Applications of Higher Special Functions to some Three-Dimensional Contact Problems in the Classical Theory of Elasticity

by

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Abstract

Three-dimensional contact problems, in the classical theory of linear elasticity, can often be regarded as mixed boundary value problems of potential theory.

In this thesis three such problems have been treated where in each case contact, between the indenting object (called a punch) and the elastic medium, is maintained over an infinite region. It is assumed that a rigid frictionless punch with a known profile has indented a homogeneous, isotropic and linearly elastic half-space.

Chapter one is intended to serve as an introduction where the basic physical assumptions and a brief discussion of the equations of elastostatics are included.

In Chapters two and three the strip punch and the parabolic punch problems are treated respectively. Applying the theory of Mathieu functions, analytic solutions of Laplace's equation are obtained through separation of variables in the appropriate curvilinear coordinate system.

In Chapter four the wedge punch problem is discussed and a partial solution is obtained in terms of Lamé functions. Also some of the mathematical and physical difficulties, inherent in the formulation of this boundary value problem are discussed.

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Contents

Chapter I - INTRODUCTION

1.1 Physical assumptions and terminology

A contact problem in the theory of elasticity is defined as a problem in which two or more bodies, at least one of which is elastic, are in contact with one another under the action of a set of forces. These forces create certain displacements and stresses within and on the boundary of some or all of the bodies in contact. Of course such displacements and stress distributions also depend on the physical characteristics of the bodies. Initially some general assumptions are made concerning the nature of the stresses and the displacements on certain regions. In addition some data pertaining to the specific physical problem is also available. Combining these assumptions and the given initial or boundary conditions, we wish to determine the stresses and the displacements at all points of the regions involved. However this may not be always possible, due to the idealized assumptions based on the nature of the problem and the approach taken to obtain a solution.

As we shall see later, for a certain class of three-dimensional problems, one frequently encounters singularities in the solution for some points of the regions involved. In particular these are stress singularities which are often of the square root type and are found on the line which defines the contact edge, that is the line separating the points of the material that are in contact with the punch from the rest. Although singularities do not appear in nature, their occurrence in such solutions is a direct consequence of the particular mathematical model. In such cases the solutions are acceptable over the regions where no singularities appear and for points where singularities are found it is common to make certain approximations based on the asymptotic behaviour of the functions involved.

Within the last hundred years or so a number of three-dimensional contact problems in the classical theory of linear elasticity have been posed and either completely or partially solved. Due to the variety of applications of contact problems they have been classified into various categories where each category is defined according to the nature of the physical assumptions and the requirements which are imposed by a particular class of physical problems. Thus the mathematical analysis in each case depends upon the initial values of the particular problem, if the problem is time-dependent, or the boundary values if it is a static problem. Throughout this work we are mainly concerned with one class of contact problems: the three-dimensional static punch (indentation) problem where a rigid body (called a punch) of known cross-section is in complete contact* with part of an infinite elastic

* The term "complete contact" is defined later in this section.

half-space under the action of some force. This problem, which can be treated as a mixed boundary value problem, will be specified in detail later. However it should be noted that by comparison with the two-dimensional contact problem, the treatment of a three-dimensional problem invariably requires more sophisticated mathematical techniques.

The literature on two-dimensional contact problems is quite extensive and as important sources of solved problems and references, Muskhelishvili [5], Love [3] and Gladwell [2] may be mentioned. These problems are usually referred to as plane contact problems and the elastic medium under consideration is assumed to occupy an infinite half-plane. The solutions are in most cases obtained in terms of functions of a complex variable.

However it is clearly desirable to treat contact problems of elasticity as genuine three-dimensional problems whenever possible rather than applying methods which only deal with a cross-section of the original problem.

Historically the development of elastic contact problems can be traced back to the works of Hertz and Boussinesq. Hertz in 1882 investigated the problem of contact between two elastic bodies under normal loading. He calculated the stress distribution within two elastic spheres which are in frictionless contact with one another. Boussinesq on the other hand developed a solution for the problem of contact between a rigid punch and an elastic half-space. In 1885 he published the work that includes the solution to the problem of determining the state of stress within and on the boundary of a homogeneous isotropic elastic half-space where part of its boundary is in frictionless contact with a rigid punch under some normal loading. He applied the methods of potential theory and was primarily concerned with axisymmetric punches, where the punch is assumed to be a solid of revolution whose axis is normal to the boundary of the half-space.

The works of Hertz and Boussinesq have been substantially extended over the past years and developed in many different ways. The problems that we have examined belong to the line of investigation originating from Boussinesq's work, and as major sources of reference in this area, Galin [1], Gladwell [2] and Luré [4] may be cited.

Most of the research in this area has been carried out for cases where the contact area (i.e. the area of contact between the surface of the base of the punch and the boundary of the elastic half-space) is finite. More recently some problems have been tackled where the contact area has been assumed to be infinite (e.g. the problem of an infinite wedge pressed against an infinite elastic half-space). Our work may be regarded as an examination of some particular examples of the latter type of contact problem. Although such problems appear to be somewhat idealized, applications do exist notably in the area of soil mechanics where the soil itself may be regarded as an elastic half-space and the interaction of certain structures or objects with the soil is the subject of inquiry (e.g. long beams, railway tracks, etc.). A number of such problems have been examined by Selvadurai [6]. On the whole since we are only concerned with deformations so small as to be regarded as infinitesimal then the idea of an infinite punch indenting an elastic medium which occupies an infinite half-space appears reasonable provided there is some restriction on the amount of energy expended in making the indentation.

As with most models in the classical theory of elasticity we need to adopt certain simplifying assumptions with respect to the elastic body and its behaviour under the pressure exerted by the punch.

It is assumed that the elastic material which occupies the entire half-space is linearly elastic, homogeneous and isotropic. An elastic medium is said to be linear if the strain-stress relationship in the medium is linear. This assumed property is sometimes referred to as the "Generalized Hooke's Law of the Proportionality of Stress and Strain", and its validity for many problems is based on experimental evidence. Love ([3], Ch. 3, sec. 64) expresses this law by the following statement: "Each of the six components of stress at any point of a body is a linear function of the six components of strain at that point". For a detailed discussion of stress and strain tensor components the reader is referred to Love [3] and Gladwell

[2]. An elastic medium is considered to be homogeneous and isotropic if for any given point within the medium the elastic constants are independent of the position of that point and also the elastic properties of the continuum are the same in each direction away from that point.

We also assume that the displacements are infinitesimal. Suppose that a point in the elastic medium with coordinates (x, y, z), with respect to the axes OX, OY, OZ which are fixed, has moved to the point $(x + u, y + v, z + w)$ as a result of deformation of the medium. If products of derivatives of u, v, w with respect to x, y and z (i.e. $(\frac{\partial u}{\partial x})(\frac{\partial v}{\partial y})$ etc.) are neglected, then we say that the assumption of infinitesimal displacement has been adopted, Gladwell ([2], sec. 1.2).

The punch itself is assumed to be a perfectly rigid body and in all that follows it is assumed that there is no friction between the punch and the surface of the elastic medium. Certain conditions must also be imposed on the shape of the punch (i.e. its profile and its cross-section). These conditions will be specified later when we consider the validity of the formal solution of the boundary value problem.

Let S be the region of contact, that is the part of the boundary of the elastic half-space consisting of those points which after deformation are in contact with the displaced surface of the base of the punch, and let $\bar{S}$ be the region of the boundary of the half-space outside S. Assuming that there is no loading outside the contact region (i.e. on $\bar{S}$) we can take the normal component of stress to be zero on $\bar{S}$.

Now the problem which has been specified up to this point, namely that of a rigid frictionless punch indenting an elastic half-space under the idealized assumptions, may further be classified into two essentially different problems (cases (a), (b) below). This distinction is made according to the shape of the base of the punch.

(a) In this case, there is smooth contact between the base of the punch and the elastic medium. Examples include punches which have convex bases so that the stress on the boundary of the elastic medium is continuous everywhere, zero outside the punch (i.e. on $\bar{S}$) and at the edge of the punch. In problems of this type the shape of the contact region S is known a priori, but its dimensions depend on the force that is applied to the punch (figures (1) and (2)). For instance S could be a circular disc whose radius increases as the punch is pushed further into the medium. Such cases are usually called incomplete contact problems.

(b) This case consists of problems where the shape and dimensions of the contact region S are fixed, and increasing the load on the punch, by a small amount, will not change the region S but rather change the distribution of stress (or pressure) under the punch as well as displacements outside the punch. Of course this situation occurs only in the infinitesimal theory of elasticity. Such a punch will have a ridge which will separate the regions S and $\bar{S}$ by a sharp edge. Figures (3) and (4). In this case the normal component of stress is still zero on $\bar{S}$ but will have a square root type singularity on the edge (i.e. the line dividing S and $\bar{S}$). For example if the base of the punch is completely flat, increasing the load on the punch will not alter S , but the pressure distribution under the base of the punch will be changed. Such cases are referred to as complete contact problems.

Figure (1) Figure (2) Figure (3) Figure (4)

Initially, in the discussion of the equations of elasticity as they apply to punch problems, there is no need to distinguish between the above cases. The point at which a distinction has to be made between the two types of problems (a) and (b) is where the boundary conditions are being outlined for a particular problem. In both cases the normal component of stress is taken to be zero on $\bar{S}$ and continuous on S . However in case (a) an extra condition is imposed, namely that the normal component of stress is zero on the edge of the region S, whereas solutions of problems of type (b) are expected to give rise to a square root singularity for the normal component of stress on the edge of the punch. It may be noted here that the problems that are discussed in the subsequent chapters will be of type (b), where the base of the punch may be oscillatory (i.e. not necessarily flat) and have a sharp edge (e.g. figure (4)).

1.2 Coordinates and the mixed boundary value problem

We start off by employing the usual Cartesian coordinate system whereby the z-axis points into the elastic medium and the origin together with the x and y axes are placed on the undisturbed surface. We also fix a set of rectangular coordinates (ξ, η, ζ) within the punch in such a way that the origins of the two systems coincide, (x, y) and (ξ, η) coincide and the ζ-axis coincides with the negative z-axis, figure (5) (in this we have followed Luré's description closely; Luré ([4], Ch. 5).

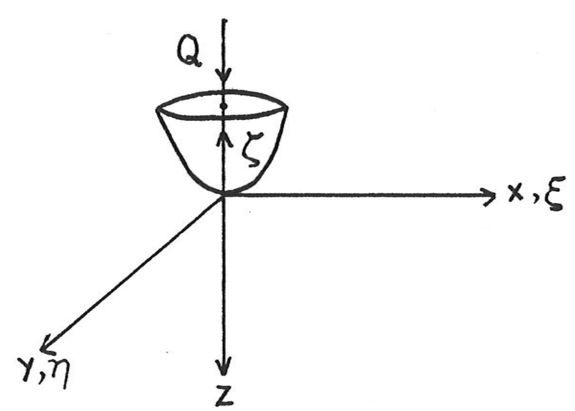

Figure (5)

Suppose under the specified conditions a punch of a given profile $\zeta = f(\xi, \eta)$ is pressed vertically into the medium under the action of a certain force Q which is applied to the punch in the direction of the z-axis. To simplify matters, the line of action of Q is taken to be the z-axis. This implies that the displacements of the punch and the points of the medium are only translational and there are no rotational displacements.

It is also assumed that

$$f(0, 0) = 0 \ , \quad \left(\frac{\partial f}{\partial \xi}\right)\bigg|_{\xi=\eta=0} = 0 \ , \quad \left(\frac{\partial f}{\partial \eta}\right)\bigg|_{\xi=\eta=0} = 0 \ .$$

We wish to determine the displacements on the boundary $z = 0$ of the infinite elastic medium on $\bar{S}$ (i.e. outside the contact region), as well as the normal component of stress on S (i.e. within the contact region).

Let (x', y', z') be the coordinates of an arbitrary point on the surface of the punch with respect to the x, y, z axes after the punch has been pressed into the medium. Then if ε is the vertical displacement of the point of the surface originally at $(0, 0, 0)$, we have

$$x' = \xi$$

$$y' = \eta$$

$$z' = \varepsilon - f(\xi, \eta) \ .$$

If we now let $(x, y, 0)$ denote the coordinates of the points of S which after the deformation of the medium correspond to (x', y', z'), then we have

$$x' = x + u$$
$$y' = y + v$$
$$z' = w$$

where u, v and w represent the components of displacement of points of the deformed body, and each of these components is a function of x, y and z.

So
$$u = \xi - x$$
$$v = \eta - y \qquad (1.1)$$
$$w = \varepsilon - f(x + u, y + v) .$$

Now if the profile function f is such that

$$f((x + u), (y + v)) - f(x, y)$$

is small compared to ε, then for $(x, y) \in S$, we can take $\varepsilon - f(x, y)$ as a close approximation to w, the normal component of displacement. We observe that since we are only concerned with small displacements then the above approximation is reasonable provided $\left|\frac{\partial f}{\partial \xi}(\xi, \eta)\right|$ and $\left|\frac{\partial f}{\partial \eta}(\xi, \eta)\right|$ are small enough. In the following chapters where certain punch problems are treated as boundary value problems, one of our boundary conditions will be

$$w = \varepsilon - f(x, y) , \quad \text{for } (x, y) \in S .$$

Clearly if the punch is flat, then $w = \varepsilon$ inside S.

The components of displacement u, v and w can be found from a set of relations derived from the Papkovich-Neuber solution to the problem of elastic equilibrium, which will be discussed shortly.

The following three conditions then give rise to a mixed boundary value problem, which is sometimes called a boundary value problem of the third kind, where the stresses are given on some part of the boundary of the body and the displacements are given on the remaining part:

(a) w is prescribed on S,

(b) the normal component of stress, i.e. τ_{zz}, is zero on $\bar{S}$,

(c) since we are assuming zero friction between the punch and the boundary of the elastic medium, the shearing stresses on the plane $z = 0$ are taken to be zero, i.e. $\tau_{xz} = \tau_{yz} = 0$ for all (x, y) on $z = 0$.

Thus a vertical force Q is applied to the punch as the result of which the punch indents the elastic medium and in order to maintain the equilibrium it is assumed that the same force is applied indefinitely (i.e. it is not altered at any time). The application of this force puts the region S under normal pressure $p(x, y)$ the distribution of which is initially unknown, but we have

$$Q = \iint_S p(x, y)\,dxdy .$$

If we neglect rotation and only consider translation of the punch into the medium, then for the case of the flat punch, which is a complete contact problem, the region S has the shape and the area of the cross-section of the punch and the normal component of stress τ_{zz} has a discontinuity on the edge of the punch, whereas for the smooth rounded punch (i.e. where $\frac{\partial f}{\partial \xi}$, $\frac{\partial f}{\partial \eta}$ are both continuous on the edge) the contour of the region S is the intersection of the surface of the punch with the plane ζ = constant . On this contour the normal pressure p is zero and τ_{zz} is continuous on the entire plane $z = 0$.

Also the direction of the line of action of the force Q is restricted since we are assuming absence of friction.

1.3 Introduction of the potential functions

The problem of static equilibrium, and in particular the problem of determining the state of stress in an elastic half-space where part of its boundary is subjected to a normal force Q , can be reduced to a boundary value problem in potential theory. The displacement and the state of stress of an elastic medium under normal loading, where the normal component of stress τ_{zz} is prescribed on part of the boundary, the normal component of displacement is given on another part of the boundary, and shear stresses are absent, can be determined when we have found a function $\psi(x, y, z)$ which is harmonic everywhere except on the region S of loading and vanishes at infinity with the following behaviour:

$$\psi \sim \frac{Q}{R} \quad \text{where} \quad Q = \iint_S p\,dx\,dy \text{ , and}$$

$$R = (x^2 + y^2 + z^2)^{1/2} .$$

Then the Papkovich-Neuber solution

$$2\mu\mathbf{d} = 4(1-\nu)\boldsymbol{\psi} - \nabla\{(\mathbf{r}\cdot\boldsymbol{\psi}) + \phi\} \tag{1.2}$$

(see Gladwell [2], Ch. 1, sec. 1.10)

to the problem of elastic equilibrium can be used to arrive at the required stress and displacement components.

In (1.2), $\mathbf{d}$ is the displacement vector, $\mathbf{r}$ is the position vector of a field point, $\boldsymbol{\psi}$ and ϕ are a pair of vector and scalar functions respectively which in the absence of body forces satisfy $\nabla^2\boldsymbol{\psi} = \mathbf{0}$, and $\nabla^2\phi = 0$; ν is Poisson's ratio and μ is the shear modulus (both constants).

Following the notation of Gladwell ([2], sec. 2.5), we consider the special case where $\boldsymbol{\psi}$ is chosen so that $\boldsymbol{\psi} = (0, 0, \psi)$. Then for the components of displacement $\mathbf{d} = (u, v, w)$ we have

$$2\mu u = -z\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial x} \tag{1.3a}$$

$$2\mu v = -z\frac{\partial\psi}{\partial y} - \frac{\partial\phi}{\partial y} \tag{1.3b}$$

and

$$2\mu w = 4(1-\nu)\psi - \left(z\frac{\partial\psi}{\partial z} + \psi + \frac{\partial\phi}{\partial z}\right) \tag{1.3c}$$.

The corresponding components of the stress tensor are given by

$$\tau_{xz} = (1-2\nu)\frac{\partial\psi}{\partial x} - z\frac{\partial^2\psi}{\partial x\,\partial z} - \frac{\partial^2\phi}{\partial x\,\partial z} \tag{1.4a}$$

$$\tau_{yz} = (1-2\nu)\frac{\partial\psi}{\partial y} - z\frac{\partial^2\psi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial y\,\partial z} \tag{1.4b}$$

$$\tau_{zz} = 2(1 - \nu) \frac{\partial\psi}{\partial z} - z \frac{\partial^2\psi}{\partial z^2} - \frac{\partial^2\phi}{\partial z^2} \qquad (1.4c) .$$

For zero shearing stress on $z = 0$ we have

$\tau_{xz}(x, y, 0) = \tau_{yz}(x, y, 0) = 0$ for all x and y .

So if $$\lim_{z \to 0} \{z \frac{\partial^2\psi}{\partial x \partial z} , z \frac{\partial^2\psi}{\partial y \partial z} , z \frac{\partial^2\psi}{\partial z^2}\} = 0 \qquad (1.5)$$

then $$(1 - 2\nu)\psi = \frac{\partial\phi}{\partial z} \qquad (1.6)$$

and consequently the normal component of stress is given by

$$\tau_{zz} = \frac{\partial\psi}{\partial z} - z \frac{\partial^2\psi}{\partial z^2} .$$

In particular for $z = 0$ we have

$$w(x, y, 0) = \frac{(1 - \nu)}{\mu} \psi(x, y, 0)$$

and $$\tau_{zz}(x, y, 0) = \frac{\partial\psi}{\partial z}(x, y, 0) \qquad (1.7a,b) .$$

It can be shown that this special case of the Papkovich-Neuber solution is satisfied by a representation of ψ in the form

$$\psi(x, y, z) = \frac{1}{2\pi} \iint_S \frac{g(x',y')}{R_1} dx'dy' \qquad (1.8)$$

where $R_1 = [(x - x')^2 + (y - y')^2 + z^2]^{1/2}$ is the distance from the point (x, y, z) of the elastic medium to the point $(x', y', 0)$ of the surface. This representation is known as the Boussinesq solution to the mentioned contact problem and provided

$$\lim_{z \to 0} (z \frac{\partial^2\psi}{\partial z^2}) = 0 \quad \text{for} \quad (x, y) \in S , \qquad (1.9)$$

then $g(x, y)$ can be taken as the normal pressure $p(x, y)$ applied to the region S of the boundary so that

$$\tau_{zz}(x, y, 0) = - p(x, y) \quad \text{for} \quad (x, y) \in S .$$

Considering various partial derivatives of ψ, where ψ is given by (1.8), it may be shown that the limits in (1.5) are all zero. For a discussion of these derivations the reader is referred to Gladwell ([2], sec. 2.5). It should also be noted that the Boussinesq solution can be obtained by first finding the solution of the elastic contact problem where a concentrated force is applied in a direction normal to the boundary of the elastic medium. This solution can then be generalized to one where a set of point forces are applied in which case the solution will have a series representation. Finally the series representation may be replaced by the integral representation (1.8) if the applied force is assumed to be distributed over a region of the boundary of the elastic medium (see Luré [4], sec. 2.5).

Now if $\psi(x, y, 0)$ is known on S (i.e. $w(x, y, 0)$ is known on S), then $p(x, y)$ can be obtained on S from (1.8) and once $p(x, y)$ is known then $\psi(x, y, z)$ can be obtained, again from (1.8), but the mathematical procedures involved in solving the integral equation (1.8) are often complicated and lengthy (see Luré [4], sec. 2.5) and we shall follow a different approach to obtain ψ, namely that of solving Laplace's equation through separation of variables by choosing a suitable coordinate system.

The function ψ represented by (1.8) may be viewed as the potential of a simple layer of intensity $P(x, y)$ distributed over the region S.

From the boundary conditions we have

$$\frac{\partial \psi}{\partial z} = 0 \quad \text{for} \quad z = 0 \quad \text{outside} \quad S .$$

$$\psi(x, y, 0) = \frac{\mu}{(1 - \nu)} w(x, y, 0) = \frac{\mu}{(1 - \nu)} [\varepsilon - f(x, y)]$$

inside S and

$$\left(\frac{\partial \psi}{\partial z}\right)\bigg|_{z \to 0\pm} = \begin{cases} \mp P(x, y) & \text{inside } S \\ 0 & \text{outside } S \end{cases}$$

Given the above boundary conditions the associated boundary value problem can be solved through a suitable choice of a curvilinear coordinate system provided S can be fitted into such a system as a normal surface.

1.4 Luré's and Shail's methods and the difference between them

Luré ([4], sec. 5.2) reduces the problem to that of finding the potential of a simple layer distributed over some surface S^* when the potential $\psi^*(x, y, z)$ on S^* is given and is continuous. This resolves into two problems which may be called the internal and the external Dirichlet problems. The internal Dirichlet problem gives $\psi_i(x, y, z)$, harmonic within S^*, such that $\psi_i(x, y, z) = \psi^*(x, y, z)$ on S^*, and the external Dirichlet problem gives

$\psi_e(x, y, z)$ which is harmonic outside S^*, and behaves like $\frac{Q}{2\pi R}$ at infinity. Also $\psi_e(x, y, z) = \psi^*(x, y, z)$ on S^*.

We than have $$\psi = \begin{cases} \psi_i & \text{inside } S^* \\ \psi_e & \text{outside } S^* \end{cases}$$

Now suppose S^* belongs to a family of surfaces depending on a single parameter s. Let S^* be given by some particular value of s, say $s = s^*$, and let S_0 be the surface that corresponds to $s^* = 0$, where it is assumed that S_0 takes the degenerate form of a flat region on the boundary of the elastic medium. It is assumed that S_0 is the same region as S, i.e. the region over which the punch is applied. Then

$$\psi(x, y, z) = \lim_{s^* \to 0} \psi_e(x, y, z; s^*)$$

represents the harmonic function which is equal to $\psi(x, y, z)$ at points of S and is a potential of a simple layer distributed over S. Hence when

$$\psi(x, y, 0) = \frac{\mu}{(1 - \nu)}[\varepsilon - f(x, y)]$$

is prescribed on S, then $\psi(x, y, z)$ will represent the solution to the particular punch problem.

Using this approach some punch problems have been solved by employing the appropriate coordinate system. Among others, the flat circular punch and the circular punch with a rounded base have been solved using oblate spheroidal coordinates (see Luré [4], sections 5.4, 5.5).

In almost all the examples solved by Luré ([4], Ch. 5), the internal and the external harmonic functions are found in each case. The reason for this seems to be that in order to find the pressure distribution under the punch he is employing the expression

$$\lim_{s^*\to 0} \left[\frac{\partial\psi_e(x, y, z; s^*)}{\partial n} - \frac{\partial\psi_i(x, y, z; s^*)}{\partial n} \right]$$

where n is the external normal to S^*. However since the pressure distribution under the punch is known when the normal component of stress, over the same region, is known, and this component of stress is given by $\left.\frac{\partial\psi}{\partial z}\right|_{z=0}$, then one could ignore the internal solution, ψ_i, and only evaluate the external solution ψ_e.

A number of similar problems have been solved by Galin ([1], Ch. 2). Furthermore Shail [7] has solved the elliptic punch problem, where complete contact is assumed, by using the ellipsoidal coordinate system.

The method of coordinate surfaces described above is limited in the sense that it can only be used where an orthogonal curvilinear coordinate system can be found to fit the punch, i.e. one of the degenerate surfaces of the coordinate system must be the same as the contact region S. Also Laplace's equation must be separable in this coordinate system. However if such a coordinate system can be found, then the punch profile can be moderately general and we may still be able to obtain a solution of the boundary value problem. On the other hand some other techniques such as the intergral equation method mentioned in section 1.3 may allow us to consider more general contact regions, but then the punch profile is usually restricted to fairly simple forms.

The difference between Luré's description of the problem and that of Shail's may be summarized as follows. Luré treats the general problem of finding a relation between the force Q acting on the punch and the displacement ε of the base of the punch at the origin. He also assumes that the cross-section of the punch is known but since the force Q is not fixed, and its variation is allowed to produce variations in the contact area S as well as in the displacement ε, then the actual contact area can be determined if Q is given and vice versa (using the condition that the normal component of stress is zero at the contact edge). That is to say he is mainly interested in problems where there is incomplete contact, although he does discuss some problems where the base of the punch is flat. Introducing such variables into the problem has certain obvious advantages, but it should be noted that one also has to restrict the shape of the punch profile. This restriction in most cases requires that the base of the punch be a surface of revolution, i.e. with axial symmetry, and be convex, so that the shape of the contact region remains fixed although its area increases as the force Q

increases (e.g. we may be dealing with an expanding circle, see Luré [4], sec. 5.5).

Shail [7] in his treatment of the elliptic punch problem assumes that there is complete contact. That is to say a force Q has been applied to a punch whose profile is described by a function $h(x, y)$ and consequently the punch has imbedded itself into the elastic half-space. This force Q is then assumed to be fixed throughout the problem so that the contact region S is known a priori and does not change. However the shape of the region S, prior to complete penetration, may go through various small changes. He also assumes that the displacement within S is known and it too does not change. So $w(x, y, 0) = h(x, y)$ for $(x, y) \in S$, where w is the vertical component of displacement of the elastic medium. However as expected in such cases where there is complete contact, his solution does involve a stress singularity, of the square root type, on the edge of the punch.

Shail gives the solution of the case where $h(x, y)$ is a polynomial of arbitrary degree in x and y so that it has a representation as a finite sum of ellipsoidal harmonics. He also remarks that his analysis applies to a wider class of functions than polynomials. This in essence is the advantage of his approach to the punch problem in comparison to Luré's.

In the problems that we have been dealing with, the contact region S is taken to be infinite which necessarily places a restriction on the shape of the base of the punch in order to avoid situations involving infinite force or infinite displacements. Clearly in such cases a punch with a convex profile will not do.

We have thus opted for Shail's approach where the contact region S and the normal displacements on S are prescribed. In fact we take $w(x, y, 0) = h(x, y)$ for $(x, y) \in S$. However the force Q which is required to keep the punch at its position of complete penetration into the elastic half-space can be determined from

$$Q = \iint_S p(x, y)\,dx\,dy$$

where $p(x, y)$ is the pressure distributed over the region S and is given by

$$p(x, y) = -\tau_{zz}(x, y, 0) \qquad \text{for} \quad (x, y) \in S ,$$

and $\tau_{zz}(x, y, 0)$ can be obtained from

$$\tau_{zz}(x, y, 0) = \frac{\partial\psi}{\partial z}(x, y, 0) .$$

Furthermore the function ϕ, which is required in the derivation of the components of displacement u and v (1.3 a, b) can be found from (1.6) and (1.8)

$$\phi(x, y, z) = \frac{(1 - 2\nu)}{2\pi} \iint_S \ln(z + R)\,p(x', y')\,dx'dy'.$$

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Chapter 2 - THE STRIP PUNCH PROBLEM

2.1 Introduction

This chapter aims at a mathematical analysis of the case where the punch occupies an infinite strip, its profile being arbitrary and subject only to the limitations of what is physically reasonable. Following the approach outlined in Chapter one, the elastic medium is assumed to occupy an infinite half-space and to satisfy the idealized conditions of linear elasticity. That is to say, it is homogeneous and isotropic, and the contact between the punch (which is assumed to be perfectly rigid) and the medium is complete and frictionless. Mathematically the problem may be regarded as an exterior mixed boundary value problem for Laplace's equation, which can be treated by transforming to the elliptic cylinder coordinate system. Applying the method of separation of variables and using Fourier transforms the general solution for this problem is obtained. As already mentioned (Ch. 1, sec. 1.4) this technique has been used for a punch of circular or elliptic cross-section, but the extension to a punch of infinite cross-sectional area introduces additional conceptual and mathematical difficulties.

From the existing literature on related problems, the works of Rvachev and Protsenko (V.L. Rvachev [8], [9]; V.S. Protsenko, and V.L. Rvachev [7]) may be mentioned. The problem which bears closest resemblance to that described in the last paragraph is to be found in Rvachev [8]. However Rvachev solves the problem for a special case and later in the same article indicates an approach for the general case. His technique differs from that which we have employed in several ways. As it is suggested by the title of Rvachev's paper, he is primarily concerned with obtaining the pressure on the elastic half-space under the punch. On the other hand we seek a harmonic function ψ which not only gives the pressure under the punch (which is the negative of the normal component of stress, $\tau_{zz}(x, y, 0) = \frac{\partial \psi}{\partial z}(x, y, 0)$) but also gives the other components of the stress tensor as well as the normal component of displacement outside the contact region, i.e. on $\bar{S}$. For a general punch problem, the regions S and $\bar{S}$ were defined in Chapter one (sec. 1.1). In this case S is an infinite strip on the xy-plane, of uniform width, and $\bar{S}$ is the region of the xy-plane outside S. As mentioned in Chapter one, since we are dealing with a complete contact problem, the solution will involve stress singularities on the edges of the punch and consequently the two lines which separate S and $\bar{S}$ are excluded from either of these regions.

Let the contact region be defined, in terms of the Cartesian coordinates (x, y, z), by $-\infty < x < \infty$, $|y| < f$, $z = 0$, where f is a constant. Rvachev [8] takes the following solution which is a relationship between the normal component of displacement, $w(x, y, 0)$, under the punch and the distribution of pressure, $p(x, y)$, over the same region:

$$w(x, y, 0) = \frac{1 - \nu}{2\pi\mu} \int_{-f}^{f} \int_{-\infty}^{\infty} \frac{p(x', y')}{[(x-x')^2 + (y-y')^2]^{1/2}} dx'dy'$$

where μ and ν are the elastic constants defined in Chapter one (sec. 1.3).

This is Boussinesq's solution with $z = 0$ (Ch. 1, equation 1.8). Rvachev solves the integral equation for $p(x, y)$ by letting

$w(x, y, 0) = g(\lambda, y)\cos\lambda x$, where λ is an arbitrary parameter.

Furthermore he assumes that $g(\lambda, y)$ can be represented as an infinite series in Chebyshev polynomials, i.e.

$$g(\lambda, y) = \sum_{n=0}^{\infty} s_n T_n(\tfrac{y}{f}) .$$

Then if $p(x, y) = \phi(\lambda, y)\cos\lambda x$, for $|y| < f$, ϕ will satisfy

$$g(\lambda, y) = \frac{1 - \nu}{\pi\mu} \int_{-f}^{f} \phi(\lambda, \xi) K_o(\lambda|y - \xi|) d\xi$$

where K_0 is the usual modified Bessel function of the third kind.

Finally ϕ is obtained as a Mathieu function series in η , where $\eta = \text{arc} \cos \frac{y}{f}$;

$$\phi(\lambda, f\cos\eta) =$$

$$\frac{\mu}{(1-\nu)\,|\sin\eta|}\left\{2s_0 \sum_{i=0}^{\infty} \frac{(-1)^i A_0^{(2i)}\,\mathrm{Fek}'_{2i}(0,-q)}{\mathrm{Fek}_{2i}(0,-q)}\,ce_{2i}(\eta,-q) \; + \right.$$

$$+ \sum_{n=1}^{\infty} (-1)^n s_{2n} \sum_{i=0}^{\infty} \frac{(-1)^i A_{2n}^{(2i)}\,\mathrm{Fek}'_{2i}(0,-q)}{\mathrm{Fek}_{2i}(0,-q)}\,ce_{2i}(\eta,-q) \; +$$

$$\left. + \sum_{n=0}^{\infty} (-1)^n s_{2n+1} \sum_{i=0}^{\infty} \frac{(-1)^i A_{2n+1}^{(2i+1)}\,\mathrm{Fek}'_{2i+1}(0,-q)}{\mathrm{Fek}_{2i+1}(0,-q)}\,ce_{2i+1}(\eta,-q)\right\}.$$

In this expansion

$$s_0 = \frac{1}{\pi}\int_0^{\pi} g(\lambda,\; f\cos\theta)\,d\theta\;, \quad s_k = \frac{2}{\pi}\int_0^{\pi} g(\lambda,\; f\cos\theta)\cos k\theta\,d\theta,$$

$q = \frac{1}{4}f^2\lambda^2$, $0 < \eta < \pi$ and, ce_i and Fek_i

are Mathieu functions (which will be discussed later in this chapter).

Theoretically one can obtain the harmonic function ψ from

$$\psi(x,\; y,\; z) = \frac{1}{2\pi}\int_{-f}^{f}\int_{-\infty}^{\infty} \frac{p(x',\; y')}{[(x-x')^2 + (y-y')^2 + z^2]^{1/2}}\,dx'dy'.$$

However, as it is pointed out by Luré ([3], sections 2.6, 2.7), this integral is not easily evaluated.

In what follows, rather than using the integral equation approach of Rvachev, we use separation of variables at the start and thus construct the solution ψ directly. Moreover instead of taking a somewhat special representation for the normal component of displacement, w, under the punch (i.e. the shape of the punch profile) we allow it to be discribed by a more general function whose behaviour with respect to the various variables is finally determined by certain conditions which have to be imposed in order to guarantee convergence of the solution. In terms of Rvachev's analysis this means $w(x, y, 0) = g(\lambda, y)\cos\lambda x$ is to be replaced by $w(x, y, 0) = \int_0^\infty g(\lambda, y)\cos\lambda x d\lambda$, and of course his final expression for the pressure under the punch must then be integrated with respect to λ from zero to infinity.

2.2 The elliptic cylinder coordinate system

The elliptic cylinder coordinates of a point are given by the variables (x, η, ξ) which are related to the Cartesian coordinates by

$$\begin{aligned} x &= x \\ y &= f \cosh\xi \cos\eta \qquad\qquad (2.2.1) \\ z &= f \sinh\xi \sin\eta \end{aligned}$$

where $-\pi < \eta \leq \pi$, and $\xi \geq 0$.

The sufaces corresponding to ξ = constant consist of a family of confocal elliptic cylinders; that for which

$\xi = \xi_0$ is such that its section by the plane $x = 0$ is an ellipse with foci $(0, \pm f, 0)$, eccentricity $\operatorname{sech}\xi_0$. For $\xi = 0$ we get the degenerate surface consisting of an infinite strip in the xy-plane of finite width $2f$. This is merely the case of an elliptic cylinder of eccentricity 1 with zero minor axis and finite major axis, $2f$. The surfaces corresponding to η = constant are portions of confocal hyperbolic cylinders which are normal to the surfaces ξ = constant .

2.3 Separation of Laplace's eqution in elliptic cylinder coordinates

In terms of (x, η, ξ) , Laplace's equation $\nabla^2\psi = 0$ is given by

$$\frac{\partial^2\psi}{\partial x^2} + \frac{2}{f^2(\cosh 2\xi - \cos 2\eta)}\left(\frac{\partial^2\psi}{\partial \xi^2} + \frac{\partial^2\psi}{\partial \eta^2}\right) = 0 \qquad (2.3.1)$$

Let $\psi = X(x)F(\xi)G(\eta)$, then

$$\frac{X''}{X} + \frac{2}{f^2(\cosh 2\xi - \cos 2\eta)}\left(\frac{F''}{F} + \frac{G''}{G}\right) = 0 \ .$$

The separated equations are:

$$X'' = \alpha X \qquad (2.3.2a)$$

$$F'' + (\tfrac{1}{2}\alpha f^2 \cosh 2\xi - \beta)F = 0 \qquad (2.3.2b)$$

$$G'' + (\beta - \tfrac{1}{2}\alpha f^2 \cos 2\eta)G = 0 \qquad (2.3.2c)$$

where α and β are separation constants.

Equation (2.3.2c) is Mathieu's equation and equation (2.3.2b) is the modified Mathieu equation.

In making use of Mathieu functions, we shall follow the notation of McLachlan [4]. He writes the standard form of Mathieu's equation as

$$\frac{d^2w}{dz^2} + (\lambda - 2q\cos 2z)w = 0 \qquad (2.3.3)$$

but the parameter q may be positive, negative or complex. It will shortly be shown that in the strip punch problem, the separation parameter α (of equations 2.3.2) must be real and negative, hence q is also real and negative.

We shall therefore write

$$q = -h^2 \qquad (h \geq 0) \qquad (2.3.4)$$

and quote the necessary information on Mathieu functions in this form, i.e.

$$\frac{d^2w}{dz^2} + (\lambda + 2h^2\cos 2z)w = 0 \qquad (2.3.5)$$

There are four types of basically periodic solutions (i.e. of period π or 2π) called Mathieu functions of integral order of the first kind. Two of these are even while the other two are odd, and they are expressed by the following expansions* (see McLachlan [4], Ch. 2):

$$ce_{2n}(z,-h^2) = \sum_{r=0}^{\infty} A_{2r}^{(2n)}(-h^2)\cos 2rz \qquad (2.3.6a)$$

$$ce_{2n+1}(z,-h^2) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(-h^2)\cos(2r+1)z \qquad (2.3.6b)$$

* Graphs of ce_0, ce_2 and ce_4, for certain values of h, are provided on page 57.

$$se_{2n+1}(z,-h^2) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(-h^2)\sin(2r+1)z \qquad (2.3.6c)$$

$$se_{2n+2}(z,-h^2) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(-h^2)\sin(2r+2)z \qquad (2.3.6d)$$

It should be noted here that the above four functions are possible solutions of equation (2.3.5) provided λ (which is dependent on h^2) takes one of the set of infinite real values called characteristic numbers. The corresponding characteristic values of λ for the expression (2.3.6a, b, c, d) are denoted respectively by $a_{2n}(-h^2)$, $a_{2n+1}(-h^2)$, $b_{2n+1}(-h^2)$ and $b_{2n+2}(-h^2)$, where n is a positive integer or zero. We also know that in this case (i.e. when the equation has as solution a periodic Mathieu function of one of the four types above) the second solution is not periodic (see Arscott [1], sec. 2.4.1).

If for equation (2.3.2b) we take the standard form as

$$\frac{d^2w}{dz^2} - (\lambda + 2h^2\cosh 2z)w = 0 \qquad (2.3.7)$$

then the four solutions with period πi and $2\pi i$ are given by McLachlan [4] in the following forms:

$$Ce_{2n}(z,-h^2) \stackrel{d.}{=} ce_{2n}(iz,-h^2) = \sum_{r=0}^{\infty} A_{2r}^{(2n)}(-h^2)\cosh 2rz \qquad (2.3.8a)$$

$$Ce_{2n+1}(z,-h^2) \stackrel{d.}{=} ce_{2n+1}(iz,-h^2) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(-h^2)\cosh(2r+1)z \qquad (2.3.8b)$$

$$Se_{2n+1}(z,-h^2) \overset{d.}{=} -i\ se_{2n+1}(iz,-h^2)$$
$$= \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(-h^2)\sinh(2r+1)z$$

(2.3.8c)

$$Se_{2n+1}(z,-h^2) \overset{d.}{=} -i\ se_{2n+1}(iz,-h^2)$$
$$= \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(-h^2)\sinh(2r+2)z$$

(2.3.8d)

and the corresponding characteristic values of λ for $Ce_m(z, -h^2)$ and $Se_m(z, -h^2)$ are the same as those for $ce_m(z, -h^2)$ and $se_m(z, -h^2)$, i.e. $a_m(-h^2)$ and $b_m(-h^2)$ respectively.

We have, of course, many possible ways of choosing a second solution independent of $Ce_m(z, -h^2)$ (or $Se_m(z, -h^2)$). One such solution is denoted by $Fek_m(z, -h^2)$ (or $Gek_m(z, -h^2)$ respectively). These functions are expressible in infinite series of the K-Bessel functions (sometimes called the modified Bessel functions K_ν). For example

$$Fek_{2n}(z,-h^2) = (-1)^n \frac{ce_{2n}(0,h^2)}{\pi A_0^{(2n)}(h^2)} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)}(h^2) K_{2r}(2h\cosh z)$$

(McLachlan [4], sec 8.30) .

The usefulness of $Fek_{2n}(z, -h^2)$ lies in its asymptotic behaviour as $z \to \infty$. The following asymptotic forms are given by McLachlan ([4], sec. 11.12):

$$Ce_{2n}(z,-h^2) \sim \frac{P'_{2n}(h)}{(2\pi)^{\frac{1}{2}}} v^{-1/2}e^{v}, \quad \text{as} \quad z \to \infty \tag{2.3.9}$$

$$Fek_{2n}(z,-h^2) \sim \frac{P'_{2n}(h)}{(2\pi)^{\frac{1}{2}}} v^{-1/2}e^{-v}, \quad \text{as} \quad z \to \infty \tag{2.3.10}$$

where

$$P'_{2n}(h) = (-1)^n ce_{2n}(0, h^2) ce_{2n}(\tfrac{\pi}{2}, h^2)/A_0^{(2n)}(h^2) ,$$

and

$$v = he^z .$$

As we shall see in section 2.5, it is precisely the behaviour of $Fek_{2n}(z, -h^2)$, as $z \to \infty$, which enables us to choose this function as one of the solutions to the boundary value problem.

Finally we note that writing $(\frac{\pi}{2} - z)$ for z in (2.3.3) (which is the standard form of Mathieu's equation) will give the relationship between $ce_{2n}(z, -h^2)$ and the standard solution:

$$\begin{aligned} ce_{2n}(z, -h^2) &= (-1)^n ce_{2n}(\tfrac{\pi}{2} - z, h^2) \\ &= (-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)}(h^2) \cos 2rz . \end{aligned}$$

It can also be shown that $a_{2n}(-h^2) = a_{2n}(h^2)$ (see Arscott [1], sec. 3.3.1).

2.4 Formulation of the general boundary value problem

Consider the half-space $z \geq 0$ occupied by a homogeneous, isotropic, linearly elastic medium. On the surface $z = 0$ let S denote the infinite strip $|y| < f$, $x \in (-\infty, \infty)$, and let $\bar{S}$ be the region outside S, $|y| > f$, $x \in (-\infty, \infty)$. A rigid frictionless punch is applied to the region S, its profile being given by a function $K(x, y)$. Contact is assumed to be complete everywhere on S, (figures (1), (2), (3)).

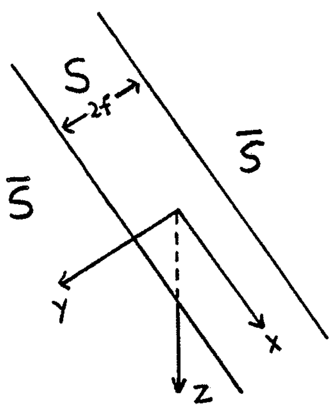


Figure (1)

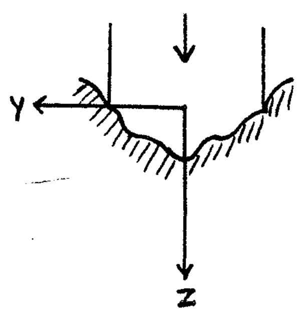


Figure (2)

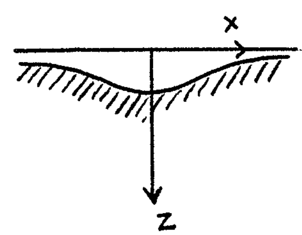


Figure (3)

The region $\bar{S}$ is assumed to be stress free ($\tau_{zz} = 0$ on $\bar{S}$). For zero shearing stress on $z = 0$ (i.e. $\tau_{xz}(x, y, 0) = \tau_{yz}(x, y, 0) = 0$, for all x and y), the set of equations (1.3 a, b, c), (1.4 a, b, c), (1.6) and (1.7 a, b) of Chapter one can be employed to solve the boundary value problem for $\nabla^2\psi = 0$. Of particular interest are (1.7 a, b) since they give the relations between ψ and the normal components of stress and displacement on the boundary $z = 0$.

The boundary condition can be stated as:

$$w(x, y, 0) = K(x, y) \qquad \text{on} \quad S$$

$$\tau_{zz}(x, y, 0) = 0 \qquad \text{on} \quad \bar{S} .$$

Hence we seek a solution to the boundary value problem for the harmonic function ψ , where for the elastic medium we have

(i) $\nabla^2\psi = 0$ for $z > 0$

(ii) $\psi \to 0$ as $r \to \infty$, $(r = (x^2 + y^2 + z^2)^{1/2})$, in $z \geq 0$

(iii) $\frac{\partial\psi}{\partial z} = 0$ on $\bar{S}$

(iv) $\frac{(1 - \nu)}{\mu} \psi(x, y, 0) = K(x, y)$ on S

where $K(x, y)$ is some prescribed function and ν , μ are elastic contants defined in Chapter one, section 1.3.

The function $K(x, y)$ can always be expressed as the sum of four functions each having symmetry or antisymmetry about one of the axes $x = 0$, $y = 0$, and because of linearity we can superpose solutions corresponding to these four functions. To simplify the analysis, therefore, we shall assume that

(v) $K(x, y)$ is symmetric about $y = 0$

(vi) $K(x, y)$ is symmetric about $x = 0$.

Consequently in terms of x and η we define

$$H(x, \eta) = \frac{\mu}{(1 - \nu)} K(x, f \cos\eta) .$$

$H(x, \eta)$ is then symmetric about $\eta = \frac{\pi}{2}$ and $x = 0$.

2.5 The general solution of the boundary value problem

We now transform to the elliptic cylinder coordinate system where η will be restricted to $0 \leq \eta \leq \pi$ since we are only concerned with the half-space occupied by the elastic medium. Separating Laplace's equation in this coordinate system we obtain equations (2.3.2 a, b, c); and conditions (i) to (vi) of section 2.4 become:

(i)' equation (2.3.1) holds for $\xi \in (0, \infty)$, $\eta \in (0, \pi)$ and $x \in (-\infty, \infty)$

(ii)' $\psi \to 0$ as $|x| \to \infty$ or $\xi \to \infty$, for $\eta \in [0, \pi]$

(iii)' since

$$\frac{\partial}{\partial z} = \frac{\cosh\xi \sin\eta}{f(\sinh^2\xi+\sin^2\eta)} \frac{\partial}{\partial\xi} + \frac{\sinh\xi \cos\eta}{f(\sinh^2\xi+\sin^2\eta)} \frac{\partial}{\partial\eta}$$

and $\bar{S}$ is the region where $\xi > 0$, $\eta = 0$ or π , then $\frac{\partial\psi}{\partial z} = 0$ on $\bar{S}$ is equivalent to $\frac{1}{f\sinh\xi} \frac{\partial\psi}{\partial\eta} = 0$ at $\eta = 0$ and $\eta = \pi$ (where $\xi \in (0, \infty)$ and $x \in (-\infty, \infty)$)

(iv)' $\psi(x, \eta, 0) = H(x, \eta)$, where $\eta \in (0, \pi)$, $x \in (-\infty, \infty)$ and

$$H(x, \eta) \overset{d.}{=} \frac{\mu}{(1 - \nu)} K(x, y)$$

(v)' $H(x, \eta)$ is symmetric about $\eta = \frac{\pi}{2}$

(vi)' $H(x, \eta)$ is symmetric about $x = 0$.

For X in (2.3.2a) to be finite α must be negative. Let $\alpha = -k^2$ so

$$X = A \cos kx + B \sin kx \qquad (2.5.1)$$

Since the solution ψ is assumed to have the form $\psi = X(x)F(\xi)G(\eta)$, we require $X(x)$ and $G(\eta)$ to have properties corresponding to the symmetries of $H(x, \eta)$ given by (v)' and (vi)' . In the first place (vi)' implies that

$$X = A \cos kx .$$

Also since $\alpha = -k^2$, if we let $k^2 f^2 = 4h^2$, then equation (2.3.2c) becomes

$$G'' + (\beta + 2h^2 \cos 2\eta) G = 0 \qquad (2.5.2)$$

but from (iii)' we have $G'(\pi) = G'(0) = 0$ which implies that G is a Mathieu function of the first kind (i.e. of period π or 2π) (see Arscott [1], sec. 2.1.1., theorem 3), and $G'(0) = 0$ implies that G must be $ce_{2n}(\eta, -h^2)$ or $ce_{2n+1}(\eta, -h^2)$. Finally from (v)', $G(\eta) = ce_{2n}(\eta, -h^2)$ and hence we can let $\beta = a_{2n}(-h^2)$.

Next equation (2.3.2b) implies that

$$F'' + (-2h^2 \cosh 2\xi - a_{2n}) F = 0 \qquad (2.5.3)$$

so we can choose $Ce_{2n}(\xi, -h^2)$ and $Fek_{2n}(\xi, -h^2)$ as a pair of linearly independent solutions of (2.5.3). However condition (ii)' requires that the general solution tends to zero as $\xi \to \infty$, and as explained in section 2.3 $Fek_{2n}(\xi, -h^2)$ is the only one which shows this behaviour (see sec. 2.3, (2.3.9) and (2.3.10)), so we must exclude the solution $Ce_{2n}(\xi, -h^2)$.

Hence a separated solution is of the form

$$\psi_n \overset{d.}{=} \psi_n(x, \eta, \xi, h) = B_n(h)\cos\frac{2hx}{f} ce_{2n}(\eta, -h^2) Fek_{2n}(\xi, -h^2) ,$$

where n is an arbitrary non-negative integer, h is an arbitrary non-negative parameter and $B_n(h)$ an arbitrary constant, written in this way since n is an integer-valued parameter while h is continuous.

The above solution is however a single separated solution and cannot be expected to satisfy the remaining boundary condition (iv)'. Since the parameter n is discrete whereas the parameter h is continuous and can take any value from zero to $+\infty$ it is natural to superpose solutions by summing over n from zero to $+\infty$ and integrating with respect to h from zero to $+\infty$. The coefficient $B_n(h)$ can then be determined if we let the general solution satisfy condition (iv)'.

A general solution is thus given by

$$\psi = \int_0^\infty \sum_{n=0}^\infty B_n(h)\cos\frac{2hx}{f} ce_{2n}(\eta, -h^2) Fek_{2n}(\xi, -h^2)\, dh \qquad (2.5.4)$$

Ignoring questions of convergence for the moment, and proceeding formally with the solution, (iv)' implies

$$H(x, \eta) = \int_0^\infty \sum_{n=0}^\infty C_n(h) ce_{2n}(\eta, -h^2) \cos \frac{2hx}{f} \, dh \tag{2.5.5}$$

where $\quad C_n(h) = B_n(h) Fek_{2n}(0, -h^2)$.

Writing (2.5.5) as

$$H(x, \eta) = \int_0^\infty g(h, \eta) \cos \frac{2hx}{f} \, dh$$

where
$$g(h, \eta) = \sum_{n=0}^\infty C_n(h) \; ce_{2n}(\eta, -h^2) \tag{2.5.6}$$

and using the Fourier cosine transform formula, we get

$$g(h, \eta) = \frac{4}{f\pi} \int_0^\infty H(x, \eta) \cos \frac{2hx}{f} \, dx \tag{2.5.7}$$

(provided the integral exists).

Multiplying both sides of the above equation by $ce_{2m}(\eta, -h^2)$ and integrating with respect to η from zero to π, we get:

$$\int_0^\pi \sum_{n=0}^\infty C_n(h) ce_{2n}(\eta, -h^2) ce_{2m}(\eta, -h^2) d\eta$$
$$= \int_0^\pi \frac{4}{f\pi} ce_{2m}(\eta, -h^2) \int_0^\infty H(x, \eta) \cos \frac{2hx}{f} \, dx \, d\eta \tag{2.5.8}$$

Still proceeding formally we interchange the order of integration and summation on the left-hand side of (2.5.8) and use the orthogonality of Mathieu functions (see McLachlan [4], sections 2.19, 2.21) to deduce that the left-hand side is equal to $\frac{\pi}{2} C_m(h)$. Therefore

$$B_n(h) = \frac{8}{f\pi^2 Fek_{2n}(0, -h^2)} \int_0^{\pi} \int_0^{\infty} H(x, \eta) ce_{2n}(\eta, -h^2) \times$$

$$\times \cos\frac{2hx}{f}\, dx\, d\eta \qquad (2.5.9)$$

Having evaluated $B_n(h)$ the solution ψ of the problem is then given by (2.5.4). From the equations of elastostatics (1.3 a, b, c) and (1.4 a, b, c) the corresponding displacements and stresses can be obtained. In particular the normal component of surface displacement outside the punch (i.e. on $\bar{S}$) is given by (1.7a), namely

$$w(x, y, 0) = \frac{(1 - \nu)}{\mu} \psi(x, y, 0)$$

$$= \frac{(1 - \nu)}{\mu} \int_0^{\infty} \sum_{n=0}^{\infty} B_n(h) \cos\frac{2hx}{f} ce_{2n}(0, -h^2) \times$$

$$\times Fek_{2n}[\cosh^{-1}\left(\frac{|y|}{f}\right), -h^2]\, dh .$$

Moreover, using (1.7b), (iii)' of section 2.5 and (2.5.4), the normal component of stress under the punch (i.e. on S) can be expressed by

$$\tau_{zz}(x, y, 0) = \frac{\partial\psi}{\partial z}(x, y, 0)$$

$$= (f^2 - y^2)^{-1/2} \int_0^{\infty} \sum_{n=0}^{\infty} B_n(h) \cos\frac{2hx}{f} \times$$

$$\times ce_{2n}[\cos^{-1}(\tfrac{y}{f}), -h^2]\, Fek'_{2n}(0, -h^2)\, dh \qquad (2.5.10)$$

where $|y| < f$.

We observe that the stress function (2.5.10) has singularities at $y = \pm f$, i.e. at the edge of the contact region. As mentioned in Chapter one, in punch problems where complete contact is assumed, one expects to find stress singularities of the square root type at the edge of the contact region.

2.6 Validity of the formal solution

As already stated the analysis of the previous section is purely formal and its validity depends on the behaviour of the prescribed function $H(x, \eta)$. For a given profile $H(x, \eta)$, naturally one can examine the iterated integral (2.5.9) for convergence and for differentiability with respect to x, ξ and η, then proceed to a corresponding investigation of the expression for ψ in (2.5.4). We can say that we have obtained an actual solution of the problem provided the profile H is sufficiently well-behaved for the formal steps indicated by (2.5.9) and (2.5.4) to be valid. This is to be expected from a "physically reasonable" profile. One would like, however, to do better than this, and here we attempt to give conditions on H which are sufficient, though not necessary, for the analysis of the last section to hold.

It may be noted that from a mathematical point of view some of the conditions stated below may be relaxed. For example most of the continuity conditions imposed on the partial derivatives of H with respect to η can be changed to piecewise continuity or if the function $H(x, \eta)$, when considered as a function of x, is of bounded variation on $[0, \infty)$ then we need not require that $\frac{\partial H}{\partial x}$ be continuous for all x on $[0, \infty)$. However since we are dealing with a physical problem where $H(x, \eta)$ describes the surface of the base of the punch, this function in almost all cases will satisfy the conditions stated below, and hence it will not be of great importance to use a more general set of conditions. It may be pointed out that if the surface of the base of the punch has sharp points then our smoothness conditions will no longer apply and must be changed to piecewise smoothness. The treatment of such cases would involve a complicated analysis of stress singularities within the contact region, which is beyond the scope of this thesis.

In what follows certain inequalities will be required to establish the various convergences. In order to avoid any digression from the main line of argument, these inequalities are derived in the appendices and only the appropriate results will be quoted here.

Let us begin by imposing the following conditions on $H(x, \eta)$:

(c.1) There exists a function $A_0(x)$, such that $|H(x, \eta)| < A_0(x)$ for all $\eta \in [0, \pi]$ and $A_0(x) \in L[0, \infty)$.

(c.2) $H(x, \eta)$ is a continuous function of both x and η for all $x \in [0, \infty)$ and all $\eta \in [0, \pi]$.

(c.3) As a function of η , $H(x, \eta)$ is four times continuously differentiable (i.e. partially with respect to η) for all $\eta \in [0, \pi]$ and all $x \in [0, \infty)$.

(c.4) For $i = 1, 2, 3, 4$, there exist functions $A_i(x)$ such that $\left|\frac{\partial^i H(x,\eta)}{\partial \eta^i}\right| < A_i(x)$, for all $\eta \in [0, \pi]$, and $A_i(x) \in L[0, \infty)$.

(c.5) $\frac{\partial H(x, \eta)}{\partial x}$ is a continuous function of both x and η for all $x \in [0, \infty)$ and all $\eta \in [0, \pi]$.

(c.6) For each $x \in [0, \infty)$, $\frac{\partial^j H(x, \eta)}{\partial \eta^j} = 0$ at $\eta = 0$, and $\eta = \pi$, for $j = 1, 3$.

(c.7) Let $M_0(h) = \max\limits_{0 \leq \eta \leq \pi} |T|$ and for $i = 1, 2, 3, 4$, let $M_i(h) = \max\limits_{0 \leq \eta \leq \pi} \left|\frac{\partial^i T}{\partial \eta^i}\right|$ where $T = T(h, \eta) \overset{d.}{=} \int_0^\infty H(x, \eta) \cos \frac{2hx}{f} dx$. The existence of these expressions is ensured by conditions (c.1) and (c.4) above. We shall assume that for integers i and m , where $0 \leq i \leq 4$ and $0 \leq m \leq 8$, $\int_0^\infty h^m M_i(h) dh < \infty$.

(c.8) There exists a constant, K, such that for $h \in [0, \infty)$ and for integers i and m, where $0 \leq i \leq 4$ and $0 \leq m \leq 8$, $h^m M_i(h) \leq K$.

In addition it should be kept in mind that $H(x, \eta)$ is assumed to be an even function of x and η.

Let

$$T(h, \eta) \stackrel{d.}{=} \int_0^\infty H(x, \eta) \cos \frac{2hx}{f} \, dx \qquad (2.6.1)$$

then (c.1) implies the existence of $T(h, \eta)$ for all $h \geq 0$ and $\eta \in [0, \pi]$.

Next we expand $T(h, \eta)$ as a Mathieu function series:

$$T(h, \eta) = \sum_{n=0}^{\infty} D_n(h) \, ce_{2n}(\eta, -h^2) \qquad (2.6.2)$$

From the general theory of Sturm-Liouville expansions (see Ince [2], sec. 11.5) we know that if, for any fixed real h, $T(h, \eta)$ is a continuous function of η, where η belongs to some finite interval, then the Fourier series and the Mathieu function series expansions of $T(h, \eta)$ are equiconvergent (i.e. the two series will converge under exactly the same conditions) on the same finite interval. Now (2.6.1), (c.1) and (c.2) together imply that $T(h, \eta)$ is a continuous function of η for each fixed $h \geq 0$. Furthermore (c.3) and (c.4) imply that $\frac{\partial T(h, \eta)}{\partial \eta}$ is also a continuous function of η for all $\eta \in [0, \pi]$ and each fixed $h \geq 0$. Hence for each $h \geq 0$, the infinite series (2.6.2) converges uniformly in η to $T(h, \eta)$. The coeffieicnts $D_n(h)$ are found by the usual technique (analogous to that for the Fourier series coefficients) as follows:

Multiplying both sides of (2.6.2) by $ce_{2m}(\eta, -h^2)$, integrating with respect to η from 0 to π and applying orthogonality properties of Mathieu functions, we get

$$\int_0^\pi ce_{2n}(\eta, -h^2)T(h, \eta)d\eta = \frac{\pi}{2} D_n(h) \tag{2.6.3}$$

(term by term integration of $\sum_{n=0}^{\infty} D_n(h)ce_{2n}(\eta, -h^2)ce_{2m}(\eta,-h^2)$ is permitted since this is a uniformly convergent series of continuous functions of η, for each fixed $h \geq 0$).

Now let

$$B_n(h) = \frac{4D_n(h)}{\pi fFek_{2n}(0, -h^2)} \tag{2.6.4}$$

and note that $Fek_{2n}(0, -h^2) \neq 0$ for any $n = 0, 1, 2, \ldots$, and any $h \geq 0$ (see appendix A). So

$$\frac{4}{f\pi} T(h, \eta) = \sum_{n=0}^{\infty} B_n(h)Fek_{2n}(0, -h^2)ce_{2n}(\eta, -h^2)$$

and

$$\begin{aligned}
&\int_0^\infty \cos \frac{2hx}{f} \sum_{n=0}^{\infty} B_n(h)Fek_{2n}(0, -h^2)ce_{2n}(\eta, -h^2)dh \\
&= \frac{4}{f\pi} \int_0^\infty \cos \frac{2hx}{f} T(h, \eta)dh \\
&= \frac{4}{f\pi} \int_0^\infty \int_0^\infty \cos \frac{2hx}{f} H(x', \eta)\cos \frac{2hx'}{f} dx'\, dh \\
&= H(x, \eta) .
\end{aligned}$$

The validity of the last step, which states that $H(x, \eta)$ is equal to the inverse cosine transform of its transform is ensured by (c.1), (c.2) and (c.5). Hence expression (2.5.5) is justified, where $C_n(h) = \frac{4}{\pi f} D_n(h)$, and $B_n(h)$ is given by (2.5.9).

It remains to show that the function $\psi(x, \eta, \xi)$, given by (2.5.4), is a continuous function of x, η and ξ , is twice partially differentiable with respect to each of x, η and ξ , and tends to zero as $|x|$ or ξ tend to infinity, for each $\eta \in [0, \pi]$.

We start by rewriting ψ as

$$\psi(x, \eta, \xi) = \frac{4}{\pi f} \int_0^\infty \cos\frac{2hx}{f} \sum_{n=0}^{\infty} D_n(h) ce_{2n}(\eta,-h^2) \frac{Fek_{2n}(\xi,-h^2)}{Fek_{2n}(0,-h^2)} dh \tag{2.6.5}$$

where $D_n(h)$ is given by (2.6.3) .

From appendix B (part I) ,

$$0 < \frac{Fek_{2n}(\xi,-h^2)}{Fek_{2n}(0,-h^2)} \leq 1$$

for all $n = 0, 1, 2, \ldots,$ all $h \geq 0$ and all $\xi \geq 0$. From appendix C , (C.1.9),

$$\left| ce_{2n}(\eta,-h^2) \right| \leq \gamma_0 + \gamma_1 h + \gamma_2 h^2 ,$$

where γ_0 , γ_1 and γ_2 are positive constants, and from appendix D , (D.1.1) ,

$$\left| \frac{\pi}{2}(a_{2n} + 2h^2 + 1) D_n(h) \right| \leq \frac{\pi}{\sqrt{2}} (1 + 4h^2) M_0(h) + \frac{\pi}{\sqrt{2}} M_2(h) ,$$

where $M_0(h)$ and $M_2(h)$ are defined by condition (c.7). From this inequality it follows that

$$\left|D_n(h)\right| \leq \frac{2[(1+4h^2)M_0(h)+M_2(h)]}{(4n^2+1)} .$$

Hence

$$\left|D_n(h)ce_{2n}(\eta,-h^2)\ \frac{Fek_{2n}(\xi,-h^2)}{Fek_{2n}(0,-h^2)}\right| \leq [M_0(h)\sum_{i=0}^{4}\alpha_i h^i +$$

$$+\ M_2(h)\sum_{j=0}^{2}\ \beta_j h^j](4n^2+1)^{-1} \qquad (2.6.6)$$

where α_i and β_j are positive constants.

Using condition (c.8) and Weierstrass's M-test we deduce that

$\sum_{n=0}^{\infty} D_n(h)ce_{2n}(\eta,-h^2)\ \frac{Fek_{2n}(\xi,-h^2)}{Fek_{2n}(0,-h^2)}$ is uniformly convergent with respect to η , ξ and h . Furthermore, from the general theory of Mathieu equations (c.f. [5] sec. 2.11) $ce_{2n}(\eta,\ -h^2)$ is continuous in η and h , $Fek_{2n}(\xi\ ,\ -h^2)$ is continuous in ξ and h , and from (2.6.1), (2.6.3), (c.1) and (c.2) $D_n(h)$ is also continuous. Hence the function represented by the above infinite series is continuous in η , ξ and h .

Next let

$$\psi_N(x,\eta,\xi) = \frac{4}{\pi f}\int_0^N \cos\frac{2hx}{f}\sum_{n=0}^{\infty} D_n(h)ce_{2n}(\eta,-h^2)\frac{Fek_{2n}(\xi,-h^2)}{Fek_{2n}(0,-h^2)}\,dh\ ,$$

then for each $N = 1, 2, \ldots, \psi_N$ is a continuous function of x, η and ξ and by condition (c.7), the sequence of functions ψ_N converges uniformly to ψ. Hence $\psi(x,\eta,\xi)$ is a continuous function of x, η and ξ.

Differentiating the integrand in the expression for ψ, (2.6.5), twice partially with respect to x only introduces factors h and h^2 in the integrand. In either case the uniform convergence of the integrand is ensured by condition (c.7) and the continuity of the integrand, as a function of x, η, ξ and h, is not affected.

To show that

$$\frac{\partial}{\partial\eta} \sum_{n=0}^{\infty} D_n(h)\, ce_{2n}(\eta,-h^2) \frac{Fek_{2n}(\xi,-h^2)}{Fek_{2n}(0,-h^2)}$$

$$= \sum_{n=0}^{\infty} D_n(h)\, ce'_{2n}(\eta,-h^2) \frac{Fek_{2n}(\xi,-h^2)}{Fek_{2n}(0,-h^2)}$$

we observe that

$$\sum_{n=0}^{\infty} \left| D_n(h)\, ce'_{2n}(\eta,-h^2) \frac{Fek_{2n}(\xi,-h^2)}{Fek_{2n}(0,-h^2)} \right|$$

$$\leq \sum_{n=0}^{\infty} \left| (a_{2n}+2h^2+1) D_n(h) \right| \left| \frac{ce'_{2n}(\eta,-h^2)}{a_{2n}+2h^2+1} \right|$$

$$\leq \sum_{n=0}^{\infty} \frac{\sqrt{2}}{(4n^2+1)} [(16h^4+16h^2+1)M_0(h)+8h^2M_1(h) +$$

$$+ 2(4h^2+1)M_2(h)+M_4(h)]\,(\gamma_0+\gamma_1 h+\gamma_2 h^2)$$

(by (D.2.7) of appendix D, and (C.2.2) of appendix C)

$$\leq M_0(h) \sum_{i=0}^{6} \rho_i h^i + M_1(h) \sum_{i=0}^{4} \lambda_i h^i + M_2(h) \sum_{i=0}^{4} \sigma_i h^i +$$

$$+ M_4(h) \sum_{i=0}^{2} \tau_i h^i$$

where the coefficients ρ_i, λ_i, σ_i and τ_i are positive constants.

So the series containing $ce'_{2n}(\eta,-h^2)$ is uniformly convergent in η, ξ and h (using condition (c.8) as before). Also the terms of the series are continuous functions of η, ξ and h. Moreover

$$\int_0^\infty \cos\frac{2hx}{f} \sum_{n=0}^{\infty} D_n(h)ce'_{2n}(\eta,-h^2) \frac{Fek_{2n}(\xi,-h^2)}{Fek_{2n}(0,-h^2)}\, dh$$

is uniformly convergent with respect to x, η and ξ and the integrand is a continuous function of x, η, ξ and h. Hence ψ can be differentiated partially with respect to η (term by term).

A similar argument can be used to show that ψ is twice partially differentiable with respect to η. Here we replace $ce''_{2n}(\eta,-h^2)$ by $-(a_{2n}+2h^2\cos 2\eta)ce_{2n}(\eta,-h^2)$. Then using (D.2.7) and (C.1.9) we have

$$\sum_{n=0}^{\infty} \left| D_n(h)ce''_{2n}(\eta,-h^2) \frac{Fek_{2n}(\xi,-h^2)}{Fek_{2n}(0,-h^2)} \right|$$

$$\leq \sum_{n=0}^{\infty} \sqrt{2}[(\gamma_0+\gamma_1 h+\gamma_2 h^2)(a_{2n}+6h^2) / (a_{2n}+2h^2+1)^2] \times$$

$$\times \quad [(16h^4+16h^2+1)M_0(h)+8h^2 M_1(h)+2(4h^2+1)M_2(h)+M_4(h)]$$

and from conditions (c.7, 8) it follows that ψ can be differentiated twice partially with respect to η (it may be noted that to obtain the required result we write
$(a_{2n}+6h^2) \leq (a_{2n}+2h^2+1) + (4h^2)$ in the last series and treat it as a sum of two series).

Next, to show that ψ is partially differentiable with respect to ξ we use (B.4) of appendix B, (C.1.9) of appendix C and (D.2.7) of appendix D. If we use (B.2) of appendix B, instead, then it can be shown that ψ is twice partially differentiable with respect to ξ. In both cases the arguments are similar to those of the differentiability of ψ with respect to η.

Finally in order to show that ψ tends to zero as $|x|$ or ξ tend to infinity, we proceed as follows. First, using the Riemann-Lebesgue lemma*, since the integral in (2.6.5) is uniformly convergent and the function represented by the series in the integrand is a continuous function of h, then for each $\eta \in [0, \pi]$ and each $\xi \geq 0$,

$$\lim_{|x| \to \infty} \psi(x, \eta, \xi) = 0 .$$

Next, for each $x \in (-\infty, \infty)$ and each $\eta \in [0, \pi]$, the integral in (2.6.5) converges uniformly for all $\xi \geq 0$ and the integrand is a continuous function of h and ξ,

* The Riemann-Lebesgue lemma is usually stated for integration over finite intervals, however it can be generalized to cases involving infinite limits of integration (see Olver [6], theorem 4.1, page 73).

so $\lim\limits_{\xi \to \infty} \int = \int \lim\limits_{\xi \to \infty}$. Furthermore the series in (2.6.5) is also uniformly convergent in h and ξ and the terms of the series are continuous functions of h and ξ . Hence $\lim\limits_{\xi \to \infty} \sum = \sum \lim\limits_{\xi \to \infty}$. From the asymptotic behaviour of $Fek_{2n}(\xi,-h^2)$ as $\xi \to \infty$ (see (2.3.10)), it follows that $\lim\limits_{\xi \to \infty} \psi(x, \eta, \xi) = 0$. Using a similar argument it can be deduced that $\frac{\partial\psi}{\partial\eta} = 0$ at $\eta = 0$ and $\eta = \pi$.

Therefore $\psi(x, \eta, \xi)$ represented by (2.5.4) is continuous and satisfies Laplace's equation as well as the boundary conditions of the stated boundary value problem, provided the function $H(x, \eta)$ satisfies conditions (c.1) to (c.8).

In passing we note that under the above smoothness conditions on H , ψ also satisfies the limits (1.5) and (1.9) of Chapter one.

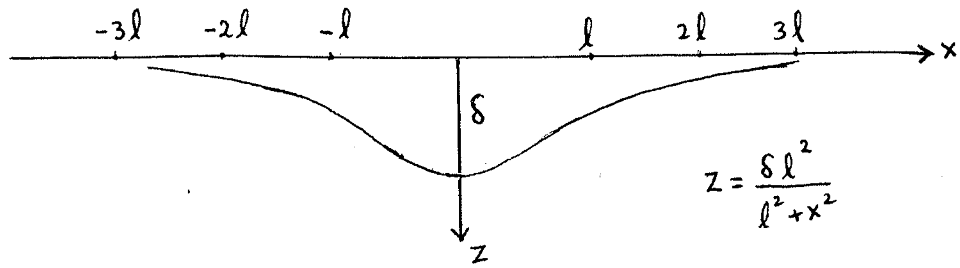
2.7 An example

As an example we consider the punch problem where the surface of the base of the punch is represented by the function

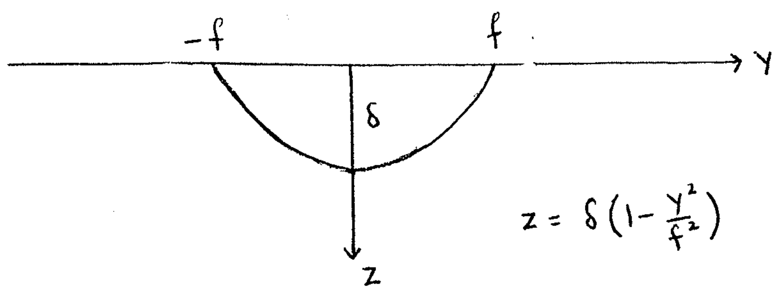
$$H(x, \eta) = \delta \frac{\ell^2 \sin^2\eta}{\ell^2+x^2}$$

where δ , ℓ are parameters having the dimensions of length, δ measuring the maximum depth of the punch which occurs at the origin. The cross-sections of the punch profile, taken through the planes of symmetry, are as shown:

longitudinal cross-section



transverse cross-section



Then $$T(h, \eta) = \delta\int_0^\infty \cos\left(\frac{2hx}{f}\right) \frac{\ell^2 \sin^2\eta}{(\ell^2+x^2)}\, dx$$
$$= \frac{2\ell\delta}{f} \sin^2\eta \exp\left(-\frac{2\ell h}{f}\right) .$$

Also $$M_0(h) = \frac{2\ell\delta}{f} \exp\left(-\frac{2\ell h}{f}\right)$$
and for $i = 1, 2, 3, 4$, $M_i(h) = 2^i \frac{\ell\delta}{f} \exp\left(-\frac{2\ell h}{f}\right)$.

Clearly conditions (c.1) to (c.8) of section 2.6 are satisfied. Moreover

$$D_n(h) = \frac{4\ell\delta}{\pi f} \exp(-\frac{2\ell h}{f}) \int_0^{\pi} ce_{2n}(\eta, -h^2) \sin^2\eta \, d\eta$$

$$= \frac{(-1)^n \ell\delta}{f} [2A_0^{(2n)}(h^2) + A_2^{(2n)}(h^2)] \exp(-\frac{2\ell h}{f}) \quad (2.7.2)$$

where $A_0^{(2n)}$ and $A_2^{(2n)}$ are the first two coefficients in the Fourier series expansion of $ce_{2n}(\eta, -h^2)$.

The above example of a punch profile has been chosen to illustrate the theory because, while being smooth and physically reasonable, it allows us to express the coefficients $D_n(h)$ explicitly. The reasons for this are (i) the expression $\frac{\ell^2}{\ell^2 + x^2}$ has a simple Fourier cosine transform and (ii) the integral $\int_0^{\pi} ce_{2n}(\eta, -h^2) \sin^2\eta \, d\eta$ is expressible in terms of only two Mathieu coefficients $A_{2r}^{(2n)}$, $r = 0, 1$. If, in place of $\sin^2\eta (= \frac{1}{2} - \frac{1}{2}\cos 2\eta)$, the expression (2.7.1) for the punch profile involved higher trigonometric terms in η (so that z , in terms of y , were given by a polynomial of degree higher than the second), then the effect would be to introduce further terms in (2.7.2), but only a finite number of these. Thus the computation involved would be of the same order of magnitude, since numerical construction of a Mathieu function generally produces all the significant coefficients $A_{2r}^{(2n)}$.

In punch problems, one of the quantities of interest is the normal component of stress under the punch (i.e. on S) , $\tau_{zz}(x, y, 0)$. Recalling that $\tau_{zz}(x, y, 0) = \left.\frac{\partial\psi}{\partial z}\right|_{z=0}$, then the normal component of stress (under the punch) for this particular example is given by

$$\tau_{zz}(x, y, 0) = \frac{4}{\pi f[f^2-y^2]^{1/2}} \int_0^\infty \cos\frac{2hx}{f} \sum_{n=0}^\infty D_n(h) \times$$

$$\times\, ce_{2n}[\cos^{-1}(\tfrac{y}{f}), -h^2]\, \frac{Fek'_{2n}(0,-h^2)}{Fek_{2n}(0,-h^2)}\, dh\ . \qquad (2.7.3)$$

The total force exerted on the punch is given, of course, by the integral of the expression (2.7.3) over the total area of the punch, i.e.

$$\int_{-\infty}^{\infty} \int_{-f}^{f} \tau_{zz}(x, y, 0)dydx$$

$$= 4f\int_0^\infty \int_0^{1/2\pi} \tau_{zz}(x, f\cos\eta, 0)\sin\eta d\eta dx \qquad (2.7.4)$$

Evaluation of the expression (2.7.2), hence of (2.7.3) and (2.7.4), can only be done numerically, but in view of recent progress in the techniques of computing Mathieu functions, this is by no means an impossible task.

To illustrate this observation, we take the particular case where $\ell = 3f$, so that the substantial part of the punch profile is long compared with its width (see the figures on p. 52 above). This has the effect that the factor $\exp(-2\ell h/f)$ tends to zero quite rapidly as h increases, so that it is only necessary to compute the $D_n(h)$ for small values of h. (Indeed, it is quite possible that tolerable accuracy could be achieved by using the perturbation formulae for Mathieu functions given in, for example, [5] sec. 2.25, but the following computations have been carried out without this approximation.)

Moreover, for such a profile, it is only necessary to consider small values of n , for the following reason: for $h = 0$, Mathieu functions reduce to trigonometric functions, namely

$$ce_0(\eta, 0) = 2^{-1/2} , \quad ce_{2n}(\eta , 0) = \cos 2\eta , \quad (n \geq 1)$$

and for small values of h the Mathieu functions remain close to these approximations. Hence, for $n > 2$ and h small, the coefficients $A_0^{(2n)}$ and $A_2^{(2n)}$ are small compared with 1, so that $D_n(h)$ is itself small (see [4], sections 3.27 to 3.35).

Using the method described in [10] the coefficients $A_0^{(2n)}$ and $A_2^{(2n)}$, and hence the $D_n(h)$ have been computed for $n = 0, 1, 2$ and $h = 0, .1, .2, .3, \ldots, 2.0$ and are shown in the following table:

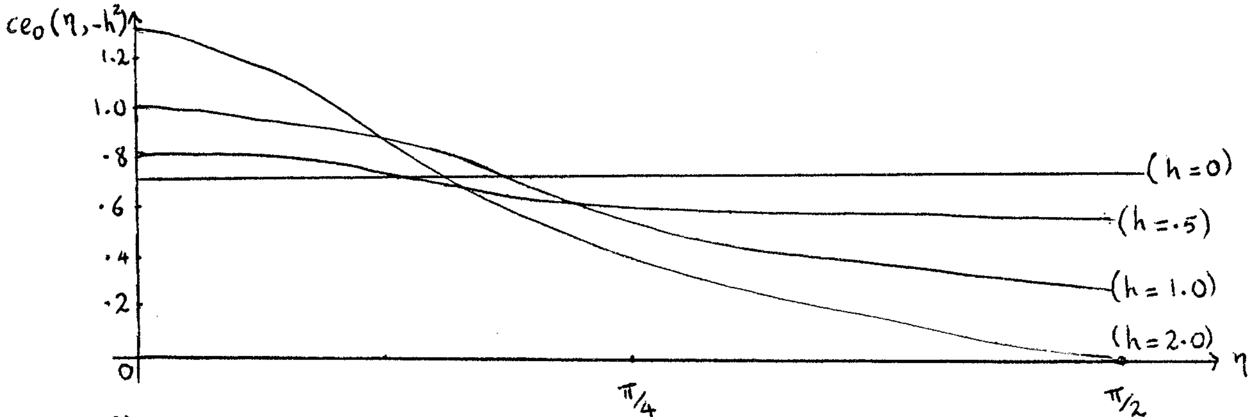
Values of $D_n(h)/\delta$

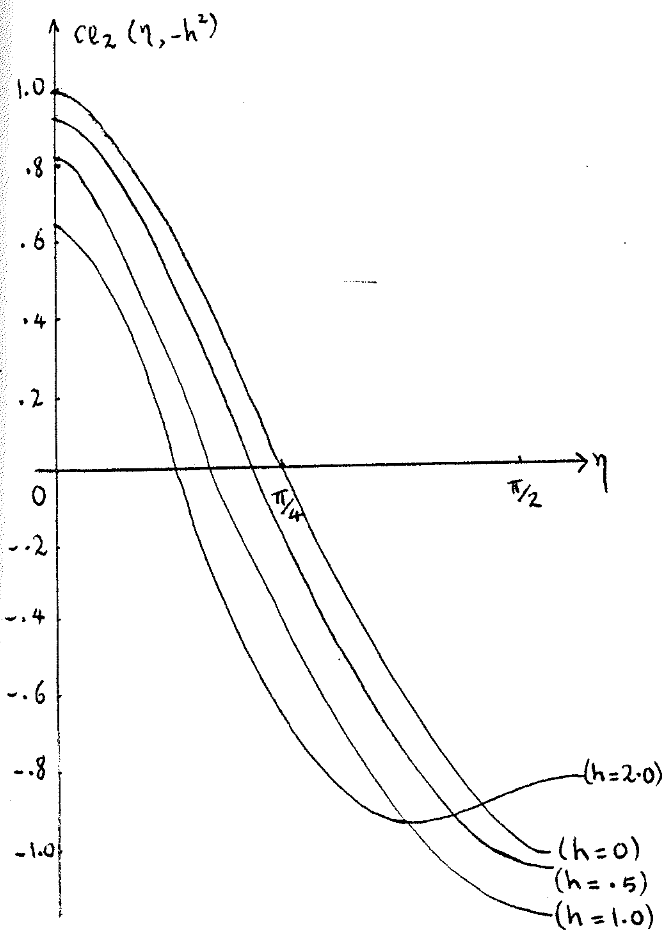
$$\frac{D_n(h)}{\delta} = \frac{(-1)^n \ell}{f}\,[2A_0^{(2n)}(h^2) + A_2^{(2n)}(h^2)]\exp(-2\ell h/f)$$

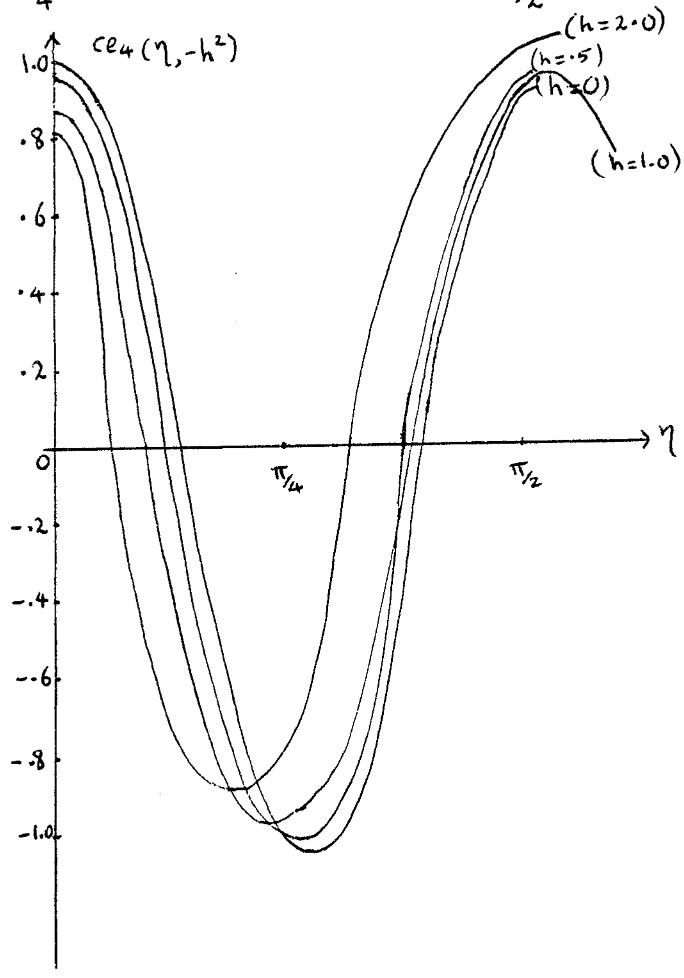
in the case $\ell = 3f$

h	n=0	n=1	n=2
.0	4.24264	-3.00000	0.00000
.1	2.32254	-1.65459	0.00137
.2	1.26495	-0.92155	0.00303
.3	0.68519	-0.51792	0.00376
.4	0.36895	0.29337	0.00370
.5	0.19736	-0.16723	0.00321
.6	0.10481	-0.09574	0.00257
.7	0.05524	-0.05492	0.00195
.8	0.02889	-0.03150	0.00142
.9	0.01499	-0.01803	0.00101
1.0	0.00773	-0.01028	0.00070
1.1	0.00397	-0.00583	0.00048
1.2	0.00203	-0.00329	0.00032
1.3	0.00103	-0.00185	0.00021
1.4	0.00053	-0.00103	0.00014
1.5	0.00027	-0.00058	0.00009
1.6	0.00014	-0.00032	0.00006
1.7	0.00007	-0.00018	0.00004
1.8	0.00004	-0.00010	0.00002
1.9	0.00002	-0.00005	0.00002
2.0	0.00001	-0.00003	0.00001

The following are graphs of $ce_{2n}(\eta, -h^2)$ for $n = 0, 1, 2,$ $h = 0, 0.5, 1.0, 2.0,$ and $\eta \in [0, \frac{\pi}{2}]$. These functions are π-periodic and symmetric about $\eta = \frac{\pi}{2}$.







A further difficulty occurs in the evaluation of the integrand of (2.7.3) by the occurrence of the factor $\mathrm{Fek}'_{2n}(0, -h^2)/\mathrm{Fek}_{2n}(0, -h^2)$, since neither numerator nor denominator is easily evaluated directly. An expression for $\mathrm{Fek}'_{2n}(0, -h^2)$ is in [4] (sec. 13.31) while $\mathrm{Fek}_{2n}(0, -h^2)$ could be obtained from the Bessel function series of [4] (sec. 8.30) or the Bessel function product series of [4] (sec. 13.30).

However, it appears to be simpler to evaluate the quotient directly, as follows:

In the modified Mathieu equation (2.5.3), which we write in the form

$$F''(\xi) = Q(\xi)F(\xi) \ , \text{ where } \ Q(\xi) = a_{2n} + 2h^2\cosh 2\xi \quad (2.7.5)$$

we make the substitution

$$v = F(\xi)/F'(\xi) \ , \quad (2.7.6)$$

obtaining the equation

$$v'(\xi) = 1 - Q(\xi)v^2(\xi) \ . \quad (2.7.7)$$

Now, from the asymptotic form of $\mathrm{Fek}_{2n}(\xi, -h^2)$ it follows that, as $\xi \to \infty$, $v \sim -h^{-1}e^{-\xi}$. We take a moderate value ξ_0 of ξ , integrate the equation (2.7.7) numerically by a standard technique back to $\xi = 0$, hence obtaining $v(0)$ which is the reciprocal of the desired quantity.

Furthermore, in the evaluation of the normal component of surface displacement outside the strip we need to find numerical approximations for $L(\xi) = \dfrac{Fek_{2n}(\xi,-h^2)}{Fek_{2n}(0,-h^2)}$. However once $v(0)$ has been obtained one can easily evaluate $L(\xi)$ by straight forward numerical integration since $L(\xi)$ tends to zero rapidly as ξ tends to infinity. The graph of $L(\xi)$ is given in Appendix B.

To evaluate the stress τ_{zz} immediately below the centre of the punch (i.e. at $x = 0$, $y = 0$) as well as at the edge of the punch (i.e. at $x = 0$, $y = f$) we proceed as follows:

$$\text{let} \qquad V_n(h) = \frac{Fek'_{2n}(0,-h^2)}{Fek_{2n}(0,-h^2)} ,$$

$$\begin{aligned} \text{and} \quad U_n(\eta, h) &= D_n(h)V_n(h)ce_{2n}(\eta, -h^2) \\ &= (-1)^n D_n(h)V_n(h)ce_{2n}(\tfrac{\pi}{2} - \eta, h^2) \end{aligned}$$

so that $\eta = 0$ corresponds to the edge of the strip and $\eta = \frac{\pi}{2}$ corresponds to the centre of the strip. Then from (2.7.3)

$$\tau_{zz}(0, y, 0) \approx \frac{4}{\pi f(f^2-y^2)^{1/2}} \int_0^\infty U(\cos^{-1}\frac{y}{f}, h)dh$$

where $U(\eta, h) = U_0(\eta, h) + U_1(\eta,h) + U_2(\eta, h)$.

Finally we truncate the above integral since the integrand is small when $h > 2$, and evaluate

$$I(\eta) = \int_0^2 U(\eta, h)dh$$

using Simpson's rule.

The following table gives the values of $V_r(h)$ for $r = 0, 1, 2$ and $h = 0, 0.1, 0.2, \ldots, 2.0$.

Values of $V_n(h)$

$$V_n(h) = \frac{Fek'_{2n}(0,-h^2)}{Fek_{2n}(0,-h^2)}$$

h	V_0	V_1	V_2
.0	0	-2	-4
.1	- .40935	-2.00670	-4.00272
.2	- .55663	-2.02606	-4.01059
.3	- .69087	-2.05766	-4.02398
.4	- .81926	-2.10221	-4.04253
.5	- .94330	-2.15503	-4.06620
.6	-1.06295	-2.22010	-4.09517
.7	-1.17776	-2.29568	-4.12899
.8	-1.28715	-2.38152	-4.16788
.9	-1.39072	-2.47721	-4.21159
1.0	-1.48832	-2.58218	4.26003
1.1	-1.57998	-2.69549	-4.31332
1.2	-1.66608	-2.81619	-4.37120
1.3	-1.74712	-2.94317	-4.43341
1.4	-1.82365	-3.07531	-4.50045
1.5	-1.89620	-3.21130	-4.57185
1.6	-1.96529	-3.35031	-4.64770
1.7	-2.03145	-3.49113	-4.72813
1.8	-2.09499	-3.63280	-4.81301
1.9	-2.15629	-3.77458	-4.90268
2.0	-2.21464	-3.91558	-4.99725

Using the above table and the appropriate values of $D_n(h)$ and $ce_{2n}(\eta, -h^2)$ we obtain the following two approximations for $I(\eta)$; namely at $\eta = \frac{\pi}{2}$ (i.e. at the centre of the punch) and at $\eta = 0$ (i.e. at the edge of the strip) .

$$I(\tfrac{\pi}{2}) \approx -1.2711$$

$$I(0) \approx 0.7968 .$$

We note that the stress at the centre of punch ($\eta = \frac{\pi}{2}$) is negative. This is in accordance with the assumption that $\tau_{zz} = -p$ where p is the normal pressure applied to the punch. Also since $I(0) > 0$ and τ_{zz} is a continuous function of η , then for some value of η in $(0, \frac{\pi}{2})$ $\tau_{zz} = 0$. This shows that the above example does not represent a complete contact problem since contact is lost near the edge of the strip.

To solve this problem completely, one has to determine the contact region. This may be done by the following method which is essentially an iterative procedure.

First we find the pressure p in the strip, where $p(x, y) = -\tau_{zz}(x, y, 0)$, and the contour in the xy-plane, on which $p = 0$. The region of the xy-plane bounded by this contour, i.e. where $p \geq 0$, is then chosen to be the new contact region, say S_1 . Next making use of the expression

$$\psi(x, y, z) = \frac{1}{2\pi} \iint_S \frac{p(x', y')}{R} \, dx' \, dy'$$

(see Ch. 1, (1.8))

we find a new $p(x, y)$ such that the prescribed displacement function $w(x, y)$, where $w(x, y) = \psi(x, y, 0)$ for $x, y \in S_1$, satisfies

$$w(x, y) = \frac{1}{2\pi} \iint_{S_1} \frac{p(x', y')}{R} \, dx' \, dy' .$$

If $p(x, y) > 0$ for all $x, y \in S_1$ then we increase the region S_1 and repeat the last step to obtain a new $p(x, y)$. On the other hand if $p(x, y) < 0$ at some point in S_1, then the region S_1 is decreased and again a new $p(x, y)$ is obtained.

The above steps are repeated until we find the region, say S_∞, such that

$$\psi(x, y, 0) = w(x, y) \qquad \text{for} \quad (x, y) \in S_\infty$$

where $w(x, y)$ is prescribed

$$p(x, y) > 0 \quad \text{in} \quad S_\infty$$

and

$$p(x, y) = 0 \quad \text{on the boundary of} \quad S_\infty ,$$

In the next example we consider a profile which is constant along the y-axis, i.e. H is independent of η.

Let $H(x, \eta) = \dfrac{\delta \ell^2}{\ell^2 + x^2}$ where ℓ and δ are as in the previous example. Then following the steps outlined above,

$$T(h, \eta) = \frac{2\ell\delta}{f} \exp\left(\frac{-2\ell h}{f}\right)$$

$$D_n(h) = \frac{4\ell\delta}{\pi f} \exp\left(\frac{-2\ell h}{f}\right) \int_0^\pi ce_{2n}(\eta, -h^2) \, d\eta$$

and $\displaystyle\int_0^\pi ce_{2n}(\eta, -h^2) \, d\eta = (-1)^n \pi A_0^{(2n)}(h^2)$.

So for $\ell = 3f$

$$D_n(h) = (-1)^n 12\delta \exp(-6h) A_0^{(2n)}(h^2)$$

and using the same numerical procedures as those used in the previous example we obtain the following table of $D_n(h)$, for $n = 0, 1, 2$ and $h = 0, 0.1, 0.2, \ldots, 2.0$.

Values of $D_n(h)/\delta$

$$\frac{D_n(h)}{\delta} = (-1)^n 12\delta \exp(-6h) A_0^{(2n)}(h^2)$$

h	n=0	n=1	n=2
.0	8.48526	0	0
.1	4.65674	− .01646	0
.2	2.55546	− .03613	.00005
.3	1.40190	− .04457	.00008
.4	.76855	− .04335	.00015
.5	.42084	− .03695	.00019
.6	.23004	− .02888	.00022
.7	.12546	− .02120	.00023
.8	.06824	− .01482	.00021
.9	.03701	− .00995	.00019
1.0	.02002	− .00645	.00016
1.1	.01080	− .00407	.00012
1.2	.00582	− .00250	.00010
1.3	.00313	− .00151	.00007
1.4	.00168	− .00089	.00005
1.5	.00091	− .00052	.00004
1.6	.00049	− .00030	.00003
1.7	.00026	− .00017	.00002
1.8	.00014	− .00010	.00001
1.9	.00008	− .00006	.00001
2.0	.00004	− .00003	.00001

The normal component of stress, under the punch, can now be approximated by

$$\tau_{zz}(0,\ y,\ 0) \approx \frac{4}{\pi f(f^2-y^2)^{1/2}}\, I(\eta)\ ,$$

where $$I(\tfrac{\pi}{2}) \approx -.4818\ ,$$

$$I(0) \approx -.4130\ ,$$

and furthermore τ_{zz} is negative everywhere under the punch.

In Chapter one it was pointed out that in punch problems where one assumes complete contact, there will be stress singularities at the edge of the contact region. The stress function obtained for the above example clearly exhibits the expected singularity at $y = \pm f$ (i.e. at the edge of the strip).

2.8 The strip-crack problem

Since crack and punch problems, when considered as boundary value problems, can be treated along similar lines, here we shall briefly discuss the crack problem corresponding to the strip-punch problem.

It is assumed that a crack has developed inside an infinite elastic medium. As in the case of the punch problem the infinite elastic medium is assumed to be homogeneous, isotropic and linearly elastic. We shall also assume that the crack is opened out symmetrically by equal normal pressures applied to its faces in the sense that if the Cartesian coordinate system is set up with the origin placed inside the crack, then the crack is opened out symmetrically with respect to each of the planes $x = 0$, $y = 0$ and $z = 0$.

In the strip crack problem the crack occupies the infinite strip S defined by $z = 0$ and $|y| < f$. This means that the region S is that part of the plane $z = 0$ which after applying the normal pressure becomes the crack-face.

Due to the assumed symmetry we need only consider an elastic medium occupying an infinite half-space where the crack-face is the region S which is now on the boundary of the half-space. The geometric aspect of this problem is similar to that of the punch problem, discussed in section 2.4, and will not be examined in detail here (see figures 1, 2 and 3 of 2.4).

For zero shear stress across the plane of the crack, the equations of elastostatics (as stated in Chapter one) hold. Namely we have the relations (1.3 a, b, c), (1.6) and (1.7 a, b). So the corresponding boundary value problem can be stated as follows. A harmonic function ψ is to be found such that

(i) $\nabla^2\psi = 0$ for $z > 0$

(ii) $\psi \to 0$ as $r \to \infty$ in $z \geq 0$

(iii) $\psi(x, y, 0) = 0$ for (x, y) outside S, i.e. the normal component of displacement is zero on the plane $z = 0$ outside the strip. This condition is due to the assumption that pressure is applied symmetrically.

(iv) $\tau_{zz}(x, y, 0) = \left.\frac{\partial\psi}{\partial z}\right|_{z=0} = -p(x, y)$, when $(x, y) \in S$.

$p(x, y)$ is some prescribed function which as before is assumed to be symmetric about $x = 0$ and $y = 0$.

In terms of elliptic cylinder coordinates, (x, η, ξ) with $x \in (-\infty, \infty)$, $\eta \in [0, \pi]$ and $\xi \geq 0$, the above problem can be restated as:

(i)' equation (2.3.1) of section (2.3) holds for

$x \in (-\infty, \infty)$, $\eta \in [0, \pi]$ and $\xi > 0$

(ii)' $\psi \to 0$ as $|x| \to \infty$ or $\xi \to \infty$, for $\eta \in [0, \pi]$

(iii)' $\psi = 0$ at $\eta = 0, \pi$ for $\xi \geq 0$ and $x \in (-\infty, \infty)$

(iv)' $\frac{1}{f\sin\eta}\left(\frac{\partial\psi}{\partial\xi}\bigg|_{\xi=0}\right) = -q(x, \eta)$, where $\eta \in (0, \pi)$, $x \in (-\infty, \infty)$ and $q(x, \eta) \stackrel{d.}{=} p(x, y)$. In addition $q(x, \eta)$ is assumed to be symmetric about $x = 0$ and $\eta = \frac{\pi}{2}$.

Let $t(x, \eta) \stackrel{d.}{=} - f \sin\eta q(x, \eta)$, then through a set of steps, similar to those outlined in section (2.5), we obtain the formal solution

$$\psi = \int_0^\infty \cos\frac{2hx}{f} \sum_{n=0}^{\infty} K_n(h)\, se_{2n+1}(\eta, -h^2)\, Gek_{2n+1}(\xi, -h^2)\, dh \tag{2.8.1}$$

where $se_{2n+1}(\eta, -h^2)$ is given by (2.3.6c) and $Gek_{2n+1}(\xi, -h^2)$ is the corresponding solution of the modified Mathieu equation which tends to zero as ξ tends to infinity (see McLachlan [4], 11.12 and 11.42).

Similarly from (iv)' (inverting the Fourier cosine transform and using the orthogonality of Mathieu functions of the first kind) the coefficients $K_n(h)$ are given by

$$K_n(h) = \frac{8}{\pi^2 f Gek'_{2n+1}(0, -h^2)} \int_0^\pi \int_0^\infty t(x, \eta)\, se_{2n+1}(\eta, -h^2) \cos\frac{2hx}{f}\, dx\, d\eta .$$

It is easy to see that if we impose a set of conditions on $t(x, \eta)$, similar to those given in section 2.6

((c.1) to (c.8)), then the formal solution ψ, expressed by (2.8.1), is continuous and satisfies (i)' to (iv)'. With some minor modifications, the inequalities obtained in the appendices can also be used here. For example it can be shown that

$$\left| \frac{\mathrm{Gek}_{2n+1}(\xi,-h^2)}{\mathrm{Gek}'_{2n+1}(0,-h^2)} \right| \leq \left| \frac{\mathrm{Gek}_{2n+1}(0,-h^2)}{\mathrm{Gek}'_{2n+1}(0,-h^2)} \right| \leq \frac{1}{2n+1}$$

for $n = 0, 1, 2, \ldots,$ and $h \geq 0$.

The normal component of stress across the xy-plane outside the crack is given by

$$\tau_{zz}(x, y, 0) = \frac{1}{f\sinh\xi}\left(\frac{\partial\psi}{\partial\eta} \Big|_{\eta=0} \right)$$

where $y > f$ (i.e. $\xi > 0$). As we approach the edge of the crack, i.e. $(y - f) \to 0+$, the stress function becomes singular since when $\eta = 0$, $f\sinh\xi = [y^2 - f^2]^{1/2}$.

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Chapter 3 - THE PARABOLIC PUNCH PROBLEM

3.1 Introduction

In this chapter the three dimensional punch problem where the contact region is parabolic is investigated using an approach similar to that of Chapter two. In general, parabolic contact problems of elasticity have not received a great deal of attention. A solution to the two dimensional problem is given by England ([2], 3.8) and the parabolic crack problem for uniform pressure has been solved by Shah and Kobayashi [6]. Also in a paper by Kassir [3], solutions for parabolic crack problems under uniform pressure, uniform shear and pure bending are given.

As mentioned in Chapter two, crack and punch problems, for the same regions, can usually be solved along similar lines. This is due to the fact that if such problems are viewed as mixed boundary value problems, then by switching the appropriate boundary conditions of one problem we can define the other. In the punch problem we have a state of zero normal stress outside the punch as well as a prescribed function for the normal component of displacement under the punch, whereas in the corresponding crack problem the normal component of displacement outside the crack is zero and the normal component of stress (pressure) is prescribed inside the crack.

Shah and Kobayashi [6] use the algebraic form of the paraboloidal coordinate system and solve the problem by integrating a differential equation and arrive at a potential function. The success of this approach is largely due to the assumption that the pressure to which the crack is subjected is uniform.

Our solution to the parabolic punch problem allows a general representation for normal displacements under the punch (within physically reasonable limits) which in terms of the corresponding crack problem amounts to allowing a general pressure distribution inside the crack.

Kassir [3] on the other hand uses existing solutions for some elliptic crack problems which are restricted in the same sense as mentioned above. He then uses a limiting process to obtain the solution for the corresponding parabolic problem. This process essentially transforms the ellipsoidal coordinate system to the paraboloidal coordinate system and the solutions to both problems are matched accordingly. It appears that the problem with a more general boundary condition, corresponding to the pressure inside the crack, will not yield easily since the limiting process becomes considerably more complicated.

In the following sections of this chapter a solution is constructed through the separation of Laplace's equation in paraboloidal coordinates. This solution will contain three types of Mathieu functions: "ordinary", "modified", and "co-Mathieu" functions, with arguments lying in the following ranges respectively, $(-\pi, \pi]$, $[0, i\infty)$, $[\frac{1}{2}\pi, \frac{1}{2}\pi+i\infty)$. Separated solutions of Laplace's equation in the paraboloidal coordinate system are discussed in section 3.3 and for a more detailed study of these solutions the reader is referred to Arscott [1].

3.2 The paraboloidal coordinate system

The paraboloidal coordinates (α, β, γ) are related to the Cartesian coordinates by

$$x = \frac{1}{2}c(\cosh 2\alpha + \cos 2\beta - \cosh 2\gamma)$$

$$y = 2c \cosh\alpha\cos\beta\sinh\gamma$$

$$z = 2c \sinh\alpha\sin\beta\cosh\gamma$$

where α, β and γ are all real, c is a dimensional parameter and

$$0 \leq \alpha < \infty, \quad -\pi < \beta \leq \pi, \quad 0 \leq \gamma < \infty .$$

The surfaces α = constant consist of a family of elliptic paraboloids. In particular if $\alpha = \alpha_0$, the vertex of the elliptic paraboloid is given by $(\frac{1}{2}c \cosh 2\alpha_0, 0, 0)$ (in the Cartesian coordinate system) and its axis is OX, so that a point with coordinates $(x, 0, 0)$ where $x \leq \frac{1}{2}c \cosh 2\alpha_0$ lies inside the elliptic paraboloid. The section of this paraboloid by a plane perpendicular to the x-axis is an ellipse, the sections by the planes $y = 0$ and $z = 0$ are both parabolas.

On such a surface $\alpha = \alpha_0$ there is a singular arc, namely the intersection with the surface $\gamma = 0$; on this singular arc the correspondence between (x, y, z) and (α, β, γ) coordinates ceases to be one-to-one. If $0 < \beta' < \frac{\pi}{2}$, the points with paraboloidal coordinates $(\alpha_0, \frac{1}{2}\pi \pm \beta', 0)$ coincide, also the pair $(\alpha_0, -\frac{1}{2}\pi \pm \beta', 0)$ represent the same point. As shown in [I] this has important consequences if we require a solution of $\nabla^2\psi = 0$ to have continuous gradient across the arc.

For $\alpha = 0$ we obtain the degenerate surface occupied by a parabolic plate in the xy-plane with vertex at $(\frac{1}{2}c, 0, 0)$. The surfaces γ = constant also consist of a family of elliptic paraboloids. For $\gamma = \gamma_0$ we have the elliptic paraboloid with vertex at $(-\frac{1}{2}c \cosh 2\gamma_0, 0, 0)$ whose axis is the x-axis, and points with Cartesian coordinates (x, y, z) where $x \geq -\frac{1}{2}c \cosh 2\gamma_0$ lie inside this paraboloid. The sections are similar to those of $\alpha = \alpha_0$. On such a surface $\gamma = \gamma_0$, the intersection with $\alpha = 0$ forms another singular arc, where the points with paraboloidal coordinates $(0, \pm\beta', \gamma_0)$, for $0 < \beta' < \frac{\pi}{2}$, coincide. This also has implications for continuity of solutions and continuity of the gradients of solutions across the arc.

When $\gamma = 0$ we obtain the parabolic plate in the xz-plane with vertex at $(-\frac{1}{2}c, 0, 0)$. Finally the surfaces β = constant consist of portions of hyperbolic paraboloids. $\beta = \beta_0$ gives one quarter of a hyperbolic paraboloid, and the complete paraboloid is given by $\beta = \pm\beta_0$, $\beta = \pm(\pi - \beta_0)$. The degenerate surfaces are $\beta = 0$, $\pm\frac{\pi}{2}$ and π. For our purposes the region formed by $\beta = 0$ and $\beta = \pi$ is of interest since it is the infinite plate with a parabolic hole in the xy-plane which occupies the exterior of the surface $\alpha = 0$ in this plane.

3.3 Separation of Laplace's equation in paraboloidal coordinates

Laplace's equation, $\nabla^2\psi = 0$, in paraboloidal coordinates becomes (1.122, Meixner and Schäfke [5])

$$(\cos 2\beta + \cosh 2\gamma)\frac{\partial^2\psi}{\partial\alpha^2} + (\cosh 2\gamma + \cosh 2\alpha)\frac{\partial^2\psi}{\partial\beta^2} +$$

$$+ (\cosh 2\alpha - \cos 2\beta)\frac{\partial^2\psi}{\partial\gamma^2} = 0 .$$

Let $\psi = A(\alpha)B(\beta)C(\gamma)$, then three ordinary differential equations emerge:

$$A''(\alpha) + (-\lambda + 2q\cosh 2\alpha)A(\alpha) = 0 \qquad (3.3.1)$$

$$B''(\beta) + (\lambda - 2q\cos 2\beta)B(\beta) = 0 \qquad (3.3.2)$$

$$C''(\gamma) + (-\lambda - 2q\cosh 2\gamma)C(\gamma) = 0 \qquad (3.3.3)$$

where λ and $2q$ are separation constants chosen so that (3.3.2) takes the standard form of Mathieu's equation. Initially λ and q are arbitrary and independent but, as we shall see later, the boundary conditions of our problem will require that q be negative, say $q = -h^2$, where $h \in [0, \infty)$. The separation constant $\lambda = \lambda(h^2)$ turns out to be one of the characteristic values a_n, b_n which were described in section 2.3.

Arscott [I] called solutions of (3.3.3) "co-Mathieu functions" and introduced a notation for them but, as will be seen, solutions of (3.3.3) can be expressed conveniently in terms of solutions of (3.3.1), namely the familiar "modified" Mathieu functions.

With the solutions of equations (3.3.1) - (3.3.3) in mind we quote various needed results from Mathieu function theory.

Consider the ordinary and modified Mathieu equations, respectively:

$$\frac{d^2w}{dz^2} + (\lambda - 2q\cos 2z)w = 0 \qquad (3.3.4)$$

$$\frac{d^2w}{dz^2} - (\lambda - 2q\cosh 2z)w = 0 \qquad (3.3.5)$$

(3.3.5) being obtained form (3.3.4) by changing z to iz.

Equation (3.3.4) has the same qualitative nature whether q is positive or negative. Indeed, it is easily seen that if $w(z, q)$ is a solution of (3.3.4), then $w(\frac{\pi}{2} - z, -q)$ is also a solution. This remark leads to the well known relations between 2π-periodic Mathieu functions of the first kind (McLachlan [4], 2.18):

$$ce_{2n}(\frac{\pi}{2} - z, q) = (-1)^n ce_{2n}(z, -q) \qquad (3.3.6a)$$

$$ce_{2n+1}(\tfrac{\pi}{2} - z, q) = (-1)^n se_{2n+1}(z, -q) \quad (3.3.6b)$$

$$se_{2n+1}(\tfrac{\pi}{2} - z, q) = (-1)^n ce_{2n+1}(z, -q) \quad (3.3.6c)$$

$$se_{2n+2}(\tfrac{\pi}{2} - z, q) = (-1)^n se_{2n+2}(z, -q) \quad (3.3.6d)$$

In the problems under consideration here, we are interested only in these 2π-periodic Mathieu functions of the first kind. The parameter λ must, of course, have the appropriate characteristic value $a_m(q)$ or $b_m(q)$.

In equation (3.3.5), on the other hand, a change of sign of q changes the qualitative nature of the equation completely. If $q > 0$, say $q = h^2$, then (at least for sufficiently large z) the coefficient of w is negative, so the equation is oscillatory. The two standard solutions are the modified Mathieu functions of the first and second kinds. To be specific, let us take the case where $\lambda = a_{2n}(h^2)$, so that the periodic solution of (3.3.4) is $ce_{2n}(z, h^2)$; the solutions of (3.3.5) are then respectively

$$Ce_{2n}(z, h^2) = ce_{2n}(iz, h^2)$$

and

$$Fey_{2n}(z, h^2) .$$

As $z \to \infty$ these are both oscillatory and tend to zero, their asymptotic behaviour being, as $z \to \infty$, ([4], 11.10)

$$Ce_{2n}(z,\ h^2) \sim p_{2n}(h^2)\left(\frac{2}{\pi v}\right)^{1/2}\sin\left(v + \frac{\pi}{4}\right)$$

$$Fey_{2n}(z,\ h^2) \sim -p_{2n}(h^2)\left(\frac{2}{\pi v}\right)^{1/2}\cos\left(v + \frac{\pi}{4}\right)$$

where

$$v = he^z \text{ and } p_{2n}(h^2) = ce_{2n}(0,h^2)ce_{2n}(\tfrac{1}{2}\pi,h^2)/A_0^{(2n)}(h^2).$$

On the other hand, if q is negative, then equation (3.3.5) is non-oscillatory (C.f. Appendix A) and the solutions are exponentially increasing or decreasing. The standard solutions are the modified Mathieu functions of the first and third kind, namely

$$Ce_{2n}(z,\ -h^2) = ce_{2n}(iz,\ -h^2)$$

and

$$Fek_{2n}(z,\ -h^2)$$

with asymptotic behaviour (for large z) ([4], 11.12)

$$Ce_{2n}(z,\ -h^2) \sim (-1)^n p_{2n}(h^2)(2\pi v)^{-1/2}e^v$$

and

$$Fek_{2n}(z,\ -h^2) \sim (-1)^n p_{2n}(h^2)(2\pi v)^{-1/2}e^{-v}\ .$$

Next we consider the "co-Mathieu equation" (3.3.3) in the form

$$\frac{d^2w}{dz^2} + (-\lambda - 2q\cosh 2z)w = 0 \qquad (3.3.7)$$

It is easily verified that if $w(z,\ q)$ satisfies (3.3.4) then $w(\frac{1}{2}\pi + iz,\ q)$ satisfies (3.3.7). Following Arscott [1] we write those solutions corresponding to $ce_{2n}(z,\ q)$ as

$$CE_{2n}(z, q) \overset{d.}{=} ce_{2n}(\tfrac{1}{2}\pi + iz, q) = Ce_{2n}(z - \tfrac{1}{2}i\pi, q)$$

$$FEY_{2n}(z, q) \overset{d.}{=} Fey_{2n}(z - \tfrac{1}{2}i\pi, q)$$

$$FEK_{2n}(z, q) \overset{d.}{=} Fek_{2n}(z - \tfrac{1}{2}i\pi, q) .$$

The qualitative nature of this equation depends also on the sign of q, but in the opposite way to that of equation (3.3.5). For $q > 0$, (3.3.7) is non-oscillatory, with CE and FEK the real solutions, CE being exponentially increasing and FEK decaying. As $z \to \infty$,

$$CE_{2n}(z, h^2) \sim p_{2n}(h^2)(2\pi v)^{-1/2}e^{v}$$

and

$$FEK_{2n}(z, h^2) \sim p_{2n}(h^2)(2\pi v)^{-1/2}e^{-v} .$$

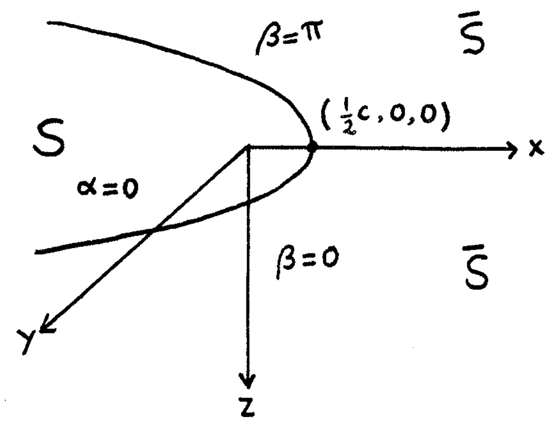
For $q < 0$, the real solutions are CE, FEY, both oscillatory and decaying. We shall not need these in the problem under discussion.

Finally we observe that formulae (3.3.6) yield simple links between the modified Mathieu and the co-Mathieu functions. Writing $(\frac{1}{2}\pi + iz)$ for z in (3.3.6a) gives

$$Ce_{2n}(z, q) = (-1)^n CE_{2n}(z, -q) \qquad (3.3.8)$$

Consequently it is possible to avoid use of the CE functions altogether. We shall retain them while putting our problem into mathematical terms but then eliminate them in favour of the Ce functions in order to analyze the solution.

3.4 Formulation of the boundary value problem



As in Chapter two, we shall assume that the half-space $z \geq 0$ is occupied by an isotropic, homogeneous, linearly elastic medium. Let S denote that part of the surface $z = 0$ which corresponds to $\alpha = 0$ (i.e. a parabolic plate) and let $\bar{S}$ be the region outside S on $z = 0$ which we assume to be stress free.

A rigid frictionless punch whose profile is defined by the function $K(x, y)$ is applied to the region S and contact is assumed to be complete everywhere on S. For convenience we shall assume that $K(x, y)$ is symmetric about $y = 0$. A general profile can be written as the sum of two functions, one symmetric and the other antisymmetric about $y = 0$ and the corresponding solutions can then be incorporated into one. For zero shearing stress on $z = 0$, the general equations of elasticity of Chapter one (in particular 1.7 a, b) can be used to reduce this problem to a mixed boundary value problem of potential theory, described by (i) to (v) of section 2.4.

3.5 The general solution of the boundary value problem

In terms of the paraboloidal coordinates (α, β, γ) we wish to find $\psi = \psi(\alpha, \beta, \gamma)$ such that it satisfies the following set of conditions:

(a) $$(\cos 2\beta+\cosh 2\gamma)\frac{\partial^2\psi}{\partial\alpha^2} + (\cosh 2\gamma+\cosh 2\alpha)\frac{\partial^2\psi}{\partial\beta^2} + (\cosh 2\alpha-\cos 2\beta)\frac{\partial^2\psi}{\partial\gamma^2} = 0$$

for $\alpha \in (0, \infty)$, $\beta \in (0, \pi)$ and $\gamma \in (0, \infty)$.

(b) $\psi \to 0$ as $\alpha \to \infty$ or $\gamma \to \infty$, for $\beta \in [0, \pi]$.

(c) Since $\frac{\partial}{\partial z} = \frac{\partial\alpha}{\partial z}\frac{\partial}{\partial\alpha} + \frac{\partial\beta}{\partial z}\frac{\partial}{\partial\beta} + \frac{\partial\gamma}{\partial z}\frac{\partial}{\partial\gamma}$, and

$$\frac{\partial\alpha}{\partial z} = 4\cosh\alpha\sin\beta\cosh\gamma(\cos^2\beta+\sinh^2\gamma)\,/\,c[\sinh^2 2\alpha(\cos 2\beta+\cosh 2\gamma) + \sin^2 2\beta(\cosh 2\alpha+\cosh 2\gamma) + \sinh^2 2\gamma(\cosh 2\alpha-\cos 2\beta)] \tag{3.5.1}$$

$$\frac{\partial\beta}{\partial z} = 4\sinh\alpha\cos\beta\cosh\gamma(\cosh^2\alpha+\sinh^2\gamma)/\,c[\sinh^2 2\alpha(\cos 2\beta+\cosh 2\gamma) + \sin^2 2\beta(\cosh 2\alpha+\cosh 2\gamma) + \sinh^2 2\gamma(\cosh 2\alpha-\cos 2\beta)] \tag{3.5.2}$$

$$\frac{\partial\gamma}{\partial z} = 4\sinh\alpha\sin\beta\sinh\gamma(\cosh^2\alpha-\cos^2\beta)\ /\ c[\sinh^2 2\alpha(\cos 2\beta+\cosh 2\gamma) + \sin^2 2\beta(\cosh 2\alpha+\cosh 2\gamma) + \sinh^2 2\gamma(\cosh 2\alpha-\cos 2\beta)] \tag{3.5.3}$$

and $\bar{S}$ is the region where $\alpha > 0$, $\gamma \geq 0$, $\beta = 0$ or π, then $\frac{\partial\psi}{\partial z} = 0$ on $\bar{S}$ is equivalent to $\frac{1}{2c\sinh\alpha\cosh\gamma}\frac{\partial\psi}{\partial\beta} = 0$ at $\beta = 0$ and $\beta = \pi$, where $\alpha \in (0, \infty)$ and $\gamma \in [0, \infty)$.

(d) $\psi(0, \beta, \gamma) = H(\beta, \gamma)$, where $H(\beta, \gamma) \stackrel{d.}{=} K(x, y)$, $\beta \in (0, \pi)$ and $\gamma \in [0, \infty)$.

(e) $H(\beta, \gamma)$ is symmetric about $\beta = \frac{\pi}{2}$.

Equation (3.3.2) together with the above conditions ((c) and (e)) imply, by the same argument as that used in Chapter two (sec. 2.5), $B(\beta) = ce_{2n}(\beta, q)$ and of course that $\lambda = a_{2n}(q)$, but with no restriction on the sign of q .

We turn to the question of what solution of (3.3.3) must be chosen. Consider the part of a surface $\alpha = \alpha_0$ ($\neq 0$) which lies inside the elastic medium. This surface is described as β and γ vary over the ranges $0 \leq \beta \leq \pi$ and $0 \leq \gamma \leq \infty$ with one-to-one correspondence except on the singular arc given by $\gamma = 0$. Here the points corresponding to the triads $(\alpha_0, \frac{1}{2}\pi \pm \beta', 0)$, for $0 < \beta' < \frac{\pi}{2}$, coincide. Now, we naturally require that our ultimate solution ψ should be continuous, with continuous gradient, throughout the interior of the elastic medium. As explained in [1], these continuity requirements lead to the conclusion that if $\psi = A(\alpha)B(\beta)C(\gamma)$ and $B(\beta) = ce_{2n}(\beta, q)$ then we must have $C(\gamma) = CE_{2n}(\gamma, q)$; the solutions $FEY_{2n}(\gamma, q)$ and $FEK_{2n}(\gamma, q)$ are ruled out.

We still have no criterion for the sign of q, but this appears when we take account of condition (b), for as noted above, $CE_{2n}(\gamma, q) \to 0$ as $\gamma \to \infty$ only if $q < 0$. We therefore set $q = -h^2$ and our separated solution is of the form

$$\psi = A(\alpha)ce_{2n}(\beta, -h^2)CE_{2n}(\gamma, -h^2) .$$

Finally, consider $A(\alpha)$ which satisfies (3.3.1) with $q = -h^2$. Hence $A(\alpha)$ may involve $Ce_{2n}(\alpha, -h^2)$ or $Fek_{2n}(\alpha, -h^2)$, but the former must be excluded because of condition (b); as $\alpha \to \infty$, $Ce_{2n}(\alpha, -h^2) \to \infty$.

So $A(\alpha) = Fek_{2n}(\alpha, -h^2)$ and our separated solution is necessarily of the form

$$\psi = Fek_{2n}(\alpha, -h^2)ce_{2n}(\beta, -h^2)CE_{2n}(\gamma, -h^2) .$$

More generally, a single separated solution can be written as

$$\psi_n \overset{d.}{=} \psi_n(\alpha,\beta,\gamma,h) = B_n(h)Fek_{2n}(\alpha,-h^2)ce_{2n}(\beta,-h^2)Ce_{2n}(\gamma,h^2)$$

where $CE_{2n}(\gamma, -h^2)$ has been replaced by $Ce_{2n}(\gamma, h^2)$ (relation (3.3.8)), and as in Chapter two, section 2.5, n is an arbitrary non-negative integer, h is an arbitrary non-negative parameter and $B_n(h)$ an arbitrary constant.(*) Consequently a general solution for the mixed boundary value problem can be written in the form (see sec. 2.5)

$$\psi = \int_0^\infty \sum_{n=0}^\infty B_n(h)\mathrm{Fek}_{2n}(\alpha,-h^2)\mathrm{ce}_{2n}(\beta,-h^2)\mathrm{Ce}_{2n}(\gamma,h^2)dh \tag{3.5.4}$$

To find the coefficients $B_n(h)$ we proceed formally. From boundary condition (d) ,

$$H(\beta, \gamma) = \int_0^\infty \sum_{n=0}^\infty C_n(h)\mathrm{ce}_{2n}(\beta,-h^2)\mathrm{Ce}_{2n}(\gamma,h^2)dh \tag{3.5.5}$$

where $C_n(h) = B_n(h)\mathrm{Fek}_{2n}(0, -h^2)$.

Since $\mathrm{ce}_{2n}(\beta, -h^2) = (-1)^n \mathrm{ce}_{2n}(\frac{\pi}{2} - \beta, h^2)$,

$$H(\beta, \gamma) = \int_0^\infty \sum_{n=0}^\infty (-1)^n C_n(h)\mathrm{ce}_{2n}(\tfrac{\pi}{2} - \beta, h^2)\mathrm{Ce}_{2n}(\gamma, h^2)dh . \tag{3.5.6}$$

We now have to invert this relationship in order to obtain $C_n(h)$ in terms of $H(\beta, \gamma)$. The problem does not appear to have been treated before. Our method is to use an integral relationship due to McLachlan ([4], sec. 10.51, (9)) which in turn is derived from Whittaker's general solution of Laplace's equation; this converts (3.5.6) into a double Fourier cosine transform.

From [4] (sec. 10.51, (9)),

$$\mathrm{ce}_{2n}(\tfrac{\pi}{2} -\beta,h^2)\mathrm{Ce}_{2n}(\gamma,h^2) = \rho_{2n}\int_0^{2\pi} \cos[F(\beta,\gamma, \theta,h)]\mathrm{ce}_{2n}(\theta,h^2)d\theta$$

(*) CE_{2n} is changed to Ce_{2n} merely for convenience, since in the evaluation of the coefficient $B_n(h)$ we shall employ a relationship involving the product $ce_{2n}Ce_{2n}$.

where $F(\beta,\gamma,\theta,h) = 2h(\cosh\gamma\sin\beta\cos\theta+\sinh\gamma\cos\beta\sin\theta)$, and

$$\rho_{2n} = ce_{2n}(0,h^2)ce_{2n}(\tfrac{1}{2}\pi,h^2) \ / \ 2\pi A_0^{(2n)}(h^2) \ .$$

So we can write

$$ce_{2n}(\tfrac{\pi}{2}-\beta,h^2)Ce_{2n}(\gamma,h^2) = 4\rho_{2n}\int_0^{\pi/2}\cos(2h\cosh\gamma\sin\beta\cos\theta)\times$$
$$\times\cos(2h\sinh\gamma\cos\beta\sin\theta)ce_{2n}(\theta,h^2)d\theta \qquad (3.5.7)$$

and

$$H(\beta,\gamma) = \int_0^\infty\sum_{n=0}^{\infty}D_n(h)\int_0^{\pi/2}\cos(2h\cosh\gamma\sin\beta\cos\theta)\times$$
$$\times\cos(2h\sinh\gamma\cos\beta\sin\theta)ce_{2n}(\theta,h^2)d\theta dh \qquad (3.5.8)$$

where $D_n(h) = (-1)^n 4\rho_{2n}Cn(h)$.

Provided we can interchange the order of summation and integration inside (3.5.8), then

$$H(\beta,\gamma) = \int_0^\infty\int_0^{\pi/2}\cos(2h\cosh\gamma\sin\beta\cos\theta)\times$$
$$\times\cos(2h\sinh\gamma\cos\beta\sin\theta)f(\theta,h)d\theta dh \qquad (3.5.9)$$

where $f(\theta, h) = \sum_{n=0}^{\infty}D_n(h)ce_{2n}(\theta, h^2)$.

Next we make the following transformations:

let $\begin{cases}\xi_1 = h\cos\theta \\ \xi_2 = h\sin\theta\end{cases}$, and $\begin{cases}x_1 = 2\cosh\gamma\sin\beta \\ x_2 = 2\sinh\gamma\cos\beta\end{cases}$

where $\theta \in [0, \frac{\pi}{2}]$, $h \in [0, \infty)$, $\gamma \in [0, \infty)$, $\beta \in [0, \pi]$, so $\xi_1 \in [0, \infty)$, $\xi_2 \in [0, \infty)$, $x_1 \in [0, \infty)$ and $x_2 \in (-\infty, \infty)$.

Then (3.5.9) becomes

$$H_1(x_1,x_2) = \int_0^\infty\int_0^\infty \cos(x_1\xi_1)\cos(x_2\xi_2)g(\xi_1,\xi_2)d\xi_1 d\xi_2 \qquad (3.5.10)$$

where $H_1(x_1,x_2) \stackrel{d.}{=} H(\beta,\gamma)$ and $g(\xi_1,\xi_2) \stackrel{d.}{=} \frac{f(\theta,h)}{h}$.

Using the two-dimensional Fourier cosine transform formula on (3.5.10) we get

$$g(\xi_1,\xi_2) = \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \cos(x_1\xi_1)\cos(x_2\xi_2)H_1(x_1,x_2)dx_1dx_2 \ .$$

Changing back to variables h and θ ,

$$\sum_{n=0}^{\infty} D_n(h)ce_{2n}(\theta,h^2) = \frac{4h}{\pi^2}\int_0^\infty\int_0^\infty \cos(x_1 h\cos\theta)\cos(x_2 h\sin\theta)H_1(x_1,x_2)dx_1dx_2 \ . \qquad (3.5.11)$$

Next we multiply both sides of (3.5.11) by $ce_{2m}(\theta, h^2)$, where m is a fixed non-negative integer, and integrate with respect to θ from 0 to $\frac{\pi}{2}$. Proceeding formally, we change the order of summation and integration, and use orthogonality of $ce_{2n}(\theta, h^2)$ to obtain

$$\frac{\pi}{4}D_m(h) = \int_0^{\pi/2} \frac{4h}{\pi^2}ce_{2m}(\theta,h^2)\int_0^\infty\int_0^\infty \cos(x_1 h\cos\theta)\cos(x_2 h\sin\theta) \times$$
$$\times H_1(x_1,x_2)dx_1dx_2d\theta \ . \qquad (3.5.12)$$

Since $dx_1dx_2 = 2(\cosh 2\gamma + \cos 2\beta)d\beta d\gamma$, then in terms of β and γ , (3.5.12) can be written as

$$D_m(h) = \frac{32h}{\pi^3}\int_0^{\pi/2} ce_{2m}(\theta,h^2)\int_0^\infty\int_0^{\pi/2} H(\beta,\gamma)\cos(2h\cosh\gamma\sin\beta\cos\theta)\times$$
$$\times \cos(2h\sinh\gamma\cos\beta\sin\theta)(\cosh 2\gamma+\cos 2\beta)d\beta d\gamma d\theta \ .$$

Then interchanging orders of integration (provided it can be justified) and using (3.5.7), we get

$$D_m(h) = \frac{8h}{\pi^3 \rho_{2m}} \int_0^\infty \int_0^{\pi/2} (\cosh 2\gamma + \cos 2\beta)\, ce_{2m}(\tfrac{\pi}{2} - \beta, h^2) \times$$

$$\times\ Ce_{2m}(\gamma, h^2) H(\beta, \gamma)\, d\beta d\gamma.$$

Hence

$$C_n(h) = \frac{2h}{\pi^3 \rho_{2n}^2} \int_0^\infty \int_0^{\pi/2} (\cosh 2\gamma + \cos 2\beta)\, ce_{2n}(\beta, -h^2) \times$$

$$\times\ Ce_{2n}(\gamma, h^2) H(\beta, \gamma)\, d\beta d\gamma \qquad (3.5.13)$$

and the inversion of the relationship (3.5.6) is complete.

It may be noted here that from (3.5.1), (3.5.2), (3.5.3) and (3.5.4), the normal component of stress under the punch, i.e. on S, is given by

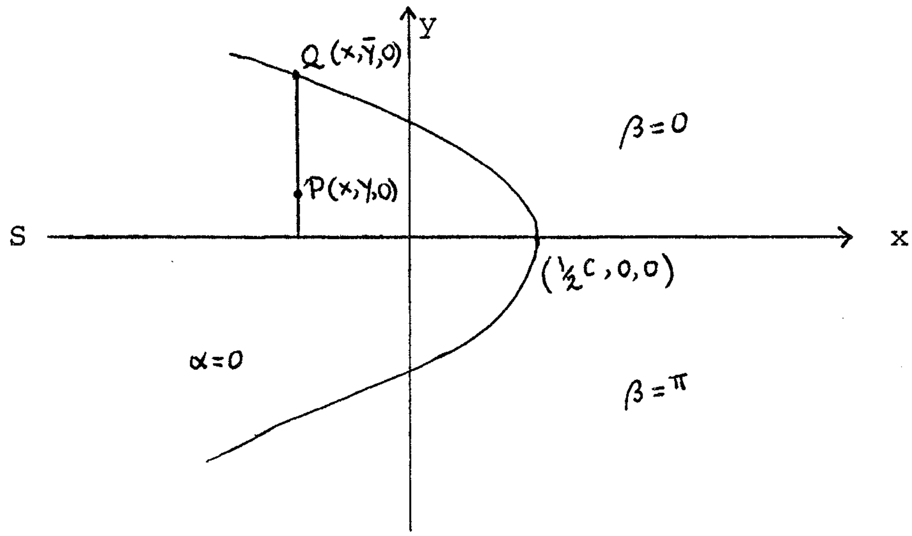
$$(2c \sin\beta \cosh\gamma)^{-1} \int_0^\infty \sum_{n=0}^{\infty} B_n(h) Fek'_{2n}(0, -h^2) \times$$

$$\times\ ce_{2n}(\beta, -h^2) Ce_{2n}(\gamma, h^2)\, dh \qquad (3.5.14)$$

where $\beta \in (0, \pi)$ and $\gamma \in [0, \infty)$.

Concerning the edge of the contact region, i.e. where $\alpha = 0$ and $\beta = 0$ or π , the presence of the term $(2c \sin\beta\cosh\gamma)^{-1}$ in (3.5.14) indicates a singularity of the function representing the normal component of stress. This singularity is of the square root type which is expected in complete contact problems.

More explicitly let P be a point inside the region S with Cartesian coordinates $(x, y, 0)$ and let Q be the point on the boundary of S with Cartesian coordinates $(x, \bar{y}, 0)$.



Also let the corresponding paraboloidal coordinates of P and Q be (α, β, γ) and $(\alpha', \beta', \gamma')$ respectively. Then $\alpha = \alpha' = 0$, and $\beta' = 0$, so

$$\bar{y}^2 - y^2 = (2c \sinh\gamma')^2 - (2c \cos\beta\sinh\gamma)^2 .$$

Since the x-coordinates of P and Q are the same,

$$\frac{1}{2}c(2 - \cosh 2\gamma') = \frac{1}{2}c(1 + \cos 2\beta - \cosh 2\gamma)$$

i.e. $$2 \sinh^2\gamma' = \cosh 2\gamma - \cos 2\beta$$

hence $$(\bar{y}^2 - y^2)^{-1/2} = (2c \sin\beta\cosh\gamma)^{-1} .$$

3.6 On the validity of the formal solution

As in Chapter two (sec. 2.6) a set of sufficient conditions can be imposed on the profile function $H(\beta, \gamma)$ in order to justify the formal steps taken in section 3.5.

However, since the subsequent analysis will be quite similar to that employed in Chapter two, we shall not pursue this in detail here. Clearly one expects the conditions on H to be somewhat stricter than those used in the previous chapter.

To justify the expression (3.5.6) the following steps may be outlined. Let

$$T(\xi_1,\xi_2) = \int_0^\infty\int_0^\infty \cos(x_1\xi_1)\cos(x_2\xi_2)H_1(x_1, x_2)dx_1dx_2 \quad (3.6.1)$$

where ξ_1, ξ_2, x_1, x_2 and $H_1(x_1, x_2)$ are as defined in section 3.5. Some conditions on $H_1(x_1, x_2)$ will be required here to ensure the existence of $T(\xi_1, \xi_2)$.

Next let

$$\xi_1 = h\cos\theta \quad , \qquad \xi_2 = h\sin\theta$$

and

$$T(\xi_1,\xi_2) \overset{d}{=}\cdot \frac{1}{h}J(h, \theta) .$$

If we expand $J(h, \theta)$ as a Mathieu function series, under similar conditions to those given in Chapter two, we obtain

$$J(h, \theta) = \sum_{n=0}^{\infty} E_n(h)ce_{2n}(\theta, h^2)$$

where, provided the series is uniformly convergent,

$$E_n(h) = \frac{4}{\pi}\int_0^{\pi/2} J(h, \theta)ce_{2n}(\theta, h^2)d\theta .$$

Now we choose the coefficients $C_n(h)$ in (3.5.6) such that

$$C_n(h) = \frac{(-1)^n E_n(h)}{\pi^2 \rho_{2n}}$$

where

$$\rho_{2n} = ce_{2n}(0,\ h^2) ce_{2n}(\tfrac{1}{2}\pi,\ h^2) \ /\ 2\pi\ A_0^{(2n)}(h^2)\ .$$

It may be noted here that since

$$ce'_{2n}(0,\ h^2) = ce'_{2n}(\tfrac{1}{2}\pi,\ h^2) = 0$$

and the zeros of basically periodic solutions of Mathieu's equation are all simple, then

$$ce_{2n}(0,\ h^2) \neq 0 \quad \text{and} \quad ce_{2n}(\tfrac{1}{2}\pi,\ h^2) \neq 0 \quad , \quad \text{i.e.} \quad \rho_{2n} \neq 0\ .$$

Finally inverting the double Fourier cosine transform (3.6.1) we obtain

$$H_1(x_1,x_2) = \frac{4}{\pi^2} \int_0^\infty \int_0^\infty \cos(x_1\xi_1)\cos(x_2\xi_2) T(\xi_1,\xi_2) d\xi_1 d\xi_2$$

and by changing the variables back to h , θ, β and γ , we get (3.5.6). Some further conditions must be imposed on H_1 to ensure that the double Fourier transform can be inverted. For example we can require H_1 to be three times continuously differentiable with respect to x_1 and x_2 .

In order to show that the function ψ represented by (3.5.4) is the solution of the boundary value problem, as before, it must be shown that ψ is continuous and satisfies Laplace's equation together with the boundary conditions. Writing ψ in the form

$$\psi(\alpha, \beta, \gamma) = \int_0^\infty \sum_{n=0}^{\infty} \frac{(-1)^n}{\pi^2} E_n(h) \frac{Fek_{2n}(\alpha,-h^2)}{Fek_{2n}(0,-h^2)} \times$$

$$\times \frac{ce_{2n}(\beta,-h^2)Ce_{2n}(\gamma,h^2)}{\rho_{2n}} dh \tag{3.6.2}$$

we observe that, from appendix B,

$$\left|\frac{Fek_{2n}(\alpha,-h^2)}{Fek_{2n}(0,-h^2)}\right| \le 1$$

and from (3.5.7) and appendix C (c.1.9),

$$\left|\frac{ce_{2n}(\beta,-h^2)Ce_{2n}(\gamma,h^2)}{\rho_{2n}}\right| \le 2\pi(\gamma_0 + \gamma_1 h + \gamma_2 h^2) .$$

Moreover, by the same technique as that used in appendix D, a suitable bound can be found for $E_n(h)$ to ensure the uniform convergence of the series and the integral in (3.6.2). The rest of the analysis, required to demonstrate that ψ is twice differentiable with respect to the three variables, can be developed by modifying the techniques used in the appendices.

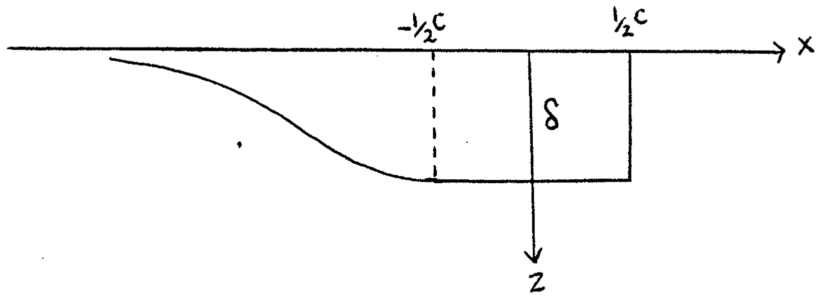
3.7 An example

We shall now consider an example where the function representing the punch profile, i.e. $H(\beta, \gamma)$, is independent of β. Let

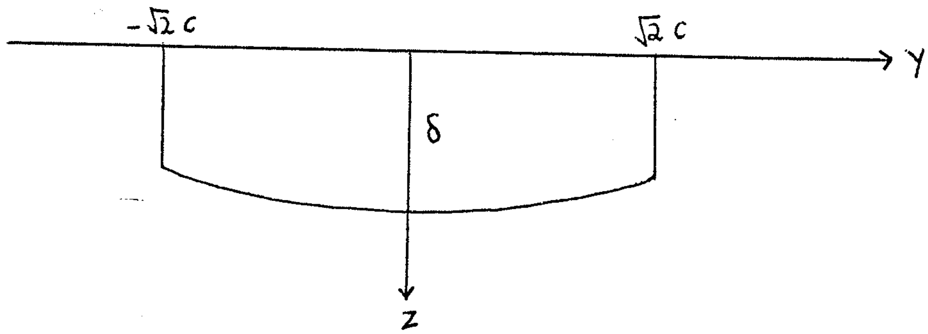
$$H(\beta, \gamma) = \delta \cosh\gamma \exp(-\sinh^2\gamma) \tag{3.7.1}$$

where δ is a dimensional parameter equal to the maximum depth of the punch. The cross-sections of the punch profile are shown below:

longitudinal cross-section



transverse cross-section



The coefficient $B_n(h)$, in the general solution (3.5.4), can be written as $\dfrac{C_n(h)}{Fek_{2n}(0,-h^2)}$ where $C_n(h)$ is given by (3.5.13) ,

$$C_n(h) = \frac{2h}{\pi^3 \rho_{2n}^2} \int_0^\infty \int_0^{\pi/2} (\cosh 2\gamma + \cos 2\beta) ce_{2n}(\beta, -h^2)$$

$$Ce_{2n}(\gamma, h^2) H(\beta, \gamma) d\beta\, d\gamma \ .$$

Using (3.7.1),

$$\frac{\pi^3 \rho_{2n}^2}{2h\delta} C_n(h) = \left[\int_0^\infty \cosh\gamma \exp(-\sinh^2\gamma) \cosh 2\gamma \, Ce_{2n}(\gamma, h^2) d\gamma\right.$$

$$\left.\int_0^{\pi/2} ce_{2n}(\beta, -h^2) d\beta\right] + \left[\int_0^\infty \cosh\gamma \exp(-\sinh^2\gamma) Ce_{2n}(\gamma, h^2) d\gamma\right.$$

$$\left.\int_0^{\pi/2} \cos 2\beta \, ce_{2n}(\beta, -h^2) d\beta\right]. \qquad (3.7.2)$$

Two of the four integrals in (3.7.2) can be evaluated easily, namely

$$\int_0^{\pi/2} ce_{2n}(\beta, -h^2) d\beta = (-1)^n \frac{\pi}{2} A_0^{(2n)} \qquad (3.7.3)$$

and

$$\int_0^{\pi/2} \cos 2\beta \, ce_{2n}(\beta, -h^2) d\beta = (-1)^{n+1} \frac{\pi}{4} A_2^{(2n)} \qquad (3.7.4)$$

where $A_0^{(2n)}$ and $A_2^{(2n)}$ are the first two coefficients in the Fourier series expansion of ce_{2n} .

To evaluate the remaining integrals in (3.7.2) we make use of the Bessel function series expansion of $Ce_{2n}(\gamma, h^2)$, where

$$Ce_{2n}(\gamma, h^2) = \frac{ce_{2n}(0, h^2)}{A_0^{(2n)}} \sum_{r=0}^{\infty} A_{2r}^{(2n)} J_{2r}(2h \sinh\gamma)$$

([4], 8.10., (17))
(3.7.4a)

This leads to the following expressions for the two integrals under consideration in (3.7.2)

$$\frac{ce_{2n}(0, h^2)}{A_0^{(2n)}} \sum_{r=0}^{\infty} A_{2r}^{(2n)} \int_0^\infty \cosh\gamma \cosh 2\gamma \times$$

$$\times \exp(-\sinh^2\gamma) J_{2r}(2h \sinh\gamma) d\gamma \qquad (3.7.5)$$

and

$$\frac{ce_{2n}(0,h^2)}{A_0^{(2n)}} \sum_{r=0}^{\infty} A_{2r}^{(2n)} \int_0^{\infty} \cosh\gamma \times$$

$$\times \exp(-\sinh^2\gamma) J_{2r}(2h\sinh\gamma)\, d\gamma \qquad (3.7.6)$$

Finally, the integrals in (3.7.5) and (3.7.6) can be evaluated using Hankel transforms ([7], 8.6, (8) and (14)). If we let $\sinh\gamma = x$, then

$$\int_0^{\infty} \cosh\gamma \cosh 2\gamma \exp(-\sinh^2\gamma) J_{2r}(2h \sinh\gamma)\, d\gamma$$

$$= \int_0^{\infty} (2x^2+1) \exp(-x^2) J_{2r}(2hx)\, dx \ .$$

The integral in (3.7.6) will reduce to the same form except for the term $(2x^2+1)$ in the integrand which will be missing. Now from tables of Hankel transforms ([7]),

$$\int_0^{\infty} x^2 \exp(-x^2) J_{2r}(2hx)\, dx = (2h)^{2r} \frac{\Gamma(r+\frac{3}{2})}{\Gamma(2r+1)} \times$$

$$\times \, {}_1F_1(\frac{2r+3}{2} \ ; \ 2r+1 \ ; \ -h^2) \qquad (3.7.7)$$

and

$$\int_0^{\infty} \exp(-x^2) J_{2r}(2hx)\, dx = \frac{\sqrt{\pi}}{2} \exp(\frac{-h^2}{2}) I_r(\frac{h^2}{2}) \qquad (3.7.8)$$

where in (3.7.7), $\Gamma(t)$ is the Gamma function, and ${}_1F_1$ is the confluent hypergometric function (also known as Kummer's confluent hypergeometric function). This function has the following series representation

$${}_1F_1(\frac{2r+3}{2} \ ; \ 2r+1 \ ; \ -h^2) = \sum_{k=0}^{\infty} \frac{(r+\frac{3}{2})_k (-h^2)^k}{(2r+1)_k \, k!}$$

where $(\alpha)_0 = 1$ and $(\alpha)_k = \alpha(\alpha+1)(\alpha+2) \ldots (\alpha+k-1)$. In (3.7.8), I_r is the usual modified Bessel function.

In section 2.7 certain numerical procedures were employed to compute the normal component of stress, τ_{zz}, under the punch. Here we shall indicate briefly how similar techniques may be used in the computation of τ_{zz} (under the punch) for the above example which is represented by the following expression

$$\tau_{zz} = (2c \sin\beta\cosh\gamma)^{-1} \int_0^\infty \sum_{n=0}^\infty C_n(h) \frac{Fek'_{2n}(0,-h^2)}{Fek_{2n}(0,-h^2)} \times$$

$$\times\, ce_{2n}(\beta, -h^2)Ce_{2n}(\gamma, h^2)\,dh \tag{3.7.9}$$

where

$$\frac{\pi^2 \rho_{2n}^2}{2h\delta} C_n(h) = (-1)^n ce_{2n}(0, h^2) \sum_{r=0}^\infty A_{2r}^{(2n)} \int_0^\infty (2x^2 + 1) \times$$

$$\times \exp(-x^2) J_{2r}(2hx)\,dx + \frac{(-1)^{n+1}}{2} \frac{A_2^{(2n)}}{A_0^{(2n)}} \times$$

$$\times\, ce_{2n}(0, h^2) \sum_{r=0}^\infty A_{2r}^{(2n)} \int_0^\infty \exp(-x^2) J_{2r}(2hx)\,dx \tag{3.7.10}$$

and

$$\rho_{2n} = \frac{ce_{2n}(0,h^2)ce_{2n}(\pi/2, h^2)}{2\pi A_0^{(2n)}}$$

For this example the integrals in (3.7.10) have been further reduced to products of Gamma functions, confluent hypergeometric functions and modified Bessel functions, however it appears that the integral representations in (3.7.10) are easier to handle as far as the actual numerical computations are concerned.

This observation is based on the fact that $|J_{2r}(2hx)| \leq 1$ and tends to zero rapidly as h, x or r increase. The numerical values of the J_{2r} Bessel functions have been extensively tabulated (e.g. M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, Dover 1968.) In addition the term $\exp(-x^2)$ causes the integrals to converge even faster. So a standard numerical method (e.g. Simpson's rule) may be employed here over a small interval of x, say $[0, 4]$.

As in section 2.7, one can obtain fairly accurate results using small values of h (say 0, 0.1, 0.2, ..., 2.0) and n (say 0, 1, 2). This is due to the occurrence of the coefficients $A_{2r}^{(2n)}$, and the functions $J_{2r}(2hx)$ in the expression for $C_n(h)$.

The coefficients $A_{2r}^{(2n)}$ and the Mathieu functions in (3.7.10) can all be computed using the method outlined in section 2.7 which is based on the techniques developed in reference [10] of Chapter 2.

Finally in order to evaluate (3.7.9) we note that the terms $\dfrac{Fek_{2n}'(0,-h^2)}{Fek_{2n}(0,-h^2)}$ and $ce_{2n}(\beta,-h^2)$ can be computed by exactly the same methods as those used for the example of section 2.7. Furthermore the non-periodic Mathieu function $Ce_{2n}(\gamma, h^2)$ can easily be computed from expression (3.7.4a) since again the series converges quite rapidly. The same set of values of n and h can be used here while the series itself may be approximated using the first 4 or 5 terms, depending on the desired accuracy.

Consequently a fairly accurate approximation of τ_{zz} can be obtained by using the first three terms of the series in (3.7.9) while h takes the values 0, 0.1, 0.2, ..., 2.0. The integral can then be computed using a standard numerical integration technique.

The above treatment can easily be modified to handle cases where the punch profile involves simple trigonometric functions of β. For instance if H is a product of $\sin^2\beta$ and the above function of γ (3.7.1) then only the expressions (3.7.3) and (3.7.4) would have to be altered to

$$\int_0^{\pi/2} \sin^2\beta\, ce_{2n}(\beta, -h^2)\, d\beta$$

and

$$\int_0^{\pi/2} \sin^2\beta \cos 2\beta\, ce_{2n}(\beta, -h^2)\, d\beta$$

both of which are easily expressible in terms of the coefficients $A_{2r}^{(2n)}$.

Thus, clearly, more general punch profiles than the one given by (3.7.1) can be treated along similar lines.

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Chapter 4 - THE WEDGE PUNCH PROBLEM

4.1 Introduction

We now consider the problem which arises when the punch contact is over an infinite wedge-shaped domain. Clearly, some assumption must be made, as was necessary for the strip punch and the parabolic punch, regarding the wedge profile at large distances from the apex. If the depth of penetration does not decrease to zero sufficiently rapidly, then theoretically an infinite amount of energy will be needed to make the indentation. The difficulties arising from this observation are, however, no more serious than in the cases considered already.

A more serious problem is posed, however, by the presence of the sharp vertex of the wedge. Intuitively, it is clear that this produces a "worse" stress singularity than the square-root type which arises at a smooth edge of a punch, and the difficulties are both physical and mathematical.

However the probem is not confined to the occurrence of stress singularities alone. In Chapter two we observed the existence of stress singularities on the edges of the strip punch, but the potential ψ which was obtained as the general solution of the boundary value problem was found to satisfy the boundedness requirement at infinity as well as being continuous within and on the boundary of the half-space. In the case of the wedge punch, with a sharp vertex, even the existence of such a well-behaved solution is in doubt. This difficulty is again due to the highly idealized condition resulting from the sharp point at the vertex. An investigation of some of the attempts by various authors in this area clearly indicates the magnitude of the difficulties involved in obtaining a solution which is mathematically as well as physically "complete". Thus the solution obtained in this chapter is by no means claimed to be the "complete" solution to such a problem.

As in the case of the strip punch problem in conjunction with the assumed idealized conditions of linear elasticity (see Ch. 2), the wedge punch problem is treated here as a mixed boundary value problem of potential theory. Basically a function ψ is to be found which is harmonic in a certain region and is prescribed on part of the boundary of this region while its normal derivative is zero on the remaining part of the boundary of the same region. In terms of the wedge punch problem this means that a function representing the punch profile is prescribed over an infinite sector in the xy-plane and the normal component of stress is taken to be zero outside this sector (on the xy-plane). We shall assume that the prescribed function which represents the punch profile is not constant and tends to zero at infinity. This is discussed in more detail in sections 4.3 and 4.4.

Here we shall briefly review some of the literature on the wedge punch and related problems. A number of different techniques have been used by researchers in this area. However these techniques will not be examined here since we are primarily concerned with the nature of the various solutions and the corresponding stress singularities obtained.

One of the earliest attempts towards a solution of the wedge punch problem can be found in the works of Galin [7]. He considers a punch with a flat profile and assumes that the first partial derivatives of ψ vanish at infinity. In his solution the normal component of stress under the punch is found to behave as r^{-1} near the apex, where r is the distance from the apex. This solution, however, indicates that there is some additional loading on the boundary of the elastic medium along the straight line which is the continuation of the line bisecting the wedge, outside the punch.

Later Rvachev [15], through an essentially different approach, attempted to solve the same problem without the additional normal pressure outside the punch. He assumed a solution of the form $\psi = r^{\nu} f(\theta, \phi)$ and concluded that the normal component of stress near the wedge apex behaves as $r^{\nu-1}$ where ν is dependent on the wedge angle and $0 \leq \nu \leq 1$.

Parihar and Keer [10], [14], Bažant and Keer [5] and Bažant [4] have also adopted the same (or very similar) boundary conditions which has inevitably led them to consider stress singularities, near the apex, of the form given by Rvachev. In [10] the boundary condition corresponding to the displacements under the punch is the same as that used by Rvachev (namely a constant), while in [14], [5] and [4] it is taken to be zero.

The wedge punch problem is mathematically very close to two other physical problem, namely (1) that of electrostatic charge distribution on a thin flat plate with a corner and (2) the "delta wing" problem relating to aerodynamic flow, at subsonic speeds, over an aerofoil in the shape of a triangular sector. Useful guidance and information is to be obtained from papers, by the following authors, dealing with these two problems: Morrison and Lewis [13], Taylor [16], Brown and Stewartson [6], and Taylor [17]. The first two relate to electrostatic potential theory and the second two to a steady-state aerodynamic problem.

All the above mentioned works (apart from that of Galin) share a common feature; namely that the stress singularity near the apex behaves like $r^{\nu-1}$, with ν dependent on the angle of the sector, and in most cases considerable emphasis is placed upon the numerical evaluation of the small values of ν. In essence they all treat the problem as an eigenvalue problem which is a direct consequence of the adoption of homogeneous boundary conditions.

Aleksandrov and Babeshko [1] treat the problem differently. They consider a wedge punch with an arbitrary profile so the boundary conditions are no longer assumed to be homogeneous. This is a significantly different problem in which the stress singularity near the apex is shown to behave as $r^{-3/2}\cos(\theta \ln r)$ where θ depends on the wedge angle. They raise an objection, on physical grounds, to the adoption of the boundary conditions which correspond to a punch with a flat (i.e. constant) profile, namely that in such cases "only a solution with infinite energy can exist". One may also argue against the case where the punch profile is taken to be zero, since it appears, at least, physically unreasonable to expect that such a punch would produce any non-zero stresses.

The problem that is treated in this chapter closely resembles that discussed by Aleksandrov and Babeshko [1]. Although the form of our solution and the techniques involved differ from those in [1], we are in close agreement with [1] as far as the behaviour of the stress singularity at the apex of the punch is concerned. It may be pointed out here that Aleksandrov and Babeshko obtain the pressure, p , under the punch and not the potential function ψ . They start with the type of integral representation discussed in Chapter two (p. 26) and apply a Mellin transform technique to find p . However, as mentioned in Chapter two (p. 27), although once the function p (representing the pressure) has been found, theoretically one can obtain the solution ψ of the boundary value problem (through a similar integral representation), usually such evaluations prove to be quite cumbersome.

One of the difficulties encountered by some of the earlier workers is the choice of a coordinate system which fits the geometry. A number of different systems have been used, but the one with most obvious advantages is the elliptic conal (sometimes called sphero conal) system, in which the infinite triangular sector, which is the area of contact between the punch and the elastic medium, is one of the normal sufaces (see Arscott and Darai [3]). Three versions of this system have been used; the "algebraic" (Brown and Stewartson [6]), the "trigonometric" (Morrison and Lewis [13]) and the "Jacobian" (Taylor [17]), but we shall use the last-named, mainly because it gives the simplest form to the basic differential equation, that of Lamé, whose solutions are needed.

4.2 The elliptic conal coordinate system

We choose an alignment of the elliptic conal system such that the elliptic conal coordinates r, α, β are related to the Cartesian coordinate by

$$x = \frac{r}{k'} \operatorname{dn}\alpha \operatorname{dn}\beta \tag{4.2.1a}$$

$$y = kr \operatorname{sn}\alpha \operatorname{sn}\beta \tag{4.2.1b}$$

$$z = \frac{ik}{k'} r \operatorname{cn}\alpha \operatorname{cn}\beta \tag{4.2.1c}$$

where sn, cn and dn are the Jacobian elliptic functions of modulus k (see [3] for a general discription). In this coordinate system r, α and β are confined to the domains

$$r \geq 0 \tag{4.2.2a}$$

$$\alpha \in (-2K,\ 2K] \tag{4.2.2b}$$

$$\beta \in [K,\ K+2iK'] \tag{4.2.2c}$$

where K is the complete elliptic integral of the first kind.

This involves a different alignment from that used by Taylor [17], but conforms to the usage elsewhere in this thesis in that the z-axis points into the elastic medium, the wedge punch lies in the xy-plane, and goes to infinity along the negative x-axis.

Taylor's [17] Cartesian coordinates X, Y, Z are thus related to our x, y, z by $X = y$, $Z = x$, $Y = z$. But the variation of the r, α, β coordinates is the same as that in Taylor's work.

With this choice of coordinates the coordinate surfaces which are of special interest in relation to our wedge problem can be described as follows:

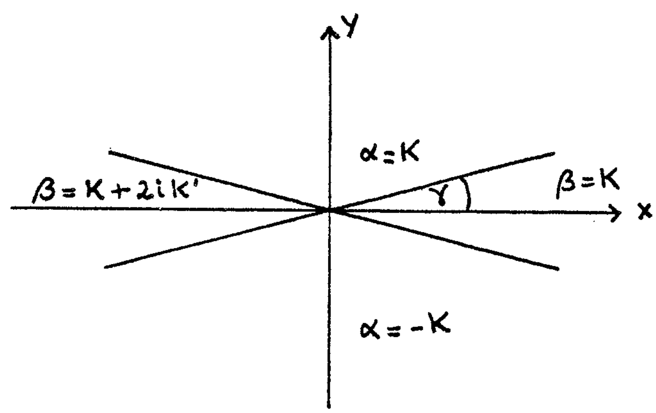
(i) The surfaces $\beta = K$, $\beta = K+2iK'$ are infinite sectors given, parametrically, by $x = \pm r \operatorname{dn}\alpha$, $y = kr \operatorname{sn}\alpha$, $z = 0$, respectively. Hence each is a sector in the xy-plane with semivertical angle γ, where

$$\sin\gamma = k \tag{4.2.3a}$$

$$\cos\gamma = k' \tag{4.2.3b}$$

These sectors have as their center-lines the positive and negative x-axes respectively, and we shall take $\beta = K+2iK'$ as the region of contact of the punch which will be denoted by S (as in the previous chapters).

(ii) The surfaces $\alpha = \pm K$ together occupy the remaining portions of the xy-plane, giving the configuration:



Since the variable β is complex, it is often convenient to replace it by the real variable u, given by (see [3])

$$\beta = K+iK'-iu \tag{4.2.4}$$

so that

$$k \operatorname{sn}\beta = \operatorname{dn}(u, k') \tag{4.2.5a}$$

$$k \operatorname{cn}\beta = - ik' \operatorname{cn}(u, k') \tag{4.2.5b}$$

$$\operatorname{dn}\beta = k' \operatorname{sn}(u, k') \tag{4.2.5c}$$

To save tedious repetition, it is to be understood that elliptic functions with argument u have modulus k' in place of k.

The relations (4.2.1a, b, c) between Cartesian and elliptic coordinates become

$$x = r \operatorname{dn}\alpha \operatorname{sn} u \tag{4.2.6a}$$

$$y = r \operatorname{sn}\alpha \operatorname{dn} u \tag{4.2.6b}$$

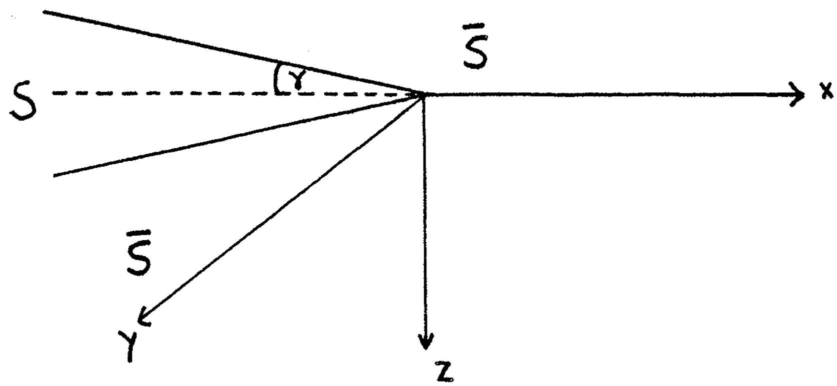
$$z = r \operatorname{cn}\alpha \operatorname{cn} u \tag{4.2.6c}$$

so $\beta = K$, $\beta = K+2iK'$ become $u = K'$, $u = -K'$ respectively.

4.3 Formulation of the boundary value problem

The mathematical formulation of the wedge punch problem is basically the same as that of the strip punch problem which was described in section 2.4.

It is assumed that a rigid frictionless punch in the shape of a wedge of semivertical angle γ indents part of the boundary of the half-space $z \geq 0$. The elastic medium which occupies this half-space is isotropic, homogeneous and linearly elastic, and there is complete contact between the punch and the elastic medium. Let S denote the contact region (i.e. in elliptic conal coordinates of section 4.2 the region $\beta = K+2iK'$) and $\bar{S}$ denote the region outside S on the xy-plane (i.e. $\alpha = \pm K$ and $\beta = K$) . We seek a function ψ which satisfies conditions (i) to (iv) of section 2.4.



4.4 The solution of the boundary value problem

Laplace's equation in elliptic conal coordinates becomes

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial\psi}{\partial r}\right) - \frac{1}{k^2(\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta)}\left(\frac{\partial^2\psi}{\partial\alpha^2} - \frac{\partial^2\psi}{\partial\beta^2}\right) = 0 \qquad (4.4.1)$$

The conditions (i) to (iv) of section 2.4 can now be stated as follows:

(a) Equation (4.4.1) holds for $r \in (0, \infty)$, $\alpha \in (-K, K)$ and $\beta \in (K, K+2iK')$.

(b) $\psi \to 0$ as $r \to \infty$, for $\alpha \in (-K, K)$ and $\beta \in (K, K+2iK')$.

(c) Since $\frac{\partial r}{\partial z} = \frac{ik}{k'} \operatorname{cn}\alpha \operatorname{cn}\beta$,

$$\frac{\partial\alpha}{\partial z} = -\frac{i \operatorname{sn}\alpha \operatorname{dn}\alpha \operatorname{cn}\beta}{kk'r(\operatorname{cn}^2\alpha - \operatorname{cn}^2\beta)} ,$$

$$\frac{\partial\beta}{\partial z} = \frac{i \operatorname{sn}\beta \operatorname{dn}\beta \operatorname{cn}\alpha}{kk'r(\operatorname{cn}^2\alpha - \operatorname{cn}^2\beta)} ,$$

and $\bar{S}$ is the region $\alpha = \pm K$ and $\beta = K$, then $\frac{\partial\psi}{\partial z} = 0$ on $\bar{S}$ is equivalent to

$$\frac{\partial\psi}{\partial\alpha} = 0 \quad \text{when} \quad \alpha = \pm K \tag{4.4.2}$$

and $$\frac{\partial\psi}{\partial\beta} = 0 \quad \text{when} \quad \beta = K \tag{4.4.3}$$

(d) $$\psi(r, \alpha, K+2iK') = F(r, \alpha) \tag{4.4.4}$$

for $\alpha \in [-K, K]$ and $r \in [0, \infty)$,

where $F(r, \alpha)$ is a prescribed function describing the punch profile. In addition, to simplify the work that follows we shall only consider the case where the punch profile is symmetric about the centre line of the punch, i.e. it is assumed that F is an even function of α .

If we separate equation (4.4.1) in the form

$$\psi = R(r)A(\alpha)B(\beta)$$

then we obtain the following ordinary differential equations:

$$r^2 \frac{d^2R}{dr^2} + 2r \frac{dR}{dr} - \nu(\nu+1)R = 0 \tag{4.4.5}$$

$$\frac{d^2A}{d\alpha^2} + (\lambda - \nu(\nu+1)k^2 sn^2\alpha)A = 0 \tag{4.4.6a}$$

$$\frac{d^2B}{d\beta^2} + (\lambda - \nu(\nu+1)k^2 sn^2\beta)B = 0 \tag{4.4.6b}$$

where λ and $\nu(\nu+1)$ are the two separation constants written in the form which allows us to express (4.4.6.a, b) as standard Lamé equations. Of course the domains of definition of the variables α and β differ in these two equations. In order to show which solutions of Lamé's equation are appropriate for our boundary value problem we first consider Lamé's equation in its general form

$$\frac{d^2w}{dz^2} + (\lambda - \nu(\nu+1)k^2 sn^2 z)w = 0 \tag{4.4.7}$$

where z is a complex variable. Since $z = K$ is an ordinary point of (4.4.7), then there is only one solution (apart from a constant multiple) such that $w'(K) = 0$. So from boundary conditions (4.4.2) and (4.4.3) and equations (4.4.6a, b), $A(\alpha)$ and $B(\beta)$ are the same functions respectively of α and β. However, since α and β belong to entirely different domains, the functions $A(\alpha)$ and $B(\beta)$ do not have the same behaviour.

To determine $A(\alpha)$ we observe that from the symmetry condition, imposed on the profile function, $A(\alpha)$ is even in α. This implies that $A'(0) = 0$, and since from condition (c) $A'(K) = 0$ then we have to solve (4.4.7) on the real axis with the boundary conditions

$$w'(0) = w'(K) = 0 \tag{4.4.8a,b}$$

From the theory of periodic differential equations it can be shown that (4.4.8a, b) are equivalent to the requirement that $w(z)$ be even with period $2K$, i.e.

$$w(-z) = w(z) \ , \quad w(z+2K) = w(z) \tag{4.4.9a,b}$$

(c.f. Arscott [2], 2.1.1 with π replaced by 2K; in particular example 2).

Now (4.4.7) and (4.4.8a, b) form a regular Sturm-Liouville eigenvalue problem which is solvable only for a denumerable infinity of eigenvalues. These may be denoted by λ_ν^{2m} with $m = 0, 1, 2, \ldots$. The corresponding solutions may be written as $Ec_\nu^{2m}(z)$ where the parameter m denotes the number of zeros of this function in $(0, K)$. The functions themselves may be determined in the form of series. Ince [8] gave them as power series

$$Ec_\nu^{2m}(z) = \sum_{r=0}^{\infty} A_{2r} sn^{2r} z \tag{4.4.10}$$

and in a later paper [9] as Fourier-Jacobi series

$$Ec_\nu^{2m}(z) = \sum_{r=0}^{\infty} C_{2r} \cos(2r \text{ am } z) \tag{4.4.11}$$

where $\cos(\text{am } z) = \text{cn } z$.

To obtain the eigenvalues λ and the coefficients A_{2r} or C_{2r} one can use the three-term recurrence relations which result from the sustitution of (4.4.10) or (4.4.11) expansions (respectively) into the differential equation (4.4.7). The method used here is similar to that used in the case of Mathieu functions. Other methods are available for the determination of λ and the corresponding solutions of (4.4.7). One such method is the perturbation technique devised by Taylor and Arscott [18] which has proved effective in the type of electrostatic and aerodynamic problems mentioned in section 4.1.

The usual phenomenon, relating to periodic differential equations, first observed for Mathieu functions occurs here (see McLachlan [12], Ch. 3, or Arscott [2], 3.6). Formal substitution of (4.4.10) or (4.4.11) in (4.4.7) yields a set of recurrence relations which can be solved for arbitrary values of λ, but the resulting series would, in general, diverge at some point on the real axis ((4.4.10) at $z = K$, (4.4.11) at $z = 0$). Choosing λ to have a characteristic value guarantees convergence at all points of the real axis, yielding a solution with the necessary boundary properties (4.4.8) or (4.4.9).

In the very exceptional case that ν is an integer, the series (4.4.10) (or (4.4.11)) terminates and yields a so-called Lamé polynomial, but this possibility does not seem to arise in our current problem.

Thus the problem of finding $A(\alpha)$ is, essentially, a matter of applying known theory and techniques.

It may be thought that since, according to our formulation of the problem, $B(\beta)$ is the same function of β as

$A(\alpha)$ is of α , and $A(\alpha)$ can be determined more-or-less explicitly, then $B(\beta)$ can also be easily determined through a similar representation. This, however, is not the case. It can be shown (Ince [8], [9]) that the series (4.4.10) or (4.4.11) converge on the entire real axis, but the region of convergence does not include all points of the interval $[K, K+2iK']$ which is the domain of definition of β . In particular, while (4.4.10) and (4.4.11) have slightly different regions of convergence and the latter converges faster than the former, both series fail to converge at $K+iK'$. Consequently, although the function $B(z)$ for $z \in [K, K+2iK']$ can be defined mathematically as the analytic continuation of $A(z)$ from a neighbourhood of $z = K$ along the line $Re(z) = K$, this alone does not lead to an explicit expression for $B(\beta)$.

For instance at $K+2iK'$ the terms of the series obtained by term-by-term differentiation of the series (4.4.10) (or (4.4.11)) are all zero, but we cannot conclude that $B'(K+2iK') = 0$ since convergence fails at this point. In particular we should note that $B(\beta)$ is not, in general, a periodic function of β .

However, there are integral representations of $B(\beta)$, in terms of other Lame functions (e.g. $A(\alpha)$ in this case), which can be handled more freely over the domain $[K, K+2iK']$ and usually such representations are used when numerical results are required. A detailed discussion of integral representations of non-periodic Lamé functions can be found in R.S. Taylor's Ph.D. thesis [19].

To continue with the formal analysis, we recall that when the partial differential equation (4.4.1) was separated the parameter ν emerged as a separation constant. Clearly, $\nu(\nu+1)$ must be real (otherwise we could get a complex expression for ψ) but there are three alternatives for $\nu(\nu+1)$ giving different forms of ν , namely

(i) $\nu(\nu+1) > -\frac{1}{4}$

(ii) $\nu(\nu+1) = -\frac{1}{4}$

(iii) $\nu(\nu+1) < -\frac{1}{4}$.

For each of the above alternatives we obtain a different type of solution $R(r)$ for equation (4.4.5). These are given by

(i) if $\nu(\nu+1) > -\frac{1}{4}$, then

$$R = Ar^{\nu} + Br^{-\nu-1} \tag{4.4.12}$$

where without loss of generality we may take $\nu > -\frac{1}{2}$

(ii) if $\nu(\nu+1) = -\frac{1}{4}$, then

$$R = r^{-1/2}(A + B \ln r) \tag{4.4.13}$$

(iii) if $\nu(\nu+1) < -\frac{1}{4}$, we set

$\nu = -\frac{1}{2} + ip \quad (p > 0)$ and then

$$R = r^{-1/2}(A\cos(p \ln r) + B\sin(p \ln r)) \;. \tag{4.4.14}$$

Thus we can write a separated solution in the form

$$\psi = R(r, \nu)Ec_\nu^{2m}(\alpha)Ec_\nu^{2m}(\beta) \qquad (4.4.15)$$

incorporating the parameter ν into the expression for R. This solution satisfies (4.4.1) and the boundary conditions (4.4.2), (4.4.3), but not (4.4.4). However, we may construct a more general solution by summing with respect to m and integrating with respect to ν, namely

$$\psi(r, \alpha, \beta) = \int \sum_{m=0}^{\infty} C_m(\nu)R(r,\nu)Ec_\nu^{2m}(\alpha)Ec_\nu^{2m}(\beta)d\nu \qquad (4.4.16)$$

with (so far) arbitrary $C_m(\nu)$. The set of values of ν over which integration is to be performed is not specified yet.

Using the remaining boundary condition (4.4.4), and the expression (4.4.16) we obtain, formally,

$$F(r,\alpha) = \int \sum_{m=0}^{\infty} D_m(\nu)R(r,\nu)Ec_\nu^{2m}(\alpha)d\nu \qquad (4.4.17)$$

where $\quad D_m(\nu) = C_m(\nu)Ec_\nu^{2m}(K+2iK')$.

Clearly, for the boundary value problem considered here, the choice of the range of integration is not immediately obvious on mathematical or physical grounds. This serious difficulty seems to stem from the nature of the problem itself. Perhaps the problem has been so idealized, physically, that it is no longer possible to obtain a "complete" mathematical solution.

In what follows we shall confine our attention to one of the alternatives mentioned above, namely the case where

$\nu = -\frac{1}{2} + ip$ with $p > 0$ (i.e. $\nu(\nu+1) < -\frac{1}{4}$) . With this choice for ν , integration takes place along the line $\mathrm{Re}\,\nu = -\frac{1}{2}$ and we can invert relation (4.4.17) to obtain a formal solution ot the problem. The solution found in this way seems to indicate that, in the immediate neighbourhood of the vertex, complete contact between the punch and the elastic medium is lost. However, due to the idealized nature of the problem, in particular the assumption of a sharp point at the vertex, an objection raised with respect to this interpretation may not be too serious.

To proceed with the formal analysis, we set

$$\nu = -\frac{1}{2} + ip \ , \quad Ec_{\nu}^{2m}(z) = \hat{E}_{p}^{2m}(z)$$

and take R in the form (4.4.14) . The expression (4.4.16) can now be written as

$$\psi(r,\alpha,\beta) = \int_0^{\infty} \sum_{m=0}^{\infty} r^{-1/2}[A_m(p)\cos(p \ln r) + $$

$$+ B_m(p)\sin(p \ln r)]\hat{E}_p^{2m}(\alpha)\hat{E}_p^{2m}(\beta)\,dp \qquad (4.4.18)$$

where the dependence on m and p of the arbitrary constants in R has been taken into account. The expression which corresponds to (4.4.17) can then be written as

$$r^{1/2}F(r,\alpha) = \int_0^{\infty} \sum_{m=0}^{\infty} [\gamma_m(p)\cos(p \ln r) + $$

$$+ \delta_m(p)\sin(p \ln r)]\hat{E}_p^{2m}(\alpha)\,dp \qquad (4.4.19)$$

where

$$\gamma_m(p) = A_m(p)\hat{E}_p^{2m}(K+2iK')$$

$$\delta_m(p) = B_m(p)\hat{E}_p^{2m}(K+2iK') \ .$$

Next we introduce a new variable s and let

$$\ln r = s \ , \qquad r^{1/2}F(r,\alpha) \stackrel{d.}{=} G(s,\alpha) \ .$$

Relation (4.4.19) now becomes

$$G(s,\alpha) = \int_0^\infty \sum_{m=0}^\infty [\gamma_m(p)\cos(ps) +$$

$$+ \quad \delta_m(p)\sin(ps)]\hat{E}_p^{2m}(\alpha)dp \qquad (4.4.20)$$

which has the form of a Fourier transform. In order to invert (4.4.20) we first split $G(s, \alpha)$ into its even and odd parts (in s), say

$$G(s, \alpha) = G_e(s, \alpha) + G_o(s, \alpha) \ .$$

So (4.4.20) now becomes

$$G_e(s, \alpha) = \int_0^\infty \sum_{m=0}^\infty \gamma_m(p)\cos(ps)\hat{E}_p^{2m}(\alpha)dp$$

$$G_o(s, \alpha) = \int_0^\infty \sum_{m=0}^\infty \delta_m(p)\sin(ps)\hat{E}_p^{2m}(\alpha)dp$$

which can be inverted by applying the inversion formulae for Fourier cosine and sine transforms. This gives

$$\sum_{m=0}^{\infty} \gamma_m(p)\hat{E}_p^{2m}(\alpha) = \frac{2}{\pi} \int_0^{\infty} \cos(ps)G_e(s, \alpha)ds \tag{4.4.21a}$$

$$\sum_{m=0}^{\infty} \delta_m(p)\hat{E}_p^{2m}(\alpha) = \frac{2}{\pi} \int_0^{\infty} \sin(ps)G_o(s, \alpha)ds \tag{4.4.21b}$$

Now since over the interval $(0, K)$, the functions $\{\hat{E}_p^{2m}(\alpha)\}$ are solutions of a regular Sturm-Liouville problem they satisfy the usual orthogonality property, i.e.

$$\int_0^{2K} \hat{E}_p^{2m}(\alpha)\hat{E}_p^{2m'}(\alpha)d\alpha = K_m(p)\,\delta_{mm'} \tag{4.4.22}$$

where $K_m(p)$ is a normalising constant (taken as π in Ince's work but we shall leave it unspecified).

Hence by applying (4.4.22) to (4.4.21a, b) the coefficients γ_m and δ_m can be determined in the following forms

$$\gamma_m(p) = \frac{2}{\pi K_m(p)} \int_0^{2K} \int_0^{\infty} \hat{E}_p^{2m}(\alpha)\cos(ps)G_e(s, \alpha)ds\,d\alpha$$

$$\delta_m(p) = \frac{2}{\pi K_m(p)} \int_0^{2K} \int_0^{\infty} \hat{E}_p^{2m}(\alpha)\sin(ps)G_o(s, \alpha)ds\,d\alpha$$

which in turn determine the coefficients A_m and B_m in the expression for ψ given by (4.4.18).

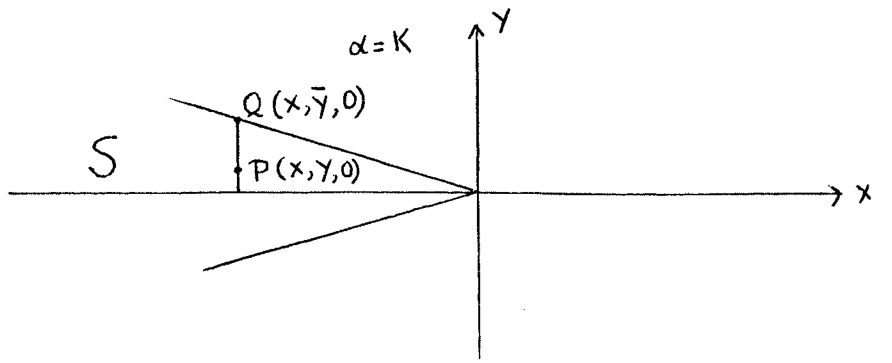
As indicated in Chapter one, the normal component of stress under the punch (inside S) is given by $\left.\frac{\partial\psi}{\partial z}\right|_{z=0}$. Here the region S is given by $\beta = K + 2iK'$, $-K < \alpha < K$ and $r > 0$.

Using (4.4.18) and the appropriate partial derivatives (see condition (c) earlier in this section), the expression for the normal component of stress under the punch becomes

$$(ikrcn\alpha)^{-1} \int \sum_{m=0}^{\infty} r^{-1/2}[A_m(p)\cos(p \ln r) + B_m(p)\sin(p \ln r)]$$

$$\hat{E}_p^{2m}(\alpha)\hat{E}_p^{2m'}(K + 2iK')dp \qquad (4.4.23)$$

Due to the presence of the term $(krcn\,\alpha)^{-1}$ in (4.4.23), the stress function exhibits a singularity at the boundary of S (i.e. the edges of the punch) where $\alpha = K$ or $\alpha = -K$. Moreover this singularity is of the square root type. To demonstrate this we let P be a point inside S with Cartesian coordinates $(x, y, 0)$ and Q be a point on the edge corresponding to $\alpha = K$ with Cartesian coordinates $(x, \bar{y}, 0)$, where $x > 0$.



If we let the elliptic conal coordinates (r, α, β) of P and Q be $(r, \alpha, K+2iK')$ and $(r', K, K+2iK')$ respectively, then

$$\bar{y}^2 - y^2 = k^2(r'^2 - r^2 sn^2\alpha) .$$

Since the x-coordinates of P and Q are the same

$$r' = \frac{r\,dn\,\alpha}{k}$$

from which it follows that

$$(\bar{y}^2 - y^2)^{-1/2} = k'(krcn\alpha)^{-1} .$$

In this section we have used a purely formal approach to obtain the general solution as well as an expression for the normal component of stress. One could conceivably justify the operations performed above by imposing a set of sufficient conditions on the profile function $F(r, \alpha)$. However, a more serious problem exists in relation to the boundary value problem itself; namely that the solution given above is not unique. This appears to be a fundamental difficulty which arises in the solution of Laplace's equation relative to a region whose boundary includes sharp corners (see Kellogg [11] Ch. 11, sec. 1).

Mathematically, the non-uniqueness of the above solution can be demonstrated in the following way. Suppose we seek a harmonic function $\phi(r, \alpha, \beta)$ which satisfies conditons (4.4.2) and (4.4.3) together with a homogeneous condition replacing (4.4.4), i.e. $\phi(r, \alpha, K+2iK') = 0$. This problem may now be viewed as an eigenvalue problem and is entirely analogous to the electrostatic and aerodynamic problems mentioned at the beginning of this chapter (in particular the trailing edge problem discussed by Taylor [17]). Various authors (as noted in section 4.1) have attempted this problem and shown that it does indeed have non-trivial solutions, for an infinity of discrete eigenvalues ν, these eigenvalues being real and numerically computable.

Now, clearly, $\psi^* = \psi + \phi$ is also a solution which satisfies our original boundary value problem (i.e. that satisfied by ψ) and if ϕ is to be rejected it cannot be on purely mathematical grounds.

As noted above, a further objection may be raised concerning the physical interpretation of the solution ψ represented by (4.4.18). This objection can be directed at our choice of values of ν (namely that $\nu = -\frac{1}{2} + ip$) which has resulted in the occurence of the functions $r^{-1/2}\cos(p \ln r)$ and $r^{-1/2}\sin(p \ln r)$ in the expression for ψ. Since these functions oscillate more and more rapidly as r approaches zero it appears then that, near the apex of the wedge, complete contact no longer exists between the punch and the elastic medium. However, due to the highly idealized assumption of a sharp point at the vertex, it is not easy to decide how seriously such an objection may be taken. A similar problem occurs in the mathematical theory of standing waves in a body of water confined between two boundaries inclined at an angle and forming a sector. In theory the waves become infinitely steep at the vertex, whereas in practice the linear theory ceases to be applicable in such a confined space so that the mathematical difficulty is not physically relevant.

Clearly the wedge punch problem merits further investigation, particularly concerning the nature of the mathematically admissible solutions.

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APPENDIX A

To show that $Fek_{2n}(0, -h^2) \neq 0$.

Consider $Fek_{2n}(\xi, -h^2)$ as a solution of the differential equation

$$F''(\xi) - (a_{2n} + 2h^2\cosh 2\xi)F(\xi) = 0 \qquad (A.1)$$

From Sturm's comparison theorm, applied to the ordinary Mathieu equation with periodic boundary conditions, the following inequality can be derived

$$(2n)^2 - 2h^2 \leq a_{2n}(h^2) \leq (2n)^2 + 2h^2$$
$$\text{for} \quad n = 0, 1, 2, \ldots, \quad \text{and} \quad h \geq 0 ,$$

(see McLachlan, Ch. 2 reference [4], sec. 12.20). So

$$a_{2n} + 2h^2\cosh 2\xi \geq 4n^2 \quad \text{for all} \quad \xi \in [0, \infty) .$$

Using Sturm's comparison theorem on (A.1) we can deduce that the solutions of the differential equation (A.1) are non-oscillatory in any interval $(0, b)$, i.e. no solution can have more than one zero in $(0, b)$, (see Ince, Ch. 2 reference [2], 10.32).

We also note that since $(a_{2n} + 2h^2\cosh 2\xi) \geq 0$, $F''(\xi)$ and $F(\xi)$ both have the same sign for all $\xi \in [0, \infty)$. In addition $Fek_{2n}(\xi, -h^2)$ tends to zero as ξ tends to infinity (the asymptotic form is given in section 2.5). So $Fek_{2n}(0, -h^2) \neq 0$ since $Fek_{2n}(\xi, -h^2) \not\equiv 0$.

APPENDIX B

I. From appendix A, $Fek_{2n}(0, -h^2) \neq 0$. Applying the same argument as that used in appendix A to the differential equation

$$L''(\xi) - (a_{2n} + 2h^2\cosh 2\xi)L(\xi) = 0 \qquad (B.1)$$

with $$L(\xi) = \frac{Fek_{2n}(\xi,-h^2)}{Fek_{2n}(0,-h^2)} ,$$

we deduce that since $L(0) = 1$, then $L(\xi)$ is a monotone decreasing function of ξ and tends to zero as ξ tends to infinity. This can be explained by the fact that since $L(\xi)$ is non-oscillatory in any interval $(0, b)$, and $L''(\xi)$ and $L(\xi)$ have the same sign, then $L(\xi)$ cannot cross the ξ-axis at any point and yet tend to zero at infinity.

II. From part I, and differentiability of $L(\xi)$, it can be deduced that $|L'(\xi)|$ and $L''(\xi)$ are both positive monotone decreasing functions of ξ , tending to zero as ξ tends to infinity. As an immediate consequence we have

$$0 < \frac{Fek''_{2n}(\xi,-h^2)}{Fek_{2n}(0,-h^2)} \leq \frac{Fek''_{2n}(0,-h^2)}{Fek_{2n}(0,-h^2)} = a_{2n} + 2h^2 \qquad (B.2)$$

Furthermore

$$\int_0^\infty L'(\xi)L''(\xi)d\xi = L(\xi)L''(\xi)\Big|_0^\infty - \int_0^\infty L(\xi)L'''(\xi)d\xi$$

and also

$$\int_0^\infty L'(\xi)L''(\xi)d\xi = \tfrac{1}{2}[L'(\xi)]^2 \Big|_0^\infty$$

so

$$\tfrac{1}{2}[L'(0)]^2 = L(0)L''(0) + \int_0^\infty L(\xi)L'''(\xi)d\xi \ . \qquad (B.3)$$

But following the argument used above, $L'''(\xi) < 0$ for all $\xi \geq 0$, and since $L(\xi) > 0$ for all $\xi \geq 0$, then the integral in (B.3) is negative. Hence

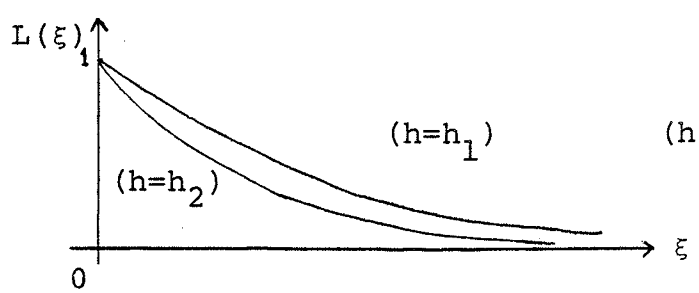
$$[L'(0)]^2 < 2L''(0)$$

and we have

$$|L'(\xi)| \leq |L'(0)| < \sqrt{2}[L''(0)]^{1/2} = \sqrt{2}(a_{2n} + 2h^2)^{1/2}$$

i.e.
$$\left|\frac{Fek'_{2n}(\xi)}{Fek_{2n}(0)}\right| < \sqrt{2}\ (a_{2n} + 2h^2)^{1/2} \ . \qquad (B.4)$$

The variation of $L(\xi)$ with ξ (for different values of h) is shown below.



APPENDIX C

I. Consider the equation

$$ce''_{2n}(\eta, -h^2) + (a_{2n} + 2h^2\cos 2\eta)ce_{2n}(\eta, -h^2) = 0$$

and define $y(\eta) = ce_{2n}(\eta, -h^2)$, then

$$y''(\eta) + (a_{2n} + 2h^2 + 1)y(\eta) = (1 + 4h^2\sin^2\eta)y(\eta). \quad (C.I.1)$$

Multiplying (C.1.1) by $y'(\eta)$ and integrating (with respect to η) from 0 to t, where $t \in [0, \eta]$ we get

$$\frac{1}{2}[y'(\eta)]^2\Big|_0^t + \frac{1}{2}(a_{2n}+2h^2+1)y^2(\eta)\Big|_0^t$$
$$= \int_0^t (1+4h^2\sin^2\eta)y(\eta)y'(\eta)d\eta \quad . \quad (C.1.2)$$

Since $ce'_{2n}(0, -h^2) = 0$,

$$[y'(t)]^2 + (a_{2n}+2h^2+1)[y^2(t)-y^2(0)]$$
$$= 2\int_0^t (1+4h^2\sin^2\eta)y(\eta)y'(\eta)d\eta \quad . \quad (C.1.3)$$

Next we integrate (C.1.3) from 0 to π (with respect to t) and note that $\int_0^\pi ce_{2n}^2(t, -h^2)dt = \frac{\pi}{2}$;

$$\int_0^\pi [y'(t)]^2dt + \frac{\pi}{2}(a_{2n}+2h^2+1)-\pi(a_{2n}+2h^2+1)y^2(0)$$
$$= 2\int_0^\pi\int_0^t (1+4h^2\sin^2\eta)y(\eta)y'(\eta)d\eta dt \quad . \quad (C.1.4)$$

Now $y^2(0)$ can be eliminated from (C.1.3) and (C.1.4) and we obtain

$$[y'(t)]^2 + (a_{2n}+2h^2+1)y^2(t) - 2\int_0^t (1+4h^2\sin^2\eta)y(\eta)y'(\eta)d\eta$$

$$= \frac{1}{\pi}\int_0^\pi [y'(t)]^2 dt + \frac{1}{2}(a_{2n}+2h^2+1) -$$

$$- \frac{2}{\pi}\int_0^\pi\int_0^t (1+4h^2\sin^2\eta)y(\eta)y'(\eta)d\eta dt \quad . \tag{C.1.5}$$

Also

$$\int_0^\pi [y'(t)]^2 dt = y(t)y'(t)\Big|_0^\pi - \int_0^\pi y(t)y''(t)dt$$

$$= \int_0^\pi (a_{2n}+2h^2\cos 2t)y^2(t)dt \tag{C.1.6}$$

since $\quad ce'_{2n}(\pi, -h^2) = 0$.

As noted in appendix A , $a_{2n} + 2h^2 \geq 4n^2$, so

$$|a_{2n}+2h^2\cos 2\eta| \leq |a_{2n}+2h^2| + 2h^2|\cos 2\eta - 1| \leq a_{2n} + 6h^2 \quad .$$

Hence

$$\int_0^\pi [y'(t)]^2 dt \leq (a_{2n}+6h^2)\int_0^\pi y^2(t)dt = \frac{\pi}{2}(a_{2n}+6h^2) \quad . \tag{C.1.7}$$

From (C.1.5) and (C.1.7) we get

$$(a_{2n}+2h^2+1)y^2(t) \leq 2\int_0^t |(1+4h^2\sin^2\eta)y(\eta)y'(\eta)|d\eta +$$

$$+ \frac{1}{2}(a_{2n}+6h^2) + \frac{1}{2}(a_{2n}+2h^2+1) +$$

$$+ \frac{2}{\pi}\int_0^\pi\int_0^t |(1+4h^2\sin^2\eta)y(\eta)y'(\eta)|d\eta dt$$

$$\leq \frac{1}{2}(a_{2n}+6h^2) + \frac{1}{2}(a_{2n}+2h^2+1) +$$

$$+ \quad 2(1+4h^2) \int_0^\pi |y(\eta)y'(\eta)|d\eta \quad +$$

$$+ \quad \frac{2}{\pi}(1+4h^2) \int_0^\pi\int_0^\pi |y(\eta)y'(\eta)|d\eta dt \ .$$

By Schwarz's inequality

$$\int_0^\pi |y(\eta)y'(\eta)|d\eta \leq \{ \int_0^\pi y^2(\eta)d\eta\}^{1/2} \{ \int_0^\pi [y'(\eta)]^2 d\eta\}^{1/2}$$

$$\leq \quad \frac{\pi}{2}(a_{2n}+6h^2)^{1/2} \qquad \text{(using (C.1.7))}$$

So

$$(a_{2n}+2h^2+1)y^2(t) \leq \frac{1}{2}(a_{2n}+6h^2) + \frac{1}{2}(a_{2n}+2h^2+1) +$$

$$+ \quad 2\pi(1+4h^2)(a_{2n}+6h^2)^{1/2}$$

and since $0 \leq a_{2n} + 6h^2 \leq 4n^2 + 8h^2$,

and $0 \leq 4n^2 + 1 \leq a_{2n} + 2h^2 + 1$,

$$y^2(t) \leq \frac{1}{2} + (4n^2+8h^2)(4n^2+1)^{-1} \quad +$$

$$+ \ [2\pi(1+4h^2)(4n^2+8h^2)^{1/2}](4n^2+1)^{-1} \ . \qquad \text{(C.1.8)}$$

Now

$(4n^2+8h^2)(4n^2+1)^{-1} \quad \leq \quad 1 + 8h^2$ and

$$(4n^2+8h^2)^{1/2}(4n^2+1)^{-1} \le (4n^2+8h^2)^{1/2}(4n^2+1)^{-1/2} \le 1 + 2\sqrt{2}h,$$

so $\quad y^2(t) \le \frac{3}{2} + 2\pi + 4\sqrt{2}\,h + (8\pi + 8)h^2 + 16\sqrt{2}\pi h^3$

$\le (\gamma_0+\gamma_1 h+\gamma_2 h^2)^2$, where γ_0, γ_1 and γ_2 are positive constants. Hence, for all $\eta \in [0, \pi]$ and $n = 0, 1, 2, \ldots$,

$$|ce_{2n}(\eta, -h^2)| \le \gamma_0 + \gamma_1 h + \gamma_2 h^2 \quad , \qquad \text{(C.1.9)}$$

It may be noted that the same inequality holds for $ce_{2n}(\eta, h^2)$.

II. If in (C.1.5), above, we drop the term $(a_{2n} + 2h^2 + 1)y^2(t)$ on the left-hand side and proceed to find an inequality for $[y'(t)]^2$ we find, using the same argument as that used above,

$$[y'(t)]^2 \le (a_{2n}+4h^2+1) + 2\pi(1+4h^2)(4n^2+8h^2)^{1/2} . \qquad \text{(C.2.1)}$$

Consequently,

$$\left(\frac{y'(t)}{a_{2n}+2h^2+1}\right)^2 \le \frac{1}{a_{2n}+2h^2+1} + \frac{2\pi(1+4h^2)(4n^2+8h^2)^{1/2}}{(a_{2n}+2h^2+1)^2}$$

$$\le 1 + 2\pi(1+4h^2)(4n^2+8h^2)^{1/2}(4n^2+1)^{-1}$$

$$\le 1 + 2\pi(1+4h^2)(1+2\sqrt{2}h) \quad \text{(by the inequality obtained in part I)}$$

$$\le (\gamma_0+\gamma_1 h+\gamma_2 h^2)^2$$

where γ_0, γ_1 and γ_2 are the same constants as those used in (C.1.9).

Hence

$$\left|\frac{ce'_{2n}(\eta,-h^2)}{a_{2n}+2h^2+1}\right| \leq \gamma_0+\gamma_1 h+\gamma_2 h^2 \quad . \tag{C.2.2}$$

APPENDIX D

In this section we shall derive two inequalities, (D.1.1) and (D.2.7), for $D_n(h)$, where

$$D_n(h) = \frac{2}{\pi}\int_0^\pi ce_{2n}(\eta,-h^2)\,T(h,\eta)\,d\eta \qquad \text{(D.1)}$$

and

$$T(h,\eta) = \int_0^\infty H(x,\eta)\cos\frac{2hx}{f}\,dx\,. \qquad \text{(D.2)}$$

I. Multiplying (C.1.1) of appendix C by $T(h,\eta)$ and integrating, with respect to η, from 0 to π we get

$$\int_0^\pi y''(\eta)\,Td\eta + (a_{2n}+2h^2+1)\int_0^\pi y(\eta)\,Td\eta = \int_0^\pi (1+4h^2\sin^2\eta)\,y(\eta)\,Td\eta$$

so

$$y'(\eta)\Big|_0^\pi - \int_0^\pi y'(\eta)\,\frac{\partial T}{\partial\eta}\,d\eta + (a_{2n}+2h^2+1)\frac{\pi}{2}\,D_n(h)$$

$$= \int_0^\pi (1+4h^2\sin^2\eta)\,y(\eta)\,Td\eta\,,$$

and since $y'(0) = y'(\pi) = 0$,

$$-\frac{\partial T}{\partial\eta}\,y(\eta)\Big|_0^\pi + \int_0^\pi \frac{\partial^2 T}{\partial\eta^2}\,y(\eta)\,d\eta + (a_{2n}+2h^2+1)\frac{\pi}{2}\,D_n(h)$$

$$= \int_0^\pi (1+4h^2\sin^2\eta)\,y(\eta)\,Td\eta\,.$$

From conditions (c.3) and (c.4) of section 2.6, for each $h \geq 0$, $\frac{\partial^2 T}{\partial\eta^2}$ is a continuous function of η, where $\eta \in [0,\pi]$, and from condition (c.6), $\frac{\partial T}{\partial\eta} = 0$ when $\eta = 0$ and when $\eta = \pi$. So

$$\frac{\pi}{2}(a_{2n}+2h^2+1)D_n(h) = \int_0^\pi (1+4h^2\sin^2\eta)y(\eta)T d\eta -$$

$$- \int_0^\pi \frac{\partial^2 T}{\partial\eta^2} y(\eta) d\eta \quad .$$

Using Schwarz's inequality and the notation of condition (c.7) of section 2.6,

$$\left| \frac{\pi}{2}(a_{2n}+2h^2+1)D_n(h) \right| \leq (1+4h^2)\left(\int_0^\pi y^2(\eta)d\eta\right)^{1/2}\left(\int_0^\pi T^2 d\eta\right)^{1/2} +$$

$$+ \left(\int_0^\pi \left(\frac{\partial^2 T}{\partial\eta^2}\right)^2 d\eta\right)^{1/2}\left(\int_0^\pi y^2(\eta)d\eta\right)^{1/2}$$

$$\leq \frac{\pi}{\sqrt{2}}(1+4h^2)M_0(h) + \frac{\pi}{\sqrt{2}} M_2(h) \quad . \tag{D.1.1}$$

II. Multiplying (C.1.1) appendix C by $\frac{\partial^2 T}{\partial\eta^2}$ and integrating with respect to η from 0 to π we obtain

$$\int_0^\pi y''(\eta)\frac{\partial^2 T}{\partial\eta^2}d\eta + (a_{2n}+2h^2+1)\int_0^\pi y(\eta)\frac{\partial^2 T}{\partial\eta^2}d\eta$$

$$= \int_0^\pi (1+4h^2\sin^2\eta)y(\eta)\frac{\partial^2 T}{\partial\eta^2}d\eta \quad .$$

So

$$y'(\eta)\frac{\partial^2 T}{\partial\eta^2}\Big|_0^\pi - \int_0^\pi y'(\eta)\frac{\partial^3 T}{\partial\eta^3}d\eta +$$

$$+ (a_{2n}+2h^2+1)\left\{y(\eta)\frac{\partial T}{\partial\eta}\Big|_0^\pi - \int_0^\pi y'(\eta)\frac{\partial T}{\partial\eta}d\eta\right\}$$

$$= \int_0^\pi (1+4h^2\sin^2\eta)y(\eta)\frac{\partial^2 T}{\partial\eta^2}d\eta \quad . \tag{D.2.1}$$

From conditions (c.3) and (c.4) of section 2.6, $\frac{\partial^3 T}{\partial \eta^3}$ is a continuous function of η for $\eta \in [0, \pi]$ and for each $h \geq 0$. Using conditions (c.4) and (c.6), we have $\frac{\partial T}{\partial \eta} = 0$ when $\eta = 0$ and when $\eta = \pi$. So (D.2.1) becomes

$$-\int_0^\pi y'(\eta) \frac{\partial^3 T}{\partial \eta^3} d\eta - (a_{2n}+2h^2+1) \int_0^\pi y'(\eta) \frac{\partial T}{\partial \eta} d\eta$$

$$= \int_0^\pi (1+4h^2 \sin^2 \eta) y(\eta) \frac{\partial^2 T}{\partial \eta^2} d\eta .$$

Integrating by parts again ,

$$- y(\eta) \frac{\partial^3 T}{\partial \eta^3} \Big|_0^\pi + \int_0^\pi y(\eta) \frac{\partial^4 T}{\partial \eta^4} d\eta -$$

$$- (a_{2n}+2h^2+1) \{ y'(\eta) T \Big|_0^\pi - \int_0^\pi y''(\eta) T d\eta \}$$

$$= \int_0^\pi (1+4h^2 \sin^2 \eta) y(\eta) \frac{\partial^2 T}{\partial \eta^2} d\eta ,$$

and using conditions (c.4) and (c.6)

$$\int_0^\pi y(\eta) \frac{\partial^4 T}{\partial \eta^4} d\eta + (a_{2n}+2h^2+1) \int_0^\pi y''(\eta) T d\eta$$

$$= \int_0^\pi (1+4h^2 \sin^2 \eta) y(\eta) \frac{\partial^2 T}{\partial \eta^2} d\eta ,$$

i.e.

$$-(a_{2n}+2h^2+1) \int_0^\pi (a_{2n}+2h^2 \cos 2\eta) y(\eta) T d\eta$$

$$= \int_0^\pi (1+4h^2 \sin^2 \eta) y(\eta) \frac{\partial^2 T}{\partial \eta^2} d\eta - \int_0^\pi y(\eta) \frac{\partial^4 T}{\partial \eta^4} d\eta \tag{D.2.2}$$

and using (D.1), we get

$$-(a_{2n}+2h^2+1)\{\frac{\pi}{2}a_{2n}D_n(h)+2h^2\int_0^\pi \cos 2\eta y(\eta)Td\eta\}$$

$$= \int_0^\pi (1+4h^2\sin^2\eta)y(\eta)\frac{\partial^2 T}{\partial\eta^2}d\eta - \int_0^\pi y(\eta)\frac{\partial^4 T}{\partial\eta^4}d\eta \quad .$$

Then

$$-\frac{\pi}{2}(a_{2n}+2h^2+1)^2 D_n(h) + (a_{2n}+2h^2+1)\{\frac{\pi}{2}(2h^2+1)D_n(h) -$$

$$- 2h^2 \int_0^\pi \cos 2\eta y(\eta)Td\eta\}$$

$$= \int_0^\pi (1+4h^2\sin^2\eta)y(\eta)\frac{\partial^2 T}{\partial\eta^2}d\eta - \int_0^\pi y(\eta)\frac{\partial^4 T}{\partial\eta^4}d\eta \quad ,$$

or

$$-\frac{\pi}{2}(a_{2n}+2h^2+1)^2 D_n(h) = -\frac{\pi}{2}(a_{2n}+2h^2+1)(2h^2+1)D_n(h) +$$

$$+ 2h^2(a_{2n}+2h^2+1)\int_0^\pi \cos 2\eta y(\eta)Td\eta +$$

$$+ \int_0^\pi (1+4h^2\sin^2\eta)y(\eta)\frac{\partial^2 T}{\partial\eta^2}d\eta - \int_0^\pi y(\eta)\frac{\partial^4 T}{\partial\eta^4}d\eta \quad . \tag{D.2.3}$$

Next multiply (C.1.1) of appendix C by $T\cos 2\eta$ and integrate with respect to η from 0 to π :

$$\int_0^{\pi} \cos 2\eta y''(\eta) T d\eta + (a_{2n}+2h^2+1) \int_0^{\pi} \cos 2\eta y(\eta) T d\eta$$

$$= \int_0^{\pi} (1+4h^2 \sin^2\eta) \cos 2\eta y(\eta) T d\eta \quad . \qquad \text{(D.2.4)}$$

Substituting for $(a_{2n}+2h^2+1) \int_0^{\pi} \cos 2\eta y(\eta) T d\eta$, in (D.2.3), from (D.2.4), we obtain

$$\frac{\pi}{2}(a_{2n}+2h^2+1)^2 D_n(h) = \frac{\pi}{2}(a_{2n}+2h^2+1)(2h^2+1) D_n(h) +$$

$$+ 2h^2 \int_0^{\pi} \cos 2\eta y''(\eta) T d\eta -$$

$$- 2h^2 \int_0^{\pi} (1+4h^2 \sin^2\eta) \cos 2\eta y(\eta) T d\eta -$$

$$- \int_0^{\pi} (1+4h^2 \sin^2\eta) y(\eta) \frac{\partial^2 T}{\partial \eta^2} d\eta + \int_0^{\pi} y(\eta) \frac{\partial^4 T}{\partial \eta^4} d\eta \quad .$$

(D.2.5)

Now

$$\int_0^{\pi} \cos 2\eta y''(\eta) T d\eta = y'(\eta) \cos 2\eta T \Big|_0^{\pi} -$$

$$- \int_0^{\pi} y'(\eta) (\cos 2\eta \frac{\partial T}{\partial \eta} - 2\sin 2\eta T) d\eta$$

$$= -y(\eta)(\cos 2\eta \frac{\partial T}{\partial \eta} - 2\sin 2\eta T) \Big|_0^{\pi} +$$

$$+ \int_0^{\pi} y(\eta) (\cos 2\eta \frac{\partial^2 T}{\partial \eta^2} - 4\sin 2\eta \frac{\partial T}{\partial \eta} - 4\cos 2\eta T) d\eta$$

$$= \int_0^{\pi} y(\eta) (\cos 2\eta \frac{\partial^2 T}{\partial \eta^2} - 4\sin 2\eta \frac{\partial T}{\partial \eta} - 4\cos 2\eta T) d\eta \quad .$$

So (D.2.5) becomes

$$\frac{\pi}{2}(a_{2n}+2h^2+1)^2 D_n(h) = \frac{\pi}{2}(a_{2n}+2h^2+1)(2h^2+1)D_n(h) +$$

$$+ 2h^2 \int_0^\pi y(\eta)\cos 2\eta \frac{\partial^2 T}{\partial\eta^2} d\eta -$$

$$- 8h^2 \int_0^\pi y(\eta)\sin 2\eta \frac{\partial T}{\partial\eta} d\eta - 8h^2 \int_0^\pi y(\eta)\cos 2\eta T d\eta -$$

$$- 2h^2 \int_0^\pi (1+4h^2\sin^2\eta) y(\eta)\cos 2\eta T d\eta -$$

$$- \int_0^\pi (1+4h^2\sin^2\eta) y(\eta) \frac{\partial^2 T}{\partial\eta^2} d\eta + \int_0^\pi y(\eta) \frac{\partial^4 T}{\partial\eta^4} d\eta$$

$$= \frac{\pi}{2}(a_{2n}+2h^2+1)(2h^2+1)D_n(h) -$$

$$- \int_0^\pi (1+4h^2\sin^2\eta) y(\eta) \frac{\partial^2 T}{\partial\eta^2} d\eta -$$

$$- 8h^2 \int_0^\pi y(\eta)\sin 2\eta \frac{\partial T}{\partial\eta} d\eta -$$

$$- 2h^2 \int_0^\pi (5+4h^2\sin^2\eta) y(\eta)\cos 2\eta T d\eta +$$

$$+ \int_0^\pi y(\eta) \frac{\partial^4 T}{\partial\eta^4} d\eta \quad . \qquad (D.2.6)$$

Now applying inequality (D.1.1) , of part I, Schwarz's inequality and condition (c.7) of section 2.6 to (D.2.6), we get

$$\frac{\pi}{2}(a_{2n}+2h^2+1)^2 |D_n(h)|$$

$$\leq \frac{\pi}{\sqrt{2}} (2h^2+1)[(1+4h^2)M_0(h)+M_2(h)] +$$

$$+ (6h^2+1)\left(\int_0^\pi y^2(\eta)d\eta\right)^{1/2}\left(\int_0^\pi \left(\frac{\partial^2 T}{\partial\eta^2}\right)^2 d\eta\right)^{1/2} +$$

$$+ 8h^2\left(\int_0^\pi y^2(\eta)d\eta\right)^{1/2}\left(\int_0^\pi \left(\frac{\partial T}{\partial\eta}\right)^2 d\eta\right)^{1/2} +$$

$$+ 2h^2(5+4h^2)\left(\int_0^\pi y^2(\eta)d\eta\right)^{1/2}\left(\int_0^\pi T^2 d\eta\right)^{1/2} +$$

$$+ \left(\int_0^\pi y^2(\eta)d\eta\right)^{1/2}\left(\int_0^\pi \left(\frac{\partial^4 T}{\partial\eta^4}\right)^2 d\eta\right)^{1/2} .$$

Hence

$$(a_{2n}+2h^2+1)^2|D_n(h)|$$

$$\leq \sqrt{2}\ (2h^2+1)[(1+4h^2)M_0(h)+M_2(h)] +$$

$$+ \sqrt{2}\ (6h^2+1)M_2(h) + 8\sqrt{2}\ h^2 M_1(h) +$$

$$+ 2\sqrt{2}h^2(5+4h^2)M_0(h) + \sqrt{2}M_4(h) ,$$

i.e.

$$(a_{2n}+2h^2+1)^2|D_n(h)|$$

$$\leq \sqrt{2}\ (16h^4+16h^2+1)M_0(h) + 8\sqrt{2}h^2 M_1(h) +$$

$$+ 2\sqrt{2}\ (4h^2+1)M_2(h) + \sqrt{2}M_4(h) . \qquad (D.2.7)$$