2}{\theta^4(2r-1)} = \frac{r^2}{(2r-1)(r-1)}$$

and for $r = 2$, ratio > 1

$r \geq 3$, ratio < 1

and we prefer $(\tilde{\theta})^*$ for $3 \leq r$ and choose $\hat{\tilde{\theta}}$ when $r = 2$, (on basis of smaller G.M.S.E.) even though bias of $\hat{\theta}$ is largest at $r = 2$. We note that bias of $\hat{a}$ is not affected by r when censoring is to the right. It is interesting to note that preference between $\tilde{\theta}^*$ and $\hat{\tilde{\theta}}$ depends only on r, the number of observations available.

Calculation of Reliability for Mission Time t

Calculation of U.M.V.U.E. of R(t)

As in previous chapter, we wish to estimate the probability of any item in our sample surviving up to some time t. However we now have a sample where only the first "r" of n items are available to us.

$$\text{We define } R(t) = \Pr[X \geq t] = \int_t^\infty f(x,|a,\theta)dx$$

and to find UMVUE of $R(t)$ we consider the ordered sample $x_{(1)},\ldots,x_{(r)}$ as a part of a random sample of size n. Now consider two independent sub samples of size n-1 and one. We wish to obtain the conditional pdf of ξ, given the joint sufficient statistics u,v. We consider the ξ range to be composed of three possibilities.

$$\xi = x_{(1)} \text{ and then } f(\xi | u,v) = \frac{1}{n}$$

Consider the other possibilities:

(a) ξ is one of the available x_i's; i.e. $x_{(1)} < \xi \leq x_{(r)}$

(b) ξ is one of the surviving x_i's; i.e. $x_{(1)} < \xi < \infty$

Then for $x_{(1)} < \xi \leq x_{(r)}$ we have that

$$g(u')\ g(v'|u',a,\theta) = \frac{n-1}{\theta} \exp\left[-\frac{(n-1)}{\theta}(u'-a)\right] \left(\frac{1}{\theta}\right)^{r-2} \exp\left[-\frac{v'}{\theta}\right] v'^{r-3} \cdot$$

$$\frac{1}{\Gamma(r-2)}\ c(u',a)\ c(v',0)$$

where

$$u' = x_{(1)}, \quad v' = \sum_{2}^{r-1} (x_{(i)} - x_{(1)}) + (n-r)\,(x_{(r)} - x_{(1)})$$

and $g(v' | u', a, \theta)$ is a direct consequence of $(r-1)$ available observations from $(n-1)$ sample size and results (3.5) and (3.6). That is $v' = \sum_{i=1}^{r-2} W_i'$ is a gamma distribution with parameters $(r-2)$, θ, and

$$g(\xi | a,\theta) = \frac{1}{\theta} \exp\left[-\frac{\xi-a}{\theta}\right] C(\xi, x_{(1)}) C(x_{(r)}, \xi) \frac{1}{\Pr[x_{(1)} < \xi \le x_{(r)}]} .$$

Now $u' = u$ if $\xi > x_{(1)}$ and $v' = v - (\xi - u)$ if $x_{(1)} < \xi \le x_{(r)}$ and so (u', v') transforms into (u,v) if ξ lies in the range defined above. Then clearly

$$g(\xi, u, v | a,\theta) = g(\psi(u') | a,\theta)\, g(\psi(v') | \psi(u') a,\theta).$$

$$.g(\xi | a,\theta).\Pr[u',v' \to u,v] \qquad (3.14)$$

$$= \frac{(n-1)}{\theta} \exp -\left[\frac{(n-1)}{\theta}(u-a)\right] C(u,a). \frac{1}{\theta^{r-2}} \cdot$$

$$\exp\left[-\frac{v-\xi+u}{\theta}\right] (v-\xi+u)^{r-3} \frac{1}{\Gamma(r-2)} C(v,0)\cdot \frac{1}{\theta} \cdot$$

$$\exp\left[-\frac{\xi+a}{\theta}\right] C(\xi, x_{(1)}) \frac{1}{\Pr[x_{(1)} < \xi \le x_{(r)}]} \cdot$$

$$\Pr[u',v' \to u,v] \qquad (3.15)$$

and from (3.7)

$$g_1(\xi | u,v,a,\theta) = g(u,v,\xi | a,\theta) \Big/ g(u,v | a,\theta)$$

$$= \frac{n-1}{n}\frac{1}{v}\left(1 - \frac{\xi-u}{v}\right)^{r-3}(r-2) \quad \text{for} \quad x_{(1)} < \xi \leq x_{(r)} \qquad (3.16)$$

Considering the possibility that $\xi > x_{(r)}$ we define

$$W_i = (n-i)\,(x_{(i)} - x_{(i-1)})$$

for $i = 2,\ldots,r$ and then $v' = \sum_2^r W_i$ = sum of $(r-1)$ exponentially distributed random variables. Given $x_{(1)} = u'$ then by m.g.F. technique

$$g(v' | u',a,\theta) = \frac{1}{\Gamma(r-1)}\,\frac{1}{\theta^{r-1}}\exp\left[-\frac{v'}{\theta}\right]v'^{\,r-2}\;C(v',0).$$

Now

$$g(u' | a,\theta) = \frac{n-1}{\theta}\exp\left[-\frac{(n-1)(u'-a)}{\theta}\right]\;C(u',a)$$

and

$$g(\xi | a,\theta) = \frac{1}{\theta}\exp\left[-\frac{\xi-a}{\theta}\right]C(\xi,x_{(r)}) \Big/ \Pr[\,\xi > x_{(r)}\,]$$

Now

$$u' = u \qquad \text{for} \quad \xi > x_{(r)} > x_{(1)}$$

$$v' = v - (x_{(r)} - u) \qquad \text{for} \quad \xi > x_{(r)}$$

so that (u',v') transforms into (u,v) if ξ lies in the range $\xi > x_{(r)}$.

By (3.14) clearly

$$g(\xi,u,v|a,\theta) = \frac{n-1}{\theta} \exp\left[-\frac{(n-1)(u-a)}{\theta}\right] C(u,a) \cdot \frac{1}{\Gamma(r-1)} \cdot \frac{1}{\theta^{r-1}} \cdot$$

$$\exp\left[-\frac{(v-x_{(r)}+u)}{\theta}\right] (v-x_{(r)}+u)^{r-2} C(v,0) \frac{1}{\theta} \cdot$$

$$\exp\left[-\frac{\xi-a}{\theta} \frac{C(\xi,x_r)}{Pr[\xi > x_{(r)}]}\right] \quad Pr[u'v' \rightarrow uv] \tag{3.17}$$

From (3.7)

$$g_2(\xi|u,v,a,\theta) = \frac{n-1}{n}\left(1-\frac{x_{(r)}-u}{v}\right)^{r-2} \frac{1}{\theta} \exp\left[-\frac{\xi-x_{(r)}}{\theta}\right] C(\xi,x_{(r)}) \tag{3.18}$$

for $\xi > x_{(r)}$

Now define $H(\xi) = 1$ if ξ lies in interval (t, ∞)

$= 0$ otherwise

and $H(\xi)$ is unbiased estimate of $R(t)$.

The UMVUE, by R.B.L.S. theorem is

$$R^*(t) = E\left[H(\xi)\,|\,g(\xi|u,v,a,\theta)\right]$$

$$= \int_t^\infty g(\xi|u,v,a,\theta)\,d\xi$$

$$= \int_t^{x_{(r)}} g_1(\xi|u,v,a,\theta)d\xi + \int_{x_{(r)}}^\infty g_2(\xi|u,v,a,\theta)d\xi$$

where $x_r < u+v$, since $(x_r - u) < v = \sum_2^r (x_i - x_{(1)}) + (n-r)(x_{(r)} - x_{(1)})$

and

$$R^*(t) = \frac{(n-1)}{n} \frac{1}{v} (r-2) \int_t^{x_{(r)}} (1 - \frac{\xi - u}{v})^{r-3} d\xi$$

$$+ \frac{(n-1)}{n} (1 - \frac{x_{(r)} - u}{v})^{r-2} \int_{x_r}^{\infty} \frac{1}{\theta} \exp\left[- \frac{\xi - x_{(r)}}{\theta} \right] d\xi$$

$$= (\frac{n-1}{n}) (1 - \frac{t-u}{v})^{r-2} \qquad \text{for } x_{(1)} < t < \infty .$$

Then the UMVUE of R(t) is

$$\begin{aligned} R^*(t) &= 1 && \text{if } t < x_{(1)} \\ &= (\frac{n-1}{n}) (1 - \frac{t-u}{v})^{r-2} && \text{if } x_{(1)} \leq t \leq u + v \\ &= 0 && \text{if } t > u+v \end{aligned} \qquad (3.19)$$

and (3.19) is in agreement with equation (14) of Basu (1964).

Calculation of MLE and BLUE for R(t)

We see that

$$E \left[\sum_1^r \ell_i x_{(i)} \right] = \sum_1^r \ell_i (a + V\theta) \neq \exp\left[- \frac{t-a}{\theta}\right]$$

so that R(t) is not estimable from any linear function of order statistics. Since no linear unbiased estimate exists, no BLUE exists for R(t).

Now $R(t) = \int_t^{\infty} f(x|a,\theta)dx$ and MLE is one which maximizes the likelihood of $f(x|a,\theta)$ for u,v given. We know $f(x|\hat{a},\hat{\theta}) \geq f(x|a,\theta)$ and then MLE of R(t) will be

$$\hat{R}(t) = \int_t^{\infty} f(x|\hat{a},\hat{\theta})dx$$

and for fixed t

$$\hat{R}(t) = 1 \qquad \text{if} \quad t < x_{(1)}$$

$$= \exp\left[-\frac{t-\hat{a}}{\hat{\theta}}\right] \quad \text{if} \quad t \geq x_{(1)} \, . \qquad (3.20)$$

Expressed in terms of u,v the sufficient statistics this becomes

$$\hat{R}(t) = 1 \qquad \text{if} \quad t < u$$

$$= \exp\left[-\frac{r\,(t-u)}{v}\right] \quad \text{if} \quad t \geq u \qquad (3.21)$$

As in Chapter II, the comparison between MLE and UMVUE is difficult for the reason that bias of MLE is very difficult to obtain. Even though UMVUE has no bias, its variance is also difficult to compute. We shall illustrate this for the bias of the MLE.

Since $\hat{R}(t)$ can be expressed as $\hat{\psi}(u,v)$, $E\,[\hat{\psi}(u,v)] =$ zero if $t < a$ so we consider the case for $t \geq a$. By definition

$$E\,[\hat{\psi}(u,v)] = \iint \psi(u,v) \quad g(u,v|a,\theta)dudv \qquad \text{for} \quad t \geq a$$

$$= \Pr[\, u \geq t] + \iint \exp[\, -\frac{r}{v}\,(t-u)] \; g(u,v|a,\theta)dudv$$

$$\text{for} \quad v > 0, \quad a \leq u \leq t.$$

$$= R^n + \int_o^\infty \int_a^t \left(\frac{n}{\theta}\right) \exp\left[-\frac{n}{\theta}\,(u-a)\right] \frac{1}{\Gamma(r-1)} \left(\frac{1}{\theta}\right)^{r-1}$$

$$\exp\left[-\frac{v}{\theta}\right] v^{r-2} \quad \exp\left[-r\frac{(t-u)}{v}\right] dudv$$

$$= R^n + \emptyset \qquad (3.22)$$

where $\emptyset$ must be evaluated

$$\phi = \frac{n}{\theta^2} \frac{1}{\Gamma(r-1)} \int_o^\infty \exp[-\frac{v}{\theta}] (\frac{v}{\theta})^{r-2} \int_a^t \exp[-\frac{n}{\theta}(u-a) - \frac{r}{v}(t-u)]dudv$$

$$\int_o^\infty \phi_c' \, dv = - \frac{n}{\Gamma(r-1)} \int_o^\infty (\frac{v}{\theta})^{r-1} \exp[-\frac{v}{\theta}] (\frac{1}{nv-r\theta}) \Big\{ \exp[-\frac{n}{\theta}(t-a)] - \exp[-\frac{r}{v}(t-a)] \Big\} dv$$

and for $W = \frac{v}{\theta}$, then we have

$$\phi_c = \frac{n}{\Gamma(r-1)} \int_o^\infty W^{r-1} \exp[-W] \frac{1}{(nw-r)} (R^{r/w} - R^n)dW \qquad (3.23)$$

where $R = \exp[-\frac{t-a}{\theta}]$ = a constant for given t, a, θ. We note that for $r = n$, (3.23) becomes identical to (2.26).

For $nw = r$, $v_o = \frac{r\theta}{n}$ and ϕ_c' at v_o is

$$\phi_c' = \frac{n R^n}{\theta^2 (r-1)} (\frac{r}{n})^{r-2} \exp[-\frac{r}{n}] (t-a)$$

and since $\theta > 0$, ϕ_c is defined at $W = \frac{r}{n}$. (for $r = n$, $\phi' = \phi_c'$).

Therefore

$$E[\hat{R}(t)] = R^n + \frac{n}{\Gamma(r-1)} \lim_{\epsilon \to o} \int_o^{r/n-\epsilon} \frac{W^{r-1}}{(nw-r)} \exp[-W](R^{r/w}-R^n)dw$$

$$+ \frac{n}{\Gamma(r-1)} \lim_{\epsilon \to o} \int_{r/n+\epsilon}^\infty W^{r-1} \exp[-W] \frac{(R^{r/w} - R^n)}{(nw-r)} dw$$

$$= R^n + \frac{n}{\Gamma(r-1)} \int_o^\infty \frac{W^{r-1}}{(nw-r)} \exp[-W] (R^{r/w} - R^n)dw \qquad (3.24)$$

As in (2.27), no standard functional form of (3.24) appears to exists.

CHAPTER IV

ANALYSIS OF LIFE TEST EXPERIMENTS IN CANCER RESEARCH

A common type of experiment in cancer research is response time studies made on samples of white mice. We analyze three different samples where the data is similar in the three cases, but amount of data available varies in each sample. In these experiments, all the sample mice are injected with a common drug and the elapsed positive reaction time for each mouse is recorded.

It is sometimes difficult to ascertain a positive reaction. In the above experiments, three criteria are required before a reaction is termed positive but only one yields accurate time estimates. However, this one criteria , a positive myeloma protein count (PMP) is known to suggest 85-90% likelihood that a positive reaction has occured. With this knowledge we then consider elapsed times from injection to PMP. From these times (t_i) we then calculate, from estimates developed in the previous chapters, the proportion of mice that develop PMP by some fixed time T.

We wish to note that in the past, reliability estimates have been calculated by counting the number of mice with known positive reactions (all necessary criterion established) by some time (t). A proportion has then been calculated 1 - Rp (t), such that

$$1 - Rp(t) = \frac{\text{Number of mice with positive reaction by time t}}{\text{Total mice in sample at time zero}}.$$

1 - Rp(t) is then the probability of having a positive reaction by time t. This estimate of the probability has severe limitations, a few of which are

a) Censoring during the experiment often makes the 'n' undefinable.

b) All experimental data is not used for estimation.

c) Estimates are easily affected by adverse experimental conditions, rendering such estimates inconsistent and unreliable for practical comparisons.

Our estimates have not these limitations but depend on some assumptions which we now describe.

The random variable t_i has an underlying negative exponential distribution with location parameter 'a' $\neq$ zero. The first part of this assumption is not new to biological experimentation [Bailey_1, 1967]. In fact, since our experiments are biologically similar to stochastic epidemics and our estimates are based on duration times, then like Bailey_2 (1967), we may claim that 'the time interval t_i before the new infection occurs has the negative-exponential distribution...'. Furthermore, when one considers length of patient-stay as duration between stimulus (patient admission) to response (patient discharge) then it has been demonstrated Saunders, M.G., Hamelin, D.G. and Martin, D. (1966) that such duration follows the negative exponential distribution. As to the second part of our assumption, researchers in this area of cancer research indicate that a dormant period after injection varies between 5 and 7 time units (t.u) (months). In our experiments any deaths (censoring) prior to 5 t.u. will be excluded from samples for purposes of estimation. Censoring is the result of deaths, before incidence of PMP has been detected. These deaths may be the results of random causes (epizootics, canibalism, etc.) or as a result of their inherent stochastic life process. Normal life expectancies of these mice are over twenty-two t.u. and we assume that for periods of less than eighteen t.u., the individual death expectancy is quite small. We then consider censoring as due to random causes for under eighteen t.u. .

With the above assumptions, we now consider three different samples and we estimate probability of any mouse developint PMP beyond some fixed time T. Such an estimator, when used in conjunction with our earlier remark of 85-90% indication of positive response, can prove quite meaningful and consistent from experiment to experiment.

Sample 1.

From a sample of size 15, the recorded times as they become available are 6,6,7,8,8,8,8,9,10,12,13,15,15,17,17 time units. From this sample we see that $u = t_{(1)} = 6$, $v = n(\bar{t}-t_{(1)}) = 69$. Since we have a full sample (no censoring) we use results of chapter two. Then

$$\text{UMVUE and BLUE for } a = \frac{nt_{(1)}-\bar{t}}{n-1} = \underline{\underline{5.67}} = a^*$$

$$\text{UMVUE and BLUE for } \theta = \frac{v}{n-1} = \underline{\underline{4.93}} = \theta^*$$

Also, 95% confidence intervals for a^* and θ^* respectively [Epstein, 1960c] are

$$\left[t_{(1)} - \frac{W_\nu \theta^*}{n},\ t_{(1)} \right] = \left[6 - \frac{(3.34)\ (4.93)}{15},\ 6 \right] = (4.90,6)$$

for W_ν at 95% confidence from F-table at (2,2n - 2) degrees of freedom, and

$$\left[\frac{2\,(n-1)\,\theta^*}{\chi^2_{.025}(2n-2)}\;,\;\frac{2\,(n-1)\,\theta^*}{\chi^2_{.975}(2n-2)}\right] = \left[\frac{2(69)}{44.46}\;,\;\frac{2(69)}{15.31}\right] = (3.11, 9.01).$$

A measure of $R(t)$ can be obtained if we consider a function of (a^*, θ^*), biased, which we define as $R_f(t)$ such that

$$R_f(t) = \exp\left[-\frac{t-a^*}{\theta^*}\right] = \exp\left[-\frac{t-5.67}{4.93}\right]$$

and a measure of 95% confidence bands about $R_f(t)$ is obtained as

$$R_{f(\text{lower})}(t) = \exp\left[-\frac{t-4.9}{3.11}\right]; \quad R_{f(\text{upper})}(t) = \exp\left[-\frac{t-6}{9.01}\right]$$

the lower and upper bounds respectively. These bounds are not exact as they do not allow for bias on one side, yet tend to give wider bands considering dual extreme values of simultaneous estimated values a^*, θ^* on the other. They do serve however, to illustrate estimators based on censored and non-censored samples as we shall later point out.

Other estimates of (a, θ) are MLE's $\hat{a}$ and $\hat{\theta}$ given by

$$\hat{a} = u = \underline{\underline{6}} \qquad \hat{\theta} = v/n = \underline{\underline{4.6}} \;.$$

We then have the MLE of $R(t)$ as

$$\hat{R}(t) = \exp\left[-\frac{t-\hat{a}}{\hat{\theta}}\right] = \exp\left[-\frac{t-6}{4.6}\right].$$

We have shown in chapter two that the UMVUE of $R(t)$ is

$$R^*(t) = \frac{n-1}{n}\left(1-\frac{t-u}{v}\right)^{n-2} = \frac{14}{15}\left(1-\frac{t-6}{69}\right)^{13}$$

and is best with respect to minimum variance estimation.

We also previously defined $R_p(t)$ so that

$$R_p(t) = \frac{\text{Number of } t_i\text{'s larger or equal to T}}{\text{Total number of } t_i\text{'s larger than 5}} = \frac{\# t_i\text{'s} \geq T}{15}.$$

Table 4.1 gives some values for the different estimates used and we point out the small differences between $R_f(t)$ and the best estimate $R^*(t)$. The location of these estimates with respect to confidence bands of $R_f(t)$ are well illustrated in Appendix 1.

Table 4.1

Estimates of R (t) given r = n = 15, t = 6(2)14

t	$R_p(t)$	$R_f(t)$	$R_{f(lower)}(t)$	$R_{f(upper)}(t)$	$\hat{R}(t)$	$R^*(t)$
6	1.000	0.935	0.703	1.000	1.000	0.934
8	0.800	0.625	0.372	0.802	0.648	0.636
10	0.467	0.417	0.196	0.642	0.421	0.429
12	0.400	0.279	0.104	0.515	0.273	0.285
14	0.266	0.186	0.060	0.413	0.177	0.190

Sample 2

We have data as in Sample 1 except that $t_{(11)}$ and $t_{(14)}$ are not available because of deaths (censoring). If we choose to discard information beyond $t_{(10)}$ we can consider the data as a right single stage censored sample and make use of theory of Chapter III. Then $u = t_{(1)} = 6$, and

$$v = \sum_{1}^{r} (t_{(i)} - t_{(1)}) + (n-r)\,(t_{(r)} - t_{(1)}) = 52 \text{ for } r = 10,\ n = 15$$

UMVUE and BLUE for $a = u - \dfrac{v}{n(r-1)} = \underline{\underline{5.61}} = a^*$

UMVUE and BLUE for $\theta = \dfrac{v}{r-1} = \underline{\underline{5.78}} = \theta^*$

and 95% confidence intervals for a^* and θ^* are given by

$$\left[u - \frac{W_\nu\ \theta^*}{n}\ ,\ u\right] = \left[6 - \frac{3.55\ (5.78)}{15}\ ,\ 6\right] = (4.63,\ 6)$$

for W_ν at 95% confidence from F-table at (2,2r-2) degrees of freedom;

and

$$\left[\frac{2\ (r-1)\ \theta^*}{\chi^2_{.025}(2r-2)}\ ,\ \frac{2\ (r-1)\ \theta^*}{\chi^2_{.975}(2r-2)}\right] = \left[\frac{18(5.78)}{31.53}\ ,\ \frac{18(5.78)}{8.23}\right] = (3.3, 12.62)$$

We define

$$R_f(t) = \exp\left[-\ \frac{t-a^*}{\theta^*}\right] = \exp\left[-\ \frac{t-5.61}{5.78}\right]$$

and

$$R_{f(lower)}(t) = \exp\left[-\ \frac{t-4.63}{3.3}\right]\ ;\quad R_{f(upper)}(t) = \exp\left[-\ \frac{t - 6}{12.62}\right]$$

MLE of (a,θ) are

$$\hat{a} = u = \underline{\underline{6}}\ ,\quad \hat{\theta} = v/r = \underline{\underline{5.2}}$$

Therefore $\hat{R}(t) = \exp\left[-\ \dfrac{t-\hat{a}}{\hat{\theta}}\right] = \exp\left[-\ \dfrac{t - 6}{5.2}\right]$

and from chapter three

$$R^*(t) = \frac{n-1}{n}\left(1 - \frac{t-u}{v}\right)^{r-2} = \frac{14}{15}\left(1 - \frac{t-6}{52}\right)^{8}\ ,$$

and is best estimate of $R(t)$ with respect to minimum variance.

$$R_f(t) = \frac{\#\ t_i\text{'s} \geq T}{15}$$

as before but is restricted to $T \leq t_{(10)}$. Table 4.2 gives some values for estimates of $R(t)$ and we again point out the small differences between $R_f(t)$ and $R^*(t)$. From Appendix II we see the wider confidence bands about $R_f(t)$ for censored case. This loss in confidence is directly attributable to the loss of sample points and thus, decreased sample space, used in calculating our estimates. In fact, the purpose of calculating confidence bands about $R_f(t)$ has been primarily to illustrate graphically the sensitivity of simultaneous estimation with respect to censoring.

Table 4.2

Estimates of $R(t)$ given $r = 10$, $n = 15$, $t = 6(2)14$

t	$R_f(t)$	$R_f(t)$	$R_{f(\text{lower})}(t)$	$R_{f(\text{upper})}(t)$	$\hat{R}(t)$	$R^*(t)$
6	1.000	0.935	0.662	1.000	1.000	0.934
8	0.800	0.663	0.362	0.854	0.682	0.682
10	0.467	0.470	0.197	0.729	0.464	0.491
12	0.400	0.333	0.108	0.623	0.317	0.351
14	? ?	0.236	0.059	0.532	0.216	0.244

Sample 3

Suppose that instead of sample 2 we have one mouse die at $t = 4$, 8,8,11 and 16, at $t_{(i)}$ values recorded are 6,6,7,8,8,9,10,13,15,17.

We arrange data as in Table 4.3 where $Z_{(i)}$ are death (censored) times.

Table 4.3

Data from multi censored experiment

Times (X_i)		Frequencies		$(K_i + d_i)(X_{(i)} - a)$
$t_{(i)}$	$Z_{(i)}$	d_i	K_i	
6	-	2	-	2
7	-	1	-	2
8	8	2	2	12
9	-	1	-	4
10	-	1	-	5
	11	-	1	6
13		1	-	8
15		1	-	10
	16	-	1	11
17		1	-	12

Then we are dealing here with a multi censored sample. We assume 'a' known = 5 and if we define

d_i = number of PMP at time $t_{(i)}$

Z_i = number of deaths at time $Z_{(i)}$

$x_{(i)}$= time unit (either $t_{(i)}$ or $Z_{(i)}$)

$y_{(i)} = x_{(i)} - a$ for 'a' known

then we may estimate θ^+ , the Scale parameter by referring to the best estimator [Herd (1956)] for known location.

$$\theta^+ = \frac{1}{\Sigma d_i} \sum_{i=1}^{n'} (K_i + d_i) y_{(i)}$$

where $n' = n - \# Z_{(i)}$ less than 'a' .

Thus for $t_{(1)} = 4$ in our present sample, $t_{(1)}$ is discarded and

$$\theta^{+} = \frac{1}{10}\,[2(1) + (2) + 4(3) + (4) + (5) + \ldots + (12)] = \underline{\underline{7.2}}$$

We define $R_f'(t)$ as

$$R_f'(t) = \exp\left[-\frac{t-a}{\theta^{+}}\right] = \exp\left[-\frac{t-5}{7.2}\right]$$

and for $t = 6(2)14$ we have $R_f'(t) = 0.871, 0.661, 0.500, 0.380$ and 0.288 respectively.

Should we wish to estimate "a", we recommend the statistic

$$u' = u - \frac{v'}{n(\Sigma d_i - 1)}, \qquad v' = \sum_{i=1}^{n'} x_{(i)}$$

which is a compromise estimate of a^{*} for the single stage censored sample. In our example we have

$$u' = 6 - \frac{142}{14(9)} = \underline{\underline{4.96}} \quad .$$

To estimate $R_f'(t)$, we then use

$$R_f''(t) = \exp\left[-\frac{t-u'}{\theta^{+}}\right] = \exp\left[-\frac{t-4.96}{7.2}\right] \doteq R_f'(t) \quad .$$

Conclusion

The similarity of data in the three samples previously discussed enable us to compare estimates of Reliability for the three samples. Appendices I and II illustrates both the decrease of precision as censoring occurs and the close approximation of $R_f(t)$ to $R^*(t)$. Further this allows for approximate comparisons between $R_f'(t)$ and $R_f(t)$ as illustrated in Appendix III. This latter illustration also indicates a tendency of $R_f(t)$ to over estimate $R(t)$ for more complex censoring to the right.

The last sample discussed indicates the difficulty one would have in order to calculate $R_p(t)$.

Finally, we compare the estimate of probability of any mouse having a PMP response by time T for the three types of samples. For our comparisons to include the multi censored sample we define this probability estimate as

$$P[\text{ dev. PMP by time T}] = 1 - R_f(t)$$

which is 0.814, 0.764, and 0.712 for t = 14 in samples one, two and three respectively. From Appendix III we note that sample estimates 2 and 3 tend to under-estimate this probability.

A question occurs when the theory of the preceeding chapters is applied to such experimental analysis as that found in the last chapter. Can any adjustment factors be derived which, when applied to estimators derived from censored samples, will tend to improve these estimators without greatly reducing the confidence of these same estimators?

To attempt to answer such a question would require the derivation of biases associated with practical estimators, and the cummulative affects of such biases dependent on the number of observations and range of elapsed times analyzed. Such an investigation is not without difficulty but the availability of data from such experimental work as done in Cancer Research would warrant that quantitative analysis in this area be done. It is only through such research that sound yet practical 'rule of thumb' adjustment factors could be defined for application by the non-statistician.

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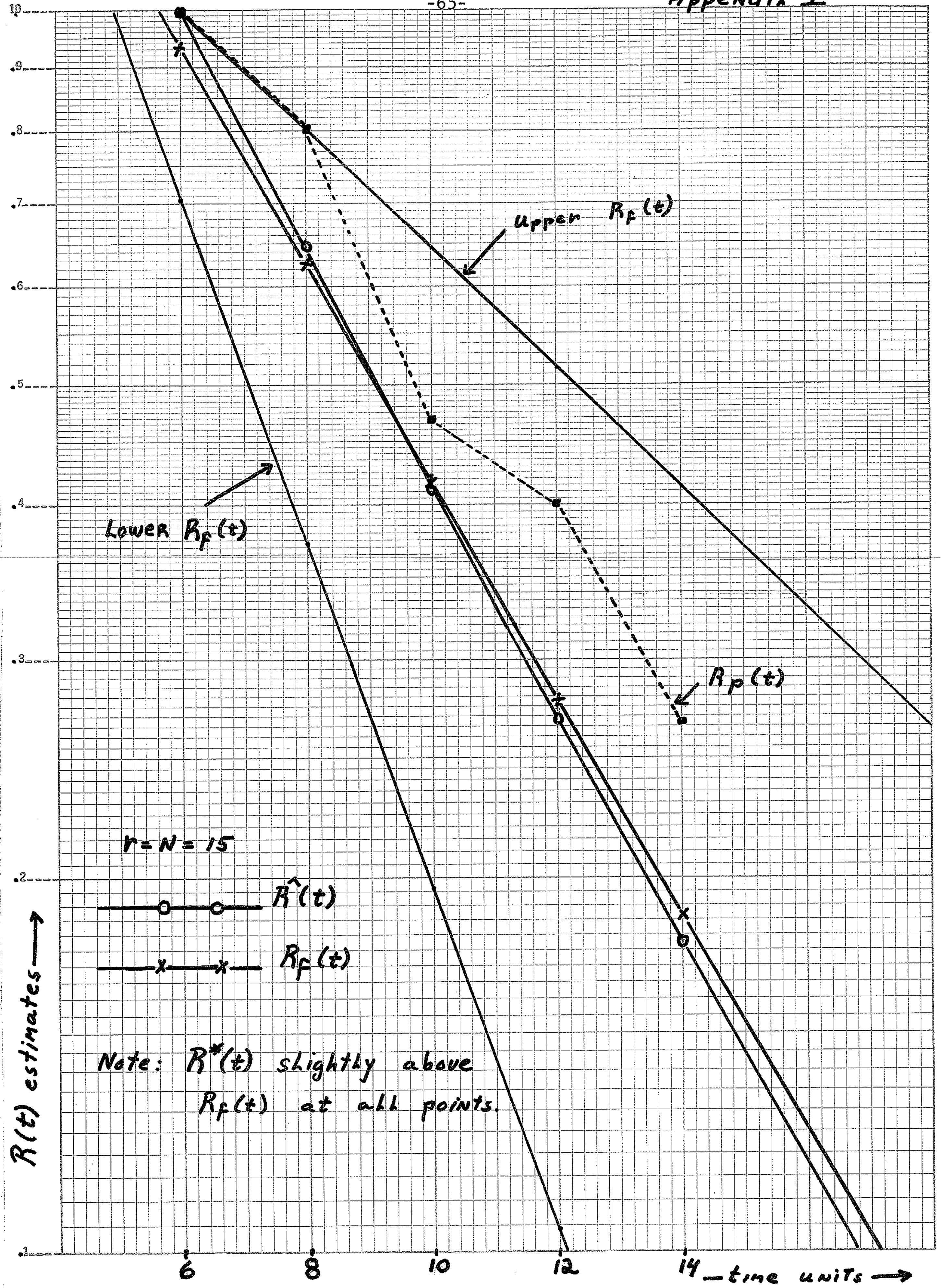
Appendix I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14 time units →
R(t) estimates →
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Lower $R_f(t)$
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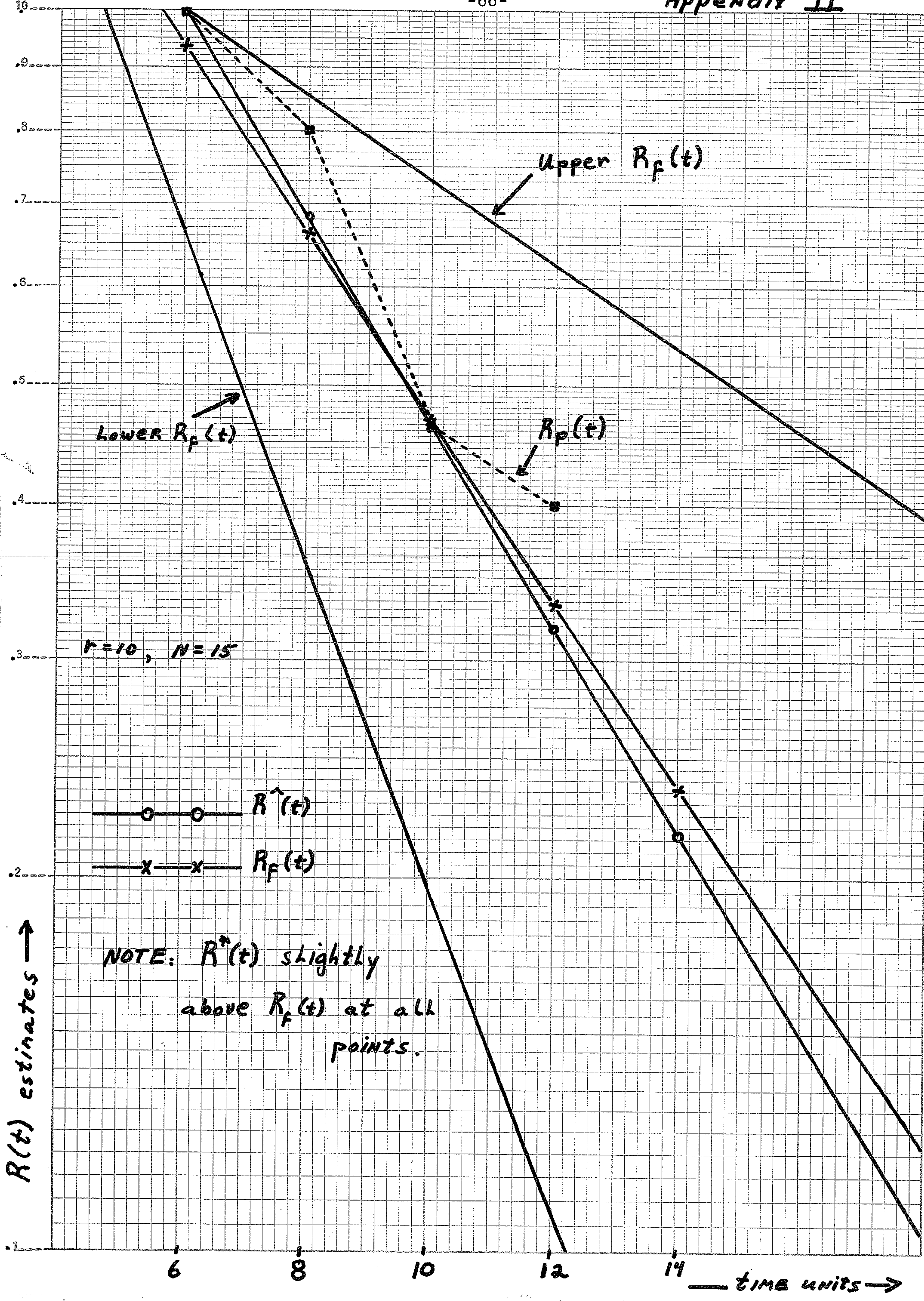
r = N = 15
$\hat{R}(t)$
$R_f(t)$
Note: $R^*(t)$ slightly above
$R_f(t)$ at all points.



Upper $R_f(t)$
Lower $R_f(t)$
$R_p(t)$
r=10, N=15
$\hat{R}(t)$
$R_f(t)$
NOTE: $\hat{R}(t)$ slightly above $R_f(t)$ at all points.
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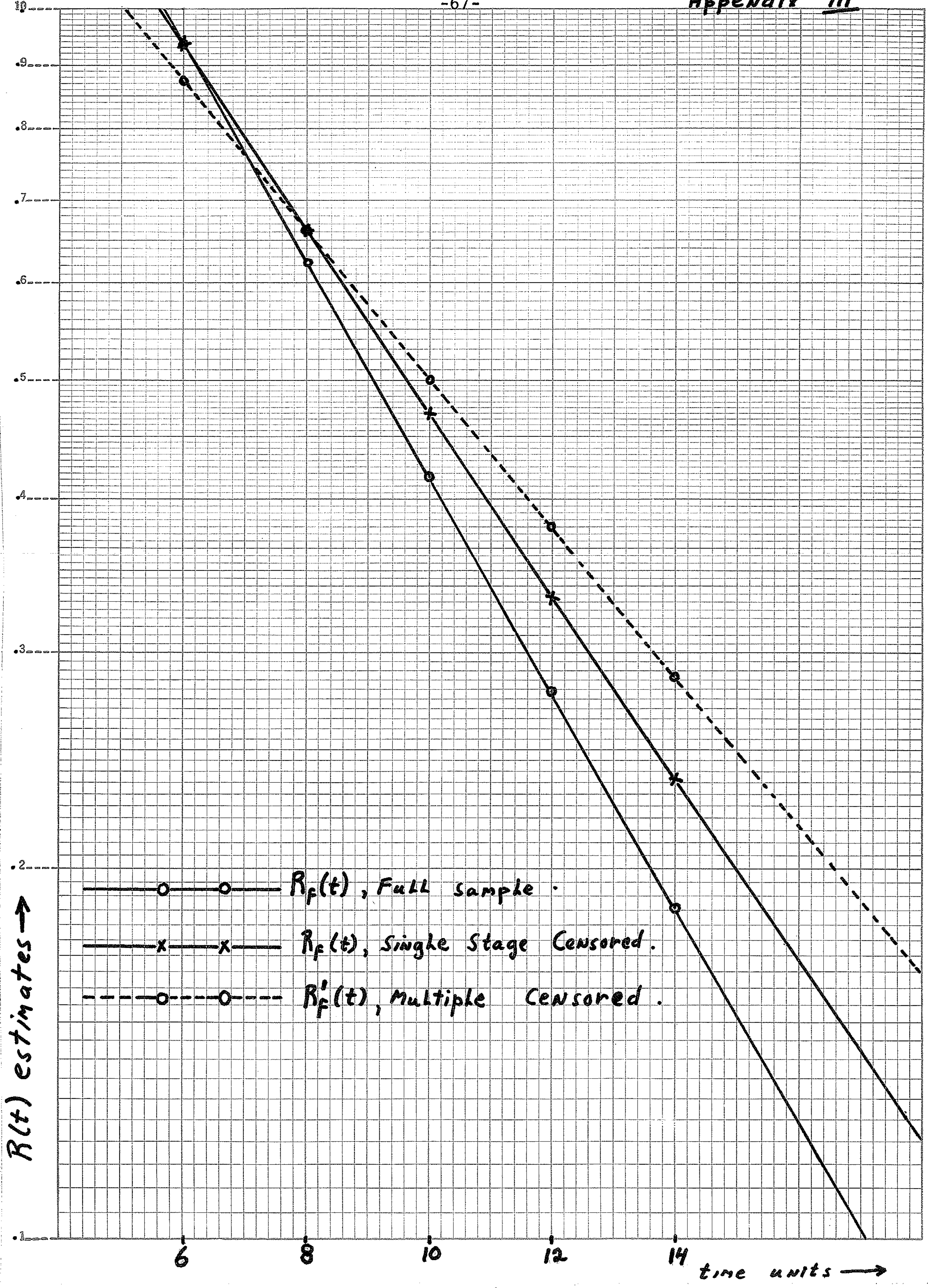
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R(t) estimates →
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time units →
$R_F(t)$, Full sample.
$R_F(t)$, Single Stage Censored.
$R'_F(t)$, Multiple Censored.