HAMILTONIAN ANALYSIS OF YANG-MILLS THEORIES IN NON-COVARIANT GAUGES

BY

KIN WAH MAK

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University of Manitoba
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ABSTRACT

A systematic Hamiltonian analysis of non-abelian gauge theories in non-covariant gauges is presented in detail. The formalism can be applied immediately to many important gauges such as the *regulated Coulomb and static temporal gauges* and the axial gauges with *Leibbrandt-Mandelstam prescription.* While the former was used in the finite temperature calculations, the latter was proposed to regularize the $(n \cdot k)$ singularity in the gluon self-energy. A review of various methods of quantizing the Yang-Mills theories are carried out in different gauges: canonical quantization in Feynman gauge and BRS quantization in covariant gauge. Path integral quantization is presented only as a convenient way to introduce the gauge fixed Lagrangian of the theories. The idea of BRS quantization in covariant gauge is then generalized to a general class of non-covariant gauges which turns out to be separable into three distinct classes. BRS Hamiltonian analysis is carried out for each class, leading to the appropriate definition of a physical subspace. Singular gauges such as the Coulomb and timelike axial gauge are considered as either belonging to class I and class II gauges or appropriate limits of the gauge parameters of class III gauges.

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1 Introduction

The idea of gauge symmetry plays an essential role in quantum field theory. One of the earliest references to gauge invariance dates back over 50 years to the pioneering classic investigations of Weyl [1], who exploited this principle in the quantization of the Maxwell-Dirac field. However, Yang and Mills [2] were the first to generalize the notion of local gauge invariance to non-abelian groups and then the notion that symmetries generate interactions was fully expressed. Their motivation stemmed from the desire to obtain a theory in which strong interactions are invariant under local isospin rotations, introduced by Heisenberg. Nevertheless, the concept of local invariance was used for the first time as a fundamental physical principle in Einstein's general theory of relativity [3]. While special relativity postulates that all inertial reference frames are equivalent, general relativity goes further and extends this postulate to arbitrary coordinate transformations. Similarly, starting with some *global internal symmetries* (the freedom in rotations in the *internal* space of quantum fields) physicists came to the concept of local gauge invariance, by allowing arbitrary rotations at different points. Like Einstein's theory, Yang-Mills theory also admits a geometrical interpretation [4]: the connection on a Principal Fibre Bundle with internal symmetry group, or gauge group, over space-time is identified as the four vector gauge potential [1].

Yet the correct application of the ideas of Yang and Mills was neither immediate nor straightforward. It is only in the past two decades that the principle

[1] If the fibre twist around the spacetime, then the topological properties of the gauge theory such as instanton number would play a significant role in non-perturbation aspects of the theory.

of local gauge invariance has blossomed into a unifying theme [5] that seems capable of embracing and even synthesizing all the elementary interactions. One reason for this delay is that in gauge theories all the interactions are mediated by massless vector bosons, whereas only a single massless vector boson, the photon, is apparent in nature. Secondly the algorithm for constructing gauge theories that we have developed applies only to exact symmetries, whereas nature exhibits numerous symmetries that are only approximate and lastly there are many situations in physics in which the exact symmetry of an interaction is concealed by circumstances. A crucial ingredient in modern gauge theories is the phenomenon of *hidden* gauge-invariance [6], which was understood in the early 1960s. It was realised that if a local gauge symmetry is spontaneously broken [7], a miraculous interplay between the would-be Goldstone boson and the normally massless gauge bosons endows the gauge bosons with mass and removes the Goldstone boson from the physical spectrum. Its application by Weinberg and Salam [8] led to a renormalisable theory of weak interactions, and to the understanding of how the electromagnetic and weak forces could after all be unified, despite the fact that the quanta of the former are massless, but of the latter are massive.

There is ample spectroscopic evidence that hadrons (baryons and mesons) are composites, made of quarks and antiquarks [2]. The interactions among them are thought to be governed by Quantum Chromodynamics (QCD) [9]. Baryons consist of three quarks; and mesons consist of a quark-antiquark pair. The

[2]Gell-Mann and Zweig [10] were the first to interpret the SU(3) multiplets of the nadrons as being many-particle states formed of one-particle states (the eightfold way [11]).

baryon resonances $\triangle^{++}$ is an (uuu) state with spin $= 3/2$ and isospin $= 3/2$, in which all the quark pairs are in relatives-waves. Thus it is apparently a symmetric state of three identical fermions. However, according to the Pauli principle a system of fermions has to be totally antisymmetric in the particle variables. This contradiction can be solved by assuming a further internal property of quarks, besides *flavour* degrees of freedom (isospin, strangeness, charm,...), quarks carry a further degree(hidden) of freedom, called *colour,* which permits the $\triangle^{++}$ wave function to be antisymmetrized. There are three possible *colour* basis states *(red, green, blue)*. Each quark flavor must exist in no fewer than three distinguishable types called colours. QCD is obtained by *gauging* the global colour symmetry. As a theoretical construction QCD is very attractive and economical, and it is also close in its foundations to Quantum Electrodynamics (QED).

A Lagrangian which is invariant under local gauge transformations is necessarily singular in the sense of Dirac and the existence of an infinite invariance group will lead to first class constraints. Thus, the quantization of Yang-Mills theory becomes non-trivial deal to the existence of unphysical modes and an indefinite metric. The quantization of Yang-Mills theory is usually carried out by a process called *gauge fixing* which in essence is a prescription for making the Lagrangian non-singular. Three kind of gauges have traditionally been used : the Coulomb gauge, axial gauges (homogeneous or inhomogeneous), and relativistic gauges. Despite the fact that relativistic or covariant gauges have an obvious advantage of maintaining the manifest Lorentz invariance of the original Lagrangian, non-covariant gauges have been examined and utilized extensively in the past few years [12]. There are several reasons for the popularity of non-covariant gauges: in many non-covariant gauges the propagator contains only physical particles, and/or the ghosts are usually decoupled from the other modes, simplifying many perturbative calculations. In addition, when one considers temperature effects, the velocity of the heat bath picks out a preferred frame, which suggests consideration of noncovariant gauges incorporating the velocity of the heat bath as a fixed four vector.

Although formal path integral quantization appears to place all three gauges on the same footing, a more careful analysis reveals important differences. In the path-integral formulation of axial gauges, for example, one needs a prescription to regulate unphysical poles in the propagator. The Principal-value prescription was first used to regulate these poles, but later calculations [13] showed inconsistencies with this regularization. Recently, a lot of attention [14] was devoted to this question. However, these unphysical poles are absent in relativistic gauges, but instead one needs to include Faddeev-Popov ghosts in Feynman diagrams. In the Hamiltonian formalism the differences between the types of gauges manifest themselves in the constraint structure, and the number of independent degrees of freedom one must quantize. It turns out that there are three distrinct classes of gauges in general:

- In class I gauges, the vector boson propagator contains 2 independent physical degrees of freedom.

- In class II gauges, the vector boson propagator contains 2 more unphysical degrees of freedom besides the physical ones.

- In class III gauges, the vector boson propagator contains 4 more unphysical degrees of freedom including the ghosts besides the physical ones.

The purpose of this thesis is to provide a systematic analysis of the essential features of these three classes of gauges. The paper is organized as follows: In chapter 2, a brief introduction to the construction of Yang-Mills theories is presented. Both the Lagrangian and Hamiltonian analysis are used to explore their dynamical structures. In chapter 3, various conventional methods of quantizing gauge fields will be presented with some standard examples: canonical quantization in Feynman gauge and BRS quantization in covariant gauge. Path integral quantization is also included as a convenient way of introducing the appropriate definition of the gauge fixed Lagrangian. Dirac constraint analysis and the BRS analysis would be present in a general setting. In Chapter 3, a Lagrangian which incorporates a general class of non-covariant gauges in non-abelian gauge theories will be studied. A general Lagrangian which incorporates a large number of Class III gauges, and also contains Class I and Class II cases for special (singular) values of the gauge parameters will be presented. In order to illustrate the utility of this general formalism, we also list examples of particular gauges that it describes. A BRS analysis for the interacting Yang-Mills theory in all three classes of gauges will be carried out. It turns out that in the case of Class III gauges one must make a distinction according to whether the associated evolution operator is hyperbolic or elliptic. If it is hyperbolic, then the usual equal-time quantization can be carried. On the other hand, if the operator is elliptic (in the linear theory), there are only physical poles in the propagator and the gauge can be called a ***physical gauge***. Finally, in Section 4, we draw some conclusions and describe some implications of these results. A brief introduction to the subject of Lie algebra and Grassmann algebra will be presented in Appendix A and B.

2 Classical Yang-Mills Theories

In this chapter we give a general introduction to non-abelian gauge theories [7] including their construction and the problems and solutions associated with the gauge symmetries. The usual Lagrangian analysis is insufficient to due with systems with gauge symmetries. The application of the constrained Hamiltonian analysis in field theories is discussed on a general footing.

2.1 Basic Ideas of The Construcion of The Gauge Theories

The theory of classical electrodynamics [15] is both the simplest gauge theory [3] and the most familiar one. Its foundations were laid down a century ago by Maxwell in his equations unifying the electric and magnetic interactions. The invariance of the electromagnetic field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.1}$$

under the *gauge transformation* of the potentials:

$$A_\mu \rightarrow A_\mu + ie\partial_\mu \alpha \tag{2.2}$$

is related to the later discovery in quantum mechanics that the phase of the electron wave function has no physical significance and can be chosen arbitrarily at all space-time points,

$$\psi(x) \rightarrow e^{-ie\alpha(x)}\psi(x). \tag{2.3}$$

[3] The so called transformations of the first kind defind the conserved quantities (the charges) whereas those of the second kind define the gauge fields which couple to the particle fields.

The combined transformations Eq.(2.2) and Eq.(2.3) are now called the gauge transformations of QED.

The conventional way of introducing the electromagnetic interaction in Hamiltonian mechanics, is via the *minimal prescription,* $p_\mu \rightarrow p_\mu - eA_\mu$. In quantum mechanics, since p_μ is replaced by $i\partial_\mu$, and thus the prescription becomes $\partial_\mu \rightarrow D_\mu$, where $D_\mu \equiv \partial_\mu + ieA_\mu$ is called the *covariant derivative* and defines parallel transport in the fibre bundle. The electromagnetic field strength $F_{\mu\nu}$ can be interpreted as the curvature tensor given by

$$F_{\mu\nu} = \frac{i}{e}[D_\mu, D_\nu]. \tag{2.4}$$

In quantum field theory, the Lagrangian [16] for a free electron of mass m is given by

$$\mathcal{L}_m = \bar{\psi}(i\gamma \cdot \partial - m)\psi \tag{2.5}$$

which becomes

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_m + \mathcal{L}_{int} \\ &= \bar{\psi}(i\gamma \cdot \partial - m)\psi - e\bar{\psi}\gamma \cdot A\psi \end{aligned} \tag{2.6}$$

after the replacement. To obtain the complete Lagrangian, we must of course add to Eq.(2.6) a part which yields the Maxwell equations for the potentials A_μ. This is simply the standard classical Lagrangian for the electromagnetic field,

$$\mathcal{L}_{em} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{2.7}$$

The extension of gauge symmetries to more than one degree of freedom is not all that obvious. One has to introduce several new parameters and generators in theory. This leads to non-Abelian gauge groups, since the different generators in general do not commute.

Consider a free particle Lagrangian for N spinor fields ψ_i,

$$\mathcal{L}_m = \sum_{i=1}^{N} \frac{i}{2}(\bar{\psi}_i \gamma \cdot \partial \psi_i - \partial \bar{\psi}_i \cdot \gamma \psi_i) - m_i \bar{\psi}_i \psi_i \tag{2.8}$$

which is invariant under global unitary phase transformations

$$\psi_i(x) \longrightarrow U[a_i]\psi_i(x) \tag{2.9}$$

where

$$U[a_i] = e^{-ig_i\alpha_i} \tag{2.10}$$

is the scalar representation of the group element a_i of some compact Lie group G and α_i is a space-time independent real constants associated with each g_i which denotes the coupling constant for the field ψ_i. However, if the masses of the matter fields are nearly degenerate, then one can incorporate the fields into a larger state

$$\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix} \tag{2.11}$$

with all the masses the same, and assume that the mass differences are caused by a weak perturbation. In other words, the phase symmetries of the $\psi_i's$ are just the residue of a larger symmetry which is broken by a weak perturbation. Thus Eq.(2.8) can then be written as

$$\mathcal{L}_m = \bar{\Psi}(i\gamma \cdot \partial - m)\Psi \tag{2.12}$$

where $m = m_1, ..., m_N$. It can be verified that $\mathcal{L}_m$ is invariant under the unitary transformation

$$\Psi(x) \longrightarrow U[g]\Psi(x) \tag{2.13}$$

with

$$U[g] = e^{-ig\alpha} \tag{2.14}$$

an appropriate representation of the group element $g \in G$ and $\alpha = \alpha^a T^a$, where α^a are some real constant and the hermitian operators T^a are the generators of the Lie algebra $\mathcal{G}$ in this representation. The coupling constants $g_i's$ are then reduced to a single coupling constant g which governs the strength of the interactions among the $\psi_i's$.

We now extend the global invariance property of $\mathcal{L}_m$ to its local generalization, i.e. Ψ transforms as Eq.(2.13) but the α^a are space-time dependent variables. Gauge fields, A_μ^a, must be introduced in the theory such that

$$D_\mu\Psi(x) \longrightarrow U[g(x)]D_\mu\Psi(x) \tag{2.15}$$

where $D_\mu = \partial_\mu - igA_\mu$ is the non-abelian generalization of the covariant derivative and $A_\mu = A_\mu^a T^a$. Thus, if ∂_μ is replaced by D_μ, the Lagrangian $\mathcal{L}_m$ would then be invariant under the local gauge transformation as desired.

Now, it is a matter of simple algebra to obtain the transformation of A_μ such that Eq.(2.15) would be satisfied. The resulting transformations are

$$A_\mu \to UA_\mu U^{-1} + \frac{i}{g}U\partial_\mu U^{-1}. \tag{2.16}$$

Infinitesimally, the transformations $\delta_\alpha A_\mu = D_\mu\alpha$ form a group,

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\, A_\mu = \delta_\gamma A_\mu \tag{2.17}$$

where $\gamma = g\alpha \times \beta$, which defines an equivalence relation among the gauge fields. The gauge fields are then separated into different classes. Fields belonging to the same class describe one and the same physical situation and are therefore physically indistinguishable. In classical electrodynamics, the transformation $A_\mu \rightarrow A_\mu + ig\partial_\mu \alpha$ establishes an equivalent relation among the gauge fields in the sense that, the physical field strength tensor, $F_{\mu\nu}$, is invariant under this transformation. Therefore it is natural to generalize the idea of field strength tensor to nonabelian theories. If the G-valued covariant derivative has been substituted into Eq.(2.4), the resulting field strength tensor becomes

$$\begin{aligned} F_{\mu\nu} &= \frac{i}{g}[\partial_\mu - igA_\mu, \partial_\nu - igA_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \end{aligned} \tag{2.18}$$

In this case $F_{\mu\nu}$ is not invariant under the gauge transformation, rather $F_{\mu\nu}$ transforms as the adjoint representation of G :

$$F_{\mu\nu} \longrightarrow UF_{\mu\nu}U^{-1}. \tag{2.19}$$

Note that $F_{\mu\nu}$ is gauge dependent (i.e. non-physical), but its transformations are very desirable when one constructs the pure gauge field Lagrangian.

2.2 Lagrangian Analysis

As a natural generalization of the Lagrangian for classical electrodynamics, we introduce the Yang-Mills Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{4}tr(F_{\mu\nu}F^{\mu\nu}) \tag{2.20}$$

with the normalization $tr(T^a T^b) = \delta^{ab}$. It is relativistically and gauge invariant (using Eq.(2.19) and the cyclic property of trace) and is also the simplest Lagrangian in the sense that $\mathcal{L}_{YM}$ contains derivatives of the field variables A^a_μ no higher than the second. Terms of higher order in the fields are not allowed if the theory is to be renormalizable. Furthermore, the form of $\mathcal{L}_{YM}$ is independent of the representation of Ψ one chooses. As a matter of fact, the form of $\mathcal{L}_{YM}$ stated in Eq.(2.20) is very compact. When expressed in terms of the gauge fields, $\mathcal{L}_{YM}$ not only contains a usual bilinear term but also trilinear and quadrilinear terms which lead to three-gluon and four-gluon vertices in Feynman diagrams respectively. Thus, $\mathcal{L}_{YM}$ essentially splits into two parts: i) the free propagating part which describe the free propagation of the quanta of the gauge fields, ii) the interacting part which describe the self interactions of the gauge fields. In other words, quanta of the Y-M fields themselves carry charges and transfer charges during interactions. Therefore, unlike QED, we can ignore the fermionic matter field that motivated the construction of the gauge fields and still have an highly nontrivial theory.

By requiring the action to be stationary with respect to small variations of the gauge fields A^a_μ, we obtain the Euler-Lagrangian equations

$$L^a_\nu \equiv D^\mu F_{\mu\nu}{}^a = 0 \qquad (2.21)$$

whose non-linear character makes their resolution non-trivial. Notice that $L'_\nu s$ are not independent but rather fulfill generalized Bianchi identities $D_\nu D_\mu F^{\mu\nu} \equiv 0$. In the limit that g, the coupling constant, goes to zero, Eq.(2.21) can be rewritten as

$$\Box A_\nu^\perp = 0 \tag{2.22}$$

where $\Box \equiv \partial_\mu \partial^\mu$ and

$$A_\nu^\perp \equiv A_\nu - \frac{\partial_\nu \partial \cdot A}{\Box} \tag{2.23}$$

is the transverse component of A_ν and satisfies the massless Klein-Gordon equation [36]. Thus, any A_ν whose transverse projection satisfies the massless Klein-Gordon equation is a solution to Eq.(2.22). All A_μ related by the gauge transformations are equivalent, since the transformations keep their transverse component unchanged. Hence, given a set of Cauchy initial condition, the equations of motion are insufficient to uniquely determine the fields A_ν. Fourier transforming Eq.(2.22) to momentum space, we have $k^2 P_{\mu\nu} A^\mu = 0$ where $P_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ is an projection operator whose inverse dose not exist, so that the standard approach of finding the propagators of the fields A_μ^a is inapplicable. The underlying physical reason is that we are trying to describe a massless vector field with two helicity states (transversely polarized) in terms of a four component field A_μ.

We have seen even in classical free field theory, one encounters various difficulties arising from the arbitrariness of the gauge fields, the *gauge freedom*, the presence of redundant variables amongst the gauge potentials and constraints on the canonical variables in the classical Lagrangian. The methods of treating this redundancy lead to conflicts with the usual particle interpretation of the quantized field modes. These difficulties are well known in the theory of electromagnetism. In general, any attempts to quantize geometrical fields by the standard techniques lead to difficulties and contradictions. The way of solving the problem is to break the gauge symmetry by imposing gauge conditions on the field variables such that only one representative will be picked out in each equivalent class of gauge fields, which on the other hand do not affect gauge invariant physical observables.

Nevertheless, the explicit form of this gauge condition is largely dictated by computational convenience, and the choice of a suitable gauge for non-abelian theory is a particularly difficult task owing to its self-interaction. Furthermore, the imposition of the gauge condition in path integeral formalism

naturally leads to the introduction of the so called Faddeev-Popov ghosts which will be considered in 3.1.

2.3 Hamiltonian analysis

To investigate the Hamiltonian structure of Yang-Mills theories, one attempts to define the canonical conjugate momenta associated with $\mathcal{L}_{YM}$ as

$$\pi^{\mu} \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_{\mu}} = F^{\mu 0}. \tag{2.24}$$

Their commutation relations [32] with A_μ are given by

$$[\pi_{\mu}(\mathrm{x}, t), A_{\nu}(\mathrm{y}, t)] = -i\delta^{3}(\mathrm{x}-\mathrm{y}). \tag{2.25}$$

Therefore, the momentum conjugate to A_0 is identically zero at all space-time which contradicts the above commutation relations. Such a Lagrangian is called *singular* [17] In the following, we will present a general framework to eliminate such degrees of freedom in a generic singular Lagrangian.

In general, singular Lagrangian density $\mathcal{L}(\Phi_a, \partial_\mu \Phi_a)$ is defined as one for which the generalized velocities $\dot{\Phi}_a$ cannot be expressed uniquely in terms of the canonical momenta

$$\Pi^a = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_a}$$

such that the canonical Hamiltonian

$$H_c = \int [\dot{\Phi}_a \Pi^a - \mathcal{L}]\, d^3x,$$

which in principle does not depend on $\dot{\Phi}_a$, is not well defined. In other words, the Lagrangian is singular if the matrix

$$\left(\frac{\partial \Pi^a}{\partial \dot{\Phi}_b} \right)$$

is not invertible, which is equivalent to the condition that

$$\det \frac{\partial^2 \mathcal{L}}{\partial \dot{\Phi}_a \partial \dot{\Phi}_b} = 0. \tag{2.26}$$

Thus, the system consists of some momenta that do not depend on the time derivatives of Φ_a and is subjected to constraints. By choosing a suitable basis for the fields, the primary constraints are then defined as

$$\phi_i = \Pi^i - \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_i} \approx 0, \quad i = 1, ..., p \tag{2.27}$$

which follow from the form of the Lagrangian alone. Notice that the primary constraints hold for all times and the symbol $\approx$ is an equivalence relation such that if $A \approx B$ then $A = B$ up to some linear combinations of all the constraints processed by the system. Therefore, Lagrange mutiplier terms, $\mathcal{H}_L = \lambda_i \phi_i$ are added to the original Hamiltonian in order to incorporate the equivalences among the canonical variables and the combined result is referred as the effective Hamiltonian, $\mathcal{H}_{eff}$, which can be split into two parts: i) $\mathcal{H}'$, which is the same as the original Hamiltonian, $\mathcal{H}_c$, with terms that contain primary constraints explicitly eliminated, ii) the Lagrange multiplier terms, H_L.

Another way of implimenting the constraints is to specify them in the initial data, e.g. the Cauchy initial data. For instance, in addition to requiring that the primary constraints ϕ_j, s be satisfied at time t_0, all orders of time derivatives of the primary constraints should weakly equal to zero at that time. i.e.

$$\phi_j^{(0)} = \phi_j(\mathbf{x}, t_0) \approx 0 \tag{2.28}$$

$$\phi_j^{(1)} = [iH_{eff}, \phi_j]_{t=t_0} \approx 0 \tag{2.29}$$

$$\phi_j^{(2)} = [iH_{eff}, [iH_{eff}, \phi_j]]_{t=t_o} \approx 0 \tag{2.30}$$

$$\vdots$$

Thus, if $[\phi_i, \phi_j^{(k)}]_{t=t_0} \approx 0 \;\; \forall \; i$ up to and including $k = n_j - 1$ and $[i\mathcal{H}', \phi_j^{(k)}]_{t=t_0} \not\approx 0$ up to and including $k = m_j - 1$, the above process would then terminate at $k = l_j = \min(n_j, m_j)$ but with different circumstances:

- $k = n_j$ for $n_j \leq m_j$
- $k = m_j$ for $n_j > m_j$.

For the first case, $n_j \leq m_j$, $\phi_j^{(o)}, ..., \phi_j^{(n_j-1)}$ would be constraint equations and $\phi_j^{(n_j)}$ would be functions of $\lambda_i' s$. On the other hand, for $n_j > m_j$, $\phi_j^{(o)}, ..., \phi_j^{(m_j)}$ would be constraint equations and $\phi_j^{(m_j+1)}$ would at least be identically weakly equal to zero. Actually, for many theories, the sequence usually ends at the first few orders. The collection of $\phi_j^{(n_j)}$ with q linearly independent terms would then be used to eliminate the number of the Lagrange mutiplier fields $\lambda_i's$ from p to $r' \equiv p - q$.

The set of all the constraints, ψ_j for $j = 1, ..., s$ where $s = \sum_{j=1}^{p} l_j + q'$, can be used to separate all functions $F(\Phi_i, \Pi_i)$ into two distinct categories: first class and second class. F would be called first class if

$$[F, \psi_j] \approx 0 \ \forall \ j \tag{2.31}$$

and F would be called second class if

$$[F, \psi_j] \not\approx 0 \tag{2.32}$$

for at least one j. Therefore, all of the constraints, ψ_j, can be divided into sets either consisting of all linearly independent first class constraints, $\Gamma_a, \quad a = 1, ..., r$ or consisting of all second class constraints, $\chi_i \quad i = 1, ..., s - r$.

The reason one does this is that there is a way to incorporate all the second class constraints implicitly into the system without actually solving the constraints beforehand. To this end, one introduces the Dirac bracket

$$\begin{aligned}[A(\mathbf{x}, t), B(\mathbf{y}, t)]' &= [A(\mathbf{x}, t), B(\mathbf{y}, t)] \\ &- \int [A(\mathbf{x}, t), \chi_i(\mathbf{z}_1, t)] {C^{-1}}_{ij}(\mathbf{z}_1, \mathbf{z}_2) [\chi_j(\mathbf{z}_2, t), B(\mathbf{y}, t)] \, d^3z_1 \, d^3z_2 \end{aligned} \tag{2.33}$$

where C^{-1} is the inverse of matrix $C_{ij}(\mathbf{x}, \mathbf{y}) = [\chi_i(\mathbf{x}, t), \chi_j(\mathbf{y}, t)]$ such that

$$\int C_{ij}(\mathbf{x}, \mathbf{z}) {C^{-1}}_{jk}(\mathbf{z}, \mathbf{y}) \, d^3z = \delta_{ik} \delta(\mathbf{x} - \mathbf{y}) = \int {C^{-1}}_{ij}(\mathbf{x}, \mathbf{z}) C_{jk}(\mathbf{z}, \mathbf{y}) \, d^3z. \tag{2.34}$$

One can immediately observe that Dirac Brackets with all the second class constraints vanish automatically. Hence, the bracket relations would then be consistent with all the second class constraints if all the commutators are replaced by their corresponding Dirac Brackets. Consequently, one postulates that the replacement of the commutators with the Dirac Bracket followed by setting all the second class constraints strongly to zero is a legitimate way to eliminate unphysical degrees of freedom associated with second class constraints. Note that the $\phi_j^{(k)}$ are defined at $t = t_0$ as the equivalent conditions to the primary constraints. However, the replacement mentioned above is extended to all time where $\chi_i(\mathbf{x}, t)$ is formally the same as $\chi_i(\mathbf{x}, t_0)$ with t_0 replaced by t. Similarly, the first class constraints are also extended to all time and thus can be directly incorporated into the effective Hamiltonian

$$H_{eff} = \int [\mathcal{H}_0 + \lambda_a \Gamma_a] \, d^3x, \tag{2.35}$$

where $\mathcal{H}_0$ will be the same as the canonical Hamiltonian except terms have explicit dependences on any constraints being eliminated. The extension, although in some sense reduntant, gives a convenient and effective way to impose the constraints into the system. The $\Gamma_a's$ plays the roles as the generators of the infinitesimal gauge transformations which are inherited from the form of the Lagrangian of the theory.

To fix the first class constraints, Γ_a, we can impose some additional constraints γ_a for $a = 1, ..., r$ such that the set of all constraints, $\Gamma_a's$ and $\gamma_a's$, cease to be first class and hence the procedure of forming the Dirac Brackets and eliminating second class constraints stated above can be used.

In order to illustrate the usage of the constrained Hamiltonian analysis, we apply the above formalism to eliminate the unphysical degrees of freedom of the gauge fields in temporal gauge. As mentioned before, associated with the usual Yang-Mills Lagrangian, the canonical momenta, $\pi^{\mu a}$, conjugate to A_μ^a are found to be $F^{\mu 0 a}$. Thus

$$\phi \equiv \pi_0 = F_{00} = 0 \tag{2.36}$$

are the only primary constraints in the theory without gauge conditions. The effective Hamiltonian will be different from the canonical Hamiltonian (which is the same as H')

$$\begin{aligned} H_c &= \int [\dot{A}_i \cdot \pi^i - \mathcal{L}]\, d^3x \\ &= \int [\mathcal{H}_0 - A_0 \cdot D_i \pi^i]\, d^3x \end{aligned} \tag{2.37}$$

by a factor:

$$\int d^3x \lambda \cdot \phi. \tag{2.38}$$

Note that

$$\mathcal{H}_0 = \tfrac{1}{2}\pi \cdot \pi + \tfrac{1}{2}\mathbf{B} \cdot \mathbf{B} \tag{2.39}$$

which corresponds to the energy density of the system and $B^i \equiv \frac{1}{2}\epsilon^{ijk}F_{jk}$ is the so called *magnetic field.* Since ϕ commutes with itself and

$$[i\mathcal{H}', \phi] = D_i \pi^i = \phi^{(1)} \tag{2.40}$$

which is not identically weakly zero, the process continue with the manipulation of the secondary contriants. It was found that the process of finding extra constriants terminated because $\phi^{(1)}$ not only commute with the primary constriants but also weakly commute with $\mathcal{H}'$ by obtaining the following identities:

$$[D_i\pi^i, D_j\pi^j] = D_k\pi^k \tag{2.41}$$

or

$$[D_i\pi^{i^a}, D_j\pi^{j^b}] = if^{abc}D_k\pi^{k^c} \tag{2.42}$$

which can be inferred by observing the group property of the gauge transformations of Eq.(2.17). It is obvious that ϕ and $\phi^{(1)}$ are first class and hence the effective Hamiltonian will be given by

$$\mathcal{H}_{eff} = \int d^3x[\mathcal{H}_0 + \alpha \cdot \pi^0 + \mathcal{H}_\beta] \tag{2.43}$$

where $\mathcal{H}_\beta \equiv -\beta \cdot D_i\pi^i$.

The introduction of the temporal gauge, $A_0 = 0$, turn the constriants $\pi_0 = 0$ into second class with the associated Dirac brackets (only the one different from the canonical) given by

$$[A_0^a(\mathbf{x}, t), \pi_0^b(\mathbf{y}, t)]' = 0. \tag{2.44}$$

By taking those second class constriants strongly zero, the effective Hamiltonian becomes

$$\int d^3x[\mathcal{H}_0 + \mathcal{H}_\beta]. \tag{2.45}$$

One can verify that

$$[i\mathcal{H}_\beta, A_i] = D_i\beta \tag{2.46a}$$

$$[i\mathcal{H}_\beta, A_0] = 0. \tag{2.46b}$$

Hence $\mathcal{H}_\beta$ generate the gauge transformations which are consistent with the gauge conditions $A_0 = 0$. One can identify β with A_0 since its conjugate momentum vanishes everywhere, and hence A_0^a itself would not be a dynamical variable or in other words A_0 play the role of Lagrange multipliers for the constraints $\Gamma : D_i\pi_i \approx 0$ consist of the canonical variables, A_i and π_i. The system then submitted to the constraints Γ which locally reduce the 3n dimensional phase space (since A_0 and π_0 do not count as dynamical variables) to a 2n dimensional manifold, Σ. Let f be a physical observable which depends on A_i and π_i, then it must be gauge independent in order to be uniquely defined, i.e. $[i\mathcal{H}_\beta, f] \approx 0$. Thus, the problem then reduce to solving the constraint equations and imposing the Gauss' law to the physical subspace as an operator. Alternatively, since any points in Σ lies in the same flow generated by $\mathcal{H}_\beta$ are equivalent, thus it is natural to choose a regular representative of S as the physical phase space which are described by the auxiliary conditions $g(A_i, \pi_i)$ such that $[i\mathcal{H}_\beta, g] \not\approx 0$ (In this sense, the imposing of Gauss' law can be considered as a special choice of gauge condition).

3 Quantization of Yang-Mills Theories

We survey various quantization methods of Yang-Mills theories in this chapter. An appropriate gauge fixing Lagrangian is introduced in the section. BRS quantization of non-abelian gauge theory in covariant gauge is presented.

3.1 Path Integral quantization

We shall now outline the general procedure for quantizing gauge fields in the formalism of the path integral [18] over all fields. The central problem is the choice of measure in the function space. In the framework of path-integral quantization, characterized by functional integration over the field $A_\mu(x)$, the gauge degrees of freedom manifest themselves in a different manner. Due to gauge invariance, there now exist infinitely many fields $A_\mu^U(x)$ that are physically equivalent to $A_\mu(x)$ and are related by transformations of the form Eq.(2.16). This suggests that equivalence classes of fields obtained from one another by gauge transformations should become the fundamental objects of the theory. Integration over these gauge-equivalent fields A_μ^U produces an infinite volume factor in group space that is proportional to $\int \mathcal{D}U$ where $\mathcal{D}U$ stand for the infinite product of invariant measures on compact group, isomorphic to the group G at every space point x, $\mathcal{D}U = \prod_x \mathcal{D}U(x)$, and whose presence in the generating functional leads to ill-defined Green functions.

For a consistent quantization in this formalism it is clearly mandatory to eliminate the troublesome gauge degrees of freedom. This may be achieved by imposing on the system a gauge condition $\mathcal{F}^a(A)$ where $\mathcal{F}^a$ is a local functional of A_μ which may involve the derivatives of the gauge fields. Faddeev-Popov procedure [19] will be used to quantize the system. Consider the functional

$$\triangle_{\mathcal{F}}^{-1}[A] = \int \mathcal{D}U \prod_x \delta[\mathcal{F}(A^U)] \tag{3.47}$$

in which the gauge choice we shall be particularly interested in is given by

$$\mathcal{F}(A) = C_{\mu\nu}\partial^\mu A^\nu. \tag{3.48}$$

If $\mathcal{F}[A^U]$ is regarded as a function of the group element $U(x)$, then by a change of variable from $U(x)$ to $\mathcal{F}^a$ and considering the integration to be restricted to an infinitesimal region we may take

$$\begin{aligned} \triangle_{\mathcal{F}}{}^{-1}[A] &= \int \mathcal{D}\alpha\delta[\mathcal{F}(D\alpha)] \\ &= \int \mathcal{D}\mathcal{F} \det{}^{-1}\left(\frac{\delta}{\delta\alpha^b(y)}\left[\mathcal{F}_a\left(D\alpha(x)\right)\right]\right)\delta[\mathcal{F}(D\alpha)] \\ &= \det{}^{-1}\mathcal{M}_{\mathcal{F}} \end{aligned} \tag{3.49}$$

with the Jacobian matrix $\mathcal{M}$ given by

$$\mathcal{M}_{\mathcal{F}}{}^{ab}(x,y) = \frac{\delta}{\delta\alpha_b(y)}\left[\mathcal{F}^a\left(D\alpha(x)\right)\right]|_{\alpha=0}. \tag{3.50}$$

These manipulations can be performed if the change of variables from U to $\mathcal{F}$ is well defined and not singular. Furthermore, any non-trivial boundary conditions could be ignored as long as we deal with the perturbative theory. Therefore, we have the identity

$$1 = \int \mathcal{D}U \det \mathcal{M}_{\mathcal{F}}\delta[\mathcal{F}(A^U)] \tag{3.51}$$

with the interpretation that $\triangle_{\mathcal{F}}[A]$ *compensates* for an infinite volume factor arising from integration over the gauge group. However, the Yang-Mills Action

$$S[A_\mu] = \int d^4x\mathcal{L}_{YM}(x) \tag{3.52}$$

and $\triangle_{\mathcal{F}}{}^{-1}[A]$ are by construction gauge invariant, i.e. $S[A_\mu] = S[A_\mu^U]$ and $\triangle_{\mathcal{F}}{}^{-1}[A] = \triangle_{\mathcal{F}}{}^{-1}[A^U]$. By inserting the identity Eq.(3.51), the naive path integral $\int \mathcal{D}A_\mu e^{iS[A_\mu]}$ becomes

$$\int \mathcal{D}A_\mu \mathcal{D}U \triangle_{\mathcal{F}}[A]\delta[\mathcal{F}(A^U)]e^{iS[A_\mu]} \tag{3.53}$$

which can be rewritten as

$$\int \mathcal{D}U \quad \mathcal{D}A_\mu \triangle_{\mathcal{F}}[A]\delta[\mathcal{F}(A)]e^{iS[A_\mu]} \tag{3.54}$$

by performing in the integrand a gauge transformation from $A_\mu^U \rightarrow A_\mu$ and using the fact that $\mathcal{D}A_\mu$ is a local gauge invariant measure, i.e. $\mathcal{D}A_\mu$ is the same path integral as $\mathcal{D}A_\mu^U$ for any U and the defining property

$$\int \mathcal{D}U f[U] = \int \mathcal{D}U f[UU'] \tag{3.55}$$

where $f[U]$ is any functional of $U(x)$, and $U'(x)$ is a fixed gauge transformation. Since nothing depends on U in the integrand, $\mathcal{D}U$ can be factored out at the cost of a multiplicative infinity, which is the infinity we wanted to take out in the first place and the gauge is being fixed by the functional δ function, as we had also intended. Hence the correct Feynman path integral for Yang-Mills is defined to be

$$\int \mathcal{D}A_\mu det \mathcal{M}_{\mathcal{F}} \delta[\mathcal{F}(A)]e^{iS[A_\mu]}. \tag{3.56}$$

In order to use the generating functional for gauge field theory to develop a perturbation theory, it is convenient to convert the functional $\delta[\mathcal{F}(A)]$ and the Jacobian determinant $det \mathcal{M}_{\mathcal{F}}$ into exponentials and incorporate them into the action. Since gauge-invariant quantities are not sensitive to changes of auxiliary conditions, the gauge condition $\mathcal{F}^a(A)$ can be modified to the form $\mathcal{F}^a(A) = C^a$

without affecting the form of the operator $\mathcal{M}$ and the ingredients in the path integral Eq.(3.56). Here C^a takes its values in the Lie algebra. Averaging the δ functional over C with a gaussian weight, we obtain the desired exponential:

$$\int \mathcal{D}C \exp\left[-\frac{i}{2\xi}\int d^4x tr\left(C^2(x)\right)\right]\delta[\mathcal{F}(A)-C]$$
$$= \exp\left[-\frac{i}{2\xi}\int d^4x\left(\mathcal{F}^2(A)\right)\right]. \tag{3.57}$$

On the other hand, the Jacobian determinant can be rewritten as path integral over of an exponential by introducing two Grassmann variables, the so called *Faddeev-Popov ghost fields,* η and its conjugate $\bar{\eta}$, which do not correspond to any physical partical but are simply a mathematical device. As will be seen in Appendix B

$$det\mathcal{M}_{\mathcal{F}} = \int \mathcal{D}\bar{\eta}\mathcal{D}\eta \exp\left[-\int d^4x d^4y \bar{\eta}^a(x)\mathcal{M}_{\mathcal{F}}{}^{ab}(x,y)\eta^b(y)\right]. \tag{3.58}$$

Therefore the path integral becomes

$$\int \mathcal{D}A_\mu \mathcal{D}\bar{\eta}\mathcal{D}\eta \exp i\int d^3x \mathcal{L}_{eff} \tag{3.59}$$

where the effective Lagrangian $\mathcal{L}_{eff}$ is given by

$$\mathcal{L}_{eff} = tr\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\mathcal{F}(A))^2 + i\,\bar{\eta}\mathcal{F}(D\eta)\right) \tag{3.60}$$

3.2 Canonical Quantization of Electromagnetic Fields in Feynman gauge

As mentioned in section 2.1, a subsidiary condition is required to factor out the physical subspace when canonical quantization of the gauge fields is carried out with a gauge fixing term adding to the classical Lagrangian. By choosing the Feynman gauge and using the effective Lagrangian obtained in last section one has the gauge fixed electromagnetic Lagrangian:

$$\mathcal{L}_{eff} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial \cdot A)^2. \tag{3.61}$$

Maxwell's equations are now replaced by

$$\Box A_\mu = 0 \tag{3.62}$$

which as expected are much more restrictive than the original Maxwell equations (Eq.(2.22)). By definition

$$\pi_0 \equiv \frac{\partial \mathcal{L}_{eff}}{\partial \dot{A}_0} = -\partial_\mu A^\mu == -\mathcal{F}(A) \tag{3.63}$$

while $\mathcal{F}(A)$ is set in the framework of path integral formalism to be a variable $C(x)$ which was averaged over a gaussian integral. Thus for the moment π_0 can be thought as an averaging value of $\mathcal{F}(A)$ with respect to a particular weighting without any physical significant. However Eq.(3.63) is dynamical equations in contrast to the constraint equations $\pi_0 = 0$ in the case without gauge fixing. Therefore, the system now does not contain any primary constraint. This does not mean that no further attention to gauge fixing is needed. On the contrary, the Lorentz condition ($\partial_\mu A^\mu = 0$) does not completely fix the gauge freedom. Gauge transformations of the form of Eq.(2.2) are still allowed provided that $\Box \alpha = 0$. This problem will be discussed later.

Let us continue by Fourier expanding A_μ into its normal modes in order to gain the photon interpretation of the quantized fields:

$$A_\mu(x) = \int d\tilde{k} \sum_{\lambda=0}^{3} \left[a^{(\lambda)}(\mathbf{k})\epsilon_\mu^{(\lambda)}(\mathbf{k})e^{-ik\cdot x} + h.c.\right] \tag{3.64}$$

with the covariant phase space measure

$$\tilde{dk} = \frac{d^3k}{(2\pi)^3 2\omega_k} = \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2 - m^2)\theta(k_0) \tag{3.65}$$

and $\omega_k = \sqrt{\mathbf{k}^2 + m}$. In the above, $h.c.$ stands for the hermitian conjugation while m regulates infrared divergences and is set to zero after all calculations. In order to find the commutation relations of the creation and annihilation operators, one expresses the commutators among A_μ and $\dot{A}_\mu$ by using the canonical commutation relations. Replacing π_0 by the dervatives of A_μ, one finds the following commutation relations:

$$[A_\mu(x), A_\nu(y)] = 0 \tag{3.66a}$$

$$[\dot{A}_\mu(x), \dot{A}_\nu(y)] = 0 \tag{3.66b}$$

$$[\dot{A}_\mu(x), A_\nu(y)] = ig_{\mu\nu}\delta(\mathbf{x} - \mathbf{y}). \tag{3.66c}$$

If the polarization vectors $\epsilon_\mu^{(\lambda)}(\mathbf{k})$, which have been chosen to be real, satisfy both orthonormality conditions

$$\epsilon^{(\lambda)}(\mathbf{k}) \cdot \epsilon^{(\lambda')}(\mathbf{k}) = g^{\lambda\lambda'} \tag{3.67}$$

and completeness relations

$$\sum_\lambda \frac{\epsilon_\mu^{(\lambda)}(\mathbf{k})\epsilon_\nu^{(\lambda)}(\mathbf{k})}{\epsilon^{(\lambda)}(\mathbf{k}) \cdot \epsilon^{(\lambda)}(\mathbf{k})} = g_{\mu\nu}, \tag{3.68}$$

then after some algebra one can find the following non-vanishing commutation relations

$$[a^{(\lambda)}(\mathbf{k}), a^{(\lambda)\dagger}(\mathbf{q})] = -g^{\lambda\lambda'} 2\omega_k (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{q}). \tag{3.69}$$

One expects the vacuum state $|0>$, without any kind of photon, to be defined by

$$a^{(\lambda)}(\mathbf{k})|0>=0 \tag{3.70}$$

for $\lambda = 0, 1, 2, 3$. An one scalar photon excitation state is given by

$$|1>=\int d\tilde{k} f(k) a^{(\lambda)}(\mathbf{k})|0> \tag{3.71}$$

with the amplitude $f(k)$ corresponding to every k. We run into trouble when we try to obtain its norm be

$$<1|1>=-<0|0>\int d\tilde{k}|f(k)|^2 \tag{3.72}$$

which is negative, i.e. the Fock space has an indefinite metric, hence the beautifully established probabilistic interpretation of quantum mechanics is ruined. In order to restore the usual arguments in quantum mechanics, we try to separate the so called *physical sector* out in which the usual interpretations hold. Observing that, up to now, we are only dealing with a modified version of Maxwell's theory. To recover it, we would like to set $\partial \cdot A$ equal to zero. However, it is impossible as an operator equation otherwise the commutation relations would be violated. However, a weaker condition

$$\partial \cdot A^{(+)}(x)|\Psi_{phys}>=0 \tag{3.73}$$

can be imposed which at least is consistent with the canonical commutation relations and elminates the unphysical states. The requirement that all *physical* states are annihilated by the positive frequency (annihilating) part of $\partial \cdot A$ ensures that Lorentz condition holds in the mean for any physical state,

$$<\Psi_{phys}|\partial \cdot A|\Psi_{phys}> . \tag{3.74}$$

Before we finish this section, we will present a different formulation of the above materials which lead to the idea of BRS quantization. The crucial difference between the new formalism and the old one is the introduction of an unphysical degree of freedom known as a Lagrange multipler (auxillary) field $S(x)$ associated with the gauge condition. The effective Lagrangian is of the form

$$\mathcal{L}_{eff} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + S\partial \cdot A + \tfrac{1}{2}S^2. \tag{3.75}$$

The variable $S(x)$ will become the canonical momentum conjugate to A_0 while the momentum conjugate to S is identically zero. The Euler-Lagrange equation for A_μ yields

$$\Box A_\mu - \partial_\mu \partial \cdot A = \partial_\mu S \tag{3.76}$$

and for S yields

$$\partial \cdot A + aS = 0. \tag{3.77}$$

Applying ∂^μ to Eq.(3.76) leads to $\Box S = 0$, and from equation Eq.(3.77) that $\Box \partial \cdot A = 0$, thus both S and $\partial \cdot A$ are free.

It is clear that both

$$\phi_1 \equiv \pi_0 - S \approx 0 \tag{3.78}$$

and

$$\phi_2 \equiv \pi_S \approx 0 \tag{3.79}$$

are second class primary constraints. It can be found that Dirac brackets associated with those constraints are essentially the same as the canonical ones. We can therefore replace S by π_0 and setting $\pi_S = 0$. Hence, the degrees of freedom associated with S are eliminated and the different formalisms become explicitly equivalent. The subsidiary condition for separating the physical sector out becomes

$$S^{(+)}(x)|\Psi_{phys}>=0 \tag{3.80}$$

which turns out to have non-abelian counterparts.

3.3 BRS Quantization

In non-abelian gauge theories canonical quantization in a singular gauge is possible, but is complicated because the equation of constraint which must now be solved is the (nonlinear) non-abelian Gauss's law. On the other hand, by a method which is a direct generalization of the Lautrup-Nakanishi [22] formalism to non-abelian gauge theories, Kugo and Ojima [20] have shown how to find a subsidiary condition which projects out the physical Hilbert space from the Fock space of asymptotic states, and which ensures the unitarity of the perturbation theory. We will recapture the essence of the work of Kugo and Ojima and generalize it to the non-covariant gauge in the next chapter.

According to section 2.3, the Lagrangian of an non-abelian gauge theory in covariant gauge is given by

$$\mathcal{L}_{eff} = -\frac{1}{4}F_{\mu\nu}\cdot F^{\mu\nu} + S\partial\cdot A + \tfrac{1}{2}aS^2 + i\,\bar{\eta}\partial\cdot D\eta \tag{3.81}$$

which apart from a total divergence can be rewritten as

$$\mathcal{L}_{eff} = -\frac{1}{4}F_{\mu\nu}\cdot F^{\mu\nu} - \partial_\mu S\cdot A^\mu + \tfrac{1}{2}aS^2 - i\,\partial_\mu\bar{\eta}\cdot D^\mu\eta \tag{3.82}$$

where $A\cdot B \equiv A^aB^a$. From the fact that the above Lagrangian is invariant under the Becchi, Rouet and Stora transformations [21],

$$\delta A_\mu = \lambda D_\mu \eta, \tag{3.83a}$$
$$\delta \eta = -\frac{\lambda}{2} g\, \eta \times \eta, \tag{3.83b}$$
$$\delta \bar{\eta} = i\, \lambda\, S, \tag{3.83c}$$
$$\delta S = 0, \tag{3.83d}$$

with $(A \times B)^a \equiv f^{abc} A^b B^c$, the BRS charge, Q_B, exists. Its positive frequence part was used in the paper of Kugo and Ojima to get rid of the unphysical states through the following subsidiary condition:

$$Q_B^{(+)} |\Psi_{phys}>= 0. \tag{3.84}$$

The procedure of finding the subsidiary condition can be generalized to non-covariant gauges and we will focus on this matter throughout the thesis.

In the generalized Lautrup-Nakanishi [22] formalism, the effective Lagrangian of Eq.(3.60) becomes

$$\mathcal{L}_{eff} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + S \mathcal{F}(A) + \tfrac{1}{2} a S^2 + i\, \bar{\eta} \mathcal{F}(D\eta). \tag{3.85}$$

Despite the fact that local gauge symmetry has been broken, it has a global invariance under the BRS transformations which is a quantum statement of the gauge-invariance of physical states since it is all that is required to guarantee the Ward identities [23]. Notice that the BRS invariance of the Lagrangian can be easily shown by verifying $\delta(D_\mu \eta) = \delta(\eta \times \eta) = 0$ and also the BRS transformation is nilpotent, i.e. $\delta^2 = 0$. The ghost, η, plays the role of gauge parameter while λ is a global parameter. Noether's first theorem [24] can then be used to construct the gauge independent BRS charge,

$$Q_B = -\int [\eta \cdot \Gamma + \tfrac{1}{2} g(\eta \times \eta) \cdot \pi_\eta - iS\pi_{\bar{\eta}}]\, d^3x \tag{3.86}$$

such that

$$\delta \Phi_I = [i\,\lambda\, Q_B, \Phi_I]^*. \tag{3.87}$$

Note that $\Gamma = D_i \pi^i$ is the first class constraint of the system without gauge fixing. The commutation relations $[A, B]^*$ is the Dirac one and from the explicit form of the BRS charge, the BRS transformation of the canonical conjugate momenta,

$$\begin{aligned}
\delta \pi_i &= \lambda\, g\, \pi_i \times \eta, && (3.88a)\\
\delta \pi_S &= -i\,\lambda\, \pi_{\bar{\eta}}, && (3.88b)\\
\delta \pi_\eta &= -\lambda (D_i \pi^i + g\, \pi_\eta \times \eta), && (3.88c)\\
\delta \pi_{\bar{\eta}} &= 0, && (3.88d)
\end{aligned}$$

can be obtained which interestingly enough are gauge independent. With these results, one then finds

$$[Q_B, Q_B]^* = 0, \tag{3.89}$$

which express the nilpotency, ${Q_B}^2 = 0$, of the BRS charge. Subsidiary conditions in general non-covariant gauges can be imposed in the operator formalism with the second quantized version of the BRS charge.

As we have seen, problems encountered in quantizing gauge theories can be solved by adding a *gauge-fixing* term to the classical Lagrangian, such as the covariant gauge condition. The requirement that Maxwell's equations hold as operator identities is dropped in favour of retaining explicit Lorentz invariance and locality. However, this is accomplished at the price of quantizing the *unphysical* longitudinal and time-like photons, a consequence of which is the occurrence of negative norm states in the Fock space of asymptotic states. The unitarity of the theory is only apparent when the *physical* subspace, in which Maxwell's equations hold in the weak sense, is projected out of this Fock space by the imposition of a suitable subsidiary condition, whose purpose is to remove the effects of the unphysically polarised photons. The subsidiary condition is a statement of the gauge invariance of physical states.

An alternative procedure involves the choice of a *singular* gauge, like Coulomb or axial gauges, in which (some of) the redundancy is removed by solving the Gauss' law, $D_i\pi^i = 0$, as an operator equation. In quantum theory, this has the advantages that Maxwell's equations hold as operator identities, the Fock space of the asymptotic states is the physical Hilbert space, and the unitarity of the theory is easily shown. Nevertheless, one loses manifest Lorentz invariance and is faced with many technical problems in this class of gauges.

4 Non-Covariant Gauges

Non-covariant gauges are frequently used. For instance, the standard canonical quantization of QED uses the Coulomb gauge, also called the radiation gauge. In QCD one often uses axial gauges which allow the quanta of the vector fields to be interpreted as partons. It is easy to check that in axial gauges the Faddeev Popov determinant is A^a_μ independent even in a non-abelian theory. The purpose of reviewing some standard materials in the previous chapter was to prepare for the study of Yang-Mills theories in a general class of non-covariant gauges in this chapter. BRS analysis is carried out in the full theory whereas an operator formalism is presented for the free field theory. The work presented here is based on material published in [25].

4.1 Classification of gauge constraints

Although there are a vast number of gauges one can use (linear or nonlinear, covariant or noncovariant, homogeneous or inhomogeneous), all known gauges can be divided into three distinct classes. They are classified by the number of independent degrees of freedom contained in the vector boson propagator. In this respect, we list the 3 classes below [26], giving particular examples in each case.

Class I gauges

In this class, the gauge boson propagator contains only physical degrees of freedom and ghosts are decoupled. Examples are the familiar Coulomb gauge

$$\partial_k A_k = 0 \tag{4.90}$$

and the pure axial gauge

$$A_3 = 0 \tag{4.91}$$

which was used by Arnowitt and Fickler [27] to investigate the quantization of non-abelian gauge theories. Since the canonical momenta are

$$\pi^k = F^{k0} = D^k A_0 - \partial_0 A^k, \tag{4.92}$$

where D is the covariant derivative and where internal symmetry indices are omited, then the requirement of validity of the gauge condition for any time implies respectively

$$\partial_k D^k A_0 - \partial_k \pi^k = 0 \tag{4.93}$$

or

$$\pi^3 - \partial^3 A_0 = 0. \tag{4.94}$$

These equations are constraints which must be solved with respect to A_0. In both cases the solution is ambiguous because, in the latter case, the definition of ∂_3^{-1} requires additional boundary conditions, while the former case suffers the Gribov ambiguity [28]:

$$\det(\partial_k D^k) = 0. \tag{4.95}$$

If one takes into account that Eq.(4.95) holds only for large values of the fields, then such an ambiguity does not affect perturbation theory. However, in the quantum theory this method leads to a difficult operator ordering problem which can be solved only with external arguments which restore the covariance of the theory and its equivalence with the temporal gauge [29]. On the other hand, the pure axial gauge is plagued with an indefinite metric although only physical degrees of freedom are involved [31]. Hence both the Coulomb and the pure axial gauges suffer from technical problems that one must take careful note of when performing explicit calculations [33,34].

Class II gauges

In class II gauges, there is, in addition to the physical degrees of freedom, a single unphysical one associated with the longitudinal polarization. One example of this type of gauge is the temporal axial gauge $A_0 = 0$. Gauss' law does not hold as an operator equation but is imposed as a condition on physical states:

$$D^k \pi^k |\Psi_{phys} >= 0. \tag{4.96}$$

This is known as Dirac quantization. In general the axial gauges $n \cdot A = 0$, are Class II unless $n_0 = 0$. Those were originally introduced by Kummer [30] in quantizing the free electromagnetic field. In Class II gauges, the unphysical degree of freedom satisfies a first-order evolution equation of the form $\partial_0 S = 0$. The solution of this Cauchy problem in the Lagrangian formalism leads to the principal-value prescription for the uphysical pole at $k_0 = 0$ in the propagator [35], which however is known to be unacceptable for Yang-Mills theory and is related to the non-realization of Gauss' law [37]. Insisting on the implementation of Gauss' law leads to a propagator which breaks time-reversal invariance and therefore cannot be written in momentum space [38]. If one follows a Hamiltonian approach, a well-defined temporal gauge propagator with the Leibbrandt-Mandelstam prescription for the unphysical pole results [39,40]. In this case, the propagator contains a vector n^*_μ that does not appear in the original Lagrangian. Thus, as with Class I gauges, there are a number of technical subtelties associated with quantization of Class II gauges.

Class III gauges

In class III gauges, there are two unphysical degrees of freedom which, in order to satisfy unitarity, are accompanied by Faddeev-Popov ghost fields. One can use BRS invariance of the classical Lagrangian to quantize the theory [41], defining physical states as cohomology classes of the BRS operator. The prototypes of Class III gauges are the relativistic gauges, but as we shall see there is a large class of non-covariant Class III gauges which have also been found useful in various circumstances. An important example of this is the Leibbrandt-Mandelstam [39] prescription for the unphysical pole in axial gauges:

$$\frac{1}{n \cdot k} \to \frac{n^* \cdot k}{n \cdot k\, n^* \cdot k + i\epsilon}, \tag{4.97}$$

which can be derived from the Class III gauge condition [42]

$$n^* \cdot \partial\, n \cdot A = 0. \tag{4.98}$$

Although the presence in general of Faddeev-Popov ghosts makes perturbative calculations somewhat more complicated algebraically, difficulties in quantization of the type found in Class I and Class II gauges are absent in Class III gauges.

In the next section we will present a unified framework for the quantization of Yang-Mills theories using all the three classes of gauge condition. As we shall see, Class III gauges are somewhat simpler to quantize since they do not possess many of the technical problems and ambiguities that plague their Class I and Class II counterparts. Since Class I and Class II gauges can in general be considered as limiting cases of Class III gauges, formalism for the class III gauge condition provides an alternate method for avoiding technical difficulties present in Class I and Class II gauges.

4.2 General Linear Gauge Action

A general linear class of gauges can be implemented by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - C_{\mu\nu}\partial^\mu S^a A^{\nu a} + \frac{1}{2}aS^2. \tag{4.99}$$

In order to show the general character of the class of gauge Eq.(4.99), we now give a partial list of particular cases which are included in it.

- The relativistic gauges have the form

  $$C_{\mu\nu} = g_{\mu\nu}$$

  The parameter a is the usual gauge parameter; its values $a = 0$ and $a = 1$ correspond respectively to the Landau and Feynman gauges.

- Regulated Coulomb and static temporal gauges have

  $$C_{\mu\nu} = n_\mu n_\nu - \alpha g_{\mu\nu},\ n^2 = 1,\ \alpha > 0\ (\neq 1)$$

  Here, a and α are the gauge parameters. The operator $\Box_C \equiv C_{\mu\nu}\partial^\mu\partial^\nu$ is elliptic for $0 < \alpha < 1$, and therefore no unphysical real pole is introduced in the propagator. For $\alpha > 1, \Box_C$ is hyperbolic, and the limit $\alpha \to \infty$ corresponds to the relativistic Feynman gauge. The limiting cases $\alpha \to 0$ and $\alpha \to 1$ correspond respectively to the static temporal and to the Coulomb gauges, and are characterized by a parabolic $\Box_C$. First introduced by Frenkel and Taylor [44], this interpolating gauge was further studied by Burnel [45] and Nyeo [46], and has been used in finite temperature calculations [47]. The choice $a = \alpha(\alpha - 1)$ turns out to be particularly convenient.

- In axial gauges with the Leibbrandt-Mandelstam prescription, we have

$$C_{\mu\nu} = n^*_\mu n_\nu, \ a = 0,$$

with arbitrary n^2. Although the LM prescription can also be derived from a Hamiltonian formalism as a class II gauge [40], in the present Lagrangian formalism it is of class III [34]. The main difference between the two approaches is that in the former, the vector n^* emerges in the course of the Hamiltonian analysis, while in the latter, it is incorporated directly into the Lagrangian.

- Gauges interpolating between axial (or planar) relativistic gauges with

$$C_{\mu\nu} = n^*_\mu n_\nu - \alpha g_{\mu\nu}$$

were introduced by Piguet, Pollak and Schweda [49] in order to regularize the non-local counter-terms occuring with the LM prescription.

- The $n^2 < 0$ Frenkel-Taylor gauge has [44]

$$C_{\mu\nu} = \frac{n_\mu n_\nu}{n^2} - \alpha g_{\mu\nu},\ n^2 < 0.$$

All the values $\alpha \neq 0$ lead to a hyperbolic $\Box_C$.

- Gauges that interpolate between space-like axial, regulated time-like and relativistic gauges have the form

$$C_{\mu\nu} = \frac{n_\mu n_\nu}{n^2} + \alpha(\frac{m_\mu m_\nu}{m^2} - \beta g_{\mu\nu}), n^2 < 0, m^2 > 0.$$

The cases $\beta > 1, \alpha > (1/\beta)$ or $\alpha < 0$ and $\beta < 0, \alpha < (1/\beta)$ or $\alpha > 0$ correspond to a hyperbolic $\Box_C$. An elliptic $\Box_C$ is obtained for $\alpha < 0, 0 < \beta < 1$. Not of easy practical use, this gauge was introduced in order to regularize the space-like axial gauge as a physical gauge [50].

Clearly, many linear gauges can be formulated in a universal way in terms of the gauges Eq.(4.99). As we shall see, the three classes of gauges described in Section 2 are distinguished by the values of the components of $C_{\mu\nu}$ which influence the constraint structure. By far the largest class of gauges is Class III, in which there are four independent degrees of freedom in the gluon propagator. This class is in a sense the simplest to deal with because of the relatively simple constraint structure. For class I and class II gauges, more constraints appear (all second class), so that the usual quantization procedure must be modified.

4.3 BRS quantization

In this section we shall examine the quantization of the non-Abelian case using the usual BRS formalism followed for relativistic gauges [41]. The Lagrangian for this particular class of gauges can be written as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu} \cdot F^{\mu\nu} - \partial S \cdot \tilde{A} + \tfrac{1}{2}aS^2 - i\,\hat{\partial}\bar{\eta} \cdot D\eta, \tag{4.100}$$

with ${F_{\mu\nu}}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g(A_\mu \times A_\nu)^a$ and $D^{ab}_\mu = \partial_\mu \delta^{ab} - g\, f^{abc} A^c_\mu$. Here we have used the shorthand notation $\tilde{A}^\mu \equiv C^\mu{}_\nu A^\nu$ and $\hat{A}_\nu \equiv A_\mu C^\mu{}_\nu$. The Grassmann variables η^a and $\bar{\eta}^a$ are the Faddeev-Popov ghosts, and left differentiation will be chosen as our convention:

$$\frac{\partial}{\partial \xi}(AB) = \left(\frac{\partial}{\partial \xi}A\right) B + (-1)^{P_A} A \left(\frac{\partial}{\partial \xi}B\right), \tag{4.101}$$

where ξ is a Grassmann variable and P_A is the parity of A, which are $P_{A^\mu} = P_S = 0$ and $P_\eta = P_{\bar{\eta}} = 1$.

The Euler-Lagrange equations derived from the Lagrangian of Eq.(4.100) are

$$D_\nu F^{\mu\nu} + \hat{\partial}^\mu S - i\,g\,\hat{\partial}^\mu \bar{\eta} \times \eta = 0, \tag{4.102a}$$

$$\partial \cdot \tilde{A} + aS = 0, \tag{4.102b}$$

$$D \cdot \hat{\partial}\bar{\eta} = 0, \tag{4.102c}$$

$$\hat{\partial} \cdot D\eta = 0. \tag{4.102d}$$

The canonical conjugate momenta are subsequently defined as

$$\pi^\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0}, \tag{4.103a}$$

$$\pi_S \equiv \frac{\partial \mathcal{L}}{\partial \dot{S}} = -\tilde{A}^0, \tag{4.103b}$$

$$\pi_\eta \equiv \frac{\partial \mathcal{L}}{\partial \dot{\eta}} = i\,\hat{\partial}_0 \bar{\eta}, \tag{4.103c}$$

$$\pi_{\bar{\eta}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\bar{\eta}}} = -i\,\tilde{D}_0 \eta, \tag{4.103d}$$

and the canonical commutation and anti-commutation relations are taken to be

$$[\pi_{\Phi^I}(\mathbf{x},t), \Phi_J(\mathbf{y},t)] = -i\,\delta^I{}_J \delta(\mathbf{x}-\mathbf{y}), \tag{4.104a}$$

$$[\pi_{\Phi^I}(\mathbf{x},t), \pi_{\Phi_J}(\mathbf{y},t)] = 0 = [\Phi_I(\mathbf{x},t), \Phi_J(\mathbf{y},t)], \tag{4.104b}$$

where Φ_I represents A^a_μ, S^a, η^a and $\bar{\eta}^a$. Note that the commutation and anti-commutation relations are defined as

$$[A, B] = AB - (-1)^{P_A P_B} BA, \tag{4.105}$$

where P_A and P_B are the parities of A and B respectively.

It can be seen from Eqs.(4.102) that the theory contains different constraint structures for different values of $C^\mu{}_\nu$. As indicated in the previous Section, there are three distinct cases, which in this context are described by

- $C_{00} \neq 0$,
- $C_{00} = 0$ but some of $C_{0i}, C_{i0} \neq 0$,
- $C_{00} = C_{0i} = C_{i0} = 0$,

which correspond to a Class III, Class II, and Class I gauge, respectively.

4.3.1 Hamiltonian Constraint Analysis

For $C_{00} \neq 0$, there exist $2n$ pairs of second class primary constraints:

$$\phi_1^a = \pi_0^a \approx 0, \tag{4.106a}$$

$$\phi_2^a = \pi_S^a + \tilde{A}_0^a \approx 0. \tag{4.106b}$$

Due to the presence of these constraints, the canonical Hamiltonian is no longer the most general evolution operator. Instead an effective Hamiltonian is required:

$$\begin{aligned}
H_e &= \int [\lambda_1 \cdot \phi_1 + \lambda_2 \cdot \phi_2 + \dot{A}_\mu \cdot \pi^\mu \\
&+ \dot{S} \cdot \pi_S + \dot{\eta} \cdot \pi_\eta + \dot{\bar{\eta}} \cdot \pi_{\bar{\eta}} - \mathcal{L}]\, d^3x \\
&= \int [\lambda_1 \cdot \phi_1 + \lambda_2 \cdot \phi_2 + \dot{A}_0 \cdot \phi_1 + \dot{S} \cdot \phi_2 \\
&+ (\frac{i}{C_{00}} \pi_{\bar{\eta}} - g A_0 \times \eta - \frac{1}{C_{00}} C_{0i} D^i \eta) \cdot \pi_\eta \\
&+ i\, \partial^i \bar{\eta} \cdot \tilde{D}_i \eta + \mathcal{H}_0 - A_0 \cdot D_i\, \pi^i + \partial_i S \cdot \tilde{A}^i - \tfrac{1}{2} a S^2]\, d^3x \\
&= \int [\lambda_1 \cdot \phi_1 + \lambda_2 \cdot \phi_2 + (\frac{i}{C_{00}} \pi_{\bar{\eta}} - g A_0 \times \eta - \frac{1}{C_{00}} C_{0i} D^i \eta) \cdot \pi_\eta \\
&+ i\, \partial^i \bar{\eta} \cdot \tilde{D}_i \eta + \mathcal{H}_0 - A_0 \cdot D_i\, \pi^i + \partial_i S \cdot \tilde{A}^i - \tfrac{1}{2} a S^2]\, d^3x
\end{aligned} \tag{4.107}$$

with

$$\mathcal{H}_0 = \tfrac{1}{2} \pi \cdot \pi + \tfrac{1}{2} \mathbf{B} \cdot \mathbf{B}, \tag{4.108}$$

so that $\dot{\mathcal{O}} = [iH, \mathcal{O}]$ for any operator $\mathcal{O}$. Notice that surface integrals are neglected and $\dot{A}_0$ and $\dot{S}$ have been absorbed into the Lagrange multiplier fields λ_1 and λ_2 respectively.

Conserving the primary constraints in time yields the relations $\lambda_1 = \dot{A}_0$ and $\lambda_2 = \dot{S}$. However, the canonical commutation and anti-commutation relations of

Eqs.(4.104) are not compatible with the constraints, and so the Dirac constraint approach must be followed [53]. In the present case, the Dirac brackets are found to be

$$[A_0^a(\mathbf{x},t), S^b(\mathbf{y},t)]^* = \frac{i}{C_{00}}\delta^{ab}\delta(\mathbf{x}-\mathbf{y}), \tag{4.109a}$$

$$[A_0^a(\mathbf{x},t), \pi_0^b(\mathbf{y},t)]^* = 0, \tag{4.109b}$$

$$[A_0^a(\mathbf{x},t), \pi_i^b(\mathbf{y},t)]^* = -\frac{i}{C_{00}}C_{0i}\delta^{ab}\delta(\mathbf{x}-\mathbf{y}), \tag{4.109c}$$

with all remaining commutators unchanged as in Eqs.(4.104).

The constraints may now be set strongly equal to zero and the Dirac brackets taken to be the equal time commutation and anti-commutation relations. The resulting reduced Hamiltonian is then

$$\begin{aligned} H \quad &= \int [\lambda_1 \cdot \phi_1 + \lambda_2 \cdot \phi_2 + \frac{1}{C_{00}}(i\pi_{\bar{\eta}} - C_{0i}D^i\eta) \cdot \pi_\eta \\ &\quad -A_0 \cdot (D_i\pi^i + g\pi_\eta \times \eta) + i\,\partial^i\bar{\eta} \cdot \tilde{D}_i\eta + \partial_i S \cdot \tilde{A}^i \\ &\quad -\tfrac{1}{2}aS^2 + \mathcal{H}_0]\, d^3x \end{aligned} \tag{4.110}$$

which is consistent with the constraints of Eqs.(4.106) and the Euler-Lagrange equations of Eqs.(4.102).

4.3.2 The BRS charge

As mentioned in a general framework before, the total Lagrangian $\mathcal{L}$ of Eq.(4.100) does not possess any local gauge invariance. However, it has a global invariance under the BRS transformations. Noether's first theorem ensure the existence of the BRS charge,

$$Q_B \;=\; \int [D_i\eta \cdot \pi^i - \tfrac{1}{2}g\,(\eta \times \eta) \cdot \pi_\eta + iS\pi_{\bar{\eta}}]\, d^3x$$

$$= \int [S \cdot \tilde{D}_0 \eta - \eta \cdot \hat{\partial}_0 S + \frac{i}{2} g\, (\eta \times \eta) \cdot \hat{\partial}_0 \bar{\eta}]\, d^3x. \qquad (4.111)$$

Note that in deriving this last relation the Euler-Lagrange equations of Eqs.(4.102) have been imposed.

From the explicit form of the BRS charge, using Eq.(3.87) and the Dirac brackets stated in Eq.(4.109), with no surprise, one can verify that the canonical momenta accually transform as Eq.(3.88). With these results, several useful transformations are readily derived:

$$\delta(\mathcal{H}_0) = 0, \qquad (4.112a)$$

$$\delta(i\, \partial^i \bar{\eta} \cdot \tilde{D}_i \eta + \partial_i S \cdot \tilde{A}^i) = 0, \qquad (4.112b)$$

$$\begin{aligned} \delta(A_0 \cdot \pi_\eta) - \frac{1}{C_{00}} \delta\phi_2 \cdot \pi_\eta = \lambda[& \frac{1}{C_{00}} (i\pi_{\bar{\eta}} - C_{0i} D^i \eta) \cdot \pi_\eta \\ & - A_0 \cdot (D_i \pi^i + g\pi_\eta \times \eta)]. \end{aligned} \qquad (4.112c)$$

One then finds

$$\begin{aligned} &[i\, \lambda\, Q_B, Q_B]^* = 0, \\ &[i\, \lambda\, Q_B, H]^* = 0, \end{aligned} \qquad (4.113)$$

which express the nilpotency ($Q_B{}^2 = 0$) and time independence ($\dot{Q}_B = 0$), respectively, of the BRS charge.

Another important conserved charge, the Faddeev-Popov ghost charge, generates the scale transformation

$$\delta_\theta \eta = \theta \eta, \qquad (4.114a)$$

$$\delta_\theta \bar{\eta} = -\theta \bar{\eta}, \qquad (4.114b)$$

$$\delta_\theta A_\mu = 0 = \delta_\theta S, \tag{4.114c}$$

and is given by

$$\begin{aligned} Q_\eta &= \int [\eta \cdot \pi_\eta - \bar{\eta} \cdot \pi_{\bar{\eta}}]\, d^3x \\ &= i \int [\eta \cdot \hat{\partial}_0 \bar{\eta} - \tilde{\partial}_0 \eta \cdot \bar{\eta} + g\, \bar{\eta} \cdot (\tilde{A}_0 \times \eta)]\, d^3x, \end{aligned} \tag{4.116}$$

such that

$$\delta_\theta \Phi_I = [i\,\theta\, Q_\eta, \Phi_I]^*. \tag{4.117}$$

It is also straightforward to show that

$$[iQ_\eta, Q_B]^* = Q_B \tag{4.118}$$

$$[iQ_\eta, Q_\eta]^* = 0 \tag{4.119}$$

which means that Q_B and Q_η carry FP ghost number one and zero respectively.

Although the local gauge invariance of $\mathcal{L}$ is broken by $\mathcal{L}_{GF}$ and $\mathcal{L}_{FP}$, the Lagrangian $\mathcal{L}$ is still invariant under the global gauge transformation

$$\delta_\varepsilon \Phi_I = g\, \Phi_I \times \varepsilon. \tag{4.120}$$

By using Noether's theorem, the associated conserved current is found to be

$$\begin{aligned} J_\mu &= gA^\nu \times F_{\mu\nu} - g\, \tilde{A}_\mu \times S - i\, g\, \hat{\partial}_\mu \bar{\eta} \times \eta \\ &+ i\, g\, \tilde{D}_\mu \eta \times \bar{\eta} + \hat{\partial}_\mu S - \tilde{\partial}_\mu S. \end{aligned} \tag{4.121}$$

The corresponding charge,

$$Q = \int J_0\, d^3x, \tag{4.122}$$

generates the global gauge transformation

$$\delta_\varepsilon \Phi_I = [i\varepsilon Q, \Phi_I]^*. \tag{4.123}$$

The current of Eq.(4.121) can be used to write the equation of motion of the first of Eqs.(4.102) in the form

$$\partial_\nu F^{\nu\mu} - J^\mu = \{Q_B, \tilde{D}^\mu \bar{\eta}\}, \tag{4.124}$$

where the curly bracket means anti-commutator, and is referred to as a generalized Maxwell equation. Note that the term $\hat{\partial}_\mu S - \tilde{\partial}_\mu S$, which has vanishing four divergence, s added to the current J_μ of Eq.(4.121) in order to make Eq.(4.124) valid even for non-symmetric $C^\mu{}_\nu$.

4.3.3 Relativistic invariance of physical states

Having constructed the BRS charge Q_B, physical states $|\Psi_{phys}\rangle$ will then be required to satisfy

$$Q_B|\Psi_{phys}\rangle = 0. \tag{4.125}$$

Although relativistic invariance is lost in general on the entire space of states, as mentioned at the end of Section 3, it is possible to verify that physical states defined in Eq.(4.125) are Lorentz invariant. To do so, we begin by considering

$$T^{\mu\nu} = \partial^\nu A^\tau \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\tau)} - g^{\mu\nu}\mathcal{L}, \tag{4.126}$$

$$\mathcal{M}^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)}A^\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}A^\rho. \tag{4.127}$$

The corresponding generators of Lorentz transformations,

$$P_\mu = \int d^3x T_{0\mu}, \tag{4.128}$$

$$M_{\mu\nu} = \int d^3x \mathcal{M}_{0\mu\nu}, \tag{4.129}$$

satisfy the commutation relations

$$[P_\mu, P_\nu] = 0, \tag{4.130}$$

$$[P^\mu, M^{\nu\lambda}] = -i\left(g^{\mu\lambda}P^\nu - g^{\mu\nu}P^\lambda\right) - i\int d^3x \partial_\tau \mathcal{M}^{\tau\nu\lambda}g^{\mu0}, \tag{4.131}$$

$$[M^{\lambda\rho}, M^{\mu\nu}] = -i\left(g^{\rho\mu}M^{\lambda\nu} + g^{\lambda\nu}M^{\rho\mu} - g^{\rho\nu}M^{\lambda\mu} - g^{\lambda\mu}M^{\rho\nu}\right) \\ -i\int d^3x \left(x^\rho g^{\lambda0} - x^\lambda g^{\rho0}\right)\partial_\tau \mathcal{M}^{\tau\mu\nu}. \tag{4.132}$$

We thus see explicitly that Lorentz invariance is lost in general. However, using Eq.(4.127) and Eq.(4.124), one can show that

$$\partial_\mu \mathcal{M}^{\mu\nu\rho} = \left\{Q_B, \partial^\rho\bar{\eta}\hat{A}^\nu - \partial^\nu\bar{\eta}\hat{A}^\rho\right\}, \tag{4.133}$$

where Q_B is the BRS charge of Eq.(4.111). In this form one can readily see that

$$\langle\Psi_{phys}|\partial_\mu\mathcal{M}^{\mu\nu\rho}|\Psi_{phys}{}'\rangle = 0, \tag{4.134}$$

and so Lorentz invariance holds in the physical subspace of Eq.(4.125).

We now verify that the Lorentz generators of Eqs.(4.128,4.129) commute with the BRS charge, which will show that the physical subspace is mapped into itself by Lorentz transformations. We first look at space-time translations, and since we already know by Eq.(4.113) that Q_B commutes with the Hamiltonian H, We only have to consider spatial translations. One can show that P_i of Eq.(4.128) can be written as

$$P_i = \int d^3x \left[\partial_i A^j \cdot \pi_j + \partial_i S \cdot \pi_S + \partial_i\eta \cdot \pi_\eta + \partial_i\bar{\eta} \cdot \pi_{\bar{\eta}}\right]. \tag{4.135}$$

Now, using Eq.(3.87), one finds

$$\delta(\partial_i S \cdot \pi_S) = -i\lambda \partial_i S \cdot (-\pi_{\bar{\eta}}), \tag{4.136a}$$

$$\begin{aligned}\delta(\partial_i \eta \cdot \pi_\eta) &= \partial_i(-\frac{\lambda}{2} g\eta \times \eta) \cdot \pi_\eta \\ &+\partial_i \eta \cdot \left[-\lambda(D_i \pi^i + g\pi_\eta \times \eta)\right],\end{aligned} \tag{4.136b}$$

$$\delta(\partial_i \bar{\eta} \cdot \pi_{\bar{\eta}}) = i\lambda \partial_i S \cdot \pi_{\bar{\eta}}, \tag{4.136c}$$

$$\begin{aligned}\delta(\partial_i A^j \cdot \pi_j) &= \partial_i \left[\lambda D^j \cdot \pi_j + \lambda A^j \cdot g(\pi_j \times \eta)\right] \\ &-\lambda D^j \eta \cdot \partial_i \pi_j - \lambda A^j \cdot g(\partial_i \pi_j \times \eta + \pi_j \times \partial_i \eta).\end{aligned} \tag{4.136d}$$

After some algebra, one arrives at the result

$$\delta P_i = [i\,\lambda Q_B, P_i]^* = 0, \tag{4.137}$$

and so with Eq.(4.113) we find

$$[i\,\lambda Q_B, P_\mu]^* = 0. \tag{4.138}$$

Similar results hold for the other Lorentz generators of Eq.(4.129). One can write these generators in the form

$$M_{0i} = \int d^3x \mathcal{M}_{00i} = \int d^3x[x_i P_0 - x_0 P_i + \pi_i A_0], \tag{4.139}$$

$$M_{ij} = \int d^3x \mathcal{M}_{0ij} = \int d^3x[x_j P_i - x_i P_j + \pi_j A_i - \pi_i A_j], \tag{4.140}$$

and again some straightforward algebra leads to the result

$$[i\,\lambda Q_B, M_{0i}]^* = 0 = [i\,\lambda Q_B, M_{ij}]^*. \tag{4.141}$$

Consequently, the BRS charge commutes with all of the generators of Lorentz transformations.

4.3.4 Class II gauges

For $C_{00} = 0$ and not all C_{0i} and C_{i0} equal to 0, there exist $4n$ primary constraints:

$$\phi_1^a \equiv \pi_0^a \approx 0, \tag{4.142a}$$

$$\phi_2^a \equiv \pi_S^a + C_{0i}A^{ia} \approx 0, \tag{4.142b}$$

$$\phi_3^a \equiv \pi_\eta^a - iC_{i0}\partial^i\bar{\eta}^a \approx 0, \tag{4.142c}$$

$$\phi_4^a \equiv \pi_{\bar{\eta}}^a + iC_{0i}D^i\eta^a \approx 0. \tag{4.142d}$$

Hence, the effective Hamiltonian can be written as

$$\begin{aligned} H_e &= \int [\lambda_i \cdot \phi_i + \dot{A}_\mu \cdot \pi^\mu + \dot{S} \cdot \pi_S + \dot{\eta} \cdot \pi_\eta + \dot{\bar{\eta}} \cdot \pi_{\bar{\eta}} - \mathcal{L}]\, d^3x \\ &= \int [\lambda_i \cdot \phi_i + \dot{A}_0 \cdot \phi_1 + \dot{S} \cdot \phi_2 + \dot{\eta} \cdot \phi_3 + \dot{\bar{\eta}}\phi_4 + \mathcal{H}_0 + C^i{}_j\partial_i S \cdot A^j \\ &+ iC^i{}_j\partial_i\bar{\eta} \cdot D^j\eta - \tfrac{1}{2}aS^2 - A_0 \cdot (D_i\pi^i - C_{i0}\partial^i S + igC_{i0}\partial^i\bar{\eta} \times \eta)]\, d^3x \\ &= \int [\lambda_i \cdot \phi_i + \mathcal{H}_0 + C^i{}_j\partial_i S \cdot A^j + iC^i{}_j\partial_i\bar{\eta} \cdot D^j\eta - \tfrac{1}{2}aS^2 \\ &- A_0 \cdot (D_i\pi^i - C_{i0}\partial^i S + igC_{i0}\partial^i\bar{\eta} \times \eta)]\, d^3x. \end{aligned} \tag{4.143}$$

Since conserving ϕ_3^a and ϕ_4^a will give the conditions on the Lagrange multiplier fields, new constraints can only be obtained by conserving ϕ_1 and ϕ_2:

$$\varphi_1 \equiv \dot{\phi}_1 = D_i\pi^i - C_{i0}\partial^i S + ig\, C_{i0}\partial^i\bar{\eta} \times \eta, \tag{4.144a}$$

$$\varphi_2 \equiv \dot{\phi}_2 = aS + C^i{}_j\partial_i A^j - C_{0i}\pi^i + \nabla A_0, \tag{4.144b}$$

where

$$\nabla^{ab} \equiv C_{0i}D^{iab} + C_{i0}\partial^i\delta^{ab}. \tag{4.145}$$

Now the $4n$ primary constraints together with the $2n$ secondary constraints all become second class. By use of the uler-Lagrangian equations, further conserving the constraints φ_1, φ_2, ϕ_3 and ϕ_4 will give $\lambda_1 = \dot{A}_0$, $\lambda_2 = \dot{S}$, $\lambda_3 = \dot{\eta}$ and $\lambda_4 = \dot{\bar{\eta}}$. Dirac brackets can then be found iteratively.

The second class pair ϕ_3 and ϕ_4 are taken to be the preliminary constraints. The Dirac brackets of them are

$$[\phi_3^a(\mathbf{x},t), \phi_3^b(\mathbf{y},t)] = 0, \tag{4.146a}$$

$$[\phi_3^a(\mathbf{x},t), \phi_4^b(\mathbf{y},t)] = -\nabla_x^{ab}\delta(\mathbf{x}-\mathbf{y}), \tag{4.146b}$$

$$[\phi_4^a(\mathbf{x},t), \phi_3^b(\mathbf{y},t)] = \nabla_x^{ab}\delta(\mathbf{x}-\mathbf{y}), \tag{4.146c}$$

$$[\phi_4^a(\mathbf{x},t), \phi_4^b(\mathbf{y},t)] = 0. \tag{4.146d}$$

The inverse of the matrix $C_{ij}^{ab}(\mathbf{x},\mathbf{y}) \equiv [\phi_i^a(\mathbf{x},t), \phi_j^b(\mathbf{y},t)]$ can be chosen as

$$[C^{-1}]_{ij}^{ab}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} 0 & K^{ab}(\mathbf{x},\mathbf{y}) \\ -K^{ab}(\mathbf{x},\mathbf{y}) & 0 \end{pmatrix}, \tag{4.147}$$

provided that

$$\nabla_x^{ac}K^{cd}(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}) = -\nabla_y^{bc}K^{ac}(\mathbf{x},\mathbf{y}). \tag{4.148}$$

Note that by symmetry $K^{ab}(\mathbf{x},\mathbf{y}) = -K^{ba}(\mathbf{y},\mathbf{x})$. We remark that here and in the following we shall assume no problems arise with respect to inverting such matrices, and thus these expressions at this level should be regarded as formal. With this in mind, the preliminary Dirac brackets can then be found to be, using Eq.(4.146),

$$[\eta^a(\mathbf{x},t), \bar{\eta}^b(\mathbf{y},t)]' = K^{ab}(\mathbf{x},\mathbf{y}), \tag{4.149a}$$

$$[\eta^a(\mathbf{x},t), \pi_\eta{}^b(\mathbf{y},t)]' = iC_{i0}\partial_y^i K^{ab}(\mathbf{x},\mathbf{y}), \tag{4.149b}$$

$$[\eta^a(\mathbf{x},t),\pi_i^b(\mathbf{y},t)]' = -iC_{0i}g\,K^{ac}(\mathbf{x},\mathbf{y})f^{cbe}\eta^e(y), \tag{4.149c}$$
$$[\pi_{\bar\eta}^a(\mathbf{x},t),\bar\eta^b(\mathbf{y},t)]' = -iC_{0i}D_x^{iac}K^{cb}(\mathbf{x},\mathbf{y}), \tag{4.149d}$$
$$[\pi_{\bar\eta}^a(\mathbf{x},t),{\pi_\eta}^b(\mathbf{y},t)]' = C_{i0}\partial_x^i C_{0j}D_y^{jbc}K^{ac}(\mathbf{x},\mathbf{y}), \tag{4.149e}$$
$$[\pi_{\bar\eta}^a(\mathbf{x},t),\pi_i^b(\mathbf{y},t)]' = iC_{j0}\partial_x^j[\eta^a(\mathbf{x},t),\pi_i^b(\mathbf{y},t)]', \tag{4.149f}$$

while others remain the same as in Eq.(4.104).

With the above preliminary Dirac brackets, the commutators of φ_1 and ϕ_2 are found to be

$$[\varphi_1^a(\mathbf{x},t),\varphi_1^b(\mathbf{y},t)]' = M^{ab}(\mathbf{x},\mathbf{y}), \tag{4.150a}$$
$$[\varphi_1^a(\mathbf{x},t),\phi_2^b(\mathbf{y},t)]' = -i\nabla_x^{ab}\delta(\mathbf{x}-\mathbf{y}), \tag{4.150b}$$
$$[\phi_2^a(\mathbf{x},t),\varphi_1^b(\mathbf{y},t)]' = i\nabla_y^{ba}\delta(\mathbf{x}-\mathbf{y}), \tag{4.150c}$$
$$[\phi_2^a(\mathbf{x},t),\phi_2^b(\mathbf{y},t)]' = 0. \tag{4.150d}$$

The inverse of the matrix $C'^{ab}_{ij}(\mathbf{x},\mathbf{y})$ can be chosen as

$$[C'^{-1}]_{ij}^{ab}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} 0 & iK^{ab}(\mathbf{x},\mathbf{y}) \\ iK^{ab}(\mathbf{x},\mathbf{y}) & -N^{ab}(\mathbf{x},\mathbf{y}) \end{pmatrix}, \tag{4.151}$$

provided that

$$\int [M^{ac}(\mathbf{x},\mathbf{z})K^{cb}(\mathbf{z},\mathbf{y})]\,d^3z = -\nabla_x^{ac}N^{cb}(\mathbf{x},\mathbf{y}), \tag{4.152a}$$
$$\int [K^{ac}(\mathbf{x},\mathbf{z})M^{cb}(\mathbf{z},\mathbf{y})]\,d^3z = \nabla_y^{bc}N^{ac}(\mathbf{x},\mathbf{y}). \tag{4.152b}$$

The secondary Dirac brackets are then found to be

$$\left[S^a(\mathbf{x},t),S^b(\mathbf{y},t)\right]'' = N^{ab}(\mathbf{x},\mathbf{y}), \tag{4.153a}$$

$$\left[\pi_i^a(\mathbf{x},t), S^b(\mathbf{y},t)\right]" = -C_{0i}K^{ab}(\mathbf{x},\mathbf{y}) + i\,g\,f^{aec}\pi_i^e(x)K^{cb}(\mathbf{x},\mathbf{y})$$
$$+iC_{0i}\int [g^2 f^{acd}\eta^c(x)K^{df}(\mathbf{x},\mathbf{z})f^{feg}\pi_\eta^e K^{gb}(\mathbf{z},\mathbf{y})]\,d^3z, \tag{4.153b}$$
$$\left[A^{ia}(\mathbf{x},t), S^b(\mathbf{y},t)\right]" = iD_x^{iac}K^{cb}(\mathbf{x},\mathbf{y}), \tag{4.153c}$$
$$\left[A^{ia}(\mathbf{x},t), \pi_j^b(\mathbf{y},t)\right]" = i\delta_j^i\delta^{ab}\delta(\mathbf{x}-\mathbf{y})$$
$$-C_{0j}[A^{ia}(\mathbf{x},t), S^b(\mathbf{y},t)]", \tag{4.153d}$$
$$\left[\pi_S^a(\mathbf{x},t), \pi_i^b(\mathbf{y},t)\right]" = -iC_{0i}C_{j0}\partial_x^j K^{ab}(\mathbf{x},\mathbf{y}), \tag{4.153e}$$
$$\left[\pi_S^a(\mathbf{x},t), S^b(\mathbf{y},t)\right]" = -iC_{0i}D_x^{iac}c^{bx}(\mathbf{y},,) \tag{4.153f}$$
$$\left[\eta^a(\mathbf{x},t), S^b(\mathbf{y},t)\right]" = i\int [K^{ac}(\mathbf{x},\mathbf{z})g\,f^{cfe}C_{i0}\partial_z^i(\eta^f(z))K^{eb}(\mathbf{z},\mathbf{y})]\,d^3z$$
$$+iK^{ac}(\mathbf{x},\mathbf{y})g\,f^{ceb}\eta^e(y), \tag{4.153g}$$
$$\left[\eta^a(\mathbf{x},t), \pi_i^b(\mathbf{y},t)\right]"$$
$$= -iC_{0i}\int [K^{ac}(\mathbf{x},\mathbf{z})gf^{cfe}C_{j0}\partial_z^j(\eta^f(z))K^{eb}(\mathbf{z},\mathbf{y})]\,d^3z, \tag{4.153h}$$
$$\left[\pi_\eta^a(\mathbf{x},t), S^b(\mathbf{y},t)\right]"$$
$$= iC_{i0}\partial_x^i\int [K^{ac}(\mathbf{x},\mathbf{z})g\,f^{cfe}\pi_\eta^f(z)K^{eb}(\mathbf{z},\mathbf{y})]\,d^3z, \tag{4.153i}$$
$$\left[\pi_\eta^a(\mathbf{x},t), \pi_i^b(\mathbf{y},t)\right]" = -C_{0i}[\pi_\eta^a(\mathbf{x},t), S^b(\mathbf{y},t)]", \tag{4.153j}$$
$$\left[\bar\eta^a(\mathbf{x},t), S^b(\mathbf{y},t)\right]" = \int [K^{ac}(\mathbf{x},\mathbf{z})gf^{cfe}\pi_\eta^f(z)K^{eb}(\mathbf{z},\mathbf{y})]\,d^3z, \tag{4.153k}$$
$$\left[\bar\eta^a(\mathbf{x},t), \pi_i^b(\mathbf{y},t)\right]" = C_{0i}C_{j0}\partial_x^j K^{ac}(\mathbf{x},\mathbf{y})gf^{cbf}\eta^f(y)$$
$$-C_{0i}[\bar\eta^a(\mathbf{x},t), S^b(\mathbf{y},t)]", \tag{4.153l}$$
$$\left[\pi_i^a(\mathbf{x},t), \pi_j^b(\mathbf{y},t)\right]"$$
$$= iC_{0i}C_{0j}\int [K^{ac}(\mathbf{x},\mathbf{z})gf^{cef}\pi_\eta^e(z)K^{fd}(\mathbf{z},\mathbf{y})gf^{dbg}\eta^g(y)]\,d^3z$$
$$-iC_{0i}K^{ac}(\mathbf{x},\mathbf{y})gf^{cbe}\pi_j^e(y) - C_{0j}[\pi_i^a(\mathbf{x},t), S^b(\mathbf{y},t)]", \tag{4.153m}$$

while others remain the same as in Eq.(4.149).

The remaining constraints that must be handled are those due to ϕ_1 and

φ_2. The Dirac brackets of them are found to be

$$[\phi_1^a(\mathbf{x},t),\phi_1^b(\mathbf{y},t)]" = 0, \tag{4.154a}$$
$$[\phi_1^a(\mathbf{x},t),\varphi_2^b(\mathbf{y},t)]" = -i\nabla_y^{ba}\delta(\mathbf{x}-\mathbf{y}), \tag{4.154b}$$
$$[\varphi_2^a(\mathbf{x},t),\phi_1^b(\mathbf{y},t)]" = i\nabla_x^{ab}\delta(\mathbf{x}-\mathbf{y}), \tag{4.154c}$$
$$[\varphi_2^a(\mathbf{x},t),\varphi_2^b(\mathbf{y},t)]" = W^{ab}(\mathbf{x},\mathbf{y}). \tag{4.154d}$$

The inverse of the matrix $C"^{ab}_{ij}(\mathbf{x},\mathbf{y})$ can be chosen as

$$[C"^{-1}]^{ab}_{ij}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} -Z^{ab}(\mathbf{x},\mathbf{y}) & -iK^{ab}(\mathbf{x},\mathbf{y}) \\ -iK^{ab}(\mathbf{x},\mathbf{y}) & 0 \end{pmatrix}, \tag{4.155}$$

provided that

$$\int [W^{ac}(\mathbf{x},\mathbf{z})K^{cb}(\mathbf{z},\mathbf{y})]\, d^3z = \nabla_x^{ac} Z^{cb}(\mathbf{x},\mathbf{y}), \tag{4.156a}$$
$$\int [K^{ac}(\mathbf{x},\mathbf{z})W^{cb}(\mathbf{z},\mathbf{y})]\, d^3z = -\nabla_y^{bc} Z^{ac}(\mathbf{x},\mathbf{y}). \tag{4.156b}$$

The final Dirac brackets are then

$$\left[A_0^a(\mathbf{x},t),\pi_0^b(\mathbf{y},t)\right]^* = 0, \tag{4.157a}$$
$$\left[A_0^a(\mathbf{x},t),A_0^b(\mathbf{y},t)\right]^* = Z^{ab}(\mathbf{x},\mathbf{y}), \tag{4.157b}$$
$$\begin{aligned}\left[A_0^a(\mathbf{x},t),\Phi^b(\mathbf{y},t)\right]^* = &-\int K^{ac}(\mathbf{x},\mathbf{z})\{[S^c(\mathbf{z},t),\Phi^b(\mathbf{y},t)]" \\ &-C_{0j}[\pi^{jc}(\mathbf{z},t),\Phi^b(\mathbf{y},t)]" + C^k{}_j\partial_k^z[A^{jc}(\mathbf{z},t),\Phi^b(\mathbf{y},t)]" \\ &+g\, f^{cde}A_0^e(z)C_{0j}[A^{jd}(z),\Phi^b(\mathbf{y},t)]"\}\, d^3z,\end{aligned} \tag{4.157c}$$
$$\begin{aligned}\left[A_0^a(\mathbf{x},t),\Psi^b(\mathbf{y},t)\right]^* = &-\int K^{ac}(\mathbf{x},\mathbf{z})\{[S^c(\mathbf{z},t),\Psi^b(\mathbf{y},t)]" \\ &-C_{0j}[\pi^{jc}(\mathbf{z},t),\Psi^b(\mathbf{y},t)]"\}\, d^3z,\end{aligned} \tag{4.157d}$$

where Φ represents S and π_i and Ψ represents A_i, π_S, η, π_η, $\bar{\eta}$ and $\pi_{\bar{\eta}}$. The remaining Dirac brackets are as in Eq.(4.153).

The effective Hamiltonian of Eq.(4.143) is then reduced to its final form by taking all the constraints strongly equal to zero. One finds

$$H = \int [\mathcal{H}_0 + C^i{}_j \partial_i S \cdot A^j + iC^i{}_j \partial_i \bar{\eta} \cdot D^j \eta - \tfrac{1}{2} a S^2] \, d^3x, \tag{4.158}$$

where

$$\mathcal{H}_0 = \tfrac{1}{2}\pi^2 + \tfrac{1}{2}\mathbf{B}^2. \tag{4.159}$$

In the same way, the BRS charge and the FP ghost charge of Section 4.2 can be written as

$$Q_B = \int [\tfrac{1}{2} g \, (\eta \times \eta) \cdot \pi_\eta + S \cdot \nabla \eta] \, d^3x, \tag{4.160}$$

$$Q_\eta = i \int \bar{\eta} \cdot \nabla \eta \; d^3x. \tag{4.161}$$

With the help of the final Dirac brackets, after some algebra it can be shown explicitly that the BRS charge generates the BRS ransformations and the FP ghost charge generates the scale transformations found in Section 4.2. Thus, relations derived in that Section and the discussion of the physical state condition and Lorentz invariance of Section 4.3 also applies in the present case.

4.3.5 Class I gauges

We now consider all of C_{00}, C_{0i} and C_{i0} equal to zero. In this case, there exist $4n$ first class primary constraints

$$\phi_1^a \equiv \pi_0^a \approx 0, \tag{4.162a}$$

$$\phi_2^a \equiv \pi_S^a \approx 0, \tag{4.162b}$$

$$\phi_3^a \equiv \pi_\eta^a \approx 0, \tag{4.162c}$$

$$\phi_4^a \equiv \pi_{\bar\eta}^a \approx 0. \tag{4.162d}$$

Hence, the effective Hamiltonian can be written as

$$\begin{aligned}
H_e &= \int [\lambda_i \cdot \phi_i + \dot{A}_\mu \cdot \pi^\mu + \dot{S} \cdot \pi_S + \dot\eta \cdot \pi_\eta + \bar{e}\dot{t}a \cdot \pi_{\bar\eta} - \mathcal{L}]\, d^3x \\
&= \int [\lambda_i \cdot \phi_i + \dot{A}_0 \cdot \phi_1 + \dot{S} \cdot \phi_2 + \dot\eta \cdot \phi_3 + \dot{\bar\eta} \cdot \phi_4 + \mathcal{H}_0 - A_0 \cdot D_i \pi^i \\
&+ C^i{}_j \partial_i S \cdot A^j + i C^i{}_j \partial_i \bar\eta \cdot D^j \eta - \tfrac{1}{2} a S^2]\, d^3x \\
&= \int [\lambda_i \cdot \phi_i + \mathcal{H}_0 - A_0 \cdot D_i \pi^i \\
&+ C^i{}_j \partial_i S \cdot A^j + i C^i{}_j \partial_i \bar\eta \cdot D^j \eta - \tfrac{1}{2} a S^2]\, d^3x.
\end{aligned} \tag{4.163}$$

secondary constraints,

$$\varphi_1 \equiv \dot\phi_1 = D_i \pi^i \approx 0, \tag{4.164a}$$

$$\varphi_2 \equiv \dot\phi_2 = aS + C^i{}_j \partial_i A^j \approx 0, \tag{4.164b}$$

$$\varphi_3 \equiv \dot\phi_3 = -i C^i{}_j D^j \partial_i \bar\eta \approx 0, \tag{4.164c}$$

$$\varphi_4 \equiv \dot\phi_4 = +i C^i{}_j \partial_i D^j \eta \approx 0, \tag{4.164d}$$

are obtained by conserving the primary constraints, assuming that the inverses of the operators $C^i{}_j D^j \partial_i$ and $C^i{}_j \partial_i D^j$ exist. Therefore, φ_3 and φ_4 can be written as $\bar\eta \approx 0$ and $\eta \approx 0$ respectively. Now, among the secondary constraints, only φ_1 is first class with respect to the primary constraints. Thus, by conserving φ_1, a further constraint $\psi_1 \equiv \dot\varphi_1 \approx 0$ is obtained. By adding a suitable combination of φ_2, φ_3 and φ_4, and assuming the invertibility of the operator $C^i{}_j D^j \partial_i$, ψ_1 can be written as $C^i{}_j \partial_i A^j$. Conserving this constraint will then lead to the further constraint $\varrho_1 \equiv \dot{\psi}_1 = -C^i{}_j\partial_i\pi^j + C^i{}_j\partial_i D^j A_0$. In summary, then, the constraint equations are defined to be

$$\phi_1 \equiv \pi_0, \tag{4.165a}$$
$$\varphi_1 \equiv D_i\pi^i, \tag{4.165b}$$
$$\psi_1 \equiv C^i{}_j\partial_i A^j, \tag{4.165c}$$
$$\varrho_1 \equiv -C^i{}_j\partial_i\pi^j + C^i{}_j\partial_i D^j A_0, \tag{4.165d}$$
$$\phi_2 \equiv \pi_S, \tag{4.165e}$$
$$\varphi_2 \equiv S, \tag{4.165f}$$
$$\phi_3 \equiv \pi_\eta, \tag{4.165g}$$
$$\varphi_3 \equiv \bar{\eta}, \tag{4.165h}$$
$$\phi_4 \equiv \pi_{\bar{\eta}}, \tag{4.165i}$$
$$\varphi_4 \equiv \eta. \tag{4.165j}$$

It is clear from the above constraint equations that the unphysical fields S, π_S, η, π_η, $\bar{\eta}$ and $\pi_{\bar{\eta}}$ are all eliminated from the theory and only those constraints with a subscript "1" must be dealt with.

The commutators of φ_1 and ψ_1 are

$$[\varphi_1^a(\mathbf{x},t), \varphi_1^b(\mathbf{y},t)] = 0, \tag{4.166a}$$
$$[\varphi_1^a(\mathbf{x},t), \psi_1^b(\mathbf{y},t)] = i\triangle_x^{ab}\delta(\mathbf{x}-\mathbf{y}), \tag{4.166b}$$
$$[\psi_1^a(\mathbf{x},t), \varphi_1^b(\mathbf{y},t)] = -i\triangle_x^{ab}\delta(\mathbf{x}-\mathbf{y}), \tag{4.166c}$$
$$[\psi_1^a(\mathbf{x},t), \psi_1^b(\mathbf{y},t)] = 0, \tag{4.166d}$$

where $\triangle^{ab} \equiv C^i{}_j\partial_i D^{jab} \approx C^i{}_j D^{jab}\partial_i$. The inverse of the matrix $C_{ij}^{ab}(\mathbf{x},\mathbf{y})$ can then be chosen as

$$[C^{-1}]^{ab}_{ij}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} 0 & iG^{ab}(\mathbf{x},\mathbf{y}) \\ -iG^{ab}(\mathbf{x},\mathbf{y}) & 0 \end{pmatrix}, \tag{4.167}$$

provided that

$$\triangle^{ac}_x G^{cb}(\mathbf{x},\mathbf{y}) = \delta^{ab}\delta(\mathbf{x}-\mathbf{y}) = \triangle^{bc}_y G^{ac}(\mathbf{x},\mathbf{y}). \tag{4.168}$$

Note that $G^{ab}(\mathbf{x},\mathbf{y}) = G^{ba}(\mathbf{y},\mathbf{x})$. The corresponding preliminary Dirac brackets are found to be

$$\begin{aligned}[\pi^a_i(\mathbf{x},t), \pi^b_j(\mathbf{y},t)]' &= i\,g\,f^{aec}\pi^e_i(x)C^k{}_j\partial^y_k G^{cb}(\mathbf{x},\mathbf{y}) \\ &\quad -i\,gC^k{}_i\partial^x_k G^{ac}(\mathbf{x},\mathbf{y})f^{cbe}\pi^e_j(y), \qquad (4.169a)\\ \left[A^{ia}(\mathbf{x},t), \pi^b_j(\mathbf{y},t)\right]' &= i\delta^i_j\delta^{ab}\delta(\mathbf{x}-\mathbf{y}) \\ &\quad +iC^k{}_j\partial^y_k D^{iac}_x G^{cb}(\mathbf{x},\mathbf{y}), \qquad (4.169b)\end{aligned}$$

while others remain the same as in Eq.(4.104).

With the above preliminary Dirac brackets, the commutators of ϕ_1 and ϱ_1 are found to be

$$[\phi^a_1(\mathbf{x},t), \phi^b_1(\mathbf{y},t)]' = 0, \tag{4.170a}$$

$$[\phi^a_1(\mathbf{x},t), \varrho^b_1(\mathbf{y},t)]' = -i\triangle^{ba}_y\delta(\mathbf{x}-\mathbf{y}), \tag{4.170b}$$

$$[\varrho^a_1(\mathbf{x},t), \phi^b_1(\mathbf{y},t)]' = i\triangle^{ab}_x\delta(\mathbf{x}-\mathbf{y}), \tag{4.170c}$$

$$[\varrho^a_1(\mathbf{x},t), \varrho^b_1(\mathbf{y},t)]' = H^{ab}(\mathbf{x},\mathbf{y}). \tag{4.170d}$$

The inverse of the matrix $C'^{ab}_{ij}(\mathbf{x},\mathbf{y})$ can be chosen as

$$[C'^{-1}]^{ab}_{ij} = \begin{pmatrix} -I^{ab}(\mathbf{x},\mathbf{y}) & -iG^{ab}(\mathbf{x},\mathbf{y}) \\ iG^{ab}(\mathbf{x},\mathbf{y}) & 0 \end{pmatrix}, \tag{4.171}$$

provided that

$$\int [H^{ac}(\mathbf{x},\mathbf{z})G^{cb}(\mathbf{z},\mathbf{y})]\ d^3z = \triangle_x^{ac} I^{cb}(\mathbf{x},\mathbf{y}), \tag{4.172a}$$

$$\int [G^{ac}(\mathbf{x},\mathbf{z})H^{cb}(\mathbf{z},\mathbf{y})]\ d^3z = \triangle_y^{bc} I^{ac}(\mathbf{x},\mathbf{y}). \tag{4.172b}$$

The final Dirac brackets obtained are thus

$$\left[A_0^a(\mathbf{x},t), \pi_0^b(\mathbf{y},t)\right]^* = 0, \tag{4.173a}$$

$$\left[A_0^a(\mathbf{x},t), A_0^b(\mathbf{y},t)\right]^* = I^{ab}(\mathbf{x},\mathbf{y}), \tag{4.173b}$$

$$\begin{aligned}\left[A_0^a(\mathbf{x},t), A^{ib}(\mathbf{y},t)\right]^* &= iC^j{}_i\partial_j^y G^{ab}(\mathbf{x},\mathbf{y}) \\ &-i\,D_y^{ibe}\int [G^{ac}(\mathbf{x},\mathbf{z})C_{jl}C^{kl}\partial_z^j\partial_k^z G^{ce}(\mathbf{z},\mathbf{y})]\ d^3z,\end{aligned} \tag{4.173c}$$

$$\begin{aligned}&\left[A_0^a(\mathbf{x},t), \pi_i^b(\mathbf{y},t)\right]^* \\ &= \int C^k{}_j\partial_k^z G^{ac}(\mathbf{x},\mathbf{z})\{g\,f^{cde}A_0^e(z)\left[A^{jd}(\mathbf{z},t), \pi_i^b(\mathbf{y},t)\right]' \\ &-[\pi^{jc}(\mathbf{z},t), \pi_i^b(\mathbf{y},t)]'\}\ d^3z.\end{aligned} \tag{4.173d}$$

The resulting Hamiltonian is then obtained from the effective amiltonian of Eq.(4.163) by taking all the constraints strongly equal to zero, and is found to be

$$H = \int \mathcal{H}_0\ d^3x, \tag{4.174}$$

where

$$\mathcal{H}_0 = \tfrac{1}{2}\pi\cdot\pi + \tfrac{1}{2}\mathbf{B}\cdot\mathbf{B}, \tag{4.175}$$

as expected in a Class I gauge where no unphysical states are present.

4.4 Quantization of The Free Field Theory

4.4.1 The canonical variables

Let us start with the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - C_{\mu\nu}\partial^{\mu}SA^{\nu} + \frac{1}{2}aS^2. \tag{4.176}$$

with $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$. Here, S is the Nakanishi-Lautrup Lagrange multiplier field [51], a and $C_{\mu\nu}$ are various gauge parameters. Note that $C_{\mu\nu}$ are not necessarily tensor quantities but they can formally be treated as such. Some restrictions on these parameters will occur in the course of the development of the theory.

The field equations resulting from Eq.(4.176) are

$$\partial^{\mu}F_{\mu\nu} - C_{\mu\nu}\partial^{\mu}S = 0 \tag{4.177a}$$

$$C^{\mu\nu}\partial_{\mu}A_{\nu} + aS = 0 \tag{4.177b}$$

Applying ∂^{ν} to the first of Eqs.(4.177) leads to

$$\Box_C S \equiv C_{\mu\nu}\partial^{\mu}\partial^{\nu}S = 0 \tag{4.178}$$

As well, applying $C^{\lambda\nu}\partial_{\lambda}$ leads to

$$\Box_C \partial \cdot A = KS \tag{4.179}$$

where

$$K = -a\Box - C^{\lambda\nu}C_{\mu\nu}\partial_{\lambda}\partial^{\mu} \tag{4.180}$$

It is useful to write the first of Eqs.(4.177) as

$$\Box A_\mu = \partial_\mu \partial \cdot A + C_{\mu\nu}\partial^\nu S \equiv V_\mu \tag{4.181}$$

The canonical momenta associated with the Lagrangian of Eq.(4.176) are, in a self-explanatory notation,

$$\pi^k = F^{k0} \tag{4.182a}$$

$$\pi^0 = 0 \tag{4.182b}$$

$$\pi_S = -C_{0\nu}A^\nu . \tag{4.182c}$$

In order to carry out canonical quantization, one must choose a particular coordinate frame in which *equal-time* will be defined. This does not matter in covariant theories because all the coordinate frames are equivalent but this choice can be crucial with $C_{\mu\nu} \neq g_{\mu\nu}$. Here, we assume that A_μ transforms as a four-vector, S as scalar. As long as $C_{00} \neq 0$, the two primary constraints of Eqs.(3.7b,3.7c) are second class so no secondary constraints are required to close the algebra. If one assumes that $C_{\mu\nu}$ transform as tensor, then one can always find a frame in which $C_{00} = 0$. In such frames, the constraints of Eqs.(3.7b,3.7c) become first class. The resulting secondary constraints imply that in a frame in which $C_{00} = 0$, there are fewer independent degrees of freedom and the gauge condition can be classified as either Class I or II, depending on the details of the final constraint algebra. For this reason frames in which $C_{00} = 0$ could be termed *singular,* and should be avoided when considering Class III gauges. The above discussion of course assumes that $C_{\mu\nu}$ transform as tensor. Since one is considering non-covariant gauges, this assumption is in a sense not necessary.

One could also adopt the viewpoint that the $C_{\mu\nu}$ are gauge parameters. In this context, a unique choice of gauge requires both a choice of frame, and the specification of all gauge parameters in that frame, including $C_{\mu\nu}$. Hence, one could in principle avoid all discussion of singular frames, and focus instead on the notion of singular gauges. Whichever point of view one prefers, in the following we restrict the discussion to non-singular frames/gauges.

Quantization now proceeds by specifying the non-vanishing canonical commutators to be

$$[A_k(x), \pi^l(y)]_{x_0=y_0} = i\delta_k{}^l\delta(\mathbf{x}-\mathbf{y}) \tag{4.183a}$$

$$[S(x), \pi^S(y)]_{x_0=y_0} = i\delta(\mathbf{x}-\mathbf{y}) \tag{4.183b}$$

in which A_0 is not considered as a canonical variable. All the commutation relations involving A_0 are obtained from the constraint $\pi_S + C_{0\nu}A^\nu \approx 0$.

This method is equivalent to the determination of Dirac brackets resulting from the pair of second-class constraints $\pi^0 \approx 0$ and $\pi_S + C_{0\nu}A^\nu \approx 0$.

4.4.2 The Cauchy problem

The field equations (4.177) define the evolution of the various fields. It is obvious that the determination of the commutation relations for any time is the solution of a Cauchy problem with evolution operators $\Box_C$ and $\Box$ and equal-time commutators as initial data. With second-order equations, the Cauchy problem is well-defined only for a hyperbolic operator, and so we shall restrict $C_{\mu\nu}$ such that $\Box_C$ is hyperbolic; the d'Alembertian operator satisfies trivially this property. The discussion of elliptic $\Box_C$ is presented in Appendix 4. Let us now consider the singular functions associated with the $\Box_C$ operator. We will require that they have properties similar to those associated with $\Box$ but, because $\Box_C$ is non-covariant, we require those properties only in particular frames. In particular, we will require that the solutions with respect to energy of $k_C^2 = C_{\mu\nu}k^\mu k^\nu = 0$ are of opposite signs. This imposes

$$C_{0k} + C_{k0} = 0 \tag{4.184}$$

A frame in which Eq.(4.184) holds will be called a *preferred frame*. In such a frame, when one goes from Minkowski to Euclidean space $t \to it'$ by a Wick rotation, the hyperbolic $\Box_C$ is transformed into an elliptic operator. Therefore, a preferred frame is a frame in which a Wick rotation can be consistently performed. Note that it is impossible to find a preferred frame with a parabolic operator. We now work in such a preferred frame where it is useful to introduce some generalized functions whose properties are collected in Appendix A. They are

$$\begin{aligned}
\Delta_C(x) &= -\frac{iC_{00}}{(2\pi)^3}\int d^4k\, \theta(k_0)\delta(k_C^2 - m^2)(e^{-ik\cdot x} - e^{ik\cdot x}) && (4.185a)\\
E_C(x) &= -\frac{iC_{00}}{(2\pi)^3}\int d^4k\, \epsilon(k_0)\delta'(k_C^2 - m^2)e^{-ik\cdot x} && (4.185b)\\
F'_\Delta(x) &= -\frac{C_{00}}{(2\pi)^3}\int d^4k\, \epsilon(k_0)\left[\frac{1}{k^2+i\epsilon}\frac{1}{k_C^2 - m^2 + i\epsilon}\right.\\
&- \left.\frac{1}{k^2 - i\epsilon}\frac{1}{k_C^2 - m^2 - i\epsilon}\right] && (4.185c)\\
F'_E(x) &= -\frac{d}{dm^2}F'_\Delta(x). && (4.185d)
\end{aligned}$$

Here, m^2 is a parameter which plays a role similar to that in the usual generalized functions, and is taken to zero at the end of the calculations. These functions satisfy

$$\begin{aligned}
(\Box_C + m^2)\Delta_C &= 0, && (4.186a)\\
(\Box_C + m^2)E_C &= \Delta_C, && (4.186b)\\
\Box F'_\Delta &= \Delta_C, && (4.186c)\\
(\Box_C + m^2)F'_\Delta &= C_{00}\Delta, && (4.186d)\\
\Box F'_E &= E_C, && (4.186e)\\
(\Box_C + m^2)F'_E &= F'_\Delta. && (4.186f)
\end{aligned}$$

We also need the corresponding causal Green functions satisfying

$$\begin{aligned}
&(\Box_C + m^2)\Delta_{F,C} = \delta^{(4)}(x) && (4.187a)\\
&\Delta_{F,C} = -\frac{1}{(2\pi)^4}\int d^4k \frac{e^{-ik\cdot x}}{k_C^2 - m^2 + i\epsilon} && (4.187b)\\
&(\Box_C + m^2)E_{F,C} = \Delta_{F,C} && (4.187c)\\
&\Box F'_{\Delta,F} = \Delta_{F,C} && (4.187d)\\
&(\Box_C + m^2)F'_{\Delta,F} = C_{00}\Delta_F && (4.187e)\\
&\Box F'_{E,F} = E_{F,C} && (4.187f)\\
&(\Box_C + m^2)F'_{E,F} = F'_{\Delta,F}. && (4.187g)
\end{aligned}$$

With the help of these functions (denoted D_C when $m^2 = 0$) as well as of those noted without the C index corresponding to the d'Alembertian operator, it is easy to write the solution of the Cauchy problem associated with the field equations (4.177). One finds

$$
\begin{aligned}
S(x) &= \int d^3y[D_C(x-y) \overleftrightarrow{\partial_0^y} S(y)] && (4.188)\\
(\partial \cdot A)(x) &= \int d^3y[D_C(x-y) \overleftrightarrow{\partial_0^y} (\partial \cdot A)(y)] + \int d^4z D_{F,C}(x-z)KS(z) \\
&- \int d^3y d^4z[D_C(x-y) \overleftrightarrow{\partial_0^y} D_{F,C}(y-z)] && (4.189)\\
A_\nu(x) &= \int d^3y[D_C(x-y) \overleftrightarrow{\partial_0^y} A_\nu(y)] + \int d^4z D_{F,C}(x-z)V_\nu(z) \\
&- \int d^3y d^4z[D_C(x-y) \overleftrightarrow{\partial_0^y} D_{F,C}(y-z)]V_\nu(z). && (4.190)
\end{aligned}
$$

where y_0 is arbitrary and $f(y) \overleftrightarrow{\partial_0^y} g(y) \equiv f(y)[\partial_0^y g(y)] - [\partial_0^y f(y)]g(y)$.

With the help of equal-time commutators, these equations allow one to get, after a straightforward but tedious calculation, the commutation relations for any time. This calculation follows exactly the same lines as the particular case of the Leibbrandt-Mandelstam regularized axial gauges[42] and for this reason will not be detailed here. The results are

$$
\begin{aligned}
[S(x), S(x')] &= 0 && (4.191)\\
[A_\mu(x), S(x')] &= -\frac{i\partial_\mu}{C_{00}} D_C(x-x') && (4.192)\\
[A_\mu(x), \partial \cdot A(x')] &= -\frac{iK}{C_{00}}\partial_\mu E_C(x-x') - \frac{i}{C_{00}} C_{\mu\nu}\partial^\nu D_C(x-x') && (4.193)\\
[A_\mu(x), A_\nu(x')] &= ig_{\mu\nu}D(x-x') + \frac{iK}{C_{00}}(C_{\lambda\mu}\partial_\nu + C_{\lambda\nu}\partial_\mu)\partial^\lambda F'_D(x-x') \\
&+ \frac{iK}{C_{00}}\partial_\mu\partial_\nu F'_E(x-x') && (4.194)
\end{aligned}
$$

4.4.3 The propagator

The propagator is defined as usual by

$$
iD_{\mu\nu}(x) = < 0|T(A_\mu(x)A_\nu(0))|0 > . \qquad (4.195)
$$

It is a straigthforward task to derive it from the commutation relations for any time. One may avoid the details by working by analogy with the relativistic case. The result, in momentum space, is:

$$\begin{aligned} D_{\mu\nu}(k) &= \frac{-1}{k^2+i\epsilon}\left[g_{\mu\nu} - \frac{C_{\lambda\mu}k_\nu + C_{\lambda\nu}k_\mu}{k_C^2+i\epsilon}k^\lambda \right. \\ &+ \left. k_\mu k_\nu \frac{ak^2 + C^{\lambda\rho}C_{\tau\rho}k_\lambda k^\tau}{(k_C^2+i\epsilon)^2}\right]. \end{aligned} \tag{4.196}$$

Another way to obtain the propagator consists of using the field equations in order to eliminate the auxiliary field S from the Lagrangian of Eq.(4.176) which can then be written up to four-divergences :

$$L = -\frac{1}{2}A_\mu (D^{-1})^{\mu\nu} A_\nu. \tag{4.197}$$

In this procedure, one assumes that the gauge parameters are such that $D^{-1} \neq 0$. Of course, when inverting k^2 and k_C^2, the causal solution is taken.

4.4.4 Creation and annihilation operators

In Fourier space, the fields satisfying Eqs.(4.177) can be written as

$$S(x) = \frac{1}{(2\pi)^3}\int d^4k s(k)\sqrt{(-k^2)}\delta(k_C^2)e^{-ik\cdot x} \tag{4.198}$$

$$\begin{aligned} (\partial \cdot A)(x) &= \frac{1}{(2\pi)^3}\int d^4k g(k)\sqrt{(-k^2)}\delta(k_C^2)e^{-ik\cdot x} \\ &- \frac{1}{(2\pi)^3}\int d^4k \tilde{K} s(k)\sqrt{(-k^2)}\delta'(k_C^2)e^{-ik\cdot x} \end{aligned} \tag{4.199}$$

where

$$\tilde{K} = -ak^2 - C^{\lambda\nu}C_{\mu\nu}k_\lambda k^\mu. \tag{4.200}$$

In a *preferred frame*, one can write

$$S(x) = \frac{1}{C_{00}(2\pi)^3}\int \frac{d^3k}{2k_0}\sqrt{(-k^2)}[s(\mathbf{k})e^{-ik\cdot x} + e^{\dagger}(\mathbf{k})e^{ik\cdot x}] \tag{4.201}$$

$$(\partial\cdot A)(x) = \frac{1}{C_{00}(2\pi)^3}\int \frac{d^3k}{2k_0}\sqrt{(-k^2)}[g(\mathbf{k})e^{-ik\cdot x} + g^{\dagger}(\mathbf{k})e^{ik\cdot x}]$$
$$+\frac{1}{C_{00}(2\pi)^3}\frac{d}{dm^2}\int \frac{d^3k}{2k_0}\tilde{K}\sqrt{(-k^2)}[s(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}} + s^{\dagger}(\mathbf{k})e^{ik\cdot x}] \tag{4.202}$$

where

$$k_0 = \sqrt{-\frac{C_{kl}k^k k^l + m^2}{C_{00}}} \tag{4.203}$$

and the limit $m^2 \to 0$ is taken at the end. Let us remark that the transition from Eqs.(4.198,4.199) to Eqs.(4.201,4.202) is possible only in preferred frames; in another frame, one loses the usual relation between particle and antiparticle.

One can now decompose A_ν as

$$A_\nu = A_\nu^T + A_\nu^L \tag{4.204}$$

where A_ν^T and A_ν^L correspond respectively to the homogeneous and inhomogeneous part of Eq.(4.181). One finds

$$A_\nu^T = \frac{1}{(2\pi)^3}\int \frac{d^3k}{2k_0}\left[\sum_{i=1}^{2} a^{(i)}(\mathbf{k})\epsilon_\nu^{(i)}(\mathbf{k})e^{-ik\cdot x} + h.c.\right] \tag{4.205}$$

where $k_0 = |\mathbf{k}|$, and

$$\begin{aligned} A_\nu^L &= \frac{1}{C_{00}(2\pi)^3}\int \frac{d^3k}{2k_0}\left[\left(k_\nu g(\mathbf{k}) - C_{\mu\nu}k^\mu s(\mathbf{k})\right)e^{-ik\cdot x} + h.c.\right] \\ &+ \frac{1}{C_{00}(2\pi)^3}\frac{d}{dm^2}\int \frac{d^3k}{2k_0}\tilde{K}k_\nu\left[s(\mathbf{k})e^{-ik\cdot x} + h.c.\right]_{m^2=0} \end{aligned} \tag{4.206}$$

where here $k_0 = \sqrt{-\frac{C_{kl}k^k k^l + m^2}{C_{00}}}$. The polarization vectors $\epsilon_\nu^{(i)}(\mathbf{k})$ satisfy be

$$\delta(k^2)k\cdot\epsilon^{(i)} = \delta(k^2)C_{\mu\nu}k^\mu\epsilon_{(i)}^\nu = 0. \tag{4.207}$$

and can be constructed from two fixed orthogonal vectors $e^{(i)}_\mu$ as

$$\epsilon^{(i)}_\mu(k) = P_{\mu\nu} e^\nu_{(i)} \tag{4.208}$$

where

$$P_{\mu\nu} = g_{\mu\nu} - \frac{C_{\lambda\mu}k_\nu + C_{\lambda\nu}k_\mu}{k_C^2 + i\epsilon} k^\lambda + k_\mu k_\nu \frac{C^{\lambda\rho}C_{\tau\rho}k_\lambda k^\tau}{(k_C^2 + i\epsilon)^2} \tag{4.209}$$

It is now a straigthforward task to get the commutation relations between creation and annihilation operators from the commutation relations for any time. The non-vanishing ones are

$$[a_{(i)}(\mathbf{k}), a^\dagger_{(j)}(\mathbf{k}')] = \delta_{ij} 2|\mathbf{k}|(2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \tag{4.210a}$$

$$[g(\mathbf{k}), s^\dagger(\mathbf{k}')] = 2C_{00}\sqrt{-C_{kl}k^k k^l / C_{00}}(2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \tag{4.210b}$$

Let us remark that these results hold only for $k^2 \neq k_C^2$; when $k^2 = k_C^2$, the equation $\partial \cdot A = aS$ implies a more subtle difference between s and g [52].

4.4.5 The Fock space structure

Let the vacuum be defined as usual by

$$s|0> = g|0> = a_i|0> = 0. \tag{4.211}$$

A one-particle state is then generated by applying appropriate creation operators to the vacuum, and multiparticle states by repeated use of the appropriate creation operators. It is obvious that the states generated by $s^\dagger$ have zero norm because

$$< 0|ss^\dagger|0> = < 0|[s, s^\dagger]|0> = 0. \tag{4.212}$$

Amongst all states in the Hilbert space one selects the physical ones by the conditions

$$s|\Psi_{phys}>=0 \tag{4.213}$$

which eliminates the states containing a particle created by $g^{\dagger}$, and

$$|\Psi'_{phys}>\approx|\Psi_{phys}>+s^{\dagger}|\Psi_{phys}> \tag{4.214}$$

which establishes an equivalence between two states differing by the presence of a particle generated by $s^{\dagger}$.

5 Conclusion

In this thesis we have concentrated on the constraint stuctures of the general classes of gauges discussed above. We have presented a general framework for the quantization of a fairly general class of linear gauges, represented by the Class III Lagrangian of Eq.(4.100). In particular, canonical quantization was performed for the free theory in the case of a hyperbolic evolution operator, and a BRS Hamiltonian constraint analysis was done for the full interacting theory. The physical subspace was then subsequently defined, and it was shown that Lorentz invariance holds in such a space. We also considered cases where such a quantization procedure is not so straightforward. In particular, a constraint analysis was presented for the case of an elliptic evolution operator in the free field case for a Class III gauge, which is not a Cauchy initial data problem and subsequently cannot be canonically quantized in the usual way. As well, a constraint analysis was done in the cases of Class I and Class II gauges, which were seen to have a significantly different and more complicated constraint structure. Thus, although the analysis is possible, the technical difficulties present in the quantization of cases such as elliptic Class III gauges and in the singular Class I and Class II gauges suggest that their quantization can be more easily considered as either an analytic continuation or as a special limiting case of the gauge parameters present in the general hyperbolic Class III gauge of Eq.(4.100).

One important feature arising from the analysis of the general Class III gauge condition was the gauge independence of the BRS charge and of the FP ghost charge. This implies that, for example, derivation of the Slavnov-Taylor identities [40] and proof of the unitarity of the S-matrix [41] and [20] follows the same lines as that used for covariant gauges. Similar comments in this regard can be made for the formulation of gauge theories at finite temperature, which as mentioned in the Introduction is a natural situation for the consideration of non-covariant gauges. Following the work of Hata and Kugo [54] for covariant gauges, one can for the general class of non-covariant gauges considered here construct the partition function, which using the BRST analysis of Section 4 will have a physical interpretation as a sum over physical states annihilated by the BRST charge. The FP ghost charge found in Section 4 can then be used to demonstrate that the ghost fields will have periodic, rather than anti-periodic, boundary conditions [54]. Such an operator construction will lead to the same results as those found by path integral techniques [55], and demonstrates in particular the connection between linear response theory in a Hamiltonian approach and the corresponding equations found in the path integral formalism.

Appendices

A Lie Groups and Lie Algebras

Definition: A Lie group G [56] is a set that is both a group in the usual algebraic sense and a differentiable manifold with the properties that taking the product of two group elements, and taking the inverse of a group element, are smooth operations.

Examples:

- The *complex general linear group* in n-dimensions is defined as

$$GL(n, C) := \{A \in M(n, C) | \det A \neq 0\} \tag{A.215}$$

  and, as an open subset of the $2n^2$ (real) dimensional manifold $M(n, C)$, it acquires the structure of a $2n^2$-dimensional Lie group and so inherits a natural topological and differentiable structure. Notice that $GL(n, C)$ is a connected space since a matrix A satisfying $\det A > 0$ can be connected continuously to a matrix B for which $\det B < 0$ by a path going around the pole, i.e. the origin of the complex plane.

- *Unitary group* $U(n)$ is defined by

$$U(n) := \{A \in GL(n, C) | A^{\dagger} A = I\} \tag{A.216}$$

  and is a compact Lie subgroup of $GL(n, C)$ with real dimension n^2. The *special unitary group* $SU(n)$ is defined as

$$SU(n) := \{A \in U(n) | \det A = 1\} \tag{A.217}$$

and is a compact Lie group of real demension $(n^2 - 1)$.

One of the most important features of a Lie Group G is the fact that there is always an associated Lie algebra which accurately encodes many of the properties of the group. However, the proof of this would be beyond our scope and we will only give the definition of Lie algebra.

Definition: A Lie algebra $\mathcal{A}$ is a vector space over a field $\mathcal{K}$ with a linear composition $[\ ,\ ]$, such that if J_i, $J_j \in \mathcal{A}$, so is $[J_i, J_j]$ and furthermore for α, $\beta \in \mathcal{K}$ the following properties of $[\ ,\ ]$ are satisfied:

$$[J_i, J_j] = -[J_j, J_i], \tag{A.218a}$$

$$[\alpha J_i + \beta J_j, J_k] = \alpha[J_i, J_k] + \beta[J_j, J_k], \tag{A.218b}$$

$$[J_i, [J_j, J_k]] + [J_j, [J_k, J_i]] + [J_k, [J_i, J_j]] = 0 \tag{A.218c}$$

where the third equation is called the *Jacobi identity.*

For the purpose of this thesis, we content ourselves by choosing a basis $\{T^a\}$ of $\mathcal{A}$ such that they satisfy

$$[T^a, T^b] = i f^{ab}{}_c T^c \tag{A.219}$$

and are traceless. Following from the Jacobi identity, the structure constants $f^{ab}{}_c$ satisfy

$$f^{ab}{}_e f^{ec}{}_d + f^{ca}{}_e f^{eb}{}_d + f^{bc}{}_e f^{ea}{}_d. \tag{A.220}$$

However, due to the possibility of the normalization $tr(T^a T^b) = \delta^{ab}$, all the indices (upper or lower) of the structure constants are totally antisymmetric. Hence, the distintion between upper and lower indices becomes unnecessary. Thus, we will write all the indices as superscripts.

Definition: Adjoint mapping of G onto itself is defined as $Ad_g(g') := gg'g^{-1}$ for each $g \in G$.

For convenience, the adjoint repersentation will be taking as

$$i(T^a)_{bc} = f^{abc}. \tag{A.221}$$

B Grassmann Algebra

We will only give a brief introduction to the subject of Grassmann alegbra [57] as a supplement to the main text. Therefore, the discussion here in any case cannot be considered to be complete. Consider the algebra generated by one real Grassmann variable θ satisfing anticommutativity

$$\{\theta, \theta\} = 0 \qquad or \qquad \theta^2 = 0. \tag{B.222}$$

Therefore, a general element of this algebra has the following form

$$P(\theta) = p_0 + \tilde{p}_1\theta \tag{B.223}$$

where p_0 and $\tilde{p}_1$ can be Grassmann or ordinary numbers in general. One defines the differential operators $\frac{\partial}{\partial\theta}$ by means of

$$\{\frac{\partial}{\partial\theta}, \theta\} = 1 \tag{B.224}$$

such that if $P(\theta)$ is taken to be an ordinary number, i.e. p_0 and $\tilde{p}_1$ are ordinary and Grassmann numbers respectively, then

$$\frac{\partial}{\partial\theta}P(\theta) = -\tilde{p}_1 \tag{B.225}$$

and

$$\frac{\partial^2}{\partial \theta^2} P(\theta) = 0. \tag{B.226}$$

Furthermore, we formally define the integration by linearity satisfying the following requirement:

$$\int d\theta = 0, \qquad \int d\theta \theta = 1 \tag{B.227}$$

which acts exactly like differentiation, i.e.

$$\int d\theta P(\theta) = \frac{\partial}{\partial \theta} P(\theta) = -\tilde{p}_1. \tag{B.228}$$

As a result, the integral of the derivative vanishes

$$\int d\theta \frac{\partial}{\partial \theta} P(\theta) = \frac{\partial^2}{\partial \theta^2} P(\theta) = 0. \tag{B.229}$$

The first relation of Eq.(B.227) follows from the requirement that the property of a convergent integral over commuting numbers

$$\int_{-\infty}^{\infty} dx f(x) = \int_{-\infty}^{\infty} dx f(x+a) \tag{B.230}$$

valid for any finite a, holds for an integral over anticommution nombers as well. The second relation of Eq.(B.227) is the normalization convention.

Consider now the change of integration variable $\theta \to \theta' = a + b\theta$. One gets

$$\int d\theta P(\theta) = \int d\theta' \left(\frac{d\theta}{d\theta'}\right)^{-1} P\left(\theta(\theta')\right) \tag{B.231}$$

i.e. the standard Jacobian appears inverted.

All the rules can be easily generalized to the case of n real Grassmann variables $\theta_i, \quad i = 1, ..., n$ which obey

$$\{\theta_i, \theta_j\} = 0. \tag{B.232}$$

Introduce their respective derivative operators $\frac{\partial}{\partial\theta_i}$ by means of

$$\{\frac{\partial}{\partial\theta_i}, \theta_j\} = \delta_{ij} \tag{B.233}$$

and

$$\{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\} = 0. \tag{B.234}$$

Integration is defined in the same way as for one variable

$$\int d\theta_i = 0 \qquad \int d\theta_i\theta_i = 1 \tag{B.235}$$

where no summation is performed over i. We now generate the idea of real Grassmann variables to a complex one. For instant, if

$$\eta_1 = \theta_1 + i\theta_2 \tag{B.236}$$

and

$$\eta_1^* = \theta_1 - i\theta_2, \tag{B.237}$$

then η_1 and η_1^* satisfy the followings:

$$\{\eta_1, \eta_1\} = \{\eta_1^*, \eta_1^*\} = 1, \tag{B.238a}$$
$$\{\eta_1, \eta_1^*\} = 0 \tag{B.238b}$$

and

$$\int d\eta_1\eta_1 = \int d\eta_1^*\eta_1^* = 1, \tag{B.239a}$$
$$\int d\eta_1\eta_1^* = \int d\eta_1^*\eta_1 = 0. \tag{B.239b}$$

Thus, η_1 and $\eta_1{}^*$ can be considered as independent generators of Grassmann algebra. For more than one complex Grassmann variables, we can easily deduce the following properties:

$$\{\eta_i, \eta_j\} = \delta_{ij}, \tag{B.240a}$$
$$\{\eta_i, \eta_j{}^*\} = 0 \tag{B.240b}$$

and

$$\int d\eta_i \eta_j = \int d\eta_i^* \eta_j^* = \delta_{ij}, \tag{B.241a}$$
$$\int d\eta_i \eta_j^* = \int d\eta_i^* \eta_j = 0. \tag{B.241b}$$

It is well known that Gaussian integral for complex Grassmann variables are very useful in quantum field theories, particular in path integral formalism, so it is worthwhile to give a brief introduction to this subject. The following identity can be obtained by using the above rules for complex Grassmann variables:

$$\int d\eta_1 d\eta_1^* ... d\eta_n d\eta_n^* e^{\eta^\dagger B \eta} = \det B \tag{B.242}$$

where

$$\eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \tag{B.243}$$

and B is an aribitary $n \times n$ square matrix. We will not prove the above statement since it should be straightforward but with some algebra. However, we would rather like to generalize the discrete set of indices to a space-time continuium x

and some internal indices a. Eq.(B.242) will then be symbolically replaced by

$$\int \mathcal{D}\eta \mathcal{D}\bar{\eta} \exp\left[\int d^4y d^4z \sum_{b,c} \bar{\eta}^b(y) B^{b,c}(y,z) \eta^c(z)\right] = \det B \tag{B.244}$$

where the measure be

$$\mathcal{D}\eta \mathcal{D}\bar{\eta} = \prod_x \sum_a d\eta^a(x) d\bar{\eta}^a(x). \tag{B.245}$$

C Generalized functions

In a way completely parallel to the one followed for the usual $\Box + m^2$ operator, we can define generalized functions associated with the operator $\Box_C + m^2$. In the following we introduce a parameter m^2 for convenience, which is taken to zero at the end of a calculation.

The positive and negative frequency solutions of the equation

$$(\Box_C + m^2) f = 0 \tag{C.246}$$

are respectively

$$\Delta_C^{(\pm)}(x, m^2) = \mp \frac{iC_{00}}{(2\pi)^3} \int d^4k\, \theta(\pm k_0) \delta(k_C^2 - m^2) e^{-ik\cdot x}. \tag{C.247}$$

where the Heaviside function $\theta(k_0)$ is equal to 1 for $k_0 \geq 0$ and 0 for $k_0 < 0$. Their sum

$$\Delta_C(x, m^2) = \Delta_C^{(+)}(x, m^2) + \Delta_C^{(-)}(x, m^2) \tag{C.248}$$

is odd and is given by

$$\Delta_C(x, m^2) = -\frac{iC_{00}}{(2\pi)^3} \int d^4k\, \epsilon(k_0) \delta(k_C^2 - m^2) e^{-ik\cdot x}. \tag{C.249}$$

Restricting now to a preferred frame in which

$$k_C^2 = k_0^2 C_{00} + C_{kl} k_k k_l, \tag{C.250}$$

we can perform the integration over k_0 and obtain

$$\Delta_C(x, m^2) = -\frac{1}{(2\pi)^3} \int \frac{d^3k}{2k_0} e^{i\mathbf{k}\cdot\mathbf{x}} \sin(k_0 x_0) \tag{C.251}$$

where $k_0 = \sqrt{(-C_{kl}k^k k^l + m^2)/C_{00}}$. From this equation, we immediately have

$$\Delta_C(x, m^2)_{x_0=0} \;=\; 0 \tag{C.252a}$$

$$[\partial_0 \Delta_C(x, m^2)]_{x_0=0} \;=\; -\delta(\mathbf{x}) \tag{C.252b}$$

The causal Green function satisfying

$$(\Box_C + m^2)\Delta_{F,C}(x, m^2) = C_{00}\delta^{(4)}(x) \tag{C.253}$$

is given by

$$\begin{aligned} i\Delta_{F,C}(x, m^2) &= \theta(x_0)\Delta_C^{(+)}(x, m^2) - \theta(-x_0)\Delta_C^{(-)}(x, m^2) \\ &= \frac{-iC_{00}}{(2\pi)^4} \int d^4k \frac{e^{-ik\cdot x}}{k_C^2 - m^2 + i\epsilon} \end{aligned} \tag{C.254}$$

Advanced and retarded Green functions can be defined in the same way as those associated with the $\Box$ operator.

The generalized functions associated with repeated applications of $\Box$ and/or $\Box_C$ satisfy

$$(\Box_C + m^2)^2 E_C = 0 \tag{C.255a}$$

$$\Box(\Box_C + m^2)F'_{\Delta,C} = 0 \tag{C.255b}$$

$$\Box(\Box_C + m^2)^2 F'_{E,C} = 0 \qquad \text{(C.255c)}$$

The corresponding equations for the causal Green functions are

$$(\Box_C + m^2)^2 E_{F,C} = C_{00}\delta(x) \qquad \text{(C.256a)}$$

$$\Box(\Box_C + m^2) F'_{\Delta,C,F} = C_{00}\delta(x) \qquad \text{(C.256b)}$$

$$\Box(\Box_C + m^2)^2 F'_{E,C,F} = C_{00}\delta(x) \qquad \text{(C.256c)}$$

The simplest way to define E_C and $F'_{E,C}$ is through derivatives with respect to the parameter m^2:

$$E_C(x, m^2) = -\frac{\partial}{\partial m^2}\Delta_C(x, m^2) \qquad \text{(C.257a)}$$

$$E_{C,F}(x, m^2) = -\frac{\partial}{\partial m^2}\Delta_F(x, m^2) \qquad \text{(C.257b)}$$

The $x_0 = 0$ properties of $E_C(x, m^2)$ and $E_{C,F}(x, m^2)$ in a preferred frame are easily deduced from their definitions:

$$E_C(x, m^2)|_{x_0=0} = \partial_0 E_C(x, m^2)|_{x_0=0} = 0 \qquad \text{(C.258)}$$

The generalized function F'_Δ is defined by

$$F'_\Delta(x) = -\frac{C_{00}}{(2\pi)^3}\int d^4k\, e^{-ik\cdot x}\epsilon(k_0)\left[\frac{1}{k^2 + i\epsilon}\frac{1}{k_C^2 - m^2 + i\epsilon} - \frac{1}{k^2 - i\epsilon}\frac{1}{k_C^2 - m^2 - i\epsilon}\right] \qquad \text{(C.259)}$$

Here we require $C_{\mu\nu} \neq g_{\mu\nu}$, as otherwise F'_Δ will be identical to E. An equivalent representation of F'_Δ is

$$F'_\Delta(x) = -\frac{iC_{00}}{(2\pi)^3}\int d^4k\, e^{-ik\cdot x}\epsilon(k_0)\left[\frac{\delta(k_C^2 - m^2)}{k^2 + i\epsilon} + \frac{\delta(k^2)}{k_C^2 - m^2 + i\epsilon}\right] \qquad \text{(C.260)}$$

or, after performing the integration over k_0 in a preferred frame,

$$F'_\Delta(x) = -\frac{1}{(2\pi)^3}\left[\int_{k_0=\sqrt{-\frac{C_{kl}k^k k^l+m^2}{C_{00}}}} d^3k \right.$$

$$\left. + \; C_{00}\int_{k_0=|\mathbf{k}|} d^3k\right]\left[\frac{e^{i\mathbf{k}\cdot\mathbf{x}}\sin(k_0 x_0)}{k_0}\right] \qquad \text{(C.261)}$$

In this form the $x_0 = 0$ properties are readily found to be

$$F'_\Delta(x, m^2)|_{x_0=0} = 0 \qquad \text{(C.262a)}$$

$$\partial_0 F'_\Delta(x, m^2)|_{x_0=0} = -[1 + C_{00}]\delta(\mathbf{x}) \qquad \text{(C.262b)}$$

The definition of the causal Green function proceeds in the same way as that associated with Δ:

$$iF'_{\Delta,F}(x, m^2) = \theta(x_0)F'^{(+)}_\Delta(x, m^2) - \theta(x_0)F'^{(-)}_\Delta(x, m^2)$$

$$= \frac{iC_{00}}{(2\pi)^4}\int d^4k \frac{e^{-ik\cdot x}}{(k^2 + i\epsilon)(k_C^2 - m^2 + i\epsilon)} \qquad \text{(C.263)}$$

In the same way as that followed for E and Δ , we define

$$F'_E(x, m^2) = -\frac{\partial}{\partial m^2}F'_\Delta(x, m^2) \qquad \text{(C.264a)}$$

$$F'_{E,F}(x, m^2) = -\frac{\partial}{\partial m^2}F'_{\Delta,F}(x, m^2) \qquad \text{(C.264b)}$$

From these definitions, the $x_0 = 0$ properties in a preferred frame are readily found to be

$$F'_E(x, m^2)|_{x_0} = \partial_0 F'_E(x, m^2)|_{x_0} = 0 \qquad \text{(C.265)}$$

The following relations are also easily obtained.

$$\Box F'_\Delta = \Delta_C \qquad \text{(C.266a)}$$

$$(\Box_C + m^2)F'_\Delta = C_{00}\Delta \tag{C.266b}$$

$$(\Box_C + m^2)E_C = \Delta_C \tag{C.266c}$$

$$\Box F'_E = E_C \tag{C.266d}$$

$$(\Box_C + m^2)F'_E = F'_\Delta \tag{C.266e}$$

These relations also hold for the causal Green functions.

All these properties are a straightforward generalization to an arbitrary but hyperbolic $\Box_C$ of the particular case $\Box_C = n \cdot k\, n^* \cdot k$ which is treated in [34].

D Elliptic Evolution operator

As the Cauchy problem with an elliptic $\Box_C$ is not well-defined, a different approach must be used. The standard one is to impose the vanishing of the fields at infinity, thereby transforming the problem into a boundary value one. The solution of the field equations (4.177) is then

$$S = 0 = \partial \cdot A \tag{D.267}$$

which however is incompatible with the equal-time commutation relations of Eqs.(4.183). Thus, the canonical formulation must be abandoned.

Using Eq.(D.267), one sees, from Eq.(4.181), that A_ν obeys the free field equation

$$\Box A_\nu = 0 \tag{D.268}$$

Taking into account Eqs.(4.198,4.199) and Eq.(D.267), one finds the solution is

$$A_\nu = A^T_\nu \tag{D.269}$$

where A_ν^T is given by Eq.(4.205). Quantization can now be carried out by imposing the usual commutation relations.

This method leads to the same result as the usual quantization in the Coulomb gauge. However, in order to avoid the trouble of the Coulomb gauge in Yang-Mills theory, one can make the following observation. Except equal-time commutators, all the equations for elliptic $\Box_C$ can be obtained from those corresponding to hyperbolic $\Box_C$ by an analytic continuation of the parameters. Indeed, in the elliptic case, the equation $k_C^2 = 0$ has no real solution, so that $\delta(k_C^2) = 0$. This observation allows one to define the quantum theory for elliptic $\Box_C$ as the analytic continuation from the region where the gauge parameters define a hyperbolic operator to the region where they define an elliptic one.

In the Abelian case, one can also carry out the quantization by considering Eq.(D.267) as constraints and using Dirac brackets to get the equal-time commutators. The details of this analysis are as follows. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - C_{\mu\nu}\partial^\mu S A_\nu - \frac{a}{2}S^2 \tag{D.270}$$

while the canonical momenta are

$$\begin{aligned}
\pi^k &= \partial^k A^0 - \partial^0 A^k && \text{(D.271a)}\\
\pi^0 &= 0 && \text{(D.271b)}\\
\pi_S &= -C_{0\nu}A^\nu. && \text{(D.271c)}
\end{aligned}$$

There are two primary constraints and the total Hamiltonian density is

$$\begin{aligned}
H_T &= \frac{1}{2}\pi^k\pi^k - \pi^k\partial^k A^0 + \frac{1}{4}F_{kl}F^{kl}\\
&+ C_{k\nu}A^\nu\partial^k S + \frac{a}{2}S^2 + \pi^0\Lambda + M(\pi_S + C_{00}A\nu)
\end{aligned} \tag{D.272}$$

where the Lagrangian multipliers Λ and M are weakly equal to $\partial_0 A_0$ and $\partial_0 S$ respectively.

Using the usual Poisson brackets

$$\{\pi_{(i)}(x), \phi^{(j)}(y)\}_{x_0=y_0} = -\delta_i^j \delta(\mathbf{x}-\mathbf{y}), \tag{D.273}$$

one gets, by taking the Poisson brackets of the total Hamiltonian with the primary constraints, the relations

$$\partial^k \pi_k + C_{00} M + C_{k0} \partial^k S \approx 0 \tag{D.274a}$$

$$C_{k\nu} \partial^k A^\nu - aS - C_{0k}(\pi^k - \partial^k A^0) + \Lambda C_{00} \approx 0 \tag{D.274b}$$

which determine the Lagrange multipliers Λ and M and therefore stop the search for additional constraints. We have however two external constraints given by Eqs.(D.267). Imposing that $S \approx 0$ is time-independent leads to

$$M \approx 0 \tag{D.275}$$

while that of $\partial \cdot A \approx 0$ imposes

$$\Lambda \approx \partial_k A^k \tag{D.276}$$

These equations thus determine the Lagrange multipliers. Imposing that such determinations are compatible leads to the new constraints

$$\partial_k \pi^k \approx 0 \tag{D.277a}$$

$$(C_{k0} + C_{0k})\partial^k A^0 + C_{kl} \partial^k A^l - C_{0k} \pi^k + C_{00} \partial^k A_k \approx 0 \tag{D.277b}$$

which are to be treated as constraints in the usual way. However, because Eq.(D.277) is fairly complicated in general, we will now restrict ourselves to the case

$$C_{\mu\nu} = n_\mu n_\nu - \alpha g_{\mu\nu},\ 0 < \alpha < 1,\ n = (1,0,0,0) \tag{D.278}$$

In this case, the second relation of Eq.(D.277) reduces to

$$\partial_l A_l \approx 0 \tag{D.279}$$

Now, time independence of the first relation in Eq.(D.277) is trivially satisfied, while that of Eq.(D.279) imposes

$$\Delta A_0 \approx 0 \tag{D.280}$$

where Δ is the Laplacian. This then leads to the trivial solution

$$\Lambda \approx \partial_0 A_0 \approx \partial_l A_l \approx 0 \tag{D.281}$$

If we now take into account the 6 second-class constraints, the non-vanishing Dirac brackets are found to be

$$\{A_k(x), \pi^l(y)\}_{x_0=y_0} = \left[\delta_k^l - \frac{\partial_k \partial^l}{\Delta}\right] \delta(\mathbf{x} - \mathbf{y}) \tag{D.282}$$

which are of course the same as those obtained in the usual formulation of the Coulomb gauge treated as a Class I gauge. One should note that for the particular case of Eq.(D.278), the theory is α -independent for $0 < \alpha < 1$.

References

[1] H. Weyl, Z. Phys. **56**, 330 (1929).

[2] C. N. Yang and R. L. Mills, Phys. Rev. **69**, 191 (1954).

[3] A. Einstein, in *Collected Works*, **1**, Nzuka, Moscow (1966).

[4] T. T. Wu and C. N. Yang, Phys. Rev. **D12**, 3843 (1975).

[5] H. Georgi, in *Unified Theory of Elementary Particle Interactions*, p. 48, Scientific American (1981).

[6] P. W. Higgs, Phys. Lett. **12**, 132 (1964); Phys. Rev. **145**, 1156 (1966).

[7] I. J. R. Aitchison and A. J. G. Hey, in *Gauge Theories in Particle Physics*, Adam Hilger Ltd, Bristol (1982);
L. H. Ryder, in *Quantum Field Theories*, Cambridge University Press (1985).

[8] S. Weinberg, Phys. Rev. Lett. **19**, 1264 (1967);
A. Salam, in *Elementary Particle Theory*, edited by N. Svartholm, Almquist and Wiksells, Stockholm (1968);
S. Weinberg, Rev. Mod. Phys. **52**, 515 (1980);
A. Salam, Rev. Mod. Phys. **52**, 525 (1980).

[9] Y. Nambu, in *Preludes in Theoretical Physiccs in Honour of V. F. Weisskopf*, edited by A. De-Shalit, H. Feshbach, and L. Van Hove, North-Holland, Amsterdam, p.133 (1966).

[10] M. Gell-Mann, Phys. Lett. **8**, 214 (1963);
G. Zweig, (unpublished CERN report (1963).

[11] M. Gell-Mann and Y. Ne'eman, in *The Eightfold Way*, Benjamin, New York (1964).

[12] For recent overviews, see the Proceedings of the Workshop on Physical and non-Standard Gauges, Vienna, 1989 (Springer-Verlag, to appear);
G. Leibbrandt, Rev. Mod. Phys. **169**, 1067 (1987).

[13] V.F. Muller and W. Ruhl, Ann. Phys. (N. Y.) **133**, 240 (1981).

[14] H. S. Chan and M. B. Halpern, Phys. Rev. **D33**, 540 (1986);
P. V. Landshoff, Phys. Lett. **B169**, 69 (1986);
H. Cheng and E. C. Tsai, Phys. rev. Lett. **57**, 511 (1988);
F. Steiner, Phys. Lett. **173B**, 321 (1986);
A. Burnel, Phys. Rev. **D36**, 1852 (1987).

[15] J. D. Jackson, in *Classical Electrodynamics*, John Wiley, New York (1975).

[16] F. Halzen and G. L. Kane, in *Quarks and Leptons*, John Wiley and Sons (1984);
J. J. Sskurai, *Modern Quantum Mechanics*, Addison-Wesley Publishing Co. (1967);
J. D. Bjorken and S. D. Drell, in *Relativistic Quantum Mechanics*, McGraw-Hill, New York (1964).

[17] P. A. M. Dirac, in *Lectures on Quantum Mechanics,* Belfer Graduate School of Science, Yeshiva University, New Yord (1964);
A. Hanson, T. Regge and C. Teitelboim, in *Constrained Hamiltonian Systems,* Accademia Nazionale dei LIncei, Rome (1976).

[18] R. P. Feynman, Rev. Mod. Phys. **20**, 367 (1948);
R. P. Feynman and A. R. Hibbs, in *Quantum Mechanics and Path Integrals,* (McGraw-Hill, New York, 1965).

[19] L. D. Faddeev and V. N. Popov, Phys. Lett. **B25**, 29 (1967).

[20] T. Kugo and I. Ojima, Prog. Theor. Phys. Supp. **66**, 1 (1979).

[21] C. Beechi, A. Rouet, R. Stora, Phys. Lett. **52B**, 344 (1974);
M. Henneaux, Phys. Rep. **126**, 1 (1985).

[22] N. Nakanashi, Prog. Theor. Phys. **35**, 111 (1966);
N. Nakanashi, Prog. Theor. Phys. **49**, 640 (1973);
Lautrup, Kon. Dan. Vid. Selsk. Mat.-Fys.Med. **35**, (11), 1 (1967).

[23] R. Brandt, Nucl. Phys. **B116**, 414 (1976);
J. C. Taylor, in *Gauge Theories of Weak Interactions,* Cambridge University Press (1978).

[24] E. Nother, in *Nachr. Kgl. Ges. Wiss. Gottingen,* p.235 (1981).

[25] A. Burnel, R. Kobes, G. Kunstatter and K. Mak, Ann. Phys. (in press).

[26] A. Burnel, Phys. Rev. **D26**, 442 (1982).

[27] R. L. Arnowitt, S. I. Fickler, Phys. Rev. **127**, 1821 (1962).

[28] V.N. Gribov, Nucl. Phys. **B139**, 1 (1978).

[29] J. Schwinger, Phys. Rev. **127**, 324 (1962); **130**, 402 (1962);
N.H. Christ and T.D. Lee, Phys. Rev. **D22**, 939 (1980);
T.N. Tudron, Phys. Rev. **D21**, 2348 (1980);
H. Cheng and E.C. Tsai, Phys. Rev. Lett. **57**, 511 (1986).

[30] W. Kummer, Acta Phys. Austriaca, **14**, 149 (1961);
W. Kummer, Acta Phys. Austriaca, **41**, 315 (1975).

[31] N. Nakanishi, Prog. Theor. Phys. **67**, 65 (1982);
Phys. Lett. **B131**, 381 (1983).

[32] N. Bohr and L. Rosenfeld, Phys. Rev. **78**, 794 (1950).

[33] I.M. Singer, Comm. Math. Phys. **60**, 7 (1978).

[34] A. Burnel, Phys. Lett. **B227**, 431 (1989).

[35] A. Burnel, Nucl. Phys. **B198**, 531 (1982).

[36] H. Feschbach and F. Villars, Rev. Mod. Phys. **30**, 24 (1958).

[37] V.F. Müller and W. Rühl, Ann. Phys.(N.Y.) **133**, 240 (1981).

[38] S. Caracciolo, G. Curci and P. Menotti, Phys. Lett. **B131**, 311 (1981).

[39] G. Leibbrandt, Phys. Rev. **D29**, 1699 (1984);
S. Mandelstam, Nucl. Phys. **B213**, 149 (1983).

[40] I. Lazzizzera, Phys. Lett. **B210**, 188 (1988);
P.V. Landshoff, Phys. Lett. **B227**, 427 (1989).

[41] C. Becchi, A. Rouet and R. Stora, in *Renormalization Theory*, edited by G. Velo and A.S. Wightman (D. Reidel, Dordrecht, Holland, 1976).

[42] A. Burnel, Phys. Rev. **D40**, 1221 (1989).

[43] A. Burnel, Phys. Rev. **D32**, 450 (1985).

[44] J. Frenkel and J.C. Taylor, Nucl. Phys. **B109**, 439 (1976).

[45] A. Burnel, Phys. Rev. **D36**, 1852 (1987).

[46] S.L. Nyeo, Phys. Rev. **D36**, 2512 (1987).

[47] R. Kobes, G. Kunstatter, and K. Mak, Phys. Lett. **B223**, 433 (1989).

[48] A. Bassetto, I. Lazzizzera, G. Nardelli and R. Soldati, Phys. Lett. **B228**, 235 (1989).

[49] O. Piguet, G. Pollak and M. Schweda, Vienna TU-preprint 1989.

[50] A. Burnel, Phys. Rev. **D36**, 1846 (1987).

[51] N. Nakanishi, Phys. Rev. **D5**, 1324 (1971);
B. Lautrup, K. Dan. Vid. Selsk. Mat. Fys. Medd. **35**, 11 (1967);
A. Burnel and H. Caprasse, Nucl. Phys. **B5**, 93 (1968).

[52] M. Hans, Liège preprint 1985.

[53] P.A.M. Dirac, Proc. Roy. Soc. **A246**, 326 (1958);
L.D. Faddeev, Theor. and Math. Phys. **1**, 1 (1969).

[54] H. Hata and T. Kugo, Phys. Rev. **D21**, 3333 (1980).

[55] C. Bernard, Phys. Rev. **D9**, 3312 (1974) ;
D.J. Gross, R.D. Pisarski and L.G. Yaffee, Rev. Mod. Phys. **53**, 43 (1981).

[56] G. D. Hochschild, in *Basic Theory of Algebraic Groups and Lie Algebras,* Springer-Verlag (1984);
Z. X. Wan, in *Lie Algebras,* Pergamon Press (1975).

[57] C. Itzykson and J.-B. Zuber, in *Quantum Field Theory,* McGraw-Hill Inc. (1980).