Analysis of variance tests for exponentiality of two distributions: Complete and censored samples

by

Hui Chen

A Thesis submitted to the Faculty of Graduate Studies of

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THE UNIVERSITY OF MANITOBA

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Abstract

Using the principles of the W-statistic for exponentiality of a single distribution (Shapiro and Wilk, 1972; Samanta and Schwarz, 1988) we develop procedures for testing a composite hypothesis of exponentiality of two distributions having the same scale parameter. The proposed V-exponential statistic for complete samples turns out to be a normalized ratio of the square of the generalized least squares estimator (also the minimum variance unbiased estimator) of the common scale parameter to a pooled sum of squares about the samples means. The V-exponential statistic is origin and scale invariant and has a null distribution that depends only on the sample sizes. We also prove some other important results relating to our proposed V-exponential statistic. Following the approach of Samanta and Schwarz (1988), the V-exponential statistic is then modified when one or both samples are censored. The modified test statistic has the same null distribution as in the uncensored case with a corresponding reduction in sample size(s).

Finally, following the approach of Stephens (1978), we propose a V^*-exponential statistic for testing exponentiality of two distributions for complete samples. In each case, we provide the empirical power results for various types of probability distributions considered under the alternative. We also compared the power results of the one-sample W-exponential test, two-sample V-exponential statistic and V^*-exponential statistic. We see that the obtained results are similar in each case. That is, the three tests seem comparable in terms of sensitivity.

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Hui Chen

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Chapter 1

Introduction

When testing for goodness-of-fit, we take an observed sample and test how well it fits a given distribution. The general procedure consists in: (i) setting the null hypothesis, denoted by H_0, which states that a given random variable X follows a probability distribution whose form is known and may depend on a number of known or unknown parameters, (ii) calculating a test statistic, which is some function of the data measuring the distance between the hypothesized distribution and the sample data, and (iii) making a decision about the acceptance or rejection of the null hypothesis.

There are many reasons to use goodness-of-fit tests. First, if the suggested probability model is correct, we have more confidence in our model for data generating process and on the parameters that describe the population. Secondly, the knowledge of the distribution for the data, allows us to use standard statistical testing and estimation procedures, such as analysis of variance and the construction and calculation of confidence and prediction intervals. Finally, with the knowledge of the distribution, extreme tail probabilities can be computed.

There are several goodness-of-fit techniques. Graphical analysis is an informal procedure. One such technique is based on the probability plot, which is defined as the ratio of the number of observations less than or equal to x_i (a realized value of the sample) to the total number of observations n. The probability distribution function is called empirical cumulative distribution function (*i.e.* ecdf). It is a simple tool which is easy to use on a piece of graph paper or using simple computer programs. It is less formal than the numerical techniques that will be presented later and usually is used to support the numerical testing procedures. It often gives a better understanding of the numerous relationships present in the data. The ecdf plot does not depend on any assumption about a hypothesized parametric distribution function. It has some advantages over the other statistical goodness-of-fit techniques since it gives immediate and direct information regarding the shape of the underlying distribution (*i.e.* skewness) and is an effective indicator of potential outliers. Also, most importantly, it can be used effectively for censored data. However, its sensitivity to random occurrences in the data sometimes leads to the wrong conclusion. We cannot solely rely on it, especially when the sample size is small.

The χ^2 goodness-of-fit test is a classical statistical procedure. It was developed by Karl Pearson. It uses the comparison between the observed cell count and the corresponding expected value under the hypothesized distribution. The test statistic asymptotically follows a χ^2 distribution with $c-k-1$ degrees of freedom. Here c is the number of cells and k is the number of estimated parameters for the distribution. The χ^2 goodness-of-fit test is applied to binned data (*i.e.* data put into classes). Therefore, the value of the χ^2 test statistic is sensitive to how the data is binned. Another disadvantage of the χ^2 test is that it requires a sufficient sample size for the

χ^2 approximation to be valid. In practice, it is usually required that, the expected frequency should be at least 5. Because of this, the test is not valid for small samples, and if some of the expected counts are less than five, one may need to combine some bins associated with the tails of the hypothesized distribution. These reductions discard some information, so that the χ^2 goodness of fit test is less powerful than the other goodness-of-fit techniques. However, this test can be used for both continuous and discrete data, as well for the univariate and multivariate data. It is the most widely used goodness-of-fit test.

The Kolmogorov-Smirnov, Cramer-von Mises and Anderson-Darling tests are based on the distance between the empirical distribution function and the hypothesized distribution function. Let us take the Kolmogorov-Smirnov test as an example. From the empirical distribution function, we have the ratio of the number of observations less than or equal to x_i to the total number of observations n and y_i are the corresponding ordered statistics. Hence the empirical distribution function for Kolmogorov-Smirnov test is a step function which will increase by $1/n$ unit at the order of each data point. Kolmogorov-Smirnov test statistic is defined as the maximum distance between the empirical distribution function and the hypothesized population distribution function. The null distribution of the Kolmogorov-Smirnov test statistic does not depend on the underlying cumulative distribution function being tested. There is no limitation for the sample size, and it is an exact test (*i.e.* not like χ^2 goodness of fit test, which requires a sufficient sample size for the χ^2 approximation to be valid).

Goodness-of-fit tests for distributional assumptions have been a major study area for statistical research, especially for testing normality. Applications of the normal distribution can be mainly classified in two categories. First they relate to the class of statistics which are taken to be normally distributed due to the applicability of large sample theorems such as the Central Limit Theorem (Rao, 1973). Secondly, when the normal distribution is assumed, it can be applied to the appropriate mathematical model for the underlying phenomenon under investigation. The tests for normality can be classified into 5 groups: χ^2 test, empirical distribution function tests, moment tests, regression tests, and miscellaneous tests. For example, as we mentioned before, the χ^2 test uses the comparison between the observed cell counts and the corresponding expected values under the null hypothesis. For the normal distribution, the χ^2 test statistic is calculated by grouping the hypothesized distribution (with known or estimated parameters) into a multinomial distribution of M cells, comparing the observed number of observations with the expected number of observations in each cell. The χ^2 test is of historical interest and is continuously being modified.

The most commonly used goodness-of-fit test of normality is the Shapiro-Wilk test (1965) (so called W-statistic). The test considers a regression of the ordered sample observations on the expected values of the order statistics of a random sample from a standardized version of the hypothesized distribution. The W-test statistic for normality is defined by dividing the square of an appropriate linear combination of the sample order statistics (using the method of generalized least squares) by the usual symmetric estimate of variance. In fact the test considers the ratio of two estimates of the population variance and hence is also known as an analysis of variance test. This ratio is invariant for both location and scale. The exact distribution of W statistic under the null hypothesis only depends on the sample size n, not on the location and scale parameters μ and σ. The exact distribution of the W-normal statistic is unknown, but Shapiro and Wilk (1965) provided the percentage points for the test using Monte Carlo simulation. Using extensive Monte Carlo studies Shapiro and Wilk suggested that the critical region of the test is the lower tail area of the null distribution of W, that is, larger values of W (*i.e.* values close to 1) indicate normality. This W-statistic is very simple to calculate when the table of appropriate linear coefficients is available. Even for small samples ($n < 20$), it is found that the test is quite sensitive (powerful) against a wide range of alternatives. Shapiro and Wilk (1965) provided the power results obtained from different goodness of fit tests. They concluded that the W-statistic is more powerful than other tests for skewed alternatives. Unfortunately, for large sample sizes, it may be difficult to determine the percentage points and may need a necessary value of the multiplier in the numerator for the test statistic.

From a general viewpoint, the procedure used to derive the W-statistic for normality can be applied to derive tests for other distributional assumptions, such as the exponential distribution. Using these principles, an analysis of variance test for exponentiality of a distribution based on a complete sample has been proposed by Shapiro and Wilk (1972). The W-exponential statistic is defined to be the ratio of the squared difference between the sample mean and the smallest observation to the usual symmetric sum of squares about the mean. Similar to the W-normal statistic, the W-exponential statistic is usable for testing various composite or simple hypotheses of exponentiality. The W-statistic for exponentiality leads to a two-tailed test. This is because the W-exponential statistic may take either low or high values depending on the properties of alternative distribution. Compared with other goodness-of-fit tests, the W-exponential statistic seems to be more powerful over a wide range of alternatives. A modified W-exponential statistic has been proposed by Samanta and Schwarz (1988). This modified statistic is applicable when the sample is censored. This will be discussed in detail in Chapter 2 of this thesis.

So far the test procedures available in the literature are all based on one single sample from one population. The proposed test statistic for testing the composite hypothesis of exponentiality of two distributions is developed in Chapter 3. We consider the null hypothesis that two independent random samples come from two exponential distributions with different unknown location parameters, but with the same unknown scale parameter, against the alternative hypothesis that the common form of these two distributions is not exponential. Using the principles used for the construction of the W-statistic for exponentiality, we propose a V-exponential statistic that turns out to be a normalized ratio of the square of the generalized least squares estimate of the common scale parameter based on the order statistics of independent random samples from the standard exponential distribution, to a pooled sum of squares about the sample means. This V-statistic is origin and scale invariant. The null distribution of the V-exponential statistic is shown to depend only on the sample sizes. We also prove some other important results relating to our proposed V-exponential statistic. Tables of empirical percentage points of the V-statistic are constructed for various combinations of sample sizes by using Monte Carlo simulation. We give some numerical examples to illustrate the applications of the proposed test. Chapter 4 provides the empirical power results for various types of probability distributions under the alternative hypothesis. Chapter 5 deals with a modified test statistic using the approach of Samanta and Schwarz (1988) when one or both samples are censored. In Chapter 6, we use Stephens' (1978) approach and propose a second test statistic called the V^*-exponential statistic, that can be used in the same context. The null distribution of V^*-exponential statistic is the same as the W-exponential statistic of Shapiro and Wilk (1972) corresponding to an appropriately modified sample size. Numerical example and power studies of the V^*-exponential statistic are also included. We also compare the power results of the one-sample W-exponential test, two-sample V-exponential and V^*-exponential tests. We see that the results are close to each other, that is, the three tests (W-exponential, V-exponential and V^*-exponential statistics) are comparable in terms of their sensitivity results. Concluding remarks are given in Chapter 7.

Chapter 2

One-sample W-statistics for Exponentiality

There has been an extensive literature on goodness-of-fit tests for exponentiality. Important references include Shapiro and Wilk (1972), Anderson and Darling (1952), Bartholomew (1957), Cox and Lewis (1966), Darling (1953), Epstein (1960), Jackson (1967), and Stephens (1978). Recently, the exponentiality testing is most used for the time-constructed problem, such as waiting time. In this chapter, we discuss the one-sample test procedures for exponentiality due to Shapiro and Wilk (1972). The principle underlying the test procedures and the properties of the test statistic are similar to the W-statistic for normality (Shapiro and Wilk, 1965, 1968).

We have the general exponential distribution which has the density function defined as follows:

$$f(x) = \begin{cases} \beta^{-1} exp\{-(x-\alpha)/\beta\}, & x \geq \alpha, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

where α is the location parameter $(-\infty < \alpha < \infty)$ and β is the scale parameter $(\beta > 0)$.

Note that a random variable X having density function f is such that

1. $\mu_x = E(X) = \alpha + \beta$,

2. $\sigma_x^2 = Var(X) = \beta^2$,

3. when $\alpha = 0$ and $\beta = 1$, X admits the standard exponential distribution,

$$f(x) = \begin{cases} e^{-x}, & x \geq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

2.1 The W-exponential statistic for uncensored data

Suppose we define $X_1 \leq X_2 \leq \ldots \leq X_n$ to be the order statistics of a sample of size n obtained from a standard exponential distribution. Let the expected value of X_i be m_i, that is,

$$E(X_i) = m_i, \quad i = 1, 2, \cdots, n.$$

We have the covariance between X_i and X_j as

$$Cov(X_i, X_j) = v_{ij} = E[(X_i - m_i)(X_j - m_j)], \quad i, j = 1, 2, \cdots, n,$$

and we write the expected values and covariances into vector and matrix form as follows

$$m' = (m_1, m_2, \cdots, m_n), \qquad \text{where } m' \text{ is the transpose of the vector } m,$$

$$V = (v_{ij}), \qquad \text{where } V \text{ is a } n \times n \text{ matrix.}$$

Further suppose $Y_1 \leq Y_2 \leq \ldots \leq Y_n$ are the corresponding ordered observations from an exponential distribution with location parameter α and scale parameter β. Then we can write Y_i in terms of X_i as

$$Y_i = \alpha + \beta X_i, \quad i = 1, 2, \cdots, n,$$

where X_i is from the standard exponential distribution with density function as given in (2.2). We also write Y in vector form for y as

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}.$$

Now, we apply the generalized least squares theory (Aitken, 1935, Lloyd, 1952) to find the least square estimators of α and β. We have

$$\begin{pmatrix} \widehat{\alpha} \\ \widehat{\beta} \end{pmatrix} = \{(1|m)'V^{-1}(1|m)\}^{-1}(1|m)'V^{-1}Y,$$

since

$$\begin{aligned} \{(1|m)'V^{-1}(1|m)\}^{-1} &= \begin{pmatrix} 1'V^{-1}1 & 1'V^{-1}m \\ 1'V^{-1}m & m'V^{-1}m \end{pmatrix}^{-1} \\ &= \frac{1}{det}\begin{pmatrix} m'V^{-1}m & -1'V^{-1}m \\ -1'V^{-1}m & 1'V^{-1}1 \end{pmatrix}, \end{aligned}$$

where

$$det = \{(1'V^{-1}1)(m'V^{-1}m) - (1'V^{-1}m)^2\},$$

$$1' = (1, 1, \cdots, 1).$$

Hence, we have

$$\widehat{\beta} = \frac{1'V^{-1}(1m' - m1')V^{-1}Y}{\{(1'V^{-1}1)(m'V^{-1}m) - (1'V^{-1}m)^2\}}. \tag{2.3}$$

From known properties of the exponential distribution (Kendall and Stuart, 1961), we have the following properties for m_i and $Cov(X_i, X_j)$

$$
\begin{aligned}
m_i &= \sum_{k=1}^{i}(n-k+1)^{-1}, \quad i = 1, 2, \cdots, n, \\
Cov(X_i, X_j) &= \sum_{k=1}^{i}(n-k+1)^{-2}, \quad i \le j, \\
Cov(X_i, X_j) &= \sum_{k=1}^{j}(n-k+1)^{-2}, \quad i > j.
\end{aligned}
$$

After some algebraic work, we have some important Lemmas (Shapiro and Wilk, 1972) as following:

1. $m_j = \sum_{i=1}^{n} v_{ij}$, that is, $m' = 1'V$,
2. $\sum_{i=1}^{n} m_i = n$, that is, $1'm = n$,
3. $1'V^{-1} = (n^2, 0, 0, \cdots, 0)$,

and Corollaries:

1. $1' = m'V^{-1}$,
2. $m'V^{-1}m = n$,
3. $1'V^{-1}m = n$,
4. $1'V^{-1}1 = n^2$,
5. $1'V^{-1}Y = n^2Y_1$,

6. $\widehat{\beta} = n(\overline{Y} - Y_1)/(n-1)$, where $\overline{Y} = \sum_{i=1}^{n} Y_i/n$.

We obtain the W-exponential statistic by standardizing the squared $\widehat{\beta}$ with S^2, so the W-statistic for testing the composite hypothesis for exponentiality is

$$W_E = \frac{n(\overline{Y} - Y_1)^2}{(n-1)S^2} \tag{2.4}$$

for the null hypothesis:

$$H_{10} : F(y) = 1 - exp\{-(y - \alpha)/\beta\}, \quad y \geq \alpha$$

where

$$S^2 = \sum_{i=1}^{n} (Y_i - \overline{Y})^2.$$

Note that

- W is used as a two-tailed statistic. It is invariant for both origin and scale parameters and hence, can be used for testing the composite hypothesis of exponentiality.
- From the mathematical point of view, the W-exponential statistic is bounded, with a maximum value of 1 and minimum value of $(n-1)^{-2}$ (see Shapiro, Wilk and Chen, 1968).
- From the statistical point of view, we know that Y_1 is a sufficient statistic for the origin parameter α and $\overline{Y}$ is the sufficient for the scale parameter β

(Lehmann, 1959). As we mentioned before, the W-statistic is invariant for both origin and scale. As a result, under the null hypothesis, we have that the W-exponential statistic is independent of Y_1 and $\overline{Y}$ (Basu, 1955). It depends only on the sample size n (Hogg and Craig, 1956).

2.2 The W-exponential statistic for censored data

As previously mentioned the W-exponential statistic is given by

$$W_E = \frac{n(\overline{Y} - Y_1)^2}{(n-1)S^2}$$

for the complete sample of size n (*i.e.* uncensored sample).

Now consider the case of a censored sample. That is, in a random sample of size n, the r_1 smallest and r_2 largest observations are censored. Then there will be $n - r_1 - r_2$ observations available. Samanta and Schwarz (1988) modified the W-exponential statistic according to two different situations, (*i.e.* origin unknown and known).

2.2.1 Origin unknown

We denote the normalized waiting times as

$$T_i = (n - i + 1)(Y_i - Y_{i-1}), \quad i = 2, 3, \cdots, n. \tag{2.5}$$

We know that when H_{10} is true, T_2, T_3, $\cdots$, T_n, are independent and identically distributed random variables from the exponential distribution function

$$F(t) = \begin{cases} 1 - e^{-t/\beta}, & t \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

Now, let

$$U_i = \frac{iY_{i+1} - (Y_1 + Y_2 + \cdots + Y_i)}{\sqrt{i(i+1)}}, \quad i = 1, 2, \cdots, n-1,$$
$$U_n = \frac{Y_1 + Y_2 + \cdots + Y_n}{\sqrt{n}} = \sqrt{n}\bar{Y},$$

which actually define an orthogonal transformation. If we write U_i in terms of T_i, we get

$$U_i = \frac{\sum_{j=1}^{i} \frac{jT_{j+1}}{n-j}}{\sqrt{i(i+1)}}, \quad i = 1, 2, \cdots, n-1.$$

Let

$$a_{ij}^{(n)} = \frac{(j-1)}{(n-j+1)}, \quad (n \geq i \geq j \geq 2)$$

with

$$a_{ij}^{(n)} = a_{ji}^{(n)} \quad \text{for } i, j = 2, 3, \cdots, n.$$

Using the newly defined variables, the original W-exponential statistic from the uncensored data (we denote $W_E(n)$ to indicate it is from a complete sample of size

n) can be written as:

$$W_E(n) = \frac{\left(\sum_{i=2}^{n} T_i\right)^2}{n(n-1)\sum_{i=1}^{n-1} U_i^2}$$

$$= \frac{\left(\sum_{i=2}^{n} T_i\right)^2}{(n-1)\sum_{i=2}^{n}\sum_{j=2}^{n} a_{ij}^{(n)} T_i T_j}$$

Now consider the censored situation as we mentioned before, that is when r_1 smallest and r_2 largest observations are censored. In other words we have an effective sample of size $n - r_1 - r_2$. That is $Y_{r_1+1} \leq Y_{r_1+2} \leq \ldots \leq Y_{n-r_2}$ available observations for testing the null hypothesis H_{10} with the origin unknown. Then, the modified test statistic (denoted as W_1) is given by

$$W_1 = \frac{\left(\sum_{i=2}^{n-r_1-r_2} T_{r_1+i}\right)^2}{(n-r_1-r_2-1)\sum_{i=2}^{n-r_1-r_2}\sum_{j=2}^{n-r_1-r_2} a_{ij}^{(n-r_1-r_2)} T_{r_1+i} T_{r_1+j}} \tag{2.7}$$

Note that

1. if there are no censored data, W_1 is equal to the original $W_E(n)$.

2. the distribution of W_1 is the same as that of $W_E(n - r_1 - r_2)$ under the null hypothesis.

2.2.2 Origin known

Let us consider the situation where the origin is known. Stephens (1978) extended the Shapiro and Wilk exponentiality test for testing the null hypothesis

$$H_{20} : F(y) = \begin{cases} 1 - exp\{-(y-\alpha_0)/\beta\}, & y \geq \alpha_0, \quad \text{with } \alpha_0 \text{ known}, \\ 0, & \text{otherwise.} \end{cases}$$

The extended W-exponential statistic is $W_E^*(n)$

$$W_E^*(n) = \frac{\left\{\sum_{i=1}^{n}(Y_i - \alpha_0)\right\}^2}{n\left\{(n+1)\sum_{i=1}^{n}(Y_i-\alpha_0)^2 - \left[\sum_{i=1}^{n}(Y_i-\alpha_0)\right]^2\right\}} \tag{2.8}$$

Let

$$Z_i = Y_i - \alpha_0, \quad i = 1, 2, \cdots, n,$$

and the normalized waiting times T_i for $i = 2, 3, \cdots, n, as$

$$T_i = (n-i+1)(Y_i - Y_{i-1}) = (n-i+1)(Z_i - Z_{i-1})$$

and define T_1 as

$$T_1 = nZ_1 = n(Y_1 - \alpha_0).$$

We also know, when H_{20} is true, T_1, T_2, $\cdots$, T_n are *iid* random variables following the exponential distribution function

$$F(t) = \begin{cases} 1 - e^{-t/\beta}, & t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

If we write $W_E^*(n)$ in terms of T_i and a_{ij}, we get

$$W_E^*(n) = \frac{\left(\sum_{i=1}^{n} T_i\right)^2}{(n+1)\sum_{i=2}^{n}\sum_{j=2}^{n} a_{ij}^{(n)} T_i T_j + \left(\sum_{i=1}^{n} T_i\right)^2}$$
$$= \frac{\left(\sum_{i=2}^{n+1} T_{i-1}\right)^2}{n \sum_{i=2}^{n+1}\sum_{j=2}^{n+1} a_{ij}^{(n+1)} T_{i-1} T_{j-1}}.$$

We can use this modified W-exponential statistic for the case where the origin α_0 is known in the context of censored data. Suppose that, in the sample of size n; the r_2 largest observations are censored. Then, we have an effective sample of $n - r_2$ available observations, that is $Y_1 \leq Y_2 \leq \ldots \leq Y_{n-r_2}$ for testing the null hypothesis H_{20} with the origin known. The modified test statistic (denotes as W_2) can be written as:

$$W_2 = \frac{\left(\sum_{i=2}^{n-r_2+1} T_{i-1}\right)^2}{(n - r_2) \sum_{i=2}^{n-r_2+1} \sum_{j=2}^{n-r_2+1} a_{ij}^{(n-r_2+1)} T_{i-1} T_{j-1}}. \tag{2.9}$$

Note that

1. if there are no censored data, W_2 is equal to the original $W_E^*(n)$.

2. following Stephens (1978), the distribution of W_2 is the same as that of $W_E^*(n - r_2 + 1)$ under the null hypothesis.

Chapter 3

The Two-sample V-statistic for Exponentiality for Complete Samples

Based on the principles used by Shapiro and Wilk(1972) to derive W-statistic for testing exponentiality and by Samanta and Schwarz(1988) to derive the modified W-exponential statistic for censored data, we develop the V-exponential statistic for testing exponentiality of two distributions.

3.1 V-exponential statistic for complete samples

Suppose for $i = 1, 2$, $Y_{i1} \leq Y_{i2} \leq \ldots \leq Y_{in_i}$ are the order statistics of a random sample from a population with distribution function $F_i(y)$. In this chapter we wish to test the null hypothesis:

$$H_{30} : \left\{ \begin{array}{lll} F_i(y) = & F\left(\frac{y-\alpha_i}{\beta}\right), & -\infty < \alpha_i < \infty, i = 1, 2, \quad 0 < \beta < \infty, \\ \text{with} & F(y) = & \left[\begin{array}{ll} 1 - e^{-y}, & \text{if } 0 < y < \infty, \\ 0, & \text{otherwise} \end{array} \right. \end{array} \right\} \tag{3.1}$$

against the alternative hypothesis

$$H_{3a}: \left\{ \begin{array}{lll} F_i(y) = & F\left(\frac{y-\alpha_i}{\beta}\right) & \\ \text{with} & F(y) \neq 1 - e^{-y}, & \text{for some } y \end{array} \right\}$$

It is known that if the null hypothesis H_{30} is true, then a test of the different null hypothesis

$$H_{40} : \alpha_1 = \alpha_2$$

can be done by using an F-test.

Let us write

$$Y_{ij} = \alpha_i + \beta X_{ij}, \quad \text{with } i = 1, 2 \quad j = 1, 2, \cdots, n_i,$$

where for each $i = 1, 2, \quad X_{i1} \leq X_{i2} \leq \ldots \leq X_{in_i}$ are the order statistics of a random sample of size n_i from a standard exponential distribution $F(x)$ as defined above (or in 2.2)

Note that x_{ij} and x_{lk} are independent when $i \neq l$.

Under H_{30}, it follows that for each $i = 1, 2, \quad Y_{i1} \leq Y_{i2} \leq \ldots \leq Y_{in_i}$ are the corresponding order statistics of a random sample of size n_i obtained from an exponential distribution with location parameter α_i and scale parameter β.

We define the two vectors Y_1 and Y_2 as

$$Y_1 = \begin{pmatrix} Y_{11} \\ Y_{12} \\ \vdots \\ Y_{1n_1} \end{pmatrix}, \quad Y_2 = \begin{pmatrix} Y_{21} \\ Y_{22} \\ \vdots \\ Y_{2n_2} \end{pmatrix}$$

and we write the general Y vector as

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}.$$

As before we denote the expected value of X_{ij} as m_{ij}, that is,

$$E(X_{ij}) = m_{ij}, \quad i = 1, 2, \quad j = 1, 2, \cdots, n_i,$$

and the covariance between X_{ij} and X_{lk} as

$$Cov(X_{ij}, X_{lk}) = \begin{cases} 0, & \text{for } i \neq l \\ w_{1jk}, & \text{for } i = l = 1, j, k = 1, 2, \cdots, n_1, \\ w_{2jk}, & \text{for } i = l = 2, j, k = 1, 2, \cdots, n_2, \end{cases}$$

So for each sample we have the covariance matrix

$$Cov_1 = (w_{1jk})_{n_1 \times n_1}$$

$$Cov_2 = (w_{2jk})_{n_2 \times n_2}.$$

For convenience, we write the covariance matrix of Y as

$$Q = \begin{pmatrix} Cov_1 & 0 \\ 0 & Cov_2 \end{pmatrix}.$$

If we take the expectation of both sides of

$$Y_{ij} = \alpha_i + \beta X_{ij},$$

we will have, in matrix form,

$$E(Y) = p\theta,$$

where

$$p = \begin{pmatrix} 1 & 0 & m_{11} \\ 1 & 0 & m_{12} \\ \vdots & \vdots & \vdots \\ 1 & 0 & m_{1n_1} \\ 0 & 1 & m_{21} \\ 0 & 1 & m_{22} \\ \vdots & \vdots & \vdots \\ 0 & 1 & m_{2n_2} \end{pmatrix},$$

and

$$\theta = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta \end{pmatrix}.$$

Now we apply the generalized least squares theory (Aitken, 1935, Lloyd, 1952) to obtain the least squares estimators for α_1, α_2 and β. We have

$$\widehat{\theta} = \begin{pmatrix} \widehat{\alpha}_1 \\ \widehat{\alpha}_2 \\ \widehat{\beta} \end{pmatrix} = (p'Q^{-1}p)^{-1}p'Q^{-1}Y$$

where p' is the transpose of p matrix.

Using Lemmas 1 to 3 and Corollaries 1 to 5, it can be shown that

$$p'Q^{-1}p = \begin{pmatrix} n_1^2 & 0 & n_1 \\ 0 & n_2^2 & n_2 \\ n_1 & n_2 & n_1+n_2 \end{pmatrix},$$

and

$$p'Q^{-1}Y = \begin{pmatrix} n_1^2 Y_{11} \\ n_2^2 Y_{21} \\ \sum_{i=1}^{2}\sum_{j=1}^{n_i} Y_{ij} \end{pmatrix}.$$

Hence, we have the estimate $\widehat{\beta}$ for β, which is the product of the last row of $(p'Q^{-1}p)^{-1}$ with the column vector $p'Q^{-1}Y$, *i.e.*

$$\widehat{\beta} = \frac{n_1(\overline{Y}_1 - Y_{11}) + n_2(\overline{Y}_2 - Y_{21})}{(n_1+n_2-2)}$$

where

$$\overline{Y}_i = \sum_{j=1}^{n_i} Y_{ij}/n_i, \quad i = 1, 2.$$

Note that $\widehat{\beta}$ is the minimum variance unbiased estimator (*i.e.* MVUE) for β (Epstein and Sobel, 1954)

We obtain the V-exponential statistic as a normalized ratio of the squared $\widehat{\beta}$ to S^2. Hence, the resulting statistic for testing the composite hypothesis of exponentiality, given in (3.1) is

$$V(n_1, n_2) = \frac{\{n_1(\overline{Y}_1 - Y_{11}) + n_2(\overline{Y}_2 - Y_{21})\}^2}{2n^*(n^* - 1)S^2} \tag{3.2}$$

where

$$S_i^2 = \sum_{j=1}^{n_i}(Y_{ij} - \overline{Y}_i)^2, \quad i = 1, 2,$$

$$S^2 = S_1^2 + S_2^2,$$

$$n^* = max(n_1, n_2).$$

Note that, as an omnibus procedure, $V(n_1, n_2)$ is to be used as a two-tailed statistic.

3.2 Properties of the V-exponential statistic

Based on the above two-sample V-statistic, we obtain the following important results.

THEOREM 1 .

$$P\{1/2(n^* - 1)^2 < V(n_1, n_2) \leq 1\} = 1$$

Proof.

According to Lemmas 5 and 6 in Shapiro and Wilk (1972), we note that for any two sample sequences

$$\{n_1(\overline{Y}_1 - Y_{11}) + n_2(\overline{Y}_2 - Y_{21})\}^2 \leq 2n^*(n^* - 1)S^2,$$

and

$$\begin{aligned}\{n_1(\overline{Y}_1 - Y_{11}) + n_2(\overline{Y}_2 - Y_{21})\}^2 &\geq \left\{\frac{n_1}{(n_1 - 1)^{1/2}} S_1 + \frac{n_2}{(n_2 - 1)^{1/2}} S_2\right\}^2 \\ &> \frac{n^* S^2}{n^* - 1}.\end{aligned}$$

Therefore, we can write

$$\begin{aligned}\frac{\{n_1(\overline{Y}_1 - Y_{11}) + n_2(\overline{Y}_2 - Y_{21})\}^2}{2n^*(n^* - 1)S^2} &> \frac{n^* S^2}{n^* - 1} \cdot \frac{1}{2n^*(n^* - 1)S^2}. \\ &> \frac{1}{2(n^* - 1)^2}\end{aligned}$$

As Shapiro and Wilk (1965) proved, the W-exponential statistic is bounded with a maximum value of 1 and minimum value of $(n-1)^{-2}$. Hence, the V-exponential statistic has the following similar property

$$1 \geq \frac{\{n_1(\overline{Y}_1 - Y_{11}) + n_2(\overline{Y}_2 - Y_{21})\}^2}{2n^*(n^* - 1)S^2} > \frac{1}{2(n^* - 1)^2}.$$

THEOREM 2 .

The null distribution of $V(n_1, n_2)$ and $V(n_2, n_1)$ are identical and depend only on n_1 and n_2, but not on α_1, α_2 and β.

Proof.

Following Samanta and Schwarz (1988) we define T_{ij} as

$$T_{ij} = (n_i - j + 1)(Y_{ij} - Y_{i,j-1}), \quad j = 2, 3, \cdots, n_i, \quad i = 1, 2. \tag{3.3}$$

Under the null hypothesis H_{30}, $T_{ij}/\beta, \quad j = 2, 3, \cdots, n_i, \quad i = 1, 2.$, are independent and identically distributed random variables with the standard exponential distribution.

Let

$$\begin{aligned}
a_{ij}^{(n_1)} &= \frac{(j-1)}{(n_1 - j + 1)}, \quad (n_1 \geq i \geq j \geq 2) \\
a_{ij}^{(n_1)} &= a_{ji}^{(n_1)} \quad \text{for } i, j = 2, 3, \cdots, n_1, \\
b_{ij}^{(n_2)} &= \frac{(j-1)}{(n_2 - j + 1)}, \quad (n_2 \geq i \geq j \geq 2) \\
b_{ij}^{(n_2)} &= b_{ji}^{(n_2)} \quad \text{for } i, j = 2, 3, \cdots, n_2,
\end{aligned}$$

Then we can write the V-exponential statistic $V(n_1, n_2)$ in terms of the T_{ij}'s as

$$V(n_1, n_2) = \frac{\left\{ \sum_{i=2}^{n_1} T_{1i} + \sum_{i=2}^{n_2} T_{2i} \right\}^2}{2n^*(n^* - 1) \left\{ \left[\sum_{i=2}^{n_1} \sum_{j=2}^{n_1} a_{ij}^{(n_1)} T_{1i} T_{1j} \right] \Big/ n_1 + \left[\sum_{i=2}^{n_2} \sum_{j=2}^{n_2} b_{ij}^{(n_2)} T_{2i} T_{2j} \right] \Big/ n_2 \right\}}$$

So we can see from the above representation that the distribution of $V(n_1, n_2)$ and $V(n_2, n_1)$ are identical and depend only on the sample sizes n_1 and n_2 but not on the parameters α_1, α_2 and β. As a result, for the exponentially distributed samples, $(Y_{11}, Y_{21}, n_1(\overline{Y}_1 - Y_{11}) + n_2(\overline{Y}_2 - Y_{21}))$ is a complete sufficient statistic for $(\alpha_1, \alpha_2, \beta)$ (Basu, 1955). Hence the statistic $V(n_1, n_2)$ is statistically independent of $(Y_{11}, Y_{21}, n_1(\overline{Y}_1 - Y_{11}) + n_2(\overline{Y}_2 - Y_{21}))$ under null hypothesis.

THEOREM 3 .

Under the null hypothesis H_{30}, the distribution function of $V(2,2)$ (*i.e.* when $n_1 = 2, n_2 = 2$) is given by

$$H(v) = 1 - (v^{-1} - 1)^{1/2}, \quad \frac{1}{2} \leq v \leq 1. \tag{3.4}$$

Proof.

We have

$$V(n_1, n_2) = \frac{\left\{\sum_{i=2}^{n_1} T_{1i} + \sum_{i=2}^{n_2} T_{2i}\right\}^2}{2n^*(n^*-1)\left\{\left[\sum_{i=2}^{n_1}\sum_{j=2}^{n_1} a_{ij}^{(n_1)} T_{1i}T_{1j}\right] \Big/ n_1 + \left[\sum_{i=2}^{n_2}\sum_{j=2}^{n_2} b_{ij}^{(n_2)} T_{2i}T_{2j}\right] \Big/ n_2\right\}}$$

When $n_1 = 2$ and $n_2 = 2$,

$$V(2,2) = \frac{(S_1 + S_2)^2}{2(S_1^2 + S_2^2)},$$

where

$$S_1 = \frac{T_{12}}{\beta},$$

$$S_2 = \frac{T_{22}}{\beta}.$$

Note that S_1 and S_2 are independent and identically distributed random variables with the standard exponential distribution.

Now, suppose it is given that

$$S_1 + S_2 = s,$$

in which case the conditional distribution of S_1 is uniform. Then, for u satisfying

$$\frac{s^2}{2} \leq u \leq s^2,$$

we have

$$P(S_1^2 + S_2^2 \leq u | S_1 + S_2 = s) = \frac{\{2(u - s^2/2)\}^{1/2}}{s}.$$

Now, let

$$r = \frac{u}{s^2},$$

and note that for any r satisfying $\frac{1}{2} \leq r \leq 1$, we have

$$\begin{aligned} P\left(\frac{S_1^2 + S_2^2}{(S_1 + S_2)^2} \leq r | S_1 + S_2 = s\right) &= P(S_1^2 + S_2^2 \leq rs^2 | S_1 + S_2 = s) \\ &= (2r - 1)^{1/2}, \end{aligned}$$

so that this probability does not depend on s.

We concluded that for any v satisfying $\frac{1}{2} \leq v \leq 1$, the distribution function $H(v)$ of

$V(2,2)$ is given by

$$\begin{aligned} H(v) &= 1 - P(V(2,2) \geq v) \\ &= 1 - P\left(\frac{S_1^2 + S_2^2}{(S_1 + S_2)^2} \leq \frac{1}{2v}\right) \\ &= 1 - (v^{-1} - 1)^{1/2}. \end{aligned}$$

3.3 Percentage Points of $V(n_1, n_2)$

The distribution of $V(n_1, n_2)$ under the null hypothesis was studied by Monte Carlo simulation. We obtained the empirical cumulative distribution of $V(n_1, n_2)$ from 100000 random samples with sizes (n_1, n_2) (*i.e.* $n_1 = 2(1)25, n_2 = 2(1)25, n_1 \leq n_2$). Then we calculated the empirical percentage points of $V(n_1, n_2)$ from this empirical distribution. The random samples were simulated using the Package R. The 0.5, 1, 2.5, 5, 95, 97.5, 99, 99.5 empirical percentage points of $V(n_1, n_2)$ are given in Table 3.1.

Table 3.1: Percentage Points of V-Exponential

n_1	n_2	0.005	0.01	0.025	0.05	0.95	0.975	0.99	0.995
2	2	0.5023	0.5047	0.5123	0.525	0.9975	0.9993	0.9998	0.99996
2	3	0.1418	0.15	0.1666	0.1865	0.6205	0.6433	0.657	0.6617
2	4	0.0752	0.0815	0.0949	0.1101	0.4575	0.499	0.5336	0.5511
2	5	0.0513	0.0569	0.0672	0.0779	0.3399	0.3871	0.4378	0.4677
2	6	0.038	0.0427	0.0515	0.0605	0.2663	0.3049	0.3509	0.3826
2	7	0.0313	0.0354	0.0428	0.05	0.2138	0.2458	0.2873	0.3214
2	8	0.0268	0.0302	0.0365	0.0427	0.1787	0.2035	0.2376	0.2661
2	9	0.0231	0.0263	0.0314	0.0367	0.1514	0.1727	0.2003	0.2234
2	10	0.0206	0.0233	0.0282	0.0331	0.1315	0.1488	0.1726	0.1902
2	11	0.0188	0.0212	0.0256	0.0297	0.1141	0.1287	0.1487	0.1633
2	12	0.0173	0.0195	0.0233	0.0269	0.1021	0.1156	0.1334	0.1469
2	13	0.0161	0.0182	0.0216	0.0249	0.0912	0.1028	0.1184	0.1304
2	14	0.0151	0.0168	0.0201	0.0232	0.0823	0.0927	0.1067	0.1167
2	15	0.0141	0.0158	0.0186	0.0215	0.0761	0.0849	0.098	0.1067
2	16	0.0132	0.0148	0.0176	0.0202	0.069	0.077	0.0876	0.0965
2	17	0.0124	0.014	0.0165	0.0189	0.0634	0.071	0.081	0.0885
2	18	0.0121	0.0136	0.0159	0.0181	0.0589	0.0651	0.074	0.0804
2	19	0.0113	0.0126	0.0149	0.017	0.0549	0.0612	0.0692	0.075
2	20	0.0108	0.0121	0.0142	0.0162	0.0513	0.0567	0.0638	0.0699
2	21	0.0104	0.0117	0.0137	0.0155	0.048	0.0532	0.0601	0.0653
2	22	0.0101	0.0112	0.0131	0.0148	0.0452	0.0497	0.0562	0.0609
2	23	0.0097	0.0108	0.0125	0.0142	0.0426	0.0471	0.053	0.0573
2	24	0.0093	0.0104	0.012	0.0136	0.0403	0.0445	0.0499	0.0537
2	25	0.0088	0.0099	0.0115	0.0131	0.0383	0.0421	0.047	0.0511

n_1	n_2	0.005	0.01	0.025	0.05	0.95	0.975	0.99	0.995
3	3	0.1623	0.1739	0.1962	0.2221	0.8278	0.8857	0.9342	0.9562
3	4	0.088	0.0969	0.1136	0.1307	0.531	0.5796	0.6329	0.6643
3	5	0.0602	0.0676	0.0787	0.0917	0.3803	0.4271	0.481	0.5156
3	6	0.0456	0.0508	0.0599	0.0698	0.2896	0.3254	0.3735	0.4067
3	7	0.0366	0.0414	0.0493	0.0571	0.2309	0.2609	0.2991	0.3285
3	8	0.0308	0.035	0.0418	0.0483	0.1909	0.2154	0.2506	0.2761
3	9	0.0266	0.0297	0.0355	0.0412	0.1606	0.1818	0.2087	0.2312
3	10	0.0239	0.0266	0.0316	0.0365	0.1383	0.1558	0.1794	0.1982
3	11	0.0206	0.0236	0.0282	0.0325	0.1204	0.1355	0.1553	0.17
3	12	0.0196	0.0219	0.0257	0.0297	0.108	0.1213	0.1395	0.1515
3	13	0.0173	0.0196	0.0233	0.0269	0.0952	0.1068	0.1216	0.1339
3	14	0.0165	0.0185	0.022	0.025	0.0863	0.0964	0.1102	0.1206
3	15	0.0153	0.0173	0.0203	0.0232	0.0787	0.0874	0.0989	0.1079
3	16	0.0141	0.016	0.0188	0.0216	0.0714	0.0793	0.0901	0.0983
3	17	0.0135	0.0151	0.0178	0.0203	0.066	0.0735	0.0831	0.0901
3	18	0.0127	0.0142	0.0167	0.0191	0.0611	0.0678	0.0769	0.0836
3	19	0.012	0.0135	0.016	0.0181	0.0568	0.0627	0.0708	0.0765
3	20	0.0117	0.0131	0.0152	0.0172	0.0529	0.0586	0.0662	0.0717
3	21	0.011	0.0123	0.0144	0.0164	0.0499	0.0549	0.0617	0.0675
3	22	0.0105	0.0118	0.0138	0.0156	0.0465	0.0512	0.0576	0.0626
3	23	0.0102	0.0114	0.0133	0.015	0.0437	0.0481	0.0536	0.0575
3	24	0.0098	0.0109	0.0127	0.0143	0.0415	0.0456	0.0509	0.0547
3	25	0.0094	0.0104	0.0121	0.0137	0.0392	0.0429	0.0481	0.052

n_1	n_2	0.005	0.01	0.025	0.05	0.95	0.975	0.99	0.995
4	4	0.1019	0.1126	0.1326	0.1528	0.6233	0.6862	0.7645	0.8136
4	5	0.0685	0.0759	0.0904	0.105	0.4322	0.4807	0.5421	0.5801
4	6	0.0519	0.0583	0.0692	0.0798	0.3216	0.3608	0.4111	0.4467
4	7	0.0416	0.0468	0.055	0.0638	0.251	0.2823	0.3236	0.3575
4	8	0.0337	0.0384	0.0457	0.0529	0.2056	0.2305	0.2634	0.288
4	9	0.0293	0.0332	0.0396	0.0455	0.1733	0.1952	0.2245	0.2441
4	10	0.0259	0.0291	0.0345	0.0398	0.1479	0.1666	0.1907	0.2077
4	11	0.0229	0.0258	0.0308	0.0355	0.1278	0.1437	0.1653	0.1798
4	12	0.0211	0.0236	0.0279	0.0321	0.1129	0.1261	0.1437	0.1559
4	13	0.0192	0.0215	0.0253	0.029	0.0997	0.1113	0.126	0.1363
4	14	0.018	0.02	0.0235	0.0269	0.0905	0.1008	0.1143	0.1256
4	15	0.0163	0.0183	0.0215	0.0247	0.082	0.0911	0.1028	0.1121
4	16	0.015	0.0169	0.0199	0.0228	0.0748	0.0828	0.0934	0.1031
4	17	0.0145	0.0163	0.0192	0.0217	0.0688	0.0761	0.0854	0.0932
4	18	0.0135	0.0152	0.0177	0.0203	0.0631	0.0697	0.0789	0.085
4	19	0.0131	0.0145	0.0169	0.0191	0.0585	0.0646	0.0721	0.0778
4	20	0.0124	0.0137	0.016	0.0181	0.0548	0.0604	0.0677	0.0729
4	21	0.0117	0.013	0.0152	0.0173	0.0509	0.0562	0.0628	0.0677
4	22	0.0112	0.0125	0.0145	0.0164	0.0478	0.0524	0.0581	0.0626
4	23	0.0107	0.012	0.014	0.0157	0.0453	0.0496	0.0551	0.0594
4	24	0.0103	0.0114	0.0132	0.0149	0.0426	0.0467	0.0519	0.0559
4	25	0.01	0.011	0.0127	0.0143	0.0401	0.0439	0.0487	0.0524
5	5	0.079	0.0873	0.103	0.1197	0.4833	0.5399	0.6093	0.6572

n_1	n_2	0.005	0.01	0.025	0.05	0.95	0.975	0.99	0.995
5	6	0.0573	0.0647	0.0768	0.0894	0.3547	0.3974	0.4496	0.4904
5	7	0.0464	0.0517	0.0616	0.0713	0.2739	0.3066	0.3524	0.3858
5	8	0.0387	0.043	0.0508	0.0584	0.2213	0.248	0.2813	0.3072
5	9	0.0327	0.0368	0.0434	0.0497	0.1833	0.2039	0.2312	0.2504
5	10	0.0282	0.0319	0.0374	0.043	0.155	0.1738	0.1967	0.2132
5	11	0.0248	0.028	0.0332	0.0383	0.1343	0.1495	0.1699	0.1876
5	12	0.0227	0.0254	0.03	0.0346	0.1186	0.1325	0.1494	0.1643
5	13	0.0206	0.0232	0.0275	0.0313	0.1048	0.1165	0.1327	0.1446
5	14	0.0185	0.0209	0.0247	0.0284	0.0943	0.1044	0.1179	0.1278
5	15	0.0177	0.0198	0.0231	0.0263	0.0848	0.094	0.1061	0.1156
5	16	0.0163	0.0184	0.0214	0.0244	0.0772	0.0855	0.0954	0.1032
5	17	0.0152	0.0169	0.0199	0.0227	0.071	0.0787	0.089	0.096
5	18	0.0145	0.0161	0.0189	0.0214	0.0654	0.0721	0.081	0.0873
5	19	0.0135	0.015	0.0177	0.0202	0.0607	0.0671	0.0747	0.0811
5	20	0.013	0.0145	0.0167	0.0189	0.0569	0.0628	0.0698	0.0756
5	21	0.0125	0.0138	0.016	0.0181	0.0528	0.0579	0.0645	0.0693
5	22	0.0118	0.0131	0.0153	0.0171	0.0495	0.0547	0.0611	0.0664
5	23	0.0114	0.0125	0.0145	0.0163	0.0465	0.051	0.0569	0.0612
5	24	0.0107	0.012	0.0139	0.0157	0.0438	0.0478	0.053	0.0566
5	25	0.0106	0.0116	0.0134	0.015	0.0414	0.0453	0.0503	0.0539
6	6	0.0642	0.0729	0.0866	0.0996	0.3847	0.4329	0.4909	0.5323
6	7	0.0519	0.0581	0.0679	0.0783	0.2939	0.3314	0.3748	0.4083
6	8	0.0424	0.0467	0.0557	0.0643	0.2359	0.2642	0.2993	0.3249

n_1	n_2	0.005	0.01	0.025	0.05	0.95	0.975	0.99	0.995
6	9	0.0351	0.0396	0.0469	0.0541	0.1946	0.2183	0.248	0.2708
6	10	0.0309	0.0346	0.0407	0.047	0.1641	0.1831	0.2079	0.226
6	11	0.0273	0.0305	0.0358	0.0413	0.1422	0.1585	0.1806	0.1951
6	12	0.0245	0.0275	0.0322	0.0368	0.124	0.1377	0.155	0.1676
6	13	0.0217	0.0249	0.0292	0.0333	0.1093	0.1218	0.1385	0.1505
6	14	0.0205	0.0227	0.0266	0.0303	0.0979	0.1091	0.1231	0.1328
6	15	0.0186	0.021	0.0247	0.0281	0.0887	0.0979	0.111	0.1196
6	16	0.0178	0.0196	0.0228	0.0259	0.0804	0.0889	0.1001	0.1087
6	17	0.0161	0.0179	0.021	0.0239	0.0733	0.080	0.0908	0.098
6	18	0.0154	0.0171	0.0199	0.0225	0.0676	0.0741	0.0829	0.089
6	19	0.0145	0.0162	0.0187	0.0211	0.0625	0.0688	0.0766	0.083
6	20	0.01387	0.0152	0.0176	0.02	0.0581	0.064	0.0714	0.0767
6	21	0.0129	0.0143	0.0166	0.0189	0.0542	0.0595	0.0664	0.0713
6	22	0.0124	0.0137	0.0158	0.0179	0.0509	0.0559	0.0624	0.0671
6	23	0.012	0.0133	0.0153	0.0171	0.048	0.0527	0.0581	0.0625
6	24	0.0114	0.0126	0.0145	0.0163	0.0449	0.049	0.0543	0.0582
6	25	0.011	0.0121	0.0139	0.0156	0.0425	0.0465	0.0514	0.055
7	7	0.056	0.0634	0.075	0.0859	0.3162	0.3539	0.402	0.4381
7	8	0.0455	0.0513	0.0607	0.0696	0.2479	0.2778	0.3174	0.3479
7	9	0.0382	0.0431	0.0509	0.0585	0.2059	0.2303	0.2612	0.2853
7	10	0.0333	0.0379	0.0446	0.0505	0.1741	0.1934	0.2178	0.2355
7	11	0.0291	0.033	0.0389	0.0441	0.1488	0.1658	0.1869	0.2029
7	12	0.0261	0.0291	0.034	0.0389	0.1297	0.1435	0.1618	0.1751

n_1	n_2	0.005	0.01	0.025	0.05	0.95	0.975	0.99	0.995
7	14	0.0214	0.024	0.0284	0.0323	0.1024	0.1135	0.1276	0.1391
7	15	0.02	0.0223	0.026	0.0295	0.0922	0.1017	0.1145	0.1234
7	16	0.0186	0.0207	0.024	0.0273	0.0836	0.0919	0.1039	0.1122
7	17	0.0167	0.019	0.0222	0.0251	0.076	0.084	0.0941	0.102
7	18	0.0159	0.0178	0.0208	0.0236	0.0702	0.0771	0.0864	0.0936
7	19	0.0151	0.0168	0.0195	0.0221	0.0647	0.0714	0.0794	0.085
7	20	0.0143	0.0159	0.0184	0.0207	0.0601	0.0657	0.073	0.0787
7	21	0.0136	0.015	0.0174	0.0197	0.0558	0.061	0.0679	0.0729
7	22	0.0131	0.0145	0.0167	0.0187	0.052	0.0569	0.0635	0.0681
7	23	0.0123	0.0137	0.0157	0.0177	0.0492	0.0539	0.0592	0.0635
7	24	0.0118	0.0131	0.0151	0.0169	0.0463	0.0507	0.0561	0.06
7	25	0.0115	0.0127	0.0144	0.0162	0.0437	0.0475	0.0524	0.0561
8	8	0.0497	0.0557	0.0652	0.0749	0.2637	0.2931	0.3331	0.3636
8	9	0.0416	0.0465	0.0551	0.0631	0.2164	0.2412	0.2755	0.2971
8	10	0.0359	0.0404	0.0468	0.0537	0.1819	0.2027	0.2292	0.2483
8	11	0.0317	0.0356	0.0413	0.0471	0.1542	0.1711	0.1937	0.2128
8	12	0.0285	0.0318	0.0369	0.0421	0.1353	0.1502	0.1691	0.184
8	13	0.0259	0.0283	0.0328	0.0374	0.1191	0.1323	0.1478	0.1596
8	14	0.023	0.0257	0.0297	0.034	0.1064	0.1173	0.1314	0.1415
8	15	0.0209	0.0235	0.0276	0.0312	0.0953	0.1049	0.1178	0.127
8	16	0.0193	0.0215	0.0251	0.0284	0.0863	0.0948	0.106	0.1146
8	17	0.0181	0.0202	0.0234	0.0264	0.0786	0.0867	0.0966	0.1036
8	18	0.0171	0.0189	0.022	0.0247	0.0723	0.0797	0.0888	0.0954

n_1	n_2	0.005	0.01	0.025	0.05	0.95	0.975	0.99	0.995
8	19	0.0159	0.0177	0.0205	0.0232	0.0666	0.0729	0.0812	0.0868
8	20	0.0152	0.0168	0.0194	0.0219	0.0619	0.0677	0.0753	0.0812
8	21	0.0143	0.0159	0.0182	0.0204	0.0575	0.0631	0.0703	0.0754
8	22	0.0136	0.015	0.0173	0.0195	0.0538	0.0588	0.0655	0.0701
8	23	0.013	0.0144	0.0165	0.0185	0.0502	0.0547	0.0606	0.0651
8	24	0.0123	0.0135	0.0156	0.0175	0.0475	0.0519	0.0572	0.0613
8	25	0.0117	0.013	0.015	0.0168	0.0449	0.0489	0.0539	0.0578
9	9	0.0446	0.0503	0.0592	0.0675	0.2268	0.252	0.2861	0.3129
9	10	0.0385	0.0429	0.0501	0.0574	0.1893	0.2092	0.2379	0.2557
9	11	0.0336	0.0377	0.0445	0.0504	0.1618	0.1793	0.2023	0.2192
9	12	0.0301	0.0337	0.0391	0.0445	0.1408	0.1555	0.176	0.1933
9	13	0.0268	0.0301	0.035	0.0399	0.1236	0.1361	0.1527	0.1655
9	14	0.0241	0.0271	0.0317	0.0359	0.1097	0.1211	0.1364	0.1471
9	15	0.0228	0.0252	0.0291	0.0329	0.0982	0.1082	0.1213	0.1311
9	16	0.0208	0.0229	0.0265	0.0301	0.089	0.098	0.1094	0.1183
9	17	0.0191	0.0211	0.0247	0.0279	0.0812	0.0891	0.0994	0.1079
9	18	0.0179	0.0199	0.023	0.026	0.0742	0.0814	0.0909	0.098
9	19	0.0169	0.0188	0.0216	0.0243	0.0688	0.0752	0.0836	0.0902
9	20	0.0159	0.0176	0.0202	0.0227	0.0636	0.0695	0.0771	0.0824
9	21	0.0151	0.0166	0.0191	0.0215	0.0592	0.0645	0.0721	0.0774
9	22	0.014	0.0157	0.0179	0.0202	0.0553	0.06	0.0655	0.0707
9	23	0.0136	0.015	0.0171	0.0192	0.0518	0.0566	0.0627	0.0672
9	24	0.013	0.0143	0.0164	0.0183	0.0487	0.053	0.0583	0.0627

n_1	n_2	0.005	0.01	0.025	0.05	0.95	0.975	0.99	0.995
9	25	0.0124	0.0135	0.0155	0.0174	0.0458	0.0497	0.0548	0.0584
10	10	0.0408	0.0461	0.054	0.0614	0.1993	0.2207	0.2482	0.2708
10	11	0.0354	0.0395	0.0463	0.0529	0.1689	0.1874	0.2122	0.2296
10	12	0.0317	0.0352	0.0414	0.047	0.1464	0.1619	0.1827	0.1981
10	13	0.0283	0.0319	0.037	0.042	0.1276	0.1413	0.1578	0.1695
10	14	0.0257	0.0289	0.0335	0.0379	0.1138	0.1258	0.1413	0.1515
10	15	0.0237	0.0263	0.0306	0.0344	0.1021	0.1127	0.1264	0.1373
10	16	0.0216	0.0238	0.0277	0.0316	0.0927	0.1015	0.1141	0.1225
10	17	0.0202	0.0225	0.026	0.0293	0.0835	0.0915	0.1021	0.1097
10	18	0.0187	0.0208	0.0239	0.027	0.0766	0.0837	0.0934	0.0999
10	19	0.0177	0.0197	0.0227	0.0254	0.0707	0.0775	0.086	0.0926
10	20	0.0167	0.0183	0.021	0.0236	0.0655	0.0715	0.0793	0.085
10	21	0.0157	0.0174	0.0198	0.0223	0.0607	0.0663	0.0735	0.0787
10	22	0.0146	0.0163	0.0188	0.021	0.0567	0.0619	0.0683	0.0731
10	23	0.0142	0.0156	0.0178	0.0198	0.0531	0.0579	0.0637	0.0682
10	24	0.0134	0.0148	0.0169	0.0189	0.0497	0.0542	0.0599	0.064
10	25	0.0126	0.0141	0.0161	0.0179	0.047	0.0512	0.0566	0.0606
11	11	0.0379	0.0422	0.0493	0.0559	0.1741	0.1928	0.2157	0.2342
11	12	0.0335	0.0375	0.0438	0.0495	0.1517	0.1681	0.1881	0.2038
11	13	0.0298	0.0331	0.0388	0.0439	0.1328	0.1457	0.1631	0.1766
11	14	0.0274	0.0305	0.035	0.0396	0.118	0.1294	0.144	0.1549
11	15	0.0249	0.0275	0.0319	0.036	0.1051	0.1156	0.1291	0.1383

n_1	n_2	0.005	0.01	0.025	0.05	0.95	0.975	0.99	0.995
11	16	0.0227	0.0254	0.0293	0.0329	0.0954	0.1046	0.1157	0.124
11	17	0.021	0.0234	0.0271	0.0304	0.0866	0.0949	0.1052	0.1131
11	18	0.0197	0.0216	0.025	0.0283	0.0788	0.0865	0.0957	0.1025
11	19	0.0185	0.0205	0.0235	0.0264	0.0728	0.0797	0.0886	0.0947
11	20	0.0174	0.019	0.022	0.0247	0.067	0.0731	0.0809	0.0867
11	21	0.0162	0.018	0.0209	0.0234	0.0621	0.0676	0.075	0.0798
11	22	0.0155	0.0171	0.0197	0.0219	0.0579	0.063	0.0691	0.074
11	23	0.0145	0.016	0.0184	0.0206	0.0543	0.0592	0.0652	0.0693
11	24	0.0139	0.0154	0.0175	0.0196	0.0509	0.0552	0.061	0.0654
11	25	0.0135	0.0147	0.0169	0.0187	0.0482	0.0522	0.0575	0.0614
12	12	0.0351	0.0395	0.0461	0.0521	0.1564	0.1723	0.1936	0.2088
12	13	0.0315	0.035	0.0407	0.0459	0.1374	0.1514	0.17	0.1824
12	14	0.0286	0.0318	0.0369	0.0416	0.121	0.1332	0.1489	0.1591
12	15	0.0258	0.0284	0.0334	0.0378	0.1087	0.1192	0.1329	0.1424
12	16	0.0243	0.0266	0.0306	0.0344	0.0977	0.1069	0.1189	0.127
12	17	0.0218	0.0245	0.0283	0.0317	0.0887	0.0969	0.1076	0.1155
12	18	0.0207	0.0229	0.0263	0.0295	0.0812	0.0888	0.0983	0.1058
12	19	0.0193	0.0212	0.0244	0.0273	0.0743	0.081	0.0902	0.097
12	20	0.0182	0.02	0.0229	0.0256	0.0688	0.0751	0.0828	0.0884
12	21	0.017	0.0186	0.0216	0.0242	0.0638	0.0695	0.0769	0.0826
12	22	0.0156	0.0174	0.0201	0.0225	0.0593	0.0648	0.0714	0.0765
12	23	0.0151	0.0166	0.0191	0.0213	0.0556	0.0607	0.0667	0.0712
12	24	0.0146	0.016	0.0182	0.0203	0.0522	0.0567	0.0622	0.0661

n_1	n_2	0.005	0.01	0.025	0.05	0.95	0.975	0.99	0.995
12	25	0.0137	0.0151	0.0172	0.0192	0.0491	0.0534	0.0586	0.0626
13	13	0.0336	0.0373	0.0433	0.0487	0.1417	0.1552	0.1732	0.1853
13	14	0.03	0.0333	0.0387	0.0439	0.1247	0.1372	0.1534	0.1651
13	15	0.0273	0.0302	0.0349	0.0394	0.1112	0.1218	0.1362	0.1469
13	16	0.0246	0.0276	0.0317	0.0359	0.1008	0.1102	0.1225	0.132
13	17	0.0228	0.0252	0.0291	0.0329	0.0914	0.0996	0.1101	0.1185
13	18	0.0214	0.0235	0.0272	0.0305	0.0832	0.0907	0.101	0.108
13	19	0.0199	0.022	0.0253	0.0284	0.0763	0.0831	0.0918	0.099
13	20	0.0185	0.0204	0.0236	0.0264	0.0705	0.0769	0.0854	0.0914
13	21	0.0176	0.0195	0.0224	0.0249	0.0652	0.0711	0.0787	0.0845
13	22	0.0167	0.0185	0.021	0.0235	0.061	0.0663	0.0735	0.0781
13	23	0.0157	0.0174	0.0199	0.0222	0.0571	0.0619	0.0683	0.0725
13	24	0.015	0.0165	0.0187	0.0209	0.0535	0.058	0.0638	0.0685
13	25	0.0142	0.0157	0.0178	0.0198	0.0503	0.0545	0.0599	0.0639
14	14	0.0318	0.0349	0.0403	0.0454	0.1292	0.1411	0.156	0.1671
14	15	0.0285	0.0315	0.0364	0.0411	0.115	0.1258	0.1398	0.1493
14	16	0.0262	0.0291	0.0335	0.0376	0.1035	0.1129	0.1265	0.137
14	17	0.0239	0.0266	0.0308	0.0344	0.0938	0.1022	0.1131	0.1201
14	18	0.0224	0.0246	0.0284	0.0318	0.0851	0.0929	0.1027	0.1094
14	19	0.0209	0.0231	0.0265	0.0296	0.0786	0.0859	0.0949	0.1008
14	20	0.0196	0.0214	0.0247	0.0277	0.0723	0.0788	0.0872	0.0933
14	21	0.0185	0.0202	0.0231	0.0258	0.0671	0.0729	0.0804	0.0862

n_1	n_2	0.005	0.01	0.025	0.05	0.95	0.975	0.99	0.995
14	22	0.0173	0.019	0.0217	0.0243	0.0624	0.0677	0.0744	0.0792
14	23	0.0162	0.0177	0.0202	0.0226	0.0584	0.0634	0.0696	0.0739
14	24	0.0154	0.017	0.0194	0.0216	0.0546	0.0589	0.0649	0.0694
14	25	0.0149	0.0163	0.0185	0.0205	0.0512	0.0554	0.0612	0.0653
15	15	0.0302	0.0333	0.0383	0.0428	0.1185	0.1296	0.1432	0.1536
15	16	0.0272	0.03	0.0351	0.0392	0.107	0.1169	0.13	0.1391
15	17	0.0253	0.0278	0.0321	0.0358	0.0964	0.1051	0.117	0.1254
15	18	0.0234	0.0257	0.0296	0.033	0.0876	0.0952	0.1049	0.1123
15	19	0.0217	0.024	0.0273	0.0305	0.0803	0.0875	0.0969	0.1029
15	20	0.0202	0.0221	0.0255	0.0283	0.0742	0.0804	0.0886	0.095
15	21	0.0187	0.0207	0.0236	0.0265	0.0685	0.0745	0.0816	0.087
15	22	0.0179	0.0197	0.0224	0.0249	0.0638	0.0691	0.076	0.0811
15	23	0.0168	0.0185	0.0212	0.0236	0.0593	0.0642	0.0702	0.0746
15	24	0.0162	0.0177	0.02	0.0222	0.0557	0.0604	0.0662	0.0706
15	25	0.0155	0.017	0.0191	0.0211	0.0523	0.0567	0.0623	0.0659
16	16	0.0283	0.0314	0.0359	0.0404	0.109	0.1188	0.1315	0.1407
16	17	0.0263	0.0285	0.0329	0.0371	0.0986	0.1074	0.1186	0.1276
16	18	0.0243	0.0268	0.0305	0.0341	0.09	0.0981	0.1075	0.1151
16	19	0.0224	0.0247	0.0283	0.0316	0.082	0.0892	0.0979	0.1049
16	20	0.021	0.023	0.0262	0.0293	0.0759	0.0821	0.0906	0.0964
16	21	0.0197	0.0216	0.0246	0.0274	0.0699	0.076	0.0835	0.0884
16	22	0.0186	0.0204	0.0233	0.0259	0.0651	0.071	0.0777	0.0827

n_1	n_2	0.005	0.01	0.025	0.05	0.95	0.975	0.99	0.995
16	23	0.0174	0.0191	0.0219	0.0242	0.0609	0.0662	0.0724	0.0774
16	24	0.0166	0.018	0.0206	0.023	0.0567	0.0613	0.0671	0.0717
16	25	0.0157	0.0172	0.0195	0.0217	0.0533	0.0574	0.0626	0.066
17	17	0.0271	0.03	0.0343	0.0383	0.101	0.1101	0.1214	0.1293
17	18	0.0247	0.0275	0.0315	0.0351	0.0922	0.1004	0.111	0.1182
17	19	0.0234	0.0255	0.0292	0.0325	0.084	0.0914	0.1007	0.1074
17	20	0.0219	0.024	0.0274	0.0306	0.0773	0.0838	0.0921	0.0977
17	21	0.0205	0.0225	0.0255	0.0284	0.0713	0.0774	0.085	0.0907
17	22	0.0193	0.0211	0.0239	0.0266	0.0664	0.0718	0.0786	0.084
17	23	0.018	0.0196	0.0224	0.0249	0.0619	0.0668	0.0738	0.0787
17	24	0.0172	0.0189	0.0212	0.0236	0.0578	0.0625	0.0684	0.073
17	25	0.0163	0.0179	0.0202	0.0224	0.0545	0.0593	0.0647	0.069
18	18	0.0262	0.0286	0.0327	0.0365	0.094	0.1018	0.1121	0.1198
18	19	0.0241	0.0264	0.0304	0.0338	0.0861	0.0935	0.1031	0.1095
18	20	0.0223	0.0245	0.028	0.0314	0.0791	0.0856	0.0943	0.1003
18	21	0.0209	0.0229	0.0262	0.0292	0.0733	0.0794	0.0873	0.0937
18	22	0.0199	0.0218	0.0247	0.0273	0.0682	0.0735	0.081	0.0868
18	23	0.0186	0.0205	0.0233	0.0257	0.0632	0.0685	0.0752	0.08
18	24	0.0179	0.0194	0.0219	0.0243	0.059	0.0639	0.0699	0.0744
18	25	0.0166	0.0183	0.0208	0.0229	0.0555	0.0601	0.0657	0.0699
19	19	0.0248	0.0275	0.0313	0.0349	0.0882	0.0958	0.1055	0.1128
19	20	0.0231	0.0254	0.029	0.0322	0.0807	0.0874	0.0954	0.1008

n_1	n_2	0.005	0.01	0.025	0.05	0.95	0.975	0.99	0.995
19	21	0.0217	0.0239	0.0271	0.0303	0.0749	0.0811	0.0886	0.094
19	22	0.0203	0.0223	0.0254	0.0282	0.0693	0.0748	0.082	0.0868
19	23	0.0192	0.0211	0.0238	0.0264	0.0646	0.0697	0.0759	0.0808
19	24	0.018	0.0196	0.0224	0.0249	0.0603	0.0653	0.0713	0.0763
19	25	0.017	0.0187	0.0213	0.0236	0.0567	0.0611	0.0665	0.0705
20	20	0.0241	0.0265	0.03	0.0333	0.0823	0.089	0.0972	0.1042
20	21	0.0226	0.0247	0.0281	0.0311	0.0762	0.0823	0.0901	0.0957
20	22	0.021	0.0231	0.0261	0.0289	0.0706	0.0764	0.0839	0.0892
20	23	0.0196	0.0216	0.0245	0.0272	0.0659	0.071	0.0778	0.0827
20	24	0.0186	0.0204	0.0232	0.0256	0.0615	0.0662	0.0726	0.0766
20	25	0.0176	0.0194	0.0219	0.0242	0.0575	0.0619	0.0675	0.0718
21	21	0.0233	0.0256	0.0289	0.032	0.0778	0.084	0.0921	0.0984
21	22	0.0221	0.024	0.0271	0.0299	0.0723	0.0779	0.0852	0.0902
21	23	0.0203	0.0223	0.0252	0.0279	0.0669	0.0723	0.0791	0.0837
21	24	0.0193	0.0211	0.0239	0.0264	0.0626	0.0674	0.0737	0.0779
21	25	0.0182	0.02	0.0226	0.0249	0.0587	0.063	0.0691	0.073
22	22	0.0222	0.0244	0.0277	0.0306	0.0736	0.0796	0.0866	0.0922
22	23	0.0212	0.0231	0.026	0.0287	0.0685	0.0738	0.0805	0.0855
22	24	0.0199	0.0217	0.0244	0.027	0.0638	0.0687	0.0748	0.079
22	25	0.019	0.0205	0.0231	0.0254	0.0597	0.064	0.0697	0.0739
23	23	0.0216	0.0236	0.0266	0.0296	0.0696	0.075	0.0821	0.0869
23	24	0.0204	0.0224	0.0251	0.0277	0.0649	0.0699	0.0764	0.0808

n_1	n_2	0.005	0.01	0.025	0.05	0.95	0.975	0.99	0.995
23	25	0.0194	0.0211	0.0237	0.0261	0.0607	0.0651	0.071	0.0752
24	24	0.0211	0.023	0.0258	0.0285	0.0661	0.0711	0.0776	0.0822
24	25	0.0199	0.0218	0.0244	0.0269	0.0619	0.0666	0.0726	0.0763
25	25	0.0204	0.0222	0.0249	0.0274	0.0627	0.0674	0.074	0.0779

3.4 Numerical Example

EXAMPLE 1 .

Proschan (1963) has given the number of successive failures of air-conditioning system of each member of a fleet of 13 Boeing 720 jet airplanes. The hours of flying time between failures are listed below for two of the planes.

Plane 7908: 413, 14, 58, 37, 100, 65, 9, 169, 447, 184, 36, 201, 118.

Plane 7911: 55, 320, 56, 104, 220, 239, 47, 246, 176, 182, 33.

For testing the null hypothesis H_{10}

$$H_{30}: \left\{ \begin{array}{lll} F_i(y) = & F\left(\frac{y-\alpha_i}{\beta}\right), & -\infty < \alpha_i < \infty, i = 1, 2, \quad 0 < \beta < \infty, \\ \text{with} & F(y) = & \left[\begin{array}{ll} 1 - e^{-y}, & \text{if } 0 < y < \infty, \\ 0, & \text{otherwise} \end{array} \right. \end{array} \right\}$$

against the alternative hypothesis

$$H_{3a}: \left\{ \begin{array}{lll} F_i(y) = & F\left(\frac{y-\alpha_i}{\beta}\right) & \\ \text{with} & F(y) \neq 1 - e^{-y}, & \text{for some } y \end{array} \right\}$$

In this example, we have

$$\begin{aligned}
n_1 &= 13, n_2 = 11, r_1 = s_1 = r_2 = s_2 = 0 \\
\overline{y}_1 &= \frac{1}{13}\sum_{j=1}^{13} y_{1j} \\
&= 142.385, \\
\overline{y}_2 &= \frac{1}{11}\sum_{j=1}^{11} y_{2j} \\
&= 152.545, \\
S_1^2 &= \sum_{j=1}^{13}(y_{1j} - \overline{y}_1)^2 \\
&= 244097.0769, \\
S_2^2 &= \sum_{j=1}^{11}(y_{2j} - \overline{y}_2)^2 \\
&= 96840.72727, \\
S^2 &= S_1^2 + S_2^2 \\
&= 244097.0769 + 96840.72727 \\
&= 340937.8042, \\
n^* &= max(13, 11) = 13.
\end{aligned}$$

Hence we calculate the V-exponential statistic as following:

$$
\begin{aligned}
V(n_1, n_2) &= \frac{\{n_1(\overline{Y}_1 - Y_{11}) + n_2(\overline{Y}_2 - Y_{21})\}^2}{2n^*(n^* - 1)S^2} \\
V(13, 11) &= \frac{\{13(142.385 - 9) + 11(152.545 - 33)\}^2}{2 \times 13 \times (13 - 1) \times 340937.8042} \\
&= 0.0874.
\end{aligned}
$$

From Table 3.1, it is seen that this value is greater than the lower 5% value 0.0439 corresponding to $n_1 = 13$ and $n_2 = 11$. Therefore, we conclude that there is no evidence of non-exponentiality of the two failure distributions with equal but unknown location parameters and same unknown scale parameters.

Chapter 4

Sensitivity Results for the Two-sample V-statistic

We used the empirical sampling results from Table 3.1 to evaluate the sensitivity properties of theV-exponential statistic for exponentiality for various alternative distributions.

As we mentioned before, the V-exponential statistic responds to nonexponentiality by assuming either small or large values, so that the test needs to be two-tailed in general. That is, for each pair of n_1 and n_2 (*i.e.* $n_1 = n_2 = 5$, $n_1 = n_2 = 10$, $n_1 = n_2 = 15$, $n_1 = n_2 = 20$, $n_1 = n_2 = 25$), we calculate the proportions which fell above the 95% point and below the 5% point for the alternative distributions. The sum of these two proportions provides the empirical power of a two-tailed test with 10% significance level. The choice of the probability distribution under the alternative hypothesis was based on the distributions commonly used as alternatives to the exponential distribution (*i.e.* the Weibull, χ^2, half-normal, and lognormal distributions) and the distributions with U-shape hazard functions (*i.e.* power function distributions). For the functional forms of the density functions of these distributions, see Brain and Shapiro (1983).

As expected, the V-exponential statistic is quite sensitive to departures form exponentiality and especially to symmetric alternative distribution. The estimated power increases as the two sample sizes increase, and is greater than the corresponding estimated power of the one sample W-exponential test at the 10% significance level for $n = 5, 10, 15, 20, 25$ (see Table 3 of Shapiro and Wilk, 1972). For example, for $n_1 = n_2 = 15$, the estimated power of the V-exponential test is quite close to the corresponding estimate for the one sample W-exponential test at the 10% significance level for $n = 30$ (see Table 1 of Samanta and Schwarz, 1988). The powers are given in Table 4.1.

Table 4.1: Estimated Powers of the V-exponential test for selected sample sizes at 10% significance level.

	n_1=n_2=5	n_1=n_2=10	n_1=n_2=15	n_1=n_2=20	n_1=n_2=25
$\chi^2(1)$	0.2044	0.3492	0.5024	0.5902	0.706
$\chi^2(4)$	0.1198	0.2488	0.3672	0.5002	0.6062
$\chi^2(6)$	0.1582	0.386	0.5992	0.7682	0.868
Beta(1,3)	0.1188	0.226	0.348	0.4914	0.5892
Lognormal(0,0.3)	0.2216	0.583	0.8032	0.9214	0.9634
Lognormal(0,0.7)	0.122	0.165	0.1962	0.2376	0.2608
Lognormal(0,1)	0.184	0.2716	0.3474	0.4114	0.465
Weibull(2.0,1)	0.2466	0.6278	0.8688	0.968	0.9908
Weibull(0.5,1)	0.4622	0.7766	0.9146	0.9694	0.9886
Half-normal	0.13	0.257	0.4144	0.5672	0.6618
Half-Cauchy	0.4848	0.7218	0.862	0.9284	0.9616
Uniform(0, 1)	0.5888	0.7876	0.9568	0.9902	0.9992
Normal	0.8488	0.8946	0.9896	0.9994	0.9999
t-distribution(2)	0.4534	0.8	0.8988	0.9408	0.9598
t-distribution(3)	0.4414	0.8406	0.9392	0.9714	0.988
t-distribution(4)	0.4432	0.8594	0.9562	0.9866	0.9962
t-distribution(6)	0.436	0.876	0.9764	0.9964	0.9988
Power Function(1/5)	0.3708	0.6148	0.7764	0.8828	0.9376
Power Function(1/3)	0.1038	0.1184	0.131	0.1438	0.1666
Power Function(1/2)	0.1088	0.1314	0.1504	0.1828	0.2094

Chapter 5

The Two-sample V-statistic for Censored Samples

Now suppose that, in a random sample of size n_1 from the distribution function $F_1(y)$ the r_1 smallest and the r_2 largest observations are censored. Therefore we have the $n_1 - r_1 - r_2$ middle observations $Y_{1,r_1+1} \leq Y_{1,r_1+2} \leq \ldots \leq Y_{1,n_1-r_2}$ available. Further suppose that in another random sample of size n_2 (first and second sample are independent) from $F_2(y)$ and similarly to the first sample, the s_1 smallest and the s_2 largest observations are censored. Therefore we have the $n_2 - s_1 - s_2$ middle observations $Y_{2,s_1+1} \leq Y_{2,s_1+2} \leq \ldots \leq Y_{2,n_2-s_2}$ available from the second sample.

Now we define

$$q_1 = (n_1 - r_1 - r_2),$$

$$q_2 = (n_2 - s_1 - s_2),$$

$$q^* = max(q_1, q_2),$$

$$A = \frac{\sum_{i=2}^{q_1}\sum_{j=2}^{q_1} a_{ij}^{(q_1)} T_{1,r_1+i} T_{1,r_1+j}}{q_1},$$

$$B = \frac{\sum_{i=2}^{q_2}\sum_{j=2}^{q_2} b_{ij}^{(q_2)} T_{2,s_1+i} T_{2,s_1+j}}{q_2},$$

and

$$a_{ij}^{(q_1)} = \frac{(j-1)}{(q_1 - j + 1)}, \quad (q_1 \geq i \geq j \geq 2),$$

$$b_{ij}^{(q_2)} = \frac{(j-1)}{(q_2 - j + 1)}, \quad (q_2 \geq i \geq j \geq 2).$$

with $a_{ij}^{(q_1)} = a_{ji}^{(q_1)}$ and $b_{ij}^{(q_2)} = b_{ji}^{(q_2)}$ as defined before.

Then, using the V-exponential statistic for the uncensored data

$$V(n_1, n_2) = \frac{\{n_1(\overline{Y}_1 - Y_{11}) + n_2(\overline{Y}_2 - Y_{21})\}^2}{2n^*(n^* - 1)S^2},$$

we propose a statistic V_1 for the null hypothesis:

$$H_{30} : \left\{ \begin{array}{lll} F_i(y) = & F\left(\frac{y-\alpha_i}{\beta}\right), & -\infty < \alpha_i < \infty, i = 1, 2, \quad 0 < \beta < \infty, \\ \text{with} & F(y) = & \left[\begin{array}{ll} 1 - e^{-y}, & \text{if } 0 < y < \infty, \\ 0, & \text{otherwise} \end{array} \right. \end{array} \right\}$$

against the alternative hypothesis

$$H_{3a} : \left\{ \begin{array}{lll} F_i(y) = & F\left(\frac{y-\alpha_i}{\beta}\right) & \\ \text{with} & F(y) \neq 1 - e^{-y}, & \text{for some } y \end{array} \right\}$$

as defined by

$$V_1 = \frac{\left\{ \sum_{i=2}^{q_1} T_{1,r_1+i} + \sum_{i=2}^{q_2} T_{2,s_1+i} \right\}^2}{2q^*(q^* - 1)(A + B)}. \tag{5.1}$$

Note that

- the null distribution of V_1 is the same as that of $V(q_1, q_2)$,
- V_1 and $V(n_1, n_2)$ are equal when there are no censored observations.

For convenience in computation, we further define

$$C = \frac{\sum_{i=2}^{q_1} \frac{i-1}{q_1 - i + 1} T_{1,r_1+i} \left(T_{1,r_1+i} + 2 \sum_{j=i+1}^{q_1} T_{1,r_1+j} \right)}{q_1},$$

$$D = \frac{\sum_{i=2}^{q_2} \frac{i-1}{q_2 - i + 1} T_{2,s_1+i} \left(T_{2,s_1+i} + 2 \sum_{j=i+1}^{q_2} T_{2,s_1+j} \right)}{q_2},$$

and point out that we can use the following expression of V_1

$$V_1 = \frac{\left\{ \sum_{i=2}^{q_1} T_{1,r_1+i} + \sum_{i=2}^{q_2} T_{2,s_1+i} \right\}^2}{2q^*(q^* - 1)(C + D)}. \tag{5.2}$$

EXAMPLE 2 .

We consider the data in Bain (1978) (Problem number 4, page 203). In a new process of making tires a certain additive is proposed for increasing the length of time of tread wear of a tire. 40 of the present tires and 40 tires made under the new process are placed in service and the experiment is continued until the 20 smallest observations (in thousands of miles) are obtained for each sample. [The data are not reported here].

In this example, $n_1 = n_2 = 40$, $r_1 = s_1 = 0$ and $r_2 = s_2 = 20$. From the data, the computed value of V_1 for testing the null hypothesis H_{30} is $V_1 = 0.0332$.

From Table 3.1, one finds this value to be greater than the lower 2.5% point of the (null) distribution of $V(20, 20)$ and very close to the 5% point of this distribution. Since both tails of this null distribution are used as critical regions defined by equal tail areas, there is no evidence against the null hypothesis H_{30} and the data do not refute the exponentiality of both distributions with equal but unknown wearing rates.

Chapter 6

Two-sample V^*-statistic for Exponentiality for Complete Samples

In this chapter, we propose another test statistic for testing exponentiality of two distributions using the one-sample approach of Stephens (1978). We show that our proposed statistic has the same null distribution as Shapiro and Wilk statistic with an appropriate sample size. We also provide an example to illustrate the application of the proposed method.

6.1 V^*-exponential statistic for complete sample

Suppose $Y_{11} \leq Y_{12} \leq \ldots \leq Y_{1n_1}$ are the order statistics of a random sample of size n_1 from a continuous distribution function $F_1(y)$ and $Y_{21} \leq Y_{22} \leq \ldots \leq Y_{2n_2}$ are the order statistics of a random sample of size n_2 from another continuous distribution function $F_2(y)$. Further suppose that the two samples are independent of each other. On the basis of these, we wish to test the null hypothesis:

$$H_{30}: \left\{ \begin{array}{lll} F_i(y) = & F\left(\frac{y-\alpha_i}{\beta}\right), & -\infty < \alpha_i < \infty, i = 1, 2, \quad 0 < \beta < \infty, \\ \text{with} & F(y) = & \left[\begin{array}{ll} 1 - e^{-y}, & \text{if } 0 < y < \infty, \\ 0, & \text{otherwise} \end{array} \right. \end{array} \right\}.$$

against the alternative hypothesis

$$H_{3a}: \left\{ \begin{array}{lll} F_i(y) = & F\left(\frac{y-\alpha_i}{\beta}\right) & \\ \text{with} & F(y) \neq 1 - e^{-y}, & \text{for some } y \end{array} \right\}$$

We define

$$\begin{aligned}
\overline{Y}_i =& \frac{1}{n_i}\sum_{j=1}^{n_i} Y_{ij}, \quad i = 1, 2, \\
S_i^2 =& \sum_{j=1}^{n_i} (Y_{ij} - \overline{Y}_i)^2, \quad i = 1, 2, \\
X_{ij} =& Y_{ij} - Y_{i1}, \quad j = 1, 2, \cdots, n_i, \quad i = 1, 2,
\end{aligned}$$

Then, we have that

$$\begin{aligned}
X_{i1} =& 0, \quad i = 1, 2, \\
\sum_{j=2}^{n_i} X_{ij} =& n_i(\overline{Y}_i - Y_{i1}), \quad i = 1, 2, \\
X_{ij} - \overline{X}_i =& Y_{ij} - \overline{Y}_i, \quad j = 1, 2, \cdots, n_i, \quad i = 1, 2,
\end{aligned}$$

where

$$\overline{X}_i = \frac{1}{n_i}\sum_{j=2}^{n_i} X_{ij}, \quad i = 1, 2.$$

From this, we can also write

$$S_i^2 = \sum_{j=2}^{n_i} (X_{ij} - \overline{X}_i)^2, \quad i = 1, 2$$

and

$$\begin{aligned}(\overline{Y}_i - Y_{i1})^2 &= \overline{X}_i^2 \\ &= \frac{1}{n_i^2}\left(\sum_{j=2}^{n_i} X_{ij}\right)^2, \quad i = 1, 2.\end{aligned}$$

We note that under the null hypothesis H_{30}, $X_{i2} \leq X_{i3} \leq \ldots \leq X_{in_i}$ are the order statistics of a random sample of size $n_i - 1$ from the exponential distribution function

$$G(x) = \begin{cases} 1 - e^{-x/\beta}, & 0 < \beta < \infty, \quad 0 < x < \infty, \\ 0, & \text{otherwise}, \end{cases} \tag{6.1}$$

Clearly, under the null hypothesis H_{30}, these two samples can be combined to form a single random sample of size $n_1 + n_2 - 2$ from the above exponential distribution $G(x)$.

If $Z_1 \leq Z_2 \leq \ldots \leq Z_{n_1+n_2-2}$ are the order statistics of this combined sample, then following Stephens (1978), we propose a two sample test statistic $V^*(n_1, n_2)$ given by

$$V^*(n_1,n_2) = \frac{\left(\sum\limits_{i=1}^{n_1+n_2-2} Z_i\right)^2}{(n_1+n_2-2)\left\{(n_1+n_2-1)\sum\limits_{i=1}^{n_1+n_2-2} Z_i^2 - \left(\sum\limits_{i=1}^{n_1+n_2-2} Z_i\right)^2\right\}}$$

$$= \frac{\left(\sum\limits_{i=1}^{2}\sum\limits_{j=2}^{n_i} X_{ij}\right)^2}{(n_1+n_2-2)\left\{(n_1+n_2-1)\left[\sum\limits_{i=1}^{2}\sum\limits_{j=2}^{n_i} X_{ij}^2\right] - \left(\sum\limits_{i=1}^{2}\sum\limits_{j=2}^{n_i} X_{ij}\right)^2\right\}}.$$

$$= \frac{\{n_1(\overline{Y}_1 - Y_{11}) + n_2(\overline{Y}_2 - Y_{21})\}^2}{(n_1+n_2-2)\left\{(n_1+n_2-1)\left[S_1^2+S_2^2\right] - \left[n_1(\overline{Y}_1 - Y_{11}) + n_2(\overline{Y}_2 - Y_{21})\right]^2\right\}}.$$

Under the null hypothesis H_{30}, $V^*(n_1,n_2)$ has the same distribution as the null distribution of the statistic $W_E(n_1+n_2-1)$ proposed by Shapiro and Wilk (1972).

We note that the above statistic $V^*(n_1,n_2)$ may be regarded as a two-sample generalization of the one-sample Shapiro-Wilk statistic.

6.2 Numerical Example

We revisit *EXAMPLE 1* from Chapter 3.

Here the test statistic is

$$V^*(n_1, n_2) = \frac{\{n_1(\overline{Y}_1 - Y_{11}) + n_2(\overline{Y}_2 - Y_{21})\}^2}{(n_1 + n_2 - 2)\left\{(n_1 + n_2 - 1)\left[S_1^2 + S_2^2\right] - \left[n_1(\overline{Y}_1 - Y_{11}) + n_2(\overline{Y}_2 - Y_{21})\right]^2\right\}}$$

where

$$\overline{Y}_i = \sum_{j=1}^{n_i} Y_{ij}/n_i,$$

$$S_i^2 = \sum_{j=1}^{n_i} (Y_{ij} - \overline{Y}_i)^2, \quad i = 1, 2.$$

In this example, we have

$$n_1 = 13, n_2 = 11, r_1 = s_1 = r_2 = s_2 = 0,$$

$$\overline{y}_1 = \frac{1}{13}\sum_{j=1}^{13} y_{1j}$$

$$= 142.385,$$

$$\overline{y}_2 = \frac{1}{11}\sum_{j=1}^{11} y_{2j}$$

$$= 152.545,$$

$$S_1^2 = \sum_{j=1}^{13} (y_{1j} - \overline{y}_1)^2$$

$$= 244097.0769,$$

$$S_2^2 = \sum_{j=1}^{11} (y_{2j} - \overline{y}_2)^2$$

$$= 96840.72727.$$

Hence we calculate the two-sample V^*-statistic as follows

$$V^*(n_1, n_2) = \frac{\{n_1(\overline{Y}_1 - Y_{11}) + n_2(\overline{Y}_2 - Y_{21})\}^2}{(n_1 + n_2 - 2)\left\{(n_1 + n_2 - 1)\left[S_1^2 + S_2^2\right] - \left[n_1(\overline{Y}_1 - Y_{11}) + n_2(\overline{Y}_2 - Y_{21})\right]^2\right\}}$$
$$= 0.05649.$$

From Table 1 of percentage points (also given in the Appendix) of the W-exponential statistic (Shapiro and Wilk, 1972), it is seen that this value is greater than the lower 5% critical value 0.0266. That is, there is no evidence of non-exponentiality and unequal failure rates of the two failure distributions. Note that we reached the same conclusion as we did in Chapter 3 using the V-exponential statistic.

6.3 Power Study

Table 6.1 gives the simulated power of the Shapiro-Wilk test based on $W_E(n)$ for $n = 20$ and the power of $V^*(n_1, n_2)$ with $n_1 = 11$ and $n_2 = 10$ for various alternatives. The reason for taking $n_1 = 11$ and $n_2 = 10$ is that $V^*(n_1, n_2)$ has the same null distribution as the null distribution of the Shapiro-Wilk statistic with $n = n_1 + n_2 - 1 = 20$, that is $W_E(20)$. The displayed results for $W_E(20)$ have been taken from Stephens (1978). From the table, it appears that the power of V^* with $n_1 = 11$ and $n_2 = 10$ is comparable to that of the one-sample Shapiro-Wilk statistic with $n = 20$ for various alternatives.

In Table 6.2 we compare the power of the one-sample W-exponential test for $n = 29$

to the power of the two-sample V-exponential and V^*-exponential tests for $n_1 = 15$, $n_2 = 15$ at the $\alpha = 10\%$ significance level.

We see that the results are close to each other. That is, the three tests (W-exponential, V-exponential and V^*-exponential statistics) seem to be comparable in terms of sensitivity.

Table 6.1: Estimated Powers of $V^*(11, 10)$ and $W_E(20)$ at 5% significance level.

	$W_E(20)$	$V^*(11, 10)$
Gamma (4)	0.40736	0.38822
$\chi^2(1)$	0.27	0.25272
$\chi^2(4)$	0.21	0.16754
$\chi^2(6)$	0.38	0.29486
Inverse-$\chi^2(6)$	0.1731	0.1698
Beta(1,3)	0.09	0.149
Lognormal(0,0.3)	0.60998	0.47034
Lognormal(0,0.7)	0.12518	0.10534
Lognormal(0,1)	0.21	0.20858
Weibull(2.0,1)	0.72	0.52654
Weibull(0.5,1)	0.63	0.698
Half-normal	0.21	0.18752
Half-Cauchy	0.69	0.67892
Uniform(0, 1)	0.76	0.72608
Normal	0.92154	0.81778
Power Function(1/5)	0.57362	0.50092
Power Function(1/3)	0.08194	0.05882
Power Function(1/2)	0.05716	0.07686

Table 6.2: Estimated Powers of V and V^* exponential tests for $n_1 = 15$, $n_2 = 15$, and W test for $n = 29$, at 10% significance level.

	$W_E(29)$	$V(15,15)$	$V^*(15,15)$
$\chi^2(1)$	0.5293	0.5024	0.49122
$\chi^2(4)$	0.45102	0.3672	0.38332
$\chi^2(6)$	0.70026	0.5992	0.60966
Beta(1,3)	0.36124	0.348	0.36228
Lognormal(0,0.3)	0.87658	0.8032	0.7935
Lognormal(0,0.7)	0.2405	0.1962	0.20392
Lognormal(0,1)	0.3395	0.3474	0.35118
Weibull(2.0,1)	0.92202	0.8688	0.87008
Weibull(0.5,1)	0.92768	0.9146	0.9154
Half-normal	0.44756	0.4144	0.42774
Half-Cauchy	0.85076	0.862	0.8575
Uniform(0, 1)	0.96218	0.9568	0.96002
Normal	0.99594	0.9896	0.98492
t-distribution(2)	0.93734	0.8988	0.80946
t-distribution(3)	0.96424	0.9392	0.90004
t-distribution(4)	0.97706	0.9562	0.93642
t-distribution(6)	0.9881	0.9764	0.9629
Power Function(1/5)	0.83426	0.7764	0.7946
Power Function(1/3)	0.1745	0.131	0.13612
Power Function(1/2)	0.1231	0.1504	0.1602

Chapter 7

Conclusion

In this thesis, using the principles of the W-statistic for exponentiality of a single distribution (Shapiro and Wilk, 1972; Samanta and Schwarz, 1988) we proposed the V-exponential statistic for testing exponentiality of two distributions for both complete and censored samples. The proposed statistic turns out to be a normalized ratio of the square of the generalized least squares estimate (also the minimum variance unbiased estimate, that is MVUE) of the common scale parameter to a pooled sum of squares about the samples means. The V-exponential statistic is origin and scale invariant. We proved some important results relating to our proposed two-sample V-exponential statistic for testing exponentiality. It has a null distribution that depends only on the sample sizes n_1 and n_2. The V-exponential statistic has been presented as two-tailed in the sense that for an unspecified alternative to the exponential, the statistic may shift to either smaller or larger values. We provided some empirical power results for various types of probability distributions under the alternative hypothesis. From these results, it is clear that it has comparative sensitive results for various alternatives. Following the approach of Samanta and Schwarz (1988) the V-exponential statistic was also modified for one or both samples being censored. The modified test statistic has the same null distribution as in the uncensored case, with a corresponding reduction in sample size(s). In each case, we considered numerical examples to illustrate the applications of the proposed test.

Further we used Stephens' (1978) approach and proposed a second test statistic called V^*-exponential statistic for testing forexponentiality of two distributions in the context of complete samples. This statistic has the same null distribution as the W-exponential statistic of Shapiro and Wilk (1972) corresponding to an appropriate sample size.

We also compared the power of the one-sample W-exponential test for $n = 29$, the two-sample V-exponential and V^*-exponential test for $n_1 = 15$, $n_2 = 15$ at the $\alpha = 10\%$ significance level. We found that the results are close to each other, that is, the three tests (W-exponential, V-exponential and V^*-exponential statistics) are comparable in terms of sensitivity.

We are not aware of any literature in which exponentiality of two distributions having the same scale parameter has been examined. In the absence of another test, the V and V^* exponential tests are useful additions to the current literature on testing exponentiality of two distributions.

Appendix

Percentage Points of W-Exponential (Shapiro and Wilk, 1972, page 361)

n	0.005	0.01	0.025	0.05	0.95	0.975	0.99	0.995
3	0.2519	0.2538	0.2596	0.2697	0.9926	0.9981	0.9997	0.99993
4	0.1241	0.1302	0.1434	0.1604	0.8581	0.9236	0.9680	0.9837
5	0.0845	0.0905	0.1048	0.1187	0.6682	0.7590	0.8600	0.9192
6	0.0610	0.0665	0.0802	0.0956	0.5089	0.5842	0.6775	0.7501
7	0.0514	0.0591	0.0700	0.0810	0.4162	0.4852	0.5706	0.6426
8	0.0454	0.0512	0.0614	0.0710	0.3497	0.4033	0.4848	0.5428
9	0.0404	0.0422	0.0537	0.0633	0.3005	0.3454	0.4015	0.4433
10	0.0369	0.0404	0.0487	0.0568	0.2525	0.2879	0.3391	0.3701
11	0.0339	0.0380	0.0447	0.0528	0.2265	0.2619	0.3039	0.3314
12	0.0311	0.0358	0.0410	0.0494	0.2019	0.2364	0.2716	0.2978
13	0.0287	0.0337	0.0382	0.0460	0.1829	0.2113	0.2422	0.2642
14	0.0265	0.0317	0.0362	0.0428	0.1647	0.1862	0.2131	0.2315
15	0.0247	0.0298	0.0334	0.0398	0.1485	0.1669	0.1926	0.2123
16	0.0233	0.0280	0.0326	0.0374	0.1355	0.1542	0.1770	0.1931
17	0.0222	0.0264	0.0310	0.0352	0.1257	0.1423	0.1614	0.1794
18	0.0212	0.0250	0.0294	0.0332	0.1164	0.1311	0.1483	0.1668
19	0.0203	0.0238	0.0278	0.0314	0.1071	0.1199	0.1374	0.1452
20	0.0196	0.0227	0.0264	0.0302	0.1002	0.1121	0.1286	0.1369
21	0.0190	0.0217	0.0250	0.0290	0.0948	0.1054	0.1198	0.1288
22	0.0185	0.0208	0.0238	0.0278	0.0894	0.0988	0.1118	0.1213

n	0.005	0.01	0.025	0.05	0.95	0.975	0.99	0.995
23	0.0181	0.0201	0.0230	0.0266	0.0836	0.0933	0.1043	0.1142
24	0.0177	0.0194	0.0224	0.0256	0.0788	0.0882	0.0984	0.1071
25	0.0173	0.0188	0.0218	0.0248	0.0749	0.0836	0.0927	0.1000
26	0.0169	0.0182	0.0213	0.0240	0.0712	0.0791	0.0885	0.0948
27	0.0165	0.0177	0.0208	0.0232	0.0687	0.0747	0.0843	0.0896
28	0.0161	0.0172	0.0203	0.0225	0.0649	0.0706	0.0801	0.0859
29	0.0157	0.0168	0.0198	0.0219	0.0621	0.0671	0.0759	0.0822
30	0.0153	0.0164	0.0193	0.0213	0.0593	0.0643	0.0719	0.0786

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