Topics in the Notion of Operator Amenability and Its Generalizations with Application in the Fourier Algebras

by

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Abstract

The operator algebraists have for a long time realized the significance of studying matrices of elements of an operator algebra in order for obtaining results about the algebra. This lead Z. J. Ruan, D. Blecher and others to introduce the notion of an abstract operator space in late 1980's. Ruan, furthermore, introduced the notion of completely contractive Banach algebras and operator-space amenability for such algebras. He showed that the Fourier algebra $A(G)$ of a locally compact group G is operator-space amenable if and only if the group G is amenable.

In this thesis we investigate further the notion of operator-space amenability and its approximate versions. In particular for the Fourier algebras. We also prove results on perturbation theory of these notions.

Furthermore we study the question of when $A\hat{\otimes}B$ (or $A\otimes^\gamma B$) is (approximately) operator-space (or weakly) amenable what conclusions can one derive about the components.

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Introduction

In his AMS Memoir of 1972, among other things B.E. Johnson introduced the notion of amenability for Banach algebras and produced some basic general theory of amenable Banach algebras. One of the important results that he obtained was that the group algebra $L^1(G)$ of a locally compact group G is amenable if and only if G is an amenable group. Since then amenability of Banach algebras has proven to be of enormous importance.

Inspite of this, it has been realized that amenability is a strong property and for certain Banach algebras it leads to finite -dimensionality. For example, it was shown by F. Ghahramani, R. J. Loy and G. A. Willis that the second dual algebra $L^1(G)^{**}$ (under Arens product) is amenable if and only if G is finite, [see 11, Theorem 1.3].

In fact even earlier it was realized by John Ringrose that for Von Neumann algebras a weaker notion of amenability is more appropriate. This notion was called Connes amenability in the context of general dual Banach algebras by Y. Helemskii [18] and extensively studied in particular classes of Banach algebras by V. Runde [29],[30].

F. Ghahramani, R. J. Loy and Y. Zhang introduced various notions of approximate amenability in [10] and [12] and [15] and since then these notions have been studied extensively.

The notion of operator amenability was introduced by Z. J. Ruan in [28], where he showed that the Fourier algebra $A(G)$ of a locally compact group G is operator amenable if and only if G is amenable.

Part of this thesis deals with approximate versions of operator amenability. The formal definitions of these notions will be given in chapters 2.1 and 2.2 . In particular, in chapters 2.4 and 2.5, we are interested in studying when some notions of amenability of tensor products of Banach algebras necessities (the same notion) of amenability for the component Banach algebras and the converse of these statements. We will show the following results for that purpose:

(i) Suppose that A and B are completely contractive Banach algebras such that $A\hat{\otimes}B$ is operator amenable. Then so are A and B.

(ii) Suppose that $A\hat{\otimes}B$ is (operator) approximately amenable and that there are elements b_0, c_0 , d_0 in B such that $b_0c_0 = c_0$ and $d_0b_0 = d_0$ and $c_0d_0 \neq 0$. Then A is also (operator) approximately amenable.

(iii) If $A\hat{\otimes}B$ is (operator) amenable and B has a bounded approximate identity, then A is (operator) approximately amenable.

Suppose that for the Banach algebras A and B, $A\hat{\otimes}B$ is weakly amenable and $Z_B(B^*) \neq \{0\}$. Then A is weakly amenable. In particular, if B is symmetrically amenable, A will be weakly amenable

Chapter 1

PRELIMINARIES

1.1 Banach algebras and amenability with its generalizations

Throughout this thesis, all our spaces are linear spaces over $\mathbb{C}$. For a normed space X, we denote the continuous dual of X by X^*.

A Banach algebra is an associative algebra over the field of complex numbers $\mathbb{C}$, that is, a Banach space with a norm $\|.\|$ such that

$$\|ab\| \leq \|a\|\|b\| \quad (a, b \in A).$$

A (bounded) net $(e_i)_i \subset A$ is called a left (right) approximate identity for A if

$$\lim_i e_i a = a \; (\lim_i a e_i = a) \;\; (a \in A).$$

A (bounded) net $(e_i) \subset A$ is called a two-sided (bounded) approximate identity for A if it is both left and right (bounded) approximate identity for A.

When A is a Banach algebra with an identity e such that $\|e\| = 1$, we call A a unital Banach algebra. We have the following elementary but useful theorem in the theory of Banach algebras

Theorem 1.1. *Suppose that A is a unital Banach algebra with identity e with $\|e\| = 1$ and $a \in A$ is such that $\|a - e\| < 1$. Then a is invertible.*

Definition 1.2. *Suppose that A is a Banach algebra and X is a Banach space which is also an A bi-module. We say that X is a Banach A-bimodule if there is a constant $C > 0$ such that*

$$\|a.x\| \leq C\|a\|\|x\|,$$

$$\|x.a\| \leq C\|a\|\|x\| \quad (a \in A, x \in X).$$

Suppose that X is a Banach A-bimodule. Then we can turn X^* into a Banach A-bimodule via the actions

$$\langle a.f, x\rangle = \langle f, x.a\rangle$$

$$\langle f.a, x\rangle = \langle f, a.x\rangle$$

$a \in A$, $f \in X^*$, $x \in X$. This is the so-called "dual module" of X.

Suppose that $F \in A^{**}$ and $f \in A^*$. We can use duality to define Ff as an element of A^* in the following way

$$\langle Ff, a\rangle = \langle F, f.a\rangle \quad (a \in A).$$

For $F, G \in A^{**}$ we define $F\Box G \in A^{**}$ by

$$\langle F\Box G, f\rangle = \langle F, Gf\rangle \quad (f \in A^*).$$

It can be shown that $\Box$ turns A^{**} into a Banach algebra, and $F\Box G$ is called the first Arens product of F and G. This product extends the product of A, as canonically embeded in A^{**}. From Goldstine's theorem, it can be easily seen that if A^{**} has a right identity with respect to the first Arens product, then A has a bounded right approximate identity.

Suppose that X, Y, Z are left, right or two-sided Banach A-modules and $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are bounded module morphisms such that $\Sigma : 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ is a short exact sequence. Σ is said to be admissible if there is a bounded linear map $F : Y \longrightarrow X$ such that $Ff = \mathrm{id_X}$ and splits if additionally F is an A-module morphism. Σ approximately splits if there is a net $(F_i)_i : Y \longrightarrow X$ of bounded left inverses of f such that $\lim_i a.F_i - F_i.a = 0 \ \ (a \in A)$, where

$$(a.F)(y) = a.F(y), (F.a)(y) = F(a.y) \quad (a \in A, y \in Y).$$

If additionally the net $(F_i)_i$ has the property that

$$\|a.F_i - F_i.a\| \leq M\|a\| \ \ (a \in A, i \in I),$$

for some $M > 0$, we say that Σ boundedly approximately splits. Suppose that A is a Banach algebra and X is a Banach A-bimodule. We say that a linear mapping $D : A \longrightarrow X$ is a derivation if

$$D(ab) = D(a).b + a.D(b) \quad (a, b \in A).$$

As an example if we fix $x \in X$ and define $\mathrm{ad}_x : A \longrightarrow X$ by

$$\mathrm{ad}_x(a) = a.x - x.a \quad (a \in A),$$

it can be easily seen that ad_x is a continuous derivation from A into X. This type of derivations is called an inner derivations. A derivation D is approximately inner if there exists a net $(x_i)_i \subset X$ such that

$$D(a) = \lim_i \mathrm{ad}_{x_i}(a) \quad (a \in A).$$

and is said to be boundedly approximately inner if furthermore for some constant $M > 0$

$$\|\mathrm{ad}_{x_i}(a)\| \leq M\|a\| \quad (a \in A, i \in I).$$

Now we define the notion of amenability.

Definition 1.3. *A Banach algebra A is said to be amenable if for every Banach A-bimodule X every continuous derivation D from A into the dual module X^* of X is inner.*

In a similar manner we can define the generalized notions of amenability.

Definition 1.4. *A Banach algebra A is said to be (boundedly) approximately amenable if every continuous derivation from A into any dual module X^* is (boundedly) approximately inner.*

Definition 1.5. *A Banach algebra A is said to be (boundedly) approximately contractible if every continuous derivation from A into any Banach A-bimodule X is (boundedly) approximately inner.*

Every boundedly approximately contractible Banach algebra has a bounded two-sided approximate identity [3, Corollary 3.4] whereas the boundedly approximately amenable Banach algebras might not have a two-sided approximate identity [12].

Suppose that A and B are Banach algebras. We define the projective tensor product of A and B, denoted by $A \otimes^{\gamma} B$, to be the completion of $A \otimes B$ under the norm

$$\|u\| = \inf\{\Sigma_{i=1}^{n}\|a_i\|\|b_i\| : u = \Sigma_{i=1}^{n} a_i \otimes b_i\}, \quad (u \in A \otimes B).$$

Definition 1.6. *For a Banach algebra A, the Banach algebra A^{op}, is the space A with product $\circ$, such that*

$$a \circ b = ba \quad (a, b \in A).$$

The mapping $\pi : A \otimes^{\gamma} A \longrightarrow A$ is specified by

$$\pi(a \otimes b) = ab \quad (a, b \in A),$$

is called the multiplication map of A and its kernel is denoted by K.

Definition 1.7. *A net $(m_i)_i \subset A \otimes^{\gamma} A$ is an approximate diagonal for A if*

$$\lim_i a.m_i - m_i.a = 0$$

$$\lim_i \pi(m_i).a = a,$$

for all $a \in A$. An element $M \in (A \otimes^{\gamma} A)^{**}$ is called a virtual diagonal if

$$a.M = M.a,$$

$$a.\pi^{**}(M) = a,$$

for $a \in A$.

There are several characterizations of amenability and its generalized notions which we briefly mention in the following theorems [19].

Theorem 1.8. *The following conditions are equivalent :*

(i) A is amenable;

(ii) A has a bounded approximate diagonal;

(iii) A has a virtual diagonal.

The flip mapping $q : A \otimes^\gamma A \longrightarrow A \otimes^\gamma A$ is specified by $q(a \otimes b) = b \otimes a \ \ (a, b \in A)$. We say that $u \in A \otimes^\gamma A$ is symmetric if $q(u) = u$. The following notion is also due to B. E. Johnson [21].

Definition 1.9. *The Banach algebra A is said to be symmetrically amenable if it has a bounded approximate diagonal comprised of symmetric elements.*

We can define an action $\circ$ of A on $A \otimes^\gamma A$ by

$$a \circ (b \otimes c) = b \otimes ac,$$

$$(b \otimes c) \circ a = ba \otimes c,$$

$a, b, c \in A$. We also define $\pi^\circ : A \otimes^\gamma A \longrightarrow A$ by

$$\pi^\circ(a \otimes b) = ba \quad (a, b \in A).$$

Theorem 1.10. *A is symmetrically amenable if and only if there is a bounded net $(m_i)_i \subset A \otimes^\gamma A$ satisfying conditions (i)-(iv) for all $a \in A$.*

(i): $\lim_i a.m_i - m_i.a = 0$

(ii): $\lim_i \pi(m_i)a = a$

(iii): $\lim_i a \circ m_i - m_i \circ a = 0$

(iv): $\lim_i a\pi^\circ(m_i) = a$.

Proof. See [21, Proposition 2.2]. $\square$

Some approximate notions of amenability have been proven to be equivalent. The following result is chosen from [12],[15].

Theorem 1.11. *The followings are equivalent :*

(i) A is approximately amenable.

(ii) A is approximately contractible.

In case of A having a central bounded approximate identity ,

(iii) A has an approximate diagonal.

It can be seen that the preceding theorem also holds for the generalized versions of operator amenability.

Let K be the kernel of the multiplication map $\pi : A \otimes^\gamma A^{op} \longrightarrow A$. The following result is a useful characterization of amenability that was done by P. C. Curtis and R. J. Loy in [4]

Theorem 1.12. *A is amenable if and only if A has a bounded approximate identity and K has a bounded right approximate identity or equivalently K^{**} has a right identity with respect to the first Arens product.*

Let $A^{\#}$ be the unitization of A and $\pi^{\#} : A^{\#} \otimes^{\gamma} A^{\#} \longrightarrow A^{\#}$ be the multiplication map. We set $K^{\#} = \ker(\pi^{\#})$.

The following characterization of approximate amenability is chosen from [10].

Theorem 1.13. *The Banach algebra A is approximately amenable if and only if $K^{\#}$ has a right approximate identity.*

Suppose that X is a Banach A-bimodule. We let $Z_A(X) := \{x \in X : a.x = x.a \quad (a \in A)\}$. The following results will be used several times throughout this thesis:

Theorem 1.14. *The following are equivalent :*

(i) The Banach algebra A is amenable.

(ii) For every Banach A-bimodule X and Y a closed submodule of X, every $f \in Z_A(Y^*)$ can be extended to an $\tilde{f} \in Z_A(X^*)$.

Proof. See [23, Theorem 1]. □

Theorem 1.15. *The following are equivalent :*

(i) The Banach algebra A is approximately amenable.

(ii) For every Banach A-bimodule X and Y a closed submodule of X, for every $f \in Z_A(Y^*)$, there is a net $(\tilde{f}_i)_i \subset X^*$ of extensions of f, such that $\lim_i a.\tilde{f}_i - \tilde{f}_i.a = 0 \quad (a \in A)$.

Proof. See [27, Theorem 3.11]. □

Theorem 1.16. *The followings are equivalent :*

(i) The Banach algebra A is boundedly approximately amenable.

(ii) For every Banach A-bimodule X and Y a closed submodule of X, for every $f \in Z_A(Y^*)$, there is a net $(\tilde{f}_i)_i \subset X^*$ of extensions of f, such that $\lim_i a.\tilde{f}_i - \tilde{f}_i.a = 0 \quad (a \in A)$. and $\|a.\tilde{f}_i - \tilde{f}_i.a\| \leq M\|a\| \quad (a \in A)$, for some $M > 0$.

Proof. See [27, Theorem 3.12] □

Amenability of Banach algebra and its generalizations can be characterized based on the splitting of short exact sequences.

Theorem 1.17. *The followings are equivalent:*

(i) The Banach algebra A is amenable.

(ii) For every Banach A-bimodules X, Y, Z the admissible short exact sequence $\Sigma : 0 \longrightarrow X^* \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ splits.

Proof. See [4, Theorem 2.5] □

Theorem 1.18. *The followings are equivalent:*

(i) The Banach algebra A is (boundedly) approximately amenable.

(ii) For everey Banach A-bimodules X, Y, Z the admissible short exact sequence $\Sigma : 0 \longrightarrow X^* \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ (boundedly) approximately splits.

Proof. See [10]. □

It is a known result that if A and B are amenable then so is $A \otimes^{\gamma} B$. The converse was established by F. Ghahramani and R. J. Loy in [9]. However, approximate amenability of A and B is not sufficient for approximate amenability of $A \otimes^{\gamma} B$. Some work on whether approximate amenability of $A \otimes^{\gamma} B$ implies approximate amenability of components is due to F. Ghahramani and R. J. Loy.

Theorem 1.19. *Suppose that $A \otimes^{\gamma} B$ is (boundedly) approximately amenable and there is a $b_0 \in B$ such that $b_0 \notin \overline{\mathrm{Lin}}\{bb_0 - b_0 b : b \in B\}$, then A is (boundedly) approximately amenable.*

Proof. See [9]. □

Theorem 1.20. *Suppose that $A \otimes^{\gamma} B$ is approximately amenable and A or B has a central bounded approximate identity. Then A and B are approximately amenable.*

Proof. See [8]. □

The Banach algebra A is said to be weakly amenable if every continuous derivation $D : A \longrightarrow A^*$ is inner. We have the following result for commutative Banach algebras A and B due to Groenbaek [17]:

Theorem 1.21. *Suppose that A and B are two weakly amenable commutative Banach algebras. Then $A \otimes^{\gamma} B$ is weakly amenable.*

1.2 Operator spaces and operator amenability

We recall the definition of the notion of the abstract operator spaces that we will be working on:

Definition 1.22. *Suppose that V is a linear space and we have a sequence $(\|.\|_n)_n$ of norms such that each $\|.\|_n : M_n(V) \longrightarrow [0, \infty)$ (where $M_n(V)$ is the space of $n \times n$ matrices with entries from V) that satisfies*

$M1. \|v \oplus w\|_{m+n} = \max\{\|v\|_n, \|w\|_m\}$

$M2. \|\alpha v \beta\|_m \leq \|\alpha\| \|v\|_n \|\beta\|,$

for $v \in M_n(V)$,$w \in M_m(V)$, $\alpha \in \mathbb{M}_{m,n}$, $\beta \in \mathbb{M}_{n,m}$. (where $\|\alpha\|$ is the operator norm when α is considered as a bounded operator from $\mathbb{C}^n \longrightarrow \mathbb{C}^m$ and so is $\|\beta\|$ when β is regarded as a bounded operator $\mathbb{C}^n \longrightarrow \mathbb{C}^m$ and $v \oplus w = [\mu_{i,j}]$, $\mu_{i,j} = v_{i,j}, 1 \leq i, j \leq n$, $\mu_{i,j} = w_{i-n,j-n}, n + 1 \leq i, j \leq m + n$, $\mu_{i,j} = 0$ otherwise.).
Then $(\|.\|_n)_n$ determines an operator space structure on V.

If we replace $M1$ with $M1' : \|v \oplus w\|_{m+n} \leq \max\{\|v\|_n, \|w\|_m\}$, we still get the same notion of an operator spaces

The following useful equality is due to E. G. Effros Z. J. Ruan [6] :

Proposition 1.23. *Suppose that V is an abstract operator space and $v \in M_n(V)$ and*

$\alpha \in M_m$. *Then we have*

$$\|v \otimes \alpha\|_{mn} = \|\alpha \otimes v\|_{mn} = \|v\|_n \|\alpha\|.$$

Suppose that V and W are abstract operator spaces and $\psi : V \longrightarrow W$ is a linear mapping. For each $n \in \mathbb{N}$, we define a mapping $\psi_n : M_n(V) \longrightarrow M_n(W)$ by

$$\psi_n([v_{i,j}]) = [\psi(v_{i,j})] \;\; ([v_{i,j}] \in M_n(V)).$$

The mapping ψ is said to be completely bounded if $\sup\{\|\psi_n\| : n \in \mathbb{N}\} < \infty$. When the supremum is finite it is called the completely bounded norm of ψ and is denoted by $\|\psi\|_{cb}$. The space of all completely bounded linear mappings from V into W is denoted by $CB(V, W)$.

Proposition 1.24. *Let V be an abstract operator space and $\psi : V \longrightarrow M_n$ be a linear map. Then*

$$\|\psi\|_{cb} = \|\psi\|_n.$$

Proof. See [6]. □

Theorem 1.25. *Let V be an abstract operator space and A be a commutative C^*-algebra. Then every bounded linear map $\psi : V \longrightarrow A$ is completely bounded and*

$$\|\psi\|_{cb} = \|\psi\|.$$

Proof. See [6]. $\square$

Suppose that V is an abstract operator space and $f = [f_{i,j}] \in M_n(V^*)$. Then we associate to f a linear mapping from V into M_n via

$$f(v) = [f_{i,j}(v)] \quad (v \in V).$$

Using this we can establish a linear isomorphism between $M_n(V^*)$ and $CB(V, M_n)$. Thus we can define the norm on $M_n(V^*)$ to be the corresponding norm from $CB(V, M_n)$. Let $f \in M_n(V^*)$ we have

$$\begin{aligned} \|f\| &= \sup\{\|f_n(v)\| : v \in M_n(V), \|v\| \leq 1\} \\ &= \sup\{\|\langle\langle f, v\rangle\rangle\| : v \in M_n(V), \|v\| \leq 1\}. \end{aligned}$$

Suppose that V and W are abstract operator spaces and $u \in M_n(V \otimes W)$. Let

$$\|u\|_\wedge := \inf\{\|\alpha\|\|v\|_p\|w\|_q\|\beta\| : \alpha \in \mathbb{M}_{n,pq}, v \in M_p(V), w \in M_q(W), \beta \in \mathbb{M}_{pq,n}, u = \alpha(v \otimes w)\beta\}.$$

In [5] it is shown that $\|.\|_\wedge$ defines an operator norm and we denote the completion of $V \otimes W$ under the norm $\|.\|_\wedge$ by $V \hat{\otimes} W$. As in Banach space case, $\|.\|_\wedge$ defines a cross matrix norm on $V \otimes W$ i.e.

$$\|v \otimes w\|_\wedge = \|v\|\|w\| \quad (v \in M_p(V), w \in M_q(W)).$$

From now on for notational convenience and for distinguishing between the projective tensor product and operator space projective tensor product, we denote the Banach space projective tensor product by $V \otimes^{\gamma} W$ and the operator space tensor product by $V \hat{\otimes} W$. The following useful completely isometric identifications for the operator spaces V, W, X is shown in [6].

$$CB(V \hat{\otimes} W, X) \cong CB(V \times W, X) \cong CB(V, CB(W, X)).$$

Definition 1.26. *Suppose that A is a Banach algebra that is also an operator space. We say that A is a completely contractive Banach algebra if the multiplication map $\pi : A \hat{\otimes} A \longrightarrow A$ is completely contractive. i.e. $\|\pi\|_{cb} \leq 1$. In other words we have*

$$\|[a_{i,j} b_{l,k}]\| \leq \|[a_{i,j}]\| \|[b_{l,k}]\| \quad ([a_{i,j}] \in M_n(A), [b_{l,k}] \in M_m(A), m, n \in \mathbb{N}).$$

Let X be a Banach A-bimodule which is an operator space. We say that X is an operator A-bimodule if the operators

$$\rho_l : A \times X \longrightarrow X : \ \ (a, x) \longmapsto a.x,$$

and

$$\rho_r : X \times A \longrightarrow X : \ \ (x, a) \longmapsto x.a,$$

are completely bounded. In other words if there is a constant $C > 0$ such that

$$\|[a_{i,j}.x_{k,l}\|, \|[x_{k,l}.a_{i,j}]\| \leq C\|[a_{i,j}]\| \|[x_{k,l}]\| \quad ([a_{i,j}] \in M_n(A), [x_{k,l}] \in M_m(X), n, m \in \mathbb{N}).$$

As an example, $A \hat{\otimes} A$ is an operator A-bimodule via the actions

$$a.(b \otimes c) = ab \otimes c$$

$$(b \otimes c).a = b \otimes ca \quad (a, b, c \in A).$$

Let X be an operator A-bimodule and X^* be the dual module of X in the canonical way. It is known that X^* is also an operator A-bimodule.

Now we can define the notion of operator amenability.

Definition 1.27. *Suppose that A is a completely contractive Banach algebra. We say that A is operator amenable if every completely bounded derivation $D : A \longrightarrow X^*$ is inner for every operator Banach A-bimodule X.*

A bounded net $(m_\alpha) \subset A\hat{\otimes}A$ is said to be a bounded (Operator) approximate diagonal for A if

$$\lim_\alpha \|a.m_\alpha - m_\alpha.a\|_\wedge = 0$$

$$\lim_\alpha \|\pi(m_\alpha)a - a\| = 0$$

An element $M \in (A\hat{\otimes}A)^{**}$ is said to be an (operator) virtual diagonal if

$$a.M = M.a$$

$$\pi^{**}(M).a = a \quad (a \in A).$$

We have the following characterization for the operator amenability:

Theorem 1.28. *The followings are equivalent for a completely contractive Banach algebra A:*

(i) A is operator amenable;

(ii) A has a bounded (operator) approximate diagonal;

(iii) A has an (operator) virtual diagonal.

Proof. See [28]. □

1.3 Fourier and Fourier Stieltjes algebras

Suppose that G is a locally compact group and H_π is a Hilbert space. A mapping $\pi : G \longrightarrow B(H_\pi)$ is called a unitary representation of G if

$$\pi(xy) = \pi(x)\pi(y)$$

$$\pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$$

The unitary representation $\pi : G \longrightarrow B(H_\pi)$ is said to be a continuous representation if for any $u \in H_\pi$, the mapping $x \longmapsto \pi(x)u$ from G into H_π is continuous.

We denote the set of all continuous unitary representations of G by Σ. We let

$$B(G) := \{\phi : G \longrightarrow \mathbb{C} : \phi(x) = \langle \pi(x)\zeta | \mu \rangle, \pi \in \Sigma, \zeta, \mu \in H_\pi, x \in G\}.$$

$B(G)$ is a Banach algebra under point-wise multiplication and the norm inherited from identifying with the dual of $C^*(G)$ [7] , where $C^*(G)$ is the completion of $L^1(G)$ under the norm

$$\|f\| = \sup\{\|\pi(f)\| : \pi \in \Sigma\} \quad (f \in L^1(G)).$$

This identification is via

$$\langle \phi, f \rangle = \int f(x)\phi(x)dx \quad (\phi \in B(G), f \in L^1(G)).$$

Thus we can define the norm on $B(G)$ as follows:

$$\|\phi\|_{B(G)} = \sup\{|\int \phi(x)f(x)dx| \quad (f \in L^1(G), \|f\|_{C^*(G)} \leq 1)\}.$$

$B(G)$ is called the Fourier-Stieltjes algebra of the locally compact group G. Let $\lambda : G \longrightarrow B(L^2(G))$ be the left regular representation of G i.e.

$$\lambda(x)(u) = L_x u, \quad (u \in L^2(G)),$$

where $L_x u(t) = u(x^{-1}t) \quad (t \in G)$.

We denote the set of all coefficient functions that are weakly contained in the left regular representation of G by $B_\lambda(G)$.

We define the norm of $B_\lambda(G)$ to be the dual norm of $C^*_\lambda(G)$. So

$$\|\phi\|_{B_\lambda(G)} = \sup\{|\int f(x)\phi(x)dx| : f \in L^1(G), \|\lambda(f)\| \leq 1\}.$$

It is known that $B(G)$ is a commutative Banach algebra with respect to point-wise additions and multiplication.

Let E be the linear span of $C_c(G) \bigcap B(G)$ and we take the closure of E in $B_\lambda(G)$. This closure is denoted by $A(G)$ and is called the Fourier algebra of G. It is shown in [7], that

$$A(G) = \{f * \check{g} : f, g \in L^2(G)\}$$

$$= \{\langle \lambda(.)\zeta|\mu\rangle : \quad \zeta, \mu \in L^2(G)\}.$$

where $\check{g}(x) = g(x^{-1})$ $(x \in G)$. The norm of $u \in A(G)$ can be obtained by

$$\|u\| = \inf\{\|f\|\|g\| : u = f * \check{g}, f, g \in L^2(G)\}.$$

Indeed this infimum will be attained [7]. It is known that $A(G)$ is a commutative Banach algebra under point-wise addition and multiplication and is a closed ideal in $B(G)$. When G is an abelian group it is proven in [7] that $A(G) \cong L^1(\hat{G})$, where $\hat{G}$ is the dual group of G consisting of continuous characters on G.

For a locally compact group G, we define the von Neumann algebra of G denoted by $VN(G)$ as

$$VN(G) := \{\lambda(x) : x \in G\}'' \subset B(L^2(G)),$$

where $''$ stands for the double commutant.

It is well-known that $VN(G)$ can be identified as the dual of $A(G)$ using the following theorem.

Theorem 1.29. *Let $\psi \in A(G)^*$. There is a unique $T_\psi \in VN(G)$ such that*

$$\langle T_\psi(f)|g\rangle = \langle \psi, \bar{g} * \check{f}\rangle \ \ (f, g \in L^2(G)).$$

The mapping $\psi \longmapsto T_\psi$ defines a linear isometery from $A(G)^*$ onto $VN(G)$.

Proof. See [7]. □

Chapter 2

ON GENERALIZED OPERATOR AMENABILITY NOTIONS AND APPLICATION FOR THE FOURIER ALGEBRAS

2.1 Generalized notions of operator amenability for the Fourier algebras

Definition 2.1. *Suppose that A is a completely contractive Banach algebra. We say that A is operator boundedly approximately contractible if for every operator A-bimodule X and every completely bounded derivation $D : A \longrightarrow X$, there is a net $(u_i)_{i\in I} \subset X$ and a constant $M > 0$ such that*

$$\lim_i a.u_i - u_i.a = D(a) \quad (a \in A) \ \text{and}$$

$$\|[a_{s,t}].u_i - u_i.[a_{s,t}]\|_n \leq M\|[a_{s,t}]\|_n \quad ([a_{s,t}] \in M_n(A), n \geq 1, i \in I),$$

where $[a_{s,t}].u = [a_{s,t}.u] \quad ([a_{s,t}] \in M_n(A), u \in X)$.

From now on A is a completely contractive Banach algebra unless otherwise is stated.

Definition 2.2. *We say that the net $(u_i) \subset A$ is a left-operator-multiplier-bounded approximate identity for A, if there is a constant $M > 0$ such that*

$$\lim_i u_i.a = a \quad (a \in A);$$

$$\|u_i.[a_{s,t}]\|_n \leq M\|[a_{s,t}]\|_n \quad ([a_{s,t}] \in M_n(A), n \geq 1).$$

We can turn $A^{\#}$ into a completely contractive Banach algebra by considering it as a subspace of $(A^* \overset{\infty}{\oplus} \mathbb{C})^*$ (where $\overset{\infty}{\oplus}$ denotes the l^∞ direct sum). It can be easily seen that for $[a_{i,j} \oplus \lambda_{i,j}] \in M_n(A^{\#})$,we have $\|[a_{i,j}]\|_n, \|[\lambda_{i,j}]\|_n \leq \|[a_{i,j} \oplus \lambda_{i,j}]\|_n$. Since

$$\begin{aligned}
\|[a_{i,j} \oplus \lambda_{i,j}]\| &= \sup\{\|\langle\langle[a_{i,j} \oplus \lambda_{i,j}], [f_{k,l} \oplus \gamma_{k,l}]\rangle\rangle\| : \|[f_{k,l} \oplus \gamma_{k,l}]\|_n \leq 1\} \\
&= \sup\{\|\langle\langle[a_{i,j}, [f_{k,l}]\rangle\rangle + \langle\langle[[\lambda_{i,j}], [\gamma_{k,l}]\rangle\rangle\| : \|[f_{k,l} \oplus \gamma_{k,l}]\|_n \leq 1\} \\
&\geq \sup\{\|\langle\langle[a_{i,j}], [f_{k,l}\rangle\rangle\| : \|f_{k,l}]\|_n \leq 1\} \\
&= \|[a_{i,j}]\|_n
\end{aligned}$$

And similarly

$$\|[a_{i,j} \oplus \lambda_{i,j}]\|_n \geq \|[\lambda_{i,j}]\|_n.$$

Proposition 2.3. *Suppose that A is operator boundedly approximately contractible. Then A has a right (and a left) operator-multiplier-bounded approximate identity.*

Proof. The completely contractible Banach algebra A can be considered as an operator A-bimodule via the actions

$$a.x = ax,$$

$$x.a = 0, \quad (a, x \in A).$$

The identity operator $i : A \longrightarrow A$ is a completely contractive derivation for these actions. Hence from our assumption there exists a net $(u_\alpha)_{\alpha\in\Lambda} \subset A$ and a constant $M > 0$ such that

$$a = i(a) = \lim_\alpha a.u_\alpha - u_\alpha.a = \lim_\alpha au_\alpha, \quad (a \in A)$$

and

$$\|(\mathrm{ad}_{u_\alpha})_n([a_{k,l}])\|_n = \|[a_{k,l}.u_\alpha]\|_n \leq M\|[a_{k,l}]\|_n \quad (\alpha \in \Lambda, n \in \mathbb{N}, [a_{k,l}] \in M_n(A)).$$

Therefore the net (u_α) is an operator-multiplier-bounded approximate identity for A . Similar argument holds for the existence of a left operator-multiplier-bounded approximate identity. $\square$

Y. Choi et al proved in [3] that a boundedly approximate contractible Banach algebra has a bounded approximate identity. We will now show that the same conclusion holds if one merely assumes that the Banach algebra is operator boundedly approximately contractible. Our proof, partly uses the same idea as in the proof of [3,Theorem 3.3].

Theorem 2.4. *Suppose that A is operator boundedly approximately contractible . Then A has a two-sided bounded approximate identity.*

Proof. From Proposition 2.3, we know that A has a right operator multiplier bounded and a left operator multiplier bounded approximate identity say (f_β) and (g_α) respectively. In the following we are going to show the existence of a bounded right approximate identity for A.

There is a constant $M > 0$ such that for all $[a_{s,t}] \in M_p(A)$, $(p \geq 1)$, and every index β, we have $\|[a_{s,t}].f_\beta\|_p \leq M\|[a_{s,t}]\|_p$. So if $\lambda \in \mathbb{M}_{1,pq}$, $[a_{s,t}] \in M_p(A)$, $[b_{k,l}] \in M_q(A)$, $\gamma \in \mathbb{M}_{pq,1}$, for $m = \lambda([a_{s,t}] \otimes [b_{k,l}])\gamma$ we have

$$\|m.f_\beta\|_\wedge = \|\lambda([a_{s,t}] \otimes [b_{k,l}].f_\beta)\gamma\|$$

$$\leq \|\lambda\|\|[a_{s,t}]\|_p\|[b_{k,l}]_q.f_\beta\|\|\gamma\|$$

$$\leq M\|\lambda\|\|[a_{s,t}]\|_p\|[b_{k,l}]\|_q\|\gamma\|$$

So that for every β,

$$\|m.f_\beta\|_\wedge \leq M\|m\|_\wedge \qquad (m \in A\hat{\otimes}A).$$

And similarly for every α,

$$\|g_\alpha.m\|_\wedge \leq M\|m\|_\wedge \qquad (m \in A\hat{\otimes}A).$$

Suppose that the net (f_β) is not bounded and suppose that the nets (m_i),(ϕ_i), (ψ_i) are as in Theorem 2.5 of [3] (by considering the operator case). Then, by letting $a = f_\beta$ in Theorem 2.5 of [3] , we have

$$\|g_\alpha(f_\beta.m_i - m_i.f_\beta - f_\beta \otimes \psi_i + \phi_i \otimes f_\beta)\| \leq CM\|f_\beta\|.$$

Therefore

$$\|g_\alpha\phi_i\|\|f_\beta\| \leq CM\|f_\beta\| + \|g_\alpha f_\beta.m_i\|_\wedge + \|g_\alpha.m_i.f_\beta\|_\wedge + \|g_\alpha f_\beta\|\|\psi_i\|$$

And hence

$$\|g_\alpha\phi_i\| \leq CM + \frac{M}{\|f_\beta\|}(\|g_\alpha\|\|m_i\| + \|g_\alpha.m_i\|_\wedge + \|g_\alpha\|\|\psi_i\|).$$

Since (f_β) is unbounded, by taking limit as $\beta \longrightarrow \infty$, we have

$$\|g_\alpha\phi_i\| \leq CM.$$

Now letting $i \longrightarrow \infty$, we get for every α that

$$\|g_\alpha\| \leq CM.$$

Therefore for every β,

$$\|f_\beta\| = \lim_\alpha \|g_\alpha f_\beta\| \leq \limsup_\alpha M\|g_\alpha\| \leq CM^2,$$

which contradicts our assumption. So the net (f_β) is bounded and similarly the net (g_α) is bounded. Therefore A has a bounded left and a bounded right approximate identity and hence it has a bounded two-sided approximate identity.

□

The following theorem can be proved in a similar way to [16, Proposition 2.1].

Theorem 2.5. *Let A be operator amenable and $D : A \longrightarrow X$ be a completely bounded derivation from A into the operator A-bimodule X. Then there is a bounded net $(u_i)_{i\in I} \subset X$ such that $D(a) = \lim_i a.u_i - u_i.a \quad (a \in A)$.*

Corollary 2.6. *Suppose that A is operator amenable. Then A is operator boundedly approximately contractible.*

Proof. Suppose that X is any operator A-bimodule and $D : A \longrightarrow X$ is a completely bounded derivation.

Let (u_i) be the net as in the preceding theorem. It suffices to show that there is $M > 0$ such that $\|[a_{s,t}].u_i - u_i.[a_{s,t}]\|_n \leq M\|[a_{s,t}]\|_n \quad ([a_{s,t}] \in M_n(A), n \geq 1, i \in I)$.

Since X is an operator A-bimodule, there is a constant $C > 0$ such that for every $[a_{s,t}] \in M_n(A)$ we have

$$\|[a_{s,t}.x\|_n, \|[x.a_{s,t}]\|_n \leq C\|[a_{s,t}]\|_n\|x\|, \quad (x \in X).$$

Also suppose that $\|u_i\| \leq K$ for some $K > 0$. Therefore

$$\|[a_{s,t}].u_i - u_i.[a_{s,t}]\|_n \leq 2C\|[a_{s,t}\|_n\|u_i\| \leq 2CK\|[a_{s,t}]\|_n, \ \ (n \geq 1).$$

This completes the proof. □

The following theorem is due to Ruan [28].

Theorem 2.7. *Let G be a locally compact group. Then $A(G)$ is operator amenable if and only if G is amenable.*

The following theorem is due to Leptin [26].

Theorem 2.8. *Let G be a locally compact group. Then $A(G)$ has a bounded approximate identity if and only if G is amenable.*

We now give an alternative proof of a result of F. Ghahramani and R. Stokke [14] that strengthens Ruan's result.

Theorem 2.9. *Let G be a locally compact group. Then the followings are equivalent:*

(i) $A(G)$ is operator amenable.

(ii) $A(G)$ is operator boundedly approximately contractible.

(iii) G is amenable.

Proof. $(i) \Longrightarrow (ii)$: It is clear from the Corollary 2.6.

$(ii) \Longrightarrow (iii)$: Since $A(G)$ is operator boundedly approximately contractible, from Theorem 2.4, it has a bounded approximate identity and whence from Theorem 2.8, G is amenable.

$(iii) \Longrightarrow (i)$: This follows from Theorem 2.7. (Ruan's Theorem) □

2.2 Some generalized operator amenability notions and their characterizations

Throughout this section, all the tensor products are operator space projective tensor products. In the next Theorem , $\pi^{\#} : A^{\#}\hat{\otimes}(A^{\#})^{op} \longrightarrow A^{\#}$ is the multiplication map and $K^{\#}$ is kernel of $\pi^{\#}$.

Theorem 2.10. *Let A be a completely contractive Banach algebra. Then A is operator amenable if and only if $K^{\#}$ has a bounded right approximate identity.*

Proof. $\Rightarrow$: Suppose that A is operator amenable. We define the completely bounded derivation $D : A \longrightarrow K^{\#}$ by

$$D(a) = a \otimes 1 - 1 \otimes a \quad (A \in A).$$

So From Theorem 2.5, there is a bounded net $(u_i)_{i\in I} \subset K^{\#}$ such that $\|u_i\| < M, \quad (i \in I)$ and $D(a) = \lim_i a.u_i - u_i.a \ \ (a \in A)$.

Let $k \in K^{\#}$. We fix ϵ, $0 < \epsilon < \frac{\|k\|}{2}$. There exists $v = \alpha([a_{s,t}]\otimes[b_{k,l}])\beta$ such that $\alpha \in \mathbb{M}_{1,pq}$, $[a_{s,t}] \in M_p(A^{\#})$, $[b_{k,l}] \in M_q(A^{\#})$, $\beta \in \mathbb{M}_{pq,1}$, $\|k - v\| < \epsilon$ and $\|\alpha\|\|[a_{s,t}]\|\|[b_{k,l}]\|\|\beta\| <$

$\|v\| + \epsilon$. Therefore

$$\|ku_i - k\| \leq \|vu_i - v\| + \|(k-v)u_i - (k-v)\|$$

$$< \|vu_i - \alpha[D(a_{s,t}).b_{k,l}]\beta + \alpha[D(a_{s,t}).b_{k,l}]\beta - v\| + (M+1)\epsilon$$

$$\leq \|vu_i - u_i.\pi^{\#}(v) - \alpha[D(a_{s,t}).b_{k,l}]\beta + \alpha[D(a_{s,t}).b_{k,l}]\beta - v\|$$

$$+ \|u_i.\pi^{\#}(v)\| + (M+1)\epsilon$$

$$< \|\alpha[a_{s,t}.u_i.b_{k,l}]\beta - \alpha[u_i.a_{s,t}b_{k,l}]\beta - \alpha[D(a_{s,t}).b_{k,l}]\beta\|$$

$$+ \|\alpha[D(a_{s,t}).b_{k,l}]\beta - v\| + (2M+1)\epsilon \quad (\text{Since}\|\pi^{\#}(v)\| < \epsilon.)$$

$$= \|\alpha[(a_{s,t}.u_i - u_i.a_{s,t} - D(a_{s,t})).b_{k,l}]\beta\| + \|\alpha[D(a_{s,t}).b_{k,l}]\beta - v\|$$

$$+ (2M+1)\epsilon. \qquad (1)$$

We have :

$$\|\alpha[(a_{s,t}.u_i - u_i.a_{s,t} - D(a_{s,t})).b_{k,l}]\beta\| \leq \|\alpha\|\|[a_{s,t}.u_i - u_i.a_{s,t}] - D_p([a_{s,t}])\|\|[b_{k,l}]\|\|\beta\|.$$

Now let $i_0 \in I$ be such that for all $i \geq i_o$,

$$\|[a_{s,t}.u_i - u_i.a_{s,t} - D(a_{s,t})]\|_p < \epsilon \frac{\|[a_{s,t}]\|_p}{\|v\|}. \qquad (2)$$

Also we have

$$\alpha[D(a_{s,t}).b_{k,l}]\beta = \alpha[(a_{s,t} \otimes 1 - 1 \otimes a_{s,t}).b_{k,l}]\beta$$

$$= \alpha([a_{s,t}] \otimes [b_{k,l}])\beta - 1 \otimes (\alpha[a_{s,t}b_{k,l}]\beta)$$

$$= \alpha([a_{s,t}] \otimes [b_{k,l}])\beta - 1 \otimes \pi^{\#}(v)$$

$$= v - 1 \otimes \pi^{\#}(v) \qquad (3)$$

Therefore, from (1), (2), (3), for $i \geq i_0$ we have ,

$$\begin{aligned}
\|ku_i - k\| &< \epsilon \frac{\|[a_{s,t}]\|}{\|v\|} \|\alpha\| \|[b_{k,l}]\| \|\beta\| + \|\pi(v)\| + (2M+1)\epsilon \\
&< \epsilon + \frac{\epsilon^2}{\|v\|} + (2M+2)\epsilon \\
&< (2M + 3 + \frac{\epsilon}{\|k\| - \epsilon})\epsilon < (2M+4)\epsilon
\end{aligned}$$

Therefore by taking F a finite subset of K, there is $u_{F,\epsilon} \in (u_i)_{i \in I}$ such that

$$\|ku_{F,\epsilon} - k\| < \epsilon, \quad (k \in F).$$

Hence there is a bounded net $(r_j) \subset K$ such that $\lim_j kr_j = k \quad (k \in K)$.

For the converse, suppose that the net (r_i) is a bounded right approximate identity for $K^{\#}$.

Then we have $\lim_i kr_i = k \quad (k \in K^{\#})$.

Let $a \in A$ and let $k = a \otimes 1 - 1 \otimes a$. Then we have $kr_i = a.r_i - r_i.a$. Therefore we have

$$\lim_i a.r_i - r_i.a = a \otimes 1 - 1 \otimes a \quad (a \in A).$$

So if we put $m_i = 1 \otimes 1 - r_i$ we have

$$\pi(m_i) = 1$$

$$\lim_i a.m_i - m_i.a = 0 \quad (a \in A).$$

Therefore the net (m_i) is a bounded approximate diagonal for $A^{\#}$ and hence from Theorem 1.28 A is operator amenable. $\square$

Definition 2.11. *We say that A is uniformly operator approximately amenable if for every operator Banach A-bimodule X and for every completely bounded derivation $D : A \longrightarrow X^*$ there is a net $(\phi_i) \subset X^*$ such that $\lim_i \|ad_{\phi_i} - D\|_{cb} = 0$.*

Similarly we can define uniform operator approximate contractiblity.

Lemma 2.12. *Suppose that A is a completely contractive Banach algebra with π its completely contractive multiplication map with $K = \ker(\pi)$. Suppose that A has a bounded left (or right) approximate identity and $k \in K$. Then for any $\epsilon > 0$, there are k_1 and k_2 such that $k = k_1 + k_2$ and $k_1 \in (A \otimes A) \cap K$ and $\|k_2\|_\wedge < \epsilon$. (Where $\|.\|_\wedge$ denotes the operator space projective tensor product.)*

Proof. Let $(e_i)_{i\in I}$ be the bounded left approximate identiy and $M > 1$ be such that $\|e_i\| \leq M \quad (i \in I)$. We can find $r \in A \otimes A$ such that $\|k - r\|_\wedge < \frac{\epsilon}{4M}$. By Cohen's factorization theorem , there are a and b in A such that $\|\pi(k-r)-b\| < \frac{\epsilon}{4M}$, $\|a\| \leq M$ and $\pi(k - r) = ab$. We let $k_2 = k - r - a \otimes b$ and $k_1 = r + a \otimes b$. Then $\|k_2\|_\wedge < \frac{\epsilon}{4M} + \frac{\epsilon}{2} < \epsilon$. Therefore k_1 and k_2 are desired ones. $\square$

The following theorem is the operator space version of [11, Theorem 3.1].

Theorem 2.13. *The completely contractive Banach algebra A is operator amenable if and only if it is uniformly operator approximately amenable.*

Proof. Since both notions of operator amenability and operator uniformly amenability hold for A if and only if they hold for $A^{\#}$, we can assume that A is unital.

Let $u = \acute{\alpha}([\acute{a}_{s,t}] \otimes [\acute{b}_{k,l}])\acute{\beta} \in A \otimes A^{op}$, where $\acute{\alpha} \in M_{1,pq}$,$[\acute{a}_{s,t}] \in M_p(A)$, $[\acute{b}_{k,l}] \in M_q(A)$, $\acute{\beta} \in M_{pq,1}$. For an elementary tensor $c \otimes d \in A \otimes A^{op}$, we have

$$(c \otimes d)u = \acute{\alpha}([c\acute{a}_{s,t}] \otimes [\acute{b}_{k,l}d])\acute{\beta} = c.u.d.$$

By taking limit, it's easy to see that the above equality holds for $u \in A\hat{\otimes}A^{op}$ as well. Now take $s = \alpha([a_{s,t}] \otimes [b_{k,l}])\beta \in A \otimes A^{op}$. Then we have

$$su = \alpha[a_{s,t}.u.b_{k,l}]\beta \quad (u \in A\hat{\otimes}A^{op}).$$

In particular the above equality holds for $k \in K \cap (A \otimes A^{op})$ and $u \in K$ (where $K = \ker \pi$.) Now we take $f \in K^*$. Then

$$\begin{aligned}\langle f, ku\rangle &= \langle f, \alpha[a_{s,t}.u.b_{k,l}]\beta\rangle \\ &= \alpha f_{pq}([a_{s,t}.u.b_{k,l}])\beta\end{aligned}$$

Let $h \in (K^{**})$. Then by Goldstein's Theorem, there is a net $(h_i) \subset K$, such that $\mathrm{wk}^* - \lim_i h_i = h$.

We define the bounded map $\psi : K^{**} \longrightarrow K^{**}$ by

$$\psi(h) = \alpha[a_{s,t}.h.b_{k,l}]\beta.$$

ψ is $\mathrm{wk}^* - \mathrm{wk}^*$ continuous since

$$\lim_i \langle \psi(h_i - h), f\rangle = \lim_i \alpha f_{pq}([a_{s,t}.(h_i - h).b_{k,l}])\beta = 0.$$

Therefore given $u = \alpha([a_{s,t}] \otimes [b_{k,l}])\beta \in A \otimes A^{op}$, we have

$$uh = \mathrm{wk}^* - \lim_i uh_i = \mathrm{wk}^* - \lim_i \psi(h_i) = \psi(h) = \alpha[a_{s,t}.h.b_{k,l}]\beta.$$

Now we fix $\epsilon > 0$ and take $k \in K$, $h \in K^{**}$. From Lemma 2.12, there exists an element $s = s_{h,k} \in K$ such that $s = \alpha([a_{s,t}] \otimes [b_{k,l}])\beta \in A \otimes A^{op}$ such that $\|k - s\| < \epsilon$, $\|kh - sh\| < \epsilon$. So we have

$$\begin{aligned} \|kh - k\| &\leq \|sh - s\| + \|s - k\| + \|sh - kh\| \\ &< \|sh - s\| + 2\epsilon \qquad (1) \end{aligned}$$

Since $\pi(s) = \alpha[a_{s,t}b_{k,l}]\beta = 0$ and $\|\pi^{**}(hs)\| = \|\pi^{**}(hs - hk)\| < \epsilon$, we have

$$\begin{aligned} \|sh - s\| &= \|\alpha[a_{s,t}.h.b_{k,l}]\beta - \alpha([a_{s,t}] \otimes [b_{k,l}])\beta\| \\ &= \|\alpha[a_{s,t}.h.b_{k,l}]\beta - \alpha[h.a_{s,t}b_{k,l}]\beta + \alpha[1 \otimes a_{s,t}b_{k,l}]\beta - \alpha[(a_{s,t} \otimes 1).b_{k,l}]\beta\| \\ &\leq \|\alpha[(a_{s,t}.h - h.a_{s,t} + 1 \otimes a_{s,t} - a_{s,t} \otimes 1).b_{k,l}]\beta\| \\ &< \|\alpha\|\|[a_{s,t}.h - h.a_{s,t} + 1 \otimes a_{s,t} - a_{s,t} \otimes 1]\|\|[b_{k,l}]\|\|\beta\| \\ &\leq \|\alpha\|\|[b_{k,l}]\|\|\beta\| \sup\{\|[a_{s,t}.h - h.a_{s,t} + 1 \otimes a_{s,t} - a_{s,t} \otimes 1]\|_p : [a_{s,t}] \in (M_p(A))_1, p \geq 1\} \\ &< (\|s\| + \epsilon) \sup\{\|[a_{s,t}.h - h.a_{s,t} + 1 \otimes a_{s,t} - a_{s,t} \otimes 1]\|_p : [a_{s,t}] \in (M_p(A))_1, p \geq 1\}, \quad (2) \end{aligned}$$

where $(M_p(A))_1$ denotes the unit ball of $M_p(A)$. We define the completely bounded derivation $D : A \longrightarrow K^{**}$ by

$$D(a) = a \otimes 1 - 1 \otimes a \quad (a \in A).$$

Since A is uniformly operator approximately amenable, there is a net $(\phi_i)_{i\in I} \subset K^{**}$ such that $\lim_i \|ad_{\phi_i} - D\|_{cb} = 0$. Therefore there is $i_0 \in I$ such for $i \geq i_0$,

$$\|[a_{s,t}.\phi_i - \phi_i.a_{s,t} + 1 \otimes a_{s,t} - a_{s,t} \otimes 1]\|_p < \epsilon, \quad ([a_{s,t}] \in (M_p(A))_1, p \geq 1). \quad (3)$$

Then for $i \geq i_0$ we have

$$\begin{aligned}
\|k\phi_i - k\| &< \|s\phi_i - s\| + 2\epsilon \qquad\qquad \text{(from (1).)}\\
&\leq (\|s\| + \epsilon) \sup\{\|[a_{s,t}.\phi_i - \phi_i.a_{s,t} + 1 \otimes a_{s,t} - a_{s,t} \otimes 1]\|_p : [a_{s,t}] \in (M_p(A))_1, p \geq 1\}\\
&+ 2\epsilon \qquad\qquad \text{(from(2))}\\
&< (\|k\| + 2\epsilon)\epsilon + 2\epsilon \qquad\qquad \text{(from (3))}\\
&= \|k\|\epsilon + 2\epsilon^2 + 2\epsilon.
\end{aligned}$$

Let $h \in K^{**}$. There is a net $(h_j) \subset K$, bounded by $\|h\|$, such that $\mathrm{wk}^* - \lim_j h_j = h$. Then since $h\phi_i - h = \mathrm{wk}^* - \lim_{\mathrm{j}} \mathrm{h_j}\phi_{\mathrm{i}} - \mathrm{h_j}$, from the preceding inequality we have

$$\|h\phi_i - h\| \leq \|h\|\epsilon + O(\epsilon),$$

where $\lim_{\epsilon \longrightarrow 0^+} O(\epsilon) = 0$.

Therefore by choosing ϵ small enough, we can find $i \in I$ such that

$$\|h\phi_i - h\| < 1 \quad (h \in (K^{**})_1).$$

So the map $\lambda : K^{**} \longrightarrow K^{**}$ by $\lambda(h) = h\phi_i$, is invertible. Thus λ is onto and hence there is $r \in K^{**}$ such that $r\phi_i = \phi_i$. So for every $h \in K^{**}$, we obtain $(hr - h)\phi_i = 0$ and hence by injectivity of λ, we have that

$$hr = r \qquad (h \in K^{**}).$$

So K^{**} has a right identity and hence $\ker(\pi)$ has a bounded right approximate identity. Therefore from Theorem 2.10, A is operator amenable.

The converse is clearly true. $\square$

A similar proof shows that uniform operator approximate contractibility is equivalent to operator contractiblity.

Definition 2.14. *Suppose that A is a completely contractive Banach algebra. The net $(u_i) \subset (A\hat{\otimes}A)^{**}$ is called a uniform operator approximate virtual diagonal if*

$$\lim_i [a_{s,t}].u_i - u_i.[a_{s,t}] = 0,$$

$$\lim_i a\pi^{**}(u_i) = a,$$

.Where the first convergence is uniform on $(M_n(A))_1, n \geq 1$ and the second convergence is uniform for $a \in (A)_1$.

Lemma 2.15. *Suppose that A is a completely contractive Banach algebra and X is an operator A-bimodule. Let $D : A \longrightarrow X^*$ be a completely bounded derivation . For every $x \in X$, we define the map $F_x \in (A\hat{\otimes}A)^*$ by*

$$\langle F_x, a \otimes b\rangle = \langle a.D(b), x\rangle \quad (a, b \in A).$$

Then the map $\varphi : X \longrightarrow (A\hat{\otimes}A)^*$, specified by

$$\varphi(x) = F_x \quad (x \in X),$$

is completely bounded.

Proof. Consider the map $\psi : A\hat{\otimes}A \longrightarrow X^*$ specified by

$$\psi(a \otimes b) = D(a).b \quad (a, b \in A).$$

Let $u = \alpha([a_{i,j}] \otimes [b_{k,l}])\beta \in M_n(A \otimes A)$, where $\alpha \in \mathbb{M}_{n,pq}, [a_{i,j}] \in M_p(A), [b_{k,l}] \in M_q(A), \beta \in \mathbb{M}_{pq,n}$. For $[x_{s,t}] \in (M_n(X))_1$, we have

$$\begin{aligned}
\langle\langle \psi_{pq}([a_{i,j}] \otimes [b_{k,l}]), [x_{s,t}]\rangle\rangle &= \langle\langle [D(a_{i,j}).b_{k,l}]_{pq}, [x_{s,t}]\rangle\rangle \\
&= [\langle D(a_{i,j}), b_{k,l}.x_{s,t}\rangle]_{pqn} \\
&= \langle\langle D_p([a_{i,j}]), [b_{k,l}.x_{s,t}]_{qn}\rangle\rangle.
\end{aligned}$$

Since $M_p(X^*) \cong CB(X, \mathbb{M}_p)$, we have

$$\begin{aligned}
\|\langle\langle \psi_{pq}([a_{i,j}] \otimes [b_{k,l}]), [x_{s,t}]\rangle\rangle\| &= \|\langle\langle D_p([a_{i,j}]), [b_{k,l}.x_{s,t}]_{qn}\rangle\rangle\| \\
&\leq \|D_p([a_{i,j}])\| \|[b_{k,l}.x_{s,t}]\|_{qn} \\
&\leq C\|D\|_{cb}\|[a_{i,j}]\|_p\|[b_{k,l}]\|_q\|[x_{s,t}]\|_n,
\end{aligned}$$

where C is the completely bounded norm of the action of A on X.

Therefore again by identifying $M_{pq}(X^*)$ with $CB(X, \mathbb{M}_{pq})$, we have

$$\|\psi_{pq}([a_{i,j}] \otimes [b_{k,l}])\|_{pq} \leq C\|D\|_{cb}\|[a_{i,j}]\|_p\|[b_{k,l}]\|_q.$$

Hence

$$\begin{aligned}
\|\psi_{pq}(u)\| &= \|\psi_{pq}(\alpha([a_{i,j}] \otimes [b_{s,t}])\beta)\| \\
&= \|\alpha\psi_{pq}([a_{i,j}] \otimes [b_{s,t}])\beta\| \\
&\leq \|\alpha\|\|\psi_{pq}([a_{i,j}] \otimes [b_{s,t}])\|\|\beta\| \\
&\leq C\|D\|_{cb}\|\alpha\|\|[a_{i,j}]\|_p\|[b_{k,l}]\|_q\|\beta\|.
\end{aligned}$$

Therefore by definition of the norm $\|.\|_\wedge$, we have that

$$\|\psi_n(u)\| \leq C\|D\|_{cb}\|u\| \quad (u \in M_n(A\hat{\otimes}A), n \geq 1).$$

Hence ψ is completely bounded and $\|\psi\|_{cb} \leq C\|D\|_{cb}$.

If we define $\phi : A\hat{\otimes}A \longrightarrow X^*$, by $\phi(a \otimes b) = a.D(b)$, then $\phi = D \circ \pi - \psi$, and hence ϕ is also completely bounded.

Now for $[x_{i,j}] \in M_n(X)$ and $[u_{k,l}] \in M_n(A\hat{\otimes}A)$ we have

$$\begin{aligned}
\langle\langle\varphi_n([x_{i,j}]), [u_{k,l}]\rangle\rangle &= \langle\langle[F_{x_{i,j}}], [u_{k,l}]\rangle\rangle \\
&= \langle\langle[x_{i,j}], \phi_n([u_{k,l}])\rangle\rangle \\
&\leq \|[x_{i,j}]\|_n\|\phi\|_n\|[u_{k,l}]\|_n \\
&\leq \|[x_{i,j}]\|_n\|\phi\|_{cb}\|[u_{k,l}]\|_n.
\end{aligned}$$

Therefore $\|\varphi\|_n \leq \|\phi\|_{cb}$ for all $n \geq 1$ and hence $\|\varphi\|_{cb} \leq \|\phi\|_{cb}$ which completes the proof. $\square$

Suppose that X is an operator Banach A-bimodule. Consider the bi-linear mapping $\rho : A \times X \longrightarrow X$ defined by

$$\rho(a, x) = a.x \quad (a \in A, x \in X).$$

We define $\rho^{**} : A^{**} \times X^{**} \longrightarrow X^{**}$ by

$$\langle \rho^{**}(\ddot{a}, \ddot{x}), F \rangle = \langle \ddot{a}, \rho^*(\ddot{x}, F) \rangle \quad (\ddot{a} \in A^{**}, \ddot{x} \in X^{**}, F \in X^*).$$

$$\langle \rho^*(\ddot{x}, F), a \rangle = \langle \ddot{x}, \rho(F, a) \rangle \quad (a \in A).$$

$$\langle \rho(F, a), x \rangle = \langle F, \rho(a, x) \rangle \quad (x \in X).$$

From the argument in [14], we have that

$$\rho^{**}(\ddot{a}, \ddot{x}) = \mathrm{wk}^* - \lim_{\alpha} - \mathrm{wk}^* - \lim_{\beta} \rho(a_\alpha, x_\beta),$$

where $(a_\alpha) \subset A$, $(x_\beta) \subset X$ are such that $\mathrm{wk}^* - \lim_\alpha \mathrm{a}_\alpha = \ddot{\mathrm{a}}$ and $\mathrm{wk}^* - \lim_\beta \mathrm{x}_\beta = \ddot{\mathrm{x}}$. Also the map ρ^{**} turns X^{**} into a left $A^{**}-$ module where A^{**} has the first Arens product. If $\square$ denotes the first Arens product on A^{**}, then we have (for convenience we denote the action of ρ^{**} and ρ^* by .)

$$\begin{aligned}
\langle \ddot{a}.(\ddot{b}.\ddot{x}), F \rangle &= \langle \ddot{a}, (\ddot{b}.\ddot{x}).F \rangle \\
&= \lim_{\alpha} \langle (\ddot{b}.\ddot{x}).F, a_\alpha \rangle \\
&= \lim_{\alpha} \langle \ddot{b}.\ddot{x}, F.a_\alpha \rangle = \\
&= \lim \alpha \lim_{\beta} \langle \ddot{x}.F.a_\alpha, b_\beta \rangle
\end{aligned}$$

$$= \lim_{\alpha} \lim_{\beta} \langle \ddot{x}.F, a_\alpha b_\beta \rangle.$$

On the other hand if for $\ddot{b}$, the net $(b_\beta) \subset A$ is such that $\mathrm{wk}^* - \lim_\beta b_\beta = \ddot{b}$, then

$$\begin{aligned} \langle (\ddot{a} \square \ddot{b}).\ddot{x}, F \rangle &= \langle \ddot{a} \square \ddot{b}, \ddot{x}.F \rangle \\ &= \lim_{\alpha} \lim_{\beta} \langle \ddot{x}.F, a_\alpha b_\beta \rangle. \end{aligned}$$

Therefore

$$(\ddot{a} \square \ddot{b}).\ddot{x} = \ddot{a}.(\ddot{b}.\ddot{x}).$$

Also it's easy to see that $\|\rho^{**}\| \leq \|\rho\|$. So by the definition above X^{**} will be a left A^{**}-module.

Indeed, in following Lemma, we will prove that the left action of A^{**} on X^{**}, defined above turns X^{**} into an operator left A^{**}-module.

In the following Lemma, ρ denotes the action of A on X which is represented by "." for convenience.

Lemma 2.16. *Suppose that A is a completely contractive Banach algebra . Consider X^{**} as a left A-module as in the discussion above. Then the mapping $\phi : A^{**} \hat{\otimes} X^{**} \longrightarrow X^{**}$, specified by*

$$\phi(\ddot{a} \otimes \ddot{x}) = \ddot{a}.\ddot{x}, \quad (\ddot{a} \in A^{**}, \ddot{x} \in X^{**}),$$

is completely bounded.

Proof. Let $[\ddot{a}_{i,j}] \in M_p(A^{**})$, $[\ddot{x}_{k,l}] \in M_q(X^{**})$, $[f_{s,t}] \in M_{pq}(X^*)$. Then we have

$$\langle\langle \phi_{pq}([\ddot{a}_{i,j}] \otimes [\ddot{x}_{k,l}]), [f_{s,t}]\rangle\rangle = \langle\langle [\ddot{a}_{i,j}.\ddot{x}_{k,l}]_{pq}, [f_{s,t}]\rangle\rangle = \langle\langle [\ddot{a}_{i,j}], [\ddot{x}_{k,l}.f_{s,t}]_{pq^2}\rangle\rangle \quad (*).$$

and for every $[a_{h,r}] \in M_{pq^2}(A)$, we have

$$\langle\langle [\ddot{x}_{k,l}.f_{s,t}], [a_{h,r}]\rangle\rangle = \langle\langle [\ddot{x}_{k,l}], [f_{s,t}.a_{h,r}]\rangle\rangle.$$

Let $C = \|\rho\|_{cb}$. So we get

$$\|\langle\langle [\ddot{x}_{k,l}.f_{s,t}], [a_{h,r}]\rangle\rangle\| = \|\langle\langle [\ddot{x}_{k,l}], [f_{s,t}.a_{h,r}]\rangle\rangle\| \leq C\|[\ddot{x}_{k,l}]\|_q\|[f_{s,t}]\|_{pq}\|[a_{h,r}]\|_{pq^2},$$

and therefore

$$\|[\ddot{x}_{k,l}.f_{s,t}]\|_{pq^2} \leq C\|[\ddot{x}_{k,l}]\|_q\|[f_{s,t}]\|_{pq} \qquad (**).$$

Hence

$$\begin{aligned}\|\langle\langle \phi_{pq}([\ddot{a}_{i,j}] \otimes [\ddot{x}_{k,l}]), [f_{s,t}]\rangle\rangle\| &= \|\langle\langle [\ddot{a}_{i,j}], [\ddot{x}_{k,l}.f_{s,t}]_{pq^2}\rangle\rangle\| \quad (\text{From } (*))\\ &\leq \|[\ddot{a}_{i,j}]\|_p\|[\ddot{x}_{k,l}.f_{s,t}]\|_{pq^2}\\ &\leq C\|[\ddot{a}_{i,j}]\|_p\|\|[\ddot{x}_{k,l}\|_q\|[f_{s,t}]\|_{pq} \quad (\text{From } (**)),\end{aligned}$$

and so

$$\|\phi_{pq}([\ddot{a}_{i,j}] \otimes [\ddot{x}_{k,l}])\|_{pq} \leq C\|\|[\ddot{a}_{i,j}]\|_p\|\|[\ddot{x}_{k,l}\|_q \qquad (***).$$

Let $v = \alpha([\ddot{a}_{i,j}] \otimes [\ddot{x}_{k,l}])\beta \in M_n(A^{**} \otimes X^{**})$, where $\alpha \in \mathbb{M}_{n,pq}$, $[\ddot{a}_{i,j}] \in M_p(A^{**})$, $[\ddot{x}_{k,l}] \in M_q(X^{**})$, $\beta \in \mathbb{M}_{pq,n}$. Then we have

$$\|\phi_n(v)\| = \|\alpha\phi_{pq}([\ddot{a}_{i,j}] \otimes [\ddot{x}_{k,l}])\beta\|$$

$$\leq \|\alpha\| \|\phi([\ddot{a}_{i,j}] \otimes [\ddot{x}_{k,l}])\|_{pq} \|\beta\|$$

$$\leq C\|\alpha\| \|[\ddot{a}_{i,j}]\|_p \| \|[\ddot{x}_{k,l}\|_q \|\beta\| \qquad \text{(From } (***)),$$

whence

$$\|\phi\|_{cb} \leq C.$$

□

Corollary 2.17. *Suppose that A is a completely contractive Banach algebra . Then so is A^{**} equipped with the first Arens product.*

Proof. The result follows from substituting X by A in the preceding Lemma. The way we defined the action of A^{**} on X^{**} is the same as definition of the first Arens product of A^{**}. □

Proposition 2.18. *Suppose that A has an operator uniform approximate virtual diagonal. Then A has a bounded approximate identity.*

Proof. We have that $\lim_i a.\pi^{**}(u_i) = a$, uniformly on A_1. Fix $\epsilon > 0$. Then there is i_0 such that for $i \geq i_0$,

$$\|a.\pi^{**}(u_i) - a\| < \epsilon \quad (a \in (A)_1).$$

Now let $\ddot{a} \in (A^{**})_1$. Then there is a net $(a_\alpha) \subset A_1$ such that

$$\ddot{a} = \text{wk}^* \lim_\alpha a_\alpha.$$

Therefore we have

$$\ddot{a}\Box\pi^{**}(u_i) - \ddot{a} = \text{wk}^* - \lim_{\alpha} a_\alpha.\pi^{**}(u_i) - a_\alpha \qquad (*).$$

Since $(a_\alpha) \subset (A)_1$, if $i \geq i_0$, for every α, we have $\|a_\alpha.\pi^{**}(u_i) - a_\alpha\| < \epsilon$. Therefore from (*), we have that

$$\|\ddot{a}\Box\pi^{**}(u_i) - \ddot{a}\| \leq \epsilon \qquad (i \geq i_0).$$

Now we define the mapping $\varphi_i : A^{**} \longrightarrow A^{**}$ by

$$\varphi_i(\ddot{a}) = \ddot{a}\Box\pi^{**}(u_i) \quad (\ddot{a} \in A^{**}).$$

We have that for $i \geq i_0$, $\|\varphi_i - Id_{A^{**}}\| < 1$ and therefore φ_{i_0} is invertible.

Hence φ_{i_0} is onto , and so, there is $\ddot{b} \in A^{**}$ such that $\varphi_{i_0}(\ddot{b}) = \ddot{b}\Box\pi^{**}(u_{i_0}) = \pi^{**}(u_{i_0})$. So we have

$$(\ddot{a}\Box\ddot{b} - \ddot{a})\Box\pi^{**}(u_{i_0}) = a\Box\pi^{**}(u_{i_0}) - a\Box\pi^{**}(u_{i_0}) = 0.$$

So by injectivity of φ_{i_0}, we have

$$\ddot{a}\Box\ddot{b} = \ddot{a} \quad (\ddot{a} \in A^{**}).$$

Therefore A^{**} has a right identity and therefore A has a bounded right approximate identity. Similarly A has a bounded left approximate identity and hence A has a two-sided bounded approximate identity. $\square$

Theorem 2.19. *A is operator amenable if and only if it has a uniform operator approxiomate virtual diagonal.*

Proof. $\Leftarrow$: Suppose that $(u_i)_{i\in I} \subset (A\hat{\otimes}A)^{**}$ be a uniform operator approximate virtual diagonal for A. Suppose that $D : A \longrightarrow X^*$ be a competely bounded derivation where X is a neo-unital operator A-bimodule. We define $F_x \in (A\hat{\otimes}A)^*$ by $\langle F_x, a\otimes b\rangle = \langle a.D(b), x\rangle$. We define $f_i \in X^*$ by

$$\langle f_i, x\rangle = \langle u_i, F_x\rangle \quad (x \in X, i \in I).$$

Let $n \geq 1$ and $[a_{s,t}] \in (M_n(A))_1$, $[x_{k,l}] \in (M_n(X))_1$. Then we have

$$\langle\langle [a_{s,t}].f_i - f_i.[a_{s,t}], [x_{k,l}]\rangle\rangle = [\langle u_i, F_{x_{k,l}.a_{s,t}-a_{s,t}.x_{k,l}}\rangle].$$

For any $m \in A\hat{\otimes}A$ we have

$$\langle F_{a.x-x.a}, m\rangle = \langle a.F_x - F_x.a, m\rangle + \langle \pi(m).D(a), x\rangle,$$

Take $u \in (A\hat{\otimes}A)^{**}$ and suppose that $u = \mathrm{wk}^* - \lim_\alpha m_\alpha$ for some bounded net $(m_\alpha) \subset (A\hat{\otimes}A)$. We obtain

$$\begin{aligned}
\langle u, F_{a.x-x.a}\rangle &= \lim_\alpha \langle F_{a.x-x.a}, m_\alpha\rangle \\
&= \lim_\alpha \langle a.F_x - F_x.a, m_\alpha\rangle + \lim_\alpha \langle \pi(m_\alpha).D(a), x\rangle \\
&= \langle u, a.F_x - F_x.a\rangle + \lim_\alpha \langle D(a), x.\pi(m_\alpha)\rangle.
\end{aligned}$$

Therefore we have

$$\langle u, F_{a.x-x.a}\rangle = \langle u, a.F_x - F_x.a\rangle + \langle x.\pi^{**}(u), D(a)\rangle,$$

and so

$$\langle f_i.a - a.f_i, x\rangle = \langle f_i, a.x - x.a\rangle$$

$$= \langle u_i, F_{a.x-x.a}\rangle$$

$$= \langle u_i, a.F_x - F_x.a\rangle + \langle x.\pi^{**}(u_i), D(a)\rangle$$

$$= \langle u_i.a - a.u_i, F_x\rangle + \langle x.\pi^{**}(u_i), D(a)\rangle.$$

Therefore

$$\langle\langle f_i.[a_{s,t}] - [a_{s,t}].f_i, [x_{k,l}]\rangle\rangle = \langle\langle [u_i.a_{s,t} - a_{s,t}.u_i], [F_{x_{k,l}}]\rangle\rangle + \langle\langle [D(a_{s,t})], [x_{k,l}].\pi^{**}(u_i)\rangle\rangle. \quad (1)$$

From the above argument we get that

$$\begin{aligned}\langle\langle (ad_{f_i} - D)_n([a_{s,t}]), [x_{k,l}]\rangle\rangle &= \langle\langle f_i.[a_{s,t}] - [a_{s,t}].f_i - D_n([a_{s,t}]), [x_{k,l}]\rangle\rangle \\ &= \langle\langle [u_i.a_{s,t} - a_{s,t}.u_i], [F_{x_{k,l}}]\rangle\rangle + \langle\langle [D(a_{s,t})], [x_{k,l}].\pi^{**}(u_i)\rangle\rangle \\ &- \langle\langle [D(a_{s,t})], [x_{k,l}]\rangle\rangle \qquad \text{(from (1))} \\ &= \langle\langle [u_i.a_{s,t} - a_{s,t}.u_i], [F_{x_{k,l}}]\rangle\rangle + \langle\langle [D(a_{s,t})], [x_{k,l}].\pi^{**}(u_i) - [x_{k,l}]\rangle\rangle. \quad (2)\end{aligned}$$

Take $[x_{k,l}] \in (M_n(X))_1$. We know that A has a bounded approximate identity (Proposition 2.18). Let M be an upper bound for the norm of the bounded approximate identity of A. Fix $\epsilon > 0$. From Cohen's Factorization Theorem (by considering $M_n(X)$ as Banach A-bimodule), there is a matrix $[y_{k,l}] \in M_n(X)$ and $c \in A$ such that $\|[y_{k,l}]\| < 1 + \epsilon$ and $\|c\| < M$ such that

$$[x_{k,l}] = [y_{k,l}].c.$$

From our assumption, there is $i_0 \in I$ such that for all $i \geq i_0$, we have

$$\|a\pi^{**}(u_i) - a\| < \epsilon \quad (a \in A_1)$$

$$\|[a_{s,t}.u_i - u_i.a_{s,t}]\| < \epsilon \quad ([a_{s,t}] \in (M_n(A))_1).$$

So for $i \geq i_0$ and all $[x_{k,l}] \in (M_n(X))_1$ we have

$$\begin{aligned}
\|[x_{k,l}].\pi^{**}(u_i) - [x_{k,l}]\| &= \|[y_{k,l}].(c\pi^{**}(u_i) - c)\| \\
&\leq \|[y_{k,l}]\|\|c\pi^{**}(u_i) - c\| \\
&< M(1+\epsilon)\epsilon. \qquad (3)
\end{aligned}$$

We consider the map $\varphi : X \longrightarrow (A\hat{\otimes}A)^*$, $\varphi(x) = F_x$. From Lemma 2.15, we know that φ is completely bounded. Let $K = \|\varphi\|_{cb}$. Therefore for $i \geq i_0$, from (2), (3), we have

$$\begin{aligned}
&\|\langle\langle(\mathrm{ad}_{f_i} - D)_n([a_{s,t}]), [x_{k,l}]\rangle\rangle = \|\langle\langle[u_i.a_{s,t} - a_{s,t}.u_i], [F_{x_{k,l}}]\rangle\rangle + \langle\langle[D(a_{s,t})], [x_{k,l}].\pi^{**}(u_i) - [x_{k,l}]\rangle\rangle\| \\
&\leq \|\langle\langle[u_i.a_{s,t} - a_{s,t}.u_i], [F_{x_{k,l}}]\rangle\rangle\| + \|\langle\langle[D(a_{s,t})], [x_{k,l}].\pi^{**}(u_i) - [x_{k,l}]\rangle\rangle\| \\
&= \|\langle\langle[u_i.a_{s,t} - a_{s,t}.u_i], \varphi_n([x_{k,l}])\rangle\rangle\| + \|\langle\langle[D(a_{s,t})], [x_{k,l}].\pi^{**}(u_i) - [x_{k,l}]\rangle\rangle\| \\
&\leq \|\varphi\|_{cb}\|[u_i.a_{s,t} - a_{s,t}.u_i]\|_n + \|D\|_{cb}\|[x_{k,l}].\pi^{**}(u_i) - [x_{k,l}]\|_n \\
&< K\epsilon + \|D\|_{cb}M(1+\epsilon)\epsilon \ \ (= O(\epsilon)),
\end{aligned}$$

where $\lim_{\epsilon \longrightarrow 0^+} O(\epsilon) = 0$. Since i_0 was independent of $[a_{s,t}] \in (M)_n(A))_1$, $[x_{k,l}] \in (M_n(X))_1$ and is also independent of $n \geq 1$, we can find a subnet $(f_\alpha) \subset (f_i)$, such that

$$\lim_\alpha \|\mathrm{ad}_{f_\alpha} - D\|_{cb} = 0.$$

Therefore A is operator uniformly approximately amenable and hence is operator amenable from Theorem 2.13.

The converse is clear since if A is operator amenable, then A has a virtual diagonal. $\square$

Definition 2.20. *We say that the net* $(m_\alpha)_{\alpha\in\Lambda} \subset A\hat{\otimes}A$ *is an operator uniform approximate diagonal for* A *if*

$$\lim_\alpha [a_{i,j}].m_\alpha - m_\alpha.[a_{i,j}] = 0 \quad (\text{uniformly on } (M_n(A))_1, n \geq 1)$$

$$\lim_\alpha \pi(m_\alpha)a = a, \qquad (\text{uniformly on } (A)_1).$$

Note that from the fact that $\lim_\alpha \pi(m_\alpha)a = a$, uniformly on A_1, using similar techniqes as in proposition 2.18, it can be shown that A will have a left identity. It can be seen from the definition of an operator uniform approximate diagonal and the definition of operator projective tensor product, that if (m_α) is an operator uniform approximate diagonal for A, we can assume without loss of generality that $(m'_\alpha) \subset A \otimes A$.

Theorem 2.21. *Suppose that* A *has a uniform operator approximate diagonal . Then* A *is operator contractible.*

Proof. Suppose that E is left identity of A and $(m_\alpha)_{\alpha\in\Lambda}$ is a uniform operator approximate diagonal for A. Without loss of generality we can assume that $(m_\alpha) \subset A \otimes A$. Suppose that $D : A \longrightarrow X$ is a completely bounded derivation where X is a neo-unital operator A-bimodule. For $f \in X^*$, we define $F_f \in (A\hat{\otimes}A)^*$ by $\langle F_f, a \otimes b\rangle = \langle f, a.D(b)\rangle$.

We define $x_\alpha \in X$ by

$$\langle x_\alpha, f\rangle = \langle F_f, m_\alpha\rangle \quad (f \in X^*, \alpha \in \Lambda).$$

(This is possible since $(A^*, \mathrm{wk}^*)^* \cong$ A.)

Using a similar argument as in the proof of the Lemma 2.15, we have that the operator $\varphi : X^* \longrightarrow (A\hat{\otimes}A)^*$, $\varphi(f) = F_f$, is completely bounded. Let $C = \|\varphi\|_{cb}$.

Let $n \geq 1$ and $[a_{i,j}] \in (M_n(A))_1$,$[f_{k,l}] \in (M_n(A^*))_1$. Then we have

$$\begin{aligned}
\langle\langle [a_{i,j}].x_\alpha - x_\alpha.[a_{i,j}], [f_{k,l}]\rangle\rangle &= [\langle f_{k,l}.a_{i,j} - a_{i,j}.f_{k,l}, x_\alpha\rangle] \\
&= [\langle F_{f_{k,l}.a_{i,j} - a_{i,j}.f_{k,l}}, m_\alpha\rangle] \\
&= [\langle F_{f_{k,l}}.a_{i,j} - a_{i,j}.F_{f_{k,l}}, m_\alpha\rangle] + \langle\langle \pi(m_\alpha).D_n([a_{i,j}], [f_{k,l}]\rangle\rangle \\
&= \langle\langle [a_{i,j}].m_\alpha - m_\alpha.[a_{i,j}], [F_{f_{k,l}}]\rangle\rangle + \langle\langle \pi(m_\alpha).D_n([a_{i,j}], [f_{k,l}]\rangle\rangle. \quad (*)
\end{aligned}$$

Let E be the left identity of A and we fix $\epsilon > 0$. There is $\alpha_0 \in \Lambda$ such that

$$\|[a_{i,j}].m_{\alpha_0} - m_{\alpha_0}.[a_{i,j}]\| < \epsilon \quad ([a_{i,j}] \in (M_n(A))_1, n \geq 1),$$

$$\|\pi(m_{\alpha_0})E - E\| < \epsilon.$$

Therefore for $[a_{i,j}] \in (M_n(A))_1$, we have

$$\begin{aligned}
\|(ad_{x_{\alpha_0}} - D)_n([a_{i,j}])\| &= \sup\{\|\langle\langle [a_{i,j}].x_{\alpha_0} - x_{\alpha_0}.[a_{i,j}], [f_{k,l}]\rangle\rangle\| : [f_{k,l}] \in (M_n(X^*))_1\} \\
&\leq \sup\{\|\langle\langle [a_{i,j}].m_{\alpha_0} - m_{\alpha_0}.[a_{i,j}], [F_{f_{k,l}}]\rangle\rangle\| : [f_{k,l}] \in (M_n(X^*))_1\} + \\
&+ \sup\{\|\langle\langle (\pi(m_{\alpha_0}E - E)D_n([a_{i,j}]), [f_{k,l}]\rangle\rangle\| : [f_{k,l}] \in (M_n(X^*))_1\} \text{ (from } (*))
\end{aligned}$$

$$\leq C\epsilon + \|D\|_{cb}\epsilon.$$

So we can find a net $(x_\alpha) \subset X$ such that

$$\lim_{\alpha} \|\mathrm{ad}_{x_\alpha} - D\|_{cb} = 0.$$

Therefore A is uniformly approximately operator contractible and hence is operator contractible. $\square$

It is an old open question asking whether every contractible Banach algebra is finite-dimensional (and it has been confirmed in some cases). Now the equivalent question is whether there is infinite-dimensional operator contractible completely contractive Banach algebra.

Here we give the operator space version of Theorem 1.11.

Theorem 2.22. *Suppose that A has a central bounded approximate identity. Then A is operator approximately contractible if and only if A has an operator approximate diagonal.*

Proof. $\Rightarrow$: Suppose that A is operator approximately contractible and let $(e_i)_{i\in I}$, be a central bounded approximate identity for A. Let $K = \ker(\pi)$ and $D_i : A \longrightarrow K$ be defined by

$$D_i(a) = ae_i \otimes e_i - e_i \otimes e_i a \quad (a \in A, i \in I)).$$

Since D_i is an inner derivation into K, it is completely bounded and hence will be approximately inner. So there is a net $(\zeta_{\alpha(i)}) \subset K$ such that

$$\lim_{\alpha(i)} a.\zeta_{\alpha(i)} - \zeta_{\alpha(i)}.a = D_i(a) \quad (a \in A).$$

We fix $\epsilon > 0$ and let F be a finite subset of A. Then there exist $i_{F,\epsilon} \in I$ such that

$$\|\pi(e_{i_{F,\epsilon}} \otimes e_{i_{F,\epsilon}})a - a\| < \epsilon \quad (a \in F).$$

Also we can find $\zeta_{F,\epsilon} \in K$ such that

$$\|a.\zeta_{F,\epsilon} - \zeta_{F,\epsilon}.a - ae_{i_{F,\epsilon}} \otimes e_{i_{F,\epsilon}} + e_{i_{F,\epsilon}} \otimes e_{i_{F,\epsilon}} a\| = \|a.\zeta_{F,\epsilon} - \zeta_{F,\epsilon}.a - D_{i_{F,\epsilon}}(a)\| < \epsilon \quad (a \in F).$$

Now let $\xi_{F,\epsilon} = e_{i_{F,\epsilon}} \otimes e_{i_{F,\epsilon}} - \zeta_{F,\epsilon}$. For $a \in F$, we have

$$\begin{aligned} \|a.\xi_{F,\epsilon} - \xi_{F,\epsilon}.a\| &= \|\zeta_{F,\epsilon}.a - a.\zeta_{F,\epsilon} - (ae_{i_{F,\epsilon}}) \otimes e_{i_{F,\epsilon}} + e_{i_{F,\epsilon}} \otimes (e_{i_{F,\epsilon}} a)\| \\ &< \epsilon, \end{aligned}$$

and

$$\begin{aligned} \|\pi(\xi_{F,\epsilon})a - a\| &= \|\pi(e_{i_{F,\epsilon}} \otimes e_{i_{F,\epsilon}})a - a\| \\ &< \epsilon. \end{aligned}$$

Therefore we can find a net $(\xi_{F,\epsilon}) \subset A \hat{\otimes} A$ such that

$$\begin{aligned} &\lim a.\xi_{F,\epsilon} - \xi_{F,\epsilon}.a = 0, \\ &\lim \pi(\xi_{F,\epsilon})a = a \quad (a \in A). \end{aligned}$$

$\Leftarrow$: Conversely suppose that the net $(m_\alpha) \subset A\hat{\otimes}A$ be an approximate diagonal for A and let D be a completely bounded derivation from $A^{\#}$ into a neo-unital operator $A^{\#}$-bimodule X. We define x_α as in Theorem 2.21 by

$$\langle x_\alpha, f\rangle = \langle F_f, m_\alpha\rangle \quad (f \in X, \alpha \in \Lambda),$$

where

$$\langle F_f, a \otimes b\rangle = \langle f, a.D(b)\rangle.$$

Using similar argument as in the Theorem 2.21, for $f \in X^*$ and $a \in A$, we have

$$\begin{aligned}\langle a.x_\alpha - x_\alpha.a, f\rangle &= \langle f.a - a.f, x_\alpha\rangle \\ &= \langle F_f, a.m_\alpha - m_\alpha.a\rangle + \langle \pi(m_\alpha).D(a), f\rangle.\end{aligned}$$

Since X is neo-unital, we have that $\lim \pi(m_\alpha).D(a) = D(a)$. Therefore

$$|\langle a.x_\alpha - x_\alpha.a - D(a), f\rangle| \leq \|D\|_{cb}\|a.m_\alpha - m_\alpha.a\|\|f\| + \|\pi(m_\alpha).D(a) - D(a)\|\|f\|.$$

Therfore

$$\lim_\alpha a.x_\alpha - x_\alpha.a - D(a) = 0 \quad (a \in A),$$

whence A is operator approximately contractible.

□

Definition 2.23. *Suppose that V is a complete operator space and W is a closed subspace of V. Then we say that W is completely complemented in V if there is a completely bounded projection P from V onto W.*

The following proposition has been shown for two-sided ideals is [31, Cor. 4]. Here we state and prove the following result for left (right) ideals from a different method.

Proposition 2.24. *Suppose that I is a completely boundedly complemented left (right) ideal in an operator amenable Banach algebra. Then I has a bounded right (left) approximate identity.*

Proof. Let $P : A \longrightarrow I$ be a completely bounded projection onto I. We define $\varphi : A \otimes A \longrightarrow I$ by

$$\varphi(\alpha([a_{i,j}]\otimes[b_{k,l}])\beta) = \alpha[a_{i,j}P(b_{k,l})]\beta \quad (\alpha \in \mathbb{M}_{1,pq}, [a_{i,j}] \in M_p(A), [b_{k,l}] \in M_q(A), \beta \in \mathbb{M}_{pq,1}).$$

We have

$$\begin{aligned}\|\varphi(\alpha([a_{i,j}] \otimes [b_{k,l}])\beta)\| &\leq \|\alpha\|\|[a_{i,j}]\|_p\|[P(b_{k,l})]\|_q\|\beta\| \\ &\leq \|P\|_{cb}\|\alpha\|\|[a_{i,j}]\|_p\|[b_{k,l}]\|_q\|\beta\|.\end{aligned}$$

Therefore φ is continuous with respect to $\|.\|_\wedge$ and hence it can be extended to a continuous linear operator from $A\hat{\otimes}A$ into I. (Indeed it is onto by Cohen's factorization Theorem.) It is also easy to see that φ is a left A-module morphism. For $a \in I$, we have

$$\begin{aligned}\varphi((\alpha([a_{i,j}] \otimes [b_{k,l}])\beta)a) &= \varphi(\alpha([a_{i,j}] \otimes [b_{k,l}a])\beta) \\ &= \alpha[a_{i,j}P(b_{k,l}a)]\beta \\ &= \alpha[a_{i,j}b_{k,l}a]\beta\end{aligned}$$

$$= \pi(\alpha([a_{i,j}] \otimes [b_{k,l}])\beta)a.$$

Therefore

$$\varphi(m.a) = \pi(m)a \quad (m \in A\hat{\otimes}A, a \in I).$$

Let $(m_\lambda) \subset A\hat{\otimes}A$ be a bounded approximate diagonal for A and we define $e_\lambda := \varphi(m_\lambda)$. For $a \in I$, we get

$$\begin{aligned}
\lim_\lambda ae_\lambda &= \lim_\lambda a\varphi(m_\lambda) \\
&= \lim_\lambda \varphi(am_\lambda) \\
&= \lim_\lambda \varphi(m_\lambda a) \\
&= \lim_\lambda \pi(m_\lambda)a \\
&= a.
\end{aligned}$$

Hence (e_λ) forms a bounded right approximate identity for I. □

The following result is contained in Cor. 6.2 of [8]. Here we give a different proof.

Theorem 2.25. *Suppose that G is a locally compact group. Then $A(G)^{**}$ is operator amenable if and only if G is finite.*

Proof. First if $A(G)^{**}$ is operator amenable, it has a two-sided bounded approximate identity say (u_α). It is easy to show that then $A(G)$ will have a bounded approximate identity say (e_i) . We let $E = \mathrm{wk}^* - \lim_i e_i$. Then E is a right identity for $A(G)^{**}$ with respect to the first Arens product. For every $F \in A(G)^{**}$, we have

$$E\square F = \lim_{\alpha} u_\alpha \square E \square F = \lim_{\alpha} u_\alpha \square F = F.$$

So $A(G)^{**}(= VN(G)^*)$ has an identity and therefore G is compact [24, Theorem 3.2]. We are going to show that G is also discrete. Then it has to be finite.

Let e be the identity of group G. From the argument in page 168 of [22], there is a state $n \in VN(G)^*(= A(G)^{**})$ such that

$$f.n = f(e)n \quad (f \in A(G)),$$

Therefore for any state $r \in VN(G)^*$, we have $r\square n = n$. So I_n defined by $I_n := n\square A(G)^{**}$ is a two-sided ideal in $A(G)^{**}$. Also the projection $P : A(G)^{**} \longrightarrow I_n$, defined by $P(\ddot{f}) = n\square \ddot{f} \quad (\ddot{f} \in A(G)^{**})$ is completely bounded.(Since $A(G)^{**}$ is a completely contractive Banach algebra.) Therefore I_n has a bounded approximate identity say (e_i). Let $m \in VN(G)^*$ be any topological invarient mean. Then we have $m = n\square m \in I_n$ and therefore

$$m = \lim_i m\square e_i = \lim_i e_i.$$

Hence $VN(G)$ has only a unique invarient mean and therefore from [25, Corollary 4.11] G is discrete.

The converse is trivial. $\square$

2.3 On the perturbation problem for the operator amenable completely contractive Banach algebras

The perturbation problem first appeared in [20], where Johnson proved that for any amenable Banach algebra, there is an $\epsilon > 0$ such that defining a new product on the algebra which norm difference is less that ϵ with the original product, will not affect the amenability. Our goal is to give operator space version of [20, Theorem 6.2] with a different approach.

Suppose that A is a completely contractive Banach algebra with respect to the product π. We can consider the product π as a completely bounded map from $A\hat{\otimes}A^{op}$ to A with $\|\pi\|_{cb} \leq 1$. The more general version of the following Lemma is in [17]. Here we state the version that we are going to use in the following theorem.

Lemma 2.26. *Suppose that X and Y are Banach spaces and $T : X \longrightarrow Y$ is a bounded operator onto Y such that there is $K > 0$ such that for every $y \in Y$ there is $x \in X$ such that $\|x\| \leq K\|y\|$ and $T(x) = y$. Suppose that $S : X \longrightarrow Y$ is another bounded operator. Let $\delta = \|S - T\|$. If $K\delta < 1$, then S is also onto and for every $y \in Y$, there is $x \in X$ such that $\|x\| \leq \frac{K}{1-K\delta}\|y\|$ and $S(x) = y$.*

Proof. Take $y \in Y$. By assumption there is $x_0 \in X$ such that $\|x_0\| \leq K\|y\|$ and $T(x_0) = y$. So we have $\|y\| \leq \|S(x_0)\| + \delta\|x_0\|$. Let $y_0 = y$ and $y_1 = y_0 - S(x_0)$. We have $\|y_1\| = \|(T - S)(x_0)\| \leq \delta\|x_0\|$. By the argument above, we can find $x_1 \in X$ such that $\|x_1\| \leq K\delta\|x_0\|$ and $T(x_1) = y_1$. For $i \geq 1$, we define the sequence $y_{i+1} = y_i - S(x_i)$ and

$x_{i+1} \in X$ such that $\|x_{i+1}\| \leq K\|y_{i+1}\|, T(x_{i+1}) = y_{i+1}$. So we have

$\|x_{i+1}\| \leq K\|y_{i+1}\| = K\|(T-S)(x_i)\| \leq K^{i+1}\delta^{i+1}\|x_0\| \leq K^{i+2}\delta^{i+1}\|y\|$. Therefore we have

$$\|y - \Sigma_{i=0}^{n} S(x_i)\| = \|y_{n+1}\| \leq K^n\delta^{n+1}\|x_0\| \leq K^{n+1}\delta^{n+1}\|y\|.$$

Let $x = \Sigma_{i=0}^{\infty} x_i$. Then we have

$$\|x\| \leq \Sigma_{i=0}^{\infty} K^{i+1}\delta^i\|y\| = \frac{K}{1-K\delta}\|y\|,$$

and $S(x) = y$. □

Suppose that X is a Banach space and Y and Z are two closed subspaces of X. The Hausdorff distance between Y and Z denoted by $d(Y, Z)$ is defined by

$$d(Y, Z) = \max\{\sup\{d(y, Z) : \|y\| \leq 1\}, \sup\{d(Y, z) : \|z\| \leq 1\}\}.$$

Lemma 2.27. *Let Y and Z be closed subspaces of a Banach space X. Suppose that there is a projection P of X onto Y with $\|P\| < d(Y, Z)^{-1} - 1$. Then P maps Z one-to-one onto Y and the inverse α of $P|_Z$ satisfies* $(d = d(Y, Z))$

$$\|\alpha\| \leq (1+d)(1 - \|P\|d)^{-1}$$

$$\|\alpha(y) - y\| \leq ((1+d)(1 - \|P\|d)^{-1} - 1)\|y\|$$

$$\|P(z) - z\| \leq d(1 + \|P\|)\|z\|.$$

Proof. See [20, Lemma 5.2]. □

Lemma 2.28. *Suppose that A is a completely contractive Banach algebra with respect to both multiplications π and ρ. Suppose that $\|\pi - \rho\|_{cb} = \delta$. Then $A^{\#}$ is also a completely contractive Banach algebra with the multiplications induced by π and ρ and we have $\|\pi^{\#} - \rho^{\#}\|_{cb} \leq \delta$.*

Proof. Let $\alpha \in \mathbb{M}_{n,pq}$, $v' = [v_{i,j} \oplus \lambda_{i,j}] \in M_p(A)$, $w' = [w_{k,l} \oplus \zeta_{k,l}] \in M_q(A)$, $\beta \in \mathbb{M}_{pq,n}$, where $\lambda_{i,j}, \zeta_{k,l} \in \mathbb{C}$. We let $v = [v_{i,j}] \in M_p(A)$ and $w = [w_{k,l}] \in M_q(A)$. We have that

$$(\pi^{\#} - \rho^{\#})_n(\alpha((v' \otimes w')\beta) = \alpha(\pi - \rho)_{pq}(v \otimes w)\beta$$

Therefore

$$\|\pi^{\#} - \rho^{\#}\|_n \leq \|\pi - \rho\|_{cb} \qquad (n \geq 1).$$

Hence

$$\|\pi^{\#} - \rho^{\#}\|_{cb} \leq \|\pi - \rho\|_{cb}.$$

□

Suppose that A is an operator amenable completely contractive Banach algebra. From Theorem 1.28, A has an operator bounded approximate diagonal. We say that A is C-operator amenable if it has an operator bounded approximate diagonal bounded by C . We define the operator amenability constant of A denoted by C_A by

$$C_A = \inf\{C \geq 0 : A \text{ is } C - \text{operator amenable}\}.$$

Theorem 2.29. *Suppose that A is an operator amenable completely contractive Banach algebra with the multiplication π and operator amenability constant C. Suppose that (A, ρ) is also a completely contractive Banach algebra such that $\delta = \|\rho - \pi\|_{cb}$ satisfies $\delta < \frac{1}{4}$ and $\delta < \frac{1}{(6C+4).(2+\frac{6}{1-3\delta})}$. Then (A, ρ) is operator amenable.*

Proof. From the preceding Lemma and using the fact that operator amenability of A is equivalent to operator amenability of $A^{\#}$ without loss of generality we can assume that A is unital with the same unit for both multiplications π and ρ. First we assume that (A, π) is an operator amenable completely contractive Banach algebra. Consider the map $\pi^{**} : (A\hat{\otimes}A^{op})^{**} \longrightarrow A^{**}$. For $\ddot{a} \in A^{**}$, there is a net $(a_i) \subset A$ such that $\|a_i\| \leq \|\ddot{a}\|$ and $\mathrm{wk}^* - \lim_i a_i = \ddot{a}$. Therefore if $M \in (A\hat{\otimes}A^{op})^{**}$ is a wk^* accumulation point of the net $(a_i \otimes 1)_i$, we have

$$\pi^{**}(M) = \mathrm{wk}^* - \lim_i \pi(a_i \otimes 1) = \mathrm{wk}^* - \lim a_i = \ddot{a}.$$

So for any $\ddot{a} \in A^{**}$, there is an $M \in (A\hat{\otimes}A^{op})^{**}$ such that $\pi^{**}(M) = \ddot{a}$ and $\|M\| \leq \|\ddot{a}\|$. If $\delta < 1$, from Lemma 2.26, we have that for any $F \in \ker(\pi^{**})$, there is a $H \in (A\hat{\otimes}A^{op})^{**}$ such that $\rho^{**}(H) = \rho^{**}(F)$ and

$$\|H\| \leq \frac{1}{1-\delta}\|\rho^{**}(F)\| = \frac{1}{1-\delta}\|\rho^{**}(F) - \pi^{**}(F)\| \leq \frac{\delta}{1-\delta}\|F\|.$$

Since $H - F \in ker(\rho^{**})$ and $\|F - (F - H)\| = \|H\| \leq \frac{\delta}{1-\delta}\|F\|$, we have

$$\sup\{d(F, \ker(\rho^{**})) : F \in (\ker(\pi^{**})_1\} \leq \frac{\delta}{1-\delta}.$$

Similarly

$$\sup\{d(H, \ker(\pi^{**})) : H \in (\ker(\rho^{**})_1\} \leq \frac{\delta}{1-\delta}.$$

Hence $d := d(\ker(\pi^{**}), \ker(\rho^{**})) \leq \frac{\delta}{1-\delta}$.

We specify the operator $P : A\hat{\otimes}A^{op} \longrightarrow A$ by $P(a \otimes b) = a \otimes b - 1 \otimes ab$. It can be easily seen that P is a projection onto $\ker \pi$. Therefore $P^{**} : (A\hat{\otimes}A^{op})^{**} \longrightarrow \ker(\pi^{**})$ is a projection onto $\ker(\pi^{**})$ with norm at most 2. So if $\delta < \frac{1}{4}$, we have

$$\|P\| \leq 2 < (\frac{\delta}{1-\delta})^{-1} - 1 \leq d^{-1} - 1,$$

and then from Lemma 2.27, P maps $\ker(\rho^{**})$ onto $\ker(\pi^{**})$ and its restriction to $\ker(\rho^{**})$ is invertible . If $\phi : \ker(\pi^{**}) \longrightarrow \ker(\rho^{**})$ satisfies the properties

$$\|\phi\| \leq (1-3\delta)^{-1},$$

$$\|F - \phi(F)\| \leq 3\delta(1-3\delta)^{-1}\|F\| \quad (F \in \ker(\pi)),$$

$$\|H - P(H)\| \leq 3\delta(1-\delta)^{-1}\|H\| \quad (H \in \ker(\rho^{**})).$$

We indicate the multiplications induced by π by index π and we use the index ρ for the multiplications induced by ρ.

Fix $\epsilon > 0$ and let $[a_{i,j}] \in M_n(A)$ and let $v = \alpha([b_{k,l}]\otimes[c_{s,t}])\beta \in A\otimes A^{op}$, where $\alpha \in \mathbb{M}_{1,pq}$, $[b_{k,l}] \in M_p(A)$, $[c_{s,t}] \in M_q(A^{op})$, $\beta \in \mathbb{M}_{pq,1}$ such that

$$\|v\|_\wedge < \|\alpha\|\|[b_{k,l}]\|_p\|[c_{s,t}]\|_q\|\beta\| + \frac{\epsilon}{\|[a_{i,j}]\|\delta}.$$

We have

$$\begin{aligned}\|[a_{i,j}]\cdot_\pi v - [a_{i,j}]\cdot_\rho v\|_n &= \|(\alpha\otimes I_n)([a_{i,j\pi}b_{k,l} - a_{i,j\rho}b_{k,l}]_{(i,j),(k,l)}\otimes[c_{s,t}])(I_n\otimes\beta)\|_n \\ &\leq \|\alpha\otimes I_n\|\|(\pi-\rho)_{np}([a_{i,j}]\otimes[b_{k,l}])\|\|[c_{s,t}]\|\|I_n\otimes\beta\| \\ &\leq \|\alpha\|\|\pi-\rho\|_{cb}\|[a_{i,j}]\|_n\|[b_{k,l}]\|_p\|[c_{s,t}]\|_q\|\beta\| \\ &< \|[a_{i,j}]\|_n\|v\|_\wedge\delta+\epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, for $m \in A\hat{\otimes}A^{op}$, we have

$$\|[a_{i,j}\cdot_\rho m - a_{i,j}\cdot_\pi m]\|_\wedge \leq \delta\|[a_{i,j}]\|_n\|m\|_\wedge,$$

and similarly

$$\|m\cdot_\rho a_{i,j} - m\cdot_\pi a_{i,j}]\|_\wedge \leq \delta\|[a_{i,j}]\|\|m\|_\wedge.$$

Now we define $D : A \longrightarrow \ker(\pi^{**})$ by

$$D(a) = a\otimes 1 - 1\otimes a \quad (a\in A).$$

D is a completely bounded derivation and hence by operator amenability of (A,π), there exists $\zeta\in\ker(\pi^{**})$ such that

$$D(a) = a\cdot_\pi\zeta - \zeta\cdot_\pi a \quad (a\in A).$$

Let M' and M'' be the implementation constants of A and $A^{\#}$ respectively defined in [10] section 5. As seen in [10], we have $M' \leq C$ and from [10, Lemma 5.1], we can choose the above ζ such that

$$\|\zeta\| \leq M''\|D\| \leq (3M'+2)\|D\| \leq (3C+2)\|D\| \leq 6C+4.$$

Let $\gamma := \phi(\zeta)$. We have

$$\begin{aligned}
\|[a_{i,j}].{}_\rho\gamma - [a_{i,j}].{}_\pi\zeta\|_n &\leq \|[a_{i,j}].{}_\rho\gamma - [a_{i,j}]_\rho\zeta\|_n + \|[a_{i,j}].{}_\rho\zeta - [a_{i,j}].{}_\pi\zeta\|_n \\
&\leq \|[a_{i,j}]\|_n\|\gamma - \zeta\| + \delta\|[a_{i,j}]\|_n\|\zeta\| \\
&= \|[a_{i,j}]\|_n\|\psi(\zeta) - \zeta\| + \delta\|[a_{i,j}]\|_n\|\zeta\| \\
&\leq \delta\|[a_{i,j}]\|_n\|\zeta\|(1 + \frac{3}{1-3\delta}). \qquad (1)
\end{aligned}$$

Similarly

$$\|\gamma.{}_\rho[a_{i,j}].{}_\rho - \zeta_\pi[a_{i,j}]\|_n \leq \delta\|[a_{i,j}]\|_n\|\zeta\|(1 + \frac{3}{1-3\delta}). \qquad (2).$$

Therefore from (1) and (2), we obtain

$$\begin{aligned}
\|[a_{i,j}.{}_\rho\gamma - \gamma.{}_\rho a_{i,j} - a_{i,j}\otimes 1 + 1\otimes a_{i,j}]\|_n &= \|[a_{i,j}.{}_\rho\gamma - \gamma.{}_\rho a_{i,j} - a_{i,j}.{}_\pi\zeta + \zeta.{}_\pi a_{i,j}]\|_n \\
&\leq \|[a_{i,j}].{}_\rho\gamma - [a_{i,j}].{}_\pi\zeta\|_n + \|\gamma.{}_\rho[a_{i,j}] - \zeta.{}_\pi[a_{i,j}]\|_n \\
&\leq (2 + \frac{6}{1-3\delta})\delta\|[a_{i,j}]\|_n\|\zeta\|. \qquad (3)
\end{aligned}$$

We fix $\epsilon > 0$ and we let $k = k_1 + k_2 \in \ker(\rho)$ where $k_1 = \alpha([a_{i,j}]\otimes[b_{k,l}])\beta$ and $k_1, k_2 \in \ker(\rho)$ and $\|k_2\| < \epsilon$. Then

$$\begin{aligned}
\|k.{}_\rho\gamma - k\| &\leq \|k_1.{}_\rho\gamma - k + 1\| + \|k_2.{}_\rho\gamma - k_2\| \\
&< \|\alpha([a_{i,j}.{}_\rho\gamma.b_{k,l}])\beta - \alpha(\gamma.{}_\rho[a_{i,j\rho}b_{k,l}])\beta \\
&\quad - \alpha[(a_{i,j}\otimes 1).{}_\rho b_{k,l}]\beta + \alpha[(1\otimes a_{i,j}).\rho b_{k,l}]\beta\| + \epsilon(\|\gamma\| + 1) \quad (\alpha([a_{i,j\rho}b_{k,l}])\beta = \rho(k_1) = 0) \\
&\leq \|\alpha\|\|[a_{i,j}.{}_\rho\gamma - \gamma.{}_\rho a_{i,j} - a_{i,j}\otimes 1 + 1\otimes a_{i,j}]\|\|[b_{k,l}]\|\|\beta\| + \epsilon(\|\gamma\| + 1)
\end{aligned}$$

$$\leq (2+\frac{6}{1-3\delta})\delta\|[a_{i,j}]\|\|\zeta\|\|\alpha\|\|[b_{k,l}]\|\|\beta\| + \epsilon(\|\gamma\|+1) \qquad \text{(from (3))}$$

$$< \delta(2+\frac{6}{1-3\delta})\|\zeta\|\|k\| + O(\epsilon), \qquad (4)$$

where $\lim_{\epsilon\longrightarrow 0^+} O(\epsilon) = 0$. Let $R \in \ker_\rho^{**}$. Then there is a net $(k_i) \subset \ker(\rho)$ such that $\mathrm{wk}^* - \lim k_i = R$ and $\|k_i\| \leq \|R\|$. So using (4) we have

$$\|R\Box_\rho\gamma - R\| \leq (2+\frac{6}{1-3\delta})\delta\|\zeta\|\|R\|. \qquad (5)$$

Therefore if $\delta < \frac{1}{(6C+4).(2+\frac{6}{1-3\delta})}$, we have

$$(2+\frac{6}{1-3\delta})\delta\|\zeta\| < 1.$$

We define the map $\lambda : \ker(\rho^{**}) \longrightarrow \ker(\rho^{**})$ by $\lambda(R) = R\Box_\rho\gamma$. Form (5), we know that $\|\lambda - \mathrm{Id}_{\ker(\rho^{**})}\| \leq (2+\frac{6}{1-3\delta})\delta\|\zeta\|$. Therefore when $\delta < \frac{1}{(6C+4).(2+\frac{6}{1-3\delta})}$, we have that $\|\lambda - Id_{\ker(\rho^{**})}\| < 1$ and therefore λ will be invertible. Since λ is onto , there exists an $E \in \ker(\rho^{**})$ such that $E\Box_\rho\gamma = \gamma$. So for every $K \in \ker(\rho^{**})$ we have

$$\lambda(K) = (K\Box E - K)\Box_\rho\gamma = K\Box_\rho\gamma - K\Box_\rho E\Box_\rho\gamma = K\Box_\rho\gamma - K\Box_\rho\gamma = 0.$$

Therefore $\ker(\rho^{**})$ has a right identity and hence (A,ρ) is operator amenable. $\Box$

Corollary 2.30. *Let G be an amenable locally compact group and denote the point-wise multiplication of $A(G)$ by π. Suppose that $A(G)$ is equipped with a completely contractive multiplication ρ such that $\|\rho - \pi\|_{cb} < \frac{1}{83}$. Then $(A(G),\rho)$ is also operator amenable.*

Proof. As seen in [28] , when G is amenable, $A(G)$ has an operator contractive approximate diagonal say (m_{ξ_α}). Therefore the operator amenability constant is $C \leq 1$ and if $\delta = \|\rho - \pi\|_{cb} < \frac{1}{83}$, we have that

$$\delta(4C + 6)(2 + \frac{6}{1 - 3\delta}) < 1.$$

and hence the result follows form the last argument in Theorem 2.29. □

2.4 On the weak amenability of the tensor product of the Banach algebras

For a Banach algebra A we define $A^2 := span\{a_1a_2 : a_1, a_2 \in A\}$. We have this well-known Lemma.

Lemma 2.31. *Suppose that A is weakly amenable Banach algebra . Then A^2 is dense in A.*

Lemma 2.32. *Suppose that $A \otimes^\gamma B$ is weakly amenable. Then $\bar{A}^2 = A$ and $\bar{B}^2 = B$.*

Proof. From Lemma 2.31, we know that $(A \otimes^\gamma B)^2$ is dense in $A \otimes^\gamma B$. Now suppose that $\bar{B}^2 \neq B$. There is a $\lambda \in B^*$ such that $\lambda(B^2) = 0$ and $\lambda(b_0) = 1$ for some $b_0 \in B$. Let $f \in A^*$ be any non-zero continuous functional. Then we have $f \otimes \lambda \in (A\hat{\otimes}B)^*$ and

$$\langle f \otimes \lambda, a_1a_2 \otimes b_1b_2\rangle = \langle f, a_1a_2\rangle\langle \lambda, b_1b_2\rangle = 0 \;\; (a_1, a_2 \in A, b_1, b_2 \in B).$$

Therefore $f \otimes \lambda = 0$ on $A \otimes^\gamma B$ and so

$$\langle f \otimes \lambda, a \otimes b_0\rangle = \langle f, a\rangle\langle \lambda, b_0\rangle = 0 \quad (a \in A).$$

Hence from our assumption on λ, we have

$$\langle f, a\rangle = 0 \quad (a \in A).$$

This is a contradiction to the choice of $f \in A^*$. Therefore $\bar{B^2} = B$ and similarly $\bar{A^2} = A$. $\square$

In what follows , for a Banach algebra B we define

$$Z_B(B^*) = \{f \in B^* : f.b = b.f \ \ (b \in B)\}.$$

Theorem 2.33. *Supose that $A \otimes^\gamma B$ is weakly amenable and that $Z_B(B^*) \neq \{0\}$. Then A is weakly amenable.*

Proof. Suppose that $\lambda \in Z_B(B^*)$ is a non-zero functional and $D : A \longrightarrow A^*$ is a continuous derivation. Then we define $\tilde{D} : A \otimes^\gamma B \longrightarrow (A \otimes^\gamma B)^*$ defined by

$$\tilde{D}(a \otimes b) = D(a) \otimes \lambda.b \ \ (a \in A, b \in B).$$

Now we show that $\tilde{D}$ is indeed a derivation . For $a_i \in A, b_i \in B, i = 1, 2, 3$ we have

$$\begin{aligned}
\langle \tilde{D}(a_1a_2 \otimes b_1b_2), a_3 \otimes b_3\rangle &= \langle D(a_1a_2), a_3\rangle\langle \lambda.b_1b_2, b_3\rangle \\
&= \langle D(a_1).a_2 + a_1.D(a_2), a_3\rangle\langle \lambda, b_1b_2b_3\rangle \\
&= \langle D(a_1), a_2a_3\rangle\langle \lambda, b_1b_2b_3\rangle + \langle D(a_2), a_3a_1\rangle\langle \lambda, b_1b_2b_3\rangle.
\end{aligned}$$

On the other hand

$$\langle \tilde{D}(a_1 \otimes b_1).(a_2 \otimes b_2) + (a_1 \otimes b_1).\tilde{D}(a_2 \otimes b_2), a_3 \otimes b_3 \rangle =$$

$$= \langle \tilde{D}(a_1 \otimes b_1), a_2a_3 \otimes b_2b_3 \rangle + \langle \tilde{D}(a_2 \otimes b_2), a_3a_1 \otimes b_3b_1 \rangle$$

$$= \langle D(a_1), a_2a_3 \rangle \langle \lambda, b_1b_2b_3 \rangle + \langle D(a_2), a_3a_1 \rangle \langle \lambda, b_2b_3b_1 \rangle.$$

Now since $\lambda.b_1 = b_1.\lambda$, we have that $\langle \lambda, b_1b_2b_3 \rangle = \langle \lambda, b_2b_3b_1 \rangle$. Therefore

$$\tilde{D}(a_1a_2 \otimes b_1b_2) = (a_1 \otimes b_1).\tilde{D}(a_2 \otimes b_2) + \tilde{D}(a_1 \otimes b_1).(a_2 \otimes b_2).$$

and then by continuity of $\tilde{D}$ we can easily see that $\tilde{D}$ is a derivation from $A \otimes^\gamma B$ into $(A \otimes^\gamma B)^*$. Therefore there exists a $\phi \in (A \otimes^\gamma B)^*$ such that

$$\tilde{D}(a \otimes b) = (a \otimes b).\phi - \phi.(a \otimes b) \quad (a \in A, b \in B). \quad (*)$$

From Lemma 2.32, and the fact that $\lambda \in Z_B(B^*)$ and $\lambda \neq 0$, there is a $b_0 \in B$ such that $\lambda(b_0^2) = 1$; for otherwise we will have $\lambda|_{B^2} = 0$. For $b = b_0$ from (*) we have

$$D(a) \otimes \lambda.b_0 = (a \otimes b_0).\phi - \phi.(a \otimes b_0) \;\; (a \in A) \quad (**).$$

We define a mapping $T : (A \otimes^\gamma B)^* \longrightarrow A^*$ by

$$\langle T(\psi), a \rangle = \langle \psi, a \otimes b_0 \rangle \quad (a \in A).$$

By applying T on both sides of $(**)$, we will get for $x \in A$,

$$\langle T(D(a) \otimes \lambda.b_0), x \rangle = \langle D(a) \otimes \lambda.b_0, x \otimes b_0 \rangle$$

$$= \langle D(a), x\rangle\langle\lambda, b_0^2\rangle$$

$$= \langle D(a), x\rangle.$$

On the other hand

$$\langle T((a \otimes b_0).\phi - \phi.(a \otimes b_0)), x\rangle = \langle\phi, (xa - ax) \otimes b_0^2\rangle \ \ (x, a \in A),$$

and therefore

$$\langle D(a), x\rangle = \langle\phi, (xa - ax) \otimes b_0^2\rangle \ \ (a, x \in A).$$

So by letting $f \in A^*$, $\langle f, x\rangle = \langle\phi, x \otimes b_0^2\rangle$, we have

$$\langle D(a), x\rangle = \langle f, ax - xa\rangle$$

$$= \langle f.a - a.f, x\rangle.$$

Hence $D = \mathrm{ad_f}$ which completes the proof and shows that A is weakly amenable. $\square$

The question is under what conditions on an algebra B can one gaurantee that $Z_B(B^*) \neq \{0\}$? If B is a commutative Banach algebra or if B has a non-zero character it is trivialy true. But are there other examples of Banach algebras B for which this result holds? We now give one such sufficient condition.

Theorem 2.34. *Suppose that A is (operator) symmetrically amenable and suppose that there exists an element $a \in A$ such that $a \notin \overline{\mathrm{Lin}}\{aa_0 - a_0a : a_0 \in A\}$. Then $Z_A(A^*) \neq \{0\}$.*

Proof. Since A is (operator) amenable, we have that the short exact sequence

$$\Pi^* : 0 \longrightarrow A^* \xrightarrow{\pi^*} (A\hat{\otimes}A)^* \xrightarrow{\iota^*} K^* \longrightarrow 0,$$

of A-bimodules splits. Therefore there exists a left inverse of π^* denoted by ζ which is also an A-module morphism. Fix an $f \in A^*$. We claim that $q^*(\pi^*(f)) \in Z_A((A\hat{\otimes}A)^*)$, where $q : A\hat{\otimes}A \longrightarrow A\hat{\otimes}A$ is the flip mapping, specified by $q(a \otimes b) = b \otimes a, \quad (a, b \in A)$. To see this we have

$$\begin{aligned} \langle d.q(\pi^*(f)), b \otimes c\rangle &= \langle \pi^*(f), cd \otimes b\rangle \\ &= \langle f, cdb\rangle. \end{aligned}$$

On the other hand

$$\begin{aligned} \langle q(\pi^*(f)).d, b \otimes c\rangle &= \langle \pi^*(f), c \otimes db\rangle \\ &= \langle f, cdb\rangle. \end{aligned}$$

Let $\phi_f := \zeta(q^*(\pi^*(f))) \in A^*$. For every $a \in A$, considering the fact that ζ is an A-module morphism, we have

$$a.\phi_f = a.\zeta(q^*(\pi^*(f))) = \zeta(a.q^*(\pi^*(f))) = \zeta(q^*(\pi^*(f)).a) = \zeta(q^*(\pi^*(f))).a = \phi_f.a.$$

Thus for every $f \in A^*$ we have that $\phi_f \in Z_A(A^*)$. Now we complete the proof by showing that $Z_A(A^*)$ is non-trivial. As shown by Curtis and Loy in [4], the mapping ζ may be defined by

$$\langle \zeta(\lambda), a\rangle = \langle M, \lambda.a\rangle, \quad (a \in A, \lambda \in (A\hat{\otimes}A)^*),$$

where $M \in (A\hat{\otimes}A)^{**}$ is a virtual diagonal of A. Now assume that $Z_A(A^*) = \{0\}$. Then for every $f \in A^*$ we have that $\phi_f = 0$. Therefore for every $a \in A, f \in A^*$,

$$\begin{aligned} 0 = \langle \phi_f, a\rangle &= \langle M, q^*(\pi^*(f)).a\rangle \\ &= \langle a.M, q^*(\pi^*(f))\rangle \\ &= \langle \pi^{**}(q^{**}(a.M)), f\rangle \\ &= \langle \pi^{\circ **}(a.M), f\rangle. \end{aligned}$$

Hence

$$\pi^{\circ **}(a.M) = 0 \quad (a \in A).$$

Let $a \in A$ be such that $a \notin \overline{\mathrm{Lin}}\{aa_0 - a_0a : a_0 \in A\}$. Then there exists an $f \in A^*$ such that $f(a) = 1$ and

$$f(aa_0) = f(a_0a) \qquad\qquad (a_0 \in A).$$

Also let $(m_i) \subset A\hat{\otimes}A$ be a symmetric approximate diagonal for A such that $M = \mathrm{wk}^* - \lim_i m_i$. For each i, we assume that $m_i = \sum_k b_k^i \otimes c_k^i$. Therefore we have

$$\begin{aligned} 0 = \langle \pi^{\circ **}(a.M), f\rangle &= \lim_i \langle f, \pi^\circ(a.m_i)\rangle \\ &\lim_i \langle f, \sum_k c_k^i a b_k^i\rangle \\ &= \lim_i \langle f, a\sum_k c_k^i b_k^i\rangle \\ &= \lim_i \langle f, a\pi^\circ(m_i)\rangle \\ &= \lim_i \langle f, a\pi(m_i)\rangle \qquad \text{(Since } m_i \text{ is symmetric)} \end{aligned}$$

$$= \langle a.\pi^{**}(M), f \rangle$$

$$= \langle f, a \rangle.$$

But the above conclusion is a contradiction to our assumption that $\langle f, a \rangle = 1$. Therefore $Z_A(A^*) \neq \{0\}$. $\square$

Corollary 2.35. *Suppose that $A \otimes^\gamma B$ is weakly amenable and B is symmetrically amenable and there is $b \in B$ such that $b \notin \overline{\mathrm{Lin}}\{bb_0 - b_0 b : b_0 \in B\}$. Then A is weakly amenable.*

Proof. Theorem 2.34 implies that $Z_B(B^*) \neq \{0\}$ and therefore from Theorem 2.33, A is weakl y amenable. $\square$

2.5 On the generalized notions of amenability for the tensor product of Banach algebras and completely contractive Banach algebras

In [9], F. Ghahramani and R. J. Loy proved that if $A \otimes^\gamma B$ is amenable then are so A and B. We will consider same question for the generalized notions of amenability as well as operator amenability.

Theorem 2.36. *Suppose that $A \otimes^\gamma A^{op}$ is boundedly approximately amenable and that A has a two-sided bounded approximate identity. Then A is boundedly approximately amenable.*

Proof. The short exact sequence: $\Sigma : 0 \longrightarrow A^* \xrightarrow{\pi^*} (A \otimes^\gamma A^{op})^* \xrightarrow{\iota^*} K^* \longrightarrow 0$ is an addmissible short exact sequence of right $A \otimes^\gamma A^{op}$-modules. Since $A \otimes^\gamma A^{op}$ is approximately amenable, from [27, Theorem 3.3], Σ boundedly approximately splits. Therefore there exists a net $(\lambda_i)_i : K^* \longrightarrow (A \otimes^\gamma A^{op})^*$ of right inverses of ι^* and a constant $M > 0$ such that

$$\lim_i u.\lambda_i - \lambda_i.u = 0 \quad (u \in A \otimes^\gamma A^{op}),$$

$$\|u.\lambda_i - \lambda_i.u\| \leq M\|u\| \quad (u \in A \otimes^\gamma A^{op}).$$

Suppose that $E = wk^* - \lim e_\alpha \otimes e_\alpha$. We let $\phi_i = \lambda_i^*(E)$. Our goal is to show that ϕ_i is a right approximate identity for $K^{**} = (\ker \pi)^{**}$, where $\pi : A\hat{\otimes}A^{op} \longrightarrow A$ is the multiplication map.

Let $k \in K$. We have :

$$\begin{aligned}\langle k.\phi_i, f\rangle &= \langle k.\phi_i.f\rangle \\ &= \langle \lambda_i^*(E), f.k\rangle \\ &= \langle E, \lambda_i(f.k)\rangle.\end{aligned}$$

On the other hand we have :

$$\begin{aligned}\langle E, \lambda_i(f).k\rangle &= \lim_\alpha \langle \lambda_i(f).k, e_\alpha \otimes e_\alpha\rangle \\ &= \langle \lambda_i(f), \iota(k)\rangle \\ &= \langle \iota^*(\lambda_i(f)), k\rangle \\ &= \langle f, k\rangle.\end{aligned}$$

Since $\lim_i \lambda_i(f.k) - \lambda_i(f).k = 0$ independent of f, we have

$$\lim_i k.\phi_i = k \quad (k \in K),$$

where the limit is taken in the norm topology. Also it is easy to verify that $\|k\phi_i\| \leq M\|k\|$.

Therefore from [12, Theorem 5.10] , A is boundedly approximately amenable. $\square$

Theorem 2.37. *Suppose that $A \otimes^\gamma B$ is approximately amenable and there are elements $b_0, c_0, d_0 \in B$ such that $b_0c_0 = c_0$ and $d_0b_0 = d_0$, $d_0c_0 \neq 0$. Then A is approximately amenable.*

Proof. Since $d_0c_0 \neq 0$, there is $\eta \in B^*$ such that $\langle \eta, d_0c_0 \rangle = 1$. We specify $\phi : (B \otimes^\gamma B)^* \longrightarrow (B \otimes^\gamma B)^*$ by

$$\langle \phi(g), b \otimes c \rangle = \langle g, c \otimes b \rangle \quad (b, c \in B, g \in (B \otimes^\gamma B)^*),$$

i.e. let $\phi = q^*$, the adjoint of the flip map. Then we claim that $\phi(\pi^*(B^*)) \subset Z_B((B \otimes^\gamma B)^*)$. We have for $d \in B$,

$$\begin{aligned} \langle d.\phi(\pi^*(g)), b \otimes c \rangle &= \langle \pi^*(g), cd \otimes b \rangle \\ &= \langle g, cdb \rangle. \end{aligned}$$

On the other hand

$$\begin{aligned} \langle \phi(\pi^*(g)).d, b \otimes c \rangle &= \langle \pi^*(g), c \otimes db \rangle \\ &= \langle g, cdb \rangle. \end{aligned}$$

Therefore $\lambda := \phi(\pi^*(\eta)) \in Z_B((B \otimes^\gamma B)^*)$.

Suppose that X is a Banach A-bimodule and Y is a closed submodule of X and $f \in Z_A(Y^*)$. Since $B \otimes^\gamma B$ is a Banach B-bimodule in standard way, we can define module actions of $A \otimes^\gamma B$ on $X \otimes^\gamma (B \otimes^\gamma B)$ as follows for :

$$(a \otimes b).(x \otimes (c \otimes d)) = a.x \otimes (bc \otimes d),$$

$$(x \hat{\otimes} (c \otimes d)).(a \otimes b) = x.a \otimes (c \otimes db),$$

for $b, c, d \in B$ and $a \in A$.

Then $f \otimes \lambda \in Z_{A \otimes^\gamma B}((Y \otimes^\gamma (B \otimes^\gamma B))^*)$. Therefore from [27, Theorem 3.11], since $A \otimes^\gamma B$ is approximately amenable, we can find a net $(\zeta)_i \subset (X \otimes^\gamma (B \otimes^\gamma B))^*$ such that

$$\lim_i u.\zeta_i - \zeta_i.u = 0 \quad (u \in A \otimes^\gamma B), \qquad (*)$$

and $\zeta_i|_{Y \otimes^\gamma (B \otimes^\gamma B)} = f \otimes \lambda$, for all i.

Now we define $\tilde{f}_i \in X^*$ by

$$\langle \tilde{f}_i, x \rangle = \langle \zeta_i, x \otimes (c_0 \otimes d_0) \rangle \quad (x \in X).$$

Then using our assumption on c_0 and d_0, we have

$$\begin{aligned} \langle a.\tilde{f}_i - \tilde{f}_i.a, x \rangle &= \langle \zeta_i, x.a \otimes (c_0 \otimes d_0 b_0) - a.x \otimes (b_0 c_0 \otimes d_0) \rangle \\ &= \langle \zeta_i, (a \otimes b_0).(x \otimes (c_0 \otimes d_0)) - (x \otimes (c_0 \otimes d_0)).(a \otimes b_0) \rangle \\ &= \langle \zeta_i.(a \otimes b_0) - (a \otimes b_0).\zeta_i, x \otimes (c_0 \otimes d_0) \rangle. \end{aligned}$$

Since the limit in (*) is norm-limit, we have

$$\lim_i a.\tilde{f}_i - \tilde{f}_i.a = 0 \quad (a \in A),$$

and for $y \in Y$,

$$\begin{aligned}
\langle \tilde{f}_i, y\rangle &= \langle \zeta_i, y \otimes (c_0 \otimes d_0)\rangle \\
&= \langle f \otimes \lambda, y \otimes (c_0 \otimes d_0)\rangle \\
&= \langle f, y\rangle\langle \lambda, c_0 \otimes d_0\rangle \\
&= \langle f, y\rangle\langle \eta, d_0c_0\rangle \\
&= \langle f, y\rangle.
\end{aligned}$$

Thus we have a net $(\tilde{f}_i) \subset X^*$ of extensions of f such that for all $a \in A$, $\lim_i a.\tilde{f}_i - \tilde{f}_i.a = 0$. So we have that A is approximately amenable from [27, Theorem 3.11]. $\square$

Throughout the remainder of this section all the tensor products are operator space tensor products, unless otherwise indicated.

Lemma 2.38. *Suppose that A is a completely contractive Banach algebra. We define right and left actions of A on $A\hat{\otimes}A$ denoted by $\circ$ as follows*

$$a \circ (b \otimes c) = b \otimes ac$$

$$(b \otimes c) \circ a = ba \otimes c,$$

$a, b, c \in A$. Then $A\hat{\otimes}A$ is an operator A-bimodule via these actions.

Proof. A straight forward calculation will show that $A\hat{\otimes}A$ is an A-bimodule via the actions $\circ$ and similar argument as showing that $A\hat{\otimes}A$ is an operator A-bimodule will show that $\circ$ defines an operator bi-module action on $A\hat{\otimes}A$. $\square$

Since $\circ$ turns $A\hat{\otimes}A$ into an A operator bi-module, we can consider its action on $(A\hat{\otimes}A)^*$ in the usual way. We still denote this action by $\circ$.

Lemma 2.39. *Suppose that A is a completely contractive Banach algebra. We let $X = (A\hat{\otimes}A)^*\hat{\otimes}(A\hat{\otimes}A)$ and we define an action of A on X by*

$$a.(f \otimes u) = f \otimes a.u$$

$$(f \otimes u).a = f \otimes u.a,$$

$f \in (A\hat{\otimes}A)^*$ and $u \in (A\hat{\otimes}A)$ and $a.u$ and $u.a$ are the standard actions of A on $A\hat{\otimes}A$. Then X is an operator A-bimodule and if we let

$$L := \overline{\text{span}}\{(a \circ f) \otimes u - f \otimes (u \circ a) : f \in (A\hat{\otimes}A)^*, u \in A\hat{\otimes}A, a \in A\}$$

and

$$R := \overline{\text{span}}\{(f \circ a) \otimes u - f \otimes (a \circ u) : f \in (A\hat{\otimes}A)^*, u \in A\hat{\otimes}A, a \in A\},$$

we have that R and L are closed submodules of X .

Proof. First we show that X is an operator A-bimodule. It can be easily checked that X is an A-bimodule. Due to the way we define the norm of the operator projective tensor product of operator spaces, if $\Omega = [x_{i,j}] = \alpha(v \otimes w)\beta \in M_n(X)$ where $\alpha \in M_{n,pq}$, $v \in M_p((A\hat{\otimes}A)^*)$, $w \in M_q(A\hat{\otimes}A)$ and $\beta \in M_{pq,n}$. Then for $[a_{s,t}] \in M_m(A)$ we have

$$[x_{i,j}.a_{s,t}]_{nm} = (\underbrace{\alpha \oplus \cdots \oplus \alpha}_{m})((v \otimes \langle\langle w, [a_{s,t}]\rangle\rangle))(\underbrace{\beta \oplus \cdots \oplus \beta}_{m}).$$

Therefore

$$\begin{aligned}\|[x_{i,j}.a_{s,t}]\|_{nm} &\leq \|\alpha\|\|v\|_p\|\langle\langle w, [a_{s,t}]\rangle\rangle\|_{qm}\|\beta\| \\ &\leq \|\alpha\|\|v\|_p\|w\|_q\|[a_{s,t}]\|_m\|\beta\|.\end{aligned}$$

Therefore by taking the infimum over all such representations of Ω, we see that

$$\|[x_{i,j}.a_{s,t}]\| \leq \|[x_{i,j}]\|\|[a_{s,t}]\|,$$

which shows that X is an operator A right- module and a similar argument shows it is an operator left-module. So X is an operator A-bimodule. We show that L is a closed submodule of X. A similar argument holds for R. For $f \in (A\hat{\otimes}A)^*$ and $u \in A\hat{\otimes}A$ we have

$$b.(a \circ f \otimes u - f \otimes u \circ a) = a \circ f \otimes b.u - f \otimes (b.u) \circ a.$$

For the right side, we get

$$(a \circ f \otimes u - f \otimes u \circ a).b = a \circ f \otimes u.b - f \otimes (u.b) \circ a.$$

Therefore L is a closed submodule of X. $\square$

Now using Lemmas 2.38, 2.39, we can prove the following characterization of operator amenability and similar characterizations for the generalized notions of operator amenability as well. The Banach space version of the following theorem is due to Lau [23, Theorem 1] although we use different techniques.

Theorem 2.40. *Suppose that A is a completely contractive Banach algebra. Then the followings are equivalent:*

(i) A is operator amenable.

(ii) For any operator A bimodule X and Y a closed submodule of X, every $f \in Z_A(Y^*)$ can be extended to a functional $\tilde{f} \in Z_A(X^*)$.

Proof. (i)⇒(ii): Suppose that $f \in Z_A(Y^*)$ and $\bar{f} \in X^*$ is an extension of f by the Hahn-Banach theorem. We define derivation $D : A \longrightarrow X^*$ by

$$D(a) = a.\bar{f} - \bar{f}.a \quad (a \in A).$$

Obviously D is a completely bounded dertivation into $Y^\perp \cong (X/Y)^*$ and therefore from the operator amenability of A, there is $\zeta \in Y^\perp$ such that

$$D(a) = a.\zeta - \zeta.a \quad (a \in A).$$

We now define $\tilde{f} = \bar{f} - \zeta$. It can be easily seen that $\tilde{f}|_Y = f$ and for all $a \in A$ we have that $a.\tilde{f} - \tilde{f}.a = 0$. So $\tilde{f} \in Z_A(X^*)$ satisfies condition (ii).

(ii)⇒(i) We let $\Omega = (A\hat{\otimes}A)^*\hat{\otimes}(A\hat{\otimes}A)$. From Lemma 2.39, we know that Ω is an operator

A-bimodule. Since the operator amenability of A and $A^{\#}$ are equivalent, by substituting A by $A^{\#}$, from now on and knowing the fact that Ω is a unital $A^{\#}$-bimodule, we will have the same result. So without loss of generality we can assume that A is unital with unit denoted by 1. We let $Z = \overline{span}\{f \otimes u : f \in Z_A((A\hat{\otimes}A)^*), u \in (A\hat{\otimes}A)\}$ and we let Y to be $\overline{span}\{L, R, Z\}$, where L, R are as defined in Lemma 2.39. So Y is a closed submodule of Ω. Now we define the functional $\phi \in Y^*$ by

$$\langle \phi, f \otimes u \rangle = \langle f, u \rangle \quad (f \otimes u \in Y).$$

Then we have that $\phi|_R = \phi|_L = 0$. Given $f \in Z_A(A\hat{\otimes}A)^*$, we have

$$\begin{aligned}
\langle a.\phi, f \otimes u \rangle &= \langle \phi, f \otimes u.a \rangle \\
&= \langle f, u.a \rangle \\
&= \langle f, a.u \rangle \\
&= \langle \phi.a, f \otimes u \rangle.
\end{aligned}$$

Therefore $\phi \in Z_A(Y^*)$. Thus there is an extention $\tilde{\phi}$ of ϕ to Ω^* such that $\tilde{\phi} \in Z_A(\Omega^*)$. We fix $\tilde{\phi}$, and define an operator $T : (A\hat{\otimes}A)^* \longrightarrow (A\hat{\otimes}A)^*$ by

$$\langle T(f), u \rangle = \langle \tilde{\phi}, f \otimes u \rangle \quad (f \in (A\hat{\otimes}A)^*, u \in A\hat{\otimes}A).$$

From the definition of $\tilde{\phi}$, we obtain

$$T(f) = f \quad (f \in Z_A(A\hat{\otimes}A)^*).$$

Furtheremore since $\phi|_L = \phi|_R = 0$, we conclude that T is a module morphism with respect to the action $\circ$ of A on $A\hat{\otimes}A$. Also T maps $(A\hat{\otimes}A)^*$ onto $Z_A((A\hat{\otimes}A)^*)$. In fact, since $\tilde{\phi} \in Z_A(\Omega^*)$, we have

$$\begin{aligned}
\langle a.T(f), u\rangle &= \langle \tilde{\phi}, f \otimes u.a\rangle \\
&= \langle \tilde{\phi}, f \otimes a.u\rangle \\
&= \langle T(f).a, u\rangle.
\end{aligned}$$

Now suppose that X is a (unital) operator A-bimodule and $D : A \longrightarrow X^*$ is a completely bounded derivation. For $x \in X$, we specify $\mu_x \in (A\hat{\otimes}A)^*$ by

$$\langle \mu_x, a \otimes b\rangle = \langle D(a), x.b\rangle \quad (a, b \in A).$$

Given $a \in A$, by evaluationg $a \circ \mu_x$ on elementary tensor $b \otimes c$, $(b, c \in A)$, we obtain

$$\begin{aligned}
\langle a \circ \mu_x, b \otimes c\rangle &= \langle \mu_x, ba \otimes c\rangle \\
&= \langle D(ba), xc\rangle \\
&= \langle b.D(a) + D(b).a, xc\rangle \\
&= \langle D(a), xcb\rangle + \langle D(b), axc\rangle \\
&= \langle \mu_{ax}, b \otimes c\rangle + \langle D(a), xcb\rangle.
\end{aligned}$$

Therefore if we specify $\psi_{a,x} \in (A\hat{\otimes}A)^*$ by acting on elemenary tensors as

$$\langle \psi_{a,x}, b \otimes c\rangle = \langle D(a), xcb\rangle \quad (a, b, c \in A, x \in X),$$

we will get

$$a \circ \mu_x = \mu_{ax} + \psi_{a,x} \quad (a \in A, x \in X).$$

On the other hand

$$\begin{aligned}\langle \mu_x \circ a, b \otimes c\rangle &= \langle \mu_x, b \otimes ac\rangle \\ &= \langle D(b), xac\rangle \\ &= \langle \mu_{xa}, b \otimes c\rangle.\end{aligned}$$

Now we define $\lambda \in X^*$ by

$$\langle \lambda, x\rangle = T(\mu_x), 1 \otimes 1\rangle \quad (x \in X).$$

We have

$$\begin{aligned}\langle \lambda.a - a.\lambda, x\rangle &= \langle T(\mu_{x.a} - \mu_{a.x}), 1 \otimes 1\rangle \\ &= \langle T(\mu_x \circ a - a \circ \mu_x), 1 \otimes 1\rangle + \langle T(\psi_{a,x}), 1 \otimes 1\rangle \\ &= \langle a \circ T(\mu_x) - T(\mu_x) \circ a, 1 \otimes 1\rangle + \langle \psi_{a,x}, 1 \otimes 1\rangle \\ &= \langle T(\mu_x), a \otimes 1 - 1 \otimes a\rangle + \langle D(a), x\rangle.\end{aligned}$$

Now since $T(\mu_x) \in Z_A((A\hat{\otimes}A)^*)$, we have that $\langle T(\mu_x), a \otimes 1 - 1 \otimes a\rangle = 0$. Therefore

$$D(a) = \lambda.a - a.\lambda \quad (a \in A),$$

which completes the proof.

$\square$

Lemma 2.41. *Suppose that A and B are completely contractive Banach algebras and X and Y are operator A and B bimodules respectively. Then $X\hat{\otimes}Y$ is an operator $(A\hat{\otimes}B)$-bimodule in a canonical way.*

Proof. Let C and $\acute{C}$ be the constants for the modules X and Y such that they fulfill the completely bounded norm inequalities for the actions of A on X and B on Y, respectively. For $[u_{i,j}] = \alpha(v \otimes w)\beta \in M_n(A\hat{\otimes}B)$, where $\alpha \in M_{n,pq}, v \in M_p(A), w \in M_q(B), \beta \in M_{pq,n}$ and for $[r_{k,l}] = \acute{\alpha}(\acute{v} \otimes \acute{w})\acute{\beta} \in M_{\acute{n}}(X\hat{\otimes}Y)$ where $\acute{\alpha} \in \mathbb{M}_{\acute{n}\acute{p}\acute{q}}, \acute{v} \in M_{\acute{p}}(X), \acute{w} \in M_{\acute{q}}(Y), \acute{\beta} \in \mathbb{M}_{\acute{p}\acute{q},\acute{n}}$, we have that

$$\begin{aligned}\langle\langle [u_{i,j}], [r_{k,l}]\rangle\rangle &= \langle\langle \alpha(v \otimes w)\beta, \acute{\alpha}(\acute{v} \otimes \acute{w})\acute{\beta}\rangle\rangle \\ &= \langle\langle \alpha, \acute{\alpha}\rangle\rangle(\langle\langle v, \acute{v}\rangle\rangle \otimes \langle\langle w, \acute{w}\rangle\rangle)\langle\langle \beta, \acute{\beta}\rangle\rangle.\end{aligned}$$

Therefore

$$\|\langle\langle u_{i,J}, r_{k,l}\rangle\rangle\|_{n\acute{n}} \leq C\acute{C}\|\alpha\|\|\acute{\alpha}\|\|v\|_p\|\acute{v}\|_{\acute{p}}\|w\|_q\|\acute{w}\|_{\acute{q}}\|\beta\|\|\acute{\beta}\|,$$

and hence

$$\|\langle [u_{i,j}], [r_{s,t}]\rangle\rangle\|_{n\acute{n}} \leq C\acute{C}\|[u_{i,j}]\|_n\|r_{s,t}\|_{\acute{n}},$$

which completes the proof.

□

Lemma 2.42. *Suppose that A and B are completely contractive Banach algebras such that $A\hat{\otimes}B$ has a bounded approximate identity . Then both A and B have a bounded approximate identity.*

Proof. Suppose that the net $(e_i)_{i\in I}$ be a bounded approximate identity for $A\hat{\otimes}B$ and let $e_i = \alpha_i(v_i \otimes w_i)\beta_i$ where $\alpha_i \in \mathbb{M}_{1,p_iq_i}, v_i \in M_{p_i}(A), w_i \in M_{q_i}(B), \beta_i \in \mathbb{M}_{p_iq_i,1}$. Let $f \in B^*$ be a non-zero functional and $b_0 \in B$ such that $f(b_0) = 1$. Then we define a mapping $T : A\hat{\otimes}B \longrightarrow A$ by

$$T(\alpha(v \otimes w)\beta) = \alpha(v \otimes f_q(w))\beta \quad (\alpha \in \mathbb{M}_{1,pq}, v \in M_p(A), w \in M_q(B), \beta \in \mathbb{M}_{pq,1}.$$

Then we have for $\alpha \in \mathbb{M}_{n,pq}, v \in M_p(A), w \in M_q(B), \beta \in \mathbb{M}_{pq,n}$.

$$\begin{aligned}
\|T_n(\alpha(v \otimes w)\beta)\| &= \|\alpha(v \otimes f_q(w))\beta\| \\
&\leq \|\alpha\|\|v \otimes f_q(w)\|\|\beta\| \\
&= \|\alpha\|\|v\|_p\|f_q(w)\|_q\|\beta\| \\
&\leq \|\alpha\|\|v\|_p\|f\|_{cb}\|w\|_q\|\beta\| \\
&= \|\alpha\|\|v\|_p\|f\|\|w\|_q\|\beta\|.
\end{aligned}$$

Thus by the way we define the operator tensor product norm, we see that $\|T_n\| \leq \|f\|$ for all $n \geq 1$. So T defines a completely bounded mapping. Fix $a \in A$. We have that $\lim_i(a \otimes b_0)e_i = a \otimes b_0$. By applying T on both sides of this identity we obtain

$$\lim_i a.(\alpha_i(v_i \otimes f_{q_i}(b_0.w_i))\beta_i) = af(b_0) = a.$$

Thus by letting $\phi_i := \alpha_i(v_i \otimes f_{q_i}(b_0.w_i))\beta_i$, we will obtain a bounded net in A that is also an approximate identity for A. A symmetric argument shows the result for B. $\square$

The Banach algebra version of the following theorem is shown in [9, Theorem 4.9]. Here we prove the operator space version from a completely new way.

Theorem 2.43. *Suppose that A and B are completely contractive Banach algebras such that $A\hat{\otimes}B$ is operator amenable and neither A or B is $\{0\}$. Then so are A and B.*

Proof. In order to show that A is operator amenable, from Theorem 2.40, it is enough to show that for any operator A-bimodule X and any closed submodule Y of X, every functional $f \in Z_A(Y^*)$ can be extended to a functional $\tilde{f} \in Z_A(X^*)$. Since $B\hat{\otimes}B$ is an operator B-bimodule in the standard way, from Lemma 2.41 , we have that $X\hat{\otimes}(B\hat{\otimes}B)$ is an operator $A\hat{\otimes}B$-bimodule in canonical way.

Let $g \in B^*$ be a non-zero functional. Then as shown in the proof of Theorem 2.37, we have that the functional $\mu_0 \in (B\hat{\otimes}B)^*$ specified by

$$\langle \mu_0, b \otimes c\rangle = \langle g, cb\rangle \quad (b, c \in B),$$

is in $Z_B((B\hat{\otimes}B)^*)$. We specify $\zeta \in (Y\hat{\otimes}(B\hat{\otimes}B))^*$ by

$$\langle \zeta, y \otimes u\rangle = \langle f, y\rangle\langle \mu_0, u\rangle \quad (y \in Y, u \in B\hat{\otimes}B).$$

It can be easily seen (from the way we define the module action of $A\hat{\otimes}B$ on $Y\hat{\otimes}(B\hat{\otimes}B)$ and given that $f \in Z_A(Y^*)$ and $\mu_0 \in Z_B((B\hat{\otimes}B)^*)$, that $\zeta \in Z_{A\hat{\otimes}B}((Y\hat{\otimes}(B\hat{\otimes}B))^*)$. So from Theorem 2.40, it can be extended to a fuctional $\phi \in Z_{A\otimes\hat{B}}((X\hat{\otimes}(B\hat{\otimes}B))^*)$. From Lemma 2.42, B has a bounded approximate identity, say $(e_\alpha)_{\alpha\in\Lambda}$. Therefore from Cohen's factorization Theorem, there exist $b_0, c_0 \in B$ such that $\langle g, c_0 b_0\rangle = 1$. We define $\psi \in X^*$ by

$$\langle \psi, x\rangle = \langle \phi, x \otimes (b_0 \otimes c_0)\rangle \quad (x \in X).$$

Therefore we have

$$\begin{aligned}
\langle \psi, a.x\rangle &= \langle \phi, ax \otimes (b_0 \otimes c_0)\rangle \\
&= \lim_\alpha \langle \phi, a.x \otimes (e_\alpha b_0 \otimes c_0)\rangle \\
&= \lim_\alpha \langle \phi.(a \otimes e_\alpha), x \otimes (b_0 \otimes c_0)\rangle. \qquad (1)
\end{aligned}$$

In a similar way we will get

$$\langle \psi, x.a\rangle = \lim_\alpha \langle (a \otimes e_\alpha).\phi, x \otimes (b_0 \otimes c_0)\rangle. \qquad (2)$$

Therefore from (1) and (2) we obtain :

$$\begin{aligned}
\langle a.\psi - \psi.a, x\rangle &= \langle \psi, x.a - a.x\rangle \\
&= \lim_\alpha \langle (a \otimes e_\alpha).\phi - \phi.(a \otimes e_\alpha), x \otimes (b_0 \otimes c_0)\rangle \\
&= 0.
\end{aligned}$$

Hence $\psi \in Z_A(X^*)$ and from the choice of $c_0, b_0 \in B$, it is easy to see that $\psi_Y = f$. So the result holds by Theorem 2.40. $\square$

It can be observed that the above also gives a new proof of [9, Theorem 4.9]. Here is a generalization of the Theorem 2.41

Theorem 2.44. *The following conditions are equivalent for a completely contractive Banach algebra A.*

(i) A is operator approximately amenable.

(ii) For any operator banach A-bimodule X and any closed submodule Y of X, if $f \in Z_A(Y^*)$, there is a net $(\tilde{f}_i) \subset X^*$ of extensions of f such that

$$\lim_i a.\tilde{f}_i - \tilde{f}_i.a = 0 \quad (a \in A).$$

Proof. (i)⇒(ii): Suppose that $f \in Z_A(Y^*)$ and $\bar{f} \in X^*$ is an extension of f by the Hahn-Banach theorem. We define a derivation $D : A \longrightarrow X^*$ by

$$D(a) = a.\bar{f} - \bar{f}.a \quad (a \in A).$$

Obviously D is a completely bounded dertivation into $Y^{\perp} \cong (X/Y)^*$ and therefore from the operator amenability of A, there is a net $(\zeta_i) \subset Y^{\perp}$ such that

$$D(a) = \lim_i a.\zeta_i - \zeta_i.a \quad (a \in A).$$

We now define $\tilde{f}_i = \bar{f} - \zeta_i$. It can be easily seen that $\tilde{f}_i|_Y = f$ and for all $a \in A$ we have that $\lim_i a.\tilde{f}_i - \tilde{f}_i.a = 0$.

(ii)⇒(i): As in the proof of Theroem 2.40, by keeping the same notations, we can find a net $(\tilde{\phi}_i)_{i \in I} \subset \Omega^*$ of extensions of ϕ such that

$$\lim_i a.\tilde{\phi}_i - \tilde{\phi}_i.a = 0 \quad (a \in A).$$

For each $i \in I$ we define $T_i : (A\hat{\otimes}A)^* \longrightarrow (A\hat{\otimes}A)^*$ by

$$\langle T(f), u\rangle = \langle \tilde{\phi}_i, f \otimes u\rangle \quad (f \in (A\hat{\otimes}A)^*, u \in (A\hat{\otimes}A)).$$

We have that

$$T_i(f) = f \quad (f \in Z_A(A\hat{\otimes}A)^*, i \in I).$$

It also can be easily seen that each T_i is a module morphism with respect to the actions $\circ$ of A on $A\hat{\otimes}A$. Furthermore, for $a \in A$, $u \in A\hat{\otimes}A$ and $f \in (A\hat{\otimes}A)^*$, we have

$$\begin{aligned}\lim_i \langle (a.T_i - T_i.a)(f), u\rangle &= \lim_i \langle \tilde{\phi}_i, f \otimes u.a - f \otimes a.u\rangle \\ &= \langle a.\tilde{\phi}_i - \tilde{\phi}_i.a, f \otimes u\rangle \\ &= 0,\end{aligned}$$

and since the last limit does not depend on the norm of u anf f, we obtain

$$\lim_i a.T_i - T_i.a = 0 \quad (a \in A).$$

Now suppose that X is a (unital) operator A-bimodule and $D : A \longrightarrow X^*$ is a completely bounded derivation. For $x \in X$ and $a \in A$ let μ_x and $\psi_{a,x}$ be as in the proof of Theorem 2.40. For all i, we define $\lambda_i \in X^*$ by

$$\langle \lambda_i, x\rangle = \langle T_i(\mu_x), 1 \otimes 1\rangle \quad (x \in X).$$

We have

$$\langle \lambda_i.a - a.\lambda_i, x\rangle = \langle T_i(\mu_{x.a} - \mu_{a.x}), 1 \otimes 1\rangle$$

$$= \langle T_i(\mu_x \circ a - a \circ \mu_x), 1 \otimes 1\rangle + \langle T_i(\psi_{a,x}), 1 \otimes 1\rangle$$

$$= \langle a \circ T_i(\mu_x) - T_i(\mu_x) \circ a, 1 \otimes 1\rangle + \langle \psi_{a,x}, 1 \otimes 1\rangle$$

$$= \langle T_i(\mu_x), a \otimes 1 - 1 \otimes a\rangle + \langle D(a), x\rangle$$

$$= \langle (T_i.a - a.T_i)(\mu_x), 1 \otimes 1\rangle + \langle D(a), x\rangle.$$

Since $\lim_i a.T_i - T_i.a = 0$, we have that

$$\lim_i \langle D(a) - (\lambda_i.a - a.\lambda_i), x\rangle = \lim_i \langle (a.T_i - T_i.a)(\mu_x), 1 \otimes 1\rangle$$

$$= 0.$$

Since the last limit only depends on the norm of μ_x (and hence the norm of x) but not x itself , we get

$$D(a) = \lim_i \lambda_i.a - a.\lambda_i \quad (a \in A),$$

which completes the proof.

$\square$

As we saw in the proof of Theorem 2.44, if we had the assumption that $\text{wk}^* - \lim_i a.\tilde{\phi}_i - \tilde{\phi}_i.a = 0$, then we would have that

$$\lim_i \langle (a.T_i - T_i.a)(\mu_x), 1 \otimes 1\rangle = 0.$$

Thus

$$\lim_i \langle D(a) - (\lambda_i.a - a.\lambda_i), x\rangle = 0 \quad (x \in X).$$

So A is wk* (operator) approximately amenable and therefore is (operator) approximately amenable .

Theorem 2.45. *Suppose that B is a (completely contractive) Banach algebra such that there exists $U \in (B\hat{\otimes}B)^{**}$ and $b_0 \in B$ such that $b_0 \cdot U = U \cdot b_0$ and $\pi^{\circ **}(b_0 \cdot U) \neq 0$. Then for any (completely contractive) Banach algebra A,(operator)(boundedly) approximate amenability of $A\hat{\otimes}B$ implies that of A.*

Proof. Let X be an (operator) A-bimodule and Y a closed submodule of X and take $f \in Z_A(A^*)$. Since $\pi^{\circ **}(b_0.U) \neq 0$, there exists $g \in B^*$ such that $\langle \pi^{\circ **}(b_0.U), g\rangle = 1$. Now we define $\lambda \in (B\hat{\otimes}B)^{***}$ by

$$\langle \lambda, V\rangle = \langle \pi^{\circ **}(V), g\rangle \qquad (V \in (B\hat{\otimes}B)^{**}).$$

It can be easily seen that

$$\pi^{\circ **}(b.V) = \pi^{\circ **}(V.b) \qquad (b \in B, V \in (B\hat{\otimes}B)^{**}.)$$

Therefore

$$\langle b.\lambda, V\rangle = \langle \pi^{\circ **}(V.b), g\rangle = \langle \pi^{\circ **}(b.V), g\rangle = \langle \lambda, b.V\rangle = \langle \lambda.b, V\rangle.$$

Thus $\lambda \in Z_B((B\hat{\otimes}B)^{***})$. Also $\langle \lambda, b_0.U\rangle = \langle \pi^{\circ **}(b_0.U), g\rangle = 1$, by the choice of g. Therefore we have $f \otimes \lambda \in Z_{A\hat{\otimes}B}((Y\hat{\otimes}(B\hat{\otimes}B)^{**})^*)$. So by Theorem 2.43, there is a net $(\phi_i) \subset (X\hat{\otimes}(B\hat{\otimes}B)^{**})^*$ of extensions of f such that

$$\lim_i u.\phi_i - \phi_i.u = 0 \qquad (u \in (A\hat{\otimes}B)).$$

Now we define a net $(\psi_i) \subset X^*$ by

$$\langle \psi_i, x \rangle = \langle \phi_i, x \otimes b_0.U \rangle \qquad (x \in X).$$

Therefore we have

$$\begin{aligned}
\lim_i \langle \psi_i.a - a.\psi_i, x \rangle &= \lim_i \langle \psi_i, a.x - x.a \rangle \\
&= \lim_i (\langle \phi_i, a.x \otimes b_0.U \rangle - \langle \phi_i, x.a \otimes U.b_0 \rangle \qquad (\text{Since } \ b_0.U = U.b_0) \\
&= \lim_i \langle \phi_i.(a \otimes b_0) - (a \otimes b_0).\phi_i, x \otimes U \rangle = 0.
\end{aligned}$$

Since the above limit only depends on the norm of x, we have

$$\lim_i a.\psi_i - \psi_i.a = 0 \qquad (a \in A).$$

In case of $A \hat{\otimes} B$ being boundedly approximately amenable , we will have the extra condition that

$$\|u.\phi_i - \phi_i.u\| \leq M\|u\| \qquad (u \in (A \hat{\otimes} B),$$

for some constant $M > 0$. Therefore we have

$$\begin{aligned}
\|\psi.a - a.\psi\| &= \sup_{\|x\| \leq 1} \{ |\langle \psi_i.a - a.\psi_i, x \rangle| \} \\
&= \sup_{\|x\| \leq 1} \{ |\langle \phi_i.(a \otimes b_0) - (a \otimes b_0).\phi_i, x \otimes U \rangle| \} \\
&\leq M\|U\|\|a\|\|b_0\|.
\end{aligned}$$

Therefore

$$\|a.\psi_i - \psi_i.a\| \leq C\|a\| \qquad (a \in A),$$

where $C = M\|U\|\|b_0\|$. Hence A is boundedly approximately amenable. $\square$

As some examples of Banach algebras B with the above property we have :

Remark 2.46. *Suppose that there are $F, G \in B^{**}$ and $b_0 \in B$ such that $b_0.F = F$ and $G.b_0 = G$ and $G \in Z_t(B^{**})$, where $Z_t(B^{**})$ is the topological centre of B^{**}, and $G \square F \neq 0$. If we let $U = \Psi(F \otimes G)$ where $\Psi : (B^{**} \otimes B^{**}) \longrightarrow (B \hat{\otimes} B)^{**}$ is the operator introduced in [9], we have $b_0.U = U.b_0$ and*

$$\pi^{\circ **}(b_0.U) = \pi^{\circ **}(\Psi(b_0.F \otimes G)) = \pi^{\circ **}(\psi(F \otimes G)).$$

Let $F = \mathrm{wk}^* - \lim_i f_i$ and $G = \mathrm{wk}^* - \lim_j g_j$. Then for $f \in B^*$, we have

$$\begin{aligned}
\langle \pi^{\circ **}(\psi(F \otimes G)), f \rangle &= \langle \psi(F \otimes G), \pi^{\circ *}(f) \rangle \\
&= \lim_i \lim_j \langle \pi^{\circ *}(f), f_i \otimes g_j \rangle \\
&= \lim_i \lim_j \langle f, \pi^{\circ}(f_i \otimes g_j) \rangle \\
&= \lim_i \lim_j \langle f, g_j f_i \rangle \\
&= \lim_i \langle G.f_i, f \rangle = \langle G \square F, f \rangle \qquad (\text{Since } \ G \in Z_t(B^{**})).
\end{aligned}$$

Therefore

$$\pi^{\circ **}(b_0.U) \neq 0.$$

In particular if there are $b, c, d \in B$ such that $bc = c$ and $db = d$ and $dc \neq 0$ then we have *the above property. (Since $B \subset Z_t(B^{**})$). This in particular says that the Theorem 2.37 is a corollary of Theorem 2.45.*

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