Application of Translational Addition Theorems to the Study of the Magnetization of Systems of Ferromagnetic Spheres

by

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A thesis submitted to the Faculty of Graduate Studies of
The University of Manitoba
in partial fulfillment of the requirements of the degree of

MASTER OF SCIENCE

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Abstract

The main objective of this research is the study of the magnetization of ferromagnetic spheres in the presence of external magnetic fields. The exact analytical solutions derived in this thesis are benchmark solutions, valuable in testing the correctness and accuracy of various approximate models and numerical methods.

First, in this study, the total scalar magnetic potential outside the spheres, related to the magnetic field intensity, is obtained by the superposition of the potentials due to all spheres and the potential corresponding to the external field. The translational addition theorems for scalar Laplacian functions in spherical coordinates are then used to solve boundary value problems for the multi-sphere systems by imposing exact boundary conditions.

On the other hand, the scalar magnetic potential inside each sphere, related to the magnetic flux density, also satisfies the Laplace equation which is solved by imposing the boundary conditions known from the solution of the outside field. Finally, the expressions derived are used to generate numerical results of controllable accuracy for various field quantities. These benchmark results are valuable as reference data to determine the accuracy of numerical methods developed to solve magnetostatic boundary value problems in real world applications.

Acknowledgments

I wish to express my gratitude to Professor I.M.R. Ciric for his advice and continuous encouragement throughout the course of this research, for the last two years. I consider myself privileged to have had the opportunity to work under his guidance.

The financial support received from the Natural Sciences and Engineering Research Council of Canada and the Faculty of Graduate Studies of the University of Manitoba is greatly appreciated. I am also grateful to Ms. Traci Hofer and Ms. Amy Dario for their administrative support during my studies.

I would like to thank Dr. Nuwan Perera, Mr. Kumara Mudun Kotuwage and my colleagues in the Department of Electrical and Computer Engineering for their valuable support.

Finally, I extend my wholehearted gratitude to my family for their encouragements that enabled me to succeed in this effort.

Gehan Anthonys

August 2014

Dedication

To my parents and teachers.

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a_i	radius of sphere i
$\mathbf{B}$	magnetic flux density
B_n	normal component of magnetic flux density
$C_{nm}, C_{n_1m_1}$	series coefficients for the potential outside the sphere 1
d	distance between the centers of two spheres
d_{ij}	distance between the centers of sphere i and sphere j
$D_{qp}, D_{n_2m_2}$	series coefficients for the potential outside the sphere 2
$E_{ts}, E_{n_3m_3}$	series coefficients for the potential outside the sphere 3
$G^{(i)}_{n_im_i}$	series coefficients for the potential inside the sphere i
$\mathbf{H_0}$	external magnetic field intensity
$\mathbf{H}$	magnetic field intensity
H_r, H_θ, H_φ	components of magnetic field intensity in spherical coordinate system
K_i	reference potentials for the sphere i
N	positive integer
$P_n^m(x)$	associated Legendre functions of first kind, degree n, order m and argument x
$P_n^{m\prime}(x)$	first derivative of associated Legendre functions with respect to argument x

r_i	distance from center of sphere i to a selected point
$(r_i, \theta_i, \varphi_i)$	spherical coordinates attached to sphere i
$(r, \theta_{ij}, \varphi_{ij})$	spherical coordinates of the center of sphere j with respect to the system attached to sphere i
$\hat{r}_i, \hat{\theta}_i, \hat{\varphi}_i$	unit vectors in the spherical coordinates attached to sphere i
V_i	potential of sphere i
(x_i, y_i, z_i)	Cartesian coordinates attached to sphere i
$Y_n^m(\theta, \varphi)$	spherical harmonics of degree n, order m and arguments θ and φ
$\delta_{n\,l}$	Kronecker delta
Φ_i	scalar magnetic potential outside of sphere i
$\Phi_j^{(i)}$	translated scalar magnetic potential due to sphere j into the coordinate system attached to sphere i
$\Psi^{(i)}$	scalar magnetic potential inside of sphere i
μ	permeability of the ferromagnetic material
μ_0	permeability of the medium outside the spheres (for free space, $4\pi \times 10^{-7}$ H/m)

Chapter 1

Introduction

1.1 Background

The magnetostatic field is a solenoidal field, i.e., the divergence of the magnetic flux density vector is always zero. A vector potential can be associated with this field. The vector potential satisfies a vector Poisson equation, in the general case, or a vector Laplace equation in the current free regions [1], [2].

A scalar magnetic potential can also be used in current-free regions since the curl of the magnetic field intensity is zero. This potential is a multiple-valued function of position if a net nonzero current is enclosed by at least one closed contour in the region considered. The scalar magnetic potential is useful in the analysis of ferromagnetic circuits in transformers, electric motors and generators, and in certain microwave ferrite devices [1]. It can also be used in the solution of magnetostatic boundary value problems when the current density is everywhere zero.

The exact analytical solution of the Laplace equation can be derived by using the method of separation of variables in spherical coordinates under both Dirichlet and Neumann boundary conditions [3]. These boundary conditions must be satisfied at the surface of each body in the system. To obtain an exact analytic solution, the surfaces of the bodies involved have to be coordinates surfaces.

When the body boundaries do not coincide with coordinate surfaces, approximate numerical methods have to be employed. The method of images [1] or the method of separation of variables [4] can be applied to solve for the field in the presence of spherical objects. In addition, for a system of only two spherical objects one can also use the bispherical coordinates [5].

By using the translational addition theorems [6], we are able to impose the boundary conditions at the surface of each sphere. To do that, it is necessary to express the field produced by all other spheres in terms of the coordinates of the system attached to one sphere. The resulting analytical expressions constitute benchmark solutions for specified geometric arrangements which are beneficial for determining the validity of numerous approximate numerical techniques.

1.2 Motivation for the Research

An exact analytical solution for the magnetic field both inside and outside the ferromagnetic spheres of many-sphere systems is not available in the literature [7], [8], [9] ,[10]. The analysis of the field outside the ferromagnetic spheres allows the evaluation of the intensification of the field, while the field inside determines the magnetization of the spheres. The investigation of the behavior of such ferromagnetic systems is useful for the construction of various models to be employed, for instances, in nano-scale engineering applications [11] and in ferrohydrodynamic application [12].

In this thesis, the problem of the magnetization of ferromagnetic spheres in the presence of external magnetic fields is solved on the basis of exact field equations by imposing the corresponding boundary conditions. The sphere system is placed in a homogeneous medium and the applied external magnetic field is taken to be uniform. The total scalar magnetic potential outside the spheres is obtained by the superposition of the potentials due to all spheres and of the potential corresponding to the external field. In order to express the potential in the coordinate system attached to a specific sphere, the translational addition theorems [13] is used.

The total scalar magnetic potential outside each sphere is considered to be constant at the surfaces of all the spheres and the total magnetic flux through the surface of each sphere is equal to zero. On the other hand, the magnetic potential inside each sphere is determined by imposing the boundary condition that the normal component of the total magnetic flux density is continuous across the surface of the sphere.

The results of the research undertaken in this thesis can be extended to the systems of prolate and oblate spheroids, which are more appropriate to approximate the real world objects in engineering applications. Numerical results generated based on the exact analytical solutions are also important in testing the efficiency of various numerical methods used for solving real world magnetostatic boundary value problems [8], [9].

1.3 Objectives of the Research

The main objective of this research is the study of the magnetization of ferromagnetic spheres in the presence of external magnetic fields. This research goal is to be achieved by accomplishing the following particular objectives:

- Derive exact analytical expressions for magnetic field quantities, both outside and inside linear arrays of ferromagnetic spheres, when placed in uniform magnetic fields;
- Derive exact analytical expressions for magnetic field quantities, both outside and inside arbitrarily located ferromagnetic spheres, when placed in uniform magnetic fields;
- Generate benchmark accurate numerical results in the above cases for various values of the characteristic parameters such as the radii of the spheres and the relative distance between the spheres, at points on the common axis of the spheres.

1.4 Overview of the Thesis

Chapter 2 presents the solution of the scalar Laplace equation in spherical coordinates. In addition, the translational addition theorems for the linear arrays and for asymmetric geometries are expressed to be used in chapter 3 and chapter 4. Next, the scalar magnetic potential for ideal ferromagnetic bodies is expressed. At the end, a solution for a single sphere in the presence of an external field along z axis is also described.

In chapters 3 and 4, exact analytical expressions are derived for the field quantities both outside and inside the ferromagnetic spheres in linear arrays and for asymmetric geometries, respectively, in the presence of external magnetic fields. Exact expressions are also derived for the field intensities in spherical coordinates, the main objective of this study being to use these expressions for generating accurate numerical results.

Accurate numerical results for each case in chapter 3 and chapter 4 are presented in chapter 5. Furthermore, discussions with some explanations of the generated results are described in this chapter. Finally, the conclusions and some recommendations to extend this research work are specified in chapter 6.

Chapter 2

Literature Review

The objective of this chapter is to briefly summarize the background theories used in the research. First, the solution of the scalar Laplace equation in spherical coordinates is given for asymmetric and for axisymmetric geometries. Then, the translational addition theorems for Laplacian fields used in asymmetric and axisymmetric geometries are described. Next, the scalar magnetic potential for ideal ferromagnetic bodies is expressed. Finally, a set of expressions for a single sphere system in the presence of an external field are also presented.

2.1 Solution of the Laplace Equation in Spherical Coordinates

In a homogeneous medium, both outside and inside of ferromagnetic bodies, the potentials satisfy the scalar Laplace equation

$$\nabla^2 f(r,\theta,\varphi) = 0 \tag{2.1.1}$$

which can be expressed in spherical coordinates as

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial}{\partial\theta}\left(sin\,\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

The general form of the solution of (2.1.1) can be expressed in the form [2], [3]

$$f(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(C'_{nm}r^{-(n+1)} + D'_{nm}r^n\right)P_n^m(cos\,\theta)\,e^{-jm\varphi} \tag{2.1.2}$$

where C'_{nm} and D'_{nm} are constants of integration, and m and n are integers. $P_n^m(cos\,\theta)$ are the associated Legendre functions of first kind, of degree n and order m.

2.1.1 Case 1: Asymmetric Systems

In the case the region is extended to infinity, the outside potential due to the spheres can be expressed as (with $f \to 0$ for $r \to \infty$)

$$\Phi(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} C'_{nm}r^{-(n+1)}P_n^m(cos\,\theta)\,e^{-jm\varphi}, \tag{2.1.3}$$

On the other hand, the potential inside is finite and can be expressed as

$$\Psi(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} D'_{nm}r^{n}P_n^m(cos\,\theta)\,e^{-jm\varphi}. \tag{2.1.4}$$

2.1.2 Case 2: Axisymmetric Systems

In the case of azimuthal symmetry (rotational symmetry about the z axis) [2], the solution can be obtained from Equation (2.1.2) with the separation constant $m = 0$ as

$$f(r,\theta) = \sum_{n=0}^{\infty} \left(C_n' r^{-(n+1)} + D_n' r^n \right) P_n(cos\,\theta) \tag{2.1.5}$$

where C_n' and D_n' are constants of integration and P_n are Legendre polynomials.

Then, the outside potential can be expressed as

$$\Phi(r,\theta) = \sum_{n=0}^{\infty} C_n' r^{-(n+1)} P_n(cos\,\theta) \tag{2.1.6}$$

and the inside potential as

$$\Psi(r,\theta) = \sum_{n=0}^{\infty} D_n' r^n P_n(cos\,\theta) \tag{2.1.7}$$

2.2 Translational Addition Theorems for Spherical Laplacian Functions

In order to impose the boundary conditions at the surface of each sphere, we need to express the functions written in the coordinates of one sphere in terms of the coordinates associated to another sphere. For this we use the translational addition theorems [6], [13].

2.2.1 Case 1: Asymmetric System

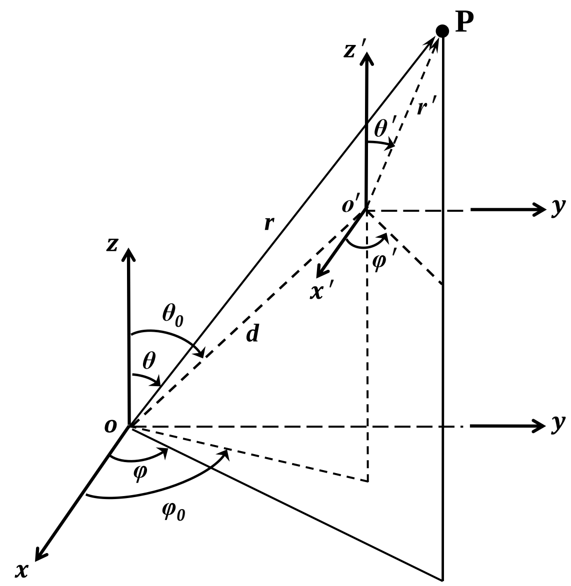

FIGURE 2.1: Coordinate translation

Figure 2.1 shows two spherical coordinate systems, (r, θ, φ) and (r', θ', φ') with the centers at o and o', respectively. The translational addition theorems for spherical scalar wave functions (corresponding to the terms in Equations (2.1.3 and 2.1.4)) can be expressed in the form [13] as

$$r^{-(n+1)} P_n^m(\cos\theta)\, e^{-jm\varphi} = \begin{cases} \sum\limits_{\nu=0}^{\infty} \sum\limits_{\mu=-\nu}^{\nu} (-1)^{\mu+\nu} \frac{(n-m+\nu+\mu)!}{(n-m)!(\nu+\mu)!} \frac{(r')^{\nu}}{d^{\nu+n+1}} P_{n+\nu}^{m-\mu}(cos\theta_0) \\ \qquad \times e^{-j(m-\mu)\varphi_0}\, P_\nu^\mu(\cos\theta')\, e^{-j\mu\varphi'}, \qquad r' \leq d \\ \\ \sum\limits_{\nu=0}^{\infty} \sum\limits_{\mu=-\nu}^{\nu} (-1)^{\nu+\mu} \frac{(n-m+\nu+\mu)!}{(n-m)!(\nu+\mu)!} \frac{d^{\nu}}{(r')^{\nu+n+1}} P_{n+\nu}^{m-\mu}(cos\theta') \\ \qquad \times e^{-j(m-\mu)\varphi'}\, P_\nu^\mu(\cos\theta_0)\, e^{-j\mu\varphi_0}, \qquad r' \geq d \end{cases} \tag{2.2.1}$$

$$r^n P_n^m(\cos\theta)\, e^{-jm\varphi} = \sum_{\nu=0}^{n}\sum_{\mu=-\nu}^{\nu} \frac{(n+m)!}{(\nu+\mu)!(n+m-\nu-\mu)!}\frac{(r')^\nu}{d^{\nu+n}} P_{n-\nu}^{m-\mu}(\cos\theta_0) \\ \times e^{-j(m-\mu)\varphi_0}\, P_\nu^\mu(\cos\theta')\, e^{-j\mu\varphi'}, \qquad r' \le d \tag{2.2.2}$$

2.2.2 Case 2: Axisymmetric System

For $m = 0$, with no dependence on φ, Equations (2.2.1) and (2.2.2) become

$$r^{-(n+1)} P_n(\cos\theta) = \begin{cases} \sum\limits_{\nu=0}^{\infty}\sum\limits_{\mu=-\nu}^{\nu}(-1)^{\nu+\mu}\frac{(n+\nu+\mu)!}{n!(\nu+\mu)!}\frac{(r')^\nu}{d^{\nu+n+1}} P_{n+\nu}^{-\mu}(\cos\theta_0)\, e^{j\mu\varphi_0} \\ \qquad \times P_\nu^\mu(\cos\theta')\, e^{-j\mu\varphi'}, \qquad r' \le d \\ \\ \sum\limits_{\nu=0}^{\infty}\sum\limits_{\mu=-\nu}^{\nu}(-1)^{\nu+\mu}\frac{(n+\nu+\mu)!}{n!(\nu+\mu)!}\frac{d^\nu}{(r')^{\nu+n+1}} P_{n+\nu}^{-\mu}(\cos\theta')\, e^{j\mu\varphi_0} \\ \qquad \times P_\nu^\mu(\cos\theta_0)\, e^{-j\mu\varphi_0'}, \qquad r' \ge d \end{cases} \tag{2.2.3}$$

$$r^n P_n(\cos\theta) = \sum_{\nu=0}^{n}\sum_{\mu=-\nu}^{\nu} \frac{n!}{(\nu+\mu)!(n-\nu-\mu)!}\frac{(r')^\nu}{d^{\nu+n}} P_{n-\nu}^{-\mu}(\cos\theta_0) \\ \times e^{j\mu\varphi_0}\, P_\nu^\mu(\cos\theta')\, e^{-j\mu\varphi'}, \qquad r' \le d \tag{2.2.4}$$

For a common z-axis, with $\theta_0 = 0$ or π, and since $P_{n\pm\nu}^{-\mu}(\pm 1) = 0$ and $P_\nu^\mu(\pm 1) = 0$ for all $\mu \ne 0$, we get

$$r^{-(n+1)} P_n(\cos\theta) = \begin{cases} \sum\limits_{\nu=0}^{\infty}(-1)^\nu \frac{(n+\nu)!}{n!\,\nu!}\frac{(r')^\nu}{d^{\nu+n+1}} P_{n+\nu}(\cos\theta_0)\, P_\nu(\cos\theta'), \;\; r' \le d \\ \\ \sum\limits_{\nu=0}^{\infty}(-1)^\nu \frac{(n+\nu)!}{n!\,\nu!}\frac{d^\nu}{(r')^{\nu+n+1}} P_\nu(\cos\theta_0)\, P_{n+\nu}(\cos\theta'), \;\; r' \ge d \end{cases} \tag{2.2.5}$$

$$r^n P_n(cos\,\theta) = \sum_{\nu=0}^{n} \frac{n!}{\nu!(n-\nu)!} \frac{(r')^\nu}{d^{\nu+n}} P_{n-\nu}(cos\,\theta_0)\, P_\nu(cos\,\theta'), \qquad r' \leq d \tag{2.2.6}$$

2.3 Scalar Magnetic Potential

Ampère's circuital law gives the curl of the magnetic field intensity **H** as [14]

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{2.3.1}$$

where **J** is the electric current density. Wherever the current density is zero Equation (2.3.1) becomes

$$\nabla \times \mathbf{H} = 0$$

Since the curl of the gradient of any scalar function is identical zero, the magnetic field intensity can be expressed as

$$\mathbf{H} = -\nabla\Phi \tag{2.3.2}$$

where Φ is the scalar magnetic potential.

For ideal ferromagnetic bodies, ie., with linear magnetic material and infinite permeability, the field lines are perpendicular to the surface. Therefore the scalar magnetic potential at the surface is constant.

2.4 Solution for a Single Sphere

A ferromagnetic sphere of radius a and permeability μ has been placed in a homogeneous medium of permeability μ_0. The external uniform magnetic field, $\mathbf{H_0}$ is oriented along the z axis as shown in Figure 2.2.

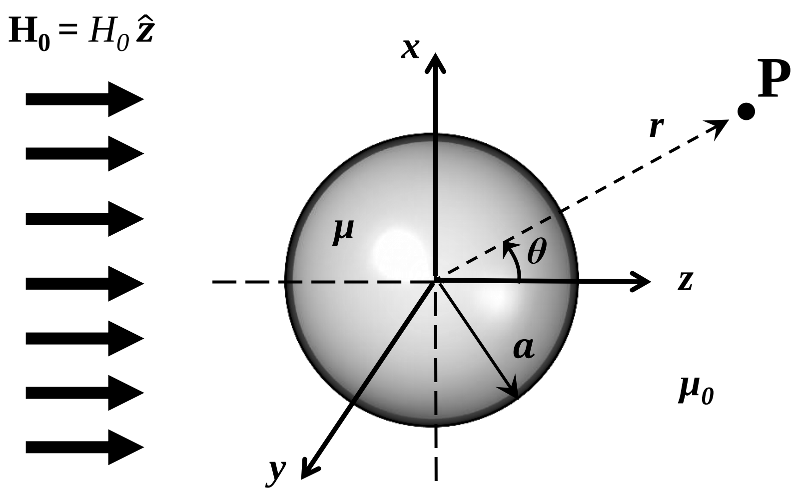

FIGURE 2.2: A sphere placed in the presence of external field along z axis

The total scalar magnetic potential at an arbitrary point $\mathbf{P}$ outside the sphere can be expressed in spherical coordinates as

$$\Phi^{tot}(r,\theta,\varphi) = \Phi(r,\theta,\varphi) + \Phi_{ext}(r,\theta,\varphi) \tag{2.4.1}$$

where Φ is the potential due to the sphere which satisfies Laplace's equation and the term Φ_{ext} is the potential corresponding to the external field.

Since the system has an axial symmetry, we use the Equation (2.1.6) written in the form

$$\Phi^{tot}(r,\theta,\varphi) = \sum_{n=0}^{\infty} C_n \left(\frac{a}{r}\right)^{n+1} P_n(cos\,\theta) - H_0\, r\, cos\,\theta, \qquad r \geq a \tag{2.4.2}$$

where C_n are constants of integration.

Now impose the condition that the total magnetic flux through the surface of the sphere is equal to zero, i.e.,

$$\oint_S B_n\, ds_r = 0$$

where $B_n = -\mu_0 \frac{\partial}{\partial r}\left\{\Phi^{tot}\right\}\Big|_S$ is the normal component of the magnetic flux density on the surface S of the sphere with $ds_r = r^2 sin\,\theta\, d\theta\, d\varphi$. Thus,

$$-\mu_0 \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \frac{\partial}{\partial r}\left\{\Phi^{tot}\right\} r^2 sin\,\theta\, d\theta\, d\varphi \Bigg|_{r\,=\,a} = -4\pi\mu a C_0 = 0 \tag{2.4.3}$$

which yields, $C_0 = 0$. Then, the Equation (2.4.2) becomes

$$\Phi^{tot}(r,\theta,\varphi) = \sum_{n=1}^{\infty} C_n \left(\frac{a}{r}\right)^{n+1} P_n(cos\,\theta) - H_0\, r\, cos\,\theta, \qquad r \geq a \tag{2.4.4}$$

At the surface of the sphere

$$\Phi^{tot}(r,\theta,\varphi)\Big|_{r\,=\,a} = V$$

where V is constant. This gives,

$$\begin{aligned} C_1 &= a\, H_0, \qquad && n = 1 \\ C_n &= 0, && n = 2, 3, \ldots \end{aligned} \tag{2.4.5}$$

By substituting, we get the total scalar magnetic potential outside

$$\Phi^{tot}(r,\theta,\varphi) = H_0\left[\left(\frac{a}{r}\right)^3 - 1\right] r\,cos\,\theta, \qquad r \geq a \tag{2.4.6}$$

The magnetic field intensity outside can be calculated from $\mathbf{H} = -\nabla\Phi^{tot}$ as [1]

$$H(r,\theta,\varphi) = H_r\hat{r} + H_\theta\hat{\theta} + H_\varphi\hat{\varphi} \tag{2.4.7}$$

where

$$\begin{aligned}
H_r &= -\frac{\partial}{\partial\,r}\left\{\Phi^{tot}\right\} = H_0\left[2\left(\frac{a}{r}\right)^3 + 1\right] cos\,\theta, \\
H_\theta &= -\frac{1}{r}\frac{\partial}{\partial\,\theta}\left\{\Phi^{tot}\right\} = H_0\left[\left(\frac{a}{r}\right)^3 - 1\right] sin\,\theta, \\
H_\varphi &= -\frac{1}{r\,sin\,\theta}\frac{\partial}{\partial\,\varphi}\left\{\Phi^{tot}\right\} = 0. \qquad\qquad r \geq a
\end{aligned}$$

On the other hand, the magnetic potential Ψ inside the sphere, defined by $\mathbf{B} = -\nabla\Psi$, can be expressed as

$$\Psi(r,\theta,\varphi) = \sum_{n=0}^{\infty} G_n\left(\frac{r}{a}\right)^n P_n(cos\,\theta), \qquad r < a \tag{2.4.8}$$

where G_n are constants of integration. Imposing the boundary condition that the normal component of the total magnetic flux density is continuous across the surface of the sphere, i.e.,

$$-\frac{\partial}{\partial\,r}\left\{\Psi(r,\theta,\varphi)\right\}\Big|_{r\;=\;a} = -\,\mu_0\frac{\partial}{\partial\,r}\left\{\Phi^{tot}(r,\theta,\varphi)\right\}\Big|_{r\;=\;a}, \tag{2.4.9}$$

we obtain

$$\begin{aligned} G_1 &= -3\mu_0\, H_0, \qquad n = 1 \\ G_n &= 0, \qquad\qquad\quad n = 2, 3, \ldots \end{aligned} \tag{2.4.10}$$

and, thus,

$$\Psi(r, \theta, \varphi) = -3\mu_0\, H_0 \left(\frac{r}{a}\right) cos\,\theta, \qquad r < a \tag{2.4.11}$$

Then, the magnetic flux density components inside the sphere are construct,

$$B_r = 3\mu_0\, H_0 \left(\frac{1}{a}\right) cos\,\theta, \qquad B_\theta = -3\mu_0\, H_0 \left(\frac{1}{a}\right) sin\,\theta, \qquad B_\varphi = 0. \tag{2.4.12}$$

2.5 Chapter Summary

The main background concepts related to the thesis were described in this chapter. This has included the solution of the scalar Laplace equation in spherical coordinates, the expressions of translational addition theorems for the solution of field problems in asymmetric and axisymmetric geometries, the scalar magnetic potential for ideal ferromagnetic bodies, and the solution for a single ferromagnetic sphere in the presence of an uniform external field. These concepts and expressions will be used in chapters 3 and 4.

Chapter 3

Linear Arrays of Ferromagnetic Spheres in External Magnetic Fields

This chapter presents the study of the scalar magnetic potential outside and inside of two-sphere and three-sphere systems. Here we limit the discussion to spheres having a common axis and in the presence of external uniform magnetic fields. The translational addition theorems [13] are used to obtain exact analytical expressions for ideal ferromagnetic spheres. The infinite series expressions are truncated to generate numerical results for some characteristic parameters, such as sphere radii, the relative distance between the spheres, etc. Finally, the potential values are used to obtain the magnetic field intensities outside and inside the spheres.

3.1 Two-Sphere System

A system of two ferromagnetic spheres of radii a_1 and a_2, with a distance d between their centers have been placed in a homogeneous medium of permeability μ_0. The centers of the spheres are on the common z axis and the system is placed in an uniform magnetic field $\mathbf{H_0}$ at an angle θ with respect to the common axis, as shown in Figure 3.1. Take $(r_1, \theta_1, \varphi_1)$ and $(r_2, \theta_2, \varphi_2)$ as the spherical coordinates attached to the sphere 1 and sphere 2, respectively.

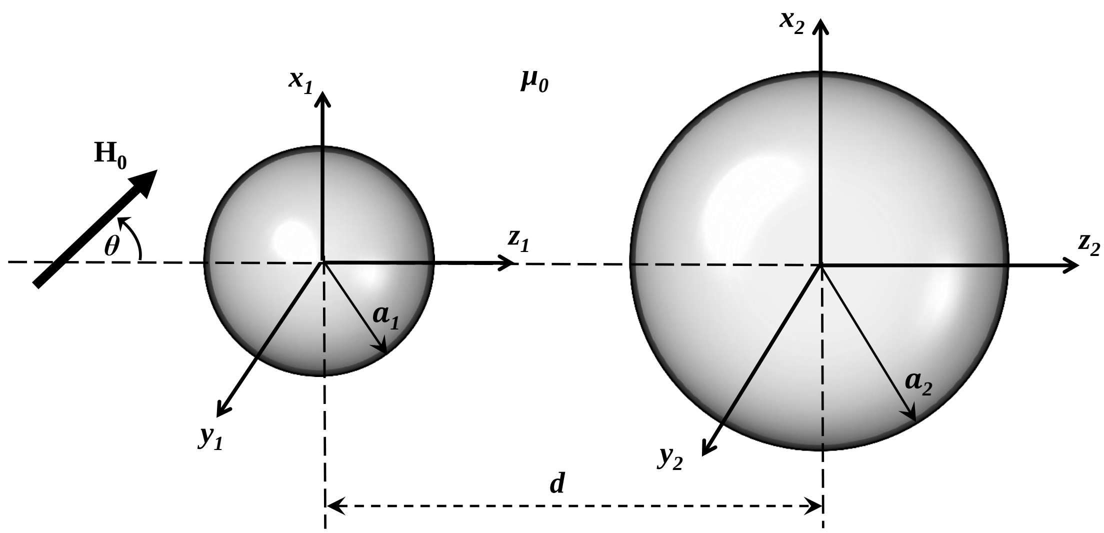

FIGURE 3.1: Two ferromagnetic spheres in an external magnetic field

3.1.1 Case I: The Magnetic Field along the Common Axis

Consider the external field to be oriented along the common axis (i.e., $\mathbf{H_0} = H_0\hat{z}$) of the spheres, as shown in Figure 3.2.

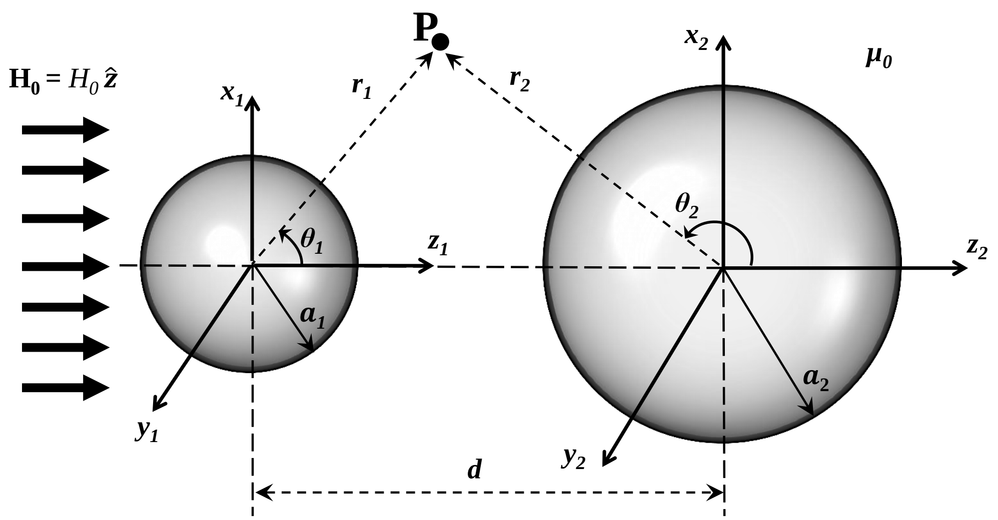

FIGURE 3.2: Two ferromagnetic spheres in the presence of an external field along z axis

The potentials outside the spheres $(r_1 \geq a_1$ ***and*** $r_2 \geq a_2)$:

The total scalar magnetic potential at an arbitrary point $\mathbf{P}$ outside the spheres can be expressed as

$$\Phi^{tot}(r_1, \theta_1, \varphi_1 \mid r_2, \theta_2, \varphi_2) = \Phi_1(r_1, \theta_1, \varphi_1) + \Phi_2(r_2, \theta_2, \varphi_2) + \Phi_{ext}(r_1, \theta_1, \varphi_1) \tag{3.1.1}$$

where Φ_1 and Φ_2 are the potentials due to sphere 1 and sphere 2, respectively, and Φ_{ext} is the potential corresponding to the external field in spherical coordinates $(r_1, \theta_1, \varphi_1)$ attached to sphere 1.

The magnetic potential is related to the magnetic field intensity by $\mathbf{H} = -\nabla\Phi$ and satisfies the Laplace equation. Since the system has an azimuthal symmetry, there is no φ dependence of the potential. Using the Equation (2.1.6) with $r = r_1, \theta = \theta_1$ for sphere 1 and $r = r_2, \theta = \theta_2$ for sphere 2, we have

$$\Phi_1(r_1, \theta_1, \varphi_1) = \sum_{n=0}^{\infty} C_n \left(\frac{a_1}{r_1}\right)^{n+1} P_n(\cos\theta_1), \qquad r_1 \geq a_1 \tag{3.1.2a}$$

$$\Phi_2(r_2, \theta_2, \varphi_2) = \sum_{m=0}^{\infty} D_m \left(\frac{a_2}{r_2}\right)^{m+1} P_m(\cos\theta_2), \qquad r_2 \geq a_2 \tag{3.1.2b}$$

where C_n and D_m are constants of integration. The potential Φ_2 is translated into the coordinate $(r_1, \theta_1, \varphi_1)$ as $\Phi_2^{(1)}(r_1, \theta_1, \varphi_1)$, and then the total scalar magnetic potential at $\mathbf{P}$ can be expressed as

$$\Phi^{tot}(r_1, \theta_1, \varphi_1) = \Phi_1(r_1, \theta_1, \varphi_1) + \Phi_2^{(1)}(r_1, \theta_1, \varphi_1) + \Phi_{ext}(r_1, \theta_1, \varphi_1) \tag{3.1.3}$$

In order to find $\Phi_2^{(1)}(r_1, \theta_1, \varphi_1)$, we use the translational addition theorem in Equation (2.2.5), with $r = r_2$, $\theta = \theta_2$, $r' = r_1$, $\theta' = \theta_1$, $\theta_0 = \pi$, and we get

$$\begin{aligned} \Phi_2^{(1)}(r_1, \theta_1, \varphi_1) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m D_m \frac{(m+n)!}{m!\, n!} \left(\frac{a_2}{d}\right)^{m+1} \left(\frac{r_1}{d}\right)^{n} P_n(\cos\theta_1), \\ d \geq r_1, \quad d \geq (a_1 + a_2) \end{aligned} \tag{3.1.4}$$

The external field is given by $\mathbf{H_0} = -\nabla \Phi_{ext}$ and, thus,

$$\Phi_{ext}(r_1, \theta_1, \varphi_1) = -H_0 z_1 + K_1$$

$$\Phi_{ext}(r_2, \theta_2, \varphi_2) = -H_0 z_2 + K_2$$

where the constants K_1 and K_2 are used to fix the reference potential. We choose $\Phi_{ext} = 0$ at $z_1 = 0$ and, then, $K_1 = 0$ and $K_2 = -H_0 d$ which yields

$$\Phi_{ext}(r_1, \theta_1, \varphi_1) = -H_0 r_1 P_1(\cos\theta_1) \tag{3.1.6a}$$

$$\Phi_{ext}(r_2, \theta_2, \varphi_2) = -H_0 r_2 P_1(\cos\theta_2) - H_0 d \tag{3.1.6b}$$

where $P_1(cos\,\theta_1)$ and $P_1(cos\,\theta_2)$ are Legendre polynomials of degree 1 and arguments $cos\,\theta_1$ and $cos\,\theta_2$, respectively. Equation (3.1.3) becomes

$$\begin{aligned}\Phi^{tot}(r_1,\theta_1,\varphi_1) = &\sum_{n=0}^{\infty} C_n\left(\frac{a_1}{r_1}\right)^{n+1} P_n(cos\,\theta_1) - H_0\,r_1 P_1(cos\,\theta_1)\\ &+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m D_m\frac{(m+n)!}{m!\,n!}\left(\frac{a_2}{d}\right)^{m+1}\left(\frac{r_1}{d}\right)^{n} P_n(cos\,\theta_1), \qquad (3.1.7)\\ &\qquad d\geq r_1\geq a_1,\quad d\geq(a_1+a_2)\end{aligned}$$

We impose the condition that the total magnetic flux through the surface of each sphere is equal to zero, i.e.,

$$\oint_S B_n\,ds_r = 0$$

where $B_n = -\mu_0\dfrac{\partial}{\partial r}\left\{\Phi^{tot}\right\}\Big|_S$ is the normal component of the magnetic flux density on the surface S of the sphere with $ds_r = r^2 sin\,\theta\,d\theta\,d\varphi$ and μ_0 is the permeability of the medium outside. Then, this condition for sphere 1 gives

$$\begin{aligned}0 = &-2\pi\mu_0 a_1^2\int_0^{\pi}\left\{\sum_{n=0}^{\infty} -(n+1)C_n a_1^{-1} P_n(cos\,\theta_1) - H_0 P_1(cos\,\theta_1)\right.\\ &\left.+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m D_m\frac{(n+m)!}{n!m!}\left(\frac{a_2}{d}\right)^{m+1}\left(\frac{n a_1^{n-1}}{d^n}\right)P_n(cos\,\theta_1)\right\} sin\,\theta_1\,d\theta_1\end{aligned} \qquad (3.1.8)$$

Applying the orthogonality of Legendre polynomials (see B.6) we obtain

$$4\pi\mu_0 a_1 C_0 = 0 \qquad (3.1.9)$$

and, thus,

$$C_0 = 0 \qquad (3.1.10)$$

Similarly, for the sphere 2, we have

$$D_0 = 0 \tag{3.1.11}$$

Thus, Equation (3.1.7) becomes

$$\begin{aligned}\Phi^{tot}(r_1,\theta_1,\varphi_1) = &\sum_{n=1}^{\infty} C_n \left(\frac{a_1}{r_1}\right)^{n+1} P_n(cos\,\theta_1) - H_0\, r_1 P_1(cos\,\theta_1)\\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m D_m \frac{(m+n)!}{m!\,n!}\left(\frac{a_2}{d}\right)^{m+1}\left(\frac{r_1}{d}\right)^n P_n(cos\,\theta_1),\\ &\qquad d \geq r_1 \geq a_1, \quad d \geq (a_1 + a_2)\end{aligned} \tag{3.1.12}$$

The potential at the surface of each ferromagnetic sphere is constant and we have

$$\Phi^{tot}(r_1,\theta_1,\varphi_1)\Big|_{r_1\,=\,a_1} = V_1$$

where V_1 is the magnetic potential at the surface of the sphere 1. From 3.1.12,

$$\begin{aligned}\sum_{n=1}^{\infty} C_n P_n(cos\,\theta_1) = V_1 - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m D_m \frac{(m+n)!}{m!\,n!}\left(\frac{a_2}{d}\right)^{m+1}\left(\frac{a_1}{d}\right)^n P_n(cos\,\theta_1)\\ +H_0\, a_1\, P_1(cos\,\theta_1)\end{aligned} \tag{3.1.13}$$

To apply the orthogonality of Legendre polynomials (see B.6), we multiply this equation by $P_l(cos\,\theta_1)\, sin\,\theta_1$ and integrate with respect to $cos\,\theta_1$ to obtain

$$\begin{aligned}C_n = &\frac{(2n+1)}{2}\int_{-1}^{1}\Big\{V_1 + a_1 H_0 P_1(cos\,\theta_1)\Big\} P_n(cos\,\theta_1)\, d(cos\,\theta_1)\\ &- \sum_{m=1}^{\infty}(-1)^m D_m \frac{(m+n)!}{m!\,n!}\left(\frac{a_2}{d}\right)^{m+1}\left(\frac{a_1}{d}\right)^n, \quad n = 1,2,3,\ldots\end{aligned} \tag{3.1.14}$$

This gives

$$C_1 + \sum_{m=1}^{\infty} (-1)^m D_m (m+1) \left(\frac{a_2}{d}\right)^{m+1} \left(\frac{a_1}{d}\right) = a_1 H_0, \qquad n = 1 \tag{3.1.15a}$$

$$C_n + \sum_{m=1}^{\infty} (-1)^m D_m \frac{(m+n)!}{m!\, n!} \left(\frac{a_2}{d}\right)^{m+1} \left(\frac{a_1}{d}\right)^{n} = 0, \qquad n = 2, 3 \ldots \tag{3.1.15b}$$

Similar steps can be followed for the sphere 2. To translate the potential $\Phi_1(r_1, \theta_1, \varphi_1)$ into $\Phi_1^{(2)}(r_2, \theta_2, \varphi_2)$, we use Equation (2.2.5) with $r = r_1$, $\theta = \theta_1$, $r' = r_2$, $\theta' = \theta_2$ and $\theta_0 = 0$. Then, the total potential at $\mathbf{P}$ can be written as

$$\begin{aligned} \Phi^{tot}(r_2, \theta_2, \varphi_2) = & \sum_{m=1}^{\infty} D_m \left(\frac{a_2}{r_2}\right)^{m+1} P_m(cos\, \theta_2) - H_0 \left[\, r_2 P_1(cos\, \theta_2) + d\, \right] \\ & + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^m C_n \frac{(n+m)!}{n!\, m!} \left(\frac{a_1}{d}\right)^{n+1} \left(\frac{r_2}{d}\right)^{m} P_m(cos\, \theta_2), \\ & \qquad d \geq r_2 \geq a_2, \quad d \geq (a_1 + a_2) \end{aligned} \tag{3.1.16}$$

Imposing the boundary condition at the surface of sphere 2 and using the orthogonality of Legendre polynomials yields

$$D_1 + \sum_{n=1}^{\infty} (-1)\, C_n (n+1) \left(\frac{a_1}{d}\right)^{n+1} \left(\frac{a_2}{d}\right) = a_2 H_0, \qquad m = 1 \tag{3.1.17a}$$

$$D_m + \sum_{n=1}^{\infty} (-1)^m C_n \frac{(n+m)!}{n!\, m!} \left(\frac{a_1}{d}\right)^{n+1} \left(\frac{a_2}{d}\right)^{m} = 0, \qquad m = 2, 3 \ldots \tag{3.1.17b}$$

The Equations (3.1.15 and 3.1.17) form an infinite system of linear algebraic equations satisfied by the constants of integration C_n and D_m. To obtain numerical solutions, this infinite set of equations is truncated to a finite number of terms N as follows.

Denoting

$$S_{n,m} \equiv (-1)^m \frac{(m+n)!}{m!\,n!} \left(\frac{a_2}{d}\right)^{m+1} \left(\frac{a_1}{d}\right)^{n},$$
$$T_{m,n} \equiv (-1)^m \frac{(n+m)!}{n!\,m!} \left(\frac{a_1}{d}\right)^{n+1} \left(\frac{a_2}{d}\right)^{m}, \qquad n,m = 1\ldots N$$

the finite system of equations can be written in a matrix form as

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & S_{1,1} & S_{1,2} & \cdots & S_{1,N} \\ 0 & 1 & \cdots & 0 & S_{2,1} & S_{2,2} & \cdots & S_{2,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & S_{N,1} & S_{N,2} & \cdots & S_{N,N} \\ T_{1,1} & T_{1,2} & \cdots & T_{1,N} & 1 & 0 & \cdots & 0 \\ T_{2,1} & T_{2,2} & \cdots & T_{2,N} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{N,1} & T_{N,2} & \cdots & T_{N,N} & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \\ D_1 \\ D_2 \\ \vdots \\ D_N \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \\ \vdots \\ 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} H_0 \qquad (3.1.18)$$

The solution of this system gives $2N$ constants of integration, namely, $C_1, C_2, \ldots, C_N$, $D_1, D_2, \ldots, D_N$.

The total scalar magnetic potential outside the spheres can be calculated from (see Equation (3.1.1))

$$\Phi^{tot}(r_1,\theta_1,\varphi_1 \mid r_2,\theta_2,\varphi_2) = \sum_{n=1}^{N} C_n \left(\frac{a_1}{r_1}\right)^{n+1} P_n(cos\,\theta_1) + \sum_{m=1}^{N} D_m \left(\frac{a_2}{r_2}\right)^{m+1} P_m(cos\,\theta_2)$$
$$- H_0\, r_1 P_1(cos\,\theta_1), \qquad r_1 \geq a_1, \quad r_2 \geq a_2 \qquad (3.1.19)$$

The magnetic field intensity outside the spheres can be calculated with $\mathbf{H} = -\nabla\Phi^{tot}$ as

$$\mathbf{H}(r,\theta,\varphi) = H_r\hat{r} + H_\theta\hat{\theta} + H_\varphi\hat{\varphi} \tag{3.1.20}$$

where

$$H_r = -\frac{\partial}{\partial r}\Big\{\Phi^{tot}\Big\}, \quad H_\theta = -\frac{1}{r}\frac{\partial}{\partial\theta}\Big\{\Phi^{tot}\Big\}, \quad H_\varphi = -\frac{1}{r\,sin\,\theta}\frac{\partial}{\partial\varphi}\Big\{\Phi^{tot}\Big\}.$$

By using the Equation (3.1.19) and the geometrical relations in Appendix (A.2), the components of the magnetic field intensity can be expressed in (r_1,θ_1,φ_1) as

$$\begin{aligned}
H_{r_1} &= \frac{1}{r_1}\sum_{n=1}^{\infty}(n+1)\,C_n\left(\frac{a_1}{r_1}\right)^{n+1}P_n(cos\,\theta_1) + H_0P_1(cos\,\theta_1)\\
&\qquad + \frac{1}{r_2^3}\sum_{m=1}^{\infty}D_m\left(\frac{a_2}{r_2}\right)^{m+1}\Big\{r_2(m+1)(r_1-d\,cos\,\theta_1)P_m(cos\,\theta_2)\\
&\qquad\qquad - r_1d\,sin^2\theta_1P_m'(cos\,\theta_2)\Big\},\\
H_{\theta_1} &= sin\,\theta_1\Bigg\{\frac{1}{r_1}\sum_{n=1}^{\infty}C_n\left(\frac{a_1}{r_1}\right)^{n+1}P_n'(cos\,\theta_1) - H_0P_1'(cos\,\theta_1)\\
&\qquad + \frac{1}{r_2^3}\sum_{m=1}^{\infty}D_m\left(\frac{a_2}{r_2}\right)^{m+1}\Big\{r_2\,d\,(m+1)P_m(cos\,\theta_2)\\
&\qquad\qquad + r_1(r_1-d\,cos\,\theta_1)P_m'(cos\,\theta_2)\Big\}\Bigg\},\\
H_{\varphi_1} &= 0, \qquad\qquad r_1\geq a_1, \quad r_2\geq a_2, \quad d\geq(a_1+a_2)
\end{aligned} \tag{3.1.21}$$

Similarly, it can be expressed in spherical coordinates $(r_2, \theta_2, \varphi_2)$ as

$$\begin{aligned}
H_{r_2} &= \frac{1}{r_1^3}\sum_{n=1}^{\infty} C_n \left(\frac{a_1}{r_1}\right)^{n+1} \Big\{ r_1(n+1)(r_2 + d\,cos\,\theta_2)P_n(cos\,\theta_1) + r_2 d\, sin^2\theta_2 P_n'(cos\,\theta_1)\Big\} \\
&\qquad + \frac{1}{r_2}\sum_{m=1}^{\infty}(m+1)\, D_m \left(\frac{a_2}{r_2}\right)^{m+1} P_m(cos\,\theta_2) + H_0 P_1(cos\,\theta_2), \\
H_{\theta_2} &= sin\,\theta_2 \Bigg\{ \frac{1}{r_1^3}\sum_{n=1}^{\infty} C_n \left(\frac{a_1}{r_1}\right)^{n+1} \Big\{ - r_1 d\,(n+1)P_n(cos\,\theta_1) + r_2(r_2 + d\,cos\,\theta_2)P_n'(cos\,\theta_1)\Big\} \\
&\qquad + \frac{1}{r_2}\sum_{m=1}^{\infty} D_m \left(\frac{a_2}{r_2}\right)^{m+1} P_m'(cos\,\theta_2) - H_0 P_1'(cos\,\theta_2)\Bigg\}, \\
H_{\varphi_2} &= 0, \qquad\qquad r_1 \geq a_1, \quad r_2 \geq a_2, \quad d \geq (a_1 + a_2)
\end{aligned} \tag{3.1.22}$$

where P_n' and P_m' are the first derivatives of the Legendre polynomials (see (B.8)) with respect to the arguments $cos\,\theta_1$ and $cos\,\theta_2$, respectively.

The potentials inside the spheres $(r_1 < a_1$ ***and*** $r_2 < a_2)$

The potential Ψ inside the spheres relates to the magnetic flux density $\mathbf{B}$ by $\mathbf{B} = -\nabla\Psi$ and satisfies the Laplace equation. By using the Equation (2.1.7), it can be expressed as

$$\Psi^{(1)}(r_1, \theta_1, \varphi_1) = \sum_{n=0}^{\infty} G_n^{(1)} \left(\frac{r_1}{a_1}\right)^n P_n(cos\,\theta_1), \qquad r_1 < a_1 \tag{3.1.23a}$$

$$\Psi^{(2)}(r_2, \theta_2, \varphi_2) = \sum_{m=0}^{\infty} G_m^{(2)} \left(\frac{r_2}{a_2}\right)^m P_m(cos\,\theta_2), \qquad r_2 < a_2 \tag{3.1.23b}$$

where $G_n^{(1)}$ and $G_m^{(2)}$ are constants of integration.

The boundary condition is that the normal component of the total magnetic flux density is continuous across the surface of the sphere, i.e.,

$$-\frac{\partial}{\partial r_i}\left\{\Psi^{(i)}(r_i,\theta_i,\varphi_i)\right\}\bigg|_{r_i\,=\,a_i} = -\,\mu_0\frac{\partial}{\partial r_i}\left\{\Phi^{tot}(r_i,\theta_i,\varphi_i)\right\}\bigg|_{r_i\,=\,a_i}, \quad i=1,2 \tag{3.1.24}$$

where μ_0 is the permeability of the medium outside the spheres.

For sphere 1, from Equations (3.1.23) and (3.1.12) we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} G_n^{(1)}\, n\, P_n(cos\,\theta_1) = \mu_0 \Bigg\{ &\sum_{n=1}^{\infty} -(n+1)C_n\, P_n(cos\,\theta_1) - a_1 H_0 P_1(cos\,\theta_1) \\ &+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m D_m \frac{(m+n)!}{m!\,n!}\left(\frac{a_2}{d}\right)^{m+1} n\left(\frac{a_1}{d}\right)^n P_n(cos\,\theta_1)\Bigg\}, \quad n\neq 0\end{aligned} \tag{3.1.25}$$

Multiplying by $P_l(cos\,\theta_1)$ and integrating with respect to $cos\,\theta_1$, we have

$$\begin{aligned}\int_{-1}^{1}\sum_{n=1}^{\infty} G_n^{(1)}\, n\, P_n(cos\,\theta_1)P_l(cos\,\theta_1)\, d(cos\,\theta_1) = \mu_0 \int_{-1}^{1}\Bigg\{&\sum_{n=1}^{\infty} -(n+1)C_n\, P_n(cos\,\theta_1) \\ &+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m D_m \frac{(m+n)!}{m!\,n!}\left(\frac{a_2}{d}\right)^{m+1} n\left(\frac{a_1}{d}\right)^n P_n(cos\,\theta_1) \\ &- a_1 H_0 P_1(cos\,\theta_1)\Bigg\} P_l(cos\,\theta_1)\, d(cos\,\theta_1)\end{aligned} \tag{3.1.26}$$

Applying the orthogonality of Legendre polynomials (see B.6) yields

$$\begin{aligned} n\, G_n^{(1)} = \mu_0 \Bigg\{ &-(n+1)C_n + \sum_{m=1}^{\infty}(-1)^m D_m \frac{(m+n)!}{m!\,n!}\left(\frac{a_2}{d}\right)^{m+1} n\left(\frac{a_1}{d}\right)^n \\ &- a_1 H_0 \frac{(2n+1)}{2}\int_{-1}^{1} P_1(cos\,\theta_1)P_n(cos\,\theta_1)\, d(cos\,\theta_1)\Bigg\}, \qquad n=1,2,3\ldots\end{aligned} \tag{3.1.27}$$

Taking into account (3.1.15) gives

$$G_1^{(1)} = 3\mu_0 \left\{ \sum_{m=1}^{\infty} (-1)^m D_m (1+m) \left(\frac{a_2}{d}\right)^{m+1} \left(\frac{a_1}{d}\right) - a_1 H_0 \right\}, \qquad n = 1 \tag{3.1.28a}$$

$$G_n^{(1)} = \frac{(2n+1)}{n} \mu_0 \sum_{m=1}^{\infty} (-1)^m D_m \frac{(m+n)!}{m!\,n!} \left(\frac{a_2}{d}\right)^{m+1} \left(\frac{a_1}{d}\right)^n, \quad n = 2, 3 \ldots \tag{3.1.28b}$$

Similarly, for sphere 2, from Equations (3.1.23), (3.1.16) and (3.1.24) we obtain

$$\begin{aligned} \sum_{m=0}^{\infty} G_m^{(2)}\, m\, P_m(cos\,\theta_2) = \mu_0 \Bigg\{ & \sum_{m=1}^{\infty} -(m+1) D_m\, P_m(cos\,\theta_2) - a_2 H_0 P_1(cos\,\theta_2) \\ & + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^m C_n \frac{(n+m)!}{n!\,m!} \left(\frac{a_1}{d}\right)^{n+1} m \left(\frac{a_2}{d}\right)^m P_m(cos\,\theta_2) \Bigg\}, \quad m \neq 0 \end{aligned} \tag{3.1.29}$$

and, finally, taking into account (3.1.17),

$$G_1^{(2)} = 3\mu_0 \left\{ \sum_{n=1}^{\infty} (-1)\, C_n (1+n) \left(\frac{a_1}{d}\right)^{n+1} \left(\frac{a_2}{d}\right) - a_2 H_0 \right\}, \qquad m = 1 \tag{3.1.30a}$$

$$G_m^{(2)} = \frac{(2m+1)}{m} \mu_0 \sum_{n=1}^{\infty} (-1)^m C_n \frac{(n+m)!}{n!\,m!} \left(\frac{a_1}{d}\right)^{n+1} \left(\frac{a_2}{d}\right)^m, \quad m = 2, 3 \ldots \tag{3.1.30b}$$

Once $G_n^{(1)}$ and $G_m^{(2)}$ are determined the Equations in (3.1.23), give the potential inside each sphere.

The magnetic flux density $\mathbf{B}$ inside the sphere can be calculated from $\mathbf{B} = -\nabla\Psi$ as

$$\mathbf{B}(r, \theta, \varphi) = B_r \hat{r} + B_\theta \hat{\theta} + B_\varphi \hat{\varphi} \tag{3.1.31}$$

Thus, for sphere 1, use the Equation (3.1.23a) to obtain

$$
\begin{aligned}
B_{r_1} &= -\frac{1}{r_1}\sum_{n=1}^{\infty} n\, G_n^{(1)} \left(\frac{r_1}{a_1}\right)^n P_n(\cos\theta_1),\\
B_{\theta_1} &= \frac{\sin\theta_1}{r_1}\sum_{n=1}^{\infty} G_n^{(1)} \left(\frac{r_1}{a_1}\right)^n P_n'(\cos\theta_1),\\
B_{\varphi_1} &= 0, \qquad\qquad r_1 < a_1
\end{aligned}
\tag{3.1.32}
$$

Similarly, for sphere 2,

$$
\begin{aligned}
B_{r_2} &= -\frac{1}{r_2}\sum_{m=1}^{\infty} m\, G_m^{(2)} \left(\frac{r_2}{a_2}\right)^m P_m(\cos\theta_2),\\
B_{\theta_2} &= \frac{\sin\theta_2}{r_2}\sum_{m=1}^{\infty} G_m^{(2)} \left(\frac{r_2}{a_2}\right)^m P_m'(\cos\theta_2),\\
B_{\varphi_2} &= 0, \qquad\qquad r_2 < a_2
\end{aligned}
\tag{3.1.33}
$$

3.1.2 Case II: The Magnetic Field perpendicular on the Common Axis

Consider a two sphere system with an external field oriented along the x axis (i.e. $\mathbf{H_0} = H_0\hat{x}$), as shown in Figure 3.3.

The potentials outside the spheres, $(r_1 \geq a_1$ ***and*** $r_2 \geq a_2)$

As in the previous case, the total scalar magnetic potential at **P** due to the two spheres and to the external field can be expressed in spherical coordinates as

$$
\Phi^{tot}(r_1,\theta_1,\varphi_1 \mid r_2,\theta_2,\varphi_2) = \Phi_1(r_1,\theta_1,\varphi_1) + \Phi_2(r_2,\theta_2,\varphi_2) + \Phi_{ext(x)}(r_1,\theta_1,\varphi_1) \tag{3.1.34}
$$

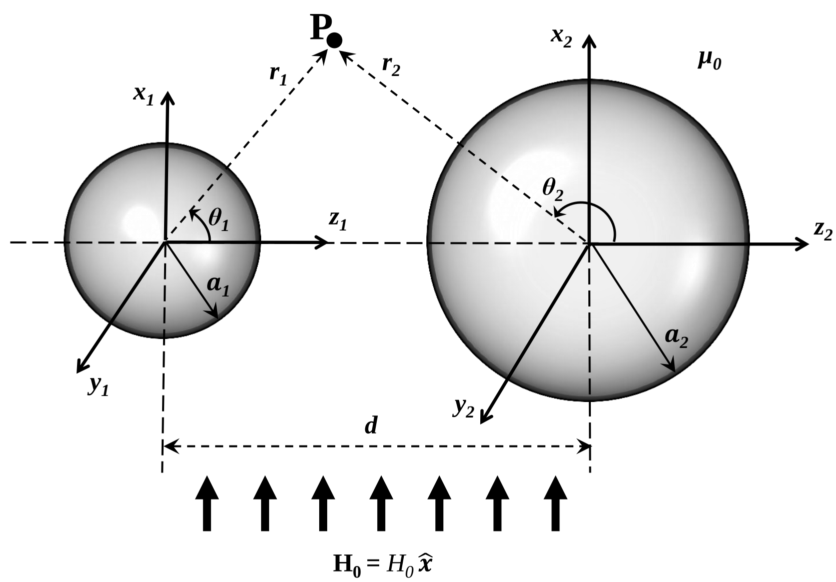

FIGURE 3.3: Two ferromagnetic spheres in the presence of an external field along x axis

where Φ_1 and Φ_2 are the potentials due to sphere 1 and sphere 2, respectively, and $\Phi_{ext(x)}$ is the potential corresponding to the external field in $(r_1, \theta_1, \varphi_1)$. Then, as in previous case, this expression is written in spherical coordinates $(r_1, \theta_1, \varphi_1)$,

$$\Phi^{tot}(r_1, \theta_1, \varphi_1) = \Phi_1(r_1, \theta_1, \varphi_1) + \Phi_2^{(1)}(r_1, \theta_1, \varphi_1) + \Phi_{ext(x)}(r_1, \theta_1, \varphi_1) \tag{3.1.35}$$

where $\Phi_2^{(1)}$ is the potential due to the sphere 2 in spherical coordinates $(r_1, \theta_1, \varphi_1)$.

Since the external field is along the x axis, the potential $\Phi_1(r_1, \theta_1, \varphi_1)$ can be written by using the real part of the Equation (2.1.3), with $r = r_1$, $\theta = \theta_1$ and $\varphi = \varphi_1$, in the form

$$\Phi_1(r_1, \theta_1, \varphi_1) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_{nm} \left(\frac{a_1}{r_1}\right)^{n+1} P_n^m(cos\,\theta_1)\, cos\,(m\varphi_1), \quad r_1 \geq a_1 \tag{3.1.36}$$

Similarly, the potential due to the sphere 2 is expressed as

$$\Phi_2(r_2,\theta_2,\varphi_2) = \sum_{q=0}^{\infty}\sum_{p=-q}^{q} D_{qp}\left(\frac{a_2}{r_2}\right)^{q+1} P_q^p(cos\,\theta_2)\,cos\,(p\varphi_2), \qquad r_2 \geq a_2 \tag{3.1.37}$$

To translate $\Phi_2(r_2,\theta_2,\varphi_2)$ into coordinates (r_1,θ_1,φ_1), we use the translational addition theorem by taking the real part of the Equation (2.2.1) with $r = r_2, \theta = \theta_2, \varphi = \varphi_2, r' = r_1, \theta' = \theta_1, \varphi' = \varphi_1$ and $\theta_0 = \pi$, i.e.,

$$\begin{aligned} {r_2}^{-(q+1)} P_q^p(cos\,\theta_2) cos(p\varphi_2) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} (-1)^{\nu+\mu}\frac{(q-p+\nu+\mu)!}{(q-p)!(\nu+\mu)!}\frac{{r_1}^{\nu}}{d^{q+\nu+1}} P_{q+\nu}^{p-\mu}(cos\,\pi) \\ \times cos\big((p-\mu)\varphi_0 + \mu\varphi_1\big) P_\nu^\mu(cos\,\theta_1), \quad r_1 \leq d \end{aligned}$$

Since (see (B.5))

$$P_{q+\nu}^{p-\mu}(\pm 1) = 0 \quad \text{for all} \quad (p-\mu) \neq 0$$

we remain with $\mu = p$, and with $P_{q+\nu}(-1) = (-1)^{q+\nu}$, the Equation (3.1.37) can be translated to

$$\begin{aligned} \Phi_2^{(1)}(r_1,\theta_1,\varphi_1) = \sum_{q=0}^{\infty}\sum_{p=-q}^{q}\sum_{\nu=0}^{\infty} (-1)^{q+p} D_{qp}\frac{(q+\nu)!}{(q-p)!(\nu+p)!}\left(\frac{a_2}{d}\right)^{q+1}\left(\frac{r_1}{d}\right)^{\nu} \\ \times P_\nu^p(cos\,\theta_1)\,cos(p\varphi_1), \qquad d \geq r_1, \quad d \geq (a_1+a_2) \end{aligned} \tag{3.1.38}$$

The potential due to the external field can be expressed as

$$\Phi_{ext(x)}(r_1,\theta_1,\varphi_1) = -H_0\,x_1 + K_1$$

$$\Phi_{ext(x)}(r_2,\theta_2,\varphi_2) = -H_0\,x_2 + K_2$$

where K_1 and K_2 are the constants of reference. At $x_1 = 0$ plane, we choose $\Phi_{ext(x)} = 0$ as reference. Then, $K_1 = 0$ and $K_2 = 0$, and with $x_1 = r_1 sin\,\theta_1\,cos\,\varphi_1$ and

$x_2 = r_2 sin\,\theta_2\, cos\,\varphi_2$, we have

$$\begin{aligned}\Phi_{ext(x)}(r_1, \theta_1, \varphi_1) &= -H_0 r_1 P_1^1(cos\,\theta_1)\, cos\,\varphi_1 \\ \Phi_{ext(x)}(r_2, \theta_2, \varphi_2) &= -H_0 r_2 P_1^1(cos\,\theta_2)\, cos\,\varphi_2\end{aligned} \tag{3.1.39}$$

The total scalar magnetic potential in (3.1.35) is finally obtained as

$$\begin{aligned}\Phi^{tot}(r_1, \theta_1, \varphi_1) = &\sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_{nm}\left(\frac{a_1}{r_1}\right)^{n+1} P_n^m(cos\,\theta_1)\, cos(m\varphi_1) - H_0\, r_1 P_1^1(cos\,\theta_1)\, cos\,\varphi_1 \\ &+ \sum_{q=0}^{\infty}\sum_{p=-q}^{q}\sum_{\nu=0}^{\infty}(-1)^{q+p} D_{qp}\frac{(q+\nu)!}{(q-p)!\,(\nu+p)!}\left(\frac{a_2}{d}\right)^{q+1}\left(\frac{r_1}{d}\right)^{\nu} P_\nu^p(cos\,\theta_1)\, cos(p\varphi_1), \\ &d \geq r_1 \geq a_1, \quad d \geq (a_1 + a_2)\end{aligned} \tag{3.1.40}$$

Imposing the condition that the total magnetic flux through the surface of each sphere is equal to zero gives

$$C_{00} = 0 \quad \text{and} \quad D_{00} = 0 \tag{3.1.41}$$

Then, the Equation (3.1.40) becomes as

$$\begin{aligned}\Phi^{tot}(r_1, \theta_1, \varphi_1) = &\sum_{n=1}^{\infty}\sum_{m=-n}^{n} C_{nm}\left(\frac{a_1}{r_1}\right)^{n+1} P_n^m(cos\,\theta_1)\, cos(m\varphi_1) - H_0\, r_1 P_1^1(cos\,\theta_1)\, cos\,\varphi_1 \\ &+ \sum_{q=1}^{\infty}\sum_{p=-q}^{q}\sum_{\nu=0}^{\infty}(-1)^{q+p} D_{qp}\frac{(q+\nu)!}{(q-p)!\,(\nu+p)!}\left(\frac{a_2}{d}\right)^{q+1}\left(\frac{r_1}{d}\right)^{\nu} P_\nu^p(cos\,\theta_1)\, cos(p\varphi_1), \\ &d \geq r_1 \geq a_1, \quad d \geq (a_1 + a_2)\end{aligned} \tag{3.1.42}$$

With

$$\Phi^{tot}(r_1, \theta_1, \varphi_1)\Big|_{r_1 = a_1} = V_1$$

where V_1 is the scalar magnetic potential of the sphere 1, Equation (3.1.42) can be written with form

$$\sum_{n=1}^{\infty}\sum_{m=-n}^{n} C_{nm}\left(\frac{a_1}{r_1}\right)^{n+1} P_n^m(cos\,\theta_1)\,cos(m\varphi_1) = V_1 + H_0\,a_1 P_1^1(cos\,\theta_1)\,cos\,\varphi_1$$
$$-\sum_{q=1}^{\infty}\sum_{p=-q}^{q}\sum_{\nu=0}^{\infty}(-1)^{q+p} D_{qp}\frac{(q+\nu)!}{(q-p)!\,(\nu+p)!}\left(\frac{a_2}{d}\right)^{q+1}\left(\frac{r_1}{d}\right)^{\nu} P_\nu^p(cos\,\theta_1)\,cos(p\varphi_1)$$

To apply the orthogonality of spherical harmonics (see C.6), we multiply this equation by $P_n^m(cos\,\theta_1)\,sin\,\theta_1\,cos\,(m\varphi_1)$ and integrate with respect to θ_1 and φ_1 to obtain

$$\sum_{n=1}^{\infty}\sum_{m=-n}^{n} C_{nm} = -\sum_{q=1}^{\infty}\sum_{m=-q}^{q}\sum_{n=1}^{\infty}(-1)^{q+m} D_{qm}\frac{(q+n)!}{(q-m)!\,(n+m)!}\left(\frac{a_2}{d}\right)^{q+1}\left(\frac{a_1}{d}\right)^{n}$$
$$+\,\varepsilon_n\frac{(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}\int_{\theta_1=0}^{\pi}\int_{\varphi_1=0}^{2\pi}\left\{V_1 + H_0\,a_1 P_1^1(cos\,\theta_1)(cos\,\varphi_1)\right\}$$
$$\times\,P_n^m(cos\,\theta_1)\,sin\,\theta_1\,cos(m\varphi_1)d\theta_1 d\varphi_1 \tag{3.1.43}$$

$$\text{where}\qquad \varepsilon_n = \begin{cases} 1, & \text{if } n=0 \\ 2, & \text{if } n=1,2,3,\ldots \end{cases}$$

This gives for $m = 0$

$$C_{n0}+\sum_{q=1}^{\infty}(-1)^q D_{q0}\frac{(q+n)!}{q!\,n!}\left(\frac{a_2}{d}\right)^{q+1}\left(\frac{a_1}{d}\right)^{n} = 0, \qquad n = 1,2,3,\ldots \tag{3.1.44a}$$

for $m = 1$

$$C_{11} + \sum_{q=1}^{\infty} (-1)^{q+1} D_{q1} \, \frac{q(q+1)}{2} \left(\frac{a_2}{d}\right)^{q+1} \left(\frac{a_1}{d}\right) = a_1 H_0, \qquad n = 1 \tag{3.1.44b}$$

$$C_{n1} + \sum_{q=1}^{\infty} (-1)^{q+1} D_{q1} \frac{(q+n)!}{(q-1)!\,(n+1)!} \left(\frac{a_2}{d}\right)^{q+1} \left(\frac{a_1}{d}\right)^{n} = 0, \qquad n = 2, 3, \ldots \tag{3.1.44c}$$

for $m = -1$

$$C_{1,-1} + \sum_{q=1}^{\infty} (-1)^{q-1} D_{q,-1} \left(\frac{a_2}{d}\right)^{q+1} \left(\frac{a_1}{d}\right) = -2\, a_1 H_0, \qquad n = 1 \tag{3.1.44d}$$

$$C_{n,-1} + \sum_{q=1}^{\infty} (-1)^{q-1} D_{q,-1} \frac{(q+n)!}{(q+1)!\,(n-1)!} \left(\frac{a_2}{d}\right)^{q+1} \left(\frac{a_1}{d}\right)^{n} = 0, \qquad n = 2, 3, \ldots \tag{3.1.44e}$$

and $C_{nm} = 0$ for all $m \neq 0, +1, -1$.

Similar steps can be applied for the sphere 2, which gives

$$\Phi^{tot}(r_2, \theta_2, \varphi_2) = \Phi_2(r_2, \theta_2, \varphi_2) + \Phi_1^{(2)}(r_2, \theta_2, \varphi_2) + \Phi_{ext(x)}(r_2, \theta_2, \varphi_2) \tag{3.1.45}$$

with

$$\Phi_1^{(2)}(r_2, \theta_2, \varphi_2) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \sum_{\eta=0}^{\infty} (-1)^{\eta+m} C_{nm} \frac{(n+\eta)!}{(n-m)!(\eta+m)!} \left(\frac{a_1}{d}\right)^{n+1} \left(\frac{r_2}{d}\right)^{\eta}$$
$$\times P_{\eta}^{m}(cos\,\theta_2)\, cos(m\varphi_2), \qquad d \geq r_2, \quad d \geq (a_1 + a_2)$$

Thus,

$$\begin{aligned} \Phi^{tot}(r_2, \theta_2, \varphi_2) = {} & \sum_{q=1}^{\infty} \sum_{p=-q}^{q} D_{qp} \left(\frac{a_2}{r_2}\right)^{q+1} P_q^p(cos\,\theta_2)\, cos(p\varphi_2) - H_0\, r_2 P_1^1(cos\,\theta_2)\, cos\,\varphi_2 \\ & + \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \sum_{\eta=0}^{\infty} (-1)^{\eta+m}\, C_{nm} \frac{(n+\eta)!}{(n-m)!(\eta+m)!} \left(\frac{a_1}{d}\right)^{n+1} \left(\frac{r_2}{d}\right)^{\eta} P_{\eta}^{m}(cos\,\theta_2)\, cos(m\varphi_2), \\ & \qquad d \geq r_2 \geq a_2, \quad d \geq (a_1 + a_2) \end{aligned} \tag{3.1.46}$$

Imposing the condition $\Phi^{tot}(a_2, \theta_2, \varphi_2) = V_2$ and the orthogonality properties of the spherical harmonics yields, finally,

for $p = 0$

$$D_{q0} + \sum_{n=1}^{\infty} (-1)^q C_{n0} \frac{(n+q)!}{n!\, q!} \left(\frac{a_1}{d}\right)^{n+1} \left(\frac{a_2}{d}\right)^q = 0, \qquad q = 1, 2, 3, \ldots \tag{3.1.47a}$$

for $p = 1$

$$D_{11} + \sum_{n=1}^{\infty} C_{n1} \frac{n(n+1)}{2} \left(\frac{a_1}{d}\right)^{n+1} \left(\frac{a_2}{d}\right) = a_2 H_0, \qquad q = 1 \tag{3.1.47b}$$

$$D_{q1} + \sum_{n=1}^{\infty} (-1)^{q+1} C_{n1} \frac{(n+q)!}{(n-1)!\,(q+1)!} \left(\frac{a_1}{d}\right)^{n+1} \left(\frac{a_2}{d}\right)^q = 0, \qquad q = 2, 3, \ldots \tag{3.1.47c}$$

for $p = -1$

$$D_{1,-1} + \sum_{n=1}^{\infty} C_{n,-1} \left(\frac{a_1}{d}\right)^{n+1} \left(\frac{a_2}{d}\right) = -2\, a_2 H_0, \qquad q = 1 \tag{3.1.47d}$$

$$D_{q,-1} + \sum_{n=1}^{\infty} (-1)^{q-1} C_{n,-1} \frac{(n+q)!}{(n+1)!\,(q-1)!} \left(\frac{a_1}{d}\right)^{n+1} \left(\frac{a_2}{d}\right)^q = 0, \quad q = 2, 3, \ldots \tag{3.1.47e}$$

and $D_{qp} = 0$ for all $p \neq 0, +1, -1$.

Equations (3.1.44 and 3.1.47) constitute an infinite system of linear algebraic equations. Numeric results for the constants of integration are derived by truncating this system as in the previous case. The system $6N \times 6N$ matrix is shown in Equation (3.1.48).

$$
\begin{pmatrix}
1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S^{(0)}_{1,1} & \cdots & S^{(0)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S^{(0)}_{N,1} & \cdots & S^{(0)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & S^{(1)}_{1,1} & S^{(1)}_{1,2} & \cdots & S^{(1)}_{1,N} & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & S^{(1)}_{2,1} & S^{(1)}_{2,2} & \cdots & S^{(1)}_{2,N} & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & S^{(1)}_{N,1} & S^{(1)}_{N,2} & \cdots & S^{(1)}_{N,N} & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S^{(-1)}_{1,1} & S^{(-1)}_{1,2} & \cdots & S^{(-1)}_{1,N} \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S^{(-1)}_{2,1} & S^{(-1)}_{2,2} & \cdots & S^{(-1)}_{2,N} \\
 & \cdots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & & \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S^{(-1)}_{N,1} & S^{(-1)}_{N,2} & \cdots & S^{(-1)}_{N,N} \\
T^{(0)}_{1,1} & \cdots & T^{(0)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
T^{(0)}_{N,1} & \cdots & T^{(0)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & T^{(1)}_{1,1} & T^{(1)}_{1,2} & \cdots & T^{(1)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & T^{(1)}_{2,1} & T^{(1)}_{2,2} & \cdots & T^{(1)}_{2,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots \\
0 & \cdots & 0 & T^{(1)}_{N,1} & T^{(1)}_{N,2} & \cdots & T^{(1)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T^{(-1)}_{1,1} & T^{(-1)}_{1,2} & \cdots & T^{(-1)}_{1,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T^{(-1)}_{2,1} & T^{(-1)}_{2,2} & \cdots & T^{(-1)}_{2,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T^{(-1)}_{N,1} & T^{(-1)}_{N,2} & \cdots & T^{(-1)}_{N,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
C_{10} \\ \vdots \\ C_{N0} \\ C_{11} \\ C_{21} \\ \vdots \\ C_{N1} \\ C_{1,-1} \\ C_{2,-1} \\ \vdots \\ C_{N,-1} \\ D_{10} \\ \vdots \\ D_{N0} \\ D_{11} \\ D_{21} \\ \vdots \\ D_{N1} \\ D_{1,-1} \\ D_{2,-1} \\ \vdots \\ D_{N,-1}
\end{pmatrix}
=
\begin{pmatrix}
0 \\ \vdots \\ 0 \\ a_1 \\ 0 \\ \vdots \\ 0 \\ -2\,a_1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \\ -2\,a_2 \\ 0 \\ \vdots \\ 0
\end{pmatrix}
H_0
\tag{3.1.48}
$$

where the entries S and T are defined as

$$
\begin{aligned}
S^{(0)}_{n,q} &= (-1)^q \frac{(q+n)!}{q!\,n!}\left(\frac{a_2}{d}\right)^{q+1}\left(\frac{a_1}{d}\right)^{n},\\
S^{(1)}_{n,q} &= (-1)^{q+1} \frac{(q+n)!}{(q-1)!\,(n+1)!}\left(\frac{a_2}{d}\right)^{q+1}\left(\frac{a_1}{d}\right)^{n},\\
S^{(-1)}_{n,q} &= (-1)^{q-1} \frac{(q+n)!}{(q+1)!\,(n-1)!}\left(\frac{a_2}{d}\right)^{q+1}\left(\frac{a_1}{d}\right)^{n},\\
T^{(0)}_{q,n} &= (-1)^q \frac{(n+q)!}{n!\,q!}\left(\frac{a_1}{d}\right)^{n+1}\left(\frac{a_2}{d}\right)^{q},\\
T^{(1)}_{q,n} &= (-1)^{q+1} \frac{(n+q)!}{(n-1)!\,(q+1)!}\left(\frac{a_1}{d}\right)^{n+1}\left(\frac{a_2}{d}\right)^{q},\\
T^{(-1)}_{q,n} &= (-1)^{q-1} \frac{(n+q)!}{(n+1)!\,(q-1)!}\left(\frac{a_1}{d}\right)^{n+1}\left(\frac{a_2}{d}\right)^{q}, \quad n,q = 1\dots N
\end{aligned}
$$

Once the system in (3.1.48) is solved, the total potential outside the spheres can be expressed (see Equation 3.1.34)) as

$$
\begin{aligned}
\Phi^{tot}(r_1,\theta_1,\varphi_1 \mid r_2,\theta_2,\varphi_2) = &\sum_{n=1}^{N}\sum_{m=-1}^{1} C_{nm}\left(\frac{a_1}{r_1}\right)^{n+1} P_n^m(cos\,\theta_1)\,cos(m\varphi_1)\\
&+\sum_{q=1}^{N}\sum_{p=-1}^{1} D_{qp}\left(\frac{a_2}{r_2}\right)^{q+1} P_q^p(cos\,\theta_2)\,cos(p\varphi_2)\\
&- H_0\, r_1 P_1^1(cos\,\theta_1)\,cos\,\varphi_1, \qquad r_1 \geq a_1, r_2 \geq a_2
\end{aligned}
\tag{3.1.49}
$$

The magnetic field intensity in the spherical coordinates attached to sphere 1 are obtained from $\mathbf{H} = -\nabla\Phi^{tot}$ and using the geometrical relation (A.2), i.e.,

$$
\mathbf{H}(r_1,\theta_1,\varphi_1) = H_{r_1}\hat{r_1} + H_{\theta_1}\hat{\theta_1} + H_{\varphi_1}\hat{\varphi}_1 \tag{3.1.50}
$$

with

$$H_{r_1} = \frac{1}{r_1}\sum_{n=1}^{\infty}\sum_{m=-1}^{1}(n+1)\,C_{nm}\left(\frac{a_1}{r_1}\right)^{n+1}P_n^m(cos\,\theta_1)\,cos(m\varphi_1) + H_0 P_1^1(cos\,\theta_1)\,cos\,\varphi_1$$
$$+ \frac{1}{{r_2}^3}\sum_{q=1}^{\infty}\sum_{p=-1}^{1}D_{qp}\left(\frac{a_2}{r_2}\right)^{q+1}\Big\{r_2(q+1)(r_1 - d\,cos\,\theta_1)P_q^p(cos\,\theta_2)$$
$$- r_1 d\,sin^2\theta_1 P_q^{p\prime}(cos\,\theta_2)\Big\}\,cos(p\varphi_2),$$

$$H_{\theta_1} = sin\theta_1\Bigg\{\frac{1}{r_1}\sum_{n=1}^{\infty}\sum_{m=-1}^{1}C_{nm}\left(\frac{a_1}{r_1}\right)^{n+1}P_n^{m\prime}(cos\,\theta_1)\,cos(m\varphi_1) - H_0 P_1^{1\prime}(cos\,\theta_1)\,cos\,\varphi_1$$
$$+ \frac{1}{{r_2}^3}\sum_{q=1}^{\infty}\sum_{p=-1}^{1}D_{qp}\left(\frac{a_2}{r_2}\right)^{q+1}\Big\{r_2 d\,(q+1)P_q^p(cos\,\theta_2)$$
$$+ r_1(r_1 - d\,cos\,\theta_1)P_q^{p\prime}(cos\,\theta_2)\Big\}\,cos(p\varphi_2)\Bigg\},$$

$$H_{\varphi_1} = \frac{1}{r_1 sin\,\theta_1}\Bigg\{\sum_{n=1}^{\infty}\sum_{m=-1}^{1}m\,C_{nm}\left(\frac{a_1}{r_1}\right)^{n+1}P_n^m(cos\,\theta_1)\,sin\,(m\varphi_1)$$
$$+ \sum_{q=1}^{\infty}\sum_{p=-1}^{1}p\,D_{qp}\left(\frac{a_2}{r_2}\right)^{q+1}P_q^p(cos\,\theta_2)\,sin(p\varphi_2) - H_0 r_1 P_1^1(cos\,\theta_1)\,sin\,\varphi_1\Bigg\},$$

$$r_1 \geq a_1, \quad r_2 \geq a_2, \quad d \geq (a_1 + a_2)$$

Similarly, we can find the magnetic field intensity in spherical coordinate $(r_2, \theta_2, \varphi_2)$ attached to sphere 2.

The potentials inside the spheres $(r_1 < a_1$ ***and*** $r_2 < a_2)$:

The magnetic potential Ψ inside each sphere is related to the magnetic flux density, $\mathbf{B}$ by $\mathbf{B} = -\nabla\Psi$ and satisfies the Laplace equation. Using the symmetry with respect to the plane $z - x$, we have

$$\Psi^{(1)}(r_1,\theta_1,\varphi_1)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}G_{nm}^{(1)}\left(\frac{r_1}{a_1}\right)^{n}P_n^m(cos\,\theta_1)\,cos\,m\varphi_1,\quad r_1<a_1 \tag{3.1.51a}$$

$$\Psi^{(2)}(r_2,\theta_2,\varphi_2)=\sum_{q=0}^{\infty}\sum_{p=-q}^{q}G_{qp}^{(2)}\left(\frac{r_2}{a_2}\right)^{m}P_q^p(cos\,\theta_2)\,cos\,p\varphi_2,\quad r_2<a_2 \tag{3.1.51b}$$

Imposing the boundary condition in (3.1.24),

$$-\frac{\partial}{\partial\,r_1}\left\{\Psi^{(1)}(r_1,\theta_1,\varphi_1)\right\}\Bigg|_{r_1\,=\,a_1}=-\mu_0\,\frac{\partial}{\partial\,r_1}\left\{\Phi^{tot}(r_1,\theta_1,\varphi_1)\right\}\Bigg|_{r_1\,=\,a_1} \tag{3.1.52}$$

Equations (3.1.51) and (3.1.42) give

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}G_{nm}^{(1)}\,nP_n^m(cos\,\theta_1)\,cos\,m\varphi_1=\mu_0\Bigg\{&\sum_{n=1}^{\infty}\sum_{m=-1}^{1}-(n+1)C_{nm}P_n^m(cos\,\theta_1)\,cos\,(m\varphi_1)\\&+\sum_{q=1}^{\infty}\sum_{p=-1}^{1}\sum_{n=1}^{\infty}(-1)^{q+p}D_{qp}\frac{(q+n)!}{(q-p)!\,(n+p)!}\left(\frac{a_2}{d}\right)^{q+1}n\left(\frac{a_1}{d}\right)^{n}P_n^p(cos\,\theta_1)\,cos(p\varphi_1)\\&-a_1H_0P_1^1(cos\,\theta_1)\,cos\,\varphi_1\Bigg\},\qquad n\neq 0\end{aligned} \tag{3.1.53}$$

Multiplying by $P_l^m(cos\,\theta_1)\,sin\,\theta_1\,cos(m\varphi_1)$ and integrating with respect to θ_1 and φ_1 yield

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}G_{nm}^{(1)}\,n\frac{2\pi}{(2n+1)}\frac{(n+m)!}{(n-m)!}=\mu_0\int_{\theta_1=0}^{\pi}\int_{\varphi_1=0}^{2\pi}\Bigg\{&\sum_{n=1}^{\infty}\sum_{m=-1}^{1}-(n+1)\,C_{nm}\\&\times P_n^m(cos\,\theta_1)\,cos(m\varphi_1)+\sum_{q=1}^{\infty}\sum_{m=-1}^{1}\sum_{n=1}^{\infty}(-1)^{q+m}D_{qm}\frac{(q+n)!}{(q-m)!\,(n+m)!}\\&\times\left(\frac{a_2}{d}\right)^{q+1}n\left(\frac{a_1}{d}\right)^{n}P_n^m(cos\,\theta_1)\,cos(m\varphi_1)-a_1H_0P_1^1(cos\,\theta_1)cos\,\varphi_1\Bigg\}\\&\times P_n^m(cos\,\theta_1)\,sin\,\theta_1\,cos(m\varphi_1)\,d\theta_1\,d\varphi_1\end{aligned} \tag{3.1.54}$$

Taking into account the orthogonality of the spherical harmonics (see (C.6)) we obtain

for $m = 0$

$$G_{n0}^{(1)} = \frac{(2n+1)}{n}\mu_0 \sum_{q=1}^{\infty}(-1)^q D_{q0}\frac{(q+n)!}{q!\,n!}\left(\frac{a_2}{d}\right)^{q+1}\left(\frac{a_1}{d}\right)^{n}, \quad n = 1, 2, 3, \ldots \tag{3.1.55a}$$

for $m = 1$

$$G_{11}^{(1)} = 3\mu_0\left\{\sum_{q=1}^{\infty}(-1)^{q+1} D_{q1}\frac{q(q+1)}{2}\left(\frac{a_2}{d}\right)^{q+1}\left(\frac{a_1}{d}\right) - a_1 H_0\right\}, \qquad n = 1 \tag{3.1.55b}$$

$$G_{n1}^{(1)} = \frac{(2n+1)}{n}\mu_0 \sum_{q=1}^{\infty}(-1)^{q+1} D_{q1}\frac{(q+n)!}{(q-1)!\,(n+1)!}\left(\frac{a_2}{d}\right)^{q+1}\left(\frac{a_1}{d}\right)^{n}, \quad n = 2, 3, \ldots \tag{3.1.55c}$$

for $m = -1$

$$G_{1,-1}^{(1)} = 3\mu_0\left\{\sum_{q=1}^{\infty}(-1)^{q-1} D_{q,-1}\left(\frac{a_2}{d}\right)^{q+1}\left(\frac{a_1}{d}\right) + 2\, a_1 H_0\right\}, \quad n = 1 \tag{3.1.55d}$$

$$G_{n,-1}^{(1)} = \frac{(2n+1)}{n}\mu_0 \sum_{q=1}^{\infty}(-1)^{q-1} D_{q,-1}\frac{(q+n)!}{(q+1)!\,(n-1)!}\left(\frac{a_2}{d}\right)^{q+1}\left(\frac{a_1}{d}\right)^{n}, \quad n = 2, 3, \ldots \tag{3.1.55e}$$

and $G_{nm}^{(1)} = 0$ for all $m \neq 0, +1, -1$.

Similarly, the constants of integration for sphere 2 are obtained as for $p = 0$

$$G_{q0}^{(2)} = \frac{(2q+1)}{q}\mu_0 \sum_{n=1}^{\infty}(-1)^q C_{n0}\frac{(n+q)!}{n!\,q!}\left(\frac{a_1}{d}\right)^{n+1}\left(\frac{a_2}{d}\right)^q, \qquad q = 1, 2, 3, \ldots \tag{3.1.56a}$$

for $p = 1$

$$G_{11}^{(2)} = 3\mu_0\left\{\sum_{n=1}^{\infty} C_{n1}\frac{n(n+1)}{2}\left(\frac{a_1}{d}\right)^{n+1}\left(\frac{a_2}{d}\right) - a_2 H_0\right\}, \qquad q = 1 \tag{3.1.56b}$$

$$G_{q1}^{(2)} = \frac{(2q+1)}{q}\mu_0 \sum_{n=1}^{\infty}(-1)^{q+1} C_{n1}\frac{(n+q)!}{(n-1)!\,(q+1)!}\left(\frac{a_1}{d}\right)^{n+1}\left(\frac{a_2}{d}\right)^q, \quad q = 2, 3, \ldots \tag{3.1.56c}$$

for $p = -1$

$$G_{1,-1}^{(2)} = 3\mu_0\left\{\sum_{n=1}^{\infty} C_{n,-1}\left(\frac{a_1}{d}\right)^{n+1}\left(\frac{a_2}{d}\right) + 2\,a_2 H_0\right\}, \quad q = 1 \tag{3.1.56d}$$

$$G_{q,-1}^{(2)} = \frac{(2q+1)}{q}\mu_0 \sum_{n=1}^{\infty}(-1)^{q-1} C_{n,-1}\frac{(n+q)!}{(n+1)!\,(q-1)!}\left(\frac{a_1}{d}\right)^{n+1}\left(\frac{a_2}{d}\right)^q, \quad q = 2, 3, \ldots \tag{3.1.56e}$$

and $G_{qp}^{(2)} = 0$ for all $p \neq 0, +1, -1$.

The potential inside each sphere is computed with Equation (3.1.51).

As before, the magnetic flux density components inside the spheres can be calculated from $\mathbf{B} = -\nabla\Psi$. Equation (3.1.51a) gives

$$
\begin{aligned}
B_{r_1} &= -\frac{1}{r_1}\sum_{n=1}^{\infty}\sum_{m=-1}^{1} n\, G_{nm}^{(1)}\left(\frac{r_1}{a_1}\right)^{n} P_n^m(cos\,\theta_1)\,cos(m\varphi_1), \\
B_{\theta_1} &= \frac{sin\,\theta_1}{r_1}\sum_{n=1}^{\infty}\sum_{m=-1}^{1} G_{nm}^{(1)}\left(\frac{r_1}{a_1}\right)^{n} P_n^{m\prime}(cos\,\theta_1)\,cos(m\varphi_1), \\
B_{\varphi_1} &= \frac{1}{r_1 sin\,\theta_1}\sum_{n=1}^{\infty}\sum_{m=-1}^{1} m\, G_{nm}^{(1)}\left(\frac{r_1}{a_1}\right)^{n} P_n^m(cos\,\theta_1)\,sin(m\varphi_1), \qquad r_1 < a_1
\end{aligned}
\tag{3.1.57}
$$

Similarly, with (3.1.51b),

$$
\begin{aligned}
B_{r_2} &= -\frac{1}{r_2}\sum_{q=1}^{\infty}\sum_{p=-1}^{1} q\, G_{qp}^{(2)}\left(\frac{r_2}{a_2}\right)^{q} P_q^p(cos\,\theta_2)\,cos(p\varphi_2), \\
B_{\theta_2} &= \frac{sin\,\theta_2}{r_2}\sum_{q=1}^{\infty}\sum_{p=-1}^{1} G_{qp}^{(2)}\left(\frac{r_2}{a_2}\right)^{q} P_q^{p\prime}(cos\,\theta_2)\,cos(p\varphi_2), \\
B_{\varphi_2} &= \frac{1}{r_2 sin\,\theta_2}\sum_{q=1}^{\infty}\sum_{p=-1}^{1} p\, G_{qp}^{(2)}\left(\frac{r_2}{a_2}\right)^{q} P_q^p(cos\,\theta_2)\,sin(p\varphi_2), \qquad r_2 < a_2
\end{aligned}
\tag{3.1.58}
$$

3.2 Three-Sphere System

A system of three coaxial ferromagnetic spheres of radii a_1, a_2 and a_3 with the separation distances d_{12}, d_{13} and d_{23} between their centers have been placed in a homogeneous medium of permeability μ_0 in the presence of a uniform magnetic field $\mathbf{H_0}$. Take $(r_1, \theta_1, \varphi_1)$, $(r_2, \theta_2, \varphi_2)$ and $(r_3, \theta_3, \varphi_3)$ as the spherical coordinates attached to sphere 1, sphere 2 and sphere 3, respectively.

3.2.1 Case I: $\mathbf{H_0}$ along z Axis

Consider the magnetic field oriented along the common axis, as shown in Figure 3.4.

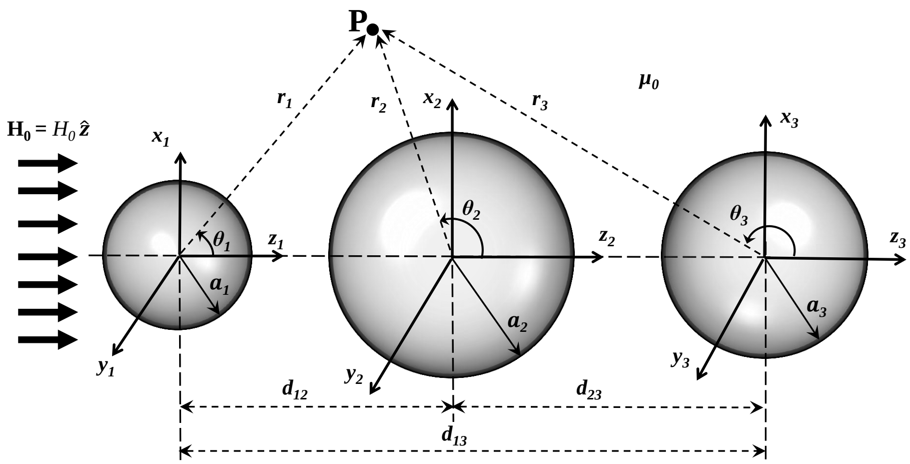

FIGURE 3.4: Three coaxial ferromagnetic spheres placed in the presence of an external magnetic field oriented along z axis

The potentials outside the spheres $(r_1 \geq a_1, r_2 \geq a_2$ ***and*** $r_3 \geq a_3)$

The total scalar magnetic potential at outside the spheres due to the three spheres and to the potential corresponding to the external field can be expressed as

$$\begin{aligned} \Phi^{tot}(r_1, \theta_1, \varphi_1 \mid r_2, \theta_2, \varphi_2 \mid r_3, \theta_3, \varphi_3) &= \Phi_1(r_1, \theta_1, \varphi_1) + \Phi_2(r_2, \theta_2, \varphi_2) \\ &+ \Phi_3(r_3, \theta_2, \varphi_3) + \Phi_{ext}(r_1, \theta_1, \varphi_1) \end{aligned} \tag{3.2.1}$$

where Φ_1, Φ_2 and Φ_3 are the potentials due to the three spheres expressed in terms of their attached spherical coordinates. $\Phi_{ext}(r_1, \theta_1, \varphi_1)$ is the potential corresponding to the external field in spherical coordinates $(r_1, \theta_1, \varphi_1)$ attached to sphere 1.

We have (see Equation (2.1.6))

$$\Phi_1(r_1,\theta_1,\varphi_1) = \sum_{n_1=0}^{\infty} C_{n_1} \left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}(cos\,\theta_1), \qquad r_1 \geq a_1 \tag{3.2.2a}$$

$$\Phi_2(r_2,\theta_2,\varphi_2) = \sum_{n_2=0}^{\infty} D_{n_2} \left(\frac{a_2}{r_2}\right)^{n_2+1} P_{n_2}(cos\,\theta_2), \qquad r_2 \geq a_2 \tag{3.2.2b}$$

$$\Phi_2(r_3,\theta_3,\varphi_3) = \sum_{n_3=0}^{\infty} E_{n_3} \left(\frac{a_3}{r_3}\right)^{n_3+1} P_{n_3}(cos\,\theta_3), \qquad r_3 \geq a_3 \tag{3.2.2c}$$

where C_{n_1}, D_{n_2} and E_{n_3} are constants of integration.

The coordinate translations are performed using the translational addition theorems in Equation (2.2.5) as follows.

In spherical coordinates (r_1,θ_1,φ_1), with $r' = r_1$, $\theta' = \theta_1$, and with $r = r_2$, $\theta = \theta_2$, $\theta_0 = \pi$, $d = d_{12}$ for sphere 2 and with $r = r_3$, $\theta = \theta_3$, $\theta_0 = \pi$, $d = d_{13}$ for sphere 3,

$$\begin{aligned}
\Phi_2^{(1)}(r_1,\theta_1,\varphi_1) &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} (-1)^{n_2} D_{n_2} \frac{(n_2+n_1)!}{n_2!\,n_1!} \left(\frac{a_2}{d_{12}}\right)^{n_2+1} \left(\frac{r_1}{d_{12}}\right)^{n_1} P_{n_1}(cos\,\theta_1),\\
\Phi_3^{(1)}(r_1,\theta_1,\varphi_1) &= \sum_{n_1=0}^{\infty}\sum_{n_3=0}^{\infty} (-1)^{n_3} E_{n_3} \frac{(n_3+n_1)!}{n_3!\,n_1!} \left(\frac{a_3}{d_{13}}\right)^{n_3+1} \left(\frac{r_1}{d_{13}}\right)^{n_1} P_{n_1}(cos\,\theta_1),\\
&\qquad d_{12},\, d_{13} \geq r_1
\end{aligned} \tag{3.2.3}$$

In spherical coordinates (r_2,θ_2,φ_2), with $r' = r_2$, $\theta' = \theta_2$, and with $r = r_1$, $\theta = \theta_1$, $\theta_0 = 0$, $d = d_{12}$ for sphere 1 and with $r = r_3$, $\theta = \theta_3$, $\theta_0 = \pi$, $d = d_{23}$ for sphere 3,

$$\begin{aligned}
\Phi_1^{(2)}(r_2,\theta_2,\varphi_2) &= \sum_{n_2=0}^{\infty}\sum_{n_1=0}^{\infty} (-1)^{n_2} C_{n_1} \frac{(n_1+n_2)!}{n_1!\,n_2!} \left(\frac{a_1}{d_{12}}\right)^{n_1+1} \left(\frac{r_2}{d_{12}}\right)^{n_2} P_{n_2}(cos\,\theta_2),\\
\Phi_3^{(2)}(r_2,\theta_2,\varphi_2) &= \sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} (-1)^{n_3} E_{n_3} \frac{(n_3+n_2)!}{n_3!\,n_2!} \left(\frac{a_3}{d_{23}}\right)^{n_3+1} \left(\frac{r_2}{d_{23}}\right)^{n_2} P_{n_2}(cos\,\theta_2),\\
&\qquad d_{12},\, d_{23} \geq r_2
\end{aligned} \tag{3.2.4}$$

In spherical coordinates $(r_3, \theta_3, \varphi_3)$, with $r' = r_3$, $\theta' = \theta_3$, and with $r = r_1$, $\theta = \theta_1$, $\theta_0 = 0$, $d = d_{13}$ for sphere 1 and with $r = r_2$, $\theta = \theta_2$, $\theta_0 = 0$, $d = d_{23}$ for sphere 2,

$$\begin{aligned}\Phi_1^{(3)}(r_3, \theta_3, \varphi_3) &= \sum_{n_3=0}^{\infty} \sum_{n_1=0}^{\infty} (-1)^{n_3} C_{n_1} \frac{(n_1+n_3)!}{n_1!\, n_3!} \left(\frac{a_1}{d_{13}}\right)^{n_1+1} \left(\frac{r_3}{d_{13}}\right)^{n_3} P_{n_3}(cos\,\theta_3),\\ \Phi_2^{(3)}(r_3, \theta_3, \varphi_3) &= \sum_{n_3=0}^{\infty} \sum_{n_2=0}^{\infty} (-1)^{n_3} D_{n_2} \frac{(n_2+n_3)!}{n_2!\, n_3!} \left(\frac{a_2}{d_{23}}\right)^{n_2+1} \left(\frac{r_3}{d_{23}}\right)^{n_3} P_{n_3}(cos\,\theta_3),\\ &\qquad d_{13},\, d_{23} \geq r_3\end{aligned} \tag{3.2.5}$$

The potential corresponding to the external field can be obtained as

$$\Phi_{ext}(r_i, \theta_i, \varphi_i) = -H_0 z_i + K_i, \qquad i = 1, 2, 3$$

where K_i are constants of reference corresponding to the coordinates attached to sphere i. As before, at $z_1 = 0$ plane, we choose $\Phi_{ext} = 0$ as reference. Then, $K_1 = 0, K_2 = -H_0 d_{12}$ and $K_3 = -H_0 d_{13}$. With $z_i = r_i P_1(cos\,\theta_i)$, we have

$$\Phi_{ext}(r_1, \theta_1, \varphi_1) = -H_0 r_1 P_1(cos\,\theta_1) \tag{3.2.6a}$$

$$\Phi_{ext}(r_2, \theta_2, \varphi_2) = -H_0 r_2 P_1(cos\,\theta_2) - H_0 d_{12} \tag{3.2.6b}$$

$$\Phi_{ext}(r_3, \theta_3, \varphi_3) = -H_0 r_3 P_1(cos\,\theta_3) - H_0 d_{13} \tag{3.2.6c}$$

where $P_1(x)$ is the Legendre polynomial of the first kind. Substituting the corresponding potentials into the Equation (3.2.1), we can express the potential in spherical coordinates $(r_1, \theta_1, \varphi_1)$ as

$$\begin{aligned}\Phi^{tot}(r_1, \theta_1, \varphi_1) &= \sum_{n_1=0}^{\infty} C_{n_1} \left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}(cos\,\theta_1) - H_0\, r_1 P_1(cos\,\theta_1)\\ &+ \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} (-1)^{n_2} D_{n_2} \frac{(n_2+n_1)!}{n_2!\, n_1!} \left(\frac{a_2}{d_{12}}\right)^{n_2+1} \left(\frac{r_1}{d_{12}}\right)^{n_1} P_{n_1}(cos\,\theta_1)\\ &+ \sum_{n_1=0}^{\infty} \sum_{n_3=0}^{\infty} (-1)^{n_3} E_{n_3} \frac{(n_3+n_1)!}{n_3!\, n_1!} \left(\frac{a_3}{d_{13}}\right)^{n_3+1} \left(\frac{r_1}{d_{13}}\right)^{n_1} P_{n_1}(cos\,\theta_1), \qquad r_1 \geq a_1\end{aligned} \tag{3.2.7}$$

By imposing the condition of zero magnetic flux through the surface of each sphere, we have

$$C_0 = 0, \quad D_0 = 0, \quad E_0 = 0 \tag{3.2.8}$$

Thus, Equation (3.2.7) becomes

$$\begin{aligned}\Phi^{tot}(r_1,\theta_1,\varphi_1) &= \sum_{n_1=1}^{\infty} C_{n_1}\left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}(cos\,\theta_1) - H_0\, r_1 P_1(cos\,\theta_1)\\ &+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}(-1)^{n_2} D_{n_2}\frac{(n_2+n_1)!}{n_2!\,n_1!}\left(\frac{a_2}{d_{12}}\right)^{n_2+1}\left(\frac{r_1}{d_{12}}\right)^{n_1} P_{n_1}(cos\,\theta_1)\\ &+ \sum_{n_1=1}^{\infty}\sum_{n_3=1}^{\infty}(-1)^{n_3} E_{n_3}\frac{(n_3+n_1)!}{n_3!\,n_1!}\left(\frac{a_3}{d_{13}}\right)^{n_3+1}\left(\frac{r_1}{d_{13}}\right)^{n_1} P_{n_1}(cos\,\theta_1), \qquad r_1 \geq a_1\end{aligned} \tag{3.2.9}$$

Similarly, in $(r_2, \theta_2, \varphi_2)$

$$\begin{aligned}\Phi^{tot}(r_2,\theta_2,\varphi_2) &= \sum_{n_2=1}^{\infty} D_{n_2}\left(\frac{a_2}{r_2}\right)^{n_2+1} P_{n_2}(cos\,\theta_2) - H_0\left[\, r_2 P_1(cos\,\theta_1) + d_{12}\,\right]\\ &+ \sum_{n_2=1}^{\infty}\sum_{n_1=1}^{\infty}(-1)^{n_2} C_{n_1}\frac{(n_1+n_2)!}{n_1!\,n_2!}\left(\frac{a_1}{d_{12}}\right)^{n_1+1}\left(\frac{r_2}{d_{12}}\right)^{n_2} P_{n_2}(cos\,\theta_2)\\ &+ \sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}(-1)^{n_3} E_{n_3}\frac{(n_3+n_2)!}{n_3!\,n_2!}\left(\frac{a_3}{d_{23}}\right)^{n_3+1}\left(\frac{r_2}{d_{23}}\right)^{n_2} P_{n_2}(cos\,\theta_2), \qquad r_2 \geq a_2\end{aligned} \tag{3.2.10}$$

and in $(r_3, \theta_3, \varphi_3)$

$$\begin{aligned}\Phi^{tot}(r_3,\theta_3,\varphi_3) &= \sum_{n_3=1}^{\infty} E_{n_3}\left(\frac{a_3}{r_3}\right)^{n_3+1} P_{n_3}(cos\,\theta_3) - H_0\left[\, r_3 P_1(cos\,\theta_1) + d_{13}\,\right]\\ &+ \sum_{n_3=1}^{\infty}\sum_{n_1=1}^{\infty}(-1)^{n_3} C_{n_1}\frac{(n_1+n_3)!}{n_1!\,n_3!}\left(\frac{a_1}{d_{13}}\right)^{n_1+1}\left(\frac{r_3}{d_{13}}\right)^{n_3} P_{n_3}(cos\,\theta_3)\\ &+ \sum_{n_3=1}^{\infty}\sum_{n_2=1}^{\infty}(-1)^{n_3} D_{n_2}\frac{(n_2+n_3)!}{n_2!\,n_3!}\left(\frac{a_2}{d_{23}}\right)^{n_2+1}\left(\frac{r_3}{d_{23}}\right)^{n_3} P_{n_3}(cos\,\theta_3), \qquad r_3 \geq a_3\end{aligned} \tag{3.2.11}$$

With

$$\Phi^{tot}(r_i,\theta_i,\varphi_i)\Big|_{r_i\,=\,a_i} = V_i \qquad i = 1, 2, 3$$

where V_i is the magnetic potential of sphere i, and applying the orthogonality of Legendre polynomials (see B.6) we obtain the following results. For sphere 1,

$$\begin{aligned} C_1 &+ \sum_{n_2=1}^{\infty}(-1)^{n_2} D_{n_2}(n_2+1)\left(\frac{a_2}{d_{12}}\right)^{n_2+1}\left(\frac{a_1}{d_{12}}\right) \\ &+ \sum_{n_3=1}^{\infty}(-1)^{n_3} E_{n_3}(n_3+1)\left(\frac{a_3}{d_{13}}\right)^{n_3+1}\left(\frac{a_1}{d_{13}}\right) = a_1 H_0, \qquad n_1 = 1 \end{aligned} \tag{3.2.12a}$$

$$\begin{aligned} C_{n_1} &+ \sum_{n_2=1}^{\infty}(-1)^{n_2} D_{n_2}\frac{(n_2+n_1)!}{n_2!\,n_1!}\left(\frac{a_2}{d_{12}}\right)^{n_2+1}\left(\frac{a_1}{d_{12}}\right)^{n_1} \\ &+ \sum_{n_3=1}^{\infty}(-1)^{n_3} E_{n_3}\frac{(n_3+n_1)!}{n_3!\,n_1!}\left(\frac{a_3}{d_{13}}\right)^{n_3+1}\left(\frac{a_1}{d_{13}}\right)^{n_1} = 0, \qquad n_1 = 2, 3\ldots \end{aligned} \tag{3.2.12b}$$

for sphere 2,

$$\begin{aligned} D_1 &+ \sum_{n_1=1}^{\infty}(-1)\, C_{n_1}(n_1+1)\left(\frac{a_1}{d_{12}}\right)^{n_1+1}\left(\frac{a_2}{d_{12}}\right) \\ &+ \sum_{n_3=1}^{\infty}(-1)^{n_3} E_{n_3}(n_3+1)\left(\frac{a_3}{d_{23}}\right)^{n_3+1}\left(\frac{a_2}{d_{23}}\right) = a_2 H_0, \qquad n_2 = 1 \end{aligned} \tag{3.2.13a}$$

$$\begin{aligned} D_{n_2} &+ \sum_{n_1=1}^{\infty}(-1)^{n_2} C_{n_1}\frac{(n_1+n_2)!}{n_1!\,n_2!}\left(\frac{a_1}{d_{12}}\right)^{n_1+1}\left(\frac{a_2}{d_{12}}\right)^{n_2} \\ &+ \sum_{n_3=1}^{\infty}(-1)^{n_3} E_{n_3}\frac{(n_3+n_2)!}{n_3!\,n_2!}\left(\frac{a_3}{d_{23}}\right)^{n_3+1}\left(\frac{a_2}{d_{23}}\right)^{n_2} = 0, \qquad n_2 = 2, 3\ldots \end{aligned} \tag{3.2.13b}$$

and for sphere 3,

$$\begin{aligned} E_1 &+ \sum_{n_1=1}^{\infty}(-1)\, C_{n_1}(n_1+1)\left(\frac{a_1}{d_{13}}\right)^{n_1+1}\left(\frac{a_3}{d_{13}}\right) \\ &+ \sum_{n_2=1}^{\infty}(-1)\, D_{n_2}(n_2+1)\left(\frac{a_2}{d_{23}}\right)^{n_2+1}\left(\frac{a_3}{d_{23}}\right) = a_3 H_0, \qquad n_3 = 1 \end{aligned} \tag{3.2.14a}$$

$$\begin{aligned} E_{n_3} &+ \sum_{n_1=1}^{\infty}(-1)^{n_3} C_{n_1}\frac{(n_1+n_3)!}{n_1!\,n_3!}\left(\frac{a_1}{d_{13}}\right)^{n_1+1}\left(\frac{a_3}{d_{13}}\right)^{n_3} \\ &+ \sum_{n_2=1}^{\infty}(-1)^{n_3} D_{n_2}\frac{(n_2+n_3)!}{n_2!\,n_3!}\left(\frac{a_2}{d_{23}}\right)^{n_2+1}\left(\frac{a_3}{d_{23}}\right)^{n_3} = 0, \qquad n_3 = 2, 3\ldots \end{aligned} \tag{3.2.14b}$$

To obtain the numerical solutions, we truncate the infinite system of equations (3.2.12) to (3.2.14) to a finite number N of terms in each of the intervening series.

Let us use the following notation:

$$\begin{aligned}
S1_{n_1,n_2} &= (-1)^{n_2}\frac{(n_2+n_1)!}{n_2!\,n_1!}\left(\frac{a_2}{d_{12}}\right)^{n_2+1}\left(\frac{a_1}{d_{12}}\right)^{n_1},\\
S2_{n_1,n_3} &= (-1)^{n_3}\frac{(n_3+n_1)!}{n_3!\,n_1!}\left(\frac{a_3}{d_{13}}\right)^{n_3+1}\left(\frac{a_1}{d_{13}}\right)^{n_1},\\
T1_{n_2,n_1} &= (-1)^{n_2}\frac{(n_1+n_2)!}{n_1!\,n_2!}\left(\frac{a_1}{d_{12}}\right)^{n_1+1}\left(\frac{a_2}{d_{12}}\right)^{n_2},\\
T2_{n_2,n_3} &= (-1)^{n_3}\frac{(n_3+n_2)!}{n_3!\,n_2!}\left(\frac{a_3}{d_{23}}\right)^{n_3+1}\left(\frac{a_2}{d_{23}}\right)^{n_2},\\
U1_{n_3,n_1} &= (-1)^{n_3}\frac{(n_1+n_3)!}{n_1!\,n_3!}\left(\frac{a_1}{d_{13}}\right)^{n_1+1}\left(\frac{a_3}{d_{13}}\right)^{n_3},\\
U2_{n_3,n_2} &= (-1)^{n_3}\frac{(n_2+n_3)!}{n_2!\,n_3!}\left(\frac{a_2}{d_{23}}\right)^{n_2+1}\left(\frac{a_3}{d_{23}}\right)^{n_3}, \qquad n_1,n_2,n_3 = 1\ldots N
\end{aligned}$$

Then, the constants of integration satisfy the $3N \times 3N$ matrix equation as shown in Equation (3.2.15).

$$\begin{pmatrix}
1 & 0 & \cdots & 0 & S1_{1,1} & S1_{1,2} & \cdots & S1_{1,N} & S2_{1,1} & S2_{1,2} & \cdots & S2_{1,N}\\
0 & 1 & \cdots & 0 & S1_{2,1} & S1_{2,2} & \cdots & S1_{2,N} & S2_{2,1} & S2_{2,2} & \cdots & S2_{2,N}\\
 & & \ddots & & & & \ddots & & & & \ddots & \\
0 & 0 & \cdots & 1 & S1_{N,1} & S1_{N,2} & \cdots & S1_{N,N} & S2_{N,1} & S2_{N,2} & \cdots & S2_{N,N}\\
T1_{1,1} & T1_{1,2} & \cdots & T1_{1,N} & 1 & 0 & \cdots & 0 & T2_{1,1} & T2_{1,2} & \cdots & T2_{1,N}\\
T1_{2,1} & T1_{2,2} & \cdots & T1_{2,N} & 0 & 1 & \cdots & 0 & T2_{2,1} & T2_{2,2} & \cdots & T2_{2,N}\\
 & & \ddots & & & & \ddots & & & & \ddots & \\
T1_{N,1} & T1_{N,2} & \cdots & T1_{N,N} & 0 & 0 & \cdots & 1 & T2_{N,1} & T2_{N,2} & \cdots & T2_{N,N}\\
U1_{1,1} & U1_{1,2} & \cdots & U1_{1,N} & U2_{1,1} & U2_{1,2} & \cdots & U2_{1,N} & 1 & 0 & \cdots & 0\\
U1_{2,1} & U1_{2,2} & \cdots & U1_{2,N} & U2_{2,1} & U2_{2,2} & \cdots & U2_{2,N} & 0 & 1 & \cdots & 0\\
 & & \ddots & & & & \ddots & & & & \ddots & \\
U1_{N,1} & U1_{N,2} & \cdots & U1_{N,N} & U2_{N,1} & U2_{N,2} & \cdots & U2_{N,N} & 0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix} C_1\\ C_2\\ \vdots\\ C_N\\ D_1\\ D_2\\ \vdots\\ D_N\\ E_1\\ E_2\\ \vdots\\ E_N \end{pmatrix}
=
\begin{pmatrix} a_1\\ 0\\ \vdots\\ 0\\ a_2\\ 0\\ \vdots\\ 0\\ a_3\\ 0\\ \vdots\\ 0 \end{pmatrix} H_0
\tag{3.2.15}$$

Once the solution of this equation is obtained, the total potential outside the spheres can be calculated in spherical coordinates as (see (3.2.2) and (3.2.6a))

$$\Phi^{tot}(r_1,\theta_1,\varphi_1 \mid r_2,\theta_2,\varphi_2 \mid r_3,\theta_3,\varphi_3) = \sum_{n_1=1}^{N} C_{n_1}\left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}(cos\,\theta_1) - H_0\,r_1 P_1(cos\,\theta_1)$$
$$+\sum_{n_2=1}^{N} D_{n_2}\left(\frac{a_2}{r_2}\right)^{n_2+1} P_{n_2}(cos\,\theta_2) + \sum_{n_3=1}^{N} E_{n_3}\left(\frac{a_3}{r_3}\right)^{n_3+1} P_{n_3}(cos\,\theta_3),$$
$$r_1 \geq a_1,\; r_2 \geq a_2,\; r_3 \geq a_3 \tag{3.2.16}$$

The magnetic field intensity components in (r_1,θ_1,φ_1) are derived in the form

$$H_{r_1} = \frac{1}{r_1}\sum_{n_1=1}^{\infty}(n_1+1)\,C_{n_1}\left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}(cos\,\theta_1) + \frac{1}{r_2^3}\sum_{n_2=1}^{\infty} D_{n_2}\left(\frac{a_2}{r_2}\right)^{n_2+1}\Big\{r_2(n_2+1)$$
$$\times (r_1 - d_{12}\,cos\,\theta_1)P_{n_2}(cos\,\theta_2) - r_1 d_{12}\,sin^2\theta_1 P'_{n_2}(cos\,\theta_2)\Big\} + \frac{1}{r_3^3}\sum_{n_3=1}^{\infty} E_{n_3}\left(\frac{a_3}{r_3}\right)^{n_3+1}$$
$$\times\Big\{r_3(n_3+1)(r_1 - d_{13}\,cos\,\theta_1)P_{n_3}(cos\,\theta_3) - r_1 d_{13}\,sin^2\theta_1 P'_{n_3}(cos\,\theta_3)\Big\} + H_0 P_1(cos\,\theta_1),$$

$$H_{\theta_1} = sin\,\theta_1\Bigg\{\frac{1}{r_1}\sum_{n_1=1}^{\infty} C_{n_1}\left(\frac{a_1}{r_1}\right)^{n_1+1} P'_{n_1}(cos\,\theta_1) + \frac{1}{r_2^3}\sum_{n_2=1}^{\infty} D_{n_2}\left(\frac{a_2}{r_2}\right)^{n_2+1}\Big\{r_2(n_2+1)$$
$$\times d_{12}P_{n_2}(cos\,\theta_2) + r_1(r_1 - d_{12}\,cos\,\theta_1)P'_{n_2}(cos\,\theta_2)\Big\} + \frac{1}{r_3^3}\sum_{n_3=1}^{\infty} E_{n_3}\left(\frac{a_3}{r_3}\right)^{n_3+1}$$
$$\times\Big\{r_3 d_{13}(n_3+1)P_{n_3}(cos\,\theta_3) + r_1(r_1 - d_{13}\,cos\,\theta_1)P'_{n_3}(cos\,\theta_3)\Big\} - H_0 P'_1(cos\,\theta_1)\Bigg\},$$

$$H_{\varphi_1} = 0, \qquad r_1 \geq a_1,\;\; r_2 \geq a_2,\;\; r_3 \geq a_3,\;\; d_{12} \geq (a_1+a_2),\;\; d_{13} \geq (a_1+a_3) \tag{3.2.17}$$

Similarly, we can find the components of the magnetic field intensity in spherical coordinates attached to the sphere 2 and sphere 3.

The potentials inside the spheres, $(r_1 < a_1, r_2 < a_2$ ***and*** $r_3 < a_3)$

As in the previous sections, the potential Ψ inside each sphere can be expressed in the coordinate attached to the respective sphere as

$$\Psi^{(1)}(r_1,\theta_1,\varphi_1) = \sum_{n_1=0}^{\infty} G_{n_1}^{(1)} \left(\frac{r_1}{a_1}\right)^{n_1} P_{n_1}(cos\,\theta_1), \qquad r_1 < a_1 \tag{3.2.18a}$$

$$\Psi^{(2)}(r_2,\theta_2,\varphi_2) = \sum_{n_2=0}^{\infty} G_{n_2}^{(2)} \left(\frac{r_2}{a_2}\right)^{n_2} P_{n_2}(cos\,\theta_2), \qquad r_2 < a_2 \tag{3.2.18b}$$

$$\Psi^{(3)}(r_3,\theta_3,\varphi_3) = \sum_{n_3=0}^{\infty} G_{n_3}^{(2)} \left(\frac{r_3}{a_3}\right)^{n_3} P_{n_3}(cos\,\theta_3), \qquad r_3 < a_3 \tag{3.2.18c}$$

where $G_{n_1}^{(1)}, G_{n_2}^{(2)}$ and $G_{n_3}^{(3)}$ are constants of integration.

To determine these constants we impose the boundary conditions

$$-\frac{\partial}{\partial\,r_i}\left\{\Psi^{(i)}(r_i,\theta_i,\varphi_i)\right\}\Bigg|_{r_i\;=\;a_i} = -\mu_0\,\frac{\partial}{\partial\,r_i}\left\{\Phi^{tot}(r_i,\theta_i,\varphi_i)\right\}\Bigg|_{r_i\;=\;a_i}, \qquad i=1,2,3 \tag{3.2.19}$$

For sphere 1, Equations (3.2.18a) and (3.2.9) yield

$$\begin{aligned}\sum_{n_1=0}^{\infty} G_{n_1}^{(1)}\, n_1\, P_{n_1}(cos\,\theta_1) = \mu_0 \Bigg\{ &\sum_{n_1=1}^{\infty} -(n_1+1)C_{n_1}P_{n_1}(cos\,\theta_1) - a_1 H_0 P_1(cos\,\theta_1) \\ &+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}(-1)^{n_2} D_{n_2}\frac{(n_2+n_1)!}{n_2!\,n_1!}\left(\frac{a_2}{d_{12}}\right)^{n_2+1} n_1 \left(\frac{a_1}{d_{12}}\right)^{n_1} P_{n_1}(cos\,\theta_1) \\ + \sum_{n_1=1}^{\infty}\sum_{n_3=1}^{\infty}(-1)^{n_3} E_{n_3}\frac{(n_3+n_1)!}{n_3!\,n_1!}&\left(\frac{a_3}{d_{13}}\right)^{n_3+1} n_1 \left(\frac{a_1}{d_{13}}\right)^{n_1} P_{n_1}(cos\,\theta_1)\Bigg\}, \qquad n_1 \neq 0\end{aligned} \tag{3.2.20}$$

Multiplying by $P_l(cos\,\theta_1)$ and integrating with respect to $cos\,\theta_1$ we have

$$\int_{-1}^{1}\sum_{n_1=1}^{\infty} G^{(1)}_{n_1}\, n_1\, P_{n_1}(cos\,\theta_1)P_l(cos\,\theta_1)d(cos\,\theta_1) = \mu_0\int_{-1}^{1}\Bigg\{\sum_{n_1=1}^{\infty} -(n_1+1)C_{n_1}P_{n_1}(cos\,\theta_1)$$
$$+\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}(-1)^{n_2}D_{n_2}\frac{(n_2+n_1)!}{n_2!\,n_1!}\left(\frac{a_2}{d_{12}}\right)^{n_2+1} n_1\left(\frac{a_1}{d_{12}}\right)^{n_1}P_{n_1}(cos\,\theta_1)$$
$$+\sum_{n_1=1}^{\infty}\sum_{n_3=1}^{\infty}(-1)^{n_3}E_{n_3}\frac{(n_3+n_1)!}{n_3!\,n_1!}\left(\frac{a_3}{d_{13}}\right)^{n_3+1} n_1\left(\frac{a_1}{d_{13}}\right)^{n_1}P_{n_1}(cos\,\theta_1)$$
$$- a_1H_0P_1(cos\,\theta_1)\Bigg\}P_l(cos\,\theta_1)\,d(cos\,\theta_1) \tag{3.2.21}$$

Applying the orthogonality of Legendre polynomials (see B.6) we obtain

$$n_1\,G^{(1)}_{n_1} = \mu_0\Bigg\{-(n_1+1)C_{n_1} - a_1H_0\frac{(2n_1+1)}{2}\int_{-1}^{1}P_1(cos\,\theta_1)P_{n_1}(cos\,\theta_1)\,d(cos\,\theta_1)$$
$$+\sum_{n_2=1}^{\infty}(-1)^{n_2}D_{n_2}\frac{(n_2+n_1)!}{n_2!\,n_1!}\left(\frac{a_2}{d_{12}}\right)^{n_2+1} n_1\left(\frac{a_1}{d_{12}}\right)^{n_1}$$
$$+\sum_{n_3=1}^{\infty}(-1)^{n_3}E_{n_3}\frac{(n_3+n_1)!}{n_3!\,n_1!}\left(\frac{a_3}{d_{13}}\right)^{n_3+1} n_1\left(\frac{a_1}{d_{13}}\right)^{n_1}\Bigg\}, \qquad n_1 = 1,2,3\ldots \tag{3.2.22}$$

This gives

$$G^{(1)}_{1} = 3\mu_0\Bigg\{\sum_{n_2=1}^{\infty}(-1)^{n_2}D_{n_2}(1+n_2)\left(\frac{a_2}{d_{12}}\right)^{n_2+1}\left(\frac{a_1}{d_{12}}\right)$$
$$+\sum_{n_3=1}^{\infty}(-1)^{n_3}E_{n_3}(1+n_3)\left(\frac{a_3}{d_{13}}\right)^{n_3+1}\left(\frac{a_1}{d_{13}}\right) - a_1H_0\Bigg\}, \qquad n_1 = 1 \tag{3.2.23a}$$

$$G^{(1)}_{n_1} = \frac{(2n_1+1)}{n_1}\mu_0\Bigg\{\sum_{n_2=1}^{\infty}(-1)^{n_2}D_{n_2}\frac{(n_1+n_2)!}{n_1!\,n_2!}\left(\frac{a_2}{d_{12}}\right)^{n_2+1}\left(\frac{a_1}{d_{12}}\right)^{n_1}$$
$$+\sum_{n_3=1}^{\infty}(-1)^{n_3}E_{n_3}\frac{(n_1+n_3)!}{n_1!\,n_3!}\left(\frac{a_3}{d_{13}}\right)^{n_3+1}\left(\frac{a_1}{d_{13}}\right)^{n_1}\Bigg\}, \qquad n_1 = 2,3\ldots \tag{3.2.23b}$$

Similarly, for sphere 2

$$G_1^{(2)} = 3\mu_0 \left\{ \sum_{n_1=1}^{\infty} (-1) C_{n_1} (1+n_1) \left(\frac{a_1}{d_{12}}\right)^{n_1+1} \left(\frac{a_2}{d_{12}}\right) + \sum_{n_3=1}^{\infty} (-1)^{n_3} E_{n_3} (1+n_3) \left(\frac{a_3}{d_{23}}\right)^{n_3+1} \left(\frac{a_2}{d_{23}}\right) - a_2 H_0 \right\}, \qquad n_2 = 1 \tag{3.2.24a}$$

$$G_{n_2}^{(2)} = \frac{(2n_2+1)}{n_2}\mu_0 \left\{ \sum_{n_1=1}^{\infty} (-1)^{n_2} C_{n_1} \frac{(n_2+n_1)!}{n_2!\, n_1!} \left(\frac{a_1}{d_{12}}\right)^{n_1+1} \left(\frac{a_2}{d_{12}}\right)^{n_2} + \sum_{n_3=1}^{\infty} (-1)^{n_3} E_{n_3} \frac{(n_2+n_3)!}{n_2!\, n_3!} \left(\frac{a_3}{d_{23}}\right)^{n_3+1} \left(\frac{a_2}{d_{23}}\right)^{n_2} \right\}, \qquad n_2 = 2, 3 \ldots \tag{3.2.24b}$$

and for sphere 3 as

$$G_1^{(3)} = 3\mu_0 \left\{ \sum_{n_1=1}^{\infty} (-1) C_{n_1} (1+n_1) \left(\frac{a_1}{d_{13}}\right)^{n_1+1} \left(\frac{a_3}{d_{13}}\right) + \sum_{n_2=1}^{\infty} (-1) D_{n_2} (1+n_2) \left(\frac{a_2}{d_{23}}\right)^{n_2+1} \left(\frac{a_3}{d_{23}}\right) - a_3 H_0 \right\}, \qquad n_3 = 1 \tag{3.2.25a}$$

$$G_{n_3}^{(3)} = \frac{(2n_3+1)}{n_3}\mu_0 \left\{ \sum_{n_1=1}^{\infty} (-1)^{n_3} C_{n_1} \frac{(n_3+n_1)!}{n_3!\, n_1!} \left(\frac{a_1}{d_{13}}\right)^{n_1+1} \left(\frac{a_3}{d_{13}}\right)^{n_3} + \sum_{n_2=1}^{\infty} (-1)^{n_3} D_{n_2} \frac{(n_3+n_2)!}{n_3!\, n_2!} \left(\frac{a_2}{d_{23}}\right)^{n_2+1} \left(\frac{a_3}{d_{23}}\right)^{n_3} \right\}, \qquad n_3 = 2, 3 \ldots \tag{3.2.25b}$$

The magnetic flux density inside the spheres is calculated with $\mathbf{B} = -\nabla\Psi$, for example inside the sphere 1 as (see (3.2.18a))

$$\begin{aligned} B_{r_1} &= -\frac{1}{r_1} \sum_{n_1=1}^{\infty} n_1\, G_{n_1}^{(1)} \left(\frac{r_1}{a_1}\right)^{n_1} P_{n_1}(\cos\theta_1), \\ B_{\theta_1} &= \frac{\sin\theta_1}{r_1} \sum_{n_1=1}^{\infty} G_{n_1}^{(1)} \left(\frac{r_1}{a_1}\right)^{n_1} P'_{n_1}(\cos\theta_1), \\ B_{\varphi_1} &= 0, \qquad\qquad r_1 < a_1 \end{aligned} \tag{3.2.26}$$

Similarly, the magnetic flux density inside the sphere 2 and sphere 3 is calculated from (3.2.18b) and (3.2.18c), respectively.

3.2.2 Case II: $\mathbf{H_0}$ perpendicular on the Common Axis

Consider the coaxial three-sphere system shown in Figure 3.5 in the presence of an external field oriented along the x axis (i.e. $\mathbf{H_0} = H_0\hat{x}$).

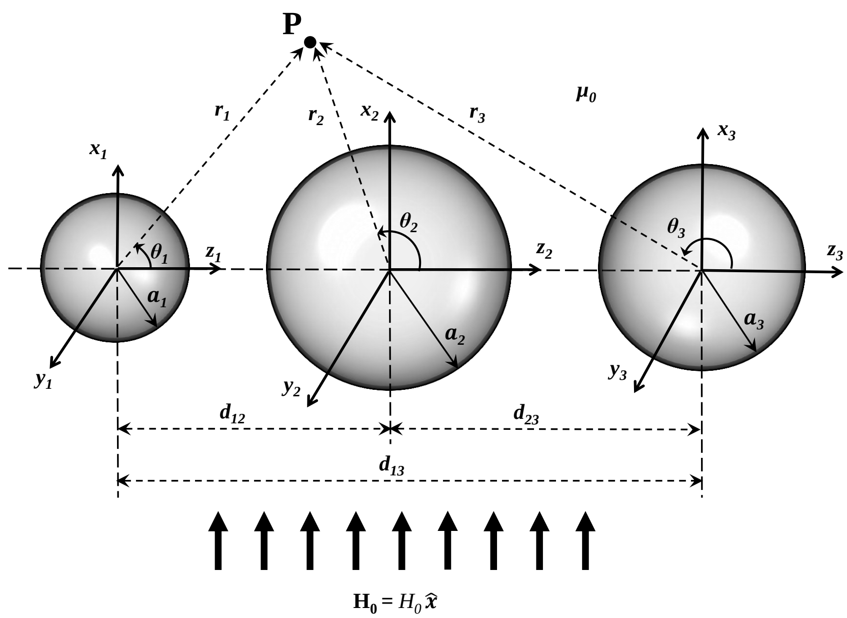

FIGURE 3.5: Three coaxial ferromagnetic spheres placed in an external field oriented along x axis

The potentials outside the spheres, $(r_1 \geq a_1, r_2 \geq a_2$ ***and*** $r_3 \geq a_3)$

The total scalar magnetic potential outside spheres can be expressed as

$$\begin{aligned}\Phi^{tot}(r_1, \theta_1, \varphi_1 \mid r_2, \theta_2, \varphi_2, \mid r_3, \theta_3, \varphi_3) = \Phi_1(r_1, \theta_1, \varphi_1) + \Phi_2(r_2, \theta_2, \varphi_2) \\ + \Phi_3(r_3, \theta_3, \varphi_3) + \Phi_{ext(x)}(r_1, \theta_1, \varphi_1)\end{aligned} \tag{3.2.27}$$

where Φ_1, Φ_2, Φ_3 are due to the three spheres, respectively, and $\Phi_{ext(x)}(r_1, \theta_1, \varphi_1)$ is the potential corresponding to the external field in spherical coordinates $(r_1, \theta_1, \varphi_1)$.

This can be written in spherical coordinates $(r_1, \theta_1, \varphi_1)$ attached to sphere 1 as

$$\Phi^{tot}(r_1, \theta_1, \varphi_1) = \Phi_1(r_1, \theta_1, \varphi_1) + \Phi_2^{(1)}(r_1, \theta_1, \varphi_1) + \Phi_3^{(1)}(r_1, \theta_1, \varphi_1) + \Phi_{ext(x)}(r_1, \theta_1, \varphi_1) \tag{3.2.28}$$

where $\Phi_2^{(1)}$ and $\Phi_3^{(1)}$ are the potentials due to the sphere 2 and sphere 3, respectively, in spherical coordinates $(r_1, \theta_1, \varphi_1)$. The potential due to each sphere can be written by using the real part of the Equation (2.1.3) corresponding to the respective coordinate system as

$$\begin{aligned}
\Phi_1(r_1, \theta_1, \varphi_1) &= \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1 m_1} \left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}^{m_1}(cos\,\theta_1)\, cos(m_1 \varphi_1), \quad r_1 \geq a_1 \\
\Phi_2(r_2, \theta_2, \varphi_2) &= \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2} \left(\frac{a_2}{r_2}\right)^{n_2+1} P_{n_2}^{m_2}(cos\,\theta_2)\, cos(m_2 \varphi_2), \quad r_2 \geq a_2 \\
\Phi_3(r_3, \theta_3, \varphi_3) &= \sum_{n_3=0}^{\infty} \sum_{m_3=-n_3}^{n_3} E_{n_3 m_3} \left(\frac{a_3}{r_3}\right)^{n_3+1} P_{n_3}^{m_3}(cos\,\theta_3)\, cos(m_3 \varphi_3), \quad r_3 \geq a_3
\end{aligned} \tag{3.2.29}$$

In order to translate $\Phi_2(r_2, \theta_2, \varphi_2)$ and $\Phi_3(r_3, \theta_3, \varphi_3)$ into coordinates $(r_1, \theta_1, \varphi_1)$, we use the translational addition theorem (see Section 2.2.1) by taking the real part of the Equation (2.2.1), with $r' = r_1$, $\theta' = \theta_1$, $\varphi' = \varphi_1$, and with $r = r_2$, $\theta = \theta_2$, $\varphi = \varphi_2$, $\theta_0 = \pi$, $d = d_{12}$ for sphere 2 and with $r = r_3$, $\theta = \theta_3$, $\varphi = \varphi_3$, $\theta_0 = \pi$, $d = d_{13}$ for sphere 3. This gives

$$\begin{aligned}
\Phi_2^{(1)}(r_1, \theta_1, \varphi_1) = &\sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} \sum_{\nu_2=0}^{\infty} (-1)^{n_2+m_2} D_{n_2 m_2} \frac{(n_2+\nu_2)!}{(n_2-m_2)!(\nu_2+m_2)!} \\
&\times \left(\frac{a_2}{d_{12}}\right)^{n_2+1} \left(\frac{r_1}{d_{12}}\right)^{\nu_2} P_{\nu_2}^{m_2}(cos\,\theta_1)\, cos(m_2 \varphi_1), \qquad r_1 \leq d_{12}
\end{aligned} \tag{3.2.30a}$$

$$\begin{aligned}
\Phi_3^{(1)}(r_1, \theta_1, \varphi_1) = &\sum_{n_3=0}^{\infty} \sum_{m_3=-n_3}^{n_3} \sum_{\nu_3=0}^{\infty} (-1)^{n_3+m_3} D_{n_3 m_3} \frac{(n_3+\nu_3)!}{(n_3-m_3)!(\nu_3+m_3)!} \\
&\times \left(\frac{a_3}{d_{13}}\right)^{n_3+1} \left(\frac{r_1}{d_{13}}\right)^{\nu_3} P_{\nu_3}^{m_3}(cos\,\theta_1)\, cos(m_3 \varphi_1), \qquad r_1 \leq d_{13}
\end{aligned} \tag{3.2.30b}$$

The potential due to the external field can be expressed as

$$\Phi_{ext(x)}(r_i, \theta_i, \varphi_i) = -H_0 x_i, \qquad i = 1, 2, 3$$

With $x_i = r_i P_1^1(cos\,\theta_i)\,cos\,\varphi_i$, we get

$$\Phi_{ext(x)}(r_i, \theta_i, \varphi_i) = -H_0\, r_i P_1^1(cos\,\theta_i)\,cos\,\varphi_i \qquad i = 1, 2, 3 \tag{3.2.31}$$

Thus,

$$\begin{aligned}\Phi^{tot}(r_1, \theta_1, \varphi_1) = \sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1} C_{n_1 m_1}\left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}^{m_1}(cos\,\theta_1)\,cos(m_1\varphi_1) - H_0\, r_1 P_1^1(cos\,\theta_1) cos(\varphi_1)\\ + \sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2}\sum_{\nu_2=0}^{\infty}(-1)^{n_2+m_2}\frac{D_{n_2 m_2}\,(n_2+\nu_2)!}{(n_2-m_2)!(\nu_2+m_2)!}\left(\frac{a_2}{d_{12}}\right)^{n_2+1}\left(\frac{r_1}{d_{12}}\right)^{\nu_2} P_{\nu_2}^{m_2}(cos\,\theta_1)\,cos(m_2\varphi_1)\\ + \sum_{n_3=0}^{\infty}\sum_{m_3=-n_3}^{n_3}\sum_{\nu_3=0}^{\infty}(-1)^{n_3+m_3}\frac{D_{n_3 m_3}\,(n_3+\nu_3)!}{(n_3-m_3)!(\nu_3+m_3)!}\left(\frac{a_3}{d_{13}}\right)^{n_3+1}\left(\frac{r_1}{d_{13}}\right)^{\nu_3} P_{\nu_3}^{m_3}(cos\,\theta_1)\,cos(m_3\varphi_1),\\ r_1 \geq a_1, \quad d_{12} \geq r_1, \quad d_{13} \geq r_1\end{aligned} \tag{3.2.32}$$

The condition of zero magnetic flux through the surface of each sphere yields, as before,

$$C_{00} = 0, \quad D_{00} = 0, \quad E_{00} = 0 \tag{3.2.33}$$

The constants of integration satisfy the following equations:

For sphere 1 we have

when $m_1 = 0$

$$\begin{aligned}C_{n_1 0} + \sum_{n_2=1}^{\infty}(-1)^{n_2} D_{n_2 0}\frac{(n_2+n_1)!}{n_2!\,n_1!}\left(\frac{a_2}{d_{12}}\right)^{n_2+1}\left(\frac{a_1}{d_{12}}\right)^{n_1}\\ + \sum_{n_3=1}^{\infty}(-1)^{n_3} E_{n_3 0}\frac{(n_3+n_1)!}{n_3!\,n_1!}\left(\frac{a_3}{d_{13}}\right)^{n_3+1}\left(\frac{a_1}{d_{13}}\right)^{n_1} = 0, \qquad n_1 = 1, 2, 3, \ldots\end{aligned} \tag{3.2.34a}$$

when $m_1 = 1$

$$C_{11}+\sum_{n_2=1}^{\infty}(-1)^{n_2+1}D_{n_2 1}\frac{n_2(n_2+1)}{2}\left(\frac{a_2}{d_{12}}\right)^{n_2+1}\left(\frac{a_1}{d_{12}}\right) + \sum_{n_3=1}^{\infty}(-1)^{n_3+1}E_{n_3 1}\frac{n_3(n_3+1)}{2}\left(\frac{a_3}{d_{13}}\right)^{n_3+1}\left(\frac{a_1}{d_{13}}\right) = a_1 H_0, \qquad n_1 = 1 \tag{3.2.34b}$$

$$C_{n_1 1}+\sum_{n_2=1}^{\infty}(-1)^{n_2+1}D_{n_2 1}\frac{(n_2+n_1)!}{(n_2-1)!\,(n_1+1)!}\left(\frac{a_2}{d_{12}}\right)^{n_2+1}\left(\frac{a_1}{d_{12}}\right)^{n_1} + \sum_{n_3=1}^{\infty}(-1)^{n_3+1}E_{n_3 1}\frac{(n_3+n_1)!}{(n_3-1)!\,(n_1+1)!}\left(\frac{a_3}{d_{13}}\right)^{n_3+1}\left(\frac{a_1}{d_{13}}\right)^{n_1} = 0, \qquad n_1 = 2,3\ldots \tag{3.2.34c}$$

when $m_1 = -1$

$$C_{1,-1}+\sum_{n_2=1}^{\infty}(-1)^{n_2-1}D_{n_2,-1}\left(\frac{a_2}{d_{12}}\right)^{n_2+1}\left(\frac{a_1}{d_{12}}\right) + \sum_{n_3=1}^{\infty}(-1)^{n_3-1}E_{n_3,-1}\left(\frac{a_3}{d_{13}}\right)^{n_3+1}\left(\frac{a_1}{d_{13}}\right) = -2\,a_1 H_0, \qquad n_1 = 1 \tag{3.2.34d}$$

$$C_{n_1,-1}+\sum_{n_2=1}^{\infty}(-1)^{n_2-1}D_{n_2,-1}\frac{(n_2+n_1)!}{(n_2+1)!\,(n_1-1)!}\left(\frac{a_2}{d_{12}}\right)^{n_2+1}\left(\frac{a_1}{d_{12}}\right)^{n_1} + \sum_{n_3=1}^{\infty}(-1)^{n_3-1}E_{n_3,-1}\frac{(n_3+n_1)!}{(n_3+1)!\,(n_1-1)!}\left(\frac{a_3}{d_{13}}\right)^{n_3+1}\left(\frac{a_1}{d_{13}}\right)^{n_1} = 0, \quad n_1 = 2,3\ldots \tag{3.2.34e}$$

Similarly, for sphere 2 we have

for $m_2 = 0$

$$D_{n_2 0}+\sum_{n_1=1}^{\infty}(-1)^{n_2}C_{n_1 0}\frac{(n_1+n_2)!}{n_1!\,n_2!}\left(\frac{a_1}{d_{12}}\right)^{n_1+1}\left(\frac{a_2}{d_{12}}\right)^{n_2} + \sum_{n_3=1}^{\infty}(-1)^{n_3}E_{n_3 0}\frac{(n_3+n_2)!}{n_3!\,n_2!}\left(\frac{a_3}{d_{23}}\right)^{n_3+1}\left(\frac{a_2}{d_{23}}\right)^{n_2} = 0, \qquad n_2 = 1,2,3,\ldots \tag{3.2.35a}$$

for $m_2 = 1$

$$D_{11} + \sum_{n_1=1}^{\infty} C_{n_1 1} \frac{n_1(n_1+1)}{2} \left(\frac{a_1}{d_{12}}\right)^{n_1+1} \left(\frac{a_2}{d_{12}}\right) + \sum_{n_3=1}^{\infty} (-1)^{n_3+1} E_{n_3 1} \frac{n_3(n_3+1)}{2} \left(\frac{a_3}{d_{23}}\right)^{n_3+1} \left(\frac{a_2}{d_{23}}\right) = a_2 H_0, \qquad n_2 = 1 \qquad (3.2.35b)$$

$$D_{n_2 1} + \sum_{n_1=1}^{\infty} (-1)^{n_2+1} C_{n_1 1} \frac{(n_1+n_2)!}{(n_1-1)!\,(n_2+1)!} \left(\frac{a_1}{d_{12}}\right)^{n_1+1} \left(\frac{a_2}{d_{12}}\right)^{n_2} + \sum_{n_3=1}^{\infty} (-1)^{n_3+1} E_{n_3 1} \frac{(n_3+n_2)!}{(n_3-1)!\,(n_2+1)!} \left(\frac{a_3}{d_{23}}\right)^{n_3+1} \left(\frac{a_2}{d_{23}}\right)^{n_2} = 0, \qquad n_2 = 2, 3 \ldots \qquad (3.2.35c)$$

for $m_2 = -1$

$$D_{1,-1} + \sum_{n_1=1}^{\infty} C_{n_1,-1} \left(\frac{a_1}{d_{12}}\right)^{n_1+1} \left(\frac{a_2}{d_{12}}\right) + \sum_{n_3=1}^{\infty} (-1)^{n_3-1} E_{n_3,-1} \left(\frac{a_3}{d_{23}}\right)^{n_3+1} \left(\frac{a_2}{d_{23}}\right) = -2\, a_2 H_0, \qquad n_2 = 1 \qquad (3.2.35d)$$

$$D_{n_2,-1} + \sum_{n_1=1}^{\infty} (-1)^{n_2-1} C_{n_1,-1} \frac{(n_1+n_2)!}{(n_1+1)!\,(n_2-1)!} \left(\frac{a_1}{d_{12}}\right)^{n_1+1} \left(\frac{a_2}{d_{12}}\right)^{n_2} + \sum_{n_3=1}^{\infty} (-1)^{n_3-1} E_{n_3,-1} \frac{(n_3+n_2)!}{(n_3+1)!\,(n_2-1)!} \left(\frac{a_3}{d_{23}}\right)^{n_3+1} \left(\frac{a_2}{d_{23}}\right)^{n_2} = 0, \quad n_2 = 2, 3 \ldots \qquad (3.2.35e)$$

and for sphere 3

for $m_3 = 0$

$$E_{n_3 0} + \sum_{n_1=1}^{\infty} (-1)^{n_3} C_{n_1 0} \frac{(n_1+n_3)!}{n_1!\,n_3!} \left(\frac{a_1}{d_{13}}\right)^{n_1+1} \left(\frac{a_3}{d_{13}}\right)^{n_3} + \sum_{n_2=1}^{\infty} (-1)^{n_3} D_{n_2 0} \frac{(n_2+n_3)!}{n_2!\,n_3!} \left(\frac{a_2}{d_{23}}\right)^{n_2+1} \left(\frac{a_3}{d_{23}}\right)^{n_3} = 0, \qquad n_3 = 1, 2, 3, \ldots \qquad (3.2.36a)$$

for $m_3 = 1$

$$E_{11} + \sum_{n_1=1}^{\infty} C_{n_1 1} \frac{n_1(n_1+1)}{2} \left(\frac{a_1}{d_{13}}\right)^{n_1+1} \left(\frac{a_3}{d_{13}}\right) + \sum_{n_2=1}^{\infty} D_{n_2 1} \frac{n_2(n_2+1)}{2} \left(\frac{a_2}{d_{23}}\right)^{n_2+1} \left(\frac{a_3}{d_{23}}\right) = a_3 H_0, \qquad n_3 = 1 \tag{3.2.36b}$$

$$E_{n_3 1} + \sum_{n_1=1}^{\infty} (-1)^{n_3+1} C_{n_1 1} \frac{(n_1+n_3)!}{(n_1-1)!\,(n_3+1)!} \left(\frac{a_1}{d_{13}}\right)^{n_1+1} \left(\frac{a_3}{d_{13}}\right)^{n_3} + \sum_{n_2=1}^{\infty} (-1)^{n_3+1} D_{n_2 1} \frac{(n_2+n_3)!}{(n_2-1)!\,(n_3+1)!} \left(\frac{a_2}{d_{23}}\right)^{n_2+1} \left(\frac{a_3}{d_{23}}\right)^{n_3} = 0, \qquad n_3 = 2, 3 \ldots \tag{3.2.36c}$$

for $m_3 = -1$

$$E_{1,-1} + \sum_{n_1=1}^{\infty} C_{n_1,-1} \left(\frac{a_1}{d_{13}}\right)^{n_1+1} \left(\frac{a_3}{d_{13}}\right) + \sum_{n_2=1}^{\infty} D_{n_2,-1} \left(\frac{a_2}{d_{23}}\right)^{n_2+1} \left(\frac{a_3}{d_{23}}\right) = -2\, a_3 H_0, \qquad n_3 = 1 \tag{3.2.36d}$$

$$E_{n_3,-1} + \sum_{n_1=1}^{\infty} (-1)^{n_3-1} C_{n_1,-1} \frac{(n_1+n_3)!}{(n_1+1)!\,(n_3-1)!} \left(\frac{a_1}{d_{13}}\right)^{n_1+1} \left(\frac{a_3}{d_{13}}\right)^{n_3} + \sum_{n_2=1}^{\infty} (-1)^{n_3-1} D_{n_2,-1} \frac{(n_2+n_3)!}{(n_2+1)!\,(n_3-1)!} \left(\frac{a_2}{d_{23}}\right)^{n_2+1} \left(\frac{a_3}{d_{23}}\right)^{n_3} = 0, \quad n_3 = 2, 3 \ldots \tag{3.2.36e}$$

For $m_1, m_2, m_3 \neq 0, +1, -1$, we have $C_{n_1 m_1} = 0, D_{n_2 m_2} = 0$ and $E_{n_3 m_3} = 0$.

To obtain numerical results for the constants of integration, the infinite system of equations in (3.2.34), (3.2.35) and (3.2.36) is truncated to retain N terms in each of the infinite series. The $9N \times 9N$ matrix of this truncated system is presented in Equation (3.2.37).

$$
\begin{pmatrix}
1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S1^{(0)}_{1,1} & \cdots & S1^{(0)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S2^{(0)}_{1,1} & \cdots & S2^{(0)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S1^{(0)}_{N,1} & \cdots & S1^{(0)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S2^{(0)}_{N,1} & \cdots & S2^{(0)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & S1^{(1)}_{1,1} & S1^{(1)}_{1,2} & \cdots & S1^{(1)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & S2^{(1)}_{1,1} & S2^{(1)}_{1,2} & \cdots & S2^{(1)}_{1,N} & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & S1^{(1)}_{2,1} & S1^{(1)}_{2,2} & \cdots & S1^{(1)}_{2,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & S2^{(1)}_{2,1} & S2^{(1)}_{2,2} & \cdots & S2^{(1)}_{2,N} & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & S1^{(1)}_{N,1} & S1^{(1)}_{N,2} & \cdots & S1^{(1)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & S2^{(1)}_{N,1} & S2^{(1)}_{N,2} & \cdots & S2^{(1)}_{N,N} & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S1^{(-1)}_{1,1} & S1^{(-1)}_{1,2} & \cdots & S1^{(-1)}_{1,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S2^{(-1)}_{1,1} & S2^{(-1)}_{1,2} & \cdots & S2^{(-1)}_{1,N} \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S1^{(-1)}_{2,1} & S1^{(-1)}_{2,2} & \cdots & S1^{(-1)}_{2,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S2^{(-1)}_{2,1} & S2^{(-1)}_{2,2} & \cdots & S2^{(-1)}_{2,N} \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S1^{(-1)}_{N,1} & S1^{(-1)}_{N,2} & \cdots & S1^{(-1)}_{N,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S2^{(-1)}_{N,1} & S2^{(-1)}_{N,2} & \cdots & S2^{(-1)}_{N,N} \\
T1^{(0)}_{(1,1)} & \cdots & T1^{(0)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T2^{(0)}_{1,1} & \cdots & T2^{(0)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
T1^{(0)}_{(N,1)} & \cdots & T1^{(0)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T2^{(0)}_{N,1} & \cdots & T2^{(0)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & T1^{(1)}_{1,1} & T1^{(1)}_{1,2} & \cdots & T1^{(1)}_{(1,N)} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & T2^{(1)}_{1,1} & T2^{(1)}_{1,2} & \cdots & T2^{(1)}_{1,N} & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & T1^{(1)}_{2,1} & T1^{(1)}_{2,2} & \cdots & T1^{(1)}_{(2,N)} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & T2^{(1)}_{2,1} & T2^{(1)}_{2,2} & \cdots & T2^{(1)}_{2,N} & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & \cdots & 0 & T1^{(1)}_{N,1} & T1^{(1)}_{N,2} & \cdots & T1^{(1)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & T2^{(1)}_{N,1} & T2^{(1)}_{N,2} & \cdots & T2^{(1)}_{N,N} & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T1^{(-1)}_{1,1} & T1^{(-1)}_{1,2} & \cdots & T1^{(-1)}_{1,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T2^{(-1)}_{1,1} & T2^{(-1)}_{1,2} & \cdots & T2^{(-1)}_{1,N} \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T1^{(-1)}_{2,1} & T1^{(-1)}_{2,2} & \cdots & T1^{(-1)}_{2,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T2^{(-1)}_{2,1} & T2^{(-1)}_{2,2} & \cdots & T2^{(-1)}_{2,N} \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T1^{(-1)}_{N,1} & T1^{(-1)}_{N,2} & \cdots & T1^{(-1)}_{N,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T2^{(-1)}_{N,1} & T2^{(-1)}_{N,2} & \cdots & T2^{(-1)}_{N,N} \\
U1^{(0)}_{1,1} & \cdots & U1^{(0)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U2^{(0)}_{1,1} & \cdots & U2^{(0)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
U1^{(0)}_{N,1} & \cdots & U1^{(0)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U2^{(0)}_{N,1} & \cdots & U2^{(0)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & U1^{(1)}_{1,1} & U1^{(1)}_{1,2} & \cdots & U1^{(1)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & U2^{(1)}_{1,1} & U2^{(1)}_{1,2} & \cdots & U2^{(1)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & U1^{(1)}_{2,1} & U1^{(1)}_{2,2} & \cdots & U1^{(1)}_{2,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & U2^{(1)}_{2,1} & U2^{(1)}_{2,2} & \cdots & U2^{(1)}_{2,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & \cdots & 0 & U1^{(1)}_{N,1} & U1^{(1)}_{N,2} & \cdots & U1^{(1)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & U2^{(1)}_{N,1} & U2^{(1)}_{N,2} & \cdots & U2^{(1)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U1^{(-1)}_{1,1} & U1^{(-1)}_{1,2} & \cdots & U1^{(-1)}_{1,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U2^{(-1)}_{1,1} & U2^{(-1)}_{1,2} & \cdots & U2^{(-1)}_{1,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U1^{(-1)}_{2,1} & U1^{(-1)}_{2,2} & \cdots & U1^{(-1)}_{2,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U2^{(-1)}_{2,1} & U2^{(-1)}_{2,2} & \cdots & U2^{(-1)}_{2,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U1^{(-1)}_{N,1} & U1^{(-1)}_{N,2} & \cdots & U1^{(-1)}_{N,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U2^{(-1)}_{N,1} & U2^{(-1)}_{N,2} & \cdots & U2^{(-1)}_{N,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
C_{10} \\ \vdots \\ C_{N0} \\ C_{11} \\ C_{21} \\ \vdots \\ C_{N1} \\ C_{1,-1} \\ C_{2,-1} \\ \vdots \\ C_{N,-1} \\ D_{10} \\ \vdots \\ D_{N0} \\ D_{11} \\ D_{21} \\ \vdots \\ D_{N1} \\ D_{1,-1} \\ D_{2,-1} \\ \vdots \\ D_{N,-1} \\ E_{10} \\ \vdots \\ E_{N0} \\ E_{11} \\ E_{21} \\ \vdots \\ E_{N1} \\ E_{1,-1} \\ E_{2,-1} \\ \vdots \\ E_{N,-1}
\end{pmatrix}
=
\begin{pmatrix}
0 \\ \vdots \\ 0 \\ a_1 \\ 0 \\ \vdots \\ 0 \\ -2\,a_1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \\ -2\,a_2 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ a_3 \\ 0 \\ \vdots \\ 0 \\ -2\,a_3 \\ 0 \\ \vdots \\ 0
\end{pmatrix}
H_0
\qquad (3.2.37)
$$

where $S1, S2, T1, T2, U1$ and $U2$ denote

$$S1^{(m_1)}_{n_1,n_2} = (-1)^{n_2+m_1} \frac{(n_2+n_1)!}{(n_2-m_1)!\,(n_1+m_1)!} \left(\frac{a_2}{d_{12}}\right)^{n_2+1} \left(\frac{a_1}{d_{12}}\right)^{n_1},$$
$$S2^{(m_1)}_{n_1,n_3} = (-1)^{n_3+m_1} \frac{(n_3+n_1)!}{(n_3-m_1)!\,(n_1+m_1)!} \left(\frac{a_3}{d_{13}}\right)^{n_3+1} \left(\frac{a_1}{d_{13}}\right)^{n_1},$$
$$T1^{(m_2)}_{n_2,n_1} = (-1)^{n_2+m_2} \frac{(n_1+n_2)!}{(n_1-m_2)!\,(n_2+m_2)!} \left(\frac{a_1}{d_{12}}\right)^{n_1+1} \left(\frac{a_2}{d_{12}}\right)^{n_2},$$
$$T2^{(m_2)}_{n_2,n_3} = (-1)^{n_3+m_2} \frac{(n_3+n_2)!}{(n_3-m_2)!\,(n_2+m_2)!} \left(\frac{a_3}{d_{23}}\right)^{n_3+1} \left(\frac{a_2}{d_{23}}\right)^{n_2},$$
$$U1^{(m_3)}_{n_3,n_1} = (-1)^{n_3+m_3} \frac{(n_1+n_3)!}{(n_1-m_3)!\,(n_3+m_3)!} \left(\frac{a_1}{d_{13}}\right)^{n_1+1} \left(\frac{a_3}{d_{13}}\right)^{n_3},$$
$$U2^{(m_3)}_{n_3,n_2} = (-1)^{n_3+m_3} \frac{(n_2+n_3)!}{(n_2-m_3)!\,(n_3+m_3)!} \left(\frac{a_2}{d_{23}}\right)^{n_2+1} \left(\frac{a_3}{d_{23}}\right)^{n_3},$$
$$m_1, m_2, m_3 = 0, +1, -1 \quad \text{and} \quad n_1, n_2, n_3 = 1 \ldots N$$

The total potential at any point outside the spheres can be calculated with (see Equation (3.2.27))

$$\begin{aligned}
\Phi^{tot}(r_1,\theta_1,\varphi_1 \mid r_2,\theta_2,\varphi_2 \mid r_3,\theta_3,\varphi_3) = &\sum_{n_1=1}^{N} \sum_{m_1=-n_1}^{n_1} C_{n_1 m_1} \left(\frac{a_1}{r_1}\right)^{n_1+1} P^{m_1}_{n_1}(cos\,\theta_1) cos\,(m_1\varphi_1) \\
&+ \sum_{n_2=1}^{N} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2} \left(\frac{a_2}{r_2}\right)^{n_2+1} P^{m_2}_{n_2}(cos\,\theta_2) cos\,(m_2\varphi_2) \\
&+ \sum_{n_3=1}^{N} \sum_{m_3=-n_3}^{n_3} E_{n_3 m_3} \left(\frac{a_3}{r_3}\right)^{n_3+1} P^{m_3}_{n_3}(cos\,\theta_3) cos\,(m_3\varphi_3) \\
&- H_0 r_1 P^1_1(cos\,\theta_1)\, cos\,\varphi_1, \quad r_1 \geq a_1,\ r_2 \geq a_2,\ r_3 \geq a_3
\end{aligned} \tag{3.2.38}$$

The magnetic field intensity components outside the spheres can be calculated from $\mathbf{H} = -\nabla\Phi^{tot}$. In the system attached to the sphere 1, for instance, these components are

$$
\begin{aligned}
H_{r_1} = &\frac{1}{r_1}\sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1}(n_1+1)\,C_{n_1m_1}\left(\frac{a_1}{r_1}\right)^{n_1+1}P_{n_1}^{m_1}(cos\,\theta_1)\,cos(m_1\varphi_1) + H_0P_1^1(cos\,\theta_1)\,cos\,\varphi_1\\
&+\frac{1}{{r_2}^3}\sum_{n_2=1}^{\infty}\sum_{m_2=-1}^{1}D_{n_2m_2}\left(\frac{a_2}{r_2}\right)^{n_2+1}\Big\{r_2(n_2+1)(r_1-d_{12}\,cos\,\theta_1)P_{n_2}^{m_2}(cos\,\theta_2)\\
&\qquad - r_1d_{12}\,sin^2\theta_1{P_{n_2}^{m_2}}'(cos\,\theta_2)\Big\}\,cos(m_2\varphi_2)\\
&+\frac{1}{{r_3}^3}\sum_{n_3=1}^{\infty}\sum_{m_3=-1}^{1}E_{n_3m_3}\left(\frac{a_3}{r_3}\right)^{n_3+1}\Big\{r_3(n_3+1)(r_1-d_{13}\,cos\,\theta_1)P_{n_3}^{m_3}(cos\,\theta_3)\\
&\qquad - r_1d_{13}\,sin^2\theta_1{P_{n_3}^{m_3}}'(cos\,\theta_3)\Big\}\,cos(m_3\varphi_3),
\end{aligned}
$$

$$
\begin{aligned}
H_{\theta_1} = sin\theta_1\Bigg\{&\frac{1}{r_1}\sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1}C_{n_1m_1}\left(\frac{a_1}{r_1}\right)^{n_1+1}{P_{n_1}^{m_1}}'(cos\,\theta_1)\,cos(m_1\varphi_1) - H_0{P_1^1}'(cos\,\theta_1)\,cos\,\varphi_1\\
&+\frac{1}{{r_2}^3}\sum_{n_2=1}^{\infty}\sum_{m_2=-1}^{1}D_{n_2m_2}\left(\frac{a_2}{r_2}\right)^{n_2+1}\Big\{r_2d_{12}(n_2+1)P_{n_2}^{m_2}(cos\,\theta_2)\\
&\qquad + r_1(r_1-d_{12}\,cos\,\theta_1){P_{n_2}^{m_2}}'(cos\,\theta_2)\Big\}\,cos(m_2\varphi_2)\\
&+\frac{1}{{r_3}^3}\sum_{n_3=1}^{\infty}\sum_{m_3=-1}^{1}E_{n_3m_3}\left(\frac{a_3}{r_3}\right)^{n_3+1}\Big\{r_3d_{13}(n_3+1)P_{n_3}^{m_3}(cos\,\theta_3)\\
&\qquad + r_1(r_1-d_{13}\,cos\,\theta_1){P_{n_3}^{m_3}}'(cos\,\theta_3)\Big\}\,cos(m_3\varphi_3)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
H_{\varphi_1} = \frac{1}{r_1 sin\,\theta_1}\Bigg\{&\sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1}m_1C_{n_1m_1}\left(\frac{a_1}{r_1}\right)^{n_1+1}P_{n_1}^{m_1}(cos\,\theta_1)\,sin\,(m_1\varphi_1) - H_0r_1P_1^1(cos\,\theta_1)\,sin\,\varphi_1\\
&+\sum_{n_2=1}^{\infty}\sum_{m_2=-1}^{1}m_2D_{n_2m_2}\left(\frac{a_2}{r_2}\right)^{n_2+1}P_{n_2}^{m_2}(cos\,\theta_2)\,sin(m_2\varphi_2)\\
&+\sum_{n_3=1}^{\infty}\sum_{m_3=-1}^{1}m_3D_{n_3m_3}\left(\frac{a_3}{r_3}\right)^{n_3+1}P_{n_3}^{m_3}(cos\,\theta_3)\,sin(m_3\varphi_3)\Bigg\},
\end{aligned}
$$

$$
r_1 \geq a_1, \quad r_2 \geq a_2, \quad r_3 \geq a_3 \tag{3.2.39}
$$

The potentials inside the spheres $(r_1 < a_1, r_2 < a_2$ ***and*** $r_3 < a_3)$

As in the previous cases, the potential inside each sphere can be expressed

$$\Psi^{(i)}(r_i,\theta_i,\varphi_i) = \sum_{n_i=0}^{\infty}\sum_{m_i=-n_i}^{n_i} G^{(i)}_{n_i m_i}\left(\frac{r_i}{a_i}\right)^{n_i} P^{m_i}_{n_i}(cos\,\theta_i)\,cos(m_i\varphi_i), \qquad r_i < a_i, \quad i = 1,2,3 \tag{3.2.40}$$

where $G^{(i)}_{n_i m_i}$ are constants of integration. To determine these constants we impose the boundary conditions at the surface of each sphere as

$$-\frac{\partial}{\partial r_i}\left\{\Psi^{(i)}\right\}\Big|_{r_i\,=\,a_i} = -\mu_0\,\frac{\partial}{\partial r_i}\left\{\Phi^{tot}\right\}\Big|_{r_i\,=\,a_i}, \qquad i = 1,2,3 \tag{3.2.41}$$

where μ_0 is the permeability of the medium outside the spheres. This yields the following expressions.

$$G^{(1)}_{n_1 0} = \frac{(2n_1+1)}{n_1}\mu_0\left\{\sum_{n_2=1}^{\infty}(-1)^{n_2}D_{n_2 0}\frac{(n_2+n_1)!}{n_2!\,n_1!}\left(\frac{a_2}{d_{12}}\right)^{n_2+1}\left(\frac{a_1}{d_{12}}\right)^{n_1} + \sum_{n_3=1}^{\infty}(-1)^{n_3}E_{n_3 0}\frac{(n_3+n_1)!}{n_3!\,n_1!}\left(\frac{a_3}{d_{13}}\right)^{n_3+1}\left(\frac{a_1}{d_{13}}\right)^{n_1}\right\}, \quad n_1 = 1,2,3,\ldots \tag{3.2.42a}$$

$$G^{(1)}_{11} = 3\mu_0\left\{\sum_{n_2=1}^{\infty}(-1)^{n_2+1}D_{n_2 1}\frac{n_2(n_2+1)}{2}\left(\frac{a_2}{d_{12}}\right)^{n_2+1}\left(\frac{a_1}{d_{12}}\right) + \sum_{n_3=1}^{\infty}(-1)^{n_3+1}E_{n_3 1}\frac{n_3(n_3+1)}{2}\left(\frac{a_3}{d_{13}}\right)^{n_3+1}\left(\frac{a_1}{d_{13}}\right) - a_1 H_0\right\}, \qquad n_1 = 1 \tag{3.2.42b}$$

$$G^{(1)}_{n_1 1} = \frac{(2n_1+1)}{n_1}\mu_0\left\{\sum_{n_2=1}^{\infty}(-1)^{n_2+1}D_{n_2 1}\frac{(n_2+n_1)!}{(n_2-1)!\,(n_1+1)!}\left(\frac{a_2}{d_{12}}\right)^{n_2+1}\left(\frac{a_1}{d_{12}}\right)^{n_1} + \sum_{n_3=1}^{\infty}(-1)^{n_3+1}E_{n_3 1}\frac{(n_3+n_1)!}{(n_3-1)!\,(n_1+1)!}\left(\frac{a_3}{d_{13}}\right)^{n_3+1}\left(\frac{a_1}{d_{13}}\right)^{n_1}\right\}, \qquad n_1 = 2,3,\ldots \tag{3.2.42c}$$

$$G^{(1)}_{1,-1} = 3\mu_0 \Bigg\{ \sum_{n_2=1}^{\infty} (-1)^{n_2-1} D_{n_2,-1} \left(\frac{a_2}{d_{12}}\right)^{n_2+1} \left(\frac{a_1}{d_{12}}\right) + \sum_{n_3=1}^{\infty} (-1)^{n_3-1} E_{n_3,-1} \left(\frac{a_3}{d_{13}}\right)^{n_3+1} \left(\frac{a_1}{d_{13}}\right) + 2\, a_1 H_0 \Bigg\}, \qquad n_1 = 1 \tag{3.2.42d}$$

$$G^{(1)}_{n_1,-1} = \frac{(2n_1+1)}{n_1} \mu_0 \Bigg\{ \sum_{n_2=1}^{\infty} (-1)^{n_2-1} D_{n_2,-1} \frac{(n_2+n_1)!}{(n_2+1)!\,(n_1-1)!} \left(\frac{a_2}{d_{12}}\right)^{n_2+1} \left(\frac{a_1}{d_{12}}\right)^{n_1} + \sum_{n_3=1}^{\infty} (-1)^{n_3-1} E_{n_3,-1} \frac{(n_3+n_1)!}{(n_3+1)!\,(n_1-1)!} \left(\frac{a_3}{d_{13}}\right)^{n_3+1} \left(\frac{a_1}{d_{13}}\right)^{n_1} \Bigg\}, \quad n_1 = 2, 3, \ldots \tag{3.2.42e}$$

$$G^{(2)}_{n_2 0} = \mu_0 \frac{(2n_2+1)}{n_2} \Bigg\{ \sum_{n_1=1}^{\infty} (-1)^{n_2} C_{n_1 0} \frac{(n_1+n_2)!}{n_1!\, n_2!} \left(\frac{a_1}{d_{12}}\right)^{n_1+1} \left(\frac{a_2}{d_{12}}\right)^{n_2} + \sum_{n_3=1}^{\infty} (-1)^{n_3} E_{n_3 0} \frac{(n_3+n_2)!}{n_3!\, n_2!} \left(\frac{a_3}{d_{23}}\right)^{n_3+1} \left(\frac{a_2}{d_{23}}\right)^{n_2} \Bigg\}, \qquad n_2 = 1, 2, 3, \ldots \tag{3.2.43a}$$

$$G^{(2)}_{11} = 3\mu_0 \Bigg\{ \sum_{n_1=1}^{\infty} C_{n_1 1} \frac{n_1(n_1+1)}{2} \left(\frac{a_1}{d_{12}}\right)^{n_1+1} \left(\frac{a_2}{d_{12}}\right) + \sum_{n_3=1}^{\infty} (-1)^{n_3+1} E_{n_3 1} \frac{n_3(n_3+1)}{2} \left(\frac{a_3}{d_{23}}\right)^{n_3+1} \left(\frac{a_2}{d_{23}}\right) - a_2 H_0 \Bigg\}, \qquad n_2 = 1 \tag{3.2.43b}$$

$$G^{(2)}_{n_2 1} = \frac{(2n_2+1)}{n_2} \mu_0 \Bigg\{ \sum_{n_1=1}^{\infty} (-1)^{n_2+1} C_{n_1 1} \frac{(n_1+n_2)!}{(n_1-1)!\,(n_2+1)!} \left(\frac{a_1}{d_{12}}\right)^{n_1+1} \left(\frac{a_2}{d_{12}}\right)^{n_2} + \sum_{n_3=1}^{\infty} (-1)^{n_3+1} E_{n_3 1} \frac{(n_3+n_2)!}{(n_3-1)!\,(n_2+1)!} \left(\frac{a_3}{d_{23}}\right)^{n_3+1} \left(\frac{a_2}{d_{23}}\right)^{n_2} \Bigg\}, \quad n_2 = 2, 3, \ldots \tag{3.2.43c}$$

$$G^{(2)}_{1,-1} = 3\mu_0 \Bigg\{ \sum_{n_1=1}^{\infty} C_{n_1,-1} \left(\frac{a_1}{d_{12}}\right)^{n_1+1} \left(\frac{a_2}{d_{12}}\right) + \sum_{n_3=1}^{\infty} (-1)^{n_3-1} E_{n_3,-1} \left(\frac{a_3}{d_{23}}\right)^{n_3+1} \left(\frac{a_2}{d_{23}}\right) + 2\, a_2 H_0 \Bigg\}, \qquad n_2 = 1 \tag{3.2.43d}$$

$$G^{(2)}_{n_2,-1} = \frac{(2n_2+1)}{n_2}\mu_0 \Bigg\{ \sum_{n_1=1}^{\infty} (-1)^{n_2-1} C_{n_1,-1} \frac{(n_1+n_2)!}{(n_1+1)!\,(n_2-1)!} \left(\frac{a_1}{d_{12}}\right)^{n_1+1} \left(\frac{a_2}{d_{12}}\right)^{n_2} + \sum_{n_3=1}^{\infty} (-1)^{n_3-1} E_{n_3,-1} \frac{(n_3+n_2)!}{(n_3+1)!\,(n_2-1)!} \left(\frac{a_3}{d_{23}}\right)^{n_3+1} \left(\frac{a_2}{d_{23}}\right)^{n_2}, \quad n_2 = 2,3,\ldots \tag{3.2.43e}$$

$$G^{(3)}_{n_3 0} = \mu_0\, \frac{(2n_3+1)}{n_3} \Bigg\{ \sum_{n_1=1}^{\infty} (-1)^{n_3} C_{n_1 0} \frac{(n_1+n_3)!}{n_1!\, n_3!} \left(\frac{a_1}{d_{13}}\right)^{n_1+1} \left(\frac{a_3}{d_{13}}\right)^{n_3} + \sum_{n_2=1}^{\infty} (-1)^{n_3} D_{n_2 0} \frac{(n_2+n_3)!}{n_2!\, n_3!} \left(\frac{a_2}{d_{23}}\right)^{n_2+1} \left(\frac{a_3}{d_{23}}\right)^{n_3} \Bigg\}, \qquad n_3 = 1,2,3,\ldots \tag{3.2.44a}$$

$$G^{(3)}_{11} = 3\mu_0 \Bigg\{ \sum_{n_1=1}^{\infty} C_{n_1 1} \frac{n_1(n_1+1)}{2} \left(\frac{a_1}{d_{13}}\right)^{n_1+1} \left(\frac{a_3}{d_{13}}\right) + \sum_{n_2=1}^{\infty} D_{n_2 1} \frac{n_2(n_2+1)}{2} \left(\frac{a_2}{d_{23}}\right)^{n_2+1} \left(\frac{a_3}{d_{23}}\right) - a_3 H_0 \Bigg\}, \qquad n_3 = 1 \tag{3.2.44b}$$

$$G^{(3)}_{n_3 1} = \frac{(2n_3+1)}{n_3}\mu_0 \Bigg\{ \sum_{n_1=1}^{\infty} (-1)^{n_3+1} C_{n_1 1} \frac{(n_1+n_3)!}{(n_1-1)!\,(n_3+1)!} \left(\frac{a_1}{d_{13}}\right)^{n_1+1} \left(\frac{a_3}{d_{13}}\right)^{n_3} + \sum_{n_2=1}^{\infty} (-1)^{n_3+1} D_{n_2 1} \frac{(n_2+n_3)!}{(n_2-1)!\,(n_3+1)!} \left(\frac{a_2}{d_{23}}\right)^{n_2+1} \left(\frac{a_3}{d_{23}}\right)^{n_3} \Bigg\}, \quad n_3 = 2,3,\ldots \tag{3.2.44c}$$

$$G^{(3)}_{1,-1} = 3\mu_0 \Bigg\{ \sum_{n_1=1}^{\infty} C_{n_1,-1} \left(\frac{a_1}{d_{13}}\right)^{n_1+1} \left(\frac{a_3}{d_{13}}\right) + \sum_{n_2=1}^{\infty} D_{n_2,-1} \left(\frac{a_2}{d_{23}}\right)^{n_2+1} \left(\frac{a_3}{d_{23}}\right) + 2\, a_3 H_0 \Bigg\}, \qquad n_3 = 1 \tag{3.2.44d}$$

$$G^{(3)}_{n_3,-1} = \frac{(2n_3+1)}{n_3} \mu_0 \Bigg\{ \sum_{n_1=1}^{\infty} (-1)^{n_3-1} C_{n_1,-1} \frac{(n_1+n_3)!}{(n_1+1)!\,(n_3-1)!} \left(\frac{a_1}{d_{13}}\right)^{n_1+1} \left(\frac{a_3}{d_{13}}\right)^{n_3} + \sum_{n_2=1}^{\infty} (-1)^{n_3-1} D_{n_2,-1} \frac{(n_2+n_3)!}{(n_2+1)!\,(n_3-1)!} \left(\frac{a_2}{d_{23}}\right)^{n_2+1} \left(\frac{a_3}{d_{23}}\right)^{n_3}, \quad n_3 = 2, 3, \ldots \tag{3.2.44e}$$

$C_{n_1 m_1} = D_{n_2 m_2} = E_{n_3 m_3} = 0$ for $m_1, m_2, m_3 \neq 0, +1, -1$.

The components of the magnetic flux density inside each sphere are calculated from $\mathbf{B}^{(i)} = -\nabla \Psi^{(i)}$, $i = 1, 2, 3$, using the corresponding Equation (3.2.40). Inside the sphere 1, for instance,

$$\begin{aligned}
B_{r_1} &= -\frac{1}{r_1} \sum_{n_1=1}^{\infty} \sum_{m_1=-1}^{1} n_1 G^{(1)}_{n_1 m_1} \left(\frac{r_1}{a_1}\right)^{n_1} P^{m_1}_{n_1}(cos\,\theta_1)\, cos(m_1 \varphi_1), \\
B_{\theta_1} &= \frac{sin\,\theta_1}{r_1} \sum_{n_1=1}^{\infty} \sum_{m_1=-1}^{1} G^{(1)}_{n_1 m_1} \left(\frac{r_1}{a_1}\right)^{n_1} P^{m_1\prime}_{n_1}(cos\,\theta_1)\, cos(m_1 \varphi_1), \\
B_{\varphi_1} &= \frac{1}{r_1 sin\,\theta_1} \sum_{n_1=1}^{\infty} \sum_{m_1=-1}^{1} m_1 G^{(1)}_{n_1 m_1} \left(\frac{r_1}{a_1}\right)^{n_1} P^{m_1}_{n_1}(cos\,\theta_1)\, sin(m_1 \varphi_1), \qquad r_1 < a_1
\end{aligned} \tag{3.2.45}$$

3.3 Chapter Summary

In this chapter, analytical expressions were derived for the magnetic scalar potential and the field quantities both inside and outside coaxial ideal ferromagnetic spheres, placed in the presence of uniform magnetic fields. The total scalar magnetic potential outside the spheres was obtained by the superposition of the potentials due to each sphere and the potential corresponding to the external field. The translational addition theorems were used in order to express the total potential in the coordinate system attached to a specific sphere. The corresponding boundary conditions were imposed to derive the constants of integration.

Chapter 4

Arbitrarily Located Ferromagnetic Spheres in External Magnetic Fields

In this chapter, the scalar magnetic potential is employed to determine the field both outside and inside two-sphere and three-sphere systems placed in arbitrary locations in the presence of external magnetic fields. The procedures in chapter 3 is extended to the analysis of these more general geometries.

4.1 Two-Sphere System

A system of two ferromagnetic spheres of radii a_1 and a_2 with a distance d between their centers are placed in a homogeneous medium of permeability μ_0, as shown in

Figure 4.1. $(r_1, \theta_1, \varphi_1)$ and $(r_2, \theta_2, \varphi_2)$ are the spherical coordinates attached to the sphere 1 and sphere 2, respectively.

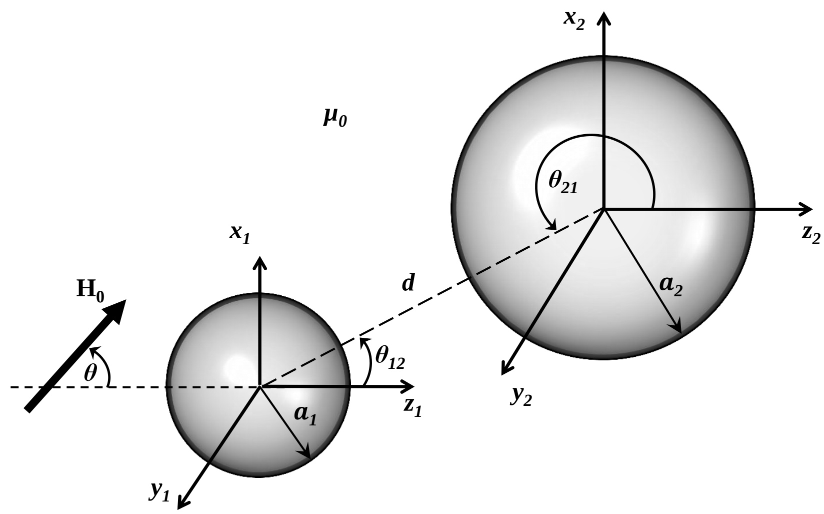

FIGURE 4.1: Two ferromagnetic spheres with common $z - x$ planes placed in an arbitrary position in the presence of external magnetic field

4.1.1 Case I: The Magnetic Field along z Axis

Consider the external field to be oriented parallel to the z axis (i.e. $\mathbf{H_0} = H_0\hat{z}$) of the spheres as shown in Figure 4.2.

The potentials outside the spheres $(r_1 \geq a_1$ ***and*** $r_2 \geq a_2)$

Take $\mathbf{P}$ to be an arbitrary point with $(a_1 \leq r_1 \leq d)$ and $(a_2 \leq r_2 \leq d)$. The total scalar magnetic potential at $\mathbf{P}$ due to the two spheres and to the potential corresponding to the external field can be expressed as

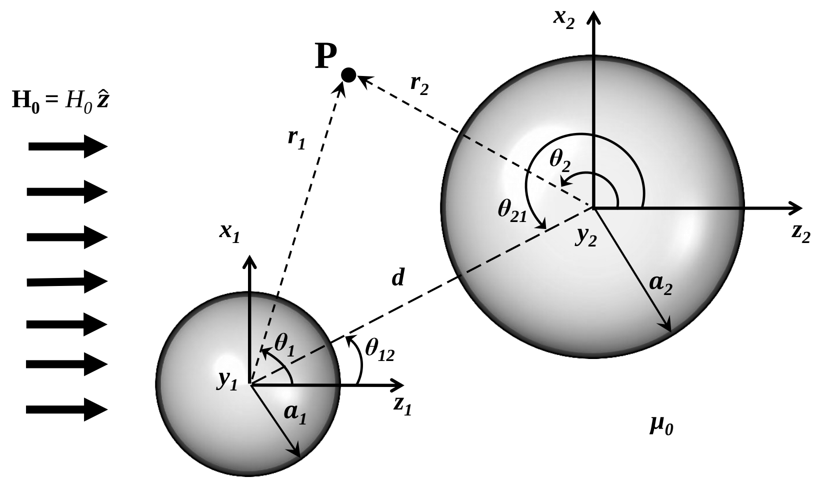

FIGURE 4.2: The two-sphere system with same $z - x$ planes placed in an arbitrary positions in the presence of external field along z axis

$$\Phi^{tot}(r_1, \theta_1, \varphi_1 \mid r_2, \theta_2, \varphi_2) = \Phi_1(r_1, \theta_1, \varphi_1) + \Phi_2(r_2, \theta_2, \varphi_2) + \Phi_{ext(z)}(r_1, \theta_1, \varphi_1) \qquad (4.1.1)$$

where $\Phi_1(r_1, \theta_1, \varphi_1)$ and $\Phi_2(r_2, \theta_2, \varphi_2)$ are the potentials produced by sphere 1 and sphere 2, respectively. $\Phi_{ext(z)}(r_1, \theta_1, \varphi_1)$ is the potential corresponding to the external field in spherical coordinate $(r_1, \theta_1, \varphi_1)$ attached to sphere 1.

By using the Equation (2.1.3) with $r = r_1, \theta = \theta_1$ and $\varphi = \varphi_1$ for sphere 1, we have

$$\Phi_1(r_1, \theta_1, \varphi_1) = \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1 m_1} \left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}^{m_1}(cos\,\theta_1)\, e^{-jm_1\varphi_1}, \quad r_1 \geq a_1 \qquad (4.1.2)$$

For sphere 2 with $r = r_2, \theta = \theta_2$ and $\varphi = \varphi_2$, we have

$$\Phi_2(r_2, \theta_2, \varphi_2) = \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2} \left(\frac{a_2}{r_2}\right)^{n_2+1} P_{n_2}^{m_2}(cos\,\theta_2)\, e^{-jm_2\varphi_2}, \quad r_2 \geq a_2 \qquad (4.1.3)$$

where $C_{n_1 m_1}$ and $D_{n_2 m_2}$ are constants of integration. We translate $\Phi_2(r_2, \theta_2, \varphi_2)$ into the coordinate $(r_1, \theta_1, \varphi_1)$ and, then, the total scalar magnetic potential at **P** in spherical coordinates $(r_1, \theta_1, \varphi_1)$ can be expressed as

$$\Phi^{tot}(r_1, \theta_1, \varphi_1) = \Phi_1(r_1, \theta_1, \varphi_1) + \Phi_2^{(1)}(r_1, \theta_1, \varphi_1) + \Phi_{ext(z)}(r_1, \theta_1, \varphi_1) \tag{4.1.4}$$

In order to find $\Phi_2^{(1)}$, we use the translational addition theorem in Equation (2.2.1), with $r' = r_1, \theta' = \theta_1, \varphi' = \varphi_1, \theta_0 = (\pi - \theta_{12}) = \theta_{21}$ and $\varphi_0 = \varphi_{21}$, which gives

$$\begin{aligned}\Phi_2^{(1)}(r_1, \theta_1, \varphi_1) = \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} \sum_{\nu_1=0}^{\infty} \sum_{\mu_1=-\nu_1}^{\nu_1} (-1)^{\mu_1+\nu_1} D_{n_2 m_2} \frac{(n_2 - m_2 + \nu_1 + \mu_1)!}{(n_2 - m_2)!(\nu_1 + \mu_1)!} \left(\frac{a_2}{d}\right)^{n_2+1} \\ \times \left(\frac{r_1}{d}\right)^{\nu_1} P_{n_2+\nu_1}^{m_2-\mu_1}(\cos\theta_{21})\, e^{-j(m_2-\mu_1)\varphi_{21}}\, P_{\nu_1}^{\mu_1}(\cos\theta_1)\, e^{-j\mu_1\varphi_1}, \qquad r_1 \leq d\end{aligned} \tag{4.1.5}$$

The potential corresponding to the external field is expressed as

$$\Phi_{ext(z)}(r_1, \theta_1, \varphi_1) = -H_0 z_1 + K_1$$

$$\Phi_{ext(z)}(r_2, \theta_2, \varphi_2) = -H_0 z_2 + K_2$$

where K_1 and K_2 are constants of references. At $z_1 = 0$, we choose $\Phi_{ext(z)} = 0$ as reference. Then $K_1 = 0$ and $K_2 = -H_0\, d \cos\theta_{12}$, are substituting with $z_1 = r_1 P_1(\cos\theta_1)$ and $z_2 = r_2 P_1(\cos\theta_2)$, we have

$$\Phi_{ext(z)}(r_1, \theta_1, \varphi_1) = -H_0\, r_1 P_1(\cos\theta_1) \tag{4.1.7a}$$

$$\Phi_{ext(z)}(r_2, \theta_2, \varphi_2) = -H_0\, r_2 P_1(\cos\theta_2) - H_0\, d \cos\theta_{12} \tag{4.1.7b}$$

Thus,

$$\begin{aligned}\Phi^{tot}(r_1, \theta_1, \varphi_1) = \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1 m_1} \left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}^{m_1}(\cos\theta_1)\, e^{-jm_1\varphi_1} \\ + \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} \sum_{\nu_1=0}^{\infty} \sum_{\mu_1=-\nu_1}^{\nu_1} D_{n_2 m_2}\, \zeta_2^{(1)}(m_2, n_2 \mid \mu_1, \nu_1 \mid d, \theta_{21}, \varphi_{21}) \\ \times \left(\frac{r_1}{d}\right)^{\nu_1} P_{\nu_1}^{\mu_1}(\cos\theta_1)\, e^{-j\mu_1\varphi_1} - H_0\, r_1 P_1(\cos\theta_1), \quad a_1 \leq r_1 \leq d\end{aligned} \tag{4.1.8}$$

where

$$\zeta_2^{(1)}(m_2, n_2 \mid \mu_1, \nu_1 \mid d, \theta_{21}, \varphi_{21}) \equiv (-1)^{\mu_1+\nu_1} \frac{(n_2 - m_2 + \nu_1 + \mu_1)!}{(n_2 - m_2)!(\nu_1 + \mu_1)!} \left(\frac{a_2}{d}\right)^{n_2+1}$$
$$\times P_{n_2+\nu_1}^{m_2-\mu_1}(cos\,\theta_{21})\, e^{-j(m_2-\mu_1)\varphi_{21}}$$

To determine the series of coefficients in (4.1.8) we first impose the condition that the total magnetic flux through the surface of each sphere is equal to zero. This yields

$$C_{00} = 0, \quad D_{00} = 0 \tag{4.1.9}$$

Secondly, we impose the condition that the surfaces of the two spheres have constant potentials, V_1 and V_2, respectively, $\Phi^{tot}(r_i, \theta_i, \varphi_i)\Big|_{r_i\, =\, a_i} = V_i, \quad i = 1, 2.$

Employing the orthoganility properties of the spherical harmonics leads to

$$C_{10} + \sum_{n_2=1}^{\infty} D_{n_2 0} \left(\frac{a_1}{d}\right) \zeta_2^{(1)}(0, n_2 \mid 0, 1 \mid d, \theta_{21}, \varphi_{21}) = a_1 H_0, \quad n_1 = 1 \tag{4.1.10a}$$

$$C_{n_1 0} + \sum_{n_2=1}^{\infty} D_{n_2 0} \left(\frac{a_1}{d}\right)^{n_1} \zeta_2^{(1)}(0, n_2 \mid 0, n_1 \mid d, \theta_{21}, \varphi_{21}) = 0, \qquad n_1 = 2, 3, \ldots \tag{4.1.10b}$$

for sphere 1 and, similarly, for sphere 2

$$D_{10} + \sum_{n_1=1}^{\infty} C_{n_1 0} \left(\frac{a_2}{d}\right) \zeta_1^{(2)}(0, n_1 \mid 0, 1 \mid d, \theta_{12}, \varphi_{12}) = a_2 H_0, \quad n_2 = 1 \tag{4.1.11a}$$

$$D_{n_2 0} + \sum_{n_1=1}^{\infty} C_{n_1 0} \left(\frac{a_2}{d}\right)^{n_2} \zeta_1^{(2)}(0, n_1 \mid 0, n_2 \mid d, \theta_{12}, \varphi_{12}) = 0, \qquad n_2 = 2, 3, \ldots \tag{4.1.11b}$$

where

$$\zeta_1^{(2)}(m_1, n_1 \mid m_2, n_2 \mid d, \theta_{12}, \varphi_{12}) \equiv (-1)^{m_2+n_2} \frac{(n_1 - m_1 + n_2 + m_2)!}{(n_1 - m_1)!(n_2 + m_2)!} \left(\frac{a_1}{d}\right)^{n_1+1}$$
$$\times\, P_{n_1+n_2}^{m_1-m_2}(cos\,\theta_{12})\, e^{-j(m_1-m_2)\varphi_{12}}$$

For $m_1, m_2 \neq 0$, we obtain $C_{n_1 m_1} = 0$ and $D_{n_2 m_2} = 0$. In order to obtain numerical results, this infinite set of equations is truncated to N terms in their series, $n_1, n_2 = 1 \ldots N$. Denoting

$$S_{n_2}^{(n_1)} \equiv \zeta_2^{(1)}(0, n_2 \mid 0, n_1 \mid d, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d}\right)^{n_1},$$
$$T_{n_1}^{(n_2)} \equiv \zeta_1^{(2)}(0, n_1 \mid 0, n_2 \mid d, \theta_{12}, \varphi_{12}) \left(\frac{a_2}{d}\right)^{n_2},$$

the unknown coefficients in (4.1.10) and (4.1.11) are obtained by solving the matrix equation

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & S_1^{(1)} & S_2^{(1)} & \cdots & S_N^{(1)} \\ 0 & 1 & \cdots & 0 & S_1^{(2)} & S_2^{(2)} & \cdots & S_N^{(2)} \\ & & \ddots & & & & \ddots & \\ 0 & 0 & \cdots & 1 & S_1^{(N)} & S_2^{(N)} & \cdots & S_N^{(N)} \\ T_1^{(1)} & T_2^{(1)} & \cdots & T_N^{(1)} & 1 & 0 & \cdots & 0 \\ T_1^{(2)} & T_2^{(2)} & \cdots & T_N^{(2)} & 0 & 1 & \cdots & 0 \\ & & \ddots & & & & \ddots & \\ T_1^{(N)} & T_2^{(N)} & \cdots & T_N^{(N)} & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} C_{10} \\ C_{20} \\ \vdots \\ C_{N0} \\ D_{10} \\ D_{20} \\ \vdots \\ D_{N0} \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \\ \vdots \\ 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} H_0 \qquad (4.1.13)$$

The total scalar magnetic potential outside the spheres can be calculated in spherical coordinates using the Equation (4.1.1) as

$$\begin{aligned} \Phi^{tot}(r_1, \theta_1, \varphi_1 \mid r_2, \theta_2, \varphi_2) = & \sum_{n_1=1}^{N} C_{n_1 0} \left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}(cos\,\theta_1) - H_0\, r_1 P_1(cos\,\theta_1) \\ & + \sum_{n_2=1}^{N} D_{n_2 0} \left(\frac{a_2}{r_2}\right)^{n_2+1} P_{n_2}(cos\,\theta_2), \qquad r_1 \geq a_1, \quad r_2 \geq a_2 \end{aligned} \qquad (4.1.14)$$

From this equation, with the geometrical relations in A.2, the components of the magnetic field intensity in the system in $(r_1, \theta_1, \varphi_1)$ are derived in the form

$$\begin{aligned}
H_{r_1} =& \frac{1}{r_1} \sum_{n_1=1}^{\infty} (n_1+1)\, C_{n_1 0} \left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}(cos\,\theta_1) + H_0 P_1(cos\,\theta_1) \\
& + \frac{1}{r_2^3} \sum_{n_2=1}^{\infty} D_{n_2 0} \left(\frac{a_2}{r_2}\right)^{n_2+1} \Bigg\{ r_2(n_2+1)\Big(r_1 - d\,cos(\theta_1-\theta_{12})\Big) P_{n_2}(cos\,\theta_2) \\
& + \Bigg\{ \Big(r_1 d\,cos(\theta_1-\theta_{12}) - d^2\Big)\, cos\,\theta_1 - \Big(r_1 - d\,cos(\theta_1-\theta_{12})\Big)\, d\,cos\,\theta_{12} \Bigg\} P'_{n_2}(cos\,\theta_2) \Bigg\}, \\
H_{\theta_1} =& \frac{sin\,\theta_1}{r_1} \sum_{n_1=1}^{\infty} C_{n_1 0} \left(\frac{a_1}{r_1}\right)^{n_1+1} P'_{n_1}(cos\,\theta_1) - H_0\, sin\,\theta_1 P'_1(cos\,\theta_1) \\
& + \frac{1}{r_2^3} \sum_{n_2=1}^{\infty} D_{n_2 0} \left(\frac{a_2}{r_2}\right)^{n_2+1} \Bigg\{ r_2\, d\,(n_2+1)\, sin(\theta_1-\theta_{12})\, P_{n_2}(cos\,\theta_2) \\
& \qquad + \Bigg\{ r_2^2\, sin\,\theta_1 + d\,sin(\theta_1-\theta_{12})\Big(r_1\, cos\,\theta_1 - d\,cos\,\theta_{12}\Big) \Bigg\} P'_{n_2}(cos\,\theta_2) \Bigg\}, \\
H_{\varphi_1} =& 0, \qquad\qquad r_1 \geq a_1, \quad r_2 \geq a_2, \quad d \geq (a_1+a_2)
\end{aligned} \tag{4.1.15}$$

The potentials inside the spheres $(r_1 < a_1$ ***and*** $r_2 < a_2)$

The magnetic potential Ψ inside the sphere ($\mathbf{B} = -\nabla\Psi$), satisfies the Laplace equation and can be expressed as

$$\Psi^{(1)}(r_1,\theta_1,\varphi_1) = \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} G^{(1)}_{n_1 m_1} \left(\frac{r_1}{a_1}\right)^{n_1} P^{m_1}_{n_1}(cos\,\theta_1) \tag{4.1.16a}$$

$$\Psi^{(2)}(r_2,\theta_2,\varphi_2) = \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} G^{(2)}_{n_2 m_2} \left(\frac{r_2}{a_2}\right)^{n_2} P^{m_2}_{n_2}(cos\,\theta_2) \tag{4.1.16b}$$

where $G^{(1)}_{n_1 m_1}$ and $G^{(2)}_{n_2 m_2}$ are constants of integration.

Imposing the boundary conditions

$$-\frac{\partial}{\partial\, r_i}\Big\{\Psi^{(i)}(r_i,\theta_i,\varphi_i)\Big\}\Bigg|_{r_i\,=\,a_i} = -\mu_0\, \frac{\partial}{\partial\, r_i}\Big\{\Phi^{tot}(r_i,\theta_i,\varphi_i)\Big\}\Bigg|_{r_i\,=\,a_i}, \quad i=1,2 \tag{4.1.17}$$

where μ_0 is the permeability of the medium outside the spheres, yields finally

$$G_{10}^{(1)} = 3\mu_0 \left\{ \sum_{n_2=1}^{\infty} D_{n_2 0} \left(\frac{a_1}{d} \right) \zeta_2^{(1)} (0, n_2 \mid 0, 1 \mid d, \theta_{21}, \varphi_{21}) - a_1 H_0 \right\}, \quad n_1 = 1 \quad (4.1.18a)$$

$$G_{n_1 0}^{(1)} = \frac{(2n_1 + 1)}{n_1} \mu_0 \sum_{n_2=1}^{\infty} D_{n_2 0} \left(\frac{a_1}{d} \right)^{n_1} \zeta_2^{(1)} (0, n_2 \mid 0, n_1 \mid d, \theta_{21}, \varphi_{21}), \quad n_1 = 2, 3, \ldots \quad (4.1.18b)$$

$$G_{10}^{(2)} = 3\mu_0 \left\{ \sum_{n_1=1}^{\infty} C_{n_1 0} \left(\frac{a_2}{d} \right) \zeta_1^{(2)} (0, n_1 \mid 0, 1 \mid d, \theta_{12}, \varphi_{12}) - a_2 H_0 \right\}, \quad n_2 = 1 \quad (4.1.19a)$$

$$G_{n_2 0}^{(2)} = \frac{(2n_2 + 1)}{n_2} \mu_0 \sum_{n_1=1}^{\infty} C_{n_1 0} \left(\frac{a_2}{d} \right)^{n_2} \zeta_1^{(2)} (0, n_1 \mid 0, n_2 \mid d, \theta_{12}, \varphi_{12}), \quad n_2 = 2, 3, \ldots \quad (4.1.19b)$$

The magnetic flux density has the components

$$\begin{aligned} B_{r_1} &= -\frac{1}{r_1} \sum_{n_1=1}^{\infty} n_1 \, G_{n_1 0}^{(1)} \left(\frac{r_1}{a_1} \right)^{n_1} P_{n_1}(cos\,\theta_1), \\ B_{\theta_1} &= \frac{sin\,\theta_1}{r_1} \sum_{n_1=1}^{\infty} G_{n_1 0}^{(1)} \left(\frac{r_1}{a_1} \right)^{n_1} P'_{n_1}(cos\,\theta_1), \\ B_{\varphi_1} &= 0, \qquad\qquad r_1 < a_1 \end{aligned} \quad (4.1.20)$$

for sphere 1, and

$$\begin{aligned} B_{r_2} &= -\frac{1}{r_2} \sum_{n_2=1}^{\infty} n_2 \, G_{n_2 0}^{(2)} \left(\frac{r_2}{a_2} \right)^{n_2} P_{n_2}(cos\,\theta_2), \\ B_{\theta_2} &= \frac{sin\,\theta_2}{r_2} \sum_{n_2=1}^{\infty} G_{n_2 0}^{(2)} \left(\frac{r_2}{a_2} \right)^{n_2} P'_{n_2}(cos\,\theta_2), \\ B_{\varphi_2} &= 0, \qquad\qquad r_2 < a_2 \end{aligned} \quad (4.1.21)$$

for sphere 2.

4.1.2 Case II: The External Magnetic Field along x Axis

Figure 4.3 shows the two spheres in the presence of a uniform external field directed along the x axis.

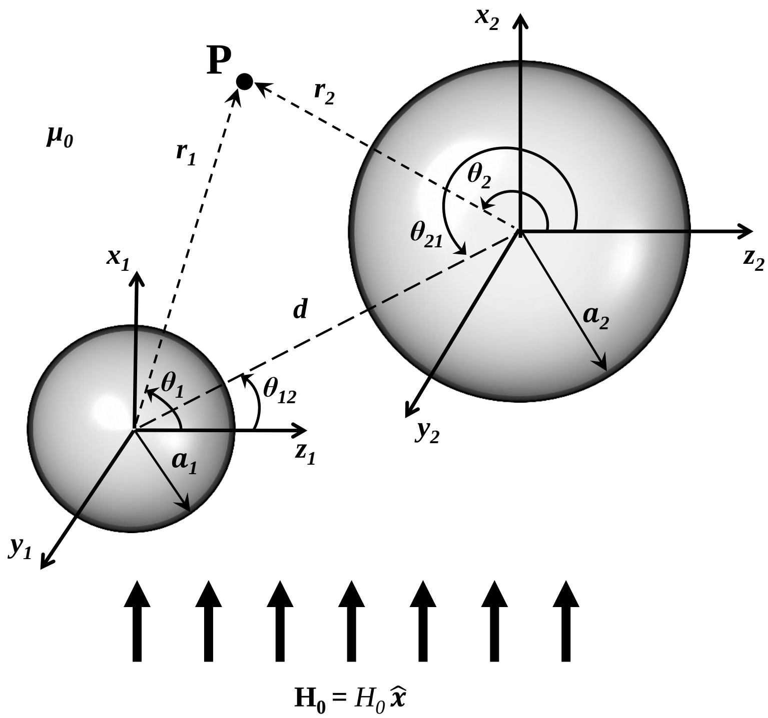

FIGURE 4.3: Two ferromagnetic spheres with same $z - x$ planes placed in an arbitrary positions in the presence of external field oriented along x axis

The potentials outside the spheres, $(r_1 \geq a_1$ ***and*** $r_2 \geq a_2)$

Again, the total scalar magnetic potential can be expressed in spherical coordinates as

$$\Phi^{tot}(r_1, \theta_1, \varphi_1 \mid r_2, \theta_2, \varphi_2) = \Phi_1(r_1, \theta_1, \varphi_1) + \Phi_2(r_2, \theta_2, \varphi_2) + \Phi_{ext(x)}(r_1, \theta_1, \varphi_1), \quad (4.1.22)$$

Taking the real part in the Equation (2.1.3), with $r = r_1, \theta = \theta_1$ and $\varphi = \varphi_1$ for sphere 1, we have

$$\Phi_1(r_1, \theta_1, \varphi_1) = \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1 m_1} \left(\frac{a_1}{r_1}\right)^{(n_1+1)} P_{n_1}^{m_1}(cos\,\theta_1)\,cos\,m_1\varphi_1, \qquad r_1 \geq a_1 \tag{4.1.23}$$

For sphere 2 with $r = r_2, \theta = \theta_2$ and $\varphi = \varphi_2$, we have

$$\Phi_2(r_2, \theta_2, \varphi_2) = \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2} \left(\frac{a_2}{r_2}\right)^{(n_2+1)} P_{n_2}^{m_2}(cos\,\theta_2)\,cos\,m_2\varphi_2, \qquad r_2 \geq a_2 \tag{4.1.24}$$

The potential corresponding to the external field is

$$\Phi_{ext(x)}(r_i, \theta_i, \varphi_i) = -H_0\,x_i + K_i \qquad i = 1, 2 \tag{4.1.25}$$

where K_i is a constant choosing $\Phi_{ext(x)} = 0$ at $x_1 = 0$ as reference, $K_1 = 0$ and $K_2 = -H_0\,d\,sin\,\theta_{12}\,cos\,\varphi_{12}$, and with $x_1 = r_1 sin\,\theta_1\,cos\,\varphi_1$ and $x_2 = r_2 sin\,\theta_2\,cos\,\varphi_2$ we have

$$\Phi_{ext(x)}(r_1, \theta_1, \varphi_1) = -H_0\,r_1\,P_1^1(cos\,\theta_1)\,cos\,\varphi_1 \tag{4.1.26a}$$

$$\Phi_{ext(x)}(r_2, \theta_2, \varphi_2) = -H_0\,r_2\,P_1^1(cos\,\theta_2)\,cos\,\varphi_2 - H_0\,d\,sin\,\theta_{12}\,cos\,\varphi_{12} \tag{4.1.26b}$$

Applying the translational addition theorems in Section (2.2.1), the Equation (4.1.22) can be expressed in spherical coordinate $(r_1, \theta_1, \varphi_1)$ as

$$\Phi^{tot}(r_1, \theta_1, \varphi_1) = \Phi_1(r_1, \theta_1, \varphi_1) + \Phi_2^{(1)}(r_1, \theta_1, \varphi_1) + \Phi_{ext(x)}(r_1, \theta_1, \varphi_1) \tag{4.1.27}$$

where

$$\Phi_2^{(1)}(r_1,\theta_1,\varphi_1) = \sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2}\sum_{\nu_1=0}^{\infty}\sum_{\mu_1=-\nu_1}^{\nu_1}(-1)^{\mu_1+\nu_1}D_{n_2m_2}\frac{(n_2-m_2+\nu_1+\mu_1)!}{(n_2-m_2)!(\nu_1+\mu_1)!}$$
$$\times\left(\frac{a_2}{d}\right)^{n_2+1}\left(\frac{r_1}{d}\right)^{\nu_1}P_{n_2+\nu_1}^{m_2-\mu_1}(cos\,\theta_{21})\,P_{\nu_1}^{\mu_1}(cos\,\theta_1)\,cos\big((m_2-\mu_1)\varphi_{21}+\mu_1\varphi_1\big), \tag{4.1.28}$$
$$r_1 \le d$$

Then, the total scalar magnetic potential at P in spherical coordinate (r_1,θ_1,φ_1) can be expressed as

$$\Phi^{tot}(r_1,\theta_1,\varphi_1) = \sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}C_{n_1m_1}\left(\frac{a_1}{r_1}\right)^{n_1+1}P_{n_1}^{m_1}(cos\,\theta_1)\,cos(m_1\varphi_1)$$
$$+\sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2}\sum_{\nu_1=0}^{\infty}\sum_{\mu_1=-\nu_1}^{\nu_1}D_{n_2m_2}\,\zeta_2^{(1)}(m_2,n_2\mid\mu_1,\nu_1\mid d,\theta_{21},\varphi_{21})\left(\frac{r_1}{d}\right)^{\nu_1} \tag{4.1.29}$$
$$\times P_{\nu_1}^{\mu_1}(cos\,\theta_1) - H_0\,r_1P_1^1(cos\,\theta_1)\,cos\,\varphi_1, \qquad a_1 \le r_1 \le d$$

where

$$\zeta_2^{(1)}(m_2,n_2\mid\mu_1,\nu_1\mid d,\theta_{21},\varphi_{21}) \equiv (-1)^{\mu_1+\nu_1}\frac{(n_2-m_2+\nu_1+\mu_1)!}{(n_2-m_2)!(\nu_1+\mu_1)!}\left(\frac{a_2}{d}\right)^{n_2+1}$$
$$\times P_{n_2+\nu_1}^{m_2-\mu_1}(cos\,\theta_{21})\,cos\big((m_2-\mu_1)\varphi_{21}+\mu_1\varphi_1\big)$$

The same condition of zero magnetic flux for each sphere and the same boundary conditions yield in this case

$$C_{00}=0, \quad D_{00}=0 \tag{4.1.30}$$

and

$$C_{n_10}+\sum_{n_2=1}^{\infty}\sum_{m_2=-n_2}^{n_2}D_{n_2m_2}\,\zeta_2^{(1)}(m_2,n_2\mid 0,n_1\mid d,\theta_{21},\varphi_{21})\left(\frac{a_1}{d}\right)^{n_1}=0,$$
$$n_1 = 1,2,3,\ldots \tag{4.1.31a}$$

$$C_{11} + \sum_{n_2=1}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2}\, \zeta_2^{(1)}(m_2, n_2 \mid 1, 1 \mid d, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d}\right) = -\frac{1}{2} a_1 H_0, \quad n_1 = 1 \tag{4.1.31b}$$

$$C_{n_1 1} + \sum_{n_2=1}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2}\, \zeta_2^{(1)}(m_2, n_2 \mid 1, n_1 \mid d, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d}\right)^{n_1} = 0, \quad n_1 = 2, 3, \ldots \tag{4.1.31c}$$

$$C_{1,-1} + \sum_{n_2=1}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2}\, \zeta_2^{(1)}(m_2, n_2 \mid -1, 1 \mid d, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d}\right) = a_1 H_0, \quad n_1 = 1 \tag{4.1.31d}$$

$$C_{n_1,-1} + \sum_{n_2=1}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2}\, \zeta_2^{(1)}(m_2, n_2 \mid -1, n_1 \mid d, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d}\right)^{n_1} = 0, \quad n_1 = 2, 3, \ldots \tag{4.1.31e}$$

$$D_{n_2 0} + \sum_{n_1=1}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1 m_1}\, \zeta_1^{(2)}(m_1, n_1 \mid 0, n_2 \mid d, \theta_{12}, \varphi_{12}) \left(\frac{a_2}{d}\right)^{n_2} = 0, \quad n_2 = 1, 2, 3, \ldots \tag{4.1.32a}$$

$$D_{11} + \sum_{n_1=1}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1 m_1}\, \zeta_1^{(2)}(m_1, n_1 \mid 1, 1 \mid d, \theta_{12}, \varphi_{12}) \left(\frac{a_2}{d}\right) = -\frac{1}{2} a_2 H_0, \quad n_2 = 1 \tag{4.1.32b}$$

$$D_{n_2 1} + \sum_{n_1=1}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1 m_1}\, \zeta_1^{(2)}(m_1, n_1 \mid 1, n_2 \mid d, \theta_{12}, \varphi_{12}) \left(\frac{a_2}{d}\right)^{n_2} = 0, \quad n_2 = 2, 3, \ldots \tag{4.1.32c}$$

$$D_{1,-1} + \sum_{n_1=1}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1 m_1}\, \zeta_1^{(2)}(m_1, n_1 \mid -1, 1 \mid d, \theta_{12}, \varphi_{12}) \left(\frac{a_2}{d}\right) = a_2 H_0, \quad n_2 = 1 \tag{4.1.32d}$$

$$D_{n_2,-1} + \sum_{n_1=1}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1 m_1} \, \zeta_1^{(2)}(m_1, n_1 \mid -1, n_2 \mid d, \theta_{12}, \varphi_{12}) \left(\frac{a_2}{d}\right)^{n_2} = 0,$$

$$n_2 = 2, 3, \ldots \qquad (4.1.32e)$$

where

$$\zeta_1^{(2)}(m_1, n_1 \mid m_2, n_2 \mid d, \theta_{12}, \varphi_{12}) \equiv (-1)^{m_2+n_2} \frac{(n_1 - m_1 + n_2 + m_2)!}{(n_1 - m_1)!(n_2 + m_2)!} \left(\frac{a_1}{d}\right)^{n_1+1}$$
$$\times P_{n_1+n_2}^{m_1-m_2}(\cos\theta_{12}) \cos\big((m_1 - m_2)\varphi_{12}\big)$$

$C_{n_1 m_1} = 0$ and $D_{n_2 m_2} = 0$ for all $m_1, m_2 \neq 0, +1, -1$.

For numerical calculations, the infinite system (4.1.31) and (4.1.32) is truncated and the corresponding matrix equation is shown in (4.1.33), with

$$S_{n_2,m_2}^{(n_1,m_1)} = \zeta_2^{(1)}(m_2, n_2 \mid m_1, n_1 \mid d, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d}\right)^{n_1},$$
$$T_{n_1,m_1}^{(n_2,m_2)} = \zeta_1^{(2)}(m_1, n_1 \mid m_2, n_2 \mid d, \theta_{12}, \varphi_{12}) \left(\frac{a_2}{d}\right)^{n_2},$$

where $m_1, m_2 = 0, +1, -1$ and $n_1, n_2 = 1 \ldots N$.

$$
\left(
\begin{array}{cccccccccccccccccccccccc}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S_{1,1}^{(1,0)} & S_{1,2}^{(1,0)} & \cdots & S_{1,N}^{(1,0)} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S_{2,1}^{(2,0)} & S_{2,2}^{(2,0)} & \cdots & S_{2,N}^{(2,0)} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S_{N,1}^{(N,0)} & S_{N,2}^{(N,0)} & \cdots & S_{N,N}^{(N,0)} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S_{1,1}^{(1,1)} & S_{1,2}^{(1,1)} & \cdots & S_{1,N}^{(1,1)} & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S_{2,1}^{(2,1)} & S_{2,2}^{(2,1)} & \cdots & S_{2,N}^{(2,1)} & 0 & 0 & \cdots & 0 \\
 & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S_{N,1}^{(N,1)} & S_{N,2}^{(N,1)} & \cdots & S_{N,N}^{(N,1)} & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S_{1,1}^{(1,-1)} & S_{1,2}^{(1,-1)} & \cdots & S_{1,N}^{(1,-1)} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S_{2,1}^{(2,-1)} & S_{2,2}^{(2,-1)} & \cdots & S_{2,N}^{(2,-1)} \\
 & & \cdots & & & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & & & & & \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S_{N,1}^{(N,-1)} & S_{N,2}^{(N,-1)} & \cdots & S_{N,N}^{(N,-1)} \\
T_{1,1}^{(1,0)} & T_{1,2}^{(1,0)} & \cdots & T_{1,N}^{(1,0)} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
T_{2,1}^{(2,0)} & T_{2,2}^{(2,0)} & \cdots & T_{2,N}^{(2,0)} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & \\
T_{N,1}^{(N,0)} & T_{N,2}^{(N,0)} & \cdots & T_{N,N}^{(N,0)} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & T_{1,1}^{(1,1)} & T_{1,2}^{(1,1)} & \cdots & T_{1,N}^{(1,1)} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & T_{2,1}^{(2,1)} & T_{2,2}^{(2,1)} & \cdots & T_{2,N}^{(2,1)} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & 0 & \cdots & 0 & T_{N,1}^{(N,1)} & T_{N,2}^{(N,1)} & \cdots & T_{N,N}^{(N,1)} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T_{1,1}^{(1,-1)} & T_{1,2}^{(1,-1)} & \cdots & T_{1,N}^{(1,-1)} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T_{2,1}^{(2,-1)} & T_{2,2}^{(2,-1)} & \cdots & T_{2,N}^{(2,-1)} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
 & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T_{N,1}^{(N,-1)} & T_{N,2}^{(N,-1)} & \cdots & T_{N,N}^{(N,-1)} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{array}
\right)
\begin{pmatrix}
C_{10} \\ C_{20} \\ \vdots \\ C_{N0} \\ C_{11} \\ C_{21} \\ \vdots \\ C_{N1} \\ C_{1,-1} \\ C_{2,-1} \\ \vdots \\ C_{N,-1} \\ D_{10} \\ D_{20} \\ \vdots \\ D_{N0} \\ D_{11} \\ D_{21} \\ \vdots \\ D_{N1} \\ D_{1,-1} \\ D_{2,-1} \\ \vdots \\ D_{N,-1}
\end{pmatrix}
=
\begin{pmatrix}
0 \\ 0 \\ \vdots \\ 0 \\ -\frac{1}{2}a_1 \\ 0 \\ \vdots \\ 0 \\ a_1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{1}{2}a_2 \\ 0 \\ \vdots \\ 0 \\ a_2 \\ 0 \\ \vdots \\ 0
\end{pmatrix}
H_0
\tag{4.1.33}
$$

Then, use the Equation (4.1.22) to find the total potential outside as

$$
\begin{aligned}
\Phi^{tot}(r_1,\theta_1,\varphi_1 \mid r_2,\theta_2,\varphi_2) = & \sum_{n_1=1}^{N}\sum_{m_1=-1}^{1} C_{n_1m_1}\left(\frac{a_1}{r_1}\right)^{(n_1+1)} P_{n_1}^{m_1}(cos\,\theta_1)\,cos(m_1\varphi_1) \\
& + \sum_{n_2=1}^{N}\sum_{m_2=-1}^{1} D_{n_2m_2}\left(\frac{a_2}{r_2}\right)^{(n_2+1)} P_{n_2}^{m_2}(cos\,\theta_2)\,cos(m_2\varphi_2) \\
& - H_0\, r_1 P_1^1(cos\,\theta_1)\,cos\,\varphi_1, \qquad r_1 \geq a_1, r_2 \geq a_2
\end{aligned}
\tag{4.1.34}
$$

The components of the magnetic field intensity in the system of coordinates $(r_1, \theta_1, \varphi_1)$ are derived in the form

$$
\begin{aligned}
H_{r_1} = & \frac{1}{r_1}\sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1}(n_1+1)\,C_{n_1m_1}\left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}^{m_1}(cos\,\theta_1)\,cos(m_1\varphi_1) \\
& - \frac{1}{r_1}\sum_{n_2=1}^{\infty}\sum_{m_2=-1}^{1}\sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1} n_1\, D_{n_2m_2}\,\zeta_2^{(1)}(m_2,n_2 \mid m_1,n_1 \mid d,\theta_{21},\varphi_{21}) \\
& \times\left(\frac{r_1}{d}\right)^{n_1} P_{n_1}^{m_1}(cos\,\theta_1)\,cos(m_1\varphi_1) + H_0 P_1^1(cos\,\theta_1)\,cos\,\varphi_1, \\
H_{\theta_1} = & \frac{sin\,\theta_1}{r_1}\Bigg\{\sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1} C_{n_1m_1}\left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}^{m_1\prime}(cos\,\theta_1)\,cos\,(m_1\varphi_1) \\
& + \sum_{n_2=1}^{\infty}\sum_{m_2=-1}^{1}\sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1} D_{n_2m_2}\,\zeta_2^{(1)}(m_2,n_2 \mid m_1,n_1 \mid d,\theta_{21},\varphi_{21}) \\
& \times\left(\frac{r_1}{d}\right)^{n_1} P_{n_1}^{m_1\prime}(cos\,\theta_1)\,cos(m_1\varphi_1) - H_0\, r_1 P_1^{1\prime}(cos\,\theta_1)\,cos\,\varphi_1\Bigg\}, \\
H_{\varphi_1} = & \frac{1}{r_1 sin\,\theta_1}\Bigg\{\sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1} m_1\, C_{n_1m_1}\left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}^{m_1}(cos\,\theta_1)\,sin\,(m_1\varphi_1) \\
& + \sum_{n_2=1}^{\infty}\sum_{m_2=-1}^{1}\sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1} m_1\, D_{n_2m_2}\,\zeta_2^{(1)}(m_2,n_2 \mid m_1,n_1 \mid d,\theta_{21},\varphi_{21}) \\
& \times\left(\frac{r_1}{d}\right)^{n_1} P_{n_1}^{m_1}(cos\,\theta_1)\,sin(m_1\varphi_1) - H_0\, r_1 P_1^1(cos\,\theta_1)\,sin\,\varphi_1\Bigg\}, \qquad d \geq r_1 \geq a_1
\end{aligned}
\tag{4.1.35}
$$

The potentials inside the spheres ($r_1 < a_1$ ***and*** $r_2 < a_2$)

The magnetic potential inside the spheres ($\mathbf{B} = -\nabla\Psi$) is expressed in the form

$$\Psi^{(1)}(r_1, \theta_1, \varphi_1) = \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} G^{(1)}_{n_1 m_1} \left(\frac{r_1}{a_1}\right)^{n_1} P_{n_1}^{m_1}(\cos\theta_1)\cos(m_1\varphi_1), \quad r_1 < a_1 \qquad (4.1.36a)$$

$$\Psi^{(2)}(r_2, \theta_2, \varphi_2) = \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} G^{(2)}_{n_2 m_2} \left(\frac{r_2}{a_2}\right)^{n_2} P_{n_2}^{m_2}(\cos\theta_2)\cos(m_2\varphi_2), \quad r_2 < a_2 \qquad (4.1.36b)$$

where the coefficients $G^{(1)}_{n_1 m_1}$ and $G^{(2)}_{n_2 m_2}$ are determined by imposing the boundary conditions at the surfaces of the two spheres,

$$G^{(1)}_{n_1 0} = \mu_0 \frac{(2n_1+1)}{n_1} \sum_{n_2=1}^{\infty} \sum_{m_2=-1}^{1} D_{n_2 m_2} \left(\frac{a_1}{d}\right)^{n_1} \zeta_2^{(1)}(m_2, n_2 \mid 0, n_1 \mid d, \theta_{21}, \varphi_{21}),$$

$$n_1 = 1, 2, 3, \ldots \qquad (4.1.37a)$$

$$G^{(1)}_{11} = 3\mu_0 \left\{ \sum_{n_2=1}^{\infty} \sum_{m_2=-1}^{1} D_{n_2 m_2} \left(\frac{a_1}{d}\right) \zeta_2^{(1)}(m_2, n_2 \mid 1, 1 \mid d, \theta_{21}, \varphi_{21}) + \frac{1}{2} a_1 H_0 \right\},$$

$$n_1 = 1 \qquad (4.1.37b)$$

$$G^{(1)}_{n_1 1} = \mu_0 \frac{(2n_1+1)}{n_1} \sum_{n_2=1}^{\infty} \sum_{m_2=-1}^{1} D_{n_2 m_2} \left(\frac{a_1}{d}\right)^{n_1} \zeta_2^{(1)}(m_2, n_2 \mid 1, n_1 \mid d, \theta_{21}, \varphi_{21}),$$

$$n_1 = 2, 3, \ldots \qquad (4.1.37c)$$

$$G^{(1)}_{1,-1} = 3\mu_0 \left\{ \sum_{n_2=1}^{\infty} \sum_{m_2=-1}^{1} D_{n_2 m_2} \left(\frac{a_1}{d}\right) \zeta_2^{(1)}(m_2, n_2 \mid -1, 1 \mid d, \theta_{21}, \varphi_{21}) - a_1 H_0 \right\},$$

$$n_1 = 1 \qquad (4.1.37d)$$

$$G^{(1)}_{n_1,-1} = \mu_0 \frac{(2n_1+1)}{n_1} \sum_{n_2=1}^{\infty} \sum_{m_2=-1}^{1} D_{n_2 m_2} \left(\frac{a_1}{d}\right)^{n_1} \zeta_2^{(1)}(m_2, n_2 \mid -1, n_1 \mid d, \theta_{21}, \varphi_{21}),$$

$$n_1 = 2, 3, \ldots \qquad (4.1.37e)$$

$$G^{(2)}_{n_2 0} = \mu_0 \, \frac{(2n_2+1)}{n_2} \sum_{n_1=1}^{\infty} \sum_{m_1=-1}^{1} C_{n_1 m_1} \left(\frac{a_2}{d}\right)^{n_2} \zeta_1^{(2)}(m_1, n_1 \mid 0, n_2 \mid d, \theta_{12}, \varphi_{12}),$$

$$n_2 = 1, 2, 3, \ldots \tag{4.1.38a}$$

$$G^{(2)}_{11} = 3\mu_0 \left\{ \sum_{n_1=1}^{\infty} \sum_{m_1=-1}^{1} C_{n_1 m_1} \left(\frac{a_2}{d}\right) \zeta_1^{(2)}(m_1, n_1 \mid 1, 1 \mid d, \theta_{12}, \varphi_{12}) + \frac{1}{2} \, a_2 H_0 \right\},$$

$$n_2 = 1 \tag{4.1.38b}$$

$$G^{(2)}_{n_2 1} = \mu_0 \, \frac{(2n_2+1)}{n_2} \sum_{n_1=1}^{\infty} \sum_{m_1=-1}^{1} C_{n_1 m_1} \left(\frac{a_2}{d}\right)^{n_2} \zeta_1^{(2)}(m_1, n_1 \mid 1, n_2 \mid d, \theta_{12}, \varphi_{12}),$$

$$n_2 = 2, 3, \ldots \tag{4.1.38c}$$

$$G^{(2)}_{1,-1} = 3\mu_0 \left\{ \sum_{n_1=1}^{\infty} \sum_{m_1=-1}^{1} C_{n_1 m_1} \left(\frac{a_2}{d}\right) \zeta_1^{(2)}(m_1, n_1 \mid -1, 1 \mid d, \theta_{12}, \varphi_{12}) - a_2 H_0 \right\},$$

$$n_2 = 1 \tag{4.1.38d}$$

$$G^{(2)}_{n_2,-1} = \mu_0 \, \frac{(2n_2+1)}{n_2} \sum_{n_1=1}^{\infty} \sum_{m_1=-1}^{1} C_{n_1 m_1} \left(\frac{a_2}{d}\right)^{n_2} \zeta_1^{(2)}(m_1, n_1 \mid -1, n_2 \mid d, \theta_{12}, \varphi_{12}),$$

$$n_2 = 2, 3, \ldots \tag{4.1.38e}$$

with $C_{n_1 m_1} = 0$ and $D_{n_2 m_2} = 0$ for $m_1, m_2 \neq 0, +1, -1$.

The components flux density inside the spheres has the following components

$$\begin{aligned}
B_{r_1} &= -\frac{1}{r_1} \sum_{n_1=1}^{\infty} \sum_{m_1=-1}^{1} n_1 \, G^{(1)}_{n_1 m_1} \left(\frac{r_1}{a_1}\right)^{n_1} P_{n_1}^{m_1}(\cos\theta_1) \cos(m_1 \varphi_1), \\
B_{\theta_1} &= \frac{\sin\theta_1}{r_1} \sum_{n_1=1}^{\infty} \sum_{m_1=-1}^{1} G^{(1)}_{n_1 m_1} \left(\frac{r_1}{a_1}\right)^{n_1} P_{n_1}^{m_1}{}'(\cos\theta_1) \cos(m_1 \varphi_1), \\
B_{\varphi_1} &= \frac{1}{r_1 \sin\theta_1} \sum_{n_1=1}^{\infty} \sum_{m_1=-1}^{1} m_1 \, G^{(1)}_{n_1 m_1} \left(\frac{r_1}{a_1}\right)^{n_1} P_{n_1}^{m_1}(\cos\theta_1) \sin(m_1 \varphi_1), \qquad r_1 < a_1
\end{aligned} \tag{4.1.39}$$

$$B_{r_2} = -\frac{1}{r_2}\sum_{n_2=1}^{\infty}\sum_{m_2=-1}^{1} n_2\, G^{(2)}_{n_2 m_2}\left(\frac{r_2}{a_2}\right)^{n_2} P^{m_2}_{n_2}(cos\,\theta_2)\, cos(m_2\varphi_2),$$
$$B_{\theta_2} = \frac{sin\,\theta_2}{r_2}\sum_{n_2=1}^{\infty}\sum_{m_2=-1}^{1} G^{(2)}_{n_2 m_2}\left(\frac{r_2}{a_2}\right)^{n_2} P^{m_2}_{n_2}{}'(cos\,\theta_2)\, cos(m_2\varphi_2),$$
$$B_{\varphi_2} = \frac{1}{r_2 sin\,\theta_2}\sum_{n_2=1}^{\infty}\sum_{m_2=-1}^{1} m_2\, G^{(2)}_{n_2 m_2}\left(\frac{r_2}{a_2}\right)^{n_1} P^{m_2}_{n_2}(cos\,\theta_2)\, sin(m_2\varphi_2), \qquad r_2 < a_2 \tag{4.1.40}$$

4.2 Three-Sphere System

A system of three ferromagnetic spheres of radii a_1, a_2 and a_3 with the separation distances d_{12}, d_{13} and d_{23} between their centers are placed in a homogeneous medium of permeability μ_0. The spheres are in arbitrary positions with the planes of the centers of the spheres chosen to be the common $z - x$ plane, and are in the presence of a uniform magnetic field $\mathbf{H_0}$.

4.2.1 Case I: The Magnetic Field along z Axis

Consider an external field oriented along the z axis (i.e. $\mathbf{H_0} = H_0\hat{z}$) as shown in Figure 4.4.

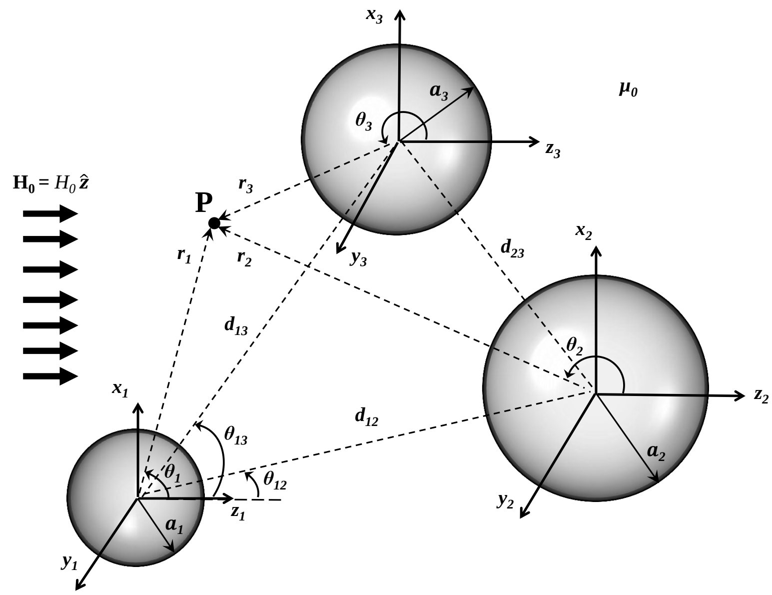

FIGURE 4.4: Three ferromagnetic spheres with same $z - x$ planes placed in an arbitrary positions in the presence of external magnetic field along z axis

The potentials outside the spheres ($r_1 \geq a_1, r_2 \geq a_2$ ***and*** $r_3 \geq a_3$)

The total scalar magnetic potential at outside the spheres due to three spheres and to the potential corresponding to the external field can be expressed as

$$\begin{aligned}\Phi^{tot}(r_1, \theta_1, \varphi_1 \mid r_2, \theta_2, \varphi_2 \mid r_3, \theta_3, \varphi_3) &= \Phi_1(r_1, \theta_1, \varphi_1) + \Phi_2(r_2, \theta_2, \varphi_2) \\ &\quad + \Phi_3(r_3, \theta_3, \varphi_3) + \Phi_{ext(z)}(r_1, \theta_1, \varphi_1)\end{aligned} \tag{4.2.1}$$

where

$$\Phi_1(r_1, \theta_1, \varphi_1) = \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1 m_1} \left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}^{m_1}(cos\,\theta_1)\, e^{-jm_1\varphi_1}, \quad r_1 \geq a_1 \tag{4.2.2a}$$

$$\Phi_2(r_2, \theta_2, \varphi_2) = \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2} \left(\frac{a_2}{r_2}\right)^{n_2+1} P_{n_2}^{m_2}(cos\,\theta_2)\, e^{-jm_2\varphi_2}, \quad r_2 \geq a_2 \tag{4.2.2b}$$

$$\Phi_3(r_3, \theta_3, \varphi_3) = \sum_{n_3=0}^{\infty} \sum_{m_3=-n_3}^{n_3} E_{n_3 m_3} \left(\frac{a_3}{r_3}\right)^{n_3+1} P_{n_3}^{m_3}(cos\,\theta_3)\, e^{-jm_3\varphi_3}, \quad r_3 \geq a_3 \tag{4.2.2c}$$

and $C_{n_1m_1}, D_{n_2m_2}$ and $E_{n_3m_3}$ are constants of integration.

Applying the translational addition theorem (see 2.2.1), the Equation (4.2.1) can be expressed in spherical coordinate $(r_1, \theta_1, \varphi_1)$ as

$$\Phi^{tot}(r_1, \theta_1, \varphi_1) = \Phi_1(r_1, \theta_1, \varphi_1) + \Phi_2^{(1)}(r_1, \theta_1, \varphi_1) + \Phi_3^{(1)}(r_1, \theta_1, \varphi_1) + \Phi_{ext(z)}(r_1, \theta_1, \varphi_1) \tag{4.2.3}$$

where

$$\begin{aligned}\Phi_2^{(1)}(r_1, \theta_1, \varphi_1) = &\sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} \sum_{\nu_1=0}^{\infty} \sum_{\mu_1=-\nu_1}^{\nu_1} (-1)^{\mu_1+\nu_1} D_{n_2m_2} \frac{(n_2-m_2+\nu_1+\mu_1)!}{(n_2-m_2)!(\nu_1+\mu_1)!} \left(\frac{a_2}{d_{12}}\right)^{n_2+1} \\ &\times \left(\frac{r_1}{d_{12}}\right)^{\nu_1} P_{n_2+\nu_1}^{m_2-\mu_1}(cos\,\theta_{21})\, e^{-j(m_2-\mu_1)\varphi_{21}}\, P_{\nu_1}^{\mu_1}(cos\,\theta_1)\, e^{-j\mu_1\varphi_1},\end{aligned} \tag{4.2.4a}$$

$$\begin{aligned}\Phi_3^{(1)}(r_1, \theta_1, \varphi_1) = &\sum_{n_3=0}^{\infty} \sum_{m_3=-n_3}^{n_3} \sum_{\nu_1=0}^{\infty} \sum_{\mu_1=-\nu_1}^{\nu_1} (-1)^{\mu_1+\nu_1} E_{n_3m_3} \frac{(n_3-m_3+\nu_1+\mu_1)!}{(n_3-m_3)!(\nu_1+\mu_1)!} \left(\frac{a_3}{d_{13}}\right)^{n_3+1} \\ &\times \left(\frac{r_1}{d_{13}}\right)^{\nu_1} P_{n_3+\nu_1}^{m_3-\mu_1}(cos\,\theta_{31})\, e^{-j(m_3-\mu_1)\varphi_{31}}\, P_{\nu_1}^{\mu_1}(cos\,\theta_1)\, e^{-j\mu_1\varphi_1},\end{aligned} \tag{4.2.4b}$$

$$r_1 \le d_{12}, \quad r_1 \le d_{13}$$

The potential corresponding to the external field can be expressed as

$$\Phi_{ext(z)}(r_i, \theta_i, \varphi_i) = -H_0\, r_i P_1(cos\,\theta_i) + K_i, \qquad i = 1, 2, 3$$

As in previous cases, choose $\Phi_{ext(z)} = 0$ at $z_1 = 0$, as reference. Then the constants $K_1 = 0, K_2 = -H_0\, d_{12}\, cos(\theta_{12})$ and $K_3 = -H_0\, d_{13}\, cos(\theta_{13})$, i.e.,

$$\begin{aligned}\Phi_{ext(z)}(r_1, \theta_1, \varphi_1) &= -H_0\, r_1 P_1(cos\,\theta_1) \\ \Phi_{ext(z)}(r_2, \theta_2, \varphi_2) &= -H_0\, r_2 P_1(cos\,\theta_2) - H_0\, d_{12}\, cos(\theta_{12}) \\ \Phi_{ext(z)}(r_3, \theta_3, \varphi_3) &= -H_0\, r_3 P_1(cos\,\theta_3) - H_0\, d_{13}\, cos(\theta_{13})\end{aligned} \tag{4.2.5}$$

Thus,

$$\begin{aligned}\Phi^{tot}(r_1,\theta_1,\varphi_1) = &\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1} C_{n_1m_1}\left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}^{m_1}(cos\,\theta_1)\, e^{-jm_1\varphi_1} - H_0\, r_1 P_1(cos\,\theta_1)\\
&+\sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2}\sum_{\nu_1=0}^{\infty}\sum_{\mu_1=-\nu_1}^{\nu_1} D_{n_2m_2}\,\zeta_2^{(1)}(m_2,n_2\mid\mu_1,\nu_1\mid d_{12},\theta_{21},\varphi_{21})\left(\frac{r_1}{d_{12}}\right)^{\nu_1} P_{\nu_1}^{\mu_1}(cos\,\theta_1)\, e^{-j\mu_1\varphi_1}\\
&+\sum_{n_3=0}^{\infty}\sum_{m_3=-n_3}^{n_3}\sum_{\nu_1=0}^{\infty}\sum_{\mu_1=-\nu_1}^{\nu_1} E_{n_3m_3}\,\zeta_3^{(1)}(m_3,n_3\mid\mu_1,\nu_1\mid d_{13},\theta_{31},\varphi_{31})\left(\frac{r_1}{d_{13}}\right)^{\nu_1} P_{\nu_1}^{\mu_1}(cos\,\theta_1)\, e^{-j\mu_1\varphi_1},\\
&\qquad r_1 \geq a_1
\end{aligned}\tag{4.2.6}$$

where

$$\zeta_2^{(1)}(m_2,n_2\mid\mu_1,\nu_1\mid d_{12},\theta_{21},\varphi_{21}) \equiv (-1)^{\mu_1+\nu_1}\frac{(n_2-m_2+\nu_1+\mu_1)!}{(n_2-m_2)!(\nu_1+\mu_1)!}\left(\frac{a_2}{d_{12}}\right)^{n_2+1} \times P_{n_2+\nu_1}^{m_2-\mu_1}(cos\,\theta_{21})\, e^{-j(m_2-\mu_1)\varphi_{21}},$$

$$\zeta_3^{(1)}(m_3,n_3\mid\mu_1,\nu_1\mid d_{13},\theta_{31},\varphi_{31}) \equiv (-1)^{\mu_1+\nu_1}\frac{(n_3-m_3+\nu_1+\mu_1)!}{(n_3-m_3)!(\nu_1+\mu_1)!}\left(\frac{a_3}{d_{13}}\right)^{n_3+1} \times P_{n_3+\nu_1}^{m_3-\mu_1}(cos\,\theta_{31})\, e^{-j(m_3-\mu_1)\varphi_{31}}$$

To determine the series coefficients in (4.2.6) we impose the same conditions as before. This gives

$$C_{00}=0,\quad D_{00}=0,\quad E_{00}=0 \tag{4.2.7}$$

and the following infinite system of linear equations

$$\begin{aligned}C_{10} &+ \sum_{n_2=0}^{\infty} D_{n_20}\left(\frac{a_1}{d_{12}}\right)\zeta_2^{(1)}(0,n_2\mid 0,1\mid d_{12},\theta_{21},\varphi_{21})\\
&+\sum_{n_3=0}^{\infty} E_{n_30}\left(\frac{a_1}{d_{13}}\right)\zeta_3^{(1)}(0,n_3\mid 0,1\mid d_{13},\theta_{31},\varphi_{31}) = a_1H_0,\quad n_1=1
\end{aligned}\tag{4.2.8a}$$

$$\begin{aligned}C_{n_10} &+ \sum_{n_2=0}^{\infty} D_{n_20}\left(\frac{a_1}{d_{12}}\right)^{n_1}\zeta_2^{(1)}(0,n_2\mid 0,n_1\mid d_{12},\theta_{21},\varphi_{21})\\
&+\sum_{n_3=0}^{\infty} E_{n_30}\left(\frac{a_1}{d_{13}}\right)^{n_1}\zeta_3^{(1)}(0,n_3\mid 0,n_1\mid d_{13},\theta_{31},\varphi_{31}) = 0,\quad n_1=2,3,\ldots
\end{aligned}\tag{4.2.8b}$$

$$
\begin{aligned}
D_{10} &+ \sum_{n_1=0}^{\infty} C_{n_1 0} \left(\frac{a_2}{d_{12}}\right) \zeta_1^{(2)}(0, n_1 \mid 0, 1 \mid d_{12}, \theta_{12}, \varphi_{12}) \\
&+ \sum_{n_3=0}^{\infty} E_{n_3 0} \left(\frac{a_2}{d_{23}}\right) \zeta_3^{(2)}(0, n_3 \mid 0, 1 \mid d_{23}, \theta_{32}, \varphi_{32}) = a_2 H_0, \qquad n_2 = 1 \qquad \text{(4.2.9a)} \\
D_{n_2 0} &+ \sum_{n_1=0}^{\infty} C_{n_1 0} \left(\frac{a_2}{d_{12}}\right)^{n_2} \zeta_1^{(2)}(0, n_1 \mid 0, n_2 \mid d_{12}, \theta_{12}, \varphi_{12}) \\
&+ \sum_{n_3=0}^{\infty} E_{n_3 0} \left(\frac{a_2}{d_{23}}\right)^{n_2} \zeta_3^{(2)}(0, n_3 \mid 0, n_2 \mid d_{23}, \theta_{32}, \varphi_{32}) = 0, \qquad n_2 = 2, 3, \ldots \qquad \text{(4.2.9b)}
\end{aligned}
$$

$$
\begin{aligned}
E_{10} &+ \sum_{n_1=0}^{\infty} C_{n_1 0} \left(\frac{a_3}{d_{13}}\right) \zeta_1^{(3)}(0, n_1 \mid 0, 1 \mid d_{13}, \theta_{13}, \varphi_{13}) \\
&+ \sum_{n_2=0}^{\infty} D_{n_2 0} \left(\frac{a_3}{d_{23}}\right) \zeta_2^{(3)}(0, n_2 \mid 0, 1 \mid d_{23}, \theta_{23}, \varphi_{23}) = a_3 H_0, \qquad n_3 = 1 \qquad \text{(4.2.10a)} \\
E_{n_3 0} &+ \sum_{n_1=0}^{\infty} C_{n_1 0} \left(\frac{a_3}{d_{13}}\right)^{n_3} \zeta_1^{(3)}(0, n_1 \mid 0, n_3 \mid d_{13}, \theta_{13}, \varphi_{13}) \\
&+ \sum_{n_2=0}^{\infty} D_{n_2 0} \left(\frac{a_3}{d_{23}}\right)^{n_3} \zeta_2^{(3)}(0, n_2 \mid 0, n_3 \mid d_{23}, \theta_{23}, \varphi_{23}) = 0, \qquad n_3 = 2, 3, \ldots \qquad \text{(4.2.10b)}
\end{aligned}
$$

where

$$
\begin{aligned}
\zeta_2^{(1)}(m_2, n_2 \mid m_1, n_1 \mid d_{12}, \theta_{21}, \varphi_{21}) &\equiv (-1)^{m_1+n_1} \frac{(n_2 - m_2 + n_1 + m_1)!}{(n_2 - m_2)!(n_1 + m_1)!} \left(\frac{a_2}{d_{12}}\right)^{n_2+1} \\
&\quad \times P_{n_2+n_1}^{m_2-m_1}(cos\,\theta_{21})\, e^{-j(m_2-m_1)\varphi_{21}}, \\
\zeta_3^{(1)}(m_3, n_3 \mid m_1, n_1 \mid d_{13}, \theta_{31}, \varphi_{31}) &\equiv (-1)^{m_1+n_1} \frac{(n_3 - m_3 + n_1 + m_1)!}{(n_3 - m_3)!(n_1 + m_1)!} \left(\frac{a_3}{d_{13}}\right)^{n_3+1} \\
&\quad \times P_{n_3+n_1}^{m_3-m_1}(cos\,\theta_{31})\, e^{-j(m_3-m_1)\varphi_{31}},
\end{aligned}
$$

$$\zeta_1^{(2)}(m_1, n_1 \mid m_2, n_2 \mid d_{12}, \theta_{12}, \varphi_{12}) \equiv (-1)^{m_2+n_2} \frac{(n_1 - m_1 + n_2 + m_2)!}{(n_1 - m_1)!(n_2 + m_2)!} \left(\frac{a_1}{d_{12}}\right)^{n_1+1} \times P_{n_1+n_2}^{m_1-m_2}(cos\,\theta_{12})\, e^{-j(m_1-m_2)\varphi_{12}},$$

$$\zeta_3^{(2)}(m_3, n_3 \mid m_2, n_2 \mid d_{23}, \theta_{32}, \varphi_{32}) \equiv (-1)^{m_2+n_2} \frac{(n_3 - m_3 + n_2 + m_2)!}{(n_3 - m_3)!(n_2 + m_2)!} \left(\frac{a_3}{d_{23}}\right)^{n_3+1} \times P_{n_3+n_2}^{m_3-m_2}(cos\,\theta_{32})\, e^{-j(m_3-m_2)\varphi_{32}},$$

$$\zeta_1^{(3)}(m_1, n_1 \mid m_3, n_3 \mid d_{13}, \theta_{13}, \varphi_{13}) \equiv (-1)^{m_3+n_3} \frac{(n_1 - m_1 + n_3 + m_3)!}{(n_1 - m_1)!(n_3 + m_3)!} \left(\frac{a_1}{d_{13}}\right)^{n_1+1} \times P_{n_1+n_3}^{m_1-m_3}(cos\,\theta_{13})\, e^{-j(m_1-m_3)\varphi_{13}},$$

$$\zeta_2^{(3)}(m_2, n_2 \mid m_3, n_3 \mid d_{23}, \theta_{23}, \varphi_{23}) \equiv (-1)^{m_3+n_3} \frac{(n_2 - m_2 + n_3 + m_3)!}{(n_2 - m_2)!(n_3 + m_3)!} \left(\frac{a_2}{d_{23}}\right)^{n_2+1} \times P_{n_2+n_3}^{m_2-m_3}(cos\,\theta_{23})\, e^{-j(m_2-m_3)\varphi_{23}}$$

$C_{n_1 m_1} = D_{n_2 m_2} = E_{n_3 m_3} = 0$ for $m_1, m_2, m_3 \neq 0$. In order to obtain numerical solutions, these infinite set of Equations (4.2.8 - 4.2.10) is truncated as in the previous cases. Using the notation

$$S1_{n_2}^{(n_1)} \equiv \zeta_2^{(1)}(0, n_2 \mid 0, n_1 \mid d_{12}, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d_{12}}\right)^{n_1},$$

$$S2_{n_3}^{(n_1)} \equiv \zeta_3^{(1)}(0, n_3 \mid 0, n_1 \mid d_{13}, \theta_{31}, \varphi_{31}) \left(\frac{a_1}{d_{13}}\right)^{n_1},$$

$$T1_{n_1}^{(n_2)} \equiv \zeta_1^{(2)}(0, n_1 \mid 0, n_2 \mid d_{12}, \theta_{12}, \varphi_{12}) \left(\frac{a_2}{d_{12}}\right)^{n_2},$$

$$T2_{n_3}^{(n_2)} \equiv \zeta_3^{(2)}(0, n_3 \mid 0, n_2 \mid d_{23}, \theta_{32}, \varphi_{32}) \left(\frac{a_2}{d_{23}}\right)^{n_2},$$

$$U1_{n_1}^{(n_3)} \equiv \zeta_1^{(3)}(0, n_1 \mid 0, n_3 \mid d_{13}, \theta_{13}, \varphi_{13}) \left(\frac{a_3}{d_{13}}\right)^{n_3},$$

$$U2_{n_2}^{(n_3)} \equiv \zeta_2^{(3)}(0, n_2 \mid 0, n_3 \mid d_{23}, \theta_{23}, \varphi_{23}) \left(\frac{a_3}{d_{23}}\right)^{n_3},$$

where $n_1, n_2, n_3 = 1, \ldots, N$

we obtain the following $3N \times 3N$ matrix equation

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & S1_1^{(1)} & S1_2^{(1)} & \cdots & S1_N^{(1)} & S2_1^{(1)} & S2_2^{(1)} & \cdots & S2_N^{(1)} \\ 0 & 1 & \cdots & 0 & S1_1^{(2)} & S1_2^{(2)} & \cdots & S1_N^{(2)} & S2_1^{(2)} & S2_2^{(2)} & \cdots & S2_N^{(2)} \\ & & \ddots & & & & \ddots & & & & \ddots & \\ 0 & 0 & \cdots & 1 & S1_1^{(N)} & S1_2^{(N)} & \cdots & S1_N^{(N)} & S2_1^{(N)} & S2_2^{(N)} & \cdots & S2_N^{(N)} \\ T1_1^{(1)} & T1_2^{(1)} & \cdots & T1_N^{(1)} & 1 & 0 & \cdots & 0 & T2_1^{(1)} & T2_2^{(1)} & \cdots & T2_N^{(1)} \\ T1_1^{(2)} & T1_2^{(2)} & \cdots & T1_N^{(2)} & 0 & 1 & \cdots & 0 & T2_1^{(2)} & T2_2^{(2)} & \cdots & T2_N^{(2)} \\ & & \ddots & & & & \ddots & & & & \ddots & \\ T1_1^{(N)} & T1_2^{(N)} & \cdots & T1_N^{(N)} & 0 & 0 & \cdots & 1 & T2_1^{(N)} & T2_2^{(N)} & \cdots & T2_N^{(N)} \\ U1_1^{(1)} & U1_2^{(1)} & \cdots & U1_N^{(1)} & U2_1^{(1)} & U2_2^{(1)} & \cdots & U2_N^{(1)} & 1 & 0 & \cdots & 0 \\ U1_1^{(2)} & U1_2^{(2)} & \cdots & U1_N^{(2)} & U2_1^{(2)} & U2_2^{(2)} & \cdots & U2_N^{(2)} & 0 & 1 & \cdots & 0 \\ & & \ddots & & & & \ddots & & & & \ddots & \\ U1_1^{(N)} & U1_2^{(N)} & \cdots & U1_N^{(N)} & U2_1^{(N)} & U2_2^{(N)} & \cdots & U2_N^{(N)} & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} C_{10} \\ C_{20} \\ \vdots \\ C_{N0} \\ D_{10} \\ D_{20} \\ \vdots \\ D_{N0} \\ E_{10} \\ E_{20} \\ \vdots \\ E_{N0} \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \\ \vdots \\ 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \\ a_3 \\ 0 \\ \vdots \\ 0 \end{pmatrix} H_0 \tag{4.2.11}$$

The solution of this equation allows the calculation of the total scalar magnetic potential at any point outside the spheres using the Equation (4.2.1) as

$$\begin{aligned} \Phi^{tot}(r_1, \theta_1, \varphi_1 \mid r_2, \theta_2, \varphi_2 \mid r_3, \theta_3, \varphi_3) = & \sum_{n_1=1}^{N} C_{n_1 0} \left(\frac{a_1}{r_1} \right)^{n_1+1} P_{n_1}(cos\,\theta_1) \\ & + \sum_{n_2=1}^{N} D_{n_2 0} \left(\frac{a_2}{r_2} \right)^{n_2+1} P_{n_2}(cos\,\theta_2) \\ & + \sum_{n_3=1}^{N} E_{n_3 0} \left(\frac{a_3}{r_3} \right)^{n_3+1} P_{n_3}(cos\,\theta_3) \\ & - H_0\, r_1 P_1(cos\,\theta_1), \qquad r_1 \geq a_1,\; r_2 \geq a_2,\; r_3 \geq a_3 \end{aligned} \tag{4.2.12}$$

The magnetic field intensity components in $(r_1, \theta_1, \varphi_1)$ are obtained in the form

$$
\begin{aligned}
H_{r_1} = & \frac{1}{r_1} \sum_{n_1=1}^{N} (n_1+1)\, C_{n_1 0} \left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}(cos\,\theta_1) + H_0 P_1(cos\,\theta_1) \\
& + \frac{1}{r_2^3} \sum_{n_2=1}^{N} D_{n_2 0} \left(\frac{a_2}{r_2}\right)^{n_2+1} \Bigg\{ r_2(n_2+1)\Big(r_1 - d_{12}\, cos(\theta_1 - \theta_{12})\Big) P_{n_2}(cos\,\theta_2) \\
& + \Bigg\{ \Big(r_1 d_{12}\, cos(\theta_1 - \theta_{12}) - d_{12}^2\Big)\, cos\,\theta_1 - \Big(r_1 - d_{12}\, cos(\theta_1 - \theta_{12})\Big)\, d_{12}\, cos\,\theta_{12} \Bigg\} P'_{n_2}(cos\,\theta_2) \\
& + \frac{1}{r_3^3} \sum_{n_3=1}^{N} E_{n_3 0} \left(\frac{a_3}{r_3}\right)^{n_3+1} \Bigg\{ r_3(n_3+1)\Big(r_1 - d_{13}\, cos(\theta_1 - \theta_{13})\Big) P_{n_3}(cos\,\theta_3) \\
& + \Bigg\{ \Big(r_1 d_{13}\, cos(\theta_1 - \theta_{13}) - d_{13}^2\Big)\, cos\,\theta_1 - \Big(r_1 - d_{13}\, cos(\theta_1 - \theta_{13})\Big)\, d_{13}\, cos\,\theta_{13} \Bigg\} P'_{n_3}(cos\,\theta_3) \Bigg\}, \\
H_{\theta_1} = & \frac{sin\,\theta_1}{r_1} \sum_{n_1=1}^{N} C_{n_1 0} \left(\frac{a_1}{r_1}\right)^{n_1+1} P'_{n_1}(cos\,\theta_1) - H_0\, sin\,\theta_1 P'_1(cos\,\theta_1) \\
& + \frac{1}{r_2^3} \sum_{n_2=1}^{N} D_{n_2 0} \left(\frac{a_2}{r_2}\right)^{n_2+1} \Bigg\{ r_2\, d_{12}\, (n_2+1)\, sin(\theta_1 - \theta_{12})\, P_{n_2}(cos\,\theta_2) \\
& \qquad + \Bigg\{ r_2^2\, sin\,\theta_1 + d_{12}\, sin(\theta_1 - \theta_{12}) \Big(r_1\, cos\,\theta_1 - d_{12}\, cos\,\theta_{12}\Big) \Bigg\} P'_{n_2}(cos\,\theta_2) \\
& + \frac{1}{r_3^3} \sum_{n_3=1}^{N} E_{n_3 0} \left(\frac{a_3}{r_3}\right)^{n_3+1} \Bigg\{ r_3\, d_{13}\, (n_3+1)\, sin(\theta_1 - \theta_{13})\, P_{n_3}(cos\,\theta_3) \\
& \qquad + \Bigg\{ r_3^2\, sin\,\theta_1 + d_{13}\, sin(\theta_1 - \theta_{13}) \Big(r_1\, cos\,\theta_1 - d_{13}\, cos\,\theta_{13}\Big) \Bigg\} P'_{n_3}(cos\,\theta_3) \Bigg\}, \\
H_{\varphi_1} = & 0, \qquad\qquad r_1 \geq a_1, \quad r_2 \geq a_2, \quad r_3 \geq a_3,
\end{aligned}
\tag{4.2.13}
$$

The potentials inside the spheres $(r_1 < a_1, r_2 < a_2$ ***and*** $r_3 < a_3)$

The potential inside the spheres can be expressed by using the Equation (2.1.4) as

$$\Psi^{(1)}(r_1,\theta_1,\varphi_1) = \sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1} G^{(1)}_{n_1m_1} r_1^{n_1} P_{n_1}^{m_1}(cos\,\theta_1)\, e^{-jm_1\varphi_1}, \quad r_1 < a_1 \tag{4.2.14a}$$

$$\Psi^{(2)}(r_2,\theta_2,\varphi_2) = \sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2} G^{(2)}_{n_2m_2} r_2^{n_2} P_{n_2}^{m_2}(cos\,\theta_2)\, e^{-jm_2\varphi_2}, \quad r_2 < a_2 \tag{4.2.14b}$$

$$\Psi^{(3)}(r_3,\theta_3,\varphi_2) = \sum_{n_3=0}^{\infty}\sum_{m_3=-n_3}^{n_3} G^{(3)}_{n_3m_3} r_3^{n_3} P_{n_3}^{m_3}(cos\,\theta_3)\, e^{-jm_3\varphi_3}, \quad r_3 < a_3 \tag{4.2.14c}$$

where $G^{(1)}_{n_1m_1}, G^{(2)}_{n_2m_2}$ and $G^{(3)}_{n_3m_3}$ are constants of integration. To determine these constants we have imposed the boundary condition as in the previous cases. I.e.,

$$-\frac{\partial}{\partial\, r_i}\left\{\Psi^{(i)}(r_i,\theta_i,\varphi_i)\right\}\Bigg|_{r_i\,=\,a_i} = -\mu_0\,\frac{\partial}{\partial\, r_i}\left\{\Phi^{tot}(r_i,\theta_i,\varphi_i)\right\}\Bigg|_{r_i\,=\,a_i}, \qquad i = 1,2,3 \tag{4.2.15}$$

where μ_0 is the permeability of the medium outside.

This gives the following results for $m_1, m_2, m_3 = 0$:

$$\begin{aligned} G^{(1)}_{10} = 3\mu_0 \Bigg\{ &\sum_{n_2=1}^{\infty} D_{n_20}\left(\frac{a_1}{d_{12}}\right)\zeta_2^{(1)}(0,n_2\mid 0,1\mid d_{12},\theta_{21},\varphi_{21}) - a_1 H_0 \\ &+ \sum_{n_3=1}^{\infty} E_{n_30}\left(\frac{a_1}{d_{13}}\right)\zeta_3^{(1)}(0,n_3\mid 0,1\mid d_{13},\theta_{31},\varphi_{31})\Bigg\}, \qquad n_1 = 1 \end{aligned} \tag{4.2.16a}$$

$$\begin{aligned} G^{(1)}_{n_10} = \mu_0\,\frac{(2n_1+1)}{n_1}\Bigg\{ &\sum_{n_2=1}^{\infty} D_{n_20}\left(\frac{a_1}{d_{12}}\right)^{n_1}\zeta_2^{(1)}(0,n_2\mid 0,n_1\mid d_{12},\theta_{21},\varphi_{21}) \\ &+ \sum_{n_3=1}^{\infty} E_{n_30}\left(\frac{a_1}{d_{13}}\right)^{n_1}\zeta_3^{(1)}(0,n_3\mid 0,n_1\mid d_{13},\theta_{31},\varphi_{31})\Bigg\}, \qquad n_1 = 2,3,\ldots \end{aligned} \tag{4.2.16b}$$

$$G_{10}^{(2)} = 3\mu_0 \Bigg\{ \sum_{n_1=1}^{\infty} C_{n_1 0} \left(\frac{a_2}{d_{12}}\right) \zeta_1^{(2)}(0, n_1 \mid 0, 1 \mid d_{12}, \theta_{12}, \varphi_{12}) - a_2 H_0$$
$$+ \sum_{n_3=1}^{\infty} E_{n_3 0} \left(\frac{a_2}{d_{23}}\right) \zeta_3^{(2)}(0, n_3 \mid 0, 1 \mid d_{23}, \theta_{32}, \varphi_{32}) \Bigg\}, \qquad n_2 = 1 \tag{4.2.17a}$$

$$G_{n_2 0}^{(2)} = \mu_0 \frac{(2n_2+1)}{n_2} \Bigg\{ \sum_{n_1=1}^{\infty} C_{n_1 0} \left(\frac{a_2}{d_{12}}\right)^{n_2} \zeta_1^{(2)}(0, n_1 \mid 0, n_2 \mid d_{12}, \theta_{12}, \varphi_{12}),$$
$$+ \sum_{n_3=1}^{\infty} E_{n_3 0} \left(\frac{a_2}{d_{23}}\right)^{n_2} \zeta_3^{(2)}(0, n_3 \mid 0, n_2 \mid d_{23}, \theta_{32}, \varphi_{32}) \Bigg\}, \qquad n_2 = 2, 3, \ldots \tag{4.2.17b}$$

$$G_{10}^{(3)} = 3\mu_0 \Bigg\{ \sum_{n_1=1}^{\infty} C_{n_1 0} \left(\frac{a_3}{d_{13}}\right) \zeta_1^{(3)}(0, n_1 \mid 0, 1 \mid d_{13}, \theta_{13}, \varphi_{13}) - a_3 H_0$$
$$+ \sum_{n_2=1}^{\infty} D_{n_2 0} \left(\frac{a_3}{d_{23}}\right) \zeta_2^{(3)}(0, n_2 \mid 0, 1 \mid d_{23}, \theta_{23}, \varphi_{23}) \Bigg\}, \qquad n_3 = 1 \tag{4.2.18a}$$

$$G_{n_3 0}^{(3)} = \mu_0 \frac{(2n_3+1)}{n_3} \Bigg\{ \sum_{n_1=1}^{\infty} C_{n_1 0} \left(\frac{a_3}{d_{13}}\right)^{n_3} \zeta_1^{(3)}(0, n_1 \mid 0, n_3 \mid d_{13}, \theta_{13}, \varphi_{13}),$$
$$+ \sum_{n_2=1}^{\infty} D_{n_2 0} \left(\frac{a_3}{d_{23}}\right)^{n_3} \zeta_2^{(3)}(0, n_2 \mid 0, n_3 \mid d_{23}, \theta_{23}, \varphi_{23}) \Bigg\}, \qquad n_3 = 2, 3, \ldots \tag{4.2.18b}$$

For $m_1, m_2, m_3 \neq 0$, we have $G_{n_1 m_1}^{(1)} = G_{n_2 m_2}^{(2)} = G_{n_3 m_3}^{(3)} = 0$. The components of $\mathbf{B}$ inside the spheres are determined from $\mathbf{B} = -\nabla\Psi$.

4.2.2 Case II: $\mathbf{H_0}$ along x Axis

Figure 4.5 shows three-sphere system in the presence of external field along x axis.

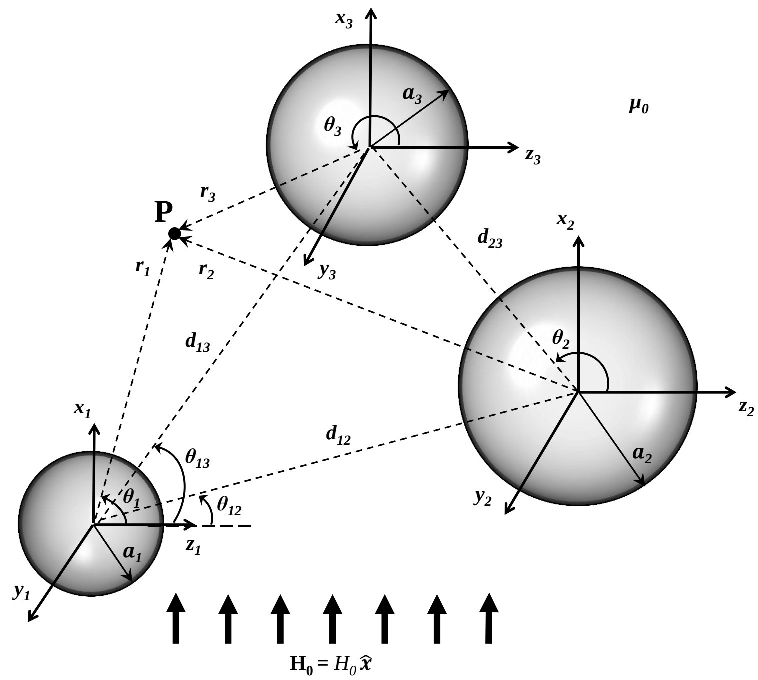

FIGURE 4.5: Three ferromagnetic spheres with same $z - x$ planes placed in an arbitrary positions in the presence of external field along x axis

The potentials outside the spheres, $(r_1 \geq a_1, r_2 \geq a_2$ ***and*** $r_3 \geq a_3)$

The total scalar magnetic potential outside can be expressed as

$$\begin{aligned}\Phi^{tot}(r_1, \theta_1, \varphi_1 \mid r_2, \theta_2, \varphi_2 \mid r_3, \theta_3, \varphi_3) = \Phi_1(r_1, \theta_1, \varphi_1) + \Phi_2(r_2, \theta_2, \varphi_2) \\ + \Phi_3(r_3, \theta_3, \varphi_3) + \Phi_{ext(x)}(r_1, \theta_1, \varphi_1)\end{aligned} \tag{4.2.19}$$

where

$$\Phi_1(r_1, \theta_1, \varphi_1) = \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1 m_1} r_1^{-(n_1+1)} P_{n_1}^{m_1}(\cos\theta_1)\, e^{-jm_1\varphi_1}, \quad r_1 \geq a_1 \tag{4.2.20a}$$

$$\Phi_2(r_2, \theta_2, \varphi_2) = \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2} r_2^{-(n_2+1)} P_{n_2}^{m_2}(\cos\theta_2)\, e^{-jm_2\varphi_2}, \quad r_2 \geq a_2 \tag{4.2.20b}$$

$$\Phi_3(r_3, \theta_3, \varphi_3) = \sum_{n_3=0}^{\infty} \sum_{m_3=-n_3}^{n_3} E_{n_3 m_3} r_3^{-(n_3+1)} P_{n_3}^{m_3}(\cos\theta_3)\, e^{-jm_3\varphi_3}, \quad r_3 \geq a_3 \tag{4.2.20c}$$

and

$$\Phi_{ext(x)}(r_1,\theta_1,\varphi_1) = -H_0 r_1 P_1^1(cos\,\theta_1)\,cos\,\varphi_1 \tag{4.2.21a}$$

$$\Phi_{ext(x)}(r_2,\theta_2,\varphi_2) = -H_0 r_2 P_1^1(cos\,\theta_1)\,cos\,\varphi_1 - H_0 d_{12}\,sin(\theta_{12})\,cos(\varphi_{12}) \tag{4.2.21b}$$

$$\Phi_{ext(x)}(r_3,\theta_3,\varphi_3) = -H_0 r_3 P_1^1(cos\,\theta_1)\,cos\,\varphi_1 - H_0 d_{13}\,sin(\theta_{13})\,cos(\varphi_{13}) \tag{4.2.21c}$$

where $C_{n_1m_1}, D_{n_2m_2}$ and $E_{n_3m_3}$ are constants of integration.

The Equation (4.2.19) can be expressed in spherical coordinate (r_1,θ_1,φ_1) as

$$\begin{aligned}\Phi^{tot}(r_1,\theta_1,\varphi_1) = \Phi_1(r_1,\theta_1,\varphi_1) + \Phi_2^{(1)}(r_1,\theta_1,\varphi_1) + \Phi_3^{(1)}(r_1,\theta_1,\varphi_1)\\ +\Phi_{ext(x)}(r_1,\theta_1,\varphi_1)\end{aligned} \tag{4.2.22}$$

where $\Phi_2^{(1)}$ and $\Phi_3^{(1)}$ are, respectively, the potentials of the sphere 2 and sphere 3 in spherical coordinate (r_1,θ_1,φ_1). Imposing again the condition of zero flux density for each sphere, we obtain

$$C_{00} = 0, \quad D_{00} = 0, \quad E_{00} = 0 \tag{4.2.23}$$

Then, the Equation (4.2.22) becomes

$$\begin{aligned}\Phi^{tot}(r_1,\theta_1,\varphi_1) = \sum_{n_1=1}^{\infty}\sum_{m_1=-n_1}^{n_1} C_{n_1m_1}\left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}^{m_1}(cos\,\theta_1)\,cos(m_1\varphi_1)\\ +\left\{\sum_{n_2=1}^{\infty}\sum_{m_2=-n_2}^{n_2}\sum_{\nu_1=0}^{\infty}\sum_{\mu_1=-\nu_1}^{\nu_1} D_{n_2m_2}\,\zeta_2^{(1)}(m_2,n_2\mid\mu_1,\nu_1\mid d_{12},\theta_{21},\varphi_{21})\left(\frac{r_1}{d_{12}}\right)^{\nu_1}\right.\\ \left.+\sum_{n_3=1}^{\infty}\sum_{m_3=-n_3}^{n_3}\sum_{\nu_1=0}^{\infty}\sum_{\mu_1=-\nu_1}^{\nu_1} E_{n_3m_3}\,\zeta_3^{(1)}(m_3,n_3\mid\mu_1,\nu_1\mid d_{13},\theta_{31},\varphi_{31})\left(\frac{r_1}{d_{13}}\right)^{\nu_1}\right\}\\ \times P_{\nu_1}^{\mu_1}(cos\,\theta_1) - H_0\,r_1 P_1^1(cos\,\theta_1)\,cos\,\varphi_1, \qquad a_1 \le r_1 \le d\end{aligned} \tag{4.2.24}$$

where

$$\zeta_2^{(1)}(m_2,n_2 \mid m_1,n_1 \mid d_{12},\theta_{21},\varphi_{21}) \equiv (-1)^{m_1+n_1}\frac{(n_2-m_2+n_1+m_1)!}{(n_2-m_2)!(n_1+m_1)!}\left(\frac{a_2}{d_{12}}\right)^{n_2+1}$$

$$\times P_{n_2+n_1}^{m_2-m_1}(cos\,\theta_{21})\,cos\big((m_2-m_1)\varphi_{21}+\mu_1\varphi_1\big),$$

$$\zeta_3^{(1)}(m_3,n_3 \mid m_1,n_1 \mid d_{13},\theta_{31},\varphi_{31}) \equiv (-1)^{m_1+n_1}\frac{(n_3-m_3+n_1+m_1)!}{(n_3-m_3)!(n_1+m_1)!}\left(\frac{a_3}{d_{13}}\right)^{n_3+1}$$

$$\times P_{n_3+n_1}^{m_3-m_1}(cos\,\theta_{31})\,cos\big((m_3-m_1)\varphi_{31}+\mu_1\varphi_1\big),$$

Now consider the potential on its surface and simplify using the normalization of the spherical harmonics (see C.6) to obtain as

$$\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}\frac{C_{n_1m_1}}{\sqrt{\frac{(2\,n_1+1)}{4\pi}\,\frac{(n_1-m_1)!}{(n_1+m_1)!}}}=$$

$$-\sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2}\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}\frac{D_{n_2m_2}}{\sqrt{\frac{(2\,n_1+1)}{4\pi}\,\frac{(n_1-m_1)!}{(n_1+m_1)!}}}\zeta_2^{(1)}(m_2,n_2 \mid m_1,n_1 \mid d_{12},\theta_{21},\varphi_{21})\left(\frac{a_1}{d_{12}}\right)^{n_1}$$

$$-\sum_{n_3=0}^{\infty}\sum_{m_3=-n_3}^{n_3}\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}\frac{E_{n_3m_3}}{\sqrt{\frac{(2\,n_1+1)}{4\pi}\,\frac{(n_1-m_1)!}{(n_1+m_1)!}}}\zeta_3^{(1)}(m_3,n_3 \mid m_1,n_1 \mid d_{13},\theta_{31},\varphi_{31})\left(\frac{a_1}{d_{13}}\right)^{n_1}$$

$$+\int_{\theta_1=0}^{\pi}\int_{\varphi_1=0}^{2\pi}\Big(V_1+H_0\,a_1\,P_1^1(cos\,\theta_1)\,cos\,\varphi_1\Big)\,\overline{Y}_{n_1}^{m_1}(\theta_1,\varphi_1)\,sin\,\theta_1 d\theta_1\,d\varphi_1 \tag{4.2.25}$$

This gives for $m_1 = 0$,

$$C_{n_1 0} + \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2}\, \zeta_2^{(1)}(m_2, n_2 \mid 0, n_1 \mid d_{12}, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d_{12}}\right)^{n_1} + \sum_{n_3=0}^{\infty} \sum_{m_3=-n_3}^{n_2} E_{n_3 m_3}\, \zeta_3^{(1)}(m_3, n_3 \mid 0, n_1 \mid d_{13}, \theta_{31}, \varphi_{31}) \left(\frac{a_1}{d_{13}}\right)^{n_1} = 0, \quad n_1 = 1, 2, 3, \ldots \tag{4.2.26a}$$

$m_1 = 1$,

$$C_{11} + \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2}\, \zeta_2^{(1)}(m_2, n_2 \mid 1, 1 \mid d_{12}, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d_{12}}\right) + \sum_{n_3=0}^{\infty} \sum_{m_3=-n_3}^{n_3} E_{n_3 m_3}\, \zeta_3^{(1)}(m_3, n_3 \mid 1, 1 \mid d_{13}, \theta_{31}, \varphi_{31}) \left(\frac{a_1}{d_{13}}\right) = -\frac{1}{2} a_1 H_0, \quad n_1 = 1 \tag{4.2.26b}$$

$$C_{n_1 1} + \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2}\, \zeta_2^{(1)}(m_2, n_2 \mid 1, n_1 \mid d_{12}, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d_{12}}\right)^{n_1} + \sum_{n_3=0}^{\infty} \sum_{m_3=-n_3}^{n_3} E_{n_3 m_3}\, \zeta_3^{(1)}(m_3, n_3 \mid 1, n_1 \mid d_{13}, \theta_{31}, \varphi_{31}) \left(\frac{a_1}{d_{13}}\right)^{n_1} = 0, \quad n_1 = 2, 3, \ldots \tag{4.2.26c}$$

$m_1 = -1$,

$$C_{1,-1} + \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2}\, \zeta_2^{(1)}(m_2, n_2 \mid -1, 1 \mid d_{12}, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d_{12}}\right) + \sum_{n_3=0}^{\infty} \sum_{m_3=-n_3}^{n_2} E_{n_3 m_3}\, \zeta_3^{(1)}(m_3, n_3 \mid -1, 1 \mid d_{13}, \theta_{31}, \varphi_{31}) \left(\frac{a_1}{d_{13}}\right) = a_1 H_0, \quad n_1 = 1 \tag{4.2.26d}$$

$$C_{n_1,-1} + \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2}\, \zeta_2^{(1)}(m_2, n_2 \mid -1, n_1 \mid d_{12}, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d_{12}}\right)^{n_1} + \sum_{n_3=0}^{\infty} \sum_{m_3=-n_3}^{n_3} E_{n_3 m_3}\, \zeta_3^{(1)}(m_3, n_3 \mid -1, n_1 \mid d_{13}, \theta_{31}, \varphi_{31}) \left(\frac{a_1}{d_{13}}\right)^{n_1} = 0, \quad n_1 = 2, 3, \ldots \tag{4.2.26e}$$

Similarly for sphere 2, $m_2 = 0$,

$$D_{n_2 0} + \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1 m_1} \, \zeta_1^{(2)}(m_1, n_1 \mid 0, n_2 \mid d_{12}, \theta_{12}, \varphi_{12}) \left(\frac{a_2}{d_{12}} \right)^{n_2} + \sum_{n_3=0}^{\infty} \sum_{m_3=-n_3}^{n_3} E_{n_3 m_3} \, \zeta_3^{(2)}(m_3, n_3 \mid 0, n_2 \mid d_{23}, \theta_{32}, \varphi_{32}) \left(\frac{a_2}{d_{23}} \right)^{n_2} = 0, \quad n_2 = 1, 2, 3, \ldots \tag{4.2.27a}$$

$m_2 = 1$,

$$D_{11} + \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1 m_1} \, \zeta_1^{(2)}(m_1, n_1 \mid 1, 1 \mid d_{12}, \theta_{12}, \varphi_{12}) \left(\frac{a_2}{d_{12}} \right) + \sum_{n_3=0}^{\infty} \sum_{m_3=-n_3}^{n_3} E_{n_3 m_3} \, \zeta_3^{(2)}(m_3, n_3 \mid 1, 1 \mid d_{23}, \theta_{32}, \varphi_{32}) \left(\frac{a_2}{d_{23}} \right) = -\frac{1}{2} a_2 H_0, \quad n_2 = 1 \tag{4.2.27b}$$

$$D_{n_2 1} + \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1 m_1} \, \zeta_1^{(2)}(m_1, n_1 \mid 1, n_2 \mid d_{12}, \theta_{12}, \varphi_{12}) \left(\frac{a_2}{d_{12}} \right)^{n_2} + \sum_{n_3=0}^{\infty} \sum_{m_3=-n_3}^{n_3} E_{n_3 m_3} \, \zeta_3^{(2)}(m_3, n_3 \mid 1, n_2 \mid d_{23}, \theta_{32}, \varphi_{32}) \left(\frac{a_2}{d_{23}} \right)^{n_2} = 0, \quad n_2 = 2, 3, \ldots \tag{4.2.27c}$$

$m_2 = -1$,

$$D_{1,-1} + \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1 m_1} \, \zeta_1^{(2)}(m_1, n_1 \mid -1, 1 \mid d_{12}, \theta_{12}, \varphi_{12}) \left(\frac{a_2}{d_{12}} \right) + \sum_{n_3=0}^{\infty} \sum_{m_3=-n_3}^{n_3} E_{n_3 m_3} \, \zeta_3^{(2)}(m_3, n_3 \mid -1, 1 \mid d_{23}, \theta_{32}, \varphi_{32}) \left(\frac{a_2}{d_{23}} \right) = a_2 H_0, \quad n_2 = 1 \tag{4.2.27d}$$

$$D_{n_2,-1} + \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1 m_1} \, \zeta_1^{(2)}(m_1, n_1 \mid -1, n_2 \mid d_{12}, \theta_{12}, \varphi_{12}) \left(\frac{a_2}{d_{12}} \right)^{n_2} + \sum_{n_3=0}^{\infty} \sum_{m_3=-n_3}^{n_3} E_{n_3 m_3} \, \zeta_3^{(2)}(m_3, n_3 \mid -1, n_2 \mid d_{23}, \theta_{32}, \varphi_{32}) \left(\frac{a_2}{d_{23}} \right)^{n_2} = 0, \quad n_2 = 2, 3, \ldots \tag{4.2.27e}$$

For sphere 3, $m_3 = 0$,

$$E_{n_30} + \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1m_1}\, \zeta_1^{(3)}(m_1, n_1 \mid 0, n_3 \mid d_{13}, \theta_{13}, \varphi_{13}) \left(\frac{a_3}{d_{13}}\right)^{n_3} + \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2m_2}\, \zeta_2^{(3)}(m_2, n_2 \mid 0, n_3 \mid d_{23}, \theta_{23}, \varphi_{23}) \left(\frac{a_3}{d_{23}}\right)^{n_3} = 0,$$

$$n_3 = 1, 2, 3, \ldots \qquad (4.2.28a)$$

$m_3 = 1$,

$$E_{11} + \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1m_1}\, \zeta_1^{(3)}(m_1, n_1 \mid 1, 1 \mid d_{13}, \theta_{13}, \varphi_{13}) \left(\frac{a_3}{d_{13}}\right) + \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2m_2}\, \zeta_2^{(3)}(m_2, n_2 \mid 1, 1 \mid d_{23}, \theta_{23}, \varphi_{23}) \left(\frac{a_3}{d_{23}}\right) = -\frac{1}{2} a_3 H_0,$$

$$n_2 = 1 \qquad (4.2.28b)$$

$$E_{n_31} + \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1m_1}\, \zeta_1^{(2)}(m_1, n_1 \mid 1, n_3 \mid d_{13}, \theta_{13}, \varphi_{13}) \left(\frac{a_3}{d_{13}}\right)^{n_3} + \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2m_2}\, \zeta_2^{(3}(m_2, n_2 \mid 1, n_3 \mid d_{23}, \theta_{23}, \varphi_{23}) \left(\frac{a_3}{d_{23}}\right)^{n_3} = 0,$$

$$n_3 = 2, 3, \ldots \qquad (4.2.28c)$$

$m_3 = -1$,

$$E_{1,-1} + \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1m_1}\, \zeta_1^{(3)}(m_1, n_1 \mid -1, 1 \mid d_{13}, \theta_{13}, \varphi_{13}) \left(\frac{a_3}{d_{13}}\right) + \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2m_2}\, \zeta_2^{(3)}(m_2, n_2 \mid -1, 1 \mid d_{23}, \theta_{23}, \varphi_{23}) \left(\frac{a_3}{d_{23}}\right) = a_3 H_0,$$

$$n_3 = 1 \qquad (4.2.28d)$$

$$E_{n_3,-1} + \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} C_{n_1m_1}\, \zeta_1^{(3)}(m_1, n_1 \mid -1, n_3 \mid d_{13}, \theta_{13}, \varphi_{13}) \left(\frac{a_3}{d_{13}}\right)^{n_3} + \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2m_2}\, \zeta_2^{(3)}(m_2, n_2 \mid -1, n_3 \mid d_{23}, \theta_{23}, \varphi_{23}) \left(\frac{a_3}{d_{23}}\right)^{n_3} = 0,$$

$$n_3 = 2, 3, \ldots \qquad (4.2.28e)$$

where

$$\zeta_1^{(2)}(m_1,n_1 \mid m_2,n_2 \mid d_{12},\theta_{12},\varphi_{12}) \equiv (-1)^{m_2+n_2}\frac{(n_1-m_1+n_2+m_2)!}{(n_1-m_1)!(n_2+m_2)!}\left(\frac{a_1}{d_{12}}\right)^{n_1+1}$$
$$\times P_{n_1+n_2}^{m_1-m_2}(cos\,\theta_{12})\,cos\big((m_1-m_2)\varphi_{12}\big),$$
$$\zeta_3^{(2)}(m_3,n_3 \mid m_2,n_2 \mid d_{23},\theta_{32},\varphi_{32}) \equiv (-1)^{m_2+n_2}\frac{(n_3-m_3+n_2+m_2)!}{(n_3-m_3)!(n_2+m_2)!}\left(\frac{a_3}{d_{23}}\right)^{n_3+1}$$
$$\times P_{n_3+n_2}^{m_3-m_2}(cos\,\theta_{32})\,cos\big((m_3-m_2)\varphi_{32}\big),$$
$$\zeta_1^{(3)}(m_1,n_1 \mid m_3,n_3 \mid d_{13},\theta_{13},\varphi_{13}) \equiv (-1)^{m_3+n_3}\frac{(n_1-m_1+n_3+m_3)!}{(n_1-m_1)!(n_3+m_3)!}\left(\frac{a_1}{d_{13}}\right)^{n_1+1}$$
$$\times P_{n_1+n_3}^{m_1-m_3}(cos\,\theta_{13})\,cos\big((m_1-m_3)\varphi_{13}\big),$$
$$\zeta_2^{(3)}(m_2,n_2 \mid m_3,n_3 \mid d_{23},\theta_{23},\varphi_{23}) \equiv (-1)^{m_3+n_3}\frac{(n_2-m_2+n_3+m_3)!}{(n_2-m_2)!(n_3+m_3)!}\left(\frac{a_2}{d_{23}}\right)^{n_2+1}$$
$$\times P_{n_2+n_3}^{m_2-m_3}(cos\,\theta_{23})\,cos\big((m_2-m_3)\varphi_{23}\big)$$

In the case of $m_1, m_2, m_3 \neq 0, \pm 1,$ we have $C_{n_1m_1} = D_{n_2m_2} = E_{n_3m_3} = 0.$

Let's denote

$$S1_{n_2,m_2}^{(n_1,m_1)} \equiv \zeta_2^{(1)}(m_2,n_2 \mid m_1,n_1 \mid d_{12},\theta_{21},\varphi_{21})\left(\frac{a_1}{d_{12}}\right)^{n_1},$$
$$S2_{n_3,m_3}^{(n_1,m_1)} \equiv \zeta_3^{(1)}(m_3,n_3 \mid m_1,n_1 \mid d_{13},\theta_{31},\varphi_{31})\left(\frac{a_1}{d_{13}}\right)^{n_1},$$
$$T1_{n_1,m_1}^{(n_2,m_2)} \equiv \zeta_1^{(2)}(m_1,n_1 \mid m_2,n_2 \mid d_{12},\theta_{12},\varphi_{12})\left(\frac{a_2}{d_{12}}\right)^{n_2},$$
$$T2_{(n_3,m_3}^{(n_2,m_2)} \equiv \zeta_3^{(2)}(m_3,n_3 \mid m_2,n_2 \mid d_{23},\theta_{32},\varphi_{32})\left(\frac{a_2}{d_{23}}\right)^{n_2},$$
$$U1_{n_1,m_1}^{(n_3,m_3)} \equiv \zeta_1^{(3)}(m_1,n_1 \mid m_3,n_3 \mid d_{13},\theta_{13},\varphi_{13})\left(\frac{a_3}{d_{13}}\right)^{n_3},$$
$$U2_{n_2,m_2}^{(n_3,m_3)} \equiv \zeta_2^{(3)}(m_2,n_2 \mid m_3,n_3 \mid d_{23},\theta_{23},\varphi_{23})\left(\frac{a_3}{d_{23}}\right)^{n_3},$$

where $m_1, m_2, m_3 = 0, \pm 1$ and $n_1, n_2, n_3 = 1, \ldots, N$.

The unknown series coefficients satisfy the $9N \times 9N$ matrix equation

$$
\left(\begin{array}{ccccccccccccccccccccccccccccccccc}
1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S1^{(1,0)}_{1,1} & \cdots & S1^{(1,0)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S2^{(1,0)}_{1,1} & \cdots & S2^{(1,0)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S1^{(N,0)}_{N,1} & \cdots & S1^{(N,0)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S2^{(N,0)}_{N,1} & \cdots & S2^{(N,0)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & S1^{(1,1)}_{1,1} & S1^{(1,1)}_{1,2} & \cdots & S1^{(1,1)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & S2^{(1,1)}_{1,1} & S2^{(1,1)}_{1,2} & \cdots & S2^{(1,1)}_{1,N} & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & S1^{(2,1)}_{2,1} & S1^{(2,1)}_{2,2} & \cdots & S1^{(2,1)}_{2,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & S2^{(2,1)}_{2,1} & S2^{(2,1)}_{2,2} & \cdots & S2^{(2,1)}_{2,N} & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & S1^{(N,1)}_{N,1} & S1^{(N,1)}_{N,2} & \cdots & S1^{(N,1)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & S2^{(N,1)}_{N,1} & S2^{(1)}_{N,2} & \cdots & S2^{(N,1)}_{N,N} & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S1^{(1,-1)}_{1,1} & S1^{(1,-1)}_{1,2} & \cdots & S1^{(1,-1)}_{1,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S2^{(1,-1)}_{1,1} & S2^{(1,-1)}_{1,2} & \cdots & S2^{(1,-1)}_{1,N} \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S1^{(2,-1)}_{2,1} & S1^{(2,-1)}_{2,2} & \cdots & S1^{(2,-1)}_{2,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S2^{(2,-1)}_{2,1} & S2^{(2,-1)}_{2,2} & \cdots & S2^{(2,-1)}_{2,N} \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S1^{(N,-1)}_{N,1} & S1^{(N,-1)}_{N,2} & \cdots & S1^{(N,-1)}_{N,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & S2^{(N,-1)}_{N,1} & S2^{(N,-1)}_{N,2} & \cdots & S2^{(N,-1)}_{N,N} \\
T1^{(1,0)}_{1,1} & \cdots & T1^{(1,0)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T2^{(1,0)}_{1,1} & \cdots & T2^{(1,0)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
T1^{(N,0)}_{N,1} & \cdots & T1^{(N,0)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T2^{(N,0)}_{N,1} & \cdots & T2^{(N,0)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & T1^{(1,1)}_{1,1} & T1^{(1,1)}_{1,2} & \cdots & T1^{(1,1)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & T2^{(1,1)}_{1,1} & T2^{(1,1)}_{1,2} & \cdots & T2^{(1,1)}_{1,N} & 0 & 0 & 0 & \cdots & 0 & \\
0 & \cdots & 0 & T1^{(2,1)}_{2,1} & T1^{(2,1)}_{2,2} & \cdots & T1^{(2,1)}_{2,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & T2^{(2,1)}_{2,1} & T2^{(2,1)}_{2,2} & \cdots & T2^{(2,1)}_{2,N} & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & \cdots & 0 & T1^{(N,1)}_{N,1} & T1^{(N,1)}_{N,2} & \cdots & T1^{(N,1)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & T2^{(N,1)}_{N,1} & T2^{(N,1)}_{N,2} & \cdots & T2^{(N,1)}_{N,N} & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T1^{(1,-1)}_{1,1} & T1^{(1,-1)}_{1,2} & \cdots & T1^{(1,-1)}_{1,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T2^{(1,-1)}_{1,1} & T2^{(1,-1)}_{1,2} & \cdots & T2^{(1,-1)}_{1,N} \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T1^{(2,-1)}_{2,1} & T1^{(2,-1)}_{2,2} & \cdots & T1^{(2,-1)}_{(2,N)} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T2^{(2,-1)}_{2,1} & T2^{(2,-1)}_{2,2} & \cdots & T2^{(2,-1)}_{2,N} \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T1^{(N,-1)}_{N,1} & T1^{(N,-1)}_{N,2} & \cdots & T1^{(N,-1)}_{N,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T2^{(N,-1)}_{N,1} & T2^{(N,-1)}_{N,2} & \cdots & T2^{(N,-1)}_{N,N} \\
U1^{(1,0)}_{1,1} & \cdots & U1^{(1,0)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U2^{(1,0)}_{1,1} & \cdots & U2^{(1,0)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
U1^{(N,0)}_{N,1} & \cdots & U1^{(N,0)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U2^{(N,0)}_{N,1} & \cdots & U2^{(N,0)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & U1^{(1,1)}_{1,1} & U1^{(1,1)}_{1,2} & \cdots & U1^{(1,1)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & U2^{(1,1)}_{1,1} & U2^{(1,1)}_{1,2} & \cdots & U2^{(1,1)}_{1,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & U1^{(2,1)}_{2,1} & U1^{(2,1)}_{2,2} & \cdots & U1^{(2,1)}_{2,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & U2^{(2,1)}_{2,1} & U2^{(2,1)}_{2,2} & \cdots & U2^{(2,1)}_{2,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & \cdots & 0 & U1^{(N,1)}_{N,1} & U1^{(N,1)}_{N,2} & \cdots & U1^{(N,1)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & U2^{(N,1)}_{N,1} & U2^{(N,1)}_{N,2} & \cdots & U2^{(N,1)}_{N,N} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U1^{(1,-1)}_{1,1} & U1^{(1,-1)}_{1,2} & \cdots & U1^{(1,-1)}_{1,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U2^{(1,-1)}_{1,1} & U2^{(1,-1)}_{1,2} & \cdots & U2^{(1,-1)}_{1,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U1^{(2,-1)}_{2,1} & U1^{(2,-1)}_{2,2} & \cdots & U1^{(2,-1)}_{2,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U2^{(2,-1)}_{2,1} & U2^{(2,-1)}_{2,2} & \cdots & U2^{(2,-1)}_{2,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
 & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & & & \ddots & & & & \ddots & & & & \ddots & \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U1^{(N,-1)}_{N,1} & U1^{(N,-1)}_{N,2} & \cdots & U1^{(N,-1)}_{N,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U2^{(N,-1)}_{N,1} & U2^{(N,-1)}_{N,2} & \cdots & U2^{(N,-1)}_{N,N} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{array}\right)
\left(\begin{array}{c}
C_{10} \\ \vdots \\ C_{N0} \\ C_{11} \\ C_{21} \\ \vdots \\ C_{N1} \\ C_{1,-1} \\ C_{2,-1} \\ \vdots \\ C_{N,-1} \\ D_{10} \\ \vdots \\ D_{N0} \\ D_{11} \\ D_{21} \\ \vdots \\ D_{N1} \\ D_{1,-1} \\ D_{2,-1} \\ \vdots \\ D_{N,-1} \\ E_{10} \\ \vdots \\ E_{N0} \\ E_{11} \\ E_{21} \\ \vdots \\ E_{N1} \\ E_{1,-1} \\ E_{2,-1} \\ \vdots \\ E_{N,-1}
\end{array}\right)
=
\left(\begin{array}{c}
0 \\ \vdots \\ 0 \\ -\frac{1}{2}a_1 \\ 0 \\ \vdots \\ 0 \\ a_1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{1}{2}a_2 \\ 0 \\ \vdots \\ 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{1}{2}a_3 \\ 0 \\ \vdots \\ 0 \\ a_3 \\ 0 \\ \vdots \\ 0
\end{array}\right) H_0
\tag{4.2.29}
$$

Then, the scalar magnetic potential outside the spheres can be calculated by using the Equation (4.2.19) as

$$\begin{aligned}\Phi^{tot}(r_1,\theta_1,\varphi_1 \mid r_2,\theta_2,\varphi_2 \mid r_3,\theta_3,\varphi_3) = &\sum_{n_1=1}^{N}\sum_{m_1=-1}^{1} C_{n_1m_1}\left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}^{m_1}(cos\,\theta_1)\,cos(m_1\varphi_1)\\ &+\sum_{n_2=1}^{N}\sum_{m_2=-1}^{1} D_{n_2m_2}\left(\frac{a_2}{r_2}\right)^{n_2+1} P_{n_2}^{m_2}(cos\,\theta_2)\,cos(m_2\varphi_2)\\ &+\sum_{n_3=1}^{N}\sum_{m_3=-1}^{1} E_{n_3m_3}\left(\frac{a_3}{r_3}\right)^{n_3+1} P_{n_3}^{m_3}(cos\,\theta_3)\,cos(m_3\varphi_3)\\ &-H_0\,r_1P_1^1(cos\,\theta_1)\,cos\,\varphi_1, \qquad r_1\geq a_1, r_2\geq a_2, r_3\geq a_3\end{aligned} \tag{4.2.30}$$

The magnetic field intensity components in the system (r_i,θ_i,φ_i) are

$$\begin{aligned}H_{r_1} = &\frac{1}{r_1}\sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1}(n_1+1)\,C_{n_1m_1}\left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}^{m_1}(cos\,\theta_1)\,cos(m_1\varphi_1) + H_0P_1^1(cos\,\theta_1)\,cos\,\varphi_1\\ &-\frac{1}{r_1}\sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1} n_1\Bigg\{\sum_{n_2=1}^{\infty}\sum_{m_2=-1}^{1} D_{n_2m_2}\,\zeta_2^{(1)}(m_2,n_2 \mid m_1,n_1 \mid d_{12},\theta_{21},\varphi_{21})\left(\frac{r_1}{d_{12}}\right)^{n_1}\\ &+\sum_{n_3=1}^{\infty}\sum_{m_3=-1}^{1} E_{n_3m_3}\,\zeta_3^{(1)}(m_3,n_3 \mid m_1,n_1 \mid d_{13},\theta_{31},\varphi_{31})\left(\frac{r_1}{d_{13}}\right)^{n_1}\Bigg\}\\ &\times P_{n_1}^{m_1}(cos\,\theta_1)\,cos(m_1\varphi_1),\end{aligned}$$

$$\begin{aligned}H_{\theta_1} = &\frac{sin\,\theta_1}{r_1}\Bigg\{\sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1} C_{n_1m_1}\left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}^{m_1\prime}(cos\,\theta_1)\,cos\,(m_1\varphi_1)\\ &+\sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1}\Bigg\{\sum_{n_2=1}^{\infty}\sum_{m_2=-1}^{1} D_{n_2m_2}\,\zeta_2^{(1)}(m_2,n_2 \mid m_1,n_1 \mid d_{12},\theta_{21},\varphi_{21})\left(\frac{r_1}{d_{12}}\right)^{n_1}\\ &+\sum_{n_3=1}^{\infty}\sum_{m_3=-1}^{1} E_{n_3m_3}\,\zeta_3^{(1)}(m_3,n_3 \mid m_1,n_1 \mid d_{13},\theta_{31},\varphi_{31})\left(\frac{r_1}{d_{13}}\right)^{n_1}\Bigg\}\\ &\times P_{n_1}^{m_1\prime}(cos\,\theta_1)\,cos(m_1\varphi_1) - H_0\,r_1P_1^{1\prime}(cos\,\theta_1)\,cos\,\varphi_1\Bigg\},\end{aligned}$$

$$
\begin{aligned}
H_{\varphi_1} = \frac{1}{r_1 sin\,\theta_1}\Bigg\{ & \sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1} m_1\, C_{n_1 m_1}\left(\frac{a_1}{r_1}\right)^{n_1+1} P_{n_1}^{m_1}(cos\,\theta_1)\, sin\,(m_1\varphi_1) \\
& + \sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1} m_1 \Bigg\{ \sum_{n_2=1}^{\infty}\sum_{m_2=-1}^{1} D_{n_2 m_2}\, \zeta_2^{(1)}(m_2, n_2 \mid m_1, n_1 \mid d_{12}, \theta_{21}, \varphi_{21}) \left(\frac{r_1}{d_{12}}\right)^{n_1} \\
& \qquad + \sum_{n_3=1}^{\infty}\sum_{m_3=-1}^{1} E_{n_3 m_3}\, \zeta_3^{(1)}(m_3, n_3 \mid m_1, n_1 \mid d_{13}, \theta_{31}, \varphi_{31}) \left(\frac{r_1}{d_{13}}\right)^{n_1} \Bigg\} \\
& \qquad \times P_{n_1}^{m_1}(cos\,\theta_1)\, sin(m_1\varphi_1) - H_0\, r_1 P_1^1(cos\,\theta_1)\, sin\,\varphi_1 \Bigg\}, \quad d_{12}, d_{13} \geq r_1 \geq a_1
\end{aligned}
\tag{4.2.31}
$$

The potentials inside the spheres $(r_1 < a_1, r_2 < a_2$ ***and*** $r_3 < a_3)$

The potential inside the spheres can be expressed by taking the real part of the Equation (2.1.4) as

$$
\Psi^{(1)}(r_1, \theta_1, \varphi_1) = \sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1} G_{n_1 m_1}^{(1)} r_1^{n_1} P_{n_1}^{m_1}(cos\,\theta_1)\, cos(m_1\varphi_1), \quad r_1 < a_1 \tag{4.2.32a}
$$

$$
\Psi^{(2)}(r_2, \theta_2, \varphi_2) = \sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2} G_{n_2 m_2}^{(2)} r_2^{n_2} P_{n_2}^{m_2}(cos\,\theta_2)\, cos(m_2\varphi_2), \quad r_2 < a_2 \tag{4.2.32b}
$$

$$
\Psi^{(3)}(r_3, \theta_3, \varphi_3) = \sum_{n_3=0}^{\infty}\sum_{m_3=-n_3}^{n_3} G_{n_3 m_3}^{(3)} r_3^{n_3} P_{n_3}^{m_3}(cos\,\theta_3)\, cos(m_3\varphi_3), \quad r_3 < a_3 \tag{4.2.32c}
$$

where $G_{n_1 m_1}^{(1)}, G_{n_2 m_2}^{(2)}$ and $G_{n_3 m_3}^{(3)}$ are the constants of integration. To determine these constants we impose the boundary condition that the normal component of the total magnetic flux density is continuous across the surface of each sphere, i.e.,

$$
-\frac{\partial}{\partial r_i}\left\{\Psi^{(i)}(r_i, \theta_i, \varphi_i)\right\}\Bigg|_{r_i\, =\, a_i} = -\mu_0 \frac{\partial}{\partial r_i}\left\{\Phi^{tot}(r_i, \theta_i, \varphi_i)\right\}\Bigg|_{r_i\, =\, a_i}, \qquad i = 1, 2, 3 \tag{4.2.33}
$$

where μ_0 is the permeability of the medium outside the spheres.

This yields for sphere 1

$$\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1} G^{(1)}_{n_1 m_1}\, n_1 P^{m_1}_{n_1}(cos\,\theta_1)\, cos(m_1\varphi_1) = \mu_0 \Bigg\{ \Bigg\{ \sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1} -(n_1+1)\, C_{n_1 m_1}$$
$$+ \sum_{n_2=1}^{\infty}\sum_{m_2=-n_2}^{n_2}\sum_{n_1=1}^{\infty}\sum_{m_1=-n_1}^{n_1} n_1\, D_{n_2 m_2}\, \zeta^{(1)}_2(m_2, n_2 \mid m_1, n_1 \mid d_{12}, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d_{12}}\right)^{n_1}$$
$$+ \sum_{n_3=1}^{\infty}\sum_{m_3=-n_3}^{n_3}\sum_{n_1=1}^{\infty}\sum_{m_1=-n_1}^{n_1} n_1\, E_{n_3 m_3}\, \zeta^{(1)}_3(m_3, n_3 \mid m_1, n_1 \mid d_{13}, \theta_{31}, \varphi_{31}) \left(\frac{a_1}{d_{13}}\right)^{n_1} \Bigg\}$$
$$\times P^{m_1}_{n_1}(cos\,\theta_1)\, e^{-jm_1\varphi_1} - a_1\, H_0 P^1_1(cos\,\theta_1)\, cos\,\varphi_1 \Bigg\}, \quad n_1 \neq 0$$

Using the orthogonality of the spherical harmonics (see C.6) we obtain

$$n_1 G^{(1)}_{n_1 m_1} = \mu_0 \Bigg\{ -(n_1+1)\, C_{n_1 m_1} - a_1\, H_0 \sqrt{\frac{(2\,n_1+1)}{4\pi}\,\frac{(n_1-m_1)!}{(n_1+m_1)!}}$$
$$\times \int_{\theta_1=0}^{\pi}\int_{\varphi_1=0}^{2\pi} P^1_1(cos\,\theta_1)\, cos\,\varphi_1\, \overline{Y}^{m_1}_{n_1}(\theta_1, \varphi_1)\, sin\,\theta_1\, d\theta_1\, d\varphi_1$$
$$+ \sum_{n_2=1}^{\infty}\sum_{m_2=-n_2}^{n_2} n_1 D_{n_2 m_2}\, \zeta^{(1)}_2(m_2, n_2 \mid m_1, n_1 \mid d_{12}, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d_{12}}\right)^{n_1}$$
$$+ \sum_{n_3=1}^{\infty}\sum_{m_3=-n_3}^{n_3} n_1 E_{n_3 m_3}\, \zeta^{(1)}_3(m_3, n_3 \mid m_1, n_1 \mid d_{13}, \theta_{31}, \varphi_{31}) \left(\frac{a_1}{d_{13}}\right)^{n_1} \Bigg\} \qquad (4.2.34)$$

and, finally, we have the following results for $m_1 = 0, +1, -1$ as

$$G^{(1)}_{n_1 0} = \mu_0 \, \frac{(2n_1+1)}{n_1} \Bigg\{ \sum_{n_2=1}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2} \, \zeta_2^{(1)}(m_2, n_2 \mid 0, n_1 \mid d_{12}, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d_{12}} \right)^{n_1}$$

$$+ \sum_{n_3=1}^{\infty} \sum_{m_3=-n_3}^{n_3} E_{n_3 m_3} \, \zeta_3^{(1)}(m_3, n_3 \mid 0, n_1 \mid d_{13}, \theta_{31}, \varphi_{31}) \left(\frac{a_1}{d_{13}} \right)^{n_1} \Bigg\},$$

$$n_1 = 1, 2, 3, \ldots \qquad (4.2.35a)$$

$$G^{(1)}_{11} = 3\mu_0 \Bigg\{ \sum_{n_2=1}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2} \, \zeta_2^{(1)}(m_2, n_2 \mid 1, 1 \mid d_{12}, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d_{12}} \right)$$

$$+ \sum_{n_3=1}^{\infty} \sum_{m_3=-n_3}^{n_3} E_{n_3 m_3} \, \zeta_3^{(1)}(m_3, n_3 \mid 1, 1 \mid d_{13}, \theta_{31}, \varphi_{31}) \left(\frac{a_1}{d_{13}} \right) + \frac{1}{2} a_1 H_0 \Bigg\},$$

$$n_1 = 1 \qquad (4.2.35b)$$

$$G^{(1)}_{n_1 1} = \mu_0 \, \frac{(2n_1+1)}{n_1} \Bigg\{ \sum_{n_2=1}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2} \, \zeta_2^{(1)}(m_2, n_2 \mid 1, n_1 \mid d_{12}, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d_{12}} \right)^{n_1}$$

$$+ \sum_{n_3=1}^{\infty} \sum_{m_3=-n_3}^{n_3} E_{n_3 m_3} \, \zeta_3^{(1)}(m_3, n_3 \mid 1, n_1 \mid d_{13}, \theta_{31}, \varphi_{31}) \left(\frac{a_1}{d_{13}} \right)^{n_1} \Bigg\},$$

$$n_1 = 2, 3, \ldots \qquad (4.2.35c)$$

$$G^{(1)}_{1,-1} = 3\mu_0 \Bigg\{ \sum_{n_2=1}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2} \, \zeta_2^{(1)}(m_2, n_2 \mid -1, 1 \mid d_{12}, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d_{12}} \right)$$

$$+ \sum_{n_3=1}^{\infty} \sum_{m_3=-n_3}^{n_3} E_{n_3 m_3} \, \zeta_3^{(1)}(m_3, n_3 \mid -1, n_1 \mid d_{13}, \theta_{31}, \varphi_{31}) \left(\frac{a_1}{d_{13}} \right) - a_1 H_0 \Bigg\},$$

$$n_1 = 1 \qquad (4.2.35d)$$

$$G^{(1)}_{n_1,-1} = \mu_0 \, \frac{(2n_1+1)}{n_1} \Bigg\{ \sum_{n_2=1}^{\infty} \sum_{m_2=-n_2}^{n_2} D_{n_2 m_2} \, \zeta_2^{(1)}(m_2, n_2 \mid -1, n_1 \mid d_{12}, \theta_{21}, \varphi_{21}) \left(\frac{a_1}{d_{12}} \right)^{n_1}$$

$$+ \sum_{n_3=1}^{\infty} \sum_{m_3=-n_3}^{n_3} E_{n_3 m_3} \, \zeta_3^{(1)}(m_3, n_3 \mid 1, n_1 \mid d_{13}, \theta_{31}, \varphi_{31}) \left(\frac{a_1}{d_{13}} \right)^{n_1} \Bigg\},$$

$$n_1 = 2, 3, \ldots \qquad (4.2.35e)$$

Similar equations are derived for sphere 2 and sphere 3. As before, for $m_1, m_2, m_3 \neq 0, \pm 1$, we obtain $C_{n_1 m_1} = D_{n_2 m_2} = E_{n_3 m_3} = 0$.

The magnetic flux density components inside the spheres are determined from $\mathbf{B} = -\nabla\Psi$ and Equation (4.2.32) as follows.

$$\begin{aligned}
B_{r_1} &= -\frac{1}{r_1}\sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1} n_1\, G^{(1)}_{n_1 m_1}\left(\frac{r_1}{a_1}\right)^{n_1} P^{m_1}_{n_1}(cos\,\theta_1)\, cos(m_1\varphi_1),\\
B_{\theta_1} &= \frac{sin\,\theta_1}{r_1}\sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1} G^{(1)}_{n_1 m_1}\left(\frac{r_1}{a_1}\right)^{n_1} P^{m_1}_{n_1}{}'(cos\,\theta_1)\, cos(m_1\varphi_1),\\
B_{\varphi_1} &= \frac{1}{r_1 sin\,\theta_1}\sum_{n_1=1}^{\infty}\sum_{m_1=-1}^{1} m_1\, G^{(1)}_{n_1 m_1}\left(\frac{r_1}{a_1}\right)^{n_1} P^{m_1}_{n_1}(cos\,\theta_1)\, sin(m_1\varphi_1), \qquad r_1 < a_1
\end{aligned} \tag{4.2.36}$$

$$\begin{aligned}
B_{r_2} &= -\frac{1}{r_2}\sum_{n_2=1}^{\infty}\sum_{m_2=-1}^{1} n_2\, G^{(2)}_{n_2 m_2}\left(\frac{r_2}{a_2}\right)^{n_2} P^{m_2}_{n_2}(cos\,\theta_2)\, cos(m_2\varphi_2),\\
B_{\theta_2} &= \frac{sin\,\theta_2}{r_2}\sum_{n_2=1}^{\infty}\sum_{m_2=-1}^{1} G^{(2)}_{n_2 m_2}\left(\frac{r_2}{a_2}\right)^{n_2} P^{m_2}_{n_2}{}'(cos\,\theta_2)\, cos(m_2\varphi_2),\\
B_{\varphi_2} &= \frac{1}{r_2 sin\,\theta_2}\sum_{n_2=1}^{\infty}\sum_{m_2=-1}^{1} m_2\, G^{(2)}_{n_2 m_2}\left(\frac{r_2}{a_2}\right)^{n_1} P^{m_2}_{n_2}(cos\,\theta_2)\, sin(m_2\varphi_2), \qquad r_2 < a_2
\end{aligned} \tag{4.2.37}$$

$$\begin{aligned}
B_{r_3} &= -\frac{1}{r_3}\sum_{n_3=1}^{\infty}\sum_{m_3=-1}^{1} n_3\, G^{(3)}_{n_3 m_3}\left(\frac{r_3}{a_3}\right)^{n_3} P^{m_3}_{n_3}(cos\,\theta_3)\, cos(m_3\varphi_3),\\
B_{\theta_3} &= \frac{sin\,\theta_3}{r_3}\sum_{n_3=1}^{\infty}\sum_{m_3=-1}^{1} G^{(3)}_{n_3 m_3}\left(\frac{r_3}{a_3}\right)^{n_3} P^{m_3}_{n_3}{}'(cos\,\theta_3)\, cos(m_3\varphi_3),\\
B_{\varphi_3} &= \frac{1}{r_3 sin\,\theta_3}\sum_{n_3=1}^{\infty}\sum_{m_3=-1}^{1} m_3\, G^{(3)}_{n_3 m_3}\left(\frac{r_3}{a_3}\right)^{n_1} P^{m_3}_{n_3}(cos\,\theta_3)\, sin(m_3\varphi_3), \qquad r_3 < a_3
\end{aligned} \tag{4.2.38}$$

4.3 Chapter Summary

In this chapter, analytical expressions were derived for the magnetic scalar potential and the field quantities for systems of arbitrarily located ideal ferromagnetic spheres, both inside and outside the spheres, in the presence of uniform magnetic fields. The total scalar magnetic potential outside the spheres was obtained by the superposition of the potentials due to all spheres and the potential corresponding to the external field. The translational addition theorems were employed in order to express the potential in the coordinate system attached to a specific sphere. The exact analytical expressions derived can be used to generate numerical results for a large range of values for sphere radii and for the relative distances between the spheres.

Chapter 5

Numerical Results and Discussions

This chapter presents numerical computations for the cases in chapter 3 and chapter 4 using the respective analytical expressions. Some discussions and explanations regarding the tabulated results are also given. Numerical results are generated by employing the mathematical tool Matlab.

5.1 Computations for Coaxial Spheres

Two-sphere system in the presence of external field along z axis

The infinite systems of linear equations in (3.1.15) and (3.1.17) have been truncated by retaining the first 20 coefficients of each C_n and D_m (excluding $C_0 = D_0 = 0$), which is found to be sufficient in order to obtain at least a 5-digit accuracy. The 40×40 matrix in (3.1.18) was used to compute the series coefficients and, then, the magnetic potentials and field intensities in spherical coordinate $(r_1, \theta_1, \varphi_1)$, using the Equations (3.1.19) and (3.1.21), respectively. Some of the numerical values are tabulated in Table 5.1.

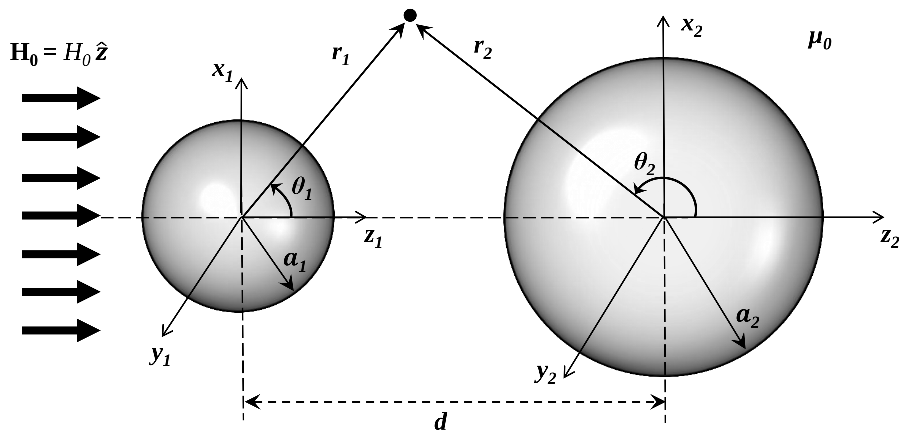

FIGURE 5.1: Two ferromagnetic spheres in an external field along z axis

TABLE 5.1: Numerical results for the magnetic potential and the field intensity in coordinates (r_1, θ_1) at some points outside a two-sphere system with $a_1 = 3$ cm, $a_2 = 5$ cm, $d = 10$ cm, in the presence of external field $H_0 = 1$ A/m along z axis.

Point	Spherical coordinate (r_i, θ_i)				Potential	Field intensity (A/m), $H_1 = \sqrt{H_{r1}^2 + H_{\theta 1}^2}$		
	r_1 (cm)	θ_1 (deg)	r_2 (cm)	θ_2 (deg)	ϕ^{tot} (A)	H_{r1}	$H_{\theta 1}$	H_1
1	3.0000	0.0000	7.0000	180.0000	-0.0139	5.2874	0.0000	5.2874
2	4.0000	0.0000	6.0000	180.0000	-0.0567	3.7792	0.0000	3.7792
3	5.0000	0.0000	5.0000	180.0000	-0.0964	4.4934	0.0000	4.4934
4	5.0000	-90.0000	11.1803	-153.4349	-0.0102	-0.1463	0.8659	0.8781
5	4.5000	-60.0000	8.6747	-153.3043	-0.0306	0.7704	0.8931	1.1795
6	4.0000	-30.0000	6.8351	-162.9858	-0.0437	2.3819	0.8975	2.5453
7	7.1517	36.3868	6.0000	135.0000	-0.0783	0.6004	-1.4568	1.5756
8	11.9269	33.0239	6.5000	90.0000	-0.0974	0.4860	-0.2548	0.5487
9	14.7986	24.1825	7.0000	60.0000	-0.1199	1.0578	0.0855	1.0613
10	18.0000	0.0000	8.0000	0.0000	-0.1584	1.5203	0.0000	1.5203

On the other hand, the Table 5.2 shows numerical values for the magnetic potentials and flux densities inside of each sphere obtained by using the Equations (3.1.23), (3.1.32) and (3.1.33). 20 coefficients $G_n^{(1)}$ and $G_m^{(2)}$ (excluding $G_0^{(1)} = G_0^{(2)} = 0$) have been used, with 20 terms in their respective series in (3.1.28) and (3.1.30). The value for the permeability of the medium outside has been taken to be $\mu_0 = 4\pi \times 10^{-7}$ H/m.

TABLE 5.2: Numerical results for the potential and flux density at selected points inside of each sphere of the two-sphere system in the presence of external field along z axis: $a_1 = 3$ cm, $a_2 = 5$ cm, $d = 10$ cm, $H_0 = 1$ A/m

Inside point	Sphere 1		Potential	Flux
	r_1 (cm)	θ_1 (deg)	$\psi^{(1)}$ (Tm)	$B^{(1)}$ (T)
1	0.2000	90.0000	2.7849E-11	4.8470E-06
2	1.5000	90.0000	1.2266E-09	4.6991E-06
3	2.9000	90.0000	1.8630E-09	4.0197E-06
4	2.5000	120.0000	5.9697E-08	4.6092E-06
5	2.0000	150.0000	8.1829E-08	4.6496E-06
6	1.5000	180.0000	7.0952E-08	4.6666E-06
7	1.2500	-135.0000	4.2263E-08	4.6826E-06
8	1.0000	-90.0000	6.2357E-10	4.7717E-06
9	0.7500	-45.0000	-2.5839E-08	5.0141E-06
10	0.5000	0.0000	-2.4696E-08	5.0508E-06

Inside point	Sphere 2		Potential	Flux
	r_2 (cm)	θ_2 (deg)	$\psi^{(2)}$ (Tm)	$B^{(2)}$ (T)
1	0.2000	0.0000	-8.0787E-09	4.0321E-06
2	2.5000	0.0000	-9.7991E-08	3.7070E-06
3	4.9000	0.0000	-1.2796E-07	1.0136E-05
4	4.5000	30.0000	-1.5503E-07	3.9134E-06
5	4.0000	60.0000	-8.2408E-08	4.0049E-06
6	3.5000	90.0000	-1.7536E-09	4.0657E-06
7	3.0000	120.0000	6.0347E-08	4.1285E-06
8	2.0000	180.0000	8.2304E-08	4.1881E-06
9	1.0000	-135.0000	2.8701E-08	4.0944E-06
10	0.5000	-90.0000	-4.3900E-11	4.0473E-06

Two-sphere system in the presence of external field along x axis

Numerical results for the potential in (3.1.49) and the field intensity in (3.1.50) at some points outside the spheres are given in Table 5.3. The infinite systems of linear equations in (3.1.44) and (3.1.47) have been truncated by retaining the coefficients C_{nm} and D_{qp} in (3.1.48) for N = 20, which is found to be sufficient in order to obtain at least a 5-digit accuracy. The matrix equation (3.1.48) has 120 unknowns. The results are generated in spherical coordinate $(r_1, \theta_1, \varphi_1)$ attached to sphere 1.

TABLE 5.3: Numerical results for the magnetic potential and field intensity in spherical coordinate $(r_1, \theta_1, \varphi_1)$ outside of a two-sphere system in the presence of external field along x axis: $a_1 = 3$ cm, $a_2 = 5$ cm, $d = 10$ cm, $H_0 = 1$ A/m

Point	Spherical coordinate (r_i, θ_i, φ_i)						Potential	Field intensity
	r_1 (cm)	θ_1 (deg)	φ_1 (deg)	r_2 (cm)	θ_2 (deg)	φ_2 (deg)	ϕ^{tot} (A)	H_1 (A/m)
1	5.0000	-90.0000	150.0000	11.1803	-153.4349	150.0000	-0.0193	1.7279
2	4.5000	-60.0000	120.0000	8.6747	-153.3043	120.0000	-0.0026	1.1686
3	4.0000	-30.0000	0.0000	6.8351	-162.9858	0.0000	-0.0077	2.6242
4	5.9150	27.7050	-30.0000	5.5000	150.0000	-30.0000	-0.0618	2.6623
5	7.1517	36.3868	-45.0000	6.0000	135.0000	-45.0000	-0.0667	2.0386
6	11.9269	33.0239	-60.0000	6.5000	90.0000	-60.0000	-0.0624	1.4160
7	14.7986	24.1825	-70.0000	7.0000	60.0000	-70.0000	-0.0360	0.8256
8	16.9161	12.8079	-85.0000	7.5000	30.0000	-85.0000	-0.0052	0.4308

In Table 5.4 we tabulate the magnetic flux density inside each sphere computed by using (3.1.57) and (3.1.58). The infinite system of linear equations in (3.1.55) and (3.1.56) has been truncated each by the same number of coefficients as for Table 5.3. Numerical values of associated Legendre functions, P_n^m have been computed using a Matlab built-in subroutine and the derivatives of P_n^m have been calculated by using the recurrence formula (see B.7) for the associated Legendre functions.

TABLE 5.4: Magnetic flux density values at selected points inside each sphere for a two-sphere system in the presence of external field along x axis: $a_1 = 3$ cm, $a_2 = 5$ cm, $d = 10$ cm, $H_0 = 1$ A/m

Inside point	Sphere-1			Flux
	r_1 (cm)	θ_1 (deg)	φ_1 (deg)	$B^{(1)}$ (T)
1	0.2000	90.0000	0.0000	9.4047E-06
2	2.9000	90.0000	0.0000	1.1684E-05
3	2.5000	120.0000	0.0000	8.4720E-06
4	2.0000	150.0000	0.0000	4.8288E-06
5	1.5000	-150.0000	0.0000	4.8246E-06
6	1.2500	-135.0000	0.0000	6.8092E-06
7	1.0000	-90.0000	0.0000	9.4865E-06
8	0.7500	-45.0000	0.0000	6.4122E-06

Inside point	Sphere-2			Flux
	r_2 (cm)	θ_2 (deg)	φ_2 (deg)	$B^{(2)}$ (T)
1	4.9000	5.0000	0.0000	2.3424E-05
2	4.5000	30.0000	0.0000	5.7529E-06
3	4.0000	60.0000	0.0000	9.1076E-06
4	3.5000	90.0000	0.0000	1.0406E-05
5	3.0000	120.0000	0.0000	8.9313E-06
6	2.0000	150.0000	0.0000	5.1347E-06
7	1.0000	-135.0000	0.0000	7.3258E-06
8	0.5000	-90.0000	0.0000	1.0419E-05

Three-sphere system in the presence of external field along z axis

Table 5.5 shows numerical values for the field intensity (by 3.2.17) in $(r_1, \theta_1, \varphi_1)$ at the points $\mathbf{P_1}, \mathbf{P_2}$ and $\mathbf{P_3}$ shown in Figure 5.2. The three spheres are coaxial and of the same radius a, with the same gap g between consecutive spheres. Results are given for six different ratios g/a. It is seen that the field values increase substantially in the gaps when the spheres are closed to each other.

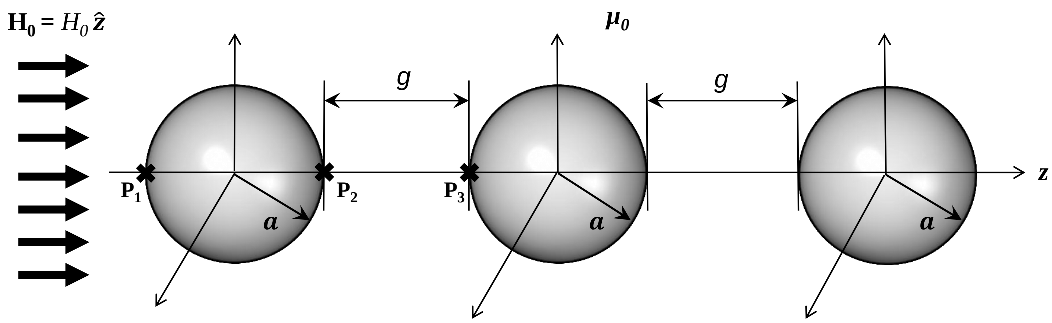

FIGURE 5.2: Three ferromagnetic spheres in an external field along z axis

TABLE 5.5: Numerical results for the magnetic field intensity at some points in the three-sphere system shown in Figure 5.2 in the presence of external field along z axis for different ratios g/a: $a_1 = a_2 = a_3 = a = 2$ cm, $H_0 = 1$ A/m, $N = 20$ in (3.2.15)

Point	Field	g/a					
		1.000	0.500	0.100	0.050	0.010	0.005
$\mathbf{P_1}$	H_z	3.1514	3.2701	3.5894	3.6913	3.6000	3.4595
$\mathbf{P_2}$		3.8068	5.2259	16.0682	27.8256	80.7905	106.2011
$\mathbf{P_3}$		3.8550	5.2568	16.0640	27.8041	80.7262	106.1484

In Table 5.6 numerical values are tabulated for the magnetic flux density inside each sphere in their attached coordinate system.

TABLE 5.6: Numerical results for the magnetic flux density at some points inside of each sphere of a three-sphere system in the presence of external field along z axis: $a_1 = a_2 = a_3 = a = 2$ cm, $g = 2$ cm, $H_0 = 1$ A/m, $N = 20$ in (3.2.23) - (3.2.25)

Inside point	Sphere-i		Flux density, $B^{(i)}$ (T) for Sphere i		
	r_i (cm)	θ_i (deg)	Sphere 1, $B^{(1)}$	Sphere 2, $B^{(2)}$	Sphere 3, $B^{(3)}$
1	0.2000	-90.0000	4.1341E-06	4.3971E-06	4.1381E-06
2	0.7500	-90.0000	4.1029E-06	4.3792E-06	4.1506E-06
3	1.0000	-30.0000	4.3688E-06	4.4553E-06	3.9703E-06
4	1.0000	60.0000	4.1394E-06	4.3609E-06	4.0963E-06
5	1.5000	90.0000	4.0478E-06	4.3479E-06	4.1723E-06
6	1.9000	150.0000	4.2626E-06	4.6192E-06	4.2446E-06

Three-sphere system in the presence of external field along x axis

Generated numerical values for the potential and for the field intensity outside the spheres in the spherical coordinate system $(r_1, \theta_1, \varphi_1)$ have been tabulated in Table 5.7.

TABLE 5.7: Numerical results for various points outside the three-sphere system in the presence of external field along x axis: $a_1 = a_2 = a_3 = a = 2$ cm, $g = 2$ cm, $H_0 = 1$ A/m, $N = 20$ in (3.2.37)

Point	Spherical coordinate (r_1,θ_1,φ_1)			Potential	Field intensity (A/m), $H_1 = \sqrt{H_{r1}^2 + H_{\theta 1}^2 + H_{\varphi 1}^2}$			
	r_1 (cm)	θ_1 (deg)	φ_1 (deg)	ϕ^{tot} (A)	H_{r1}	$H_{\theta 1}$	$H\varphi_1$	H_1 (A/m)
1	4.0000	-90.0000	150.0000	-0.0247	1.4474	-0.0213	-0.3561	1.4907
2	3.5000	-60.0000	120.0000	-0.0080	0.9028	-0.4038	-0.4577	1.0897
3	3.0000	-30.0000	0.0000	0.0022	-1.3329	1.7100	0.0000	2.1681
4	3.7179	23.7940	-30.0000	-0.0241	-0.8218	-1.6624	-0.0716	1.8558
5	5.2202	24.5036	-45.0000	-0.0320	-0.8993	-1.9323	0.0654	2.1323
6	6.3246	18.4349	-60.0000	-0.0300	-1.1036	-2.3374	0.8660	2.7261
7	12.3693	14.0362	-45.0000	-0.0345	-0.4029	-1.5119	-0.2628	1.5866
8	13.7477	10.8934	-30.0000	-0.0363	3.0190	-1.6984	-0.1941	3.4694
9	14.6749	5.8667	0.0000	-0.0241	3.4553	-1.7161	0.0000	3.8580

Numerical values for the magnetic flux density inside each sphere, in their attached coordinate system, are given in Table 5.8. The points inside were selected to be on the $z - x$ plane of each sphere.

TABLE 5.8: Numerical results for the magnetic flux density at selected points inside of each sphere of the three-sphere system in the presence of external field along x axis: $a_1 = a_2 = a_3 = a = 2$ cm, $g = 2$ cm, $H_0 = 1$ A/m with $N = 20$ in (3.2.42) - (3.2.44)

Inside point	Spherical coordinate (r_i, θ_i, ϕ_i)			Flux density, $B^{(i)}$ (T) for Sphere i		
	r_i (cm)	θ_i (deg)	ϕ_i (deg)	Sphere 1, $B^{(1)}$	Sphere 2, $B^{(2)}$	Sphere 3, $B^{(3)}$
1	0.2000	90.0000	0.0000	1.0254E-05	9.9176E-06	1.0249E-05
2	2.9000	90.0000	0.0000	1.8142E-03	8.8781E-04	9.0780E-04
3	2.5000	120.0000	0.0000	1.3367E-04	6.8306E-05	5.8232E-05
4	2.0000	150.0000	0.0000	3.9858E-06	4.1974E-06	5.6704E-06
5	1.5000	-150.0000	0.0000	5.1638E-06	4.8700E-06	4.9980E-06
6	1.2500	-135.0000	0.0000	7.3106E-06	6.9558E-06	7.1296E-06
7	1.0000	-90.0000	0.0000	1.0308E-05	9.9506E-06	1.0229E-05
8	0.7500	-45.0000	0.0000	7.1138E-06	6.9854E-06	7.3567E-06

The computer program and the numerical results generated have been validated by considering some limiting geometric configurations and also the symmetry of the structures involved.

For instance, by increasing the distance between the spheres to large values, the field about each sphere is practically the same as in the case of only one sphere in the presence of the external field, the numerical results being approximately those given by the expressions in Section 2.4.

As well, the field quantities at some points are equal due to symmetry of the corresponding system. For example, the magnetic flux densities are the same at the points $\mathbf{P_1}$ and $\mathbf{P_4}$, and $\mathbf{P_2}$ and $\mathbf{P_3}$ shown in Figure 5.3. Both the magnetic field intensities and magnetic flux densities are identical at $\mathbf{P_5}$ and $\mathbf{P_8}$, and $\mathbf{P_6}$ and $\mathbf{P_7}$, respectively.

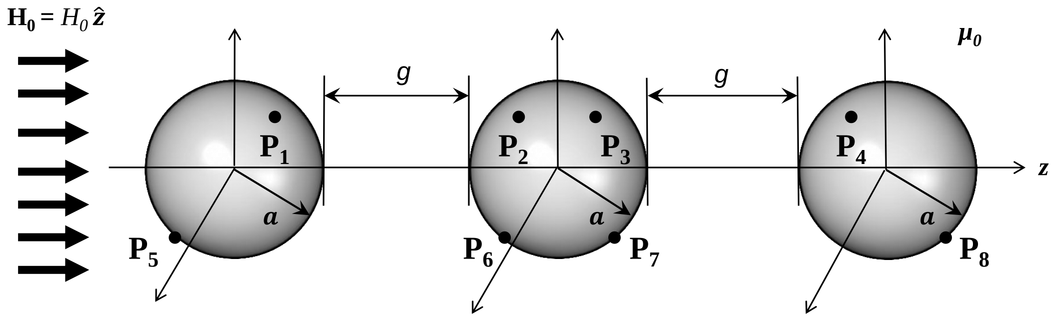

FIGURE 5.3: Symmetrical points for a three-sphere system

Another check of the numerical results has been performed by computing the field quantities at various points outside the spheres using coordinates attached to only one sphere.

5.2 Computations for Arbitrarily Located Spheres

Two-sphere system in the presence of external field along z axis

Using (4.1.15), numerical values for field intensity in spherical coordinate $(r_1, \theta_1, \varphi_1)$ were generated at the points $\mathbf{P_1}, \cdots, \mathbf{P_4}$ shown in Figure 5.4 for different ratios g/a. The results are presented in Table 5.9. The infinite system of linear equations in (4.1.10) and (4.1.11) has been truncated by retaining the first 20 coefficients of each C_{n0} and D_{q0}, (excluding $C_{00} = D_{00} = 0$), and these coefficients were computed using the matrix in (4.1.13), with $N = 20$.

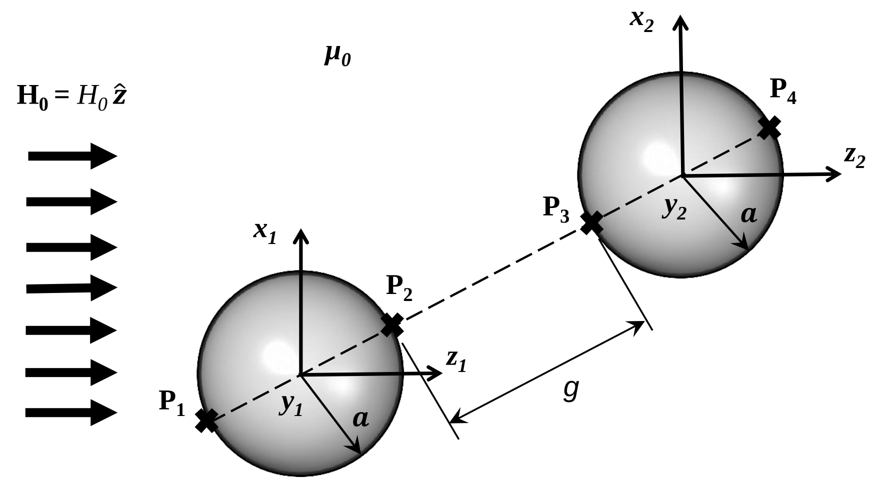

FIGURE 5.4: Two ferromagnetic spheres with same $z - x$ planes placed in an external field along z axis

TABLE 5.9: Magnetic field intensity values in coordinates $(r_1, \theta_1, \varphi_1)$ at $\mathbf{P_1}, \cdots, \mathbf{P_4}$ for a two-sphere system in the presence of external field along z axis for different ratios g/a: $a_1 = a_2 = a = 2$ cm, $\theta_{12} = 30$ deg, $\varphi_{12} = 0$ deg, $H_0 = 1$ A/m, $N = 20$ in (4.1.13)

Point	Field	g / a					
		1.000	0.500	0.100	0.050	0.010	0.005
$\mathbf{P_1}$	H_z	2.8694	2.9246	3.0158	3.0334	3.0490	3.0511
$\mathbf{P_2}$		2.9400	3.3926	4.7061	5.0747	5.4516	5.5056
$\mathbf{P_3}$		2.9400	3.3926	4.7061	5.0747	5.4516	5.5056
$\mathbf{P_4}$		2.6886	2.7472	2.8442	2.8628	2.8794	2.8816

Table 5.10 shows the numerical values of magnetic flux density inside each sphere computed with (4.1.20) and (4.1.21). The infinite system of linear equations in (4.1.18) and (4.1.19) has been truncated by the same number of coefficients, i.e., $N = 20$, for $G_{n0}^{(1)}$ and $G_{q0}^{(2)}$, (excluding $G_{00}^{(1)} = G_{00}^{(2)} = 0$).

TABLE 5.10: Numerical results for flux density inside each sphere for the two-sphere system in the presence of external field along z axis: $a_1 = a_2 = a = 2$ cm, $\theta_{12} = 30$ deg, $\varphi_{12} = 0$ deg, $g = 2$ cm, $H_0 = 1$ A/m, $N = 20$ in (4.1.18) and (4.1.19)

Inside point	Sphere 1		Flux
	r_1 (cm)	θ_1 (deg)	$B^{(1)}$ (T)
1	0.2000	-90.0000	4.4830E-06
2	0.7500	-90.0000	4.4424E-06
3	1.0000	-30.0000	4.8199E-06
4	1.0000	60.0000	4.5445E-06
5	1.5000	90.0000	4.3417E-06
6	1.9000	150.0000	4.3581E-06

Inside point	Sphere 2		Flux
	r_2 (cm)	θ_2 (deg)	$B^{(2)}$ (T)
1	0.2000	-90.0000	4.4895E-06
2	0.7500	-90.0000	4.5245E-06
3	1.0000	-30.0000	4.1882E-06
4	1.0000	60.0000	4.4305E-06
5	1.5000	90.0000	4.5897E-06
6	1.9000	150.0000	4.4966E-06

Two-sphere system in the presence of external field along x axis

Table 5.11 presents numerical values of field intensity at the points $\mathbf{P_1}, \mathbf{P_2}, \mathbf{P_3}$ outside the spheres for different ratios g/a in spherical coordinate $(r_1, \theta_1, \varphi_1)$. The infinite system of linear equations in (4.1.31) and (4.1.32) has been truncated by retaining 60 coefficients of each C_{nm} and D_{qp} (excluding $C_{00} = D_{00} = 0$), which are computed using the matrix equation in (4.1.33), with 120 unknown constants.

TABLE 5.11: Field intensity values at the points $\mathbf{P_1}, \mathbf{P_2}, \mathbf{P_3}$ in coordinate $(r_1, \theta_1, \varphi_1)$ of the two-sphere system in the presence of external field along x axis for different ratios g/a: $a_1 = a_2 = a = 2$ cm, $\theta_{12} = 30$ deg, $\varphi_{12} = 0$ deg, $H_0 = 1$ A/m, $N = 20$ in (4.1.33)

Point	Field	g/a					
		1.000	0.500	0.100	0.050	0.010	0.005
$\mathbf{P_1}$	H_x	1.6757	1.6628	1.6777	1.6796	1.6823	1.6827
$\mathbf{P_2}$		3.5509	5.7204	10.2037	11.2895	12.2300	12.3544
$\mathbf{P_3}$		7.7284	9.8172	11.6563	12.0964	12.4040	12.4422

Numerical values of magnetic flux density inside the spheres are given in Table 5.12. The expressions in (4.1.37) and (4.1.38) have been truncated each to the same number of coefficients, i.e., 60, for $G_{nm}^{(1)}$ and for $G_{qp}^{(2)}$ (excluding $G_{00}^{(1)} = G_{00}^{(2)} = 0$).

TABLE 5.12: Flux density at selected points inside each sphere of a two-sphere system in the presence of external field along x axis: $a_1 = a_2 = a = 2$ cm, $\theta_{12} = 30$ deg, $\varphi_{12} = 0$ deg, $\varphi_i = 0$ deg, $H_0 = 1$ A/m, $N = 20$ in (4.1.37) and (4.1.38)

Inside point	Sphere-1		Flux
	r_1 (cm)	θ_1 (deg)	$B^{(1)}$ (T)
1	0.2000	-90.0000	4.9315E-06
2	0.7500	-90.0000	4.9759E-06
3	1.0000	-30.0000	2.0306E-06
4	1.0000	60.0000	4.2409E-06
5	1.5000	90.0000	5.0653E-06
6	1.9000	150.0000	1.6913E-06

Inside point	Sphere-2		Flux
	r_2 (cm)	θ_2 (deg)	$B^{(2)}$ (T)
1	0.2000	-90.0000	4.9210E-06
2	0.7500	-90.0000	4.8896E-06
3	1.0000	-30.0000	2.7907E-06
4	1.0000	60.0000	4.3266E-06
5	1.5000	90.0000	4.8305E-06
6	1.9000	150.0000	3.4233E-06

Three-sphere system in the presence of external field along z axis

Table 5.13 shows the numerical values of the potential (see (4.2.12)) and field intensity (see (4.2.13)) at the points $\mathbf{P_1}, \mathbf{P_2}, \cdots, \mathbf{P_{10}}$ shown in Figure 5.5. The infinite system of linear equations in (4.2.8) - (4.2.10) have been truncated by retaining the first 20 coefficients of each C_{n0}, D_{q0} and E_{t0} (excluding $C_{00} = D_{00} = E_{00} = 0$), which is found to be sufficient in order to obtain at least a 5-digit accuracy for the numerical results. The numerical values for the required coefficients are computed using the matrix in (4.2.11), which has 60 unknowns.

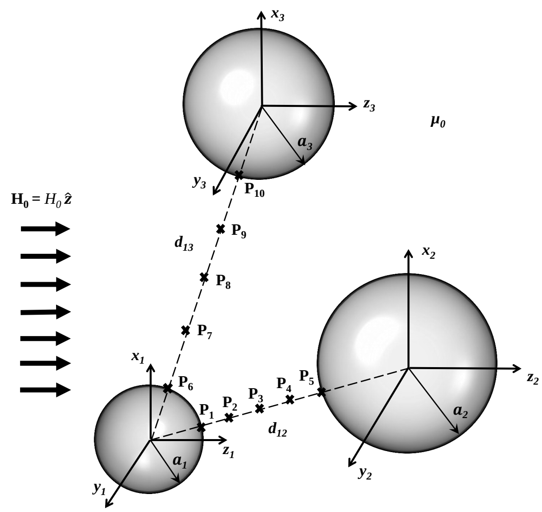

FIGURE 5.5: Three ferromagnetic spheres in arbitrary positions with the common $z-x$ plane in the presence of external magnetic field along z axis

TABLE 5.13: Numerical results for the potential and the field intensity outside of a three-sphere system in coordinates (r_1, θ_1) in the presence of external field along z axis: $a_1 = 3$ cm, $a_2 = 5$ cm, $a_3 = 4$ cm, $d_{12} = 10$ cm, $d_{13} = 20$ cm, $\theta_{12} = 30°$, $\theta_{13} = 60°$, $\varphi_{12} = \varphi_{13} = 0$, $H_0 = 1$ A/m, $N = 20$ in (4.2.11)

Point	Spherical coordinate (r_i, θ_i)						Potential	Field (A/m), $H_1 = \sqrt{H_{r1}^2 + H_{\theta 1}^2}$		
	r_1 (cm)	θ_1 (deg)	r_2 (cm)	θ_2 (deg)	r_3 (cm)	θ_3 (deg)	ϕ^{tot} (A)	H_{r1}	$H_{\theta 1}$	H_1
P_1	3.0000	30.0000	7.0000	210.0000	17.4665	244.9266	-0.0188	3.6195	0.3008	3.6319
P_2	3.5000	30.0000	6.5000	210.0000	17.0589	245.8881	-0.0351	3.0060	0.1211	3.0084
P_3	4.0000	30.0000	6.0000	210.0000	16.6564	246.8964	-0.0494	2.7991	0.0573	2.7997
P_4	4.5000	30.0000	5.5000	210.0000	16.2593	247.9542	-0.0634	2.8578	0.0687	2.8586
P_5	5.0000	30.0000	5.0000	210.0000	15.8680	249.0647	-0.0782	3.1568	0.1447	3.1602
P_6	3.0000	60.0000	7.5524	198.5441	17.0000	240.0000	-0.0202	2.0231	-0.1008	2.0256
P_7	6.0000	60.0000	5.6637	178.0152	14.0000	240.0000	-0.0674	1.3496	-1.9766	2.3934
P_8	9.0000	60.0000	5.0115	146.1128	11.0000	240.0000	-0.0885	1.0530	-2.4914	2.7048
P_9	12.0000	60.0000	6.0128	116.2591	8.0000	240.0000	-0.0791	0.8021	-0.8730	1.1855
P_{10}	16.0000	60.0000	8.8810	94.2636	4.0000	240.0000	-0.0991	1.5092	0.0740	1.5110

Flux density values inside in spherical coordinates $(r_i, \theta_i, \varphi_i)$ are given in Table 5.14. The expressions in (4.2.16) - (4.2.18) are truncated to the same number $N = 20$ coefficients for each of $G_{n0}^{(1)}, G_{q0}^{(2)}$ and $G_{t0}^{(3)}$ (excluding $G_{00}^{(1)} = G_{00}^{(2)} = G_{00}^{(3)} = 0$).

TABLE 5.14: Numerical results for the flux density inside the spheres of a three-sphere system in the presence of external field along z axis: $a_1 = 3$ cm, $a_2 = 5$ cm, $a_3 = 4$ cm, $d_{12} = 10$ cm, $d_{13} = 20$ cm, $\theta_{12} = 30°$, $\theta_{13} = 60°$, $\varphi_{12} = \varphi_{13} = 0$, $H_0 =$ 1 A/m, $N = 20$ in (4.2.16) - (4.2.18)

Inside point	Sphere-1		Flux	Sphere-2		Flux	Sphere-3		Flux
	r_1 (cm)	θ_1 (deg)	$B^{(1)}$ (T)	r_2 (cm)	θ_2 (deg)	$B^{(2)}$ (T)	r_3 (cm)	θ_3 (deg)	$B^{(3)}$ (T)
1	0.2000	90.0000	4.3697E-06	0.2000	0.0000	3.7990E-06	0.2000	0.0000	3.5273E-06
2	1.5000	90.0000	4.3378E-06	2.5000	0.0000	3.6708E-06	2.5000	0.0000	3.8201E-06
3	2.9000	90.0000	4.2165E-06	4.9000	0.0000	1.2135E-06	3.9000	0.0000	1.0669E-05
4	2.5000	120.0000	4.2923E-06	4.5000	30.0000	3.7266E-06	3.5000	30.0000	3.5765E-06
5	2.0000	150.0000	4.3006E-06	4.0000	60.0000	3.7781E-06	3.0000	60.0000	3.5030E-06
6	1.5000	180.0000	4.3082E-06	3.5000	90.0000	3.8243E-06	2.5000	90.0000	3.4971E-06
7	1.2500	-135.0000	4.3190E-06	3.0000	120.0000	3.8486E-06	2.0000	120.0000	3.5064E-06
8	1.0000	-90.0000	4.3542E-06	2.0000	180.0000	3.8405E-06	1.5000	180.0000	3.5134E-06
9	0.7500	-45.0000	4.4138E-06	1.0000	-135.0000	3.8224E-06	1.0000	-135.0000	3.5133E-06
10	0.5000	0.0000	4.4194E-06	0.5000	-90.0000	3.8052E-06	0.5000	-90.0000	3.5217E-06

Three-sphere system in the presence of external field along x axis

Numerical values of the potential calculated from (4.2.24) and of the field intensity from (4.2.31) at the same points $\mathbf{P_1}, \mathbf{P_2}, \ldots \mathbf{P_{10}}$ have been tabulated in Table 5.15. The infinite system of linear equations in (4.2.26) - (4.2.28) has been truncated by retaining the first 60 coefficients of each of C_{nm}, D_{qp} and E_{ts} (excluding $C_{00} = D_{00} = E_{00} = 0$). These coefficients were computed using the matrix in (4.2.29), which has now 180 unknowns.

Numerical values of the magnetic flux density at some points inside the spheres are shown in Table 5.16. The series expressions in (4.2.35) have been truncated each by the same number $N = 60$ of coefficients of each of $G^{(1)}_{nm}, G^{(2)}_{qp}$ and $G^{(3)}_{ts}$ (excluding $G^{(1)}_{00} = G^{(2)}_{00} = G^{(3)}_{00} = 0$).

TABLE 5.15: Potential and field intensity in coordinates $(r_1, \theta_1, \varphi_1)$ outside a three-sphere system in the presence of external field along x axis: $a_1 = 3$ cm, $a_2 = 5$ cm, $a_3 = 4$ cm, $d_{12} = 10$ cm, $d_{13} = 20$ cm, $\theta_{12} = 30°$, $\theta_{13} = 60°$, $\varphi_{12} = \varphi_{13} = 0$, $H_0 = 1$ A/m, $N = 20$ in (4.2.29)

Point	Spherical coordinate (r_1, θ_1)			Potential	Field intensity (A/m), $H_1 = \sqrt{H_{r1}^2 + H_{\theta 1}^2 + H_{\varphi 1}^2}$			
	r_1 (cm)	θ_1 (deg)	φ_1 (deg)	ϕ^{tot} (A)	H_{r1}	$H_{\theta 1}$	$H\varphi_1$	H_1
P_1	3.0000	30.0000	0.0000	0.0116	0.6932	-1.8939	0.0000	2.0167
P_2	3.5000	30.0000	0.0000	0.0079	0.4215	-1.6351	0.0000	1.6885
P_3	4.0000	30.0000	0.0000	0.0058	0.2740	-1.4952	0.0000	1.5201
P_4	4.5000	30.0000	0.0000	0.0051	0.1910	-1.4162	0.0000	1.4291
P_5	5.0000	30.0000	0.0000	0.0058	0.1442	-1.3710	0.0000	1.3786
P_6	3.0000	60.0000	0.0000	0.0051	0.7449	-1.3926	0.0000	1.5793
P_7	6.0000	60.0000	0.0000	-0.0423	-0.5363	-1.0594	0.0000	1.1874
P_8	9.0000	60.0000	0.0000	-0.0446	-0.6649	-1.1184	0.0000	1.3011
P_9	12.0000	60.0000	0.0000	-0.0646	-0.6927	-1.2094	0.0000	1.3937
P_{10}	16.0000	60.0000	0.0000	-0.0906	-0.7000	-1.3399	0.0000	1.5117

TABLE 5.16: Magnetic flux density inside each sphere of the three-sphere system in the presence of external field along x axis: $a_1 = 3$ cm, $a_2 = 5$ cm, $a_3 = 4$ cm, $d_{12} = 10$ cm, $d_{13} = 20$ cm, $\theta_{12} = 30°$, $\theta_{13} = 60°$, $\varphi_{12} = \varphi_{13} = 0$, $\varphi_1 = \varphi_2 = \varphi_3 = 0$, $H_0 = 1$ A/m, $N = 20$ in (4.2.36) - (4.2.38)

Inside point	Sphere-1		Flux	Sphere-2		Flux	Sphere-3		Flux
	r_1 (cm)	θ_1 (deg)	$B^{(1)}$ (T)	r_2 (cm)	θ_2 (deg)	$B^{(2)}$ (T)	r_3 (cm)	θ_3 (deg)	$B^{(3)}$ (T)
1	0.2000	90.0000	4.7703E-06	0.2000	90.0000	5.0398E-06	0.2000	90.0000	4.9007E-06
2	1.5000	90.0000	4.7703E-06	2.5000	90.0000	5.0398E-06	2.5000	90.0000	4.9007E-06
3	2.9000	90.0000	4.7703E-06	4.9000	90.0000	5.0398E-06	3.9000	90.0000	4.9007E-06
4	2.5000	120.0000	4.3196E-06	4.5000	120.0000	4.3384E-06	3.5000	120.0000	4.1790E-06
5	2.0000	150.0000	2.5360E-06	4.0000	150.0000	2.4966E-06	3.0000	150.0000	2.3945E-06
6	1.2500	-135.0000	3.4818E-06	3.0000	-135.0000	3.5435E-06	2.0000	-135.0000	3.4223E-06
7	1.0000	-90.0000	4.7703E-06	2.0000	-90.0000	5.0398E-06	1.5000	-90.0000	4.9007E-06
8	0.7500	-45.0000	3.3080E-06	1.0000	-45.0000	3.5704E-06	1.0000	-45.0000	3.4869E-06

5.3 Chapter Summary

Numerical results for the magnetic scalar potential and for the field quantities both outside and inside ferromagnetic spheres in various configurations in the presence of a uniform external magnetic field have been presented. The results were obtained by using exact analytical expressions derived in chapter 3 and chapter 4 for linear arrays of and also for arbitrarily located spheres. A large range of values for the sphere radii and for the relative distance between the spheres have been considered.

For the quantities outside the spheres the infinite systems of linear equations satisfied by the constants of integration were truncated such that are imposed accuracy of the numerical results was achieved. Then the solution of the corresponding matrix equations were solved employing the mathematical tool Matlab. Next, these constants of integration were used to obtain the magnetic scalar potential and the field quantities.

The constants of integration from the field quantities for the region outside were, then, used to obtain the field quantities inside the spheres by imposing the boundary conditions. Numerical values of associated Legendre functions [4], were computed using a Matlab built-in subroutine and their derivatives were calculated by using the recurrence formulas, as shown in (B.7). For all numerical calculations, the permeability of the medium outside was taken to be that of free space, i.e., $\mu_0 = 4\pi \times 10^{-7}$ H/m.

Chapter 6

Conclusions and Future Work

6.1 Conclusions

In this thesis, exact analytical expressions were derived regarding the magnetization of ferromagnetic spheres in the presence of external magnetic fields on the basis of the exact field equations and by imposing the boundary conditions. Translational addition theorems for Laplacian fields in spherical coordinates are used to solve various boundary value problems for multi-sphere systems in linear arrays or of arbitrary geometries. In the case of axisymmetric geometries, the general expressions for the translational addition theorems are simplified as shown in Section 2.2.2. The field quantities are now independent of the coordinate φ, which considerably reduces the complexity of the equations. As a result, the generated matrices for such geometries are relatively simple which facilitates the computation of the numerical results. Legendre polynomials are used when dealing with systems of linear arrays of spheres, while spherical harmonics are needed for systems with arbitrarily located spheres.

Then, these expressions were applied to generate numerical results of sufficient accuracy for the field quantities. When computing the field quantities for the whole system, the superposition of the field quantities due to all spheres and to the external field can be performed using either the attached coordinate systems (with sets of single series) or a single coordinate system (with sets of multiple series, after using the translational addition theorems). In the former case, the coordinate relations in A.2 are needed.

It should be noted that numerical values of the field quantities were not exactly the same when using the two procedures. This is due to the summation of the terms in the respective series. For example, let's consider N terms in the truncated single series. Then, when using the translation to only one coordinate system, the number of terms in the translated portion of each expression is N^2, which increases usually the computational errors. Thus, it is expected that the expressions containing single series yield more accurate numerical results than the expressions containing multiple series.

As shown in Tables: 5.5, 5.9 and 5.11 the magnetic field intensity increases when the spheres are close to each other. The high field intensities determine high values for the interaction forces. As a result, the magnetization of the spheres increases now considerably.

The benchmark numerical results generated in this thesis are valuable as reference data to inspect the accuracy of different numerical methods developed to solve magnetostatic boundary value problems in the presence of external magnetic fields for real world applications.

6.2 Future Work

The research regarding the magnetization of ferromagnetic systems of spheres was conducted considering uniform magnetic fields and homogeneous materials. This work can be extended to nonuniform fields and to non homogeneous media.

On the other hand, it is of more generality if the methodology herein could be adopted to the systems of prolate and oblate spheroids, which better approximate real world ferromagnetic objects. Translational addition theorems for spheroidal Laplacian functions are now needed in order to derive exact analytical expressions for the magnetostatic field quantities.

Appendix A

Geometric Relations

A.1 Spherical Coordinates

The spherical coordinate (r, θ, φ) are related to the Cartesian coordinates as

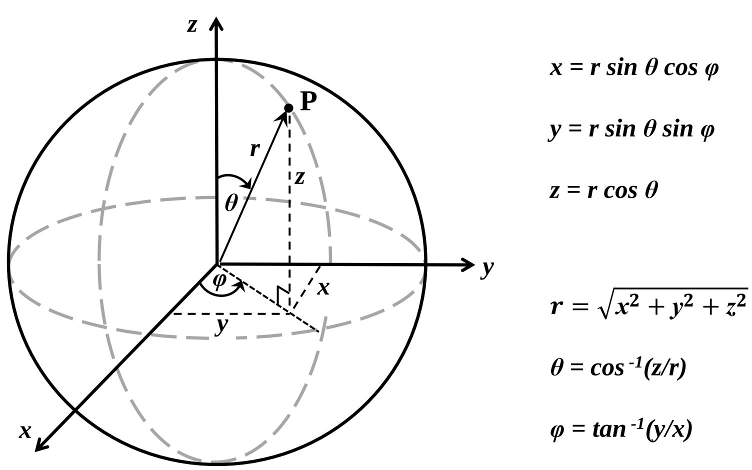

Figure A.1: Spherical coordinates

A.2 Coordinates Relations

The relation of two coordinate systems with common $z-x$ planes as

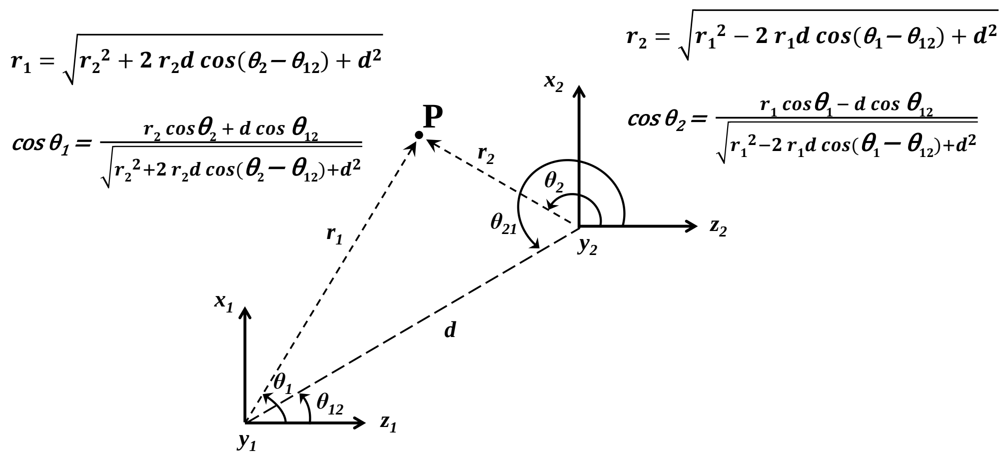

FIGURE A.2: Geometrical relations for point **P** on the common $z-x$ planes

Appendix B

Legendre Functions

The associated Legendre differential equation is given by [4]

$$\frac{d}{dx}\left[(1-x^2)\frac{d\,y}{dx}\right] + \left[n\,(n+1) - \frac{m^2}{1-x^2}\right] y = 0 \tag{B.1}$$

where $x = cos\,\theta, (-1 < x < 1)$ and the solution of Equation (B.1) can be written as

$$\begin{gathered} y = A_{nm} P_n^m(x) + B_{nm} Q_n^m(x), \\ n = 0, 1, 2, \ldots \text{ and } m = 0, \pm 1, \pm 2, \ldots, \pm n \end{gathered} \tag{B.2}$$

where A_{nm} and B_{nm} are arbitrary constants. $P_n^m(x)$ and $Q_n^m(x)$ are associated Legendre functions of degree n, order m, of the first and the second kind, respectively, defined by

$$P_n^m(x) = (1-x^2)^{\frac{1}{2}m} \frac{d^m}{dx^m} P_n(x) \quad \text{and} \quad Q_n^m(x) = (1-x^2)^{\frac{1}{2}m} \frac{d^m}{dx^m} Q_n(x) \tag{B.3}$$

Legendre polynomials are

$$P_n(x) \equiv P_n^0(x)$$

Explicit expressions: $(-1 < x < 1)$

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x, \\ &P_1^1(x) = (1-x^2)^{1/2}, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \ \ldots \end{aligned} \tag{B.4}$$

Special values [4]: $n = 1, 2, 3, \ldots$ and $m = 0, 1, 2, \ldots$

$$\begin{aligned} P_n^m(x) &= 0, \qquad \text{if } |m| > n \\ P_n^{-m}(x) &= (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x), \\ P_{-n}^m(x) &= P_{n-1}^m(x), \\ P_n^m(-x) &= (-1)^{n+m} P_n^m(x), \\ P_n^m(0) &= \begin{cases} 0, & \text{if } (n+m) \text{ odd} \\ (-1)^{\frac{1}{2}(n-m)} \dfrac{1 \cdot 3 \cdot 5 \cdots (n+m-1)}{2 \cdot 4 \cdot 6 \cdots (n-m)}, & \text{if } (n+m) \text{ even} \end{cases} \\ P_n^m(\pm 1) &= 0, \qquad \text{if } m \neq 0 \\ P_n(1) &= 1, \\ P_n(-1) &= (-1)^n, \end{aligned} \tag{B.5}$$

Orthogonality property:

$$\int_{-1}^{1} P_n^m(x) P_l^m(x)\,dx = \frac{2}{(2n+1)}\frac{(n+m)!}{(n-m)!}\,\delta_{n\,l}, \qquad 0 \le m \le n$$

$$\text{where} \quad \delta_{n\,l} = \begin{cases} 0, & \text{if} \quad l \neq n \\ 1, & \text{if} \quad l = n \end{cases} \quad \equiv \text{Kronecker delta} \tag{B.6}$$

Recurrence formulas [4]:

$$(m-n-1)P_{n+1}^m + (2n+1)xP_n^m - (m+n)P_{n-1}^m = 0, \qquad 0 \le m \le n$$

$$P_n^{m\,\prime} = \frac{\mp nxP_n^m \pm (n+m)P_{n-1}^m}{1-x^2} \tag{B.7a}$$

$$\left[\frac{d^r}{dx^r}P_n^m(x)\right]_{x\,=\,0} = P_n^{m+r}(0), \qquad n = 1, 2, 3, \ldots \tag{B.7b}$$

$$(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0, \qquad m = 0$$

$$P_n' = \frac{-n(xP_n - P_{n-1})}{1-x^2} \tag{B.8a}$$

$$P_n'(0) = -\,(n+1)P_{n+1}(0), \tag{B.8b}$$

$$P_n'(1) = \frac{1}{2}\,n(n+1), \qquad n = 1, 2, 3, \ldots \tag{B.8c}$$

Appendix C

Laplace's Spherical Harmonics

The spherical harmonic differential equation is given by [4]

$$\frac{1}{sin\,\theta}\frac{\partial}{\partial\theta}\Big(sin\,\theta\frac{\partial u}{\partial\theta}\Big)+\frac{1}{sin^2\,\theta}\frac{\partial^2 u}{\partial\varphi^2}+n(n+1)\,u=0 \tag{C.1}$$

and its solutions are called spherical harmonics which can be written as

$$Y_n^m(\theta,\varphi)=\sqrt{\frac{(2\,n+1)}{4\pi}\,\frac{(n-m)!}{(n+m)!}}\,P_n^m(cos\theta)\,e^{-jm\varphi},$$
$$n=0,1,2,\ldots \quad \text{and} \quad -n\leq m\leq n \tag{C.2}$$

where P_n^m is the associated Legendre function of the first kind, of degree n and order m. $Y_n^m(\theta,\varphi)$ is the spherical harmonics of the arguments θ and φ.

The first few spherical harmonics:

$$
\begin{aligned}
Y_0^0(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{1}{\pi}}, \qquad Y_1^0(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\, cos\,\theta, \\
Y_1^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\, sin\,\theta\, e^{-j\varphi}, \\
Y_1^{1}(\theta,\varphi) &= \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\, sin\,\theta\, e^{j\varphi}, \ \cdots
\end{aligned}
\tag{C.3}
$$

Some properties:

$$
\begin{aligned}
Y_n^0(\theta,\varphi) &= \sqrt{\frac{(2n+1)}{4\pi}}\, P_n(cos\,\theta), \\
Y_n^{-n}(\theta,\varphi) &= \frac{1}{2^n\, n!}\sqrt{\frac{(2n+1)!}{4\pi}}\, sin^n\theta\, e^{-jn\varphi}, \\
Y_n^{-m}(\theta,\varphi) &= (-1)^m\, \overline{Y}_n^{\,m}(\theta,\varphi),
\end{aligned}
\tag{C.4}
$$

where P_n is the Legendre polynomial and $\overline{Y}$ denotes the complex conjugate.

Normalization:

$$
\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} Y_n^m(\theta,\varphi)\,\overline{Y}_{n'}^{\,m'}(\theta,\varphi)\, sin\,\theta\, d\theta\, d\varphi = \frac{4\pi}{(2\,n+1)}\,\delta_{n\,n'}\,\delta_{m\,m'}
\tag{C.5}
$$

where $\delta_{i\,j}$ is the Kronecker delta.

$$
\begin{gathered}
\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \left|Y_n^m(\theta,\varphi)\right|^2 sin\,\theta\, d\theta\, d\varphi = \frac{4\pi}{\varepsilon_n(2\,n+1)}\,\frac{(n+m)!}{(n-m)!} \\
\text{where} \quad \varepsilon_n = \begin{cases} 1, & n=0 \\ 2, & n=1,2,3,\ldots \end{cases}
\end{gathered}
\tag{C.6}
$$

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