

Extremal Problems about Bootstrap Percolation on Graphs

by

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Abstract

Bootstrap percolation is a dynamic process on configurations of two possible states of vertices in a graph. For a given graph and a fixed non-negative integer threshold, some vertices are initially “activated” whilst others are “inactivated”. At each step, every inactivated vertex will become activated if the number of their activated neighbours is at least the threshold. If all the vertices are eventually activated, then say the set of initially activated vertices percolates. In this thesis, I focus on percolating sets containing as few vertices as possible – minimal and minimum percolating sets. Specifically, I survey exact and approximate cardinalities of minimal/minimum percolating sets of some classes of graphs, and obtain several new bounds and constructions for them. The main tools used in the thesis are structural properties and invariants of graphs. Throughout the thesis, I also present the major outstanding new problems in this area.

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Chapter 1

Introduction

1.1 Fundamental Concepts of Bootstrap Percolation

Imagine such a picture – an organization wants to spread an opinion and make all the people in the social network accept it. Although the organization can post a lot of advertisements on the internet or TV channels, people will not easily accept it unless at least 3 friends of theirs do. So, the organization plans to hire some people as initial accepters and starting points of the spread. Then comes a question – how to choose the initial accepters by using as low cost as possible? Similar questions occur in many scientific and technological fields. For instance, in the study of magnetic materials, an atom will alter to a particular state if it has a sufficient number of neighbours of the same state, then at least how many atoms of the particular state can assimilate all the atoms in an object? In the study of some infectious diseases, healthy people

will be more likely infected if they are in closely physical contact with a certain number of infected people, then determining the smallest number of initially infected people is quite helpful to trace back to the origin of the disease. All these questions have similar essence – spread of a state on a network under some threshold conditions, which coincides with the theme of this thesis – bootstrap percolation on graphs.

Bootstrap percolation is a dynamic process on a graph $G = (V, E)$ related to a non-negative integer r called the threshold. Suppose each vertex has one out of two possible states – *activated (infected)* or *inactivated (clean)*, and let A_0 be the set of activated vertices at the beginning or at time 0. In each discrete time step $i \in \{k \in \mathbb{Z} : k \geq 0\}$, given the set $\bigcup_{j=0}^i A_j$ of activated vertices after time i , if a vertex $v \in V \setminus (\bigcup_{j=0}^i A_j)$ has at least r activated neighbours in $\bigcup_{j=0}^i A_j$, then the state of v will alter from inactivated to activated, and define a set A_{i+1} consisting of such vertices. If no vertices in $V \setminus (\bigcup_{j=0}^i A_j)$ have at least r (activated) neighbours in $\bigcup_{j=0}^i A_j$, then we say the r -threshold activation process starting from A_0 stops. Once a vertex is activated, it will remain in this state forever. An example of bootstrap percolation is as follows. On the graph illustrated in Figure 1.1, a 3-threshold activation process occurs from three initially activated vertices in A_0 . Then, at the time 1, the four vertices in A_1 are activated because each of them has at least 3 activated neighbours at the time 1; and at the time 2, two vertices in A_2 get activated. Thereafter, since no inactivated vertices get at least 3 activated neighbours, the activation

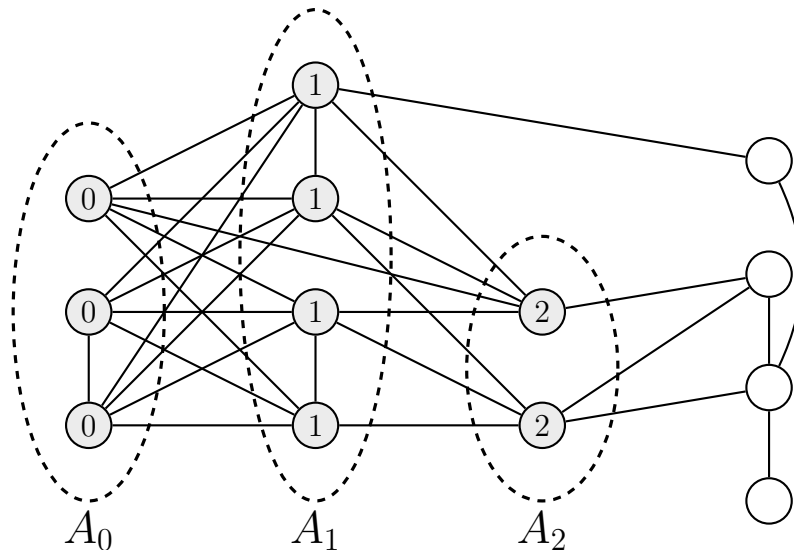


Figure 1.1: A 3-threshold activation process starting from A_0 on a graph G . The numbers attached to the vertices represent the time at which the vertices get activated.

process ends.

The time at which the activation process ends is called the *activation time* (or *infection time*) of A_0 for G . Every initially activated vertex in A_0 is called a *seed*, and the union $\bigcup_{i=0}^{\infty} A_i$ is called the *r -threshold activation closure* (or *infection closure*, or *span*) of A_0 and denoted by $\langle A_0 \rangle_r$ (or $\langle A_0 \rangle$ if the threshold is understood). For a finite graph G , the activation closure $\langle A_0 \rangle_r$ is obviously a union of finitely many sets A_i 's. For the set A_0 of seeds, if $\langle A_0 \rangle_r = V$ (in other words, all the vertices of the graph G are finally activated), then A_0 is called an *r -threshold percolating set* of G (or A_0 *percolates on* G).

For a given graph $G = (V, E)$ and a threshold r , the overall questions about bootstrap percolation are “which sets of vertices percolate on G ?”, and “what

can we say about the percolating sets?”. Large percolating sets are easy to find. For example, the vertex set V or $V \setminus \{x\}$ for some vertex x with $\deg_G(x) \geq r$ can all percolate on G . In fact, if $A_0 \subsetneq V$ is an r -threshold percolating set of G , then any superset B of A_0 is also an r -threshold percolating set of G because $\langle A_0 \rangle \subseteq \langle B \rangle$. Therefore, we shall focus on another extreme in this thesis – small percolating sets, and discuss their cardinalities, structures, and procedures for generating them.

1.2 History and Applications

The r -threshold bootstrap percolation was formally introduced and first studied by Chalupa, Leath, and Reich [11] as a model of the dynamics of ferromagnetism regarding changes of magnetic states of particles relevant to states of their neighbours, whilst bootstrap percolation is an example of cellular automata introduced earlier by von Neumann [25] based on a suggestion of Ulam [34].

Later, Poljak and Sûra [30], Chen [12], Amini and Fountoulakis [3] studied some problems about spread of opinions in social networks by using bootstrap percolation models. Newman [26], Liu, Lai, Ye and Liu [20] conducted extensive analysis about spread of contagious diseases with bootstrap percolation models. Peleg [28], Flocchini, Lodi, Luccio, Pagli, and Santoro [16], Luccio, Pagli, and Sanossian [21] studied applications of bootstrap percolation to distributed computing.

MacGillivray and Wang [22], Ng and Raff [27], Bazgan, Chopin and Ries [8] discussed some applications to firefighting-type problems of a model like bootstrap percolation in which activated vertices can become inactivated under some conditions.

Adler and Lev [1] described some applications of bootstrap percolation into the orientational ordering process of quadrupoles in solid molecules, fluid flow in rocks, and computer storage systems.

The rest of the thesis is arranged as follows. In Chapter 2, we introduce some basic facts and tools that are used through the thesis. In Chapter 3, we shall show the exact values for percolating sets of some special classes of graphs. In Chapter 4, we shall first present the bounds for minimum percolating sets of regular graphs, and then discuss the minimum 2-threshold percolating sets of a particular type of regular graphs – expander graphs. In Chapter 5, we shall first introduce approximate and exact results for minimum percolating sets of square grid graphs, and then turn to 2-threshold minimum/minimal percolating sets of general grid graphs. Finally, in Chapter 6, we shall show the bounds for minimum/minimal percolating sets of trees, and present an algorithm for generating minimum percolating sets of trees.

1.3 Notation

In this section, we introduce some notation used throughout the thesis.

Let $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers, respectively. For a set X , $|X|$ or $\#(X)$ denotes the cardinality of X , and $\mathcal{P}(X)$ denotes the power set of X . For a positive integer n , $[n]$ denotes the set $\{i \in \mathbb{Z} : 1 \leq i \leq n\}$.

For a graph $G = (V, E)$, if two vertices u, v are adjacent, then uv or $u - v$ denotes the edge joining u and v . If $v_1, \dots, v_r \in V$, $e_1, \dots, e_s \in E$, then $G - \{v_1, \dots, v_r, e_1, \dots, e_s\}$ denotes the graph obtained from G by removing $v_1, \dots, v_r, e_1, \dots, e_s$ and the edges incident with any one of v_1, \dots, v_r from G . In particular, for a single vertex v and a single edge e of G , write $G - v$ for $G - \{v\}$, and write $G - e$ for $G - \{e\}$ in short. For two subsets $S, W \subseteq V$ of vertices, define $N_W(S) = \bigcup_{v \in S} N_W(v)$ as the set of neighbours of S in W , and define $\partial_W(S) = N_W(S) \setminus S$ as the set of outer neighbours of S in W . In particular, let $N_G(S) = \bigcup_{v \in V} N_G(v)$ and $\partial_G(S) = N_G(S) \setminus S$.

For every positive integer n , P_n denotes the path of n vertices, C_n denotes the cycle of n vertices, and K_n denotes the complete graph of n vertices. For any m positive integers p_1, \dots, p_m , K_{p_1, \dots, p_m} denotes the complete m -partite graph.

When talking about asymptotic results, we use standard Landau notation. Suppose $f(x), g(x)$ are two positive real functions defined on \mathbb{R} . If there exists $M, c_1, c_2 \in \mathbb{R}$ with $c_1 > 0$ such that for every $x \geq M$, $f(x) \leq c_1 g(x) + c_2$, then we say $f(x) = O(g(x))$. If there exists $M, c_1, c_2 \in \mathbb{R}$ with $c_1 > 0$ such that for every $x \geq M$, $f(x) \geq c_1 g(x) + c_2$, then we say $f(x) = \Omega(g(x))$. If

both $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$, then we say $f(x) = \Theta(g(x))$. If $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$, then we say $f(x) = o(g(x))$. If $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \infty$, then we say $f(x) = \omega(g(x))$. Finally, if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$, then we say $f(x) \sim g(x)$.

For a given graph $G = (V, E)$ and a threshold r , if A is an r -threshold percolating set of G and for every vertex $u \in A$, the set $A \setminus \{u\}$ does not percolate on G , then A is called a *minimal r -threshold percolating set* of G . Besides, an r -threshold percolating set of the smallest cardinality is called a *minimum r -threshold percolating set* of G . Usually, $E(G, r)$ and $m(G, r)$ denote the cardinalities of the largest minimal and minimum r -threshold percolating sets of G , respectively.

Chapter 2

General Tools for Bootstrap Percolation

In this chapter, we introduce general facts and tools that are useful for bootstrap percolation on any graph.

2.1 Basic Facts about Bootstrap Percolation

This section is dedicated to some general facts and fundamental properties of bootstrap percolation.

By the description of activation process on graphs, loops and multiple edges have no effect on activation or percolation. Besides, activation runs independently on each connected component of a graph. So, we shall only consider connected simple graphs without loops in this thesis.

Recall that $E(G, r)$ and $m(G, r)$ denote the cardinalities of the largest minimal r -threshold percolating sets and the minimum r -threshold percolating

sets of a graph G , respectively. For any connected simple graph G , $m(G, 0) = E(G, 0) = 0$ and the unique minimal 0-threshold percolating set of G is just empty set. Meanwhile, for threshold 1, $m(G, 1) = E(G, 1) = 1$ and the singleton containing any individual vertex of G is a minimal 1-threshold percolating set of G because any single seed is capable of activating all the vertices. Thus, we will only consider the cases when threshold $r \geq 2$ in this thesis.

For a simple graph $G = (V, E)$ of order n , if A is an r -threshold percolating set of G , then A must contain all the vertices of degree smaller than r because every such vertex can not be activated by other vertices. In particular, if $r \geq n$, then every vertex has degree less than $n - 1 = r - 1$, and the unique r -threshold percolating set is just V . Hence, it remains to consider r -threshold percolation in nontrivial cases – that is when $2 \leq r < n$. In addition, when $r < n$, A has to contain at least r seeds because otherwise, it could not activate any clean vertex. We summarize this paragraph in the following remark.

Remark 2.1. *Let $G = (V, E)$ be a simple graph. If A is an r -threshold percolating set of G , then A contains all the vertices of degree smaller than r and $\min\{|V|, r\} \leq |A| \leq |V|$. In addition, we usually consider the percolation in non-trivial cases when $2 \leq r < |V|$.*

At some time t during the activation process starting from A_0 on $G = (V, E)$, if all the vertices in a set $B \subseteq V$ are activated, then we say B is activated (or infected). For two sets $B, C \subseteq V$, if $\langle B \rangle_r \supseteq C$, then we say B activates (or infects, or spans) C ; whilst if $\langle B \cap C \rangle_r \supseteq C$, then we say B internally activates

(or infects, or spans) C . Note that B internally activates C implies B activates C because $\langle B \cap C \rangle_r \subseteq \langle B \rangle_r$, but the converse is not true.

Although percolation or activation process on the graph G is a dynamic configuration of states of the vertices, edges still play an important role for it. Recall that an inactivated vertex v can get activated at some time t if and only if it has at least r activated neighbours at time t . Alternatively, we can describe the activation of v as “these r activated neighbours help v get activated via the r edges between them and v ”. From this point of view, each edge of G can also be attached with one out of several states – *used*, *unused* and *wasted*. More specifically, during the r -threshold activation (or percolation) process starting from $A_0 \subseteq V$, each edge is attached with a state used or unused. At time 0, all edges are assigned unused. Then, at some time $t \geq 1$ if an inactivated vertex v has at least r activated neighbours, change the state of r edges between v and its r activated neighbours from unused to used. Note that once an edge becomes used, it will remain in this state forever. Finally, after the activation process ends, change the state of all unused edges to wasted. The most important difference between the vertex states and the edge states is – for a given graph G and a threshold r , the state of every vertex is merely determined by the set A_0 of seeds, while the state of each edge depends not only on A_0 but also on the choice of used edges at each time (in other words, the activation process itself). For instance, at time 2 of the activation process showed in Figure 1.1, a vertex in A_2 has 4 activated neighbours, but it only need help from 3 out of these 4 neighbours to get activated. Consequently, we

have 4 different choices of unused edges, which also causes different states of edges after the activation process ends. In spite of this fact, the total number of used edges and the number of wasted edges are not affected by the choice of used edges. This is because for each time t before the activation process ends, A_t always needs $r|A_t|$ edges to get activated, and so the total number of used edges is $\sum_{i=1}^{\infty} r|A_i| = r \sum_{i=1}^{\infty} |A_i|$. Consequently, the number of wasted edges is $|E| - r \sum_{i=1}^{\infty} |A_i|$.

The following lemmas show how the states of edges help estimate cardinality of percolating sets of graphs.

Lemma 2.2. *Let $G = (V, E)$ be a connected simple graph of order n , r be an integer with $2 \leq r \leq n - 1$, and A_0 be an r -threshold percolating set of G . Then, $|A_0| \geq n - \frac{|E|}{r}$.*

Proof. Suppose the percolation time of A_0 is T , and let u denote the total number of used edges in the percolation process starting from A_0 on G . Then, $\sum_{i=0}^T |A_i| = n$ and $u = r \sum_{i=1}^T |A_i|$. On the other hand, it is straightforward that $u \leq |E|$. Hence, we get

$$r(n - |A_0|) = r \sum_{i=1}^T |A_i| = u \leq |E|,$$

and so $|A_0| \geq n - \frac{|E|}{r}$. □

Lemma 2.3 (Riedl [33]). *Let $G = (V, E)$ be a connected simple graph of order n , r be an integer with $2 \leq r \leq n - 1$, $L = \{x \in V : \deg_G(x) < r\}$, A be a minimal r -threshold percolating set of G , and u be the total number of used edges of some r -threshold percolation process starting from A . Then, $u \geq |A| - |L|$ and $|A| \leq \frac{rn+|L|}{r+1}$.*

Proof. By Remark 2.1, $L \subseteq A$.

On one hand, since A is a minimal r -threshold percolating set of G , every seed $v \in A$ of degree at least r must be incident with at least 1 used edge because otherwise, v would contribute nothing to the activation of V and $A \setminus \{v\}$ could percolate on G which would contradict the minimality of A . On the other hand, every used edge is incident with at most 1 seed in A because any edge between seeds will definitely be wasted. Note that $u = r(n - |A|)$. Hence, $r(n - |A|) = u \geq |A| - |L|$, and the lemma follows. \square

2.2 A Degree-Based Bound for Minimum Percolating Sets

In [32], Reichman obtained an upper bound for the cardinality of minimum percolating set together with an algorithm for generating such a set for a given graph. This upper bound is only dependent on the degree of vertices and is valid for any graph.

Theorem 2.4 (Reichman [32]). *For a graph $G = (V, E)$, there is a r -threshold percolating set of cardinality at most $\sum_{v \in V} \min\{1, \frac{r}{\deg(v)+1}\}$. Furthermore, such a set can be found in deterministic polynomial time.*

Proof. We give two different proofs – a probabilistic one and an algorithmic one – for existence of the percolating set in the theorem.

First, we prove existence of an r -threshold percolating set of cardinality at most $\sum_{v \in V} \min\{1, \frac{r}{\deg(v)+1}\}$ by probabilistic method. Consider a uniformly random permutation σ over $V = \{v_1, \dots, v_n\}$. For each $i = 1, \dots, n$, let $L_i(\sigma)$ be set of all the vertices that appear on the i -th position among themselves and their neighbours, where the order of vertices is with respect to the string $\sigma(v_1), \dots, \sigma(v_n)$. For every $v \in V$, we have

$$\Pr[v \in L_i(\sigma)] = \begin{cases} \frac{1}{\deg(v)+1}, & i \leq \deg(v) + 1 \\ 0, & \text{otherwise} \end{cases}.$$

Since $L_1(\sigma), \dots, L_n(\sigma)$ are pairwise disjoint, then $\Pr[v \in \bigcup_{i=1}^r L_i(\sigma)] \leq \min\{1, \frac{r}{\deg(v)+1}\}$. Hence, by linearity of expectation, the expected cardinality of $\bigcup_{i=1}^r L_i(\sigma)$ is at most $\sum_{v \in V} \min\{1, \frac{r}{\deg(v)+1}\}$. Thus, there exists a permutation τ over $V = \{v_1, \dots, v_n\}$ with

$$\left| \bigcup_{i=1}^r L_i(\tau) \right| \leq \sum_{v \in V} \min\left\{1, \frac{r}{\deg(v) + 1}\right\}.$$

Now suppose the set $\bigcup_{i=1}^r L_i(\tau)$ is infected, and observe the infection process of $\bigcup_{i=1}^r L_i(\tau)$ in the order (from the left end to the right end) of the string $\tau(v_1), \dots, \tau(v_n)$. For $j \geq 1$, label the j -th vertex not in $\bigcup_{i=1}^r L_i(\tau)$ with y_j . Then, y_1 has at least r neighbours in front of it. But now, all the vertices in front of y_1 are infected because they are in $\bigcup_{i=1}^r L_i(\tau)$. So, y_1 will be infected. Next, assume y_1, \dots, y_s are infected. Then, since y_{s+1} has at least r neighbours in front of it and all the vertices in front of y_{s+1} are infected, then y_{s+1} will get infected. By induction, every vertex not in $\bigcup_{i=1}^r L_i(\tau)$ can get activated sometime. Therefore, $\bigcup_{i=1}^r L_i(\tau)$ percolates on G , which shows existence of an r -threshold percolating set of cardinality at most $\sum_{v \in V} \min\{1, \frac{r}{\deg(v)+1}\}$.

Second, we give a deterministic polynomial time algorithm for finding such a percolating set. Let the graph $G_0 = (V_0, E_0) = G$, and consider the following iterative procedure. For the given graph $G_i = (V_i, E_i)$, if $\{v \in V_i : \deg_{G_i}(v) \geq r\} \neq \emptyset$, then choose a vertex $x_i \in \{v \in V_i : \deg_{G_i}(v) \geq r\}$ of the smallest degree, and define a new graph $G_{i+1} = (V_{i+1}, E_{i+1}) = G_i - x_i$ (the graph derived from G_i by removing x_i and all the edges incident with x_i). Otherwise, if $\{v \in V_i : \deg_{G_i}(v) \geq r\} = \emptyset$, then the iteration procedure stops, set $T = i$, and return V_T as a set of seeds. This procedure is formally described in Algorithm 1 with pseudocodes. By the above description, for each $i = T-1, \dots, 0$, x_i can be infected by V_{i+1} because it has at least r neighbours in V_{i+1} . Thus, V_T is an r -threshold percolating set of G . It remains to show $|V_T| \leq \sum_{v \in V} \min\{1, \frac{r}{\deg(v)+1}\}$.

For each G_i , define $W(G_i) = \sum_{v \in V_i} \min\{1, \frac{r}{\deg_{G_i}(v)+1}\}$. Consider $W(G_i)$ and $W(G_{i+1})$. Let $\deg_{G_i}(x_i) = d$, and x_i 's neighbours with degree at least r in G_i be $\{u_1, \dots, u_l\}$. Then, since x_i has the smallest degree amongst all the vertices of degree at least r , we get

$$\begin{aligned} W(G_{i+1}) - W(G_i) &= -\frac{r}{d+1} + \sum_{i=1}^l \left(\frac{r}{\deg_{G_i}(u_i)} - \frac{r}{\deg_{G_{i+1}}(u_i) + 1} \right) \\ &= -\frac{r}{d+1} + \sum_{i=1}^l \frac{r}{\deg_{G_i}(u_i)(\deg_{G_i}(u_i) + 1)} \\ &\leq -\frac{r}{d+1} + \frac{dr}{d(d+1)} \\ &= 0. \end{aligned}$$

So, $W(G_{i+1}) \leq W(G_i)$, which implies

$$W(G_T) \leq W(G_{T-1}) \leq \dots \leq W(G_1) \leq W(G_0) = \sum_{v \in V} \min \left\{ 1, \frac{r}{\deg_G(v) + 1} \right\}.$$

On the other hand, since each vertex in G_T has degree smaller than r ,

$$|V_T| = \sum_{v \in V_T} \min \left\{ 1, \frac{r}{\deg_{G_T}(v) + 1} \right\} = W(G_T).$$

Therefore, V_T is just an r -threshold percolating set satisfying the upper bound in the theorem.

To estimate the time complexity of the above procedure, represent the graph by its adjacency matrix and use a hash table (integrated in the most of modern

programming languages) to record vertex-degree pairs. Since we remove a vertex from the graph G_i in each round of the outer loop, the procedure runs at most n rounds. Besides, in every round, we can determine the vertex x_i of the smallest degree amongst all the vertices of degree at least r in $O(n)$ time by searching the hash table, then update the adjacency matrix (alter the entries on the row and the column corresponding to x_i) and update the hash table (alter the degrees of the neighbours of x_i and finally remove the pair $[x_i, \deg(x_i)]$ from the table) in $O(n)$ time. Therefore, the time complexity of the whole procedure is at most $O(n^2)$. \square

We also give some explanations for Algorithm 1 as follows.

1. In each round of the loop between line 1 and line 20, we are looking for a vertex of the smallest degree amongst all the vertices of degree at least r in the current graph G , where d is not only a flag for determining whether there is a vertex of degree at least r but also a variable saving the smallest degree amongst all the vertices of degree at least r .
2. In the loop between line 4 and line 16, we are looking for a vertex of the smallest degree amongst all the vertices of degree at least r in the current graph G by going through $V(G)$ only once.
3. Between line 17 and line 19, if there exists vertices of degree at least r in the current graph G (indicated by $d > 0$), then we update the graph by removing the vertex of the smallest degree amongst them from G .

Algorithm 1: ComputeSmallPercSet

Input : A graph $G = (V, E)$, the threshold $r > 0$ **Output**: An r -threshold percolating set of G

```

1 repeat
2    $d \leftarrow 0$ ;
3    $x \leftarrow \text{null}$ ;
4   for  $v \in V(G)$  do
5     if  $\text{deg}(v) \geq r$  then
6       if  $d = 0$  then
7          $d \leftarrow \text{deg}(v)$ ;
8          $x \leftarrow v$ ;
9       else
10        if  $\text{deg}(v) < d$  then
11           $d \leftarrow \text{deg}(v)$ ;
12           $x \leftarrow v$ ;
13        end
14      end
15    end
16  end
17  if  $d > 0$  then
18     $G \leftarrow G - x$ ;
19  end
20 until  $d > 0$ ;
21 return  $V(G)$ ;

```

In this chapter, we introduced general facts and universal tools for bootstrap percolation on any graph. Then, in the following chapters, we shall discuss bootstrap percolation on some specific classes of graphs.

Chapter 3

Bootstrap Percolation on Some Special Graphs

In this section, we shall present some results about bootstrap percolation on some special classes of graphs including paths, cycles, and complete multipartite graphs for which we can determine the precise values for minimal or minimum percolating sets.

3.1 Bootstrap Percolation on Paths and Cycles

Since the largest degree of any path and any cycle is 2, by Remark 2.1, it suffices to consider 2-threshold percolation on paths and cycles of order n with $n \geq 3$.

Let P_n denote the path of n vertices, and C_n denote the cycle of n vertices. The next theorem was partially presented by Dreyer and Roberts [15], where

the result for the largest minimal percolating sets is new work presented in this thesis.

Theorem 3.1. *For every integer $n \geq 3$,*

$$m(P_n, 2) = \left\lceil \frac{n+1}{2} \right\rceil, \quad E(P_n, 2) = \left\lfloor \frac{2(n+1)}{3} \right\rfloor.$$

Proof. Let P_n be $v_1 - v_2 - \cdots - v_n$.

Since the two end vertices of P_n have degree smaller than 2, they have to be seeds. When $n = 3$, the unique minimal percolating set of P_3 is $\{v_1, v_3\}$, and so the theorem follows.

From now, suppose $n \geq 4$.

First, let us consider the minimum percolating sets of P_n . Let A be a minimum percolating set of P_n . Then, A has to contain v_1, v_n and at least 1 seed out of each two consecutive vertices from v_2 to v_{n-1} (otherwise, A would not percolate). Thus,

$$|A| \geq \left\lceil 2 + \frac{n-2}{2} \right\rceil \geq \left\lceil \frac{n+1}{2} \right\rceil.$$

Moreover, when n is even,

$$\{v_1\} \cup \left\{ v_{2i} : 1 \leq i \leq \frac{n}{2} \right\}$$

is a minimal percolating set of cardinality $\left\lceil \frac{n+1}{2} \right\rceil$; when n is odd,

$$\left\{ v_{2i-1} : 1 \leq i \leq \frac{n+1}{2} \right\}$$

is a minimal percolating set of cardinality $\lceil \frac{n+1}{2} \rceil$. Therefore, $m(P_n, 2) = \lceil \frac{n+1}{2} \rceil$.

Next, consider the largest minimal percolating sets of P_n . Again, let A be a minimal percolating set of P_n . Then, A must contain v_1, v_n and at most 2 out of every three consecutive vertices from v_2 to v_{n-1} (otherwise, the middle one would be wasted). Thus,

$$|A| \leq \left\lfloor 2 + \frac{2}{3}(n-2) \right\rfloor = \left\lfloor \frac{2(n+1)}{3} \right\rfloor.$$

Moreover, when $n \equiv 0 \pmod{3}$,

$$\{v_i : 1 \leq i \leq n, i \equiv 0 \text{ or } 1 \pmod{3}\}$$

is a minimal percolating set of cardinality $\lfloor \frac{2(n+1)}{3} \rfloor$; when $n \equiv 1 \pmod{3}$,

$$\{v_i : 1 \leq i \leq n-1, i \equiv 1 \text{ or } 2 \pmod{3}\} \cup \{v_n\}$$

is a minimal percolating set of cardinality $\lfloor \frac{2(n+1)}{3} \rfloor$; when $n \equiv 2 \pmod{3}$,

$$\{v_i : 1 \leq i \leq n, i \equiv 1 \text{ or } 2 \pmod{3}\}$$

is a minimal percolating set of cardinality $\lfloor \frac{2(n+1)}{3} \rfloor$. Therefore, $E(P_n, 2) =$

$$\left\lfloor \frac{2(n+1)}{3} \right\rfloor. \quad \square$$

The next theorem was partially mentioned by Dreyer and Roberts [15] too, where the result for the largest minimal percolating sets is new work presented in this thesis.

Theorem 3.2. *For every integer $n \geq 3$,*

$$m(C_n, 2) = \left\lceil \frac{n}{2} \right\rceil, \quad E(C_n, 2) = \left\lfloor \frac{2n}{3} \right\rfloor.$$

Proof. Let C_n be $v_0 - v_1 - \cdots - v_{n-1} - v_0$.

It is trivial that any 2-threshold minimal percolating set of C_3 consists of two adjacent vertices, which implies the theorem. Similarly, any 2-threshold minimal percolating set of C_4 consists of two vertices whose distance is 2, which implies the theorem too.

Suppose $n > 4$ from now.

First, consider minimum percolating sets of C_n . Without loss of generality, let A be a 2-threshold minimum percolating set containing v_0 of C_n . Since A has to contain at least 1 seed in each two consecutive vertices from v_1 to v_{n-1} via v_1 (otherwise, A would not percolate), we get

$$|A| \geq 1 + \left\lceil \frac{n-1}{2} \right\rceil \geq \left\lceil \frac{n}{2} \right\rceil.$$

Moreover, when n is even,

$$\{v_i : 0 \leq i \leq n-1, i \equiv 0 \pmod{2}\}$$

is a minimal percolating set of cardinality $\left\lceil \frac{n}{2} \right\rceil$; when n is odd,

$$\{v_i : 0 \leq i \leq n-1, i \equiv 0 \pmod{2}\}$$

is a minimal percolating set of cardinality $\lfloor \frac{n}{2} \rfloor$. Therefore, $m(C_n, 2) = \lfloor \frac{n}{2} \rfloor$.

Next, consider the largest minimal percolating sets of C_n . Without loss of generality, let A be a minimal 2-threshold percolating set containing v_0 of C_n . Since A must contain at most 2 out of every 3 consecutive vertices from v_1 to v_{n-1} via v_1 (otherwise, the middle one would be wasted and A would not be minimal), we get

$$|A| \leq 1 + \left\lfloor \frac{2}{3}(n-1) \right\rfloor \leq \left\lfloor \frac{2n}{3} \right\rfloor.$$

Moreover, when $n \equiv 0 \pmod{3}$,

$$\{v_i : 0 \leq i \leq n-1, i \equiv 0 \text{ or } 1 \pmod{3}\}$$

is a minimal percolating set of cardinality $\lfloor \frac{2n}{3} \rfloor$; when $n \equiv 1 \pmod{3}$,

$$\{v_i : 0 \leq i \leq n-4, i \equiv 0 \text{ or } 1 \pmod{3}\} \cup \{v_{n-2}\}$$

is a minimal percolating set of cardinality $\lfloor \frac{2n}{3} \rfloor$; when $n \equiv 2 \pmod{3}$,

$$\{v_i : 0 \leq i \leq n-3, i \equiv 0 \text{ or } 1 \pmod{3}\} \cup \{v_{n-2}\}$$

is a minimal percolating set of cardinality $\lfloor \frac{2n}{3} \rfloor$. Therefore, $E(C_n, 2) = \lfloor \frac{2n}{3} \rfloor$.

□

3.2 Bootstrap Percolation on Complete Multipartite Graphs

In this section, we shall study another family of graphs – complete multipartite graphs – for which we can determine the precise values about minimum or minimal percolating sets. As described in Section 2.1, it suffices to consider r -threshold percolation on multipartite graphs of order n with $2 \leq r \leq n - 1$.

Let us start from an easier case: bootstrap percolation on complete bipartite graphs. The next theorem partly comes from Dreyer and Roberts [15], where the result for largest minimal percolating sets is new work presented in this thesis.

Theorem 3.3. *Let $K_{m,n} = (X \cup Y, E)$ be the complete bipartite graph with $m \geq n$, and r be an integer with $2 \leq r < m + n$. Then,*

$$\begin{cases} m(K_{m,n}, r) = E(K_{m,n}, r) = r, & \text{if } r \leq n; \\ m(K_{m,n}, r) = E(K_{m,n}, r) = m, & \text{if } n < r \leq m; \\ m(K_{m,n}, r) = E(K_{m,n}, r) = m + n, & \text{if } m < r. \end{cases}$$

Proof. First, when $r \leq n$, since every r -threshold minimum percolating set contains at least r seeds and every r -subset of Y can percolate in $K_{m,n}$, every r -subset of X or Y is a minimum percolating set and $m(K_{m,n}, r) = r$. Let B be a largest r -threshold minimal percolating set of $K_{m,n}$. Then, either $|B \cap X| \geq r$ or $|B \cap Y| \geq r$ because otherwise B could not infect any vertex in $(X \cup Y) \setminus B$.

Without loss of generality, suppose $|B \cap X| \geq r$. Then, due to minimality of B , $|B \cap X| = r$ and $B \cap Y = \emptyset$. Thus, $E(K_{m,n}, r) = r$ too.

Next, when $n < r \leq m$, since every vertex in X has degree smaller than r , by Remark 2.1, any percolating set has to include X . Moreover, since every vertex in Y has degree $m \geq r$, the unique r -threshold minimal percolating set of $K_{m,n}$ is X . Thus, $m(K_{m,n}, r) = E(K_{m,n}, r) = m$.

Finally, when $m < r$, since every vertex of $K_{m,n}$ has degree smaller than r , by Remark 2.1, the unique r -threshold percolating set of $K_{m,n}$ is the vertex set $X \cup Y$. Thus, $m(K_{m,n}, r) = E(K_{m,n}, r) = m + n$. \square

Next, consider bootstrap percolation on complete multipartite graphs. For a complete m -partite graph $G = (\bigcup_{i=1}^m V_i, E) = K_{p_1, p_2, \dots, p_m}$ with $p_1 \geq p_2 \geq \dots \geq p_m$, an integer k and a subset $J \subseteq [m] = \{1, 2, \dots, m\}$, if $|\bigcup_{j \in J} V_j| \geq k$ and $|\bigcup_{j \in J \setminus \{\max(J)\}} V_j| < k$, then we say *the cardinality of the union $\bigcup_{j \in J} V_j$ of partite sets is just above k* . For instance, if $m > 5$, $|V_1 \cup V_3 \cup V_5| \geq k$ and $|V_1 \cup V_3| < k$, then we say the cardinality of the union $V_1 \cup V_3 \cup V_5$ is just above k .

Dreyer and Roberts [15] obtained the next exact result about minimum percolating sets of complete multipartite graphs. Here, our proof gives substantially more details than the original one does.

Theorem 3.4 (Dreyer and Roberts [15]). *Let $G = (\bigcup_{i=1}^m V_i, E)$ be the complete m -partite graph K_{p_1, p_2, \dots, p_m} with $p_1 \geq p_2 \geq \dots \geq p_m$, $n = \sum_{i=1}^m p_i$ and $2 \leq r <$*

n. If there is a union $V_J = \bigcup_{j \in J} V_j$ of partite sets of cardinality just above r satisfying $n - |V_{\max(J)}| \geq r$, then $m(G, r) = r$; otherwise,

$$m(G, r) = \min \left\{ \left| \bigcup_{j \in I} V_j \right| : I \subseteq [m], \left| \bigcup_{j \in I} V_j \right| \text{ is just above } r \right\}.$$

Proof. Since $r < n$, there always exists a union $\bigcup_{j \in I} V_j$ of partite sets of cardinality just above r .

First, suppose there exists a union $V_J = \bigcup_{j \in J} V_j$ of cardinality just above r satisfying $n - |V_{\max(J)}| \geq r$. Let W be any subset of $V_{\max(J)}$ with $|W| = r - |\bigcup_{j \in J \setminus \{\max(J)\}} V_j|$ (such a set W exists because $|\bigcup_{j \in J} V_j|$ is just above r), and $A = (\bigcup_{j \in J \setminus \{\max(J)\}} V_j) \cup W$. Then, $A \subseteq V_J = \bigcup_{j \in J} V_j$ and $|A| = r$. Suppose A is initially infected. In the first step of the infection starting from A , since every vertex (if there are still some) in the complement of $V_J = \bigcup_{j \in J} V_j$ has r infected neighbours in A , it will get infected. Then, in the second step of the infection, since $n - |V_{\max(J)}| \geq r$, every vertex in $V_{\max(J)} \setminus W$ has at least r infected neighbours in the complement of $V_{\max(J)}$, and so it will get infected. Thus, A is a r -threshold percolating set of cardinality r . On the other hand, any r -threshold percolating set of G has to contain at least r seeds. So, A is exactly a minimum r -threshold percolating set of G , and $m(G, k) = r$.

Otherwise, suppose every union $V_I = \bigcup_{j \in I} V_j$ of cardinality just above r satisfies $n - |V_{\max(I)}| < r$. Let \mathcal{U} be the set of all the unions $V_I = \bigcup_{j \in I} V_j$ of

cardinality just above r , and let $b = \min\{|V_I| : V_I \in \mathcal{U}\}$. We shall show a correspondence between \mathcal{U} and the minimum r -threshold percolating sets of G . Let $V_J = \bigcup_{j \in J} V_j$ be any union in \mathcal{U} . Then, V_J can infect all the vertices (if there are still some) in the complement of it in one step because every vertex in the complement of V_J has $|V_J| \geq r$ infected neighbours, that is, all the vertices in V_J . Hence, V_J is an r -threshold percolating set of G , and so $m(G, r) \leq b$.

On the other hand, we are going to show any minimum r -threshold percolating set A of G has the same cardinality as some union in \mathcal{U} . Let $\bigcup_{j \in I} V_j$ be the smallest union containing A . For each $j \in I$, define $A_j = V_j \cap A$. If there is at least one $j \in I$ such that $\emptyset \subsetneq A_j \subsetneq V_j$, let $H = \{i_1, \dots, i_h\} = \{j \in I : A_j \subsetneq V_j\}$ with $|A_{i_1}| \geq \dots \geq |A_{i_h}|$. Since $|A| \geq r$, all the vertices in the complement of $\bigcup_{j \in H} V_j$ will get infected in the first step of the percolation process starting from A . Then, after the first step of the percolation process, for each $k = 1, \dots, h$, every clean vertex in V_{i_k} has degree $|\bigcup_{j \notin I} V_j| + |\bigcup_{j=1}^h A_{i_j}| - |A_{i_k}|$. Since $|A_{i_1}| \geq \dots \geq |A_{i_h}|$, every clean vertex in V_{i_h} has a most number of infected neighbours amongst clean vertices in $\bigcup_{j \in H} V_j$. Thus, the clean vertices in V_{i_h} will get infected no later than other clean vertices in $\bigcup_{j \in H} V_j$. Hence, each vertex in $V_{i_h} \setminus A_{i_h}$ has at least r infected neighbours after the first step of the percolation process (note that all these infected neighbours lie in partite sets other than V_{i_h} because every partite set is an independent set). Now, exchange the seeds in A_{i_h} with a same number of clean vertices in $V_{i_1} \cup \dots \cup V_{i_{h-1}}$ (but

keep the cardinality of each partite set V_i) to make $V_{i_1} \cup \dots \cup V_{i_{h-1}}$ contain as many seeds as possible (illustrated in Figure 3.1). Then, for each $j = 1, \dots, h$, update A_{i_j} to the set A'_{i_j} of seeds in V_{i_j} , and so we get a new set $A' = \bigcup_{j=1}^h A'_{i_j}$ of seeds. It is trivial that $|A'| = |A|$. If $h > 1$, then we conduct the following operations depending on two different cases.

1. If $|A_{i_h}| > \left| \bigcup_{j=1}^{h-1} (V_{i_j} \setminus A_{i_j}) \right|$, then for each $j = 1, \dots, h-1$, $A'_{i_j} = V_{i_j}$ and V_{i_h} will still remain some seeds. Moreover, A' will infect all the vertices in the complement of $\bigcup_{j \in H} V_j$ in the first step of infection process starting from A' because A percolates on G and $|A'| = |A|$. Note that after the first step of infection process, the clean vertices are just those clean vertices in $V_{i_h} \setminus A'_{i_h}$. Then, since we exchanged the seeds in A_{i_h} with the clean vertices in $V_{i_1} \cup \dots \cup V_{i_{h-1}}$, every vertex in $V_{i_h} \setminus A'_{i_h}$ has more infected neighbours than the vertices in $V_{i_h} \setminus A_{i_h}$ do, and so it will get infected in the second step of the infection process. Thus, A' is also a minimum percolating set such that there is a unique $i \in [m]$ satisfying $0 < |V_i \cap A'| < |V_i|$.
2. Otherwise, if $|A_{i_h}| \leq \left| \bigcup_{j=1}^{h-1} (V_{i_j} \setminus A_{i_j}) \right|$, then $V_{i_h} \cap A' = \emptyset$. Since A percolates on G and $|A'| = |A|$, all the vertices in the complement of $\bigcup_{j=1}^{h-1} V_{i_j}$ will be infected after the first step of the infection process starting from A' . Note that after the first step of infection process, the clean

vertices are just those clean vertices in $\bigcup_{j=1}^{h-1} (V_{i_j} \setminus A'_{i_j})$. Then, since we exchanged the seeds in A_{i_h} with the clean vertices in $V_{i_1} \cup \dots \cup V_{i_{h-1}}$, for each $j = 1, \dots, h-1$, every vertex in $V_{i_j} \setminus A'_{i_j}$ has more infected neighbours than the vertices in $V_{i_j} \setminus A_{i_j}$ do, and so it will get infected at some time. Thus, A' is also a minimum r -threshold percolating set such that there are at most $h-1$ partite sets V_j 's of G satisfying $\emptyset \subsetneq A' \cap V_j \subsetneq V_j$. If $h-1 > 1$, then repeat the same operations on A' as we did on A .

Thus, in all cases, we can always get a minimum r -threshold percolating set A such that there is at most one partite set V_i of G satisfying $0 < |V_i \cap A| < |V_i|$.

For such a minimum r -threshold percolating set A , without loss of generality, suppose $V_I = \bigcup_{j \in I} V_j$ be the smallest union containing all the elements of A with $I = \{1, \dots, s\}$. If for every $j \in I$, $A \cap V_j = V_j$, then A is a union of some partite sets of G . Since A is a minimum percolating set, $|\bigcup_{j \in I} V_j| = |A|$ is just above r , and so $A \in \mathcal{U}$, which implies $m(G, r) \geq b$. Otherwise, suppose there is some $i \in I$ such that $\emptyset \subsetneq V_i \cap A \subsetneq V_i$. Since $|A| \geq r$, all the vertices in the complement of V_i will get infected in the first step of the percolation process starting from A . If $i < s$, then exchange the clean vertices in V_i with a same number of seeds in $V_s, V_{s-1}, \dots, V_{i+1}$ in turn (but not change cardinality of each V_j) to let V_i contain as many seeds as possible (illustrated in Figure 3.2). Then, we will get a new set A' of seeds with $|A'| = |A|$. To be precise, we distinguish two cases.

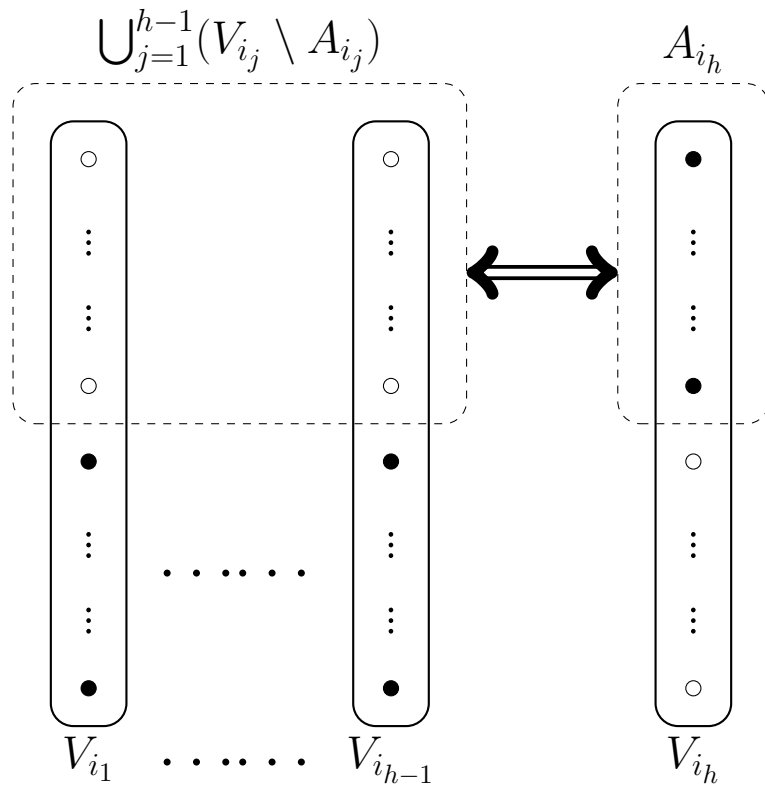


Figure 3.1: Exchange of seeds in A_{i_h} and clean vertices in $V_{i_1} \cup \dots \cup V_{i_{h-1}}$

1. If $|V_s| > |V_i \setminus A|$, exchange all the clean vertices in $V_i \setminus A$ with a same number of seeds in V_s , then we will get a new set A' of seeds such that $|A'| = |A|$, the smallest union of partite sets containing all the elements of A' is also $\bigcup_{j=1}^s V_j$, and V_s is the only partite set of G satisfying $\emptyset \subsetneq A' \cap V_s \subsetneq V_s$. Now, since $|A'| = |A| \geq r$, A' will infect all the vertices in the complement of $\bigcup_{j=1}^s V_j$ in the first step of infection process starting from A' . Note that after the first step of infection process, the clean vertices (if there are still some) are merely those clean vertices in V_s . Then, since A percolates on G , every vertex in $V_s \setminus A'$ has degree $n - |V_s| \geq n - |V_i| \geq r$, and so these clean vertices will get infected in the second step of infection process. Thus, A' is a minimum r -threshold percolating set of G such that in the smallest union $\bigcup_{j \in I} V_j$ of partite sets containing all the vertices of A' , every partite set V_j other than $V_{\max(I)}$ satisfies $A' \cap V_j = V_j$.

2. If $|V_s| \leq |V_i \setminus A|$, exchange all seeds in V_s with a same number of the clean vertices in $V_i \setminus A$, then we will get a new set A' of seeds such that $|A'| = |A|$, the smallest union of partite sets containing all the elements of A' is $\bigcup_{j=1}^{s-1} V_j$, and every partite set V_j of G other than V_i satisfies either $A' \cap V_j = V_j$ or $A' \cap V_j = \emptyset$. Now, since $|A'| = |A| \geq r$, A' will infect all the vertices in the complement of $\bigcup_{j=1}^{s-1} V_j$ in the first step of infection process starting from A' . Note that after the first step of infection process,

the clean vertices (if there are still some) are merely those clean vertices in V_i . Then, since A percolates on G , every vertex in $V_i \setminus A'$ has degree $n - |V_i| \geq r$, and so these clean vertices will get infected in the second step of infection process. Thus, A' is a minimum r -threshold percolating set of G too. If $i = s - 1$, then A' is a minimum r -threshold percolating set of G such that in the smallest union $\bigcup_{j \in I} V_j$ of partite sets containing all the vertices of A' , every partite set V_j other than $V_{\max(I)}$ satisfies $A' \cap V_j = V_j$. Otherwise, if $i < s - 1$ and $A' \cap V_i \subsetneq V_i$, then repeat the same operation on A' as we did on A .

So, in all cases, we can get a minimum r -threshold percolating set A of G such that in the smallest union $\bigcup_{j \in I} V_j$ of partite sets containing all the vertices of A , every partite set V_j other than $V_{\max(I)}$ satisfies $A \cap V_j = V_j$.

At this point, we are facing the following two cases.

1. $V_{\max(I)} \cap A = V_{\max(I)}$. In this case, $A = \bigcup_{j \in I} V_j$. Since A is a minimum r -threshold percolating set, $|\bigcup_{j \in I} V_j| = |A|$ is just above r , and so $A \in \mathcal{U}$, which implies $m(G, r) \geq b$.
2. $\emptyset \subsetneq V_{\max(I)} \cap A \subsetneq V_{\max(I)}$. In this case, since A is a minimum percolating set of G , $|\bigcup_{j \in I} V_j| > |A| \geq r$ and $|\bigcup_{j \in I \setminus \{\max(I)\}} V_j| < r$. In other words, $|\bigcup_{j \in I} V_j|$ is just above r . Then, by our conjecture, $n - |V_{\max(I)}| < r$. Now, let us observe the infection process starting from A . Since $|A| \geq r$, all the

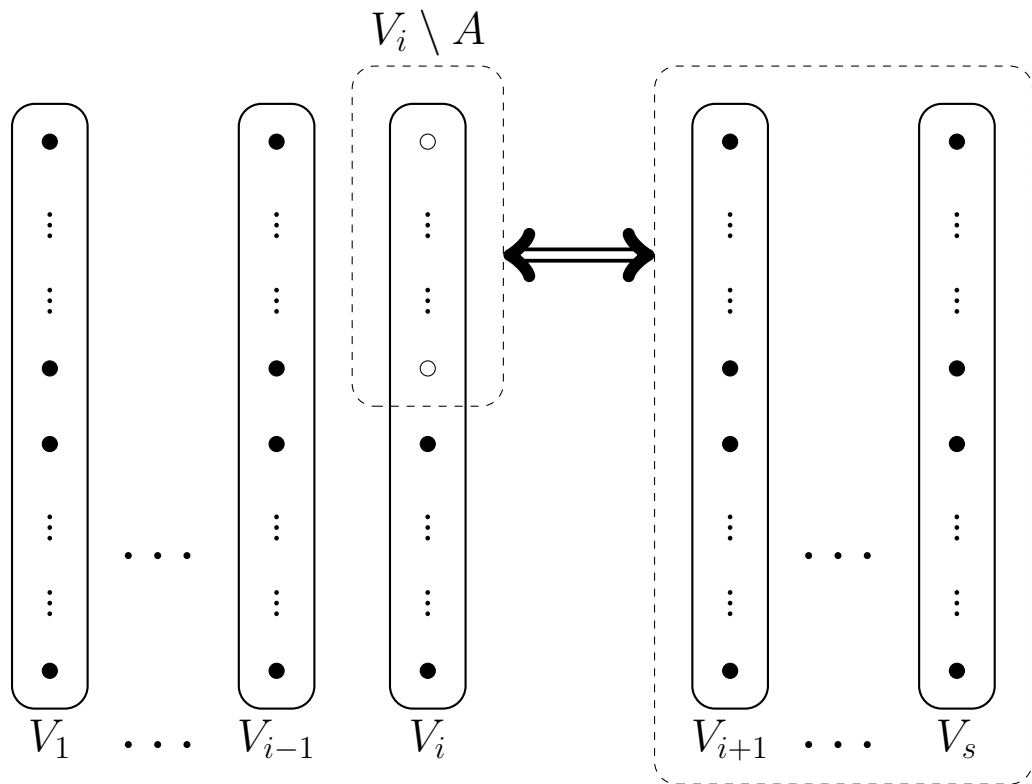


Figure 3.2: Exchange of clean vertices in V_i and seeds in $V_s, V_{s-1}, \dots, V_{i+1}$ in turn

vertices except those in $V_{\max(I)} \setminus A$ will be infected after the first step of the infection process. Then, unfortunately, the infection process cannot proceed because every vertex in $V_{\max(I)} \setminus A$ has degree $n - |V_{\max(I)}| < r$, which contradicts the fact that A percolates on G . Thus, this case will never happen.

Therefore, we can always establish $m(G, r) = b$, and the theorem follows. \square

In this chapter, we showed the precise values for minimum and largest minimal percolating sets of paths, cycles and complete multipartite graphs. However, it seems quite hard to work out the precise values for minimum or minimal percolating sets of most classes of graphs. Hence, we shall mainly focus on the bounds for minimum or minimal percolating sets (but they are still proper reflection of the key word “extremal problems” of the thesis title) in the following chapters. Besides, we shall also indicate that these bounds are tight for some classes of graphs.

Chapter 4

Bootstrap Percolation on Regular Graphs

This chapter is dedicated to bootstrap percolation on regular graphs. First, we shall present some results about the bounds for minimum percolating sets of regular graphs. Then, we are going to show some regular graphs having “optimum” minimum percolating set. Finally, we will focus on percolation process on a special class of regular graphs – expander graphs.

When it comes to percolating sets of regular graphs, it suffices to consider r -threshold percolation on d -regular connected graphs of order n with $2 \leq r \leq d < n$ because other cases are determined trivially as described in Section 2.1.

4.1 Bounds for Minimum Percolating Sets of Regular Connected Graphs

The main results of this section are bounds for minimum percolating sets of regular graphs. The lower bound for minimum percolating sets is obtained with the technique of counting used/wasted edges of the percolation process that was used in some research papers such as Riedl [33], whilst the upper bound for minimum percolating sets comes from Reichman's technique of recursively removing vertices in [32]. The bounds in the next theorem are the best known in the literature for arbitrarily regular graphs.

Theorem 4.1. *Let $G = (V, E)$ be a d -regular graph of order n , and r be an integer with $2 \leq r \leq d < n$. Then,*

$$\max \left\{ \left\lceil n \left(1 - \frac{d}{2r} \right) + \frac{d}{r} - 1 \right\rceil, r \right\} \leq m(G, r) \leq \left\lfloor \frac{nr}{d+1} \right\rfloor.$$

Proof. The upper bound directly comes from Theorem 2.4 with $d_1 = \dots = d_n = d$.

To obtain the lower bound, let A_0 be an arbitrary r -threshold percolating set of G . Then, $|A_0| \geq r$ because otherwise no clean vertices can be infected in the first step. To show $|A_0| \geq n \left(1 - \frac{d}{2r} \right) + \frac{d}{r} - 1$, let T be the percolation time of A_0 on G , for each $i = 1, \dots, T$, let A_i be the set of vertices infected at time i , and let U be the total number of used edges in the percolation process

starting from A_0 on G . For each $i = 1, \dots, T$, since the vertices in A_i needs exactly $r|A_i|$ edges to get infected, $U = \sum_{i=1}^T (r|A_i|) = r \sum_{i=1}^T |A_i|$. Looking at the last step of the percolation, since every vertex in A_T only needs r edges to get infected, at least $d - r$ wasted edges are incident with some vertex in A_T . Therefore, $U \leq |E| - (d - r)$. Besides, we already have $2|E| = nd$ and $\sum_{i=0}^T |A_i| = n$. Hence,

$$r(n - |A_0|) = r \sum_{i=1}^T |A_i| = U \leq |E| - (d - r) = \frac{nd}{2} - d + r,$$

and so $|A_0| \geq n \left(1 - \frac{d}{2r}\right) + \frac{d}{r} - 1$. Therefore, the theorem follows. \square

The following fact will be used in Section 4.2 when analyzing whether the lower bound in Theorem 4.1 is the best one we can achieve.

Remark 4.2. *Note that the bound $n \left(1 - \frac{d}{2r}\right) + \frac{d}{r} - 1$ can be rewritten as $\left(1 - \frac{n}{2}\right)\frac{d}{r} + n - 1$. Since $2 \leq r \leq d \leq n - 1$, then $1 - \frac{n}{2} < 0$, and so $n \left(1 - \frac{d}{2r}\right) + \frac{d}{r} - 1$ is decreasing when d is increasing. Consequently, if $d \geq 2r$, then*

$$n \left(1 - \frac{d}{2r}\right) + \frac{d}{r} - 1 \leq n \left(1 - \frac{2r}{2r}\right) + \frac{2r}{r} - 1 = 1 < r$$

and so the lower bound in Theorem 4.1 takes the value r .

4.2 Regular Connected Graphs with Small Minimum Percolating Sets

In this section, our main purpose is to show the lower bound in Theorem 4.1 is the best one we can achieve in some cases. All the results in this section are new work presented in this thesis.

First, we describe some constructions of d -regular connected simple graphs of order n (illustrated in Figure 4.1) which are useful for the following theorems.

1. **Construction $\mathbf{G}_{rc}^1(\mathbf{n}, \mathbf{d})$** for the case when n is even and $d \leq \frac{n}{2}$. In this case, we construct a graph $G = (V, E)$ as follows. Let $V = X \cup Y$ with $X \cap Y = \emptyset$, $m = |X| = |Y| = \frac{n}{2}$. Let $X = \{x_0, \dots, x_{m-1}\}$, $Y = \{y_0, \dots, y_{m-1}\}$, both X and Y be independent sets of G . For each $i = 0, \dots, m-1$, define $N_G(x_i) = \{y_{i+j \pmod{m}} : 0 \leq j \leq d-1\}$.
2. **Construction $\mathbf{G}_{rc}^2(\mathbf{n}, \mathbf{d})$** for the case when n is even and $d > \frac{n}{2}$. In this case, we construct a graph $G = (V, E)$ as follows. Let $V = X \cup Y$ with $X \cap Y = \emptyset$, $m = |X| = |Y| = \frac{n}{2}$. Let $X = \{x_0, \dots, x_{m-1}\}$, $Y = \{y_0, \dots, y_{m-1}\}$, both X and Y be cliques of G . For each $i = 0, \dots, m-1$, define $N_G(x_i) \cap Y = \{y_{i+j \pmod{m}} : 0 \leq j \leq d-m\}$.
3. **Construction $\mathbf{G}_{rc}^3(\mathbf{n}, \mathbf{d})$** for the case when n is odd and $d \leq \frac{n-1}{2}$. In this case, d has to be even because dn is even and n is odd. Let $n = 2m + 1$.

We construct a graph $G = (V, E)$ as follows. Let $V = X \cup Y \cup \{z\}$ with $X, Y, \{z\}$ are pairwise disjoint, and $|X| = |Y| = m$. Let $X = \{x_0, \dots, x_{m-1}\}$, $Y = \{y_0, \dots, y_{m-1}\}$, both X and Y be independent sets of G . To set the edges of G , first, for each $i = 0, \dots, m-1$, set $N_G(x_i) = \{y_{i+j \pmod{m}} : 0 \leq j \leq d-1\}$. Hereby, we get a graph whose vertices in $X \cup Y$ have degree d along with a single isolated vertex z . Then, for every $i = 0, \dots, \frac{d}{2} - 1$, delete the edges $x_i - y_{i+d-1 \pmod{m}}$ and join each of the end vertices of these deleted edges with z to get G . Note that this operation does not change the degree of the vertices in $X \cup Y$ but gives z degree d .

4. **Construction $\mathbf{G}_{rc}^4(\mathbf{n}, \mathbf{d})$** for the case when n is odd and $d > \frac{n-1}{2}$. In this case, again, d has to be even because dn is even and n is odd. Let $n = 2m + 1$. We construct a graph $G = (V, E)$ as follows. Let $V = X \cup Y \cup \{z\}$ with $X, Y, \{z\}$ are pairwise disjoint, and $|X| = |Y| = m$. Let $X = \{x_0, \dots, x_{m-1}\}$, $Y = \{y_0, \dots, y_{m-1}\}$, both X and Y be cliques of G . To obtain the edges of G , for each $i = 0, \dots, m-1$, set $N_G(x_i) \cap Y = \{y_{i+j \pmod{m}} : 0 \leq j \leq d-m\}$. Hereby, we get a graph whose vertices in $X \cup Y$ have degree d along with a single isolated vertex z . Then, for every $i = 0, \dots, \frac{d}{2} - 1$, delete the edge $x_i - y_{i+d-m \pmod{m}}$ (we can always do this because $\frac{d}{2} \leq m$) and join each of the end vertices of these deleted edges with z to get G . Note that this operation does not

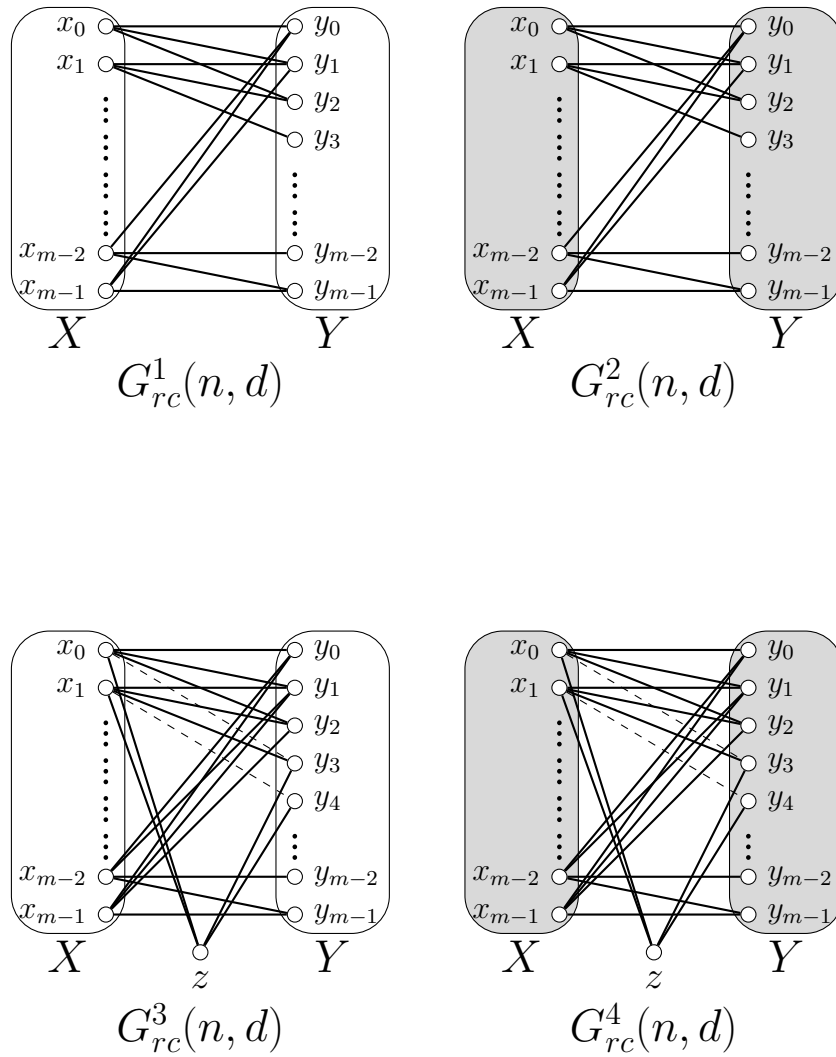


Figure 4.1: Constructions of d -regular connected graphs of order n with $d < n$. $G_{rc}^4(n, d)$ is illustrated with $n = 9$ and $d = 6$

change the degree of the vertices in $X \cup Y$ but gives z degree d .

It is straightforward to check each of these constructions is d -regular and connected.

Next, we shall show the lower bound in Theorem 4.1 is tight in some cases. Let us start from the case when $d = r$.

Theorem 4.3. *Let $G = (V, E)$ be a d -regular connected simple graph of order n with $2 \leq d < n$. Then,*

$$m(G, d) \geq \max \left\{ \left\lceil \frac{n}{2} \right\rceil, d \right\}.$$

Moreover, for any positive integer n, d with $2 \leq d \leq n - 1$ and $2 \mid nd$, there always exists d -regular connected simple graphs of order n for which the lower bound is tight.

Proof. This lower bound comes from Theorem 4.1 with $r = d$. Hence, it remains to show some d -regular connected simple graphs of order n reaching this bound.

Since the complete graph K_n is an $(n - 1)$ -regular connected simple graph of order n and

$$m(K_n, n - 1) = n - 1 = \max \left\{ \left\lceil \frac{n}{2} \right\rceil, n - 1 \right\},$$

the theorem is true when $d = n - 1$.

Now, suppose that $2 \leq d \leq n - 2$, so that $n \geq 4$. If $d = 2$, then $\max \left\{ \left\lceil \frac{n}{2} \right\rceil, d \right\} = \left\lceil \frac{n}{2} \right\rceil$. Since the cycle C_n of n vertices is a 2-regular connected simple graphs of order n and $m(C_n, 2) = \left\lceil \frac{n}{2} \right\rceil$, the theorem follows in this case.

From now, suppose that $3 \leq d \leq n - 2$, so that $n \geq 5$. We distinguish several cases. First, consider the cases when n is even.

Case 1. Suppose that $n \geq 5$ is even and $d \leq \lceil \frac{n}{2} \rceil = \frac{n}{2}$. In this case, $m(G, d) \geq \max \{ \lceil \frac{n}{2} \rceil, d \} = \frac{n}{2}$. We construct a graph $G = (V, E)$ with Construction $G_{rc}^1(n, d)$ in Figure 4.1, then G is a d -regular connected graph of order n . Now, suppose X is initially infected. Then, since every vertex in Y has d neighbours in X , it can be infected in the first step of the infection process starting from X , which implies X is a d -threshold percolating set of cardinality $\frac{n}{2}$. Therefore, the lower bound is tight in this case.

Case 2. Suppose that $n \geq 5$ is even and $\lceil \frac{n}{2} \rceil = \frac{n}{2} < d \leq n - 2$. In this case, $m(G, d) \geq \max \{ \lceil \frac{n}{2} \rceil, d \} = d$. We construct a graph $G = (V, E)$ as follows. Let $V = X \cup Y$ with $X \cap Y = \emptyset$, $|X| = d$, $|Y| = n - d$. Let $X = \{x_1, \dots, x_d\}$ and $Y = \{y_1, \dots, y_{n-d}\}$. Let Y be an independent set of G whilst the induced subgraph $G[X]$ be a $(2d - n)$ -regular connected simple graph of order d (it can pick one of the constructions $G_{rc}^i(d, 2d - n)$ with $i \in \{1, 2, 3, 4\}$ in Figure 4.1, which depends on the values of d and $(2d - n)$), and each vertex $x_i \in X$ be adjacent to all the vertices in Y . For such a construction, each vertex $x_i \in X$ has degree $(2d - n) + (n - d) = d$, and each vertex $y_j \in Y$ has degree d too. Thus, G is d -regular. In addition, since each vertex $x_i \in X$ is adjacent to all the vertices in Y , G is connected. Now, suppose X is initially infected. Since every vertex in Y has d neighbours in X , it can be infected in the first step of the

infection process starting from X , which implies X is a d -threshold percolating set of cardinality d . Therefore, the lower bound is tight in this case.

Next, consider the cases when n is odd.

Case 3. Suppose that $n \geq 5$ is odd, $3 \leq d \leq n - 2$ and $d \leq \frac{n-1}{2}$. In this case, $m(G, d) \geq \max \left\{ \left\lceil \frac{n}{2} \right\rceil, d \right\} = \frac{n+1}{2}$, and d has to be even because $dn = 2|E|$ is even and n is odd. Let $n = 2m + 1$. We construct a graph $G = (V, E)$ with Construction $G_{rc}^3(n, d)$ in Figure 4.1, then G is a d -regular connected graph of order n . Now, suppose $X \cup \{z\}$ is initially infected. Then, since every vertex in Y has d neighbours in $X \cup \{z\}$, it can be infected in the first step of the infection process starting from $X \cup \{z\}$, which implies $X \cup \{z\}$ is a d -threshold percolating set of cardinality $\frac{n+1}{2}$. Therefore, the lower bound is tight in this case.

Case 4. Suppose that $n \geq 5$ is odd, $3 \leq d \leq n - 2$ and $d = \frac{n+1}{2}$. In this case, $m(G, d) \geq \max \left\{ \left\lceil \frac{n}{2} \right\rceil, d \right\} = d$, and d has to be even because dn is even and n is odd. We construct a graph $G = (V, E)$ as follows. Let $V = X \cup Y$ with $X \cap Y = \emptyset$, $|X| = d$, $|Y| = n - d = d - 1$. Let $X = \{x_1, \dots, x_d\}$, $Y = \{y_1, \dots, y_{n-d}\}$. Let Y be an independent set of G whilst the induced subgraph $G[X]$ be a 1-regular simple graph of order d (such a graph exists because d is even), and each vertex $x_i \in X$ be adjacent to all the vertices in Y (illustrated in Figure 4.2). For such a construction, each vertex $x_i \in X$ has degree $(d - 1) + 1 = d$, and each vertex $y_j \in Y$ has degree d too. Thus, G is

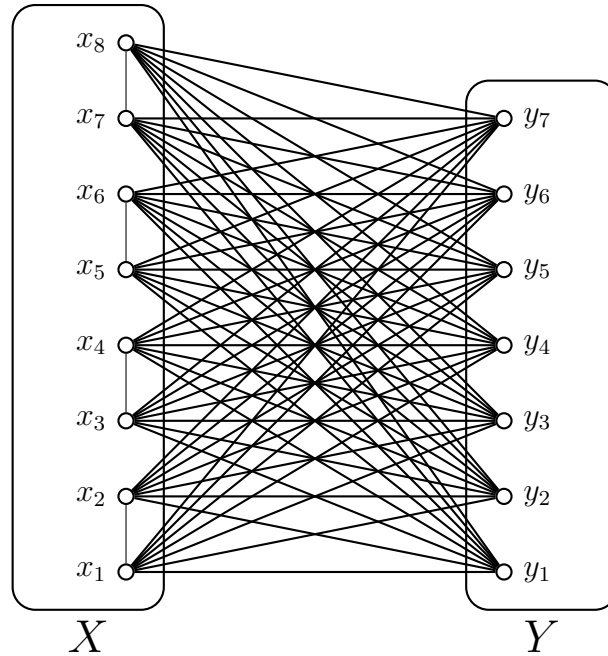


Figure 4.2: A d -regular connected graph of order n with $n \geq 5$ is odd, $3 \leq d \leq n - 2$, $d = \frac{n+1}{2}$ when $n = 15$ and $d = 8$

d -regular. In addition, since each vertex $x_i \in X$ is adjacent to all the vertices in Y , G is connected. Now, suppose X is initially infected. Since every vertex in Y has d neighbours in X , it can be infected in the first step of the infection process starting from X , which implies X is a d -threshold percolating set of cardinality d . Therefore, the lower bound is tight in this case.

Case 5. Suppose that $n \geq 5$ is odd, $3 \leq d \leq n - 2$ and $d > \frac{n+1}{2}$. In this case, $m(G, d) \geq \max \left\{ \left\lceil \frac{n}{2} \right\rceil, d \right\} = d$, and d has to be even because dn is even and n is odd. We construct a graph $G = (V, E)$ as follows. Let $V = X \cup Y$ with $X \cap Y = \emptyset$, $|X| = d$, $|Y| = n - d$. Let $X = \{x_1, \dots, x_d\}$, $Y = \{y_1, \dots, y_{n-d}\}$.

Let Y be an independent set of G whilst the induced subgraph $G[X]$ be a $(2d - n)$ -regular connected simple graph of order d (it can pick one of the constructions $G_{rc}^i(d, 2d - n)$ with $i \in \{1, 2\}$ in Figure 4.1, which depends on the values of d and $(2d - n)$), and each vertex $x_i \in X$ be adjacent to all the vertices in Y . For such a construction, each vertex $x_i \in X$ has degree $(2d - n) + (n - d) = d$, and each vertex $y_j \in Y$ has degree d too. Thus, G is d -regular. In addition, since each vertex $x_i \in X$ is adjacent to all the vertices in Y , G is connected. Now, suppose X is initially infected. Since every vertex in Y has d neighbours in X , it can be infected in the first step of the infection process starting from X , which implies X is a d -threshold percolating set of cardinality d . Therefore, the lower bound is tight in this case.

In conclusion, for any positive integer n, d with $2 \leq d \leq n - 1$ and $2|nd$, there exists d -regular graphs of order n for which the lower bound is tight. \square

Next, let us turn to the case when $d \geq 2r$.

Theorem 4.4. *Let $G = (V, E)$ be a d -regular connected simple graph of order n , and r be a positive integer with $2 \leq r < 2r \leq d$. Then, $m(G, r) \geq r$. Moreover, for any integers n, d, r with $2 \leq r < 2r \leq d \leq n - 1$ and $2|dn$, there always exists d -regular connected simple graphs of order n for which the lower bound is tight.*

Proof. By Theorem 4.1 and Remark 4.2,

$$m(G, r) \geq \max \left\{ \left\lceil n \left(1 - \frac{d}{2r} \right) + \frac{d}{r} - 1 \right\rceil, r \right\} = r.$$

So, it remains to show some d -regular connected simple graphs of order n reaching this lower bound.

Since the complete graph K_n is an $(n - 1)$ -regular connected simple graph of order n and any r seeds can infect all of its vertices in one step, the theorem is true when $d = n - 1$.

From now, suppose $2 \leq r < 2r \leq d \leq n - 2$. We distinguish several cases.

If d is even, let $d = 2m$.

Case 1. Suppose that $d = 2m$ is even and $n = 2m + b$ with $1 \leq b \leq m - 1$. In this case, $m \geq r$ and $2 \leq b \leq m - 1$. We construct a graph $G = (V, E)$ as follows. Let $V = X_1 \cup X_2 \cup Y$ with X_1, X_2, Y are pairwise disjoint, $|X_1| = |X_2| = m$, $|Y| = b$. Let $X_1 = \{x_0^1, \dots, x_{m-1}^1\}$, $X_2 = \{x_0^2, \dots, x_{m-1}^2\}$. Let X_1, X_2 be two cliques whilst Y be an independent set. Let each vertex $y \in Y$ be adjacent to all the vertices in $X_1 \cup X_2$, and for each $i = 0, \dots, m - 1$, let $N_G(x_i^1) \cap X_2 = \{x_{i+j \pmod{m}}^2 : 0 \leq j \leq m - b\}$ (illustrated in Figure 4.3). By such a construction, every vertex in $X_1 \cup X_2$ has degree $(m - 1) + b + (m - b + 1) = 2m = d$, and every vertex in Y has degree d too. So, G is d -regular. Since each vertex $y \in Y$ is adjacent to all the vertices in $X_1 \cup X_2$, G is connected. Now, let A be an r -subset of X_1 and be initially infected. Since X_1 is a clique of G and each

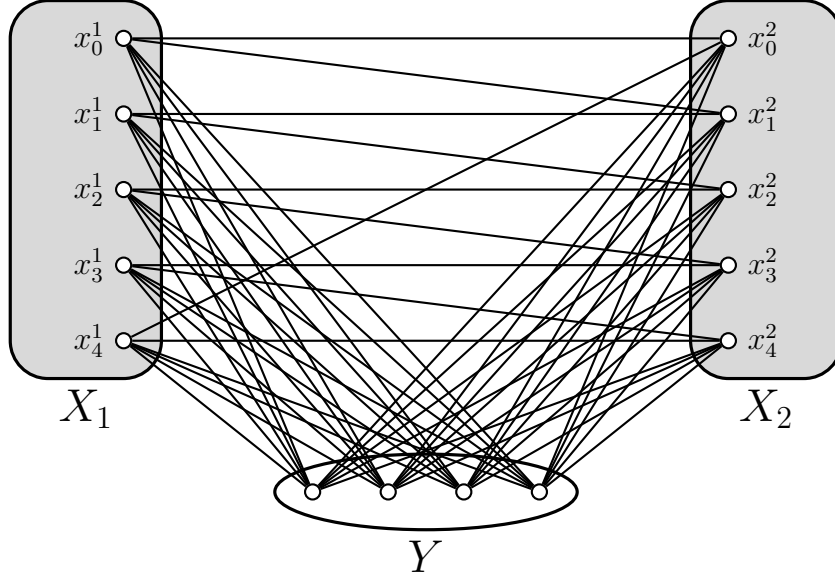


Figure 4.3: A d -regular connected graph of order n with $d = 2m$ is even, and $n = 2m + b$ when $m = 5$, and $b = 4$

vertex $y \in Y$ is adjacent to all the vertices in X_1 , $X_1 \cup Y$ will get infected in at most the first step of the infection process starting from A . Then, since every vertex in X_2 has $(m - b + 1) + b = m + 1 > r$ neighbours in $X_1 \cup Y$, X_2 can be infected in at most the second step of the infection process. Thus, A is an r -threshold percolating set of cardinality r . Therefore, the lower bound is tight in this case.

Case 2. Suppose that $d = 2m$ is even and $n = km$ with an integer $k \geq 3$. In this case, $m \geq r$. We construct a k -partite graph $G = (V, E)$. Specifically, let $V = \bigcup_{i=0}^{k-1} X_i$ with X_0, X_1, \dots, X_{k-1} are pairwise disjoint, every partite set X_i be an independent m -set of G . For each $i = 0, \dots, k - 1$, let every vertex $x^i \in X_i$ is adjacent to all the vertices in $X_{i+1 \pmod k}$ (illustrated in Figure 4.4).

By such a construction, it is straightforward that G is d -regular and connected. Now, suppose A is an r -subset of X_0 and is initially infected. Since every vertex in $X_1 \cup X_{k-1}$ is adjacent to all the vertices in X_0 , $X_1 \cup X_{k-1}$ will get infected in at most the first step of the infection starting from A . Then, since every vertex in X_0 is adjacent to all the vertices in X_1 , all the clean vertices in X_0 will get infected in at most the second step of the infection process. Moreover, since every vertex in X_2 is adjacent to all the vertices in X_1 , and every vertex in X_{k-2} is adjacent to all the vertices in X_{k-1} , $X_2 \cup X_{k-2}$ can be infected in at most the third step of the infection process. Continuing this process, A can infect all the vertices of G . Thus, A is an r -threshold percolating set of cardinality r . Therefore, the lower bound is tight in this case.

Case 3. Suppose that $d = 2m$ is even and $n = km + b$ with an integer $k \geq 3$ and $1 \leq b \leq m - 1$. In this case, $m \geq r$. We construct a $(k + 1)$ -partite graph $G = (V, E)$. Specifically, let $V = (\bigcup_{i=1}^k X_i) \cup Y$ with X_1, X_2, \dots, X_k and Y are pairwise disjoint, let every partite set X_i be an independent m -set, and let Y be an independent b -set. Let each $X_i = \{x_0^i, x_1^i, \dots, x_{m-1}^i\}$. For each $i = 1, \dots, k-1$, let every vertex $x_j^i \in X_i$ be adjacent to all the vertices in X_{i+1} , let every vertex $y \in Y$ be adjacent to all the vertices in $X_1 \cup X_k$, and for every vertex $x_j^1 \in X_1$, let $N_G(x_j^1) \cap X_k = \{x_{j+t}^k : 0 \leq t \leq m-b-1\}$ (illustrated in Figure 4.5). By such a construction, every vertex in $(\bigcup_{i=2}^{k-1} X_i) \cup Y$ has degree d , and every vertex in $X_1 \cup X_k$ has degree $m + b + (m - b) = 2m = d$ too. So, G is d -regular. In addition, connectedness of G is trivial. Now, suppose A is

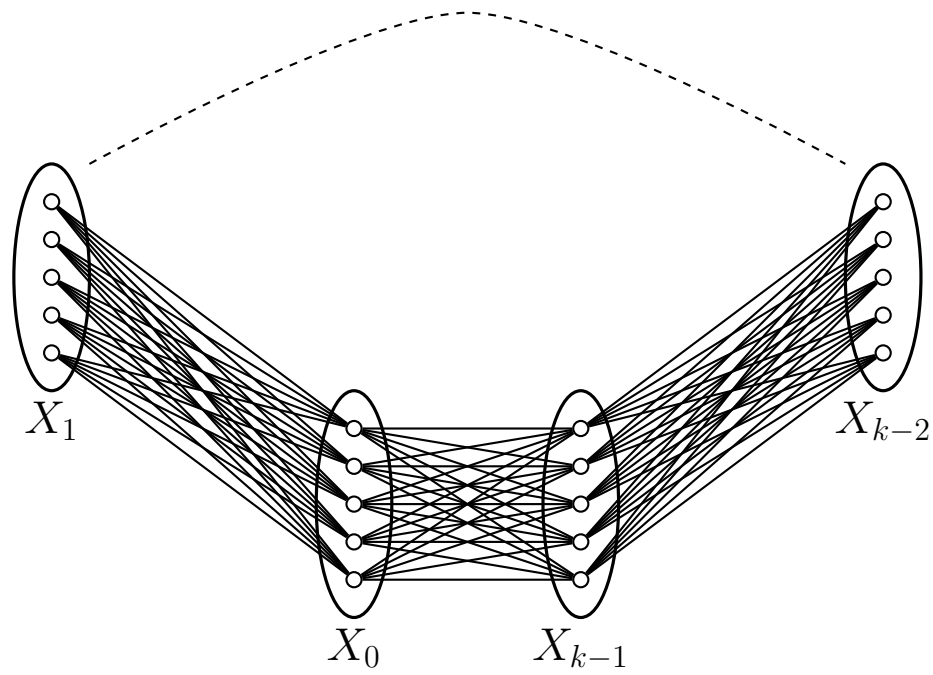


Figure 4.4: A d -regular connected graph of order n with $d = 2m$ is even, $n = km$, and $k \geq 3$ when $m = 5$

an r -subset of X_1 and is initially infected. Since every vertex in X_2 is adjacent to all the vertices in X_1 , X_2 will get infected in at most the first step of the infection process starting from A . Then, since every vertex in X_3 is adjacent to all the vertices in X_2 , X_3 will get infected in at most the second step of the infection process. Likewise, each X_i can get infected in at most the $(i - 1)$ -th step of the infection process. Besides, since every vertex in Y is adjacent to all the vertices in X_1 , Y will get infected in at most the first step of the infection process. Finally, since every vertex $x^1 \in X_1$ is adjacent to all the vertices in X_2 , every clean vertex in X_1 can get infected once X_2 becomes infected. Thus, A is an r -threshold percolating set of cardinality r . Therefore, the lower bound is tight in this case.

If d is odd, let $d = 2m + 1$.

Case 4. Suppose that $d = 2m + 1$ is odd and $n = 2m + b$ with $1 \leq b \leq m - 1$. In this case, $m \geq r$, $3 \leq b \leq m - 1$, and both n and b are even because dn is even. We construct a graph $G = (V, E)$ as follows. Let $V = X_1 \cup X_2 \cup Y$ with X_1, X_2, Y are pairwise disjoint, $|X_1| = |X_2| = m$ and $|Y| = b$. Let each $X_i = \{x_0^i, x_1^i, \dots, x_{m-1}^i\}$ and $Y = \{y_0, \dots, y_{b-1}\}$. Let X_1 and X_2 be two cliques. Let every vertex $y_j \in Y$ with even j be adjacent to y_{j+1} , let every vertex $y_j \in Y$ be adjacent to all the vertices in $X_1 \cup X_2$, and for every vertex $x_j^1 \in X_1$, let $N_G(x_j^1) \cap X_2 = \{x_{j+t}^2 \pmod{m} : 0 \leq t \leq m - b + 1\}$ (illustrated in Figure 4.6). By such a construction, every vertex $y_j \in Y$ has degree $2m + 1 = d$, and every vertex in $X_1 \cup X_2$ has degree $(m - 1) + b + (m - b + 2) = 2m + 1 = d$

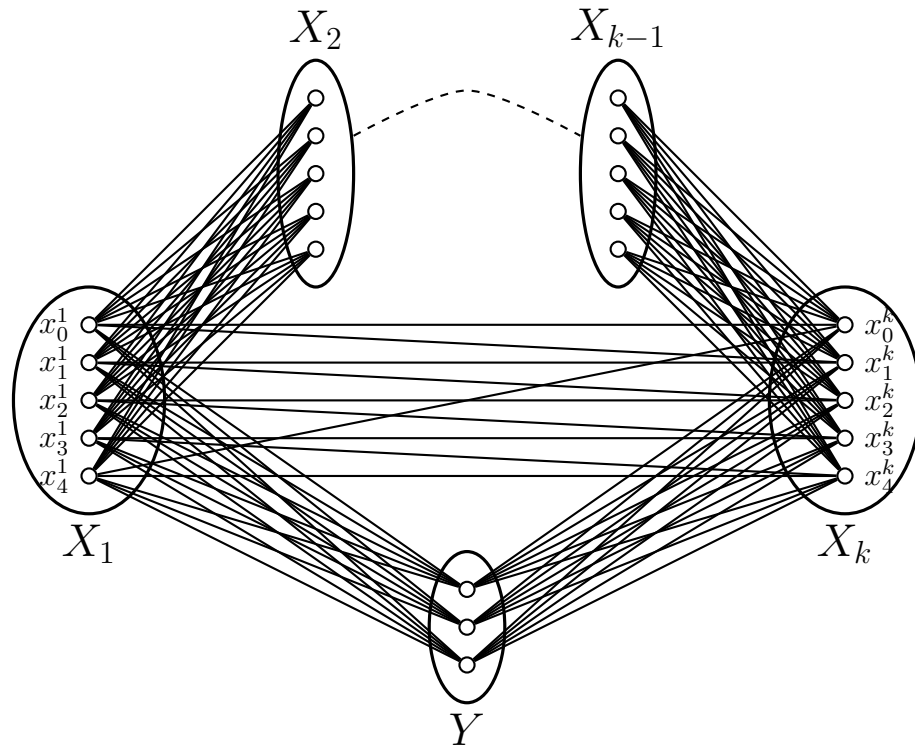


Figure 4.5: A d -regular connected graph of order n with $d = 2m$ is even, $n = km + b$, and $k \geq 3$ when $m = 5$ and $b = 3$

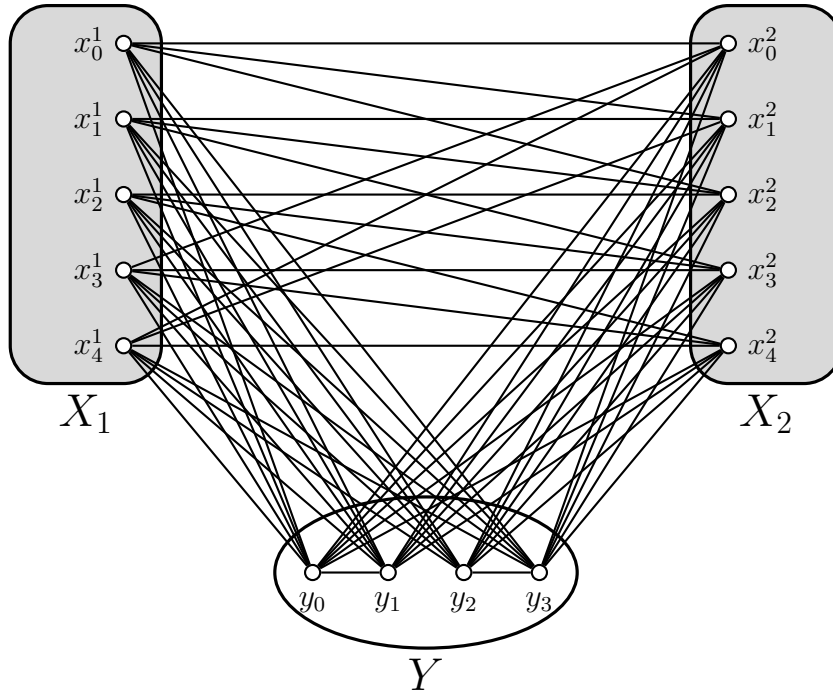


Figure 4.6: A d -regular connected graph of order n with $d = 2m + 1$ is odd, and $n = 2m + b$ is even when $m = 5$ and $b = 4$

too. So, G is d -regular. In addition, since every vertex $y_j \in Y$ is adjacent to all the vertices in $X_1 \cup X_2$, G is connected. Now, suppose A is an r -subset of X_1 and is initially infected. Since X_1 is a clique and every vertex in Y is adjacent to all the vertices in X_1 , $X_1 \cup Y$ will get infected in at most the first step of the infection process starting from A . Then, since every vertex $x_j^2 \in X_2$ will have $(m - b + 2) + b = m + 2 > r$ infected neighbours in $X_1 \cup Y$, X_2 will get infected in at most the second step of the infection process. Thus, A is an r -threshold percolating set of cardinality r . Therefore, the lower bound is tight in this case.

Case 5. Suppose that $d = 2m + 1$ is odd and $n = km$ with an even integer $k \geq 3$. In this case, $m \geq r$ and n is even because dn is even. We construct a k -partite graph $G = (V, E)$. Specifically, let $V = \bigcup_{i=0}^{k-1} X_i$ with X_0, \dots, X_{k-1} are pairwise disjoint, and let each X_i be an independent m -set. Let each $X_i = \{x_0^i, x_1^i, \dots, x_{m-1}^i\}$. Let every vertex $x_j^i \in X_i$ be adjacent to all the vertices in $X_{i-1 \pmod m} \cup X_{i+1 \pmod m}$, and for every $0 \leq i \leq \frac{k}{2} - 1$, let every vertex $x_j^i \in X_i$ be adjacent to $x_j^{i+\frac{k}{2}}$ (illustrated in Figure 4.7). By such a construction, every vertex has degree $2m + 1 = d$, and G is connected. Now, suppose A is an r -subset of X_0 and is initially infected. Since every vertex in $X_1 \cup X_{k-1}$ has $m \geq r$ neighbours in X_0 , $X_1 \cup X_{k-1}$ will get infected in at most the first step of infection process starting from A . Then, since every vertex in X_2 has $m \geq r$ neighbours in X_1 , and every vertex in X_{k-2} has $m \geq r$ neighbours in X_{k-1} , $X_2 \cup X_{k-2}$ will get infected in at most the second step of infection process. Continue this procedure, $X_1 \cup \dots \cup X_{k-1}$ can get infected. In addition, since every vertex in X_0 has $m \geq r$ neighbours in X_1 , all the clean vertices in X_0 will get infected once X_1 is infected. Thus, A is an r -threshold percolating set of cardinality r . Therefore, the lower bound is tight in this case.

Case 6. Suppose that $d = 2m + 1$ is odd and $n = km$ with an odd integer $k \geq 3$. In this case, $m \geq r$ and n is even, and so m is even too. We construct a graph $G = (V, E)$ as follows. Let $V = \bigcup_{i=0}^{k-1} X_i$ with X_0, \dots, X_{k-1} are pairwise disjoint. Let each X_i contain m vertices and $X_i = \{x_0^i, x_1^i, \dots, x_{m-1}^i\}$. Let every

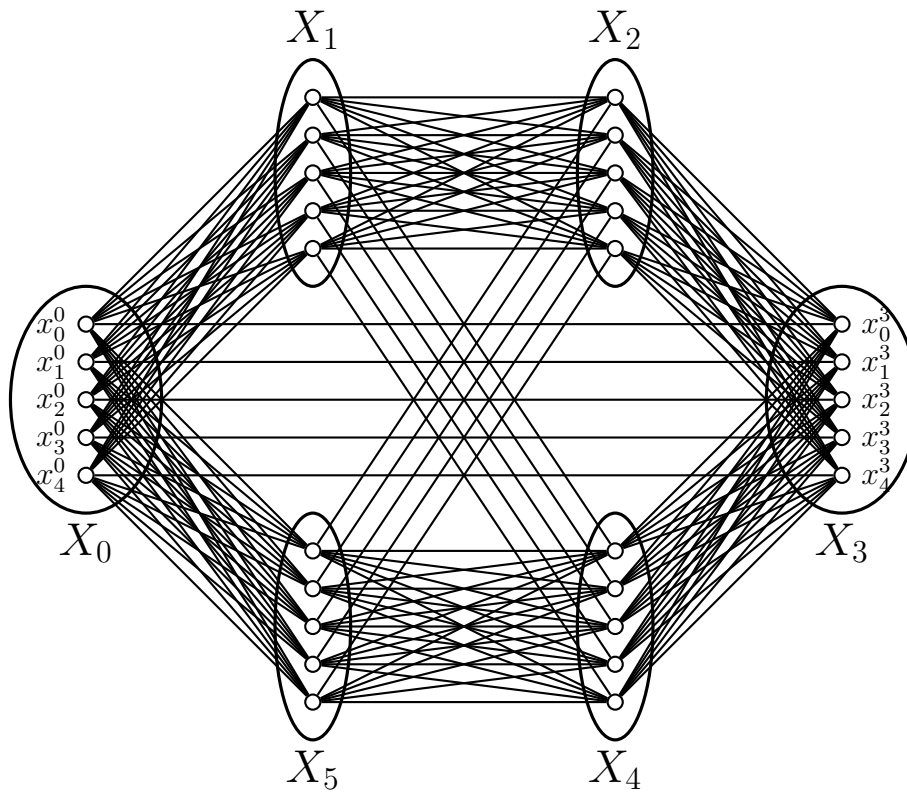


Figure 4.7: A d -regular connected graph of order n with $d = 2m + 1$ is odd, $n = km$, and $k \geq 3$ is even when $m = 5$ and $k = 6$

vertex $x_j^i \in X_i$ be adjacent to all the vertices in $X_{i-1 \pmod m} \cup X_{i+1 \pmod m}$, and for every i and every even j , let $x_j^i - x_{j+1}^i \in E$ (illustrated in Figure 4.8). By such a construction, every vertex has degree $2m + 1 = d$, and G is connected. Now, suppose A is an r -subset of X_0 and is initially infected. Since every vertex in $X_1 \cup X_{k-1}$ has $m \geq r$ neighbours in X_0 , $X_1 \cup X_{k-1}$ will get infected in at most the first step of infection process starting from A . Then, since every vertex in X_2 has $m \geq r$ neighbours in X_1 , and every vertex in X_{k-2} has $m \geq r$ neighbours in X_{k-1} , $X_2 \cup X_{k-2}$ will get infected in at most the second step of infection process. Continue this procedure, $X_1 \cup \dots \cup X_{k-1}$ can get infected. In addition, since every vertex in X_0 has $m \geq r$ neighbours in X_1 , all the clean vertices in X_0 will get infected once X_1 is infected. Thus, A is an r -threshold percolating set of cardinality r . Therefore, the lower bound is tight in this case.

Case 7. Suppose that $d = 2m + 1$ is odd and $n = km + b$ with an even integer $k \geq 3$ and $1 \leq b \leq m - 1$. In this case, $m \geq r$ and n is even, and so b must be even too. We construct a graph $G = (V, E)$ as follows. Let $V = (\bigcup_{i=0}^{k-1} X_i) \cup Y$ with X_0, \dots, X_{k-1} and Y are pairwise disjoint, let each X_i be an independent m -set, and $|Y| = b$. Let each $X_i = \{x_0^i, x_1^i, \dots, x_{m-1}^i\}$ and $Y = \{y_0, y_1, \dots, y_{b-1}\}$. For each $i = 0, \dots, k - 2$, let every vertex $x_j^i \in X_i$ be adjacent to all the vertices in X_{i+1} , let every vertex $y_j \in Y$ be adjacent to all the vertices in $X_0 \cup X_{k-1}$, and for each even j , let $y_j - y_{j+1} \in E$. For each vertex $x_j^0 \in X_0$, let $N_G(x_j^0) \cap X_{k-1} = \{x_{j+t}^{k-1} : 0 \leq t \leq m - b - 1\}$,

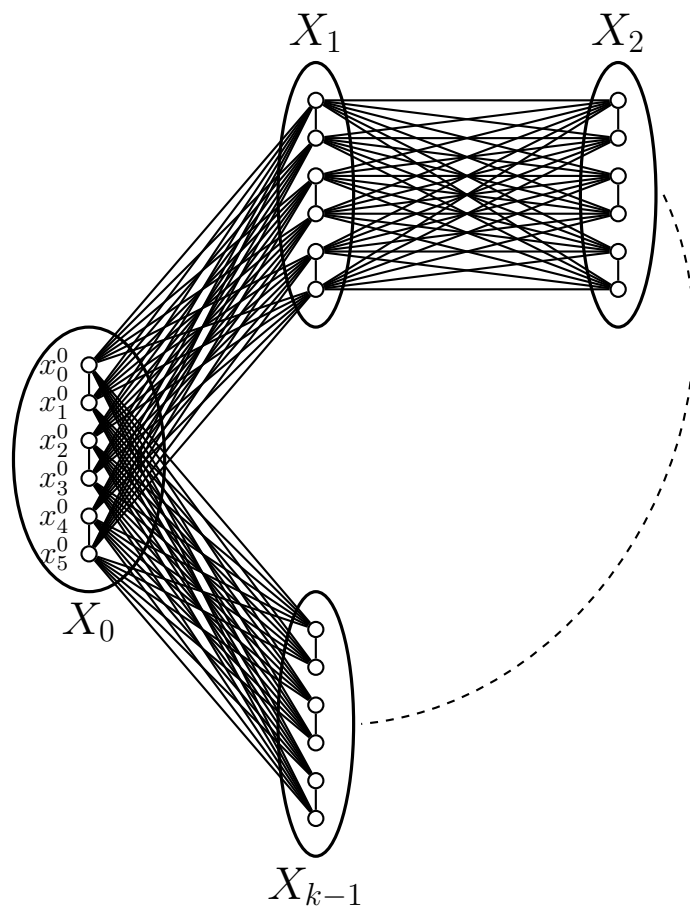


Figure 4.8: A d -regular connected graph of order n with $d = 2m + 1$ is odd, $n = km$, and $k \geq 3$ is odd when $m = 6$

and for every $i = 0, \dots, \frac{k}{2} - 1$, let every vertex $x_j^i \in X_i$ be adjacent to $x_j^{i+\frac{k}{2}}$ (illustrated in Figure 4.9). For such a construction, every vertex in $(\bigcup_{i=1}^{k-2} X_i) \cup Y$ has degree $2m + 1 = d$, and every vertex in $X_1 \cup X_{k-1}$ has degree $m + b + (m - b) + 1 = 2m + 1 = d$, which indicates G is d -regular. The connectedness of G is trivial. Now, suppose A is an r -subset of X_0 and is initially infected. Since every vertex in X_1 has r infected neighbours in X_0 , X_1 will get infected in at most the first step of infection process starting from A . Then, since every vertex in X_2 has r infected neighbours in X_1 , X_2 will get infected in at most the second step of the infection process. Continue this procedure, each X_i can get infected. Besides, Y can be infected in at most the first step of the infection process because every vertex in Y has r infected neighbours in X_0 initially. In addition, since every vertex in X_0 has $m \geq r$ neighbours in X_1 , all the clean vertices in X_0 will get infected once X_1 is infected. Thus, A is an r -threshold percolating set of cardinality r . Therefore, the lower bound is tight in this case.

Case 8. Suppose that $d = 2m + 1$ is odd and $n = km + b$ with an odd integer $k \geq 3$ and $1 \leq b \leq m - 1$. In this case, $m \geq r$ and n is even, and so $2|(m - b)$. We construct a graph $G = (V, E)$ as follows. Let $V = (\bigcup_{i=0}^{k-1} X_i) \cup Y$ with X_0, \dots, X_{k-1} and Y are pairwise disjoint, each X_i contain m vertices, and $|Y| = b$. Let each $X_i = \{x_0^i, x_1^i, \dots, x_{m-1}^i\}$ and $Y = \{y_0, y_1, \dots, y_{b-1}\}$. For each $i = 0, \dots, k - 2$, let every vertex $x_j^i \in X_i$ be adjacent to all the vertices in X_{i+1} , and let every vertex $y_j \in Y$ be adjacent to all the vertices in $X_0 \cup X_{k-1}$.

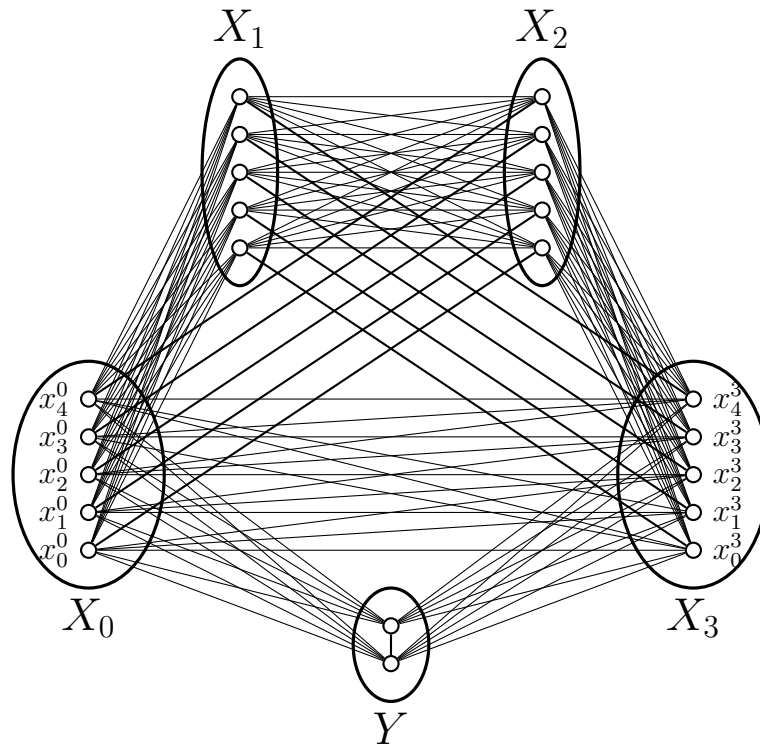


Figure 4.9: A d -regular connected graph of order n with $d = 2m + 1$ is odd, $n = km + b$ and $k \geq 3$ is even when $m = 5$, $k = 4$ and $b = 2$

For each vertex $x_j^0 \in X_0$, let $N_G(x_j^0) \cap X_{k-1} = \{x_{j+t}^{k-1} \pmod{m} : 0 \leq t \leq m-b-1\}$.

For every $i = 0, \dots, \frac{k-1}{2} - 1$ and every $j = 0, \dots, m-1$, let $x_j^i - x_j^{i+\frac{k+1}{2}} \in E$. For

every $j = 0, \dots, b-1$, let $y_j - x_j^{\frac{k-1}{2}} \in E$, and for each even $j = 0, \dots, m-b-1$,

let $x_{b+j}^{\frac{k-1}{2}} - x_{b+j+1}^{\frac{k-1}{2}} \in E$ (illustrated in Figure 4.10). For such a construction,

every vertex in $(\bigcup_{i=1}^{k-2} X_i) \cup Y$ has degree $2m+1 = d$, and every vertex in

$X_0 \cup X_{k-1}$ has degree $m+b+(m-b)+1 = 2m+1 = d$, which indicates G is

d -regular. The connectedness of G is trivial. Now, suppose A is an r -subset of

X_0 and is initially infected. Since every vertex in X_1 has r infected neighbours

in X_0 , X_1 will get infected in at most the first step of infection process starting

from A . Then, Since every vertex in X_2 has r infected neighbours in X_1 , X_2

will get infected in at most the second step of infection process. Continue

this procedure, each X_i can get infected. Besides, Y can be infected in at

most the first step of the infection process because every vertex in Y has r

infected neighbours in X_0 initially. In addition, since every vertex in X_0 has

$m \geq r$ neighbours in X_1 , all the clean vertices in X_0 will get infected once X_1 is

infected. Thus, A is an r -threshold percolating set of cardinality r . Therefore,

the lower bound is tight in this case.

In conclusion, for any integers n, d, r with $2 \leq r < 2r \leq d \leq n-1$ and $2|dn$,

there exists d -regular connected simple graphs of order n for which the lower

bound is tight. \square

After investigating minimum r -threshold percolating sets of d -regular con-

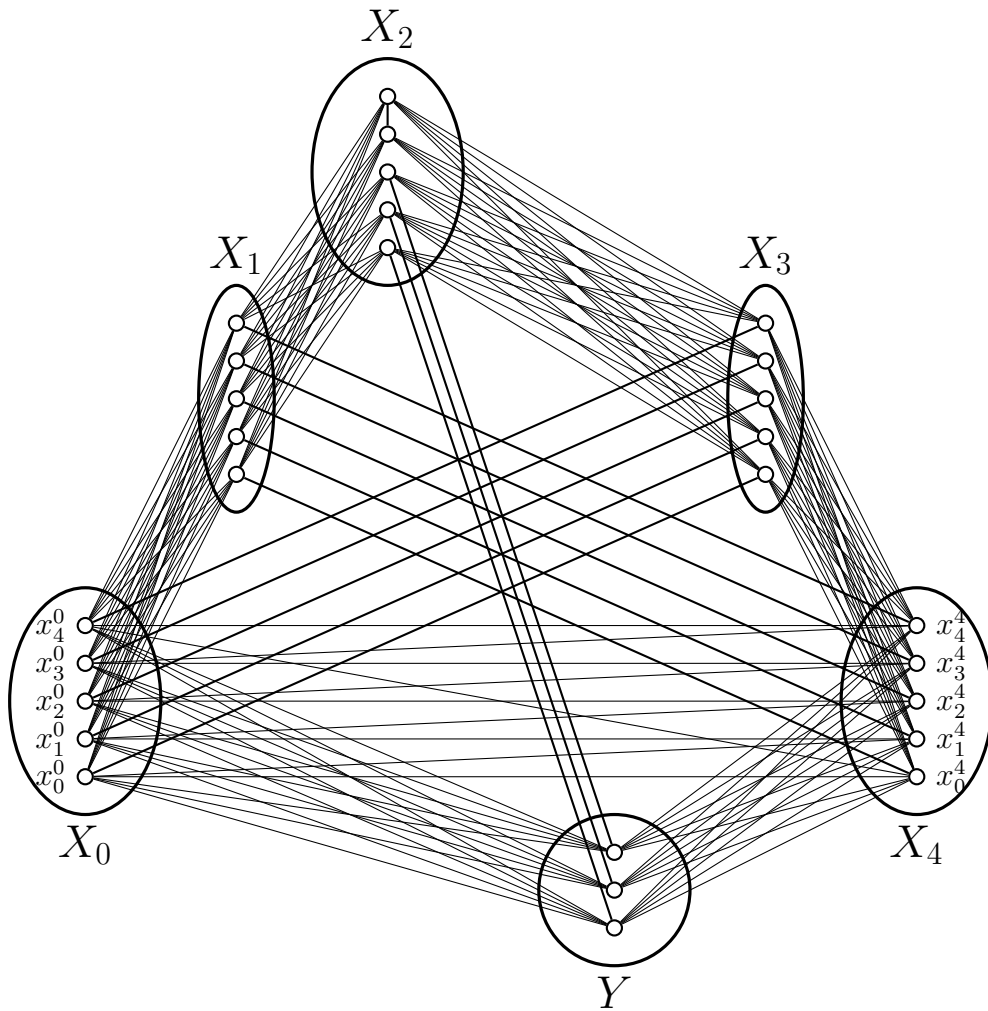


Figure 4.10: A d -regular connected graph of order n with $d = 2m + 1$ is odd, $n = km + b$ and $k \geq 3$ is odd when $m = 5$, $k = 5$ and $b = 3$

nected graphs with $r = d$ or $2r \leq d$, we finish this section by discussing a special case when $r < d < 2r$.

Theorem 4.5. *Let G be a 3-regular connected simple graph of order n . Then, $\lceil \frac{n+2}{4} \rceil \leq m(G, 2) \leq \frac{n}{2}$. Moreover, for any even integer $n \geq 4$, there always exists 3-regular connected simple graphs of order n for which the lower bound is tight.*

Proof. Note that for a 3-regular connected simple graph $G = (V, E)$ of order n , $n \geq 4$ and n is even because $2|E| = 3n$.

The bounds directly come from Theorem 4.1 when assigning $r = 2$ and $d = 3$. So, it remains to show some 3-regular connected graphs of order n reaching the lower bound.

When $n = 4$, the complete graph K_4 is a 3-regular connected graph with

$$\left\lceil \frac{4+2}{4} \right\rceil = 2 = m(K_4, 2).$$

So the lower bound is tight in this case.

From now, suppose $n \geq 6$. We distinguish several cases.

Case 1. Suppose that $n \equiv 2 \pmod{4}$. In this case, $\lceil \frac{n+2}{4} \rceil = \frac{n+2}{4}$. Let $k = \frac{n-6}{4} \in \mathbb{Z}$. We construct a graph $G = (V, E)$ as follows. Let $V = (\bigcup_{i=0}^k X_i) \cup \{z\}$ with X_0, \dots, X_k and $\{z\}$ are pairwise disjoint, $|X_0| = 5$, and $|X_1| = \dots = |X_k| = 4$ (if they exist). Let $X_0 = \{x_j^0 : j = 0, \dots, 4\}$, and for

each $i = 1, \dots, k$, let $X_i = \{x_j^i : j = 0, \dots, 3\}$. In X_0 , for each $j = 0, \dots, 2$, let $x_j^0 - x_3^0 \in E$ and $x_j^0 - x_4^0 \in E$. If $k > 0$, then for each $i = 1, \dots, k$ and each $j = 0, \dots, 2$, let $x_j^i - x_3^i, x_j^i - x_j^{i-1}, x_j^k - z \in E$; otherwise, if $k = 0$, then for each $j = 0, \dots, 2$, let $x_j^0 - z \in E$ (illustrated in Figure 4.11). For such a construction, every vertex in $(\bigcup_{i=0}^k X_i)$ has degree 3, and $\deg(z) = 3$. Thus, G is 3-regular. In addition, G is connected because for every $i = 0, \dots, k$, the induced subgraph $G[X_i]$ is connected, for every $i = 0, \dots, k-1$, $G[X_i]$ is connected to $G[X_{i+1}]$ and z is connected to $G[X_k]$. Now, let $A = (\bigcup_{i=0}^k \{x_3^i\}) \cup \{x_4^0\}$ is initially infected. Then $|A| = \frac{n-6}{4} + 1 + 1 = \frac{n+2}{4}$. Since every vertex amongst x_0^0, x_1^0, x_2^0 has 2 infected neighbours initially, X_0 will get infected in the first step of infection process starting from A . Then, since every vertex amongst x_0^1, x_1^1, x_2^1 has 2 infected neighbours, X_1 will get infected in the second step of infection process. Continue this procedure, each X_i can get infected. Finally, z can get infected once X_k is infected. Thus, A is a 2-threshold percolating set of cardinality $\frac{n+2}{4}$. Therefore, the lower bound is tight in this case.

If $n \not\equiv 2 \pmod{4}$, then $n \geq 8$, and either $n \equiv 0 \pmod{8}$ or $n \equiv 4 \pmod{8}$.

Case 2. Suppose that $n \equiv 0 \pmod{8}$. In this case, $\lceil \frac{n+2}{4} \rceil = \frac{n}{4} + 1$. Let $k = \frac{n-8}{8} \in \mathbb{Z}$. We construct a graph $G = (V, E)$ as follows. Let $V = (\bigcup_{i=0}^k X_i) \cup Y$ with X_0, \dots, X_k and Y are pairwise disjoint, let $|X_0| = |Y| = 4$, and let $|X_1| = \dots = |X_k| = 8$ (if they exist). Let $X_0 = \{x_j^0 : j = 0, \dots, 3\}$, $Y = \{y_j : j = 0, \dots, 3\}$, and for each $i = 1, \dots, k$, let $X_i = \{x_j^i : j =$

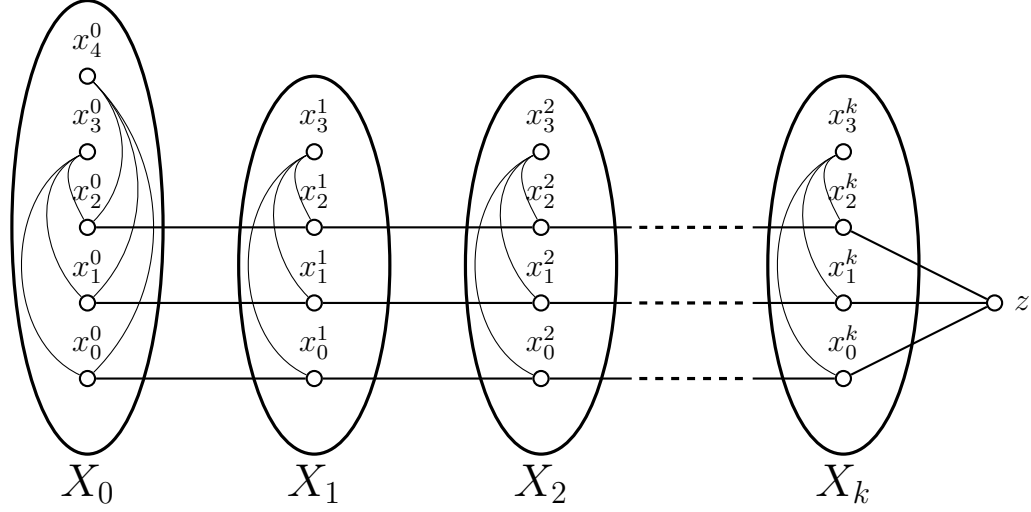


Figure 4.11: A 3-regular connected graph of order $n \geq 6$ with $n \equiv 2 \pmod{4}$

$0, \dots, 7\}$. For X_0 , let $x_0^0 - x_1^0, x_0^0 - x_2^0, x_0^0 - x_3^0, x_1^0 - x_2^0, x_1^0 - x_3^0 \in E$. For Y , let $y_0 - y_2, y_0 - y_3, y_1 - y_2, y_1 - y_3, y_2 - y_3 \in E$. If $k > 0$, then for each $i = 1, \dots, k$, let $x_0^i - x_2^i, x_2^i - x_4^i, x_1^i - x_3^i, x_3^i - x_5^i, x_6^i - x_0^i, x_6^i - x_2^i, x_6^i - x_4^i, x_7^i - x_1^i, x_7^i - x_3^i, x_7^i - x_5^i \in E$, for $i = 1, \dots, k-1$, let $x_4^i - x_0^{i+1}, x_5^i - x_1^{i+1} \in E$, and let $x_2^0 - x_0^1, x_3^0 - x_1^1, y_0 - x_4^k, y_1 - x_5^k \in E$; otherwise, if $k = 0$, then just let $x_2^0 - y_0, x_3^0 - y_1 \in E$ (illustrated in Figure 4.12). For such a construction, G is 3-regular and connected. Now, let

$$A = \begin{cases} \{x_0^0, x_1^0, y_2\}, & k = 0 \\ \{x_0^0, x_1^0, y_2\} \cup (\bigcup_{i=1}^k \{x_6^i, x_7^i\}), & k > 0 \end{cases}.$$

be initially infected. Then $|A| = \frac{(n-8)^2}{8} + 3 = \frac{n}{4} + 1$. Since both of x_2^0, x_3^0 have 2 infected neighbours initially, X_0 will get infected in the first step of infection process starting from A . Then, X_0 will infect X_1, \dots, X_k in turn

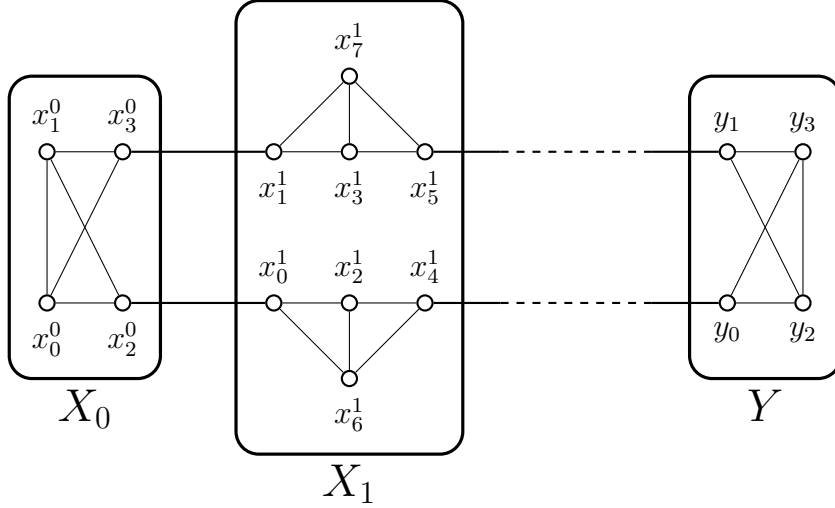


Figure 4.12: A 3-regular connected graph of order $n \geq 8$ with $n \equiv 0 \pmod{8}$

(if they exist). Finally, it is trivial that Y will get infected. Thus, A is an 2-threshold percolating set of cardinality $\frac{n}{4} + 1$. Therefore, the lower bound is tight in this case.

Case 3. Suppose $n \equiv 4 \pmod{8}$. In this case, $\lceil \frac{n+2}{4} \rceil = \frac{n}{4} + 1$ too. Let $k = \frac{n-4}{8} \in \mathbb{Z} \geq 1$. We construct a graph $G = (V, E)$ as follows. Let $V = \bigcup_{i=0}^k X_i$ with X_0, \dots, X_k are pairwise disjoint, let $|X_0| = 4$, and let $|X_1| = \dots = |X_k| = 8$. Let $X_0 = \{x_j^0 : j = 0, \dots, 3\}$, and for each $i = 1, \dots, k$, let $X_i = \{x_j^i : j = 0, \dots, 7\}$. For X_0 , let $x_0^0 - x_1^0, x_0^0 - x_2^0, x_0^0 - x_3^0, x_1^0 - x_2^0, x_1^0 - x_3^0 \in E$. For each $i = 1, \dots, k$, let $x_0^i - x_2^i, x_2^i - x_4^i, x_1^i - x_3^i, x_3^i - x_5^i, x_6^i - x_0^i, x_6^i - x_2^i, x_6^i - x_4^i, x_7^i - x_1^i, x_7^i - x_3^i, x_7^i - x_5^i \in E$. For each $i = 1, \dots, k-1$, let $x_4^i - x_0^{i+1}, x_5^i - x_1^{i+1} \in E$. Finally, let $x_2^0 - x_0^1, x_3^0 - x_1^1, x_4^k - x_5^k \in E$ (illustrated in Figure 4.13). For such a construction, G is 3-regular and connected. Now, let

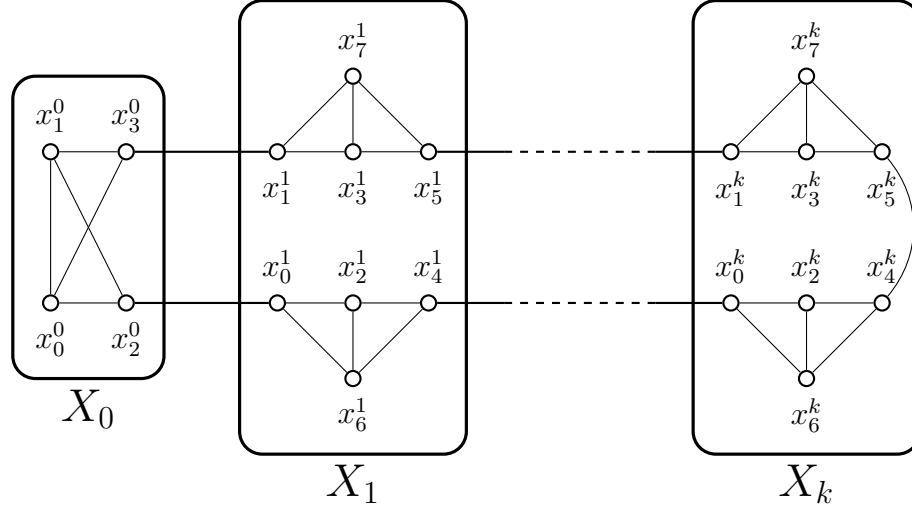


Figure 4.13: A 3-regular connected graph of order $n \geq 8$ with $n \equiv 4 \pmod{8}$

$A = \{x_0^0, x_1^0\} \cup (\bigcup_{i=1}^k \{x_6^i, x_7^i\})$ be initially infected. Then $|A| = \frac{(n-4)2}{8} + 2 = \frac{n}{4} + 1$. Since both of x_2^0, x_3^0 have 2 infected neighbours initially, X_0 will get infected in the first step of infection process starting from A . Then, X_0 will infect X_1, \dots, X_k in turn. Thus, A is an 2-threshold percolating set of cardinality $\frac{n}{4} + 1$. Therefore, the lower bound is tight in this case.

In conclusion, for any even integer $n \geq 4$, there exists 3-regular connected simple graphs of order n for which the lower bound is tight. \square

In this section, we have shown that the lower bound in Theorem 4.1 is the best one that can be achieved in cases when $d = r$ and when $d \geq 2r$. In addition, we also displayed its tightness in a particular case of $r < d < 2r$. However, it is still pending whether this lower bound is the best one for other cases when $r < d < 2r$. Besides, whether Reichman's upper bound can be

improved for regular connected graphs is also unsolved. So, we list them as open problems here for further research.

Problem 4.6. *For a d -regular connected graph G of order n and an integer r with $2 \leq r \leq d < n$, what are the best lower bound and upper bound for $m(G, r)$?*

Problem 4.7. *For any integers n, r with $2 \leq r < n$, is there a d -regular connected graph G of order n other than the complete graph K_{d+1} that can reach Reichman's upper bound $\lfloor \frac{nr}{d+1} \rfloor$?*

4.3 Minimum 2-Threshold Percolating Sets of Expander Graphs

In this section, we shall study how minimum percolating sets depend on *expansion properties* of regular graphs. The main results in this section come from Coja-Oghlan, Feige, Krivelevich and Reichman [13].

For a d -regular graph G , we focus on *spectral expanders*, namely, if the second largest eigenvalue λ of G satisfies $\lambda < \delta d$ for some real number $\delta \in (0, 1)$, then G is called a spectral expander or an (n, d, λ) -graph, where n is the order of G . Note that when studying expander graphs, the order n is usually much larger than d (even $d = o(n)$ sometimes). Expanders are rich mathematical objects with diverse applications in algebra, combinatorics, probability and theoretical computer science (see Hoory, Linial and Wigderson [18]).

Table 4.1: Asymptotic upper bounds for minimum 2-threshold percolating sets of expander graphs satisfying different conditions, where n is the order, d is the degree, and λ is the second largest eigenvalue of the graph.

Conditions	Upper Bound
The girth is bigger than $2 \log \log d$	$O\left(\frac{n}{d^2}\right)$
Without 4-cycles	$O(nd^{-7/4})$
The girth is at least 7	$O\left(\frac{n \log d}{d^2}\right)$
$\lambda \leq O(\sqrt{d})$	$O\left(\frac{n}{d^2}\right)$

The next theorem gives lower bounds for minimum percolating sets of expander graphs having properties about both the second largest eigenvalues and girths.

Theorem 4.8 (Coja-Oghlan, Feige, Krivelevich and Reichman [13]). *For a large enough integer d , there exists a connected (n, d, λ) -graph $G = (V, E)$ with $\lambda = O(\sqrt{d})$, the girth at least $\Omega(\log \log d)$ and $m(G, 2) \geq \Omega\left(\frac{n}{d^2 \log d}\right)$.*

Meanwhile, Coja-Oghlan, Feige, Krivelevich and Reichman [13] also presented some results that are listed in Table 4.1 about the upper bounds for minimum percolating sets of different types of expander graphs.

In the rest of this section, we focus on minimum 2-threshold percolating sets of (n, d, λ) -graphs without 4-cycles. We shall show how to get asymptotic upper bounds for minimum percolating sets of this type of expander graphs by an algorithmic method.

To prepare for discussion about percolating sets of (n, d, λ) -graphs, we need

to introduce some algebraic results about regular graphs first.

Theorem 4.9 (see Cvetković, Doob and Sachs [14]). *Let G be a connected graph of n vertices with adjacency matrix A and eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$, then $\delta(G) \leq \lambda_n \leq \Delta(G)$, and equality holds if and only if G is regular.*

Theorem 4.10 (see Cvetković, Doob and Sachs [14]). *Let A be an adjacency matrix for a regular graph G of n vertices with (real) eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. Then, the number of connected components of G is equal to the algebraic multiplicity of λ_n .*

For a graph $G = (V, E)$ and two (not necessarily disjoint) subsets $B, C \subseteq V$, let $e(B, C)$ be the number of edges $u - v$ with $u \in B$ and $v \in C$. Note that if B and C are disjoint, then $e(B, C)$ is just the number of edges between them.

Theorem 4.11 (see Alon and Spencer [2]). *Let $G = (V, E)$ be a d -regular connected graph of order n with an adjacency matrix A , and let λ be the second largest eigenvalue of A . Then, for every partition of V into two disjoint subsets B and C ,*

$$e(B, C) \geq \frac{(d - \lambda)|B||C|}{n}.$$

Proof. Let $V = \{v_1, \dots, v_n\}$, $b = |B|$ and $c = |C|$. Then $b + c = n$.

First, by Theorem 4.9 and Theorem 4.10, d is the largest eigenvalue of A , and with algebraic multiplicity 1. Let $\mathbf{1}$ denote the all-1 column vector of

length n . Since $A\mathbf{1} = d\mathbf{1}$, then $(d, \mathbf{1})$ is an eigenvalue pair of A . Let

$$\lambda_1 \leq \cdots \leq \lambda_{n-2} \leq \lambda_{n-1} = \lambda \leq \lambda_n = d$$

be eigenvalues and $X_1, \dots, X_{n-1}, X_n = \mathbf{1}$ be corresponding pairwise orthogonal eigenvectors of A (they exist because A is real symmetric). For any $\mu \in \mathbb{R}$ and $X \in \mathbb{R}^n$, it is trivial that $AX = \mu X$ if and only if $(dI_n - A)X = (d - \mu)X$, where I_n denotes the $(n \times n)$ identity matrix. Thus, (μ, X) is an eigenpair of A if and only if $(d - \mu, X)$ is an eigenpair of $dI_n - A$. Hence,

$$d - \lambda_1 \geq \cdots \geq d - \lambda_{n-2} \geq d - \lambda_{n-1} = d - \lambda \geq d - \lambda_n = d - d$$

are eigenvalues and $X_1, \dots, X_{n-1}, X_n = \mathbf{1}$ are corresponding pairwise orthogonal eigenvectors of $dI_n - A$, and the smallest eigenvalue $d - d = 0$ has algebraic multiplicity 1.

Second, observe that for any real vector $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$, we can consider it as a mapping from V to \mathbb{R} by $\mathbf{y}(v_i) = y_i$. Let $\langle *, * \rangle$ be the usual Euclidean inner product over \mathbb{R}^n . Then, we have

$$\begin{aligned} \langle (dI_n - A)\mathbf{y}, \mathbf{y} \rangle &= d\langle \mathbf{y}, \mathbf{y} \rangle - \langle A\mathbf{y}, \mathbf{y} \rangle \\ &= \sum_{u \in V} d\mathbf{y}(u)^2 - \sum_{u \in V} \sum_{\substack{v \in V: \\ uv \in E}} \mathbf{y}(u)\mathbf{y}(v) \\ &= d \sum_{u \in V} \mathbf{y}(u)^2 - 2 \sum_{uv \in E} \mathbf{y}(u)\mathbf{y}(v) \\ &= \sum_{uv \in E} (\mathbf{y}(u) - \mathbf{y}(v))^2. \end{aligned}$$

Now, define a vector $\mathbf{x} \in \mathbb{R}^n$ by

$$\mathbf{x}(v) = \begin{cases} -c, & v \in B \\ b, & v \in C \end{cases}.$$

Then, $\langle \mathbf{1}, \mathbf{x} \rangle = \sum_{v \in V} \mathbf{x}(v) = -bc + cb = 0$, which indicates that \mathbf{x} is orthogonal to $\mathbf{1}$, and so it is a linear combination of X_1, \dots, X_{n-1} . Let $\mathbf{x} = \sum_{i=1}^{n-1} c_i X_i$.

Since $d - \lambda_{n-1} = d - \lambda$ is the second smallest eigenvalue of $dI_n - A$, then

$$\begin{aligned} \langle (dI_n - A)\mathbf{x}, \mathbf{x} \rangle &= \left\langle (dI_n - A) \left(\sum_{i=1}^{n-1} c_i X_i \right), \left(\sum_{i=1}^{n-1} c_i X_i \right) \right\rangle \\ &= \sum_{i=1}^{n-1} c_i^2 \langle (dI_n - A)X_i, X_i \rangle \\ &= \sum_{i=1}^{n-1} c_i^2 (d - \lambda_i) \langle X_i, X_i \rangle \\ &\geq (d - \lambda_{n-1}) \left(\sum_{i=1}^{n-1} c_i^2 \langle X_i, X_i \rangle \right) \\ &= (d - \lambda) \langle \mathbf{x}, \mathbf{x} \rangle \\ &= (d - \lambda)(bc^2 + cb^2) \\ &= (d - \lambda)bcn. \end{aligned}$$

Therefore, by the definition of $e(B, C)$, we have

$$\begin{aligned} (d - \lambda)bcn &\leq \langle (dI_n - A)\mathbf{x}, \mathbf{x} \rangle \\ &= \sum_{uv \in E} (\mathbf{x}(u) - \mathbf{x}(v))^2 \\ &= e(B, C)(-c - b)^2 \\ &= e(B, C)n^2, \end{aligned}$$

and the theorem follows. \square

Now, we can turn to the bounds for percolating sets of (n, d, λ) -graphs. Our main goal is to find small 2-threshold percolating sets of 4-cycle free connected (n, d, λ) -graphs. Let us start from the next lemma.

Lemma 4.12 (Coja-Oghlan, Feige, Krivelevich and Reichman [13]). *Let $G = (V, E)$ be a connected (n, d, λ) -graph such that $\lambda < \delta d$ for some real number $\delta \in (0, 1)$, and let the threshold for percolation be $r = 2$. Then, every subset $A \subseteq V$ with $|A| > \frac{n}{(1-\delta)d}$ percolates on G .*

Proof. Suppose $B \subseteq V$ does not percolate on G . Let $\langle B \rangle = S$. Then $S \subsetneq V$, and for every $u \in V \setminus S$, $\deg_S(u) \leq 1$. Thus, $e(S, V \setminus S) \leq |V \setminus S| = n - |S|$. On the other hand, by Theorem 4.11, we have

$$e(S, V \setminus S) \geq \frac{(1 - \delta)d|S|(n - |S|)}{n},$$

and so

$$n - |S| \geq \frac{(1 - \delta)d|S|(n - |S|)}{n}.$$

Therefore, $|B| \leq |S| \leq \frac{n}{(1-\delta)d}$, and the lemma follows. \square

Next, we describe a procedure *SmallPercSetExpander* presented by Coja-Oghlan, Feige, Krivelevich and Reichman [13] for building a small 2-threshold percolating set of a given 4-cycle free connected (n, d, λ) -graph $G = (V, E)$ of

order n . The strategy is choose seeds one by one in a greedy manner. For an integer $t \geq 0$, let s_t denote the seed chosen in the round t , S_t be the set of seeds in the round t , define $A_t = \langle S_t \rangle$, $B_t = A_t \cup \partial_G(A_t)$, and $R_t = V \setminus B_t$ (illustrated in Figure 4.14). The procedure *SmallPercSetExpander* takes G and the threshold $r = 2$ as input and runs as follows.

1. Initially, let $S_0 = A_0 = B_0 = \emptyset$, and let $R_0 = V$.
2. For $t \geq 1$ and the given $S_{t-1}, A_{t-1}, B_{t-1}, R_{t-1}$, if $A_{t-1} = V$, then the procedure ends; Otherwise, go to the next step.
3. If $|B_{t-1}| < \frac{n}{2}$, then select a vertex $v \in V \setminus A_{t-1}$ as the seed s_t such that $S_t = S_{t-1} \cup \{v\}$ maximizes $|B_t|$. Otherwise, if $|B_{t-1}| \geq \frac{n}{2}$, then select a vertex $v \in V \setminus A_{t-1}$ as the seed s_t such that $S_t = S_{t-1} \cup \{v\}$ maximizes $|A_t|$.

It is straightforward that for every t in the process of *SmallPercSetExpander*, $S_t \subsetneq S_{t+1}$, $A_t \subseteq A_{t+1}$, $B_t \subseteq B_{t+1}$, and $R_t \supseteq R_{t+1}$. Let T be the smallest integer with $A_T = V$. Then, we get a percolating set S_T containing T seeds. Hence, we can estimate the cardinality of the set of seeds via investigation of the number of rounds the procedure *SmallPercSetExpander* runs. The faster A_t (or B_t) grows, the fewer rounds *SmallPercSetExpander* runs, and then the smaller set of seeds one can obtain.

The following lemmas show properties of *SmallPercSetExpander*.

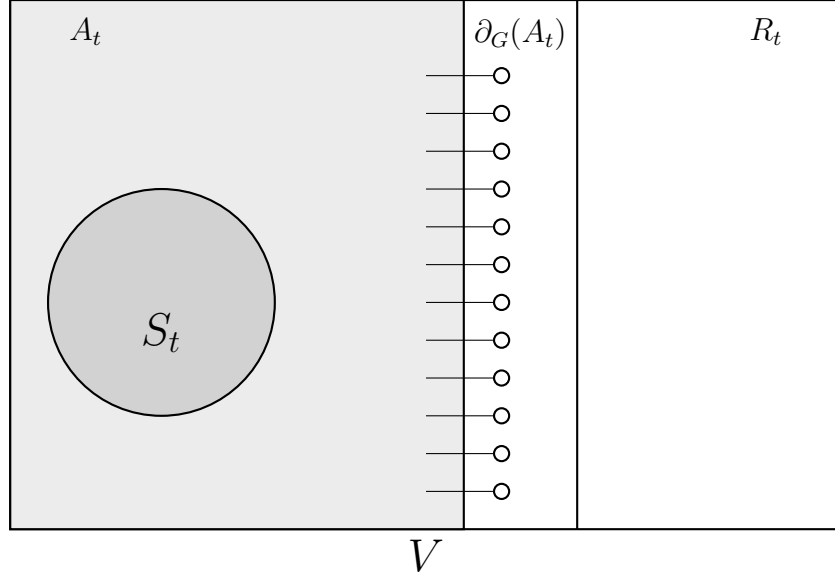


Figure 4.14: The state of vertices at the round t of *SmallPercSetExpander*

Lemma 4.13 (Coja-Oghlan, Feige, Krivelevich and Reichman [13]). *Let $G = (V, E)$ be a d -regular graph. Then for any $0 < t \leq \frac{n}{d}$ in *SmallPercSetExpander*, $|B_t| \geq \frac{d}{2}t$.*

Proof. We prove the lemma by induction on t .

For the base step when $t = 1$, we have $S_1 = A_1 = \{s_1\}$ where s_1 can be any vertex of G , and $|B_1| = 1 + d \geq \frac{d}{2}$.

For the inductive step, assume the lemma is true for $t < \frac{n}{d}$ and $t + 1 \leq \frac{n}{d}$. If $|B_t| \geq \frac{d}{2}(t+1)$, then $|B_{t+1}| \geq |B_t| \geq \frac{d}{2}(t+1)$, and the lemma follows. Otherwise, if $|B_t| < \frac{d}{2}(t+1)$, then $|B_t| < \frac{d}{2}(t+1) \leq \frac{n}{2}$, which implies $R_t \geq \frac{n}{2}$. Since G is d -regular, we have $\sum_{v \in V} \deg_{R_t}(v) = \sum_{u \in R_t} \deg_G(u) \geq \frac{nd}{2}$, and so there exist

a vertex $v \in V$ satisfying $\deg_{R_t}(v) \geq \frac{d}{2}$. Note that such a vertex v cannot be in A_t because every vertex in A_t has no neighbours in R_t . If we add v into S_t , then all the vertices in $N_{R_t}(v)$ will be included into $\langle S_t \cup \{v\} \rangle \cup \partial_G(\langle S_t \cup \{v\} \rangle)$, and so

$$|\langle S_t \cup \{v\} \rangle \cup \partial_G(\langle S_t \cup \{v\} \rangle)| - |B_t| \geq \frac{d}{2}.$$

Hence, since in the $(t + 1)$ -th step of the procedure *SmallPercSetExpander*, the choice of s_{t+1} maximizes $|B_{t+1}|$, we get $|B_{t+1} \setminus B_t| = |B_{t+1}| - |B_t| \geq \frac{d}{2}$. Therefore,

$$|B_{t+1}| \geq |B_t| + \frac{d}{2} \geq \frac{d}{2}t + \frac{d}{2} = \frac{d}{2}(t + 1),$$

and the lemma follows. \square

Lemma 4.14 (Coja-Oghlan, Feige, Krivelevich and Reichman [13]). *For a real number $\varepsilon \in (0, 1)$, let $G = (V, E)$ be a connected (n, d, λ) -graph with $\lambda \leq (1 - \varepsilon)d$ and without 4-cycles. Let $\frac{4}{\varepsilon^2 d} \leq c \leq \frac{d}{2}$ and let the threshold $r = 2$. For a set A of infected vertices of G , let $B = A \cup \partial_G(A)$ and let $R = V \setminus B$. If $|B| = \frac{cn}{d}$, then there is a vertex $u \in R$ such that $|N_R(N_B(u))| \geq \frac{2|B|}{|R|} \binom{\varepsilon d|R|/n}{2}$.*

Proof. We call triple vertices u, v, w a triplet if $w \in B$, $u, v \in R$ and $u - w, v - w \in E$. Let f denote the number of triplets on G . Then, $f = \sum_{w \in B} \binom{\deg_R(w)}{2}$.

By Theorem 4.11, we have

$$\sum_{w \in B} \deg_R(w) = e(B, R) \geq \frac{\varepsilon d |B| |R|}{n}.$$

Since $\frac{4}{\varepsilon^2 d} \leq c \leq \frac{d}{2}$ and $B = \frac{cn}{d}$, then $\varepsilon d \geq 2\sqrt{2}$, $|B| \leq \frac{n}{2}$ and $|R| \geq \frac{n}{2}$. Hence,

$$\frac{\sum_{w \in B} \deg_R(w)}{|B|} \geq \frac{\varepsilon d |R|}{n} \geq \frac{1}{2}.$$

Then, since the function $y = \binom{x}{2} = \frac{x(x-1)}{2}$ is convex on \mathbb{R} and increasing on the interval $[\frac{1}{2}, \infty)$, we get

$$\sum_{w \in B} \frac{1}{|B|} \binom{\deg_R(w)}{2} \geq \binom{\sum_{w \in B} \frac{\deg_R(w)}{|B|}}{2} \geq \binom{\frac{\varepsilon d |R|}{n}}{2},$$

and so $f \geq |B| \binom{\varepsilon d |R|/n}{2}$.

On the other hand, since every triplet involves two vertices in R , each vertex in R gets involved in $\frac{2f}{|R|}$ triplets in average. Hence, there exists some vertex $u \in R$ involved in at least $\frac{2f}{|R|}$ triplets. For any two such triplets $\{u, v_1, w_1\}$ and $\{u, v_2, w_2\}$, the vertices v_1, v_2 must be distinct because G is 4-cycle free. Therefore,

$$|N_R(N_B(u))| \geq \frac{2f}{|R|} \geq \frac{2|B|}{|R|} \binom{\varepsilon d |R|/n}{2},$$

and the lemma follows. \square

Now, we can present the main result about an asymptotic upper bounds for minimum 2-threshold percolating sets of connected (n, d, λ) -graphs without 4-cycles. The next theorem is a corrected version of a theorem by Coja-Oghlan, Feige, Krivelevich and Reichman [13].

Theorem 4.15. *For a real number $\varepsilon \in (0, 1)$, let $G = (V, E)$ be a connected (n, d, λ) -graph with $\lambda \leq (1 - \varepsilon)d$ and without 4-cycles. If $\varepsilon^2 d > 8$ and $d \in o(\sqrt{n})$, then $m(G, 2) \leq O\left(\frac{n \log d}{\varepsilon d(\varepsilon d - 2)}\right)$.*

Proof. We divide the procedure *SmallPercSetExpander* into three phases by the cardinality of B_t .

In the first phase, $|B_t|$ grows from 0 to $\frac{4n}{\varepsilon^2 d^2}$. Since $\varepsilon^2 d > 8$ and $d \in o(\sqrt{n})$, then $\lceil \frac{8n}{\varepsilon^2 d^2} \rceil \leq \frac{n}{d}$ when n is sufficiently large. By Lemma 4.13, if we take $t_0 = \lceil \frac{8n}{\varepsilon^2 d^2} \rceil$, then $|B_{t_0}| \geq \frac{d}{2} t_0 \geq \frac{4n}{\varepsilon^2 d} \geq \frac{4n}{\varepsilon^2 d^2}$. Therefore, the first phase can be finished within $O\left(\frac{n}{\varepsilon^2 d^2}\right)$ iterations.

In the second phase, $|B_t|$ grows from $\frac{4n}{\varepsilon^2 d^2}$ to $\frac{n}{2}$. Once we get B_{t_0} , suppose $|B_{t_0}| < \frac{n}{2}$, otherwise we can directly go to the next phase. Then, for each $i = 1, 2, \dots$, by Lemma 4.14, there exists a vertex $u_{t_0+i} \in R_{t_0+i-1}$ such that

$$|N(N(u_{t_0+i}) \cap B_{t_0+i-1}) \cap R_{t_0+i-1}| \geq \frac{2|B_{t_0+i-1}|}{|R_{t_0+i-1}|} \binom{\varepsilon d |R_{t_0+i-1}| / n}{2}.$$

Consequently, if we add u_{t_0+i} to S_{t_0+i-1} , then all the vertices in $N(u_{t_0+i}) \cap B_{t_0+i-1}$ will be contained in $\langle S_{t_0+i-1} \cup \{u_{t_0+i}\} \rangle$ (because the threshold is 2), and so

$$\begin{aligned} & N(N(u_{t_0+i}) \cap B_{t_0+i-1}) \cap R_{t_0+i-1} \\ & \subseteq \langle S_{t_0+i-1} \cup \{u_{t_0+i}\} \rangle \cup \partial(\langle S_{t_0+i-1} \cup \{u_{t_0+i}\} \rangle). \end{aligned}$$

Note that the vertices in $N(N(u_{t_0+i}) \cap B_{t_0+i-1}) \cap R_{t_0+i-1}$ are not in B_{t_0+i-1} .

Thus,

$$\begin{aligned} & |\langle S_{t_0+i-1} \cup \{u_{t_0+i}\} \rangle \cup \partial(\langle S_{t_0+i-1} \cup \{u_{t_0+i}\} \rangle)| \\ & \geq |B_{t_0+i-1}| + \frac{2|B_{t_0+i-1}|}{|R_{t_0+i-1}|} \binom{\varepsilon d |R_{t_0+i-1}|/n}{2}. \end{aligned}$$

Recall that the choice of s_{t_0+i} maximizes $|B_{t_0+i}|$. Hence, we can choose a vertex v_{t_0+i} as s_{t_0+i} such that

$$\begin{aligned} |B_{t_0+i}| & \geq |B_{t_0+i-1}| + \frac{2|B_{t_0+i-1}|}{|R_{t_0+i-1}|} \binom{\varepsilon d |R_{t_0+i-1}|/n}{2} \\ & = \left[1 + \frac{2}{|R_{t_0+i-1}|} \binom{\varepsilon d |R_{t_0+i-1}|/n}{2} \right] |B_{t_0+i-1}| \\ & \geq \left[1 + \frac{2}{|R_{t_0}|} \binom{\varepsilon d |R_{t_0}|/n}{2} \right] |B_{t_0+i-1}| \end{aligned}$$

until for some integer $M > 0$, $|B_{t_0+M}|$ exceeds $\frac{n}{2}$. Now, let $c = \frac{8}{\varepsilon^2 d^2}$. Hence,

since $|R_{t_0}| \geq \frac{n}{2}$, we get

$$\begin{aligned} |B_{t_0+M}| & \geq \left[1 + \frac{2}{|R_{t_0}|} \binom{\varepsilon d |R_{t_0}|/n}{2} \right]^M |B_{t_0}| \\ & = \left[1 + \frac{\varepsilon d}{n} \left(\frac{\varepsilon d |R_{t_0}|}{n} - 1 \right) \right]^M |B_{t_0}| \\ & \geq \left[1 + \frac{\varepsilon d}{n} \left(\frac{\varepsilon d}{2} - 1 \right) \right]^M |B_{t_0}| \\ & \geq \left[1 + \frac{\varepsilon d}{n} \left(\frac{\varepsilon d}{2} - 1 \right) \right]^M \frac{dt_0}{2} \\ & = \left[1 + \frac{\varepsilon d}{n} \left(\frac{\varepsilon d}{2} - 1 \right) \right]^M \frac{dcn}{2}. \end{aligned}$$

Moreover, note that for any real number $x \in [0, 1]$, we have $e^{x/2} \leq 1 + x \leq e^x$. Since $d = o(\sqrt{n})$ and $\varepsilon^2 d > 8$, when n is sufficiently large, we have $0 < \frac{\varepsilon d}{n} \left(\frac{\varepsilon d}{2} - 1 \right) < 1$, and so

$$|B_{t_0+M}| \geq \left[1 + \frac{\varepsilon d}{n} \left(\frac{\varepsilon d}{2} - 1 \right) \right]^M \frac{dcn}{2} \geq \exp \left[\frac{\varepsilon d}{2n} \left(\frac{\varepsilon d}{2} - 1 \right) M \right] \frac{dcn}{2}.$$

To seek an upper bound of M , let

$$\exp \left[\frac{\varepsilon d}{2n} \left(\frac{\varepsilon d}{2} - 1 \right) M \right] \frac{dcn}{2} = \frac{n}{2}.$$

Then, we get

$$M = \frac{4n \log(\frac{1}{cd})}{\varepsilon d(\varepsilon d - 2)} = \frac{4n \log(\varepsilon^2 d/8)}{\varepsilon d(\varepsilon d - 2)} \leq \frac{4n \log d}{\varepsilon d(\varepsilon d - 2)}.$$

Therefore, the second phase can be finished within $O\left(\frac{n \log d}{\varepsilon d(\varepsilon d - 2)}\right)$ iterations.

The third phase is when $|B_t| \geq \frac{n}{2}$. If $|A_t| > \frac{n}{\varepsilon d}$, then by Lemma 4.12, A_t will infect all the vertices of G , and so S_t will be a percolating set of G . So, suppose $|A_t| \leq \frac{n}{\varepsilon d}$. Moreover, we may suppose $d > \frac{10}{3\varepsilon}$, because otherwise we could get

$$O\left(\frac{n \log d}{\varepsilon d(\varepsilon d - 2)}\right) \geq O(n \log d) \geq m(G, 2),$$

and so the theorem directly follows. For these parameters, we have

$$|\partial_G(A_t)| = |B_t| - |A_t| \geq \frac{n}{2} - \frac{n}{\varepsilon d} \geq \frac{n}{2} - \frac{3n}{10} = \frac{n}{5}.$$

Since every vertex in $\partial_G(A_t)$ has exactly one neighbour in A_t and $d \geq 2$, then

$$e(\partial_G(A_t), V \setminus A_t) \geq (d-1) \frac{n}{5} \geq \frac{dn}{10},$$

and so

$$\frac{e(\partial_G(A_t), V \setminus A_t)}{|V \setminus A_t|} \geq \frac{e(\partial_G(A_t), V \setminus A_t)}{n} \geq \frac{d}{10}.$$

This implies there is a vertex in $V \setminus A_t$ having at least $\frac{d}{10}$ neighbours in $\partial_G(A_t)$. Hence, $|A_t|$ grows at least $\frac{d}{10}$ in each iteration of this phase until $|A_t|$ exceeds $\frac{n}{\varepsilon d}$. Once $|A_t| \geq \frac{n}{\varepsilon d}$, by Lemma 4.12 again, A_t can infect all the vertices of G , which implies we do not need any more seeds and the corresponding S_t is indeed a percolating set of G . Therefore, the third phase can be finished within $O(\frac{n}{\varepsilon d} / \frac{d}{10}) = O(\frac{n}{\varepsilon d^2})$ iterations.

In conclusion, we can obtain a 2-threshold percolating set of G within $O\left(\frac{n \log d}{\varepsilon d(\varepsilon d - 2)}\right)$ by the procedure *SmallPercSetExpander* because $\frac{n}{\varepsilon^2 d^2} = O\left(\frac{n \log d}{\varepsilon d(\varepsilon d - 2)}\right)$ and $\frac{n}{\varepsilon d^2} = O\left(\frac{n \log d}{\varepsilon d(\varepsilon d - 2)}\right)$, and the theorem follows. \square

In this chapter, we mainly introduced some approximate or exact results for minimum percolating sets of regular graphs. Meanwhile, we also listed some interesting questions worth further investigation.

Chapter 5

Bootstrap Percolation on Grid Graphs

In this chapter, we focus on bootstrap percolation on grid graphs whose vertices are some integral vectors in \mathbb{Z}^k for some positive integer k . Percolation process on this class of graphs has been extensively studied in random case ([11], [7], [6], [5]), and the extremal bounds for minimum or minimal percolating sets of grid graphs are often tools in the random threshold results. For bootstrap percolation on grid graphs, one of the big questions is for what values of parameters (such as dimension, the length of each side and the threshold) of grid graphs we understand the minimum percolating sets? More precisely, in what cases we have the exact value for minimum percolating sets and in what cases we have only obtained approximate or asymptotic values? These questions are the most important motivation for the contents in this chapter.

First, we introduce some concepts and notation used in this chapter.

For positive integers k and n_1, \dots, n_k , let $[n_1] \times \dots \times [n_k]$ or $\prod_{i=1}^k [n_i]$ denote the k -dimensional grid graph whose vertex set is $[n_1] \times \dots \times [n_k]$, and two vertices $u = (a_1, \dots, a_k)$ and $v = (b_1, \dots, b_k)$ are adjacent if and only if there exists a unit vector e_j (the j -th co-ordinate is 1 and any other coordinate is 0) for some $j = 1, \dots, k$ such that $u - v = e_j$. Then, the number of edges of $\prod_{i=1}^k [n_i]$ is

$$\begin{aligned} & (n_1 - 1)n_2 \cdots n_k + n_1(n_2 - 1)n_3 \cdots n_k + \cdots + n_1 \cdots n_{k-1}(n_k - 1) \\ &= \left(\prod_{i=1}^k n_i \right) \left(\frac{n_1 - 1}{n_1} + \cdots + \frac{n_k - 1}{n_k} \right) \\ &= \left(\prod_{i=1}^k n_i \right) \left(k - \sum_{i=1}^k \frac{1}{n_i} \right). \end{aligned}$$

It is straightforward that any permutation over the coordinates of $\prod_{i=1}^k [n_i]$ gives an isomorphic graph. In addition, the grid graph $[n_1] \times \dots \times [n_k] \times [1]$ is obviously isomorphic to $[n_1] \times \dots \times [n_k]$. Hence, it suffices in most time to discuss the grid graph $\prod_{i=1}^k [n_i]$ with every side length $n_i > 1$.

In the grid $\prod_{i=1}^k [n_i]$, for every $i = 1, \dots, k$, if $P_i \subseteq [n_i]$ be a set of m_i consecutive integers, then $\prod_{i=1}^k P_i = P_1 \times \dots \times P_k$ is called a k -dimensional $(m_1 \times \dots \times m_k)$ rectangle. In particular, each vertex of $\prod_{i=1}^k [n_i]$ is a k -dimensional $(1 \times \dots \times 1)$ rectangle. For a set $S \subseteq V(\prod_{i=1}^k [n_i])$, let $[S]$ denote the smallest rectangle containing S . For two rectangles $P = \prod_{i=1}^k P_i$ and $Q = \prod_{i=1}^k Q_i$, if for each $i = 1, \dots, k$, C_i is the smallest set of consecutive

integers containing $P_i \cup Q_i$, then it is straightforward that the rectangle $C = \prod_{i=1}^k C_i = [P \cup Q]$. In addition, for a set $A \subseteq \mathbb{Z}^k$ and a vector $\mathbf{y} \in \mathbb{Z}^k$, define $\mathbf{y} + A = A + \mathbf{y} = \{\mathbf{y} + \mathbf{x} : \mathbf{x} \in A\}$.

Since the maximum degree of the grid graph $\prod_{i=1}^k [n_i]$ is $2k$, by the facts mentioned in Section 2.1, it suffices to consider r -threshold bootstrap percolation on $\prod_{i=1}^k [n_i]$ for $2 \leq r \leq 2k$.

5.1 Minimum Percolating Sets of Square Grid Graphs

In this section, we shall focus on the minimum percolating sets of square grid graphs. For a grid graph $\prod_{i=1}^k [n_i]$, if $n_1 = \dots = n_k = n$, then we call it a square grid graph, and denote it by $[n]^k$. As we mentioned above, the big question in this case is – for different values of the parameters n, k, r , how much information can we obtain about $m([n]^k, r)$ approximately or exactly? To answer this question, we are going to consider two aspects of $m([n]^k, r)$, namely, approximate results and exact results. Most results we are going to show in this section come from Balogh and Pete [7], and Pete [29].

We consider approximate results about $m([n]^k, r)$ first and start from some bounds for minimum percolating sets of square grid graphs.

Lemma 5.1 (Pete [29], Perimeter Lemma). *Let n, k, r be positive integers with $r \leq 2k$. Then*

$$m([n]^k, r) \geq \frac{r-k}{r}n^k + \frac{k}{r}n^{k-1}.$$

Proof. Let A be a minimum r -threshold percolating set of $[n]^k$. Note that $[n]^k$ has n^k vertices and $kn^{k-1}(n-1)$ edges. Then, by Lemma 2.2, we get

$$m([n]^k, r) = |A| \geq \frac{r-k}{r}n^k + \frac{k}{r}n^{k-1},$$

and the lemma follows. \square

Note that although Lemma 5.1 is true for all $r \leq 2k$, it only makes sense when $r \geq k$ because when $r < k$, the lower bound in Lemma 5.1 is non-positive.

Lemma 5.2 (Pete [29], Projection Lemma). *Let n, k, r be positive integers with $r \leq 2k$. Then, $m([n]^k, r) \geq m([n]^{k-1}, r)$.*

Proof. Let $\pi : V([n]^k) \rightarrow V([n]^{k-1})$ be the projection defined by

$$\pi(x_1, \dots, x_{k-1}, x_k) = (x_1, \dots, x_{k-1}).$$

Then, π is surjective and for each $(x_1, \dots, x_{k-1}) \in [n]^{k-1}$,

$$\pi^{-1}(x_1, \dots, x_{k-1}) = \{(x_1, \dots, x_{k-1}, x_k) : x_k \in [n]\}.$$

Call each set $\pi^{-1}(x_1, \dots, x_{k-1})$ a class of $V([n]^k)$ and identify it with the vertex (x_1, \dots, x_{k-1}) of $[n]^{k-1}$. For any two vertices, say $(x_1, \dots, x_{k-1}, x_k)$

and $(x_1, \dots, x_{k-1}, y_k)$, in a same class of $V([n]^k)$, suppose (z_1, \dots, z_k) is one of their common neighbours in $[n]^k$. If there exists a j with $1 \leq j \leq k-1$ such that $x_j \neq z_j$, then $z_k = x_k \neq y_k$ and $z_j \neq x_j$ which would lead to (x_1, \dots, y_k) and (z_1, \dots, z_k) are non-adjacent in $[n]^k$, and so would contradict their adjacency. So, if two vertices in a class $\pi^{-1}(x_1, \dots, x_{k-1})$ of $V([n]^k)$ have common neighbours, then their common neighbours are in $\pi^{-1}(x_1, \dots, x_{k-1})$ too. In addition, if two vertices (x_1, \dots, x_k) and (x'_1, \dots, x'_k) are adjacent in $[n]^k$ but from different classes, then they differ at exactly one of the first $k-1$ coordinates. In other words, their image (x_1, \dots, x_{k-1}) and (x'_1, \dots, x'_{k-1}) under π are adjacent in $[n]^{k-1}$.

Now, let A_0 be a minimum r -threshold percolating set of $[n]^k$, A_i be the set of activated vertices after time i in the percolation process starting from A_0 , and T be the smallest non-negative integer such that $A_T = V([n]^k)$. For each $i = 0, \dots, T$, define $B_i = \pi(A_i)$. Then, $|B_0| \leq |A_0|$, $B_T = V([n]^{k-1})$, and for each $i = 0, \dots, T-1$, $B_i \subseteq B_{i+1}$. We are going to show B_0 is an r -threshold percolating set of $[n]^{k-1}$. To do this, it suffices to show for $i = 0, \dots, T-1$, if $B_{i+1} \setminus B_i \neq \emptyset$, then every vertex in $B_{i+1} \setminus B_i$ has at least r neighbours in B_i . Fix an i , suppose $(y_1, \dots, y_{k-1}) \in B_{i+1} \setminus B_i$. Then, there exists a vertex $(y_1, \dots, y_{k-1}, y_k) \in A_{i+1}$ and for any $y'_k \in [n]$, $(y_1, \dots, y_{k-1}, y'_k) \notin A_i$. Thus, $(y_1, \dots, y_{k-1}, y_k)$ has r neighbours $X_1, \dots, X_r \in A_i$ that are not in the class $\pi^{-1}(y_1, \dots, y_{k-1})$. Since each pair of X_1, \dots, X_r has a common neighbour

$(y_1, \dots, y_{k-1}, y_k)$ in a class other than any class of theirs, by the argument in the preceding paragraph, $\pi(X_1), \dots, \pi(X_r)$ are distinct and all adjacent to (y_1, \dots, y_{k-1}) . Thus, B_0 is indeed an r -threshold percolating set of $[n]^{k-1}$. Therefore,

$$m([n]^k, r) = |A_0| \geq |B_0| \geq m([n]^{k-1}, r).$$

□

The following upper bound of $m([n]^k, r)$ comes from splitting $V([n]^k)$ to some lower dimensional faces and recursively building a percolating set.

Lemma 5.3 (Pete [29], Recursive Lemma). *Let n, k, r be positive integers with $n \geq 3$ and $r \leq 2k$, and $f_k(j) = \binom{k}{k-j} 2^{k-j}$ be the number of j -dimensional faces of the k -dimensional cube. Then,*

$$m([n]^k, r) \leq \sum_{i=0}^k f_k(k-i) m([n-2]^{k-i}, r-i) = \sum_{i=0}^k \binom{k}{i} 2^i m([n-2]^{k-i}, r-i).$$

Proof. Let $G = [n]^k$ and $S = \{1, n\}$. Then, for $i = 0, \dots, k$, there are $f_k(k-i) = \binom{k}{i} 2^i$ pairwise disjoint subsets $W_{i,1}, \dots, W_{i,f_k(k-i)}$ of vertices of $V(G)$ that have exactly i co-ordinates in S . Define $V_i = \bigcup_{j=1}^{f_k(k-i)} W_{i,j}$ for $i = 0, \dots, k$. Then, it is easy to check $\bigcup_{i=0}^k V_i$ is a partition of $V(G)$ (an 2-dimensional example is illustrated in Figure 5.1). Note that for each $W_{i,j}$, the induced subgraph $G[W_{i,j}]$ is isomorphic to the grid $[n-2]^{k-i}$.

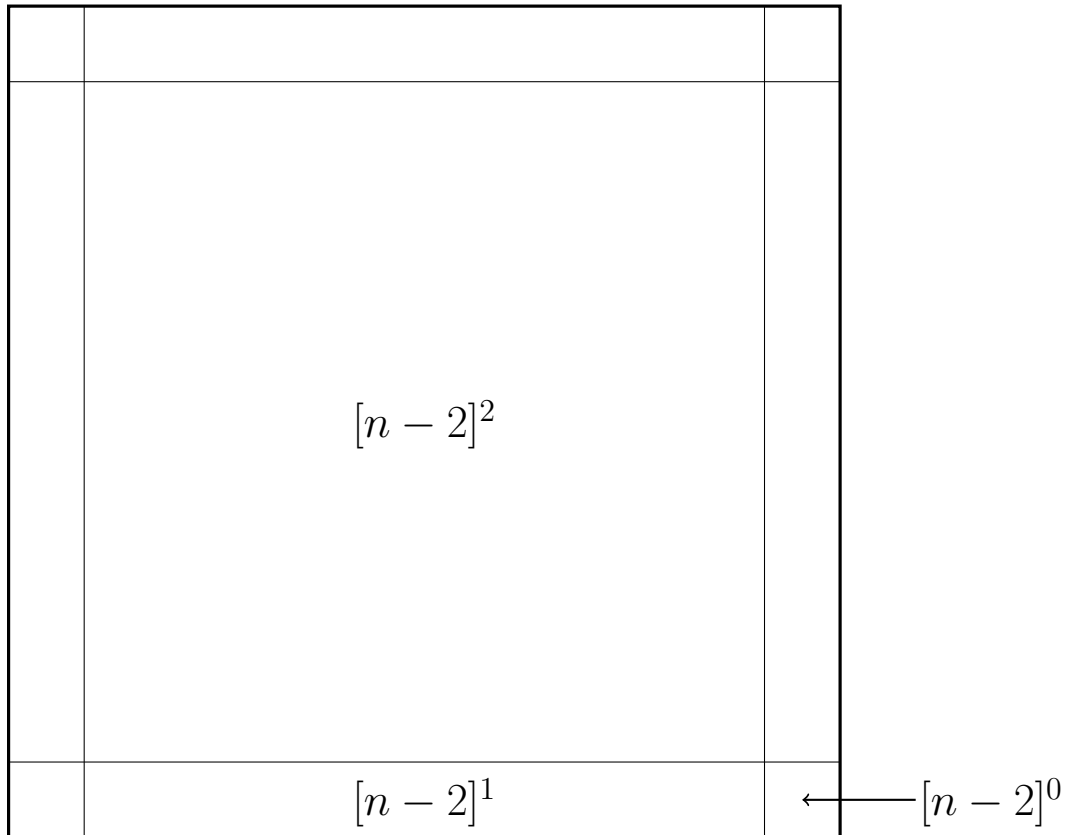


Figure 5.1: Decomposition of the grid graph $[n]^2$ to 1 copy of 2-dimensional grid graph $[n - 2]^2$, 4 copies of 1-dimensional grid graph $[n - 2]^1$ and 4 copies of 0-dimensional grid graph $[n - 2]^0$

Now, for each $W_{i,j}$, pick an $(r - i)$ -threshold minimum percolating set $A_{i,j}$ of $G[W_{i,j}] \cong [n - 2]^{k-i}$. We are going to show $\bigcup_{i,j} A_{i,j}$ is an r -threshold percolating set of G . First, since $A_{0,1}$ is an r -threshold percolating set of $G[W_{0,1}] = G[V_0]$, V_0 can be activated by $\bigcup_j A_{0,j}$. Next, since each vertex in each $W_{1,j}$ has a neighbour in V_0 , each vertex in $W_{1,j}$ will get an activated neighbour out of $W_{1,j}$ after V_0 is activated. Meanwhile, since $A_{1,j}$ is an $(r - 1)$ -threshold percolating set of $G[W_{1,j}]$, $W_{1,j}$ will eventually get activated by V_0 and $A_{1,j}$, which implies $V_0 \cup V_1$ can be activated by $\bigcup_{i=0}^1 \bigcup_j A_{i,j}$. Likewise, since each vertex in each $W_{2,j}$ has two neighbours in V_1 , each vertex in $W_{2,j}$ will get two activated neighbours out of $W_{2,j}$ after V_1 is activated. Meanwhile, since $A_{2,j}$ is an $(r - 2)$ -threshold percolating set of $G[W_{2,j}]$, $W_{2,j}$ will eventually get activated by V_1 and $A_{2,j}$, which implies $V_0 \cup V_1 \cup V_2$ can be activated by $\bigcup_{i=0}^2 \bigcup_j A_{i,j}$. Continuing the process, all the vertices in $\bigcup_{i=0}^k V_i = V(G)$ can finally become activated. So, $\bigcup_{i,j} A_{i,j}$ is indeed an r -threshold percolating set of G . Therefore,

$$m([n]^k, r) \leq \left| \bigcup_{i,j} A_{i,j} \right| = \sum_{i=0}^k f_k(k-i) m([n-2]^{k-i}, r-i).$$

□

Combining the above lemmas, we have an asymptotic result about $m([n]^k, r)$ given by Pete [29] when n is large enough. Our proof gives much more details

than the original one did.

Theorem 5.4 (Pete [29]). *Let n, k, r be positive integers with $n \geq 3$ and $r \leq 2k$.*

Then,

$$m([n]^k, r) = \begin{cases} \Theta(n^{r-1}), & r \leq k \\ \Theta(n^k), & k+1 \leq r \leq 2k \end{cases} .$$

Proof. If $k = 1$, then $[n]^k$ is just a path of n vertices. It is straightforward that $m([n], 1) = 1$. Besides, by Theorem 3.1, $m([n], 2) = \lceil \frac{n+1}{2} \rceil$. Hence, the theorem is true when $k = 1$.

From now, suppose $k \geq 2$.

First, consider the case when $r \leq k$. It is straightforward that $m([n]^k, 0) = 0$ and $m([n]^k, 1) = 1$, which implies the theorem is true when $r = 0$ and $r = 1$. So, suppose $r > 1$. On one hand, by Lemma 5.2 and Lemma 5.1,

$$m([n]^k, r) \geq m([n]^{k-1}, r) \geq \cdots \geq m([n]^r, r) \geq n^{r-1}.$$

On the other hand, define a sequence of positive real numbers $\langle c_i \rangle_{i \geq 2}$ recursively by $c_2 = 2^k k$ and $c_{i+1} = 2^{k-1} 3^i c_i$, and we are going to show

$$m([n]^k, r) \leq \binom{k}{r-1} c_r n^{r-1} \tag{5.1}$$

by induction on r . Note that the sequence $\langle c_i \rangle_{i \geq 2}$ is increasing, and each c_i does not depend on n .

For the base step when $r = 2$, we distinguish two cases by whether n is odd or even. If n is odd, then by Lemma 5.3, we get

$$\begin{aligned}
m([n]^k, 2) &\leq m([n-2]^k, 2) + 2 \binom{k}{1} m([n-2]^{k-1}, 1) \\
&= m([n-2]^k, 2) + 2k \\
&\leq m([n-4]^k, 2) + 2k + 2k \\
&\leq \dots\dots \\
&\leq m([1]^k, 2) + 2k \frac{n-1}{2} \\
&= 1 + k(n-1) \\
&\leq kn \\
&\leq \binom{k}{1} k 2^k n \\
&= \binom{k}{2-1} c_2 n^{2-1}.
\end{aligned}$$

If n is even, then by Lemma 5.3 again, we get

$$\begin{aligned}
m([n]^k, 2) &\leq m([n-2]^k, 2) + 2 \binom{k}{1} m([n-2]^{k-1}, 1) \\
&= m([n-2]^k, 2) + 2k \\
&\leq m([n-4]^k, 2) + 2k + 2k \\
&\leq \dots\dots \\
&\leq m([2]^k, 2) + 2k \left(\frac{n}{2} - 1\right) \\
&\leq 2^k + k(n-2) \\
&\leq k 2^{k-1} + kn \\
&\leq \left(k 2^{k-1} + \binom{k}{1}\right) n
\end{aligned}$$

$$\begin{aligned}
&\leq \left(k2^{k-1} + 2\binom{k}{1} \right) n \\
&\leq \left(k2^{k-1}2\binom{k}{1} \right) n \\
&= k2^k \binom{k}{1} n \\
&= \binom{k}{2-1} c_2 n^{2-1}.
\end{aligned}$$

Therefore, the inequality 5.1 is correct when $r = 2$.

For the inductive step, assume the inequality 5.1 holds for $r = 2, \dots, j < k$ and consider the case when $r = j + 1$. We distinguish two cases by whether n is odd or even. If n is odd, then by Lemma 5.3 and induction assumption, we get

$$\begin{aligned}
m([n]^k, j+1) &\leq m([n-2]^k, j+1) + \sum_{i=1}^j \binom{k}{i} 2^i m([n-2]^{k-i}, j+1-i) \\
&\leq m([n-2]^k, j+1) + \sum_{i=1}^j \binom{k}{i} 2^i \binom{k-i}{j-i} c_{j+1-i} n^{j-i} \\
&= m([n-2]^k, j+1) + \sum_{i=1}^j \binom{k}{i} \binom{k-i}{j-i} 2^i c_{j+1-i} n^{j-i} \\
&= m([n-2]^k, j+1) + \sum_{i=1}^j \binom{k}{j} \binom{j}{i} 2^i c_{j+1-i} n^{j-i} \\
&\leq m([n-2]^k, j+1) + \sum_{i=1}^j \binom{k}{j} \binom{j}{i} 2^i c_j n^{j-1} \\
&\leq m([n-2]^k, j+1) + \binom{k}{j} c_j n^{j-1} \left(\sum_{i=0}^j \binom{j}{i} 2^i \right)
\end{aligned}$$

$$\begin{aligned}
&= m([n-2]^k, j+1) + \binom{k}{j} 3^j c_j n^{j-1} \\
&\leq \dots\dots \\
&\leq m([1]^k, j+1) + \binom{k}{j} 3^j c_j n^{j-1} \frac{n-1}{2} \\
&= 1 + \binom{k}{j} 3^j c_j n^{j-1} \frac{n-1}{2} \\
&\leq \binom{k}{j} 3^j c_j n^{j-1} \frac{n}{2} \\
&= \binom{k}{j} \frac{3^j}{2} c_j n^j \\
&\leq \binom{k}{j} 2^k \frac{3^j}{2} c_j n^j \\
&= \binom{k}{j} 2^{k-1} 3^j c_j n^j \\
&= \binom{k}{j} c_{j+1} n^j.
\end{aligned}$$

If n is even, then by Lemma 5.3 and induction assumption again, we get

$$\begin{aligned}
m([n]^k, j+1) &\leq m([n-2]^k, j+1) + \binom{k}{j} 3^j c_j n^{j-1} \\
&\leq \dots\dots \\
&\leq m([2]^k, j+1) + \binom{k}{j} 3^j c_j n^{j-1} \left(\frac{n}{2} - 1\right) \\
&\leq 2^k + \binom{k}{j} 3^j c_j n^{j-1} \left(\frac{n}{2} - 1\right) \\
&\leq 2^k + \binom{k}{j} 3^j c_j n^{j-1} \frac{n}{2} \\
&= 2^k + \binom{k}{j} \frac{3^j}{2} c_j n^j
\end{aligned}$$

$$\begin{aligned}
&\leq \left(2^k + \binom{k}{j} \frac{3^j}{2} c_j\right) n^j \\
&\leq \binom{k}{j} 2^k \frac{3^j}{2} c_j n^j \\
&= \binom{k}{j} 2^{k-1} 3^j c_j n^j \\
&= \binom{k}{j} c_{j+1} n^j.
\end{aligned}$$

So, the inequality 5.1 is correct for any $r = 2, \dots, k$. Therefore, $m([n]^k, r) = \Theta(n^{r-1})$ when $r \leq k$.

Next, consider the case when $k+1 \leq r \leq 2k$. On one hand, by Lemma 5.1,

$$m([n]^k, r) \geq \frac{r-k}{r} n^k + \frac{k}{r} n^{k-1} \geq \frac{r-k}{r} n^k.$$

On the other hand, $m([n]^k, r) \leq |V([n]^k)| = n^k$. Therefore, $m([n]^k, r) = \Theta(n^k)$ when $k+1 \leq r \leq 2k$. \square

Besides asymptotic results, exact results about $m([n]^k, r)$ under some conditions are also known. So, next, we turn to those exact results.

By Lemma 5.1, $m([n]^k, k) \geq n^{k-1}$. It is straightforward to check a diagonal $\{(x_1, x_1) : x_1 \in [n]\}$ is a 2-threshold percolating set of $[n]^2$, and a 3-dimensional diagonal $\{(x_1, x_1 + x_2, x_2) : x_1 \in [n], x_2 \in [n]\}$ is a 3-threshold percolating set of $[n]^3$. Consequently, $m([n]^2, 2) = n$ and $m([n]^3, 3) = n^2$, which indicates the

lower bound in Lemma 5.1 is tight to 2-dimensional and 3-dimensional square grid graphs with $k = r$. But is it tight for higher dimensional grid graphs? The formula $m([n]^k, k) = n^{k-1}$ has been sort of “folklore knowledge” in the area of bootstrap percolation since Pete stated it in [29]. However, there has been no formal proof for this formula until one was given by Przykucki and Shelton [31] recently. Their proof, which we will not repeat here, relied on the investigation of the subsets $V_l = \left\{ (v_1, \dots, v_k) \in V([n]^k) : \sum_{i=1}^k v_i = l \right\}$ of $V([n]^k)$ for $l = n, 2n, \dots, kn$ and the set $A = \bigcup_{i=1}^k V_{in}$. Briefly speaking, they first indicated $|A| = n^{k-1}$. Next, they showed $V_{\lceil \frac{k}{n} \rceil n}$ can activate all the vertices in $\bigcup_{j < \lceil \frac{k}{n} \rceil n} V_j$. Finally, for $s = \lceil \frac{k}{n} \rceil + 1, \dots, k$, they showed $V_{(s-1)n} \cup V_{sn}$ can activate all the vertices in $\bigcup_{j=(s-1)n+1}^{sn-1} V_j$ by constructing a tree whose vertices are labelled by vertices in $\bigcup_{j=(s-1)n}^{sn} V_j$. Therefore, they finally made sure that the set A is indeed a k -threshold percolating set for $[n]^k$ and presented the following theorem.

Theorem 5.5 (Przykucki and Shelton [31]). *Let n and k be positive integers. Then, $m([n]^k, k) = n^{k-1}$.*

In [7], Balogh and Pete summarized exact and asymptotic results about minimum percolating set of 2-dimensional and 3-dimensional square grid graphs as follows, where we modify the result for $m([n]^3, 5)$.

Theorem 5.6 (Balogh and Pete [7]). *For 2-dimensional square grid graphs,*

we have

1. $m([n]^2, 1) = 1$;
2. $m([n]^2, 2) = n$;
3. $m([n]^2, 3) \sim \frac{1}{3}n^2$;
4. $m([n]^2, 4) \sim \frac{1}{2}n^2$.

Theorem 5.7 (Balogh and Pete [7]). *For 3-dimensional square grid graphs, we have*

1. $m([n]^3, 1) = 1$;
2. $m([n]^3, 2) = \lfloor \frac{3}{2}n \rfloor$;
3. $m([n]^3, 3) = n^2$;
4. $m([n]^3, 4) \sim \frac{1}{4}n^3$;
5. *For any $\varepsilon \in (0, 1)$, $\frac{2}{5}n^3 \leq m([n]^3, 5) \leq \frac{5}{12}n^3 + \varepsilon$ when n is big enough;*
6. $m([n]^3, 6) \sim \frac{1}{2}n^3$.

Instead of thorough proofs for Theorem 5.6 and Theorem 5.7, we just give brief explanations for these results here.

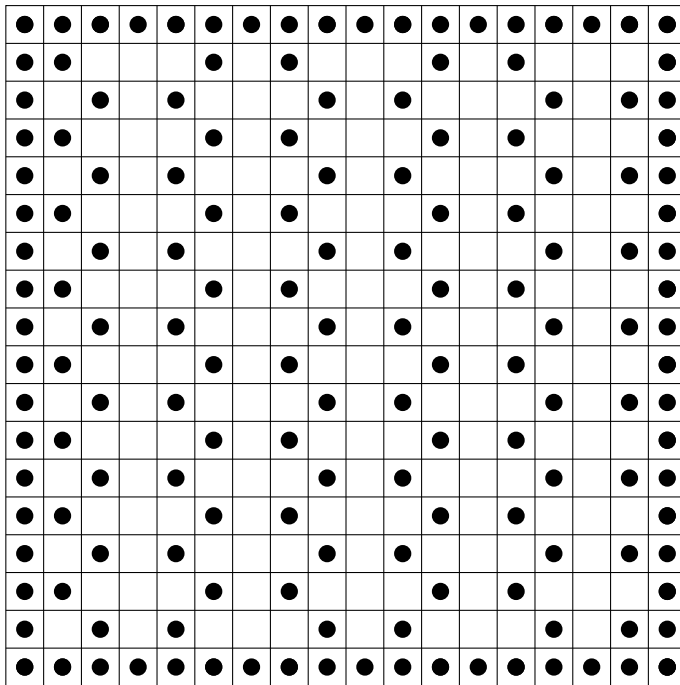


Figure 5.2: A 3-threshold percolating set A of $[n]^2$ with $|A| \sim \frac{1}{3}n^2$

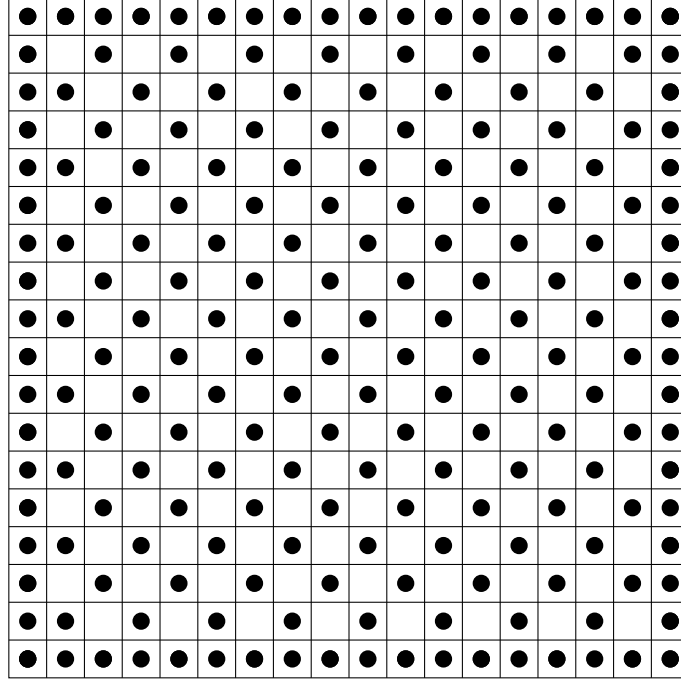


Figure 5.3: A 4-threshold percolating set A of $[n]^2$ with $|A| \sim \frac{1}{2}n^2$

For $m([n]^2, 3)$, by Lemma 5.1, we have $m([n]^2, 3) \geq \frac{1}{3}n^2 + \frac{2}{3}n$. Besides, a 3-threshold percolating set A of $[n]^2$ with $|A| \sim \frac{1}{3}n^2$ is illustrated in Figure 5.2.

For $m([n]^2, 4)$, by Lemma 5.1, we have $m([n]^2, 4) \geq \frac{1}{2}n^2 + \frac{1}{2}n$. Besides, a 4-threshold percolating set A of $[n]^2$ with $|A| \sim \frac{1}{2}n^2$ is illustrated in Figure 5.3.

For $m([n]^3, 2)$, by *rectangle process* we are going to introduce in the next section, we have $m([n]^3, 2) \geq \lfloor \frac{3n}{2} \rfloor = \lceil \frac{3n-1}{2} \rceil$. Besides, let A be a diagonal of the bottom layer $\{(x, y, 1) : x, y \in [n]\}$ of $[n]^3$, then A is a minimum 2-threshold percolating set of the bottom layer and $|A| = n$. Moreover, since the subgraph

of $[n]^3$ induced by the set $\{(1, 1, z) : 3 \leq z \leq n\}$ is just a path of length $n - 2$, it has a minimum percolating set B of cardinality $\lceil \frac{n-1}{2} \rceil$. It is straightforward that $A \cup B$ can percolate on $[n]^3$ and $|A \cup B| = \lceil \frac{3n-1}{2} \rceil$. Therefore, $m([n]^3, 2) = \lfloor \frac{3n}{2} \rfloor$.

For $m([n]^3, 4)$, by Lemma 5.1, we have $m([n]^3, 4) \geq \frac{1}{4}n^3 + \frac{3}{4}n^2$. On the other hand, for each $i = 1, \dots, n$, define a layer $L(i) = \{(x, y, i) : x, y \in [n]\}$ and separate $[n]^3$ into n layers from the bottom to the top. By discussion about $m([n]^2, 4)$, for each odd i and $i = n$, the layer $L(i)$ has a minimum 4-threshold percolating set A_i with $|A_i| \sim \frac{1}{2}n^2$. For each even i , define a set A_i as a diagonal of $L(i)$ (illustrated in Figure 5.4), then $|A_i| = n$. Let $A = \bigcup_{i=1}^n A_i$ be the set of seeds. In the 4-threshold percolation process starting from A , the odd layers and $L(n)$ can be internally infected by A . Thereafter, for each even layer $L(i)$, since every vertex in $L(i)$ has two infected neighbours from the upper layer and the lower layer and A_i is a 2-threshold percolating set of $L(i)$, then $L(i)$ will get infected. Therefore, A is a 4-threshold percolating set of $[n]^3$ with

$$|A| \sim \frac{n}{2} (|A_1| + |A_2|) \sim \frac{n}{2} \left(\frac{1}{2}n^2 + n \right) \sim \frac{1}{4}n^3.$$

For $m([n]^3, 5)$, by Lemma 5.1, we have $m([n]^3, 5) \geq \frac{2}{5}n^3 + \frac{3}{5}n^2$. On the other hand, we define 4 subsets of $V([n]^2)$ as follows.

1. Define $B_1 \subseteq V([n]^2)$ as the set illustrated in Figure 5.3. Then B_1 is a

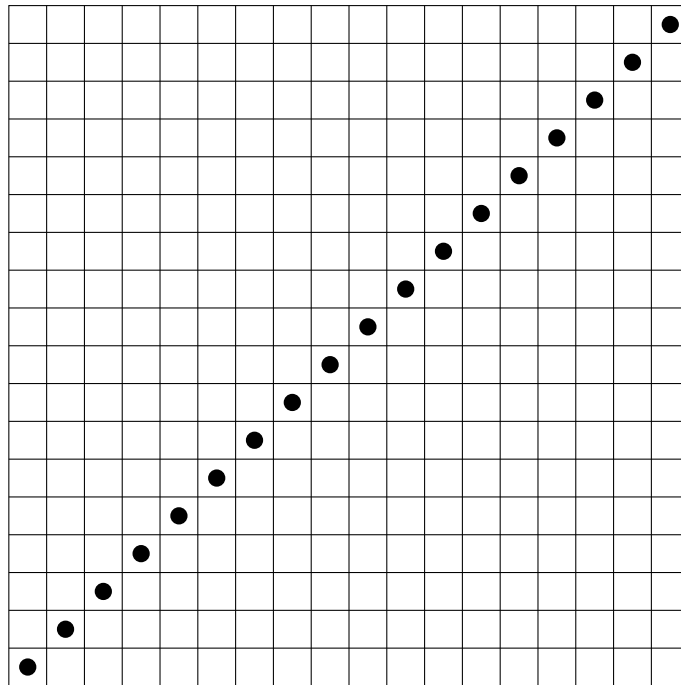


Figure 5.4: A set of seeds in the layer $L(2)$ of $[n]^3$ for 4-threshold percolation

- 4-threshold percolating set of $[n^2]$ with $|B_1| \sim \frac{1}{2}n^2$.
2. Define $B_2 \subseteq V([n]^2)$ as the set illustrated in Figure 5.5. Then B_2 is a 4-threshold percolating set of $[n^2]$ with $|B_2| \sim \frac{1}{2}n^2$.
 3. Define $B_3 \subseteq V([n]^2)$ as the set illustrated in Figure 5.2. Then B_3 is a 3-threshold percolating set of $[n^2]$ with $|B_3| \sim \frac{1}{3}n^2$.
 4. Define $B_4 \subseteq V([n]^2)$ as the set illustrated in Figure 5.6. Then B_4 is a 3-threshold percolating set of $[n^2]$ with $|B_4| \sim \frac{1}{3}n^2$.

Then, separate $[n]^3$ into n layers $L(1), \dots, L(n)$ from the bottom to the top again as we did for $m([n]^3, 4)$, and give each layer $L(i)$ a set A_i of seeds as follows. For the bottom two and the top two layers, let $A_1 = B_1$, $A_2 = B_2$, $A_{n-1} = B_1$ and $A_n = B_2$; for each odd $i = 3, \dots, n-3$, assign B_3 and B_4 to A_i alternately; for each even $i = 3, \dots, n-3$, let $A_i = B_1$. Finally, let $A = \bigcup_{i=1}^n A_i$ be the set of seeds. In the 5-threshold percolation process starting from A on $[n]^3$, first, $L(1), L(2)$ can help each other get infected, and so do $L(n-1), L(n)$; second, for each even $i = 3, \dots, n-3$, $L(i)$ can get infected with help of $L(i-1)$ and $L(i+1)$; thereafter, since for each odd $i = 3, \dots, n-3$, A_i is a 3-threshold percolating set of $L(i)$ and $L(i-1), L(i+1)$ are already infected, then $L(i)$ will get infected. Therefore, A is a 5-threshold percolating set with

$$|A| \sim 2n^2 + \frac{n-4}{2} \left(\frac{1}{3}n^2 + \frac{1}{2}n^2 \right) \sim \frac{5}{12}n^3 < \frac{3}{7}n^3.$$

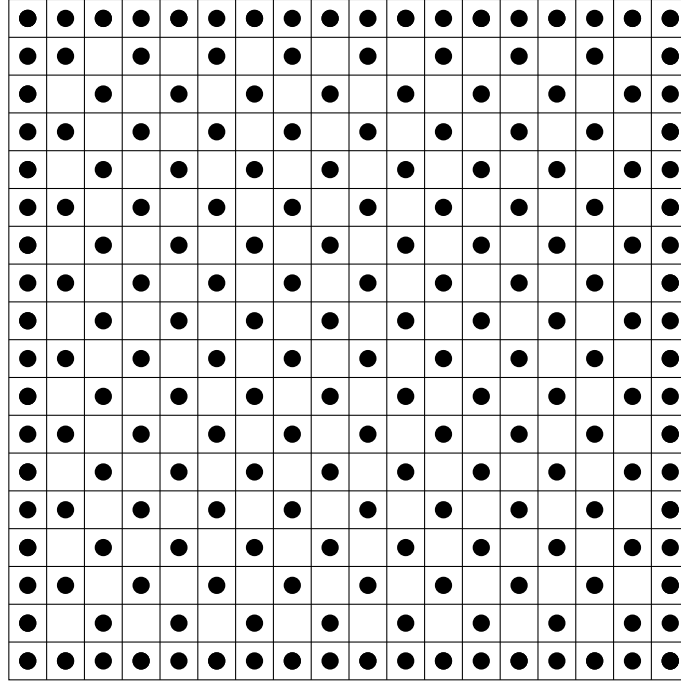


Figure 5.5: A 4-threshold percolating set B_2 of $[n]^2$ with $|B_2| \sim \frac{1}{2}n^2$

For $m([n]^3, 6)$, by Lemma 5.1, we have $m([n]^3, 6) \geq \frac{1}{2}n^3 + \frac{1}{2}n^2$. On the other hand, for the bottom layer and the top layer, define $A_1 = L(1)$ and $A_n = L(n)$; for each odd $i = 1, \dots, n$, give the layer $L(i)$ a set A_i of seeds illustrated in Figure 5.3; whilst for each even $i = 1, \dots, n$, give the layer $L(i)$ a set A_i of seeds illustrated in Figure 5.5. Then, the set $A = \bigcup_{i=1}^n A_i$ will be a 6-threshold percolating set of $[n]^3$ with

$$|A| \sim 2n^2 + (n-2) \cdot \frac{1}{2}n^2 \sim \frac{1}{2}n^3.$$

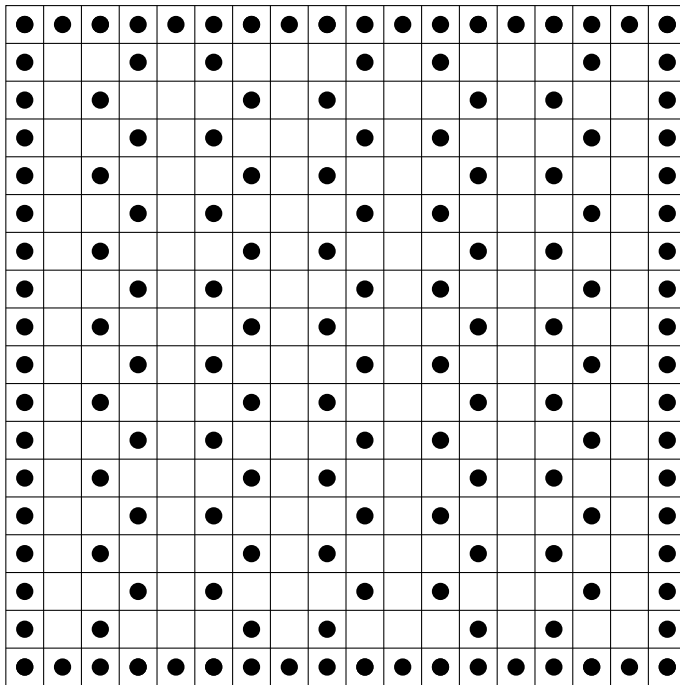


Figure 5.6: A 3-threshold percolating set B_4 of $[n]^2$ with $|B_4| \sim \frac{1}{3}n^2$

A special case of square grid graphs is when each side length $n_i = 2$, and $\prod_{i=1}^k [2]$ is called the k -dimensional hypercube and usually denoted by Q_k . A lot of problems about bootstrap percolation on hypercubes have been studied, and the most significant questions are also centred at minimum percolating sets. For example, Balogh and Bollobás [4] presented the following conjecture.

Conjecture 5.8 (Balogh and Bollobás [4]). *For a fixed threshold r and a large enough dimension k ,*

$$m(Q_k, r) = \frac{1 + o(1)}{r} \binom{k}{r-1}.$$

Then, Balogh, Bollobás and Morris [6] gave an upper bound for Conjecture 5.8 by letting the set of seeds A_0 consist of all vertices on “level $r - 2$ ” of Q_k and an approximate Steiner system on “level r ”. Moreover, by the celebrated result of Keevash [19], the approximate Steiner system in this construction can be replaced by an exact Steiner system under some conditions on k and r . Hence, they got an r -threshold percolating set of cardinality $\frac{1}{r} \binom{k}{r-1} + \binom{k}{r-2}$ which yields

$$m(Q_k, r) \leq \frac{k^{r-1}}{r!} + \frac{k^{r-2}(r+2)}{2r(r-2)!} + O(k^{r-3}).$$

Later, by utilizing a relationship between bootstrap percolation and the notion of weak saturation introduced by Bollobás [9], Morrison and Noel [24]

showed that for any $k \geq r \geq 1$,

$$m(Q_k, r) \geq 2^{r-1} + \sum_{j=1}^{r-1} \binom{k-j-1}{r-j} \frac{j2^{j-1}}{r}.$$

By this inequality, they got a lower bound

$$m(Q_k, r) \geq \frac{k^{r-1}}{r!} + \frac{k^{r-2}(6-r)}{2r(r-2)!} + \Omega(k^{r-3})$$

for Conjecture 5.8 and established the correctness of it. Besides, by a recursive approach, Morrison and Noel [24] also obtained an exact value for minimum 3-threshold percolating sets of hypercubes. We only state their result for this special case in the next theorem without repeating their proof.

Theorem 5.9 (Morrison and Noel [24]). *For any dimension $k \geq 3$, we have*

$$m(Q_k, 3) = \left\lceil \frac{k(k+3)}{6} \right\rceil + 1.$$

In this section, we talked about some extremal results for percolation on square grid graphs. In the following sections, we are going to introduce results about percolation on general grid graphs.

5.2 2-Threshold Minimum Percolating Sets of Grid Graphs

Besides minimum percolating sets of square grid graphs, many results about minimum 2-threshold percolating sets of general grid graphs are also known.

So, we shall mainly introduce such results in this section. As mentioned in Section 5.1, there are also two aspects for this topic – approximate values and exact values, where the former are reflected with bounds.

Let us start at lower bounds for minimum 2-threshold percolating sets of general grid graphs. To get the lower bounds, we need to talk about a concept called *rectangle process* that was introduced by Holroyd [17], Balogh and Bollobás [4], and was utilized in Balogh, Bollobás and Morris [6] later.

Now let $A = \{v_1, \dots, v_{s_0}\}$ be a 2-threshold percolating set of the grid graph $\prod_{i=1}^k [n_i] = (V, E)$ with each $n_i > 1$, and observe the percolation process starting from A . Note that $s_0 \geq 2$, and every vertex $v_i \in A$ is just a k -dimensional $(1 \times \dots \times 1)$ rectangle internally infected by A . For each $i = 1, \dots, s_0$, define a rectangle $P_i^0 = \{v_i\}$. Then, we get rectangles $P_1^0, \dots, P_{s_0}^0 \subsetneq V$ such that they can infect V , each rectangle P_i^0 is internally infected by some set $A_i^0 \subseteq P_i^0 \cap A$, and $A_1^0, \dots, A_{s_0}^0$ are pairwise disjoint. Since $P_1^0, \dots, P_{s_0}^0$ can infect V , at least two rectangles P_i^0 and P_j^0 of these infected rectangles $P_1^0, \dots, P_{s_0}^0$ has distance at most 2 and can internally infect a bigger rectangle $[P_i^0 \cup P_j^0]$ (we shall explain shortly why it must be $[P_i^0 \cup P_j^0]$), which is equivalent to say $A_i^0 \cup A_j^0$ internally infects $[P_i^0 \cup P_j^0]$. If $[P_i^0 \cup P_j^0] = V$, then we get two rectangles $P_i^0, P_j^0 \subsetneq V$ such that they can infect V , the rectangle P_i^0 is internally infected by $A_i^0 \subseteq P_i^0 \cap A$, the rectangle P_j^0 is internally infected by $A_j^0 \subseteq P_j^0 \cap A$, and $A_i^0 \cap A_j^0 = \emptyset$. Otherwise, if $[P_i^0 \cup P_j^0] \subsetneq V$, then let $P_i^0 \cup P_j^0$ internally infect $[P_i^0 \cup P_j^0]$ (note

that it is also internally infected by $A_i^0 \cap A_j^0$), and we get a new family of infected rectangles $P_1^1, \dots, P_{s_1}^1 \subsetneq V$ such that $s_1 < s_0$, they can infect V , each P_i^1 is internally infected by a set $A_i^1 \subseteq P_i^1 \cap A$, and $A_1^1, \dots, A_{s_1}^1$ are pairwise disjoint. Thereafter, since $P_1^1, \dots, P_{s_1}^1$ can infect V , at least two rectangles P_i^1 and P_j^1 of $P_1^1, \dots, P_{s_1}^1$ has distance at most 2 and can internally infect a bigger rectangle $[P_i^1 \cup P_j^1]$, which is equivalent to say $A_i^1 \cup A_j^1$ internally infects $[P_i^1 \cup P_j^1]$. If $[P_i^1 \cup P_j^1] = V$, then we get two rectangles $P_i^1, P_j^1 \subsetneq V$ such that they can infect V , the rectangle P_i^1 is internally infected by $A_i^1 \subseteq P_i^1 \cap A$, the rectangle P_j^1 is internally infected by $A_j^1 \subseteq P_j^1 \cap A$, and $A_i^1 \cap A_j^1 = \emptyset$. Otherwise, if $[P_i^1 \cup P_j^1] \subsetneq V$, then let $P_i^1 \cup P_j^1$ internally infect $[P_i^1 \cup P_j^1]$, and we get a new family of infected rectangles $P_1^2, \dots, P_{s_2}^2 \subsetneq V$ such that $s_2 < s_1$, they can infect V , each P_i^2 is internally infected by a set $A_i^2 \subseteq P_i^2 \cap A$, and $A_1^2, \dots, A_{s_2}^2$ are pairwise disjoint. Continue this process, at some time, we will finally get two infected rectangles $S, T \subsetneq V$ such that they can infect V , the rectangle S is internally infected by a set $A_1 \subseteq S \cap A$, whilst T is internally infected by another set $A_2 \subseteq T \cap A$, and A_1, A_2 are disjoint. Therefore, in summary of this paragraph, the percolation process starting from A can be viewed as a series of 2-rectangle internally spanning. This notion about percolation process on grid graphs is called *rectangle process*.

By the above description, the key step of rectangle process is that two infected rectangle P_i, P_j internally infect a bigger rectangle $[P_i \cup P_j]$ – the

smallest rectangle containing P_i and P_j . Recall that the largest set internally infected by P_i and P_j is their infection closure $\langle P_i \cup P_j \rangle_2$. So, one may wonder why P_i, P_j can internally infect $[P_i \cup P_j]$? Are $[P_i \cup P_j]$ and $\langle P_i \cup P_j \rangle_2$ the same set in this case? Let us take a little more insight to this key step by starting from the following lemmas.

Lemma 5.10. *Let P be a rectangle in the grid graph $\prod_{i=1}^k [n_i] = (V, E)$ and $x \in V \setminus P$. Then, x has at most one neighbour in P .*

Proof. Let $P = \prod_{i=1}^k P_i$, and let $x = (x_1, \dots, x_k)$.

Suppose x has neighbours in P . Since $x \notin P$, there exists $j \in \{1, \dots, k\}$ such that $x_j \notin P_j$, and for any $l \in \{1, \dots, k\} \setminus \{j\}$, $x_l \in P_l$. Since P_j consists of consecutive integers, there are only one element $y_j \in P_j$ such that $|x_j - y_j| = 1$. So, x has only one neighbour in P , namely, $(x_1, \dots, x_{j-1}, y_j, x_{j+1}, \dots, x_k)$. \square

Lemma 5.11. *Let P be a rectangle in the grid graph $\prod_{i=1}^k [n_i]$, and let x be a vertex of $\prod_{i=1}^k [n_i]$. If x has a neighbour in P , then $\langle P \cup \{x\} \rangle_2 = [P \cup \{x\}]$.*

Proof. Let $P = \prod_{i=1}^k P_i$, let $x = (x_1, \dots, x_k)$, and let $[P \cup \{x\}] = \bigcup_{i=1}^k C_i$.

If $x \in P$, then $P \cup \{x\} = P$. By Lemma 5.10, every vertex $y \notin P$ has at most 1 neighbour in P , so it cannot be infected by P . Hence, $\langle P \cup \{x\} \rangle_2 = P = [P \cup \{x\}]$. So, we suppose $x \notin P$ in the following argument.

Without loss of generality, suppose $x_1 \notin P_1$. By Lemma 5.10, x has a unique neighbour $y = (y_1, x_2, \dots, x_k)$ in P . It is straightforward that $y_1 = x_1 + 1$ or $y_1 = x_1 - 1$. Thus, $C_1 = P_1 \cup \{x_1\}$ and $[P \cup \{x\}] = C_1 \times P_2 \times \dots \times P_k$. Let us suppose $y_1 = x_1 + 1$ (the argument is similar when $y_1 = x_1 - 1$) and $P \cup \{x\}$ is infected. Then, it suffices to show $\{x_1\} \times P_2 \times \dots \times P_k$ can be infected.

If x_2 is not the largest number in P_2 , then $(x_1, x_2 + 1, x_3, \dots, x_k)$ can get infected because it has two infected neighbours $(x_1, x_2, x_3, \dots, x_k)$ and $(y_1, x_2 + 1, x_3, \dots, x_k)$. If $x_2 + 1$ is not the largest number in P_2 , then $(x_1, x_2 + 2, x_3, \dots, x_k)$ can get infected because it has two infected neighbours $(x_1, x_2 + 1, x_3, \dots, x_k)$ and $(y_1, x_2 + 2, x_3, \dots, x_k)$. Continue this process, all the vertices of the form $(x_1, y_2, x_3, \dots, x_k)$ with $y_2 \in P_2$ and $y_2 \geq x_2$ will become infected. Similarly, if x_2 is not the smallest number in P_2 , then $(x_1, x_2 - 1, x_3, \dots, x_k)$ can get infected because it has two infected neighbours $(x_1, x_2, x_3, \dots, x_k)$ and $(y_1, x_2 - 1, x_3, \dots, x_k)$. If $x_2 - 1$ is not the smallest number in P_2 , then $(x_1, x_2 - 2, x_3, \dots, x_k)$ can get infected because it has two infected neighbours $(x_1, x_2 - 1, x_3, \dots, x_k)$ and $(y_1, x_2 - 2, x_3, \dots, x_k)$. Continue this process, all the vertices of the form $(x_1, y_2, x_3, \dots, x_k)$ with $y_2 \in P_2$ and $y_2 \leq x_2$ will become infected. Thus, $\{x_1\} \times P_2 \times \{x_3\} \times \dots \times \{x_k\}$ can be infected.

By the same argument as expanding x_2 to P_2 , $\{x_1\} \times P_2 \times P_3 \times \{x_4\} \times \dots \times \{x_k\}$ can also be infected by P and $\{x_1\} \times P_2 \times \{x_3\} \times \dots \times \{x_k\}$. Continue this process, $\{x_1\} \times P_2 \times \dots \times P_k$ can be eventually infected, and the lemma

follows. □

For every positive integer i , let d_i denote the distance between vertices in i -dimensional grid graphs. By Dijkstra's Algorithm for computing distance in graphs, for any two vertices $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$ in a k -dimensional grid $\prod_{i=1}^k [n_i]$, the distance between them is

$$d_k(x, y) = \sum_{i=1}^k d_1(x_i, y_i) = \sum_{i=1}^k |x_i - y_i|.$$

Hence, for any two rectangles $P = \prod_{i=1}^k P_i$ and $Q = \prod_{i=1}^k Q_i$ in the k -dimensional grid graph $\prod_{i=1}^k [n_i]$, the distance between them is computed by

$$d_k(P, Q) = \sum_{i=1}^k d_1(P_i, Q_i).$$

Now, we can readily explain the key step in rectangle process – two rectangles P and Q internally infect a bigger one $[P \cup Q]$.

Theorem 5.12 (2-Rectangle Spanning). *Let P and Q be two rectangles in the grid graph $\prod_{i=1}^k [n_i]$. Then,*

$$\langle P \cup Q \rangle_2 = \begin{cases} P \cup Q, & \text{if } d_k(P, Q) > 2; \\ [P \cup Q], & \text{if } d_k(P, Q) \leq 2. \end{cases}$$

Proof. Let $P = \prod_{i=1}^k P_i$, $Q = \prod_{i=1}^k Q_i$, and let $[P \cup Q] = \prod_{i=1}^k C_i$. Suppose P and Q are infected, and the threshold of percolation is 2.

If $d_k(P, Q) > 2$, then $P \cup Q \subsetneq [P \cup Q]$. Let y be any vertex out of $P \cup Q$. Since $d_k(P, Q) > 2$, then y does not have two neighbours such that one is in P and another is in Q . In addition, by Lemma 5.10, y has neither two neighbours in P nor two neighbours in Q . Thus, any vertex out of $P \cup Q$ cannot get infected, and so $\langle P \cup Q \rangle_2 = P \cup Q$.

Next, suppose $d_k(P, Q) \leq 2$. We shall show P and Q can internally infect $[P \cup Q]$ in the case when $d_k(P, Q) = 2$. The argument works too for other cases when $d_k(P, Q) = 1$ and when $d_k(P, Q) = 0$.

Since $d_k(P, Q) = 2$, there is a vertex $x^0 \in [P \cup Q]$ such that x is adjacent to a vertex in P and adjacent to a vertex in Q . Then, x can be infected by P and Q . Thereafter, by Lemma 5.11, the infected rectangle Q and the infected vertex x^0 can internally infect a bigger rectangle $Q^1 = \langle Q \cup \{x^0\} \rangle_2 = [Q \cup \{x^0\}] \supsetneq Q$, and so $Q \subsetneq Q^1 \subseteq \langle P \cup Q \rangle_2$. If $P \subseteq Q^1$, then $[P \cup Q] \subseteq Q^1 \subseteq \langle P \cup Q \rangle_2$; otherwise, if $P \not\subseteq Q^1$, then there is a vertex $x^1 \in P \setminus Q^1$ adjacent to a vertex in Q^1 because $d_k(P, Q^1) \leq 1$. Hence, by Lemma 5.11, the infected rectangle Q^1 and the infected vertex x^1 can internally infect a bigger rectangle $Q^2 = \langle Q^1 \cup \{x^1\} \rangle_2 = [Q^1 \cup \{x^1\}] \supsetneq Q^1$, and so $Q \subsetneq Q^2 \subseteq \langle P \cup Q \rangle_2$. If $P \subseteq Q^2$, then $[P \cup Q] \subseteq Q^2 \subseteq \langle P \cup Q \rangle_2$; otherwise, If $P \not\subseteq Q^2$, then there is a vertex $x^2 \in P \setminus Q^2$ adjacent to a vertex in Q^2 because $d_k(P, Q^2) \leq 1$ (in fact, $d_k(P, Q^2) = 0$). Continue this operation (illustrated in Figure 5.7), since P is a finite set, we will finally obtain a rectangle Q^t such that $[P \cup Q] \subseteq Q^t \subseteq \langle P \cup Q \rangle_2$.

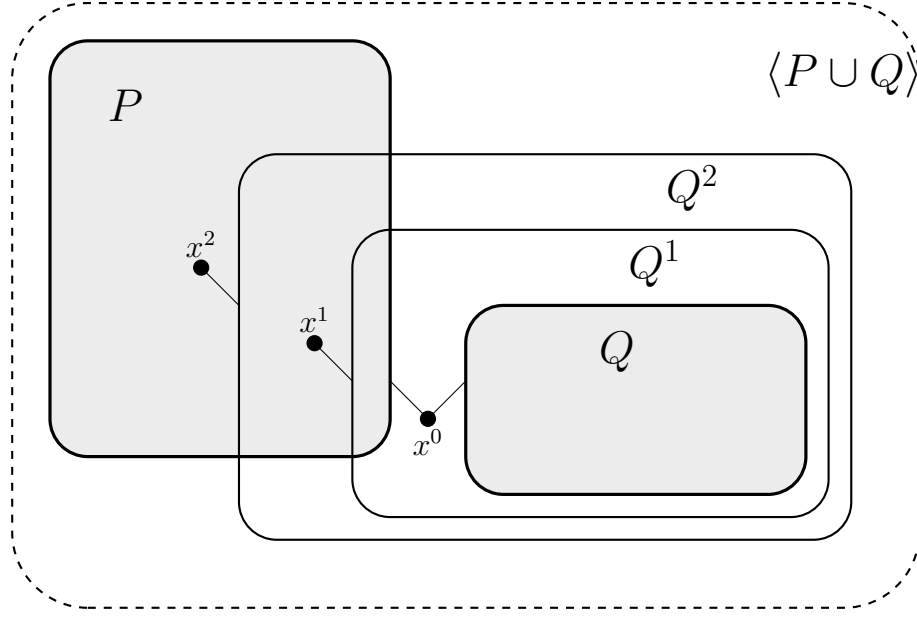


Figure 5.7: The 2-rectangle spanning in the case when $d_k(P, Q) = 2$

On the other hand, by Lemma 5.10, any vertex out of the rectangle $[P \cup Q]$ will never get infected because it has at most 1 neighbour in $[P \cup Q]$, and so $\langle P \cup Q \rangle_2 \subseteq [P \cup Q]$. Therefore, $\langle P \cup Q \rangle_2 = [P \cup Q]$, and the theorem follows. \square

Now, combining the description of rectangle process and Theorem 5.12, we have the next theorem presented by Balogh and Bollobás [4].

Theorem 5.13 (Balogh and Bollobás [4]). *Let Q be a rectangle in the grid graph $\prod_{i=1}^k [n_i]$, and $A \subseteq V(\prod_{i=1}^k [n_i])$ be a set of seeds. If Q is internally infected by A , then there exists two rectangles $S, T \subsetneq Q$ such that $Q = [S \cup T] = \langle S \cup T \rangle_2$, the rectangle S can be internally infected by a set $A_1 \subseteq S \cap A$, the rectangle T*

can be internally infected by a set $A_2 \subseteq T \cap A$, and $A_1 \cap A_2 = \emptyset$.

In addition, Balogh, Bollobás and Morris mentioned the next result in [6] as a tool for obtaining a lower bound of $m(\prod_{i=1}^k [n_i], 2)$.

Lemma 5.14 (Balogh, Bollobás and Morris [6]). *Let $P = \prod_{i=1}^k P_i$ and $Q = \prod_{i=1}^k Q_i$ be two rectangles in the grid graph $\prod_{i=1}^k [n_i]$, and let $[P \cup Q] = \prod_{i=1}^k C_i$. If $\langle P \cup Q \rangle_2 = [P \cup Q]$, then*

$$\left(\sum_{i=1}^k |P_i| \right) + \left(\sum_{i=1}^k |Q_i| \right) \geq \left(\sum_{i=1}^k |C_i| \right) + k - 2.$$

Proof. Since C can be internally infected by P and Q , by Theorem 5.12, $d_k(P, Q) = \sum_{i=1}^k d_1(P_i, Q_i) \leq 2$. Note that each $d_1(P_i, Q_i)$ and $d_k(P, Q)$ are non-negative integers. Hence, it suffices to consider the following three cases.

1. If $d_k(P, Q) = 0$, then for each $i = 1, \dots, k$, $d_1(P_i, Q_i) = 0$ too. Thus, for each $i = 1, \dots, k$, $P_i \cap Q_i \neq \emptyset$, which implies $|P_i| + |Q_i| \geq |C_i| + 1$. So, we have

$$\begin{aligned} \left(\sum_{i=1}^k |P_i| \right) + \left(\sum_{i=1}^k |Q_i| \right) &\geq \left(\sum_{i=1}^k |C_i| \right) + k \\ &\geq \left(\sum_{i=1}^k |C_i| \right) + k - 2. \end{aligned}$$

2. If $d_k(P, Q) = 1$, then exactly one amongst $d_1(P_1, Q_1), \dots, d_1(P_k, Q_k)$ is equal to 1, while others are all 0. Without loss of generality, suppose $d_1(P_1, Q_1) = 1$. Then, $|P_1| + |Q_1| = |C_1|$, and for each $i = 2, \dots, k$, $|P_i| + |Q_i| \geq |C_i| + 1$. So, we have

$$\begin{aligned} \left(\sum_{i=1}^k |P_i| \right) + \left(\sum_{i=1}^k |Q_i| \right) &= |C_1| + \left(\sum_{i=2}^k |P_i| \right) + \left(\sum_{i=2}^k |Q_i| \right) \\ &\geq |C_1| + \left(\sum_{i=2}^k |C_i| \right) + k - 1 \\ &\geq \left(\sum_{i=1}^k |C_i| \right) + k - 2. \end{aligned}$$

3. Finally, if $d_k(P, Q) = 2$, then either one of $d_1(P_1, Q_1), \dots, d_1(P_k, Q_k)$ is equal to 2 and others are all 0, or two of $d_1(P_1, Q_1), \dots, d_1(P_k, Q_k)$ are equal to 1 and others are all 0. For the former case, without loss of generality, suppose $d_1(P_1, Q_1) = 2$, then $|P_1| + |Q_1| = |C_1| - 1$, and so

$$\begin{aligned} \left(\sum_{i=1}^k |P_i| \right) + \left(\sum_{i=1}^k |Q_i| \right) &= |C_1| - 1 + \left(\sum_{i=2}^k |P_i| \right) + \left(\sum_{i=2}^k |Q_i| \right) \\ &\geq |C_1| - 1 + \left(\sum_{i=2}^k |C_i| \right) + k - 1 \\ &\geq \left(\sum_{i=1}^k |C_i| \right) + k - 2. \end{aligned}$$

Whilst for the latter case, without loss of generality, suppose $d_1(P_1, Q_1) =$

$d_1(P_2, Q_2) = 1$, then $|P_1| + |Q_1| = |C_1|$ and $|P_2| + |Q_2| = |C_2|$, and so

$$\begin{aligned} \left(\sum_{i=1}^k |P_i| \right) + \left(\sum_{i=1}^k |Q_i| \right) &= |C_1| + |C_2| + \left(\sum_{i=3}^k |P_i| \right) + \left(\sum_{i=2}^k |Q_i| \right) \\ &\geq |C_1| + |C_2| + \left(\sum_{i=3}^k |C_i| \right) + k - 2 \\ &\geq \left(\sum_{i=1}^k |C_i| \right) + k - 2. \end{aligned}$$

Therefore, the lemma follows. \square

Combining the above lemmas and theorems, a lower bound for $m(\prod_{i=1}^k [n_i], 2)$ was obtained by Balogh, Bollobás and Morris [6].

Theorem 5.15 (Balogh, Bollobás and Morris [6]). *Let $\prod_{i=1}^k [n_i]$ be a grid graph.*

Then,

$$m \left(\prod_{i=1}^k [n_i], 2 \right) \geq \left\lceil \frac{\left(\sum_{i=1}^k n_i \right) - k + 2}{2} \right\rceil = \left\lceil \frac{\sum_{i=1}^k (n_i - 1)}{2} \right\rceil + 1.$$

Proof. We shall prove the inequality

$$m \left(\prod_{i=1}^k [n_i], 2 \right) \geq \frac{\left(\sum_{i=1}^k n_i \right) - k + 2}{2} \quad (5.2)$$

by induction on $\sum_{i=1}^k n_i$.

The base step when $\sum_{i=1}^k n_i = k$ (each $n_i = 1$) is true because

$$m \left(\prod_{i=1}^k [n_i], 2 \right) = 1 \geq 1 = \frac{\left(\sum_{i=1}^k n_i \right) - k + 2}{2}.$$

For the inductive step, assume inequality (5.2) holds for all the grid graphs $\prod_{i=1}^k [n_i]$ with $\sum_{i=1}^k n_i = k, k+1, \dots, m$. Now, consider the grid graph $\prod_{i=1}^k [n_i]$ with $\sum_{i=1}^k n_i = m+1$. Let A be a minimum 2-threshold percolating set of $\prod_{i=1}^k [n_i]$. Then, by Theorem 5.13, there exists two rectangles $S, T \subsetneq V(\prod_{i=1}^k [n_i])$ such that $V(\prod_{i=1}^k [n_i]) = \langle S \cup T \rangle_2 = [S \cup T]$, S can be internally infected by a set $A_1 \subseteq S \cap A$, T can be internally infected by a set $A_2 \subseteq T \cap A$, and $A_1 \cap A_2 = \emptyset$. Let $S = \prod_{i=1}^k S_i$ and $T = \prod_{i=1}^k T_i$, then both $\sum_{i=1}^k |S_i| \leq m$ and $\sum_{i=1}^k |T_i| \leq m$. Hence, by induction assumption and Lemma 5.14, we get

$$\begin{aligned} |A| &\geq |A_1| + |A_2| \\ &\geq m(S, 2) + m(T, 2) \\ &\geq \frac{(\sum_{i=1}^k |S_i|) - k + 2}{2} + \frac{(\sum_{i=1}^k |T_i|) - k + 2}{2} \\ &= \frac{(\sum_{i=1}^k |S_i|) + (\sum_{i=1}^k |T_i|) - 2k + 4}{2} \\ &\geq \frac{(\sum_{i=1}^k n_i) + k - 2 - 2k + 4}{2} \\ &= \frac{(\sum_{i=1}^k n_i) - k + 2}{2}. \end{aligned}$$

Therefore, the inequality 5.2 holds for any grid graph, and the theorem follows. \square

A natural question following Theorem 5.15 is – in what cases is the lower bound tight? Let us check some simple cases.

For the 1-dimensional grid graph $[n_1]$, by Theorem 5.15, $m([n_1], 2) \geq \lceil \frac{n_1+1}{2} \rceil$. On one hand, if n_1 is odd, then $\{i \in [n_1] : 2 \nmid i\}$ is a 2-threshold percolating set with cardinality $\lceil \frac{n_1+1}{2} \rceil = \frac{n_1+1}{2}$. On the other hand, if n_1 is even, then $\{i \in [n_1] : 2 \mid i\} \cup \{1\}$ is a 2-threshold percolating set with cardinality $\lceil \frac{n_1+1}{2} \rceil = \frac{n_1}{2} + 1$. Therefore, the lower bound in Theorem 5.15 is tight to any 1-dimensional grid graph $[n_1]$, and $[n_1]$ has a minimum 2-threshold percolating set containing the 1-dimensional all 1 vector $\mathbf{1}_1 = (1)$.

For 2-dimensional grid graph $[n_1] \times [n_2]$, by Theorem 5.15 again, $m([n_1] \times [n_2], 2) \geq \lceil \frac{n_1+n_2}{2} \rceil$. If $n_1 = n_2$, then it is straightforward that a diagonal of $[n_1] \times [n_2]$ (illustrated in Figure 5.4) is a 2-threshold percolating set of cardinality $\lceil \frac{n_1+n_2}{2} \rceil = n_1$ and containing the 2-dimensional all 1 vector $\mathbf{1}_2 = (1, 1)$. If $n_1 \neq n_2$, without loss of generality, suppose $n_1 < n_2$, then no matter whether $n_2 - n_1$ is odd or even, there always exists a 2-threshold percolating set of cardinality $\lceil \frac{n_1+n_2}{2} \rceil = n_1 + \lceil \frac{n_2-n_1}{2} \rceil$ and containing $\mathbf{1}_2 = (1, 1)$ (illustrated in Figure 5.8). Therefore, the lower bound in Theorem 5.15 is tight to any 2-dimensional grid graph $[n_1] \times [n_2]$, and $[n_1] \times [n_2]$ has a minimum 2-threshold percolating set containing the all 1 vector $\mathbf{1}_2 = (1, 1)$.

By the above discussion, the lower bound in Theorem 5.15 is tight to any 1 dimensional or 2-dimensional grid graph. Will this preferable property

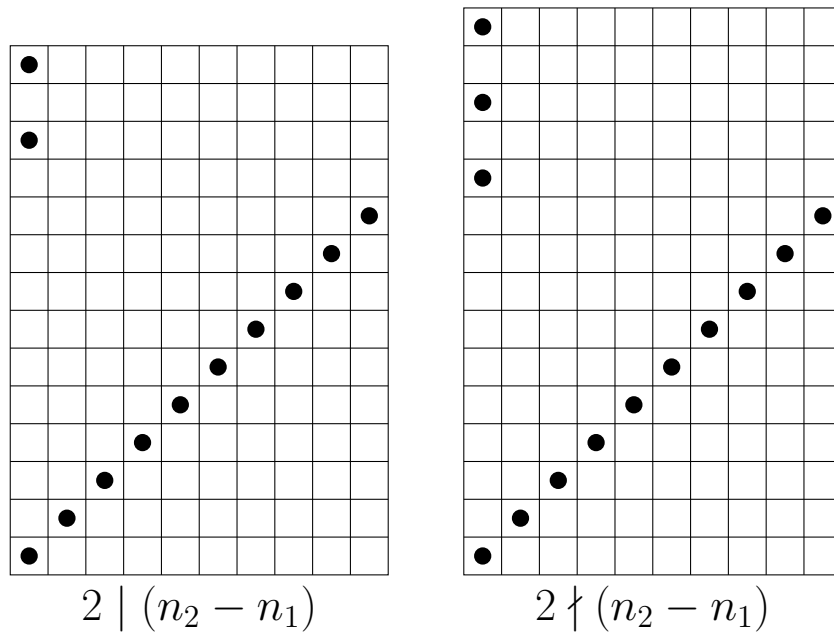


Figure 5.8: A 2-threshold percolating set of the grid graph $[n_1] \times [n_2]$ of cardinality $n_1 + \lceil \frac{n_2 - n_1}{2} \rceil$

be retained in any higher dimensional grid graph? If the answer is positive, then this lower bound will become the exact value for $m(\prod_{i=1}^k [n_i], 2)$. In fact, Bollobás already presented a thorough solution for this question in [10]. The key trick is based on an elementary fact – for any $l \geq 3$ integers, by the Pigeonhole principle, at least $\lceil \frac{l}{2} \rceil$ of them are all even or all odd, and so there exists two integers having an even sum amongst these l integers. The next theorem gives everything we want about minimum 2-threshold percolating sets of general grid graphs. Although our proof is different from the original one, they have the same essence.

Theorem 5.16 (Bollobás [10], Problem 35). *For any k -dimensional grid graph $\prod_{i=1}^k [n_i]$, we have*

$$m\left(\prod_{i=1}^k [n_i], 2\right) = \left\lceil \frac{\sum_{i=1}^k (n_i - 1)}{2} \right\rceil + 1.$$

Moreover, $\prod_{i=1}^k [n_i]$ has a minimum 2-threshold percolating set containing k -dimensional all 1 vector $\mathbf{1}_k = (1, \dots, 1)$.

Proof. We may suppose for each i , the dimension length $n_i \geq 2$.

The paragraphs preceding the theorem already show the theorem is true for $k = 1$ and $k = 2$. So, suppose $k \geq 3$ from now.

By Theorem 5.15, we already got $m\left(\prod_{i=1}^k [n_i], 2\right) \geq \left\lceil \frac{\sum_{i=1}^k (n_i - 1)}{2} \right\rceil + 1$. So,

it suffices to show a 2-threshold percolating set reaching this lower bound for any k -dimensional grid graph.

By the above argument, at least 2 integers out of n_1, \dots, n_k have an even sum. Without loss of generality, suppose $n_1 + n_2$ is even. Thereafter, if $k - 2 \geq 3$, then at least 2 integers out of n_3, \dots, n_k have an even sum. Without loss of generality, suppose $n_3 + n_4$ is even. Continue this operation, there are only two different cases as follows.

1. If k is odd, then every one of $n_1 + n_2, n_3 + n_4, \dots, n_{k-2} + n_{k-1}$ is even.

We split $\prod_{i=1}^k [n_i]$ to some 2-dimensional grid graphs $[n_1] \times [n_2], [n_3] \times [n_4], \dots, [n_{k-2}] \times [n_{k-1}]$ and a 1-dimensional grid graph $[n_k]$. Then, by the above discussion about minimum 2-threshold percolating set of 1-dimensional and 2-dimensional grid graphs, each 2-dimensional grid graph $[n_{2i-1}] \times [n_{2i}]$ has a minimum 2-threshold percolating set A_i of cardinality $\lceil \frac{n_{2i-1}-1+n_{2i}-1}{2} \rceil + 1$ and containing the 2-dimensional all 1 vector $\mathbf{1}_2$, whilst the 1-dimensional grid graph $[n_k]$ has a minimum 2-threshold percolating set A_k of cardinality $\lceil \frac{n_k-1}{2} \rceil + 1$ and containing the 1-dimensional all 1 vector $\mathbf{1}_1$. Since $[n_1] \times [n_2]$ is isomorphic to the subgraph of $\prod_{i=1}^k [n_i]$ induced by the rectangle $P_1 = [n_1] \times [n_2] \times [1]^{k-2}$, the set A_1 can be lifted to a subset $B_1 = \{(x_1, x_2, 1, \dots, 1) \in \mathbb{Z}^k : (x_1, x_2) \in A_1\}$ of $V(\prod_{i=1}^k [n_i])$. Then, the set B_1 can internally infect P_1 and contains the all 1 vector $\mathbf{1}_k$, and $|B_1| = |A_1| = \lceil \frac{n_1-1+n_2-1}{2} \rceil + 1$.

By the same argument, for each $i = 2, \dots, \frac{k-1}{2}$, the set A_i can be lifted to a subset $B_i = \{(1, \dots, 1, x_{2i-1}, x_{2i}, 1, \dots, 1) \in \mathbb{Z}^k : (x_{2i-1}, x_{2i}) \in A_i\}$ of $V(\prod_{i=1}^k [n_i])$. Then, the set B_i can internally infect the rectangle $P_i = [1]^{2i-2} \times [n_{2i-1}] \times [n_{2i}] \times [1]^{k-2i}$ and contains the all 1 vector $\mathbf{1}_k$, and $|B_i| = |A_i| = \lceil \frac{n_{2i-1}-1+n_{2i}-1}{2} \rceil + 1$. Besides, the set A_k can be lifted to a subset $B_k = \{(1, \dots, 1, x_k) \in \mathbb{Z}^k : x_k \in A_k\}$ of $V(\prod_{i=1}^k [n_i])$. Then, the set B_k can internally infect the rectangle $P_k = [1]^{k-1} \times [n_k]$ and contains all 1 vector $\mathbf{1}_k$, and $|B_k| = |A_k| = \lceil \frac{n_k-1}{2} \rceil + 1$. Now, define $B = \left(\bigcup_{i=1}^{\frac{k-1}{2}} B_i \right) \cup B_k$. Since the intersection of each pair of the rectangles $P_1, \dots, P_{\frac{k-1}{2}}, P_k$ is $\{\mathbf{1}_k\}$, then the distance between each pair of them is 0. Hence, by Theorem 5.12, we have

$$\left\langle \left(\bigcup_{i=1}^{\frac{k-1}{2}} P_i \right) \cup P_k \right\rangle_2 = \left[\left(\bigcup_{i=1}^{\frac{k-1}{2}} P_i \right) \cup P_k \right] = V \left(\prod_{i=1}^k [n_i] \right),$$

which indicates B is a 2-threshold percolating set of $\prod_{i=1}^k [n_i]$, and B contains the all 1 vector $\mathbf{1}_k$. Moreover, since every one of $n_1 + n_2, n_3 + n_4, \dots, n_{k-2} + n_{k-1}$ is even and the intersection of each pair of the sets $B_1, \dots, B_{\frac{k-1}{2}}, B_k$ is $\{\mathbf{1}_k\}$, we have

$$|B| = \left(\sum_{i=1}^{\frac{k-1}{2}} |B_i| - 1 \right) + (|B_k| - 1) + 1$$

$$\begin{aligned}
&= \left(\sum_{i=1}^{\frac{k-1}{2}} \left\lceil \frac{n_{2i-1} - 1 + n_{2i} - 1}{2} \right\rceil \right) + \left\lceil \frac{n_k - 1}{2} \right\rceil + 1 \\
&= \left\lceil \frac{\sum_{i=1}^k (n_i - 1)}{2} \right\rceil + 1.
\end{aligned}$$

Therefore, the theorem follows in this case.

2. If k is even, then every one of $n_1 + n_2, n_3 + n_4, \dots, n_{k-3} + n_{k-2}$ is even.

By the same argument as in the preceding case, for each $i = 1, \dots, \frac{k}{2}$,

we can get a set B_i in the rectangle $P_i = [1]^{2i-2} \times [n_{2i-1}] \times [n_{2i}] \times [1]^{k-2i}$ such that B_i can internally infect P_i and contains the all 1 vector $\mathbf{1}_k$, and

$|B_i| = \lceil \frac{n_{2i-1}-1+n_{2i}-1}{2} \rceil + 1$. Now, define $B = \bigcup_{i=1}^{\frac{k}{2}} B_i$. Again, since the

intersection of each pair of the rectangles $P_1, \dots, P_{\frac{k}{2}}$ is $\{\mathbf{1}_k\}$, then the

distance between each pair of them is 0. Hence, by Theorem 5.12, B is a

2-threshold percolating set of $\prod_{i=1}^k [n_i]$, and B contains the all 1 vector

$\mathbf{1}_k$. Moreover, since every one of $n_1 + n_2, n_3 + n_4, \dots, n_{k-3} + n_{k-2}$ is even

and the intersection of each pair of the sets $B_1, \dots, B_{\frac{k}{2}}$ is $\{\mathbf{1}_k\}$, we have

$$\begin{aligned}
|B| &= \left(\sum_{i=1}^{\frac{k-2}{2}} |B_i| - 1 \right) + (|B_{\frac{k}{2}}| - 1) + 1 \\
&= \left(\sum_{i=1}^{\frac{k-2}{2}} \left\lceil \frac{n_{2i-1} - 1 + n_{2i} - 1}{2} \right\rceil \right) + \left\lceil \frac{n_{k-1} - 1 + n_k - 1}{2} \right\rceil + 1 \\
&= \left\lceil \frac{\sum_{i=1}^k (n_i - 1)}{2} \right\rceil + 1.
\end{aligned}$$

Therefore, the theorem follows in this case too.

□

The proof of Theorem 5.16 not only shows the exact cardinality of minimum 2-threshold percolating sets of general grid graphs, but also clarifies the construction of them. By now, we can confidently claim that the important aspects about minimum 2-threshold percolating sets of general grid graphs are thoroughly clear.

5.3 Largest Minimal 2-Threshold Percolating Sets of 2-Dimensional Grid

In the last section, we have clearly characterized the minimum 2-threshold percolating sets of general grid graphs. Recall that a minimum percolating set of a graph is a minimal percolating set of the smallest cardinality. A given graph may have more than one minimum percolating sets that have the same cardinality. Then, another interesting question slightly different from those about minimum percolating sets arises – is it true that all the minimal percolating sets of a given graph have roughly the same cardinality? This is a big question still pending. However, some results for some specific graphs are already known. In this section, we shall introduce such results about a

special kind of largest minimal percolating sets – 2-threshold largest minimal percolating sets of 2-dimensional grid graphs. Although no exact values have been guaranteed for this kind of percolating sets, upper bounds and lower bounds for them is already known. The results introduced in this section come from Morris [23].

5.3.1 Corner-Avoiding Minimal Percolating Sets

To obtain the bounds for the largest minimal 2-threshold percolating sets of 2-dimensional grid graphs, we need to explain first a concept called “corner-avoiding” introduced by Morris [23].

In the grid $[m] \times [n]$, denote the top-left 2×2 corner by

$$J_{TL} = \{(1, n-1), (1, n), (2, n-1), (2, n)\},$$

denote the bottom-right 2×2 corner by

$$J_{BR} = \{(m-1, 1), (m, 1), (m-1, 2), (m, 2)\}.$$

Definition 5.17 (Morris [23], Corner-avoiding). *For a 2-threshold minimal percolating set A of $[m] \times [n]$, if for every $v \in A$, $\langle A \setminus \{v\} \rangle \cap (J_{TL} \cup J_{BR}) = \emptyset$, then we say A is corner-avoiding. Moreover, if $[m] \times [n]$ has a corner-avoiding minimal percolating set, then let $E_C([m] \times [n], 2)$ denote the maximum cardinality of all corner-avoiding minimal percolating sets. Otherwise, if $[m] \times [n]$ has no corner-avoiding minimal percolating set, then let $E_C([m] \times [n], 2) = 0$.*

Apparently, the family of corner-avoiding minimal percolating sets of the grid $[m] \times [n]$ is a special class of minimal percolating sets. Thus, $E_C([m] \times [n], 2) \leq E([m] \times [n], 2)$. To get the bounds for $E([m] \times [n], 2)$, the following properties of $E_C([m] \times [n], 2)$ are needed.

Lemma 5.18 (Morris [23]). *Let k be a positive integer. Then, $E_C([8] \times [3k + 2], 2) \geq 4k + 4$.*

Proof. Let $P = \{(1, 1), (1, 3)\}$, and $L(k) = \bigcup_{i=0}^{k-1} (P + (0, 3i))$. Then,

$$A = L(k) \cup \{(2, 3k), (4, 1), (5, 3k + 2), (7, 3)\} \cup (L(k) + (7, 2))$$

is a corner-avoiding minimal percolating set of $[8] \times [3k + 2]$ (illustrated in Figure 5.9) with $|A| = 4k + 4$. \square

Lemma 5.19 (Morris [23]). *Let m, n, m', n' be positive integers with $n' \geq n$, and suppose $E_C([m] \times [n], 2) > 0$, $E_C([m'] \times [n'], 2) > 0$. Then,*

$$E_C([m + m' + 3] \times [n' + 2], 2) \geq E_C([m] \times [n], 2) + E_C([m'] \times [n'], 2) + 2.$$

Proof. Let B, C be largest minimal percolating sets of $[m] \times [n]$ and $[m'] \times [n']$, respectively. Define

$$C' = C + (m + 3, 2) \subseteq [m + m' + 3] \times [n' + 2],$$

and

$$A = B \cup C' \cup \{(m + 1, 1), (m + 3, n' + 2)\}.$$

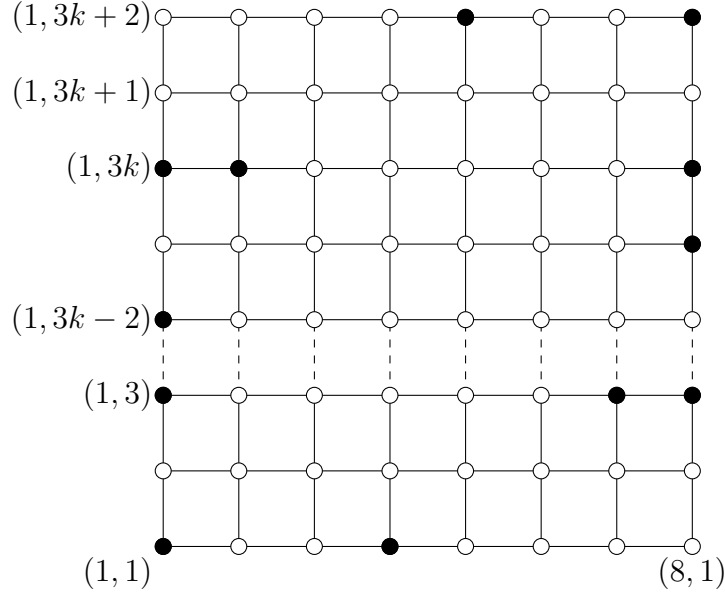


Figure 5.9: A corner-avoiding minimal percolating set A of $[8] \times [3k + 2]$. The vertices in A are coloured black.

Then, A is a corner-avoiding minimal percolating set of $[m + m' + 3] \times [n' + 2]$ (illustrated in Figure 5.10) with $|A| = E_C([m] \times [n], 2) + E_C([m'] \times [n'], 2) + 2$. \square

Lemma 5.20 (Morris [23]). *Let m, n, k be positive integers. Then*

$$E_C([km + 3(k - 1)] \times [n + 2(k - 1)], 2) \geq kE_C([m] \times [n], 2).$$

Proof. The result is trivial if $E_C([m] \times [n], 2) = 0$. So, suppose $E_C([m] \times [n], 2) > 0$, and prove the lemma by induction on k .

For the base step when $k = 1$, we get the equality.

For the induction step, assume the lemma holds for $k - 1$. Let $m' =$

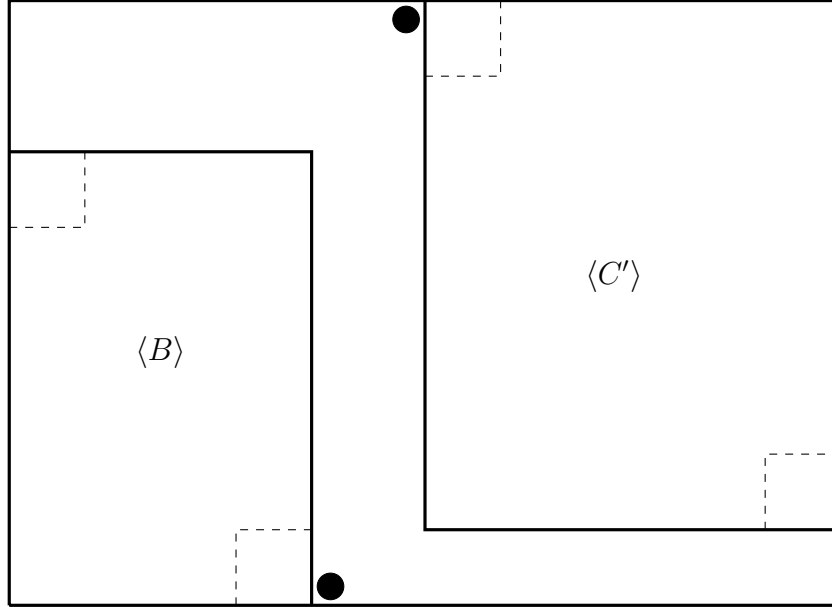


Figure 5.10: A corner-avoiding minimal percolating set A of $[m+m'+3] \times [n'+2]$.

$(k-1)m + 3(k-2)$ and $n' = n + 2(k-2)$. So, by induction assumption,

$$E_C([m'] \times [n'], 2) \geq (k-1)E_C([m] \times [n], 2).$$

Then, Lemma 5.19 gives

$$\begin{aligned} & E_C([km + 3(k-1)] \times [n + 2(k-1)], 2) \\ &= E_C([m + m' + 3] \times [n' + 2], 2) \\ &\geq E_C([m] \times [n], 2) + E_C([m'] \times [n'], 2) \\ &\geq E_C([m] \times [n], 2) + (k-1)E_C([m] \times [n], 2) \\ &= kE_C([m] \times [n], 2). \end{aligned}$$

□

Lemma 5.21 (Morris [23]). *Let m, n, t be positive integers. Then*

$$E_C([2^t(m+3) - 3] \times [n + 2t], 2) \geq 2^t E_C([m] \times [n], 2).$$

Proof. The result is immediately true if $E_C([m] \times [n], 2) = 0$. So, suppose $E_C([m] \times [n], 2) > 0$.

By Lemma 5.19 with $m' = m$ and $n' = n$, we get

$$E_C([2m + 3] \times [n + 2], 2) \geq 2E_C([m] \times [n], 2).$$

Then, use this inequality to $E_C([2^t(m + 3) - 3] \times [n + 2t], 2)$ t times, we obtain

$$\begin{aligned} & E_C([2^t(m + 3) - 3] \times [n + 2t], 2) \\ & \geq 2E_C([2^{t-1}(m + 3) - 3] \times [n + 2t - 2], 2) \\ & \geq 2^2 E_C([2^{t-2}(m + 3) - 3] \times [n + 2t - 2 \cdot 2], 2) \\ & \geq \dots \\ & \geq 2^t E_C([2^{t-t}(m + 3) - 3] \times [n + 2t - 2 \cdot t], 2) \\ & = 2^t E_C([m] \times [n], 2). \end{aligned}$$

□

5.3.2 Bounds of $E([m] \times [n], 2)$

Now, we can turn to the bounds for $E([m] \times [n], 2)$. We start with the lower bound.

Lemma 5.22 (Morris [23]). *Let m, n, k, l be positive integers with $k \leq m$ and $l \leq n$. Then, $E([k] \times [l], 2) \leq E([m] \times [n], 2)$.*

Proof. First, consider $[k] \times [l]$ and $[k + 1] \times [l]$. Let A be a largest minimal percolating set of $[k] \times [l]$. So, $|A| = E([k] \times [l], 2)$. Since A percolates in $[k] \times [l]$,

there exists $a \in [l]$ such that $(k, a) \in A$. Define $C = A \cup \{(k+1, a)\} \setminus \{(k, a)\}$. Then, $|C| = |A| = E([k] \times [l], 2)$. We are going to show for any vertex $u \in C$, $C \setminus \{u\}$ does not percolate in $[k+1] \times [l]$. Indeed, if $u = (k+1, a)$, then $C \setminus \{u\}$ does not percolate in $[k+1] \times [l]$ because $(k+1, a)$ is the only vertex of C in the last column of $[k+1] \times [l]$. If $u \neq (k+1, a)$, assume $C \setminus \{u\}$ could percolate in $[k+1] \times [l]$. Then, since every vertex $(k+1, j)$ in the last column of $[k+1] \times [l]$ except $(k+1, a)$ have to get infected by help of (k, j) , $C \setminus \{u\}$ could also percolate in a subgraph H of $[k+1] \times [l]$ induced by $V([k] \times [l]) \cup \{(k+1, a)\}$ (illustrated in Figure 5.11). So, $C \cup \{(k, a)\} \setminus \{u\}$ could percolate in H too, which would cause $C \cup \{(k, a)\} \setminus \{u, (k+1, a)\} = A \setminus \{u\}$ to percolate in $H - (k+1, a) = [k] \times [l]$. But this contradicts that A is a minimal percolating set of $[k] \times [l]$. Thus, for any vertex $u \in C$, $C \setminus \{u\}$ does not percolate in $[k+1] \times [l]$. Then, we can derive a minimal percolating set D of $[k+1] \times [l]$ by adding vertices in $V([k+1] \times [l]) \setminus C$ to C one by one ($D = C$ if C percolates in $[k+1] \times [l]$ itself). Subsequently,

$$E([k] \times [l], 2) = |C| \leq |D| \leq E([k+1] \times [l], 2).$$

In addition, by symmetry of 2-dimensional grid, we can also obtain

$$E([k] \times [l], 2) \leq E([k] \times [l+1], 2).$$

Finally, by expanding $[k] \times [l]$ to $[m] \times [n]$ step by step, we get

$$E([k] \times [l], 2) \leq E([k+1] \times [l], 2)$$

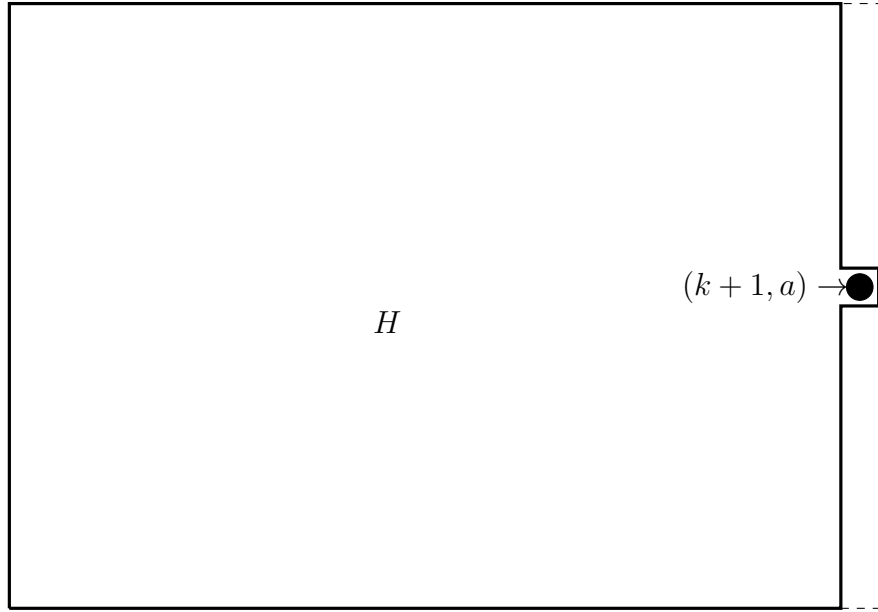


Figure 5.11: $C \setminus \{u\}$ could also percolate in an induced subgraph H of $[k+1] \times [l]$.

$$\begin{aligned}
 &\leq E([k+2] \times [l], 2) \\
 &\leq \dots \\
 &\leq E([m] \times [l], 2) \\
 &\leq E([m] \times [l+1], 2) \\
 &\leq E([m] \times [l+2], 2) \\
 &\leq \dots \\
 &\leq E([m] \times [n], 2).
 \end{aligned}$$

□

Lemma 5.23 (Morris [23]). *For any positive integers M, N , we have*

$$E_C([11M - 3] \times [3N + 2M], 2) \geq 4M(N + 1).$$

Hence, for any positive integers m, n , we have

$$E([m] \times [n], 2) \geq 4 \left\lfloor \frac{m+3}{11} \right\rfloor \left\lfloor \frac{n - 2 \lfloor \frac{m+3}{11} \rfloor + 3}{3} \right\rfloor.$$

Proof. First, apply Lemma 5.20 with $m = 8$, $n = 3N + 2$ and $k = M$ together with Lemma 5.18, we get

$$E_C([11M - 3] \times [3N + 2M], 2) \geq M E_C([8] \times [3N + 2], 2) \geq 4M(N + 1).$$

Second, let $M = \lfloor \frac{m+3}{11} \rfloor$ and $N = \lfloor \frac{n-2M}{3} \rfloor$. Then, the right hand side of the second inequality is just $4M(N + 1)$. If $M(N + 1) \leq 0$, the lemma is obviously true. If $N = 0$ and $M \geq 1$, then $m \geq 8$. Since

$$\{(i, 1) : i \in [m], i \equiv 0 \text{ or } 2 \pmod{3}\} \cup \{(1, j) : j \in [n], j \equiv 0 \text{ or } 2 \pmod{3}\}$$

is a minimal percolating set of $[m] \times [n]$, we obtain

$$E([m] \times [n], 2) \geq \frac{2(m+n)}{3} \geq \frac{4(m+3)}{11} \geq 4M(N + 1).$$

Finally, if $M \geq 1$ and $N \geq 1$, then $m \geq 11M - 3$ and $n \geq 3N + 2M$. So, by Lemma 5.22 and the first inequality, we get

$$\begin{aligned} E([m] \times [n], 2) &\geq E([11M - 3] \times [3N + 2M], 2) \\ &\geq E_C([11M - 3] \times [3N + 2M], 2) \\ &\geq 4M(N + 1). \end{aligned}$$

□

Next, let us consider the upper bound of $E([m] \times [n], 2)$. We start from some simple grid graphs.

Theorem 5.24 (Morris [23]). *Let m be any positive integer. Then*

$$1. E([m] \times [1], 2) = \left\lfloor \frac{2(m+1)}{3} \right\rfloor.$$

$$2. E([m] \times [2], 2) = \left\lfloor \frac{2(m+2)}{3} \right\rfloor.$$

$$3. E([m] \times [3], 2) = \left\lfloor \frac{2(m+3)}{3} \right\rfloor.$$

Proof. To calculate $E([m] \times [1], 2)$, note that for any minimal percolating set A , it contains at most two out of each three consecutive vertices of $[m] \times [1]$ (otherwise the middle seed would contradict minimality of A), which implies

$$E([m] \times [1], 2) \leq \frac{2(m+1)}{3}.$$

Moreover, when $m \equiv 0 \pmod{3}$,

$$\{(i, 1) : i \in [m-2], i \equiv 1 \text{ or } 2 \pmod{3}\} \cup \{(m, 1)\}$$

is a minimal percolating set of cardinality $\left\lfloor \frac{2(m+1)}{3} \right\rfloor$. On the other hand, when

$m \equiv 1 \text{ or } 2 \pmod{3}$,

$$\{(i, 1) : i \in [m], i \equiv 1 \text{ or } 2 \pmod{3}\}$$

is a minimal percolating set of cardinality $\left\lfloor \frac{2(m+1)}{3} \right\rfloor$. Therefore, we get the first equation.

To calculate $E([m] \times [2], 2)$, let A be a minimal percolating set of $[m] \times [2]$. Then, A has to contain two seeds $(s, 1), (t, 2)$ such that $|s - t| \leq 1$ (otherwise, A would not percolate in $[m] \times [2]$). If $s = t$ and $(s, 1)$ is one of the end vertex of the first row of $[m] \times [2]$, without loss of generality, suppose A contains $(1, 1), (1, 2)$, then A does not contain any vertex in the second column because its minimality, and A can contain at most 2 seeds in every consecutive three columns starting from the third column to the right end of $[m] \times [2]$ (otherwise, if A contained more than 2 seeds in such consecutive three columns, at least one of them would be wasted, which contradicts minimality of A), as illustrated in Figure 5.12. Thus,

$$|A| \leq 2 + \left\lfloor \frac{2(m-2+1)}{3} \right\rfloor \leq \left\lfloor \frac{2(m+2)}{3} \right\rfloor.$$

If $1 < s = t < m$, then A contains vertices neither in the $(s-1)$ -th column nor in the $(s+1)$ -th column because its minimality, A can contain at most 2 seeds in every consecutive three columns starting from the $(s+2)$ -th column to the right end of $[m] \times [2]$, and A can contain at most 2 seeds in every consecutive three columns starting from the $(s-2)$ -th column to the left end of $[m] \times [2]$, as illustrated in Figure 5.13. Thus,

$$|A| \leq \left\lfloor \frac{2(s-2+1)}{3} \right\rfloor + 2 + \left\lfloor \frac{2(m-s-1+1)}{3} \right\rfloor \leq \left\lfloor \frac{2(m+2)}{3} \right\rfloor.$$

If $2 < s < s + 1 = t < m - 1$, then by minimality of A , A contains at most 2 more seeds in

$$\{(i, j) : s - 2 \leq i \leq s + 3, 1 \leq j \leq 2\},$$

other than $(s, 1), (t, 2)$ (precisely, A contains at most 1 seed in the $(s - 2)$ -th and $(s - 1)$ -th columns, and at most 1 seed in the $(s + 2)$ -th and $(s + 3)$ -th columns), and A can contain at most 2 seeds in every consecutive three columns starting from the $(s + 4)$ -th column to the right end of $[m] \times [2]$, and A can contain at most 2 seeds in every consecutive three columns starting from the $(s - 3)$ -th column to the left end of $[m] \times [2]$, as illustrated in Figure 5.14. Thus,

$$|A| \leq \left\lfloor \frac{2(s - 3 + 1)}{3} \right\rfloor + 4 + \left\lfloor \frac{2(m - s - 3 + 1)}{3} \right\rfloor \leq \left\lfloor \frac{2(m + 2)}{3} \right\rfloor.$$

Finally, by the same argument as above, we can also obtain $|A| \leq \left\lfloor \frac{2(m+2)}{3} \right\rfloor$ for other trivial cases. Meanwhile, if $m \equiv 0 \pmod{3}$, then

$$\{(i, 1) : 1 \leq i \leq m - 1, i \equiv 1 \text{ or } 2 \pmod{3}\} \cup \{(m, 2)\}$$

is a minimal percolating set of cardinality $\left\lfloor \frac{2(m+2)}{3} \right\rfloor$. If $m \equiv 1 \pmod{3}$, then

$$\{(i, 1) : 1 \leq i \leq m - 1, i \equiv 1 \text{ or } 2 \pmod{3}\} \cup \{(m, 1)\} \cup \{(m, 2)\}$$

is a minimal percolating set of cardinality $\left\lfloor \frac{2(m+2)}{3} \right\rfloor$. If $m \equiv 2 \pmod{3}$, then

$$\{(i, 1) : 1 \leq i \leq m - 2, i \equiv 1 \text{ or } 2 \pmod{3}\} \cup \{(m - 1, 1)\} \cup \{(m, 2)\}$$

is a minimal percolating set of cardinality $\left\lfloor \frac{2(m+2)}{3} \right\rfloor$. Therefore, the second equation is proved.

To calculate $E([m] \times [3], 2)$, again let A be a minimal percolating set of $[m] \times [3]$. We prove the third equation by induction on m . If $m \leq 2$, then the third equation immediately follows the first one and the second one. If $m = 3$, note that every consecutive 2 columns contain at most 3 seeds. If the first 2 columns contain 3 seeds, then they are internally A -spanned and the 3-rd column can contain at most 1 seed, which implies $|A| \leq 4$; if the first 2 columns contain at most 2 seeds, then we can get $|A| \leq 4$ too because the 3-rd column can have at most 2 seeds. Moreover, $\{(1, 1), (1, 2), (3, 2), (3, 3)\}$ is a minimal percolating set of $[3] \times [3]$. Thus, $E([3] \times [3], 2) = 4 = \left\lfloor \frac{2(3+3)}{3} \right\rfloor$. If $m = 4$, then again every consecutive 2 columns contain at most 3 seeds. If both the first 2 columns and the last 2 columns contains at most 2 seeds, then $|A| \leq 4$; if the first 2 columns (the 1-st and the 2-nd columns) contain exactly 3 seeds, then they can get internally infected by A , and so A needs to have only one more seed in the 4-th column, which implies $|A| \leq 4$; likewise, if the last 2 columns contain exactly 3 seeds, we can obtain $|A| \leq 4$ too. Moreover, $\{(1,1),(1,3),(2,2),(4,1)\}$ is a minimal percolating set of $[4] \times [3]$. Thus,

$$E([4] \times [3], 2) = 4 = \left\lfloor \frac{2(4+3)}{3} \right\rfloor.$$

Now, suppose $m \geq 5$, and assume the theorem is true for any grid $[p] \times [3]$ with

$p < m$. Suppose first there exists an internally A -spanned $(k \times 3)$ rectangle R that does not include the last two columns (the $(m-1)$ -th and the m -th columns) of $[m] \times [3]$. Then, the vertices to the left of R do not need the help from the vertices to the right of R . Besides, since any two consecutive columns contain at least 1 seed, R can be spanned to right until $(m-3)$ -th column or $(m-2)$ -th column get infected. So, $[m-3] \times [3]$ or $[m-2] \times [3]$ must be internally A -spanned. For the former case, A can contain exactly 2 more seeds in the last three columns, which implies

$$|A| \leq E([m-3] \times [3], 2) + 2 \leq \left\lfloor \frac{2m}{3} \right\rfloor + 2 = \left\lfloor \frac{2(m+3)}{3} \right\rfloor;$$

for the latter case, A can contain exactly 1 more seeds in the last two columns, which implies

$$|A| \leq E([m-2] \times [3], 2) + 1 \leq \left\lfloor \frac{2(m+1)}{3} \right\rfloor + 1 = \left\lfloor \frac{2(m+3)}{3} \right\rfloor.$$

Next, suppose no such a rectangle R exists. Since A percolates in $[m] \times [3]$, it has to contain two seeds with distance at most 2, which implies there exists an internally A -spanned (1×2) or (2×2) rectangle T . By symmetry of the grid, we can assume T is included in the first two rows and does not intersect the last two columns of $[m] \times [3]$ because $m \geq 5$. Let the bottom-left corner vertex and bottom-right corner vertex of T are $(t_1, 1)$ and $(t_2, 1)$, respectively. Note that the (t_1-1) -th column and (t_1-2) -th column contain at least 1 seed. Due to non-existence of R , $(t_1-1, 3)$ is not a seed. If $(t_1-2, 3)$ is a seed, then by minimality

of A , any other vertex in the $(t_1 - 1)$ -th column and $(t_1 - 2)$ -th column cannot be a seed. If neither $(t_1 - 1, 3)$ nor $(t_1 - 2, 3)$ is a seed, then again by minimality of A , exactly one vertex amongst $\{(t_1 - 1, 1), (t_1 - 1, 2), (t_1 - 2, 1), (t_1 - 2, 2)\}$ is a seed, and T can be extended to the left to a bigger internally A -spanned $((t_2 - t_1 + 1 + 1) \times 2)$ or $((t_2 - t_1 + 1 + 2) \times 2)$ rectangle. The same argument is valid too to the columns to the right of T . In the process of spanning T to both directions, if we got an internally A -spanned rectangle

$$S = \{(i, j) : s_1 \leq i \leq s_2, 1 \leq j \leq 2\}$$

including T such that both $(s_1 - 2, 3)$ and $(s_2 + 2, 3)$ are seeds, then neither of the two sets $\{(i, j) : i \leq s_1 - 3, 1 \leq j \leq 3\}$ and $\{(i, j) : i \leq s_2 + 3, 1 \leq j \leq 3\}$ is internally A -spanned (because of non-existence of rectangle R), and at least one of the two sets

$$\begin{aligned} S_1 &= \{(i, j) : i \leq s_1 - 3, 2 \leq j \leq 3\} \cup \{(s_1 - 2, 3)\}, \\ S_2 &= \{(i, j) : i \geq s_2 + 3, 2 \leq j \leq 3\} \cup \{(s_2 + 2, 3)\} \end{aligned}$$

(illustrated in Figure 5.15) is internally A -spanned (otherwise, A could not percolate in $[m] \times [3]$). Without loss of generality, assume S_1 was internally A -spanned, then the rectangle $\{(i, j) : i \leq s_2, 1 \leq j \leq 3\}$ would be internally A -spanned, which contradicts non-existence of R . Therefore, when T spanned to the left and the right, arise only three possible cases:

1. $[m - 3] \times [2]$ is internally A -spanned;

2. $[m - 2] \times [2]$ is internally A -spanned;
3. there exists an internally A -spanned rectangle $S = V([l] \times [2])$ such that $l \leq m - 4$ and $d(S, A \setminus S) \geq 3$.

If case 1 (but neither case 2 nor case 3) happens, then the $(m - 2)$ -th column contains no seed, and the last two columns contain at most 2 seeds (otherwise, the last two columns would form an internally A -spanned (2×3) rectangle, which would contradict the assumption that no such a rectangle R exists), which implies

$$|A| \leq E([m - 3] \times [2], 2) + 2 \leq \left\lfloor \frac{2(m - 1)}{3} \right\rfloor + 2 = \left\lfloor \frac{2(m + 3)}{3} \right\rfloor.$$

If case 2 (but neither case 1 nor case 3) happens, then the last two columns contain at most 2 seeds (because of minimality of A), which implies

$$|A| \leq E([m - 2] \times [2], 2) + 2 \leq \left\lfloor \frac{2m}{3} \right\rfloor + 2 = \left\lfloor \frac{2(m + 3)}{3} \right\rfloor.$$

If case 3 happens, then the set

$$S_2 = \{(i, j) : l + 2 \leq i, 2 \leq j \leq 3\} \cup \{(l + 2, 3)\}$$

(illustrated in Figure 5.16) has to be internally A -spanned (otherwise A would not percolate in $[m] \times [3]$), and no seed lies out of $S \cup S_2$ (because of non-existence of the rectangle R). Hence, we obtain two internally A -spanned

rectangles S and $S_2 \cup \{(l+2, 2)\}$ containing all the seeds. Therefore, by the induction assumption,

$$\begin{aligned} |A| &\leq E([l] \times [2], 2) + E([m-l-1] \times [2], 2) \\ &\leq \left\lfloor \frac{2(l+2)}{3} \right\rfloor + \left\lfloor \frac{2(m-l+1)}{3} \right\rfloor \\ &\leq \left\lfloor \frac{2(m+3)}{3} \right\rfloor. \end{aligned}$$

By now, we have shown that $E([m] \times [3], 2) \leq \left\lfloor \frac{2(m+3)}{3} \right\rfloor$ for any positive integer m . Furthermore, if $m \equiv 0 \pmod{3}$, then the set

$$\{(i, 1), 1 \leq i \leq m, i \equiv 1 \text{ or } 2 \pmod{3}\} \cup \{(m, 2), (m, 3)\}$$

is a minimal percolating set of cardinality $\left\lfloor \frac{2(m+3)}{3} \right\rfloor$. If $m \equiv 1 \pmod{3}$, then the set

$$\{(i, 1), 1 \leq i \leq m-1, i \equiv 1 \text{ or } 2 \pmod{3}\} \cup \{(m-1, 2), (m, 3)\}$$

is a minimal percolating set of cardinality $\left\lfloor \frac{2(m+3)}{3} \right\rfloor$. Finally, if $m \equiv 2 \pmod{3}$, then the set

$$\{(i, 1), 1 \leq i \leq m-2, i \equiv 1 \text{ or } 2 \pmod{3}\} \cup \{(m-2, 2), (m, 2), (m, 3)\}$$

is a minimal percolating set of cardinality $\left\lfloor \frac{2(m+3)}{3} \right\rfloor$, which finishes the proof for the theorem. □

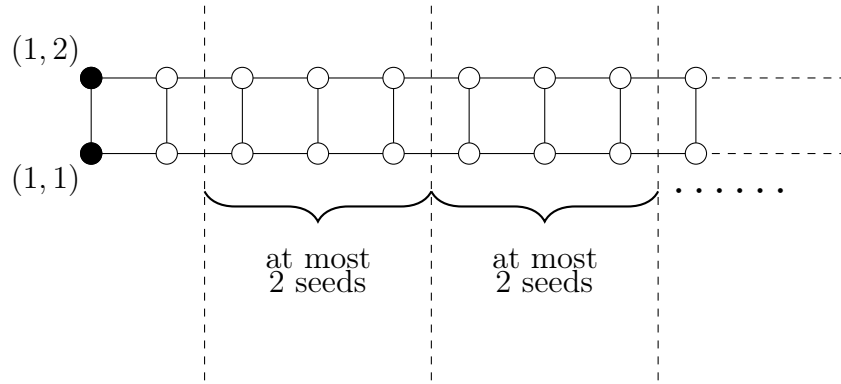


Figure 5.12: A minimal percolating set A of $[m] \times [2]$ containing $(1, 1), (1, 2)$

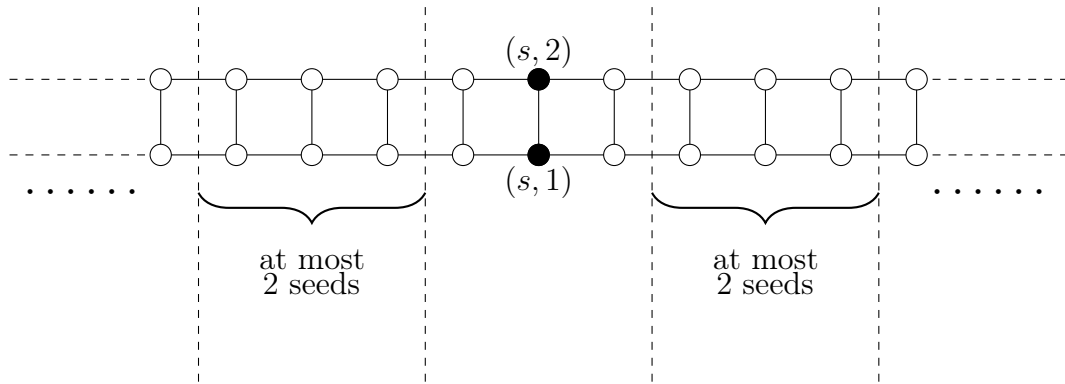


Figure 5.13: A minimal percolating set A of $[m] \times [2]$ containing $(s, 1), (s, 2)$ with $1 < s < m$

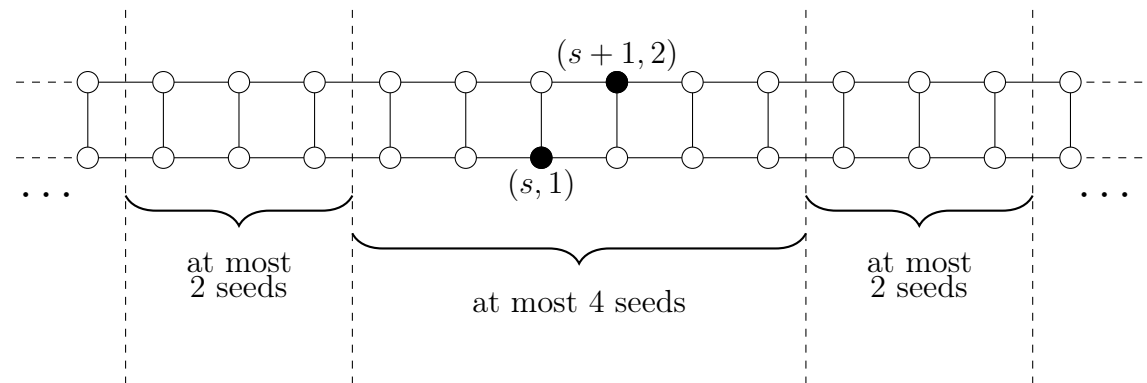


Figure 5.14: A minimal percolating set A of $[m] \times [2]$ containing $(s, 1), (s + 1, 2)$

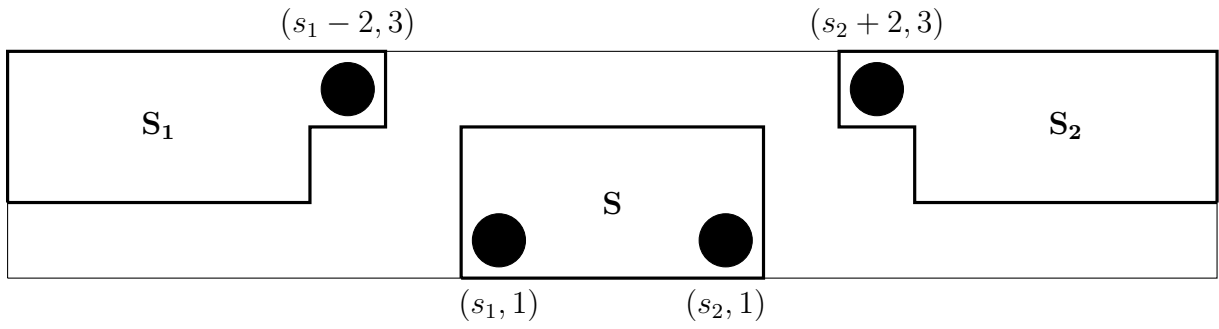


Figure 5.15: An internally A -spanned rectangle S and two other sets S_1, S_2

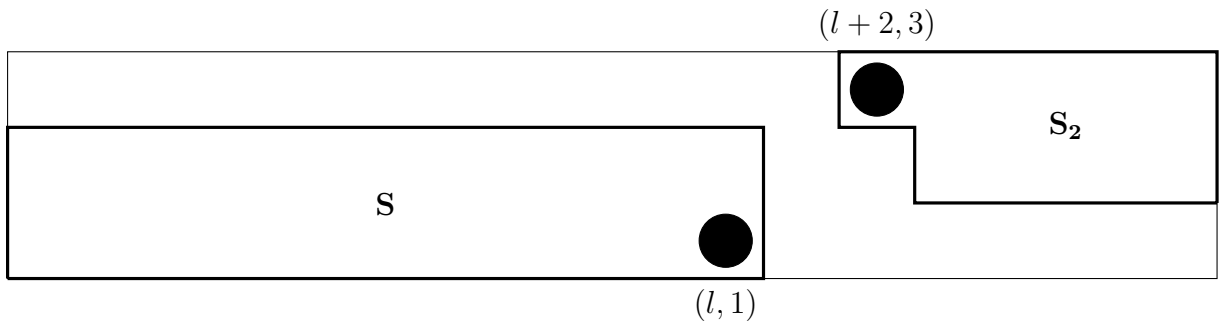


Figure 5.16: An internally A -spanned rectangle $S = [l] \times [2]$ and another set S_2

Now, we introduce a binary relation over all the rectangles in $[m] \times [n]$. For an $(a \times b)$ rectangle S and a $(c \times d)$ rectangle T in $[m] \times [n]$, define $S <_R T$ if and only if

$$\min\{m - a, n - b\} > \min\{m - c, n - d\},$$

or

$$\min\{m - a, n - b\} = \min\{m - c, n - d\} \wedge \max\{m - a, n - b\} > \max\{m - c, n - d\}.$$

Obviously, this relation depends not on the positions but on the widths and heights of rectangles. It is trivial to check the relation “ $<_R$ ” is transitive. Moreover, if a rectangle S is properly included in another rectangle T , then $S <_R T$.

Lemma 5.25 (Morris [23]). *Let S be a $(p \times q)$ rectangle, and T be a $(k \times l)$ rectangle in the grid $[m] \times [n]$. If a $S <_R T$, then*

$$kl + p(n - l) + q(m - k) \leq mn.$$

Proof. Note that

$$kl + p(n - l) + q(m - k) = mn + (m - k)(n - l) - (m - p)(n - l) - (m - k)(n - q).$$

If $p \leq k$, then $(m - k)(n - l) \leq (m - p)(n - l)$. While if $p > k$, then $q < l$, and so $(m - k)(n - l) \leq (m - k)(n - q)$. In both cases, the lemma follows. \square

For general 2-dimensional grid $[m] \times [n]$, the following theorem gives an upper bound of $E([m] \times [n], 2)$.

Theorem 5.26 (Morris [23]). *Let $m, n \geq 2$ be two integers. Then*

$$E([m] \times [n], 2) \leq \frac{(m+2)(n+2)}{6} = \frac{1}{6}mn + \frac{1}{3}(m+n) + \frac{2}{3}.$$

Proof. We prove the theorem by induction on $m+n$.

If $2 \leq \min\{m, n\} \leq 3$, then the result immediately follows Theorem 5.24.

Now, let $m, n \geq 4$, let A be a minimal percolating set of $[m] \times [n]$, and assume for any $[p] \times [q]$ with $p, q \geq 2$ and $p+q < m+n$, $E([p] \times [q], 2) \leq \frac{(p+2)(q+2)}{6}$. By induction assumption, for any subgrid $[p] \times [q]$ of $[m] \times [n]$ with $p, q \geq 2$, $E([p] \times [q], 2) \leq \frac{(p+2)(q+2)}{6}$. Let $S \subset V([m] \times [n])$ be a maximal (under the relation $<_R$) internally A -spanned $(k \times l)$ rectangle with $k, l \geq 2$ (such a rectangle must exist due to transitivity of $<_R$ and $m, n \geq 4$). We shall distinguish several distinct cases.

Case 1. Either $k = m$ or $l = n$. Without loss of generality, suppose $k = m$. Since any two consecutive rows of $[m] \times [n]$ contain at least 1 seed, S is maximal, and A is minimal, we must get $l = n - 1$ or $n - 2$, and S covers the 1st or the n -th row of $[m] \times [n]$, and $|A \setminus S| = 1$. Hence, by the induction assumption and $m \geq 4$, we get

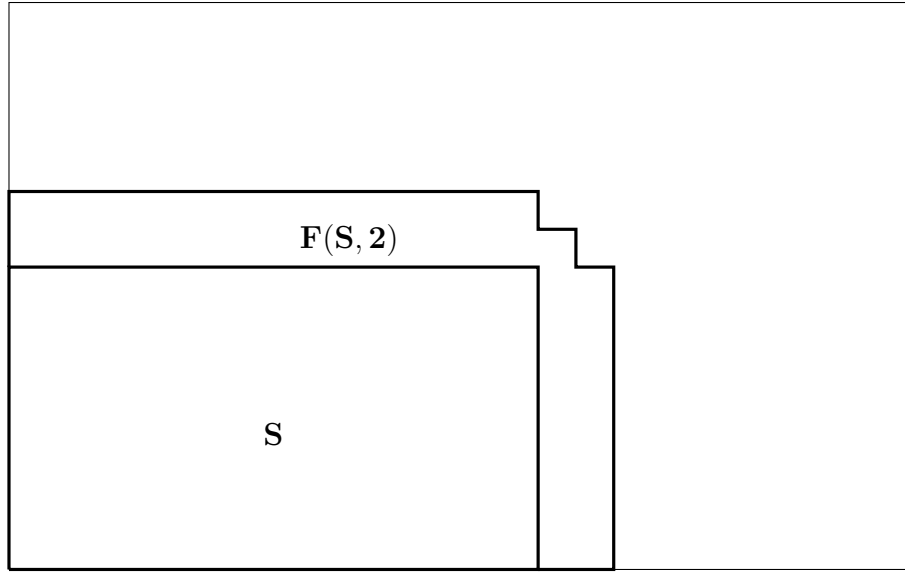
$$\begin{aligned} |A| &\leq E([m] \times [n-1], 2) + 1 \\ &\leq \frac{1}{6}m(n-1) + \frac{1}{3}(m+n-1) + \frac{2}{3} + 1 \\ &= \frac{1}{6}mn + \frac{1}{3}(m+n) + \frac{2}{3} + (1 - \frac{1}{6}m - \frac{1}{3}) \end{aligned}$$

$$\leq \frac{1}{6}mn + \frac{1}{3}(m+n) + \frac{2}{3}.$$

If $k < m$ and $l < n$, let $B = A \setminus S$ and $T = \langle B \rangle$. Define the 2-fringe of S as

$$F(S, 2) = \{v \in V([m] \times [n]) : d(v, S) \leq 2\}.$$

If $|B| > 1$, then by maximality of S and minimality of A , no seed in B will lie in $F(S, 2)$. So, either $|B| = 1$ or $d(S, B) \geq 3$. Since the infection process of any set of seeds just consists of a series of rectangle span (two rectangles with distance at most 2 span a bigger rectangle), T is the union of some rectangles (with pairwise distance at most 3). If T was a union of more than 1 rectangles, that is $T = \cup_{i=1}^t T_i$, then at least one of T_1, \dots, T_t must have distance at most 2 with S because A percolates in $[m] \times [n]$. Without loss of generality, let T_1 be such a rectangle included in T . Then, by maximality of S , $\langle S, T_1 \rangle = V([m] \times [n])$, which implies the seeds in T_2, \dots, T_t would be wasted, and would contradict minimality of A . Thus, T must be a single rectangle internally spanned by B , the sets $A \setminus S$ and B are minimal percolating sets of S and T respectively, $d(S, T) \leq 2$, and $\langle S, T \rangle = V([m] \times [n])$. Note that once $|B| \geq 2$, since $d(S, B) \geq 3$ and $d(S, T) \leq 2$, B has to contain seeds in distinct rows and distinct columns, and so both the width and height of T are at least 2. Moreover, if S contained no corner vertex of $[m] \times [n]$, then since $\langle S, T \rangle = V([m] \times [n])$, T would have to contain two corner vertices of $[m] \times [n]$, and so $S <_R T$, which would contradict maximality of S . Thus, S

Figure 5.17: $S = [k] \times [l]$ and its 2-fringe $F(S, 2)$

must contain at least one of the corner vertices of $[m] \times [n]$. Without loss of generality, suppose $S = [k] \times [l]$ (illustrated in Figure 5.17).

Case 2. $k < m$, $l < n$, and $|B| = 1$. In this case, $k = m - 1$, $l = n - 1$, and $T = B = \{(m, n)\}$. So, by the induction assumption and $m \geq 4$, we get

$$\begin{aligned}
 |A| &\leq E([m-1] \times [n-1], 2) + 1 \\
 &\leq \frac{1}{6}(m-1)(n-1) + \frac{1}{3}(m-1+n-1) + \frac{2}{3} + 1 \\
 &\leq \frac{1}{6}m(n-1) + \frac{1}{3}(m+n-1) + \frac{2}{3} + 1 \\
 &= \frac{1}{6}mn + \frac{1}{3}(m+n) + \frac{2}{3} + \left(1 - \frac{1}{6}m - \frac{1}{3}\right) \\
 &\leq \frac{1}{6}mn + \frac{1}{3}(m+n) + \frac{2}{3}.
 \end{aligned}$$

Hence, suppose $|B| \geq 2$ from now.

Case 3. $k < m$, $l < n$, $|B| \geq 2$, and S, T neither overlap rows, nor overlap columns. In this case, T can be included in neither a single row nor a single column of $[m] \times [n]$ (because $d(S, B) \geq 3$), and $T = [k + 1, m] \times [l + 1, n]$ (because $d(S, T) \leq 2$). Thus,

$$\begin{aligned} |A| &\leq E([k] \times [l], 2) + E([m - k] \times [n - l], 2) \\ &\leq \frac{1}{6}kl + \frac{1}{3}(k + l) + \frac{2}{3} + \frac{1}{6}(m - k)(n - l) + \frac{1}{3}(m - k + n - l) + \frac{2}{3} \\ &= \frac{1}{6}mn + \frac{1}{3}(m + n) + \frac{2}{3} + \left[\frac{2}{3} - \frac{1}{6}(k(n - l) + l(m - k)) \right] \\ &\leq \frac{1}{6}mn + \frac{1}{3}(m + n) + \frac{2}{3}, \end{aligned}$$

because $k, l, m - k, n - l \geq 2$.

Other than Case 3, S, T will overlap columns or rows. Without loss of generality, suppose S, T overlap columns.

Case 4. $k < m$, $l < n$, $|B| = 2$, and S, T overlap columns. In this case, there must be at least a seed in B lying in the first k columns and at least a seed in B lying in the last $m - k$ columns of $[m] \times [n]$. Consequently, $k \leq m - 1$ and $l \leq n - 3$. So,

$$\begin{aligned} |A| &\leq E([m - 1] \times [n - 3], 2) + 2 \\ &\leq \frac{1}{6}(m - 1)(n - 3) + \frac{1}{3}(m - 1 + n - 3) + \frac{2}{3} + 2 \\ &= \frac{1}{6}mn + \frac{1}{3}(m + n) + \frac{2}{3} + \frac{1}{6}[-3m - n + 7] \\ &\leq \frac{1}{6}mn + \frac{1}{3}(m + n) + \frac{2}{3}, \end{aligned}$$

because $m, n \geq 4$.

Case 5. $k < m$, $l < n$, $|B| = 3$, and S, T overlap columns. In this case, again, there must be at least a seed in B lying in the first k columns and at least a seed in B lying in the last $m - k$ columns of $[m] \times [n]$. Consequently, $k \leq m - 1$ and $l \leq n - 3$. So,

$$\begin{aligned} |A| &\leq E([m-1] \times [n-3], 2) + 3 \\ &\leq \frac{1}{6}(m-1)(n-3) + \frac{1}{3}(m-1+n-3) + \frac{2}{3} + 3 \\ &= \frac{1}{6}mn + \frac{1}{3}(m+n) + \frac{2}{3} + \frac{1}{6}[-3m-n+13] \\ &\leq \frac{1}{6}mn + \frac{1}{3}(m+n) + \frac{2}{3}, \end{aligned}$$

because $m, n \geq 4$.

Other than the above cases, suppose $|B| \geq 4$. Then, there exists two disjoint internally B -spanned rectangles P, Q such that $d(P, Q) \leq 2$ and $\langle P, Q \rangle = T$. If one of P, Q had distance at most 2 with S , then the seeds in the other one would be wasted due to maximality of S . Thus, $d(S, P) \geq 3$ and $d(S, Q) \geq 3$. Let P be a $(p \times s)$ rectangle, and Q be a $(t \times q)$ rectangle. If $\min\{p, s, t, q\} = 1$, without loss of generality, suppose $\min\{t, q\} = 1$ but $|B \cap Q| \geq 2$, and let u_1, u_2 be two end vertices of Q , define $Q_1 = \langle B \cap Q \setminus u_1 \rangle$ and $Q_2 = \langle B \cap Q \setminus u_2 \rangle$. Since $d(P, Q_1) \geq 3$ and $d(P, Q_2) \geq 3$ would cause $d(P, Q) \geq 3$, at least one of $d(P, Q_1), d(P, Q_2)$ is at most 2. Then, if $d(P, Q_1) \leq 2$, we can update P, Q as $\langle P, Q_1 \rangle$ and $\langle u_1 \rangle$, respectively. Anyway, we can get two disjoint internally B -spanned rectangles P, Q such that $d(P, Q) \leq 2$, $\langle P, Q \rangle = T$, $|Q| = 1$ and both the width and the height of P are at least 2.

Case 6. $k < m$, $l < n$, $|B| \geq 4$, S, T overlap columns, $d(P, Q) \leq 2$, $\langle P, Q \rangle = T$, $d(S, P) \geq 3$, $d(S, Q) \geq 3$, $|Q| = 1$, both the width and the height of P are at least 2, and S, P overlap columns. In this case, it is trivial to check

$$\frac{1}{6}n(m-p) + \frac{1}{3}(m-k) \geq \frac{1}{6}l(k-p) + 1$$

by considering cases when $kp \geq$ and when $k < p$. Hence, by induction assumption, we get

$$\begin{aligned} |A| &\leq E([k] \times [l], 2) + E([p] \times [n-l-2], 2) + 1 \\ &\leq \frac{1}{6}(kl + pn - pl - 2p) + \frac{1}{3}(k + p + n - 2) + \frac{4}{3} + 1 \\ &= \frac{1}{6}[mn - n(m-p) + l(k-p)] + \frac{1}{3}[m + n - (m-k)] + \frac{2}{3} + 1 \\ &\leq \frac{1}{6}mn + \frac{1}{3}(m+n) + \frac{2}{3}. \end{aligned}$$

Case 7. $k < m$, $l < n$, $|B| \geq 4$, S, T overlap columns, $d(P, Q) \leq 2$, $\langle P, Q \rangle = T$, $d(S, P) \geq 3$, $d(S, Q) \geq 3$, $|Q| = 1$, both the width and the height of P are at least 2, and S, P do not overlap columns. In this case, since S, T overlap columns, Q has to overlap a column with S . Since $d(P, Q) \leq 2$ and $d(S, P) \geq 3$, P does not overlap rows with S either (observe Figure 5.17). Subsequently, either $p = m - k$ and $s \leq n - l - 1$ or $p = m - k - 1$ and $s \leq n - l$. Besides, it is trivial to check

$$(k+1)(n-l) + l(m-k) \geq m + 2k + 3 \geq 9,$$

and

$$k(n-l) + (l+1)(m-k) \geq 2m + k \geq 9.$$

So, by the induction assumption, either

$$\begin{aligned}
|A| &\leq E([k] \times [l], 2) + E([m-k] \times [n-l-1], 2) + 1 \\
&\leq \frac{1}{6}(kl + mn - ml - m - kn + kl + k) + \frac{1}{3}(m+n-1) + \frac{4}{3} + 1 \\
&= \frac{1}{6}mn + \frac{1}{3}(m+n) + \frac{2}{3} - \frac{1}{6}[k(n-l) + (l+1)(m-k)] - \frac{1}{3} + \frac{2}{3} + 1 \\
&\leq \frac{1}{6}mn + \frac{1}{3}(m+n) + \frac{2}{3},
\end{aligned}$$

or

$$\begin{aligned}
|A| &\leq E([k] \times [l], 2) + E([m-k-1] \times [n-l], 2) + 1 \\
&\leq \frac{1}{6}[kl + (m-k-1)(n-l)] + \frac{1}{3}(m+n-1) + \frac{4}{3} + 1 \\
&= \frac{1}{6}mn + \frac{1}{3}(m+n) + \frac{2}{3} - \frac{1}{6}[(k+1)(n-l) + l(m-k)] - \frac{1}{3} + \frac{2}{3} + 1 \\
&\leq \frac{1}{6}mn + \frac{1}{3}(m+n) + \frac{2}{3}.
\end{aligned}$$

Now suppose $\min\{p, s, t, q\} \geq 2$. Since S, T overlap columns and $\langle P, Q \rangle = T$, we may suppose without loss of generality that S, P overlap columns. Since $d(S, P) \geq 3$ and $s \geq 2$, we get $n-l \geq 4$ and $s \leq n-l-2$. If the height of T is equal to q , then Q would contain two corner vertices of T , and we would choose two rectangles P, Q in T with $|P| = 1$, and so it would go to above cases. Thus, the height of T is bigger than q . Similarly, the width of T is bigger than p . Therefore, it suffices to investigate the last two cases.

Case 8. $k < m, l < n, |B| \geq 4, d(P, Q) \leq 2, \langle P, Q \rangle = T, d(S, P) \geq 3, d(S, Q) \geq 3, \min\{p, s, t, q\} \geq 2, S, P$ overlap columns, and S, T overlap both

columns and rows. In this case, Q has to overlap rows with S , and so $t \leq m - k - 2$. Thus, by the induction assumption and Lemma 5.25, we get

$$\begin{aligned} |A| &\leq E([k] \times [l], 2) + E([p] \times [n - l - 2], 2) + E([m - k - 2] \times [q], 2) \\ &\leq \frac{1}{6}[kl + p(n - l - 2) + q(m - k - 2)] + \frac{1}{3}(m + n + p + q - 4) + 2 \\ &= \frac{1}{6}[kl + p(n - l) + q(m - k)] - \frac{1}{3}(p + q) + \frac{1}{3}(m + n + p + q) + \frac{2}{3} \\ &\leq \frac{1}{6}mn + \frac{1}{3}(m + n) + \frac{2}{3}. \end{aligned}$$

Case 9. $k < m$, $l < n$, $|B| \geq 4$, $d(P, Q) \leq 2$, $\langle P, Q \rangle = T$, $d(S, P) \geq 3$, $d(S, Q) \geq 3$, $\min\{p, s, t, q\} \geq 2$, S, P overlap columns, and S, T overlap columns but not rows. In this case, since $d(S, Q) \geq 3$ and $d(S, T) \leq 2$, Q overlaps neither columns nor rows with S , and so $t \leq m - k$ and $q \leq n - l$. Let

$$M = (k - p)(n - l) + (m - k)(n - q) = (m - p)(n - l) + (m - k)(l - q).$$

If $k \geq p + 2$, then by $q \leq n - l - 1$ and $p \leq m - 2$, we have

$$M \geq (k - p)(n - l) + (m - k)(l + 1) \geq 2(n - l) + 2 \geq 2q + 4.$$

If $k \leq p + 1$, then by $T \leq_R S$, we have $l \geq q - 1$. If $k \leq m - 2$, then

$$M \geq (m - p)(n - l) + (m - k) \geq 2(n - l) + 2 \geq 2q + 4.$$

On the other hand, if $k = m - 1$, then since $d(S, B) \geq 3$, the set B contains no vertex in the $(l + 1)$ -th row of $[m] \times [n]$, which implies $q \leq n - l - 2$, and so

$$M \geq (m - p)(n - l) \geq 2(n - l) \geq 2q + 4.$$

In summary, we always have $M \geq 2q + 4$. Therefore, by the induction assumption, we get

$$\begin{aligned}
|A| &\leq E([k] \times [l], 2) + E([p] \times [n - l - 2], 2) + E([m - k] \times [q], 2) \\
&\leq \frac{1}{6}[kl + p(n - l - 2) + q(m - k)] + \frac{1}{3}(m + n + p + q - 2) + 2 \\
&= \frac{1}{6}(mn - M - 2p) + \frac{1}{3}(m + n) + \frac{1}{3}(p + q) + \frac{4}{3} \\
&\leq \frac{1}{6}(mn - 2q - 4 - 2p) + \frac{1}{3}(m + n) + \frac{1}{3}(p + q) + \frac{4}{3} \\
&= \frac{1}{6}mn + \frac{1}{3}(m + n) + \frac{2}{3},
\end{aligned}$$

which finishes the proof of the theorem. \square

In this chapter, we introduced a lot of results about percolating sets of grid graphs. Our discussion is consistently centred at the big question mentioned at the beginning of the chapter – what information about minimum/minimal percolating sets of grid graphs can we get exactly or approximately under different selection of parameters? The results in this chapter partially answered this question.

Chapter 6

Bootstrap Percolation on Trees

Trees are minimal connected graphs. Their importance has grown considerably in view of their wide applicability in theoretical computer science. In this chapter, we shall introduce some results about bootstrap percolation on trees. As in previous chapters, our discussion will be concentrated on minimum or minimal percolating sets too. We are going to work out the bounds for minimum/minimal percolating sets of trees first, then discuss under what conditions these approximate values can become exact values. In addition, we shall show a recursive algorithm for constructing the minimum percolating sets of trees. The results in this chapter mainly come from the paper [33] by Riedl.

By the facts mentioned in Section 2.1, it suffices to consider r -threshold percolation on trees of order n with $2 \leq r \leq n - 1$.

Throughout the chapter, we call every non-leaf vertex of a tree T an *internal vertex* of T .

6.1 Bounds for Minimal Percolating Sets of Trees

This section is for the cardinality of minimum or minimal percolating sets of trees. We shall first discuss the bounds for minimum/minimal percolating sets of trees, then check whether these bounds are the optimal results under the given parameters. In addition, as mentioned in Section 5.3, another interesting question in bootstrap percolation is what is the range for cardinalities of minimal percolating set. Hence, we will try to answer this question on trees by investigate the gap between minimum and largest minimal percolating sets of trees.

First, let us consider the bounds for minimal percolating sets of trees. The next theorem partially comes from Riedl [33], where the argument about tightness of the lower bound is new work presented in this thesis.

Theorem 6.1. *Let $T = (V, E)$ be a tree of n vertices, r be an integer with $2 \leq r \leq n - 1$, and $L = \{x \in V : \deg_G(x) < r\}$. Then,*

$$\left\lceil \frac{rn - n + 1}{r} \right\rceil \leq m(T, r) \leq E(T, r) \leq \left\lfloor \frac{rn + |L|}{r + 1} \right\rfloor.$$

Moreover, for any integers n, r satisfying $2 \leq r \leq n - 1$, both the lower bound and the upper bound are tight for some trees with n vertices.

Proof. These two bounds come from Lemma 2.2 and Lemma 2.3 with $|E| = n - 1$. Hence, we just show some trees of order n reaching these bounds.

To show the tightness of the lower bound for any integers n, r satisfying $2 \leq r \leq n - 1$, we distinguish several cases.

Case 1. $n = r + b$ with $1 \leq b \leq r - 1$. In this case,

$$\left\lceil \frac{rn - n + 1}{r} \right\rceil = n + \left\lceil \frac{-n + 1}{r} \right\rceil = n - 1 + \left\lceil \frac{-b + 1}{r} \right\rceil = n - 1.$$

We construct a tree T as a star with $n - 1$ leaves. Then, the set of all the leaves of T is an r -threshold percolating set of cardinality $n - 1$. Therefore, the lower bound is tight in this case.

Case 2. $n = kr$ for some integer $k \geq 2$. In this case,

$$\left\lceil \frac{rn - n + 1}{r} \right\rceil = n + \left\lceil \frac{-kr + 1}{r} \right\rceil = n - k + 1 = k(r - 1) + 1.$$

We construct a tree $T = (V, E)$ as follows. Let $V = \bigcup_{i=1}^k V_i$ where V_1, \dots, V_k are pairwise disjoint, each V_i contains r vertices, and label the vertices in $V_i = \{x_j^i : j = 1, \dots, r\}$. For each $i = 1, \dots, k$, let the induced subgraph $T[V_i]$ be a star centred at x_1^i . Finally, for each $i = 2, \dots, k$, add $x_1^1 - x_1^i$ to E (illustrated in Figure 6.1). By such construction, T is a tree of order $n = kr$. Now, let $A = V_1 \cup \{x_j^i : 2 \leq i \leq k, 2 \leq j \leq r\}$ be initially infected. Then, $|A| = k(r - 1) + 1$ and the initially clean vertices are just x_1^i 's with $2 \leq i \leq k$. For each $2 \leq i \leq k$, x_1^i can get infected in the first step of infection process starting from A because it has r infected neighbours x_2^i, \dots, x_r^i and x_1^1 . Thus,

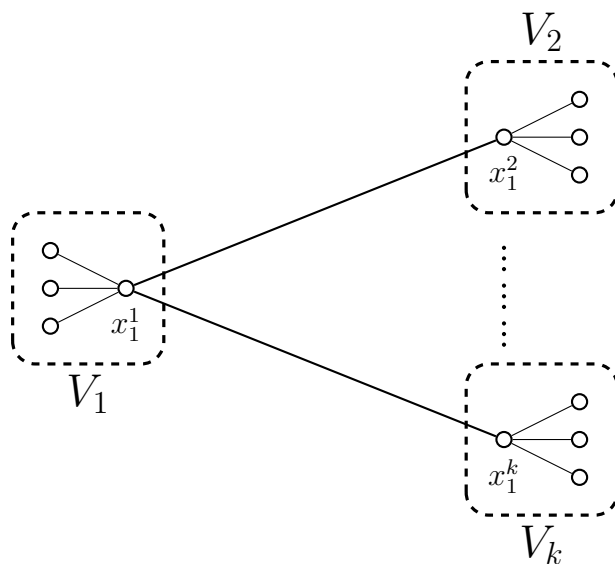


Figure 6.1: A tree T of order $n = kr$ with $k \geq 2$ and $r = 4$.

A is an r -threshold percolating set of cardinality $k(r - 1) + 1$. Therefore, the lower bound is tight in this case.

Case 3. $n = kr + b$ for integers k, b with $k \geq 2$ and $1 \leq b \leq r - 1$. In this case,

$$\left\lceil \frac{rn - n + 1}{r} \right\rceil = n + \left\lceil \frac{-kr - b + 1}{r} \right\rceil = n - k + \left\lceil \frac{-b + 1}{r} \right\rceil = k(r - 1) + b.$$

We construct a tree $T = (V, E)$ as follows. Let $V = (\bigcup_{i=1}^k X_i) \cup Y$ where X_1, \dots, X_k and Y are pairwise disjoint, each X_i contain r vertices, and $|Y| = b$. Label the vertices in $X_i = \{x_j^i : j = 1, \dots, r\}$ and in $Y = \{y_j : j = 1, \dots, b\}$. For each $i = 1, \dots, k$, let the induced subgraph $T[X_i]$ be a star centred at x_1^i , and let $T[Y]$ be a star centred at y_1 . Finally, for each $i = 1, \dots, k$, add $y_1 - x_1^i$

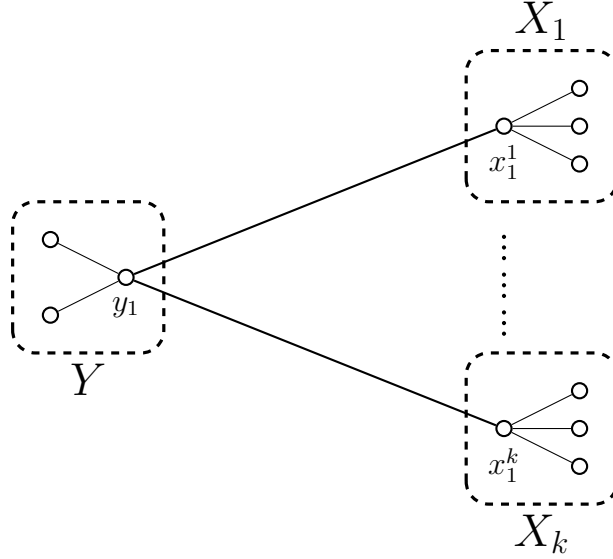


Figure 6.2: A tree T of order $n = kr + b$ with $k \geq 2$ and $1 \leq b \leq r - 1$ when $r = 4$ and $b = 3$.

to E (illustrated in Figure 6.2). By such construction, T is a tree of order $n = kr + b$. Now, let $A = Y \cup \{x_j^i : 1 \leq i \leq k, 2 \leq j \leq r\}$ be initially infected. Then, $|A| = k(r - 1) + b$ and the initially clean vertices are just x_1^i 's with $1 \leq i \leq k$. For each $1 \leq i \leq k$, x_1^i can get infected in the first step of infection starting from A because it has r infected neighbours x_2^i, \dots, x_r^i and y_1 . Thus, A is an r -threshold percolating set of cardinality $k(r - 1) + b$. Therefore, the lower bound is tight in this case.

To show the tightness of the upper bound, for every integer n, r with $2 \leq r \leq n - 1$, construct a tree T as a star of order n . In this case, $|L| = n - 1$, and so $\left\lfloor \frac{rn + |L|}{r + 1} \right\rfloor = n - 1$. Besides, the set of leaves of T is an r -threshold

minimal percolating set of cardinality $n - 1$. Therefore, the upper bound is tight too. \square

Tightness in Theorem 6.1 indicates that those bounds are the best ones one can achieve for trees. Besides the bounds, whether all the minimal percolating sets have roughly the same cardinality is another interesting question. So next, let us consider this question by investigating the gap between minimum percolating sets and largest minimal percolating sets of trees.

Lemma 6.2 (Riedl [33]). *Let T be a tree of order n , r be an integer with $2 \leq r \leq n - 1$, L be the set of leaves of T , A be a minimal r -threshold percolating set of T , and w be the number of wasted edges of the percolation process starting from A . If $\langle L \rangle_r = L$ and every internal vertex has degree at least r , then $w \leq \frac{(n-1)(r-1)}{r}$.*

Proof. Since A percolates on T and $\langle L \rangle_r = L$, A has to contain internal vertices of T . Let $B \subseteq A$ be the set of all the internal vertices in A , and u be the total number of used edges after A percolates. Then, $u + w = n - 1$.

First, we shall show that we can always choose a percolation process starting from A such that every wasted edge is incident with a vertex in B . Suppose E_W is a set of all the wasted edges of some percolation process \mathcal{P} starting from A , and k edges in E_W are incident with vertices in B . If $k < w$, then suppose $x - y \in E_W$ is an edge not incident with any vertex in B and x is an internal vertex of T (x, y cannot be leaves simultaneously because $n \geq 3$).

Note that deleting the edge $x - y$ would give us two subtrees of T . Let T_x be one subtree containing x , and T_y be another subtree containing y . Since $x - y$ is wasted, T_x can be internally infected by $A \cap V(T_x)$, and T_y can be internally infected by $A \cap V(T_y)$. Since $\langle L \rangle_r = L$, x must get infected with help of a seed in $B \cap V(T_x)$. Suppose $x_0 - x_1 - \dots - x_d = x$ is the path of used edges (in the percolation process \mathcal{P}) connecting a seed $x_0 \in B \cap V(T_x)$ and x (illustrated in Figure 6.3). Now, define $E'_W = E_W \cup \{x_0 - x_1\} \setminus \{x - y\}$ and construct a new percolation process \mathcal{P}' also starting from A as follows (illustrated in Figure 6.4).

1. $V(T_y)$ is internally spanned by $A \cap V(T_y)$. This phase works just as it does in the percolation process \mathcal{P} .
2. x is infected by some of its neighbours and y . This phase works well because at the time when x get infected in the process \mathcal{P} , x has at least r infected neighbours in $V(T_x)$ and the infection of these neighbours does not rely on the infection of x . Now in the process \mathcal{P}' , since $x_0 - x_1$ becomes a wasted edge, x_1 cannot get infected with help from x_0 , then x_2 cannot get infected with help from x_1 , etc. Finally, $x = x_d$ cannot get infected with help from x_{d-1} . That is equivalent to having x lost an infected neighbour in $V(T_x)$ but got a new infected neighbour y . Thus, x can get infected before the vertices x_1, \dots, x_{d-1} are infected.
3. $V(T_x)$ is internally spanned by $A \cap V(T_x)$ and x . By the same argument

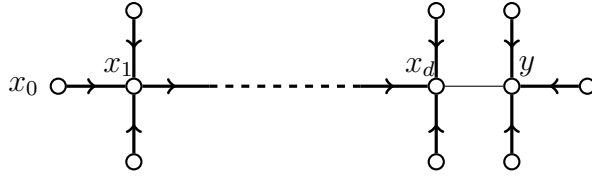


Figure 6.3: The r -threshold infection process of $x_1, \dots, x_d = x$ and y in \mathcal{P} where $r = 3$, the thick lines represent used edges, the thin lines represent wasted edges, and the arrows represent infection directions.

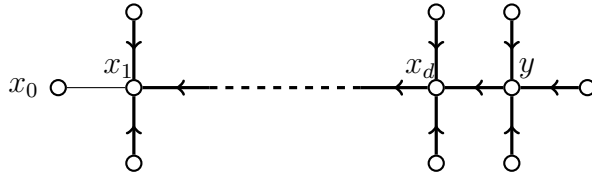


Figure 6.4: The r -threshold infection process of $y, x_d = x, \dots, x_1$ in \mathcal{P}' where $r = 3$, the thick lines represent used edges, the thin lines represent wasted edges, and the arrows represent infection directions.

in the preceding phase, x_{d-1}, \dots, x_1 will get infected in turn. Then, other vertices of T_x can get infected just as they do in the process \mathcal{P} .

By the description of \mathcal{P}' , it is a percolation process starting from A whose wasted edges are E'_W . Moreover, there are exactly $k + 1$ edges in E'_W incident with vertices in B (because we just remove $x - y$ from E_W and add a new wasted edge $x_0 - x_1$ to E_W). Thus, from E_W , we get a new percolation process along with a new set of wasted edges containing one more edge incident with vertices in B . Repeating this operation, we can finally get a set E_W of all the wasted edges of some percolation process starting from A such that every edge in E_W is incident with a vertex in B .

Next, we assert that every seed in A is incident with at most $r - 1$ wasted edges in E_W . Otherwise, if $v \in A$ was incident with r wasted edges $v -$

$x_1, \dots, v - x_r \in E_W$, for each $i = 1, \dots, r$, let T_i be the one of two subtrees of T containing x_i when deleting the edge $v - x_i$. Then, for each $i = 1, \dots, r$, since $v - x_i$ is wasted, x_i could be infected by $A \cap V(T_i)$. Subsequently, v could be infected by r infected neighbours x_1, \dots, x_r at some time, which implies $A \setminus \{v\}$ can percolate on T and contradicts minimality of A . Thus, every seed in A is incident with at most $r - 1$ wasted edges in E_W , and so $w \leq (r - 1)|B|$. On the other hand, by Lemma 2.3, $|B| \leq u$. Thus,

$$w \leq (r - 1)|B| \leq (r - 1)u = (r - 1)(n - 1 - w),$$

and the lemma follows. \square

Lemma 6.2 gives us a bound for the gap between minimum and largest minimal percolating sets of trees with some special properties. We introduce it in the following lemma.

Lemma 6.3 (Riedl [33]). *Let T be a tree of order n , r be an integer with $2 \leq r \leq n - 1$, and L be the set of leaves of T . If $\langle L \rangle_r = L$ and every internal vertex has degree at least r , then*

$$E(T, r) \leq \frac{(r^2 - 1)n + 1}{r^2},$$

and

$$E(T, r) - m(T, r) \leq \frac{(r - 1)(n - 1)}{r^2}.$$

Proof. Let A be a minimal r -threshold percolating set of T . Then, by Lemma 6.2 and the relation between used edges and the corresponding percolating set mentioned in Section 1.1, we have

$$(n-1) - r(n - |A|) = (n-1) - u = w \leq \frac{(n-1)(r-1)}{r},$$

and so

$$|A| \leq \frac{(r^2 - 1)n + 1}{r^2},$$

which proves the first inequality of the lemma.

On the other hand, by Lemma 2.2, we have $n - \frac{n-1}{r} \leq m(T, r)$. Thus,

$$\begin{aligned} E(T, r) - m(T, r) &\leq \frac{(r^2 - 1)n + 1}{r^2} - \left(n - \frac{n-1}{r} \right) \\ &= \frac{(r-1)(n-1)}{r^2}. \end{aligned}$$

□

The next lemma is useful to the following argument.

Lemma 6.4. *Let $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_m}{b_m}$ be a series of real numbers with every $b_i > 0$.*

Then

$$\frac{\sum_{i=1}^m a_i}{\sum_{i=1}^m b_i} \leq \max_{i \in [m]} \left\{ \frac{a_i}{b_i} \right\}.$$

Proof. Let $M = \max_{i \in [m]} \left\{ \frac{a_i}{b_i} \right\}$. Then, for every $i \in [m]$, we have $\frac{a_i}{b_i} \leq M$, and

so $a_i \leq b_i M$. Hence, we get

$$\sum_{i=1}^m a_i \leq \sum_{i=1}^m b_i M = M \left(\sum_{i=1}^m b_i \right),$$

and the lemma follows. \square

Now, we extend the result about the gap between minimum percolating sets and largest minimal percolating sets in Lemma 6.3 with a strategy of removing the conditions on leaves and degrees of internal vertices.

Theorem 6.5 (Riedl [33]). *Let T be a tree of order n , r be an integer with $2 \leq r \leq n - 1$. Then,*

$$E(T, r) - m(T, r) \leq \frac{(r-1)(n-1)}{r^2}.$$

Proof. First, we dispose of the condition imposed on leaves in Lemma 6.3. Suppose, in hopes of a contradiction, that $T = (V, E)$ is a tree of the smallest order such that $\langle L \rangle_r \supsetneq L$, every internal vertex has degree at least r and $E(T, r) - m(T, r) > \frac{(r-1)(n-1)}{r^2}$. Then, there is a vertex $v \in V$ adjacent to at least r leaves of T . Let L_v be the set of leaves adjacent to v in T , and define T'_1, \dots, T'_k be the connected components of $T - (\{v\} \cup L_v)$ (the theorem is obviously true if no such components exist), and for each $i = 1, \dots, k$, let T_i be the subgraph of T induced by $V(T'_i) \cup \{v\}$. Now, we shall construct a bijection between the r -threshold percolating sets of T_1, \dots, T_k and the percolating sets

of T . For a sequence of sets (A_1, \dots, A_k) where each A_i is an r -threshold percolating set of T_i , define

$$\eta(A_1, \dots, A_k) = \left(\bigcup_{i=1}^k A_i \right) \cup L_v \setminus \{v\}.$$

Since each A_i is an r -threshold percolating set of T_i and v is adjacent to at least r leaves of T , $\eta(A_1, \dots, A_k)$ percolates on T . Conversely, for an r -threshold percolating set A of T , each component $A \cap V(T_i) \cup \{v\}$ of A 's inverse image

$$\eta^{-1}(A) = (A \cap V(T_1) \cup \{v\}, \dots, A \cap V(T_k) \cup \{v\})$$

percolates in T_i because v is a leaf of T_i . Hence, (A_1, \dots, A_k) is a sequence of r -threshold percolating sets of T_1, \dots, T_k if and only if $\eta(A_1, \dots, A_k)$ is an r -threshold percolating set of T . In addition, for any sequence (A_1, \dots, A_k) of r -threshold percolating sets of T_1, \dots, T_k ,

$$|\eta(A_1, \dots, A_k)| = \left(\sum_{i=1}^k (|A_i| - 1) \right) + |L_v|.$$

In particular, suppose for every $i = 1, \dots, k$, A_i is an r -threshold minimal percolating set of T_i , and consider $\eta(A_1, \dots, A_k)$. If we remove a vertex $x \in L_v$ from $\eta(A_1, \dots, A_k)$, then $\eta(A_1, \dots, A_k) \setminus \{x\}$ cannot percolate on T because x will never get infected. If we remove a vertex $x \in \eta(A_1, \dots, A_k) \setminus L_v$ from $\eta(A_1, \dots, A_k)$, then there exist some j such that $x \in A_j \setminus \{v\}$. If

$\eta(A_1, \dots, A_k) \setminus \{x\}$ percolated in T , then $A_j \setminus \{x\}$ would percolate in T_j because the leaf v is in A_j and $x \neq v$, which would contradict minimality of A_j on T_j . So, $\eta(A_1, \dots, A_k)$ is a minimal percolating set of T .

In converse, suppose A is a minimal r -threshold percolating set of T . If some component $A \cap V(T_j) \cup \{v\}$ of $\eta^{-1}(A)$ was not a minimal percolating set of T_j , then there exists a vertex $x_j \in A \cap V(T_j) \cup \{v\}$ such that $A \cap V(T_j) \cup \{v\} \setminus \{x_j\}$ could percolate in T_j . Note that in this case, x_j had to be an internal vertex of T_j because any leaf will never be infected by other vertices. So, $x_j \neq v$, and so $A \setminus \{x_j\} \subsetneq A$ could percolate on T which would contradict the minimality of A . Thus, for every $i = 1, \dots, k$, the i -th component of $\eta^{-1}(A)$ is a minimal percolating set of T_i .

Hence, (A_1, \dots, A_k) is a sequence of minimal r -threshold percolating sets of T_1, \dots, T_k if and only if $\eta(A_1, \dots, A_k)$ is a minimal r -threshold percolating sets of T . By the same argument, (A_1, \dots, A_k) is a sequence of minimum (or largest minimal) r -threshold percolating sets of T_1, \dots, T_k if and only if $\eta(A_1, \dots, A_k)$ is a minimum (or largest minimal) r -threshold percolating sets of T . Hence, we get $E(T, r) - m(T, r) = \sum_{i=1}^k (E(T_i, r) - m(T_i, r))$.

Suppose each T_i has order n_i . Then, $n = (\sum_{i=1}^k (n_i - 1)) + 1 + |L_v|$. Besides, by the choice of T , for each $i = 1, \dots, k$, every internal vertex of T_i has degree at least r and $E(T_i, r) - m(T_i, r) \leq \frac{(r-1)(n_i-1)}{r^2}$. Hence, using Lemma 6.4, we

get

$$\begin{aligned}
\frac{E(T, r) - m(T, r)}{n - 1} &= \frac{\sum_{i=1}^k (E(T_i, r) - m(T_i, r))}{(\sum_{i=1}^k (n_i - 1)) + |L_v|} \\
&< \frac{\sum_{i=1}^k (E(T_i, r) - m(T_i, r))}{\sum_{i=1}^k (n_i - 1)} \\
&\leq \max_{i \in [k]} \left\{ \frac{E(T_i, r) - m(T_i, r)}{n_i - 1} \right\} \\
&\leq \frac{r - 1}{r^2},
\end{aligned}$$

which contradicts the assumption $E(T, r) - m(T, r) > \frac{(r-1)(n-1)}{r^2}$. Therefore, for every tree T of order n with every internal vertex has degree at least r , we have $E(T, r) - m(T, r) \leq \frac{(r-1)(n-1)}{r^2}$.

Next, we dispose of the condition “every internal vertex has degree at least r ” in Lemma 6.3. Again, in hopes of a contradiction, suppose T is a tree of the smallest order such that T has some internal vertices of degree less than r and $E(T, r) - m(T, r) > \frac{(r-1)(n-1)}{r^2}$. Suppose $v \in V$ is an internal vertex of degree less than r . Then, by Remark 2.1, every r -threshold percolating set must contain v . Now, let T'_1, \dots, T'_k be the connected components of $T - v$ (the theorem is obviously true if no such components exist), and for each $i = 1, \dots, k$, let T_i be the subgraph of T induced by $V(T'_i) \cup \{v\}$. Following a similar argument as above, we construct a bijection between the r -threshold percolating sets of T_1, \dots, T_k and the percolating sets of T . For a sequence of

sets (A_1, \dots, A_k) where each A_i is an r -threshold percolating set of T_i , define

$$\theta(A_1, \dots, A_k) = \left(\bigcup_{i=1}^k A_i \right).$$

Since each A_i is an r -threshold percolating set of T_i , $\theta(A_1, \dots, A_k)$ percolates on T . Conversely, for an r -threshold percolating set A of T , each component $A \cap V(T_i)$ of A 's inverse image

$$\theta^{-1}(A) = (A \cap V(T_1), \dots, A \cap V(T_k))$$

percolates in T_i because $v \in A$ and v is a leaf of T_i . Hence, (A_1, \dots, A_k) is a sequence of r -threshold percolating sets of T_1, \dots, T_k if and only if $\theta(A_1, \dots, A_k)$ is an r -threshold percolating set of T . In addition, for any sequence (A_1, \dots, A_k) of r -threshold percolating sets of T_1, \dots, T_k ,

$$|\theta(A_1, \dots, A_k)| = \left(\sum_{i=1}^k (|A_i| - 1) \right) + 1.$$

By the same argument as above, (A_1, \dots, A_k) is a sequence of minimal (or minimum or largest minimal) r -threshold percolating sets of T_1, \dots, T_k if and only if $\theta(A_1, \dots, A_k)$ is a minimal (or minimum or largest minimal) r -threshold percolating sets of T . Hence, we get $E(T, r) - m(T, r) = \sum_{i=1}^k (E(T_i, r) - m(T_i, r))$.

Suppose each T_i has order n_i . Then, $n = (\sum_{i=1}^k (n_i - 1)) + 1$. Besides, by the choice of T , for each $i = 1, \dots, k$, $E(T_i, r) - m(T_i, r) \leq \frac{(r-1)(n_i-1)}{r^2}$. Hence,

we have

$$\begin{aligned} \frac{E(T, r) - m(T, r)}{n - 1} &= \frac{\sum_{i=1}^k (E(T_i, r) - m(T_i, r))}{\sum_{i=1}^k (n_i - 1)} \\ &\leq \max_{i \in [k]} \left\{ \frac{E(T_i, r) - m(T_i, r)}{n_i - 1} \right\} \\ &\leq \frac{r - 1}{r^2}, \end{aligned}$$

which contradicts $E(T, r) - m(T, r) > \frac{(r-1)(n-1)}{r^2}$. Therefore, for every tree T of order n with $2 \leq r \leq n - 1$, we finally have $E(T, r) - m(T, r) \leq \frac{(r-1)(n-1)}{r^2}$, and the theorem follows. \square

The upper bound in Theorem 6.5 is tight for some n and r as in the following example.

Example 6.6. Construct a tree $T = (V, E)$ of order n for a given integer $r \geq 2$ as follows. Start at a vertex $v \in V$ as the root of T . Let v be adjacent to r internal vertices v_1, \dots, v_r , and for each $i = 1, \dots, r$, let v_i be adjacent to $r - 1$ leaves (illustrated in Figure 6.5). For such a construction, $n = 1 + r + r(r - 1) = 1 + r^2$. Let L be the set of all the leaves of T . Then, $L \cup \{v_1, \dots, v_r\}$ is a largest minimal r -threshold percolating set while $L \cup \{v\}$ is a minimum r -threshold percolating set of T . Hence,

$$\begin{aligned} E(T, r) - m(T, r) &= [r(r - 1) + r] - [r(r - 1) + 1] \\ &= r - 1 \\ &= \frac{(r - 1)(n - 1)}{r^2}, \end{aligned}$$

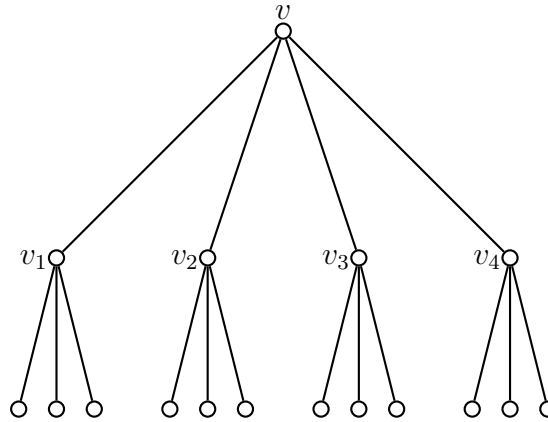


Figure 6.5: A tree T for a given integer $r = 4$

and so the upper bound of the gap between $E(T, r)$ and $m(T, r)$ is tight for T .

We mainly introduced the results about bounds for cardinality of minimum/minimal percolating sets of trees. Although these results do not give exact values for all cases, the tightness of those bounds indicated that they are the best values one can achieve. In the next section, we will present a construction of percolating sets of trees.

6.2 Constructing Minimum Percolating Sets of Trees

As mentioned above, for a given graph, how to get a percolating set containing as few seeds as possible is not only interesting in theory but also practical for applications of bootstrap percolation. Fortunately, for trees, it is absolutely possible to construct percolating sets of a smallest number of seeds – minimum

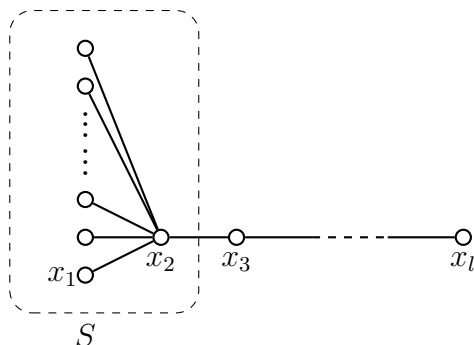
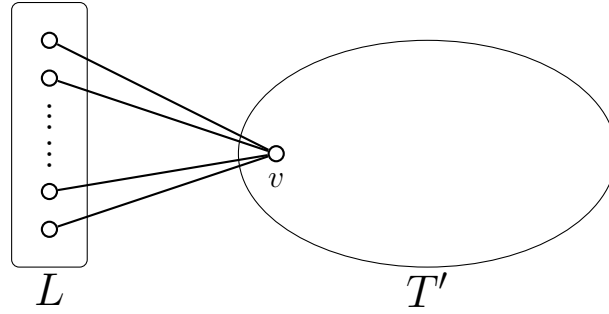


Figure 6.6: Finding a trailing star S of T via identifying a longest path in T

percolating sets – with an algorithm presented by Riedl [33]. So, in this section, we shall introduce Riedl’s method for constructing minimum percolating sets of trees.

A critical concept for finding the minimum percolating sets of trees is *trailing stars*. For a tree T other than a single vertex or a single edge (in other words, the diameter of T is at least 2), if a star S centred at v is a subtree of T such that S is connected to $G - S$ by a single edge incident with v , then S is called a *trailing star* of T . To find a trailing star S of any given tree T whose diameter is at least 2, one can first identify a longest path $x_1 - x_2 - x_3 - \dots - x_l$ in T and define $L_{x_2} = N_T(x_2) \setminus \{x_3\}$. Note that every neighbour (if there are still some) of x_2 other than x_1, x_3 must be a leaf of T because $x_1 - x_2 - x_3 - \dots - x_l$ is a longest path in T . Then, the induced subgraph $S = T[\{x_2\} \cup L_{x_2}]$ is exactly a trailing star of T (illustrated in Figure 6.6).

To explain the procedure of constructing the minimum percolating sets of trees more clearly, we first make preparation by introducing some lemmas.

Figure 6.7: The tree T in Lemma 6.7

Lemma 6.7 (Riedl [33]). *Suppose T is a tree of order n whose diameter is at least 2, r is an integer with $2 \leq r < n$, S is a trailing star of T centred at v , $L = \{N_S(v)\}$, $T' = T - L$ is a subtree of T , and A' is a minimum r -threshold percolating set of T' . If $|L| \geq r$, then $A' \cup L \setminus \{v\}$ is a minimum r -threshold percolating set of T .*

Proof. Let $T = (V, E)$ (illustrated in Figure 6.7). Note that $v \in A'$ since v is a leaf of T' .

First, consider infection process starting from $A' \cup L \setminus \{v\} = (A' \setminus \{v\}) \cup L$ in T . Since the vertices in L are infected neighbours of v and $|L| \geq r$, v can get infected in the first step of the infection process. Then, since A' percolates on $T' = T - L$ and $v \in A'$, $\{v\} \cup (A' \setminus \{v\}) = A'$ can infect all the vertices of $T - L$. Thus, $A' \cup L \setminus \{v\}$ percolates on T .

Next, consider whether $A' \cup L \setminus \{v\}$ is a minimum r -threshold percolating set of T . In hope of a contradiction, assume B is a minimum percolating set of T with $|B| < |A' \cup L \setminus \{v\}|$. Then, $L \subseteq B$ because every vertex in L is a

leaf of T , but $v \notin B$ because v can be infected by the vertices in L . Hence, $(B \setminus L) \cup \{v\}$ percolates on T' . However, since $|B| < |A' \cup L \setminus \{v\}|$, then

$$|(B \setminus L) \cup \{v\}| < |[(A' \cup L \setminus \{v\}) \setminus L] \cup \{v\}| = |A'|,$$

which contradicts A' is a minimum percolating set of T' . Therefore, $A' \cup L \setminus \{v\}$ is a minimum r -threshold percolating set of T , and the lemma follows. \square

Lemma 6.8 (Riedl [33]). *Let T be a tree of order n , x be a leaf of T , and r be an integer with $2 \leq r < n$. Then,*

$$m(T - x, r) \leq m(T, r) \leq m(T - x, r) + 1.$$

Proof. Let x be adjacent to the vertex v of T .

To prove the first inequality, suppose A is a minimum r -threshold percolating set of T , then $x \in A$ because it is a leaf of T . We are facing two cases.

1. If $v \in A$, then the edge $x-v$ will be wasted (because it joins two seeds) and $A \setminus \{x\}$ can percolate on $T-x$, and so $m(T-x, r) \leq |A \setminus \{x\}| = m(T, r) - 1$.
2. If $v \notin A$, let a_1, \dots, a_r be the vertices other than x and get infected after v does in the percolation process of A on T , and let b_1, \dots, b_s be the vertices other than x and get infected before v does in the percolation process of A on T . In the forest $T - \{v, x\}$, for each i, j , let A_i be the tree containing a_i , and let B_j be the tree containing b_j . Then, first, $\bigcup_{j=1}^s V(B_j)$ will be internally infected by A ; second, v will be

infected by r infected neighbours amongst b_1, \dots, b_s, x ; finally, $\bigcup_{i=1}^r V(A_i)$ will be infected by $A \cap [\bigcup_{i=1}^r V(A_i)]$ and v . Now, consider a new set $A \cup \{v\} \setminus \{x\}$ of seeds and its infection process on $T - x$. Note that $A_1, \dots, A_r, B_1, \dots, B_s$ are all the trees of the forest $T - x - v$. First, since $(A \cup \{v\} \setminus \{x\}) \cap (\bigcup_{j=1}^s V(B_j)) = A \cap (\bigcup_{j=1}^s V(B_j))$, then $\bigcup_{j=1}^s V(B_j)$ can be internally infected by $A \cup \{v\} \setminus \{x\}$ too. At this time, v does not need to be infected because it is a seed itself. Second, since $(A \cup \{v\} \setminus \{x\}) \cap (\bigcup_{i=1}^r V(A_i)) = A \cap (\bigcup_{i=1}^r V(A_i))$, then $\bigcup_{i=1}^r V(A_i)$ can be infected by $A \cup \{v\} \setminus \{x\}$ and v . Hence, $A \cup \{v\} \setminus \{x\}$ can percolate on $T - x$, and so

$$m(T - x, r) \leq |A \cup \{v\} \setminus \{x\}| = |A| = m(T, r).$$

Therefore, $m(T - x, r) \leq m(T, r)$.

To prove the second inequality, suppose B is a minimum r -threshold percolating set of $T - x$. Then, $B \cup \{x\}$ is a percolating set of T . So,

$$m(T, r) \leq |B \cup \{x\}| = m(T - x, r) + 1,$$

and the lemma follows. \square

Lemma 6.9 (Riedl [33]). *Suppose T is a tree of order n whose diameter is at least 2, r is an integer with $2 \leq r < n$, S is a trailing star of T centred at v , $L = \{N_S(v)\}$, $T' = T - L - v$ is a subtree of T , and A' is a minimum r -threshold percolating set of T' . If $|L| = r - 1$, then $A' \cup L$ is a minimum r -threshold percolating set of T .*

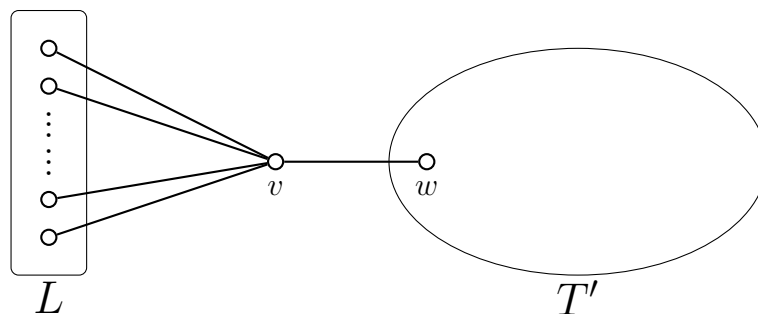


Figure 6.8: The tree T in Lemma 6.9

Proof. Let $T = (V, E)$ (illustrated in Figure 6.8) and let w be the neighbour of v in $V(T')$.

To show $A' \cup L$ percolates on T , consider the infection process of $A' \cup L$ on T . First, A' can percolate on T' . Thereafter, the vertex v will be the unique clean vertex of T . Then, since v has r infected neighbours, namely, $L \cup \{w\}$, it will get infected. So, $A' \cup L$ is indeed an r -threshold percolating set of T .

Next, to show $A' \cup L$ is an r -threshold minimum percolating set of T , assume B is an r -threshold minimum percolating set of T with $|B| < |A' \cup L|$. Then, $L \subseteq B$ because all the vertices in L are leaves of T . Now, we are facing two cases.

1. If $v \notin B$, then since $|L| = r - 1 < r$, v has to get infected after w does. Consequently, B can internally infect $V(T')$. In other words, $B \setminus L$ is a percolating set of T' . However, since $|B| < |A' \cup L|$, then $|B \setminus L| < |A'|$, which contradicts A' is a minimal percolating set of T' .
2. If $v \in B$, let us observe $T - L$ and T' . Note that $T' = (T - L) - v$.

Since $v \in B$, all the edges between L and v will be wasted in the percolation process of B on T , and $B \setminus L$ can percolate on $T - L$. Hence, $m(T - L, r) \leq |B \setminus L|$. However, since $|B| < |A' \cup L|$, then $|B \setminus L| < |A'|$, and so

$$m(T - L, r) \leq |B \setminus L| < |A'| = m(T', r),$$

which contradicts Lemma 6.8.

Therefore, $A' \cup L$ is a minimum r -threshold percolating set of T , and the lemma follows. \square

Now, we shall readily describe the procedure *MinimumPercSetTree* presented by Riedl [33] for constructing minimum percolating sets of trees as follows. Meanwhile, we characterize the procedure with pseudocode in Algorithm 2.

1. The procedure *MinimumPercSetTree* has a recursive structure. It takes a tree $T = (V, E)$ of order n and a threshold r with $2 \leq r < n$ as input, and returns a minimum r -threshold percolating set A of T as output.
2. For the given tree $T = (V, E)$ of order n and the threshold r with $2 \leq r < n$, define a set A for a minimum r -threshold percolating set of T , and initialize A as the empty set.
3. Define $L_T = \{v \in V : \deg_T(v) < r\}$ and assign L_T to A because A must contain the vertices of degree smaller than r . Thereafter, if $L_T = V$, then return A because it is already a minimum r -threshold percolating set of T ;

Algorithm 2: MinimumPercSetTree

Input : A tree $T = (V, E)$ of order n , the threshold r with $2 \leq r < n$ **Output**: A minimum r -threshold percolating set A of T

```

1 define a set  $A$  and initialize it as empty;
2  $A \leftarrow \{v \in V : \deg_T(v) < r\}$ ;
3 if  $|A| = n$  then
4   | return  $A$ ;
5 end
6 decompose  $T$  to a series of subtrees  $T_1, \dots, T_s$  of  $T$  such that
    $\bigcup_{i=1}^s V(T_i) = V$ ,  $\bigcup_{i=1}^s E(T_i) = E$ , for each  $i = 1, \dots, s$ , every internal
   vertex of  $T_i$  has degree at least  $r$  and every leaf of  $T_i$  is in  $A$ ;
7 for  $i = 1, \dots, s$  do
8   | define a set  $B$  and initialize it as empty;
9   |  $B \leftarrow \{v \in V(T_i) : \deg_{T_i}(v) < r\}$ ;
10  | if  $|B| = |V(T_i)|$  then
11  |   |  $A \leftarrow A \cup B$ ;
12  |   | continue;
13  | end
14  | identify a trailing star  $S$  centred at  $v$  and with leaves  $L$ ;
15  | if  $|L| < r$  then
16  |   |  $B \leftarrow L \cup \text{MinimumPercSetTree}(T - L - v, r)$ ;
17  |   else
18  |     |  $B \leftarrow L \cup \text{MinimumPercSetTree}(T - L, r) \setminus \{v\}$ ;
19  |   end
20  |  $A \leftarrow A \cup B$ ;
21 end
22 return  $A$ ;

```

otherwise, decompose T to a series of subtrees T_1, \dots, T_s of T such that $\bigcup_{i=1}^s V(T_i) = V$, $\bigcup_{i=1}^s E(T_i) = E$, for each $i = 1, \dots, s$, every internal vertex of T_i has degree at least r and every leaf of T_i is in L_T (illustrated in Figure 6.9). Now, define a mapping $\eta : \prod_{i=1}^s \mathcal{P}(V(T_i)) \rightarrow \mathcal{P}(V)$ by $\eta(A_1, \dots, A_s) = \bigcup_{i=1}^s A_i$. Then, by the same argument as that in the proof of Theorem 6.5, (A_1, \dots, A_s) is a sequence of minimum r -threshold percolating sets of T_1, \dots, T_s if and only if $\eta(A_1, \dots, A_s)$ is a minimum r -threshold percolating set of T . Thus, if for each $i = 1, \dots, s$, we can obtain a minimum r -threshold percolating set A_i of T_i , then $\bigcup_{i=1}^s A_i$ will be a minimum r -threshold percolating set of T . Hence, our next objective is to find a minimum r -threshold percolating set of each T_i .

4. The decomposition of T to T_1, \dots, T_s can be performed iteratively as follows. Define $IL_T = \{v \in V : 1 < \deg_T(v) < r\} = \{x_1, \dots, x_a\}$. First, we decompose T to the branches T'_1, \dots, T'_{b_1} rooted at x_1 (every branch contains x_1 as a leaf). Second, without loss of generality, suppose x_2 is in the tree T'_2 , then decompose T'_2 to the branches rooted at x_2 and replace T'_2 by these new branches to get a new set T'_1, \dots, T'_{b_2} of subtrees of T . Continue this process, one can finally decompose T to a series of subtrees T_1, \dots, T_s of T such that $\bigcup_{i=1}^s V(T_i) = V$, $\bigcup_{i=1}^s E(T_i) = E$, for each $i = 1, \dots, s$, every internal vertex of T_i has degree at least r and every leaf of T_i is in L_T .

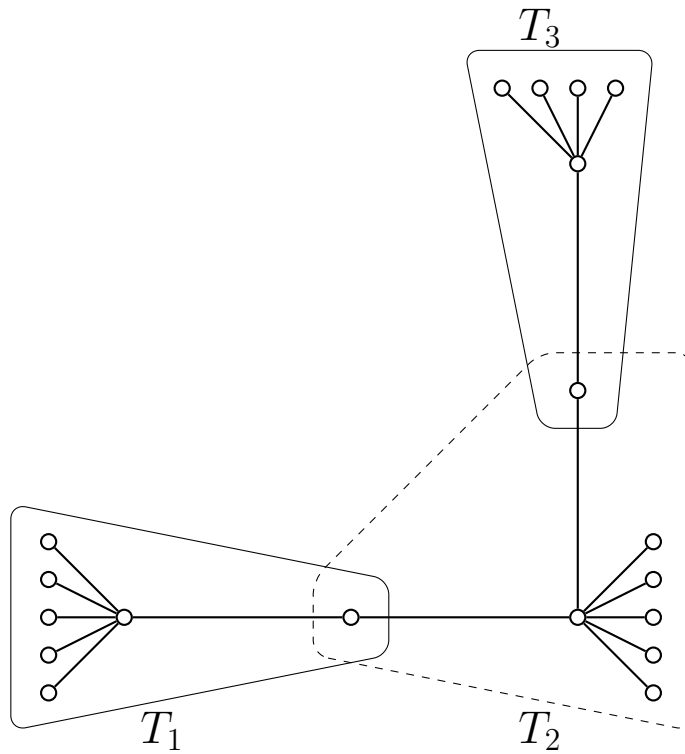


Figure 6.9: Decomposition of a tree $T = (V, E)$ to subtrees T_1, T_2, T_3 of T such that $\bigcup_{i=1}^3 V(T_i) = V$, $\bigcup_{i=1}^3 E(T_i) = E$, for each $i = 1, \dots, 3$, every internal vertex of T_i has degree at least 4 and every leaf of T_i is in the set $L_T = \{v \in V : \deg_T(v) < 4\}$

5. For each $i = 1, \dots, s$, to find a minimum r -threshold percolating set of T_i , first identify a trailing star S centred at v and with leaves set L . Note that $\deg_{T_i}(v) = \deg_T(v) \geq r$ by the definition of T_i . Hence, either $|L| = r - 1$ or $|L| \geq r$. If $|L| = r - 1$, then by Lemma 6.9, we can first recursively invoke *MinimumPercSetTree* and compute a minimum percolating set A' of the subtree $T - L - v$, thereby obtain a minimum percolating set $A = A' \cup L$ of T_i . Otherwise, if $|L| \geq r$, then by Lemma 6.7, we can first recursively invoke *MinimumPercSetTree* and compute a minimum percolating set A' of the subtree $T - L$, thereby obtain a minimum percolating set $A = A' \cup L \setminus \{v\}$ of T_i .

In this chapter, we studied bootstrap percolation on trees from the bounds to finding the percolating sets. By now, we can say the most important questions about bootstrap percolation on trees are answered satisfactorily.

Bibliography

- [1] Joan Adler and Uri Lev. Bootstrap percolation: visualizations and applications. *Brazilian Journal of Physics*, 33(3):641–644, 2003. (Cited on page 5.)

- [2] Noga Alon and Joel H. Spencer. *The Probabilistic Method*. Wiley Series in Discrete Mathematics and Optimization. John Wiley and Sons, Inc., Hoboken, New Jersey, USA, 4th edition, 2015. (Cited on page 70.)

- [3] Hamed Amini and Nikolaos Fountoulakis. What I tell you three times is true: Bootstrap percolation in small worlds. In *Internet and Network Economics - 8th International Workshop, WINE 2012, Liverpool, UK, December 10-12, 2012, Proceedings*, pages 462–474, 2012. (Cited on page 4.)

- [4] József Balogh and Béla Bollobás. Bootstrap percolation on the hypercube. *Probability Theory and Related Fields*, 134(4):624–648, 2006. (Cited on pages 105, 107 and 113.)

- [5] József Balogh, Béla Bollobás, Hugo Duminil-Copin, and Robert Morris. The sharp threshold for bootstrap percolation in all dimensions. *Transactions of the American Mathematical Society*, 364(5):2667–2701, 2012. (Cited on page 83.)
- [6] József Balogh, Béla Bollobás, and Robert Morris. Bootstrap percolation in high dimensions. *Combinatorics, Probability and Computing*, 19(5–6):643–692, 2010. (Cited on pages 83, 105, 107, 114 and 116.)
- [7] József Balogh and Gábor Pete. Random disease on the square grid. *Random Structures and Algorithms*, 13(3–4):409–422, 1998. (Cited on pages 83, 85, 96 and 97.)
- [8] Cristina Bazgan, Morgan Chopin, and Bernard Ries. The firefighter problem with more than one firefighter on trees. *Discrete Applied Mathematics*, 161(7-8):899–908, 2013. (Cited on page 5.)
- [9] Béla Bollobás. Weakly k -saturated graphs. *Beiträge zur Graphentheorie, Kolloquium*:25–31, 1968. (Cited on page 105.)
- [10] Béla Bollobás. *The Art of Mathematics – Coffee Time in Memphis*. Cambridge University Press, 2006. (Cited on page 120.)
- [11] J. Chalupa, Paul L. Leath, and G. R. Reich. Bootstrap percolation on a bethe lattice. *Journal of Physics C: Solid State Physics*, 12(1):L31–L35, 1979. (Cited on pages 4 and 83.)

- [12] Ning Chen. On the approximability of influence in social networks. *SIAM Journal on Discrete Mathematics*, 23(3):1400–1415, 2009. (Cited on page 4.)
- [13] Amin Coja-Oghlan, Uriel Feige, Michael Krivelevich, and Daniel Reichman. Contagious sets in expanders. In *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1953–1987, 2015. (Cited on pages 68, 69, 73, 75, 76 and 77.)
- [14] Dragoš M. Cvetković, Michael Doob, and Horst Sachs. *Spectra of Graphs: Theory and Application*, volume 87 of *Pure and applied mathematics, a series of monographs and textbooks*. Academic Press, New York, 2nd edition, 1982. (Cited on page 70.)
- [15] Paul A. Dreyer and Fred S. Roberts. Irreversible k-threshold processes: graph-theoretical threshold models of the spread of disease and of opinion. *Discrete Applied Mathematics*, 157(7):1615–1627, 2009. (Cited on pages 21, 23, 26 and 27.)
- [16] Paola Flocchini, Elena Lodi, Fabrizio Luccio, Linda Pagli, and Nicola Santoro. Irreversible dynamos in tori. In *Euro-Par'98 Parallel Processing - 4th International Euro-Par Conference Southampton, UK, September 1-4, 1998 Proceedings*, pages 554–562, 1998. (Cited on page 4.)
- [17] Alexander E. Holroyd. Sharp metastability threshold for two-dimensional

- bootstrap percolation. *Probability Theory and Related Fields*, 125:195–224, 2003. (Cited on page 107.)
- [18] Shlomo Hoory, Nathan Linial, and Avi Wigderson. Expander graphs and their applications. *Bulletin of the American Mathematical Society*, 43(4):439–561, 2006. (Cited on page 68.)
- [19] Peter Keevash. The existence of designs. *arXiv: 1401.3665v3*, August 2019. (Cited on page 105.)
- [20] Zonghua Liu, Ying-Cheng Lai, Nong Ye, and Zonghua Liu. Propagation and immunization of infection on general networks with both homogeneous and heterogeneous components. *Physical Review. E, Statistical, Nonlinear, and Soft Matter Physics*, 67(3 Pt 1):031911–031911, 2003. (Cited on page 4.)
- [21] Fabrizio Luccio, Linda Pagli, and Hermineh Sanossian. Irreversible dynamos in butterflies. In *Proc. 6th International Colloq. on Structural Information and Communication Complexity*, pages 204–218. Carleton University Press, 1999. (Cited on page 4.)
- [22] Gary MacGillivray and Ping Wang. On the firefighter problem. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 47:83–96, 2003. (Cited on page 5.)
- [23] Robert Morris. Minimal percolating sets in bootstrap percolation. *The*

- Electronic Journal of Combinatorics*, 16(1):2–20, 2009. (Cited on pages 125, 126, 127, 128, 129, 131, 133, 143 and 144.)
- [24] Natasha Morrison and Jonathan A. Noel. Extremal bounds for bootstrap percolation in the hypercube. *Journal of Combinatorial Theory, Series A*, 156:61–84, 2018. (Cited on pages 105 and 106.)
- [25] John Von Neumann. *Theory of Self-Reproducing Automata*. University of Illinois Press, Champaign, IL, USA, 1966. (Cited on page 4.)
- [26] Mark E. J. Newman. Spread of epidemic disease on networks. *Physical Review. E, Statistical, Nonlinear, and Soft Matter Physics*, 66(1 Pt 2):016128–016128, 2002. (Cited on page 4.)
- [27] Kahloon Ng and Patricia Raff. A generalization of the firefighter problem on $\mathbb{Z} \times \mathbb{Z}$. *Discrete Applied Mathematics*, 156(5):730–745, 2008. (Cited on page 5.)
- [28] David Peleg. Size bounds for dynamic monopolies. *Discrete Applied Mathematics*, 86(2–3):263–273, 1998. (Cited on page 4.)
- [29] Gábor Pete. How to make the cube weedy? *Polygon*, VII(1):69–80, 1997. (Cited on pages 85, 86, 88, 90, 91 and 96.)
- [30] Svatopluk Poljak and Miroslav Sûra. On periodical behaviour in societies with symmetric influences. *Combinatorica*, 3(1):119–121, 1983. (Cited on page 4.)

-
- [31] Michał Przykucki and Thomas Shelton. Smallest percolating sets in bootstrap percolation on grids. *arXiv*, 1907.01940v1 [math.CO], 2019. (Cited on page 96.)
- [32] Daniel Reichman. New bounds for contagious sets. *Discrete Mathematics*, 312(10):1812–1814, 2012. (Cited on pages 13, 14 and 38.)
- [33] Eric Riedl. Largest and smallest minimal percolating sets in trees. *The Electronic Journal of Combinatorics*, 19(1):64–81, 2012. (Cited on pages 13, 38, 153, 154, 158, 161, 163, 169, 170, 172, 173 and 175.)
- [34] Stanislaw Ulam. Random processes and transformations. In *Proceedings of the International Congress of Mathematicians*, volume 2, pages 264–275, 1950. (Cited on page 4.)